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Perspectives

Hereditary classes of graphs: A parametric approach ☆

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ABSTRACT

The world of hereditary classes is rich and diverse and it contains a variety of classes of theoretical and practical importance. Thousands of results in the literature are devoted to individual classes and only a few of them analyse the universe of hereditary classes as a whole. To shift the analysis into a new level, in the present paper we exploit an approach, where we operate by infinite families of classes, rather than individual classes. Each family is associated with a graph parameter and is characterized by classes that are critical with respect to the parameter. In particular, we obtain a complete parametric description of the bottom of the lattice of hereditary classes and discuss a number of open questions related to this approach.

Contents

☆ Some results presented in this paper appeared in the extended abstract [32] published in the proceedings of the 28th International Workshop on Combinatorial Algorithms, IWOCA 2017.

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1. Introduction

We begin our journey to the land of hereditary classes with two short stories.

> In 1928, Frank Ramsey published the paper "On a problem of formal logic" [41], where he proved a result, which nowadays bears his name and is known as Ramsey's Theorem. He proved it parenthetically, as a minor lemma on the way to his true goal in the paper, solving a special case of the decision problem for first-order logic. In fact, this "minor" result turned out to be so fruitful that it led to the emergence of an entire branch of mathematics, known as Ramsey Theory.
>
> In 1977, Václav Chvátal and Peter Hammer published the paper "Aggregation of inequalities in integer programming" [18] and interpreted the main result of this paper in terms of graph theory and threshold logic. This interpretation became the origin of the notion of threshold graphs, which gave rise to an ocean of results and a vast literature, including a book devoted to this class of graphs [35].

An obvious moral uniting both stories is that the main result is not necessarily the most valued one. To reveal more connections between the two stories, we observe the following.

According to Ramsey, in the universe of hereditary classes there exist precisely two minimal classes containing graphs with an arbitrarily large number of vertices: complete graphs and edgeless graphs. The number of vertices is the main graph parameter and Ramsey's Theorem characterizes the family of hereditary classes where this parameter is bounded by means of two minimal obstructions, or "forbidden" elements, i.e. two minimal classes that do not belong to the family.

The number of vertices is the main graph parameter, but not the only one, and in some cases not the most valued one. With the advent of parameterized complexity, the world of graph parameters has been enriched with a myriad of new representatives, and many of them admit a Ramsey-type characterization in terms of minimal hereditary classes where these parameters are unbounded. In particular, as we shall see later, the class of threshold graphs is one of the nine minimal hereditary classes of unbounded neighbourhood diversity.

These examples explain the main idea of our approach. We explore the universe of hereditary classes by analysing families of classes where some graph parameters are bounded, and characterize these families by means of minimal forbidden elements, i.e. minimal classes that do not belong to the families. The idea of minimal forbidden elements for *families of classes* generalizes the idea of minimal forbidden induced subgraphs for individual classes. It is well-known (and not difficult to see) that a class of graphs is hereditary if and only if it admits a characterization in terms of minimal forbidden induced subgraphs, i.e. minimal graphs that do not belong to the class. To emphasize the importance of such a characterization, we present one more story.

> In 1969, Journal of Combinatorial Theory published the paper entitled "An interval graph is a comparability graph" [26]. One year later, the same journal published another paper entitled "An interval graph is not a comparability graph" [24]. Each of the two classes mentioned in these papers is hereditary, and for each of them, there exists a list of minimal forbidden induced subgraphs. Apparently, in 1969 this list was not available for at least one of them, because minimal forbidden induced subgraphs provide a simple way of comparing two classes.

Now let us shift our discussion from classes of graphs to *families of classes* and ask whether a family $\mathcal{A}$ of classes closed under taking subclasses can be characterized in terms of minimal classes that do not belong to $\mathcal{A}$. In a well-quasi-ordered world, the answer to this question is 'yes'. For instance, the graph minor relation is a well-quasi-order [43], and in the family of minor-closed classes of graphs the class of planar graphs constitutes a unique minimal class of unbounded tree-width [42]. The induced subgraph relation is not a well-quasi-order, because it contains infinite antichains of graphs, for instance, cycles. As a result, the lattice of hereditary classes contains infinite strictly descending chains of classes, and hence minimal classes outside of a particular family may not exist. In this case, we employ the notion of boundary classes. This is a relaxation of the notion of minimal classes and it is defined as follows.

Let $\mathcal{A}$ be a family of hereditary classes closed under taking subclasses, and let $X_1 \supseteq X_2 \supseteq X_3 \supseteq \dots$ be a chain of classes not in $\mathcal{A}$. The intersection of all classes in this chain is called a *limit* class and a minimal limit class is called a *boundary* class for $\mathcal{A}$. The importance of this notion is due to the fact that it plays the role of minimal forbidden elements for hereditary classes defined by *finitely many* forbidden induced subgraphs (*finitely defined* classes, for short). More precisely, a finitely defined class X belongs to $\mathcal{A}$ if and only if X contains none of the boundary classes for $\mathcal{A}$.

The notion of boundary classes was formally introduced by Alekseev in [6], where he proved basic results related to this notion. However, implicitly it appeared earlier. In particular, the famous Gyárfás–Sumner conjecture [17] can be stated in the terminology of boundary classes as follows: the class of forests is the only boundary class for the family of hereditary classes of bounded chromatic number. Notice that for this family there is one more obstruction, the class of complete graphs, which is a minimal hereditary class of unbounded chromatic number.

Together, minimal and boundary classes, are known as critical properties, and they provide a universal and uniform approach to the description of various families of classes. In particular, in [33] this approach was applied to hereditary classes of graphs of bounded tree-width, providing a complete dichotomy for finitely defined classes with respect to bounded/unbounded tree-width.

In the present paper, we apply this approach in order to provide a complete parametric description of several lower layers in the lattice of hereditary classes. This part of the universe consists of classes that are well-quasi-ordered by induced subgraphs. As a result, each of the lower layers admits a description in terms of minimal hereditary classes that do not belong to the layer. We obtain this description by means of a number of Ramsey-type results derived in Section 3. In Section 4, we show that our parametric characterization of the lower layers is consistent with the quantitative description of this part of the universe, in the sense that for every jump in the speed of hereditary classes, there is a parameter responsible for the jump. Our parametric description covers all hereditary classes with speeds up to the Bell number (Section 5). Beyond this level, only partial information is available, which we present in Section 6 together with some open problems and conjectures. All preliminary information related to the topic of the paper can be found in Section 2.

2. Graphs, classes of graphs and graph parameters

We consider only simple undirected graphs without loops and multiple edges and denote the vertex set and the edge set of a graph G by $V(G)$ and $E(G)$, respectively. If v is a vertex of G, then $N(v)$ is its *neighbourhood*, i.e. the set of vertices of G adjacent to v. The *closed neighbourhood* of v is defined and is denoted as $N[v] = N(v) \cup \{v\}$. The *degree* of v is $|N(v)|$.

Given a graph G and a subset $U \subseteq V(G)$, we denote by $G[U]$ the subgraph of G induced by U, i.e. the subgraph obtained from G by deleting all the vertices not in U. We say that a graph G contains a graph H as an induced subgraph if H is isomorphic to an induced subgraph of G. If G contains no induced subgraphs isomorphic to H, then we say that G is H-free and call H a forbidden induced subgraph for G. As usual, we denote by K_n a complete graph with n vertices and by P_n a chordless path with n vertices. By nG we denote the disjoint union of n copies of a graph G.

For a graph G, we denote by $\overline{G}$ the complement of G. Similarly, for a set X of graphs, we denote by $\overline{X}$ the set of complements of graphs in X. A graph G is said to be *n-universal* for a set X of graphs if G contains all n-vertex graphs from X as induced subgraphs.

In a graph,

- an *independent set* is a set of vertices no two of which are adjacent,
- a *clique* is a set of vertices every two of which are adjacent,
- a *matching* is a set of edges no two of which share a vertex.

A *class of graphs* is a set of graphs closed under isomorphism. A class is *hereditary* if it is closed under taking induced subgraphs. Two hereditary classes of particular interest in this paper are split graphs and bipartite graphs.

A graph G is a *split* graph if $V(G)$ can be partitioned into an independent set and a clique, and G is *bipartite* if $V(G)$ can be partitioned into two independent sets. A bipartite graph G given together with a bipartition of its vertices into independent sets A and B will be denoted $G = (A, B, E)$, in which case we will say that A and B are the colour classes or simply the parts of G. If every vertex of A is adjacent to every vertex of B, then $G = (A, B, E)$ is *complete bipartite*, also known as a *biclique*. A *star* is a complete bipartite graph with one of the colour classes being of size 1. The star with the second colour class being of size n is denoted $K_{1,n}$. Throughout the paper we denote by $2K_{1,n}$ the bipartite graph in which the centres of the two stars belong to the *same* part of its bipartition and call any graph of this form a *double star*.

The *bipartite complement* of a bipartite graph $G = (A, B, E)$ is the bipartite graph $G^{bc} = (A, B, E')$, where $ab \in E'$ if and only if $ab \notin E$ for any $a \in A$ and $b \in B$. Observe that the bipartite complement of $2K_{1,n}$ is $2K_{1,n}$ again. The *bi-codegree* of a vertex v in a bipartite graph $G = (A, B, E)$ is the degree of v in the bipartite complement of G. Clearly, by creating a clique in one of the colour classes of a bipartite graph, we transform it into a split graph, and vice versa.

Given a graph G and two disjoint subset A, B of its vertices, we denote by $G[A, B]$ the bipartite (not necessarily induced) subgraph with colour classes A and B formed by the edges of G between A and B.

A graph G is *complete multipartite* if the complement of G is a disjoint union of cliques. It is well-known (and not difficult to see) that G is complete multipartite if and only if $\overline{G}$ is P_3-free.

For a graph G,

- the *independence number* $\alpha(G)$ of G is the size of a maximum independent set in G,
- the *clique number* $\omega(G)$ of G is the size of a maximum clique in G,
- the *matching number* $\mu(G)$ of G is the size of a maximum matching in G,
- the *chromatic number* of G is the minimum number of sets in a partition of $V(G)$ into independent sets,
- the *co-chromatic number* of G is the minimum number of sets in a partition of $V(G)$ such that every subset is either a clique or an independent set,
- the *biclique number* of a graph G is the size of a maximum complete bipartite (not necessarily induced) subgraph of G with equal parts.

More graph classes and graph parameters will be introduced in the subsequent sections.

3. Graph parameters and Ramsey theory

In 1930, a 26 years old British mathematician Frank Ramsey proved the following theorem, known nowadays as Ramsey's Theorem.

Theorem 1 ([41]). *For any positive integers k, r, p, there exists a minimum positive integer $F = F(k, r, p)$ such that if the k-subsets of an F-set are coloured with r colours, then there is a monochromatic p-set, i.e. a p-set all of whose k-subsets have the same colour.*

We will refer to the number $F(k, r, p)$ defined in this theorem as the Frank Ramsey number.

It is not difficult to see that with $k = 1$ Ramsey's theorem coincides with Pigeonhole Principle. For $k = 2$, the theorem admits an interpretation in the terminology of graph theory, since colouring 2-subsets can be viewed as colouring the edges of a complete graph: for any positive integers r and p, there is a positive integer $n = n(r, p)$ such that if the edges of an n-vertex complete graph are coloured with r colours, then there is a monochromatic clique of size p, i.e. a clique all of whose edges have the same colour.

In the case of $r = 2$ colours, the graph-theoretic interpretation of Ramsey's Theorem can be further rephrased as follows.

Theorem 2. *For any positive integer p, there is a minimum positive integer $R(p)$ such that every graph with at least $R(p)$ vertices has either a clique of size p or an independent set of size p.*

The number $R(p)$ is known as the *symmetric Ramsey number*. In other words, $R(p) = F(2, 2, p)$, where $F(k, r, p)$ is the Frank Ramsey number. Theorem 2 also admits a non-symmetric formulation as follows.

Theorem 3. *For any positive integers p and q, there is a minimum positive integer $R(p, q)$ such that every graph with at least $R(p, q)$ vertices has either a clique of size p or an independent set of size q.*

The number $R(p, q)$ is known as the *Ramsey number* $R(p, q)$. In particular, $R(p, p) = R(p)$.

Theorem 3 (or Theorem 2) allows us to make the following conclusion: if X is a hereditary class that does not contain a complete graph K_p and an edgeless graph $\overline{K}_q$, then graphs in X have fewer than $R(p, q)$ vertices, i.e. X is finite. More formally, Theorem 3 implies the following conclusion.

Theorem 4. *The class of complete graphs and the class of edgeless graphs are the only two minimal infinite hereditary classes of graphs.*

On the other hand, it is not difficult to see that the reverse is also true: Theorem 4 implies Theorem 3. In other words, these two theorems are equivalent.

Theorem 4 characterizes the family of hereditary classes containing graphs with a bounded number of vertices in terms of minimal "forbidden" elements, i.e. minimal classes with an arbitrarily large number of vertices. It turns out that various other parameters admit a similar characterization. For instance, directly from Ramsey's Theorem it follows that

- the class of complete graphs and the class of stars (and all their induced subgraphs) are the only two minimal hereditary classes of graphs of unbounded vertex degree,
- the class of complete graphs and the class of complete bipartite graphs are the only two minimal hereditary classes of graphs of unbounded biclique number.

One more example of this type was obtained in [17] and deals with the *splitness* of a graph G, which is the minimum k such that $V(G)$ can be partitioned into two subsets A and B with $\alpha(G[A]) \le k$ and $\omega(G[B]) \le k$. According to a result proved in [17], the class of complete multipartite graphs and the class of their complements are the only two minimal hereditary classes of graphs of unbounded splitness.

In the next section, we use Ramsey-type arguments in order to characterize several other graph parameters by means of minimal hereditary classes where these parameters are unbounded. In addition to the original Ramsey's Theorem we will need its bipartite version, which can be stated as follows.

Theorem 5. *For every positive integer s, there is a positive integer $n = n(s)$ such that every bipartite graph G with at least n vertices in each part contains either a biclique with colour classes of size s or its bipartite complement.*

3.1. Independence number, clique number and complex number

We repeat that $\alpha(G)$ stands for the size of a maximum independent set in G, and $\omega(G)$ for the size of a maximum clique in G. Now we introduce one more parameter:

$c(G) = \min(\alpha(G), \omega(G))$ is the *complex number* of G.

In what follows we give a Ramsey-type characterization of this parameter, i.e. we characterize it in terms of minimal hereditary classes where this parameter is unbounded. To this end, let us denote by

$\mathcal{S}$ the class of graphs partitionable into a clique and a set of isolated vertices. Also, let S_n be a graph in $\mathcal{S}$ with a clique of size n and a set of isolated vertices of size n. Obviously, S_n is n-universal for graphs in $\mathcal{S}$, i.e. it contains every n-vertex graph from $\mathcal{S}$ as an induced subgraph.

Theorem 6. *$\mathcal{S}$ and $\overline{\mathcal{S}}$ are the only two minimal hereditary classes of graphs of unbounded complex number.*

Proof. Obviously, the complex number of graphs in $\mathcal{S}$ and $\overline{\mathcal{S}}$ can be arbitrarily large. Conversely, let X be a hereditary class containing a graph G with $c(G) \geq k$ for each value of k. Then G contains a clique C of size k and an independent set I of size k. Since C and I have at most one vertex in common, we may assume without loss of generality that they are disjoint, and since k can be arbitrarily large, in the bipartite graph $G[C, I]$ we can find an arbitrarily large biclique or its bipartite complement (Theorem 5). Therefore, graphs in X contain either S_n or $\overline{S}_n$ for arbitrarily large values of n, i.e. X contains either $\mathcal{S}$ or $\overline{\mathcal{S}}$. □

3.2. Degree, co-degree and complex degree

Let G be a graph and v a vertex of G. We denote by $d(v)$ the degree of v and by $\overline{d}(v)$ the *co-degree* of v, i.e. the degree of v in the complement of G. The c-degree of v is denoted and defined as follows: $cd(v) = \min(d(v), \overline{d}(v))$.

As usual, $\Delta(G)$ is the maximum vertex degree in G. Also, we denote by $\overline{\Delta}(G)$ the maximum co-degree and by $c\Delta(G)$ the maximum c-degree in G. We call $c\Delta(G)$ the *complex degree* of G. In order to characterize this parameter by means of minimal hereditary classes where complex degree is unbounded, let us denote by

- $\mathcal{Q}$ the class of graphs whose vertices can be partitioned into a set inducing a star and a set of isolated vertices. Also, let Q_n be a graph in $\mathcal{Q}$ whose vertices can be partitioned into an induced star $K_{1,n}$ and a set of isolated vertices of size n. Obviously, Q_n is n-universal for graphs in $\mathcal{Q}$, i.e. it contains every n-vertex graph from $\mathcal{Q}$ as an induced subgraph.
- $\mathcal{B}$ the class of complete bipartite graphs (an edgeless graph is counted as complete bipartite with one part being empty). For consistency of notation with previously defined classes, we will denote a biclique (complete bipartite graph) with n vertices in each part of its bipartition by B_n. Clearly, B_n is n-universal for graphs in $\mathcal{B}$, i.e. it contains all n-vertex graphs in $\mathcal{B}$ as induced subgraphs.

Theorem 7. *$\mathcal{S}$, $\overline{\mathcal{S}}$, $\mathcal{Q}$, $\overline{\mathcal{Q}}$, $\mathcal{B}$ and $\overline{\mathcal{B}}$ are the only minimal hereditary classes of graphs of unbounded complex degree.*

Proof. Obviously, the complex degree of graphs in $\mathcal{S}$, $\overline{\mathcal{S}}$, $\mathcal{Q}$, $\overline{\mathcal{Q}}$, $\mathcal{B}$ and $\overline{\mathcal{B}}$ can be arbitrarily large. Conversely, let X be a hereditary class containing a graph G with $c\Delta(G) \geq k$ for each value of k. Then G contains a vertex v such that $d(v) \geq k$ and $\overline{d}(v) \geq k$. Let A be the set of neighbours of v and B the set of its non-neighbours. Since both A and B can be arbitrarily large, each of them contains either a big clique or a big independent set (Theorem 3), and $G[A, B]$ contains either a big biclique or its bipartite complement (Theorem 5). Therefore, graphs in X contain either S_n or $\overline{S}_n$ or Q_n or $\overline{Q}_n$ or B_n or $\overline{B}_n$ for arbitrarily large values of n. As a result, X contains at least one of $\mathcal{S}$, $\overline{\mathcal{S}}$, $\mathcal{Q}$, $\overline{\mathcal{Q}}$, $\mathcal{B}$ and $\overline{\mathcal{B}}$. □

3.3. Matching number, co-matching number and c-matching number

We repeat that the *matching number* $\mu(G)$ of a graph G is the size of a maximum matching in G. Also, we define the *co-matching number* of G to be the size of a maximum matching in the complement of G and denote it by $\overline{\mu}(G)$. The *c-matching number* of G is defined and denoted as follows: $c\mu(G) = \min(\mu(G), \overline{\mu}(G))$. In this section, we characterize all three parameters in terms of minimal hereditary classes where these parameters are unbounded. To this end, we denote by

- $\mathcal{M}$ the class of graphs of vertex degree at most 1. By M_n we denote an induced matching of size n, i.e. the up to an isomorphism unique graph from this class with $2n$ vertices each of which has degree 1. Clearly, M_n is n-universal for graphs in $\mathcal{M}$.
- $\mathcal{Z}$ the class of chain graphs. These are bipartite graphs in which the vertices in each colour class can be linearly ordered under inclusion of their neighbourhoods, i.e. the neighbourhoods form a chain. By Z_n we denote a chain graph such that for each $i \in \{1, 2, \ldots, n\}$, each part of the graph contains exactly one vertex of degree i (this graph is also known as a half-graph, see e.g. [20]). Fig. 1 represents the graph Z_n for $n = 5$. It is known [34] that Z_n is n-universal for graphs in $\mathcal{Z}$.

Lemma 1. *For any two positive integers s, t, there exists a positive integer $q = q(s, t)$ such that every bipartite graph G with a matching of size q contains either an induced M_s or an induced B_t.*

Proof. Let us denote $m = 2\max(s, t)$ and $q = F(2, 4, m)$, where $F(k, r, p)$ is the Frank Ramsey number (Theorem 1). Consider a matching $M = \{x_1y_1, \ldots, x_qy_q\}$ of size q. To each pair x_iy_i, x_jy_j $(i < j)$ of edges of M we assign one of the four colours as follows:

- colour 1 if G contains no edges between x_iy_i and x_jy_j,
- colour 2 if G contains both possible edges between x_iy_i and x_jy_j,
- colour 3 if G contains the edge x_iy_j but not the edge y_ix_j,
- colour 4 if G contains the edge y_ix_j but not the edge x_iy_j.

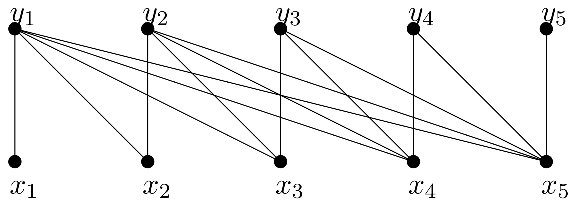


Fig. 1. The graph Z_5.

By Ramsey's Theorem, M contains a monochromatic set M' of edges of size m. If the colour of each pair in M' is

1, then M' is an induced matching of size $m \geq 2s > s$,
2, then the vertices of M' induce a complete bipartite graph B_m with $m \geq 2t > t$,
3 or 4, then the vertices of M' induce a Z_m and hence G contains a complete bipartite graph $B_{m/2}$ with $m/2 \geq t$. □

Lemma 2. *For any positive integers s, t, p, there exists a positive integer $Q = Q(s, t, p)$ such that every graph G with a matching of size Q contains either an induced M_s or an induced B_t or a clique K_p.*

Proof. Let us denote $Q = R(p, R(p, q))$, where R is the (non-symmetric) Ramsey number and $q = F(2, 4, 2\max(s, t))$ is the value defined in the proof of Lemma 1. We consider a matching M of size Q in G and colour the endpoints of each edge of M in two colours, say white and black, arbitrarily. Since the set of white vertices has size Q, it must contain either a clique K_p, in which case we are done, or an independent set A of size $R(p, q)$. In the latter case, we look at the black vertices matched with the vertices of A. According to the size of this set, it must contain either a clique K_p, in which case we are done, or an independent set A' of size q. In the latter case, we denote by A'' the set of white vertices matched with the vertices of A'. Then A' and A'' induce a bipartite graph with a matching of size q, in which case, by Lemma 1, G contains either an induced matching of size s or an induced complete bipartite graph B_t. □

The above sequence of results allows us to make the following conclusions, the first two of which follow directly from Lemma 2.

Theorem 8. *$\mathcal{M}$, $\mathcal{B}$ and the class of complete graphs are the only three minimal hereditary classes of graphs of unbounded matching number.*

Theorem 9. *$\overline{\mathcal{M}}$, $\overline{\mathcal{B}}$ and the class of edgeless graphs are the only three minimal hereditary classes of graphs of unbounded co-matching number.*

Theorem 10. *$\mathcal{M}$, $\mathcal{B}$, $\mathcal{S}$, $\overline{\mathcal{M}}$, $\overline{\mathcal{B}}$ and $\overline{\mathcal{S}}$ are the only six minimal hereditary classes of graphs of unbounded c-matching number.*

Proof. Clearly, graphs in $\mathcal{M}$, $\mathcal{B}$, $\mathcal{S}$, $\overline{\mathcal{M}}$, $\overline{\mathcal{B}}$ and $\overline{\mathcal{S}}$ can have arbitrarily large c-matching number. Conversely, let X be a hereditary class with unbounded c-matching number. Assume X contains none of $\mathcal{M}$, $\mathcal{B}$, $\overline{\mathcal{M}}$, $\overline{\mathcal{B}}$, i.e. there is a value of p such that none of M_p, B_p, $\overline{M}_p$, $\overline{B}_p$ belongs to X. By assumption, X contains a graph G with $c\mu(G) \geq k$ for each value of k, i.e. G contains a matching and a co-matching of size k. Since k can be arbitrarily large and both M_p and B_p are forbidden, G contains a large clique (Lemma 2). Similarly, G contains a large independent set. Therefore, X contains graphs with arbitrarily large complex number. But then X contains either $\mathcal{S}$ or $\overline{\mathcal{S}}$ (Theorem 6). □

3.4. Neighbourhood diversity

The neighbourhood diversity of a graph was introduced in [31]. This parameter can be defined as follows.

Definition 1. We say that two vertices x and y are *similar* if there is no vertex z distinguishing them, i.e. if there is no vertex z adjacent to exactly one of x and y. Clearly, the similarity is an equivalence relation. We denote by $nd(G)$ the number of similarity classes in G and call it the *neighbourhood diversity* of G.

In order to characterize the neighbourhood diversity by means of minimal hereditary classes of graphs where this parameter is unbounded, we denote by

- $\mathcal{M}^{bc}$ the class of bipartite complements of graphs in $\mathcal{M}$. The bipartite complement of the graph M_n will be denoted M_n^{bc}. Clearly, M_n^{bc} is n-universal for graphs in $\mathcal{M}^{bc}$.
- $\mathcal{M}^*$ the class of split graphs obtained from graphs in $\mathcal{M}$ by creating a clique in one of the colour classes. The graph obtained from M_n by creating a clique in one of its colour classes will be denoted by M_n^*. Clearly, M_n^* is n-universal for graphs in $\mathcal{M}^*$.

$\mathcal{Z}^*$ the class of split graphs obtained from graphs in $\mathcal{Z}$ by creating a clique in one of the colour classes. This class is known in the literature as the class of *threshold graphs*. The graph obtained from Z_n by creating a clique in one of its colour classes will be denoted Z_n^*. This graph is n-universal for threshold graphs [25].

Before we provide a Ramsey-type characterization of the neighbourhood diversity, we introduce an auxiliary parameter.

Definition 2. A *skew matching* in a graph G is a matching $\{x_1y_1, \dots, x_qy_q\}$ such that y_i is not adjacent to x_j for all $i < j$. The *complement of a skew matching* is a sequence of pairs of vertices that create a skew matching in the complement of G.

Lemma 3. *For any positive integer m, there exists a positive integer $r = r(m)$ such that any bipartite graph $G = (A, B, E)$ of neighbourhood diversity r contains either a skew matching of size m or its complement.*

Proof. Define $r = 2^{2m}$ and let D be a set of pairwise non-similar vertices of size $r/2$ chosen from the same colour class of G, say from A. Let y_1 be a vertex in B distinguishing the set D (i.e. y_1 has both a neighbour and a non-neighbour in D) and let us say that y_1 is *big* if the number of its neighbours in D is larger than the number of its non-neighbours in D, and *small* otherwise. If

y_1 is small, we arbitrarily choose its neighbour in D, denote it by x_1 and remove all neighbours of y_1 from D.
y_1 is big, we arbitrarily choose a non-neighbour of y_1 in D, denote it by x_1 and remove all non-neighbours of y_1 from D.

Observe that y_1 does not distinguish the vertices in the updated set D.

We apply the above procedure to the set D $2m-1$ times and obtain in this way a sequence of $2m-1$ pairs x_iy_i. If m of these pairs contain small vertices y_i, then these pairs create a skew matching (of size m). Otherwise, there is a set of m pairs containing big vertices y_i, in which case these pairs create the complement of a skew matching. □

Lemma 4. *For any positive integer p, there exists a positive integer $q = q(p)$ such that any bipartite graph $G = (A, B, E)$ of neighbourhood diversity q contains either an induced M_p or an induced Z_p or an induced M_p^{bc}.*

Proof. Let $m = R(p + 1)$ (where R is the symmetric Ramsey number) and $q = 2^{2m}$. According to the proof of Lemma 3, G contains a skew matching of size m or its complement. If G contains a skew matching M, we colour each pair x_iy_i, x_jy_j $(i < j)$ of edges of M in two colours as follows:

- colour 1 if x_i is not adjacent to y_j,
- colour 2 if x_i is adjacent to y_j.

By Ramsey's Theorem, M contains a monochromatic set M' of edges of size $p + 1$. If the colour of each pair of edges in M' is

1, then M' is an induced matching of size $p + 1$,
2, then the vertices of M' induce a Z_{p+1}.

Analogously, in the case when G contains the complement of a skew matching, we find either an induced M_{p+1}^{bc} or an induced Z_p (observe that the bipartite complement of Z_{p+1} contains an induced Z_p). □

Lemma 5. *For any positive integer p, there exists a positive integer $Q = Q(p)$ such that every graph G of neighbourhood diversity Q contains one of the following nine graphs as an induced subgraph: M_p, M_p^{bc}, Z_p, $\overline{M}_p$, $\overline{M}_p^{bc}$, $\overline{Z}_p$, M_p^*, $\overline{M}_p^*$, Z_p^*.*

Proof. Let $Q = R(q)$, where $q = 2^{2m}$ and $m = R(R(p) + 1)$ (R is the symmetric Ramsey number). We choose one vertex from each similarity class of G and find in the chosen set a subset A of vertices that form an independent set or a clique of size $q = 2^{2m}$. Let us call the vertices of A white. We denote the remaining vertices of G by B and call them black. Let $G' = G[A, B]$. By the choice of A, all vertices of this set have pairwise different neighbourhoods in G'. Therefore, according to the proof of Lemma 4, G' contains a subgraph G'' inducing either M_n, or M_n^{bc} or Z_n with $n = R(p)$. Among the n black vertices of G'', we can find a subset B' of vertices that form either a clique or an independent set of size p in the graph G. Then B' together with a subset of A of size p induce in G one of the nine graphs listed in the statement of the theorem. □

Since the nine graphs of Lemma 5 are universal for their respective classes, we make the following conclusion.

Theorem 11. *There exist exactly nine minimal hereditary classes of graphs of unbounded neighbourhood diversity: $\mathcal{M}$, $\mathcal{M}^{bc}$, $\mathcal{Z}$, $\overline{\mathcal{M}}$, $\overline{\mathcal{M}}^{bc}$, $\overline{\mathcal{Z}}$, $\mathcal{M}^*$, $\overline{\mathcal{M}}^*$, $\mathcal{Z}^*$.*

3.5. VC-dimension

A set system (X, S) consists of a set X and a family S of subsets of X. A subset $A \subseteq X$ is *shattered* if for every subset $B \subseteq A$ there is a set $C \in S$ such that $B = A \cap C$. The VC-dimension of (X, S) is the cardinality of a largest shattered subset of X.

The VC-dimension of a graph $G = (V, E)$ was defined in [8] as the VC-dimension of the set system (V, S), where S the family of closed neighbourhoods of vertices of G, i.e. $S = \{N[v] \ : \ v \in V(G)\}$. Let us denote the VC-dimension of G by $vc[G]$.

In this section, we characterize VC-dimension by means of three minimal hereditary classes where this parameter is unbounded. To this end, we first redefine it in terms of open neighbourhoods as follows. Let $vc(G)$ be the size of a largest set A of vertices of G such that for any subset $B \subseteq A$ there is a vertex v *outside of* A with $B = A \cap N(v)$. In other words, $vc(G)$ is the size of a largest subset of vertices shattered by *open* neighbourhoods of vertices of G.

We start by showing that the two definitions are equivalent in the sense that they both are either bounded or unbounded in a hereditary class. To prove this, we introduce the following terminology. Let A be a set of vertices that is shattered by a collection of neighbourhoods (open or closed). For a subset $B \subseteq A$ we will denote by $v(B)$ the vertex whose neighbourhood (open or closed) intersects A at B. We will say that B is *closed* if $v(B)$ belongs to B, and *open* otherwise.

Lemma 6. $vc(G) \leq vc[G] \leq vc(G)(vc(G) + 1) + 1$

Proof. The first inequality is obvious. To prove the second one, let A be a subset of $V(G)$ of size $vc[G]$ that is shattered by a collection of closed neighbourhoods. If A has no closed subsets, then $vc[G] = vc(G)$. Otherwise, let B be a closed subset of A.

Assume first that $|B| = 1$. Then $B = \{v(B)\}$ and $v(B)$ is isolated in $G[A]$, i.e. it has no neighbours in A. Let C be the set of all such vertices, i.e. the set of vertices each of which is a closed subset of A. By removing from A any vertex $x \in C$ we obtain a new set A and may assume that it has no closed subsets of size 1. Indeed, for any vertex $y \in C$ different from x, there must exist a vertex $y' \notin A$ such that $N(y') \cap A = \{x, y\}$ (since A is shattered). After the removal of x from A, we have $N(y') \cap A = \{y\}$ and hence $\{y\}$ is not a closed subset anymore. This discussion allows us to assume in what follows that A has no closed subsets of size 1, in which case we only need to show that $vc[G] \leq vc(G)(vc(G) + 1)$.

Assume now that B is a closed subset of A of size at least 2. Suppose that $B - v(B)$ contains a closed subset C, i.e. $v(C) \in C$. Observe that $v(C)$ is adjacent to $v(B)$, as every vertex of $B - v(B)$ is adjacent to $v(B)$. But then $N[v(C)] \cap A$ contains $v(B)$ contradicting the fact that $N[v(C)] \cap A = C$. This contradiction shows that every subset of $B - v(B)$ is open, i.e. $|B - v(B)| \leq vc(G)$.

The above observation allows us to apply the following procedure: as long as A contains a closed subset B with at least two vertices, delete from A all vertices of B except for $v(B)$. Denote the resulting set by A^*. Assume the procedure was applied p times and let $B_1, \dots, B_p$ be the closed subsets it was applied to. It is not difficult to see that the set $\{v(B_1), \dots, v(B_p)\}$ has no closed subsets and hence its size cannot be larger than $vc(G)$, i.e. $p \leq vc(G)$. Combining, we conclude:

$$vc[G] = |A| \leq |A^*| + \sum_{i=1}^{p} |B_i - v(B_i)| \leq vc(G) + p \cdot vc(G) \leq vc(G)(vc(G) + 1). \quad \square$$

This lemma allows us to assume that if A is shattered, then there is a set C *disjoint* from A such that for any subset $B \subseteq A$ there is a vertex $v \in C$ with $B = A \cap N(v)$, in which case we will say that A is shattered by C, or C shatters A.

Let $W_n = (A, B, E)$ be the bipartite graph with $|A| = n$ and $|B| = 2^n$ such that all vertices of B have pairwise different neighbourhoods in A. Also, let D_n be the split graph obtained from W_n by creating a clique in A.

Lemma 7. *The graph W_n is an n-universal bipartite graph, i.e. it contains every bipartite graph with n vertices as an induced subgraph.*

Proof. Let G be a bipartite graph with n vertices and with parts A and B of size n_1 and n_2, respectively. By adding at most n_2 vertices to A, we can guarantee that all vertices of B have pairwise different neighbourhoods in A. Clearly, W_n contains the extended graph and hence it also contains G as an induced subgraph. $\square$

Corollary 1. *Every co-bipartite graph with at most n vertices is contained in $\overline{W_n}$ and every split graph with at most n vertices is contained in both D_n and $\overline{D}_n$.*

Lemma 8. *If a set A shatters a set B with $|B| = 2^n$, then B shatters a subset A^* of A with $|A^*| = n$.*

Proof. Without loss of generality we assume that B is the set of all binary sequences of length n. Then every vertex $a \in A$ defines a Boolean function of n variables (the neighbourhood of a consists of the binary sequences, where the function takes value 1). For each $i = 1, \dots, n$, let us denote by a_i the Boolean function such that $a_i(x_1, \dots, x_n) = 1$ if and only if $x_i = 1$. Let A' be an arbitrary subset of $A^* = \{a_1, \dots, a_n\}$ and $\alpha = (\alpha_1, \dots, \alpha_n)$ its characteristic vector, i.e. $\alpha_i = 1$ if and only if $a_i \in A'$. Clearly, $\alpha \in B$ and $N(\alpha) \cap A^* = A'$. Therefore, B shatters A^*. $\square$

Lemma 9. *For every n, there exists a $k = k(n)$ such that every graph G with $vc(G) = k$ contains one of $W_n, \overline{W}_n, D_n, \overline{D}_n$ as an induced subgraph.*

Proof. Define $k = R(2^{R(n)})$, where R is the symmetric Ramsey number. Since $vc(G) = k$, there are two subsets A and B of $V(G)$ such that $|A| = k$ and B shatters A. By definition of k, A must have a subset A' of size $2^{R(n)}$ which is a clique or an independent set. Clearly, B shatters A' and hence, by Lemma 8, A' shatters a subset B' of B of size $R(n)$. Then B' must have a subset B'' of size n which is either a clique or an independent set. Now $G[A' \cup B'']$ is either bipartite or co-bipartite or split graph, $|B''| = n$ and A' shatters B''. Therefore, $G[A' \cup B'']$ contains one of $W_n, \overline{W}_n, D_n, \overline{D}_n$ as an induced subgraph. □

Theorem 12. *The classes of bipartite, co-bipartite and split graphs are the only three minimal hereditary classes of graphs of unbounded VC-dimension.*

Proof. Clearly these three classes have unbounded VC-dimension, since they contain $W_n, \overline{W}_n, D_n, \overline{D}_n$ with arbitrarily large values of n.

Now let X be a hereditary class containing none of these three classes. Therefore, there is a bipartite graph G_1, a co-bipartite graph G_2 and a split graph G_3 which are forbidden for X. Denote by n the maximum number of vertices in these graphs.

Assume that VC-dimension is not bounded for graphs in X and let $G \in X$ be a graph with $vc(G) = k$, where $k = k(n)$ is from Lemma 9. Then G contains one of $W_n, \overline{W}_n, D_n, \overline{D}_n$, say W_n. Since W_n is n-universal (Lemma 7), it contains G_1 as an induced subgraph, which is impossible because G_1 is forbidden for graphs in X. This contradiction shows that VC-dimension is bounded in the class X. □

4. Graph parameters and the speed of hereditary classes

Given a hereditary class X, we denote by X_n the number of n-vertex labelled graphs in X and call it the *speed* of X.

According to Ramsey's Theorem, there exist precisely two minimal infinite hereditary classes: the complete graphs and the edgeless graphs. In both cases, the speed is obviously $X_n = 1$. On the other extreme, lies the set of all simple graphs, in which case the speed is $X_n = 2^{\binom{n}{2}}$. Between these two extremes, there are uncountably many other hereditary classes and their speeds have been extensively studied, originally in the special case of a single forbidden subgraph, and more recently in general. For example, Erdős et al. [23] and Kolaitis et al. [29] studied K_r-free graphs, Erdős et al. [21] studied classes where a single graph is forbidden as a subgraph (not necessarily induced), and Prömel and Steger obtained a number of results [38–40] for classes defined by a single forbidden induced subgraph. This line of research culminated in a breakthrough result stating that for every hereditary class X different from the set of all finite graphs, the entropy $\lim_{n\to\infty} \frac{\log_2 X_n}{\binom{n}{2}}$ satisfies

$$\lim_{n\to\infty} \frac{\log_2 X_n}{\binom{n}{2}} = 1 - \frac{1}{k(X)}, \tag{1}$$

where $k(X)$ is a positive integer called the *index* of X. To define this notion, let us denote by $\mathcal{E}_{i,j}$ the class of graphs whose vertices can be partitioned into at most i independent sets and j cliques. In particular, $\mathcal{E}_{2,0}$ is the class of bipartite graphs, $\mathcal{E}_{0,2}$ is the class of co-bipartite (i.e. complements of bipartite) graphs and $\mathcal{E}_{1,1}$ is the class of split graphs. Then $k(X)$ is the largest k such that X contains $\mathcal{E}_{i,j}$ for some i, j with $i + j = k$. This result was obtained independently by Alekseev [4] and Bollobás and Thomason [15,16] and is known nowadays as the Alekseev–Bollobás–Thomason Theorem (see e.g. [7]). This theorem shows that the set of possible values for the entropy is not continuous, but in fact undergoes a series of discrete 'jumps'. In particular, the entropy jumps from 0 to 1/2.

A systematic study of hereditary classes of low speed (0 entropy) was initiated by Scheinerman and Zito in [44]. In particular, they revealed that the speed jumps

- from constant (classes X with $X_n = \Theta(1)$) to polynomial ($X_n = n^{\Theta(1)}$),
- from polynomial to exponential ($X_n = 2^{\Theta(n)}$) and
- from exponential to factorial ($X_n = n^{\Theta(n)}$).

Independently, similar results have been obtained by Alekseev in [5]. Moreover, Alekseev described the set of minimal classes in all the four lower layers and the asymptotic structure of classes in the first three of them.

In this section, we complement this line of research by identifying graph parameters responsible for the above mentioned jumps. In particular, Theorem 12 together with Alekseev–Bollobás–Thomason Theorem implies the following conclusion.

Theorem 13. *The entropy of a hereditary class X equals zero if and only if the VC-dimension of graphs in X is bounded by a constant.*

In the rest of the section we characterize other jumps of the speed by means of graph parameters. In the proofs we use the results and notation of Section 3.

4.1. Hereditary classes of constant speed

The family of hereditary classes of constant speed constitutes the lowest layer of the lattice of hereditary classes. To characterize classes in this layer, we introduce one more parameter as follows. Let $S(G)$ and $s(G)$ denote the size of a largest and a smallest similarity class (see Definition 1) of vertices in G, respectively. Then the *similarity difference* of G is $S(G) - s(G)$.

Theorem 14. *The speed of a hereditary class X is constant if and only if the similarity difference of graphs in X is bounded by a constant.*

Proof. To prove the theorem we introduce the following classes of graphs:

$\mathcal{R}$ the class of graphs each of which is either an edgeless graph or a star,
$\mathcal{E}^1$ the class of graphs with at most one edge.

The proof of the theorem is based on the following claim.

(*) *If none of $\mathcal{R}$, $\mathcal{E}^1$, $\overline{\mathcal{R}}$, $\overline{\mathcal{E}}^1$ is a subclass of X, then X contains finitely many graphs different from complete and empty graphs.*
Indeed, if none of the four classes is a subclass of X, then there is a number p such that none of the following graphs belongs to X: $K_{1,p}$, $\overline{K}_{1,p}$, H_p^1, $\overline{H}_p^1$, where H_p^1 is a graph from $\mathcal{E}^1$ containing one edge and p isolated vertices.
Let G be a graph in X which is neither complete nor edgeless. Then G contains a vertex x which has both a neighbour y and a non-neighbour z. The remaining vertices of G can be partitioned (with respect to x, y, z) into at most eight subsets. For $U \subseteq \{x, y, z\}$, we denote $V_U = \{v \notin \{x, y, z\} \ : \ N(v) \cap \{x, y, z\} = U\}$. To prove the claim, let us show that each of V_U contains at most $R(p)$ vertices, where $R(p)$ is the symmetric Ramsey number.
If $U = \emptyset$, then $|V_U| < R(p)$ because a clique of size p in V_U together with x creates an induced $\overline{K}_{1,p}$, while an independent set of size p in V_U together with x and y creates an induced H_p^1, which is impossible because both graphs are forbidden in X.
If $|U| = 1$, say $U = \{x\}$, then $|V_U| < R(p)$ because a clique of size p in V_U together with y or z creates an induced $\overline{K}_{1,p}$, while an independent set of size p in V_U together with x creates an induced $K_{1,p}$, which is impossible because both graphs are forbidden in X.
For $|U| > 1$, the result follows by complementary arguments.

Now we turn to the proof of the theorem and assume first that X has a constant speed, then none of $\mathcal{R}$, $\mathcal{E}^1$, $\overline{\mathcal{R}}$, $\overline{\mathcal{E}}^1$ is a subclass of X, since $\mathcal{R}_n = \overline{\mathcal{R}}_n = n + 1$ and $\mathcal{E}_n^1 = \overline{\mathcal{E}}_n^1 = \binom{n}{2} + 1$. Therefore, by Claim (*), X contains finitely many graphs different from complete and empty graphs, and hence the similarity difference of graphs in X is bounded by a constant.

Conversely, assume that the similarity difference of graphs in X is bounded by a constant. Then none of $\mathcal{R}$, $\mathcal{E}^1$, $\overline{\mathcal{R}}$, $\overline{\mathcal{E}}^1$ is a subclass of X, because this parameter is unbounded in each of the four listed classes. Therefore, by Claim (*), X contains finitely many graphs different from complete and empty graphs, implying that X has a constant speed. $\square$

4.2. Hereditary classes with a polynomial speed of growth

According to the proof of Theorem 14, the four minimal classes above the constant layer are $\mathcal{R}$, $\mathcal{E}^1$, $\overline{\mathcal{R}}$ and $\overline{\mathcal{E}}^1$. Each of them contains polynomially many n-vertex labelled graphs and hence the layer following the constant one is polynomial.

Theorem 15. *Let X be a hereditary class above the constant layer, then the speed of X is polynomial if and only if the complex degree and c-matching number are bounded for graphs in X by a constant.*

Proof. If the speed of a hereditary class X is polynomial, then it contains none of the classes $\mathcal{B}, \mathcal{S}, \mathcal{Q}, \mathcal{M}, \overline{\mathcal{B}}, \overline{\mathcal{S}}, \overline{\mathcal{Q}}, \overline{\mathcal{M}}$, because $\mathcal{S}_n = 2^n - n$, $\mathcal{B}_n = 2^{n-1}$, $\mathcal{Q}_n = n2^{n-1} - n(n+1)/2 + 1$ and $\mathcal{M}_n$ is at least $\lfloor n/2 \rfloor !$. This implies by Theorems 7 and 10 that the complex degree and c-matching number are bounded for graphs in X.

Conversely, assume that the complex degree and c-matching number are bounded for graphs in X by a constant k and let G be a graph in X with n vertices. Without loss of generality, let $\mu(G) \leq k$. Therefore, $n - 2k$ vertices form an independent set I in G. Also, since the complex degree is bounded in G, every vertex v outside of I has at most k neighbours or at most k non-neighbours in I. By removing from I either k neighbours or k non-neighbours of v, for each $v \notin I$, we transform I into a similarity class of size $n - c$, where $c \leq 2k(k+1)$. It is not difficult to see that the number of labelled graphs on n vertices with a similarity class of size $n - c$ is $\binom{n}{c} 2^{\binom{c+1}{2}+1}$ and hence the speed of X is polynomial. $\square$

4.3. Hereditary classes with an exponential speed of growth

The proof of Theorem 15 tells us that there are eight minimal classes above the polynomial layer, namely $\mathcal{B}, \mathcal{S}, \mathcal{Q}, \mathcal{M}, \overline{\mathcal{B}}, \overline{\mathcal{S}}, \overline{\mathcal{Q}}$, and $\overline{\mathcal{M}}$. Each of these classes contains at least exponentially many n-vertex labelled graphs and hence the next layer is the exponential one.

Theorem 16. *Let X be a hereditary class above the polynomial layer, then the speed of X is exponential if and only if the neighbourhood diversity is bounded for graphs in X by a constant.*

Proof. If the speed of X is exponential, then X contains none of the following classes as a subclass:

$$\mathcal{M}, \mathcal{M}^{bc}, \mathcal{Z}, \overline{\mathcal{M}}, \overline{\mathcal{M}}^{bc}, \overline{\mathcal{Z}}, \mathcal{M}^*, \overline{\mathcal{M}}^*, \mathcal{Z}^*.$$

Indeed, as stated previously, the number of n-vertex labelled graphs in $\mathcal{M}$ is at least $\lfloor n/2 \rfloor !$. It is not difficult to see that the same is true for each of the other listed classes. This implies, by Theorem 11, that the neighbourhood diversity is bounded for graphs in X.

Conversely, assume the neighbourhood diversity is bounded for graphs in X by a constant k. It is not difficult to see that the number of labelled graphs on n vertices with at most k similarity classes is at most $k^n 2^{\binom{k}{2}+k}$. Therefore, the speed of X is exponential. □

5. Beyond bounded neighbourhood diversity

We observe that among the nine minimal classes of unbounded neighbourhood diversity there are two basic structures: chain graphs and graphs of degree at most 1 (matchings, for short). Both are subclasses of bipartite graphs. The remaining seven classes can be obtained from these two by flipping (complementing) the edges within or between the two parts in their bipartition. We will refer to all these classes as *flipped chain graphs* and *flipped matchings*. Note that chain graphs are self-complementary in the bipartite sense, i.e. flipping the edges between the two parts produces a chain graph again.

The two basic minimal structures of unbounded neighbourhood diversity (chain graphs and matchings) suggest an idea of two basic parameters that follow neighbourhood diversity in the hierarchy of graph complexity measures. They are known as uniformicity and lettericity. A common property of these parameters is that boundedness of any of them implies well-quasi-orderability by induced subgraphs. As a result, each of them admits a characterization in terms of minimal hereditary classes where the respective parameter is unbounded. For uniformicity, this characterization can be derived, with some care, from the results in [11,13,14] and we present it in Section 5.2. For lettericity, such a characterization is known only partially, and in Section 5.3 we propose a conjecture concerning the structure of minimal hereditary classes of unbounded lettericity.

For both parameters (uniformicity and lettericity), the set of minimal classes is infinite. For uniformicity, we describe this set in two steps starting with a related parameter, known as distinguishing number.

5.1. Distinguishing number

The notion of distinguishing number appeared implicitly, without a name and a formal definition, in [13]. It was given its name in [11] and later it was also studied in [10]. Originally, distinguishing number was defined as a parameter associated with *classes* of graphs. Below we define this parameter with respect to *graphs*. The two definitions are consistent in the sense that the distinguishing number in a class of graphs is either bounded in both definitions or unbounded in both of them.

Given a graph G and a set $U \subseteq V(G)$, we say that the disjoint subsets $U_1, \ldots, U_m$ of $V(G)$ (also disjoint from U) are *distinguished* by U if for each i, all vertices of U_i have the same neighbourhood in U, and for each $i \neq j$, vertices $x \in U_i$ and $y \in U_j$ have different neighbourhoods in U.

Definition 3. The *distinguishing number* of G is the maximum k such that G contains a subset $U \subset V(G)$ that distinguishes at least k subsets of $V(G)$, each of size at least k.

The paper [14] provides a complete description of minimal classes of unbounded distinguishing number, of which there are precisely 13. In Theorem 17 we list all of them with an alternative proof, which is substantially shorter and which is based on the characterization of neighbourhood diversity given in Theorem 11. One of these classes is known as *star forests*, i.e. graphs every connected component of which is a star. In the context of minimal classes we additionally require that the centres of all stars belong to the same part of the bipartition of a star forest.

Theorem 17. *Let X be a hereditary graph class where the distinguishing number is unbounded. Then X contains at least one of the following* 13 *minimal classes:*

- *the class of P_3-free graphs;*
- *the class of chain graphs;*
- *the class of threshold graphs;*
- *the class of star forests;*
- *the class of graphs obtained from star forests by creating a clique on the leaves of the stars;*
- *the class of graphs obtained from star forests by creating a clique on the centres of the stars;*
- *the class of graphs obtained from star forests by creating a clique on the leaves of the stars and a clique on the centres of the stars;*

- *the classes of complements of graphs in the above listed classes (note that the complements of threshold graphs are threshold graphs).*

Proof. Let X be a class of unbounded distinguishing number. Then for each k there is a graph G_k in X containing a subset U that distinguishes k subsets $U_1, \ldots, U_k$ of $V(G)$, each of size k. Since k can be arbitrarily large, we may assume (by a Ramsey argument) that each U_i is either a clique or an independent set and that for any two sets U_i and U_j, there are either all possible edges or no edges between them. For each i, we choose arbitrarily a vertex $u_i \in U_i$, and denote the subgraph of G_k induced by $\{u_1, \ldots, u_k\} \cup U$ by G_k^*. Also, let X^* be the subclass of X consisting of all graphs G_k^* and all their induced subgraphs.

By definition, the vertices $u_1, \ldots, u_k$ have pairwise different neighbourhoods in U. Therefore, the neighbourhood diversity is unbounded in X^* and hence it contains one of the nine minimal classes of unbounded neighbourhood diversity.

Assume first that X^* contains the class $\mathcal{M}$, i.e. we assume that the graphs G_k^* contain induced copies of pK_2 for arbitrarily large values of p. Therefore, the graphs G_k induce either arbitrarily large star forests (if arbitrarily many sets U_i are independent) or arbitrarily large P_3-free graphs (if arbitrarily many sets U_i are cliques). Therefore, if X^* contains the class $\mathcal{M}$, then X contains either the class of all star forests or the class of all P_3-free graphs.

If X^* contains any minimal class of unbounded neighbourhood diversity obtained from $\mathcal{M}$ by various complementation operations, then the situation is similar, i.e. in this case X contains one of the listed classes obtained from P_3-free graphs or from star forests by various complementation operations.

If X^* contains the class of chain graphs, then X contains either all chain graphs (if arbitrarily many sets U_i are independent) or all P_3-free graphs (if arbitrarily many sets U_i are cliques). Similarly, if X^* contains the class of threshold graphs, then X contains either all threshold graphs or all P_3-free graphs or all $\overline{P}_3$-free graphs. □

Theorem 17 describes what is "forbidden" for graphs of bounded distinguishing number. The next result can be viewed as a counterpart of Theorem 17 explaining what is "allowed" for graphs of bounded distinguishing number.

Theorem 18. *The distinguishing number is bounded in a hereditary class X if and only if there are positive integers $b = b(X)$ and $d = d(X)$ such that the vertices of any graph G in X can be partitioned into at most b subsets, called bags, in such a way that any bag is either a clique or an independent set and for any two bags U and W the graph $G[U, V]$ has either degree or bi-codegree bounded by d.*

Proof. Let us call a partition satisfying the statement of the theorem a *proper* (b, d)-partition. To prove the theorem in the 'only if' direction, we will show that none of the minimal classes of unbounded distinguishing number admits a proper partition. Clearly, it suffices to show this only for the following three classes listed in Theorem 17: P_3-free graphs, chain graphs and star forests.

The class of P_3-free graphs does not admit a proper partition, because the co-chromatic number is unbounded in this class. Indeed, consider the graph pK_p in this class for some p. Clearly, $\alpha(pK_p) = \omega(pK_p) = p$. Therefore, if pK_p admits a partition into t cliques and s independent sets, then $p^2 = |V(pK_p)| \le (t+s)p$ and hence $p \le t+s$. Since p can be arbitrarily large, we conclude that the co-chromatic number is unbounded in the class of P_3-free graphs.

Assume now that the class of star forests admits a proper (b, d)-partition. Without loss of generality we may assume that every bag in this partition is an independent set, since the clique number is bounded in this class. Consider the graph $G = (b+1)K_{1,(b-1)d+1}$ in this class. Then there are two bags in the partition of $V(G)$, say U and W, such that U contains the centres of at least two stars and W contains at least $d+1$ leaves of one of these stars. But then neither the degree nor bi-codegree is bounded by d in $G[U, V]$. This contradiction shows that the class of star forests does not admit a proper partition.

Finally, assume that the class of chain graphs admits a proper (b, d)-partition, and again without loss of generality we assume that every bag in this partition is an independent set. Consider the graph $G = Z_{b((b-1)d+1)+1}$ in this class. Then there are two bags in the partition of $V(G)$, say U and W, such that U contains two vertices x_i, x_j with $j - i \ge (b-1)d + 1$ (in the notation of Fig. 1) and W contains at least $d+1$ neighbours of x_j, which are not adjacent to x_i. But then neither the degree nor bi-codegree is bounded by d in $G[U, V]$. This contradiction shows that the class of chain graphs does not admit a proper partition.

Now we turn to the proof of the 'if' part of the theorem and consider a class X of graphs of bounded distinguishing number. Then X excludes a graph from each of the 13 minimal classes identified in Theorem 17. By excluding (forbidding) a P_3-free graph, a star forest and their complements we bound the co-chromatic number of any graph G in X, which follows from the results in [17,28]. Therefore, the vertices of G can be partitioned into finitely many bags, each of which is either a clique or an independent set. Now we consider two independent sets in this partition and show that they can be further partitioned into finitely many subsets any two of which induce a graph of bounded degree or bi-codegree. If two bags induce a co-bipartite graph or a split graph, the arguments are similar.

It was shown in [9] that the vertices of any bipartite graph excluding a star forest and the bipartite complement of a star forest can be partitioned into finitely many subsets so that any two subsets induce a bipartite graph, which is $2K_{1,p}$-free for a constant p. We repeat that the bipartite complement of $2K_{1,p}$ is again $2K_{1,p}$, i.e. the class of $2K_{1,p}$-free bipartite graphs is self-complementary in the bipartite sense.

It remains to show that the vertices of any $2K_{1,p}$-free bipartite graph $H = (A, B, E)$, which also excludes a chain graph Z, can be partitioned into finitely many subsets any two of which induce a graph of bounded degree or bi-codegree. Without loss of generality, we assume that $Z = Z_t$ for some t (since Z is an induced subgraph of Z_t for $t \geq |V(Z)|$) and prove the statement by induction on t. For $t = 1$, the result is obvious. To prove the result for larger values of t, we assume by induction that it is valid for Z_{t-1}, as well as for the bipartite complement of Z_{t-1} (since both chain graphs and $2K_{1,p}$-free bipartite graphs are self-complementary in the bipartite sense) and we let H be a Z_t-free graph.

Let a be a vertex of maximum degree in A and let b be a vertex of maximum degree in B, and assume first that a is adjacent to b. We denote by B_1 the set of neighbours (different from b) and by B_0 the set of non-neighbours of a in B. Also, let A_1 be the set of neighbours (different from a) and let A_0 be the set of non-neighbours of b in A.

First, we observe that every vertex $a' \in A$ has at most $p-1$ neighbours in B_0, because if a' has p neighbours non-adjacent to a, then a has p neighbours non-adjacent to a' (since $deg(a) \geq deg(a')$), in which case an induced $2K_{1,p}$ arises. Similarly, every vertex $b' \in B$ has at most $p - 1$ neighbours in A_0.

Second, we note that the subgraph of H induced by $A_0 \cup B_1$ does not contain Z_{t-1} (since otherwise together with a and b it induces a Z_t) and the subgraph of H induced by $A_1 \cup B_1$ does not contain the bipartite complement of Z_{t-1} (since otherwise together with a and b it induces a Z_t). Similarly, the subgraph of H induced by $A_1 \cup B_0$ does not contain Z_{t-1}. Applying induction to the subgraphs of H induced by $A_0 \cup B_1$, by $A_1 \cup B_1$, and by $A_1 \cup B_0$, we conclude that the vertices of H can be partitioned into finitely many subsets any two of which induce a graph of bounded degree or bi-codegree.

The case when a is not adjacent to b can be analysed similarly by involving into the analysis a vertex $b' \in B$ adjacent to a and a vertex $a' \in A$ adjacent to b. If such vertices do not exist, then H is edgeless and there is nothing to prove. Otherwise, the number of subsets in the partition of A and B doubles, but the arguments remain the same. □

Let us observe that if $d = 0$ in Theorem 18, then bounded distinguishing number coincides with bounded neighbourhood diversity. According to this theorem, by complementing dense bags (cliques) and dense pairs of bags (of bounded bi-codegree) in a graph G of bounded distinguishing number we transform this graph into a graph of bounded degree. We will refer to this operation as *sparsification* of G.

5.2. Uniformicity

Uniformicity is one more parameter that implicitly appeared in [13] without a name and a formal definition. It was formally defined in [30] as follows.

Let k be a natural number, F a simple graph on the vertex set $\{a_1, \dots, a_k\}$, and K a graph with loops allowed on the vertex set $\{a_1, \dots, a_k\}$. Let $U(F)$ be the disjoint union of infinitely many copies of F, and for $i = 1, \dots, k$, let A_i be the subset of vertices of $U(F)$ containing vertex a_i from each copy of F. We call $A_1, \dots, A_k$ the *bags* of $U(F)$. Now we construct from $U(F)$ an infinite graph $U(F, K)$ on the same vertex set by complementing (flipping) the edges between two bags A_i and A_j (or within A_i), whenever a_ia_j (or a_ia_i) is an edge in K. We call K the *flipping graph*. Finally, let $\mathcal{C}(F, K)$ be the hereditary class consisting of all the finite induced subgraphs of $U(F, K)$.

To give an example, let $k = 2$, $F = K_2$ and $E(K) = \emptyset$. Then $\mathcal{C}(F, K)$ is the class $\mathcal{M}$ of graphs of vertex degree at most 1 defined in Section 3.3. By adding to K the edge a_1a_2, we transform $\mathcal{M}$ into the class $\mathcal{M}^{bc}$ of bipartite complements of graphs in $\mathcal{M}$.

Definition 4. A graph G is called *k-uniform* if there is a number k such that $G \in \mathcal{C}(F, K)$ for some F and K. The minimum k such that G is k-uniform is the *uniformicity* of G.

By Theorem 18, bounded uniformicity implies bounded distinguishing number. However, the inverse implication is not valid. Indeed, sparsification transforms any graph of bounded uniformicity into a graph with connected components of bounded size, while the distinguishing number can be bounded even for sparse graphs with arbitrarily large connected components, for instance, for *linear forests*, i.e. graphs every connected component of which is a path.

The importance of the notion of uniformicity is due to the fact that all classes of bounded uniformicity are well-quasi-ordered under the labelled induced subgraph relation, which was shown in [30]. Moreover, in case of finite distinguishing number a hereditary class is well-quasi-ordered under the labelled induced subgraph relation if and only if it is of bounded uniformicity, which was shown in [10].

In the context of speeds of hereditary classes the importance of this notion is due to the fact that the family of classes of bounded uniformicity coincides with the family of classes with speeds below the Bell number, which follows from the results in [13,14]. The latter family was characterized in [11] by means of minimal hereditary classes that do not belong to the family in the following way.

First, we observe that all the 13 minimal classes of unbounded distinguishing number (Theorem 17) are also minimal classes with speeds at least the Bell number, which was shown in [14]. Therefore, all the 13 minimal classes of unbounded distinguishing number are also minimal classes of unbounded uniformicity. To illustrate this, consider, for instance, any proper hereditary subclass X of star forests, and let $pK_{1,p}$ be a star forest not in X for some p. Then every graph in X has at most $p-1$ connected components (stars) of arbitrarily large size and arbitrarily many components $K_{1,s}$ with $s \leq p-1$. Therefore, the uniformicity of graphs in X is bounded by $3p - 2$: take the graph F on the vertex set $\{a_1, \dots, a_{3p-2}\}$ in which the vertices $a_1, \ldots, a_p$ induce a $K_{1,p-1}$ and the remaining vertices are isolated, and define the edges of the flipping graph K to be $a_{p+1}a_{p+2}, a_{p+3}a_{p+4}, \ldots, a_{3p-3}a_{3p-2}$; then clearly every graph in X is an induced subgraph of $\mathcal{C}(F, K)$.

In addition to the 13 minimal classes of unbounded distinguishing number, the list of minimal hereditary classes of unbounded uniformicity also includes an infinite collection of classes with bounded distinguishing number. Exactly one of them consists of sparse graphs: the class of linear forests. The minimality of this class follows from the obvious fact that in any proper hereditary subclass of linear forests the size of connected graphs is bounded. All other minimal classes are not sparse, because paths are the only *connected* sparse (i.e. of bounded degree) structures unavoidable in large graphs (according to Ramsey). All other minimal classes can be obtained from linear forests by various complementation operations. The complete list of these classes was described in [11] in the terminology of language theory as follows.

Let A be a finite alphabet, w an infinite word over A, and K an undirected graph with loops allowed and with vertex set $V(K) = A$ (the flipping graph). Also, let $P(w)$ be the infinite path running through the letters of w and let $P(w, K)$ be the graph obtained from $P(w)$ by flipping in accordance with the edges of K, i.e. by changing the adjacency between two letters whenever these letters form an edge in K. Finally, let $\mathcal{C}(w, K)$ be the hereditary class consisting of all the finite induced subgraphs of $P(w, K)$. In particular, if K has no edges, then $\mathcal{C}(w, K)$ is the class of linear forests.

A *factor* in a word w is a contiguous subword, i.e. a subword whose letters appear consecutively in w. A word w is called *almost periodic* if for any factor f of w there is a constant k_f such that any factor of w of size at least k_f contains f as a factor.

Theorem 19 (*[11]*)**.** *Let X be a class of graphs with finite distinguishing number. Then X is a minimal hereditary class with speed at least the Bell number if and only if there exists a finite graph K with loops allowed and an infinite almost periodic word w over $V(K)$ such that $X = \mathcal{C}(w, K)$.*

We refer to any class of the form $\mathcal{C}(w, K)$, where K is a finite graph with loops allowed and w is an infinite almost periodic word over $V(K)$, as a *class of flipped linear forests*, and summarize the above discussion in the following statement.

Theorem 20. *A hereditary class X has bounded uniformicity if and only if it has speed below the Bell number. The list of minimal hereditary classes of unbounded uniformicity consists of the 13 classes of unbounded distinguishing number (Theorem 17) and all the classes of flipped linear forests (Theorem 19).*

5.3. Lettericity

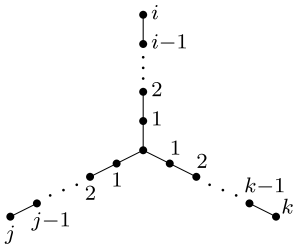
The notion of graph lettericity was introduced in [37] and it is closely related to the notion of geometric grid classes of permutations introduced in [1], as was shown in [3]. This notion can be defined as follows.

Let A be a finite alphabet, D a subset of A^2 (a decoder) and $w = w_1w_2 \ldots w_n$ a word over A. The *letter graph* $G(D, w)$ associated with w has $\{1, 2, \ldots, n\}$ as its vertex set, and two vertices $i < j$ are adjacent if and only if the ordered pair (w_i, w_j) belongs to D. The *lettericity* of G is the minimum k such that G is representable as a letter graph over an alphabet of k letters. We denote the set of all graphs representable over an alphabet A with a decoder D by $\mathcal{C}(A, D)$.

The class of graphs representable over the alphabet $A = \{a, b\}$ with the decoder $D = \{(a, b)\}$ is precisely the class of chain graphs. In other words, chain graphs are "easy" for lettericity. On the other hand, they are "hard" for uniformicity (uniformicity is unbounded for chain graphs). We also recall that matchings are easy for uniformicity, and from [37] we know that they are hard for lettericity. In other words, these two notions seem to be "orthogonal" to each other. We claim that there is, in fact, a close relationship between lettericity and uniformicity.

In order to describe this relationship, we observe that for each fixed k there is a bijection between the set of pairs (F, K) defining classes of bounded uniformicity and the set of decoders D defining classes of bounded lettericity. This bijection can be defined as follows. For each i, we include the pair (a_i, a_i) in the decoder D whenever a_ia_i is a loop in K. If $i < j$ and $a_ia_j \notin E(F)$, then $a_ia_j \in E(K) \iff (a_i, a_j), (a_j, a_i) \in D$ and $a_ia_j \notin E(K) \iff (a_i, a_j), (a_j, a_i) \notin D$. If $i < j$ and $a_ia_j \in E(F)$, then $a_ia_j \in E(K) \iff (a_i, a_j) \in D, (a_j, a_i) \notin D$ and $a_ia_j \notin E(K) \iff (a_j, a_i) \in D, (a_i, a_j) \notin D$. We will refer to this bijection as *duality*. This notion, together with Theorem 20, suggests an approach to identifying minimal classes of unbounded lettericity.

First, we observe that matchings and their flipped versions play the same role for lettericity as the 13 classes of unbounded distinguishing number play for uniformicity, i.e., they reduce the analysis to graphs with a bounded number of bags, as was shown in [9] (see Theorem 25 in the next section). In case of finite distinguishing number, every minimal class $\mathcal{C}(w, K)$ of unbounded uniformicity is a class of flipped linear forests. The underlying graph of $P(w, K)$ is an infinite path $P(w)$ running through an infinite almost periodic word w over a finite alphabet. To simplify the description, we assume that w is strictly periodic (which is always the case for finitely defined classes) with period $a_1 \ldots a_k$. The two bags A_1 and A_k (formed by the letters a_1 and a_k) induce in $P(w)$ a matching with edges $a_k^{(i)}a_1^{(i+1)}$, where the upper index in parentheses indicates the order in which the letters appear in the word. By shifting the upper indices, we switch this matching to $a_k^{(i)}a_1^{(i)}$ and obtain the infinite graph $U(C_k)$ consisting of an infinite collection of finite cycles C_k. In this way, we transform a minimal class $\mathcal{C}(w, K)$ of unbounded uniformicity into a class $\mathcal{C}(C_k, K)$ of bounded uniformicity. Next, we map the class $\mathcal{C}(C_k, K)$ to the dual class $\mathcal{C}(A, D)$ of bounded lettericity. The infinite periodic word w, read with the decoder D, now defines an infinite graph of lettericity k. In this graph, letters a_1 and a_k induce a chain graph. By shifting the indices



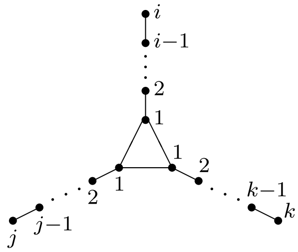


Fig. 2. The graphs $T_{i,j,k}$ (left) and $T^{\Delta}_{i,j,k}$ (right).

again in the same manner as described above, it is not difficult to show that the class of finite induced subgraphs of this graph becomes a minimal class of unbounded lettericity. We call this procedure of transforming a class of flipped linear forests into a class of unbounded lettericity the shifting procedure and conjecture that all minimal classes of unbounded lettericity can be obtained in this way.

Conjecture 1. *The set of minimal hereditary classes of unbounded lettericity consists of matchings and their flipped versions and the infinite collection of classes obtained from flipped linear forests by the shifting procedure.*

6. More parameters and open problems

We conclude the paper by reporting more results characterizing graph parameters in terms of critical hereditary classes and propose a number of open problems.

- *Tree-depth and path number*. Let us define the *path number* of a graph G to be the length of a longest (not necessarily induced) path in G. From the results in [12] it follows that there are precisely three minimal hereditary classes of graphs of unbounded path number: complete graphs, complete bipartite graphs and linear forests. In [36], based on the results in [12], it was reported that the same is true for tree-depth, i.e. the tree-depth is unbounded in a hereditary class X if and only if the path number is unbounded in X, i.e. if and only if X contains either the class of complete graphs or the class of complete bipartite graphs or the class of linear forests.
- *h-index*. The *h-index* of G is the maximum k such that G contains at least k vertices of degree at least k [19]. This parameter was characterized in [2] by means of three minimal classes as follows: the h-index of graphs in a hereditary class X is unbounded if and only if X contains either the class of complete graphs or the class of complete bipartite graphs or the class of star forests.

Comparing the characterization of tree-depth and h-index with Theorem 8, we conclude that both of them generalize matching number in the sense that the family of hereditary classes of bounded matching number is contained in the family of classes of bounded tree-depth and in the family of classes of bounded h-index. On the other hand, distinguishing number generalizes h-index in the same sense, which follows from Theorem 17 and the above characterization of h-index.

An important generalization of tree-depth is tree-width, which was characterized [33] in terms of four critical classes as follows.

- *Tree-width*. It is well-known that the class of complete graphs and the class of complete bipartite graphs are two minimal hereditary classes of unbounded tree-width. However, tree-width remains unbounded even in hereditary classes of bounded biclique number, i.e. classes excluding a complete graph and a complete bipartite graph as induced subgraphs. In this family, minimal hereditary classes of unbounded tree-width do not exist, but this family contains two boundary classes:

  $\mathcal{T}$ the class of graphs every connected component of which has the form $T_{i,j,k}$ represented in Fig. 2 (left),
  $\mathcal{T}^{\Delta}$ the class of graphs every connected component of which has the form $T^{\Delta}_{i,j,k}$ represented in Fig. 2 (right).

  We call graphs in $\mathcal{T}$ the *tripods*. It is not difficult to see that graphs in $\mathcal{T}^{\Delta}$ are the line graphs of tripods. In [33], it was shown that $\mathcal{T}$ and $\mathcal{T}^{\Delta}$ are the only boundary classes for the family of classes of bounded tree-width, resulting in the following tree-width dichotomy.

  Theorem 21. *Let X be a hereditary class defined by a finite set F of forbidden induced subgraphs. There is a constant bounding the tree-width of graphs in X if and only if F includes a complete graph, a complete bipartite graph, a tripod and the line graph of a tripod.*

A parameter generalizing both tree-width and h-index is known as degeneracy.

- *Degeneracy*. The *degeneracy* of a graph G is the minimum k such that every induced subgraph of G contains a vertex of degree at most k. The results in [27] imply the following characterization of degeneracy in terms of critical classes.

  Theorem 22. *The class of complete graphs and the class of complete bipartite graphs are the only two minimal hereditary classes of unbounded degeneracy, while the class of forests is the only boundary class for this parameter. In other words, if X is a hereditary class defined by a finite set F of forbidden induced subgraphs, then degeneracy is bounded for graphs in X if and only if F includes a complete graph, a complete bipartite graph, and a forest.*

It is well-known that bounded degeneracy implies bounded chromatic number, which in turn implies bounded co-chromatic number.

- *Chromatic and co-chromatic number*. According to the Gyárfás–Sumner conjecture, there are two critical hereditary classes for chromatic number, one of which is a minimal class of unbounded chromatic number (complete graphs) and the other is a boundary class (forests). The authors of [17] showed that the Gyárfás–Sumner conjecture is equivalent to a conjecture concerning co-chromatic number stating that there are four critical classes for co-chromatic number: two minimal classes (complete multipartite graphs and their complements) and two boundary classes (forests and their complements).

According to Theorem 18, a special case of bounded co-chromatic number is given by graphs of bounded distinguishing number. We now propose one more parameter, intermediate between co-chromatic number and distinguishing number, which implicitly appeared in the proof of Theorem 18 and which can be defined as follows.

- *Double-star partition number*. The *double-star partition number* of a graph G is the minimum p such that the vertices of G can be partitioned into at most p subsets, each of which is either a clique or an independent set, and the edges between any pair of subsets form a bipartite graph excluding a double star $2K_{1,p}$ as an induced subgraph. The following theorem proved in [9] characterizes the double-star partition number in terms of ten minimal hereditary classes of graphs where this parameter is unbounded.

  Theorem 23. *A hereditary class X is of unbounded double-star partition number if and only if X contains at least one of the ten classes of Theorem 17 different from chain graphs, co-chain graphs and threshold graphs.*
  This theorem and Theorem 17 characterize their respective parameters (double-star partition number and distinguishing number) in terms of "forbidden" structures. Theorem 18 is a counterpart of Theorem 17 in the sense that it describes what is "allowed" in graphs of bounded distinguishing number. The following result provides a similar counterpart for Theorem 23 and follows from a characterization of bipartite graphs excluding a double star obtained in [2].

  Theorem 24. *If the double-star partition number is bounded in a hereditary class X, then there are positive integers $b = b(X)$ and $d = d(X)$ such that the vertices of any graph G in X can be partitioned into at most b subsets, each of which is either a clique or an independent set, and the edges between any pair of subsets form a bipartite graph H such that every induced subgraph of H has a vertex of degree or bi-codegree bounded by d.*
  We observe, however, that Theorem 24, unlike Theorem 18, is valid only in the 'if' direction, i.e. bounding double-star partition number is sufficient for partitioning the graph in the way described in the theorem, but it is not necessary, because star forests admit such a partition. Finding minimal (critical) classes that are necessary obstructions for obtaining a partition described in Theorem 24 is an open problem.

Finally, we propose one more parameter which is intermediate between double-star partition and lettericity. It deals with the restriction of double-star partition number to the case when the edges between any pair of subsets in a partition of G form a $2K_{1,1}$-free bipartite graph, i.e. a chain graph. We refer to this parameter as chain partition number.

- *Chain partition number* of a graph G is the minimum number of subsets in a partition of $V(G)$ into cliques and independent sets such that the edges between any pair of subsets in the partition form a chain graph.

  Theorem 25 ([9]). *The chain partition number is unbounded in a hereditary class X if and only if X contains at least one of the six classes of Theorem 11 different from chain graphs, co-chain graphs and threshold graphs.*
  The importance of this parameter is due to the fact that graphs of bounded chain partition number can be viewed as an analog of monotone grid classes of permutations [45], when translated to the language of permutation graphs. More precisely, let P be a permutation class and $G(P)$ the corresponding class of permutation graphs. Then the chain partition number is bounded in $G(P)$ if and only if P is a monotone grid class.

We summarize our discussion in Fig. 3 which contains most of the parameters mentioned in this paper along with the respective critical classes.

To conclude the paper, let us observe that there is an intriguing relationship between the three unavoidable classes of bipartite graphs of unbounded neighbourhood diversity ($\mathcal{M}$, $\mathcal{M}^{bc}$ and $\mathcal{Z}$, see Lemma 4) and the three unavoidable structures in the Canonical Ramsey Theorem [22] in case of colouring 2-subsets. In this case, the theorem can be stated as follows: for every positive integer ℓ, there exists a positive integer $n = n(\ell)$ such that if the 2-subsets of an n-set are coloured with arbitrarily many colours, then the set contains a subset of size ℓ which is either

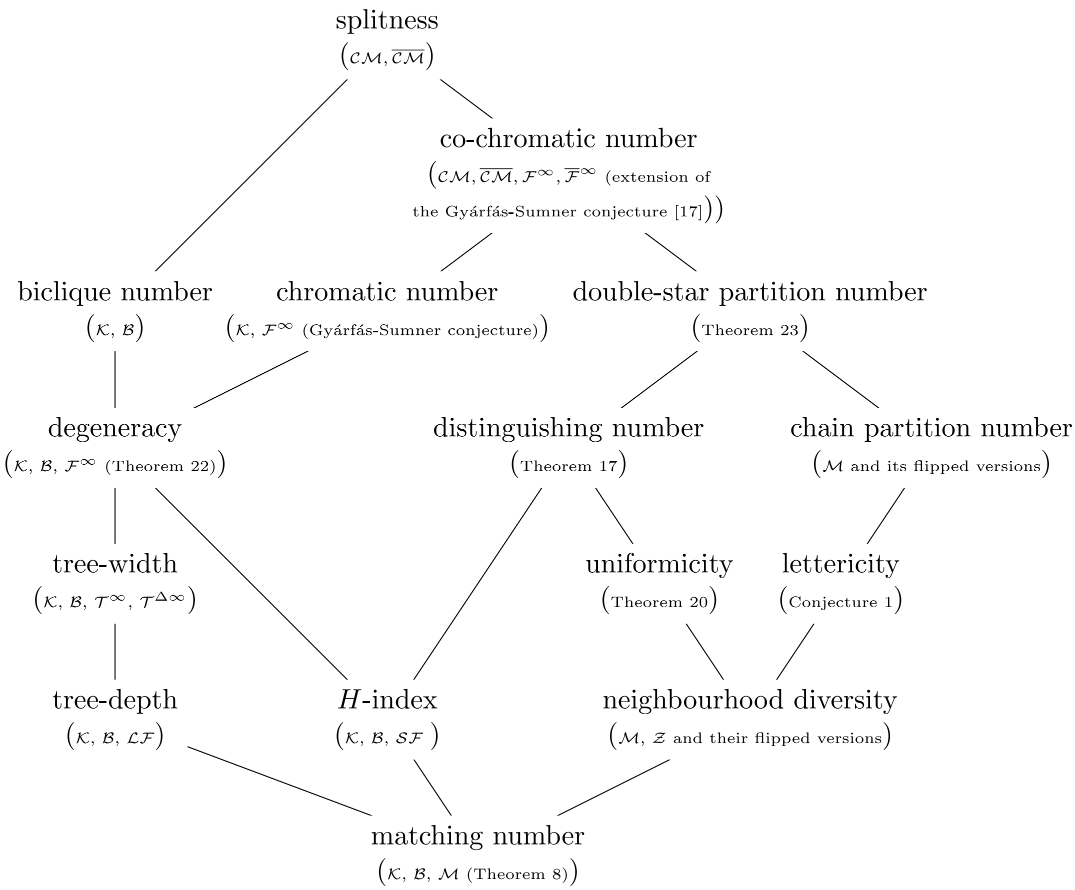


Fig. 3. A Hasse diagram describing the following relation on graph parameters: a parameter p_1 is weaker than parameter p_2 if for every hereditary class, boundedness of p_1 implies boundedness of p_2. In parentheses, for each parameter, we indicate either the list of critical hereditary classes or/and the result (conjecture) describing these classes. We indicate boundary classes with the upper index ∞. The classes that are not upper-indexed by ∞ are minimal. In addition to the notation and terminology introduced earlier, we denote by $\mathcal{K}$ the class of complete graphs, by $\mathcal{L}F$ the class of linear forest, by $\mathcal{S}F$ the class of star forests, by $\mathcal{C}M$ the class of complete multipartite graphs.

- monochromatic, i.e. all the 2-subsets have the same colour, or
- rainbow, i.e. the 2-subsets have pairwise different colours, or
- skew, i.e. the elements of the subset can be ordered $x_1, x_2, \ldots, x_\ell$ in such a way that the subsets $\{x_i, x_j\}$ $(i < j)$ and $\{x_p, x_t\}$ $(p < t)$ have the same colour if and only if $i = p$.

In this statement, monochromatic and rainbow colourings are complementary to each other, similarly to classes $\mathcal{M}$ and $\mathcal{M}^{bc}$, while a skew colouring is self-complementary, as in the case of chain graphs (the class $\mathcal{Z}$). We ask whether the two results (the existence of three unavoidable classes of bipartite graphs of unbounded neighbourhood diversity and three unavoidable colourings in the Canonical Ramsey Theorem) can be derived from each other.

Data availability

No data was used for the research described in the article.

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