Log del Pezzo Surfaces, Degenerations and Torus Actions

by

Edwin Kutas

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Contents

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Declarations

I declare that, to the best of my knowledge, the material in this thesis is original and my own work unless otherwise indicated.

We explicitly state here that chapters one and two are purely expository and contain no original work.

The material in this thesis has not been submitted for any other degree either at the University of Warwick or any other University.

Abstract

We provide an explicit classification for a number of large classes of complex projective surfaces. In particular we classify all log del Pezzo surfaces whose singularities have small discrepancy, and in the process we make precise the notion of small discrepancy. We also provide an explicit description of the morphisms relating these surfaces.

In a different vein, we consider surfaces admitting an action by a torus, which does not necessarily have full dimension. We provide a terminating algorithm that classifies all such log del Pezzo surfaces whose singularities have bounded index.

Chapter 1

Introduction

This thesis solves a range of classification problems for singular surfaces.

Throughout this thesis we consider varieties X with $-K_X$ ample and various restrictions on the singularities. These are particular instances of Fano varieties. A two dimensional Fano variety is called a log del Pezzo surface, see 1.0.1. Smooth log del Pezzo surfaces were described in work in the late 19th century and early 20th century. These surface are all of the form $\mathbb{P}^2$ blown up in k points where $k < 9$, or the exceptional case $\mathbb{P}^1 \times \mathbb{P}^1$. Such an elegant classification does not exist in the case of log del Pezzo surfaces with singularities; as represented by [5], [16]. A lot of work has been done on extending this classification to singular surfaces. In particular recent approaches have been interested in using of machinery of toric degeneration. This technique involves constructing a family $\mathcal{X}$ over $\mathbb{A}^1$ such that the fiber over 0 is a normal surface that contains $(\mathbb{C}^*)^2$ as a dense subvariety for which the natural action of the torus extends to the variety.

Work of [5] and [6] have established a one to one correspondence between log del Pezzo surfaces with $h^0(-K_X) \neq 0$ and toric degenerations up to mutation subject to assumptions on the singularities of the surface. It has been conjectured that this one to one correspondence extends to other classes of singularities beyond those considered in the above papers.

In the case of surfaces it is interesting to study the log del Pezzo surfaces with log terminal singularities. In the full generality, a log terminal surface singularity is a group quotient by a subgroup of $\mathrm{GL}_2(\mathbb{C})$. The case when the subgroup is cyclic is particularly important, and we refer to these as cyclic quotients. In the case of cyclic quotient singularities it has been conjectured that these admit toric degeneration.

This thesis is about log del Pezzo surfaces. The formal definition is:

Definition 1.0.1. A log del Pezzo surface is a normal two dimensional variety over $\mathbb{C}$ which has only log terminal singularities and has $-K_X$ ample.

Our motivating aim is the classification of such surfaces. This is an absolutely hopeless task in full generality. Nevertheless, we can classify special cases as follows.

1.0.1 Log del Pezzo surfaces of complexity 1

The Gorenstein index of a singularity S is the smallest value n such that nK_S is Cartier. We define the Gorenstein index of a surface X to be the smallest value n such that nK_X is Cartier. For any given $i \in \mathbb{N}$, the set of deformation families of log del Pezzo surfaces X with Gorenstein index $i_X = i$ is finite.

It is worth noting that the number of families increases enormously as the Gorenstein index of the surface increases. For example, only in the toric case, the start of the classification is

index i	number of toric surfaces
1	16
2	30
3	99
4	91
5	250
6	379
7	429
8	307
9	690
10	916

We consider these surfaces from three different and related points of view.

In particular, we classify log del Pezzo surfaces that admit a $\mathbb{C}^*$ action. In this thesis we give an algorithm to classify log del Pezzo surfaces that admit a $\mathbb{C}^*$ action and which have only log terminal singularities with fixed Gorenstein index.

A normal variety X of dimension n equipped with an effective action of a torus of dimension $n-k$ is referred to as a variety of complexity k. To illustrate the notion, note first that a toric variety X has an action of its n-dimensional 'big torus' $T \subset X$, and equipped with this action X is a variety of complexity 0. One could also give consider X equipped with the natural action of a k-dimensional subtorus $T' \subset T$, and then X is a variety of complexity $n-k$. (See §4.4.1.)

However, there are many varieties of complexity $k < n$ whose torus action does not extend to a toric variety. This is one of the main themes of this thesis: we study and classify surfaces of complexity 1 that are not toric.

In this way, complexity provides a way of grading the difficulty of a classification problem. Significant progress has been made on this problem before: Süss [17] classifies log del Pezzo surfaces admitting a $\mathbb{C}^*$ action which have Picard rank one and Gorenstein index less than 3. Huggenberger [9] classifies log del Pezzo surfaces of complexity 1 that have index 1 and arbitrary Picard rank. Ilten, Mishna and Trainor [12] recover the same classification and extend it into higher dimension. The methods and language used are broadly the same (though, analogous to the language of toric geometry, it varies whether papers work in the lattice N or its dual lattice M), though Huggenberger exploits Hausen's anticanonical complex technology to describe the Cox ring in detail.

We extend these results by presenting an algorithm that classifies log del Pezzo surfaces of complexity 1 with given index. The algorithm works and **terminates for any** index, though since the index is an unbounded invariant, there is no hope of a closed-form classification via methods of this type for all such del Pezzo with a torus action.

1.0.2 Bounded singularity content of log del Pezzo surfaces

We can consider log del Pezzo surfaces from a completely different point of view. Rather than considering the global invariants, we can consider the local invariants of the singularities. It follows from the definition 2.2.1 that the singularities are all finite quotient singularities, but this class of singularities itself is an infinite set. The *discrepancies* associated to a singularity form a measure of its complexity expressed as a collection of rational numbers, one for each curve in a resolution. When these numbers are small, but still greater than -1, the singularity may be regarded as 'more complicated'. However surfaces that have only these more complicated singularities can be classified explicitly. Informally, the basic reason is that it is hard to impose many of these singularities onto a single surface.

These conditions naturally arise as soon as you start to consider singularities in families. The first place this was considered was in [4] where they considered the case of $\frac{1}{p}(1,1)$ singularities, where $p \geq 5$. We extend this by

Theorem 1.0.2 (Theorem 3.4.9)**.** *Let X be a log del Pezzo surface with singularities of only small discrepancy. Then X has at most one singularity except for one sporadic* *family. All of these log del Pezzo surfaces admit a toric degeneration.*

This reproves the results of [4] who classified log del Pezzo surfaces with only $\frac{1}{p}$ singularities and extends results in [6] who classified surfaces with $\frac{1}{5}(1,2)$ and $\frac{1}{3}(1,1)$ singularities.

We also consider how the cascade of these surfaces behaves. This notion was introduced in [16] and is essentially asking for the birational relations between the surfaces. We prove that once our singularity is sufficiently complicated then you get a subset of the following series of birational relations

$$\begin{tikzcd}
 & & & & & & X''_{a''-k''_1-k''_2-2} \arrow[r] & \cdots \arrow[r] & X''_0 & \\
 & & & & & X''_{a''-k''_1-k''_2-1} \arrow[ur] \arrow[dr] & & Y^{1''}_1 \arrow[r] & \cdots \arrow[r] & Y^{1}_{n''_1}{}'' \\
 & & & & & & Y'' \arrow[ur] \arrow[r] & Y^{2''}_1 \arrow[r] & \cdots \arrow[r] & Y^{2}_{n''_2}{}'' \\
X = X_0 & X_1 \arrow[l] & \cdots \arrow[l] & X_{a-k_1-k_2-1} \arrow[l] \arrow[d] & X_{a-k_1-k_2} \arrow[l] \arrow[uur] \arrow[ddr] & & & & & \\
 & & & Y \arrow[dl] \arrow[dr] & & & Y' \arrow[r] \arrow[dr] & Y^{2'}_1 \arrow[r] & \cdots \arrow[r] & Y^{2}_{n'_2}{}' \\
 & & Y^1_1 \arrow[d] & & Y^2_1 \arrow[d] & X'_{a'-k'_1-k'_2-1} \arrow[ur] \arrow[dr] & & Y^{1'}_1 \arrow[r] & \cdots \arrow[r] & Y^{1}_{n'_1}{}' \\
 & & \vdots \arrow[d] & & \vdots \arrow[d] & & X'_{a'-k'_1-k'_2-2} \arrow[r] & \cdots \arrow[r] & X'_0 & \\
 & & Y^1_{n_1} & & Y^2_{n_2} & & & & &
\end{tikzcd}$$

In certain cases not all branches of this diagram may exist, and examples of this are provided in 3.6.1. In addition we provide simple examples outside of small discrepancy where toric degeneration do not exist.

This thesis is organised in the following way: Chapter 2 provides a technical background on toric varieties, quotient singularities, classification curves on surfaces. Chapter 3: This sets up the notions of small discrepancy and provides the associated proofs. Chapter 4: This introduces the technical description of complexity one varieties and then provides the algorithm to classify log del Pezzo surfaces with a torus action. An example of how the algorithm works is provided at the end of the chapter.

Chapter 2

Technical details

Nothing in this chapter is original work, and references are provided. Throughout this thesis we work only over the field $\mathbb{C}$.

2.1 Toric Geometry

We use the traditional language set up in [7]. In particular we use the fact that a normal toric variety X of dimension n can be associated with a (non unique) fan $\Sigma \subset N \cong \mathbb{Z}^n$. We consider the dual lattice to N, denoted M. Any $m \in M$ corresponds to a character of the torus which in turn corresponds to a monomial function in the function field of X. We denote the 1-skeleton of one dimensional cones in Σ by Σ^1. We also use this to refer to the set of primitive vectors generating those rays; this is a small abuse of notation that is always clear in context. We say a fan is complete if every lattice point $u \in N$ lies inside some cone $\sigma \in \Sigma$. The associated variety to a complete fan is complete.

2.1.1 Cox rings

Given a toric variety X, we wish to construct it as a GIT quotient. We follow the construction of [7]. Given a complete fan Σ with $\Sigma^1 = \{v_1, \ldots, v_m\}$, we consider the toric variety given by a fan $\overline{\Sigma} \subset \mathbb{Z}^m$, with $\overline{\Sigma}^1 = \{e_i\}$, where e_i are the standard basis vectors. A set $\{e_i\}_{i\in S}$ spans a cone in $\overline{\Sigma}$ if and only if the set $\{v_i\}_{i\in S}$ is contained in a cone of Σ. The variety Y associated to $\overline{\Sigma}$ is a subset of K^m. By construction we have a well defined map of fans $\phi\colon \overline{\Sigma} \to \Sigma$ corresponding to a linear projection. This induces a map $\widetilde{\phi}\colon Y \to X$ which can be seen as a GIT quotient with weights corresponding to the linear dependencies of Σ^1, and a finite group corresponding to the index of the sublattice of N generated by Σ^1. Now Y is equal to $\mathbb{C}^m_{x_i} - U$, the coordinate ring of Y is called the Cox ring. We note that the set U is comprised of the toric strata such that the corresponding cone $\{e_{i_1}, e_{i_k}\}$ does not correspond to a cone of $\overline{\Sigma}$.

2.1.2 Cyclic quotient singularities and singularity content

We also make frequent use of the following concepts introduced in [14] and [1]. Suppose given a cyclic quotient singularity $S = \frac{1}{r}(a, b)$ in two dimensions. Here S is the quotient of $\mathbb{C}^2$ by the group $G \cong \frac{\mathbb{Z}}{r\mathbb{Z}}$, with action defined by the matrix

$$\begin{pmatrix} \zeta^a & 0 \\ 0 & \zeta^b \end{pmatrix}$$

where $\zeta = e^{\frac{2\pi i}{r}}$. Without loss of generality a and b are coprime to r. This in turn implies that, by change of basis, we can write S as $\frac{1}{r}(1, u)$. The minimal resolution of this singularity is a chain of curves $C_1, \ldots, C_n$ with self intersections equal to $[a_1, \ldots, a_n]$, where these values a_i are equal to the coefficients of the Hirzebruch Jung continued fraction of $\frac{r}{u}$, as laid out in [14].

We are mainly interested in studying the restricted class of deformations known as $\mathbb{Q}$-Gorenstein as given in [11].

Definition 2.1.1. For X a normal projective surface with quotient singularities, a $\mathbb{Q}$-Gorenstein smoothing is a one parameter flat family $\mathcal{X} \to \mathcal{D}$ such that the total space is $\mathbb{Q}$-Gorenstein and the general fibre is smooth.

Singularity content is a concept introduced in [1] as a $\mathbb{Q}$-Gorenstein deformation invariant of a surface. Given a cyclic quotient surface singularity S, with associated group G, we define the index one cover S_1 to be the quotient of $\mathbb{C}^2$ by the subgroup $H = G \cap SL_2(\mathbb{C})$. We say $|H| = n$ and $|G| = r$. This gives $\mathbb{C}^2 \to S_1 \to S$ where S_1 has a singularity of type A_n, and this has equation $xy = z^{n+1}$. The group G/H acts on S_1 with quotient S. That is, this group acts on $xy = z^{n+1}$ with some weight k; this means $G/H \cong \frac{\mathbb{Z}}{\frac{r}{n}\mathbb{Z}}$ acts naturally by some weights on the x, y, z and the equation has weight k. This gives us the $\mathbb{Q}$-Gorenstein deformations of S are the quotients of the equivariant deformations $xy = \sum a_i z^{k+i\frac{r}{n}}$. This is smooth if and only if $k = 0$. On the other hand if $k \neq 0$ the deformation has a residual singularity $\frac{1}{r'}(a', b')$. We call the pair $(n, \frac{1}{r'}(a', b'))$ the singularity content. If $n = 0$ we say the singularity is $\mathbb{Q}$-Gorenstein rigid. The value n can be seen to be equal to the topological Euler number of the $\mathbb{Q}$-Gorenstein smoothing with the singular point removed, although this is not used in this thesis.

Given a log del Pezzo surface X with only $\mathbb{Q}$-Gorenstein rigid singularities, we define the singularity content $(n, \{S_1, \ldots, S_n\})$ where S_i are the singularities of X and n is once again the topological Euler number of $X^0 = X - \{\text{Singular locus of } X\}$.

2.2 Log del Pezzo background

2.2.1 Definitions

We here relate some basic definitions and facts about surfaces.

Given a normal surface singularity S and minimal resolution $\pi\colon \widetilde{S} \to S$ then we have

$$K_{\widetilde{S}} = \pi^*(K_S) + \sum a_i E_i$$

Definition 2.2.1. Throughout this thesis a log del Pezzo surface is a normal complex projective surface with log terminal singularities and $-K_X$ ample.

Where we say a singularity is

- terminal singularities if $a_i > 0$
- canonical singularities if $a_i \geq 0$
- log terminal singularities if $a_i \geq -1$
- log canonical singularities is $a_i \geq -1$

A surface singuarity is log terminal if and only if it can be constructed as a quotient of $\mathbb{C}^2$ by a, not necessarily cyclic, group action [10]. The smooth log del Pezzo surfaces have been classified as the blowups of $\mathbb{P}^2$ at less than 9 general points or $\mathbb{P}^1 \times \mathbb{P}^1$.

Given an orbifold log del Pezzo surface we frequently use the invariants $-K_X^2$ and $h^0(-K_X)$. These can be computed via orbifold Riemann Roch as set out in [15]. For a rough sketch of how we do these calculations, given a singular X, with minimal resolution Y.

In most of the cases in this thesis, these can be calculated by toric geometry. As shown in [7] the value $-K_X^2$ corresponds to the area of the full dimensional lattice cones of the fan. In the case of $h^0(-K_X)$ this is equal to a count of lattice points inside the dual polytope contained inside the lattice M. These invariants are invariant under $\mathbb{Q}$-Gorenstein deformation.

2.2.2 Hirzebruch Surfaces

We briefly state some basic results about Hirzebruch surfaces [13]. A Hirzebruch surface is a rational scroll defined as the quotient of $\mathbb{C}^4$ by $(\mathbb{C}^*)^2$ with weights $(1, -1, 0, 0)$ and $(n, 0, 1, 1)$. Alternatively it is the minimal resolution of $\mathbb{P}(1, 1, n)$. From this we see that we have the Picard group generated by B and F, where $B^2 = -n$ and F

is the strict of the transform of the generator of the Picard group of $\mathbb{P}(1, 1, n)$. The rational map that sends $(x; y; z) \in \mathbb{P}(1, 1, n)$ to $(x; y)$ is resolved by this blowup and gives a projection from the Hirzebruch surface to mbP^1.

2.2.3 Basic Surfaces

We finish with a very brief overview of [5], [6] and [4] as some of the methods we employ are similar. Respectively these paper classify log del Pezzos with singularities with minimal resolution [3] in [5], $[3, 2]$ and [3] in [6], and finally one singularity with resolution $[n]$ in [4]. The structure is similar, classify the possible surfaces X which admit no Mori contractions to another surface which could arise from these choices of singularities. These are called basic surfaces. Then study their blowups and their birational relations, often in the context of cascades as introduced by [16]. Via these explicit classifications they have been able to give explicit coordinate constructions and their toric degenerations (when they exist). In Chapter 2 we classify log del Pezzo surfaces with singularities that have Hirzebruch-Jung of the form $[a_1, \ldots, a_n]$ with both a_1 and a_n greater than two via similar, although modified methods.

Chapter 3

Small Discrepancy

3.1 Context

We aim to classify all possible log del Pezzo surface whose singularities have discrepancies sufficiently small.

3.2 Standard notions and notation for quotient singularities

We reference back to chapter 2 for a lot of the notation. We use the following definitions liberally throughout the chapter.

Definition 3.2.1. Let X be a log del Pezzo surface. If X cannot be constructed as the blowup of a surface X' then we say X is a basic surface.

Definition 3.2.2. A floating -1-curve on a surface X is a curve $C \subset X$, such that the exist a map $\pi \colon X \to X'$ which contracts C and is a blowdown.

We consider the germ S of a cyclic quotient singularity appearing at a point P on a projective surface X. The minimal resolution of X is denoted $f\colon Y \longrightarrow X$. It contains a chain of exceptional (smooth, rational) curves $C_1, \ldots, C_n$, entirely determined by S itself, which are ordered so that the only intersections between these curves are $C_i \cap C_{i+1}$ which is a single transverse intersection for each $i = 1, \ldots, n-1$; in other words, C_1 and C_n are the two 'ends' of the chain. We also denote the discrepancies of each C_i (as curves in Y) by $d_i \in \mathbb{Q}$: thus

$$K_Y = f^*(K_X) + \sum_{i=1}^{n} d_i C_i.$$

We denote by a_i the value C_i^2. The values a_i uniquely define S via Hirzebruch-Jung fractions [14] and the notation $S = \frac{1}{r}(1, a) = [a_1, \ldots, a_n]$ indicates the singularity such that the resolution has these self intersections.

3.3 Singularities with small discrepancy

We introduce a property of cyclic quotient singularities that is central to the rest of the chapter.

Definition 3.3.1. Let S be a cyclic quotient singularity, and $C_1, \ldots, C_n$ the exceptional curves of the minimal resolution of S and $d_1, \ldots, d_n$ their discrepancies, as above. We say that S is a *singularity with small discrepancy* if $d_i \leq -\frac{1}{2}$ for all $i = 1, \ldots, n$.

To simplify our calculations we introduce to the notation of the log discrepancy $e_i = d_i + 1$.

Proposition 3.3.2. *In the notation above, a singularity S has small discrepancy if and only if $C_1^2 \neq -2$ and $C_n^2 \neq -2$ and $S \ncong \frac{1}{3}(1, 1)$.*

Proof. We use the fact that the discrepancies form a sequence which at first is strictly decreasing and then strictly increasing. So it suffices to show this for C_1 and C_n and then apply this to show it for the intermediate values. We use the following formula for the log discrepancy $e_i = \frac{e_{i-1}+e_{i+1}}{a_i}$. We note that if $a_1 \geq 4$, then as $e_0 = 1$ and $e_2 \leq 1$ we have $e_1 \leq \frac{2}{-4}$. This implies the inequality for small discrepancy. In the case where $a_1 = 3$ this results in the following, as $e_1 \geq e_2$ by substituting e_2 into $e_1 = \frac{1+e_2}{-3}$ we get $e_2 \leq \frac{1+e_2}{3}$ which rearranges to $2e_2 - 1 < 0$. Hence $e_2 \leq \frac{1}{2}$. Substituting this back into the equation for e_1 we get $e_1 \leq \frac{1+\frac{1}{2}}{3} = \frac{1}{2}$. □

Throughout the rest of this chapter we restrict the class of singularities we consider as follows:

Assumption 3.3.3. Any singularity germ S that appears in this chapter is assumed to be a cyclic quotient singularity with small discrepancy.

3.4 Log del Pezzo surfaces and small discrepancy

Lemma 3.4.1. *Let X be a surface having cyclic quotient singularities of small discrepancy, and let $f\colon Y \to X$ be the minimal resolution of X. Let $C \subset X$ be a rational curve whose strict transform $\widetilde{C} \subset Y$ is smooth. Let $\{E_i\}$ be the exceptional locus of f. Suppose in addition that $\widetilde{C} \cdot \sum E_i \geq 2$. Then $\widetilde{C}^2 = -1$ implies $-K_X \cdot C \leq 0$.*

Proof. We use the genus formula that states $g(C) = C^2 + \frac{1}{2}(C^2 + C \cdot K_S)$ for any smooth curve C contained in a smooth surface Y. Applying this to $\widetilde{C} \subset Y$ we see that $K_Y \cdot \widetilde{C} = -1$. If $\widetilde{C}$ intersects two distinct exceptional curves E_i, E_j, with discrepancy d_i, d_j respectively, then $K_X \cdot C = f^*(K_X) \cdot \widetilde{C} \geq -1 - d_i - d_j \geq 0$, as X has only singularities with small discrepancy. If, on the other hand, $\widetilde{C}$ meets only one exceptional curve E_i, but with intersection multiplicity m_i, then $K_X \cdot C = f^*(K_X) \cdot \widetilde{C} \geq -1 - m_i d_i \geq 0$. □

We show next that in fact such rational curves cannot lie on a log del Pezzo. We need a preliminary lemma.

Lemma 3.4.2. *Let X be a log del Pezzo and $f\colon Y \to X$ its minimal resolution. Let*

$C \subset Y$ be a smooth rational curve. If $C^2 \leq -2$ then C is contracted by f to a point of X.

Proof. We prove this by contradiction. Assume there is a curve C that is not contracted. Then

$$K_X \cdot f(C) = f^*(K_X) \cdot C \geq K_Y \cdot C \geq 0$$

The first inequality follows as $f^*(K_X) - K_Y$ is an effective divisor. The second inequality follows as $K_Y \cdot C = -2 - C^2$, once again by the genus formula. □

We use the following definition introduced in [3].

Definition 3.4.3. We say that a non empty collection of curves $C_1, \ldots, C_n$ contained in a surface X is a *zero cycle* if there is a morphism π, that only contracts the curves C_i and $\pi_*(\cup C_i)$ is a smooth point.

Lemma 3.4.4. *Let $\mathcal{C} = \{C_1, \ldots, C_n\} \subset X$ be a zero cycle. Let $\pi \colon X \to Y$ be the contraction of the zero cycle, and P the point to which they are contracted. If P lies on the intersection of two curves D_1 and D_2, with $D_i^2 \leq -1$ then either*

1. *The strict transform of D_1 and D_2 still intersect.*
2. *The strict transform of D_1 and D_2 are joined connected by a collection of curves $\{C_{a_1}, \ldots, C_{a_n}\} \subset \mathcal{C}$ with $C_{a_i}^2 \leq -2$, and $C_{a_i} \cdot C_{a_j}$ equals one if $j = \pm 1$ and is zero otherwise.*
3. *There is a -1-curve C in $\mathcal{C}$ such that C intersects two curves, which both have self intersection less than or equal to -2.*

Proof. It is clear that if the strict transform of D_1 and D_2 do not intersect then the zero cycle, $\mathcal{C}$, gives rise to a chain of curves $\{C_{a_1}, \ldots, C_{a_n}\} \subset \mathcal{C}$ connecting the two curves. If there is a -1-curve in this chain it will have to intersect two curves with self intersection less than -1, otherwise we would have two -1-curves intersecting and this cannot be contracted. □

This is usually applied in the following context.

Lemma 3.4.5. *Let C_1 and C_2 be two curves on a surface X with $C_i^2 \leq -1$ intersecting in a smooth point of the surface. Then every surface Y with $\pi\colon Y \to X$ a dominant morphism satisfies at least one of the following*

1. *The strict transform of C_1 intersects the strict transform of C_2.*

2. *The strict transform of C_1 is connected by a chain of curves with self intersection less than -1 to the the strict transform of C_2.*

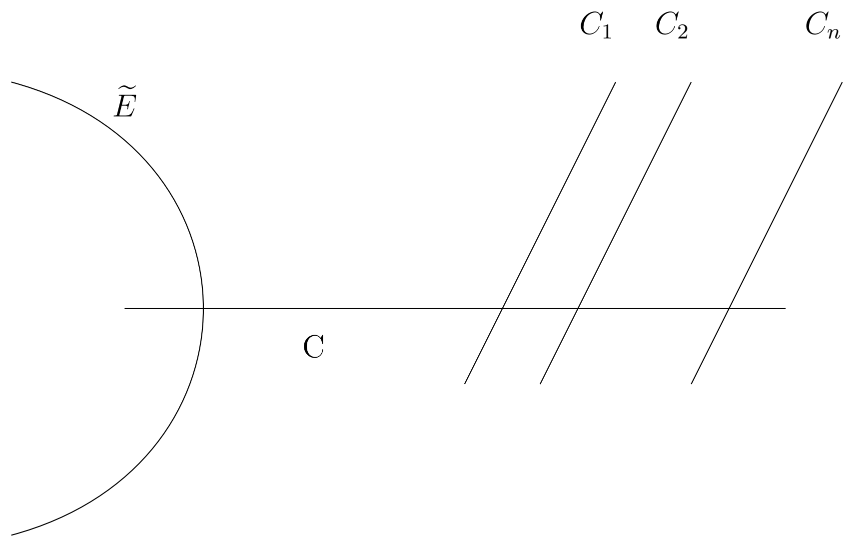
3. *There a -1 curve intersecting two curves with self intersection less than -1.*

Proof. This clearly follows from the previous Lemma 3.4.4. □

Proposition 3.4.6. *Let X be a log del Pezzo surface with singularities of small discrepancy. Let Y be its minimal resolution. Let Z be a smooth surface and $\pi\colon Y \to Z$ be a dominant morphism. Let $C \subset Y$ be a curve with $C^2 \leq -2$. Then $\pi_* C$ is either a smooth curve or a point.*

Proof. We proof this by contradiction. Assume that $\pi_* C$ has a singular point S, either S has multiple branches or it a cuspidal singularity. If there are multiple branches then locally at S it resembles two curves intersecting a point P. Via Lemma 3.4.5 the curve C on Y would either still be singular, or it would be connected by a chain of curves with self intersection less than -1 (which would give rise to a non cyclic quotient singularity) or there would be a -1 curve joining two curves with self intersection less than -1 which would contradict X being a log del Pezzo surface via Lemma 3.4.1. Hence S cannot have multiple branches.

The case of S only having one branch follows via similar logic. On Y every curve with self intersection less than -1 is smooth and every -1-curve only intersects at most one curve with self intersection less than -1 and the intersections are transverse, this is via Lemma 3.4.1. Taking the first stage when every intersection is transverse and no more than two curves intersect a given point we have the following curve configuration.

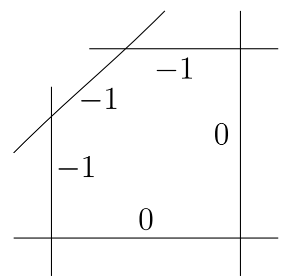


Here $C^2 = -1$ and $C_i^2 \leq -2$. Once again applying Lemma 3.4.5 to the curves we see that there is no way that this can occur from a contraction of Y. Hence there cannot be a singularity on the push forwards of the curve. $\square$

Proposition 3.4.7. *There is a unique family of non smooth log del Pezzo surfaces S_p, indexed by $p \in \mathbb{N}$, such that given the minimal resolution Y of S_p, Y does not admit a map to $\mathbb{F}_i$ with $i \geq 2$. Here S_p has one $\frac{1}{p}(1,1)$ singularity.*

Proof. The first case is that Y only admits a map to $\mathbb{F}_0$. Then Y must be $\mathbb{F}_0$, since a blow up of any point of $\mathbb{F}_0$ also permits a map to $\mathbb{F}_1$; but then $X = Y$ is smooth, contradicting the assumption.

If Y only admits a map to $\mathbb{F}_1$ other cases arise. Clearly if we blow up a point on the -1-curve we get a map to $\mathbb{F}_2$. So the only option is a blowup at a general point. At this point you get the following toric variety



This results in three adjacent -1-curves. We now split into three cases, dependent on which -1-curve we blow up, or if we blow up a general point.

Case 1: Our next blow up is a blow up at a general point. This results in DP_6. Let Z be DP_6 blown up at k general points. Then Z has the property that for any -1-curve C there is map $\pi\colon Z \to \mathbb{F}_1$ that sends C to the -1-curve on $\mathbb{F}_1$. Hence if the map Y to $\mathbb{F}_1$ factors through Z, then given a $-n$-curve on Y, with $n \geq 2$, then there will be a map sending it to the -1-section of $\mathbb{F}_1$. Hence there will also be a map sending it to the negative section of $\mathbb{F}_2$.

Case 2: Blowing up any point on the curve which is the pullback of the negative section B of $\mathbb{F}_1$. Once again this results in an obvious map to $\mathbb{F}_2$. We note that our surface admits two maps to $\mathbb{F}_1$ via the symmetry of the surface, so there are two possible pullbacks.

Case 3: The final case gives rise to an infinite family of surfaces with Y only admitting a map to $\mathbb{F}_1$. This is blowing up $p-1$ general points on the strict transform of the fiber that went through the point we blew up. This results in an infinite family of surfaces Y_p with a single $-p$-curve. Any further blowups of Y_p in any position not on this curve will admit a map to a Hirzebruch surface $\mathbb{F}_n$ with $n \geq 2$ via analogous arguments to case 1 and case 2. $\square$

Lemma 3.4.8. *Let X be a log del Pezzo with only singularities of small discrepancy, and let $f\colon Y \to X$ be the minimal resolution. We suppose that Y admits a map π to $\mathbb{F}_l$ where $l \geq 2$.*

For a germ S of a singularity of X, denote by $E_i^S \subset Y$ the exceptional curves in the resolution of S. For each singularity S on X:

1. *Every exceptional curve E_i^S is either contracted to a point of $\mathbb{F}_l$ by π, or the pushdown $\pi_* E_i^S \subset \mathrm{F}_l$ is a smooth rational curve with self-intersection one of $-l$, 0, l, $l+2, 4l$.*

2. *In addition there is always a curve E_j^S not contracted by π for all singularities S.*

Proof. To prove the first statement note that $\pi_* E_i^S$ cannot be a singular curve by Proposition 3.4.6, hence it is a smooth rational curve. The Picard group of a Hirzebruch Surface $\mathbb{F}_l$ is generated by the curves B, with $B^2 = -l$, F with $F^2 = 0$, We note that the curve $\pi_* E_i^S$ is one of the following: B itself or intersects B once or it is disjoint from B, by Proposition 3.4.6. Hence it either B or $mF + n(lF + B)$ for m and n positive integers. By Lemma 3.4.1 we see m cannot be greater than 1 as $C \cdot B = m$. If m is 0 and then n has to be less than 3, otherwise the curve would not be smooth and rational. Finally if m is one, we use the genus formula, let $C = \pi_* E_i^S$ then

$$-1 = \frac{1}{2}(C^2 + K_{\mathbb{F}_l} \cdot C) = \frac{1}{2}((n^2 l + 2) - ((2+l)n + 2)) = \frac{1}{2}(n^2 l - ln - 2n)$$

We clearly see that this cannot occur if $n \geq 3$, as $l \geq 2$. Enumerating these case of $m \in 0, 1$ and $n \in 0, 1, 2$ gives us the desired intersections.

To show that not all the curves E_j^S can be contracted to a point if $l \geq 2$, we go for a proof by contradiction. Assume $l \geq 2$ and every exceptional curve in a singularity S is contracted to a point $P \in \mathbb{F}_l$. Then P lies on a fiber F which intersects the curve B. First we consider $P \notin B$. We have $E_i^S \in \pi^{-1}P$ for all i. Hence we have to blow up P several times. However the strict transform of the fiber F, denoted $\widetilde{F}$ now has $\widetilde{F}^2 \leq -1$. If $\widetilde{F}^2 \leq -2$ then it has to be contracted, meaning $\widetilde{F}$, $B \in \{E_i^S\}$ which would be curves not contracted to a point. If $\widetilde{F}^2 = -1$, then the only -1-

curves in $\pi^{-1}P$ cannot intersect $\widetilde{F}$. This is because, after the first blowup we have an exceptional curve E and the fiber $\widetilde{F}$. These both have square -1. If we blow up the intersection point of $\widetilde{F}$ and E then $\widetilde{F}^2 \leq -2$, hence we can only blowup general points on E. At this point we have none of the -1-curves intersecting $\widetilde{F}$. If we blowup no points on E then clearly we are not introducing a singularity so this does not occur. Now finally we note that our curve configuration would contradict Lemma 3.4.1 if we had $\widetilde{F}^2 = -1$ as it would connect B to an exceptional curve E_i^S.

□

Remark. In the case where the length, n, of the singularity is 1 or 2, Lemma 3.4.1 follows via easy toric geometry as any curve joining two singularities is a locally toric configuration. This corresponds to the associated fan occurring as the face fan of a non convex polygon.

Now we can classify these log del Pezzo surfaces in a straightforwards way.

Theorem 3.4.9. *Let X be a non-smooth log del Pezzo with only singularities of small discrepancy. Then X has either one singularity or two singularities, and if there are two one of the singularities is a $\frac{1}{r_1}(1,1)$ and the other singularity is a $\frac{1}{r_2}(1,1)$. In addition we can explicitly describe all possible basic surfaces.*

Proof. Given a log del Pezzo X_0 we start by contracting all floating -1-curves. This gives rise to a log del Pezzo X_1; note that X_1 is not $\mathbb{P}^2$ since the contraction map is an isomorphism in the neighbourhood of any singularity of X_0. Let $\sigma\colon Y \to X_1$ be the minimal resolution of X_1. We know that there is a map $\pi\colon Y \to \mathbb{F}_l$, and we may suppose l is maximal with this property. There is a curve $B \subset \mathbb{F}_l$ with $B^2 = -l$. If $l \geq 2$ then B has to be the image of a σ-exceptional curve E_i inside Y.

We first show that π cannot contract a curve to a point on B. If on the contrary there is a curve contracted to B, then without loss of generality we may assume that it is the exceptional curve of the final blowdown $Y \to Y_2 \to \mathrm{F}_l$. In that case, there two curves C_1, C_2 on Y_2, both -1-curves, with C_2 being the strict transform of 0 fiber.

But then we could instead contract C_2 from Y_2 and get a map to $\mathbb{F}_{l+1}$, contradicting maximality of l. Hence π is indeed an isomorphism in a neighbourhood of B.

We note that $l \leq 1$ has been classified in Proposition 3.4.7. So we restrict to $l \geq 2$. Now there is a singularity S such that $B \in \{\pi_* E_i^S\}$. Assume first that S is not a $\frac{1}{p}(1,1)$ singularity. Note that there is a curve E_j^S such that $\pi_* E_j^S$ is B. The adjacent (one or two) exceptional curves $E_{j\pm 1}^S$ cannot be contracted to a point via π (by the argument of the previous paragraph). We suppose there are two adjacent curves $E_{j\pm 1}^S$; the case where E_j^S is at the end of a chain of blowups with only one adjacent exceptional curve works in exactly the same way. Thus each of $\pi_* E_{j\pm 1}^S$ is either a 0 curve (a fiber) or an $l+2$ curve on F_l (by the classification of smooth rational curves on F_l). Note, by $l+2$ curve we mean the curve has self intersection $l+2$. Denote these two adjacent curves by C_1 and C_2 respectively. Assume there was another singularity with exceptional curves $\{E_i^{S'}\}_{i=0}^{m_{S'}}$ on Y. Then by Lemma 3.4.8 there would be a curve $E_j^{S'}$ such that $\pi_* E_j^{S'}$ is a curve with self-intersection 0, l, $l+2$. However these curves would necessarily intersect C_1 and C_2 meaning either S' is not distinct from S or there is a -1-curve in Y connecting two of their curves in the minimal resolution. Hence X has precisely one singularity.

To complete the analysis of this step, suppose S is a $\frac{1}{p}(1,1)$ singularity and that its unique exceptional curve is mapped to the negative section B. Then consider the possibility of there being another singularity S' on X. By Lemma 3.4.8, there is a curve $E_j^{S'}$ such that $A = \pi_* E_j^{S'}$ and A has self-intersection l or $4l$; it cannot be 0 or $l+2$ as it must not meet B. If S' is not a $\frac{1}{p}(1,1)$ then there is at least one exceptional curve among the $E_k^{S'}$ that is contracted to a point on $A \subset \mathrm{F}_l$. However each blowup of a point $Q \in A$ introduces a -1-curve D which is joined to curve B by another -1-curve, the birational transform of the fiber through Q. Hence none of these curves $E_k^{S'}$ can be mapped to D, as otherwise it would be adjoined to B by a -1-curve, contradicting Lemma 3.4.1. Thus any other singularity on X is also of type $\frac{1}{p}(1,1)$ (though possibly for a different p).

Suppose now that there was a third singularity of type $\frac{1}{p}(1,1)$. Once again, its exceptional curve would have to be sent to a 0, l, $l+2$ curve. Any smooth rational curve on F_l with one of these intersection numbers intersects the curve A. Thus on Y it must either meet the birational transform of A or meet some curve that intersects A. Once again in the second case it will result in two singularities connected by a -1-curve. This is a contradiction to small discrepancy by Lemma 3.4.1.

Thus X has exactly one or two singularities of type $\frac{1}{p}(1,1)$, and part (3.4.9) is complete in the case $l \geq 2$.

To help explicitly describe these basic surfaces we first observe that neither of the adjacent curves $E^S_{j\pm1}$ can map to an $l+2$ curve, as this would result in X having a floating -1-curve. This is because $l+1$ points on an $l+2$-curve can be cut out as the intersection of the $l+2$-curve with an l-curve and we would have to blowup at least $l+1$ times. Hence the strict transform of this l-curve would be a -1-curve.

Because of this we see that the only possibilities for $\pi_*(E^S_{j\pm1})$ are two different 0 curves. (Again we suppose there are two adjacent curves; the case of one adjacent curve follows via the same logic.) We can then proceed to construct the configuration of all exceptional curves inductively. This means that when a surface of this form is able to be constructed we can obtain it by doing two weighted blowups at two general points of a Hirzebruch surface with weights k_1 and k_2. We then do a series of toric blowups, and then finally do a series of non toric blowups on the boundary. The following surface is one example, arising from blowing up two general points of a Hirzebruch surface with k_1 and k_2 times.

□

In Figure 3.1 the picture is where the map to the Hirzebruch surface is an isomorphism on an exceptional curve E_i, where $1 < i < n$. Here the red curves indicate -1-curves and the blue curves indicate curves with positive self-intersection. The blue curve has self-intersection $a_i - k_1 - k_2$. This value is dependent on the map to the Hirzebruch surface $\mathbb{F}_{a_i}$. If $i = 1$ or $i = n$ then we would have a similar looking configuration except with positive curve now having self-intersection $a_i + 2 - k_1$ as we consider the curve in the linear system $|(l+1)F+B|$ which would be the nodal curve inside $|-K_X|$.

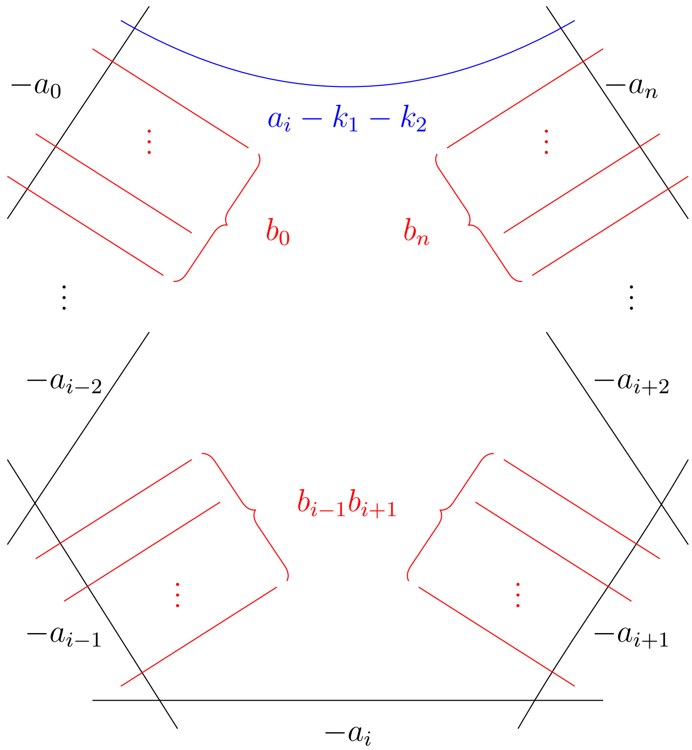


Figure 3.1: Example of a minimal surface with invariants $S = [a_1, \ldots, a_n]$ and the number of blowups being k_1 and k_2.

Remark. In our construction of our surface we blew up two points, P_1 and P_2, of a Hirzebruch surface k_1 and k_2 times. If for every possible value of k_1 and k_2 we have $k_1 + k_2 > a_i$ for all i. Then there are no log del Pezzo surfaces with singularity S. This follows because the positive section going through P_1 and P_2 would now have negative self intersection. For instance the singularity $S_n = [3, 2, \cdots, 2, 3]$ of length n cannot be the only singularity on a log del Pezzo surface when $n \geq 7$. This means, heuristically, that there are no surfaces with only singularities of small discrepancy where the length of the singularity is a lot larger than the largest self intersection of an exceptional curve.

We use these results to classify log del Pezzo surfaces with certain singularities. This leads to the following corollaries in which we classify all log del Pezzo surfaces with singularities of small discrepancy, each of which is resolved by a one or two exceptional curves.

3.5 Examples

We start by specifying how we deviate from the previous literature. In [16] they construct cascades as explicit maps between log del Pezzo surfaces, mirroring the classical constructions. We do not do this, however we explicitly state when the map exists and give good embeddings. If a variety admits no floating -1-curves, so a basic surface, we call it the root of a cascade. If a surface X cannot be blown up at any point while preserving $-K_X$ ample we call it the head of a cascade.

Corollary 3.5.1. *Let X be a log del Pezzo surface with small discrepancy and singularities $\{\frac{1}{p_1}(1,1),\ \ldots,\ \frac{1}{p_n}(1,1)\}$ for $n \geq 0$. Then $n \leq 2$ and moreover*

1. *if $n \leq 1$ then either X is a smooth del Pezzo surface or lies in a cascade over $\mathrm{P}(1,1,k)$ (see [4]);*
2. *if $n = 2$ then let c be the highest common factor of p and q and $a = \frac{p}{c}$, $b = \frac{q}{c}$. Then X is isomorphic to a quasismooth weighted hypersurface $X_{a+b} \subset \mathbb{P}(1,\ 1,\ a,\ b)$ quotiented out by μ_c acting with weights $(1,1,0,1)$. Conversely any such hypersurface with $p, q \geq 4$ is a log del Pezzo surface with small discrepancy.*

In particular, in the case of two singularities there is no cascade.

The small discrepancy condition is equivalent to the condition that $p_i \geq 4$ for each $i = 1, \ldots, n$. For the sake of completeness, we outline the classification result of [4] that describes part 1, which also follows independently from Propostion 3.4.7 and Theorem 3.4.9.

Proof. With these restrictions on singularities, it fits the criterion for the above theorem. The explicit classification was done in the proof of Theorem 3.4.9. The case of one singularity was done in [4]. The only examples of these surfaces with more than one singularity are constructed by blowing up a Hirzebruch surface in several points along the positive section and then contracting the resulting negative curve and the negative section, as was prove in Theorem 3.4.9. Denote this surface by X. Note X admits a $\mathbb{C}^*$ action and an equivariant degeneration to the toric variety with rays $\{(-p_1, -1), (0, 1), (p_2, 1)\}$. This is $\mathbb{P}(a+b, a, b)$ quotiented out by μ_c acting with weights $(1, 1, 0, 1)$. Taking the veronese embedding of degree $a + b$, denoted v_{a+b}, gives us the desired embedding. . We have $-K_X^2 = \frac{4}{p_1} + \frac{4}{p_2}$. Even in cases where $-K_X^2 > 1$ we see that X cannot be blown up while preserving $-K_X$ ample. If X admitted a blow up at a general point P then there is a fiber F such that $P \in F$. Then $\widetilde{F}$ is a -1-curve on the minimal resolution connecting the $-p_1$ curve with the $-p_2$ curve. This is a contradiction. Hence there is only one element in the cascade.

We note that this surface can be see as a hypersurface of degree $p + q$ inside $\mathbb{P}(1,\, 1,\, p,\, q)$.

□

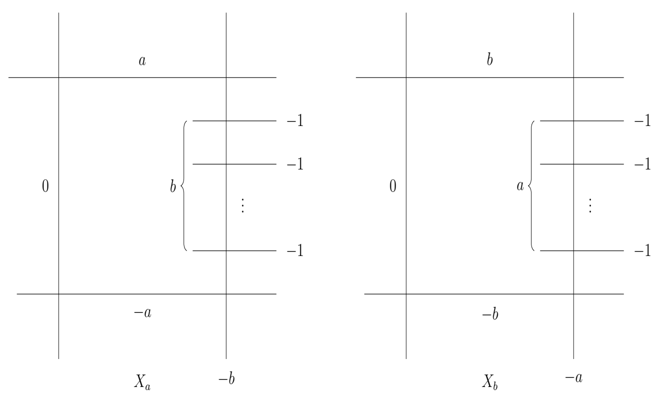
We now do a more difficult example by classifying the log del Pezzo's with singularities $S_{a,b}$ with resolution E_1, E_2 with $E_1^2 = -a$, $E_2^2 = -b$. To make sure that this obeys they conditions on the theorem we insist a, $b \neq 2$. We note that the case of $S_{3,3}$ does satisfy the conditions for the theorem. However we are interested in $\mathbb{Q}$-Gorenstein smoothings and $S_{3,3}$ is not $\mathbb{Q}$-Gorenstein rigid and admits a partial smoothing to $\frac{1}{6}(1, 1)$ singularity. These were classified above. This is the only one of these singularities which is not $\mathbb{Q}$-Gorenstein rigid. This is a more complicated example of how the above theorem can be used.

Corollary 3.5.2. *Let X be a surface such that the basket is $(\{S_{a_1,b_1}, \ldots, S_{a_m,b_m}\},\, n)$, with the condition that a_i, $b_i \geq 3$ and we exclude the case $a_i = b_i = 3$. Then there is at most one singularity $S_{a,b}$.*

Moreover all such surfaces are related by morphisms in the following way:

$$\begin{tikzcd}
X_a^0 & X_a^1 \arrow[l, "\phi_a^0"'] & \cdots \arrow[l, "\phi_a^1"'] & X_a^{a-1} \arrow[l, "\phi_a^{a-2}"'] & & & \\
& & & & X_1 \arrow[ul, "\phi_a^{a-1}"] \arrow[dl, "\phi_b^{b-1}"] & X_2 \arrow[l, "\Phi_1"'] & X_3 \arrow[l, "\Phi_2"'] \\
X_b^0 & X_b^1 \arrow[l, "\phi_b^0"] & \cdots \arrow[l, "\phi_b^1"] & X_b^{b-1} \arrow[l, "\phi_b^{b-2}"] & & &
\end{tikzcd}$$

Proof. Once again by Theorem 3.4.9 there are two basic surfaces, upto deformation, with a $S_{a,b}$ singularity. These are given by the following two surfaces: X_a is the surface constructed by taking $\mathbb{F}_a$ and blowing up b point on a fiber F, then contracting the $-a$ curve and the $-b$ curve. The surface X_b is symmetric to X_a and is constructed as a blowup of a points on a fiber F inside $\mathbb{F}_b$.



From construction, we see the following formula for the anticanonical degree of X_a:

$$-K_{X_a}^2 = 8 - b + a\left(1 - \tfrac{b+1}{ab-1}\right)^2 + b\left(1 - \tfrac{a+1}{ab-1}\right)^2 - 2\left(1 - \tfrac{a+1}{ab-1}\right)\left(1 - \tfrac{b+1}{ab-1}\right)$$

We note that once we blow up X_a exactly a times our formula will be completely symmetric in a and b.

Both X_a and X_b admit a toric degeneration, to $\mathbb{P}(1, b, ab-1)$ and $\mathbb{P}(1, a, ab-1)$

respectively. We can see this as an equivariant toric degeneration of a complexity one variety, alternatively this follows by considering the numerical invariants in [1]. We only consider the case of X_a since X_b is completely symmetric. We see that we can smooth the weighted projective space by taking the bth Veronese embedding of $\mathbb{P}(1, b, ab-1)$ getting $\mathbb{P}_{u,v,w,t}(1, 1, ab-1, a)$ with the relation $uw = t^b$. This admits a smoothing, giving us the surface lying as a hyper surface of degree ab inside $\mathbb{P}(1,\,1,\,ab-1,\,a)$.

Given a surface X_a^u with $0 \leq u \leq a$ with $X_a^a = X_1$ there is a corresponding toric degeneration X_Σ where Σ is the fan with rays $(-1, b)$, $(-1, 0)$, $(a, -1)$, $(a-u, -1)$. This has an A_{b-1} singularity and an A_{u-1} singularity. Via Cox rings this can be viewed as $\mathbb{C}^4_{\{x,y,z,t\}}$ with a quotient

$$\begin{array}{cccc} x & y & z & t \\ \end{array}$$
$$\begin{pmatrix} u & 0 & bu-(ab-1) & ab-1 \\ 1 & ab-1 & b & 0 \end{pmatrix}$$

Taking the Veronese embedding of degree $\binom{u}{b}$ we get a codimension 2 complete intersection with weights $\begin{matrix} b & b^2u \\ ab & u \end{matrix}$ inside the toric variety with weights

$$\begin{array}{cccccc} x^b & y^b & xy & z^u & t^u & tz \end{array}$$
$$\begin{pmatrix} b & 0 & 1 & bu-(ab-1) & ab-1 & b \\ 1 & ab-1 & a & u & 0 & 1 \end{pmatrix}$$

We can see the smoothing of both the A_b and A_{u-1} singularities inside this embedding, so this gives this a good coordinate construction as a codimension 2 subvariety.

After $a+1$ blowups the surface admits a toric degeneration to the toric variety whose fan has rays $(-1, b)$, $(-1, -1)$, $(a, -1)$. This is because upto deformation there is only one such surface, and so any potential toric degeneration must be a degeneration. A toric degeneration after $a + 2$ blowups is given by the fan whose rays are $\{(-1, 0), (-a, -1 - a), (-1, -1 - a), (b, ab - 1)\}$. This can be verified by calculating anticanonical degree of the toric surfaces.

We now show that the surfaces $X_a = X_a^0$ through to $X_3 = X_a^{a+2}$ are log del Pezzo surfaces. We start by showing X_a is a log del Pezzo surface, let Y be the minimal resolution of X_a. We know $-K_Y = A + F + E_a + E_b$ where E_a and E_b are the $-a$-curve and the $-b$-curve respectively. Here $A^2 = n$ and $F^2 = 0$, also, let D_1 through to D_b be the -1-curves inside Y. In addition let $C = A + F$. We wish to consider every possible decomposition of C into effective curves not in the exceptional locus, so that we can apply the Hodge index theorem [13]. We first note $F - D_i$ is linearly equivalent to $E_a + \sum_{j \neq i} D_j$, as this contains an exceptional curve we do not consider it further. Now note $F - 2D_i$ is not an effective curve as there is a curve U in $|A - D_i|$ such that $U \cdot (F - 2D_i) = A \cdot F - 2D_i = -1$. A similar analysis holds for $F - D_i - D_j$. Hence F cannot be split into a sum of effective divisors outside of the exceptional locus. We note that A can only be split into $(A - D_i) + D_i$ as any curve inside $|A - 2D_i|$ would intersect $F - D_i$, which is an effective divisor, with intersection -2. A similar anlysis holds for $A - D_i - D_j$. Hence the only ways to decompose $A + F$ outside of the exceptional locus are

1. C
2. $(C - D_i) + D_i$
3. $A + F$
4. $(A - D_i) + D_i + F$

Calculating the intersection with $-K_Y$ of all these possible effective curves we get

1. $-K_Y \cdot C = n + 2 - d_1 - d_2$
2. $-K_Y \cdot D_i = 1$
3. $-K_Y \cdot (C - D_i) = n + 1 - d_1 - d_2$

4. $K_Y \cdot A = n + 1 - d_2$

5. $-K_Y \cdot F = 1 - d_1$

6. $-K_Y \cdot (A - D_i) = n - d_2$

As the d_i are greater than -1 all these intersections are positive. Via the Hodge index theorem $-K_Y$ is a big divisor and the corresponding rational map contracts E_a and E_b as it is negative only on these curves. Hence the pushforwards of $-K_Y$, which is $-K_{X_a}$ is an ample divisor. The same analysis holds for X_a^{a+2}, except there are some extra exceptional curves but these do not effect the calculations.

To see that this marks the end of the cascade, as $(-K_{X_a^{a+2}})^2 = 2 + d_a + d_b < 1$, if we blow up one more time $(-K_{X_a^{a+3}})^2 = 1 + d_a + d_b < 0$ via small discrepancy and hence cannot be an ample divisor.

We now show that if we blow up X_a exactly a times we can blow down to X_b. Assume $a > b$, if we show that blowing up a points on X_a introduces b disjoint floating -1-curves, then we can contract them all. This gives rise to a log del Pezzo surface with the same numerics as X_b. Hence via Theorem 3.4.9 it has to be deformation equivalent to X_b. On X_a there are b curves D_i going through the singularity corresponding to the b blow ups we did on the Hirzebruch surface $\mathbb{F}_a$ to construct X_a. To each of these curves D_i there is a linear system of curves $|C_i|$ intersecting them with every element of $|C_i|$ being disjoint from the singularity and having self intersection $a-1$. Hence there is a unique curve $C_i \in |C_i|$ which contain the points P_1 through to P_a with self intersection $a-1$. In addition $A - D_i$ intersects C_j only at the points P_k and at each point the intersection is transverse. Hence after blowing up all the points P_k, these curves are now floating -1-curves all of which are disjoint and hence can all be contracted. Via the argument at the beginning of the paragraph this gives rise to a surface which is, upto deformation, the same as X_b.

□

This structure of these birational relationships can be put in more general terms.

3.6 Web of maps

Given a singularity with small discrepancy such that the minimal resolution is $a_1, \dots a_n$, there are a finite number of basic surfaces classified in Theorem 3.4.9. Attached to any one of these basic surfaces X with minimal resolution Y we have the following invariants a_i and k_1, k_2. The a_i indicates that Y admits a map to $\mathbb{F}_{a_i}$ which is an isomorphism on the negative section. The invariants k_1 and k_2 are the number of times we blew up general points on $\mathbb{F}_{a_i}$ to obtain Y.

Theorem 3.6.1. *Let X_i be one of the basic surfaces constructed in Theorem 3.4.9. We can describe how their cascades. In the case when the singularities are of the form $\frac{1}{p}(1,1)$ the cascades have been classified by [4] and the above example.*

For the general case we split in to cases. We note X_i is constructed by blowing up a Hirzebruch surface at a point k_1 times, and at another point k_2 times. We label the strict transform of these curves by E_i^S and E_i^T respectively. Let s_i, t_i be the number of -1-curves intersecting E_i^S and E_i^T respectively. We then classify the cascade via these invariants

There is the sporadic case where the singularity is length one where we have the cascade arising in [4] and the single surface with two singularities.

Outside of the above case, consider X a surface with k_1 and k_2 not equal to zero. In *addition s_1, t_1, s_2 and t_2 are all non zero. Then we have the following cascade*

$$\begin{tikzcd}
 & & & & & & X''_{a''-k_1''-k_2''-2} \arrow[r] & \cdots \arrow[r] & X''_0 & \\
 & & & & & X''_{a''-k_1''-k_2''-1} \arrow[ur] \arrow[dr] & & Y_1^{1''} \arrow[r] & \cdots \arrow[r] & {Y^1_{n_1''}}'' \\
 & & & & & & Y'' \arrow[ur] \arrow[r] & Y_1^{2''} \arrow[r] & \cdots \arrow[r] & {Y^2_{n_2''}}'' \\
X = X_0 & X_1 \arrow[l] & \cdots \arrow[l] & X_{a-k_1-k_2-1} \arrow[l] \arrow[d] & X_{a-k_1-k_2} \arrow[l] \arrow[uur] \arrow[ddr] & & & & & \\
 & & & Y \arrow[dl] \arrow[dr] & & & Y' \arrow[r] \arrow[dr] & Y_1^{2'} \arrow[r] & \cdots \arrow[r] & {Y^2_{n_2'}}' \\
 & & Y_1^1 \arrow[d] & & Y_1^2 \arrow[d] & X'_{a'-k_1'-k_2'-1} \arrow[ur] \arrow[dr] & & Y_1^{1'} \arrow[r] & \cdots \arrow[r] & {Y^1_{n_1'}}' \\
 & & \vdots \arrow[d] & & \vdots \arrow[d] & & X'_{a'-k_1'-k_2'-2} \arrow[r] & \cdots \arrow[r] & X'_0 & \\
 & & Y^1_{n_1} & & Y^2_{n_2} & & & & &
\end{tikzcd}$$

where we have

1. *X is a surface with k_1 and k_2 not equal to zero. In addition s_1, t_1, s_2 and t_2 are all non zero.*

2. *The $Y^1_{n_1}$ or $Y^2_{n_2}$ are surfaces such that $k_1 = 0$ and $k_2 = a_1 + 1 - s_1$ or $k_2 = a_1 + 1 - t_1$ respectively. In each of the Y_i there is only one degenerate fiber in the $\mathbb{P}^1$ fibration induced by the map to the Hirzebruch surface.*

3. *The surfaces X_0' and X_0'' are of the same form as X and the picture has 3-fold symmetry. Indeed $X_{a-k_1-k_2} \cong X_{a'-k_1'-k_2'} \cong X_{a''-k_1''-k_2''}$.*

There are a variety of subcases of this which occur

1. *The next case is X is a surface with $k_1 = k_2 = 0$ and there two non general*

fibers or $s_2 = t_2 = 0$. Then the cascade is

$$
\begin{tikzcd}
X = X_0 & X_1 \arrow[l] & \cdots \arrow[l] & X_{a-k_1-k_2-1} \arrow[l] \arrow[d] & X_{a-k_1-k_2} \arrow[l] \\
 & & & Y \arrow[dl] \arrow[dr] & \\
 & & Y_1^1 \arrow[d] & & Y_1^2 \arrow[d] \\
 & & \vdots \arrow[d] & & \vdots \arrow[d] \\
 & & Y_{n_1}^1 & & Y_{n_2}^2
\end{tikzcd}
$$

2. *The next case is X is a surface with $k_1 = 0$, $k_2 \neq 0$ and there are two non general fibers or $t_2 = 0$.*

$$
\begin{tikzcd}[column sep=small]
 & & & & & & Y' \arrow[r] \arrow[dr] & Y_1^{2\prime} \arrow[r] & \cdots \arrow[r] & {Y_{n_2'}^{2}}' \\
X = X_0 & X_1 \arrow[l] & \cdots \arrow[l] & X_{a-k_1-k_2-1} \arrow[l] \arrow[d] & X_{a-k_1-k_2} \arrow[l] \arrow[r] & X'_{a'-k_1'-k_2'-1} \arrow[ur] \arrow[dr] & & Y_1^{1\prime} & \cdots \arrow[r] & {Y_{n_1'}^{1}}' \\
 & & & Y \arrow[dl] \arrow[dr] & & & X'_{a'-k_1'-k_2'-2} \arrow[r] & \cdots \arrow[r] & X_0' & \\
 & & Y_1^1 \arrow[d] & & Y_1^2 \arrow[d] & & & & & \\
 & & \vdots \arrow[d] & & \vdots \arrow[d] & & & & & \\
 & & Y_{n_1}^1 & & Y_{n_2}^2 & & & & &
\end{tikzcd}
$$

3. *The next case is $s_1 = 0$ and two non general fibers*

$$\begin{tikzcd}
& & & & & & Y' \arrow[r] \arrow[dr] & Y_1^{2\prime} \arrow[r] & \cdots \arrow[r] & {Y^{2}_{n_2'}}' \\
X = X_0 & X_1 \arrow[l] & \cdots \arrow[l] & X_{a-k_1-k_2-1} \arrow[l] \arrow[d] & X_{a-k_1-k_2} \arrow[l] \arrow[r] & X'_{a'-k_1'-k_2'-1} \arrow[ur] \arrow[dr] & & Y_1^{1\prime} & \cdots \arrow[r] & {Y^{1}_{n_1'}}' \\
& & & Y \arrow[dl] \arrow[dr] & & & X'_{a'-k_1'-k_2'-2} \arrow[r] & \cdots \arrow[r] & X'_0 & \\
& & Y_1^1 \arrow[d] & & Y_1^2 \arrow[d] & & & & & \\
& & \vdots \arrow[d] & & \vdots \arrow[d] & & & & & \\
& & Y^1_{n_1} & & Y^2_{n_2} & & & & &
\end{tikzcd}$$

4. *The next case is $s_1 = t_1 = 0$ and two non general fibers*

$$X = X_0 \longleftarrow X_1 \longleftarrow \cdots \longleftarrow X_{a-k_1-k_2-1} \longleftarrow X_{a-k_1-k_2}$$

5. *We also have the case $s_2 = 0$ or $k_1 = k_2 = 0$ with one non general fiber, which has cascade*

$$\begin{tikzcd}
X_a^0 & X_a^1 \arrow[l] & \cdots \arrow[l] & X_a^{a-1} \arrow[l] & & & \\
& & & & X_1 \arrow[ul] \arrow[dl] & X_2 \arrow[l] & X_3 \arrow[l] \\
X_b^0 & X_b^1 \arrow[l] & \cdots \arrow[l] & X_b^{b-1} \arrow[l] & & &
\end{tikzcd}$$

6. *Finally case $s_1 = s_2 = 0$ with one non general fiber, which has cascade*

$$X = X_0 \longleftarrow X_1 \longleftarrow \cdots \longleftarrow X_{n-1} \longleftarrow X_n$$

We note that $s_1 = 0$ and $s_2 = 0$ with one non general fiber would occur in case 3 above.

Proof. Given a singularity S of small discrepancy with length $m > 1$. Then a basic surface X with singularity S has minimal resolution Y. The surface Y is constructed by taking a Hirzebruch surface $\mathbb{F}_{a_i}$ picking two points P_1, P_2 and blowing them up k_1 and k_2 times, this gives rise to an intermediate surface Z and then X is constructed by doing further blow ups. We can assume $k_1 \leq k_2$ and this gives the relations that either $k_1 = k_2 = 0$ and $m \in \{1, 2, 3\}$ or $k_1 = 0$ and $k_1 = m - 2$ or $k_1 + k_2 = m - 3$. These cases arise by considering the case where the strict transform of both/ one/ none of the fibers are exceptional curves. The case where no fiber becomes an exceptional curve has been classified in Corollary 3.5.1 and we will note mention it further.

We note that k_1 and k_2 should not be viewed as invariant of the surface X but an invariant of the given map to $\mathbb{F}_{a_i}$:

$$X \xleftarrow[f]{} Y \xrightarrow{\pi} \mathbb{F}_{a_i}$$

This is because the map to a Hirzebruch surface is non unique even with the moderate restrictions we have placed on these maps.

We now note that by construction in the case where the strict transform of both fibers are exceptional curves we get a curve C on Y with self-intersection $a_i - k_1 - k_2$. Via construction C was a toric curve and $f_*(C) \in |-K_X|$. We have that the class group of X is generated by C and D_i where D_i are the curves arising from the non toric blowups of Y. This implies that the cascade of X is of length $L = a_i - k_1 - k_2$ as blowups in general position do not affect the $-K_X \cdot D_i$ and when we blow up $a_i - k_1 - k_2 + 1$ times $K_X^2 \leq 0$ via the small discrepancy condition. If $L < 0$ then this surface is not a log del Pezzo surface.

In the case where one of the fibers is not exceptional, so $k_2 = 0$, we have that the class group is generated by the same D_i, the fiber class F and a final curve C. Here

$-K_X = C + F$ and $F^2 = 0$, $C^2 = a_i - k_1$. This surface admits a cascade of length $L = a_i - k_1 + 2$. This is because, we only need to calculate the intersections on the subgroup generated by C and F. If we blowup L times we can assume that we blew up one point on the fiber F and $a_i - k_1 + 1$ points on the curve C. After this process the strict transform of both these curves would have self-intersection -1. As these are both on through a singularity with multiplicity one we see that $-K_X$ has positive intersection with these curves. If we blow up one more time we can assume all $L + 1$ points lie on a curve in the class $|C + F|$ this has self-intersection L and hence after all these blowups would be a -1-curve intersecting the singularity twice. This would not be a log del Pezzo surface via small discrepancy.

We now wish to explore the birational relationships between these surfaces. The first stage is to show that the only possible -1-curves on any basic surface arise from the class $|B + a_iF|$ on the Hirzebruch surface $\mathbb{F}_{a_i}$. To show this we note that it is impossible for any curve that intersects $|B|$ to end up being a floating curve as it will always intersect the curve B. So any floating curve C lies in the class $n|B + a_iF|$. To show that $n = 1$ we compute the self intersection of these curves. If $n = 2$ then the smallest possible self intersection of a curve not going through the singularity is $4a_i - k_1 - k_2 - 4$ if there are two non general fibers and $4a_i - k_1 - 2$ if there is only one. As in the first case $L = a_i - k_1 - k_2$ as $a_i \geq 2$ we have $4a_i - k_1 - k_2 - 4 > L$ so we cannot blow up enough to make this a -1-curve. In the second case $L = a_i - k_1 + 2$ so once again as $a_i \geq 2$ we cannot blow up enough to make it a -1-curve. As n increases the size of the self intersection increases and hence they can never occur as floating -1-curves.

Now we explicitly state how the cascade structure occurs. We note that we can restrict our analysis to the exceptional curves $E_i \subset Y$ that arose as part of the original k_1 and k_2 blowups. These are the only curves that can be intersected by curves in the class $|B + a_iF|$ as otherwise they would have to intersect the fiber with multiplicity greater than 1. Label these exceptional curves $S_1, \ldots, S_{k_1}$ and $T_1, \ldots, T_{k_2}$ with S_1 and T_1 being the strict transforms of the fibers. Hence any potential floating curve intersects a -1-curve coming out of a curve S_i and a -1-curve coming out of

T_j. We now denote by $C_{i,j}$ the curve intersecting a -1-curve coming from S_i and a -1-curve coming from T_j. Then in the case of both fibers becoming exceptional curves $C_{i,j}^2 = L - 4 + i + j$ so in order for it to become a -1-curve it needs to be blown up in $L - 3 + i + j$ points. However as the length of the cascade is L this implies $i + j \leq 3$. So $\{(i, j)\} \in \{(1, 1), (1, 2), (2, 1)\}$. We now go on a case by case analysis:

Case 1: We start with $(i, j) = (1, 1)$. It takes $L - 1$ blowups for these curves to become -1-curves. Denoting the number of -1-curves intersecting S_1 and T_1 by s, t, we label these curves D_u^S and D_v^T respectively. We have st possible curves which would give rise to a -1-curve after $L - 1$ blowups. We denote by $C_{u,v}$ the curve intersecting D_u^S and D_v^T. These curves originally lay in $|B + a_i F|$ so on the Hirzebruch surface they intersected b times. By repeating the same calculation on this on Y blown up $L-1$ times, we see that $C_{u,v}$ intersects $C_{u',v'}$ if and only if $u \neq u'$ and $v \neq v'$. So fixing $C_{u,v}$ we get the curve configuration in Figure 3.2

Hence if we choose a floating -1-curve $C_{u,v}$ to contract the only remaining floating curves are $C_{u,\beta}$ and $C_{\alpha,v}$ where $\alpha \in \{1, \ldots, s\}$ and $\beta \in \{1, \ldots, t\}$. When the second floating curve is contracted this uniquely defines whether we are iterating over s or over t. So after two contractions the cascade is uniquely defined. To see where it ends up we note that a basic surface is uniquely defined by the number of -1-curves coming out of each curve on the boundary. Picking one of these chains of leads to either s blowdowns or t blowdowns. These cases behave symmetrically, so focusing on the case of s blowdowns we get a curve $S_1 = E_1 \subset Y$ with self intersection $-a_1$ and no -1-curves coming out of it. So it admits a map to $\mathbb{F}_{a_1}$. We note that in our notation the basic surface $Y_{n_1}^1$ which it blows down to has to have new value $k_1^{Y_{n_1}^1} = 0$ as there is only one exceptional curve intersecting it. Hence there is only one non general fiber of the fibration. As we are doing s blow downs the self intersection of the positive section is $1 + s = a_1 - k_2^{Y_{n_1}^1} + 2$. This gives us $k_2^{Y_{n_1}^1} = a_1 + 1 - s$.

Case 2: The second case is $(i, j) = (1, 2)$. We do not spell this out in the same level of detail. Replicating the above arguments we see that we now get exactly the same

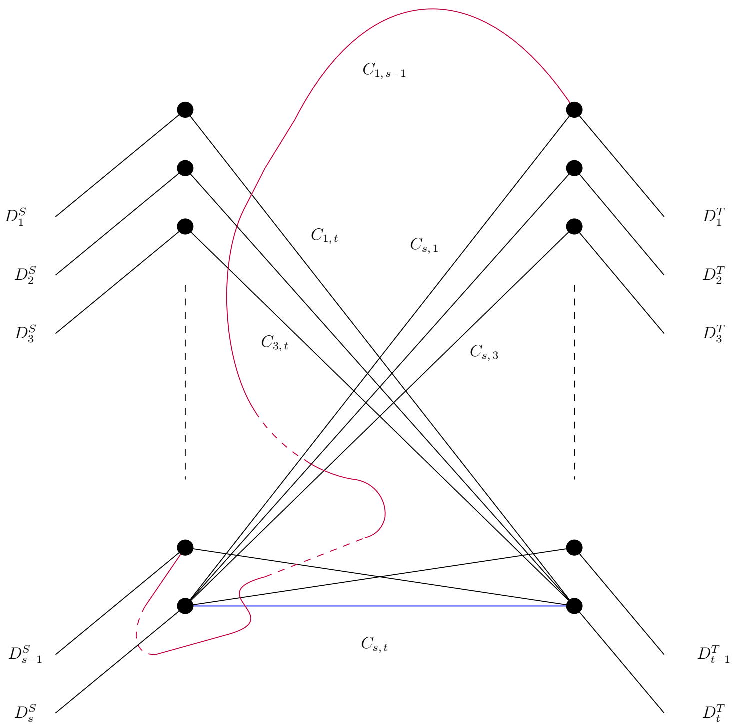


Figure 3.2: This is a diagram illustrating the intersections that occur on our bipartite graph. The curve $C_{1,\,s-1}$ has its intersections being shown with every curve intersecting $C_{s,\,t}$.

curve configuration as in Figure 2 except now connecting the curves E_1^S to E_2^T. So once again this leads to a set of two branching contractions. However the choice of contractions now give different basic surfaces at the end. If you contract the curves $C_{u,\,1}$ you get a surface Z_{m_1} with $k_1^{Z_{m_1}} = 0$ and only one non general fiber. However now via the same calculations as previously $k_2^{Z_{m_1}} = a_1 + 2 - s$. In the other case we are contracting all curves of the form D_v^T this gives rise to a surface W_{m_2} with $k_1 = 0$ but two exceptional fibers. This has invariant $k_2^{W_{m_2}} = t$.

Case 3: The final case is $(i,\, j) = (2,\, 1)$. This is symmetrical to case 2.

A crucial point in this proof is that each case behaves independently of the others as two floating curves don't intersect each other only if they lie in the same case. This means that upon any contraction we have limited ourselves to a set of -1-curves.

We note that if there are no -1-curves coming out of the curves E_1^S and E_1^T then the cascade is a straight line as the above discussion is entirely predicated on their existence. Via similar logic we get the following cases where not all of these maps occur. Let s_i, t_i equal the number of -1-curve going through E_i^S and E_i^T respectively. These conditions are symmetrical in s and t

1. $k_1 = k_2 = 0$ and there two non general fibers or $s_2 = t_2 = 0$. Then neither case 2 or 3 can occur.

2. $k_1 = 0$, $k_2 \neq 0$ and there are two non general fibers or $t_2 = 0$. Then case 3 cannot occur.

3. $s_1 = 0$. Then case 1 and 2 cannot occur.

4. $s_1 = t_1 = 0$ or $s_1 = s_2 = 0$. Then the cascade is a straight line.

This concludes the cascade for basic surfaces of this type.

We make a quick mention of what happens in the case where there is only one non generic fiber. These surfaces all start by blowing up a point k times. Label the exceptional curves that arose from blowing up this point $E_k, \ldots E_1$ and we denote the strict transform of the fiber by E_0. Once again $L = a_1 - k + 2$ and we have curves with self intersection $a_1 - i - 1$ intersecting the -1-curves coming out of E_i. To get a -1-curve we need $a_1 - i - 1 < L = a_1 - k + 2$ giving $k - i < 3$ so $i \in \{k,\, k-1,\, k-2\}$.

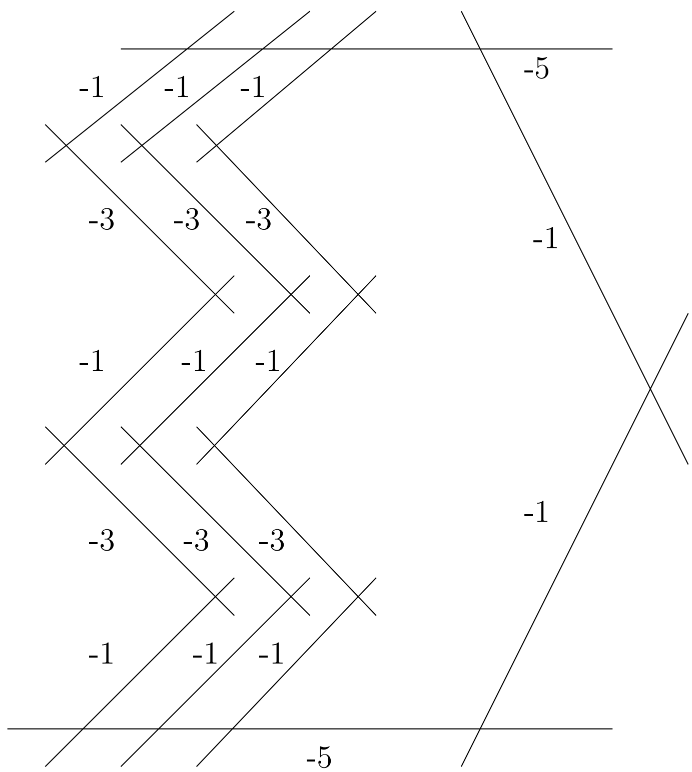
Considering the case where $i = k - 1$ then the curve intersects a -1-curve going through the curve E_{k-1}, which has self intersection $-a_{k-1}$. Blowing down all these floating curves gives rise to a basic surface X which has a map to $\mathbb{F}_{a_{k-1}}$, sending E_k to the negative section. Hence this is a surface in one of the above cases, as there are two exceptional curves adjacent to the negative section. So we it lies in one of the above diagrams. $\square$

The smallest possible length of a singularity where this full cascade can be seen is if the singularity is length 5 or more.

3.7 Outside of the small discrepancy

If you consider singularities of the type $\frac{1}{p}(1,1)$ we note that if $p \geq 7$ then a $\frac{1}{p}(1,1)$ singularity cannot be joined to any other $\frac{1}{p}(1,1)$ singularity by a -1-curve. Hence a similar analysis to Theorem 3.4.9 gives us the bound that there cannot be a log del Pezzo surface X with singularities $\frac{1}{p_1}(1,1), \ldots, \frac{1}{p_n}(1,1)$ and $p_1 \geq 7$ and more than 2 different singularities.

However when we enter the case where $p_1 < 7$ you can get surfaces with many more singularities. Not if $p \geq 7$ then a -1-curve intersecting a $\frac{1}{p}(1,1)$ singularity and $\frac{1}{k}(1,1)$ singularity, where $k \geq 3$ would have negative intersection with the canonical divisor. For instance consider the surface X with the following minimal resolution:



This surface has six $\frac{1}{3}(1,1)$ singularities and two $\frac{1}{5}(1,1)$ singularities and the following invariants

- $-K_X^2 = \frac{2}{5}$
- $h^0(-K_X) = 1$

This is a complexity one surface, and the calculations can be done via polyhedral divisors as set out in [2]. Alternatively we know this is constructed by 13 blowups of a Hirzebruch surface and then 8 subsequent contractions. This gives $-K_X^2 = 8-13+6v_{\frac{1}{3}(1,1)}+2v_{\frac{1}{5}(1,1)}$ where the v_i are correction terms in orbifold Riemann-Roch [15]. We calculate them to be $v_{\frac{1}{3}(1,1)} = \frac{5}{3}$ and $v_{\frac{1}{5}(1,1)} = \frac{1}{5}$. This gives $-K_X^2 = \frac{2}{5}$. We see once again via [15] that we can calculate $h^0(-K_X)$ from this and get $h^0(-K_X) = 1$.

We can construct this surface as a toric complete intersection via Cox rings [8] and we see that it lies as a complete intersection in the toric variety given by the GIT quotient

$$\begin{array}{c} \begin{array}{ccccccccccc} T_1 & T_2 & T_3 & T_4 & T_5 & T_6 & T_7 & T_8 & T_9 & T_{10} & T_{11} \end{array} \\ \begin{pmatrix} 1 & -2 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & -6 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & -4 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & -6 & 0 & 0 & 0 & -2 & 1 & 0 & 0 & 0 \\ 0 & 2 & -4 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & -3 & 7 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \end{pmatrix} \end{array}$$

with the equations

$$T_1T_2^2T_3 + T_4T_5^2T_6 + T_7T_8^2T_9 = T_1T_2^2T_3 + T_4T_5^2T_6 + \lambda T_{10}T_{11} = 0$$

To do this we note that this surface has polyhedral divisor, as introduced in Chapter 4

$$\left[0,\, \frac{3}{2},\, 3\right] \otimes P_1 + \left[0,\, \frac{3}{2},\, 3\right] \otimes P_2 + \left[0,\, \frac{3}{2},\, 3\right] \otimes P_3 + [-5,\, -4] \otimes P_4$$

Where λ is a deformation parameter. This gives us equations for the surface, to obtain the weights we calculate the action of the Picard torus via taking the Smith normal form of the matrix representing relations on the class group. We do not provide this calculation as it is largely disjoint from the rest of this Thesis. We note that there are also surfaces which admit a toric degeneration with the same numerics.

Chapter 4

Complexity One log del Pezzo Surfaces

4.1 Abstract

In this chapter we introduce an algorithm which enables us to classify log del Pezzos with a $\mathbb{C}^*$ action and a given index n.

4.2 Introduction

All varieties we consider are normal and projective. Here we give an algorithm to classify log del Pezzo surfaces with only log terminal singularities that admit a $\mathbb{C}^*$ action. A variety X of dimension n which admits a torus action of dimension $n-k$ is referred to as complexity k. Here complexity 0 is the study of purely toric varieties, and complexity n is the study of varities with no possible torus action. This provides essentailly a way of grading the difficulty of your problem. Significant progress has been made on this problem before: Süss [17] classifies log del Pezzo surfaces admitting said action with Picard rank one and index less than 3. Huggenberger [9] classifies the anticanonical complex of the Cox ring of log del Pezzo surfaces with index 1, this classification was later finished by Ilten, Mishna and Trainor [12] with a view towards higher dimension. This was achieved by looking at polarised complexity one log del Pezzo surfaces. We will show their work fits into our algorithm.

4.3 Polyhedral divisors

Recall that a toric variety is a normal variety of dimension n containing a dense torus $(\mathbb{C}^*)^n$ with the natural action extending to the variety, there is a correspondence between these varieties and fans inside a lattice $N \cong \mathbb{Z}^n$. In [2] the authors establish a similar correspondence for varieties with $T = (\mathbb{C}^*)^{n-k}$ actions where $k \leq n$. We say that this is a torus action of complexity k. They introduce the notion of a polyhedral divisor to recover some of the geometry that a fan encodes in the toric case. We now outline how this theory occurs.

Let N be a lattice and M its dual. Given a polyhedron $\Delta \subset N_{\mathbb{Q}}$ we say the *tailcone* is the set $\mathrm{tail}(\Delta) = \{v \in N_{\mathbb{Q}} | v + \Delta \subset \Delta\}$. Here the addition is a Minkowski sum. Let Y be a normal variety over $\mathbb{C}$. Fix a cone $\sigma \subset N_{\mathbb{Q}}$ we say a polyhedral divisor on Y with tail cone σ is a formal sum

$$\mathcal{D} = \sum \mathcal{D}_P \otimes P$$

over all prime divisors P of Y. Here the $\mathcal{D}_P$ are either polyhedra in $N_{\mathbb{Q}}$ with tailcone σ or the empty set. In addition only finitely many $\mathcal{D}_P$ differ from σ. We say the locus of the $\mathcal{D}$ is the Y minus the divisors where the associated polyhedron is the empty set. We now consider the dual cone of σ denote ω. For any $u \in \omega$ there is an associated Weil divisor $\mathcal{D}(u)$ with coefficients over $\mathbb{Q} \cup \infty$ on Y where

$$\mathcal{D}(u) = \sum_P \inf\langle \mathcal{D}_P, u\rangle \cdot P$$

Here the infimum over the empty set is ∞. This map is piecewise linear and convex. Given $D = \mathcal{D}(u)$ we define the set $L(D) = \{f \in \mathbb{C}(X)^* | \ \mathrm{div}(f) + D \geq 0\} \cup \{0\}$.

Definition 4.3.1. Given a Polyhedral divisor $\mathcal{D}$ we give the following properties:

1. $\mathcal{D}$ is $\mathbb{Q}$-Cartier if for all $u \in \omega$, $\mathcal{D}(u)$ is $\mathbb{Q}$-Cartier.
2. $\mathcal{D}$ is semiample if for all $u \in \omega$, $\mathcal{D}(u)$ is semiample.
3. $\mathcal{D}$ is big if $\mathcal{D}(u)$ is big for u in the interior of ω
4. $\mathcal{D}$ is a p-divisor if the locus of $\mathcal{D}$ is semiprojective, ω is full dimensional and $\mathcal{D}$ is $\mathbb{Q}$-Cartier, semi ample and big.
5. If $\mathcal{D}$ is a p-divisor, we call ω its weight cone.

To any polyhedral divisor we can associate an affine scheme

$$X(\mathcal{D}) = \mathrm{Spec}\left(\bigoplus L(\mathcal{D}(u) \cdot \chi^u\right)$$

Here χ^u is a character of the torus. This scheme comes with an action by the torus $T = \mathrm{Spec}(\mathbb{C}[M])$ and a rational map to the locus of $\mathcal{D}$. We use the following result from [2]:

Theorem 4.3.2. *If $\mathcal{D}$ is a p-divisor, then $X(\mathcal{D})$ is T-variety of complexity equal to the dimension of Y. Conversely, for any complexity k normal affine T-variety X, there is a p-divisor $\mathcal{D}$ on some normal k-dimensional variety Y with $X(\mathcal{D})$ equivariantly isomorphic to X.*

To deal with the situation of no-affine T-varieties we need to replace the p-divisor $\mathcal{D}$, which is analogous to a cone in toric geometry, with what is known as a divisorial fan $\mathcal{S}$ consisting of a finite set of p-divisors on a variety Y. Each associated affine variety glues together to give an invariant affine cover of the T-variety. In order for all these glueings to behave coherently, the elements of $\mathcal{S}$ must satisfy certain compatibility condition's. In particular, given a prime divisor P on Y, the set of polyhedra

$$\mathcal{S}_P = \{\mathcal{D}_P \subset N_{\mathbb{Q}} | \, \mathcal{D} \in \mathcal{S}\}$$

must form a polyhedral complex in $N_{\mathbb{Q}}$, this is called the *slice* at P. In particular this implies that the fan which is the union of the tail cones must span $N_{\mathbb{Q}}$, this fan is called the tail fan. For a complete and rigorous statement of the compatibility conditions can be found in [2].

We now restrict to the case that will let us study log del Pezzo surfaces, namely $Y \cong \mathbb{P}^1$ and $N \cong \mathbb{Z}$. We wish to study projective surfaces in this context so we need a notion of gueing together polyhedral divisors to make a polyhedral fan. In the above context this is a lot more straightforwards than in the general case. In this case a polyhedral fan can be described in the following way:
Given points $P_1, \ldots, P_n \in \mathbb{P}^1$ and a point $P \neq P_i$. Associated to each point P_i there is list of rational numbers $a^i_1, a^i_{n_i}$ with $a^i_j < a^i_{j+1}$, then we get polyhedral divisors $[a^i_k, a^i_{k+1}] \otimes P_i + \varnothing \otimes P$, this p-divisor has tail cone 0. For the tail cone $[0, \infty)$ our polyhedral divisors have an affine locus or a projective locus. If the locus is projective, we say that our polyhedral fan has a marking of $\mathbb{Q}^+$. The analagous construction can be done for the tail cone $(-\infty, 0]$ and once again if a polyhedral divisor with tail cone $(-\infty, 0]$ has projective locus we say the polyhdral fan has a marking of $\mathbb{Q}^-$. In addition given a polyhedral fan $\mathcal{S}$ with some choice of markings, we insit every polyhedral divisor is a p-divisor. In practice this means the following:

1. If there are no markings, then there is no constraint.

2. If there is a marking of $\mathbb{Q}^+$ then $\sum a^i_{n_i} > 0$

3. If there is a marking of $\mathbb{Q}^-$ then $\sum a^i_1 < 0$

4.4 Examples

Example 4.4.1. The first is the polyhedral divisor given by the unmarked polyhedral divisor $[0,1] \otimes P_0$ over $Y = \mathbb{P}^1$. The tail cone δ is equal to 0. We will show how we can construct from this an affine variety X. We denote the polyhedral divisor by $\mathcal{D}$. So the dual of the tail cone $\hat{\delta}$ is the lattice M itself. Given an element $m > 0$ of M we have $\mathcal{D}(m) = -mP_0$ as the minimum value is obtained on -1. If $m < 0$ we get $\mathcal{D}(m) = mP_0$ as the minimum is attained on 1. Finally if $n = 0$ we get $\mathcal{D}(m) = 0$ as a divisor on $\mathbb{A}^1$ as the function is 0 everywhere.

Hence we have the M graded ring

$$\bigoplus_{m \in M} \mathcal{O}_{\mathbb{A}^1}(-|m|P)$$

In degree 0 the ring is generated by the constant function on $\mathbb{A}^1$ which we denote x and the function y which is zero at the origin. We note that the function 1 in degree 0 is the multiplicative identity of our ring. In degree one every element is of the form $f \cdot (a_0 x + a_1 y + a_2 y^2 + \cdots + a_n y^n)$, here f is the same function on $\mathbb{A}^1$ as y but now in degree 1. Every element of degree $m > 0$ is of the form $f^m \cdot (a_0 x + a_1 y + a_2 y^2 + \cdots + a_n y^n)$. Hence this ring is generated in degree 1. The calculation on the ring graded in negative degree is exactly the same, except with a function g. We note that $fg = y^2$ as f and g are both equal to y as functions on $\mathbb{A}^1$. We finally discuss the function x. Given a function $F \in \mathbb{C}(X)$, then $F = \sum F_i \chi^{m_i}$, here the F_i are functions on the curve Y and χ^{m_i} are monomials in the lattice M. By construction the function $x = \mathbb{1}_Y \chi^0$, hence x is the constant function on the variety. Hence we get the ring $\mathbb{C}[f, g, y]/(fg = y^2)$, so an A_1 singularity.

We can more generally describe what occurs with polyhedral divisors of the form $[\frac{a}{b}, \frac{c}{d}] \otimes P_0$ with $\frac{a}{b} < \frac{c}{d}$. We note that the tail cone δ is always 0 and so the dual tail cone is all of M. From the definitions we get the ring $R = \bigoplus_{m \in M} R_m \chi^m$ where

$$R_m = \begin{cases} \mathcal{O}_{\mathbb{A}^1_z}(\frac{ma}{b}P) & \text{if } m \geq 0, \\ \mathcal{O}_{\mathbb{A}^1_z}(\frac{mc}{d}P) & \text{if } m \leq 0. \end{cases}$$

We can associate the monomial $z^u\chi^v$ with the lattice point (u, v) inside a two dimensional monomial lattice. This gives rise to a cone σ. The set of monomials with poles of order at worst $\frac{a}{b}$ gives rise to the vector $(a, -b)$. Similarly the other side of the graded ring gives rise to the vector $(-c, d)$. These are boundary rays of σ, this means that as toric variety they can be described as the cone (a, b), (c, d) inside the lattice $N \cong \mathbb{Z}^2$ with torus action corresponding to $(1, 0)$. Hence we get a toric variety of the form $\frac{1}{r}(\alpha,\, \beta)$ where $r = bc - ad$ and α and β are generators of the kernel of the matrix M modulo r where

$$M = \begin{pmatrix} c & a \\ d & b \end{pmatrix}$$

Example 4.4.2. We now show how this behaves in the case of a divisor with tail cone $\delta = [0,\, \infty)$. Consider the divisor $\mathcal{D} = \left[\frac{1}{2},\, \infty\right) \otimes P_0 + \left[\frac{1}{2},\, \infty\right) \otimes P_1 + \left[\frac{-1}{2},\, \infty\right) \otimes P_2$ over $\mathbb{P}^1$ with coordinates x_1, x_2. Here the varieties X and $\widetilde{X}$ are different. To start with, we look at how to construct X. For simplicity we assume $P_0 = (1;\, 0)$, $P_1 = (1;\, 1)$ and $P_2 = (0;\, 1)$.

The tail cone is $\delta = [0,\, \infty)$ so $\hat{\delta} = [0,\, \infty)$. By calculating $\mathcal{D}(m)$ we get the following ring

$$\bigoplus_{m \in M|_{\geq 0}} \mathcal{O}_{\mathbb{P}^1}\left(\frac{m}{2}P_0 + \frac{m}{2}P_1 + \frac{-m}{2}P_2\right)$$

Once again in degree 0 we get the constant function. In degree one we get no functions. In degree 2 we get 2 functions $\frac{x_2}{x_1}\chi^2$ and $\frac{x_2}{x_1 - x_2}\chi^2$ denote these u and v. In degree 3 we get the function $\frac{x_2^2}{x_1(x_1 - x_2)}\chi^3$ denote this by w. We then have the relations $w^2 = uv(v - u)$. This gives rise to a D_4 singularity.

To calculate $\widetilde{X}$ we do all these calculations as relative spec. In particular this means we can replace our above graded ring with the following three graded rings

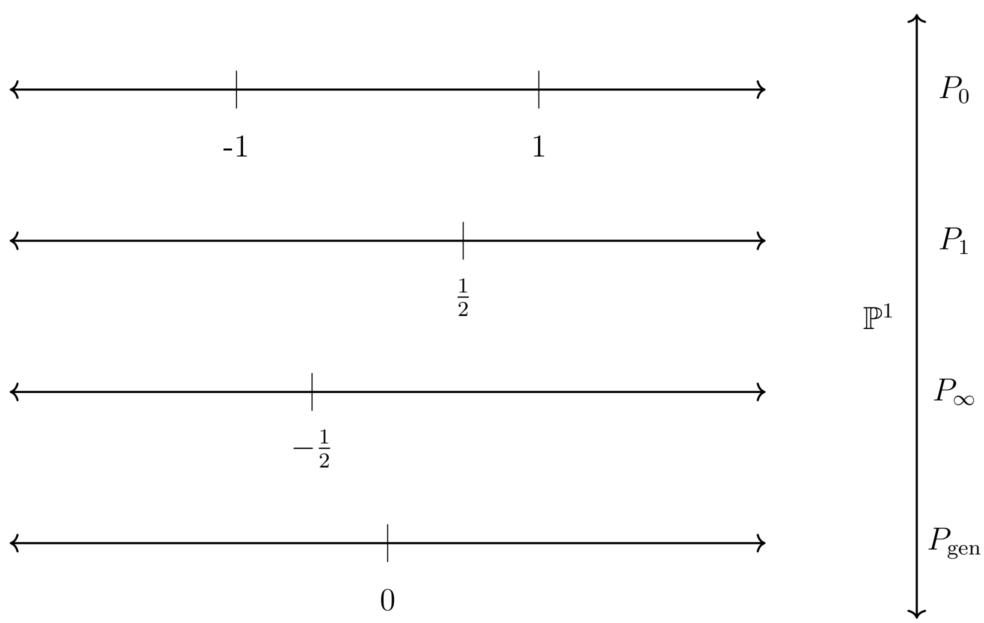
$$\bigoplus_{m \in M|_{\geq 0}} \mathcal{O}_{(\mathbb{P}^1 - P_i - P_j)}\left(\frac{m}{2}P_0 + \frac{m}{2}P_1 + \frac{-m}{2}P_2\right)$$

For all choices of i and j. We then glue together on the intersection. Calculating in the case $i = 1$ and $j = 2$. We have

$$\bigoplus_{m \in M|_{\geq 0}} \mathcal{O}_{(\mathbb{P}^1 - P_1 - P_2)}\left(\frac{m}{2}P_0 + \frac{m}{2}P_1 + \frac{-m}{2}P_2\right) \cong \bigoplus_{m \in M|_{\geq 0}} \mathcal{O}_{(\mathbb{A}^1 - P_1)}(\frac{m}{2}P_0)$$

This gives us the ring $\mathbb{C}[x, \frac{1}{x+1}, \frac{1}{x}\chi^2, \chi^1] = \mathbb{C}[u, v, w, t]/[v(u+1) = 1,\ uw = t^2]$. Hence this is an A_1 singularity. The calculations on the other 3 patches are the same, so we have taken the partial resolution of the D_4 singularity by extracting the trivalent curve.

Example 4.4.3. Consider the following polyhedral fan with marking $\mathbb{Q}^{\pm}$



We now go through each polyhedral divisor and calculate the associated rings. There are three polyhedral divisors contained inside this polyhedral fan which are one dimensional cones. There are three polyhedral divisors which correspond to two dimensional cones.

a) The unmarked polyhedron $[-1, 1]$

b) The marked cone $1 \otimes P_0 + \frac{1}{2} \otimes P_1 + \frac{-1}{2} \otimes P_\infty$ with tail cone $[0, \infty)$.

c) The marked cone $-1 \otimes P_0 + \frac{1}{2} \otimes P_1 + \frac{-1}{2} \otimes P_\infty$ with tail cone $(-\infty, 0]$.

In case a the tail cone δ is equal to 0. We denote the polyhedral divisor by $\mathcal{D}$. So the dual of the tail cone $\hat{\delta}$ is the lattice M itself. Given an element $m > 0$ of M we have $\mathcal{D}(m) = -mP_0$. If $m < 0$ we get $\mathcal{D}(m) = mP_0$ and if $n = 0$ we get $\mathcal{D}(m) = 0$ as a divisor on $\mathbb{A}^1$.

Hence we have the M graded ring

$$\bigoplus_{m \in M} \mathcal{O}_{\mathbb{A}^1}(-|m|P)$$

In degree 0 the ring is generated by the constant function x and the function y which is zero at the origin. We note that the function 1 in degree 0 is the multiplicative identity of our ring. In degree one every element is of the form $f(a_0x + a_1y + a_2y^2 + \ldots a_ny^n)$, here f is the same function as y but now in degree 1. Every element of degree $m > 0$ is of the form $f^m(a_0x + a_1y + a_2y^2 + \ldots a_ny^n)$. Hence this is generated in degree 1. The calculation on the ring graded in negative degree is exactly the same. Hence we get the ring $\mathbb{C}[f, g, y]/(fg = y^2)$, so an A_1 singularity.

In case a the tail cone δ is equal to $[0, \infty)$. We denote the polyhedral divisor by $\mathcal{D}$. So the dual of the tail cone $\hat{\delta}$ is $[0, \infty) \subset M$. Given an element $m \geq 0$ of M we have $\mathcal{D}(m) = mP_0 + \frac{m}{2}P_1 + \frac{-m}{2}P_2$.

Hence we have the M graded ring

$$\bigoplus_{m \in M|_{\geq 0}} \mathcal{O}_{\mathbb{P}^1}\left(mP_0 + \frac{m}{2}P_1 + \frac{-m}{2}P_2\right) \cong \bigoplus_{m \in M|_{\geq 0}} \mathcal{O}_{\mathbb{P}^1}\left(\frac{m}{2}P_1 + \frac{m}{2}P_2\right)$$

Here the isomorphism follows via linear equivalence of divisors on $\mathbb{P}^1$.

In degree 0 we just get the constant function. In degree 1, once again it is only the constant function, denoted this by x. In degree 2 we have the function with a pole at P_1 and a zero at P_2, denote this by f and the function $g = \frac{1}{f}$. This generates the ring hence we have $\mathbb{C}[x,\, f,\, g]/(fg = x^4)$, so an A_3 singularity.

Case c is exactly the same as case b but with the grading negative instead of positive. So it is also an A_3 singularity.

We finish by discussing the unmarked polyhedral divisors

A) The unmarked cone $-1 \otimes P_0$

B) The unmarked cone $1 \otimes P_0$

C) The unmarked cone $\frac{1}{2} \otimes P_1$

D) The unmarked cone $-\frac{1}{2} \otimes P_2$

In all case the tail cone δ is equal to 0. In case A we have the following ring graded by M

$$\bigoplus_{m \in M} \mathcal{O}_{\mathbb{A}^1}(-mP) \cong \mathbb{C}[x,\, y,\, z]/(yz = 1)$$

Here x is of degree 0, y is of degree 1 and z is of degree -1. Case B is symmetrical to case A.

Cases C and D are also symmetrical, and give rise to the rings $\mathbb{C}[x, y_1, y_2, z_1, z_2]$ with relations induced by the second Veronese embedding of $C[x,\, y,\, z]/(yz = 1)$.

We also briefly describe the inclusion maps. We note that

$$\bigoplus_{m\in M|_{\leq 0}} \mathcal{O}_{\mathbb{P}^1}\left(-mP_0+\frac{-m}{2}P_1+\frac{m}{2}P_2\right)$$

is trivially contained inside

$$\bigoplus_{m\in M} \mathcal{O}_{\mathbb{A}^1}(-mP_0)$$

Hence we get an associated map of affine patches.

In toric varieties full dimensional cones give rise to torus fixed point. Analogously, the same way for varieties of higher complexity every full dimensional subdivision of the plane gives rise to a toric fixed point. In the case of surfaces these fixed points can be classified giving rise to three cases

- **Elliptic** - Around the fixed point in local coordinates, the torus behaves on all coordinates with positive or negative degree. These points are isolated.
- **Parabolic** - These always arise as blowups of elliptic points, these occur when in local coordinates, one of the coordinates is acted trivially upon by the torus. These points lie on a section of the map to Y
- **Hyperbolic** - These are where the the local coordinates are acted in positive and negative degree.

It is easy to see that hyperbolic points correspond to a subdivision with $\delta = 0$, parabolic correspond to an unmarked edge going to infinity and elliptic to a marked point going to infinity.

4.5 Divisors in complexity one

We now limit ourselves strictly to complexity one, and the Chow quotient Y will now be $\mathbb{P}^1$. In the torus setting we know that divisors correspond to rays of the associated fan. Almost exactly the same is true in complexity one: divisors occur as torus invariant divisor, these correspond the codimension 1 polyhedral divisors or they are premimages of the $\mathbb{P}^1$. These correspond to a polyhedral divisor $\mathcal{D}$ going of to infinity in a direction, with $\dim(\delta) = \infty$ which for all $P \in \mathbb{P}^1$ we do not have $\mathcal{D}|_P = \varnothing$. Note that this also holds for higer dimensions, with a little bit of extra work. From this it is easy to derive the following theorem

Theorem 4.5.1. *[17] The Picard rank of a complexity one surface defined by a polyhedral fan $\mathcal{S}$ is*

$$\rho_X = 2 - \textit{Number of markings} + \sum_{P \in Y} (\#\mathcal{S}_P^{(0)} - 1)$$

where n is the dimension and $\#\mathcal{S}_P^{(0)}$ is the number of points on this slice of the fan. In a similar style to this we can classify Cartier divisors, we here make no pretense at proof or justification.

Definition 4.5.2. A divisorial support function h on a divisorial fan $\mathcal{S}$ is a piecewise linear function on each component of the fan such that

- On every polyhedron $\delta \in \mathcal{S}_{P_i}$ it is a linear function
- h is continuous
- at all points h has integer slope and integer translation
- if $\mathcal{D}_1$ and $\mathcal{D}_2$ have the same tail cone, then the linear part of h restricted to them is equal

We call a support function principal if it is of the form $h(v) = \langle u, v \rangle + D$, this corresponds to a principal Cartier divisor. We call a support function Cartier, if on every component with complete locus the support function is principal. In the case of Fansy divisors, this just correspond to the edge with a marking. We denote

h restricted to a component by h_P. We refer to a piecewise linear function with rational slope and rational translation as a $\mathbb{Q}$ support function.

Theorem 4.5.3. *[17] Let X be the variety associated with the divisorial fan $\mathcal{S}$. There exists a one to correspondence between support functions quotiented by principal support functions and Cartier divisors on the complexity one variety. In addition there exists a one to correspondence between $\mathbb{Q}$ support functions support function quotiented by principal support functions and $\mathbb{Q}$-Cartier divisors on the complexity one variety*

Using the above languages we represent the canonical divisor as a Weil divisor, it has the following form

Theorem 4.5.4. *[17] The canonical divisor of a complexity one surface can be represented in the following form*

$$K_X = \sum_{(P,v)} (\mu(v) K_Y(P) + \mu(v) - 1) \cdot D_{(P,v)} - \sum_{\rho} D_\rho$$

Here $K_Y(P)$ is the degree of K_Y at P, and $\mu(v)$ is the smallest value k such that $k \cdot v \in \mathbb{N}$. While I have not stated the conditions for linear equivalence these can be seen in [17], and using these you can show that it does not depend on the choice of representative of K_Y. Note that given the singularities and varieties we are working with we know that our K_X will be $\mathbb{Q}$-Cartier. The Fano index is clear and easy to derive from the singularities we have, so all that remains is to check on the conditions for a complexity one divisor to be ample.

Theorem 4.5.5. *[17] A suppport function h is ample iff for all P we have h_P is strictly concave, and for all polyhedral divisors $\mathcal{D}$ defined on an affine curve we have*

$$-\sum_{P \in \mathbb{P}^1} h_P|_{\mathcal{D}}(0) \in Weil_{\mathbb{Q}}(Y)$$

is an ample $\mathbb{Q}$-Cartier divisor.

Note that in reality $h_P|_{\mathcal{D}}$ may not be defined at 0 but we can extend the affine function to 0. We finish this recap on divisors by describing the Weil divisor corresponding to a Cartier divisor

Theorem 4.5.6. *[17] Let $h = \sum_P h_P$ be a Cartier divisor on $\mathcal{S}$ then the corresponding Weil divisor is*

$$-\sum_{\rho} h_t(n_\rho)D_\rho - \sum_{(P,v)} \mu(v)h_P(v)D_{(P,v)}$$

Here n_ρ is the generator of the ray inside the tail fan and $\mu(v)$ is as before. Note that is easy to see why we need this μ function. If you start with a closed subinterval $[a, b]$ and try to work out what the corresponding affine variety is, we see that it just the toric variety defined by the cone $(a, 1)$, $(b, 1)$, and then all the calculations can be done in the realm of toric varieties, however there you use the generator of your rays in the lattice.

We use the above note to easily calculate the minimal resolution of a complexity one surface. Note that we can split this across affine charts, in the first case if we have the affine chart corresponding to the polyhedral divisor $[a, b]$ then using the above point we can calculate this by the toric methods. In case two where we have a non marked edge going to infinity, we can split this into affine charts $[a_i, \infty)$ this is also a toric chart corresponding to the cone $(a, 1)$, $(1, 0)$, so once again the resolution is toric. The final case is with a marked edge, however we can take a weighted blowup to resolve the elliptic point, then resolve the resulting singularities by the above methods. To calculate the intersection numbers on the resolution you can either use [17] or you can note that the only part that is not toric is the parabolic line, this is defined by glueing together charts coming from $[a'_i, \infty)$ Now by smoothness we know all the $a'_i \in \mathbb{Z}$, hence we get an isomorphism of local charts to the charts defined by $[\sum(a_i'), \infty)$ at P_1 and $[0, \infty)$ for all other P_i. Hence we see that the parabolic line is define torically as the fan $(\sum(a_i'), 1)$, $(1, 0)$, $(0, -1)$ from this an easy derivation of the intersection number follows.

4.6 Algorithm

We begin with the following lemma:

Lemma 4.6.1. *[17] Let S be a non cyclic complexity one log terminal surface singularity. Then S has, up to isomorphism, a fan over $\mathbb{P}^1$ with coefficients*

$$\left[\frac{p_1}{q_1}, \infty\right) \otimes P_1 + \left[\frac{p_2}{q_2}, \infty\right) \otimes P_2 + \left[\frac{p_3}{q_3}, \infty\right) \otimes P_3$$

with (q_1, q_2, q_3) satisfying $\sum(1 - \frac{1}{q_i}) < 2$.

Proof. See [17] □

In particular this means the only possible denominators are $(2, 2, m)$, $(2, 3, 3)$, $(2, 3, 4)$ and $(2, 3, 5)$, we call these of type D_n, E_6, E_7 and E_8 respectively.

We now state the following lemma

Lemma 4.6.2. *Let S be a log terminal surface singularity of Gorenstein index l. Let E be an exceptional curve in the minimal resolution. Then $E^2 \geq -2l$ if it is not a trivalent curve and $E^2 \geq -3l$ if it is trivalent.*

Proof. Via the classification of log terminal singularities we have that E intersects at most three other exceptional curves. Denote the discrepancies of these curves d_1, d_2, d_3, note that any d_i could be equal to zero. Also note that $0 \geq d_i \geq -1$. Denote the discrepancy of E by d. Then we have the formula $dE^2 + \sum d_i = 0$. This rearranges to $d = \frac{(\sum d_i)}{E^2} \leq \frac{-3}{E^2}$ as the singularity is log terminal. As $d \in \frac{1}{l}\mathbb{Z}$ we get

$E^2 \geq -3l$. In the case of a non trivalent curve, we can assume $d_3 = 0$ and we see that $E^2 \geq -2l$. □

Lemma 4.6.3. *Given a complexity one log del Pezzo surface of index l then there cannot be more than $6l$ points where the polyhedral fan is not the tail fan*

Proof. Taking the minimal resolution of our log del Pezzo, this admits a map to a Hirzebruch surface $\mathbb{F}_n$. As we are contracting -1-curves our map is invariant under the torus action. Hence this is a torus action on the Hirzebruch surface. Any series of complexity one non toric blowups on a toric surface correspond to blowing up points on a line of invariant points. We note that by the above lemma we cannot get a map to $\mathbb{F}_n$ when $n > 3l$. Hence the largest possible self intersection of a torus invariant curve on our Hirzebruch surface is $3l$ and the smallest possible intersection on our minimal resolution is $-3l$ so there can only be $6l$ blowups on the curve. □

Remark. In the case of index one, we know Du Val singularities only have -2 curves in the resolution hence this bound can be refined to four non general fibers.

Lemma 4.6.4. *Consider the following polyhedral fan:*

$$\mathcal{S} = \begin{array}{cc} \left[a_1^1,\, a_2^1,\, \ldots,\, a_{n_1-1}^1,\, a_{n_1}^1\right] & P_1 \\ \left[a_1^2,\, a_2^2,\, \ldots,\, a_{n_2-1}^2,\, a_{n_2}^2\right] & P_2 \\ \left[a_1^3,\, a_2^3,\, \ldots,\, a_{n_3-1}^3,\, a_{n_3}^3\right] & P_3 \\ \left[a_1^4,\, a_2^4,\, \ldots,\, a_{n_4-1}^4,\, a_{n_4}^4\right] & P_4 \\ \vdots & \\ \left[a_1^{6k},\, a_2^{6k},\, \ldots,\, a_{n_{6k-1}}^{6k},\, a_{n_{6k}}^{6k}\right] & P_{6k} \end{array}$$

If $\mathcal{S}$ defines a log del Pezzo surface X of index k then $\sum_j \lfloor a_1^j \rfloor \geq -6k$ and $\sum_j \lceil a_{n_j}^j \rceil \leq 6k$.

Proof. Let Y be the minimal resolution of X, and consider $\widetilde{Y}$. This has two parabolic curves E_1 and E_2 corresponding to $[-0, \infty)$ and $(-\infty, 0]$. The self intersections of these curves are $\sum_j \lceil a^j_{n_j} \rceil \leq 6$ and $\sum_j \lfloor a^j_1 \rfloor \geq -6$ respectively, and these values have to be less than $6k$. $\square$

Just as in the case of Gorenstein index one, where the singularities are formed of -2 curves. There is an explicit way to classify the resolutions of singularities of higher index.

Lemma 4.6.5. *A singularity of index n which is non toric log terminal singularities can be described by one of the following polyhedral divisors*

1. *Corresponding to D_n singularities we have $\left[-\frac{1}{2}, \infty\right) \otimes P_1 + \left[\frac{1}{2}, \infty\right) \otimes P_2 + \left[\frac{n}{m}, \infty\right) \otimes P_3$ if m is odd. If m is divisible by 4 we have $\left[-\frac{1}{2}, \infty\right) \otimes P_1 + \left[\frac{1}{2}, \infty\right) \otimes P_2 + \left[\frac{2n}{m}, \infty\right) \otimes P_3$ and if m is even but not divisible by four this case does not occur.*
2. *Corresponding to E_6 singularity we have $\left[-\frac{1}{2}, \infty\right) \otimes P_1 + \left[\frac{p}{3}, \infty\right) \otimes P_2 + \left[\frac{q}{3}, \infty\right) \otimes P_3$ where either $p = q$ mod 3 and $2(p+q) = n+3$, or $p \neq q$ mod 3 and $6(p+q) - 9 = n$*
3. *Corresponding to E_7 singularity we have $\left[-\frac{1}{2}, \infty\right) \otimes P_1 + \left[\frac{p}{3}, \infty\right) \otimes P_2 + \left[\frac{q}{4}, \infty\right) \otimes P_3$ where either q is even and $2(4p+3q-6) = n$ or q is odd and $(4p+3q-6) = n$.*
4. *Corresponding to E_8 singularity we have $\left[-\frac{1}{2}, \infty\right) \otimes P_1 + \left[\frac{p}{3}, \infty\right) \otimes P_2 + \left[\frac{q}{5}, \infty\right) \otimes P_3$ where $10p + 6q - 15 = n$.*

Proof. We note that for a multiple of the canonical divisor to be Cartier the corresponding slope function has to have integral slope and $h_P(0) \in \mathbb{Z}$ for all $P \in \mathbb{P}^1$. We note the slope of the divisorial polytope corresponding to K_X for a polyhedral divisor of the form $\left[-\frac{1}{2}, \infty\right) \otimes P_1 + \left[\frac{1}{2}, \infty\right) \otimes P_2 + \left[\frac{p}{q}, \infty\right) \otimes P_3$ is $\frac{1}{p}$. Hence the index of the singularity, n, is a multiple of p for the slope to be integer. We now note $h_{P_1}(0) = \frac{-1}{2} + \frac{1}{2}\frac{1}{p} = \frac{1}{2}(-1 + \frac{1}{p})$ and, via the same logic, $h_{P_2}(0) = \frac{1}{2}(1 + \frac{1}{p})$. Now if p is odd this becomes a fraction $\frac{a}{p}$ and if p is even this becomes $\frac{a}{2p}$. Now $h_{P_3}(0) = \frac{q-1}{q} + \frac{p}{q}\frac{1}{p}$, this is always just 1. Hence, for $kh_P(n)$ to be an integer for every

$P \in \mathbb{P}^1$ and $n \in \mathbb{Z}$, we need k to be a multiple of p if p is odd and a multiple of $2p$ if p is even.

For the E_6 case consider the following divisor $\left[-\frac{1}{2}, \infty\right) \otimes P_1 + \left[\frac{p}{3}, \infty\right) \otimes P_2 + \left[\frac{q}{3}, \infty\right) \otimes P_3$. Once again we calculate the slope corresponding to the canonical divisor and get $\frac{1}{2(p+q)-3}$. We have the following calculations $h_{P_1}(0) = \frac{-1}{2}(1 + \frac{1}{2(p+q)-3}) = -\frac{p+q-1}{2(p+q)-3}$, $h_{P_2}(0) = \frac{1}{3}\frac{3-p-2q}{2(p+q)-3}$ and $h_{P_3}(0) = \frac{1}{3}\frac{3-2p-q}{2(p+q)-3} + 1$. Hence this has index $3(2(p+q)-3)$ if p is not equal to q mod 3, and $2(p+q)-3$ otherwise.

For the E_7 case consider the following divisor $\left[-\frac{1}{2}, \infty\right) \otimes P_1 + \left[\frac{p}{3}, \infty\right) \otimes P_2 + \left[\frac{q}{4}, \infty\right) \otimes P_3$. Then the slope is $\frac{1}{4p+3q-6}$. We have the following calculations $h_{P_1}(0) = \frac{1}{2}\frac{4p+3q-7}{4p+3q-6}$, $h_{P_2}(0) = \frac{1}{3}\frac{6-3p-3q}{4p+3q-6} = \frac{2-p-q}{4p+3q-6}$ and $h_{P_3}(0) = \frac{1}{4}\frac{6-4p-2q}{4p+3q-6} = \frac{1}{2}\frac{3-2p-q}{4p+3q-6}$. Hence this has index $2(4p+3q-6)$ if q is even and index $(4p+3q-6)$ otherwise.

We finish with E_8 case. Consider the following divisor $\left[-\frac{1}{2}, \infty\right) \otimes P_1 + \left[\frac{p}{3}, \infty\right) \otimes P_2 + \left[\frac{q}{5}, \infty\right) \otimes P_3$. Then the slope is $\frac{1}{10p+6q-15}$. We have the following calculations $h_{P_1}(0) = \frac{1}{2}\frac{10p+6q-16}{10p+6q-15} = \frac{5p+3q-8}{10p+6q-15}$, $h_{P_2}(0) = \frac{5-3p-2q}{10p+6q-15}$ and $h_{P_3}(0) = \frac{3-2p-q}{10p+6q-15}$. Hence this has index $10p+6q-15$. □

Remark. This imposes stricter bounds on the number of non generic fibers a complexity one surface of index n can have than Lemma 4.6.5 and enables you to put individual bounds given a singularity.

We also need a bound on what possible toric singularities can occur and the possible actions on them.

Lemma 4.6.6. *Let X be a non toric non smooth complexity one log del Pezzo surface described by a polyhedral fan Ξ. Then the map to the invariant $\mathbb{P}^1$ is not a morphism and the locus of indeterminacy contains a singular point. In addition, consider the cones σ_1 and σ_2 with tail fan $[0, \infty)$ and $(-\infty, 0]$, and the associated affine charts X_{σ_1} and X_{σ_2}. Then for at least one of these charts, X_i, the minimal resolution Y_i admits a morphism to $\widetilde{X}_i$.*

Proof. Let Y be the minimal resolution of X. Then X admits an equivariant map to $\mathbb{F}_i$ as every -1-curve on Y is torus invariant. Hence Y is constructed by a non toric blowup of a toric surface. Considering the first non toric blowup, we are blowing up a point P on a torus invariant curve C. Clearly C is a torus invariant curve in the minimal resolution of X.

To show C is not a curve on the surface we note that if C has to be a torus invariant curve on the Hirzebruch surface $\mathbb{F}_i$, otherwise we would blow up a -1-curve at a general point and the resulting curve would have to be contracted. If C was the positive section then the contraction of the negative section would give a singularity with the desired properties. If C was a fiber, C needs to be blownup at least twice to be non toric, which would make it have negative self intersection, meaning it would have be contracted. This would once again give a curve with the desired properties. $\square$

We finish with the following observation that helps shorten calculations. Let X be a complexity one surface with the following polyhedral fan:

$$\begin{array}{cc} P_1 & [a_1, \ldots, a_n] \\ P_2 & [b_1, \ldots, b_m] \\ P_3 & [c_1, \ldots, c_o,] \\ P_4 & [d_1, \ldots, d_{p-1}, 0] \end{array}$$

Also assume that the minimal resolution of X contains a curve E which is pointwise fixed by the action and which is contracted to point corresponding to the tail cone $[0, \infty)$.

Let h be the piecewise linear function corresponding to $-K_X$. Then $h_P|_{[0,\infty)}$ is defined by a unique value $u \in \mathbb{Q}$. If $-K_X$ is ample then the only possible values for d_{p-1} are the values $\frac{p}{q}$, with $q > 0$, such that $\frac{q-1}{q} < u\frac{p}{q}$, this is to preserve convexity.

As there are a finite number of log del Pezzo surfaces, up to deformation, with Gorenstein index k there can only be a finite number of singularities S which can occur on such a surface.

Algorithm 1. We set out an algorithm to classify toric log del Pezzo surfaces of index K. This works by classifying the possible convex polytopes $C \subset N$ such that the spanning fan Σ gives rise to a toric log del Pezzo surface of degree K.

This algorithm is split into two parts, the first part classifies how, given the first n vertices of a polytope to find a potential $(n+1)$'rst vertex which satisfies all the conditions. There is then a second part which bounds all potential starting edges.

The first algorithm start with an initial edge of the polygon, which upto the the action of the lattice transformations can be assumed to be a horizontal line so $P_{n-1} = (a, h)$, $P_n = (b, h)$ with $-h < a < 0$ and $b > a$. Given a point $P_{n+1} \in N$ such that the cone σ with rays P_n, P_{n+1} corresponds to a toric singularity of we associate a triple (i, j, k) in the following way:

- i is the Gorenstein index of the toric singularity corresponding to σ.
- Consider the line segment $L_i = \{P_n + \frac{i}{h}(1,0) - uP_n | \, u \in \mathbb{Q}_{\geq 0}\}$. Now considering the edge$E$ connecting P_n and P_{n+1} there is a unique point $v = L_i \cap E \cap N$. So $v = P_n + \frac{i}{h}(1,0) - uP_n$ for some $u \in \mathbb{Q}$. Then $j = u$.
- k is $\lfloor \frac{n}{i} \rfloor$ where n is the number of points on the edge P_n, P_{n+1} (This is the Picard rank of the T-resolution, as in [1]).

To every point there is a unique triple, however there are triples which do not have an associated point. Clearly there are only finite options for i as the Gorenstein index has to divide K. We say a point P_{n+1} is not valid if for every P_{n+2} such that the cone between P_{n+1} and P_{n+2} is of index dividing K then the connecting edge E_1 from P_{n+1} to P_{n+2} intersects the edge connecting P_{n-1} and P_n.

Give a triple (i, j, k) we say $P_{(i,j,k)}$ is the point associated to the triple (i, j, k). We now note that if $P_{(i,j,k)}$ is not valid then neither is $P_{(i,j+a,k+b)}$. This is proved by induction, and it suffices to show that $P_{(i,j+1,k)}$ and $P_{(i,j,k+1)}$ are not valid.

To prove that $P_{(i,j,k)}$ not valid implies $P_{(i,j+1,k)}$ not valid, assume that $P_{(i,j+1,k)}$ is valid and there is an associated point P which can be constructed from $P_{(i,j+1,k)}$. Let $w \in M$ be such that w is constant on the line L_i and assume it takes value $m > 0$ on every point on L_i. Also let us say $a = w \cdot P$. Then $P - (a + m - 1)P_2$ is a valid point for $P_{(i,j,k)}$ and hence $P_{(i,j,k)}$ is valid.

Similarly assume $P_{(i,j,k+1)}$ is valid, and P and there is an associated point P which can be constructed from $P_{(i,j+1,k)}$. Let $w \in M$ be the vector such that $w \cdot P_n = w \cdot P_{(i,j,k+1)} = m > 0$. Let v be the vector $\lfloor \frac{P_n - P_{(i,j,k+1)}}{i} \rfloor$ and let $a = w \cdot P$. Then $P - (a + m - 1)v$ is a valid point for $P_{(i,j,k)}$ and hence $P_{(i,j,k)}$ is valid.

All that remains is to prove that if j or k are large enough there are no valid points. Given P_1, P_2 and $P_{(i,j,k)}$, construct the associated toric variety X. Let $\widetilde{X}$ be the minimal resolution, and $D_{i,j,k} \subset \widetilde{X}$ be the divisor corresponding to the ray with primitive vector P_2. As we are considering log del Pezzo surfaces $D_{i,j,k}^2 \geq -1$, and as we are considering surfaces of index K, $D_{i,j,k}^2 \leq 3K$ by Lemma 4.6.3. By construction, we can see that $D_{i,j+1,k}^2 = D_{i,j,k}^2 + 1$ so j is bounded. Given a point $P_{(i,j,k)}$, consider all possible points P connecting to $P_{(i,j,k)}$. Denote by v_P the intersection of the line connecting P and $P_{(i,j,k)}$ with the line $y = h$. Finally let $v = min(v_E)$, it is straightforwards to verify, by the previous affine transformation, that v limits towards b. If $v > a$ then there are no valid points, hence if k is large enough there are no potential points. Hence given the beginning of a polytope this gives only a finite amount of ways to extend it.

We now bound the potential starting edges. Once again we can assume up to lattice transformations we can assume the first edge is (a, h), (b, h) where $h|K$ and $-h \geq a < 0$ and $b \geq 0$. For any value of b there are at least h valid points. These are the points with y coordinate in the range 0 through to $h - 1$. We denote these points $P_{b,y}$, where y is the y-coordinate. We wish to show that if there are no valid points connecting to $P_{b,y}$ then there are no valid points connecting to $P_{b+h,y}$. To prove this, assume there is a valid point connecting to $P_{b+h,y}$, denote it $P = (x_2, y_2)$.

Then $P' = (x_2 - y_2, y_2)$ is a valid point connecting to P. To show that $P_{b,y} = (x, y)$ eventually gives rise to no further valid points if b is large enough follows a similar proof to the asymptotics of k above. It can be shown that if you consider the next if b is large enough let P be a vertex connecting to $P_{b,y}$ and P' a vertex connecting to P. Then eventually the edge E connecting P to P' will always intersect the line $y = h$ at a point greater than a. This would violate convexity and so would not give rise to any options.

We note that this algorithm allows us to classify any convex fan of Gorenstein index K connecting any two points in the plane. This occurs by exactly the same principles, namely bounding the initial edges length and then doing a growing algorithm from that initial side. We now state the complexity one algorithm.

Algorithm 2. By Lemma 4.6.6 there is a singularity whose minimal resolution is mapped to the $\mathbb{P}^1$ by a morphism. We start by bounding this singularity. We start with the non-cyclic case, by Lemma 4.6.5 there are only finitely many possibilities and so there is nothing to bound. In the cyclic case there is an infinite family for each potential index and for a given singularity there are finitely many ways of picking a sub torus action so that it is compatible with Lemma 4.6.6.

Without loss of generality we can assume there are no no cyclic singularities otherwise we could start from the non cyclic singularity and grow from there. Because of this we can assume there are only cyclic quotient singularities. Given this starting cyclic quotient singularity is equivalent to the polyhedral divisor

$$\mathcal{D} = \left[\frac{h}{a}, \infty\right) \otimes P_1 + \left[\frac{h}{b}, \infty\right) \otimes P_2 + \sum_{i=3}^{6K} [0, \infty) \otimes P_i$$

Note $h|K$ as h is the index of the singularity. Now by Lemma 4.6.5 if this singularity lies on a log del Pezzo surface it the sum of the smallest values in each slice must be greater than $-6K$. We are now split this into three cases.

Case 1: The polyhedral divisor with tail cone $(-\infty, 0]$ has affine locus. For this to lie on a log del Pezzo surface, is a stronger condition than asking for a convex polytope connecting $(-b, h)$, (a, h) to $(-1, 0)$ and a polytope connecting $(-a, h)$, (b, h) to $(-1, 0)$. These are both classifiable by the second part of the toric algorithm and hence there are only finitely many cases.

Now we consider the cases where this map to $\mathbb{P}^1$ is not a morphism at two points of the variety. Then our polyhedral divisor with negative tailfan is

$$\mathcal{D}_2 = \sum(-\infty, a_1^i] \otimes P_i$$

Here at most two a_i are non integers. This leaves us the following cases
Case 2: a_1^i for $i > 2$ is an integer. Then if X had polyhedral slice $a_1^i, \cdots, a_{n_i}^i$ at point P_i. Let $a_i^1 = \frac{a_i}{b_i}$ and $a_i^2 = \frac{c_i}{d_i}$. In addition let $k = \sum_3^{6K} a_1^i$. then a necessary condition for X to be a log del Pezzo surface is that the toric surface whose fan has rays generated by the vectors $(c_i, -d_i)$ and $(a - kb_i, b_i)$ must also be a log del Pezzo surface. These toric surfaces have a bounded Gorenstein index and are classifiable by the toric algorithm.

Case 3: One of a_1^i for $i > 2$ is not an integer, denote this value j. Without loss of generality a_1^2 is an integer. Assume $a > h$, we can do this as we are trying to bound the value of a. Then via the same principles as case 2, we need the spanning fan of the polytope with vertices connecting (a, h) to $(-k, -1)$ to have index dividing K. Once again this fits into the second part of the toric algorithm, and there are finitely many possibilities. To bound b we note that if b is large enough there is only one possible slice at P_2 with the given restraints which is the slice with subdivisions made by $[0, \frac{h}{b}]$. If a surface can arise with this value of b then there also be a log del Pezzo surface with every other being the same but P_2 replaced with $[0, \frac{h}{b-h}]$. Hence this can be checked iteratively for failure

Case 4: Two of a_1^i for $i > 2$ are not integers. Then if a and b are large enough the only possible slices at P_1 and P_2 are $[0, \frac{h}{a}]$ and $[0, \frac{h}{b}]$ respectively. Once again if these are log del Pezzo surfaces then the same would be true for $\frac{h}{a-h}$ and $\frac{h}{b-h}$ so we can proceed inductively.

This shows how we can bound the singularity. We now show how given a singularity we "grow" this into a complete surface. We start by classifying all possible slices of the form $[a^i_1, \ldots, a^i_{n_i-1}, 0]$ with $a^i_1 \in \mathbb{Z}_{\geq -6K}$, this can be done via the toric algorithm as this equivalent to finding all partial fans connecting the coordinates $(a^i_1, 1)$ to $(0, 1)$. We call these degenerate fibers. We now split into cases.

Case 1: The surface has two non cyclic singularities. The polyhedral divisor with tail cone $[0, \infty)$ is of the form specified in Lemma 4.6.6. There is only finitely many ways of having the polyhedral divisor with tail cone $(-\infty, 0]$ being a non cyclic log terminal singularity with index dividing K without it interfering with the values already specified. Consider all these possible ways. This specifies at most 6 points which not degenerate fibers. At a given one of these points P_i the slice is one of the following $[a^i_1, \ldots, a^i_{n_i-1}, 0]$, $[0, \ldots, a^i_{n_i}]$ or $[a^i_1, \ldots, a^i_{n_i},]$. Here a^i_1 and $a^i_{n_i}$ are known so these can be completed via the toric algorithm.

Case 2: One non cyclic singularity. Once again we start with this having tail cone $[0, \infty)$. There will be at most two fibers P_i and P_j with $a^i_1 \notin \mathbb{Z}$ and $a^j_1 \notin \mathbb{Z}$. Let $k = \sum_{u \neq = i,j,1,2} a^u_1 \in [-6K, 6K]$, we iterate over all choices of k. Let $a^i_{n_i} = \frac{a}{b}$ and $a^j_{n_j} = \frac{c}{d}$. Then we start by finding all toric varieties connecting $(a - kb, b)$ to $(c, -d)$ via an anticlockwise direction. Taking the torus downgrade and shifting this gives us the fiber for P_i and P_j. Note, this includes the cases $i, j \in \{1, 2, 3\}$. Every other fiber other than the P_i, P_j, P_1 and P_2 are a degenerate fiber and are classified. If $i, j \in \{1, 2, 3\}$ then the fibers are classified from earlier. Otherwise we classify P_1 and P_2 by assuming a^1_1 and a^2_1 are somewhere in the range $[-6K, 0]$ and enumerating over every possible choice.

Case 3: One non cyclic singularity with tailcone $[0, \infty)$. This is highly similar to case 2 however as $a^3_{n_3} = 0$ we just have to remove studying the case of i, or j equaling

3.

This shows how we can grow these surfaces from an original singularity. The large majority of the work is in studying the original singularities themselves, and then everything else is repeated applications of the toric algorithm.

As an example we illustrate how this can classify Gorenstein log del Pezzo surfaces which have complexity one and only cyclic quotient singularities. This is a simpler task than the general algorithm as the possible fibers are so restricted a lot of the analysis can be done via Lemma 4.6.4 instead of using the full algorithm.

Example 4.6.7. As the Gorenstein index $k = 1$, i.e the only singularities are Du Val, and we are restricting to the case of A_n singularities. We know there is a singularity which has an action which corresponds to a curve on the minimal resolution via Lemma 4.6.6. We iterate over all possible singularities and all possible actions until we can show they do not exist on a projective log del Pezzo surface. We assume that this singularity has polyhedral divisor with tail cone $[0, \infty)$. From now on we assume this without stating it.

To make this easier we make two observations. The first is that if a fiber over a point P_i is $[a_1 = \frac{p_1}{q_1}, \ldots, a_n = \frac{p_n}{q_n}, 0]$ then $q_i = 1$. This follows via the above remarks, first, we know we get an A_m singularity given by $(0, 1)$, (p_n, q_n). Then, via the stated properties of our singularity, we have the slope of $h_P|_{[0,\infty)}$ is -1. So we need $\frac{q_n - 1}{q_n} < \frac{-p_n}{q_n}$ which implies $p_n + q_n < 1$, hence if $q_n > 1$ then $p_n < -1$. Now every point on the line connecting (p_n, q_n) to $(0, 1)$ has to satisfy this inequality. In particular there is a point $(-1, a)$ on this line, with $a \in \mathbb{Z}$, satisfying the inequality, hence $a \leq 1$. As $a > 0$ for $q_n > 0$ we have the only possible value of a is one. This corresponds to $q_n = 1$.

The second observation is, let $\mathcal{D} = \frac{1}{u} \otimes P_1 + \frac{1}{v} \otimes P_2$ be a polyhedral divisor with tail cone $[0, \infty)$. We can assume without loss of generality that $u \geq v$. Then if $u > v + 2$ and $v \geq 2$ then there are no complexity one Gorenstein log del Pezzo surfaces which contain $\mathcal{D}$ as a polyhedral cone. We note as $u, v \geq 2$ we have the only possible polyhedral fans over P_1 are $[0, \frac{1}{u}]$ and $[\frac{1}{u}]$, similarly for P_2. Viewing this from a toric perspective the only points that can be connected to $(1, u)$ or $(0, 1)$ while preserving the necessary convexity are less than $-v$. This implies as a complexity one surface this would need a denominator on another fiber less than v, but this cannot happen.

- A_1 **Singularity**

  This has to have polyhedral divisor

$$\begin{array}{ll} P_1 & [a_1, \ldots, a_n, 1] \\ P_2 & [b_1, \ldots, b_m, 1] \\ P_3 & [c_1, \ldots, c_o, 0] \\ P_4 & [d_1, \ldots, d_p, 0] \end{array}$$

  This gives rise to the following surfaces

$$[1] \otimes P_1 + [0,\ 1] \otimes P_2 + [-1,\ 0] \otimes P_3 + [-1,\ 0] \otimes P_4$$

  and

$$[1] \otimes P_1 + [0,\ 1] \otimes P_2 + [-1,\ 0] \otimes P_3 + [-2,\ 0] \otimes P_4$$

  and finally

$$[1] \otimes P_1 + [0,\ 1] \otimes P_2 + [-1,\ 0] \otimes P_3 + [-1,\ 0] \otimes P_4 + [-1,\ 0] \otimes P_5$$

  We note that the point P_5 is involved, however this does not contradict Lemma 4.6.3 as the fiber over P_1 is the general fiber. So the number of non general fibers is still four.

- A_2 **Singularity**

The only case is $u = v + 1$ this corresponds to the polyhedral divisor

$$\begin{array}{ll} P_1 & [a_1, \ldots, a_n, \frac{1}{2}] \\ P_2 & [b_1, \ldots, b_m, 1] \\ P_3 & [c_1, \ldots, c_o, 0] \\ P_4 & [d_1, \ldots, d_p, 0] \end{array}$$

We have the following two cases for P_1, $[0, \frac{1}{2}]$ and $[\frac{1}{2}]$. In the first case we get the following surface

$$\left[0, \frac{1}{2}\right] \otimes P_1 + [0,\ 1] \otimes P_2 + [-1,\ 0] \otimes P_3$$

In the second case we have the following three surfaces

$$\left[\frac{1}{2}\right] \otimes P_1 + [0,\ 1] \otimes P_2 + [-1,\ 0] \otimes P_3$$

With a marking of $-$, and the surface

$$\left[\frac{1}{2}\right] \otimes P_1 + [0,\ 1] \otimes P_2 + [-2,\ 0] \otimes P_3$$

and finally

$$\left[\frac{1}{2}\right] \otimes P_1 + [0,\ 1] \otimes P_2 + [-1,\ 0] \otimes P_3 + [-1,\ 0] \otimes P_4$$

- A_3 **Singularity**

We start with the case $u = v + 2$ and we have

$$\begin{array}{ll} P_1 & \left[a_1,\, \ldots,\, a_n, \frac{1}{3}\right] \\ P_2 & [b_1,\, \ldots,\, b_m, 1] \\ P_3 & [c_1,\, \ldots,\, c_o, 0] \\ P_4 & [d_1,\, \ldots,\, d_p, 0] \end{array}$$

There are two cases for P_1, $[0, \frac{1}{3}]$ and $[\frac{1}{3}]$. If it is the first case then the denominator has to be greater than 2 so does not occur. In the second case there is only one possibility: the other elliptic singularity is given by toric coordinates $(1,\, 3)$, $(-1, -1)$. This can be constructed by the non toric surface with polyhedral divisor

$$\left[\frac{1}{3}\right] \otimes P_1 + [0, 1] \otimes P_2 + [-1,\, 0] \otimes P_3$$

The second case is $u = v = \frac{1}{2}$ this has the following polyhedral divisor

$$\begin{array}{ll} P_1 & \left[a_1,\, \ldots,\, a_n, \frac{1}{2}\right] \\ P_2 & \left[b_1,\, \ldots,\, b_m, \frac{1}{2}\right] \\ P_3 & [c_1,\, \ldots,\, c_o, 0] \\ P_4 & [d_1,\, \ldots,\, d_p, 0] \end{array}$$

There are two cases for P_1, $[0, \frac{1}{2}]$ and $[\frac{1}{2}]$. The first case gives rise to one surface with the following polyhedral divisor

$$\left[-\infty,\, 0, \frac{1}{2}\right] \otimes P_1 + \left[-\infty,\, \frac{1}{2}\right] \otimes P_2 + [-\infty,\, 0,\, -1] \otimes P_3$$

In the second case we have the following, we get the following three surfaces:

$$\left[-\infty,\, \frac{1}{2}\right] \otimes P_1 + \left[-\infty,\, \frac{1}{2}\right] \otimes P_2 + [-\infty,\, -1,\, 0] \otimes P_3$$

and

$$\left[\frac{1}{2}\right] \otimes P_1 + \left[\frac{1}{2}\right] \otimes P_2 + [-2,\, 0] \otimes P_3$$

and finally

$$\left[\frac{1}{2}\right] \otimes P_1 + \left[\frac{1}{2}\right] \otimes P_2 + [-1,\, 0] \otimes P_3 + [-1,\, 0] \otimes P_4$$

- A_4 **Singularity**

The only case is $u = v + 1$ and we have

$$\begin{array}{cc} P_1 & \left[a_1,\, \ldots,\, a_n, \frac{1}{3}\right] \\ P_2 & \left[b_1,\, \ldots,\, b_m, \frac{1}{2}\right] \\ P_3 & [c_1,\, \ldots,\, c_o, 0] \\ P_4 & [d_1,\, \ldots,\, d_p, 0] \end{array}$$

There are two cases for P_1, $[0, \frac{1}{3}]$ and $[\frac{1}{3}]$. If it is the first case then the denominator has to be greater than 2 so does not occur. So only the second case occurs. In the second case we can have it connecting to a point with y coordinate 1 or y coordinate 2. In the first case this leads to a surface

$$\left[\frac{1}{3}\right] \otimes P_1 + \left[\frac{1}{2}, 0\right] \otimes P_2 + [0,\, -1] \otimes P_3$$

- A_5 **Singularity**

Starting with the case is $u = v + 2$ and we have

$$\begin{array}{ll} P_1 & \left[a_1, \ldots, a_n, \frac{1}{4}\right] \\ P_2 & \left[b_1, \ldots, b_m, \frac{1}{2}\right] \\ P_3 & [c_1, \ldots, c_o, 0] \\ P_4 & [d_1, \ldots, d_p, 0] \end{array}$$

Once again two cases for P_1, $[0, \frac{1}{4}]$ and $[\frac{1}{4}]$. If it is the first case then the denominator has to be greater than 2 so does not occur. In the second case there is only one possibility: the other elliptic singularity is given by toric coordinates (1, 4), $(-1, -2)$. The only way to get a denominator greater than 1 is on P_2 and the only choice is if this is the original $\frac{1}{2}$ hence we get

$$\left[\frac{1}{4}\right] \otimes P_1 + \left[\frac{1}{2}\right] \otimes P_2 + [0, -1] \otimes P_3$$

The final case is

$$\begin{array}{ll} P_1 & \left[a_1, \ldots, a_n, \frac{1}{3}\right] \\ P_2 & \left[b_1, \ldots, b_m, \frac{1}{3}\right] \\ P_3 & [c_1, \ldots, c_o, 0] \\ P_4 & [d_1, \ldots, d_p, 0] \end{array}$$

This leads to a lot more cases, as follows. Once again possible choices for P_1 and P_2 are $\left[0, \frac{1}{3}\right]$ or $\left[\frac{1}{3}\right]$. Once again denoting these by case a and case b. In case a it needs to connect to a point with denominator $u \in \{2, 5, 8 \ldots\}$, however it is impossible to construct any of these denominators, so this does no occur. In the case b, we either have the A_1 singularity with coordinates $(1, 3)$, $(-1, -1)$ or the A_2 singularity given by $(1, 3)$, $(-2, -3)$, other values result in non compatible denominators. If we had the singularity given by $(1, 3)$, $(-1, -1)$, then the polyhedral divisor over P_2 would have to be $\left[0, \frac{1}{3}\right]$ which we have already shown cannot occur. If we have the A_2 singularity then this involves the only way this can be constructed is via

$$\left[\frac{1}{3}\right] \otimes P_1 + \left[\frac{1}{3}\right] \otimes P_2 + [0, -1] \otimes P_3$$

This finishes the A_5 case.

- A_N **Singularity for** $N \geq 6$

This results in a singularity given by surface given by

$$\begin{array}{ll} P_1 & \left[a_1, \ldots, a_n, \frac{1}{u}\right] \\ P_2 & \left[b_1, \ldots, b_m, \frac{1}{v}\right] \\ P_3 & [c_1, \ldots, c_o, 0] \\ P_4 & [d_1, \ldots, d_p, 0] \end{array}$$

We split this into three case

a) $u = v$

b) $u = v + 1$

c) $u = v + 2$

Every other case is covered by our discussion at the beginning or the example. We note that which of these cases occur on the value of N. We note as $N \geq 6$ this implies $u \geq 4$ and $v \geq 3$.

In case a), the only denominators that occur which are less than or equal to v are $\frac{1}{u-2}$ and $\frac{1}{u-1}$ however as $u \geq 3$, as $N \geq 6$ we have neither of these values are equal to u or 1 so this cannot occur.

In case b), we have potential denominators $\frac{1}{v-1}$, $\frac{1}{v-2}$, $\frac{1}{u-1}$ and $\frac{1}{u-2}$. This equals $\frac{1}{u}$, $\frac{1}{u-1}$, $\frac{1}{u-2}$. As these all have denominators greater than 1 they cannot occur on a fiber.

Exactly the same logic holds in case c).

Bibliography

[1] Mohammad Akhtar and Alexander Kasprzyk. Singularity content. *arXiv:1401.5458 [math.AG], 2014.*

[2] Klaus Altmann and Jürgen Hausen. The geometry of t-varieties. *In Contributions to Algebraic Geometry - EMS - pages 17-69.*

[3] Michael Artin. On isolated rational singularities of surfaces. *American Journal of Mathematics Vol. 88, No. 1 (Jan., 1966), pp. 129-136 (8 pages).*

[4] Daniel Cavey and Thomas Prince. Del pezzo surfaces with a single 1/k(1,1) singularity. *J. Math. Soc. Japan 72(2): 465-505 (April, 2020). DOI: 10.2969/jmsj/79337933.*

[5] Alessio Corti and Liana Heuberger. Del pezzo surfaces with 1/3(1,1) points. *manuscripta math. 153, 71–118 (2017).*

[6] Alice Cuzzucoli. On the classification of orbifold del pezzo surfaces. *PhD Thesis,University of Warwick. http://wrap.warwick.ac.uk/141307/, 2019.*

[7] Henry K. Schenck. David A. Cox, John B. Little. Toric varieties. *In American-Maths Society, 2011.*

[8] Jurgen Hausen and Hendrik Suss. The cox ring of an algebraic variety with torusaction.

[9] Elaine Huggenberger. Fano varieties with torus action of complexityone.

[10] Yujiro Kawamata. Crepant blowing-up of 3-dimensional canonical singularities and its application to degenerations of surfaces. *Annals of Mathematics Second Series, Vol. 127, No. 1 (Jan., 1988), pp. 93-163.*

[11] Janos Kollar and Nicholas Shepher-Barron. Threefolds and deformations of surfacesingularities. *Inventiones mathematicae (1988), Vol 91 (2), 299?338.*

[12] Marni Mishna Nathan Ilten and Charlotte Trainor. Classifying fano complexity-one t-varieties via divisorial polytopes.

[13] Miles Reid. Chapters on algebraic surfaces. *Complex algebraic varieties, J.Kollar Ed., IAS/Park City lecture notes series (1993 volume), AMS, 1997*, .

[14] Miles Reid. Surface cyclic quotient singularities and hirzebruch-jung resolutions. *http://homepages.warwick.ac.uk/masda/surf/more/cyclic.pdf*, .

[15] Miles Reid. Young person's guide to canonical singularities. *InAlgebraic geometry, Bowdoin, 1985 (Brunswick, Maine, 1985), volume 46 ofProc. Sympos.Pure Math., pages 345–414. Amer. Math. Soc., Providence, RI, 1987*, .

[16] Miles Reid and Kaori Suzuki. Cascades of projections from log del pezzo surfaces. *InNumber theory and algebraic geometry, volume 303 ofLondon Math.Soc. Lecture Note Ser., pages 227–249. Cambridge Univ. Press, Cambridge, 2003.*

[17] Hendrik Suss. Canonical divisors on t-varieties. *arXiv:0811.0626 [math.AG].*