Depinning Transition of Travelling Waves for Particle Chains

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Abstract

In this paper we apply Aubry-Mather theory for equilibria of 1D Hamiltonian lattice systems and the theory of invariant ordered circles to investigate the depinning transition of travelling waves for particle chains. Assume $A < B$ are two critical values such that the particle chain has three homogeneous equilibria if the driving force $F \in (A, B)$. It is already known that there exist transition thresholds $F_c^- \le F_c^+$ of the driving force such that the particle chain has stationary fronts but no travelling fronts for $F_c^- \le F \le F_c^+$ and travelling fronts but no stationary fronts if $A < F < F_c^-$ or $F_c^+ < F < B$.

The novelty of our approach is that we prove the transition threshold F_c^+ (F_c^-) coincides with the upper (lower) limit of the upper (lower) depinning force as the rotation number tends to zero from the right. Based on this conclusion, we demonstrate that when the driving force $F \in (F_c^-, F_c^+)$, besides stationary fronts there are various kinds of equilibria with rotation numbers close to zero such that the spatial shift map has positive topological entropy on the set of equilibria. Furthermore, we give a necessary and sufficient condition for the absence of propagation failure, i.e., $F_c^- = F_c^+$, in terms of a minimal foliation. Finally we show that $F_c^\pm$ are continuous with respect to potential functions in C^1 topology.

Keywords: Stationary Front, Travelling Front, Depinning Transition, Particle Chain, Aubry-Mather Theory, Invariant Ordered Circle.
MSC: 34C12, 34C37, 37L60, 70F45, 70G60.

*Corresponding author: qinwx@suda.edu.cn, supported by the National Natural Science Foundation of China (12171347, 11790274, 11771316).

1 Introduction

The existence of a depinning transition for travelling waves is thought to be a distinct feature of spatially discrete media. There is a large amount of research work on existence, uniqueness, and stability of travelling waves with various nonlinearities for lattice systems. Mallet-Paret [28](see also Carpio et al. [14]) employed a global continuation method to obtain the existence, uniqueness, and continuous dependence on parameters of travelling waves for bistable nonlinearities, while Zinner [38] relied on a fixed point theorem and Hankerson and Zinner [21] used the idea of an integer-valued Lyapunov function.

Travelling waves were also studied by Chow, Mallet-Paret, and Shen for lattice dynamical systems and coupled map lattices [16]. Chen, Guo, and Wu investigated travelling waves in more general cases for which the reference equilibria supporting wave propagation are periodic, not just homogeneous [15]. In [1], Al Haj, Forcadel, and Monneau studied a model not covered by Mallet-Paret's work [28]. The Frenkel-Kontorova model with an inertial term was considered by Forcadel, Ghorbel, and Walha [19]. For the conservative case of the Frenkel-Kontorova chain, that is, without damping term, Buffoni, Schwetlick, and Zimmer [10, 11] studied the travelling heteroclinic waves connecting one well of the on-site potential to the other.

We focus in this paper on the depinning transition of fronts between two homogeneous equilibria for particle chains with potential function h with range $r \geq 1$, paying particular attention to treat $r > 1$, as follows.

Let $r \geq 1$ and $h : \mathbb{R}^{r+1} \to \mathbb{R}$ be a C^2 function satisfying hypotheses (H1)-(H4) specified in Section 2.1. Let $\mathbf{x} = (x_n) \in \mathbb{R}^{\mathbb{Z}}$ be a configuration with x_n denoting the position of the n-th particle. We denote by $W(\mathbf{x})$ the formal energy of the particle chain with potential function h:

$$W(\mathbf{x}) = \sum_{n \in \mathbb{Z}} h(x_n, \cdots, x_{n+r}).$$

The equations of motion of the particle chain are taken to be

$$\dot{x}_n = -\partial_n W(\mathbf{x}) + F, \tag{1.1}$$

where

$$\partial_n W(\mathbf{x}) = \sum_{j=1}^{r+1} \partial_j h(x_{n+1-j}, \cdots, x_{n+r+1-j})$$

denotes the partial derivative of W with respect to the variable x_n, and $F \in \mathbb{R}$ is an external driving force.

Let $V(x) = h(x, \cdots, x)$. Then $V(x+1) = V(x), x \in \mathbb{R}$. We always assume in this paper that $V'(x)$ has three zeros $u^1 = 0 < u^2 < u^3 = 1$, and there exist e_1 and e_2 with $u^1 < e_1 < u^2 < e_2 < u^3$ such that (see Figure 1)

$$-V''(u^i) < 0,\ i = 1,\ 3,\quad -V''(u^2) > 0,\quad V''(e_i) = 0,\ i = 1,\ 2,$$

and

$$-V''(x) < 0,\quad x \in (u^1, e_1) \cup (e_2, u^3),\quad -V''(x) > 0,\quad x \in (e_1, e_2).$$

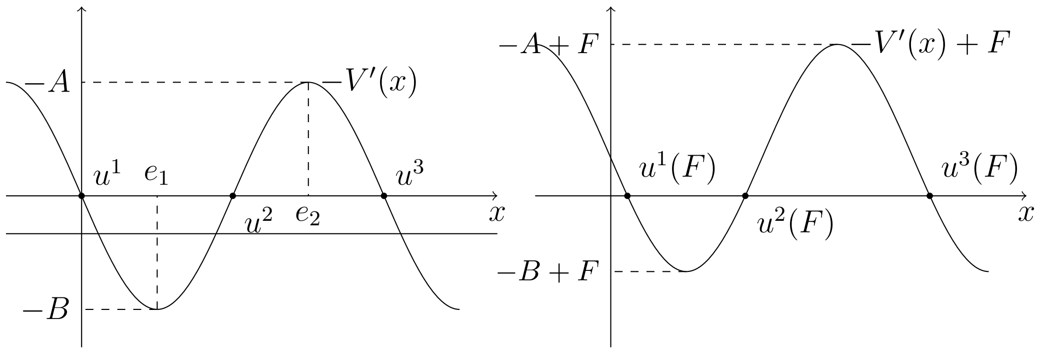

Figure 1: The graph of $-V'(x)$ and $-V'(x)+F$.

Let $A = V'(e_2) < 0$ and $B = V'(e_1) > 0$. Then for $A < F < B$, $-V'(x)+F$ also has three zeros $u^1(F) < u^2(F) < u^3(F) = u^1(F) + 1$ satisfying $-V''(u^i(F)) < 0$ for $i = 1, 3$, $-V''(u^2(F)) > 0$, and for $A < F_1 < F_2 < B$,

$$u^1(F_1) < u^1(F_2) < u^2(F_2) < u^2(F_1) < u^3(F_1) < u^3(F_2). \tag{1.2}$$

Let $\mathbf{u}^i(F) = u^i(F) \cdot \mathbf{1}$ $(i = 1, 2, 3)$ denote the homogeneous equilibria of system (1.1), where $\mathbf{1}$ denotes the configuration with each component being 1.

If we take in particular the potential function

$$h(x_1, x_2) = \frac{1}{2}(x_2 - x_1)^2 + \frac{k}{4\pi^2}\cos 2\pi x_1,$$

then we have the classical tilted Frenkel-Kontorova model [3].

We restrict attention in this paper to a particular class of fronts, as follows.

A travelling front of (1.1) is a solution of form $x_n(t) = u(n - ct)$ with $c \neq 0$, where the profile function $u : \mathbb{R} \to (u^1(F), u^3(F))$ is C^1 smooth with $u'(s) > 0$, $s \in \mathbb{R}$, and satisfies

$$-cu'(s) = -\partial_1 h(u(s), \cdots, u(s+r)) - \cdots - \partial_{r+1} h(u(s-r), \cdots, u(s)) + F \tag{1.3}$$

and

$$\lim_{s \to -\infty} u(s) = u^1(F), \quad \lim_{s \to \infty} u(s) = u^3(F). \tag{1.4}$$

A stationary front is an equilibrium $\mathbf{x} = (x_n)$ of (1.1) of form $x_n = u(n)$, where $u : \mathbb{R} \to (u^1(F), u^3(F))$ is strictly increasing and satisfies (1.4).

It has already been obtained, see [28] (or [14] for linear and nearest-neighbor coupling), that there exist transition thresholds $F_c^- \leq F_c^+$ of the driving force such that the particle chain has stationary fronts but no travelling fronts for $F_c^- \leq F \leq F_c^+$ and travelling fronts but no stationary fronts if $A < F < F_c^-$ or $F_c^+ < F < B$.

We associate the transition thresholds $F_c^\pm$ with the depinning force for fronts of the driven particle chain, which is a physical quantity describing the transition from pinning to sliding for particle systems, see [3, 18, 27, 36] and the references therein. By analogy to [3], depinning of a front can be expected generically to be a saddle-node of invariant circle (SNIC) bifurcation [5].

The upper (lower) depinning force $F_d^+(p/q)$ ($F_d^-(p/q)$) for spatially periodic equilibria of (1.1) is defined to be (see Section 2.3) the critical value of the driving force below (beyond) which there exists a (p, q)-periodic equilibrium and beyond (below) which there is none; see [36] for the details of discussions on depinning force. Our first conclusion is that the transition threshold F_c^+ (F_c^-) is identical to the upper (lower) limit of the upper (lower) depinning force as the rotation number approaches zero from the right.

Theorem A.

$$F_c^- = \liminf_{n \to +\infty} F_d^-(1/n) \geq A \quad \text{and} \quad F_c^+ = \limsup_{n \to +\infty} F_d^+(1/n) \leq B. \tag{1.5}$$

A stationary or travelling front can be regarded as a heteroclinic connection joining two stable equilibria $\mathbf{u}^1(F)$ and $\mathbf{u}^3(F)$ for the spatial shift map. However, an added difficulty arises due to the presence of the intermediate equilibrium $\mathbf{u}^2(F)$, and overcoming the difficulties presented by this equilibrium is a significant task, as remarked by Mallet-Paret [28], see also [1]. In Section 4, we use a result in monotone dynamical systems [37] to show that $\mathbf{u}^2(F)$ does not but $\mathbf{u}^1(F)$ does lie in an invariant ordered circle (IOC, see Section 2.4 for the definition) so that $\mathbf{u}^1(F)$ and $\mathbf{u}^3(F)$ are neighboring homogeneous equilibria on this IOC, guaranteeing the existence of heteroclinic connections from $\mathbf{u}^1(F)$ to $\mathbf{u}^3(F)$.

If $F_c^- < F_c^+$, we say that pinning occurs for $F \in [F_c^-, F_c^+]$ or the particle chain admits propagation failure. The front is pinned and cannot propagate, i.e., there are no travelling fronts, when the driving force F lies in $[F_c^-, F_c^+]$, which is called the pinning interval or propagation failure interval. In [25] Keener proved that the discrete Nagumo equation admits propagation failure provided the diffusion coefficient is small enough. Propagation failure in discrete media has attracted much attention, see [12, 16, 22, 23, 28, 33] and the references therein.

Hoffman and Mallet-Paret [22] proposed a generic condition on the nonlinearity for occurrence of propagation failure for high-dimensional lattice systems.

A "depinning criterion" was also provided by Carpio and Bonilla [13] using the smallest eigenvalue of an operator for chains of linearly coupled overdamped oscillators. Hupkes, Pelinovsky, and Sandstede [23] presented an explicit criterion that can determine whether propagation failure occurs or not: Roughly speaking, if there is a smooth one-parameter branch of "translationally invariant" stationary fronts when the detuning parameter $a = a_*$, then there are no stationary fronts for $a \neq a_*$, and hence no propagation failure. We remark that this is a sufficient condition, see [33] for Pelinovsky's alternative proof. Here we give for particle chains a necessary and sufficient condition for no propagation failure.

Theorem B. *$F_c^- = F_c^+$ if and only if the equation*

$$\partial_1 h(u(s), \cdots, u(s+r)) + \cdots + \partial_{r+1} h(u(s-r), \cdots, u(s)) = 0 \tag{1.6}$$

has a strictly increasing and continuous solution $u : \mathbb{R} \to (0,1)$ satisfying

$$\lim_{s \to -\infty} u(s) = 0, \quad \textit{and} \quad \lim_{s \to +\infty} u(s) = 1. \tag{1.7}$$

Remark: Moser [30] called a connected, strictly ordered, and shift-invariant family of equilibria of (1.1) with $F = 0$ a minimal foliation, and he proved the solutions in this family are minimizers (see Section 2.1), see also Theorem 10.1 in [31] and Theorem (7.7) in [7]. We actually show that $F_c^- = F_c^+$ if and only if there exists a minimal foliation consisting of homogeneous and heteroclinic minimizers, see Remark 1 following the proof of Theorem B.

One natural question is how many equilibria there are in the region of propagation failure besides the homogeneous equilibria and the stationary fronts. Keener [25] proved for the discrete Nagumo equation with small diffusion coefficient, the propagation is blocked by a family of non-monotone stationary solutions. Bates and Chmaj obtained similar results for a discrete convolution model by applying the Implicit Function Theorem [8]. For coupled map lattices, Chow and Shen [17] studied the relation between propagation failure and spatial topological chaos. We remark that our approach is different from the perturbation method. We even obtain the existence of equilibria with rotation numbers in an interval including 0, see the discussion following the proof of Theorem C in Section 6.

Theorem C. *Assume $F_c^- < F_c^+$ and $F_c^- < F < F_c^+$. Then the spatial shift map has positive topological entropy on the set of equilibria of (1.1).*

To the best of our knowledge, there is no conclusion concerning continuous dependence of the transition thresholds $F_c^\pm$ on parameters. The following conclusion is based on the continuous dependence of the depinning force on system parameters [36].

Theorem D. *$F_c^{\pm}$ are continuous with respect to h in C^1 topology.*

Now we explain why it is useful to apply techniques from Aubry-Mather theory, and its extension to the theory of invariant ordered circles, to discuss the depinning transition of travelling waves for driven particle chains. An equilibrium $\mathbf{x} = (x_n)$ of (1.1) satisfies a difference equation (2.4) from which we define a homeomorphism $\mathcal{F}_\Delta : \mathbb{R}^{2r} \to \mathbb{R}^{2r}$ (see Section 2.5). Let $a^* = (a, \cdots, a), b^* = (b, \cdots, b) \in \mathbb{R}^{2r}$ denote two fixed points of $\mathcal{F}_\Delta$, where $a = u^1(F), b = a+1$ are two zeros of $-V'(x)+F = 0$. A stationary front corresponds to a heteroclinic orbit of $\mathcal{F}_\Delta$ connecting a^* to b^*. Therefore, in order to show the existence of stationary fronts of (1.1), it suffices to establish the existence of heteroclinic orbits connecting a^* to b^*.

If the range $r = 1$ then it is relatively easy to detect the intersection of the unstable manifold of a^* and the stable manifold of b^*, but for $r > 1$ it is not so straightforward. The advantage of the Aubry-Mather theory is to guarantee heteroclinic connections without assuming intersection of invariant manifolds or applying a perturbation method. For $F = 0$, there always exists a minimizer, so an equilibrium, forming a heteroclinic connection between two neighboring periodic minimizers with the same rotation number, see, for example, Section 5 in [7] or Theorem 13.5 in [29] for monotone twist maps, or [32] for general Hamiltonian lattices.

We should emphasize that for the driven particle chains, i.e., $F \neq 0$, we do not have minimizers. Nevertheless, there are invariant ordered circles (IOCs) for the temporal dynamics for each $F \in \mathbb{R}$. In particular (see Section 2.4), there is an IOC containing a heteroclinic connection between two neighboring homogeneous equilibria of (1.1) on the same IOC, leading to the existence of stationary or travelling fronts.

We remark that we do not consider the existence, uniqueness, and continuous (smooth) dependence on system parameters of the travelling speed c and monotone solution $u = P(s)$ of (1.3) and (1.4). It is by constructing suitable IOCs containing heteroclinic configurations that we prove Theorem A, from which we actually obtain the existence of the depinning transition thresholds $F_c^{\pm}$. Meanwhile, it is the relation between $F_c^{\pm}$ and the depinning force that makes it possible for us to use the tools developed in [35, 36] to derive spatial chaos of the shift map on the set of equilibria, a necessary and sufficient condition for the absence of propagation failure, and continuous dependence of $F_c^{\pm}$ on system parameters.

2 Preliminaries

2.1 The Aubry-Mather Theory

Hypotheses: We assume that the C^2 potential function h satisfies the following hypotheses. We remark that these hypotheses are standard for the discussion of lattice Aubry-Mather theory [26, 31, 32].

(H1) $h(\xi_1+1,\cdots,\xi_{r+1}+1)=h(\xi_1,\cdots,\xi_{r+1})$;

(H2) $\max_{1\le i\le j\le r+1}\|\partial_{i,j}h\|_{\sup}\le K$;

(H3) h is bounded from below and $h(\xi_1,\cdots,\xi_{r+1})\to\infty$ if $|\xi_2-\xi_1|\to\infty$;

(H4) Twist condition:

$$\partial_{1,k}h(\xi_1,\cdots,\xi_{r+1})\le-\lambda<0 \text{ for } 2\le k\le r+1, \text{ and } \partial_{i,k}h(\xi_1,\cdots,\xi_{r+1})\le 0 \text{ for } k\neq i.$$

We say that $B\subset\mathbb{Z}$ is a connected component of $\mathbb{Z}$ if B consists of consecutive integers. We denote by $B=[i_0-r,i_1]$ an arbitrary finite connected component of $\mathbb{Z}$ with $i_1\ge i_0$, $\mathrm{int}(B)=[i_0,i_1]$ the interior of B, $\bar{B}=[i_0-r,i_1+r]$ its closure, $\partial B=\bar{B}\setminus\mathrm{int}(B)$ the boundary of B. Let

$$W_B(\mathbf{x})=\sum_{i\in B}h(x_i,\cdots,x_{i+r}),$$

which is a function of coordinates of $\mathbf{x}$ with indices in $\bar{B}$. Denote by $\mathrm{supp}(\mathbf{v})$ the support of $\mathbf{v}=(v_i)\in\mathbb{R}^{\mathbb{Z}}$, i.e., $\mathrm{supp}(\mathbf{v})=\{i\in\mathbb{Z}\,|\,v_i\neq 0\}$.

Definition 2.1. *A configuration $\mathbf{x}$ is called a minimizer if $W_B(\mathbf{x})\le W_B(\mathbf{x}+\mathbf{v})$ for all finite connected components $B\subset\mathbb{Z}$ and all $\mathbf{v}$ with $supp(\mathbf{v})\subset int(B)$.*

For configurations $\mathbf{x}=(x_i)$ and $\mathbf{y}=(y_i)\in\mathbb{R}^{\mathbb{Z}}$, we say $\mathbf{x}\le\mathbf{y}$ if $x_i\le y_i$ for all $i\in\mathbb{Z}$, $\mathbf{x}<\mathbf{y}$ if $\mathbf{x}\le\mathbf{y}$ and $\mathbf{x}\neq\mathbf{y}$, and $\mathbf{x}\ll\mathbf{y}$ if $x_i<y_i$ for all $i\in\mathbb{Z}$. Similarly for $\ge$, $>$, and $\gg$. Two configurations $\mathbf{x}\neq\mathbf{y}$ are said to be ordered if $\mathbf{x}\le\mathbf{y}$ or $\mathbf{x}\ge\mathbf{y}$, to be strictly ordered if $\mathbf{x}\ll\mathbf{y}$ or $\mathbf{x}\gg\mathbf{y}$. For elements in $\mathbb{R}^n$ $(n\ge 1)$, we define $\le,<,\ll,\ge,>,\gg$ as above.

Let $\{\tau_{k,l}|\,k,l\in\mathbb{Z}\}$ denote the translation group on $\mathbb{R}^{\mathbb{Z}}$ defined by

$$(\tau_{k,l}\mathbf{x})_i=x_{i-k}+l,\quad i\in\mathbb{Z}.$$

A configuration $\mathbf{x}$ is said to be (p,q)-periodic if $\tau_{q,p}\mathbf{x}=\mathbf{x}$, where p and $q>0$ are integers. A configuration $\mathbf{x}$ is said to be Birkhoff if for all $k,l\in\mathbb{Z}$, either $\tau_{k,l}\mathbf{x}\le\mathbf{x}$ or $\tau_{k,l}\mathbf{x}\ge\mathbf{x}$. We denote by $\mathcal{B}_{p,q}$ the set of Birkhoff (p,q)-periodic configurations and $\mathcal{B}$ all Birkhoff configurations.

If $\mathbf{x}=(x_i)$ is Birkhoff, then [7, 20] there is a unique $\omega=\omega(\mathbf{x})\in\mathbb{R}$, called the rotation number of $\mathbf{x}$, such that

$$|x_i-x_j-(i-j)\omega|\le 1,\quad\text{for all }\ i,j\in\mathbb{Z}.\tag{2.1}$$

Let $\mathcal{B}_\omega$ denote the set of Birkhoff configurations with rotation number ω. Then $\mathcal{B}_\omega$ is closed in the topology of pointwise convergence. Moreover, the rotation number $\omega(\mathbf{x})$ depends continuously [7, 20] on $\mathbf{x} \in \mathcal{B}$.

We denote by $\mathscr{M}_{p,q}$ the set of (p,q)-periodic Birkhoff minimizers and $\mathscr{M}_\omega$ the set of Birkhoff minimizers with rotation number ω. It is clear that each minimizer is an equilibrium of (1.1) with $F = 0$. Under the conditions (H1)-(H4) of h, we have [26, 31, 32]

Lemma 2.2. *$\mathscr{M}_{p,q} \neq \emptyset$ and $\mathscr{M}_\omega \neq \emptyset$ for each $\omega \in \mathbb{R}\backslash\mathbb{Q}$, and $\mathscr{M}_\omega$ is totally ordered for irrational ω.*

Let $\mathcal{B}_{[a,b]} = \cup_{\alpha\in[a,b]}\mathcal{B}_\alpha$. The following lemma is obtained by Tychonoff's Theorem. We refer to [20, 31, 34] for a full proof.

Lemma 2.3. *For $-\infty < a < b < +\infty$, $\mathcal{B}_{[a,b]}/\langle \mathbf{1} \rangle$ is compact in the topology of pointwise convergence.*

2.2 Gradient Flow and Strict Monotonicity

We need to study the solutions of (1.1) with initial conditions in Banach space

$$\mathscr{X} = \left\{ \mathbf{x} \in \mathbb{R}^{\mathbb{Z}} \;\middle|\; \|\mathbf{x}\| = \sum_{n\in\mathbb{Z}} 2^{-|n|}|x_n| < \infty \right\}.$$

The condition (H2) of h ensures the existence of a unique solution $\mathbf{x}(t)$ of (1.1) with $\mathbf{x}(0) \in \mathscr{X}$ for all $t \in \mathbb{R}$ so that we can define a flow $\{\phi_F^t\}_{t\in\mathbb{R}}$ on $\mathscr{X}$. The periodic condition (H1) makes it possible to consider $\{\phi_F^t\}$ in $\mathscr{X}/\langle \mathbf{1} \rangle$. Moreover, ϕ_F^t commutes with $\tau_{k,l}$:

$$\phi_F^t(\tau_{k,l}\mathbf{x}) = \tau_{k,l}(\phi_F^t\mathbf{x}), \quad \text{for all} \quad t \in \mathbb{R}, \quad \text{and} \quad k, l \in \mathbb{Z}. \tag{2.2}$$

The twist condition (H4) of h guarantees the strict monotonicity of ϕ_F^t for $t > 0$, i.e., $\mathbf{x} < \mathbf{y}$ implies $\phi_F^t\mathbf{x} \ll \phi_F^t\mathbf{y}$ for all $t > 0$.

Lemma 2.4. *Let $\mathbf{x} = (x_i)$ and $\mathbf{y} = (y_i) \in \mathbb{R}^{\mathbb{Z}}$. Assume $x_{i_0} = y_{i_0}$ for some $i_0 \in \mathbb{Z}$ and $x_i \leq y_i$ for $i \in [i_0 - r, i_0 + r]$. Then it follows that $\partial_{i_0}W(\mathbf{y}) \leq \partial_{i_0}W(\mathbf{x})$. Furthermore, if there is $k \in [i_0 - r, i_0 + r] \setminus \{i_0\}$ such that $x_k < y_k$, then $\partial_{i_0}W(\mathbf{y}) < \partial_{i_0}W(\mathbf{x})$.*

Proof: It is easy to check that

$$-\partial_{i_0}W(\mathbf{x}) = -\sum_{j=1}^{r+1} \partial_j h(x_{i_0+1-j}, \cdots, x_{i_0+r+1-j})$$

is strictly increasing in x_{i_0+i} for $i = -r, \cdots, r, i \neq 0$, by twist condition (H4). $\square$

2.3 Lower and Upper Depinning Force

In this section we define for rational numbers the lower and upper depinning force for the driven particle chain (1.1).

Let p and $q > 0$ be relatively prime integers, and

$$\mathcal{A}_{p,q} = \{F \in \mathbb{R} \,|\, \exists\, \mathbf{x} \in \mathcal{B}_{p,q} \text{ such that } F = \partial_i W(\mathbf{x}),\ i \in \mathbb{Z}\}.$$

We know that $0 \in \mathcal{A}_{p,q}$ by Lemma 2.2 and hence $\mathcal{A}_{p,q}$ is nonempty.

Lemma 2.5. $\mathcal{A}_{p,q}$ *is compact.*

Proof: Since $\partial_i W(\mathbf{x})$ is continuous with respect to $\mathbf{x}$ and $\mathcal{B}_{p,q}/\langle \mathbf{1} \rangle$ is compact by Lemma 2.3, then $\mathcal{A}_{p,q}$ is bounded.

To prove $\mathcal{A}_{p,q}$ is closed, let $F_n \in \mathcal{A}_{p,q}$ and $F_n \to F$ as $n \to \infty$. We shall show that $F \in \mathcal{A}_{p,q}$. Indeed, for each F_n, there exists $\mathbf{x}^n = (x_i^n) \in \mathcal{B}_{p,q}$ such that $x_0^n \in [0,1]$, $n \geq 1$, and $F_n = \partial_i W(\mathbf{x}^n)$, for all $i \in \mathbb{Z}$. Then an accumulation point $\mathbf{x} \in \mathcal{B}_{p,q}$ of $\{\mathbf{x}^n\}$ satisfies $\partial_i W(\mathbf{x}) = F$, for all $i \in \mathbb{Z}$, implying $F \in \mathcal{A}_{p,q}$. □

Lemma 2.6. *Let $\underline{\mathbf{x}} \leq \overline{\mathbf{x}}$ be two (p,q)-periodic configurations, one of which is Birkhoff. Assume $\partial_i W(\underline{\mathbf{x}}) \leq F \leq \partial_i W(\overline{\mathbf{x}})$ for all $i \in \mathbb{Z}$. Then there exists a Birkhoff (p,q)-periodic configuration $\mathbf{y}$ such that $\partial_i W(\mathbf{y}) = F$ for all $i \in \mathbb{Z}$.*

Proof: This is a straightforward consequence of Theorem 4.2 together with Addenda 4.3 and 4.4 in [2]. □

Definition 2.7. *Define the upper and lower depinning force respectively as*

$$F_d^+(p/q) = \sup \mathcal{A}_{p,q} \geq 0 \quad \textit{and} \quad F_d^-(p/q) = \inf \mathcal{A}_{p,q} \leq 0. \tag{2.3}$$

Lemma 2.8. $[F_d^-(p/q), F_d^+(p/q)] = \mathcal{A}_{p,q}$.

Proof: It suffices to show "$\subset$". Indeed, assume $F_d^-(p/q) \leq F \leq F_d^+(p/q)$. Then there exist $\underline{\mathbf{x}}, \overline{\mathbf{x}} \in \mathcal{B}_{p,q}$ such that

$$\partial_i W(\underline{\mathbf{x}}) = F_d^-(p/q) \leq F \leq F_d^+(p/q) = \partial_i W(\overline{\mathbf{x}}),\ i \in \mathbb{Z}.$$

We assume $\underline{\mathbf{x}} \leq \overline{\mathbf{x}}$ by the periodicity assumption (H1). Then from Lemma 2.6 we deduce the existence of $\mathbf{x} \in \mathcal{B}_{p,q}$ satisfying $\partial_i W(\mathbf{x}) = F$ for all $i \in \mathbb{Z}$, and hence $F \in \mathcal{A}_{p,q}$.

Lemma 2.9. *Assume $F \in \mathbb{R}$ and there exists (p,q)-periodic configuration $\mathbf{x}$ such that $-\partial_i W(\mathbf{x}) + F \leq 0$ (≥ 0) for all $i \in \mathbb{Z}$. Then $F_d^+(p/q) \geq F$ $(F_d^-(p/q) \leq F)$.*

Proof: Assume $F_d^-(p/q) \le F$. Then there exists $\underline{\mathbf{x}} \in \mathcal{B}_{p,q}$ such that

$$\partial_i W(\underline{\mathbf{x}}) = F_d^-(p/q) \le F \le \partial_i W(\mathbf{x}),\ i \in \mathbb{Z}.$$

We assume $\underline{\mathbf{x}} \le \mathbf{x}$ by the periodicity assumption (H1). Then from Lemma 2.6 we obtain the existence of $\mathbf{y} \in \mathcal{B}_{p,q}$ satisfying $\partial_i W(\mathbf{y}) = F$ for all $i \in \mathbb{Z}$, implying $F \in \mathcal{A}_{p,q}$, and hence $F \le F_d^+(p/q)$. The other conclusion is proved similarly. $\square$

We mention that one can also define $F_d^{\pm}(\omega)$ for irrational ω as in [36], but we do not need them in this paper.

2.4 Invariant Ordered Circles (IOCs)

A solution $\mathbf{x}(t) = (x_n(t))$ of (1.1) is said to be a uniform sliding solution if $x_n(t) = u(n\omega + \nu t)$, for $n \in \mathbb{Z}$, where $\nu \neq 0$ and the C^1 function $u:\ \mathbb{R} \to \mathbb{R}$ satisfies $u(t+1) = u(t) + 1$, $t \in \mathbb{R}$. Uniform sliding solutions are special form of travelling waves with the profile function u (also called the dynamical hull function). The parameter ω indicates the mean spacing of particles and ν the average velocity.

Proposition 2.10. *For $F_d^-(p/q) \le F \le F_d^+(p/q)$, there exist for (1.1) Birkhoff equilibria in $\mathcal{B}_{p,q}$ and there are no (p,q)-periodic equilibria when $F > F_d^+(p/q)$ or $F < F_d^-(p/q)$. Moreover, for $F > F_d^+(p/q)$ ($F < F_d^-(p/q)$) there exists a uniform sliding solution $\mathbf{x}(t) \in \mathcal{B}_{p,q}$ with the average velocity $\nu > 0$ (< 0).*

Baesens and MacKay [3] proved this in the case that h has two variables, i.e., $r = 1$. We remark that the proof for the general case $r > 1$ is the same as that of Theorem 3.1 in [4] which depends upon the strict monotonicity.

Let $\mathbf{x} = (x_n) \in \mathscr{X}$ and denote $\dot{\mathbf{x}} = (\dot{x}_n)$. We say that $\dot{\mathbf{x}} \ge \mathbf{0}$ ($\gg \mathbf{0}, \le \mathbf{0}, \ll \mathbf{0}$) for (1.1) if $\dot{x}_n = -\partial_n W(\mathbf{x}) + F \ge 0$ ($> 0, \le 0, < 0$) for all $n \in \mathbb{Z}$.

For each uniform sliding solution $\mathbf{x}(t) = (x_n(t)) \in \mathcal{B}_{p,q}$, it follows that $\dot{\mathbf{x}}(t) \gg \mathbf{0}$ ($\ll \mathbf{0}$) for all $t \in \mathbb{R}$ if $F > F_d^+(p/q)$ ($< F_d^-(p/q)$). Also, each uniform sliding solution is invariant for all translations $\tau_{k,l}$, $k, l \in \mathbb{Z}$.

Definition 2.11. *Suppose $g : \mathbb{R} \to \mathscr{X}$ is continuous and satisfies $g(s+1) = g(s) + \mathbf{1}$. We say the image $\ell = g(\mathbb{R})$ is a strictly ordered circle if $g(s_1) \ll g(s_2)$ for $s_1 < s_2$. A strictly ordered circle is called an invariant ordered circle (IOC) if it is invariant both for $\{\tau_{k,l}\}$ and the flow ϕ_F^t of (1.1) for all $t \in \mathbb{R}$.*

We should remark that IOCs for (1.1) without driving force are called ghost circles [20, 31]. It's clear that an IOC ℓ is a nonempty closed set in $\mathcal{B}$ with $p_0(\ell) = \mathbb{R}$, where p_0 denotes the projection from $\mathbb{R}^{\mathbb{Z}}$ to $\mathbb{R}$: $p_0(\mathbf{x}) = x_0$ for $\mathbf{x} = (x_n)$. Its quotient by $\tau_{0,1}$ is homeomorphic to a circle. Moreover, thanks to the invariance of ℓ for $\tau_{k,l}$ for $k, l \in \mathbb{Z}$, there exists a lift of a circle homeomorphism G with rotation number ω such that for each $\mathbf{x} = (x_n) \in \ell$, $x_n = G^n(x_0)$ for all $n \in \mathbb{Z}$, i.e., an IOC ℓ can be expressed as

$$\ell = \{\mathbf{x} = (x_n) \,|\, x_n = G^n(x_0),\ x_0 \in \mathbb{R}\},$$

and hence $\ell \subset \mathcal{B}_\omega$.

Proposition 2.12. *For $p/q \in \mathbb{Q}$ in lowest terms and $F \in \mathbb{R}$, there exists for (1.1) an IOC $\ell \subseteq \mathcal{B}_{p,q}$. Moreover, for $F \in [F_d^-(p/q), F_d^+(p/q)]$, there exists on ℓ at least one Birkhoff (p,q)-periodic equilibrium of (1.1). If $\omega \in \mathbb{R}\backslash\mathbb{Q}$ and $F \in \mathbb{R}$, there exists for (1.1) an IOC $\ell \subseteq \mathcal{B}_\omega$.*

For the case $r = 1$, the above proposition was proved in [35] using the Schauder fixed point theorem. The proof for the general case $r > 1$ is the same and we omit the proof here.

We remark that for $F > F_d^+(p/q)$ ($F < F_d^-(p/q)$), the IOC ℓ obtained by the above proposition is exactly a uniform sliding solution with average velocity $\nu > 0$ (< 0), and hence $\dot{\mathbf{x}} \gg \mathbf{0}$ ($\ll \mathbf{0}$) for each $\mathbf{x} \in \ell$.

The Hausdorff metric is defined by setting [24]

$$\mathrm{d}(A, B) = \sup\{\mathrm{d}(x, B) \,|\, x \in A\} + \sup\{\mathrm{d}(A, y) \,|\, y \in B\}$$

for any two closed sets A, B in a metric space X. Then we have the following conclusion (see [24]): The Hausdorff metric on the closed subsets of a compact metric space defines a compact topology.

Assume that $\omega_n \in [a, b]$ with $\omega_n \to \omega \in [a, b]$, and $F_n \to F \geq 0$. Let $\ell_n \subset \mathcal{B}_{\omega_n}$ be IOCs for (1.1) with the driving force being F_n.

We remark that on bounded subsets of $\mathcal{B}_{[a,b]}$, the topology induced by the norm $\|\cdot\|$ is equivalent to the pointwise convergence topology, see [34]. It follows from Lemma 2.3 that there is a subsequence of $\hat{\ell}_n = \ell_n/\langle\mathbf{1}\rangle$, not relabeled, converging to a closed set $\hat{\ell}$ of $\mathcal{B}_{[a,b]}/\langle\mathbf{1}\rangle$ in the Hausdorff metric defined for compact sets in $\mathcal{B}_{[a,b]}/\langle\mathbf{1}\rangle$.

Note that ℓ_n is periodic with period $\mathbf{1}$ and $\hat{\ell}_n$ is in fact a segment in ℓ_n. We lift $\hat{\ell}$ to ℓ by periodicity along $\mathbf{1}$. Then $\ell_n \to \ell$ as $n \to \infty$ in the Hausdorff metric and hence for each $\mathbf{x} \in \ell$, there exists a sequence of points $\mathbf{x}^n \in \ell_n$ such that $\lim_{n\to\infty} \|\mathbf{x}^n - \mathbf{x}\| = 0$ which is equivalent to $\mathbf{x}^n \to \mathbf{x}$ as $n \to \infty$ in the pointwise convergence topology.

Proposition 2.13. *Let $(\omega_n, F_n) \to (\omega, F)$ as $n \to \infty$ and assume that $\{\ell_n\} \subseteq \mathcal{B}_{\omega_n}$ are IOCs of (1.1) with driving force F_n. Then there exist a subsequence $\{\ell_{n_i}\} \subseteq \{\ell_n\}$ and an IOC $\ell \subseteq \mathcal{B}_\omega$ for (1.1) such that $\ell_{n_i} \to \ell$ in the Hausdorff metric.*

The case $r = 1$ was proved in [35], see Lemma 3.9 in [35]. The proof for the general case $r > 1$ is totally the same.

Remark: For $F > F_d^+(p/q)$, we know from Proposition 2.10 that there exists a uniform sliding solution $\mathbf{x}(t) = (x_n(t))$ with $\dot{x}_n(t) > 0$ for all $n \in \mathbb{Z}$ and $t \in \mathbb{R}$. In fact, $\ell_F = \{\mathbf{x}(t) \,|\, t \in \mathbb{R}\}$ is an IOC for (1.1) with $F > F_d^+(p/q)$. Letting $F \to F_d^+(p/q)$ and applying Proposition 2.13 we obtain an IOC ℓ for (1.1) with driving force $F_d^+(p/q)$ such that for each $\mathbf{z} \in \ell$, $\dot{\mathbf{z}} \geq \mathbf{0}$ for (1.1) with $F = F_d^+(p/q)$. Similarly, we have an IOC ℓ' such that $\dot{\mathbf{z}} \leq \mathbf{0}$ for (1.1) with $F = F_d^-(p/q)$ for each $\mathbf{z} \in \ell'$.

2.5 A Criterion for Positive Topological Entropy

Let $\Delta : \mathbb{R}^{2r+1} \to \mathbb{R}$ be defined as

$$\begin{aligned} \Delta(x_{-r}, \cdots, x_0, \cdots, x_r) &= -\partial_0 W(\mathbf{x}) = -\sum_{j=1}^{r+1} \partial_j h(x_{1-j}, \cdots, x_{r+1-j}) \\ &= -\partial_1 h(x_0, \cdots, x_r) - \cdots - \partial_{r+1} h(x_{-r}, \cdots, x_0). \end{aligned}$$

Then the function Δ is C^1 and satisfies the following properties owing to our assumptions (H1)-(H4).

(C1) $\Delta(x_{-r}, \cdots, x_0, \cdots, x_r)$ is strictly increasing for the x_j except x_0,

(C2) $\Delta(x_{-r} + 1, \cdots, x_r + 1) = \Delta(x_{-r}, \cdots, x_r)$,

(C3) $\lim_{x_{-r} \to \pm\infty} \Delta(x_{-r}, \cdots, x_r) = \pm\infty$ and $\lim_{x_r \to \pm\infty} \Delta(x_{-r}, \cdots, x_r) = \pm\infty$.

Note that an equilibrium $\mathbf{x} = (x_n) \in \mathbb{R}^{\mathbb{Z}}$ of (1.1) is now a solution of the monotone recurrence relation defined by (see [2])

$$\Delta(x_{n-r}, \cdots, x_n, \cdots, x_{n+r}) + F = 0, \quad \text{for} \quad n \in \mathbb{Z}. \tag{2.4}$$

By virtue of (C1) and (C3), we can solve (2.4) for x_{n+r} if $(x_{n-r}, \cdots, x_{n+r-1})$ is given. Therefore we define a continuous map $\mathcal{F}_\Delta$ from $\mathbb{R}^{2r}$ to $\mathbb{R}^{2r}$ by

$$\mathcal{F}_\Delta(x_{n-r}, \cdots, x_{n+r-1}) = (x_{n-r+1}, \cdots, x_{n+r}).$$

The map $\mathcal{F}_\Delta$ is a homeomorphism of $\mathbb{R}^{2r}$ onto itself since we can solve (2.4) for x_{n-r} if $(x_{n-r+1}, \cdots, x_{n+r})$ is given. Taking into account the periodicity property (C2), we define a homeomorphism φ_Δ on the $2r$-dimensional cylinder $S^1 \times \mathbb{R}^{2r-1}$ which is a generalization of the class of monotone twist maps of the annulus or two-dimensional cylinder [2].

Let $\tilde{S}$ denote the set of solutions of (2.4), i.e., the set of equilibria of (1.1), $S = \tilde{S}/\langle \mathbf{1} \rangle$, and $\sigma = \tau_{-1,0}/\langle \mathbf{1} \rangle$. Then the system generated by σ on S, i.e., the spatial shift map on the set of equilibria of the driven particle chain (1.1), is equivalent to that by φ_Δ on the $2r$-dimensional cylinder.

Let $\alpha > 0$. A configuration $\mathbf{x} = (x_n)$ is said to be an α-pseudo solution of (2.4) if

$$|\Delta(x_{n-r}, \cdots, x_n, \cdots, x_{n+r}) + F| \leq \alpha \quad \text{for} \quad n \in \mathbb{Z}.$$

A configuration $\mathbf{x} = (x_n)$ is said to be a supersolution (subsolution) of (2.4) if

$$\Delta(x_{n-r}, \cdots, x_n, \cdots, x_{n+r}) + F \leq 0 \quad (\geq 0) \quad \text{for} \quad n \in \mathbb{Z}.$$

Assume $\overline{\mathbf{x}} = (\overline{x}_n)$ and $\underline{\mathbf{x}} = (\underline{x}_n)$ are a supersolution and subsolution of (1.1) respectively. It is said they exchange rotation numbers if

$$\liminf_{n\to+\infty} \frac{\overline{x}_n}{n} \geq \omega_2, \ \limsup_{n\to-\infty} \frac{\overline{x}_n}{n} \leq \omega_1, \ \limsup_{n\to+\infty} \frac{\underline{x}_n}{n} \leq \omega_1, \ \liminf_{n\to-\infty} \frac{\underline{x}_n}{n} \geq \omega_2, \tag{2.5}$$

hold for some $\omega_1 < \omega_2$, see Section 6 in [2].

A criterion proposed by Angenent in [2] shows that if there exist a supersolution and a subsolution of (2.4) exchanging rotation numbers, then the homeomorphism φ_Δ defined by (2.4), or σ on S, has positive topological entropy, see Theorem 7.1 in [2].

3 Depinning Thresholds

Let $F_c^{\pm}$ be defined by (1.5). Then $F_c^- \leq 0 \leq F_c^+$. We shall show that $F_c^- \geq A$ is the depinning threshold from travelling ($c > 0$) to stationary fronts, and $F_c^+ \leq B$ from stationary to travelling fronts ($c < 0$).

We should mention that the limits $\lim_{n\to+\infty} F_d^{\pm}(1/n)$ exist. In fact, for each rational p/q, the limits of $F_d^{\pm}(\omega)$ as ω approaches p/q from the left or right exist, see [6], but we do not need such results in this paper.

Let a and b be two distinct solutions of $-V'(x) + F = 0$.

Definition 3.1. *An equilibrium $\mathbf{x} = (x_n)$ of (1.1) is said to be heteroclinic if $x_n \to a$ as $n \to -\infty$ and $x_n \to b$ as $n \to +\infty$.*

We also say that a configuration $\mathbf{x} = (x_n)$ is heteroclinic if $x_n \to a$ as $n \to -\infty$ and $x_n \to b$ as $n \to +\infty$. Note that $a^* = (a, \cdots, a), b^* = (b, \cdots, b) \in \mathbb{R}^{2r}$ are two fixed points of $\mathcal{F}_\Delta$ defined in Section 2.5. Moreover, the orbit corresponding to the heteroclinic equilibrium $\mathbf{x} = (x_n)$ is a heteroclinic orbit asymptotic to a^* as $n \to -\infty$ and b^* as $n \to +\infty$. As we shall see in Section 5, there is a close relation between heteroclinic equilibria and stationary fronts of (1.1).

Theorem 3.2. *(i) For $\bar{F} \in [F_c^-, F_c^+]$, there exists an IOC ℓ for (1.1) with $F = \bar{F}$ such that there is a heteroclinic equilibrium between any two neighboring homogeneous equilibria on ℓ.*
(ii) For $\bar{F} \in [F_c^+, B)$ ($(A, F_c^-]$), there exists an IOC ℓ such that for each $\mathbf{z} \in \ell$, $\dot{\mathbf{z}} \geq \mathbf{0}$ ($\leq \mathbf{0}$) for (1.1) with $F = \bar{F}$, and either it is a homogeneous equilibria, or it is heteroclinic connecting two neighboring homogeneous equilibria.

Proof: (i) Let $\bar{F} \in [0, F_c^+]$ and $F_n \to \bar{F}$ as $n \to \infty$. The proof for $\bar{F} \in [F_c^-, 0]$ is similar. We know that there is a sequence of positive integers $\{q_n\}$ such that $\lim_{n\to\infty} F_d^+(1/q_n) = F_c^+$. If $\bar{F} = F_c^+$, take $F_n = F_d^+(1/q_n)$. If $0 \leq \bar{F} < F_c^+$, then take $F_n = \bar{F}$. In either case we have $F_d^-(1/q_n) \leq F_n \leq F_d^+(1/q_n)$ for n large enough, say $n \geq n_0$.

Let $\ell_n \subset \mathcal{B}_{1,q_n}$ be IOCs for (1.1) with $F = F_n$ due to Proposition 2.12. Thanks to Proposition 2.13, there is an IOC ℓ which is an accumulation point of $\{\ell_n\}$ in Hausdorff metric for (1.1) with $F = \bar{F}$, and each element on ℓ has rotation number zero. Moreover, we have $\tau_{1,0}\mathbf{y} \leq \mathbf{y}$ for each $\mathbf{y} \in \ell_n$ ($n \geq n_0$) since $\mathbf{y}$ is Birkhoff with rotation number $1/q_n > 0$, implying that $\tau_{1,0}\mathbf{x} \leq \mathbf{x}$ for each $\mathbf{x} \in \ell$ since it is a limit in product topology of $\mathbf{y}^n \in \ell_n$.

Since there exists at least one $(1, q_n)$-periodic equilibrium $\mathbf{y}^n \in \ell_n$ for (1.1) with $F = F_n$ by Proposition 2.12, we have an equilibrium on ℓ for (1.1) with $F = \bar{F}$ which is an accumulation point of $\{\mathbf{y}^n\}$. Furthermore, there is at least one homogeneous equilibrium on ℓ. Indeed, for any equilibrium $\mathbf{y} \in \ell$, it follows that $\mathbf{y} - \mathbf{1} \leq \tau_{k,0}\mathbf{y} \leq \mathbf{y}$ since $\mathbf{y}$ has rotation number 0, hence the limit $\lim_{k\to\infty} \tau_{k,0}\mathbf{y}$ is a homogeneous equilibrium on ℓ. This also proves $F_c^+ \leq B$.

It is not possible that ℓ is composed of homogeneous equilibria since we only have two, $\mathbf{u}^1(\bar{F})$ and $\mathbf{u}^2(\bar{F})$, in one period. We therefore assume $\mathbf{u}^1(\bar{F})$ and $\mathbf{u}^3(\bar{F})$ are two neighboring homogeneous equilibria on ℓ (the proof for the case that $\mathbf{u}^1(\bar{F})$ and $\mathbf{u}^2(\bar{F})$ are neighboring is similar).

Let $\mathbf{x}^n = (x_i^n) \in \ell_n \subset \mathcal{B}_{1,q_n}$ be Birkhoff $(1, q_n)$-periodic equilibria of (1.1) with $F = F_n$. Then $x_i^n \to -\infty$ as $i \to -\infty$ and $x_i^n \to +\infty$ as $i \to +\infty$. Therefore, for each $n \geq 1$, there exists $i(n)$ such that

$$x_i^n \leq u^1(\bar{F}) + \varepsilon_0 \quad \text{for} \quad i < i(n) \quad \text{and} \quad x_{i(n)}^n > u^1(\bar{F}) + \varepsilon_0,$$

where $\varepsilon_0 = (u^3(\bar{F}) - u^1(\bar{F}))/2 = 1/2$.

Let $\tilde{\mathbf{x}}^n = \tau_{-i(n),0}\mathbf{x}^n \in \ell_n$. Then

$$\tilde{x}_i^n \leq u^1(\bar{F}) + \varepsilon_0 \quad \text{for} \quad i < 0 \quad \text{and} \quad \tilde{x}_0^n > u^1(\bar{F}) + \varepsilon_0.$$

Consequently, there exists a convergent subsequence of $\{\tilde{\mathbf{x}}^n\}$ with limit $\mathbf{x} = (x_i) \in \ell$ which is an equilibrium of (1.1) with $F = \bar{F}$ such that

$$x_i \leq u^1(\bar{F}) + \varepsilon_0 < u^3(\bar{F}) \quad \text{for} \quad i < 0 \quad \text{and} \quad x_0 \geq u^1(\bar{F}) + \varepsilon_0,$$

implying $\mathbf{u}^1(\bar{F}) \ll \mathbf{x} \ll \mathbf{u}^3(\bar{F})$ since they are strictly ordered on ℓ.

It remains to show that $x_i \to u^1(\bar{F})$ as $i \to -\infty$ and $x_i \to u^3(\bar{F})$ as $i \to +\infty$. Assume $x_i \to e < u^3(\bar{F})$ as $i \to +\infty$. Then one can check that $e \cdot \mathbf{1}$ is a homogeneous equilibrium between $\mathbf{u}^1(\bar{F})$ and $\mathbf{u}^3(\bar{F})$, yielding a contradiction since we assume they are neighboring homogeneous equilibria on ℓ. The proof for $x_i \to u^1(\bar{F})$ as $i \to -\infty$ is similar. Therefore, there exists a heteroclinic equilibrium connecting any two neighboring homogeneous equilibria on ℓ.

(ii) Let $F_c^+ \leq \bar{F} < B$ and $F_n \to \bar{F}$ as $n \to \infty$. We have a sequence of positive integers $\{q_n\}$ such that $\lim_{n\to\infty} F_d^+(1/q_n) = F_c^+$. If $\bar{F} = F_c^+$, take $F_n = F_d^+(1/q_n)$. Take $F_n = \bar{F}$ if $\bar{F} > F_c^+$. Then we always have $F_n \geq F_d^+(1/q_n)$ for $n \geq n_0$. From the remark following Proposition 2.13 we obtain the IOCs ℓ_n for (1.1) with $F = F_n$ such that $\dot{\mathbf{z}} \geq \mathbf{0}$ for (1.1) with $F = F_n$ for each $\mathbf{z} \in \ell_n$, $n \geq n_0$. Let ℓ denote the IOC obtained by Proposition 2.13 of (1.1) with $F = \bar{F}$. Then for each $\mathbf{x} \in \ell$, we have $\dot{\mathbf{x}} \geq \mathbf{0}$ for (1.1) with $F = \bar{F}$ since it is the limit of $\mathbf{y}^n \in \ell_n$ in product topology.

Let $\mathbf{x} \in \ell$. Then $\tau_{1,0}\mathbf{x} \leq \mathbf{x}$. The reason is the same as in the proof of part (i). Since each element on ℓ has rotation number 0, then $\mathbf{x} - \mathbf{1} \leq \tau_{k,0}\mathbf{x} \leq \mathbf{x}$ for $k \geq 1$. It then follows that $\lim_{k\to\infty} \tau_{k,0}\mathbf{x} = \mathbf{y} \in \ell$ is homogeneous. Let $\mathbf{y}(t)$ be the solution with $\mathbf{y}(0) = \mathbf{y}$ of (1.1) with $F = \bar{F}$. Then $\mathbf{y}(t) \in \ell$, $\dot{\mathbf{y}}(t) \geq \mathbf{0}$, and $\mathbf{y}(t)$ is homogeneous for $t \in \mathbb{R}$. If there is no homogeneous equilibrium on ℓ, then $\ell = \{\mathbf{y}(t) \,|\, t \in \mathbb{R}\}$, implying each configuration on ℓ is homogeneous and hence $\mathbf{z} = e_1 \cdot \mathbf{1}$, where e_1 satisfies $-V'(e_1) + B = 0$ (see Figure 1), lies on ℓ. Note that

$$\dot{z}_i = -\partial_i W(\mathbf{z}) + \bar{F} \geq 0 \quad \text{for} \quad i \in \mathbb{Z},$$

since $\mathbf{z} \in \ell$. On the other hand,

$$\dot{z}_i = -\partial_i W(\mathbf{z}) + \bar{F} = -V'(e_1) + \bar{F} < -V'(e_1) + B = 0,$$

yielding a contradiction. Consequently, there is at least one homogeneous equilibrium on ℓ.

Assume $\mathbf{u}^1(\bar{F})$ and $\mathbf{u}^2(\bar{F})$ are two neighboring homogeneous equilibria on ℓ (the proof is similar for the case $\mathbf{u}^2(\bar{F})$ and $\mathbf{u}^3(\bar{F})$ are neighboring or $\mathbf{u}^1(\bar{F})$ and $\mathbf{u}^3(\bar{F})$ are neighboring). We prove by contradiction that there is no homogeneous configuration on the segment of ℓ between $\mathbf{u}^1(\bar{F})$ and $\mathbf{u}^2(\bar{F})$.

Assume $\mathbf{y} \in \ell$ is homogeneous and $\mathbf{u}^1(\bar{F}) \ll \mathbf{y} \ll \mathbf{u}^2(\bar{F})$. Then the solution $\mathbf{y}(t)$ with $\mathbf{y}(0) = \mathbf{y}$ of (1.1) with driving force $\bar{F}$ lies on ℓ for $t \in \mathbb{R}$. Furthermore, $\mathbf{u}^1(\bar{F}) \ll \mathbf{y}(t) \ll \mathbf{u}^2(\bar{F})$ for $t \in \mathbb{R}$ and $\mathbf{y}(t) \to \mathbf{u}^1(\bar{F})$ as $t \to -\infty$ and $\mathbf{y}(t) \to \mathbf{u}^2(\bar{F})$ as $t \to \infty$, leading to a contradiction by considering $e_1 \cdot \mathbf{1}$ as above. Consequently, either $\mathbf{x} \in \ell$ is a homogeneous equilibrium, or it is heteroclinic connecting two neighboring homogeneous equilibria. The proof for $F \in (A, F_c^-]$ is similar. $\square$

4 The "Unstable" Equilibrium

Let
$$h^F(x_1, \cdots, x_{r+1}) = h(x_1, \cdots, x_{r+1}) - Fx_1,$$
and
$$W^F(\mathbf{x}) = \sum_{j\in\mathbb{Z}} h^F(x_j, \cdots, x_{j+r}).$$
Then the driven particle system (1.1) is exactly
$$\dot{x}_i = -\partial_i W^F(\mathbf{x}), \quad i \in \mathbb{Z}. \tag{4.1}$$

In this section, we always assume $n > 2r+1$. Let $F \in (A, B)$. Denote for simplicity $a = u^2(F)$, where $u^2(F)$ is the intermediate zero of $-V'(x) + F = 0$ with $-V''(u^2(F)) > 0$ (see Figure 1). Let
$$\alpha_{i,j}(\mathbf{x}) = -\partial_{i,j} W^F(\mathbf{x}), \quad i, j \in \mathbb{Z}.$$
Then $\alpha_{i,j}(\mathbf{x}) = 0$ for $j > i + r$ or $j < i - r$. Meanwhile, one can check that for $0 \leq m \leq r$,
$$\alpha_{i,i+m}(\mathbf{x}) = -\sum_{k=1}^{r+1-m} \partial_{k,k+m} h(x_{i+1-k}, \cdots, x_{i+r+1-k}),$$
$$\alpha_{i,i-m}(\mathbf{x}) = -\sum_{k=1}^{r+1-m} \partial_{k+m,k} h(x_{i+1-m-k}, \cdots, x_{i+r+1-m-k}),$$
and hence
$$\alpha_{i,i+m}(\mathbf{x}) = \alpha_{i+m,i}(\mathbf{x}) > 0 \ \text{ for } \ i \in \mathbb{Z}, 1 \leq m \leq r, \tag{4.2}$$
since h is C^2 satisfying the twist condition (H4). Let
$$\alpha_{i,j} = \alpha_{i,j}(\mathbf{u}^2(F)), \ i \in \mathbb{Z}, \ j \in \mathbb{Z}.$$
Then we have $\alpha_{i,j} = \alpha_{i+1,j+1}$ for $i, j \in \mathbb{Z}$. One can check that for each $i \in \mathbb{Z}$,
$$\sum_{j=i-r}^{i+r} \alpha_{i,j} = -\sum_{j=1}^{r+1}\sum_{i=1}^{r+1} \partial_{i,j} h(a, \cdots, a) = -V''(a) > 0. \tag{4.3}$$

Let $A_n = (a_{i,j})$ be an $n \times n$ matrix with
$$\begin{aligned} a_{i,j} &= \alpha_{i,j}, && 1 \leq i \leq n, \ \ 1 \leq i - r \leq j \leq i + r \leq n, \\ a_{i,j} &= \alpha_{i,j-n}, && 1 \leq i \leq n, \ \ i + r < j \leq n, \\ a_{i,j} &= \alpha_{i,j+n}, && 1 \leq i \leq n, \ \ 1 \leq j < i - r. \end{aligned} \tag{4.4}$$

Then the off-diagonal elements of A_n are non-negative and A_n is symmetric due to (4.2). Meanwhile, the sum of the elements in each row is $\mu_0 = -V''(a) > 0$ thanks to (4.3). Then the biggest eigenvalue of A_n is $\mu_0 > 0$ with eigenvector $(1, \cdots, 1) \in \mathbb{R}^n$.

Letting the elements in the upper right corner and lower left corner in A_n be zero, we construct an $n \times n$ symmetric matrix $\overline{A}_n = (\bar{a}_{i,j})$ as follows.

$$\bar{a}_{i,j} = a_{i,j}, \quad \text{for} \;\; 1 \leq i \leq n, \;\; 1 \leq i - r \leq j \leq i + r \leq n, \quad \text{otherwise} \;\; \bar{a}_{i,j} = 0.$$

We shall show that the biggest eigenvalue of $\overline{A}_n$ is positive provided n is large enough. Let $\lambda_n^n \leq \lambda_{n-1}^n \leq \cdots \leq \lambda_1^n$ denote the eigenvalues of $\overline{A}_n$.

Lemma 4.1. *$\lambda_1^n \to \mu_0$ as $n \to \infty$.*

Proof: From the Perron-Frobenius Theorem it follows that μ_0 is the biggest eigenvalue of A_n for all $n > 2r+1$ with corresponding eigenvector $(1, \cdots, 1) \in \mathbb{R}^n$. Note that $\overline{A}_n$ is a leading principal submatrix of A_{2n}. We deduce by Cauchy's interlacing theorem that $\lambda_1^n \leq \mu_0$. Meanwhile, for $2r + 1 < n_1 < n_2$, $\overline{A}_{n_1}$ is a leading principal submatrix of $\overline{A}_{n_2}$. We deduce again by Cauchy's interlacing theorem that $\lambda_1^{n_1} \leq \lambda_1^{n_2}$, implying the limit $\lim_{n\to\infty} \lambda_1^n = \lambda_0$ exists. We claim that $\lambda_0 = \mu_0$.

Indeed, if $\lambda_0 < \mu_0$, then there exists $\varepsilon_0 > 0$ such that $\lambda_1^n \leq \mu_0 - \varepsilon_0$ for $n > 2r+1$, and hence μ_0 is in the resolvent set of $\overline{A}_n$. Consequently, for each $\bar{\xi} \in \mathbb{R}^n$, there exists $\bar{\eta} \in \mathbb{R}^n$ such that

$$(\mu_0 I_n - \overline{A}_n)\bar{\eta} = \bar{\xi},$$

where I_n denotes the identity of the Euclidean space $\mathbb{R}^n$ with norm $\|\cdot\|_n$.

There exists an orthonormal matrix P_n of order n such that

$$P_n^{-1}\overline{A}_n P_n = \operatorname{diag}(\lambda_n^n, \cdots, \lambda_1^n).$$

Let

$$\tilde{\eta} = P_n^{-1}\bar{\eta} \;\; \text{and} \;\; \tilde{\xi} = P_n^{-1}\bar{\xi}.$$

Then

$$P_n^{-1}(\mu_0 I_n - \overline{A}_n)P_n\tilde{\eta} = \tilde{\xi}, \quad \text{hence} \;\; \operatorname{diag}(\mu_0 - \lambda_n^n, \cdots, \mu_0 - \lambda_1^n)\tilde{\eta} = \tilde{\xi}.$$

Note that $\mu_0 - \lambda_i^n \geq \varepsilon_0$, $i = 1, \cdots, n$. Then $\varepsilon_0\|\tilde{\eta}\|_n \leq \|\tilde{\xi}\|_n$, implying

$$\|\bar{\eta}\|_n = \|\tilde{\eta}\|_n \leq \|\tilde{\xi}\|_n/\varepsilon_0 = \|\bar{\xi}\|_n/\varepsilon_0. \tag{4.5}$$

Let

$$\hat{\eta} = (1/\sqrt{n}, \cdots, 1/\sqrt{n}) \in \mathbb{R}^n, \quad \text{and} \quad (\mu_0 I_n - \overline{A}_n)\hat{\eta} = \hat{\xi}.$$

Then $\|\hat{\eta}\|_n = 1$. Since $A_n\hat{\eta} = \mu_0\hat{\eta}$, then

$$\hat{\xi} = (A_n - \overline{A}_n)\hat{\eta} = (\bar{\zeta}_1, \cdots, \bar{\zeta}_r, 0, \cdots, 0, \hat{\zeta}_1, \cdots, \hat{\zeta}_r),$$

where

$$\bar{\zeta}_1 = (a_{1,n-r+1} + \cdots + a_{1,n})/\sqrt{n}, \cdots, \hat{\zeta}_r = (a_{n,1} + \cdots + a_{n,r})/\sqrt{n}.$$

Note that $|a_{i,j}| \leq K$ for $i, j \in \{1, \cdots, n\}$ by (4.4) and the hypothesis (H2). Then

$$\|\hat{\xi}\|_n^2 = \bar{\zeta}_1^2 + \cdots + \bar{\zeta}_r^2 + \hat{\zeta}_1^2 + \cdots + \hat{\zeta}_r^2 \leq \ 2r^3K^2/n.$$

It then follows from (4.5) that

$$1 = \|\hat{\eta}\|_n \leq \|\hat{\xi}\|_n/\varepsilon_0 \leq rK\sqrt{2r}/(\sqrt{n}\varepsilon_0),$$

leading to a contradiction for sufficiently large n. $\square$

Let $a^* = (a, \cdots, a) \in \mathbb{R}^n$, where $a = u^2(F)$. Define for $u = (u_1, \cdots, u_n) \in \mathbb{R}^n$

$$\begin{aligned}\overline{W}_n^F(u) \quad &= \quad h^F(a, \cdots, a, u_1) + \cdots + h^F(a, u_1, \cdots, u_r) + h^F(u_1, \cdots, u_{r+1}) \\ &\quad + \cdots + h^F(u_{n-r+1}, \cdots, u_n, a) + \cdots + h^F(u_n, a, \cdots, a),\end{aligned}$$

and consider the truncated system

$$\dot{u}_i = -\partial_i \overline{W}_n^F(u), \ i = 1, \cdots, n. \tag{4.6}$$

Note that a^* is an equilibrium of (4.6). Let $-D^2\overline{W}_n^F(a^*)$ denote the Hessian matrix at a^*. Then it coincides with $\overline{A}_n$ by simple calculations. We then deduce that a^* is an unstable equlibrium of system (4.6) for n large enough by Lemma 4.1. Note that $\overline{A}_n$ is irreducible by (4.2). Then by the Perron-Frobenius Theorem the eigenvector $v = (v_1, \cdots, v_n) \in \mathbb{R}^n$ corresponding to the biggest eigenvalue $\lambda_1^n > 0$ is strictly positive, i.e., $v_i > 0$ for $i = 1, \cdots, n$.

Note also that system (4.6) is strongly monotone by the twist condition (H4). We have the following conclusion owing to the results in Section 5, Chapter 2, in [37].

Lemma 4.2. *Let n be large enough such that $\lambda_1^n > 0$. Then there exists a neighborhood $U \subset \mathbb{R}^n$ of a^* such that for each $u \in U$ with $u \ll a^*$ ($u \gg a^*$), it follows that $u(t) \ll u$ ($u(t) \gg u$) for $t > 0$, where $u(t)$ is a solution of (4.6) with initial value $u(0) = u$.*

Theorem 4.3. *Let $A < F < B$ and ℓ be an IOC obtained by Theorem 3.2. Then $\mathbf{u}^2(F) \notin \ell$.*

Proof: We prove by contradiction. Assume $\mathbf{u}^2(F) \in \ell$. Let n be large enough such that $a^* \in \mathbb{R}^n$ is an equilibrium of system (4.6) with the Hessian matrix $\overline{A}_n$ at a^* having a positive eigenvalue $\lambda_1^n > 0$, the corresponding eigenvector being strictly positive.

From Lemma 4.2 we deduce the existence of a neighborhood $U \subset \mathbb{R}^n$ of a^* such that for each $u \in U$ with $u \ll a^*$, $u(t) \ll u$ for $t > 0$, in which $u(t)$ is a solution of the truncated system (4.6) with the initial value $u(0) = u$.

Let $F_c^- \leq F < B$. Then we deduce by Theorem 3.2 the existence of a heteroclinic configuration $\mathbf{z} = (z_i) \in \ell$ such that $\dot{\mathbf{z}} \geq \mathbf{0}$ ($= \mathbf{0}$ if $F_c^- \leq F \leq F_c^+$) for (1.1), $\mathbf{z} \ll \mathbf{u}^2(F)$ and $u = (z_1, \cdots, z_n) \in U$. Since $z_i < a = u^2(F)$ for $i = -r, \cdots, n, \cdots, n+r$, and $-\partial_i W^F(\mathbf{z}) \geq 0$ for $i \in \mathbb{Z}$, we derive by the monotonicity (see Lemma 2.4) that $-\partial_i \overline{W}_n^F(u) > 0$ for $i = 1$, implying $u_1(t) > u_1(0) = z_1$ for $0 < t \leq t_0$ with t_0 small, in which $u(t) = (u_1(t), \cdots, u_n(t))$ is a solution of (4.6) with $u(0) = u$. This is a contradiction to Lemma 4.2 since $u \ll a^*$. The case for $A < F < F_c^-$ is proved by choosing $\mathbf{z} \gg \mathbf{u}^2(F)$. This completes the proof. □

5 Stationary and Travelling Fronts

To prove our conclusions, we need to associate the existence of IOCs of (1.1) with travelling and stationary fronts.

Lemma 5.1. *Equation (1.6) has a strictly increasing and continuous solution u satisfying (1.7) if and only if there exists an IOC for (1.1) with $F = 0$ consisting of homogeneous equilibria $\mathbf{0}$, $\mathbf{1}$, and heteroclinic equilibria connecting $\mathbf{0}$ to $\mathbf{1}$.*

Proof: Let ℓ be the IOC for (1.1) with $F = 0$ consisting of $\mathbf{0}$, $\mathbf{1}$, and heteroclinic equilibria connecting $\mathbf{0}$ to $\mathbf{1}$. Let $\mathbf{y} = (y_n) \in \ell$ be a heteroclinic equilibrium. Note that we have a strictly increasing and continuous function G satisfying $G(t+1) = G(t) + 1$ for $t \in \mathbb{R}$ such that

$$\ell = \{\mathbf{x} = (x_n) \,|\, x_n = G^n(t),\ n \in \mathbb{Z},\ t \in \mathbb{R}\}.$$

Since $\mathbf{0}$, $\mathbf{1} \in \ell$, $G(0) = 0$, $G(1) = 1$. Meanwhile, $y_n = G^n(y_0)$ for $n \in \mathbb{Z}$.

Note that $y_n < y_{n+1}$, $n \in \mathbb{Z}$. Define $\eta : (0,1) \to (-\infty, +\infty)$ as follows. Let

$$\begin{cases} \eta(y_0) = 0, \quad \eta(y_n) = n, \quad \text{for} \quad n \in \mathbb{Z}, \\ \eta(t) = (t - y_0)/(y_1 - y_0), \quad \text{for} \quad t \in (y_0, y_1), \\ \eta(t) = \eta(G^{-n}(t)) + n, \text{ for } t \in (y_n, y_{n+1}),\ n \in \mathbb{Z},\ n \neq 0. \end{cases}$$

Then η is strictly increasing and continuous, and satisfies $\lim_{t\to 0} \eta(t) = -\infty$, and $\lim_{t\to 1} \eta(t) = +\infty$.

Let $u(s) = \eta^{-1}(s)$ for $s \in (-\infty, +\infty)$. Then $u : \mathbb{R} \to (0, 1)$ is a strictly increasing and continuous function satisfying (1.7) and $y_n = u(n)$, $n \in \mathbb{Z}$. Furthermore, for each $s \in \mathbb{R}$, from the definition of η it follows that

$$u(s + n) = G^n(u(s)) \ \text{ for } \ n \in \mathbb{Z}.$$

Let $s \in \mathbb{R}$, $x_0 = u(s)$, and $x_n = u(s + n)$, $n \in \mathbb{Z}$. Then $\mathbf{x} = (x_n) \in \ell$ satisfies $\mathbf{0} \ll \mathbf{x} \ll \mathbf{1}$ and is a stationary front of (1.1), implying that the function u satisfies (1.6) and (1.7).

Assume there exists a strictly increasing and continuous function u satisfying (1.6) and (1.7). Let $\eta : (0, 1) \to \mathbb{R}$ denote a strictly increasing and continuous function satisfying $\lim_{t\to 0} \eta(t) = -\infty$, and $\lim_{t\to 1} \eta(t) = +\infty$. We remark that such a function exists. For $n \in \mathbb{Z}$, let

$$g_n(s) = u(n + \eta(s)), \ s \in (0, 1), \ g_n(0) = 0, \ g_n(1) = 1,$$

and

$$g_n(s') = g_n(s) + k, \quad \text{where } \ s' = s + k, \ k \in \mathbb{Z}, \ s \in [0, 1).$$

Then we have a continuous function $g : \mathbb{R} \to \mathscr{X}$, $g(s) = (g_n(s))$, the image of which, $\ell = \{g(s) \,|\, s \in \mathbb{R}\}$ is a strictly ordered circle. Meanwhile, each element on ℓ is an equilibrium of (1.1) with $F = 0$, and hence ℓ in invariant for the gradient flow of (1.1) with $F = 0$. It remains to show that ℓ is invariant for $\tau_{k,l}$ for k, $l \in \mathbb{Z}$.

Indeed, for each $s \in (0, 1)$, let $\tilde{s} \in (0, 1)$ such that $\eta(\tilde{s}) = \eta(s) - k$. Let $s' = \tilde{s} + l$. Then $\tau_{k,l} g(s) = g(s')$, implying ℓ is invariant for $\tau_{k,l}$. Note also that the intermediate homogeneous equilibrium $\mathbf{u}^2 \notin \ell$ since $g_n(s) \neq g_{n+1}(s)$ for $s \in (0, 1)$. Therefore, ℓ is an IOC for (1.1) with $F = 0$ consisting of $\mathbf{0}$, $\mathbf{1}$, and heteroclinic equilibria connecting $\mathbf{0}$ to $\mathbf{1}$. $\square$

Lemma 5.2. *If there exists an IOC ℓ for (1.1) such that $\mathbf{u}^1(F), \mathbf{u}^3(F) \in \ell$, and there is a heteroclinic equilibrium $\mathbf{y}$ on ℓ connecting $\mathbf{u}^1(F)$ to $\mathbf{u}^3(F)$, then $\mathbf{y}$ is a stationary front.*

Proof: The proof is similar to the first part of that of the above lemma, with $\mathbf{0}$ and $\mathbf{1}$ being replaced by $\mathbf{u}^1(F)$ and $\mathbf{u}^3(F)$ respectively. $\square$

Lemma 5.3. *System (1.1) has a travelling front with the profile function u satisfying (1.3) with $c < 0$ ($c > 0$) and (1.4) if and only if there exists an IOC ℓ for (1.1) such that $\mathbf{u}^1(F), \mathbf{u}^3(F) \in \ell$ and for each $\mathbf{z} \in \ell \backslash \{\mathbf{u}^1(F), \mathbf{u}^3(F)\}$, $\mathbf{z}$ is heteroclinic connecting $\mathbf{u}^1(F)$ to $\mathbf{u}^3(F)$ and satisfying $\dot{\mathbf{z}} \gg \mathbf{0}$ ($\ll \mathbf{0}$) for (1.1).*

Proof: Assume there is a travelling front with the profile function u satisfying (1.3) with $c < 0$ (the proof for $c > 0$ is similar) and (1.4). The construction of a continuous function $g : \mathbb{R} \to \mathscr{X}$ is the same as that in the proof of Lemma 5.1 using the profile function u with 0 and 1 being replaced by $u^1(F)$ and $u^3(F)$ respectively. Then $\ell = \{g(s) \mid s \in \mathbb{R}\}$ is a strictly ordered circle invariant for $\tau_{k,l}$, $k, l \in \mathbb{Z}$. It remains to check the invariance of ℓ for the flow ϕ_F^t of (1.1).

Indeed, if we denote by $\mathbf{x}(t) = (x_n(t))$ the travelling front, i.e., $x_n(t) = u(n-ct)$, then for each $\mathbf{z} \in \ell \backslash \{\mathbf{u}^1(F), \mathbf{u}^3(F)\}$, there is $s \in (u^1(F), u^3(F))$, such that

$$z_n = g_n(s) = u(n + \eta(s)) = u(n - ct_0) = x_n(t_0), \quad \text{where} \ \ t_0 = -\eta(s)/c.$$

As a consequence, $\phi_F^t \mathbf{z} = \mathbf{x}(t_0 + t) \in \ell$ since

$$x_n(t_0 + t) = u(n - c(t_0 + t)) = u(n + \eta(s')) = g_n(s'),$$

where $s' \in (u^1(F), u^3(F))$ such that $\eta(s') = -c(t_0 + t)$. This implies that ℓ is invariant for ϕ_F^t for all $t \in \mathbb{R}$. Moreover, $z_n \to u^1(F)$ as $n \to -\infty$, $z_n \to u^3(F)$ as $n \to +\infty$, and $\dot{\mathbf{z}} = \dot{\mathbf{x}}(t_0) \gg \mathbf{0}$ since $\dot{x}_n(t_0) = -cu'(n - ct_0) > 0$ for all $n \in \mathbb{Z}$.

Assume there exists an IOC ℓ for (1.1) such that $\mathbf{u}^1(F), \mathbf{u}^3(F) \in \ell$ and for each $\mathbf{z} \in \ell \backslash \{\mathbf{u}^1(F), \mathbf{u}^3(F)\}$, $\mathbf{z}$ is heteroclinic connecting $\mathbf{u}^1(F)$ to $\mathbf{u}^3(F)$, and $\dot{\mathbf{z}} \gg \mathbf{0}$ for (1.1). Let $\mathbf{x}^0 \in \ell \backslash \{\mathbf{u}^1(F), \mathbf{u}^3(F)\}$ and $\mathbf{x}(t)$ be the solution of (1.1) with initial value $\mathbf{x}(0) = \mathbf{x}^0$.

Since ℓ is invariant for ϕ_F^t ($t \in \mathbb{R}$) and $\tau_{k,l}$, $k, l \in \mathbb{Z}$, then $\mathbf{x}(t) \in \ell$ for all $t \in \mathbb{R}$ and $\tau_{-1,0}\mathbf{x}(0) \in \ell$. Note that ℓ is in fact composed of two equilibria $\mathbf{u}^1(F)$, $\mathbf{u}^3(F)$, and the solution $\{\mathbf{x}(t) \mid t \in \mathbb{R}\}$. Furthermore, $\mathbf{x}(t) \to \mathbf{u}^1(F)$ as $t \to -\infty$ and $\mathbf{x}(t) \to \mathbf{u}^3(F)$ as $t \to +\infty$ in product topology. Since $\mathbf{x}(0)$ is left asymptotic to $\mathbf{u}^1(F)$ and right asymptotic to $\mathbf{u}^3(F)$, implying $\tau_{-1,0}\mathbf{x}(0) \gg \mathbf{x}(0)$, then there exists $T > 0$ such that $\tau_{-1,0}\mathbf{x}(0) = \mathbf{x}(T)$. Meanwhile, from (2.2) we have $\tau_{-1,0}\mathbf{x}(t) = \mathbf{x}(t + T)$ for all $t \in \mathbb{R}$, i.e., $x_{n+1}(t) = x_n(t + T)$ and hence $x_n(t) = x_0(t + nT)$ for all $t \in \mathbb{R}$ and $n \in \mathbb{Z}$.

Let $c = -1/T < 0$ and $u(s) = x_0(Ts)$ for $s \in \mathbb{R}$. Then u is C^1 smooth with $u'(s) > 0$ and satisfies (1.3) and (1.4). Moreover, $x_n(t) = u(n - ct)$ for $n \in \mathbb{Z}$ and $t \in \mathbb{R}$ and hence $\mathbf{x}(t)$ is a travelling front solution. $\square$

Lemma 5.4. *If there exists an IOC ℓ_F such that $\mathbf{u}^1(F), \mathbf{u}^3(F) \in \ell_F$ and for each $\mathbf{z} \in \ell_F \backslash \{\mathbf{u}^1(F), \mathbf{u}^3(F)\}$, $\mathbf{z}$ is heteroclinic connecting $\mathbf{u}^1(F)$ to $\mathbf{u}^3(F)$ and satisfying $\dot{\mathbf{z}} \geq \mathbf{0}$ ($\leq \mathbf{0}$) for (1.1), then there is no heteroclinic equilibrium of (1.1) with driving force $F_1 > F$ ($F_1 < F$) connecting $\mathbf{u}^1(F_1)$ to $\mathbf{u}^3(F_1)$.*

Proof: Denote $v^1 = u^1(F)$, $\mathbf{v}^1 = \mathbf{u}^1(F)$, $u^1 = u^1(F_1)$, and $\mathbf{u}^1 = \mathbf{u}^1(F_1)$ for simplicity. Then $v^3 = v^1 + 1$, $\mathbf{v}^3 = \mathbf{v}^1 + \mathbf{1}$, $u^3 = u^3(F_1) = u^1 + 1$ and $\mathbf{u}^3 = \mathbf{u}^1 + \mathbf{1}$. We know that $\mathbf{v}^1$ and $\mathbf{v}^3$ are two homogeneous equilibria of (1.1) with driving force F, and $\mathbf{u}^1$ and $\mathbf{u}^3$ are two homogeneous equilibria of (1.1) with driving force F_1, satisfying $\mathbf{v}^1 \ll \mathbf{u}^1 \ll \mathbf{v}^3 \ll \mathbf{u}^3$ by (1.2).

We prove the case $F_1 > F$ by contradiction (the proof for $F_1 < F$ is similar). Assume there exists heteroclinic equilibrium $\mathbf{x} = (x_n)$ of (1.1) with driving force $F_1 > F$ connecting $\mathbf{u}^1$ to $\mathbf{u}^3$., i.e., $x_n \to u^1$ as $n \to -\infty$, and $x_n \to u^3$ as $n \to +\infty$.

Meanwhile, there exists a lift of a circle homeomorphism G such that

$$\ell_F = \{\mathbf{w} = (w_n) \,|\, w_n = G^n(w_0),\ n \in \mathbb{Z},\ w_0 \in \mathbb{R}\},$$

$G(v^1) = v^1$, $G(v^3) = (v^3)$. Let $w_0 \in (v^1, v^3)$ and $\mathbf{w} = (w_n) \in \ell_F$, where $w_n = G^n(w_0)$, $n \in \mathbb{Z}$. Then $w_n \to v^1$ as $n \to -\infty$ and $w_n \to v^3$ as $n \to +\infty$. Consequently, there exist $n_1 < n_2 < n_3$ such that

$$w_n < x_n \text{ for } n \leq n_1, \quad w_{n_2} > x_{n_2}, \quad \text{and} \quad w_n < x_n \text{ for } n \geq n_3.$$

Let

$$J = \{v^1 < s \leq w_0 \,|\, G^n(s) \leq x_n,\ n_1 \leq n \leq n_3\}, \quad \text{and} \quad s_1 = \sup J.$$

Since $G^n(s) \to v^1$ as $s \to v^1$ uniformly for $n_1 \leq n \leq n_3$ and G is continuous in $\mathbb{R}$, the set J is nonempty. Let

$$\mathbf{z} = (z_n) \in \ell_F, \quad \text{where} \quad z_n = G^n(s_1),\ n \neq 0,\ z_0 = s_1.$$

Then it follows that there exists $m \in \mathbb{Z}$ with $n_1 < m < n_3$, such that

$$z_m = x_m, \quad z_{m-i} \leq x_{m-i}, \quad \text{and} \quad z_{m+i} \leq x_{m+i},\ 1 \leq i \leq r.$$

We deduce by Lemma 2.4 together with $\dot{\mathbf{z}} \geq \mathbf{0}$ for (1.1) that

$$0 \leq -\partial_m W(\mathbf{z}) + F < -\partial_m W(\mathbf{z}) + F_1 \leq -\partial_m W(\mathbf{x}) + F_1 = 0,$$

since $\mathbf{x}$ is an equilibrium of (1.1) with driving force F_1, a contradiction. □

Corollary 5.5. *Let $F \in (A, F_c^-) \cup (F_c^+, B)$. Then there is no heteroclinic equilibrium of (1.1) connecting $\mathbf{u}^1(F)$ to $\mathbf{u}^3(F)$.*

Proof: This follows from Theorem 3.2, Theorem 4.3, and Lemma 5.4. □

6 Proof of Main Conclusions

Proof of Theorem A:

Let $F_c^{\pm}$ be defined by (1.5) and $F_c^- < F < F_c^+$. Then by Theorem 3.2 we have $F_c^- \geq A$, $F_c^+ \leq B$, and an IOC ℓ_F for (1.1) such that $\mathbf{u}^1(F), \mathbf{u}^3(F) \in \ell_F$ but $\mathbf{u}^2(F) \notin \ell_F$ due to Theorem 4.3. Meanwhile, there exists a heteroclinic equilibrium

$\mathbf{z} \in \ell_F$ connecting $\mathbf{u}^1(F)$ to $\mathbf{u}^3(F)$, leading to the existence of a stationary front by Lemma 5.2.

If there exist travelling fronts, then from Lemma 5.3, we obtain an IOC ℓ for (1.1) such that $\mathbf{u}^1(F), \mathbf{u}^3(F) \in \ell$ and for each $\mathbf{z} \in \ell \backslash \{\mathbf{u}^1(F), \mathbf{u}^3(F)\}$, $\dot{\mathbf{z}} \gg \mathbf{0}$ ($\ll \mathbf{0}$) for (1.1), implying immediately by Lemma 5.4 $F_c^+ \leq F$ ($F \leq F_c^-$) since (1.1) with driving force F_c^+ (F_c^-) has heteroclinic equilibrium by Theorem 3.2, a contradiction. Consequently, there are stationary fronts but no travelling fronts for (1.1) with $F \in (F_c^-, F_c^+)$.

Let $F_c^+ < F < B$. Then we have an IOC ℓ_F by Theorem 3.2 such that for each $\mathbf{z} \in \ell$, $\dot{\mathbf{z}} \geq \mathbf{0}$. Meanwhile, from Theorem 4.3, we know that $\mathbf{u}^2(F) \notin \ell_F$, implying that $\mathbf{u}^1(F), \mathbf{u}^3(F) \in \ell_F$ and each $\mathbf{z} \in \ell_F \backslash \{\mathbf{u}^1(F), \mathbf{u}^3(F)\}$ is heteroclinic connecting $\mathbf{u}^1(F)$ to $\mathbf{u}^3(F)$ by Theorem 3.2. Moreover, we deduce by Corollary 5.5 that $\dot{\mathbf{z}} \gg \mathbf{0}$ for each $\mathbf{z} \in \ell_F \backslash \{\mathbf{u}^1(F), \mathbf{u}^3(F)\}$, and hence we obtain the existence of travelling fronts by Lemma 5.3 and nonexistence of stationary fronts by Corollary 5.5. The proof for the case $A < F < F_c^-$ is similar. Therefore, the critical values $F_c^\pm$ defined by (1.5) are identical to those obtained in [28], which characterize the depinning transition from travelling ($c > 0$) to stationary fronts, and stationary to travelling ($c < 0$) fronts. □

Proof of Theorem B:

Assume $u : \mathbb{R} \to (0, 1)$ is a strictly increasing and continuous function satisfying (1.6) and (1.7). Then we deduce by Lemma 5.1 the existence of an IOC ℓ for (1.1) with $F = 0$ such that $\mathbf{u}^1(0), \mathbf{u}^3(0) \in \ell$, $\mathbf{u}^2(0) \notin \ell$, and $\dot{\mathbf{z}} = \mathbf{0}$ for each $\mathbf{z} \in \ell$. If $F_c^+ > 0$, then by Theorem 3.2 there exists a heteroclinic equilibrium connecting $\mathbf{u}^1(F_c^+)$ to $\mathbf{u}^3(F_c^+)$, a contradiction to Lemma 5.4. Therefore, $F_c^- = F_c^+ = 0$.

Assume $F_c^+ = F_c^-$. Then $F_c^+ = 0$ since $F_c^- \leq 0$ and $F_c^+ \geq 0$. Let $\{q_n\}$ be a sequence of positive integers such that $\lim_{n\to\infty} F_d^+(1/q_n) = F_c^+ = 0$. Let $\ell_n \subset \mathcal{B}_{1,q_n}$ denote IOCs for (1.1) with $F = F_d^+(1/q_n) \geq 0$ such that (see the remark following Proposition 2.13) for each $\mathbf{y} \in \ell_n$, $\dot{\mathbf{y}} \geq \mathbf{0}$ for (1.1) with $F = F_d^+(1/q_n)$. Let ℓ denote a lift of an accumulation point of $\ell_n/\langle \mathbf{1} \rangle$ in the Hausdorff metric. Then ℓ is an IOC of (1.1) with $F = 0$ by Proposition 2.13, and for each $\mathbf{x} \in \ell$, $\dot{\mathbf{x}} \geq \mathbf{0}$ for (1.1) with $F = 0$. We shall show in what follows that each $\mathbf{x} \in \ell$ is an equilibrium of (1.1) with $F = 0$, i.e., $-\partial_n W(\mathbf{x}) = 0$ for all $n \in \mathbb{Z}$.

The strategy is similar to that in the first part of the proof of Theorem B in [36]. Assume there exists $\mathbf{x}' \in \ell$ and $i_0 \in \mathbb{Z}$ such that $\partial_{i_0} W(\mathbf{x}') < 0$. Then owing to the continuity of ℓ, there exists a neighborhood $U \subset \mathscr{X}$ of $\mathbf{x}'$ such that $\partial_{i_0} W(\mathbf{x}) < 0$ for all $\mathbf{x} \in U \cap \ell$, implying

$$\int_{\ell/\langle \mathbf{1} \rangle} \partial_{i_0} W(\mathbf{x}) \mathrm{d}x_{i_0} < 0. \tag{6.1}$$

Thanks to the translation-invariance of ℓ, i.e., $\tau_{-j+1,0}\ell = \ell$, we make a transformation of variables, $\mathbf{z} = \tau_{-j+1,0}\mathbf{x}$, i.e., $z_i = x_{i+j-1}$ for $i \in \mathbb{Z}$, to calculate the integral for $1 \le j \le r+1$,

$$\begin{aligned}
&\int_{\ell/\langle \mathbf{1}\rangle} \partial_j h(x_{i_0}, \cdots, x_{i_0+j-1}, \cdots, x_{i_0+r}) \mathrm{d}x_{i_0+j-1} \\
=& \int_{\tau_{-j+1,0}(\ell/\langle \mathbf{1}\rangle)} \partial_j h(z_{i_0+1-j}, \cdots, z_{i_0}, \cdots, z_{i_0+r+1-j}) \mathrm{d}z_{i_0} \\
=& \int_{\tau_{-j+1,0}\ell/\langle \mathbf{1}\rangle} \partial_j h(z_{i_0+1-j}, \cdots, z_{i_0}, \cdots, z_{i_0+r+1-j}) \mathrm{d}z_{i_0} \\
=& \int_{\ell/\langle \mathbf{1}\rangle} \partial_j h(x_{i_0+1-j}, \cdots, x_{i_0}, \cdots, x_{i_0+r+1-j}) \mathrm{d}x_{i_0}.
\end{aligned}$$

Note that

$$\partial_{i_0} W(\mathbf{x}) = \sum_{j=1}^{r+1} \partial_j h(x_{i_0+1-j}, \cdots, x_{i_0}, \cdots, x_{i_0+r+1-j}).$$

Due to the periodicity hypothesis (H1) and the fact $\tau_{0,1}\ell = \ell$, we derive that

$$\begin{aligned}
0 =& \int_{\ell/\langle \mathbf{1}\rangle} \mathrm{d}h(x_{i_0}, \cdots, x_{i_0+r}) = \sum_{j=1}^{r+1} \int_{\ell/\langle \mathbf{1}\rangle} \partial_j h(x_{i_0}, \cdots, x_{i_0+j-1}, \cdots, x_{i_0+r}) \mathrm{d}x_{i_0+j-1} \\
=& \sum_{j=1}^{r+1} \int_{\ell/\langle \mathbf{1}\rangle} \partial_j h(x_{i_0+1-j}, \cdots, x_{i_0}, \cdots, x_{i_0+r+1-j}) \mathrm{d}x_{i_0} = \int_{\ell/\langle \mathbf{1}\rangle} \partial_{i_0} W(\mathbf{x}) \mathrm{d}x_{i_0},
\end{aligned}$$

which is a contradiction to (6.1).

Therefore, each $\mathbf{x} \in \ell$ is an equilibrium of (1.1) with $F = 0$. Furthermore, we deduce that $\mathbf{u}^2(0) \notin \ell$ by Theorem 4.3 and hence ℓ is composed of $\mathbf{u}^1(0)$, $\mathbf{u}^3(0)$, and heteroclinic equilibria connecting $\mathbf{u}^1(0)$ to $\mathbf{u}^3(0)$. Consequently, we have by Lemma 5.1 and its proof a strictly increasing and continuous function u satisfying (1.6) and (1.7). □

Remark 1: By the above proof we know that $F_c^- = F_c^+$ if and only if there exists an IOC ℓ for (1.1) with $F = 0$ such that each $\mathbf{z} \in \ell$ is an equilibrium of (1.1) with $F = 0$, and hence ℓ is a minimal foliation since it is connected, strictly ordered, and invariant for shifts $\{\tau_{k,l}\}$. Moreover, each $\mathbf{z} \in \ell$ is a minimizer according to Theorem 10.1 in [31] or Lemma 3.8 in [36].

Remark 2: If we restrict our attention to the case $r = 1$, then the projection of the IOC ℓ with each $\mathbf{z} \in \ell$ being an equilibrium of (1.1) obtained for $F_c^- = F_c^+$ is a homotopically non-trivial invariant circle with rotation number 0 consisting of fixed points and their right-going separatices on the cylinder for the corresponding twist map. Here we give an example of twist maps showing the existence of the invariant circle consisting of fixed points and heteroclinic orbits. Let

$$f(x) = x + k\sin 2\pi x,\ \ k \in (0, 1/(2\pi)),\ \text{ and } \psi(x) = f(x) + f^{-1}(x) - 2x, x \in \mathbb{R}.$$

Construct a twist map $\Psi : (x, y) \mapsto (x', y')$ by $x' = x + y + \psi(x)$, $y' = y + \psi(x)$. Then one can check that Ψ has invariant circles

$$\Gamma_1 = \{(x, y) \,|\, y = x - f^{-1}(x), x \in \mathbb{R}\},\ \text{ and }\ \Gamma_2 = \{(x, y) \,|\, y = x - f(x), x \in \mathbb{R}\},$$

and the restrictions of Ψ on Γ_1 and Γ_2 are f and f^{-1}, respectively. Let Γ be the union of the piece on Γ_1 for $x \in [0, 1/2]$ and the piece on Γ_2 for $x \in [1/2, 1]$. Then Γ is an invariant circle consisting of fixed points and their right-going separatices.

Proof of Theorem C:

Let $0 < \alpha < \min\{F_c^+ - F, F - F_c^-\}$ and take $F_1 = F + \alpha$ and $F_2 = F - \alpha$. From Theorems 3.2 and 4.3, we deduce the existence of an IOC ℓ_{F_1} for (1.1) with driving force F_1 and a heteroclinic equilibrium $\mathbf{x} = (x_n) \in \ell_{F_1}$ such that $x_n \to u^1(F_1)$ as $n \to -\infty$ and $x_n \to u^3(F_1)$ as $n \to +\infty$. Then for each $\delta > 0$, there exists $N_1 > 2r$ such that

$$|x_n - u^1(F_1)| < \delta \ \text{ for }\ n \le -N_1 \ \text{ and }\ |x_n - u^3(F_1)| < \delta \ \text{ for }\ n \ge N_1.$$

Let $q_1 = 2N_1 + 2r$. We construct an α-pseudo solution $\bar{\mathbf{x}} = (\bar{x}_i)$ of (2.4) with driving force F_1 as follows. Let

$$\bar{x}_i = \begin{cases} u^1(F_1), & i < -N_1 - r, \\ x_i, & -N_1 - r \le i \le N_1 + r - 1, \\ (\tau^k_{q_1,1}\mathbf{x})_i, & -N_1 - r + kq_1 \le i \le N_1 + r - 1 + kq_1,\ k \ge 1. \end{cases}$$

Then one can check that $\bar{\mathbf{x}}$ is an α-pseudo solution of (2.4) with driving force F_1 provided δ is small enough, implying

$$\Delta(\bar{x}_{i-r}, \cdots, \bar{x}_{i+r}) + F = \Delta(\bar{x}_{i-r}, \cdots, \bar{x}_{i+r}) + F_1 + F - F_1 \le 0,\ i \in \mathbb{Z},$$

and hence $\bar{\mathbf{x}}$ is a supersolution of (2.4) with driving force F satisfying

$$\lim_{i\to-\infty} (\bar{x}_i - \bar{x}_0)/i = 0 \ \text{ and }\ \lim_{i\to+\infty} (\bar{x}_i - \bar{x}_0)/i = 1/q_1.$$

Similarly, there exists an IOC ℓ_{F_2} for (1.1) with driving force F_2 and a heteroclinic equilibrium $\mathbf{y} = (y_n) \in \ell_{F_2}$ such that $y_n \to u^1(F_2)$ as $n \to -\infty$ and $y_n \to u^3(F_2)$ as $n \to +\infty$. Then for each $\delta > 0$, there exists $N_2 > 2r$ such that

$$|y_n - u^1(F_2)| < \delta \ \text{ for }\ n \le -N_2 \ \text{ and }\ |y_n - u^3(F_2)| < \delta \ \text{ for }\ n \ge N_2.$$

Let $q_2 = 2N_2 + 2r$ and

$$\underline{x}_i = \begin{cases} u^1(F_2), & i > N_2 + r, \\ y_i, & -N_2 - r + 1 \le i \le N_2 + r, \\ (\tau^k_{-q_2,-1}\mathbf{y})_i, & -N_2 - r + 1 - kq_2 \le i \le N_2 + r - kq_2,\ k \ge 1. \end{cases}$$

Then one can also check that $\underline{\mathbf{x}} = (\underline{x}_i)$ is an α-pseudo solution of (2.4) with driving force F_2 if δ is small enough, implying

$$\Delta(\underline{x}_{i-r}, \cdots, \underline{x}_{i+r}) + F = \Delta(\underline{x}_{i-r}, \cdots, \underline{x}_{i+r}) + F_2 + F - F_2 \ge 0,\ i \in \mathbb{Z},$$

and hence $\underline{\mathbf{x}}$ is a subsolution of (2.4) satisfying

$$\lim_{i\to-\infty} (\underline{x}_i - \underline{x}_0)/i = 1/q_2 \ \text{ and } \ \lim_{i\to+\infty} (\underline{x}_i - \underline{x}_0)/i = 0.$$

Taking $\omega_1 = 0 < \min\{1/q_1, 1/q_2\} = \omega_2$, we obtain a supersolution and a subsolution of (2.4) exchanging rotation numbers, implying positive topological entropy for σ on S by Theorem 7.1 in [2]. □

Remark: Since we have constructed a supersolution and a subsolution of (2.4) exchanging rotation numbers 0 and $\omega_2 > 0$, then we deduce by Theorem 6.1 in [2] that for each $\omega \in [0, \omega_2]$, there exists a Birkhoff solution of (2.4) with rotation number ω, i.e., an equilibrium of (1.1) with $F_c^- < F < F_c^+$.

Proof of Theorem D:

Let $p/q \in \mathbb{Q}$ in lowest terms and denote by $F_d^\pm(p/q, h)$ the lower and upper depinning force depending on the potential function h for (1.1). We first show that $F_d^\pm(p/q, h)$ are continuous with respect to h in C^1 topology.

Let $h, \tilde{h}$ be two potential functions satisfying the assumptions (H1)-(H4). We shall show that for each $\varepsilon > 0$, there exists $\delta > 0$ independent of p/q such that

$$|F_d^+(p/q, h) - F_d^+(p/q, \tilde{h})| < \varepsilon, \quad \text{if} \quad \|h - \tilde{h}\|_{C^1} < \delta.$$

From Lemma 2.8 we deduce the existence of $\mathbf{x}$, $\tilde{\mathbf{x}} \in \mathcal{B}_{p,q}$ such that

$$-\partial_i W(\mathbf{x}, h) + F_d^+(p/q, h) = 0 \ \text{ and } \ -\partial_i W(\tilde{\mathbf{x}}, \tilde{h}) + F_d^+(p/q, \tilde{h}) = 0,\ i \in \mathbb{Z}.$$

We claim that there exists $i_0 \in \mathbb{Z}$ such that

$$-\partial_{i_0} W(\tilde{\mathbf{x}}, h) \ge -\partial_{i_0} W(\mathbf{x}, h).$$

Otherwise, we have for all $i \in \mathbb{Z}$,

$$-\partial_i W(\tilde{\mathbf{x}}, h) + F_d^+(p/q, h) < -\partial_i W(\mathbf{x}, h) + F_d^+(p/q, h) = 0,$$

and hence by the periodicity $\partial_{i+q}W(\tilde{\mathbf{x}},h)=\partial_iW(\tilde{\mathbf{x}},h)$ for all $i\in\mathbb{Z}$, we have some $c>0$ such that

$$-\partial_iW(\tilde{\mathbf{x}},h)+F_d^+(p/q,h)+c\le 0,\quad\text{for all}\quad i\in\mathbb{Z},$$

a contradiction to Lemma 2.9.

Meanwhile, we have

$$|\partial_{i_0}W(\tilde{\mathbf{x}},\tilde{h})-\partial_{i_0}W(\tilde{\mathbf{x}},h)|<\delta'=(r+1)\delta\ \text{ since }\ \|h-\tilde{h}\|_{C^1}<\delta,$$

and hence

$$F_d^+(p/q,\tilde{h})-\delta'=\partial_{i_0}W(\tilde{\mathbf{x}},\tilde{h})-\delta'<\partial_{i_0}W(\tilde{\mathbf{x}},h)\le\partial_{i_0}W(\mathbf{x},h)=F_d^+(p/q,h).$$

Similarly we have if $\|h-\tilde{h}\|_{C^1}<\delta$

$$F_d^+(p/q,h)-\delta'<F_d^+(p/q,\tilde{h}),\quad\text{implying}\quad|F_d^+(p/q,h)-F_d^+(p/q,\tilde{h})|<\delta'.$$

Taking $\delta<\varepsilon/(r+1)$ we show the continuity of $F_d^+(p/q,h)$ with respect to h in C^1 topology. The proof for F_d^- is similar. We remark that δ is independent of p/q.

Note that $F_c^+(h)=\limsup_{n\to\infty}F_d^+(1/n,h)$. Meanwhile, for each $\varepsilon>0$, there exists $0<\delta<\varepsilon/(r+1)$ such that

$$|F_d^+(1/n,h)-F_d^+(1/n,\tilde{h})|<\varepsilon,$$

implying $|F_c^+(h)-F_c^+(\tilde{h})|\le\varepsilon$ provided $\|h-\tilde{h}\|_{C^1}<\delta$. Consequently, F_c^+ is continuous with respect to h in C^1 topology. The proof for F_c^- is similar. □

7 Discussions

1. Our approach also applies to general lattice systems not generated by potential functions:

$$\dot{x}_n=\Delta(x_{n-k},\cdots,x_n,\cdots,x_{n+l})+F,$$

where $\Delta:\mathbb{R}^{k+l+1}\to\mathbb{R}$ is C^1 satisfying (C1)-(C3) in Section 2.5 and some other assumptions to guarantee the existence, uniqueness, and continuous dependence on system parameters of global solutions of the above system. We define lower and upper depinning force as in Section 2.3. This time the nonemptiness of $\mathcal{A}_{p,q}$ is not from the Aubry-Mather theory. It is obtained by Theorem 9.1 in [2] and it is not necessary that $0\in\mathcal{A}_{p,q}$. Meanwhile, we also have IOCs for each $F\in\mathbb{R}$ owing to the periodicity and monotonicity conditions. Therefore, most of the main conclusions hold true for the above general lattice systems, including some dissipative systems.

2. We should mention that the context here of periodic potential is a special case of the bistable potential treated by many authors, but the periodic case allows to consider (p, q)-periodic equilibria, which do not exist in the general bistable case.

3. The stationary and travelling fronts are sometimes called kink solutions. We can also investigate the depinning transition of anti-kink solutions using the approach of this paper. A travelling wave solution $\mathbf{x}(t) = (x_n(t))$ is said to be anti-kink if $x_n(t) = u(n-ct)$ with $c \neq 0$ and the profile function $u : \mathbb{R} \to (u^1(F), u^3(F))$ being C^1 smooth, $u'(s) < 0$ for $s \in \mathbb{R}$, and satisfying (1.3) and (1.4) with $u^1(F)$ and $u^3(F)$ interchanged. An equilibrium $\mathbf{x} = (x_n)$ of (1.1) is said to be anti-kink if there exists a strictly decreasing function u satisfying (1.4) interchanging $u^1(F)$ and $u^3(F)$ such that $x_n = u(n)$ for $n \in \mathbb{Z}$.

We define

$$\hat{F}_c^- = \liminf_{n\to+\infty} F_d^-(-1/n) \quad \text{and} \quad \hat{F}_c^+ = \limsup_{n\to+\infty} F_d^+(-1/n).$$

Then conclusions similar to those in Theorems A, B, C, and D hold true if we are concerned with the depinning transition of anti-kink equilibria and travelling waves of (1.1).

If $\max\{\hat{F}_c^-, F_c^-\} < F < \min\{\hat{F}_c^+, F_c^+\}$, then there are neither kink nor anti-kink travelling fronts, leading to yet more equilibria having a rotation interval with 0 in the interior.

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