Received: 16 November 2021 | Revised: 11 October 2022 | Accepted: 11 December 2022

DOI: 10.1002/nav.22093

RESEARCH ARTICLE

WILEY

Ranking and selection for pairwise comparison

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Funding information
National Natural Science Foundation of China, Grant/Award Numbers: 71971176, 71701123, 71725001, 71910107002; Fundamental Research Funds for the Central Universities, Grant/Award Number: JBK2103010.

Abstract

In many real-world applications, designs can only be evaluated pairwise, relative to each other. Nevertheless, in the simulation literature, almost all the ranking and selection procedures are developed based on the individual performances of each design. This research considers the statistical ranking and selection problem when the design performance can only be simulated pairwise. We formulate this new problem using the optimal computing budget allocation approach and derive the asymptotic optimality condition based on some approximations. The numerical study indicates that our approach can reduce the number of simulations required to confidently identify the best design.

KEYWORDS

OCBA, pairwise comparison, ranking and selection, simulation

1 | INTRODUCTION

Simulation is more and more widely used in evaluating discrete event dynamic systems due to the increasing complexity of these systems. Examples of using simulation can be found in supervisory control (Deng & Qiu, 2015), logistics and supply chain (Tako & Robinson, 2012), and virtual reality (Turner et al., 2016). Although computing power has improved significantly in recent years due to the rapid development of information technology, efficiency is still the bottleneck in using simulation. This is because a single simulation replication of a complex system may take several hours or days, and a large number of simulation replications are usually needed to obtain a reliable mean performance value (Chen & Lee, 2010). As a result, how to improve the efficiency of simulation experiments remains important in the simulation area (Xu et al., 2016).

For a discrete event dynamic system, the number of designs (alternatives) for selection is usually fixed and finite. Therefore, finding the design with the best expected performance through simulation and within a limited number of simulation replications essentially requires optimally allocating these simulation replications among all the alternatives. This problem falls into a popular research area known as ranking and selection, which has received substantial attention in recent years (Hong et al., 2021; Peng et al., 2017). In general, there exist three major approaches for studying ranking and selection problems, the indifference zone (IZ) approach, the optimal computing budget allocation (OCBA), and the value of information (VIP) approach. The IZ approach provides a guarantee for selecting the best design with a predetermined probability of correct selection (PCS), assuming that the user is indifferent to performance differences below a given threshold (Kim & Nelson, 2001; Rinott, 1978). The OCBA approach allocates a fixed number of simulation replications sequentially based on the performance mean and variance of each design such that the probability of correctly selecting the best design can be maximized (Chen et al., 2000). The VIP approach adopts the Bayesian framework and determines the future simulation budget allocation by maximizing the value information such as expected improvement and knowledge gradient (Chick et al., 2010; Ding et al., 2022; Frazier et al., 2008; Peng & Fu, 2017; Ryzhov, 2016).

Research in ranking and selection has been developed and extended in a variety of ways. For example, several studies proposed efficient computing budget allocation rules for selecting the optimal subset (Chen et al., 2008; Gao & Chen, 2016; Zhang et al., 2015). In the case of designs with multiple performance measures, Lee et al. (2010) and Branke and Zhang (2019) developed efficient simulation allocation procedures for selecting the nondominated Pareto set. For stochastic constrained simulation optimization problems, simulation budget allocation rules have been developed in Lee et al. (2012), Hunter and Pasupathy (2013), and Xiao et al. (2019). In the case that there exist multiple scenarios for each design, robust ranking and selection procedures have been developed from both the IZ approach and OCBA approach (Fan et al., 2020; Gao, Xiao, et al., 2017; Xiao & Gao, 2018). Some other extensions and applications of the ranking and selection problem include feasibility determination (Gao & Chen, 2017; Peng et al., 2020), minimizing the opportunity cost (Gao, Chen, & Shi, 2017), large-scale problems (Zhong & Hong, 2022), and multifidelity models (Peng et al., 2019; Song et al., 2019).

All the above-mentioned research has made an implicit assumption that the performance of each design can be estimated individually. However, in many real-world problems, only pairwise information is available instead of individual performance (Groves & Branke, 2019). A common phenomenon in human decision-making is the lack of transitivity due to factors like the threshold effect, disturbance in concentration, and errors of input data. As a result, pairwise comparison is a fundamental tool in multicriteria decision-making for making judgments about alternatives and has wide applications connected to human activity, including manufacturing, service industry, research, and surveys (Kou et al., 2016; Rácz, 2022; Wang et al., 2021). For example, a round-robin tournament is a typical pairwise comparison format in which the winner is determined by points counting. Round-robin tournaments not only exist in sports but also in public choice models like voting schemes and decision rules in committees (Harary & Moser, 1966; Ryvkin & Ortmann, 2008). The ranking of NBA teams can only be estimated via playing the teams against each other is a typical example of pairwise comparison in sports. Another example can be found in market research, a respondent is usually asked to indicate his preference for each pair of brands in order to know the structure of customers' preferences among the competing brands of a product.

Given the popularity of pairwise comparison in the real application, different approaches have been used to rank and select the best design (designs) for pairwise comparison in a noisy environment. Negahban et al. (2016) introduced rank centrality algorithm for discovering scores for items from pairwise comparisons. This algorithm constructs a Markov chain on the pairwise observations under the Bradley–Terry–Luce (BTL) model (Bradley & Terry, 1952), and then returns its stationary distribution by computing the top left eigenvector of the associated probability transition matrix. Li et al. (2021) considered the problem of selecting important nodes in a random network. In this article, Markov chains are used to characterize random networks, and the importance of each node is described by stationary probability. Simulation sampling is carried out on the interaction parameter between each pair of nodes to calculate the transition probability and estimate the stationary probability. Chen and Suh (2015) proposed a nearly linear-time two-stage procedure called the Spectral MLE (maximum likelihood estimation) for exact recovering the top k items based on rank centrality. They assume that the pairwise comparison outcomes are generated according to the BTL model. Chen, Gopi, et al. (2017) and Chen, Wang, et al. (2017) assume that each comparison has noise constrained by the strongly stochastically transitive model (Fishburn, 1973), and present a linear time algorithm for selecting the top k items. Heckel et al. (2019) proposed an active ranking algorithm partitioning the items into sets of prespecified size, where ranking is based on the Borda score and does not make any assumptions on the pairwise comparison probabilities. Compared to the parametric models that often fail in real-world pairwise comparison data (Ballinger & Wilcox, 1997), using the Borda score does not need to make any assumptions. Shah and Wainwright (2018) prove that the Borda score has three good properties including simplicity, optimality, and robustness. Recently, the work in Groves and Branke (2019) proposed two active sampling methods that adapted the OCBA framework and knowledge gradient framework to the pairwise sampling setting to select the top-m designs based on the Borda score. Thus, this work adopts the Borda score as the performance measure in developing the simulation budget allocation rules.

In this research, we consider this ranking and selection problem with pairwise comparisons. The performance of each design cannot be estimated individually by simulation. Instead, we only have the pairwise comparison information between two alternatives via simulation. Thus, the number of simulation replications needed for a single output increases from k to $k(k-1)/2$ given that there are designs for comparisons. Since the pairwise comparison information can only be obtained from simulation, a good estimation needs multiple simulation replications. The objective of this research is to derive an efficient computing budget allocation rule that distributes a fixed and finite number of simulation replications such that we select the best design or top m designs as accurately as possible.

The contribution of this research is threefold. First, we formulate the pairwise comparison ranking and selection problem as an optimal computing budget allocation model based on the performance measure of the Borda score. Second, we extended the research in Groves and Branke (2019) by developing an easily implementable asymptotically optimal allocation rule. Finally, the proposed pairwise comparison simulation procedure broadens the use of simulation in decision making and market research where only pairwise information is available.

The rest of the article is organized as follows. Section 2 formulates the new optimal budget computing allocation model. Section 3 derives the asymptotically optimal allocation rule.

Section 4 considers the problem of selecting the top m designs. In Section 5, a heuristic sequential simulation algorithm is proposed, and numerical experiments are carried out. Finally, we conclude this research in Section 6.

2 | PROBLEM FORMULATION

We consider the problem of selecting the best design from a finite number of k alternatives, which can be represented by the set $x = \{x_1, x_2, \dots ., x_k\}$. The best design is defined as the design with the minimum performance. However, the performance of any design $x_i, i = 1, 2, \dots, k$ is unknown and cannot be estimated individually. Instead, for each pair of designs $\{x_i, x_j\}$, $X_{i,j}$ is a random variable representing the result of the comparison between the two designs. Let $\mu_{i,j}$ denote the expected outcome of a "pairwise comparison," that is, $\mu_{i,j} = E(X_{i,j})$. For example, in decision-making, the expert is asked to indicate the preference of two alternatives numerically. The numerical preference is usually denoted by a real number from −100 to 100, that is, $X_{i,j}$ can take any value from −100 to 100. Thus, $X_{i,j}$ denotes how much alternative i is better than alternative j.

In the case of pairwise comparison, we use the Borda score S_i to denote the performance of the design x_i (Borda, 1784), that is,

$$S_i = \sum_{j \neq i} \mu_{i,j}. \tag{1}$$

To facilitate the presentation, we use the following notations.

T	the total number of simulation replications
$X_{i,j,t}$	the output of the tth simulation replication for pairwise comparison between x_i and x_j, where $i, j \in \{1, 2, \dots, k\}$ and $i \neq j$
$\mu_{i,j}$	the mean for $X_{i,j}$, that is, $E(X_{i,j,t}) = E(X_{i,j}) = \mu_{i,j}$
$\sigma^2_{i,j}$	the variance for $X_{i,j}$
$\overline{X}_{i,j}$	$\overline{X}_{i,j} = (1/n_{i,j}) \sum_{t=1}^{n_{i,j}} X_{i,j,t}$ denotes the sample mean of $X_{i,j}$
$\overline{S}_i$	the sum of the sample mean of $X_{i,j}$ for all $j \neq i$, that is, $\overline{S}_i = \sum_{j \neq i} \overline{X}_{i,j}$
$n_{i,j}$	the number of simulation replications allocated to the pair of designs $\{x_i, x_j\}$

We make the following assumptions in developing the model.

Assumption 1 Comparing x_i to x_j has the same effect as comparing x_j to x_i, that is, $X_{j,i} = -X_{i,j}$.

Assumption 1 states that performing a single replication of simulation on a pair of designs $\{x_i, x_j\}$ affects the estimation of the Borda scores of both designs. Thus, we restrict that $n_{i,j} = 0$ if $i > j$ or $i = j$.

Assumption 2 The simulation output samples are normally distributed. They are identically distributed and independent from replication to replication, as well as independent across different pairs of designs.

Assumption 2 is commonly used in the area of simulation since the normality assumption can always be met if batch means are used due to the well-known central limit theorem. Although this assumption can be removed, the Borda score which is a sum of random variables, the distribution of the Borda score may be unknown if the simulation output samples are non-normal distribution.

Assumption 3 The outcomes of the pairwise comparisons need not be transitive. It means that from A better than B, B better than C, we cannot get A better than C.

In practice, there are many examples where pairwise comparisons are transitive. For instance, comparing the output of different production lines, the protein content of different milk, the lifetime of different electronical devices. However, due to the threshold effects, disturbance in concentration, errors of input data, and other factors, the lack of transitivity in pairwise comparison is a common phenomenon. There are some traditional examples like finding the full ranking of teams in a tournament, identifying the best player in online video games, and even in rock-paper-scissors games. The analytic hierarchy process is often used in multi-attribute decision-making in the fields of manufacturing, service industry, research, surveys, and some others, in which pairwise comparisons between elements form a judgment matrix. Theoretically, the judgment matrix is required to be consistent. However, in practice, the lack of transitivity is common (Kou et al., 2016; Rácz, 2022; Wang et al., 2021). Recently, the internet era has led to a variety of applications involving pairwise comparisons. For example, recommender systems for rating movies, books, images, and other items (Aggarwal, 2016; Chen, Gopi, et al., 2017; Chen, Wang, et al., 2017), peer grading for ranking students in massive open online courses (MOOCs) (Piech et al., 2013; Shah et al., 2013), and so on. The transitivity of preferences is one of the most important assumptions in the recommender system research. However, the intransitive relations have also been widely observed in practice (Chen, Gopi, et al., 2017, Chen, Wang, et al., 2017). This assumption is important in formulating this pairwise ranking and selection problem. Otherwise, the pairwise ranking and selection problem can reduce to the original OCBA problem (Chen et al., 2000).

The OCBA framework is developed based on Bayesian setting. (Chen, 1996; Chen et al., 2000; Chen et al., 2008). We formulate the new optimal budget computing allocation problem using the Bayesian framework, where the mean of the simulation output for each pair of designs, $\mu_{i,j}$, is assumed unknown and treated as a random variable. After the simulation is performed, a posterior distribution for the unknown mean $\mu_{i,j}$ is constructed based on two pieces of information: (i) prior knowledge about $\mu_{i,j}$, and (ii) simulation output. We introduce $\tilde{X}_{i,j}$ denotes the posterior estimate of $\mu_{i,j}$, and $\tilde{S}_i$ denotes the sum of the posterior estimate of $\mu_{i,j}$ for all $j \neq i$. As in Chen (1996), we assume that the unknown mean $\mu_{i,j}$ has a conjugate normal prior distribution and consider non-informative prior distributions, which implies that no prior knowledge is available about the performance of any pairwise comparison before conducting the simulation. Thus, the posterior distribution of $\mu_{i,j}$ is $\tilde{X}_{i,j} \sim N\left(\overline{X}_{i,j}, \sigma_{i,j}^2/n_{i,j}\right)$, and the posterior distribution of S_i is $\tilde{S}_i \sim N\left(\sum_{j,j\neq i}\overline{X}_{i,j}, \sum_{j,j\neq i}\sigma_{i,j}^2/n_{i,j}\right)$ (DeGroot, 1970).

In this problem, the objective is to select the design with the minimum performance. Note that the design with the minimum performance has the largest Borda score. Let the random variable μ_i denote the unknown performance of the design x_i for each $i \in \{1, 2, \dots, k\}$. Without loss of generality, we assume that $\mu_1 < \mu_2 < \dots < \mu_k$. Based on the definition of the Borda score, $\mu_1 < \mu_2 < \dots < \mu_k$ results in $S_1 > S_2 > \dots > S_k$, where S_i is the Borda score of the design x_i for each $i \in \{1, 2, \dots, k\}$. Given a fixed number of simulation replications, the best design cannot be selected with certainty. A commonly used performance measure is the PCS. A correct selection occurs when the posterior estimate of Borda score of the true best design is better than that of any other design. Then,

$$PCS_1 = P\left\{\bigcap_{i=2}^{k}\left(\tilde{S}_1 > \tilde{S}_i\right)\right\}. \tag{2}$$

The selection problem is

$$\begin{aligned} &\max PCS_1 \\ s.t.\quad &\sum_{i<j} n_{i,j} = T, i = 1, \dots, k-1, j = 2, \dots, k \\ &n_{i,j} \geq 0, \quad \text{for all } i \text{ and } j. \end{aligned} \tag{3}$$

In this research, we ignore the minor technicalities associated with $n_{i,j}$ being not integer. The major difficulty of solving the optimization model in (3) is that PCS_1 is computationally intractable. To deal with this issue, we give an approximation on PCS_1.

Theorem 1 *The PCS_1 in (2) is lower bounded by*

$$APCS_1 = 1 - P\left(\tilde{S}_1 \leq c_1\right) - \sum_{i=2}^{k} P\left(\tilde{S}_i \geq c_1\right),$$

where c_1 is a constant defined as $c_1 = \left(S_1 + S_2\right)/2$.

Proof For a constant c_1,

$$\begin{aligned} PCS_1 &= P\left\{\bigcap_{i=2}^{k}\left(\tilde{S}_1 > \tilde{S}_i\right)\right\} \\ &\geq P\left[\left(\tilde{S}_1 > c_1\right) \cap \left(\bigcap_{i=2}^{k}\tilde{S}_i < c_1\right)\right] \\ &\geq P\left(\tilde{S}_1 > c_1\right) + \sum_{i=2}^{k} P\left(\tilde{S}_i < c_1\right) - (k-1) \\ &= 1 - P\left(\tilde{S}_1 \leq c_1\right) - \sum_{i=2}^{k} P\left(\tilde{S}_i \geq c_1\right) = APCS_1 \end{aligned}$$

■

In this research, we consider the following optimization model instead of the model given in (3).

$$\begin{aligned} &\max APCS_1 \\ s.t.\quad &\sum_{i<j} n_{i,j} = T, i = 1, \dots, k-1, j = 2, \dots, k \\ &n_{i,j} \geq 0, \quad \text{for all } i \text{ and } j. \end{aligned} \tag{4}$$

This problem falls into the OCBA framework. It maximizes the approximated PCS by determining the optimal allocation of the simulation replications $n_{i,j}$ (the decision variables). Although it is an approximated PCS, it is much more computationally tractable compared with (3). Further development of simulation budget allocation is then based on (4).

3 | SELECTION PROCEDURE

This section aims to derive the asymptotically optimal allocation rule based on (4) and suggests a sequential allocation algorithm to implement the proposed rule.

3.1 | Asymptotical optimality condition

Based on Assumptions 1–3, $\tilde{S}_i$ is normally distributed as $\tilde{S}_i \sim N\left(\sum_{j,j\neq i}\overline{X}_{i,j}, \sum_{j,j\neq i}\sigma_{i,j}^2/n_{i,j}\right)$.

Then, we have

$$P\left(\tilde{S}_1 \leq c_1\right) = \Phi\left(\left(c_1 - \sum_{j=2}^{k}\overline{X}_{1,j}\right) \Big/ \sqrt{\sum_{j=2}^{k}\left(\sigma_{1,j}^2/n_{1,j}\right)}\right), \tag{5}$$

$$\begin{aligned} P\left(\tilde{S}_i \geq c_1\right) &= \Phi\left(\left(-c_1 + \sum_{j,j\neq i}\overline{X}_{i,j}\right) \Big/ \sqrt{\sum_{j,j\neq i}\left(\sigma_{i,j}^2/n_{i,j}\right)}\right) \\ &= \Phi\left(\left(-c_1 + \sum_{j,j\neq i}\overline{X}_{i,j}\right) \Big/ \sqrt{\sum_{j,j>i}\left(\sigma_{i,j}^2/n_{i,j}\right) + \sum_{j,j<i}\left(\sigma_{i,j}^2/n_{j,i}\right)}\right). \end{aligned} \tag{6}$$

Thus,

$$\begin{aligned} APCS_1 = 1 &- \Phi\left(\left(c_1 - \sum_{j=2}^{k}\overline{X}_{1,j}\right) \Big/ \sqrt{\sum_{j=2}^{k}\left(\sigma_{1,j}^2/n_{1,j}\right)}\right) \\ &- \sum_{i=2}^{k}\Phi\left(\left(-c_1 + \sum_{j,j\neq i}\overline{X}_{i,j}\right) \Big/ \sqrt{\sum_{j,j>i}\left(\sigma_{i,j}^2/n_{i,j}\right) + \sum_{j,j<i}\left(\sigma_{i,j}^2/n_{j,i}\right)}\right). \end{aligned} \tag{7}$$

Before deriving the optimality condition, we first present two lemmas that are needed in deriving the asymptotical optimality condition.

Lemma 1 (Gao & Chen, 2016) *Consider $a_i, b_j, c_i, d_j \in R$ with $a_i, b_j > 0$ and $c_i, d_j < 0$. Suppose that $\sum_{i=1}^{\eta_1} a_i \exp(c_i n) = \sum_{j=1}^{\eta_2} b_j \exp(d_j n)$ for $\eta_1, \eta_2 > 1$ as $n \to \infty$. Then $\max_{i \in \{1,\dots,\eta_1\}} c_i = \max_{j \in \{1,\dots,\eta_2\}} d_j$.*

Lemma 2 *The optimization problem defined in (4) is asymptotically concave as the total computing budget T goes to infinity, where $APCS_1$ is given in (7).*

Proof See Appendix A. ■

Given the concavity, we can use the Karush–Kuhn–Tucker (KKT) conditions to solve the optimization problem in (4), whose objective function $APCS_1$ is given by (7).

Theorem 2 *Let $\alpha_{i,j} = n_{i,j}/T$ for all $i < j$ and $i,j \in \{1, 2, \dots, k\}$ denote the proportion of simulation budget allocated to a pair of designs (i,j). Let $\overline{S}_i = \sum_{j,j \neq i} \overline{X}_{i,j}$ denote the sample mean of Borda score of the design x_i for each $i \in \{1, 2, \dots, k\}$. Let $R_i = \left(c_1 - \overline{S}_i\right)^2 / 2\left(\sum_{j,j>i} \sigma_{i,j}^2/\alpha_{i,j} + \sum_{j,j<i} \sigma_{i,j}^2/\alpha_{j,i}\right)$ for each $i \in \{1, 2, \dots, k\}$. The optimization problem defined in (4) is asymptotically optimized if for any two pairs of designs (i, i') and (l, l') as the total computing budget T goes to infinity, we have*

$$\lim_{T\to\infty} \min\{R_i, R_{i'}\} = \lim_{T\to\infty} \min\{R_l, R_{l'}\}, \tag{8}$$

where $l < l'; i < i'; i, i', l, l' \in \{1, 2, \dots, k\}$. According to the law of large numbers, the sample mean $\overline{X}_{i,j}$ converges to the mean $\mu_{i,j}$ in probability. Then, R_i converges to $\left(c_1 - \sum_{j,j\neq i} \mu_{i,j}\right)^2 / 2\left(\sum_{j,j>i} \sigma_{i,j}^2/\alpha_{i,j} + \sum_{j,j<i} \sigma_{i,j}^2/\alpha_{j,i}\right)$ in probability.

Proof Let F be the Lagrangian function with Lagrange multipliers λ_1 and $v_{i,j}$, $i<j$ and $i,j \in \{1, 2, \dots, k\}$. That is,

$$F = 1 - \Phi\left(\left(c_1 - \sum_{j=2}^{k} \overline{X}_{1,j}\right) \Big/ \sqrt{\sum_{j=2}^{k} \frac{\sigma_{1,j}^2}{n_{1,j}}}\right) - \sum_{i=2}^{k} \Phi\left(\left(-c_1 + \sum_{j,j\neq i} \overline{X}_{i,j}\right) \Big/ \sqrt{\sum_{j,j>i} \frac{\sigma_{i,j}^2}{n_{i,j}} + \sum_{j,j<i} \frac{\sigma_{i,j}^2}{n_{j,i}}}\right) - \lambda_1\left(\sum_{i<j} n_{i,j} - T\right) - \sum_{\substack{i<j \\ i,j\in\{1,\dots,k\}}} v_{i,j} n_{i,j}.$$ ■

The stationarity conditions can be stated as follows.

For $i = 1, i' \neq 1, i < i'$,

$$\begin{aligned}\frac{\partial F}{\partial n_{1,i'}} &= \phi\left(\left(c_1 - \sum_{j=2}^{k} \overline{X}_{1,j}\right) \Big/ \sqrt{\sum_{j=2}^{k} \left(\sigma_{1,j}^2/n_{1,j}\right)}\right) \\ &\times \left[-\frac{1}{2}\left(c_1 - \sum_{j=2}^{k} \overline{X}_{1,j}\right)\left(\sum_{j=2}^{k} \frac{\sigma_{1,j}^2}{n_{1,j}}\right)^{-\frac{3}{2}} \sigma_{1,i'}^2 n_{1,i'}^{-2}\right] \\ &+ \phi\left(\left(\sum_{j,j\neq i'} \overline{X}_{i',j} - c_1\right) \Big/ \sqrt{\sum_{j,j>i'} \frac{\sigma_{i',j}^2}{n_{i',j}} + \sum_{j,j<i'} \frac{\sigma_{i',j}^2}{n_{j,i'}}}\right) \\ &\times \left[\frac{1}{2}\left(c_1 - \sum_{j,j\neq i'} \overline{X}_{i',j}\right)\left(\sum_{j,j>i'} \frac{\sigma_{i',j}^2}{n_{i',j}} + \sum_{j,j<i'} \frac{\sigma_{i',j}^2}{n_{j,i'}}\right)^{-\frac{3}{2}} \sigma_{i',1}^2 n_{1,i'}^{-2}\right] \\ &= \lambda_1 + v_{1,i'}.\end{aligned} \tag{10}$$

For $i \neq 1,\ \ i' \neq 1,\ \ i < i'$

$$\begin{aligned}\frac{\partial F}{\partial n_{i,i'}} &= \phi\left(\left(-c_1 + \sum_{j,j\neq i} \overline{X}_{i,j}\right) \Big/ \sqrt{\sum_{j,j>i} \frac{\sigma_{i,j}^2}{n_{i,j}} + \sum_{j,j<i} \frac{\sigma_{i,j}^2}{n_{j,i}}}\right) \\ &\times \left[\frac{1}{2}\left(c_1 - \sum_{j,j\neq i} \overline{X}_{i,j}\right)\left(\sum_{j,j>i} \frac{\sigma_{i,j}^2}{n_{i,j}} + \sum_{j,j<i} \frac{\sigma_{i,j}^2}{n_{j,i}}\right)^{-\frac{3}{2}} \sigma_{i,i'}^2 n_{i,i'}^{-2}\right] \\ &+ \phi\left(\left(\sum_{j,j\neq i'} \overline{X}_{i',j} - c_1\right) \Big/ \sqrt{\sum_{j,j>i'} \frac{\sigma_{i',j}^2}{n_{i',j}} + \sum_{j,j<i'} \frac{\sigma_{i',j}^2}{n_{j,i'}}}\right) \\ &\times \left[\frac{1}{2}\left(c_1 - \sum_{j,j\neq i'} \overline{X}_{i',j}\right)\left(\sum_{j,j>i'} \frac{\sigma_{i',j}^2}{n_{i',j}} + \sum_{j,j<i'} \frac{\sigma_{i',j}^2}{n_{j,i'}}\right)^{-\frac{3}{2}} \sigma_{i',i}^2 n_{i,i'}^{-2}\right] \\ &= \lambda_1 + v_{i,i'}\end{aligned} \tag{11}$$

The complementary slackness conditions can be stated as follows.

$$v_{i,j} n_{i,j} = 0, \quad \text{for all } i < j. \tag{12}$$

As $n_{i,j}$, $i < j$ are in the denominator in Equation (10) and (11), they cannot be zero. Then in Equation (12), it must hold that $v_{i,j} = 0$ for all $i < j$. Thus, the KKT conditions can be simplified as follows.

For $i = 1, i' \neq 1, i < i'$,

$$\begin{aligned}\frac{\partial F}{\partial n_{1,i'}} &= \phi\left(\left(c_1 - \sum_{j=2}^{k} \overline{X}_{1,j}\right) \Big/ \sqrt{\sum_{j=2}^{k} \left(\sigma_{1,j}^2/n_{1,j}\right)}\right) \\ &\times \left[-\frac{1}{2}\left(c_1 - \sum_{j=2}^{k} \overline{X}_{1,j}\right)\left(\sum_{j=2}^{k} \frac{\sigma_{1,j}^2}{n_{1,j}}\right)^{-\frac{3}{2}} \sigma_{1,i'}^2 n_{1,i'}^{-2}\right] \\ &+ \phi\left(\left(\sum_{j,j\neq i'} \overline{X}_{i',j} - c_1\right) \Big/ \sqrt{\sum_{j,j>i'} \frac{\sigma_{i',j}^2}{n_{i',j}} + \sum_{j,j<i'} \frac{\sigma_{i',j}^2}{n_{j,i'}}}\right) \\ &\times \left[\frac{1}{2}\left(c_1 - \sum_{j,j\neq i'} \overline{X}_{i',j}\right)\left(\sum_{j,j>i'} \frac{\sigma_{i',j}^2}{n_{i',j}} + \sum_{j,j<i'} \frac{\sigma_{i',j}^2}{n_{j,i'}}\right)^{-\frac{3}{2}} \sigma_{i',1}^2 n_{1,i'}^{-2}\right] \\ &= \lambda_1.\end{aligned} \tag{13}$$

For $i \neq 1, i' \neq 1, i < i'$

$$\frac{\partial F}{\partial n_{i,i'}} = \phi\left(\left(-c_1 + \sum_{j,j\neq i}\overline{X}_{i,j}\right) \Big/ \sqrt{\sum_{j,j>i}\frac{\sigma_{i,j}^2}{n_{i,j}} + \sum_{j,j<i}\frac{\sigma_{i,j}^2}{n_{j,i}}}\right)$$
$$\times\left[\frac{1}{2}\left(c_1 - \sum_{j,j\neq i}\overline{X}_{i,j}\right)\left(\sum_{j,j>i}\frac{\sigma_{i,j}^2}{n_{i,j}} + \sum_{j,j<i}\frac{\sigma_{i,j}^2}{n_{j,i}}\right)^{-\frac{3}{2}}\sigma_{i,i'}^2 n_{i,i'}^{-2}\right]$$
$$+\ \phi\left(\left(\sum_{j,j\neq i'}\overline{X}_{i',j} - c_1\right) \Big/ \sqrt{\sum_{j,j>i'}\frac{\sigma_{i',j}^2}{n_{i',j}} + \sum_{j,j<i'}\frac{\sigma_{i',j}^2}{n_{j,i'}}}\right)$$
$$\times\left[\frac{1}{2}\left(c_1 - \sum_{j,j\neq i'}\overline{X}_{i',j}\right)\left(\sum_{j,j>i'}\frac{\sigma_{i',j}^2}{n_{i',j}} + \sum_{j,j<i'}\frac{\sigma_{i',j}^2}{n_{j,i'}}\right)^{-\frac{3}{2}}\sigma_{i',i}^2 n_{i,i'}^{-2}\right]$$
$$= \lambda_1 \tag{14}$$

To analyze optimality, we consider the following three cases.

Case 1 For $(1, i')$ and $(1, l')$, $i' \neq l'$, $i', l' > 1$. From (13), using the Lemma 1, as $T \to \infty$,

$$\max\left\{\frac{\left(c_1 - \sum_{j=2}^{k}\overline{X}_{1,j}\right)^2}{-2\left(\sum_{j=2}^{k}\frac{\sigma_{1,j}^2}{\alpha_{1,j}}\right)}, \frac{\left(-c_1 + \sum_{j,j\neq i'}\overline{X}_{i',j}\right)^2}{-2\left(\sum_{j,j>i'}\frac{\sigma_{i',j}^2}{\alpha_{i',j}} + \sum_{j,j<i'}\frac{\sigma_{i',j}^2}{\alpha_{j,i'}}\right)}\right\}$$
$$= \max\left\{\frac{\left(c_1 - \sum_{j=2}^{k}\overline{X}_{1,j}\right)^2}{-2\left(\sum_{j=2}^{k}\frac{\sigma_{1,j}^2}{\alpha_{1,j}}\right)}, \frac{\left(-c_1 + \sum_{j,j\neq l'}\overline{X}_{l',j}\right)^2}{-2\left(\sum_{j,j>l'}\frac{\sigma_{l',j}^2}{\alpha_{l',j}} + \sum_{j,j<l'}\frac{\sigma_{l',j}^2}{\alpha_{j,l'}}\right)}\right\}. \tag{15}$$

Case 2 For $(i, i') \neq (l, l')$, $i, i', l, l' \neq 1$, $i < i'$, $l < l'$. From (14), using the Lemma 1, as $T \to \infty$,

$$\max\left\{\frac{-\left(\sum_{j,j\neq i}\overline{X}_{i,j} - c_1\right)^2\Big/2}{\sum_{j,j>i}\frac{\sigma_{i,j}^2}{\alpha_{i,j}} + \sum_{j,j<i}\frac{\sigma_{i,j}^2}{\alpha_{j,i}}}, \frac{-\left(\sum_{j,j\neq i'}\overline{X}_{i',j} - c_1\right)^2\Big/2}{\sum_{j,j>i'}\frac{\sigma_{i',j}^2}{\alpha_{i',j}} + \sum_{j,j<i'}\frac{\sigma_{i',j}^2}{\alpha_{j,i'}}}\right\}$$
$$= \max\left\{\frac{-\left(\sum_{j,j\neq l}\overline{X}_{l,j} - c_1\right)^2\Big/2}{\sum_{j,j>l}\frac{\sigma_{l,j}^2}{\alpha_{l,j}} + \sum_{j,j<l}\frac{\sigma_{l,j}^2}{\alpha_{j,l}}}, \frac{-\left(\sum_{j,j\neq l'}\overline{X}_{l',j} - c_1\right)^2\Big/2}{\sum_{j,j>l'}\frac{\sigma_{l',j}^2}{\alpha_{l',j}} + \sum_{j,j<l'}\frac{\sigma_{l',j}^2}{\alpha_{j,l'}}}\right\}. \tag{16}$$

Case 3 For $(1, i')$ and (l, l'), $i', l, l' > 1, l < l'$. From (13) and (14), using the Lemma 1, as $T \to \infty$,

$$\max\left\{\frac{\left(c_1 - \sum_{j=2}^{k}\overline{X}_{1,j}\right)^2}{-2\left(\sum_{j=2}^{k}\frac{\sigma_{1,j}^2}{\alpha_{1,j}}\right)}, \frac{\left(-c_1 + \sum_{j,j\neq i'}\overline{X}_{i',j}\right)^2}{-2\left(\sum_{j,j>i'}\frac{\sigma_{i',j}^2}{\alpha_{i',j}} + \sum_{j,j<i'}\frac{\sigma_{i',j}^2}{\alpha_{j,i'}}\right)}\right\}$$
$$= \max\left\{\frac{-\left(\sum_{j,j\neq l}\overline{X}_{l,j} - c_1\right)^2\Big/2}{\sum_{j,j>l}\frac{\sigma_{l,j}^2}{\alpha_{l,j}} + \sum_{j,j<l}\frac{\sigma_{l,j}^2}{\alpha_{j,l}}}, \frac{-\left(\sum_{j,j\neq l'}\overline{X}_{l',j} - c_1\right)^2\Big/2}{\sum_{j,j>l'}\frac{\sigma_{l',j}^2}{\alpha_{l',j}} + \sum_{j,j<l'}\frac{\sigma_{l',j}^2}{\alpha_{j,l'}}}\right\}. \tag{17}$$

Let $\overline{S}_i = \sum_{j,j\neq i}\overline{X}_{i,j}$ denote the sample mean of the Borda score of a design x_i for each $i \in \{1, 2, \ldots, k\}$. Let $R_i = \left(c_1 - \overline{S}_i\right)^2/2\left(\sum_{j,j>i}\sigma_{i,j}^2/\alpha_{i,j} + \sum_{j,j<i}\sigma_{i,j}^2/\alpha_{j,i}\right)$ for each $i \in \{1, 2, \ldots, k\}$. The equalities in (15)–(17) can be simplified to

$$\begin{cases} \lim_{T\to\infty}\max\{-R_1, -R_{i'}\} = \lim_{T\to\infty} \\ \max\{-R_1, -R_{l'}\}, i' \neq l', i', l' \in \{2, 3, \ldots, k\} \\ \lim_{T\to\infty}\max\{-R_i, -R_{i'}\} = \lim_{T\to\infty} \\ \max\{-R_l, -R_{l'}\}, i, l, i', l' \in \{2, 3, \ldots, k\}, \\ (i, i') \neq (l, l'), i < i', l < l' \\ \lim_{T\to\infty}\max\{-R_1, -R_{i'}\} = \lim_{T\to\infty} \\ \max\{-R_l, -R_{l'}\}, l, i', l' \in \{2, 3, \ldots, k\}, l < l', \end{cases} \tag{18}$$

which is equivalent to

$$\begin{cases} \lim_{T\to\infty}\min\{R_1, R_{i'}\} = \lim_{T\to\infty} \\ \min\{R_1, R_{l'}\}, i' \neq l', i', l' \in \{2, 3, \ldots, k\} \\ \lim_{T\to\infty}\min\{R_i, R_{i'}\} = \lim_{T\to\infty} \\ \min\{R_l, R_{l'}\}, i, l, i', l' \in \{2, 3, \ldots, k\}, \\ (i, i') \neq (l, l'), i < i', l < l' \\ \lim_{T\to\infty}\min\{R_1, R_{i'}\} = \lim_{T\to\infty} \\ \min\{R_l, R_{l'}\}, l, i', l' \in \{2, 3, \ldots, k\}, l < l' \end{cases}. \tag{19}$$

Summarizing the three cases, we can obtain the equality in (8).

Remark 1 Note that (13) and (14) correspond to the KKT conditions of the concave optimization problem in (4). The optimal allocation is achieved asymptotically if solutions satisfying (13) and (14) are implemented. Theorem 2 provides useful insights on characterizing the optimality condition for the problem (4).

Let $\overline{S}_i = \sum_{j,j\neq i}\overline{X}_{i,j}$ denote the sample mean of the Borda score of a design x_i for each $i \in \{1, 2, \ldots, k\}$. Let $R_i = \left(c_1 - \overline{S}_i\right)^2/2\left(\sum_{j,j>i}\sigma_{i,j}^2/\alpha_{i,j} + \sum_{j,j<i}\sigma_{i,j}^2/\alpha_{j,i}\right)$ for

each $i \in \{1,2,\dots,k\}$. R_1 is the rate function of probability $P\left(\tilde{S}_1 \le c_1\right)$ and R_i is the rate function of probability $P\left(\tilde{S}_i \ge c_1\right)$ (Glynn & Juneja, 2004). As the total simulation budget T goes to infinity, it shows an exponential decaying rate in which the probability of a wrong comparison $\tilde{S}_i \ge c_1$ (or $\tilde{S}_1 \le c_1$) goes to zero. This research considers pairwise comparison, where $\min\{R_i, R_{i'}\}$ denotes the minimum convergence rate for a pair of designs x_i and $x_{i'}$. The optimality condition indicates that the simulation budget allocated to each pair of designs is such that the resulting minimum convergence rates are all equal for each pair of designs, where there are $k(k-1)/2$ pairs of designs.

Remark 2 Assumption 2 states that the simulation output samples are normally distributed. This assumption is commonly used in the ranking and selection literature. Based on the law of large numbers, the normality assumption can always be met if batch means are used. However, this assumption can be removed if we adopt the large deviation theory to derive the optimality conditions for determining the optimal allocation rule as shown in Glynn and Juneja (2004). Using the large deviation theory, we can derive a more general optimality condition. However, the rate function can be difficult to estimate for some underlying distributions. The optimal allocation rule derived based on the normality assumption is easy to estimate and implement. Further, the optimality condition given in Theorem 2 can be easily extended to the case where the underlying distribution is non-normal even though the results are derived based on the normality assumption.

3.2 | Sequential simulation procedure

Based on the optimality condition above, we present a sequential simulation budget allocation procedure to implement the optimality condition in Theorem 4. In each iteration, we determine the pair of designs that results in the minimum convergence rate and provide a small incremental budget Δ to this pair of designs. That is, we find $\left(i^*, i^{*\prime}\right) = \text{argmin}_{(i,i'),i<i',i,i'\in\{1,2,\dots,k\}} \min\{R_i, R_{i'}\}$, and increase $n_{i^*,i^{*\prime}}$ by Δ.

Note that the conclusion above has implicitly assumed that we know R_i for each $i \in \{1,2,\dots,k\}$. In practice, they are unknown and need to be estimated. Thus, we need to simulate each pair of designs initially, and estimate R_i. Let $\widehat{R}_i$ denote the estimate of R_i. In the following iterations, the simulation is allocated based on the proposed allocation rule in Theorem 4. We name it as OCBA-PC (pairwise comparison OCBA) procedure.

BOX 1. OCBA-PC procedure

1. Specify the total simulation budget T, the initial number of simulation replications n_0 and the incremental budget Δ. Let the iteration counter $\rho \leftarrow 1$. Perform n_0 replications to each of $k(k-1)/2$ pairs of designs. $n_{i,j}^{\rho} = n_0$ for all $i,j \in \{1,2,\dots,k\}, i<j$.
2. Calculate the sample mean $\overline{X}_{i,j}$ and sample variance $\overline{\omega}_{i,j}^2$ for all $i,j \in \{1,2,\dots,k\}$, $i<j$.
3. while $\sum_{i<j} n_{i,j}^{\rho} \le T$
4. Calculate the estimated Borda score $\overline{S}_i$ based on $\overline{X}_{i,j}$ for all $i \in \{1,2,\dots,k\}$.
5. Estimate $c_1 = \overline{S}_1 + \overline{S}_2$ and calculate $\widehat{R}_i$ based on c_1, $\overline{S}_i$ and $\overline{\omega}_{i,j}^2$ for all $i \in \{1,2,\dots,k\}$.
6. Find the pair $\left(i^*, i^{*\prime}\right) = \text{argmin}_{(i,i'),i<i',i,i'\in\{1,2,\dots,k\}} \min\left\{\widehat{R}_i, \widehat{R}_{i'}\right\}$ that has the minimum convergence rate.
7. Provide $\min\left\{\Delta, T - \sum_{i<j} n_{i,j}^{\rho}\right\}$ replications to the pair of designs $\left(i^*, i^{*\prime}\right)$, $n_{i^*,i^{*\prime}}^{\rho+1} = n_{i^*,i^{*\prime}}^{\rho} + \min\left\{\Delta, T - \sum_{i<j} n_{i,j}^{\rho}\right\}$.
8. Update $\overline{X}_{i,j}$ and $\overline{\omega}_{i,j}^2$ for all $i,j \in \{1,2,\dots,k\}, i<j$.
9. $\rho \leftarrow \rho + 1$.
10. end while
11. Select the design with the largest Borda score.

Remark 3 In the algorithm above, n_0 is the initial number of replications allocated to each pair of designs. Since no information on the design performance is available before simulation, the goal of simulating each pair of designs n_0 times is to obtain the initial sample information. The value of n_0 is usually set to be 5–15 depending on the specific problem. Δ is the number of replications increased per iteration. A large Δ may result in an excessive allocation to some pair of designs, but a small Δ increases the iterations for simulation. Note that c_1, which is set to be $S_1 + S_2$, is unknown. In this sequential simulation algorithm, we let $c_1 = \overline{S}_1 + \overline{S}_2$ as an estimate. In the later iterations, c_1 is updated as $\overline{S}_i, i = 1,2,\dots,k$ are updated.

4 | EXTENSION TO THE SUBSET SELECTION PROBLEM

In ranking and selection, besides selecting the single design, selecting the optimal subset is also an important problem that is well studied. In this section, we extend the pairwise ranking and selection procedure to the problem of selecting the top m designs from a finite number of k alternatives. Let Ω denote the set of top m designs. The top m subset selection problem is finding the set

$$\Omega = \left\{ x_p \mid \min_{p\in\Omega} S_p > \max_{q\notin\Omega} S_q, p, q \in \{1, 2, \ldots, k\}, |\, \Omega \,|= m \right\}. \tag{20}$$

Without loss of generality, we still assume that $S_1 > S_2 > \ldots > S_k$. Let $\tilde{S}_{(1)} > \tilde{S}_{(2)} > \ldots > \tilde{S}_{(k)}$ denote the ordered realization of the posterior estimate of the Borda scores. Thus, correct selection occurs if $\{(1), (2), \ldots, (m)\} = \{1, 2, \ldots, m\}$. Thus, we can define the PCS as

$$PCS_m = P\left\{ \bigcap_{p=1}^{m} \bigcap_{q=m+1}^{k} \tilde{S}_p > \tilde{S}_q \right\}. \tag{21}$$

The objective of this section is to maximize this PCS_m given the total computing budget T.

$$\begin{aligned} &\max PCS_m \\ s.t.\ &\sum_{i<j} n_{i,j} = T, i = 1, \ldots, k-1, j = 2, \ldots, k \\ &n_{i,j} \geq 0, \ \text{ for all } i \text{ and } j. \end{aligned} \tag{22}$$

Due to the same difficulty as we face in Section 3, we develop a lower bound on PCS_m.

Theorem 3 *The PCS_m in (21) is lower bounded by*

$$APCS_m = 1 - \sum_{p=1}^{m} P\left(\tilde{S}_p \leq c_m\right) - \sum_{q=m+1}^{k} P\left(\tilde{S}_q \geq c_m\right), \tag{23}$$

where $c_m = (S_m + S_{m+1})/2$.

Proof For a constant c_m,

$$\begin{aligned} PCS_m &= P\left\{ \bigcap_{p=1}^{m} \bigcap_{q=m+1}^{k} \tilde{S}_p > \tilde{S}_q \right\} \\ &\geq P\left[\left(\bigcap_{p=1}^{m} \{\tilde{S}_p > c_m\} \right) \cap \left(\bigcap_{q=m+1}^{k} \{\tilde{S}_q < c_m\} \right) \right] \\ &\geq \sum_{p=1}^{m} P\left(\tilde{S}_p > c_m\right) + \sum_{q=m+1}^{k} P\left(\tilde{S}_q < c_m\right) - (k-1) \\ &= 1 - \sum_{p=1}^{m} P\left(\tilde{S}_p \leq c_m\right) - \sum_{q=m+1}^{k} P\left(\tilde{S}_q \geq c_m\right) \\ &= APCS_m. \end{aligned} \tag{24}$$

■

According to (5) and (6),

$$\begin{aligned} APCS_m = 1 - &\sum_{p=1}^{m} \Phi\left(\left(c_m - \sum_{j,j\neq p} \overline{X}_{p,j} \right) \Big/ \sqrt{\sum_{j,j>p} \frac{\sigma_{p,j}^2}{n_{p,j}} + \sum_{j,j<p} \frac{\sigma_{p,j}^2}{n_{j,p}}} \right) \\ - &\sum_{q=m+1}^{k} \Phi\left(\left(-c_m + \sum_{j,j\neq q} \overline{X}_{q,j} \right) \Big/ \sqrt{\sum_{j,j>q} \frac{\sigma_{q,j}^2}{n_{q,j}} + \sum_{j,j<q} \frac{\sigma_{q,j}^2}{n_{j,q}}} \right). \end{aligned} \tag{25}$$

Similar to Section 3, we consider the following optimization model instead of the model given in (22).

$$\begin{aligned} &\max APCS \\ s.t.\ &\sum_{i<j} n_{i,j} = T, i = 1, \ldots, k-1, j = 2, \ldots, k \\ &n_{i,j} \geq 0, \ \text{ for all } i \text{ and } j. \end{aligned} \tag{26}$$

Lemma 3 *The optimization problem defined in (26) is asymptotically concave as the total computing budget T goes to infinity, where APCS is given in (23).*

Proof The proof of Lemma 3 is similar to the proof of Lemma 2. We ignore the proof to avoid repetition.

Given the concavity, we can use the KKT conditions to solve the problem (26). ■

Theorem 4 *Let $\alpha_{i,j} = n_{i,j}/T$ for all $i < j$ and $i,j \in \{1, 2, \ldots, k\}$ denote the proportion of simulation budget allocated to a pair of designs (i,j). Let $\overline{S}_i = \sum_{j,j\neq i} \overline{X}_{i,j}$ denote the sample mean of the Borda score of a design x_i for each $i \in \{1, 2, \ldots, k\}$. Let $G_i = \left(c_m - \overline{S}_i\right)^2 / 2\left(\sum_{j,j>i} \sigma_{i,j}^2/\alpha_{i,j} + \sum_{j,j<i} \sigma_{i,j}^2/\alpha_{j,i}\right)$ for each $i \in \{1, 2, \ldots, k\}$. The optimization problem in (26) is asymptotically optimized if for any two pairs of designs (i, i') and (l, l') as the total computing budget T goes to infinity, we have*

$$\lim_{T\to\infty} \min\{G_i, G_{i'}\} = \lim_{T\to\infty} \min\{G_l, G_{l'}\}, \tag{27}$$

where $i < i'$, $l < l'$, $i, i', l, l' \in \{1, 2, \ldots, k\}$. According to the law of large numbers, the sample mean $\overline{X}_{i,j}$ converges to the mean $\mu_{i,j}$ in probability. Then, R_i converges to $\left(c_1 - \sum_{j,j\neq i} \mu_{i,j}\right)^2 / 2\left(\sum_{j,j>i} \sigma_{i,j}^2/\alpha_{i,j} + \sum_{j,j<i} \sigma_{i,j}^2/\alpha_{j,i}\right)$ in probability.

Proof See Appendix B. ■

It is interesting to note that the optimality condition in Theorem 4 is exactly the same as that in Theorem 2 except that a different constant threshold c_m is used. Thus, we can conclude that the optimality condition applies to the selection of the top m designs no matter the value of m. Therefore, the OCBA-PC procedure can also be used for implementing the asymptotical optimality condition in Theorem 4.

5 | NUMERICAL EXPERIMENTS

To test the effectiveness of the proposed OCBA-PC procedure, we compare OCBA-PC with equal allocation

TABLE 1 Parameter setting for different experiments

Experiments		*m*	*k*	Distribution of design *i*
1	Equal variance	1	10	$N\left(i, 8^2\right)$
2	Equal variance	3	10	$N\left(i, 8^2\right)$
3	Increasing variance	1	10	$N\left(i, (i+4)^2\right)$
4	Increasing variance	3	10	$N\left(i, (i+4)^2\right)$
5	Decreasing variance	1	10	$N\left(i, (15-i)^2\right)$
6	Decreasing variance	3	10	$N\left(i, (15-i)^2\right)$
7	Large scale	1	50	$N\left(i, 8^2\right)$
8	Large scale	5	50	$N\left(i, 8^2\right)$
9	Exponential distribution	1	10	$\exp(1/i)$
10	Exponential distribution	3	10	$\exp(1/i)$
11	Uniform distribution	1	10	$Uniform\,(i-10, i+10)$
12	Uniform distribution	3	10	$Uniform\,(i-10, i+10)$

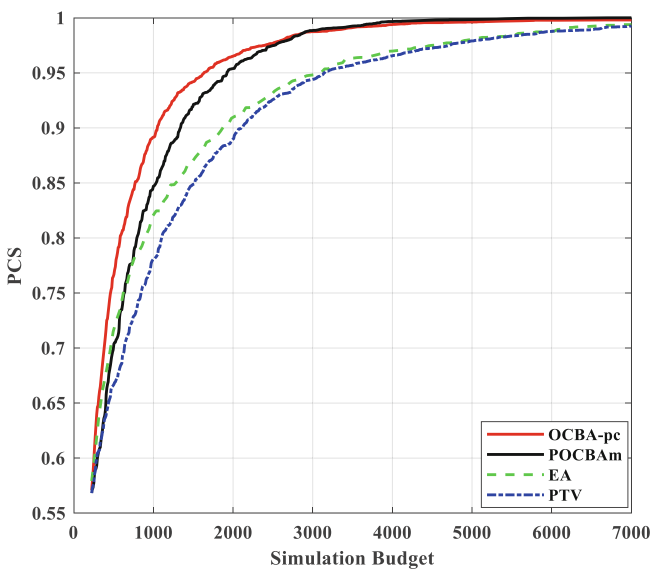

FIGURE 1 PCS comparison for Experiment 1.

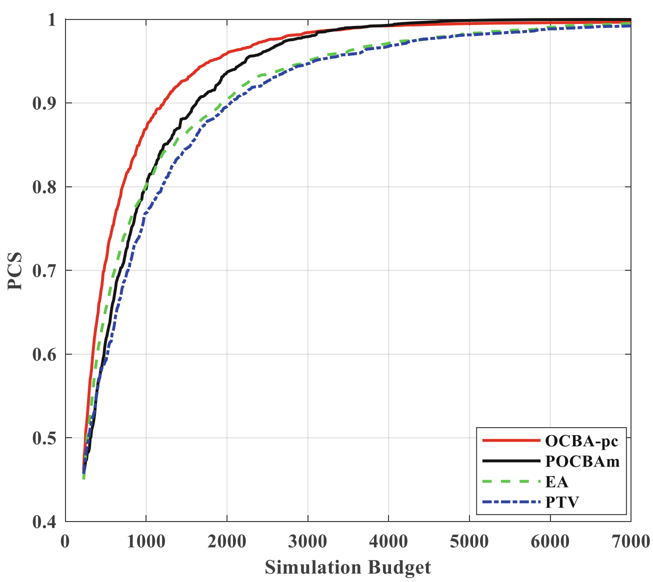

FIGURE 2 PCS comparison for Experiment 2.

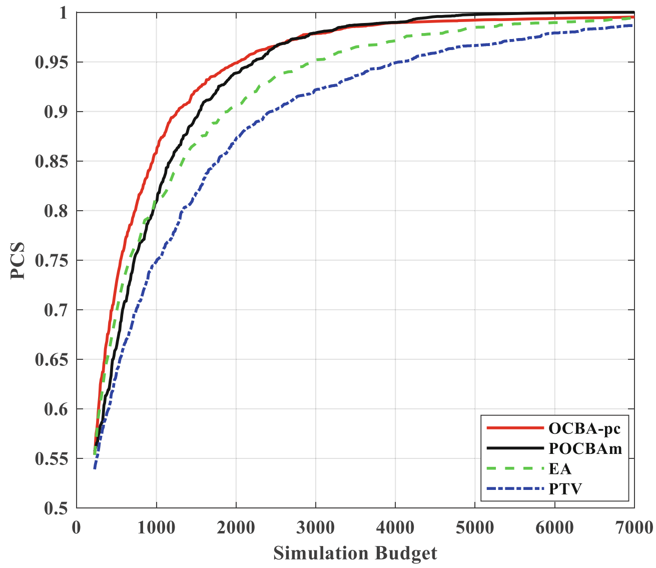

FIGURE 3 PCS comparison for Experiment 3.

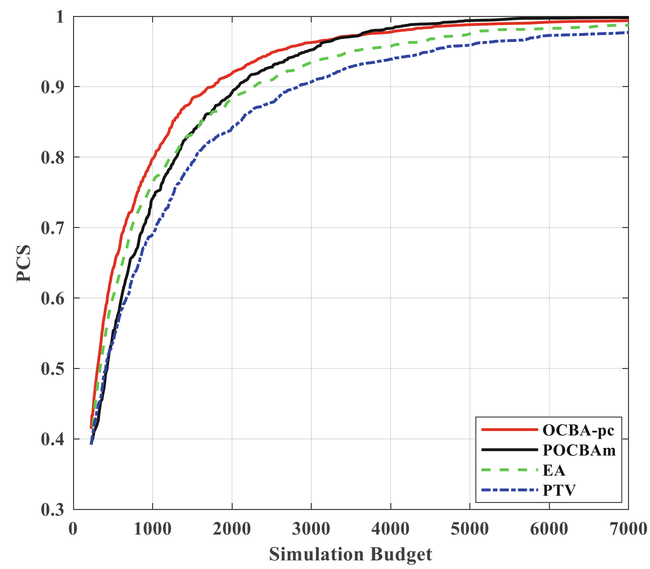

FIGURE 4 PCS comparison for Experiment 4.

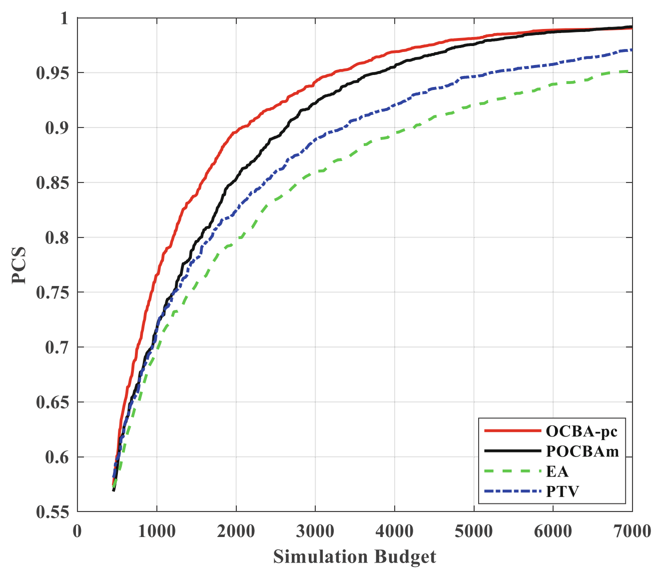

FIGURE 5 PCS comparison for Experiment 5.

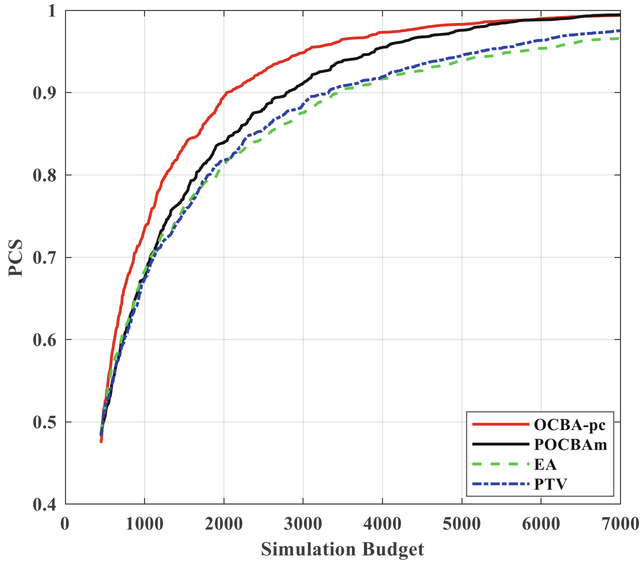

FIGURE 6 PCS comparison for Experiment 6.

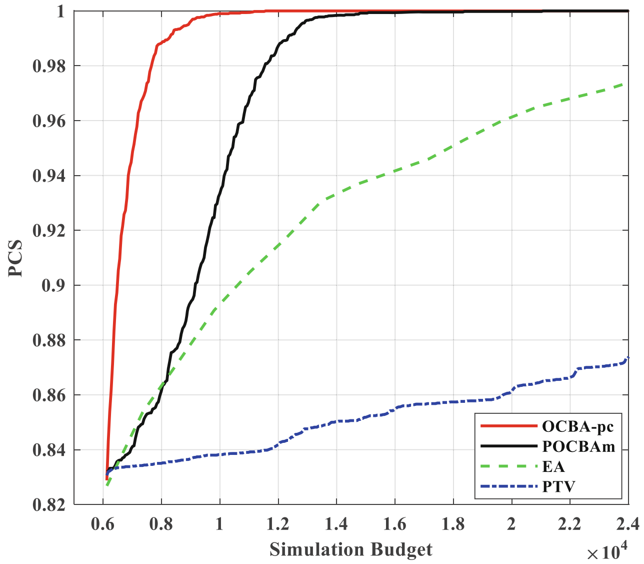

FIGURE 7 PCS comparison for Experiment 7.

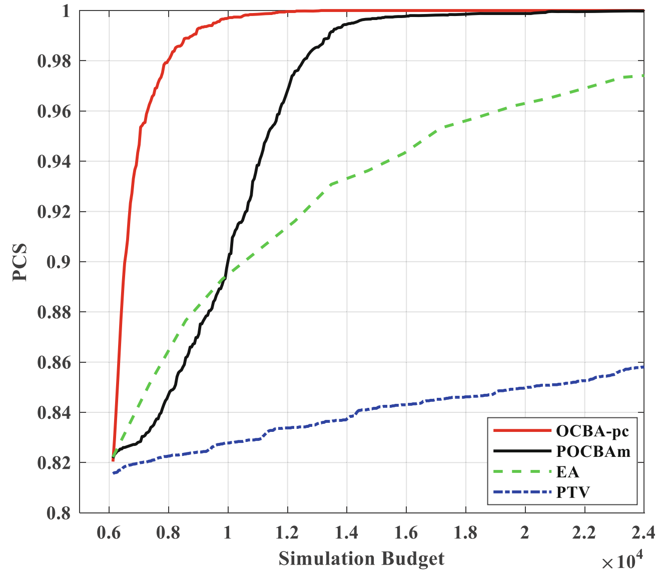

FIGURE 8 PCS comparison for Experiment 8.

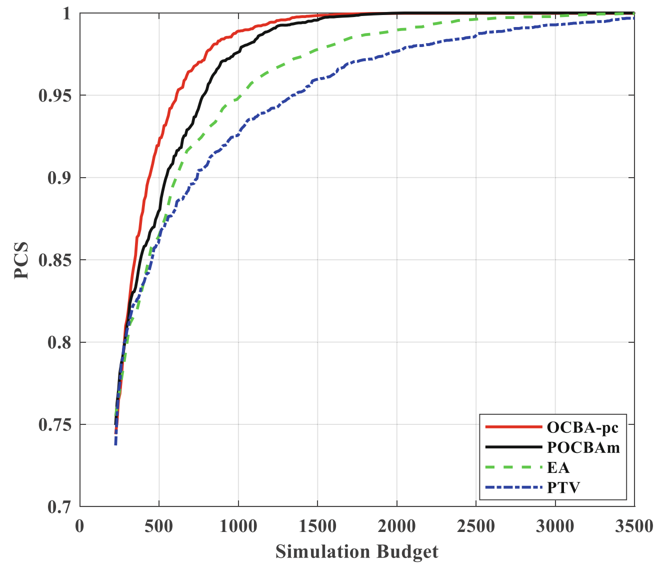

FIGURE 9 PCS comparison for Experiment 9.

(EA), proportional to variance (PTV), and the POCBAm proposed in Groves and Branke (2019) in a series of experiments. EA and PTV are commonly chosen as benchmarking procedures due to their simplicity. POCBAm is a recently proposed allocation rule that aims to select the top m designs through pairwise comparisons. In our experiments, EA allocates the same number of simulation replications to each pair of designs. PTV allocates the number of simulation replications to each pair of designs proportionally to their variances. Both EA and PTV are implemented sequentially. POCBAm is implemented based on the POCBAm procedure given in Groves and Branke (2019).

In all experiments, the PCS is estimated by counting the number of times that the desired designs are correctly selected out of 5000 independent applications of each procedure. PCS is then obtained by dividing this number by 5000, denoting the frequency of correct selection. The initial number of simulation replications n_0 is 5 for all experiments. Note that n_0 is the initial number of simulation replications allocated to each pair of designs. Thus, the total budget spent during the first iteration of simulation depends on the number of designs in that experiment. The incremental budget Δ is 10 for all experiments except experiments 7 and 8, where Δ is set to be 50 since the problem scale is large. In the case when there are multiple pairs of designs with the same minimum rate during an iteration, the Δ replications are equally distributed among all such pairs. The performance distributions of each design in the 12 experiments we conducted are shown in Table 1.

In Table 1, $m = 1$ means selecting the best design, while $m > 1$ refers to the top subset selection. The performance of each design cannot be simulated individually.

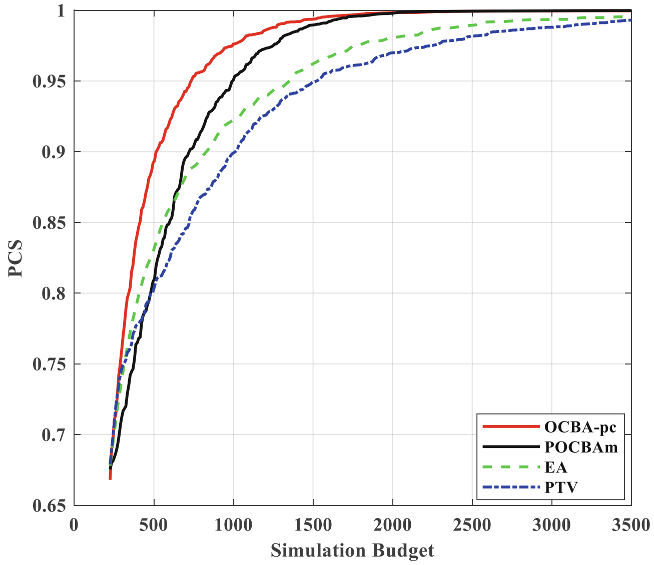

FIGURE 10 PCS comparison for Experiment 10.

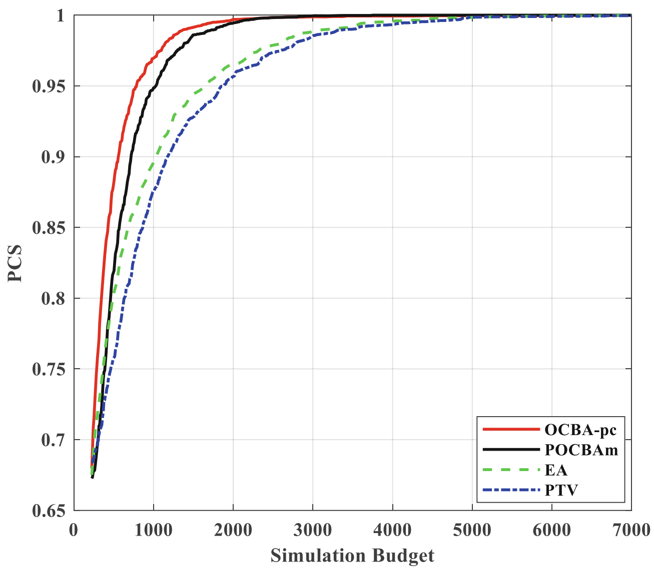

FIGURE 11 PCS comparison for Experiment 11.

We only have the simulation outcome of pairwise comparison. We have carried out experiments under different settings. The PCS comparison of the four simulation procedures are shown in Figures 1–12 for experiments 1–12, respectively.

The numerical results in Figures 1–12 show that the PCS increases with increasing the simulation budget in all experiments. In particular, we can see that the proposed OCBA-PC procedure outperforms POCBAm, PTV, and EA in all experiments. Comparing Figures 7–8 with other figures, we find that the advantage of using OCBA-PC becomes more significant when the number of designs is larger. OCBA-PC can perform better than PTV and EA because it concentrates on those critical pairs of designs that allow the desired design (subset) to be selected more accurately. In the case of a large

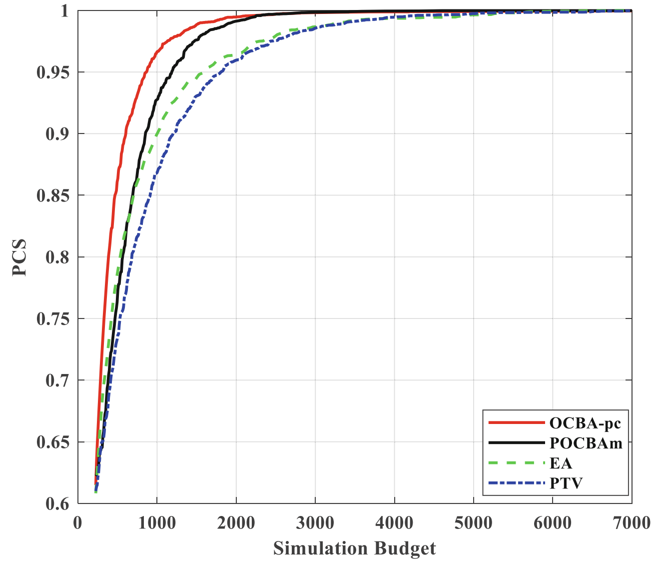

FIGURE 12 PCS comparison for Experiment 12.

number of designs, more budget is wasted on those noncritical pairs. This is why the PCS of OCBA-PC is much larger than that of EA and PTV given the same number of simulation budget in Experiments 7 and 8. In all experiments, we see that OCBA-PC performs slightly better than POCBAm proposed in Groves and Branke (2019), which is a heuristic allocation rule. This demonstrates the importance of deriving an asymptotically optimal allocation rule.

6 | CONCLUSIONS

Motivated by the restriction of only having access to pairwise comparisons in many real-life problems, this research proposes an efficient ranking and selection procedure when the performance of designs can only be simulated pairwise instead of individually. We formulate this new ranking and selection problem using the optimal computing budget allocation framework and derive the asymptotical optimality conditions for selecting the best design and the top subset respectively. We devise a sequential budget allocation algorithm to implement the asymptotical optimality condition of the allocation rule. The numerical experiments confirm that the proposed OCBA-PC rule can improve the probability of correctly selecting the best design or the top subset given the same simulation budget compared with other rules from the literature.

ACKNOWLEDGMENTS

This research is supported in part by National Natural Science Foundation of China under Grant numbers 71971176, 71701123, 71725001, and 71910107002, and the Fundamental Research Funds for the Central Universities under Grant JBK2103010.

DATA AVAILABILITY STATEMENT

The data that support the findings of this study are available from the corresponding author upon reasonable request.

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How to cite this article: Xiao, H., Zhang, Y., Kou, G., Zhang, S., & Branke, J. (2023). Ranking and selection for pairwise comparison. *Naval Research Logistics (NRL)*, 1–19. https://doi.org/10.1002/nav.22093

APPENDIX A

Proof $APCS_1$ given in Theorem 1 is concave if and only if the second order derivative matrices of $-P\left(\tilde{S}_1 \leq c_1\right)$ and $-\sum_{i=2}^{k} P\left(\tilde{S}_i \geq c_1\right)$ are negative semidefinite for all possible values of $n_{i,j}$, $i = 1, \ldots, k-1, j = 2, \ldots, k, i < j$. Based on the derivation in (5), we let

$$f = -P\left(\tilde{S}_1 \leq c_1\right) = -\Phi\left(\left(c_1 - \sum_{j=2}^{k} \overline{X}_{1,j}\right) \Big/ \sqrt{\sum_{j=2}^{k}\left(\sigma_{1,j}^2/n_{1,j}\right)}\right). \tag{A1}$$

The first order derivative of f is

$$\nabla f = \left(\frac{\partial f}{\partial n_{1,2}}, \ldots, \frac{\partial f}{\partial n_{1,k}}, \frac{\partial f}{\partial n_{2,3}}, \ldots, \frac{\partial f}{\partial n_{2,k}}, \ldots, \frac{\partial f}{\partial n_{k-1,k}}\right),$$

where the first $k-1$ terms $\partial f/\partial n_{1,2}, \ldots, \partial f/\partial n_{1,k}$ are nonzero, the other terms are all zeros, and for i $= 2, 3, \ldots, k$, we have

$$\begin{aligned}\frac{\partial f}{\partial n_{1,i}} = &-\frac{1}{2}\phi\left(\left(c_1 - \sum_{j=2}^{k} \overline{X}_{1,j}\right) \Big/ \sqrt{\sum_{j=2}^{k}\left(\sigma_{1,j}^2/n_{1,j}\right)}\right) \\ &\times \left(c_1 - \sum_{j=2}^{k} \overline{X}_{1,j}\right) \times \left(\sum_{j=2}^{k}\left(\sigma_{1,j}^2/n_{1,j}\right)\right)^{-\frac{3}{2}} \sigma_{1,i}^2 n_{1,i}^{-2}.\end{aligned} \tag{A2}$$

Since only the first $k-1$ terms in ∇f are nonzero, the second order derivative of f can be expressed as

$$\nabla^2 f = \begin{pmatrix} \frac{\partial^2 f}{\partial n_{1,2}\partial n_{1,2}} & \cdots & \frac{\partial^2 f}{\partial n_{1,2}\partial n_{1,k}} & 0 & 0 & \cdots & 0 \\ \frac{\partial^2 f}{\partial n_{1,3}\partial n_{1,2}} & \cdots & \frac{\partial^2 f}{\partial n_{1,3}\partial n_{1,k}} & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial n_{1,k}\partial n_{1,2}} & \cdots & \frac{\partial^2 f}{\partial n_{1,k}\partial n_{1,k}} & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & 0 & \vdots & \vdots & \ddots & \vdots \\ 0 & \ldots & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Let $\partial^2 f/\partial^2 n_{1,i}$ denote an arbitrary diagonal nonzero term in $\nabla^2 f$, and $\partial^2 f/\partial n_{1,i}\partial n_{1,l}$ denote an arbitrary non-diagonal nonzero term in $\nabla^2 f$, where $i \neq l$, $i, l \in \{2, 3, \ldots, k\}$. From (A2) we have

$$\begin{aligned}\frac{\partial^2 f}{\partial^2 n_{1,i}} = &\frac{1}{4}\left(c_1 - \sum_{j=2}^{k} \overline{X}_{1,j}\right) \times \phi\left(\left(c_1 - \sum_{j=2}^{k} \overline{X}_{1,j}\right) \Big/ \sqrt{\sum_{j=2}^{k}\frac{\sigma_{1,j}^2}{n_{1,j}}}\right) \times \left[\left(\sum_{j=2}^{k}\left(\sigma_{1,j}^2/n_{1,j}\right)\right)^{-7/2} \sigma_{1,i}^4 n_{1,i}^{-4}\right] \\ &\times \left[\left(c_1 - \sum_{j=2}^{k} \overline{X}_{1,j}\right)^2 - 3\left(\sum_{j=2}^{k}\left(\sigma_{1,j}^2/n_{1,j}\right)\right) + 4\left(\sum_{j=2}^{k}\left(\sigma_{1,j}^2/n_{1,j}\right)\right)^2 \sigma_{1,i}^{-2} n_{1,i}\right],\end{aligned} \tag{A3}$$

$$\begin{aligned}\frac{\partial^2 f}{\partial n_{1,i}\partial n_{1,l}} = &\frac{1}{4}\left(c_1 - \sum_{j=2}^{k} \overline{X}_{1,j}\right) \times \phi\left(\left(c_1 - \sum_{j=2}^{k} \overline{X}_{1,j}\right) \Big/ \sqrt{\sum_{j=2}^{k}\frac{\sigma_{1,j}^2}{n_{1,j}}}\right) \times \left[\left(\sum_{j=2}^{k}\left(\sigma_{1,j}^2/n_{1,j}\right)\right)^{-7/2} \sigma_{1,i}^2 \times n_{1,i}^{-2} \times \sigma_{1,l}^2 \times n_{1,l}^{-2}\right] \\ &\times \left[\left(c_1 - \sum_{j=2}^{k} \overline{X}_{1,j}\right)^2 - 3\left(\sum_{j=2}^{k}\left(\sigma_{1,j}^2/n_{1,j}\right)\right)\right].\end{aligned} \tag{A4}$$

For further consideration, we make an equivalent transformation to $\nabla^2 f$,

$$\nabla^2 f = a \times A + b \times B,$$

where

$$a = \frac{1}{4}\left(c_1 - \sum_{j=2}^{k}\overline{X}_{1,j}\right)\times\phi\left(\left(c_1 - \sum_{j=2}^{k}\overline{X}_{1,j}\right)\Big/\sqrt{\sum_{j=2}^{k}\frac{\sigma_{1,j}^2}{n_{1,j}}}\right)$$
$$\times\left(\sum_{j=2}^{k}\left(\sigma_{1,j}^2/n_{1,j}\right)\right)^{-7/2}\times\left[\left(c_1 - \sum_{j=2}^{k}\overline{X}_{1,j}\right)^2 - 3\left(\sum_{j=2}^{k}\left(\sigma_{1,j}^2/n_{1,j}\right)\right)\right]$$

,

$$A = \begin{pmatrix} \sigma_{1,2}^4 n_{1,2}^{-4} & \cdots & \sigma_{1,2}^2\times n_{1,2}^{-2}\times\sigma_{1,k}^2\times n_{1,k}^{-2} & 0 & 0 & \cdots & 0 \\ \sigma_{1,3}^2\times n_{1,3}^{-2}\times\sigma_{1,2}^2\times n_{1,2}^{-2} & \cdots & \sigma_{1,3}^2\times n_{1,3}^{-2}\times\sigma_{1,k}^2\times n_{1,k}^{-2} & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \sigma_{1,k}^2\times n_{1,k}^{-2}\times\sigma_{1,2}^2\times n_{1,2}^{-2} & \cdots & \sigma_{1,k}^4 n_{1,k}^{-4} & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & 0 & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 0 & 0 & \cdots & 0 \end{pmatrix},$$

$$b = \left(c_1 - \sum_{j=2}^{k}\overline{X}_{1,j}\right)\times\phi\left(\left(c_1 - \sum_{j=2}^{k}\overline{X}_{1,j}\right)/\sqrt{\sum_{j=2}^{k}\frac{\sigma_{1,j}^2}{n_{1,j}}}\right)\times\left(\sum_{j=2}^{k}\left(\sigma_{1,j}^2/n_{1,j}\right)\right)^{-3/2},$$

$$B = \begin{pmatrix} \sigma_{1,2}^2 n_{1,2}^{-3} & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \sigma_{1,k}^2 n_{1,k}^{-3} & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & 0 & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

To prove that the second order derivative matrix of $-P\left(\tilde{S}_1 \le c_1\right)$ is negative semidefinite, we only need to prove that a and b are nonpositive, and both A and B are positive semidefinite.

Since $\overline{S}_1 = \sum_{j=2}^{k}\overline{X}_{1,j}$, $\overline{S}_1$ converges to S_1 when the total simulation budget T goes to infinity. As defined in Theorem 1, $c_1 = (S_1 + S_2)/2$. Thus, $c_1 - \sum_{j=2}^{k}\overline{X}_{1,j}$ is equivalent to $\frac{(S_1+S_2)}{2} - S_1$ when T goes to infinity. Based on the definition of Borda score and the assumption in Section 2, we know that $S_1 > S_2$. Therefore, $c_1 - \sum_{j=2}^{k}\overline{X}_{1,j} < 0$ when T goes to infinity. Furthermore, it is easy to see that

$$\phi\left(\left(c_1 - \sum_{j=2}^{k}\overline{X}_{1,j}\right)\Big/\sqrt{\sum_{j=2}^{k}\frac{\sigma_{1,j}^2}{n_{1,j}}}\right) > 0,$$

and

$$\left(\sum_{j=2}^{k}\left(\sigma_{1,j}^2/n_{1,j}\right)\right)^{-3/2} > 0,\quad \left(\sum_{j=2}^{k}\left(\sigma_{1,j}^2/n_{1,j}\right)\right)^{-7/2} > 0.$$

Let $n_{1,j} = \alpha_{1,j}T$ where $\alpha_{1,j}$ denotes the proportion of simulation budget allocated to the pair of designs 1 and j. Then,

$$\left(c_1 - \sum_{j=2}^{k}\overline{X}_{1,j}\right)^2 - 3\times\sum_{j=2}^{k}\left(\sigma_{1,j}^2/n_{1,j}\right) \ge 0$$

is equivalent to

$$\left(c_1 - \sum_{j=2}^{k}\overline{X}_{1,j}\right)^2 - (3/T)\times\sum_{j=2}^{k}\left(\sigma_{1,j}^2/\alpha_{1,j}\right) \ge 0. \tag{A5}$$

(A5) follows if we let

$$T \geq 3 \times \sum_{j=2}^{k} \left(\sigma_{1,j}^2/\alpha_{1,j}\right)\left(c_1 - \sum_{j=2}^{k} \overline{X}_{1,j}\right)^2.$$

Thus, we have proven that both a and b are nonpositive when T goes to infinity.

A matrix is positive semidefinite if and only if all of its principal minors are nonnegative.

For matrix A, the first order principal minors are made up of each diagonal element individually. It is easy to see that $\sigma_{1,i}^4 n_{1,i}^{-4} > 0$, $i = 2, 3, \ldots, k$. Thus, all of its first order principal minors are nonnegative. Because any two rows of matrix A are proportional to each other, for all principal minors above second order, any two rows are also proportional to each other. Thus, all principal minors above second order of matrix A are equal to zero.

Matrix B is a diagonal matrix and all terms on the diagonal are nonnegative. Obviously, all of its first order principal minors are not less than zero. Because B is a diagonal matrix, all of its principal minors above second order are also triangular determinants, and the diagonal elements of the principal minors are part of matrix B. Thus the principal minors above second order of B are nonnegative.

Thus, we have proven that both A and B are positive semidefinite. Then, we can conclude that $\nabla^2 f$ is a negative semidefinite matrix.

Following almost the same procedure, we can prove that the second order derivative matrix of $-P\left(\tilde{S}_i \geq c_1\right)$ is a negative semidefinite for each $i = 2, 3, .., k$. Then, we can conclude that the second order derivative matrix of $APCS_1$ is negative semidefinite. ∎

APPENDIX B

Let F be the Lagrangian function with Lagrange multiplier λ and $\xi_{i,j}$, $i < j$ and $i, j \in \{1, 2, \ldots, k\}$. That is,

$$\begin{aligned} F = 1 &- \sum_{p\in\Omega} \Phi\left(\left(c_m - \sum_{j,j\neq p} \overline{X}_{p,j}\right) \Big/ \sqrt{\sum_{j,j>p} \frac{\sigma_{p,j}^2}{n_{p,j}} + \sum_{j,j<p} \frac{\sigma_{p,j}^2}{n_{j,p}}}\right) \\ &- \sum_{q\notin\Omega} \Phi\left(\left(-c_m + \sum_{j,j\neq q} \overline{X}_{q,j}\right) \Big/ \sqrt{\sum_{j,j>q} \frac{\sigma_{q,j}^2}{n_{q,j}} + \sum_{j,j<q} \frac{\sigma_{q,j}^2}{n_{j,q}}}\right) - \lambda\left(\sum_{i<j} n_{i,j} - T\right) - \sum_{\substack{i<j \\ i,j\in\{1,\ldots,k\}}} \xi_{i,j} n_{i,j}. \end{aligned}$$

The stationarity conditions can be stated as follows.

For $i \in \Omega$, $i' \notin \Omega$, $i < i'$,

$$\begin{aligned} \frac{\partial F}{\partial n_{i,i'}} &= \phi\left(\left(c_m - \sum_{j,j\neq i} \overline{X}_{i,j}\right) \Big/ \sqrt{\sum_{j,j>i} \frac{\sigma_{i,j}^2}{n_{i,j}} + \sum_{j,j<i} \frac{\sigma_{i,j}^2}{n_{j,i}}}\right) \\ &\quad \times \left[-\frac{1}{2}\left(c_m - \sum_{j,j\neq i} \overline{X}_{i,j}\right)\left(\sum_{j,j>i} \frac{\sigma_{i,j}^2}{n_{i,j}} + \sum_{j,j<i} \frac{\sigma_{i,j}^2}{n_{j,i}}\right)^{-3/2} \sigma_{i,i'}^2 n_{i,i'}^{-2}\right] \\ &\quad + \phi\left(\left(-c_m + \sum_{j,j\neq i'} \overline{X}_{i',j}\right) \Big/ \sqrt{\sum_{j,j>i'} \frac{\sigma_{i',j}^2}{n_{i',j}} + \sum_{j,j<i'} \frac{\sigma_{i',j}^2}{n_{j,i'}}}\right) \\ &\quad \times \left[\frac{1}{2}\left(c_m - \sum_{j,j\neq i'} \overline{X}_{i',j}\right)\left(\sum_{j,j>i'} \frac{\sigma_{i',j}^2}{n_{i',j}} + \sum_{j,j<i'} \frac{\sigma_{i',j}^2}{n_{j,i'}}\right)^{-3/2} \sigma_{i',i}^2 n_{i,i'}^{-2}\right] \\ &= \lambda + \xi_{i,i'}. \end{aligned} \tag{B1}$$

For $i \in \Omega$, $i' \in \Omega$, $i < i'$,

$$\begin{aligned} \frac{\partial F}{\partial n_{i,i'}} &= \phi\left(\left(c_m - \sum_{j,j\neq i} \overline{X}_{i,j}\right) \Big/ \sqrt{\sum_{j,j>i} \frac{\sigma_{i,j}^2}{n_{i,j}} + \sum_{j,j<i} \frac{\sigma_{i,j}^2}{n_{j,i}}}\right) \times \left[-\frac{1}{2}\left(c_m - \sum_{j,j\neq i} \overline{X}_{i,j}\right)\left(\sum_{j,j>i} \frac{\sigma_{i,j}^2}{n_{i,j}} + \sum_{j,j<i} \frac{\sigma_{i,j}^2}{n_{j,i}}\right)^{-3/2} \sigma_{i,i'}^2 n_{i,i'}^{-2}\right] \\ &\quad + \phi\left(\left(c_m - \sum_{j,j\neq i'} \overline{X}_{i',j}\right) \Big/ \sqrt{\sum_{j,j>i'} \frac{\sigma_{i',j}^2}{n_{i',j}} + \sum_{j,j<i'} \frac{\sigma_{i',j}^2}{n_{j,i'}}}\right) \times \left[-\frac{1}{2}\left(c_m - \sum_{j,j\neq i'} \overline{X}_{i',j}\right)\left(\sum_{j,j>i'} \frac{\sigma_{i',j}^2}{n_{i',j}} + \sum_{j,j<i'} \frac{\sigma_{i',j}^2}{n_{j,i'}}\right)^{-3/2} \sigma_{i',i}^2 n_{i,i'}^{-2}\right] \\ &= \lambda + \xi_{i,i'}. \end{aligned} \tag{B2}$$

For $i \notin \Omega, i' \notin \Omega, i < i'$,

$$\begin{aligned}\frac{\partial F}{\partial n_{i,i'}} &= \phi\left(\left(-c_m + \sum_{j,j\neq i}\overline{X}_{i,j}\right)\Big/\sqrt{\sum_{j,j>i}\frac{\sigma_{i,j}^2}{n_{i,j}} + \sum_{j,j<i}\frac{\sigma_{i,j}^2}{n_{j,i}}}\right) \times \left[\frac{1}{2}\left(c_m - \sum_{j,j\neq i}\overline{X}_{i,j}\right)\left(\sum_{j,j>i}\frac{\sigma_{i,j}^2}{n_{i,j}} + \sum_{j,j<i}\frac{\sigma_{i,j}^2}{n_{j,i}}\right)^{-3/2}\sigma_{i,i'}^2 n_{i,i'}^{-2}\right]\\ &+ \phi\left(\left(-c_m + \sum_{j,j\neq i'}\overline{X}_{i',j}\right)\Big/\sqrt{\sum_{j,j>i'}\frac{\sigma_{i',j}^2}{n_{i',j}} + \sum_{j,j<i'}\frac{\sigma_{i',j}^2}{n_{j,i'}}}\right) \times \left[\frac{1}{2}\left(c_m - \sum_{j,j\neq i'}\overline{X}_{i',j}\right)\left(\sum_{j,j>i'}\frac{\sigma_{i',j}^2}{n_{i',j}} + \sum_{j,j<i'}\frac{\sigma_{i',j}^2}{n_{j,i'}}\right)^{-\frac{3}{2}}\sigma_{i',i}^2 n_{i,i'}^{-2}\right]\\ &= \lambda + \xi_{i,i'}. \end{aligned} \tag{B3}$$

The complementary slackness conditions can be stated as follows.

$$\xi_{i,j} n_{i,j} = 0, \quad \text{for all } i < j. \tag{B4}$$

As $n_{i,j}$, $i < j$ are in the denominator in Equations (B1)–(B3), they cannot be zero. Then in Equation (B4), it must hold that $\xi_{i,j} = 0$ for all $i < j$. Thus, the KKT conditions can be simplified as follows.

For $i \in \Omega, i' \notin \Omega, i < i'$,

$$\begin{aligned}\frac{\partial F}{\partial n_{i,i'}} &= \phi\left(\left(c_m - \sum_{j,j\neq i}\overline{X}_{i,j}\right)\Big/\sqrt{\sum_{j,j>i}\frac{\sigma_{i,j}^2}{n_{i,j}} + \sum_{j,j<i}\frac{\sigma_{i,j}^2}{n_{j,i}}}\right) \times \left[-\frac{1}{2}\left(c_m - \sum_{j,j\neq i}\overline{X}_{i,j}\right)\left(\sum_{j,j>i}\frac{\sigma_{i,j}^2}{n_{i,j}} + \sum_{j,j<i}\frac{\sigma_{i,j}^2}{n_{j,i}}\right)^{-3/2}\sigma_{i,i'}^2 n_{i,i'}^{-2}\right]\\ &+ \phi\left(\left(-c_m + \sum_{j,j\neq i'}\overline{X}_{i',j}\right)\Big/\sqrt{\sum_{j,j>i'}\frac{\sigma_{i',j}^2}{n_{i',j}} + \sum_{j,j<i'}\frac{\sigma_{i',j}^2}{n_{j,i'}}}\right) \times \left[\frac{1}{2}\left(c_m - \sum_{j,j\neq i'}\overline{X}_{i',j}\right)\left(\sum_{j,j>i'}\frac{\sigma_{i',j}^2}{n_{i',j}} + \sum_{j,j<i'}\frac{\sigma_{i',j}^2}{n_{j,i'}}\right)^{-3/2}\sigma_{i',i}^2 n_{i,i'}^{-2}\right]\\ &= \lambda. \end{aligned} \tag{B5}$$

For $i \in \Omega, i' \in \Omega, i < i'$,

$$\begin{aligned}\frac{\partial F}{\partial n_{i,i'}} &= \phi\left(\left(c_m - \sum_{j,j\neq i}\overline{X}_{i,j}\right)\Big/\sqrt{\sum_{j,j>i}\frac{\sigma_{i,j}^2}{n_{i,j}} + \sum_{j,j<i}\frac{\sigma_{i,j}^2}{n_{j,i}}}\right) \times \left[-\frac{1}{2}\left(c_m - \sum_{j,j\neq i}\overline{X}_{i,j}\right)\left(\sum_{j,j>i}\frac{\sigma_{i,j}^2}{n_{i,j}} + \sum_{j,j<i}\frac{\sigma_{i,j}^2}{n_{j,i}}\right)^{-3/2}\sigma_{i,i'}^2 n_{i,i'}^{-2}\right]\\ &+ \phi\left(\left(c_m - \sum_{j,j\neq i'}\overline{X}_{i',j}\right)\Big/\sqrt{\sum_{j,j>i'}\frac{\sigma_{i',j}^2}{n_{i',j}} + \sum_{j,j<i'}\frac{\sigma_{i',j}^2}{n_{j,i'}}}\right) \times \left[-\frac{1}{2}\left(c_m - \sum_{j,j\neq i'}\overline{X}_{i',j}\right)\left(\sum_{j,j>i'}\frac{\sigma_{i',j}^2}{n_{i',j}} + \sum_{j,j<i'}\frac{\sigma_{i',j}^2}{n_{j,i'}}\right)^{-3/2}\sigma_{i',i}^2 n_{i,i'}^{-2}\right]\\ &= \lambda. \end{aligned} \tag{B6}$$

For $i \notin \Omega, i' \notin \Omega, i < i'$,

$$\begin{aligned}\frac{\partial F}{\partial n_{i,i'}} &= \phi\left(\left(-c_m + \sum_{j,j\neq i}\overline{X}_{i,j}\right)\Big/\sqrt{\sum_{j,j>i}\frac{\sigma_{i,j}^2}{n_{i,j}} + \sum_{j,j<i}\frac{\sigma_{i,j}^2}{n_{j,i}}}\right) \times \left[\frac{1}{2}\left(c_m - \sum_{j,j\neq i}\overline{X}_{i,j}\right)\left(\sum_{j,j>i}\frac{\sigma_{i,j}^2}{n_{i,j}} + \sum_{j,j<i}\frac{\sigma_{i,j}^2}{n_{j,i}}\right)^{-3/2}\sigma_{i,i'}^2 n_{i,i'}^{-2}\right]\\ &+ \phi\left(\left(-c_m + \sum_{j,j\neq i'}\overline{X}_{i',j}\right)\Big/\sqrt{\sum_{j,j>i'}\frac{\sigma_{i',j}^2}{n_{i',j}} + \sum_{j,j<i'}\frac{\sigma_{i',j}^2}{n_{j,i'}}}\right) \times \left[\frac{1}{2}\left(c_m - \sum_{j,j\neq i'}\overline{X}_{i',j}\right)\left(\sum_{j,j>i'}\frac{\sigma_{i',j}^2}{n_{i',j}} + \sum_{j,j<i'}\frac{\sigma_{i',j}^2}{n_{j,i'}}\right)^{-\frac{3}{2}}\sigma_{i',i}^2 n_{i,i'}^{-2}\right]\\ &= \lambda. \end{aligned} \tag{B7}$$

There are six cases about the relationship between the simulation replications allocated to any two pairs of designs.

Case 1 For $\left(i, i'\right) \neq \left(l, l'\right), i, l \in \Omega, i', l' \notin \Omega, i < i', l < l'$. From (B5), using the Lemma 1, as $T \to \infty$, we have

$$\begin{aligned}&\max\left\{\frac{\left(c_m - \sum_{j,j\neq i}\overline{X}_{i,j}\right)^2}{-2\left(\sum_{j,j>i}\frac{\sigma_{i,j}^2}{\alpha_{i,j}} + \sum_{j,j<i}\frac{\sigma_{i,j}^2}{\alpha_{j,i}}\right)}, \frac{\left(-c_m + \sum_{j,j\neq i'}\overline{X}_{i',j}\right)^2}{-2\left(\sum_{j,j>i'}\frac{\sigma_{i',j}^2}{\alpha_{i',j}} + \sum_{j,j<i'}\frac{\sigma_{i',j}^2}{\alpha_{j,i'}}\right)}\right\}\\ &= \max\left\{\frac{\left(c_m - \sum_{j,j\neq l}\overline{X}_{l,j}\right)^2}{-2\left(\sum_{j,j>l}\frac{\sigma_{l,j}^2}{\alpha_{l,j}} + \sum_{j,j<l}\frac{\sigma_{l,j}^2}{\alpha_{j,l}}\right)}, \frac{\left(-c_m + \sum_{j,j\neq l'}\overline{X}_{l',j}\right)^2}{-2\left(\sum_{j,j>l'}\frac{\sigma_{l',j}^2}{\alpha_{l',j}} + \sum_{j,j<l'}\frac{\sigma_{l',j}^2}{\alpha_{j,l'}}\right)}\right\}.\end{aligned}$$

Case 2 For $(i, i') \neq (l, l')$, $i, l, i', l' \in \Omega$, $i < i'$, $l < l'$, from (B6), using the Lemma 1, as $T \to \infty$, we have

$$\max\left\{\frac{\left(c_m - \sum_{j,j\neq i}\overline{X}_{i,j}\right)^2}{-2\left(\sum_{j,j>i}\frac{\sigma_{i,j}^2}{\alpha_{i,j}} + \sum_{j,j<i}\frac{\sigma_{i,j}^2}{\alpha_{j,i}}\right)}, \frac{\left(c_m - \sum_{j,j\neq i'}\overline{X}_{i',j}\right)^2}{-2\left(\sum_{j,j>i'}\frac{\sigma_{i',j}^2}{\alpha_{i',j}} + \sum_{j,j<i'}\frac{\sigma_{i',j}^2}{\alpha_{j,i'}}\right)}\right\}$$
$$= \max\left\{\frac{\left(c_m - \sum_{j,j\neq l}\overline{X}_{l,j}\right)^2}{-2\left(\sum_{j,j>l}\frac{\sigma_{l,j}^2}{\alpha_{l,j}} + \sum_{j,j<l}\frac{\sigma_{l,j}^2}{\alpha_{j,l}}\right)}, \frac{\left(c_m - \sum_{j,j\neq l'}\overline{X}_{l',j}\right)^2}{-2\left(\sum_{j,j>l'}\frac{\sigma_{l',j}^2}{\alpha_{l',j}} + \sum_{j,j<l'}\frac{\sigma_{l',j}^2}{\alpha_{j,l'}}\right)}\right\}.$$

Case 3 For $(i, i') \neq (l, l')$, $i, l, i', l' \notin \Omega$, $i < i'$, $l < l'$, from (B7), using the Lemma 1, as $T \to \infty$, we have

$$\max\left\{\frac{\left(-c_m + \sum_{j,j\neq i}\overline{X}_{i,j}\right)^2}{-2\left(\sum_{j,j>i}\frac{\sigma_{i,j}^2}{\alpha_{i,j}} + \sum_{j,j<i}\frac{\sigma_{i,j}^2}{\alpha_{j,i}}\right)}, \frac{\left(-c_m + \sum_{j,j\neq i'}\overline{X}_{i',j}\right)^2}{-2\left(\sum_{j,j>i'}\frac{\sigma_{i',j}^2}{\alpha_{i',j}} + \sum_{j,j<i'}\frac{\sigma_{i',j}^2}{\alpha_{j,i'}}\right)}\right\}$$
$$= \max\left\{\frac{\left(-c_m + \sum_{j,j\neq l}\overline{X}_{l,j}\right)^2}{-2\left(\sum_{j,j>l}\frac{\sigma_{l,j}^2}{\alpha_{l,j}} + \sum_{j,j<l}\frac{\sigma_{l,j}^2}{\alpha_{j,l}}\right)}, \frac{\left(-c_m + \sum_{j,j\neq l'}\overline{X}_{l',j}\right)^2}{-2\left(\sum_{j,j>l'}\frac{\sigma_{l',j}^2}{\alpha_{l',j}} + \sum_{j,j<l'}\frac{\sigma_{l',j}^2}{\alpha_{j,l'}}\right)}\right\}.$$

Case 4 For (i, i') and (l, l'), $i \in \Omega$, $i', l, l' \notin \Omega$, $i < i'$, $l < l'$. From (B5) and (B7), using the Lemma 1, as $T \to \infty$, we have

$$\max\left\{\frac{\left(c_m - \sum_{j,j\neq i}\overline{X}_{i,j}\right)^2}{-2\left(\sum_{j,j>i}\frac{\sigma_{i,j}^2}{\alpha_{i,j}} + \sum_{j,j<i}\frac{\sigma_{i,j}^2}{\alpha_{j,i}}\right)}, \frac{\left(-c_m + \sum_{j,j\neq i'}\overline{X}_{i',j}\right)^2}{-2\left(\sum_{j,j>i'}\frac{\sigma_{i',j}^2}{\alpha_{i',j}} + \sum_{j,j<i'}\frac{\sigma_{i',j}^2}{\alpha_{j,i'}}\right)}\right\}$$
$$= \max\left\{\frac{\left(-c_m + \sum_{j,j\neq l}\overline{X}_{l,j}\right)^2}{-2\left(\sum_{j,j>l}\frac{\sigma_{l,j}^2}{\alpha_{l,j}} + \sum_{j,j<l}\frac{\sigma_{l,j}^2}{\alpha_{j,l}}\right)}, \frac{\left(-c_m + \sum_{j,j\neq l'}\overline{X}_{l',j}\right)^2}{-2\left(\sum_{j,j>l'}\frac{\sigma_{l',j}^2}{\alpha_{l',j}} + \sum_{j,j<l'}\frac{\sigma_{l',j}^2}{\alpha_{j,l'}}\right)}\right\}.$$

Case 5 For (i, i') and (l, l'), $i, l, l' \in \Omega$, $i' \notin \Omega$, $i < i'$, $l < l'$. From (B5) and (B6), using the Lemma 1, as $T \to \infty$, we have

$$\max\left\{\frac{\left(c_m - \sum_{j,j\neq i}\overline{X}_{i,j}\right)^2}{-2\left(\sum_{j,j>i}\frac{\sigma_{i,j}^2}{\alpha_{i,j}} + \sum_{j,j<i}\frac{\sigma_{i,j}^2}{\alpha_{j,i}}\right)}, \frac{\left(-c_m + \sum_{j,j\neq i'}\overline{X}_{i',j}\right)^2}{-2\left(\sum_{j,j>i'}\frac{\sigma_{i',j}^2}{\alpha_{i',j}} + \sum_{j,j<i'}\frac{\sigma_{i',j}^2}{\alpha_{j,i'}}\right)}\right\}$$
$$= \max\left\{\frac{\left(c_m - \sum_{j,j\neq l}\overline{X}_{l,j}\right)^2}{-2\left(\sum_{j,j>l}\frac{\sigma_{l,j}^2}{\alpha_{l,j}} + \sum_{j,j<l}\frac{\sigma_{l,j}^2}{\alpha_{j,l}}\right)}, \frac{\left(c_m - \sum_{j,j\neq l'}\overline{X}_{l',j}\right)^2}{-2\left(\sum_{j,j>l'}\frac{\sigma_{l',j}^2}{\alpha_{l',j}} + \sum_{j,j<l'}\frac{\sigma_{l',j}^2}{\alpha_{j,l'}}\right)}\right\}.$$

Case 6 For (i, i') and (l, l'), $i, i' \in \Omega, l, l' \notin \Omega, i < i', l < l'$. From (B6) and (B7), using the Lemma 1, as $T \to \infty$, we have

$$\max\left\{\frac{\left(c_m - \sum_{j,j\neq i}\overline{X}_{i,j}\right)^2}{-2\left(\sum_{j,j>i}\frac{\sigma_{i,j}^2}{\alpha_{i,j}} + \sum_{j,j<i}\frac{\sigma_{i,j}^2}{\alpha_{j,i}}\right)}, \frac{\left(c_m - \sum_{j,j\neq i'}\overline{X}_{i',j}\right)^2}{-2\left(\sum_{j,j>i'}\frac{\sigma_{i',j}^2}{\alpha_{i',j}} + \sum_{j,j<i'}\frac{\sigma_{i',j}^2}{\alpha_{j,i'}}\right)}\right\}$$

$$= \max\left\{\frac{\left(-c_m + \sum_{j,j\neq l}\overline{X}_{l,j}\right)^2}{-2\left(\sum_{j,j>l}\frac{\sigma_{l,j}^2}{\alpha_{l,j}} + \sum_{j,j<l}\frac{\sigma_{l,j}^2}{\alpha_{j,l}}\right)}, \frac{\left(-c_m + \sum_{j,j\neq l'}\overline{X}_{l',j}\right)^2}{-2\left(\sum_{j,j>l'}\frac{\sigma_{l',j}^2}{\alpha_{l',j}} + \sum_{j,j<l'}\frac{\sigma_{l',j}^2}{\alpha_{j,l'}}\right)}\right\}.$$

Summarizing the six cases, we can obtain the equality in Theorem 4.