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RESEARCH ARTICLE

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Formality of cochains on *BG*

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Abstract

Let G be a compact Lie group with maximal torus T. If $|N_G(T)/T|$ is invertible in the field k, then the algebra of cochains $C^*(BG;\mathsf{k})$ is formal as an A_∞ algebra, or equivalently as a differential graded algebra.

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1 | INTRODUCTION

It is well known that over a field of characteristic zero, the algebra of cochains on the classifying space of a connected compact Lie group is formal as an A_∞ algebra, or equivalently as a differential graded (DG) algebra. We prove that this is the case for a compact Lie group G that is not necessarily connected, over any field in which the order of $N_G(T)/T$ is invertible, where T is a maximal torus in G.

In contrast, even for a finite group with cyclic Sylow subgroups of order $p^n \geqslant 3$ in characteristic p, the algebra of cochains on the classifying space is not formal. The A_∞ structure in this case is computed in [3].

Our notation is as follows. We write G for a compact Lie group, T for a maximal torus in T and W for the finite group $N_G(T)/T$. This a finite group acting on T by conjugation, and the kernel of the action is $C_G(T)/T$. If G is connected, then $C_G(T) = T$, and W is the Weyl group of G. If p is a prime, we write $X^\wedge_p$ for the Bousfield–Kan p-completion of a space X, see [10].

Our strategy is quite different to the usual characteristic zero proof. After reducing to the case of a normal torus, we make an approximation to G at the prime p by a locally finite group $\breve{G}$ with $B\breve{G}^\wedge_p$ homotopy equivalent to $BG^\wedge_p$, and then we put an internal grading on the group algebra $\mathbb{F}_p\breve{G}$. This gives us a second, internal grading on cochains $C^*(B\breve{G};\mathbb{F}_p)$, and the A_∞ structure maps

m_i provided by Kadeishvili's theorem on $H^*(B\breve{G};\mathbb{F}_p)$ have to preserve the internal grading. This then proves that they are all zero apart from the multiplication map m_2. Our main theorem is the following.

Theorem 1.1. *If G is a compact Lie group with maximal torus T, and p does not divide $|N_G(T)/T|$, then $C^*(BG;\mathbb{F}_p)$ is a formal A_∞ algebra.*

In Section 2 we recall the proof of the characteristic zero theorem. The proof of Theorem 1.1 occupies Sections 3 and 4. In Section 5 we make some remarks that put our result in context.

2 | CHARACTERISTIC ZERO

We include the well-known proof for the connected case in characteristic zero for the sake of convenience of comparison. It uses the fact that the algebra of rational cochains on a simply connected space has a commutative model as a DG algebra.

Theorem 2.1. *Let G be a connected Lie group (or more generally, any path connected topological group with finite-dimensional rational homology) and let k be a field containing $\mathbb{Q}$, the field of rational numbers. Then $C^*(BG;\mathsf{k})$ is formal.*

Proof. In rational homotopy theory, if the Sullivan minimal model of a space has zero differential, then the rational homotopy type is formal. This is true for simply connected spaces whose rational cohomology is a polynomial ring on even degree generators tensored with an exterior algebra on odd degree generators. The reason is that we can choose arbitrary cocycles representing the generators of cohomology, and there are no relations to satisfy apart from (graded) commutativity, which is automatic because of the commutativity of the minimal model. In the case of a connected Lie group, the rational cohomology is isomorphic to the invariant ring $H^*(BT;\mathbb{Q})^W$, where T is a maximal (compact) torus and W is the Weyl group. This is a polynomial ring on even degree generators. So $C^*(BG;\mathbb{Q})$ is formal, and hence so is $C^*(BG;\mathsf{k})$ by extension of scalars. For more details and background on rational homotopy theory, we refer the reader to Proposition 15.5 of Félix, Halperin and Thomas [13], or Example 2.67 in the book of Félix, Oprea and Tanré [14]. □

Remark 2.2. In the theorem, for G connected, $\Omega BG^\wedge_\mathbb{Q} \simeq G^\wedge_\mathbb{Q}$ has homology an exterior algebra on odd degree generators, so it is again formal. In fact, the proof shows that $BG^\wedge_\mathbb{Q}$ and $G^\wedge_\mathbb{Q}$ are both intrinsically formal.

The obstruction to extending the proof to other characteristics is that the algebra of mod p cochains on a space usually does not have a commutative DG algebra model. However, we can give a modified version of this theorem using the method of internal gradings. We do this in several steps, in the rest of this paper.

3 | FINITE APPROXIMATIONS

In this section, we show how to approximate compact Lie groups by finite groups. This is closely related to the work of Dwyer and Wilkerson [12], but better suited to the internal grading method.

We begin with case of a circle group $T = S^1$. We have $H^*(BT;\mathbb{F}_p) = \mathbb{F}_p[x]$ with $|x| = -2$. Let μ_m be the finite subgroup of T consisting of elements whose mth power is the identity. Then assuming that $p^n \geqslant 3$, we have $H^*(B\mu_{p^n};\mathbb{F}_p) = \mathbb{F}_p[x_n] \otimes \Lambda(t_n)$, with $|x_n| = -2$ and $|t| = -1$, and x_n is the restriction of x. The restriction map from $H^*(B\mu_{p^{n+1}};\mathbb{F}_p)$ to $H^*(B\mu_{p^n};\mathbb{F}_p)$ sends the element x_{n+1} to x_n and t_{n+1} to zero. Let μ_{p^∞} be the union of the chain of subgroups $\mu_p \subseteq \mu_{p^2} \subseteq \cdots$ of T, which we regard as a discrete group isomorphic to $\mathbb{Z}/p^\infty$. Then the restriction map $H^*(BT;\mathbb{F}_p) \to H^*(B\mu_{p^\infty};\mathbb{F}_p)$ is an isomorphism.

For a torus $T = (S^1)^r$ of rank r it works similarly. The inclusion of $\mu^r_{p^\infty}$ in T induces an isomorphism

$$H^*(BT;\mathbb{F}_p) \to H^*(B\mu^r_{p^\infty};\mathbb{F}_p).$$

Lemma 3.1. *Let T be a torus of rank r acted on by a finite group W, let $\alpha \in H^2(W,T)$ be an element of order $m \geqslant 1$ and let μ^r_m be the finite subgroup of T consisting of elements whose mth power is the identity. Then α is in the image of $H^2(W,\mu^r_m) \to H^2(W,T)$.*

Proof. We have a short exact sequence

$$1 \to \mu^r_m \to T \xrightarrow{m} T \to 1,$$

which gives an exact sequence

$$\cdots \to H^1(W,T) \to H^2(W,\mu^r_m) \to H^2(W,T) \xrightarrow{m} H^2(W,T) \to \cdots$$

If $\alpha \in H^2(W,T)$ has order m, then it is in the kernel of multiplication by m on $H^2(W,T)$, and hence, it is in the image of of $H^2(W,\mu^r_m)$. □

Theorem 3.2. *Let G be a compact Lie group with normal maximal torus T of rank r and let $W = G/T$. Then the Bousfield–Kan p-completion $BG^\wedge_p$ is homotopy equivalent to $B\breve{G}^\wedge_p$, where $\breve{G}$ is a locally finite group sitting in a short exact sequence*

$$1 \to \breve{T} \to \breve{G} \to W \to 1$$

with $\breve{T}$ isomorphic to $\mu^r_{p^\infty}$. If $|W|$ is coprime to p, then this sequence splits.

Proof. The group G sits in a short exact sequence

$$1 \to T \to G \to W \to 1.$$

This extension defines an element α of $H^2(W,T)$. Since $|W|$ annihilates $H^2(W,T)$, the order m of α is a divisor of $|W|$. So by the lemma, α is in the image of $H^2(W,\mu^r_m) \to H^2(W,T)$. It follows that if we quotient out by μ^r_m, we have a split short exact sequence

$$1 \to T/\mu^r_m \to G/\mu^r_m \to W \to 1.$$

Choosing a splitting and taking the inverse image in G, we obtain a subgroup $\tilde{W}$ of G that sits in a diagram

$$\begin{tikzcd}
1 \arrow[r] & \mu_m^r \arrow[r] \arrow[d] & \tilde{W} \arrow[r] \arrow[d] & W \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & T \arrow[r] & G \arrow[r] & W \arrow[r] & 1.
\end{tikzcd}$$

Let $\tilde{G}$ be the subgroup of G generated by $\tilde{W}$ and $\mu_{p^\infty}^r$, regarded as a discrete locally finite group, and let $\tilde{T}$ be the discrete locally finite subgroup of $\tilde{G}$ generated by μ_m^r and $\mu_{p^\infty}^r$. Then we have a diagram

$$\begin{tikzcd}
1 \arrow[r] & \tilde{T} \arrow[r] \arrow[d] & \tilde{G} \arrow[r] \arrow[d] & W \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & T \arrow[r] & G \arrow[r] & W \arrow[r] & 1.
\end{tikzcd}$$

The comparison map of spectral sequences

$$\begin{tikzcd}
H^*(BW; H^*(BT; \mathbb{F}_p)) \arrow[r, Rightarrow] \arrow[d] & H^*(BG; \mathbb{F}_p) \arrow[d] \\
H^*(BW; H^*(B\tilde{T}; \mathbb{F}_p)) \arrow[r, Rightarrow] & H^*(B\tilde{G}; \mathbb{F}_p)
\end{tikzcd}$$

is an isomorphism on the E_2 page, and hence, the map $\tilde{G} \to G$ induces an isomorphism $H^*(BG; \mathbb{F}_p) \to H^*(B\tilde{G}; \mathbb{F}_p)$. It therefore induces a homology equivalence $H_*(B\tilde{G}; \mathbb{F}_p) \to H_*(BG; \mathbb{F}_p)$ and a homotopy equivalence of Bousfield–Kan completions $B\tilde{G}_p^\wedge \to BG_p^\wedge$.

Now the p' elements of $\tilde{T}$ are the p' elements of μ_m^r, and they form a characteristic finite subgroup $O_{p'}\mu_m^r$ of $\tilde{T}$, and hence also of $\tilde{G}$. Set $\breve{G} = \tilde{G}/O_{p'}\mu_m^r$ and $\breve{T} = \tilde{T}/O_{p'}\mu_m^r \cong \mu_{p^\infty}^r$. Then the quotient map $\tilde{G} \to \breve{G}$ is mod p cohomology isomorphism, and $\breve{G}$ sits in a short exact sequence

$$1 \to \breve{T} \to \breve{G} \to W \to 1.$$

The space $B\breve{G}_p^\wedge$ is homotopy equivalent to $BG_p^\wedge$.

For the final statement about splitting, the group $H^2(W, \mu_{p^n}^r)$ is annihilated by $|W|$ and by p^n, and is hence zero. So $H^2(W, \breve{T}) \cong H^2(W, \mu_{p^\infty}^r) \cong \varinjlim_n H^2(W, \mu_{p^n}^r) = 0$. □

We shall make use of the theorem by means of the following proposition.

Proposition 3.3. *Let G be a compact Lie group with maximal torus T, and let $W = N_G(T)/T$. If $p = 0$ or $(p, |W|) = 1$, then the inclusion $N_G(T) \to G$ induces a homotopy equivalence of p-completions $BG_p^\wedge \simeq BN_G(T)_p^\wedge$.*

Proof. In Theorem 20.3 of Borel [8], it is proved that for G connected, $H^*(BG; \mathbb{Q}) \to H^*(BT; \mathbb{Q})^W$ is an isomorphism. Theorem I.5 of Feshbach [15] improves this to the statement that whether or not G is connected, the inclusion $T \to G$ induces an isomorphism

$$H^*(BG;\mathbb{Z}) \otimes \mathbb{Z}[1/|W|] \to H^*(BT,\mathbb{Z})^W \otimes \mathbb{Z}[1/|W|].$$

Feshbach attributes this theorem to Borel without reference, but the argument is simple. The restriction followed by the Becker–Gottlieb transfer is equal to multiplication by the Euler characteristic $\chi(G/T)$, which is equal to $|W|$, and transfer followed by restriction is the sum of the conjugates under $N_G(T)$, by the double coset formula.

If $(p,|W|)=1$, then it follows using the five lemma on the long exact sequence in cohomology associated to the short exact sequence of coefficients

$$0 \to \mathbb{Z}[1/|W|] \xrightarrow{p} \mathbb{Z}[1/|W|] \to \mathbb{F}_p \to 0$$

that $H^*(BG;\mathbb{F}_p) \to H^*(BT;\mathbb{F}_p)^W$ is an isomorphism. This applies just as well to $N_G(T)$, and so $H^*(BG;\mathbb{F}_p) \to H^*(BN_G(T);\mathbb{F}_p)$ is an isomorphism. Now complete at p. □

4 | FORMALITY

Let G be a compact Lie group with maximal torus T of rank r, and let $W = N_G(T)/T$. According to Theorem 3.2 and Proposition 3.3, there is a locally finite discrete group $\breve{G}$ sitting in a short exact sequence

$$1 \to \breve{T} \to \breve{G} \to W \to 1$$

with $\breve{T} \cong \mu_{p^\infty}^r$, and a homotopy equivalence $B\breve{G}^\wedge_p \simeq BG^\wedge_p$.

For the proof of formality, we shall need to make use of Kőnig's lemma from graph theory, so we begin by stating this.

Lemma 4.1. *Let Γ be a locally finite, connected infinite graph. Then Γ contains a ray, namely an infinite sequence of vertices without repetitions $v_0, v_1, v_2, \ldots$ such that there is an edge from each v_i to v_{i+1}.*

Proof. This is proved in Kőnig [17]. For a modern reference in English, see, for example, Diestel [11, Lemma 8.1.2]. An equivalent and possibly more familiar formulation is that an inverse limit of a sequence of non-empty finite sets is non-empty (see, for example, Bourbaki [9, III.7.4]). □

Theorem 4.2. *If $(p,|W|)=1$, then the A_∞ algebra $C^*(B\breve{G};\mathbb{F}_p)$ is formal.*

Proof. We firstly examine the case $\breve{G} = \breve{T} = \mu_{p^\infty}$. Regarding S^1 as the unit circle in $\mathbb{C}$, let $g_n = e^{2\pi i/p^n}$ be a generator for μ_{p^n} as a subgroup of S^1, so that $g_n^p = g_{n-1}$. So μ_{p^∞} is the union of these subgroups, regarded as an infinite discrete group.

Set $X_n = g_n - 1$, a generator for the radical $J(\mathbb{F}_p\mu_{p^n})$. Then $\mathbb{F}_p\mu_{p^n} = \mathbb{F}_p[X_n]/(X_n^{p^n})$. We put an internal $\mathbb{Z}[\frac{1}{p}]$-grading on the group algebra $\mathsf{k}\mu_{p^n}$ by setting $|X_n| = 1/p^n$. Then the fact that $X_n^p = X_{n-1}$ implies that the inclusion $\mathbb{F}_p\mu_{p^n} \hookrightarrow \mathbb{F}_p\mu_{p^{n+1}}$ preserves the internal grading. Thus the union $\mathbb{F}_p\mu_{p^\infty}$ is $\mathbb{Z}[\frac{1}{p}]$-graded, with all degrees lying in the interval [0,1).

Assuming that $p^n \geqslant 3$, we have $H^*(B\mu_{p^n}; \mathbb{F}_p) = \mathbb{F}_p[x_n] \otimes \Lambda(t_n)$. This ring inherits an internal grading from the group algebra, and we have $|x_n| = (-2, -1)$ and $|t_n| = (-1, -\frac{1}{p^n})$. The restriction map from $H^*(B\mu_{p^{n+1}}; \mathbb{F}_p)$ to $H^*(B\mu_{p^n}; \mathbb{F}_p)$ sends x_{n+1} to x_n and t_{n+1} to zero. Therefore $H^*(B\mu_{p^\infty}; \mathbb{F}_p) = \mathsf{k}[x]$ with $|x| = (-2, -1)$.

Now the A_∞ structure maps $m_i : H^*(B\mu_{p^\infty}; \mathbb{F}_p)^{\otimes i} \to H^*(B\mu_{p^\infty}; \mathbb{F}_p)$ given by Kadeishvili's theorem [16] preserve the internal grading, and increase the homological grading by $i - 2$. To prove that $C^*(B\mu_{p^\infty}; \mathbb{F}_p)$ is formal, we must show that we may take m_i to be zero for $i \neq 2$. But for every non-zero element of $H^*(B\mu_{p^\infty}; \mathbb{F}_p)$, the homological grading is twice the internal grading. So for m_i to be non-zero, we must have $i = 2$. Thus $C^*(B\mu_{p^\infty}; \mathbb{F}_p)$ is a formal A_∞ algebra.

Similarly, for a larger rank torus, $\check{T} \cong \mu^r_{p^\infty}$, we put an internal $\mathbb{Z}[\frac{1}{p}]$-grading on $\mathbb{F}_p\mu^r_{p^\infty}$ by adding the internal gradings on the factors μ_{p^∞}, so that all degrees lie in the interval $[0, r)$. This puts an internal grading on $H^*(B\mu^r_{p^\infty}; \mathbb{F}_p)$, so that it is a polynomial ring on r indeterminates, all in degree $(-2, -1)$. The same argument as in the case of μ_{p^∞} now implies that $C^*(B\check{T}; \mathbb{F}_p)$ is formal.

Next, we come to the case where $\check{T} \cong \mu^r_{p^\infty}$ is normal in $\check{G}$ and $W = \check{G}/\check{T}$ is a finite p'-group. In this case, the extension

$$1 \to \check{T} \to \check{G} \to W \to 1$$

splits by Theorem 3.2.

In this case, we need to be more careful about the invariance under W of the grading. The group W acts on the subgroup μ^r_p of elements of T of order p, and this defines an $\mathbb{F}_pW$-module M of dimension r. The action of W on $J(\mathbb{F}_p\mu^r_{p^n})/J^2(\mathbb{F}_p\mu^r_{p^n})$ makes it a $\mathbb{F}_pW$-module canonically isomorphic to M. Since $\mathbb{F}_pW$ is semisimple, the short exact sequence of $\mathbb{F}_pW$-modules

$$0 \to J^2(\mathbb{F}_p\mu^r_{p^n}) \to J(\mathbb{F}_p\mu^r_{p^n}) \to J(\mathbb{F}_p\mu^r_{p^n})/J^2(\mathbb{F}_p\mu^r_{p^n}) \to 0$$

splits. Such a splitting gives us a W-invariant grading on $\mathbb{F}_p\mu^r_{p^n}$, by lifting a basis. The problem is that it is not unique, so we need to worry about compatibility.

Now the diagram

$$\begin{tikzcd}
\mathbb{F}_p\mu^r_{p^n} \arrow[r, hook] \arrow[dr, "x \mapsto x^p"'] & \mathbb{F}_p\mu^r_{p^{n+1}} \arrow[d, two heads] \\
& \mathbb{F}_p\mu^r_{p^n}
\end{tikzcd}$$

commutes, because over $\mathbb{F}_p$, we have $(\sum_i a_i x_i)^p = \sum_i a_i x_i^p$. So a splitting for $n+1$ gives a splitting for n by taking pth powers. Since there are only finitely many splittings at each stage, it follows using Kőnig's lemma that we may choose consistent splittings for all $n > 0$. The graph to which the lemma is applied has as vertices the pairs consisting of a value of n and a splitting for the above sequence. The edges go from the pairs with $n+1$ to the pairs with n, by taking pth powers. A ray in this graph consists of a consistent set of splittings for all $n > 0$.

Let $X_{n,1}, \dots, X_{n,r}$ be bases for such a consistent set of splittings, so that $X^p_{n+1,i} = X_{n,i}$. We may now put a grading on $\mathbb{F}_p\mu^r_{p^\infty}$ in such a way that the degree of $X_{n,i} \in \mathbb{F}_p\mu^r_{p^n}$ is equal to $1/p^n$, and W preserves the grading.

Now we can put a grading on $\mathbb{F}_p\check{G}$ by choosing a copy of W in $\check{G}$ complementary to $\check{T}$, and putting it in degree zero. Then

$$H^*(B\check{G}; \mathbb{F}_p) \cong H^*(B\check{T}; \mathbb{F}_p)^W$$

inherits an internal grading, and the homological grading of all elements is again twice the internal grading. So as before, all m_i with $i \neq 2$ in the A_∞ structure on $H^*(B\breve{G};\mathbb{F}_p)$ are forced to be zero. It follows that $C^*(B\breve{G};\mathbb{F}_p)$ is a formal A_∞ algebra. □

Proof of Theorem 1.1. By Theorem 3.2, the A_∞ algebra $C^*(BG;\mathbb{F}_p)$ is quasi-isomorphic to $C^*(B\breve{G};\mathbb{F}_p)$, which by Theorem 4.2 is formal. □

5 | FINAL REMARKS

It follows from Stasheff and Halperin [19, Theorem 9] that if X is a space and R is a commutative ring of coefficients such that $H^*(X;R)$ is a polynomial algebra, then $C^*(X;R)$ is formal. For example, $H^*(BSU(n);\mathbb{Z})$ and $H^*(BU(n);\mathbb{Z})$ are polynomial rings on Chern classes, and $H^*(BSp(n);\mathbb{Z})$ is a polynomial ring on Pontryagin classes, so the algebras of cochains are formal for all commutative coefficients in these cases. Similarly, $H^*(BO(n);\mathbb{F}_2)$ and $H^*(BSO(n);\mathbb{F}_2)$ are polynomial rings on Stiefel–Whitney classes, so the algebras of cochains are formal over any commutative ring in which $2 = 0$.

In the case of a connected, simply connected compact Lie groups G, Borel [4–8] and Steinberg [20] have investigated torsion in $H^*(BG;\mathbb{Z})$. Borel comments at the end of Volume II of his *Œuvres* that the upshot of these papers is that the following are equivalent.

(1) $H^*(G;\mathbb{Z})$ has no p-torsion.
(2) $H^*(G;\mathbb{F}_p)$ is an exterior algebra on odd degree classes.
(3) $H^*(BG;\mathbb{F}_p)$ is a polynomial algebra on even classes.
(4) $H^*(BG;\mathbb{Z})$ has no p-torsion.
(5) $H^*(G/T;\mathbb{F}_p)$ is generated by elements of degree two, where T is a maximal torus.
(6) Every elementary abelian p-subgroup is contained in a torus.
(7) Every elementary abelian p-subgroup of rank at most three is contained in a torus.
(8) The multiplicity of some fundamental root in the dominant coroot is divisible by p.

The primes p for which this occurs are called the *torsion primes* for G. They are a subset of the *bad primes*, which are those for which the multiplicity of some fundamental root in the dominant root is divisible by p. But these are not the same set. For example, if $G = Sp(n)$, then there are no torsion primes, but 2 is a bad prime. If $G = G_2$, then the only torsion prime is 2 but the bad primes are 2 and 3.

Putting these together, if G is a connected, simply connected compact Lie group, k is a field of characteristic p and p is not a torsion prime, then $H^*(BG;\mathsf{k})$ is a polynomial ring on even classes, and $C^*(BG;\mathsf{k})$ is formal.

Here is a table of the torsion primes and bad primes.

type	bad	torsion	type	bad	torsion
A_n	$\varnothing$	$\varnothing$	E_6	$\{2,3\}$	$\{2,3\}$
B_n	$\{2\}$	$\{2\}$	E_7	$\{2,3\}$	$\{2,3\}$
C_n	$\{2\}$	$\varnothing$	E_8	$\{2,3,5\}$	$\{2,3,5\}$
D_n	$\{2\}$	$\{2\}$	F_4	$\{2,3\}$	$\{2,3\}$
			G_2	$\{2,3\}$	$\{2\}$

If G is a compact Lie group which is connected but not simply connected, then there exists a torus T^n and a simply connected group H, such that if π is the torsion subgroup of $\pi_1(G)$, then G sits in a short exact sequence

$$1 \to \pi \to T^n \times H \to G \to 1.$$

So $H^*(BG;\mathbb{Z})$ has p-torsion if and only if $H^*(G;\mathbb{Z})$ has p-torsion, which happens if and only if either p divides $|\pi|$ or p is a torsion prime for H. Otherwise, $H^*(BG;\mathbb{F}_p)$ is a polynomial ring in even degree generators, and $C^*(BG;\mathbb{F}_p)$ is formal. The case where p divides $|\pi|$ is studied in Borel [5], Mimura and Toda [18].

For non-connected compact Lie groups, the situation is quite different. For example, let G be a semidirect product $T^2 \rtimes \mathbb{Z}/2$, where $\mathbb{Z}/2$ acts on the 2-torus T^2 by inverting every element. Then $H^*(BG;\mathbb{F}_3)$ is isomorphic to $H^*(BT^2;\mathbb{F}_3)^{\mathbb{Z}/2}$. We have $H^*(BT^2;\mathbb{F}_3) = \mathbb{F}_3[x,y]$ with $|x| = |y| = 2$, and $H^*(BG;\mathbb{F}_3)$ is the subring generated by x^2, xy, y^2, which is not a polynomial ring. On the other hand, 3 is not a torsion prime for $H^*(BG;\mathbb{Z})$, and $C^*(BG;\mathbb{F}_3)$ is formal by Theorem 1.1.

For a finite group G and a prime p dividing $|G|$, it follows from Benson and Carlson [2] that $H^*(BG;\mathbb{F}_p)$ is a polynomial ring if and only if $p = 2$, $G/O(G)$ is an elementary abelian 2-group and the polynomial generators are in degree one; $C^*(BG;\mathbb{F}_2)$ is formal in this case. More generally, if $p = 2$ and G has elementary abelian Sylow 2-subgroups, then $C^*(BG;\mathbb{F}_2)$ is formal, even though the cohomology ring does not have to be polynomial. There are also other formal examples. For example, it is shown in Benson [1] that $C^*(BM_{11};\mathbb{F}_2)$ is formal, whereas $H^*(BM_{11};\mathbb{F}_2) \cong \mathbb{F}_2[x,y,z]/(x^2y + z^2)$ with $|x| = 3$, $|y| = 4$, $|z| = 5$ is not a polynomial ring. It would be interesting to know whether there are any formal examples in odd characteristic.

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