A Thesis Submitted for the Degree of PhD at the University of Warwick

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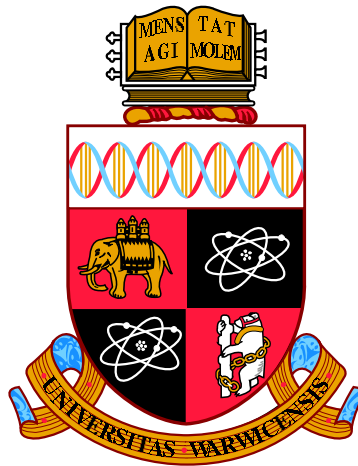http://wrap.warwick.ac.uk/173085

One-relator hierarchies

by

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Thesis

Submitted to the University of Warwick

for the degree of

Doctor of Philosophy

Mathematics

June 2022

Contents

Acknowledgments

Firstly, I am grateful to my supervisor Saul Schleimer for providing me with constant guidance and support throughout my PhD. The endless advice, anecdotes and mathematical discussions have contributed in many ways to the work behind this thesis.

I would also like to thank the many friends I made along this journey. You have all made these years amazing. I am sorry for forcing everyone to get into climbing (but not really). I am also grateful to all the Warwick maths community for never leaving me with a dull moment.

Of course, without my family I would be nowhere. I especially thank my parents for always being there. Finally, Sara, I am deeply grateful to you for your love and wholehearted support.

Declarations

The main results from Chapters 3, 4, 6 and Section 7.1 appeared in the preprint [Lin22], which has been submitted for publication.

The material presented in this thesis is entirely my own and has not been submitted elsewhere for any other degree or qualification.

Abstract

The Magnus hierarchy has been used for almost a century to study one-relator groups. Taking a topological viewpoint, we refine the Magnus hierarchy. With this new tool, we characterise quasi-convex one-relator hierarchies in the sense of Wise. This new characterisation has several applications: we confirm a conjecture of Louder and Wilton on one-relator groups with negative immersions, we characterise hyperbolic one-relator groups with exceptional intersection, and answer a question of Baumslag's on parafree one-relator groups. Finally, we introduce two new families of two-generator one-relator groups and prove that Gersten's hyperbolicity conjecture is true for all one-relator groups if and only if it is true for these families.

Chapter 1

Introduction

Almost a century ago, the theory of one-relator groups began when Magnus proved the Freiheitssatz [Mag30]. An early triumph of the theory came with the solution of the word problem [Mag32]. Despite their long history, one-relator groups have evaded any form of geometric characterisation. With this thesis, we aim to partially remedy this. We are motivated by the following, known as Gersten's conjecture [Ger92b].

Conjecture. A one-relator group is hyperbolic if and only if it contains no Baumslag–Solitar subgroups:

$$\mathrm{BS}(m,n) = \langle a, t \mid t^{-1}a^m t = a^n \rangle \, .$$

The introduction of hyperbolic groups in [Gro87] revolutionised the study of infinite groups; what was formerly known as combinatorial group theory, transformed into geometric group theory. Hyperbolic groups are now well understood and constitute a major component in any standard text [GdlH90, ABC⁺91, BH99, DK18]. Therefore, it can be very useful, when studying a class of groups, to seek a characterisation of the hyperbolic groups within that class. Indeed, Gersten's conjecture forms part of a broader theme in geometric group theory to characterise hyperbolicity.

A more general conjecture, attributed to Gromov, states that a group with finite $K(G,1)$ is hyperbolic if and only if it does not contain any Baumslag–Solitar subgroups [Bes04]. Although this has recently been disproved [IMM21], there are many subclasses of groups where this dichotomy holds. For example: Coxeter groups [Mou88], free-by-cyclic groups [Bri00], three-manifold groups, special groups [CH09] and ascending HNN-extensions of free groups [Mut21b]. Moreover, hyperbolicity is decidable within several of these classes[1].

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[1] See [Mou88, Theorem 17.1] for Coxeter groups, [Bri00] and [FH18, Corollary 16.4] for free-by-cyclic groups, [AFW15, Theorem 4.24] and the discussion preceding it for three-manifold groups, and [Mut21a] for ascending HNN-extensions of free groups.

1.1 One-relator groups and the isoperimetric function

The isoperimetric function of a finitely presented group roughly measures the complexity of the word problem. A geometric characterisation of hyperbolic groups is that they are precisely the finitely presented groups with linear isoperimetric function. By contrast, some one-relator groups have isoperimetric function not bounded above by any finite tower of exponentials [Pla04]. The current best known uniform upper bound is Ackermannian [Ber94].

A large class of one-relator groups that are known to be hyperbolic are those with torsion. This is a corollary of Newman's Spelling Theorem [New68]. For torsion-free one-relator groups, hyperbolicity criteria generally rely on the Bestvina-Feighn combination theorem [BF92], as in the case of cyclically and conjugacy pinched one-relator groups [FRS97a, Theorem B12], or on variations of small-cancellation conditions [GS90, IS98, BM22].

Gersten also asked whether one-relator groups are automatic if and only if they do not contain $BS(m, n)$ subgroups with $m \neq \pm n$ [Ger92b]. If this were true, it would imply a gap in the isoperimetric spectrum: that is, all one-relator groups would have isoperimetric function either at most quadratic or at least exponential by [Ger92a, Theorem C]. However, by building on the work in [BB00], Gardam and Woodhouse provide counterexamples, showing in [GW19] that the isoperimetric spectrum of one-relator groups is as rich as possible: there exists a family of one-relator groups with isoperimetric function $\sim n^{\alpha}$ where the values of α are dense in $[2, \infty)$. These are also counterexamples to Wise's conjecture that such one-relator groups should act freely on a CAT(0) cube complex [Wis14, Conjecture 1.9]. Since these groups contain copies of $\mathbb{Z}^2 \cong BS(1, 1)$, they are not counterexamples to the hyperbolicity conjecture.

1.2 One-relator hierarchies

The most useful tool of this thesis, Theorem 7.1.2, is a criterion for when a one-relator group is hyperbolic and has a quasi-convex hierarchy. The starting point for this theorem is a new hierarchy for *one-relator complexes*: combinatorial 2-complexes with precisely one 2-cell. We show that for any one-relator complex X, there is a finite sequence of immersions of one-relator complexes

$$X_N \looparrowright ... \looparrowright X_1 \looparrowright X_0 = X \ ,$$

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such that $\pi_1(X_i)$ splits as an HNN-extension over $\pi_1(X_{i+1})$ and $\pi_1(X_N)$ splits as a free product of cyclic groups. This refines the well-known Magnus–Moldavanskii hierarchy for one-relator groups. See Theorem 3.3.2 for the full statement and the beginning of Chapter 3 for a more detailed discussion.

One of the main results in [Wis21] states that hyperbolic groups with a quasi-convex hierarchy are virtually special. In [Wis21], Wise also shows that one-relator groups with torsion have quasi-convex one-relator hierarchies and hence, coupled with the Newman Spelling Theorem, that they are virtually special. Since special groups embed in right-angled Artin groups, Wise obtains residual finiteness for one-relator groups with torsion, answering an old question of Baumslag.

We here provide the first quasi-convex hierarchy criterion for all one-relator groups. See the beginning of Chapters 4 and 7 for the relevant definitions appearing in the following statement.

Theorem 7.1.2. *Let X be a one-relator complex and let $X_N \rightsquigarrow ... \rightsquigarrow X_1 \rightsquigarrow X_0 = X$ be a one-relator hierarchy. The following are equivalent:*

1. *$X_N \rightsquigarrow ... \rightsquigarrow X_1 \rightsquigarrow X_0 = X$ is a quasi-convex hierarchy and $\pi_1(X)$ is hyperbolic,*

2. *$X_N \rightsquigarrow ... \rightsquigarrow X_1 \rightsquigarrow X_0 = X$ is an acylindrical hierarchy,*

3. *$X_N \rightsquigarrow ... \rightsquigarrow X_1 \rightsquigarrow X_0 = X$ is a stable hierarchy and $\pi_1(X)$ contains no Baumslag–Solitar subgroups.*

Given as input a finitely presented group, it is undecidable whether it is hyperbolic. This follows from the Adian–Rabin Theorem and the fact that hyperbolicity is a Markov property. On the other hand, within the class of one-relator groups, it is open whether hyperbolicity is decidable [KS05]. As a consequence of Theorem 7.1.7, we show that hyperbolicity is decidable for one-relator groups that possess a stable hierarchy. Although we do not know whether it is decidable if a one-relator hierarchy is stable, we provide a simple algorithm to verify stability with Proposition 6.1.7.

1.3 One-relator groups with negative immersions

Recently, Louder and Wilton have shown a number of strong results for one-relator groups [LW22]. In order to describe them, we first need a definition, due to Puder [Pud14]. The *primitivity rank* of an element $w \in F(\Sigma)$ is the following quantity:

$$\pi(w) = \min \{\mathrm{rk}(K) \mid w \in K < F(\Sigma) \text{ and } w \text{ is imprimitive in } K\} \in \mathbb{N} \cup \{\infty\} .$$

A one-relator group $G = F(\Sigma)/\langle\langle w \rangle\rangle$ has torsion if and only if $\pi(w) = 1$ by [KMS60]. Thus, by Newman's Spelling Theorem, the case of interest is when $\pi(w) \geqslant 2$.

Louder and Wilton showed that if $\pi(w) = k + 1 \geqslant 2$, then G is k-free. Thus, when $\pi(w) \geqslant 3$, we have that G cannot contain Baumslag–Solitar subgroups; this led them to conjecture that such groups are hyperbolic [LW22, Conjecture 1.9]. Since G also has negative immersions if $\pi(w) \geqslant 3$ [LW22, Theorem 1.3], their conjecture can be considered as a special case of an older conjecture of Wise [Wis04, Conjecture 14.2]. Indeed, their results grew out of earlier ideas of Wise [Wis03, Wis04]. Louder and Wilton's conjecture has been experimentally verified for all one-relator groups with negative immersions that admit a one-relator presentation with relator of length less than 17 [CH21].

In this thesis, we verify their conjecture and in fact prove more.

Theorem 7.1.3. *One-relator groups with negative immersions are hyperbolic and virtually special.*

Before moving on to our main hyperbolicity result, we first discuss two applications of Theorem 7.1.3.

Exceptional intersection groups

A one-relator group $G = F(\Sigma)/\langle\langle w \rangle\rangle$ is an *exceptional intersection group* if there are subsets $A, B \subset \Sigma$ such that $\langle A \rangle \cap \langle B \rangle \neq \langle A \cap B \rangle$. Such groups were first studied in [Col04, How05]. There, the intersections of Magnus subgroups of one-relator groups were characterised. As a consequence of Theorem 5.2.8 and Theorem 7.1.3, we prove that Gersten's conjecture holds for exceptional intersection groups.

Corollary 7.1.5. *An exceptional intersection group is hyperbolic (and virtually special) if and only if it contains no Baumslag–Solitar subgroups.*

Parafree groups

A group G is *parafree* if the following hold:

1. G is residually nilpotent,

2. there is some finitely generated free group F such that $G/\gamma_n(G) \cong F/\gamma_n(F)$ for all $n \geqslant 1$, where $\gamma_i(G) = [\gamma_{i-1}(G), G]$ and $\gamma_1(G) = [G, G]$.

First introduced in [Bau67a], examples of parafree one-relator groups abound [Bau69, BC06]. Often mentioned as examples demonstrating the difficulty of the isomorphism problem for one-relator groups [CM82, BFR19], Baumslag asked in [Bau86, Problem

4] whether the isomorphism problem for parafree one-relator groups is solvable. Since then, several authors have solved the isomorphism problem for certain subfamilies [FRS97b, HK17, HK20, Che21] and carried out computational experiments [LL94, BCH04]. By showing that parafree one-relator groups have negative immersions, we use Theorem 7.1.3 to answer Baumslag's question.

Corollary 7.1.6. *Parafree one-relator groups are hyperbolic and virtually special. In particular, their isomorphism problem is decidable.*

1.4 Gersten's conjecture and primitive extension groups

In order to state our main hyperbolicity result, we now introduce two new families of one-relator groups. The relators of these families are Christoffel words. First appearing in [Chr73], these words have since been shown to have connections in several different areas of mathematics. Christoffel words are parametrised by a positive rational $p/q \in \mathbb{Q}_{>0}$ and we denote them by:

$$\mathrm{pr}_{p/q}(x, y) \in F(x, y) \ .$$

See Section 2.1.3 for their definition and a geometric interpretation. Now, for each $i \leqslant j \in \mathbb{Z}$, denote by

$$A_{i,j} = \{t^{-i}at^i, t^{-i-1}at^{i+1}, ..., t^{-j}at^j\} \subset F(a, t)$$

and let

$$x \in \langle A_{0,k-1} \rangle - \langle A_{1,k-1} \rangle \ ,$$
$$y \in \langle A_{1,k} \rangle - \langle A_{1,k-1} \rangle \ ,$$
$$z \in \langle A_{1,k-1} \rangle$$

be non-trivial elements (with some extra conditions explained in Section 7.2). Then, a *primitive extension group* has one of the following presentations:

$$E_{p/q}(x, y) = \langle a, t \mid \mathrm{pr}_{p/q}(x, y) \rangle \ ,$$
$$F_{p/q}(x, y, z) = \langle a, t \mid \mathrm{pr}_{p/q}(xy, z) \rangle \ .$$

Examples of primitive extension groups include free-by-cyclic one-relator groups and some Baumslag–Solitar groups. The primitive extension groups $E_{p/q}(x, y)$ all split as graphs of free groups. The groups $F_{p/q}(x, y, z)$ also admit a similar graph of groups

decomposition. In particular, the isoperimetric function of primitive extension groups is at most exponential.

Theorem 7.2.3. *A one-relator group is hyperbolic if and only if its primitive extension subgroups are hyperbolic.*

We conclude with a corollary.

Corollary 7.2.4. *Gersten's conjecture is true if and only if it is true for primitive extension groups.*

1.5   Structure of the thesis

In Chapter 2, we introduce the necessary notation and definitions. We then present an overview of one-relator groups, stating the most important results that we will need. In the remaining sections, we discuss some combinatorics on words, inert subgroups of free groups, and graphs of spaces.

In Chapter 3, we refine the Magnus hierarchy using 2-complexes. We also define the hierarchy length of a one-relator group and show that it is computable. Finally, we prove the existence of hierarchies relative to w-subgroups.

In Chapter 4, we prove a criterion for when a one-relator group has a quasi-convex hierarchy in the sense of Wise [Wis21]. In order to do this, we show that hyperbolic one-relator groups with quasi-convex hierarchies have quasi-convex Magnus subgroups.

In Chapter 5, we define exceptional intersection groups. In the first section, we introduce primitive exceptional intersection groups and show that they have negative immersions. In the second section, we prove that all other exceptional intersection groups contain Baumslag–Solitar subgroups, establishing Theorem 5.2.8.

In Chapter 6, we introduce inertial one-relator extensions and develop their Bass-Serre theory. We also introduce the graph of cyclic stabilisers and use it to prove a criterion for when an inertial one-relator extension contains a Baumslag–Solitar subgroup. We then show that, under certain conditions, the graph of cyclic stabilisers is computable. Finally, we apply these results to the HNN-extensions that arise in our one-relator hierarchies.

In Chapter 7, we combine all our main results from each chapter and prove Theorem 7.1.2, our hierarchy equivalence theorem. Using this, we then prove Theorems 7.1.3 and 7.2.2. Finally, we conclude with several open questions.

Chapter 2

Preliminaries

2.1 Definitions and notation

In this section, we fix notation and gather definitions that we will be using throughout this thesis. The notions described here are standard and can be found in most combinatorial and geometric group theory textbooks such as [MKS66] and [BH99]. Those that differ from the literature will be pointed out.

2.1.1 Graphs and combinatorial complexes

A *graph* Γ is a 1-dimensional CW-complex. We will write $V(\Gamma)$ for the collection of 0-cells or *vertices* and $E(\Gamma)$ for the collection of 1-cells or *edges*. We will usually assume Γ to be oriented. That is, Γ comes equipped with maps $o : E(\Gamma) \to V(\Gamma)$ and $t : E(\Gamma) \to V(\Gamma)$, the *origin* and *target* maps. The *degree* $\deg(v)$ of a vertex $v \in V(\Gamma)$ is the number of adjacent edges, counted with multiplicity. In other words, $\deg(v) = |o^{-1}(v)| + |t^{-1}(v)|$. For simplicity, we will write I to denote any connected graph whose vertices all have degree two, except for two vertices of degree one. Then S^1 will denote a connected graph, all of whose vertices have degree precisely two.

A map between graphs $f : \Gamma \to \Gamma'$ is *combinatorial* if it sends vertices to vertices and edges (homeomorphically) to edges, preserving orientations. We will always assume that our maps between graphs are combinatorial. A combinatorial map is an *immersion* if it is also locally injective. Such maps will be denoted by the arrow $\looparrowright$. Combinatorial graph maps $\lambda : I \to \Gamma$, $\lambda : S^1 \to \Gamma$ will be called *paths* and *cycles* respectively. The *reverse* of a path $\lambda : I \to \Gamma$ is the path obtained from λ by reversing direction and it is denoted by $\bar{\lambda}$. If $\lambda_1, \lambda_2 : I \to \Gamma$ are two paths with $t(\lambda_1) = o(\lambda_2)$, then the path obtained by concatenation is denoted by $\lambda_1 * \lambda_2$. The *length* of a combinatorial path $\lambda : I \to \Gamma$ is the number of edges in I and is

denoted by $|\lambda|$. If $\lambda : I \to X$ is a path, we may often identify the vertices of I with the integers $0, 1, ..., |\lambda|$ so that $\lambda(i)$ is the ith vertex that λ traverses. We also put $o(\lambda) = \lambda(0)$ and $t(\lambda) = \lambda(|\lambda|)$. We similarly define the length of a cycle $\lambda : S^1 \to \Gamma$. If $\lambda : S^1 \looparrowright \Gamma$ is an immersed cycle, we will write $\deg(\lambda)$ to denote the maximal degree of a covering map $S^1 \looparrowright S^1$ that λ factors through. We say λ is *primitive* if $\deg(\lambda) = 1$, *imprimitive* otherwise. Note that λ being imprimitive is not the same as $[\lambda]$ being imprimitive in $\pi_1(\Gamma)$.

The *core* of a graph Γ is the subgraph consisting of the union of all the images of immersed cycles $S^1 \looparrowright \Gamma$ and will be denoted by $\text{Core}(\Gamma)$. Note that if Γ is a forest, then $\text{Core}(\Gamma) = \varnothing$. A graph Γ is core if $\Gamma = \text{Core}(\Gamma)$.

If $\gamma : \Gamma \looparrowright \Delta$ and $\lambda : \Lambda \looparrowright \Delta$ are graph immersions, then their *fibre product* is the graph $\Gamma \times_\Delta \Lambda$ with:

$$V(\Gamma \times_\Delta \Lambda) = \{(v, w) \in V(\Gamma) \times V(\Lambda) \mid \gamma(v) = \lambda(w)\} \, ,$$
$$E(\Gamma \times_\Delta \Lambda) = \{(e, f) \in E(\Gamma) \times E(\Lambda) \mid \gamma(e) = \gamma(f)\} \, ,$$

and where $o(e, f) = (o(e), o(f))$ and $t(e, f) = (t(e), t(f))$ for each $(e, f) \in E(\Gamma \times_\Delta \Lambda)$. There are natural maps $p_\Gamma : \Gamma \times_\Delta \Lambda \looparrowright \Gamma$ and $p_\Lambda : \Gamma \times_\Delta \Lambda \looparrowright \Lambda$ given by projecting to the first and second factor. It is an easy exercise to see that these projection maps are immersions.

A *combinatorial 2-complex* X is a 2-dimensional CW-complex whose attaching maps are all immersions. We will usually write $X = (\Gamma, \lambda)$ where Γ is the 1-skeleton and $\lambda : \mathbb{S} = \sqcup S^1 \looparrowright \Gamma$ are the attaching maps. We say X has a *free face* if there is an edge $e \in E(\Gamma)$ that is traversed precisely once by λ; that is, if $\lambda^{-1}(e)$ consists of a single edge. If e is a free face of X, then X is homotopy equivalent to the 2-complex $(\Gamma - e, \lambda')$ where λ' is obtained from λ by removing the component of $\mathbb{S}$ containing $\lambda^{-1}(e)$.

A *combinatorial map* of combinatorial 2-complexes $f : Y \to X$ is a map that restricts to a combinatorial map of graphs $f_\Gamma : \Gamma_Y \to \Gamma_X$ and induces a combinatorial map $f_\mathbb{S} : \mathbb{S}_Y \to \mathbb{S}_X$ such that $f_\Gamma \circ \lambda_Y = \lambda_X \circ f_\mathbb{S}$. We say that f is an *immersion* if f_Γ is an immersion and $f_\mathbb{S}$ restricts to a homeomorphism on each component.

Since we will always be assuming that our maps are combinatorial, we will often neglect to use the descriptor.

2.1.2 Hyperbolic spaces

Fix a geodesic metric space X. If $\delta \geqslant 0$ is some constant, we say X is δ-*hyperbolic* if, for every triple of geodesic paths $\alpha, \beta, \gamma : I \to X$ that form the sides of a triangle,

each of $\mathrm{Im}(\alpha), \mathrm{Im}(\beta), \mathrm{Im}(\gamma)$ is contained in the δ-neighbourhood of the other two. We say that X is *hyperbolic* if it is δ-hyperbolic for some $\delta \geqslant 0$.

Let $f : X \to Y$ be a function between geodesic metric spaces and $K > 0$ a constant. We call f a K-*quasi-isometric embedding* if for all points $x, y \in X$, the following is satisfied:

$$\frac{1}{K} d_X(x, y) - K \leqslant d_Y(f(x), f(y)) \leqslant K d_X(x, y) + K .$$

We call f a K-*quasi-isometry* if, additionally, for all $y \in Y$, there is some $x \in X$ such that:

$$d_Y(y, f(x)) \leqslant K ,$$

We call f a *quasi-isometric embedding* or a quasi-isometry if it is a K-quasi-isometric embedding or a K-quasi-isometry for some $K > 0$.

A *quasi-geodesic* is a map $c : I \to X$ that is a quasi-isometric embedding. We write $|c|$ to denote the length of c.

The definition of quasi-geodesics may be generalised in the following way. Let X be a metric space and $f : \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ a monotonic increasing function. A path $c : I \to X$, is an f-*quasi-geodesic* if, for all $0 \leqslant p \leqslant q \leqslant |c|$, the following is satisfied:

$$q - p = d_I(p, q) \leqslant f(d_X(c(p), c(q))) .$$

If f is bounded above by a linear function, then c is simply a quasi-geodesic. The following theorem is essentially due to Gromov [Gro87]. We include a proof for completeness.

Theorem 2.1.1. *Let X be a geodesic δ-hyperbolic metric space and $f : \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ a monotonic increasing subexponential function. There is a constant $K(f)$ such that all f-quasi-geodesics are $K(f)$-quasi-geodesics.*

Proof. By [Gro87, Corollary 7.1.B], if f is a subexponential function, then there exists some constant $C \geqslant 0$, depending only on δ and f, such that any f-quasi-geodesic remains in the C-neighbourhood of any geodesic connecting its endpoints. In fact, by taking C large enough, any geodesic remains in the C-neighbourhood of any f-quasi-geodesic connecting its endpoints. So now let $\gamma : [0, q] \to X$ be an f-quasi-geodesic and let $\gamma' : [0, p] \to X$ be a geodesic connecting the endpoints of γ. Then there is a sequence $i_1, i_2, ..., i_{\lfloor p \rfloor}$ such that $d(\gamma(i_j), \gamma'(j)) \leqslant C$ for all $j \in \mathbb{N} \cap [0, p]$. Hence $d(\gamma(i_j), \gamma(i_{j+1})) \leqslant 2C + 1$ and so $i_{j+1} - i_j \leqslant f(2C + 1)$ for all j. Now we obtain that $q \leqslant (p + 1)f(2C + 1)$ and so f-quasi-geodesics are $f(2C + 1)$-quasi-geodesics. $\quad\square$

Now let $Y \subset X$ be a subset and $K \geqslant 0$. We say Y is K-*quasi-convex* if every geodesic in X joining two points in Y remains at distance at most K from Y. We say it is *quasi-convex* if it is K-quasi-convex for some $K \geqslant 0$. If Y is hyperbolic, Y being a quasi-convex subset is equivalent to Y being quasi-isometrically embedded (see [BH99, Theorem III.H.1.7]).

2.1.3 Groups

Let G be a group. If $g, h \in G$, we will adopt the usual convention that $h^g = g^{-1}hg$ and $[h, g] = h^{-1}h^g$. If $H < G$ is a subgroup, we will write $[H]$ to denote the conjugacy class of H in G. More generally, if $X, Y \subset G$ are subsets, then define the Y-*conjugacy class* of X to be the following:

$$[X]_Y = \{X^y \mid y \in Y\} .$$

A subgroup $H < G$ is said to be *malnormal* if $H \cap H^g \neq 1$ implies that $g \in H$. A collection of subgroups $H_1, ..., H_k < G$ is said to be a *malnormal family* if $H_i \cap H_j^g \neq 1$ implies that $i = j$ and $g \in H_i$.

Denote by $\mathrm{rk}(G)$ the *rank* of G, that is, the smallest number of elements needed to generate G. Then we denote by $\mathrm{rr}(G) = \max\{0, \mathrm{rk}(G) - 1\}$ the *reduced rank* of G.

Free groups

If Σ is a set, then we write Σ^{-1} for the set of formal inverses of elements in Σ. We call each element of Σ a *letter*. A *word* over Σ is simply an element of the free monoid Σ^*. Denote by ϵ the *empty word*. We say a word $w \in (\Sigma \sqcup \Sigma^{-1})^*$ is *freely reduced* if in w there does not appear a subword of the form $\sigma\sigma^{-1}$ or $\sigma^{-1}\sigma$ for any $\sigma \in \Sigma$. We call it *cyclically reduced* if the first and last letters of w are not inverses of each other.

The *free group* generated by Σ is denoted by $F(\Sigma)$. We will often conflate elements of the free group $F(\Sigma)$ with words in the free monoid $(\Sigma \sqcup \Sigma^{-1})^*$. A subset of elements $S \subset F(\Sigma)$ *freely generates* the subgroup $\langle S \rangle < F(\Sigma)$ if the natural homomorphism $F(S) \to F(\Sigma)$ is injective. If S freely generates $\langle S \rangle$, we say that S is a *free basis* for $\langle S \rangle$. An element $\sigma \in F(\Sigma)$ is *primitive* if it forms part of a free basis for $F(\Sigma)$, *imprimitive* otherwise.

We now explicitly describe a subset of the primitive elements of $F(a, b)$, that we will be particularly interested in. These elements are known as Christoffel words and were first introduced in [Chr73]. They are parametrised by a rational slope

Figure 2.1: L is in green with slope $5/6$. The a edges are in blue and the b edges are in red, so $\mathrm{pr}_{5/6}(a,b) = a^2bababab$.

$p/q \in \mathbb{Q}_{\geqslant 0} \cup \{\infty\}$. Let $\Gamma \subset \mathbb{R}^2$ denote the Cayley graph for $\mathbb{Z}^2$ on the generating set $a = (1,0)$, $b = (0,1)$. Let $L \subset \mathbb{R}^2$ be the line segment beginning at the origin and ending at the vertex (q,p). Now let $P \subset \Gamma$ be the shortest length edge-path connecting the endpoints of L, remaining below L and such that there are no integral points contained in the region enclosed by $L \cup P$. See Figure 2.1 for an example. The word in a and b traced out by P is denoted by:

$$\mathrm{pr}_{p/q}(a,b) \ .$$

By [OZ81, Theorem 1.2], every primitive element of $F(a,b)$ is conjugate into the set

$$\left\{ \mathrm{pr}_{p/q}\left(a^{\pm 1}, b^{\pm 1}\right) \ \middle| \ \frac{p}{q} \in \mathbb{Q}_{\geqslant 0} \cup \{\infty\} \right\} \ .$$

We provide an alternative proof of this fact with Corollary 3.3.10.

Using graph immersions to understand the subgroups of free groups is a powerful tool that was popularised by Stallings in [Sta83]. We will take advantage of this idea several times throughout this thesis. The main facts that we shall use repeatedly are the following:

1. If Δ is a finite graph, there is a bijection between immersions of finite core graphs $\Gamma \looparrowright \Delta$ and conjugacy classes of finitely generated subgroups of $\pi_1(\Delta)$, induced by the π_1 functor. Moreover, this bijection is algorithmic, see [Sta83, Algorithm 5.4].

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2. If $\Gamma \hookrightarrow \Delta$ and $\Lambda \hookrightarrow \Delta$ are a pair of immersions of finite graphs, then there is a bijection between components of $\mathrm{Core}(\Gamma \times_\Delta \Lambda)$ and double cosets $(\pi_1(\Gamma))g(\pi_1(\Lambda))$ such that $\pi_1(\Gamma)^g \cap \pi_1(\Lambda) \neq 1$, induced by the π_1 functor. Moreover, this bijection is algorithmic, see [Sta83, Theorem 5.5].

Group presentations

Let Σ be a set, $R \subset F(\Sigma) - \{1\}$ a subset and denote by $\langle\langle R \rangle\rangle$ the *normal closure* of R in $F(\Sigma)$. A group G has *presentation* $\langle \Sigma \mid R \rangle$ if:

$$G \cong F(\Sigma)/\langle\langle R \rangle\rangle \ .$$

There is a natural 2-complex $X = (\Gamma, \lambda)$ associated with a group presentation $\langle \Sigma \mid R \rangle$ called the *presentation complex*: it has one vertex, one edge for each generator $\sigma \in \Sigma$ and one 2-cell for each relator $r \in R$, with attaching map spelling out the corresponding relation.

Hyperbolic groups

A finitely generated group G is *hyperbolic* if its Cayley graph with respect to some finite generating set is hyperbolic with the path metric. Since hyperbolicity is a quasi-isometry invariant [BH99, Theorem III.H.1.9], if G has a hyperbolic locally finite Cayley graph, then every locally finite Cayley graph for G must be hyperbolic.

A subgroup $H < G$ of a hyperbolic group is *quasi-convex* if for some finite generating set $S \subset G$, H is a quasi-convex subspace of $\mathrm{Cay}(G, S)$. The property of being quasi-convex does not depend on the chosen generating sets. Quasi-convex subgroups of hyperbolic groups are finitely generated and hyperbolic [BH99, Proposition III.H.3.7].

Distortion

Let $H < G$ be a pair of finitely generated groups and fix finite generating sets for H and G. Let d_H and d_G be given by the path metrics in the Cayley graphs for H and G respectively. The *distortion* of H in G is given by the function:

$$\delta_G^H(n) = \max \{d_H(1, g) \mid g \in H, d_G(1, g) \leqslant n\} \ .$$

Varying generating sets for H and G produce Lipschitz equivalent distortion functions. A finitely generated subgroup $H < G$ is *undistorted* if it has a distortion function

bounded above by a linear function. The quasi-convex subgroups of hyperbolic groups are precisely those that are undistorted [BH99, Corollary III.H.3.6].

It follows from Theorem 2.1.1 that there is a gap in the spectrum of possible distortion functions of subgroups of hyperbolic groups. This is known by work of Gromov [Gro87]. It also appears as [Kap01, Proposition 2.1].

Corollary 2.1.2. *Subexponentially distorted subgroups of hyperbolic groups are undistorted and hence, quasi-convex.*

2.1.4 Graphs of groups

See [Ser03] for a comprehensive treatment of graphs of groups. Our definitions here deviate from the standard ones only in that we do not use Serre graphs. A *graph of groups* is a tuple

$$\mathcal{G} = \left(\Gamma, \{G_v\}_{v \in V(\Gamma)}, \{G_e\}_{e \in E(\Gamma)}, \{\partial_e^\pm\}_{e \in E(\Gamma)}\right) ,$$

where Γ is the *underlying graph*, G_v are the *vertex groups*, G_e are the *edge groups* and

$$\partial_e^- : G_e \hookrightarrow G_{o(e)} ,$$
$$\partial_e^+ : G_e \hookrightarrow G_{t(e)} ,$$

are monomorphisms called the *edge maps*. We abuse notation and write $o(e^{-1}) = t(e)$ and $t(e^{-1}) = o(e)$ and $\partial_{e^{\pm 1}}^+ = \partial_e^\pm$ and $\partial_{e^\pm}^- = \partial_e^\mp$.

Consider the group

$$F(E(\Gamma)) * \left(\underset{v \in V(\Gamma)}{\ast} G_v \right) .$$

Then the *path groupoid* is the subgroupoid consisting of elements

$$g = g_0 \cdot e_1^{\epsilon_1} \cdot g_1 \cdot ... \cdot e_n^{\epsilon_n} \cdot g_n ,$$

where the following hold:

1. $\epsilon_i = \pm 1$ and $e_i \in E(\Gamma)$ for all $1 \leqslant i \leqslant n$,

2. $v_0 = o(e_1^{\epsilon_1})$, $v_n = t(e_n^{\epsilon_n})$ and $t\left(e_i^{\epsilon_i}\right) = v_i = o\left(e_{i+1}^{\epsilon_{i+1}}\right)$ for all $1 \leqslant i < n$,

3. $g_i \in G_{v_i}$ for all $0 \leqslant i \leqslant n$.

We say g is *reduced* if for each $1 \leqslant i < n$ such that $e_i^{\epsilon_i} = e_{i+1}^{\epsilon_{i+1}}$, we have

$$g_i \notin \operatorname{Im}\left(\partial^+_{e_i^{\epsilon_i}}\right) \ .$$

If $v_n = v_0$, then we say g is *cyclically reduced* if g is reduced and when $e_n^{\epsilon_n} = e_1^{-\epsilon_1}$, we have

$$g_n g_0 \notin \operatorname{Im}\left(\partial^+_{e_n^{\epsilon_n}}\right) \ .$$

We call $e_1^{\epsilon_1} * \ldots * e_n^{\epsilon_n}$ the path associated with g.

The *fundamental groupoid* $\pi_1(\mathcal{G})$ is the quotient of the path groupoid by the normal closure of the elements:

$$\partial^-_e(g) = e \cdot \partial^+_e(g) \cdot e^{-1} \ ,$$

for all $e \in E(\Gamma)$ and $g \in G_e$. Then, given a vertex $v \in V(\Gamma)$, the *fundamental group* $\pi_1(\mathcal{G}, v)$ is the subgroup of $\pi_1(\mathcal{G})$ consisting only of those elements whose associated path begins and ends at v.

Following [Ser03, §5.3], there is a tree associated with a graph of groups, known as the *Bass-Serre tree*. If T is the Bass-Serre tree associated with $\mathcal{G}$, then $\pi_1(\mathcal{G}, v)$ acts on T with vertex stabilisers the conjugates of vertex groups, edge stabilisers the conjugates of edge groups, and with $\pi_1(\mathcal{G}, v) \backslash T = \Gamma$. We say an element $g \in \pi_1(\mathcal{G}, v)$ acts *elliptically* on T if it fixes a vertex. We say it acts *hyperbolically* if $\inf_{t \in T} d(t, g \cdot t) > 0$. An element g acting hyperbolically on a tree has a unique embedded line on which it acts by translations by $\inf_{t \in T} d(t, g \cdot t)$, called its *axis*.

We will mostly be concerning ourselves with the case in which Γ is a graph with a single edge e. If $V(\Gamma) = \{v, w\}$, then

$$\pi_1(\mathcal{G}, v) = G_v \underset{G_e}{*} G_w$$

is the *amalgamated free product* of G_v and G_w along G_e. If $V(\Gamma) = \{v\}$, then

$$\pi_1(\mathcal{G}, v) = G_v *_{G_e}$$

is the *HNN-extension* of G_v along G_e. We will also use the notation $G_v *_\psi$ where $\psi = \partial^+_e \circ (\partial^-_e)^{-1} : \operatorname{Im}(\partial^-_e) \to \operatorname{Im}(\partial^+_e)$.

2.2 Counting subwords

In this section, we prove a couple of lemmas regarding subwords of powers of elements in the free group $F(\Sigma)$. These will be of use to us in Chapter 5. The first lemma we prove essentially follows from the Fine and Wilf periodicity Theorem (see [FW65] and [Lot97, Proposition 1.3.5]). The second is a little more specialised.

Let $v, w \in F(\Sigma)$. We say that w is a *subword* of v if v is equal to a freely reduced word $v_1 w v_2 \in F(\Sigma)$. If $v_1 = 1$, we say w is a *prefix*. If $v_2 = 1$, we say w is a *suffix*. We say that $v = v_1 w v_2 = v_1' w v_2'$ are distinct occurrences of w as a subword of v if $|v_1| \neq |v_1'|$.

Lemma 2.2.1. *Let $1 \neq y^{-1} z y \in F(A)$ be a freely reduced word with z cyclically reduced and not equal to a proper power. Then:*

1. *if $y = 1$ and z appears as a subword of z^i, then $i \geqslant 1$ and z is a subword in precisely i different ways,*

2. *if $y \neq 1$ and $y^{-1} z^i y$ occurs as a subword of $y^{-1} z^j y$ for some $i, j \in \mathbb{Z} - \{0\}$, then $i = j$.*

Proof. Let us first suppose that $y = 1$. By comparing prefixes and suffixes, if z appears as a subword of z^2 in three different ways, this implies that $z = z_1 z_2$ and $z = z_2 z_1$. Then by [LS62, Lemma 2], we contradict the assumption that z is not a proper power. If z appears as a subword of z^{-2}, then $z = z_1 z_2$ and $z^{-1} = z_2 z_1$. This implies that $z_1, z_2 = 1$ and so $z = 1$. Thus, z appears as a subword of z^i in i different ways when $i \geqslant 1$ and does not appear as a subword of z^i when $i \leqslant -1$. Now suppose that $y \neq 1$ and that $y^{-1} z^i y$ occurs as a subword of $y^{-1} z^j y$ for some non-zero integers $i < j$. By length considerations, z must occur as a subword of z^2 or z^{-2} in at least three distinct ways and we have completed the proof. $\qquad\square$

Lemma 2.2.2. *Let $zb \in F(A, B)$ be a cyclically reduced word such that z begins and ends with an element in $A \sqcup A^{-1}$ and $b \in \langle B \rangle$. If zb is not a proper power and z appears as a subword of $(zb)^i$, then $i \geqslant 1$ and z appears as a subword in precisely i different ways.*

Proof. By Lemma 2.2.1, we may assume that $b \neq 1$. It suffices to show that z cannot be a subword of $(zbz)^{-1}$ and can only be a subword of zbz in two ways. If z is a subword of $(zbz)^{-1}$, then either $z = z^{-1}$, which is not possible, or there is a copy of z in $z^{-1} b^{-1} z^{-1}$ that overlaps with both copies of z^{-1}. But this would only be possible if z had a non-trivial suffix equal to its own inverse and so z does not appear

as a subword of $(zbz)^{-1}$. Now suppose that z is a subword of zbz in more than two ways. Then:

$$z = z_1 z_2 z_3 \ ,$$
$$z = z_3 b z_1 \ ,$$

for some $z_1, z_2, z_3 \in F(A, B)$. If $|z_1| = |z_3|$, then $z_2 = b$ and we have $zb = z_1 b z_1 b = (z_1 b)^2$, contradicting our assumption. So suppose that $|z_1| < |z_3|$. Note that z_1 and z_3 are cyclically reduced since z was. Since the first letter of z_3 is in $A \sqcup A^{-1}$, we have that $|z_3| > |z_1 z_2|$. Now by [LS62, Lemma 1], we have $z_3 = (z_1 z_2)^i z'$ for some $i \geqslant 1$ and where z' is a proper prefix of $z_1 z_2$. Then we have:

$$z = (z_1 z_2)^{i+1} z' \ ,$$
$$z = (z_1 z_2)^i z' b z_1 \ .$$

By comparing suffixes, we have:

$$z_1 z_2 z' = z' b z_1 \ .$$

But now we obtain equalities of the same form as before, with z' playing the role of z_3. Since $|z'| < |z_3|$, by induction on length, we see that $z_2 = b$ and so we obtain a contradiction as before. $\qquad\square$

2.3   One-relator groups

The study of one-relator groups was initiated in 1930 by Magnus [Mag30] with his proof of the Freiheitssatz.

Definition 2.3.1. A *Magnus subgroup* of a one-relator group $F(\Sigma)/\langle\langle w \rangle\rangle$ is a subgroup generated by a subset $A \subset \Sigma$, such that w is not conjugate into $\langle A \rangle$ within $F(\Sigma)$.

The Freiheitssatz can now be stated as follows.

Theorem 2.3.2. *Let $G = F(\Sigma)/\langle\langle w \rangle\rangle$ be a one-relator group and $A \subset \Sigma$ a subset generating a Magnus subgroup. Then $\langle A \rangle < G$ is a free group, freely generated by A.*

In order to prove the Freiheitssatz, Magnus proved the existence of a hierarchy of one-relator groups. Using the hierarchy, Magnus also proved the following in [Mag32].

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Theorem 2.3.3. *The membership problem is solvable for Magnus subgroups of one-relator groups.*

Corollary 2.3.4. *The word problem is solvable for one-relator groups.*

As will be discussed in Chapter 3, Magnus' hierarchy was refined by Moldavanskii in [Mol67] and by Masters in [Mas06]. A key tool in this thesis is a simplification of these hierarchies using 2-complexes. See Theorem 3.3.2 for the statement. Our main hierarchy results, Theorem 7.1.2 and Theorem 7.2.2, use this hierarchy and induction on hierarchy length as the overarching proof strategy.

2.3.1   Primitivity rank

First introduced in [Pud14, Definition 1.7], the *primitivity rank* of a word $w \in F(\Sigma)$ is the following quantity:

$$\pi(w) = \min \left\{ \mathrm{rk}(K) \mid w \in K < F(\Sigma) \text{ and } w \text{ is imprimitive in } K \right\} \in \mathbb{N} \cup \{\infty\} \, ,$$

with $\pi(w) = \infty$ if w is primitive in $F(\Sigma)$. In a recent series of articles [LW17, LW20, LW21, LW22], the primitivity rank of w has been related with several properties of the subgroup structure of the one-relator group $F(\Sigma)/\langle\langle w \rangle\rangle$. In this, and the following subsection, we state some of these results. The first result we shall need is the following freedom theorem, appearing as [LW22, Theorem 1.5 and Corollary 6.18].

Theorem 2.3.5. *Let $k \geqslant 2$ be an integer and $w \in F(\Sigma)$. The following are equivalent:*

- *$F(\Sigma)/\langle\langle w \rangle\rangle$ is k-free,*

- *$\pi(w) = k + 1$.*

Note that being 1-free is equivalent to being torsion-free, so Theorem 2.3.5 also recovers the main result of [KMS60].

In [Bau74], Baumslag conjectured that all one-relator groups are coherent: that is, that every finitely generated subgroup of a one-relator group is also finitely presented. This conjecture has motivated several important advances in the theory of one-relator groups. For instance, Wise and, independently, Louder and Wilton showed that one-relator groups with torsion are coherent in [Wis22] and [LW20] respectively. Moreover, Louder and Wilton showed in [LW22] that if $w \in F(\Sigma)$ with $\pi(w) \geqslant 3$, then $G = F(\Sigma)/\langle\langle w \rangle\rangle$ is coherent. In the same article, they also prove several strong results about the subgroups of G. We collect below the statements from [LW22, Theorem A and Theorem B] that we make use of.

Theorem 2.3.6. *Let $w \in F(\Sigma)$ such that $\pi(w) \geqslant 3$ and $G = F(\Sigma)/\langle\langle w \rangle\rangle$. Then the following hold:*

1. *G is coherent.*

2. *Every finitely generated non-cyclic freely indecomposable subgroup of G is not isomorphic to any proper subgroup of itself.*

3. *For any natural number n, there are only finitely many conjugacy classes of non-cyclic subgroups $H < G$ with abelianised rank at most n.*

Theorem 2.3.6 is an essential ingredient in the proof of Theorem 6.3.3, and hence in our resolution of [LW22, Conjecture 1.9]. The bound on the number of conjugacy classes of subgroups of fixed rank will allow us to make strong conclusions about the action of such a one-relator group on its Bass-Serre tree.

2.3.2 w-subgroups

If $w \in F(\Sigma)$ is a word with $\pi(w) < \infty$, a *w-subgroup* is a finitely generated subgroup $K < F(\Sigma)$, satisfying the following:

1. $w \in K$ and w is not a primitive element of K,

2. $\mathrm{rk}(K) = \pi(w)$,

3. If $K' < F(\Sigma)$ is a subgroup properly containing K, then $\mathrm{rk}(K') > \mathrm{rk}(K)$.

By [LW22, Lemma 6.4], for any fixed w, there are only finitely many conjugacy classes of w-subgroups. Moreover, a complete list of these subgroups, up to conjugacy, is computable. By [LW22, Theorem 6.17], if $\pi(w) < \infty$ and $K < F(\Sigma)$ is a w-subgroup, then the natural map

$$K/\langle\langle w \rangle\rangle \to F(\Sigma)/\langle\langle w \rangle\rangle$$

is an injective homomorphism. This allows us to treat $P = K/\langle\langle w \rangle\rangle$ as a subgroup of $G = F(\Sigma)/\langle\langle w \rangle\rangle$. In a slight abuse of notation, we will also call P a *w-subgroup* of G. The following result connects w-subgroups with the subgroup structure of $F(\Sigma)/\langle\langle w \rangle\rangle$. It appears as [LW22, Theorem 1.5 and Corollary 1.10].

Theorem 2.3.7. *Let $w \in F(\Sigma)$ and $G = F(\Sigma)/\langle\langle w \rangle\rangle$. Then G has finitely many conjugacy classes of w-subgroups $P_1, ..., P_n < G$ and, for every subgroup $H < G$ of rank $\pi(w)$, one of the following hold:*

1. *H is free,*

2. *H is conjugate into some P_i.*

Moreover, if $\pi(w) = 2$, then there exists only one conjugacy class of w-subgroup $P < G$.

We prove the existence of a one-relator hierarchy relative to any w-subgroup in Chapter 3, see Theorem 3.3.16. This allows us to prove that the membership problem is decidable for w-subgroups (see Corollary 3.3.17).

2.4   One-relator complexes

The principal objects of study in this thesis will be one-relator complexes.

Definition 2.4.1. A *one-relator complex* is a combinatorial 2-complex of the form $X = (\Gamma, \lambda)$ where $\lambda : S^1 \looparrowright \Gamma$ is an immersion of a single cycle. We denote by $X_\lambda \subset X$ the smallest one-relator subcomplex of X.

The first example of a one-relator complex is the presentation complex of a one-relator group. As we shall see in Chapter 3, more general one-relator complexes arise naturally.

2.4.1   Magnus subcomplexes

A *Magnus subcomplex* $A \subset X = (\Gamma, \lambda)$ of a one-relator complex is a connected subgraph of Γ in which the attaching map λ is not supported. This is the topological analogue of Magnus subgroups of one-relator groups. The classic Freiheitssatz may be restated as follows for one-relator complexes.

Theorem 2.4.2. *Let $X = (\Gamma, \lambda)$ be a one-relator complex. If $A \subset \Gamma$ is a Magnus subcomplex, then the induced map $\pi_1(A) \to \pi_1(X)$ is injective.*

In the Magnus–Moldavanskii hierarchy, a one-relator group is decomposed as an HNN-extension over another one-relator group with shorter relator length, after possibly passing to a one-relator overgroup. At each step in this hierarchy, a pair of Magnus subgroups are identified. In our version of the hierarchy, we will instead be identifying subgroups that correspond to fundamental groups of Magnus subcomplexes of a one-relator complex.

Let X be a one-relator complex and let $A, B \subset X$ be a pair of Magnus subcomplexes. If $A \cap B$ is connected, we may obtain a one-relator presentation for $\pi_1(X)$ in which both $\pi_1(A)$ and $\pi_1(B)$ are Magnus subgroups. If $A \cap B$ is not connected, this may no longer be true. However, we have the following lemma.

Lemma 2.4.3. *Let $X = (\Gamma, \lambda)$ be a one-relator complex and let $A, B \subset X$ be Magnus subcomplexes. Then $\pi_1(A)$ and $\pi_1(B)$ are free factors of a pair of Magnus subgroups for some one-relator presentation of $\pi_1(X) * F$, where F is some free group.*

Proof. If $A \cap B$ is connected, there is nothing to prove. So suppose that $A \cap B$ is not connected. We may add edges $\{e_1, e_2, ...\} = E$ to Γ so that $(A \cap B) \cup E$ is connected. Let Z be the resulting one-relator complex. Now $A \cup E, B \cup E \subset Z$ are Magnus subcomplexes with connected intersection. Since $\pi_1(Z) = \pi_1(X) * F$ for some free group F, the result follows. $\qquad\square$

2.4.2 Non-positive and negative immersions

The non-positive immersion property for 2-complexes was first introduced by Wise in [Wis03]: a finite 2-complex X is said to have *non-positive immersions* if for every immersion of finite connected 2-complexes $Z \looparrowright X$, either $\chi(Z) \leqslant 0$, or Z is contractible.

The connection with one-relator complexes is provided by the w-cycles conjecture, stated in [Wis03]. This conjecture was resolved independently by Helfer and Wise in [HW16] and by Louder and Wilton in [LW17]. A direct consequence was a characterisation of one-relator complexes with non-positive immersions.

Theorem 2.4.4. *Let $X = (\Gamma, \lambda)$ be a one-relator complex. Then X has non-positive immersions if and only if $\pi_1(X)$ is torsion-free. Equivalently, X has non-positive immersions if and only if λ is primitive.*

Wise later introduced a stronger property in [Wis04], called negative immersions. In [Wis20], Wise proved several strong results for 2-complexes with negative immersions such as coherence and 2-freeness.

The version of negative immersions that we shall be using is due to Louder and Wilton [LW22]: a finite 2-complex X is said to have *negative immersions* if for every immersion of finite connected 2-complexes $Z \looparrowright X$, either $\chi(Z) < 0$, or Z is reducible. We will not define what a reducible complex is as we shall not need it. The interested reader is referred to [LW22, Definition 3.5]. The important property to keep in mind is that if Z is reducible, then Z is homotopy equivalent to a graph. The following characterisation of one-relator complexes with negative immersions is proved in [LW22, Theorem 1.3].

Theorem 2.4.5. *Let X be a one-relator complex. Then X has negative immersions if and only if $\pi_1(X)$ is 2-free.*

By Theorems 2.4.4 and 2.4.5, it makes sense to talk about one-relator groups with non-positive immersions or negative immersions.

If $X = (\Gamma, \lambda)$ is a one-relator complex, then every conjugacy class of w-subgroup in $\pi_1(X)$ can be realised by an immersion:

$$Q \hookrightarrow X$$

of one-relator complexes (see [LW22, Section 6.1]). By Theorems 2.3.7 and 2.4.5, if X has non-positive immersions, but does not have negative immersions, then there exists precisely one such immersion.

2.5 Strongly inert subgroups

A subgroup $H < G$ is called *inert* if for every subgroup $K < G$, we have:

$$\mathrm{rk}(H \cap K) \leqslant \mathrm{rk}(K) .$$

This definition was first introduced in [DV96], motivated by the study of fixed subgroups of endomorphisms of free groups. More generally, as defined in the introduction of [Iva18], we say that H is *strongly inert* if for every subgroup $K < G$, we have:

$$\sum_{[H \cap K^g]} \mathrm{rr}(H \cap K^g) \leqslant \mathrm{rr}(K) .$$

Ivanov shows in [Iva18, Theorem 1.2] that, given as input a collection of elements of a free group, it is decidable whether the subgroup they generates is strongly inert. Examples of strongly inert subgroups of free groups are:

1. subgroups of rank at most two [Tar92, Theorem 1],

2. subgroups that are the fixed subgroups of an injective endomorphism [DV96, Theorem IV.5.5],

3. subgroups that are images of immersions of free groups, as defined in [Kap00, Definition 3.1].

The latter example follows by observing that the fibre product of two rose graphs is again a disjoint union of a rose graph and some cycles. It is unknown whether inert subgroups of free groups are strongly inert or whether there exists an algorithm to decide if a subgroup of a free group is inert.

The following lemma will produce more examples of strongly inert subgroups of free groups.

Lemma 2.5.1. *Let F be a free group and $A < F$ a subgroup. If $A * B < F * B$ is inert for all free groups B, then A is strongly inert.*

Proof. Assuming that A is finitely generated, let us represent $A < F$ by an immersion of graphs $\gamma : \Gamma \hookrightarrow \Delta$. Then A is strongly inert if and only if for every graph immersion $\lambda : \Lambda \hookrightarrow \Delta$, we have $\chi(\text{Core}(\Gamma \times_\Delta \Lambda)) \geq \chi(\text{Core}(\Lambda))$. So suppose for a contradiction that there exists a graph immersion $\lambda : \Lambda \hookrightarrow \Delta$ such that:

$$\chi(\text{Core}(\Gamma \times_\Delta \Lambda)) < \chi(\text{Core}(\Lambda)) \ .$$

Let $\Theta_1, ..., \Theta_k \subset \Gamma \times_\Delta \Lambda$ be the distinct connected core subgraphs such that $\chi(\Theta_i) \leq -1$ for all i and $\sum_{i=1}^k \chi(\Theta_i) = \chi(\text{Core}(\Gamma \times_\Delta \Lambda))$. Then each Θ_i has a vertex $(v_i, w_i) \in V(\Gamma) \times V(\Lambda)$ such that $\deg(v_i, w_i) \geq 3$. Let Δ' be the graph obtained from Δ by attaching edges $e_1, ..., e_{k-1}$ along their endpoints to the basepoint of Δ. Then let Γ' be the graph obtained from Γ by attaching edges $f_1, ..., f_{k-1}$ connecting the basepoint of Γ with itself. Denote by $\gamma' : \Gamma' \hookrightarrow \Delta$ the graph immersion obtained by extending γ, mapping f_i to e_i for each i. By assumption, $\gamma'_* \pi_1(\Gamma')$ is inert in $\pi_1(\Delta')$.

Now let $g_1, ..., g_{k-1} : I \hookrightarrow \Gamma$ be paths such that for each i, g_i begins at v_i and ends at v_{i+1}, traversing f_i. Let Λ' be the graph obtained from Λ by attaching segments connecting w_i with w_{i+1} for each $i < k$. Let $\lambda' : \Lambda' \to \Delta$ be the map obtained by extending λ, mapping each extra segment to the path $\gamma \circ g_i$. Finally, let $\lambda'' : \Lambda'' \hookrightarrow \Delta$ be the graph immersion obtained from λ' by folding. By construction, the folding map $\Lambda' \to \Lambda''$ is a homotopy equivalence. We have $\chi(\Lambda'') = \chi(\Lambda) - k + 1$, and so:

$$\chi(\text{Core}(\Gamma' \times_{\Delta'} \Lambda'')) = \chi(\text{Core}(\Gamma' \times_{\Delta'} \Lambda)) - k + 1$$
$$< \chi(\text{Core}(\Lambda'')) \ .$$

However, $\text{Core}(\Gamma' \times_\Delta \Lambda'')$ only has one component with non-zero Euler characteristic by construction, contradicting the fact that $\gamma'_* \pi_1(\Gamma')$ is inert in $\pi_1(\Delta')$. $\qquad\square$

Let F be a finitely generated free group and $A < F$ a subgroup. We say A is an *echelon subgroup* if there exists a free basis $x_1, x_2, ..., x_n$ for F, such that $\text{rk}(A \cap \langle x_1, ..., x_k \rangle) - \text{rk}(A \cap \langle x_1, ..., x_{k-1} \rangle) \leq 1$ for all $1 \leq i \leq n$, where x_0 is understood to be 1.

Echelon subgroups of free groups were introduced in [Ros13, Definition 3.2] and were shown to be inert in [Ros13, Theorem 3.2]. As a corollary of Lemma 2.5.1, we establish that they are also strongly inert.

Corollary 2.5.2. *Echelon subgroups of free groups are strongly inert.*

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2.6 Graphs of spaces

A *graph of spaces* is a tuple:

$$\mathcal{X} = \left(\Gamma, \{X_v\}_{v \in V(\Gamma)}, \{X_e\}_{e \in E(\Gamma)}, \{\partial_e^\pm\}_{e \in E(\Gamma)}\right) \ ,$$

where Γ is a connected graph called the *underlying graph*, X_v are CW-complexes called the *vertex spaces*, X_e are CW-complexes called the *edge spaces*, and

$$\partial_e^- : X_e \to X_{o(e)} \ ,$$
$$\partial_e^+ : X_e \to X_{t(e)}$$

are π_1 injective continuous maps called the *edge maps*. The *geometric realisation* of $\mathcal{X}$ is the following space:

$$X_\mathcal{X} = \left(\bigsqcup_{v \in V(\Gamma)} X_v \sqcup \bigsqcup_{e \in E(\Gamma)} (X_e \times [-1,1])\right) \Big/ (x, \pm 1) \sim \partial_e^\pm(x) \ .$$

This space has an induced CW-complex structure. We will say a cell $c \subset X_\mathcal{X}$ is *horizontal* if its attaching map is supported in a vertex space, *vertical* otherwise.

There is a natural *vertical map*

$$v : X_\mathcal{X} \to \Gamma \ ,$$

where X_v maps to v and $X_e \times (-1,1)$ maps to the open edge e in the obvious way. The following fact about the vertical map v is well known, see [Ser03].

Lemma 2.6.1. *The map:*
$$v_* : \pi_1(X_\mathcal{X}) \to \pi_1(\Gamma)$$

is surjective.

Define $H_\mathcal{X}$ to be the space obtained from $X_\mathcal{X}$ by identifying $\{x\} \times [-1,1]$ to a point for each $e \in E(\Gamma)$ and each $x \in X_e$. Then we call the quotient map

$$h : X_\mathcal{X} \to H_\mathcal{X} \ ,$$

the *horizontal map*. A path $p : I \to X_\mathcal{X}$ is a *vertical path* if $h \circ p$ is the constant path. It is a *horizontal* path if $v \circ p$ is the constant path.

In general, not much can be said about the horizontal map. However, with sufficient restrictions on the edge maps, we can show that h is a homotopy equivalence.

Proposition 2.6.2. *Let $\mathcal{X} = (\Gamma, \{X_v\}, \{X_e\}, \{\partial_e^{\pm}\})$ be a graph of spaces. Suppose that $\partial_e^{\pm}$ are given by inclusions of subcomplexes. Then the following hold:*

1. *$h : X_{\mathcal{X}} \to H_{\mathcal{X}}$ is a homotopy equivalence if and only if $X_{\mathcal{X}}^{(1)}$ does not support any vertical loops,*

2. *if h is a homotopy equivalence, then $H_{\mathcal{X}}$ has a CW-structure inherited from $X_{\mathcal{X}}$ and $h \mid X_v$ is an immersion for all $v \in V(\Gamma)$.*

Proof. Let c be a horizontal 0-cell. Then $h^{-1}(h(c))$ is a graph we denote by Γ_c. We have that Γ_c is a tree for all horizontal 0-cells c if and only if $X_{\mathcal{X}}^{(1)}$ supports no vertical loops. So first suppose that Γ_c is not a tree for some 0-cell c. Let $p : I \to \Gamma_c \to X_{\mathcal{X}}$ be a non-trivial loop. Then $v \circ p$ is a non-trivial loop in the graph Γ. Thus, by Lemma 2.6.1, $[p] \in \pi_1(X_{\mathcal{X}})$ is a non-trivial element. But then the map h_* sends $[p]$ to a trivial element and so h cannot be a homotopy equivalence. So now suppose that Γ_c is a tree for all 0-cells c.

Since each edge map $\partial_e^{\pm}$ is an inclusion of subcomplexes, if c is any horizontal cell, then $h^{-1}(h(c))$ has a product decomposition $c \times \Gamma_c$ where Γ_c is a graph immersing into Γ. Denote by $\gamma_c : \Gamma_c \looparrowright \Gamma$ this immersion. If the image of the attaching map of the cell c has non-empty intersection with an open cell d, then Γ_c immerses into Γ_d. Hence, since Γ_c is a tree for all 0-cells c, this implies that Γ_c must be a tree for all higher dimensional cells c.

Denote by $h_0 : X_{\mathcal{X}} \to X_0$ the map obtained by collapsing each component of $\bigcup_{c \in X_{\mathcal{X}}^{(0)}} c \times \Gamma_c$ to a point. Then we may inductively define the maps $h_i : X_{i-1} \to X_i$ by collapsing each $c \times \Gamma_c$ to c for each horizontal i-cell c. By definition, h factors through $h_i \circ ... \circ h_1 \circ h_0$ for all i. If $X_{\mathcal{X}}$ is finite dimensional, then for some n we have $h_n \circ ... \circ h_1 \circ h_0 = h$. If $X_{\mathcal{X}}$ is infinite dimensional, we may take the direct limit and we see that it coincides with h.

By the following lemma, each h_i is a homotopy equivalence and thus, so is h. This proves (1).

Lemma 2.6.3. *Let X be a CW-complex and let $B \subset A \subset X$ be subcomplexes. Suppose there exists a deformation retraction $f_t : A \to A$ such that $f_1(A) = B$. Then*

$$q : X \to X/(A \sim f_1(A))$$

is a homotopy equivalence.

Proof. Exactly the same proof as [Hat02, Proposition 0.17]. $\qquad\square$

By construction of the maps h_i, we see that $H_{\mathcal{X}}$ has a CW-structure. By induction on dimension, we can also see that if Γ_c is a tree for all horizontal cells c, then X_v immerses into $H_{\mathcal{X}}$ for all $v \in V(\Gamma)$. $\qquad\qquad\square$

If $\mathcal{X}$ is a graph of spaces, then the universal cover $\tilde{X}_{\mathcal{X}} \to X_{\mathcal{X}}$ also has a graph of spaces structure where each vertex space is the universal cover of some vertex space of $\mathcal{X}$ and each edge space is the universal cover of some edge space of $\mathcal{X}$. We will denote this by

$$\tilde{\mathcal{X}} = \left(T, \left\{ \tilde{X}_v \right\}, \left\{ \tilde{X}_e \right\}, \left\{ \tilde{\partial}_e^{\pm} \right\} \right) ,$$

where T is the Bass-Serre tree of the graph of groups $\pi_1(X_{\mathcal{X}})$. There is a natural covering action of $\pi_1(X_{\mathcal{X}})$ on $\tilde{X}_{\mathcal{X}}$ which pushes forward to an action on T. Indeed, we have the following $\pi_1(X_{\mathcal{X}})$-equivariant commuting diagram:

$$
\begin{array}{ccc}
\tilde{X}_{\mathcal{X}} = X_{\tilde{\mathcal{X}}} & \longrightarrow & X_{\mathcal{X}} \\
\downarrow & & \downarrow \\
T & \longrightarrow & \Gamma
\end{array}
$$

If $\mathcal{X}$ satisfies the hypothesis of Proposition 2.6.2, then we also have a $\pi_1(X_{\mathcal{X}})$-equivariant commuting diagram:

$$
\begin{array}{ccc}
X_{\tilde{\mathcal{X}}} & \longrightarrow & H_{\tilde{\mathcal{X}}} \\
\downarrow & \searrow & \searrow \\
T & X_{\mathcal{X}} & \longrightarrow H_{\mathcal{X}} \\
& \searrow & \downarrow \\
& & \Gamma
\end{array}
$$

where $H_{\tilde{\mathcal{X}}} = \tilde{H}_{\mathcal{X}}$.

2.6.1 Hyperbolic graphs of spaces

Acylindrical actions were first defined by Sela in [Sel97].

Definition 2.6.4. Let G be a group acting on a tree T and let $k \geqslant 0$. We say the action is *k-acylindrical* if every finite segment in T of length at least k has finite stabiliser. We will say the action is *acylindrical* if it is k-acylindrical for some $k \geqslant 0$.

By putting constraints on the geometry of the vertex and edge spaces of a graph of spaces $\mathcal{X}$, we may deduce acylindricity of the action of $\pi_1(X_{\mathcal{X}})$ on its

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Bass-Serre tree. The following is essentially well-known, we include a proof for completeness.

Proposition 2.6.5. *Let $\mathcal{X} = (\Gamma, \{X_v\}, \{X_e\}, \{\partial_e^\pm\})$ be a finite graph of finite spaces such that $\tilde{X}_{\mathcal{X}}^{(1)}$ is hyperbolic with the path metric and $\tilde{X}_e^{(1)} \to \tilde{X}_{\mathcal{X}}^{(1)}$ is a quasi-isometric embedding for each $e \in E(\Gamma)$. Then $\pi_1(X_{\mathcal{X}})$ acts acylindrically on its Bass-Serre tree T.*

Proof. Suppose not. Then there exist segments $S_i \subset T$ of length i for all $i \in \mathbb{N}$, such that $\mathrm{Stab}(S_i)$ is infinite. Let δ be the hyperbolicity constant for $\tilde{X}_{\mathcal{X}}$. Since $\tilde{X}_e$ is quasi-isometrically embedded in $\tilde{X}_{\mathcal{X}}$ for each $e \in E(\Gamma)$, there exists a constant $M > 0$ with the following property: if $I \to \tilde{X}_e$ is a geodesic, then any geodesic $I \to \tilde{X}_{\mathcal{X}}$ with the same endpoints is contained in its M-neighbourhood and vice versa. This is known as the Morse property, see [BH99, Theorem III.H.1.7]. Now let $n > 2\delta + M$ be an integer. Let $g \in \mathrm{Stab}(S_{2n})$ be an element of infinite order. Since $\pi_1(X_{\mathcal{X}})$ is hyperbolic, such an element exists by [GdlH90, Corollary 36]. Denote by $\tilde{X}_e, \tilde{X}_f \to \tilde{X}_{\mathcal{X}}$ the two translates of edge spaces associated with the endpoints of S_{2n}. Let $\gamma : I \to \tilde{X}_{\mathcal{X}}$ be a shortest path connecting $\tilde{X}_e$ with $\tilde{X}_f$. We have $|\gamma| \geqslant 2n$. Let $\alpha_i : I \to \tilde{X}_e$ and $\beta_i : I \to \tilde{X}_f$ be geodesics such that $t(\alpha_i) = g^i \cdot o(\alpha_i) = g^i \cdot o(\gamma)$ and $t(\beta_i) = g^i \cdot o(\beta_i) = g^i \cdot t(\gamma)$. Let k be large enough so that $d(o(\beta_k), t(\beta_k)) > |\gamma| + 2\delta$. By construction, we have that $g^i \cdot \gamma$ is a geodesic connecting the endpoints of α_i and β_i. Now consider the quadrangle $\gamma \cup \beta_k \cup g^k \cdot \gamma \cup \alpha_k$. The midpoint of γ is either at distance at most 2δ from $g^k \cdot \gamma$ or at distance at most $2\delta + M$ from α_k or β_k. In the first case, we get a contradiction by our choice of k. In the second case, we get a contradiction by our choice of n. Thus, $\pi_1(X_{\mathcal{X}})$ must act acylindrically on T. $\square$

Proposition 2.6.5 can be thought of as a converse to [Kap01, Theorem 1.2]. Putting the two together, we obtain the following statement.

Theorem 2.6.6. *Let $\mathcal{X} = (\Gamma, \{X_v\}, \{X_e\}, \{\partial_e^\pm\})$ be a finite graph of finite spaces where $\tilde{X}_v^{(1)}$ is hyperbolic with the path metric for all $v \in V(\Gamma)$ and $\partial_e^\pm$ is a quasi-isometric embedding for all $e \in E(\Gamma)$. Then $\pi_1(X_{\mathcal{X}})$ acts acylindrically on the Bass-Serre tree T, if and only if $\tilde{X}_{\mathcal{X}}^{(1)}$ is hyperbolic with the path metric and $\tilde{X}_v^{(1)} \to \tilde{X}_{\mathcal{X}}^{(1)}$ is a quasi-isometric embeddings for all $v \in V(\Gamma)$.*

Chapter 3

One-relator hierarchies

This chapter is dedicated to refining the Magnus–Moldavanskii hierarchy using 2-complexes. First conceived by Magnus in his thesis [Mag30], the Magnus hierarchy is possibly the oldest general tool in the theory of one-relator groups. After the introduction of HNN-extensions, the hierarchy was later refined to be called the Magnus–Moldavanskii hierarchy [Mol67]: if G is a one-relator group, then there is a diagram of monomorphisms of one-relator groups

$$
\begin{array}{ccc}
G = G_0 & \hookrightarrow & G'_0 \\
& \nearrow & \\
G_1 & \hookrightarrow & G'_1 \\
& \nearrow & \\
\cdots & \hookrightarrow & \cdots \\
& \nearrow & \\
G_N & &
\end{array}
$$

such that $G'_i \cong G_{i+1} *_{\psi_i}$ where ψ_i identifies two Magnus subgroups of G_{i+1}, and G_N splits as a free product of cyclic groups. The proofs of many results for one-relator groups then proceed by induction on the length of such a hierarchy. See [MKS66] for a classical introduction to one-relator groups with many such examples.

In the preprint [Mas06], Masters showed that we can dispense with the horizontal homomorphisms. In other words, if G is a one-relator group, there is a sequence of monomorphisms of one-relator groups:

$$
G_N \hookrightarrow ... \hookrightarrow G_1 \hookrightarrow G_0 = G
$$

such that $G_i \cong G_{i+1} *_{\psi_i}$ where ψ_i identifies two Magnus subgroups of G_{i+1}, and G_N

splits as a free product of cyclic groups. However, the one-relator presentation of G_{i+1} is not easily obtainable from the one-relator presentation of G_i.

The versatility of the Magnus–Moldavanskii hierachy comes from the fact that it may be described very explicitly in terms of one-relator presentations. Masters' hierarchy is conceptually simpler, but is not so explicit. By working with 2-complexes, we may reconcile both of these advantages. Our version of the hierarchy can be stated as follows.

Theorem 3.3.2. *Let X be a one-relator complex. There exists a finite sequence of immersions of one-relator complexes:*

$$X_N \looparrowright \ldots \looparrowright X_1 \looparrowright X_0 = X \ ,$$

such that $\pi_1(X_i) \cong \pi_1(X_{i+1}) *_{\psi_i}$ where ψ_i is induced by an identification of Magnus subcomplexes, and such that $\pi_1(X_N)$ splits as a free product of cyclic groups.

3.1   Tree domains

Let X be a CW-complex and let F be a free group. An F-*cover* is a regular cover $p : Y \to X$ such that $\mathrm{Deck}(p) \cong F$. A *free cover* is an F-cover for some free group F. We now introduce tree domains. These are subcomplexes of Y that allow us to construct a homotopy equivalence between X and a graph of spaces. Thus, tree domains will correspond to (multiple) HNN-splittings of $\pi_1(X)$, forming the basis for our one-relator hierarchies.

Let $p : Y \to X$ be a free cover, $F = \mathrm{Deck}(p)$ and $S \subset F$ a free generating set. A subcomplex $D \subset Y$ is *S-connected* if, for every open cell $c \subset D$ and every freely reduced word $s_1 \ldots s_n \in (S \sqcup S^{-1})^*$ such that $s_1 \ldots s_n \cdot c \subset D$, we also have that $s_1 \ldots s_i \cdot c \subset D$ for all $i < n$.

Definition 3.1.1. Let $p : Y \to X$ be a free cover, $F = \mathrm{Deck}(p)$ and $S \subset F$ a free generating set. A subcomplex $D \subset Y$ is an *S-tree domain* if the following hold:

1. $F \cdot D = Y$,

2. D is S-connected,

3. $D \cap s \cdot D$ is connected and non-empty for all $s \in S$.

We will denote by $\mathrm{TD}(p, S)$ the set of all S-tree domains. A *minimal S-tree domain* is an S-tree domain, minimal under the partial order of inclusion.

When the generating set is clear from context, we will often simply write $\mathrm{TD}(p)$ and refer to $D \in \mathrm{TD}(p)$ as a tree domain. For example, this is always clear when $F = \mathbb{Z}$.

Example 3.1.2. Let S_g be the orientable surface of genus $g \geqslant 1$ and $\gamma : S^1 \to S_g$ a non-separating simple closed curve. We may give S_g a CW-structure so that γ is in the 1-skeleton. The curve γ determines an epimorphism $\pi_1(S_g) \to \mathbb{Z}$ via the intersection form. Let $p : Y \to S_g$ be the induced cyclic cover. Consider the subspace $Z \subset Y$ obtained by taking the closure of some component of $p^{-1}(S_g - \mathrm{Im}(\gamma))$. Its translates cover Y and intersect in lifts of γ. Hence, Z is a (minimal) tree domain for p.

Now let $p : Y \to X$ be a free cover, $S \subset F \cong \mathrm{Deck}(p)$ a free generating set and $D \subset Y$ an S-tree domain. Consider the following families of spaces:

$$\{D_f\}_{f \in F} \ ,$$

$$\{D_{f,s}\}_{f \in F, s \in S} \ ,$$

where $D_f \cong D$ and $D_{f,s} \cong D \cap s \cdot D$. There are natural inclusion maps:

$$\iota_{f,s}^- : D_{f,s} \hookrightarrow D_f$$

and

$$\iota_{f,s}^+ : D_{f,s} \hookrightarrow D_{fs} \ ,$$

given by the inclusions $D \cap s \cdot D \hookrightarrow D$ and $D \cap s \cdot D \hookrightarrow s \cdot D$. With this data we define the space:

$$\Theta(p, S, D) = \left(\bigsqcup_{f \in F} D_f \right) \sqcup \left(\bigsqcup_{f \in F, s \in S} D_{f,s} \times [-1, 1] \right) \Big/ (x, \pm 1) \sim \iota_{f,s}^{\pm}(x) \ .$$

We also have an action of F on $\Theta(p, S, D)$ induced by the action on $\{D_f\}_{f \in F}$.

Lemma 3.1.3. *If the map $\pi_1(D \cap s \cdot D) \to \pi_1(D)$ is injective for all $s \in S$, then $\Theta(p, S, D)$ is a graph of spaces. Moreover, in this case, the horizontal map is an F-equivariant homotopy equivalence to Y:*

$$h : \Theta(p, S, D) \to Y \ ,$$

and the vertical map is an F-equivariant map to the Cayley graph of F with generating

set S:

$$v : \Theta(p, S, D) \to \mathrm{Cay}(F, S) \ .$$

Proof. The fact that the horizontal map is an F-equivariant map to Y is by construction. Similarly for the vertical map. By Proposition 2.6.2, the map $h : \Theta(p, S, D) \to Y$ is a homotopy equivalence if $\mathrm{Cay}(F, S)$ is a tree. Since S is a free basis, the result now follows. $\qquad\qquad\square$

Proposition 3.1.4. *Let* $p : Y \to X$ *be a free cover and let* $D \subset Y$ *be an S-tree domain for some free generating set* $S \subset F = \mathrm{Deck}(p)$. *If* $\Theta(p, S, D)$ *is a graph of spaces, then*

$$\pi_1(X) \cong \pi_1(F\backslash\Theta(p, S, D)) \ .$$

In particular, $\pi_1(X)$ *splits as a multiple HNN-extension with vertex group* $\pi_1(D)$ *and edge groups* $\pi_1(D \cap s \cdot D)$ *for each* $s \in S$.

Proof. Since $\mathrm{Cay}(F, S)$ is a tree and $\Theta(p, S, D)$ is a graph of spaces, it follows from Proposition 2.6.2 and Lemma 3.1.3 that the map $h : \Theta(p, S, D) \to Y$ is a homotopy equivalence. By equivariance, this descends to a homotopy equivalence $F\backslash\Theta(p, S, D) \to X$. The space $F\backslash\Theta(p, S, D)$ is a graph of spaces over $F\backslash\mathrm{Cay}(F, S)$, which is a rose on $|S|$ petals. Thus, $\pi_1(F(S)\backslash\Theta(p, S, D))$ splits as a multiple HNN-extension over $\pi_1(D)$ with edge groups $\pi_1(D \cap s \cdot D)$ for each $s \in S$. $\qquad\square$

3.2   Minimal cyclic tree domains

3.2.1   Cyclic covers

From now on, we will only focus on cyclic covers. That is, free covers whose deck group is $\mathbb{Z}$. In this section we will show that tree domains always exist for cyclic covers of finite CW-complexes.

Let us first set up some notation. Let $p : Y \to X$ be a cyclic cover. Choose some spanning tree $T \subset X^{(1)}$ and some orientation on $X^{(1)}$, this induces an identification of $\pi_1(X^{(1)})$ with $F(\Sigma)$, the free group on

$$\Sigma = \{e\}_{e \in E(X^{(1)})\backslash E(T)} \ .$$

For any subset $A \subset \Sigma$, define:

$$T^A = T \cup \left(\bigcup_{e \in A} e \right) \ .$$

Choose some lift of T to Y and denote this by $T_0 \subset Y$. Then, since p is regular, every lift of T is obtained by translating T_0 by an element of $\mathbb{Z}$. We introduce the notation:

$$T_{i+j} = i \cdot T_j .$$

For each $e \in E\left(X^{(1)}\right)$, denote by e_i the lift of e such that $o(e_i) \in T_i$. If $e \notin \Sigma$, then $t(e_i) \in T_i$. If $e \in \Sigma$, then $t(e_i) \in T_{i+\iota(e)}$ where

$$\iota : \Sigma \to \pi_1\left(X^{(1)}, x\right) \to \mathbb{Z}$$

is the induced map.

If $e \in \Sigma$ and $C \subset \mathbb{Z}$, define $C_e \subset C$ to be the subset consisting of the elements $i \in C$ such that $i + \iota(e) \in C$. Then, if $A \subset \Sigma$, we define the following subcomplex of the 1-skeleton of Y:

$$T_C^A = \left(\bigcup_{i \in C} T_i\right) \cup \left(\bigcup_{e \in A} \bigcup_{i \in C_e} e_i\right) .$$

Examples of such subcomplexes can be seen in Figures 3.1 and 3.2. We also write

$$K_A = \langle \iota(A) \rangle < \mathbb{Z} ,$$
$$k_A = [\mathbb{Z} : K_A] .$$

Remark 3.2.1. The quantity k_A is precisely the number of connected components of $T_\mathbb{Z}^A$. In fact, if we contract all the lifts of T in $Y^{(1)}$, then the resulting graph is the Cayley graph for $\mathbb{Z}$ over the generating set $\iota(\Sigma)$. So the connected components of $T_\mathbb{Z}^A$ correspond to K_A cosets in $\mathbb{Z}$.

We call a subset $C \subset \mathbb{Z}$ *connected* if $C = \{i, i+1, ..., j\}$ for some $i \leqslant j \in \mathbb{Z}$.

Proposition 3.2.2. *Let $p : Y \to X$ be a cyclic cover such that $|\iota(\Sigma)| < \infty$. For any subset $A \subset \Sigma$ and any connected subset $C \subset \mathbb{Z}$, the following hold:*

1. *If $k_A = \infty$, then T_C^A consists of $|C|$ connected components.*

2. *If $k_A < \infty$, then there exists some constant $k = k(A) \geqslant 0$ such that T_C^A consists of k_A connected components whenever $|C| \geqslant k$.*

Proof. For (1), note that we have $K_A = \{0\}$. Hence, each edge e_j with $e \in A$ has both endpoints in T_j. So for each $j \in C$, the subcomplexes $T_j^A \subset T_C^A$ are all pairwise disjoint and cover T_C^A.

By assumption $|\iota(A)|$ is bounded above. So for (2), we use induction on $|\iota(A)|$. Denote by $A_i = \iota^{-1}(i) \cap A$. For the base case, suppose $\iota(A) = \{i\}$. We show the

result holds for all C with $|C| \geqslant |i| = k_A$. For each $e \in A_i$, we have $o(e_j) \in T_j$ and $t(e_j) = o(e_{j+i}) \in T_{j+i}$ by definition. Hence, T_C^A consists of k_A connected components when $|C| \geqslant k_A + |i|$.

Now let $|\iota(A)| \geqslant 2$, by induction we may assume the result holds for $A' \subsetneq A$ with $\iota(A') = \iota(A) \backslash \{i\}$. So for A' there exists some $k' = k(A') \geqslant 0$ such that $T_C^{A'}$ consists of $k_{A'} \geqslant k_A$ connected components for all $C \subset \mathbb{Z}$ with $|C| \geqslant k'$. If $k_{A'} = k_A$ then the result is clear and we may take $k = k'$, so assume $k_{A'} > k_A$. We claim that if

$$|C| \geqslant k = \max \left\{ k', (k_{A'} - k_A) + |i| \right\},$$

then T_C^A consists of k_A connected components. By Remark 3.2.1, we note that for each $j, j + i \in C$, we must have T_j and T_{j+i} are in distinct components of $T_C^{A'}$. For any $e \in A_i$, by adding e_j edges to $T_C^{A'}$ with both endpoints in $T_C^{A'}$, we decrease the number of connected components. If there are $k_{A'}$ connected components, then after adding $k_{A'} - k_A$ of the e_j edges, we obtain a graph with k_A connected components. Since the minimal number of connected components is k_A by Remark 3.2.1, this number is attained. $\qquad\square$

Proposition 3.2.3. *Cyclic covers of finite CW-complexes have finite tree domains.*

Proof. Let X be a finite CW-complex and $p : Y \to X$ a cyclic cover. We prove this by induction on n-skeleta of X. Denote by $p_n : Y^{(n)} \to X^{(n)}$ the restriction of p to the n-skeleton of Y. Since X is finite, $|\iota(\Sigma)|$ is finite. As $k_\Sigma = 1$, we may apply Proposition 3.2.2 and obtain an integer $k(\Sigma) \geqslant 0$ such that T_C^Σ is connected for all $|C| \geqslant k$. So if we choose C so that $|C| \geqslant k + 1$, then T_C^Σ is a tree domain for $p_1 : Y^{(1)} \to X^{(1)}$.

Now suppose we have a tree domain D_{n-1} for $p_{n-1} : Y^{(n-1)} \to X^{(n-1)}$. Then for any connected subset $C \subset \mathbb{Z}$, the subcomplex $\bigcup_{j \in C} j \cdot D_{n-1}$ is also a tree domain for p_{n-1}. We may choose C large enough so that $\bigcup_{j \in C} j \cdot D_{n-1}$ contains a lift of each attaching map of n-cells in X. Then the full subcomplex of $Y^{(n)}$ containing $\bigcup_{j \in C} j \cdot D_{n-1}$ is a tree domain for p_n. $\qquad\square$

3.2.2 Graphs

If we restrict our attention to graphs, the topology of minimal tree domains is much simpler.

Lemma 3.2.4. *Let Γ be a finite graph and let $p : Y \to \Gamma$ be a cyclic cover. If*

$D \in \mathrm{TD}(p)$ *is a minimal tree domain, then*

$$\chi(D) = \chi(\Gamma) + 1 \ ,$$
$$\chi(D \cap 1 \cdot D) = 1 \ .$$

Proof. Up to contracting a spanning tree, we may assume that Γ has a single vertex and n edges. Let $D \in \mathrm{TD}(p)$ be a minimal tree domain. We have $\chi(D \cap 1 \cdot D) = \chi(D) + n - 1$ by definition. Since D is minimal, we must have that $D \cap 1 \cdot D$ is actually a tree, otherwise we could remove some edge. Thus $\chi(D) = \chi(\Gamma) + 1$. $\square$

As a consequence of Lemma 3.2.4 and Proposition 3.1.4, we see that minimal tree domains of cyclic covers of graphs Γ correspond to certain free product decompositions of $\pi_1(\Gamma)$. In the case where $\chi(\Gamma) = -1$, the free product decompositions are much more rigid.

Lemma 3.2.5. *Let Γ be a rose graph and let $p : Y \to \Gamma$ be a cyclic cover. If $\chi(\Gamma) = -1$, then, up translations and changes of orientations, $\mathrm{TD}(p)$ has a unique minimal element.*

Proof. As Γ is a rose and $\chi(\Gamma) = -1$, we have $E(\Gamma) = \Sigma = \{e, f\}$. Since $\iota(\Sigma)$ generates $\mathbb{Z}$, $\iota(e)$ and $\iota(f)$ must be coprime. Up to changing the orientations of e and f, we may assume that $\iota(e) \geqslant \iota(f) \geqslant 0$. Let $D \in \mathrm{TD}(p)$ be a minimal tree domain. If $\iota(f) = 0$, then we must have $\iota(e) = 1$. In this case, $e_0 \cup f_0$ is a minimal tree domain and is unique, up translations and changes of orientations. So now suppose that $\iota(e) \geqslant \iota(f) \geqslant 1$. If $C = \{0, ..., \iota(e) + \iota(f) - 1\}$, it is not hard to see that:

$$D = T^e_{C_e} \cup T^f_{C_f} = T^\Sigma_C$$

is the unique minimal tree domain. $\square$

In Section 3.3.2, using Lemma 3.2.5, we provide an alternative proof of [OZ81, Theorem 1.2]. That is, that there is a correspondence between proper free factors of $\pi_1(\Gamma)$ and cyclic covers of Γ, when $\chi(\Gamma) = -1$. See Corollary 3.3.10 for the statement.

Example 3.2.6. Let Γ be a graph with a single vertex and two edges, so $\chi(\Gamma) = -1$. The spanning tree is the unique vertex $v \in \Gamma$. Let $\Sigma = \{a, b\}$ be the two edges and let $\phi : F(a, b) \to \mathbb{Z}$ be the homomorphism that sends a to 5 and b to 3. Then let $p : Y \to \Gamma$ be the corresponding cyclic cover. Let $C = \{0, 1, ..., 6\}$, then T^Σ_C can be seen in Figure 3.1. The numbers i in the diagram correspond to T_i, the lower black edges are lifts of a and the upper red edges are lifts of b. This is not quite a tree
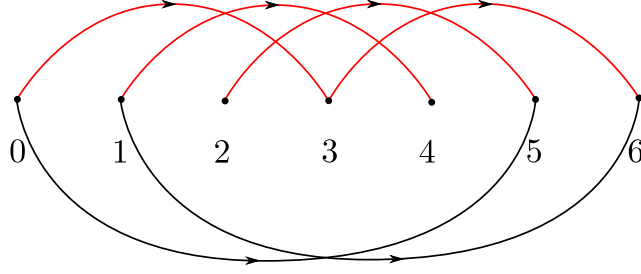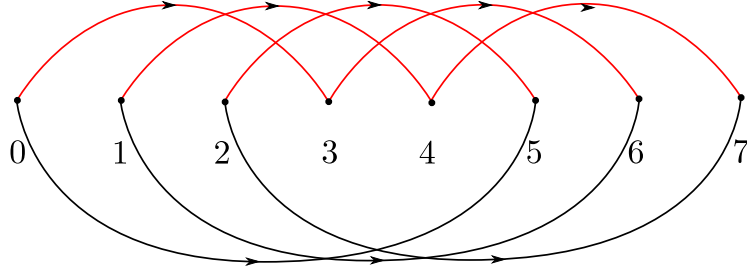
Figure 3.1: Not quite a tree domain.

Figure 3.2: A minimal tree domain.

domain as its intersection with a translate is disconnected. However, we can see that $T_{C\cup 7}^{\Sigma}$ is a genuine tree domain, see Figure 3.2. In fact, it is the unique minimal tree domain for p. The unique primitive cycle that factors through $T_{C\cup 7}^{\Sigma}$, represents the unique conjugacy class of primitive element in $\ker(\phi)$.

3.2.3 One-relator complexes

Another special case is that of one-relator complexes. The following two propositions are essential ingredients for our one-relator hierarchies.

Proposition 3.2.7. *If $X = (\Gamma, \lambda)$ is a finite one-relator complex and $p : Y \to X$ a cyclic cover, then there exists a finite one-relator tree domain $D \in \mathrm{TD}(p)$.*

Proof. Let $T \subset X$ be a spanning tree and let Σ be the remaining edges. Fix a lift $\tilde{\lambda} : S^1 \hookrightarrow Y$ of λ. If λ traverses $e \in \Sigma$, then define:

$$M_e = \max \{k | e_j \cap \mathrm{Im}(\tilde{\lambda}) \neq \varnothing, e_{j+k} \cap \mathrm{Im}(\tilde{\lambda}) \neq \varnothing\} .$$

This quantity is independent of the choice of lift $\tilde{\lambda}$. Denote by $\Sigma_e = \Sigma - \{e\}$.

Let $\alpha : I \to Y$ be a path of shortest length that factors through $\tilde{\lambda}$, traverses a lift of an edge in Σ and such that $o(\alpha), t(\alpha) \in T_j$ for some $j \in \mathbb{Z}$. Let $e \in \Sigma$ be the edge whose lift α traverses first. Up to changing orientation, we may assume that α

traverses e_j first. If $k_{\Sigma_e} = 1$, then by Proposition 3.2.2, there is a finite tree domain $D \in \mathrm{TD}(p)$ with precisely M_e lifts of e. Hence, D can only support one lift of λ and so must be one-relator. So now suppose that $k_{\Sigma_e} \neq 1$, and so $t(e_j) = T_{j+i}$ for some $i \neq 0$. Then α must remain within a component of $T_{\mathbb{Z}}^{\Sigma_e}$, distinct to that of T_j, until it traverses another lift of e. So α either traverses e_j in the reverse direction, a contradiction to minimality of $|\alpha|$, or traverses some $e_{j+lk_{\Sigma_e}}$ for some $l \neq 0$. Thus, $M_e \geqslant k_{\Sigma_e}$ and so by Proposition 3.2.2 there is a finite tree domain $D \in \mathrm{TD}(p)$ with precisely M_e many lifts of e. As before, D must be one-relator. $\qquad \square$

Remark 3.2.8. More generally, by [How87, Lemma 2], if X is a staggered 2-complex and $p : Y \to X$ a cyclic cover, then every tree domain $D \in \mathrm{TD}(p)$ is also a staggered 2-complex. Furthermore, by [HW01, Corollary 6.2] and Proposition 3.1.4, each $D \in \mathrm{TD}(p)$ induces a HNN-splitting of $\pi_1(X)$ over $\pi_1(D)$.

Define the *complexity* of a one-relator complex $X = (\Gamma, \lambda)$ to be the following quantity:

$$c(X) = \left(\frac{|\lambda|}{\deg(\lambda)} - \left| X_\lambda^{(0)} \right|, -\chi(X) \right) .$$

We endow the complexity of one-relator complexes with the dictionary order so that $(q, r) < (s, t)$ if $q < s$ or $q = s$ and $r < t$.

Proposition 3.2.9. *Let X be a one-relator complex and $p : Y \to X$ a cyclic cover. If $D \in \mathrm{TD}(p)$ is a minimal one-relator tree domain, then:*

$$c(D) < c(X) .$$

Proof. It is clear that if $c(D) = (q, r)$ and $c(X) = (s, t)$, we cannot have $q > s$. Suppose that for some one-relator tree domain $D \in \mathrm{TD}(p)$, we have $q = s$, then we must have that X_λ actually lifts to D. But then this implies that X_λ is a subcomplex of D. So applying Lemma 3.2.4 to the induced cyclic cover of the graph $X/(X_\lambda \sim \mathrm{pt})$, we see that $\chi(D) = \chi(X) + 1$ when D is minimal. Thus, $r < t$ and $c(D) < c(X)$ when D is minimal. $\qquad \square$

3.3 One-relator hierarchies

Let $X = (\Gamma, \lambda)$ be a one-relator complex. By Proposition 3.2.7, if $p : Y \to X$ is a cyclic cover, there exists a one-relator complex $X_1 \in \mathrm{TD}(p)$. If X_1 admits a cyclic cover, we may repeat this and obtain a sequence of immersions of one-relator complexes $X_N \looparrowright \dots \looparrowright X_1 \looparrowright X_0 = X$ where X_i is a one-relator tree domain of a

cyclic cover of X_{i-1}. We will call this a *one-relator tower*. Note that each immersion is a tower map in the sense of [How81]. If X_N does not admit any cyclic covers, we will call this a *maximal* one-relator tower.

Proposition 3.3.1. *Every finite one-relator complex X has a maximal one-relator tower $X_N \looparrowright ... \looparrowright X_1 \looparrowright X_0 = X$.*

Proof. The proof is by induction on $c(X)$, the base case is $c(X) = (0,0)$. We have $c(X) = (0,0)$ precisely when $\Gamma \simeq S^1$. This is the only case where X does not admit any cyclic cover. This is because if $c(X) > (0,0)$, then $\chi(X) \leqslant 0$ and so $\mathrm{rk}(H_1(X, \mathbb{Z})) \geqslant 1$. Hence the base case holds. The inductive step is simply Proposition 3.2.9. $\qquad\square$

Now we are ready to prove our version of the Magnus–Moldavanskii hierarchy for one-relator complexes.

Theorem 3.3.2. *Let X be a one-relator complex. There exists a finite sequence of immersions of one-relator complexes:*

$$X_N \looparrowright ... \looparrowright X_1 \looparrowright X_0 = X \ ,$$

such that $\pi_1(X_i) \cong \pi_1(X_{i+1})*_{\psi_i}$ where ψ_i is induced by an identification of Magnus subcomplexes, and such that $\pi_1(X_N)$ splits as a free product of cyclic groups.

Proof. By Proposition 3.3.1 there is a maximal one-relator tower $X_N \looparrowright ... \looparrowright X_1 \looparrowright X_0 = X$. By Theorem 2.4.2, the inclusions $X_i \cap 1 \cdot X_i \hookrightarrow X_i$ are all injective on π_1. Thus, by Lemma 3.1.3 and Proposition 3.1.4, $\pi_1(X_i) \cong \pi_1(X_{i+1})*_{\psi_i}$ for some isomorphisms ψ_i induced by an identification of Magnus subcomplexes. $\qquad\square$

We call a splitting $\pi_1(X_i) \cong \pi_1(X_{i+1})*_{\psi}$ as in Theorem 3.3.2 a *one-relator splitting*. Then the *associated Magnus subcomplexes* inducing ψ are $A = -1 \cdot X_{i+1} \cap X_{i+1}$ and $B = 1 \cdot A$.

3.3.1 The hierarchy length of a one-relator complex

A one-relator tower $X_N \looparrowright ... \looparrowright X_1 \looparrowright X_0 = X$ is a *hierarchy of length N* if $\pi_1(X_N)$ splits as a free product of cyclic groups. Denote by $h(X)$ the *hierarchy length* of X. That is, the smallest integer N such that a hierarchy for X of length N exists. We extend this definition also to one-relator presentations $\langle \Sigma \mid w \rangle$ by saying that the hierarchy length of a one-relator presentation is the hierarchy length of its presentation complex. A hierarchy $X_N \looparrowright ... \looparrowright X_1 \looparrowright X_0 = X$ is *minimal* if

$N = h(X)$. In this section we show that we can actually compute $h(X)$ effectively, and that an upper bound is given by the length of the attaching map.

Lemma 3.3.3. *Let $X = (\Gamma, \lambda)$ and $X' = (\Gamma', \lambda')$ be one-relator complexes and let $q : X \hookrightarrow X'$ be an immersion. Then $h(X) \leqslant h(X')$ with equality if q is an embedding.*

Proof. The proof is by induction on $h(X')$. We may assume that λ and λ' are primitive. The fact that $h(X) = h(X')$ if q is an embedding is clear. Hence, since $h(X) = h(X_\lambda)$, without loss we assume $X = X_\lambda$. The base case is when $h(X') = 0$. If $h(X') = 0$, then $\pi_1(X')$ is free and $\pi_1(X)$ must also be free. This is because $[\lambda']$ would have to be a primitive element of $\pi_1(\Gamma')$ and since $q \circ \lambda = \lambda'$, $[\lambda]$ must be a primitive element of $\pi_1(\Gamma)$. Hence $h(X) = h(X') = 0$. Now suppose $h(X') \geqslant 1$.

Let $X'_N \hookrightarrow ... \hookrightarrow X'_0 = X'$ be some hierarchy for X' with $h(X') = N$. Let $\phi' : \pi_1(X') \to \mathbb{Z}$ be the homomorphism inducing the first cyclic cover $p : Y' \to X'$. Let us first consider the case that $q_*(\pi_1(X)) < \ker(\phi')$. Then there is a lift $q_1 : X \to Y'$ of q. Since q was an immersion, we must actually have that q_1 restricts to a lift of X to X'_1. The inductive hypothesis applies and we have $h(X) \leqslant h(X'_1) < h(X')$. Now assume that $q_*(\pi_1(X))$ is not a subgroup of $\ker(\phi')$. Then the map $\phi' \circ q_*$ induces an epimorphism $\pi_1(X) \to \mathbb{Z}$. Let $p : Y \to X$ be the associated cyclic cover and $X_1 \in \mathrm{TD}(p)$ a one-relator tree domain. Then $(q \circ p)_*(\pi_1(X_1)) < \ker(\phi')$ and, as before, we get $h(X_1) \leqslant h(X'_1) < h(X)$. Since $h(X) = h(X_1)$ or $h(X) = h(X_1) + 1$, we are done. $\qquad\square$

An *abelian cover* is a regular cover $p : Y \to X$ where $\mathrm{Deck}(p)$ is abelian. The *maximal torsion-free abelian cover* is the cover $a : A \to X$ with $\mathrm{Deck}(a) \cong H_1(X, \mathbb{Z})/L$ where $L < H_1(X, \mathbb{Z})$ denotes the torsion subgroup.

Infinite cyclic covers of X are in correspondence with codimension one subgroups of $\mathrm{Deck}(a)$. Indeed, for any codimension one subgroup $K < \mathrm{Deck}(a)$, we obtain a cyclic covering $p : K \backslash A = Y \to X$. If we denote by $f_p : A \to Y$ the quotient map, it is not hard to see that for any finite subcomplex $Z \subset A$, we may choose a codimension one subgroup $K < \mathrm{Deck}(a)$ such that $Z \hookrightarrow A \to K \backslash A$ is actually an embedding. Thus, we obtain the following.

Lemma 3.3.4. *There exists a cyclic cover $p : Y \to X$ such that for any attaching map $\tilde{\lambda} : S^1 \to A$, the map*

$$f_p \mid A_{\tilde{\lambda}} : A_{\tilde{\lambda}} \to Y$$

is an embedding.

Proposition 3.3.5. *Let $X = (\Gamma, \lambda)$ be a one-relator complex with $h(X) \geqslant 1$, let $A \to X$ be the maximal torsion-free abelian cover and let $\tilde{\lambda} : S^1 \to A$ be a lift of λ. Then we have:*

$$h(X) = h(A_{\tilde{\lambda}}) + 1 \leqslant |\lambda|/\deg(\lambda) \ .$$

Proof. Let $X_1 \hookrightarrow X$ be a one-relator tower with $h(X) = h(X_1) + 1$. Now we have an immersion $A_{\tilde{\lambda}} \hookrightarrow X_1$, induced by $f_p : A \to Y$. So by Lemma 3.3.3, we have $h(A_{\tilde{\lambda}}) \leqslant h(X_1)$. By Lemma 3.3.4, we have $h(A_{\tilde{\lambda}}) = h(X_1)$. Since $A_{\tilde{\lambda}} \hookrightarrow X$ is not an embedding, we see by induction on $h(X)$ that $h(X) \leqslant |\lambda|/\deg(\lambda)$. $\qquad\square$

Corollary 3.3.6. *If $X = (\Gamma, \lambda)$ is a one-relator complex, then $h(X)$ is computable in time polynomial in $|\lambda|$.*

Proof. Let $\lambda' : S^1 \to \Gamma$ be the unique primitive immersion that λ factors through. By using the Knuth–Morris–Pratt algorithm [KMP77], for instance, this may be computed in linear time. Since $h(X') = h(X)$, where $X' = (\Gamma, \lambda')$, we may as well assume that λ is primitive. The proof is by induction on $h(X)$. In the base case, $h(X) = 0$ which may be decided in polynomial time [RVW07, Corollary 3.10]. For the inductive step, we may use the equality $h(X) = h(A_{\tilde{\lambda}}) + 1$ from Proposition 3.3.5. $\qquad\square$

The hierarchy length of a one-relator complex is not preserved under homotopy equivalence. This is illustrated by the following examples.

Example 3.3.7. Let $p, q \in \mathbb{Z} - \{-1, 0, 1\}$ be a pair of coprime integers. The Baumslag–Solitar group $\mathrm{BS}(p, q)$ has (at least) two one-relator presentations:

$$\mathrm{BS}(p, q) \cong \langle a, t | t^{-1}a^p t = a^q \rangle \cong \langle b, s | \mathrm{pr}_{p/q}(b, s^{-1}b^{-1}s) \rangle \ ,$$

such that the hierarchy length of the first presentation is two, but of the second it is one. We see this by noting that

$$\langle b, s \mid \mathrm{pr}_{p/q}(b, s^{-1}b^{-1}s) \rangle \cong \langle b_0, b_1, s \mid s^{-1}b_1 s = b_0, \mathrm{pr}_{p/q}(b_1, b_0^{-1}) \rangle$$

and that $\langle b_0, b_1 \rangle \cong \mathbb{Z}$ by definition of $\mathrm{pr}_{p/q}$. Since presentation complexes of torsion-free one-relator groups are aspherical by Lyndon's Identity Theorem [Lyn50], the two associated presentation complexes are homotopy equivalent.

Note that neither $t^{-1}a^p t a^{-q}$ nor $(t^{-1}a^p t a^{-q})^{-1}$ are in the same $\mathrm{Aut}(F(a, t))$ orbit as $\mathrm{pr}_{p/q}(a, t^{-1}a^{-1}t)$. One can see this by applying Whiteheads algorithm [Whi36].

We may do the same with the Baumslag–Gersten groups:

$$\mathrm{BG}(p,q) \cong \langle a, t \mid t^{-1}ata^p t^{-1}a^{-1}t = a^q \rangle \cong \langle b, s \mid \mathrm{pr}_{p/q}(b, s^{-1}bsb^{-1}s^{-1}b^{-1}s) \rangle \ .$$

The hierarchy length of the first presentation is three, but of the second it is two.

More generally, by reverse engineering the hierarchy, we may produce examples of one-relator groups with one-relator presentations of hierarchy length n and $n+1$ for all $n \geqslant 1$.

We now move on to one-relator complexes that are low down in the hierarchy.

3.3.2 Hierarchy length zero and primitives of the free group of rank two

The one-relator complexes $X = (\Gamma, \lambda)$ with $h(X) = 0$ are precisely those in which $[\lambda]$ represents a power of a primitive element of $\pi_1(\Gamma)$. Such elements are well understood thanks to the large body of work that followed Whiteheads seminal paper [Whi36]. In this section we relate one-relator towers with primitive elements of free groups.

Proposition 3.3.8. *Let $X = (\Gamma, \lambda)$ be a one-relator complex. If $1 - \chi(X) = N$ and X possesses a maximal one-relator tower of length N, then $h(X) = 0$.*

Proof. Without loss, we may assume that λ is primitive. The proof is by induction on N. If $N = 0$, then $\pi_1(X)$ is finite cyclic and so $h(X) = 0$. Now assume the inductive hypothesis. Let $X_N \looparrowright ... \looparrowright X_1 \looparrowright X_0 = X$ be a maximal one-relator tower for X. Since $\chi(X) = 1 - N$, $\chi(X_N) = 1$ and $\chi(X_{i+1}) \leqslant \chi(X_i) + 1$, it follows that $\chi(X_1) = \chi(X) + 1$. But then:

$$\chi(X_1 \cap 1 \cdot X_1) = \chi(X_1) + N - 1 = \chi(X) + N = 1 \ ,$$

Thus, $X_1 \cap 1 \cdot X_1$ is a tree and so $\pi_1(X) \cong \pi_1(X_1) * \mathbb{Z}$. $\qquad\square$

When $\chi(X) = 0$, the converse to Proposition 3.3.8 is also true.

Lemma 3.3.9. *Let $X = (\Gamma, \lambda)$ be a one-relator complex with $\chi(X) = 0$. Then X has a maximal one-relator tower of length one if and only if $h(X) = 0$.*

Proof. One direction is Proposition 3.3.8. If $\chi(X) = 0$ and $h(X) = 0$, then there is precisely one cyclic cover $p : Y \to X$. Let $D \in \mathrm{TD}(p)$ be a minimal one-relator tree domain. Since $\pi_1(Y) \cong *_{\mathbb{Z}} C_{\deg(\lambda)}$, where $C_{\deg(\lambda)}$ denotes the cyclic group of order $\deg(\lambda)$, we must have $\pi_1(D) \cong C_{\deg(\lambda)}$. $\qquad\square$

As an application of Lemma 3.3.9, we provide a topological proof of [OZ81, Theorem 1.2], characterising the primitive elements of the free group of rank two.

Corollary 3.3.10. *Conjugacy classes of primitive elements of F_2 are in bijection with primitive elements of $\mathbb{Z}^2$ via the abelianisation map* $\mathrm{ab} : F_2 \to \mathbb{Z}^2$.

Proof. Let Γ be a rose graph with $\pi_1(\Gamma) \cong F_2$. Then each primitive element $x \in \mathbb{Z}^2$ corresponds to a cyclic cover $p : Y \to \Gamma$ induced by the homomorphism $\pi_1(\Gamma) \to \mathbb{Z}^2 \to \mathbb{Z}^2/\langle\langle x \rangle\rangle$, where the first map is the abelianisation map. Now let $x \in \mathbb{Z}^2$ be primitive and let $\lambda : S^1 \looparrowright \Gamma$ be an immersion representing a conjugacy class of a primitive element in $\mathrm{ab}^{-1}(x)$. Consider the one-relator complex $X = (\Gamma, \lambda)$ and its unique cyclic cover $p : Y \to X$. Since $\pi_1(Y) = 1$, it follows that $\pi_1(D) = 1$ for any tree domain $D \in \mathrm{TD}(p)$. Now this must mean that D is a disc and so $D^{(1)}$ is a minimal tree domain for the cyclic cover $Y^{(1)} \to \Gamma$. Lemma 3.2.5 tells us that this is unique. Since $D^{(1)} \simeq S^1$, the immersion λ is also uniquely defined by x. $\square$

The proof also gives us a straightforward way of computing the conjugacy class of primitive elements in $\mathrm{ab}^{-1}(p,q)$ for any primitive $(p,q) \in \mathbb{Z}^2$. See Example 3.2.6.

3.3.3 Hierarchy length one and Brown's criterion

A one-relator complex $X = (\Gamma, \lambda)$ satisfies $h(X) = 1$ only if $\pi_1(X)$ splits as an HNN-extension of a free product of cyclic groups. By Example 3.3.7, the converse is not necessarily true. However, by Brown's criterion [Bro87], if $\pi_1(X)$ splits as an ascending HNN-extension of a free group, then $h(X) = 1$. Brown's criterion is moreover straightforward to use in practice. In this section we will prove a sufficient condition for when $h(X) = 1$, without making use of Whiteheads algorithm. Before stating the criterion, we will need to make some definitions.

Let $F(\Sigma)$ be the free group freely generated by Σ and let $r \in F(\Sigma)$ be some cyclically reduced word. Let $\Gamma \subset \mathbb{R}^{|\Sigma|}$ be the Cayley graph of $\mathbb{Z}^{|\Sigma|}$ embedded in the obvious way. The word r determines a path in Γ, starting at 0, via the identification of Σ with a basis for $\mathbb{Z}^{|\Sigma|}$. Denote this path by $w_r : I \to \mathbb{R}^{|\Sigma|}$ and call it the *trace* of r. A vertex or edge in Γ traversed exactly once by w_r is called *simple*. With this language, Brown's criterion [Bro87, Theorem 4.2] can be stated as follows.

Theorem 3.3.11. *Let $G \cong F_2/\langle\langle r \rangle\rangle$ be a one-relator group.*

1. *If $r \in [F_2, F_2]$, then G splits as an ascending HNN extension of a free group if and only if there exists a line $L \subset \mathbb{R}^2$ on one side of the trace of r, intersecting it in a single simple vertex or edge.*

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2. *If $r \notin [F_2, F_2]$ and the trace of r ends at the vertex (p,q), then G splits as an ascending HNN-extension of a free group if and only if there exists a line $L \subset \mathbb{R}^2$ with slope p/q, on one side of the trace of r, intersecting it in a single simple vertex or edge.*

Brown also shows that G is free-by-cyclic if and only if there are two parallel lines L and L' on opposite sides of the trace as above, each intersecting the trace in a single simple vertex or edge.

In [DT06] it is shown that there is a definite proportion of two generator one-relator groups that do not split as ascending HNN-extensions of free groups. Moreover, one-relator groups with three or more generators cannot split as ascending HNN-extensions of free groups. We provide a result, similar in flavour to Brown's criterion, that applies to a generic one-relator complex.

Theorem 3.3.12. *Let $X = (\Gamma, \lambda)$ be a one-relator complex. Let $T \subset \Gamma$ be a spanning tree and $\Sigma = E(\Gamma) - E(T)$ so that $\pi_1(X) \cong F(\Sigma)/\langle\langle r \rangle\rangle$ with $[r] = [\lambda]$. Let $P \subset \mathbb{R}^{|\Sigma|}$ be a hyperplane parallel to $t(w_r)$. If P intersects w_r in a single simple vertex or simple edge, then X has a one-relator hierarchy*

$$X_1 \hookrightarrow X_0 = X \ ,$$

such that X_1 has a free face.

Proof. Let $n = |\Sigma|$. Given some vector with rational coordinates $v \in \mathbb{R}^n$, this determines a homomorphism $\mathbb{Z}^n \to \mathbb{Z}$ in the following way: let P be a hyperplane orthogonal to v, then since v has rational coordinates, P can be spanned by vectors with integer coordinates. These vectors are linearly independent so $P \cap \mathbb{Z}^n \cong \mathbb{Z}^{n-1}$ and the homomorphism is given by $\mathbb{Z}^n \to \mathbb{Z}^n/P \cap \mathbb{Z}^n \cong \mathbb{Z}$. Now there is some vector w, orthogonal to P, such that $\langle w, P \cap \mathbb{Z}^n \rangle = \mathbb{Z}^n$. Geometrically, this homomorphism can be thought as a height function in the direction of v. More specifically, $kw + P \cap \mathbb{Z}^n \to k$.

If there exists a hyperplane parallel to the endpoint of w_r that intersects w_r in a single simple vertex or simple edge, then there exists such a hyperplane P, spanned by integral vectors. So by the above, this determines a homomorphism $\phi : F(\Sigma) \to \mathbb{Z}$. Furthermore, since P was parallel to the endpoint of w_r, we have $\phi(r) = 0$. This descends to a homomorphism $\bar{\phi} : \pi_1(X) \to \mathbb{Z}$. Let $|r| = k$ and r_i be the prefix of r of length i. If P intersects w_r in a single simple vertex, then there is some i such that $\phi(r_i) \neq \phi(r_j)$ for all $j \neq i$. If P intersects w_r in a single simple edge, then there is some i such that $\phi(r_i) = \phi(r_{i+1})$ and there is no $j \neq i$ such that $\phi(r_i) = \phi(r_j) = \phi(r_{j+1})$.

Let $p : Y \to X$ be the cyclic cover associated with $\bar{\phi}$. By construction of $\bar{\phi}$, if P intersects w_r in a single simple vertex or simple edge, the attaching map $\tilde{\lambda}$ of any one-relator tree domain $D \in \mathrm{TD}(p)$ traverses some edge precisely once, thus $X_1 = D$ has a free face. $\qquad\square$

Remark 3.3.13. Let $F = F(\Sigma)$ be a free group of rank at least two. By [Fu06, Theorem 3.1.1] and [Sv11, Lemma 3.5], for generic $r \in F$, there is a hyperplane $P \subset \mathbb{R}^{|\Sigma|}$ satisfying the hypothesis of Theorem 3.3.12. By [Pud15, Corollary 8.3], r is generically imprimitive. So by Theorem 3.3.12, we see that generic one-relator presentations $\langle \Sigma \mid r \rangle$ have hierarchy length one.

3.3.4 Primitivity rank and w-subgroups

Recall the definition of w-subgroups from Section 2.3.2. We now relate w-subgroups to one-relator hierarchies. First, we shall need the following lemma.

Lemma 3.3.14. *Let $G \cong F(\Sigma)/\langle\langle w \rangle\rangle$ be a one-relator group with $\pi(w) < |\Sigma|$. If $H < G$ is a finitely generated subgroup with $\mathrm{rk}(H) \leqslant \pi(w)$, then either H is free or there exists some surjective homomorphism $\phi : G \to \mathbb{Z}$ such that $H < \ker(\phi)$.*

Proof. Since $\mathrm{rk}(H) \leqslant \pi(w)$, we have that either H is free, or H is conjugate into some w-subgroup of G by Theorems 2.3.5 and 2.3.7. Hence, we may assume in fact that H is a w-subgroup for G, so $H \cong F(\Sigma')/\langle\langle w_H \rangle\rangle$ with $|\Sigma'| = \pi(w)$. Let $\iota : H \to G$ be the inclusion homomorphism. If $w_H \in [F(\Sigma'), F(\Sigma')]$, then $w \in [F(\Sigma), F(\Sigma)]$. In particular, $\mathrm{rk}(H_1(H, \mathbb{Z})) < \mathrm{rk}(H_1(G, \mathbb{Z}))$ and so there is some codimension one subgroup $A < H_1(G, \mathbb{Z})$ such that the image of $H_1(H, \mathbb{Z})$ in $H_1(G, \mathbb{Z})$ is contained in A. Now $G \to H_1(G, \mathbb{Z})/A \cong \mathbb{Z}$ is a homomorphism of the required form. $\qquad\square$

If X is a one-relator complex, then the w-subgroups of $\pi_1(X)$ are represented by a finite collection of immersions $\{q_i : Q_i \hookrightarrow X\}$ where each Q_i are also one-relator complexes (see Section 2.3.2).

Corollary 3.3.15. *Let X be a one-relator complex and let $q : Q \hookrightarrow X$ represent some w-subgroup. If $Q \neq X$, there exists a cyclic cover $p : Y \to X$ such that Q lifts to Y.*

Proof. If $Q \neq X$ then $\mathrm{rk}(\pi_1(Q)) < \mathrm{rk}(\pi_1(X))$, so $\pi_1(X)$ is indicable and by Lemma 3.3.14, there is a homomorphism $\phi : \pi_1(X) \to \mathbb{Z}$ such that $(q)_*(\pi_1(Q)) < \ker(\phi)$. The cyclic cover $p : Y \to X$ associated with ϕ is hence the required cover. $\qquad\square$

Theorem 3.3.16. *Let X be a one-relator complex and let $q : Q \looparrowright X$ represent some w-subgroup. Then there exists a one-relator tower*

$$Q = X_K \looparrowright ... \looparrowright X_0 = X \ .$$

Proof. The proof is by induction on $c(X)$. The base case is trivial, so let us assume the inductive hypothesis. By Corollary 3.3.15, there is a cyclic cover $p : Y \to X$ such that Q lifts to Y. Let $D \in \mathrm{TD}(p)$ be a minimal tree domain. Then D is one-relator and $c(D) < c(X)$ by Proposition 3.2.9. Since q is a combinatorial immersion of one-relator complexes and $Q = Q_\lambda$, some lift of Q factors through D. So now either $Q = D$, in which case we are done, or Q immerses properly into D and so the inductive hypothesis applies. $\square$

As a consequence of Magnus' classical result that the membership problem is solvable for Magnus subgroups of one-relator groups, Theorem 2.3.3, we obtain the following.

Corollary 3.3.17. *The membership problem for w-subgroups in one-relator groups is solvable.*

Chapter 4

Quasi-convex one-relator hierarchies

In [Wis21], Wise introduced the notion of a quasi-convex hierarchy. There, he proved that one-relator groups with torsion have a quasi-convex hierarchy, resolving an old conjecture of Baumslag. In this chapter, we aim to generalise his results, providing criteria for when an arbitrary one-relator group has a quasi-convex hierarchy. First, let us specialise the definition of a quasi-convex hierarchy to one-relator complexes.

Definition 4.0.1. A one-relator tower (hierarchy) $X_N \rightsquigarrow ... \rightsquigarrow X_1 \rightsquigarrow X_0$ is a *quasi-convex one-relator tower (hierarchy)* if, for each $i < N$, the induced maps $\tilde{A}_{i+1}, \tilde{B}_{i+1} \hookrightarrow \tilde{X}_i$ are quasi-isometric embeddings, where $A_{i+1}, B_{i+1} \subset X_{i+1}$ are the associated Magnus subcomplexes.

Most of this chapter will be dedicated towards proving the main ingredient needed for our quasi-convex hierarchy criterion, Corollary 4.3.2: if X has a quasi-convex one-relator hierarchy and $\pi_1(X)$ is hyperbolic, then $\tilde{A} \hookrightarrow \tilde{X}$ is a quasi-isometric embedding for all subcomplexes $A \subset X$.

To state our result, we need one more definition.

Definition 4.0.2. A one-relator tower (hierarchy) $X_N \rightsquigarrow ... \rightsquigarrow X_1 \rightsquigarrow X_0$ is an *acylindrical one-relator tower (hierarchy)* if $\pi_1(X_i)$ acts acylindrically on the Bass-Serre tree associated with the splitting $\pi_1(X_{i+1})*_{\psi_i}$ for all $i < N$.

The main result of this chapter is the following.

Theorem 4.4.1. *Let X be a one-relator complex and let $X_N \rightsquigarrow ... \rightsquigarrow X_1 \rightsquigarrow X_0 = X$ be a one-relator hierarchy. The following are equivalent:*

1. $X_N \rightsquigarrow ... \rightsquigarrow X_1 \rightsquigarrow X_0 = X$ is a quasi-convex hierarchy and $\pi_1(X)$ is hyperbolic,

2. $X_N \leftrightarrow ... \leftrightarrow X_1 \leftrightarrow X_0 = X$ *is an acylindrical hierarchy.*

In Chapter 7 we prove a more general statement, Theorem 7.1.2.

4.1 Normal forms

Computable normal forms for one-relator complexes can be derived from the original Magnus hierarchy [Mag30]. Following the same idea, we build normal forms for universal covers of one-relator complexes. The geometry of these normal forms will then allow us to prove our main theorem.

Let X be a combinatorial 2-complex. We define $I(X)$ to be the set of all combinatorial immersed paths $I \looparrowright X^{(1)}$.

Definition 4.1.1. A *normal form* for X is a map $\eta : X^{(0)} \times X^{(0)} \to I(X)$ where $o(\eta_{p,q}) = p$ and $t(\eta_{p,q}) = q$, for all $p, q \in X^{(0)}$. We say that η is *prefix-closed* if for every $p, q \in X^{(0)}$ and $i \in \{0, 1, ..., |\eta_{p,q}|\}$, we have that $\eta_{p,r} = \eta_{p,q} \mid [0, i]$ where $r = \eta_{p,q}(i)$.

We will be particularly interested in certain kinds of normal forms, defined below. The first being quasi-geodesic normal forms and the second being normal forms relative to a given subcomplex.

Definition 4.1.2. Let $K > 0$. A normal form $\eta : X^{(0)} \times X^{(0)} \to I(X)$ is *K-quasi-geodesic* if every path in $\mathrm{Im}(\eta)$ is a K-quasi-geodesic. We will call η *quasi-geodesic* if it is K-quasi-geodesic for some $K > 0$.

Definition 4.1.3. Suppose $Z \subset X$ is a subcomplex. A normal form $\eta : X^{(0)} \times X^{(0)} \to I(X)$ is a *normal form relative to Z* if, for any $r \in X^{(0)}$ and p, q contained in the same connected component of Z, the following hold:

1. $\eta_{p,q}$ is supported in Z,

2. $\eta_{p,r} * \bar{\eta}_{q,r}$ is supported in Z after removing backtracking.

The following proposition will be of use to us in Chapter 7.

Proposition 4.1.4. *Let X be a finite connected combinatorial complex and $A \subset X$ a connected subcomplex. Then $\tilde{X}$ admits $\pi_1(X)$-equivariant quasi-geodesic prefix-closed normal forms relative to $\tilde{A}$ if and only if $\tilde{A}$ is quasi-isometrically embedded in $\tilde{X}$.*

Proof. One direction is clear. So let us suppose now that $\tilde{A}$ is quasi-isometrically embedded in $\tilde{X}$. Let $T \subset A$ be a spanning tree and $T' \subset X$ a spanning tree extending

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T. If there exist normal forms of the required form for the combinatorial complex obtained from X by contracting T to a point, then we may extend these to normal forms of the required form for X. So without loss, assume that X has a single vertex.

Fix a vertex $x \in \tilde{A}^{(0)}$. We inductively define sets P_i as follows: $P_0 = \{x\}$ and P_{i+1} is obtained from P_i by adding $\pi_1(A)$ orbit representatives of the vertices p such that, for some $p' \in P_i$, we have

$$d(x, p) = d(x, p') + 1 = d(x, p') + d(p', p) = d(\tilde{A}, p) \ .$$

Then we can see that $\pi_1(A) \cdot P_i$ consists of all vertices at distance precisely i from $\tilde{A}$ and contains each vertex precisely once. Similarly to the construction of the sets P_i, we may inductively define a map:

$$\eta_i : P_i \to I(X)$$

as follows. The map η_0 simply sends x to the constant path. So assume we have defined η_i and we want to define η_{i+1}. For each $p \in P_{i+1}$, choose a neighbouring vertex $p' \in P_i$ and define $\eta_{i+1}(p) = \eta_i(p') * e$ where $e \in E(X)$ is an edge connecting p' with p.

Let us first assume that A is a point. Then for all $p \in \bigcup_{i \in \mathbb{N}} P_i$, define $\eta(x, p) = \eta_{d(x,p)}(p)$. Extending $\pi_1(X)$-equivariantly, we obtain quasi-geodesic prefix-closed normal forms relative to x.

Now suppose that A is not a point. By the above, we may construct $\pi_1(A)$-equivariant quasi-geodesic prefix-closed normal forms for $\tilde{A}$, relative to a point. For each $a \in \pi_1(A)$, we define $\eta(x, a \cdot p)$ to be the concatenation of the normal form for $\tilde{A}$ joining x with $a \cdot x$ and the path $a \cdot \eta_{d(x,p)}(x, p)$. Since $\tilde{A}$ is quasi-isometrically embedded in $\tilde{X}$, these paths are always quasi-geodesics. Furthermore, they are prefix-closed. By extending $\pi_1(X)$-equivariantly, we obtain normal forms for X that are quasi-geodesic prefix-closed relative to $\tilde{A}$. $\qquad\square$

4.2   Graph of spaces normal forms

For simplicity, we only discuss normal forms for graphs of spaces with underlying graph containing a single vertex and a single edge. However, the construction is essentially the same for any graph.

Let $\mathcal{X} = (\Gamma, \{X\}, \{C\}, \partial^{\pm})$ be a graph of spaces where Γ consists of a single vertex and a single edge, X and C are combinatorial 2-complexes and $\partial^{\pm}$ are inclusions of subcomplexes. Denote by $A = \text{Im}(\partial^-)$ and $B = \text{Im}(\partial^+)$. We may

orient the vertical edges so that they go from the subcomplex A to the subcomplex B. When it is clear from context which path we mean, we will abuse notation and write t^k for all $k \geqslant 0$, to denote a path that follows k vertical edges consecutively, respecting this orientation. We will write t^{-k} for all $k \geqslant 0$, for the path that follows k vertical edges in the opposite direction. Since $\partial^{\pm}$ are embeddings, these paths are uniquely defined given an initial vertex.

The universal cover $\tilde{X}_{\mathcal{X}} \to X_{\mathcal{X}}$ also has a graph of spaces structure with underlying graph the Bass-Serre tree of the associated splitting, the vertex spaces are copies of the universal cover $\tilde{X}$, the edge spaces are copies of the universal cover $\tilde{C}$ and the edge maps are $\pi_1(X)$-translates of lifts $\tilde{\partial}^{\pm} : \tilde{C} \to \tilde{X}$ of the maps $\partial^{\pm}$. We will denote by $\tilde{\mathcal{X}}$ the underlying graph of spaces so that $X_{\tilde{\mathcal{X}}} = \tilde{X}_{\mathcal{X}}$. Every path $c \in I\left(X_{\tilde{\mathcal{X}}}\right)$ can be uniquely factorised:

$$c = c_0 * t^{\epsilon_1} * c_1 * \ldots * t^{\epsilon_n} * c_n \ ,$$

with $\epsilon_i = \pm 1$ and where each c_i is supported in some copy of $\tilde{X}$. Note that each c_i can be the empty path. We say c is *reduced* if there are no two subpaths c_i, c_j, with $i \neq j$, that are both supported in the same copy of $\tilde{X}$ in $X_{\tilde{\mathcal{X}}}$. In other words, the path $\tilde{v} \circ c$ is immersed, where $\tilde{v} : \tilde{X}_{\mathcal{X}} \to T$ is the vertical map to the Bass-Serre tree T.

A *vertical square* in $\tilde{X}_{\mathcal{X}}$ is a 2-cell with boundary path $e * t^{\epsilon} * f * t^{-\epsilon}$ where e and f are edges in two different copies of $\tilde{X}$. We will call the paths $e * t^{\epsilon}$, $t^{\epsilon} * f$, $f * t^{-\epsilon}$, $t^{-\epsilon} * e$ and their inverses, *corners* of this square. We may pair up each corner so that their concatenation forms the boundary path of the square. In this way, each corner exhibits a vertical homotopy to the opposing corner it is paired up with. For instance, there is a vertical homotopy through the square between $t^{\epsilon} * \bar{f}$ and $e * t^{\epsilon}$.

Now let

$$\eta_{\tilde{A}} : \tilde{X}^{(0)} \times \tilde{X}^{(0)} \to I(\tilde{X}) \ ,$$

$$\eta_{\tilde{B}} : \tilde{X}^{(0)} \times \tilde{X}^{(0)} \to I(\tilde{X})$$

be $\pi_1(X)$-equivariant normal forms relative to $\tilde{A}$ and $\tilde{B}$ respectively. Since these are $\pi_1(X)$-equivariant, it is not important which lift of $\tilde{A}$ and $\tilde{B}$ we choose. Let

$$\eta_{\tilde{X}} : \tilde{X}^{(0)} \times \tilde{X}^{(0)} \to I(\tilde{X}) \ ,$$

be a $\pi_1(X)$-equivariant normal form for $\tilde{X}$. From this data, we may define normal

forms for $X_{\tilde{\mathcal{X}}}$ as follows. We say a normal form

$$\eta : X_{\tilde{\mathcal{X}}}^{(0)} \times X_{\tilde{\mathcal{X}}}^{(0)} \to I\left(X_{\tilde{\mathcal{X}}}\right)$$

is *a graph of spaces normal form, induced by* $\left(\{\eta_{\tilde{X}}\}, \{\eta_{\tilde{A}}, \eta_{\tilde{B}}\}\right)$, if the following holds for all $c = c_0 * t^{\epsilon_1} * c_1 * ... * t^{\epsilon_n} * c_n \in \mathrm{Im}(\eta)$:

1. $c_0 = \eta_{\tilde{X}}(o(c_0), t(c_0))$.

2. If $\epsilon_i = 1$ then $c_i = \eta_{\tilde{B}}(o(c_i), t(c_i))$. Furthermore, if e is the first edge c_i traverses, then $t * e$ does not form a corner of a vertical square.

3. If $\epsilon_i = -1$ then $c_i = \eta_{\tilde{A}}(o(c_i), t(c_i))$. Furthermore, if e is the first edge c_i traverses, then $t^{-1} * e$ does not form a corner of a vertical square.

Our definition of graph of spaces normal form and the following theorem are the topological translations of the algebraic definition and normal form theorem [LS01, Chapter IV, Section 2].

Theorem 4.2.1. *If $\eta_{\tilde{A}}, \eta_{\tilde{B}}$ and $\eta_{\tilde{X}}$ are as above, there is a unique graph of spaces normal form η, induced by $\left(\{\eta_{\tilde{X}}\}, \{\eta_{\tilde{A}}, \eta_{\tilde{B}}\}\right)$. Moreover, the following are satisfied:*

1. *η is $\pi_1(X_{\mathcal{X}})$-equivariant,*

2. *the action of $\pi_1(X)$ on η and $\eta_{\tilde{X}}$ coincide,*

3. *if $\eta_{\tilde{X}}$, $\eta_{\tilde{A}}$ and $\eta_{\tilde{B}}$ are prefix-closed, then so is η.*

Now suppose further that the horizontal map $h : X_{\mathcal{X}} \to H_{\mathcal{X}}$ is a homotopy equivalence (see Proposition 2.6.2). Let $s : H_{\mathcal{X}}^{(0)} \hookrightarrow X_{\mathcal{X}}^{(0)}$ be a section and let $\tilde{s} : H_{\tilde{\mathcal{X}}} \to X_{\tilde{\mathcal{X}}}$ be a lift of s, extended $\pi_1(X)$-equivariantly. Then, given graph of space normal forms $\eta : X_{\mathcal{X}}^{(0)} \times X_{\mathcal{X}}^{(0)} \to I(X_{\mathcal{X}})$, we may define normal forms

$$\mu : H_{\tilde{\mathcal{X}}}^{(0)} \times H_{\tilde{\mathcal{X}}}^{(0)} \to I\left(H_{\tilde{\mathcal{X}}}\right) ,$$

by setting:

$$\mu(p, q) = h \circ \eta(\tilde{s}(p), \tilde{s}(q)) .$$

We call normal forms obtained in such a way, *collapsed graph of spaces normal forms.* These normal forms will be important for the proof of Theorem 4.3.1.

4.2.1 Quasi-geodesic normal forms for graphs of hyperbolic spaces

The following result can be thought of as a refinement of [Kap01, Corollary 1.1]. We make use of Theorem 2.1.1.

Theorem 4.2.2. *Let* $\mathcal{X} = (\Gamma, \{X\}, \{C\}, \{\partial^{\pm}\})$ *be a graph of spaces with* Γ *consisting of a single vertex and edge. Let* $\eta_{\tilde{A}} : \tilde{X}^{(0)} \times \tilde{X}^{(0)} \to I(\tilde{X})$, $\eta_{\tilde{B}} : \tilde{X}^{(0)} \times \tilde{X}^{(0)} \to I(\tilde{X})$ *and* $\eta_{\tilde{X}} : \tilde{X}^{(0)} \times \tilde{X}^{(0)} \to I(\tilde{X})$ *be the graph of spaces normal forms induced by* $\left(\{\eta_{\tilde{X}}\}, \{\eta_{\tilde{A}}, \eta_{\tilde{B}}\}\right)$. *Suppose that the following are satisfied:*

1. *$\tilde{X}^{(1)}$ is hyperbolic with the path metric,*

2. *$\pi_1(X_{\mathcal{X}})$ acts acylindrically on the Bass-Serre tree T,*

3. *$\eta_{\tilde{A}}$, $\eta_{\tilde{B}}$ and $\eta_{\tilde{X}}$ are prefix-closed quasi-geodesic normal forms.*

Then the graph of spaces normal form induced by $\left(\{\eta_{\tilde{X}}\}, \{\eta_{\tilde{A}}, \eta_{\tilde{B}}\}\right)$:

$$\eta : \tilde{X}_{\mathcal{X}}^{(0)} \times \tilde{X}_{\mathcal{X}}^{(0)} \to I(\tilde{X}_{\mathcal{X}})$$

is prefix-closed and quasi-geodesic.

Proof. By Theorem 2.6.6, we have that $\tilde{X}_{\mathcal{X}}$ is δ-hyperbolic and $\tilde{X} \hookrightarrow X_{\tilde{\mathcal{X}}}$ is a quasi-isometric embedding. Thus, there is a constant $K \geqslant 1$ such that the images of $\eta_{\tilde{A}}$, $\eta_{\tilde{B}}$ and $\eta_{\tilde{X}}$ are K-quasi-geodesics in $\tilde{X}_{\mathcal{X}}$. Now let $x, y \in \tilde{X}_{\mathcal{X}}^{(0)}$ be any two points and γ a geodesic connecting them. We may factorise this geodesic

$$\gamma = \gamma_0 * t^{\epsilon_1} * \gamma_1 * \ldots * t^{\epsilon_k} * \gamma_k \ ,$$

such that $\epsilon_i = \pm 1$, each γ_i is path homotopic into a copy of $\tilde{X}$ and there is no subpath $\gamma_i * t^{\epsilon_{i+1}} * \ldots * \gamma_j$ with $i < j$ that is path homotopic into some copy of $\tilde{X}$. If $\rho \in I(\tilde{X}_{\mathcal{X}})$, we write $\tilde{X}_{\rho}$ to denote the copy of $\tilde{X}$ in $\tilde{X}_{\mathcal{X}}$ that ρ ends in. When it makes sense, we do the same for $\tilde{A}$ and $\tilde{B}$. This is well defined since all the $\pi_1(X_{\mathcal{X}})$ translates of these subcomplexes are disjoint or equal.

Let $p \in \tilde{X}_{\gamma}^{(0)}$, let α be a geodesic connecting y with p and let β be a geodesic connecting $o(\gamma_k)$ with p. Denoting by $L = \delta + M + 1$, we claim that

$$|\eta_{x,p}| \leqslant K(k+1)\left(kL + |\gamma| + |\alpha|\right) \ ,$$

where $M = M(K, \delta)$ is the maximal Hausdorff distance of all $\eta_{\tilde{X}}$, $\eta_{\tilde{A}}$ and $\eta_{\tilde{B}}$ normal forms from their respective geodesics in $\tilde{X}_{\mathcal{X}}$.

The proof is by induction on k. First suppose that $k = 0$. We will write $\simeq$ to mean path homotopic. Since $\beta \simeq \gamma_k * \alpha$ and $o(\beta), t(\beta) \in \tilde{X}_\gamma$, we get that $\eta(x, p) = \eta_{\tilde{X}}(x, p)$ is actually a K-quasi-geodesic path. This establishes the base case.

Now suppose it is true for all $i < k$. Suppose $\epsilon_k = -1$, the other case is the same. Let $\beta' = \eta_{\tilde{A}}(o(\gamma_k), p) = a * c$ where $a \subset \tilde{A}_{\gamma * \bar{\gamma}_k}$ and the first edge of c is not in $\tilde{A}_{\gamma * \bar{\gamma}_k}$. By assumption, c is a K-quasi-geodesic. Hence there is a point $q \in \beta$ such that $d(t(a), q) \leqslant M$. But then since $\beta \cup \alpha \cup \gamma_k$ forms a geodesic triangle, there is a point $z \in \alpha \cup \gamma_k$ such that $d(t(a), z) \leqslant L$. We may now divide into two subcases: either $z \in \alpha$ or $z \in \gamma_k$.

First suppose z is a vertex traversed by α, see Figure 4.1. It follows that

$$d(o(a), t(a)) \leqslant |\gamma_k| + L + |\alpha| \ ,$$
$$|c| \leqslant K(L + |\alpha|) \ .$$

Since a is supported in a copy of $\tilde{A}$, the path $t^{-1} * a$ may be homotoped through vertical squares to a path $b * t^{-1}$, where b is supported in $\tilde{B}_{\gamma * \bar{\gamma}_k * t}$. Let α' be the geodesic connecting $o(b)$ to $t(b)$. We have

$$\left|\alpha'\right| \leqslant d(o(a), t(a)) + 2 \leqslant |\gamma_k| + L + |\alpha| + 2 \ .$$

Let $t(\alpha') = p'$. By definition of graph of space normal forms we have

$$\eta_{x,p} = \eta_{x,p'} * t^{-1} * c \ .$$

Hence, we may apply the inductive hypothesis with $\gamma_0 * t^{\epsilon_1} * ... * \gamma_{k-1}$ playing the role of γ and α' playing the role of α. Putting everything together, we obtain the following inequality:

$$\begin{aligned}
|\eta_{x,p}| &\leqslant Kk \left((k-1)L + |\gamma_0 * t^{\epsilon_1} * ... * \gamma_{k-1}| + |\alpha'|\right) + |c| + 1 \\
&\leqslant Kk \left((k-1)L + |\gamma| + L + |\alpha| + 1\right) + K(L + |\alpha|) + 1 \\
&\leqslant K(k+1) \left(kL + |\gamma| + |\alpha|\right) \ .
\end{aligned}$$

Now we deal with the other case, see Figure 4.2. If z is a vertex traversed by γ_k, then

$$d(o(a), t(a)) \leqslant d(o(\gamma_k), z) + L \ ,$$
$$|c| \leqslant K(L + d(z, t(\gamma)) + |\alpha|) \ .$$

Just as before, we may define the paths b and α'. We have

$$|\alpha'| \leqslant d(o(a), t(a)) + 2 \leqslant d(o(\gamma_k), z) + L + 2 .$$

So we apply the inductive hypothesis again:

$$\begin{aligned}
|\eta_{x,p}| \leqslant & Kk((k-1)L + |\gamma_0 * t^{\epsilon_1} * ... * \gamma_{k-1}| + d(o(\gamma_k), z) + L + 2) \\
& + K(L + d(z, t(\gamma)) + |\alpha|) + 1 \\
\leqslant & K(k+1)(kL + |\gamma| + |\alpha|) .
\end{aligned}$$

This concludes the proof of the claim.

Consider the polynomial function $f : \mathbb{N} \to \mathbb{R}$ given by

$$f(n) = (K)^2 n(n+1)(L+1) .$$

We have shown that for all $x, y \in \tilde{X}_\mathcal{X}^{(0)}$, we have $|\eta_{x,y}| \leqslant f(d(x,y))/K$. We now want to show that the graph of spaces normal forms are actually f-quasi-geodesics.

Let $h = h_0 * t^{\epsilon_1} * h_1 * ... * t^{\epsilon_k} * h_k \in \mathrm{Im}(\eta)$ be a normal form and let $h' = h_i' * t^{\epsilon_{i+1}} * h_{i+1} * ... * t^{\epsilon_j} * h_j'$ be a subpath of h. Then we have that

$$\eta_{o(h'), t(h')} = \eta_{o(h_i'), t(h_i')} * t^{\epsilon_{i+1}} * h_{i+1} * ... * t^{\epsilon_j} * h_j'$$

is the corresponding normal form since $\eta_{\tilde{A}}$, $\eta_{\tilde{B}}$ and $\eta_{\tilde{X}}$ are prefix closed. We showed that:

$$\left|\eta_{o(h'), t(h')}\right| \leqslant \frac{f(d(o(h'), t(h')))}{K}$$

and so we know that

$$\begin{aligned}
|h'| &= |\eta_{o(h'), t(h')}| - |\eta_{o(h_i'), t(h_i')}| + |h_i'| \\
&\leqslant K|\eta_{o(h'), t(h')}| \\
&\leqslant f(d(o(h'), t(h'))) .
\end{aligned}$$

By Theorem 2.1.1, it follows that there is some constant $K' = K'(f)$ such that η is a K'-quasi-geodesic normal form. $\qquad\square$
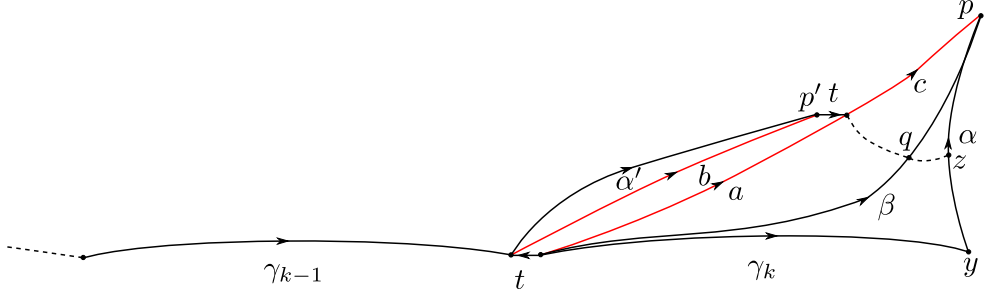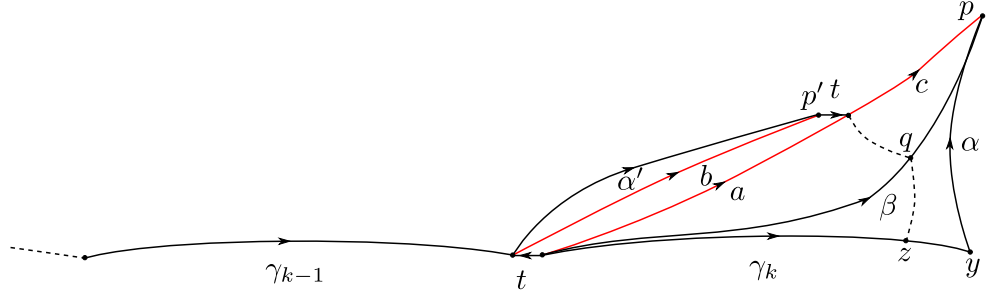
Figure 4.1: First case, $z \in \alpha$.

Figure 4.2: Second case, $z \in \gamma_k$.

4.3 Quasi-convex Magnus subcomplexes

In order to prove that one-relator groups with torsion have quasi-convex (one-relator) hierarchies, Wise showed in [Wis21, Lemma 19.8] that their Magnus subgroups are quasi-convex. When the torsion assumption is dropped, this is certainly no longer true. However, under additional hypotheses, we may recover quasi-convexity of Magnus subgroups.

Theorem 4.3.1. *Let* $X = (\Gamma, \lambda)$ *be a one-relator complex,* $p : Y \to X$ *a cyclic cover and* $Z \in \mathrm{TD}(p)$ *a finite one-relator tree domain. Suppose further that the following hold:*

1. $\pi_1(X)$ *is hyperbolic,*

2. $\pi_1(C)$ *is quasi-convex in* $\pi_1(X)$ *for all connected subcomplexes* $C \subset Z$

Then $\pi_1(C)$ *is quasi-convex in* $\pi_1(X)$ *for all connected subcomplexes* $C \subset X$.

Proof. Let $C \subset X$ be a connected subcomplex. Denote by $Z = (\Lambda, \tilde{\lambda})$. We may assume that C is a Magnus subcomplex and that $E(C) = E(\Gamma) - \{f\}$ with f non separating.

Now let $T \subset X$ be a spanning tree not containing f. After choosing a lift of T in Y as in Section 3.2.1, since Z is a subcomplex of Y, we have that each edge in

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$E(Z)$ is of the form e_i where $e \in E(\Gamma)$ and i is an integer. Denote by m_e the smallest integer such that e_{m_e} is traversed by $\tilde{\lambda}$, and by M_e the largest (as in the proof of Proposition 3.2.7). We have two cases to consider.

Case 1. For all $f_i \in E(\Lambda)$, we have that $m_f \leqslant i \leqslant M_f$ and f_i is non separating in Λ.

The action of $\text{Deck}(p)$ on Y gives us a graph of spaces $\mathcal{X} = (S^1, \{Z\}, \{Z \cap 1 \cdot Z\}, \{\partial^{\pm}\})$ as in Proposition 3.1.4. Moreover, we have have a map

$$h : X_{\mathcal{X}} \to X$$

that is a homotopy equivalence obtained by collapsing the edge space onto the vertex space as in Proposition 2.6.2. This lifts to a map in the universal covers:

$$\tilde{h} : X_{\tilde{\mathcal{X}}} \to \tilde{X}.$$

Denote by $A, B \subset Z$ the Magnus subcomplexes that are the images of $\partial^{\pm}$. That is, $A = (-1) \cdot (Z \cap 1 \cdot Z)$ and $B = Z \cap 1 \cdot Z$.

Denote by $A' = \Lambda - f_{M_f}$ and $B' = \Lambda - f_{m_f}$. These are Magnus subcomplexes containing A and B respectively. In particular, $p^{-1}(C) \cap Z \subset A' \cap B'$. By Proposition 4.1.4, there are prefix-closed, quasi-geodesic, $\pi_1(Z)$-equivariant normal forms $\eta_{\tilde{A}}$, $\eta_{\tilde{B}}$ for $\tilde{Z}$, relative to $\tilde{A}'$ and $\tilde{B}'$ respectively. By the Freiheitssatz, these are also normal forms relative to $\tilde{A}$ and $\tilde{B}$ so that we may apply Theorem 4.2.1 to obtain unique prefix-closed $\pi_1(X_{\mathcal{X}})$-equivariant graph of spaces normal forms η for $X_{\tilde{\mathcal{X}}}$, induced by $(\{\eta_{\tilde{A}}\}, \{\eta_{\tilde{A}}, \eta_{\tilde{B}}\})$. By Theorems 2.6.5 and 4.2.2, η is also quasi-geodesic.

So now let $s : X^{(0)} \to Z^{(0)}$ be a section. This section, along with the graph of spaces normal forms η, induce collapsed graph of spaces normal forms $\mu : \tilde{X}^{(0)} \times \tilde{X}^{(0)} \to I(\tilde{X})$ as in Section 4.2. By construction, since η was prefix-closed, quasi-geodesic and $\pi_1(X)$-equivariant, then so is μ. We now show that μ is a normal form relative to $\tilde{C}$, from which the result in this case will follow.

Let $c = c_0 * t^{\epsilon_1} * c_1 * ... * t^{\epsilon_n} * c_n$ be a path in $X_{\tilde{\mathcal{X}}}$ such that each c_i is supported in lifts of $p^{-1}(C) \cap Z$. Note that each c_i is supported in lifts of $A' \cap B'$. Thus, up to homotoping one corner of vertical squares $t^{\epsilon_i} * c_i'$ to another $c_i'' * t^{\epsilon_i}$, this path is actually in normal form by construction. These vertical homotopies do not change the image of the path in $\tilde{X}$. Therefore, $\mu_{p,q}$ is supported in $\tilde{C}$ if p, q are in the same $\tilde{C}$ component.

Now let $c = c_0 * t^{\epsilon_1} * c_1 * ... * t^{\epsilon_n} * c_n$ and $d = d_0 * t^{\nu_1} * d_1 * ... * t^{\nu_k} * d_k$ be normal form paths in $X_{\tilde{\mathcal{X}}}$ such that the origins of c and d are contained in the same copy of $\tilde{C}$ and their endpoints coincide. We want to show that, up to removing

backtracking, the path $c * \bar{d}$ projects to a path supported in $\tilde{C} \subset \tilde{X}$. Note that $c * \bar{d}$ is homotopic to the path $\eta_{c(0),d(0)}$ that projects to $\tilde{C}$. Hence, unless $c * \bar{d}$ already projects to $\tilde{C}$, we must have that $\epsilon_n = -\nu_k$ and $c_n * \bar{d}_k$ is supported in some $\tilde{A}'$ or $\tilde{B}'$ component, depending upon the sign of ϵ_n and up to removing backtracking. Then, by homotoping $t^{\epsilon_n} * c_n * \bar{d}_k * t^{-\nu_k}$ through vertical squares, we obtain a new path $c'_n * \bar{d}'_k$ that projects to the same path in $\tilde{X}$. By taking this reasoning forward until no more such vertical homotopies may be performed, we obtain a path that is supported in the preimage of $\tilde{C}$. Therefore we may conclude that $\mu = \eta_{\tilde{C}}$ is a normal form relative to $\tilde{C}$. Thus, since μ was a quasi-geodesic normal form, $\pi_1(C)$ is quasi-convex in $\pi_1(X)$.

Case 2. There is some $f_i \in E(\Lambda)$ such that either $i < m_f$, or $i > M_f$, or f_i is separating in Λ.

Let X' be the one-relator complex obtained from X by adding a single edge, d. Thus, we have $\pi_1(X') = \pi_1(X) * \langle x \rangle$. If $\phi : \pi_1(X) \to \mathbb{Z}$ is the epimorphism inducing p, denote by $\phi' : \pi_1(X') \to \mathbb{Z}$ the epimorphism such that $\phi' \mid \pi_1(X) = \phi$ and $\phi'(x) = 1$. Let $Z'' \subset Z$ be the subcomplex obtained from Z by removing each f_i with $i < m_f$ or $i > M_f$. If $p' : Y' \to X'$ is the cyclic cover induced by ϕ', then Z'' lifts to Y'. Moreover, for appropriately chosen integers $k \leqslant l$, we see that

$$(\Lambda', \tilde{\lambda}') = Z' = Z'' \cup \left(\bigcup_{i=k}^{l} d_i \right)$$

is a one-relator tree domain for p'. By construction, for all $f_i \in E(\Lambda')$, we have that $m_f \leqslant i \leqslant M_f$ and that f_i is non separating in Λ'. Moreover:

$$\pi_1(Z') = \underset{i=0}{\overset{k-l}{\bigstar}} \pi_1(A_i)^{x^i} ,$$

where each A_i is a (possible empty) subcomplex of Z. By hypothesis, $\pi_1(A')$ is quasi-convex in $\pi_1(X')$ for all subcomplexes $A' \subset Z'$. By the proof of the first case, we see that $\pi_1(C)$ is quasi-convex in $\pi_1(X')$. But then $\pi_1(C)$ must also be quasi-convex in $\pi_1(X)$ and we are done.

$\square$

Corollary 4.3.2. *Let $X = (\Gamma, \lambda)$ be a one-relator complex. Suppose that $\pi_1(X)$ is hyperbolic and X admits a quasi-convex one-relator hierarchy. Then $\pi_1(A)$ is quasi-convex in $\pi_1(X)$ for all subcomplexes $A \subset X$.*

Proof. The proof is by induction on hierarchy length. The base case is clear. Proposition 2.6.5 and Theorems 2.6.6 and 4.3.1 handle the inductive step. $\square$

4.4 Quasi-convex one-relator hierarchies

We conclude this chapter with the first part of our main hierarchy equivalence result, Theorem 7.1.2.

Theorem 4.4.1. *Let X be a one-relator complex and let $X_N \looparrowright ... \looparrowright X_1 \looparrowright X_0 = X$ be a one-relator hierarchy. The following are equivalent:*

1. $X_N \looparrowright ... \looparrowright X_1 \looparrowright X_0 = X$ is a quasi-convex hierarchy and $\pi_1(X)$ is hyperbolic,

2. $X_N \looparrowright ... \looparrowright X_1 \looparrowright X_0 = X$ is an acylindrical hierarchy.

Moreover, if either of the above is satisfied, then $\pi_1(X)$ is virtually special and $\pi_1(A)$ is quasi-convex in $\pi_1(X)$ for any subcomplex $A \subset X_i$.

Proof. If $\pi_1(X)$ is hyperbolic, then $\pi_1(X_i)$ is hyperbolic for all i by [Ger96, Corollary 7.8]. Then the fact that (1) implies (2) is Proposition 2.6.5.

So let us show that (2) implies (1). The proof is by induction. Clearly X_N is hyperbolic and each subcomplex of X_N has quasi-convex fundamental group. So now suppose that the result is true for $X_N \looparrowright ... \looparrowright X_1$. By Corollary 4.3.2, $\tilde{A}_1$ and $\tilde{B}_1$ quasi-isometrically embed in $\tilde{X}_1$. By Theorem 2.6.6, $\pi_1(X)$ is hyperbolic and $\tilde{A}_1$ and $\tilde{B}_1$ quasi-isometrically embed in $\tilde{X}$.

Virtual specialness is [Wis21, Theorem 13.3]. The fact that $\pi_1(A)$ is quasi-convex in $\pi_1(X)$ for any subcomplex $A \subset X_i$ follows from Corollary 4.3.2. $\square$

Remark 4.4.2. In [Kap01] it is remarked that one-relator groups with torsion have acylindrical one-relator hierarchies. Thus, Theorem 4.4.1 also encompasses the torsion case, proved by Wise [Wis21].

Chapter 5

Exceptional intersection groups

The interactions between Magnus subgroups of one-relator groups are well understood. The following is [Col04, Theorem 2].

Theorem 5.0.1. *Let $F(\Sigma)/\langle\langle w \rangle\rangle$ be a one-relator group and suppose $\Sigma = A \sqcup B \sqcup C$. If $\langle A, B \rangle$ and $\langle B, C \rangle$ are Magnus subgroups, then one of the following holds:*

1. $\langle A, B \rangle \cap \langle B, C \rangle = \langle B \rangle$,

2. $\langle A, B \rangle \cap \langle B, C \rangle = \langle B \rangle * \mathbb{Z}$.

We say $\langle A, B \rangle$ and $\langle B, C \rangle$ have *exceptional intersection* if the latter situation occurs.

Definition 5.0.2. A one-relator group G is an *exceptional intersection group* if, for some one-relator presentation of G, it has a pair of Magnus subgroups with exceptional intersection.

The following result appears as [Col04, Corollary 2.3].

Corollary 5.0.3. *Exceptional intersection groups are torsion-free.*

By Theorem 2.4.4, it follows that exceptional intersection groups have non-positive immersions. The aim of this chapter is to characterise when exceptional intersection groups have negative immersions; see Theorem 5.2.8.

5.1   Primitive exceptional intersection groups

In this section, we introduce primitive exceptional intersection groups and show that they have negative immersions.

Let $F(A, B, C)$ be a free group, freely generated by disjoint sets A, B, C. Let $p/q \in \mathbb{Q}_{>0}$ and let:

$$x \in \langle A, B \rangle - \langle B \rangle \ ,$$
$$y \in \langle B, C \rangle - \langle B \rangle \ .$$

Then we call $\mathrm{pr}_{p/q}(x, y)$ a *primitive exceptional intersection word of the first type* if the following hold:

1. $\{\langle x \rangle, \langle y \rangle\}$ is a malnormal family,

2. if $p/q = 1$, then there is no $a \in \langle A, B \rangle - \langle B \rangle$, $c \in \langle B, C \rangle - \langle B \rangle$ such that $\mathrm{pr}_1(x, y) = \mathrm{pr}_1(a, c)$ and $\{\langle a \rangle, \langle c \rangle\}$ is not a malnormal family.

By definition, we see that $\langle x, y \rangle$ is an infinite cyclic subgroup of $G = F(A, B, C)/\langle\langle w \rangle\rangle$ where $w = \mathrm{pr}_{p/q}(x, y)$. We find that G has the following exceptional intersection:

$$\langle A, B \rangle \cap \langle B, C \rangle =_G \langle B \rangle * \langle x^p \rangle = \langle B \rangle * \langle y^{-q} \rangle \ .$$

The following example demonstrates why we require the second condition in the definition.

Example 5.1.1. Consider the word $\mathrm{pr}_1(a^2 b^{-1}, bc^2) \in F(a, b, c)$. Although the subgroups $\{\langle a^2 b^{-1} \rangle, \langle bc^2 \rangle\}$ form a malnormal family, we have

$$\mathrm{pr}_1(a^2 b^{-1}, bc^2) = \mathrm{pr}_1(a^2, c^2) = a^2 c^2 \ ,$$

where $\{\langle a^2 \rangle, \langle c^2 \rangle\}$ is not a malnormal family. Hence, $\mathrm{pr}_1(a^2 b^{-1}, bc^2)$ is not a primitive exceptional intersection word of the first type.

Now let $z \in \langle B \rangle - 1$. We call $\mathrm{pr}_{p/q}(xy, z)$ a *primitive exceptional intersection word of the second type* if the following hold:

1. $\langle z \rangle$ is malnormal,

2. if $p/q = 1/k$, then there is no $a \in \langle A, B \rangle - \langle B \rangle$, $c \in \langle B, C \rangle - \langle B \rangle$ such that $\mathrm{pr}_{1/k}(xy, z) = \mathrm{pr}_1(a, c)$ and $\{\langle a \rangle, \langle c \rangle\}$ is not a malnormal family.

By definition, we see that $\langle xy, z \rangle$ is an infinite cyclic subgroup of $G = F(A, B, C)/\langle\langle w \rangle\rangle$, where $w = \mathrm{pr}_{p/q}(xy, z)$. Hence, $(xy)^{-1} z (xy) =_G z$ and we find that G has the following exceptional intersection:

$$\langle A, B \rangle \cap \langle B, C \rangle =_G \langle B \rangle * \langle x^{-1} z x \rangle = \langle B \rangle * \langle yzy^{-1} \rangle \ .$$

A word $w \in F(A, B, C)$ is a *primitive exceptional intersection word* if w is conjugate to a primitive exceptional intersection word of the first or second type.

Definition 5.1.2. A group G is a *primitive exceptional intersection group* if $G \cong F(\Sigma)/\langle\langle w \rangle\rangle$ where w is a primitive exceptional intersection word.

Example 5.1.3. Consider the word $\mathrm{pr}_{1/k}(a^2b^2c^2, b) = a^2b^2c^2b^k \in F(a, b, c)$. Although this is equal to $\mathrm{pr}_1(a^2b^{2-k}, b^{-k}c^2b^k)$ where $\langle b^{-k}c^2b^k \rangle$ is not malnormal, it is also equal to the word $\mathrm{pr}_{1/2}(a^2b^{2-k}, b^{-k}cb^k)$ which is a primitive exceptional intersection word of the first type. Thus

$$G = F(a, b, c)/\langle\langle a^2b^2c^2b^k \rangle\rangle$$

is a primitive exceptional intersection group for all $k \geqslant 1$.

The proof of the following theorem is rather involved and will take up the remainder of this section.

Theorem 5.1.4. *Primitive exceptional intersection groups have negative immersions.*

Proof. If G is a primitive exceptional intersection group, then there is a free group $F(A, B, C)$, a rational number $p/q \in \mathbb{Q}_{>0}$, and elements:

$$x \in \langle A, B \rangle - \langle B \rangle \ ,$$
$$y \in \langle B, C \rangle - \langle B \rangle \ ,$$
$$z \in \langle B \rangle - 1 \ ,$$

such that $G \cong F(A, B, C)/\langle\langle w \rangle\rangle$ where one of the following holds:

1. $w = \mathrm{pr}_{p/q}(x, y)$ is a primitive exceptional intersection word of the first type,

2. $w = \mathrm{pr}_{p/q}(xy, z)$ is a primitive exceptional intersection word of the second type.

Let us assume for sake of contradiction that G does not have negative immersions. Then by Corollary 5.0.3 and Theorem 2.4.5, $\pi(w) = 2$. We will handle only the first case in full detail as the two cases are very similar.

Let $X = (\Delta, \omega)$ be the one-relator presentation complex for $\langle A, B, C \mid w \rangle$. The edges of Δ will be given labels according to the generators they correspond to and can be partitioned into *A-edges*, *B-edges* and *C-edges*. By Theorem 2.3.7, the w-subgroup of $\pi_1(X)$ is represented by an immersion of one-relator complexes $Q \looparrowright X$ where $Q = (\Lambda, \lambda)$ and $\chi(Q) = 0$. Now let $\Gamma \looparrowright \Delta$ be the graph immersion of core graphs representing $\langle x, y \rangle < \pi_1(\Delta)$. Then there is a lift $\gamma : S^1 \looparrowright \Gamma$ of ω,

making $Z = (\Gamma, \gamma)$ a one-relator complex, immersing into X. Note that Γ and Λ are core graphs.

Lemma 5.1.5. *There is some connected component $\Theta \subset \text{Core}(\Gamma \times_\Delta \Lambda)$ such that $\Theta \cong S^1$ and $\lambda = p_\Lambda \mid \Theta$.*

Proof. By [Ken09, Theorem 1], either $\chi(\text{Core}(\Gamma \times_\Delta \Lambda)) = 0$, or $\text{rk}(\langle \pi_1(\Lambda), \pi_1(\Gamma)^g \rangle) = 2$ for some $g \in \pi_1(\Delta)$. In the first case, we must have that λ factors through some component $\Theta \subset \text{Core}(\Gamma \times_\Delta \Lambda)$. Since X has non-positive immersions, ω and λ are primitive by Theorem 2.4.4 and we are done.

Now suppose that $\text{rk}(\langle \pi_1(\Lambda), \pi_1(\Gamma)^g \rangle) = 2$. We claim that Z factors through Q. Since $\pi_1(\Lambda)$ is a w-subgroup for ω, Γ must factor through Λ. Since $\pi_1(Q)$ is not free, Q cannot have a free face. Thus, by [LW17, Theorem 1.2], $\Lambda \times_\Delta S^1$, where the factor on the right is given by λ, must consist of a single cycle of degree one as $\chi(\Lambda) = -1$. If Γ factors through Λ, then the cycle

$$ S^1 \overset{\omega}{\looparrowright} \Gamma \looparrowright \Lambda \ , $$

must be equal to λ. Hence, Z must actually factor through Q. We split the remainder of the proof into two parts, according to whether w is of the first or second type.

Suppose that w is of the first type. Then since Z factors through Q, there must be loops based at the same vertex in Λ with labels x and y, covering Λ. Since a path labelled by x cannot traverse any C-edges and a path labelled by y cannot traverse any A-edges, it follows that there is a decomposition $Q^{(1)} = Q_1 \cup Q_2$ where $\chi(Q_1), \chi(Q_2) = 0$ and Q_1 only contains A-edges and B-edges and Q_2 only contains B-edges and C-edges. Moreover, the path labelled by x is supported in Q_1 and the path labelled by y is supported in Q_2. If $Q_1 \cap Q_2$ is connected, then since x and y are not proper powers by assumption, it follows that $\pi_1(Q) = \langle x, y \rangle$. But this contradicts the fact that $\pi_1(Q)$ is not free. If $Q_1 \cap Q_2$ is not connected, then x would be conjugate to $y^{\pm 1}$, contradicting malnormality of $\{\langle x \rangle, \langle y \rangle\}$.

Now suppose that w is of the second type. Since Z factors through Q, there must be loops based at the same vertex in Λ with labels xy and z, covering Λ. Since z only traverses B-edges, it follows that there is a decomposition $Q^{(1)} = Q_1 \cup Q_2 \cup Q_3$ where $\chi(Q_1) = 0$, Q_1 only contains B-edges, $\chi(Q_2) = \chi(Q_3) = 1$, Q_2 only contains A-edges and B-edges and Q_3 only contains B-edges and C-edges. Moreover, z is supported in Q_1, x is supported in $Q_1 \cup Q_2$ and y is supported in $Q_1 \cup Q_3$. Since z is not a proper power by assumption, it follows that $\pi_1(Q) = \langle xy, z \rangle$, contradicting the assumption that $\pi_1(Q)$ was not free. $\square$

We will make use of the following factorisations of x and y:

$$x = b_1 \cdot x_1^{-1} \cdot x_2 \cdot x_1 \cdot b_2 \ ,$$
$$y = b_3 \cdot y_1^{-1} \cdot y_2 \cdot y_1 \cdot b_4 \ ,$$

as freely reduced words, where $b_1, b_2, b_3, b_4 \in \langle B \rangle$, $x_1^{-1} \cdot x_2 \cdot x_1$ and $y_1^{-1} \cdot y_2 \cdot y_1$ do not begin or end with a B-letter and x_2 and y_2 are cyclically reduced.

If $n \geqslant 1$, denote by u_n the free reduction of $x_1^{-1} x_2 x_1 (b_2 b_1 x_1^{-1} x_2 x_1)^{n-1}$. Similarly, denote by v_n the free reduction of $y_1^{-1} y_2 y_1 (b_4 b_3 y_1^{-1} y_2 y_1)^{n-1}$.

Lemma 5.1.6. *Let $n \geqslant 1$ and let $\alpha : I \looparrowright \Lambda$ be a path labelled by u_n or v_n. Then α must traverse a vertex of degree at least three, other than at its endpoints.*

Proof. We shall prove the result for u_n as the other case is identical. Firstly, we show that Γ supports precisely one path with label u_n. By definition, Γ has at least one, so let us suppose that there are two paths in Γ with label u_n. Recall that $\pi_1(\Gamma) = \langle x, y \rangle$. Since u_n begins and ends with an A-letter and does not contain any C-letters, it follows that:

1. if $b_2 b_1 \neq 1$ or $b_2 b_1 = x_1 = 1$, then u_n must be a subword of u_{n+1} that is not a prefix or a suffix, or u_n is a subword of u_{n+1}^{-1},

2. if $b_2 b_1 = 1$ and $x_1 \neq 1$, then u_n must be a subword of $u_m^{\pm 1}$ for some $m > n$.

Since $\langle x \rangle$ is malnormal, x is not a proper power. The first situation cannot happen by Lemma 2.2.2. The second situation cannot happen by Lemma 2.2.1.

Now, since there is precisely one path in Γ with label u_n, there can be at most one lift of α to $\mathrm{Core}(\Gamma \times_\Delta \Lambda)$ by definition of the fibre product graph. If α does not traverse vertices of degree three or more, except possibly at its endpoints, then it must factor through any cycle in Λ whose image shares an edge with the image of α. Since α is injective on edges, any edge in the image of α has at most one preimage in $E(\mathrm{Core}(\Gamma \times_\Delta \Lambda))$. So by Lemma 5.1.5, λ traverses some edge precisely once. But then Q must have a free face and so cannot represent a w-subgroup. $\square$

We now use Lemma 5.1.6 to derive a contradiction to the definitions of primitive exceptional intersection words. Let $\alpha : I \looparrowright \Lambda$ be a path satisfying the following:

1. α factors through λ,

2. α is labelled by u_n for some $n > 0$,

3. there is no path $\alpha' : I \looparrowright \Lambda$, strictly extending α and satisfying the above.

We similarly define $\beta : I \looparrowright \Lambda$, replacing u_n with v_n. Such paths exist by definition of w.

By Lemma 5.1.6, α and β must traverse a vertex of degree at least three, away from their endpoints. Now the idea is to use this fact to divide Λ according to where α or β are supported. Since α does not traverse any C-edges and β does not traverse any A-edges, they will block each other from traversing certain regions of Λ.

Since $\chi(\Lambda) = -1$, and Λ is a core graph, we only have three topologically distinct cases to consider:

1. Λ is a rose graph, see Figure 5.1,

2. Λ is a theta graph, see Figure 5.2,

3. Λ is a spectacles graph, see Figure 5.4.

Figures 5.1, 5.2 and 5.4 contain all the different cases, up to symmetry. Before proceeding with the case analysis, we briefly explain the diagrams. The red regions indicate sections that α traverses and must contain an A-edge; β cannot traverse any edge in a red region. The blue regions indicate sections that β traverses and must contain a C-edge; α cannot traverse any edge in a blue region. The yellow regions indicate sections that α or β or both α and β traverse. In any case, the yellow regions must only contain B-edges, but are allowed to have length zero when this does not change the topology of the underlying graph. The black regions indicate sections that are not traversed by either α or β and are also allowed to have length zero when this does not change the topology of the underlying graph. The red vertices and blue vertices indicate the start and endpoints of α and β respectively.

Topologically, in each graph there can be at most three edges. The path α must leave one of these edges by Lemma 5.1.6 and re-enter another edge, leaving enough space for β to do the same elsewhere. Given these constraints, the reader should check that these are indeed all the cases to consider.

Case 1. We handle this case more in detail than the others as the arguments are mostly identical. We have three subcases to consider, according to Figure 5.1.

Suppose we are in the situation of the first subdiagram. When λ traverses a red segment from a red vertex, it must then be followed by the other red segment. Otherwise we would obtain a contradiction to Lemma 5.1.6. Similarly for the blue segments. Thus, since λ is primitive, it must traverse each edge precisely once. Hence, Q would have a free face which is a contradiction.
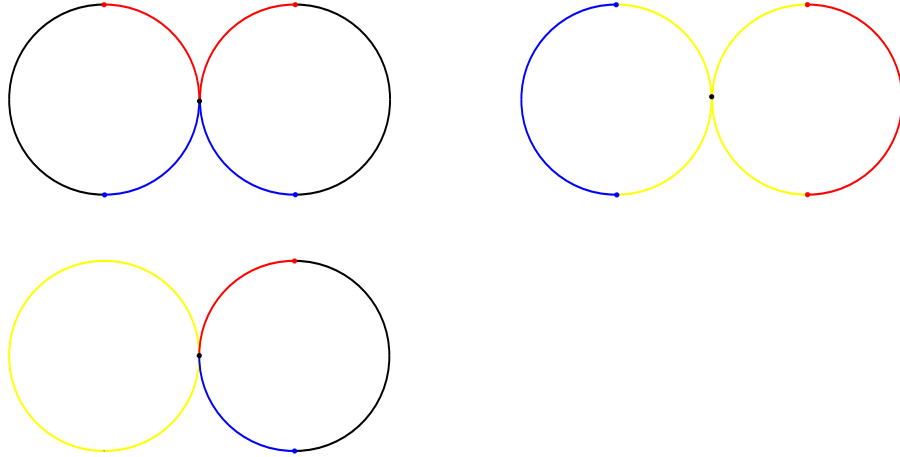
Figure 5.1: Rose graph cases.

Now consider the second subdiagram. Any (maximal) u_n labelled path must begin at one red vertex and end at the other red vertex. Similarly for the v_n labelled paths. But this then implies that only one power of x and one power of y appears in w. Thus, if w is of the first type, then it must be equal to xy. If w is of the second type, it must be equal to xyz^i for some $i \geqslant 1$. By Lemma 5.1.6, λ must traverse both loops at least twice. We may now deduce that there exist elements:

$$a \in \langle A, B \rangle - \langle B \rangle \,,$$
$$b \in \langle B \rangle - 1 \,,$$
$$c \in \langle B, C \rangle - \langle B \rangle \,,$$

such that $w = (ab^{-1})(bc)$ and that $\pi(a), \pi(c) \neq 1$. Hence, $\{\langle a \rangle, \langle c \rangle\}$ is not a malnormal family, contradicting our assumptions on w.

Let us move onto the third subdiagram. Similarly to the second subcase, we see that if w is of the first type, it must be equal to xy and if w is of the second type, it must be equal to xyz^i for some $i \geqslant 1$. From the diagram we may now deduce that there exist elements:

$$a \in \langle A, B \rangle - \langle B \rangle \,,$$
$$b \in \langle B \rangle \,,$$
$$c \in \langle B, C \rangle - \langle B \rangle \,,$$

such that $w = (ab^{-1})(bc)$ and that $\langle a \rangle^g \cap \langle c \rangle \neq 1$ for some $g \in F(A, B, C)$. Hence, $\{\langle a \rangle, \langle c \rangle\}$ is not a malnormal family, contradicting our assumptions on w. It follows
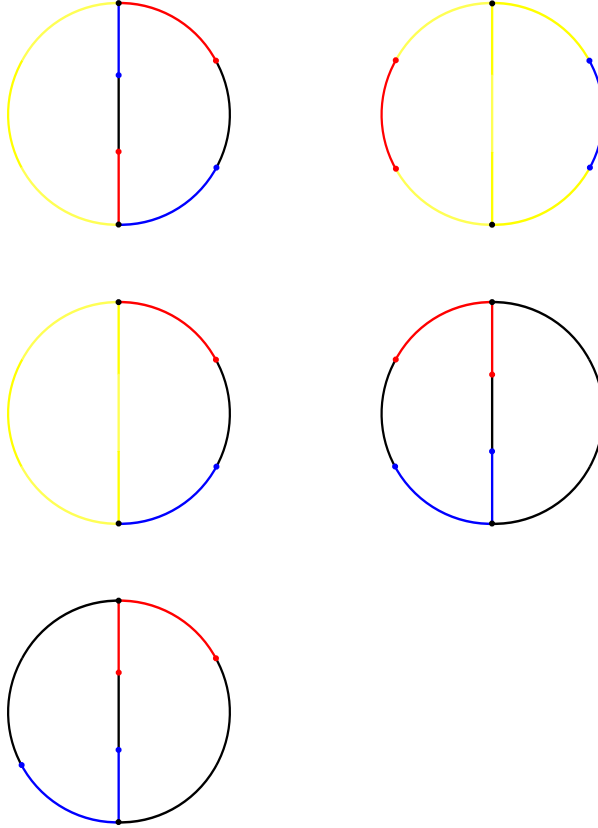
Figure 5.2: Theta graph cases.

that Λ cannot be a rose graph.

Case 2. Consider the first subdiagram in Figure 5.2. By collapsing the yellow edge, we see that we may handle this case in the same way as the first subcase of the rose case. Similarly, we may reduce the second and third subcases to the second and third subcases of the rose case.

So now let us consider the new cases appearing in the fourth and fifth subdiagrams. Now if $b_2 b_1 \neq 1$ and α was labelled by u_n with $n \geqslant 2$, then we would have that Λ would support a path labelled by u_1 that does not traverse a vertex of degree at least three, away from its endpoints. But this contradicts Lemma 5.1.6. So if $b_2 b_1 \neq 1$ and $n = 1$, then there can be no other u_m-labelled path beginning or ending at a red vertex as $\Lambda \looparrowright \Delta$ is an immersion. This would imply that there can be no other u_m-labelled paths in Λ for any $m \geqslant 1$ and so λ traverses the red segments only once by Lemma 5.1.5. Hence, Q would have a free face, a contradiction. So now we may assume that $b_2 b_1 = 1$. By a symmetric argument, we may assume that $b_4 b_3 = 1$. Similarly, we must have $x_1, y_1 \neq 1$. As before, there must be at least one other path $\alpha' : I \looparrowright \Lambda$ labelled by u_m for some $m \geqslant 1$. We may assume that
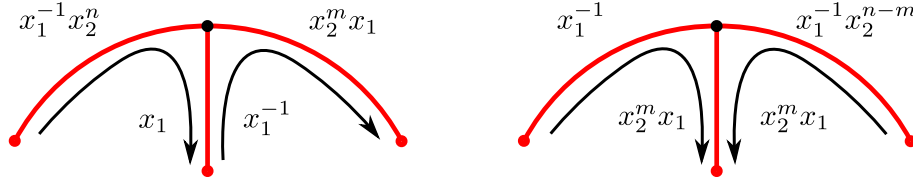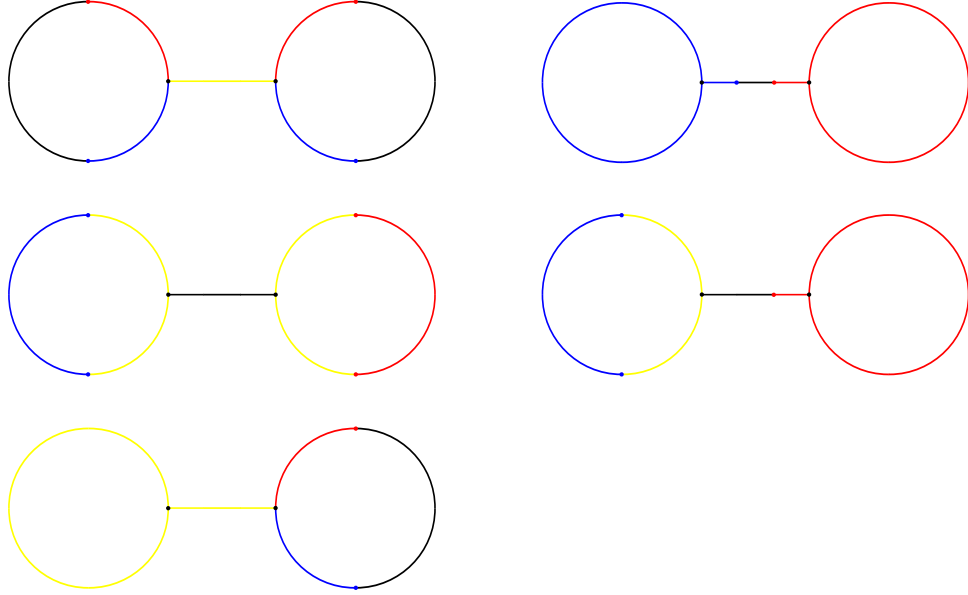
Figure 5.3: Extra theta cases.

Figure 5.4: Spectacles graph cases.

α' is maximal. We see that α and α' must traverse a common segment. We now have two subcases to consider, up to symmetry, depending on whether α and α' traverse a common segment in the same direction or the opposite direction. See Figure 5.3. In either case, there can only be one lift of the segment with label $x_2^m x_1$ to $\mathrm{Core}(\Gamma \times_\Delta \Lambda)$ for the following reason: the projection of any loop in $\mathrm{Core}(\Gamma \times_\Delta \Lambda)$ traversing the segment labelled $x_2^m x_1$, must then traverse a blue segment labelled by $y_1^{-1} y_2^k$ for some $k \geqslant 0$. Since there is only one path in Γ with this label, there can be only one lift of these segments to $\mathrm{Core}(\Gamma \times_\Delta \Lambda)$. Now Lemma 5.1.5 tells us that Q has a free face. It follows that Λ cannot be a theta graph.

Case 3. The first subdiagram in Figure 5.4 is analogous to the first subdiagram of the rose case. The second, third and fourth subdiagrams are analogous to the second subdiagram of the rose case. The fifth subdiagram is analogous to the third subdiagram of the rose case. Hence, Λ cannot be a spectacles graph.

Now we may conclude that G has no w-subgroups of rank two and hence, must have negative immersions. $\qquad\square$

5.2 The general case

In this section, we show that primitive exceptional intersection groups are the only exceptional intersection groups which have negative immersions. Moreover, we show that exceptional intersection groups that do not have negative immersions, must contain a Baumslag–Solitar subgroup.

Theorem 5.0.1 was generalised to one-relator products in [How05]. As a consequence, we have the following strengthening of Theorem 5.0.1, appearing as [How05, Theorem C].

Theorem 5.2.1. *Let $F(\Sigma)/\langle\langle w \rangle\rangle$ be a one-relator group and suppose $\Sigma = A \sqcup B \sqcup C$. If $\langle A, B \rangle$ and $\langle B, C \rangle$ are Magnus subgroups with exceptional intersection, then there is a monomorphism of free groups*

$$\iota : F(a,c) \hookrightarrow F(\Sigma) \;,$$

with the following properties. There is some $r \in F(a,c)$ with $\iota(r)$ conjugate to w and some $m, n \neq 0$, such that one of the following holds:

1. $a^m = c^n$ in $F(a,c)/\langle\langle r \rangle\rangle$ with $\iota(a) = x$, $\iota(c) = y$ and

$$x \in \langle A, B \rangle - \langle B \rangle \;,$$
$$y \in \langle B, C \rangle - \langle B \rangle \;.$$

2. $ac^m a^{-1} = c^n$ in $F(a,c)/\langle\langle r \rangle\rangle$ with $\iota(a) = xy$, $\iota(c) = z$ and

$$x \in \langle A, B \rangle - \langle B \rangle \;,$$
$$y \in \langle B, C \rangle - \langle B \rangle \;,$$
$$z \in \langle B \rangle - 1 \;.$$

Using this result and the algorithm to compute the centre of a one-relator group from [Bau67b], Howie also showed that a generating set for the intersection of given Magnus subgroups is computable [How05, Theorem E].

In the discussion following [Col04, Theorem 4], Collins points out the following.

Corollary 5.2.2. *Assume the notation of Theorem 5.2.1 and suppose that we are in the first case. Denote by*

$$H = \langle B \rangle * F(a,c)/\langle\langle r \rangle\rangle \;.$$

Then we have:

$$G \cong \langle A, B \rangle \underset{\langle B, \iota(a) \rangle = \langle B, a \rangle}{*} H \underset{\langle B, c \rangle = \langle B, \iota(c) \rangle}{*} \langle B, C \rangle \ .$$

Corollary 5.2.3. *Assume the notation of Theorem 5.2.1 and suppose that we are in the second case. Let $\iota(a) = x \cdot y$ where $x \in \langle A, B \rangle - \langle B \rangle$ and $y \in \langle B, C \rangle - \langle B \rangle$ and denote by*

$$H = \langle B \rangle \underset{\langle \iota(c) \rangle = \langle c \rangle}{*} F(a, c) / \langle\langle r \rangle\rangle \underset{\langle a \rangle = \langle de \rangle}{*} F(d, e) \ .$$

Then we have:

$$G \cong \langle A, B \rangle \underset{\langle B, x \rangle = \langle B, d \rangle}{*} H \underset{\langle B, e \rangle = \langle B, y \rangle}{*} \langle B, C \rangle \ .$$

Remark 5.2.4. There is a minor typographical error in the splitting provided by Collins for the second case. Corollary 5.2.3 is the corrected version.

The following follows directly from Corollaries 5.2.2 and 5.2.3.

Corollary 5.2.5. *Assume the notation of Theorem 5.2.1. The monomorphism ι descends to a monomorphism of one-relator groups:*

$$\bar{\iota} : F(a, c) / \langle\langle r \rangle\rangle \hookrightarrow F(\Sigma) / \langle\langle w \rangle\rangle \ .$$

Theorem 5.2.1, coupled with Corollary 5.2.5, finds us two-generator one-relator subgroups of exceptional intersection groups. We now characterise precisely what these subgroups can be.

Lemma 5.2.6. *Let $H = F(a, c) / \langle\langle r \rangle\rangle$ be torsion-free such that $a^m = c^n$ in H for some $m, n \neq 0$. Then one of the following hold:*

1. *$r \in F(a, c)$ is primitive and so $H \cong \mathbb{Z}$,*

2. *H is non-cyclic with non-trivial centre.*

Proof. Note that H has non-trivial centre as $\langle a^m \rangle$ is an infinite subgroup contained in the centre. By [LS01, Chapter II Proposition 5.11], H is cyclic if and only if r is primitive. $\qquad\square$

Lemma 5.2.7. *Let $H = F(a, c) / \langle\langle r \rangle\rangle$ be torsion-free such that $a^{-1} c^m a = c^n$ in H for some $m, n \neq 0$. Then one of the following hold:*

1. *$r \in F(a, c)$ is primitive and so $H \cong \mathbb{Z}$,*

2. *H is non-cyclic with non-trivial centre,*

3. r or r^{-1} is conjugate to $\mathrm{pr}_{p/q}(c^{-1}, a^{-1}ca)$ for some $p/q \in \mathbb{Q}_{>0}$ and so $H \cong BS(p, q)$.

4. r or r^{-1} is conjugate in $F(a, c)$ to an element $r' \in \langle c, a^{-1}ca \rangle$ such that $\langle c, a^{-1}ca \rangle \cong F(c, a^{-1}ca)/\langle\langle r' \rangle\rangle$ is non-cyclic with non-trivial centre.

Proof. Suppose that $|m|$ is smallest possible. If $n = m = \pm 1$, then either r is primitive or $H \cong \mathbb{Z}^2$. If $n = m \neq \pm 1$, then H has non-trivial centre generated by c^n. So if $n = m$, we have obtained conclusion (1) or (2). Now suppose that $n \neq m$. Therefore, the exponent sum of c in r must be non-zero. There is a single epimorphism, up to sign change, $\phi : F(a, c) \to \mathbb{Z}$ such that $\phi(r) = 0$, given by

$$a \to \frac{-\sigma_c(r)}{\gcd(\sigma_a(r), \sigma_c(r))} \ ,$$
$$c \to \frac{\sigma_a(r)}{\gcd(\sigma_a(r), \sigma_c(r))} \ ,$$

where $\sigma_a(r), \sigma_c(r)$ denote the exponent sum of a and c in r respectively. But now $\phi(a^{-1}c^m a c^{-n}) = 0$ from which it follows that

$$0 = \phi(c^m) - \phi(c^n) = (m - n)\sigma_a(r) \ .$$

Now let X be the presentation complex for $F(a, c)/\langle\langle r \rangle\rangle$ and $p : Y \to X$ the cyclic cover associated with the homomorphism $\bar{\phi} : H \to \mathbb{Z}$ given by $\bar{\phi}(a) = 1$, $\bar{\phi}(c) = 0$. Let $Z \in \mathrm{TD}(p)$ be a one-relator tree domain. We see that $\pi_1(Z)$ is generated as a one-relator group by $c, a^{-1}ca, ..., a^{-k}ca^k$ for some $k \in \mathbb{N}$. By the Freiheitssatz, it follows that $k = 1$. If $\pi_1(Z) \cong \mathbb{Z}$, then we have obtained conclusion (3). If not, then we have obtained conclusion (4). □

We are now ready to prove the main result of this chapter.

Theorem 5.2.8. *Let G be an exceptional intersection group. The following are equivalent:*

1. *G contains a Baumlsag–Solitar subgroup,*

2. *G is not a primitive exceptional intersection group,*

3. *G does not have negative immersions.*

Proof. By Theorem 2.4.5, (1) implies (3). By Theorem 5.1.4, (3) implies (2). So all that is left to show is that (2) implies (1). Suppose for a contradiction that G is not a primitive exceptional intersection group and does not contain Baumslag–Solitar

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subgroups. Let $H < G$ be the two-generator subgroup from Theorem 5.2.1. We may assume that H is maximal in the sense that there is no subgroup properly containing H and that is of the same form as the two-generator subgroup from Theorem 5.2.1. By Lemmas 5.2.6 and 5.2.7 and Corollary 5.2.5, H is infinite cyclic. Thus, G has one of the following presentations:

1. $G \cong \langle \Sigma \mid \mathrm{pr}_{p/q}(x, y) \rangle$ for some $x \in \langle A, B \rangle - \langle B \rangle$ and $y \in \langle B, C \rangle - \langle B \rangle$,

2. $G \cong \langle \Sigma \mid \mathrm{pr}_{p/q}(xy, z) \rangle$ for some $x \in \langle A, B \rangle - \langle B \rangle$, $y \in \langle B, C \rangle - \langle B \rangle$ and $z \in \langle B \rangle$.

where $H = \langle x, y \rangle$ in the first case, and $H = \langle xy, z \rangle$ in the second case.

Suppose that we are in the first situation. By Definition 5.1.2, we may assume that either $\{\langle x \rangle, \langle y \rangle\}$ is not a malnormal family, or that $p/q = 1$ and that there exist elements $a \in \langle A, B \rangle - \langle B \rangle$ and $c \in \langle B, C \rangle - \langle B \rangle$ such that $\mathrm{pr}_1(x, y) = \mathrm{pr}_1(a, c)$ and $\{\langle a \rangle, \langle c \rangle\}$ is not a malnormal family. As the two cases are identical, it suffices to assume that $\{\langle x \rangle, \langle y \rangle\}$ is not a malnormal family. Now, if $\{\langle x \rangle, \langle y \rangle\}$ is not a malnormal family, then either x is a proper power, y is a proper power, or a conjugate of $\langle x \rangle$ intersects $\langle y \rangle$ non-trivially. If either x or y is a proper power, by adjoining a root of x or y to H, we obtain a contradiction to maximality of H. If $\langle x \rangle^f \cap \langle y \rangle \neq 1$ for some $f \in F(\Sigma)$, it follows that there must be elements $g \in \langle A, B \rangle - \langle B \rangle$, $h \in \langle B, C \rangle - \langle B \rangle$, $d \in \langle B \rangle$ such that $\langle x \rangle^g \cap \langle y \rangle^{h^{-1}} < \langle d \rangle < \langle B \rangle$. Now H would be properly contained in $\langle gh, h^{-1}dh \rangle$. However, since $(gh)(h^{-1}dh)^n(gh)^{-1} = (h^{-1}dh)^m$ holds for some $m, n \neq 0$, we obtain a contradiction to maximality of H.

Finally, suppose that we are in the second situation. As before, we may assume that $\langle z \rangle$ is not malnormal by Definition 5.1.2. Then z is a proper power and we contradict maximality of H. $\qquad\square$

Example 5.2.9. We give two examples of groups with exceptional intersection that do not have negative immersions. Let $p/q \in \mathbb{Q}_{>0}$ and $n, m \neq 0$. Consider the group with presentation:

$$G \cong \langle c_0, c_1 \mid \mathrm{pr}_{p/q}(c_0^m, c_1^{-n}) \rangle .$$

The relation $c_0^{pm} = c_1^{qn}$ holds in G and so it has an exceptional intersection of the first type. In [MPS73], this group was shown to be isomorphic to a generalised Baumslag–Solitar group with presentation:

$$\langle c_0, b, c_1 \mid c_0^m = b^q, b^p = c_1^n \rangle ,$$

and so does not have negative immersions.

Now consider the HNN-extension:

$$\langle c_0, b, c_1, a \mid c_0^m = b^q, b^p = c_1^n, ac_0a^{-1} = c_1 \rangle \cong \langle c_0, c_1, a \mid ac_0a^{-1} = c_1, \mathrm{pr}_{p/q}(c_0^m, c_1^{-n}) \rangle$$
$$\cong \langle a, c \mid \mathrm{pr}_{p/q}(c^m, a^{-1}c^{-n}a) \rangle$$

The relation $a^{-1}c^{mq}a = c^{np}$ holds in this group and so

$$\langle a_0, a_1, c \mid \mathrm{pr}_{p/q}(c^m, (a_0a_1)^{-1}c^{-n}(a_0a_1)) \rangle$$

has an exceptional intersection of the second type. This group also contains a generalised Baumslag–Solitar group and so does not have negative immersions.

Remark 5.2.10. More generally, by [Pie74] and [MPS73, Theorem 1], if G is an exceptional intersection group that does not have negative immersions, G has a w-subgroup isomorphic to a non-cyclic generalised Baumslag–Solitar group.

Chapter 6

Bass-Serre theory of one-relator splittings

In this chapter, we aim to understand more in detail the action of a one-relator group on the Bass-Serre tree associated with one of its one-relator splittings. Since stabilisers of (non-trivial) segments in the Bass-Serre tree are subgroups of intersections of conjugates of Magnus subgroups, the first step towards this goal is to understand such intersections. The following is [Col08, Theorem 2].

Theorem 6.0.1. *Let $F(\Sigma)/\langle\langle w \rangle\rangle$ be a one-relator group and let $A, B < G$ be Magnus subgroups. For all $g \in G$, one of the following holds:*

1. $g \in B \cdot A$,

2. $A \cap B^g = 1$,

3. $A \cap B^g \cong \mathbb{Z}$.

We may, in some sense, strengthen Theorems 5.0.1 and 6.0.1 to incorporate intersections of subgroups of Magnus subgroups. First, we will need the following lemma.

Lemma 6.0.2. *Let $F(\Sigma)/\langle\langle w \rangle\rangle$ be a one-relator group and let A, B be Magnus subgroups. Then $A \cap B$ is strongly inert in A and B.*

Proof. By Theorem 5.0.1, $A \cap B$ is an echelon subgroup of A and B. Now Corollary 2.5.2 implies the result. $\square$

Theorem 6.0.3. *Let $G = F(\Sigma)/\langle\langle w \rangle\rangle$ be a one-relator group and let $A, B < G$ be Magnus subgroups. If $C < A$, $D < B$ are finitely generated subgroups, then the*

following is satisfied:

$$\sum_{\substack{[C \cap D^g]_C \\ g \in G}} \mathrm{rr}(C \cap D^g) = \sum_{\substack{[C \cap D^g]_C \\ g \in B \cdot A}} \mathrm{rr}(C \cap D^g) \leqslant \mathrm{rr}(C) \, \mathrm{rr}(D) \ .$$

Moreover, if C is a strongly inert subgroup of A, then

$$\sum_{\substack{[C \cap D^g]_C \\ g \in G}} \mathrm{rr}(C \cap D^g) = \sum_{\substack{[C \cap D^g]_C \\ g \in B \cdot A}} \mathrm{rr}(C \cap D^g) \leqslant \mathrm{rr}(D) \ .$$

Proof. The first equality in both cases follows from Theorem 6.0.1. Lemma 6.0.2 implies the other inequalities when $C = A$. Since

$$\sum_{\substack{[C \cap D^g]_C \\ g \in B \cdot A}} \mathrm{rr}(C \cap D^g) = \sum_{\substack{[C \cap D^{ba}]_C \\ b \in B, a \in A}} \mathrm{rr}(C \cap ((A \cap B) \cap D^b)^a) \ ,$$

the first inequality now follows from the Hanna Neumann inequality [Min12, Fri15]. The second inequality follows from the definition of strongly inert subgroups. $\qquad\square$

6.1   Inertial one-relator extensions

Let H be a group and $\psi : A \to B$ an isomorphism between subgroups $A, B < H$. Inductively define

$$\mathcal{A}_0^\psi = \{[H]\}, \quad \mathcal{A}_1^\psi = \{[B]\}, \quad \mathcal{A}_{i+1}^\psi = \{[\psi(A \cap A_i)]\}_{A_i \in [A_i] \in \mathcal{A}_i^\psi} \ .$$

Then we denote by $\bar{\mathcal{A}}_i^\psi \subset \mathcal{A}_i^\psi$ the subset corresponding to the conjugacy classes of non-cyclic subgroups. Define the *stable number $s(\psi)$* of ψ as

$$s(\psi) = \sup \{k + 1 \mid \bar{\mathcal{A}}_k^\psi \neq \varnothing\} \in \mathbb{N} \cup \{\infty\} \ ,$$

where $s(\psi) = \infty$ if $\bar{\mathcal{A}}_i^\psi \neq \varnothing$ for all $i \in \mathbb{N}$. We say that $H*_\psi$ is *stable* if $s(\psi) < \infty$.

Definition 6.1.1. Let H be a finitely generated one-relator group and let $\psi : A \to B$ be an isomorphism between finitely generated, strongly inert subgroups of Magnus subgroups of H. We call $H*_\psi$ an *inertial one-relator extension.*

Remark 6.1.2. By Lemma 2.4.3, all one-relator splittings of a one-relator group are inertial one-relator extensions, up to possibly adding a free factor.

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For the purposes of this section, let us fix an inertial one-relator extension $H *_\psi$. Let T denote the associated Bass-Serre tree. See Section 2.1.4 and [Ser03] for the relevant notions in Bass-Serre theory. Each vertex of T can be identified with a left coset of H. Denote by

$$\mathcal{S}_{gH}^n = \{S \mid S \text{ a geodesic segment of length } n \text{ with endpoint at } gH\} \ ,$$
$$\mathcal{S}_{gH} = \bigcup_i \mathcal{S}_{gH}^i \ .$$

Each edge in T has an orientation induced by ψ. The elements $[A_n] \in \bar{\mathcal{A}}_n^\psi$ correspond to stabilisers of segments $S \in \mathcal{S}_H^n$ such that $\mathrm{rr}(\mathrm{Stab}(S)) \neq 0$ and such that S consists of edges only oriented towards H.

For inertial one-relator extensions, the ranks of stabilisers of elements in $\mathcal{S}_{gH}^n$ are bounded in a strong sense.

Lemma 6.1.3. *For all $n \geqslant 1$, the following holds:*

$$\sum_{\substack{S \in \mathcal{S}_H^n \\ [\mathrm{Stab}(S)]_H}} \mathrm{rr}(\mathrm{Stab}(S)) = 2 \sum_{[A_n] \in \bar{\mathcal{A}}_n^\psi} \mathrm{rr}(A_n) \leqslant 2\,\mathrm{rr}(A) \ .$$

Proof. Let $S \in \mathcal{S}_H^n$, then $\mathrm{Stab}(S) = H \cap H^c$ where c is equal to a reduced word of the form:

$$c = c_0 t^{\epsilon_1} c_1 t^{\epsilon_2} ... t^{\epsilon_n} c_n \ ,$$

with $\epsilon_i = \pm 1$ and $c_i \in H$. By Theorem 6.0.3, if there is some i such that $\epsilon_i = -\epsilon_{i+1}$, then $\mathrm{Stab}(S)$ is either cyclic or trivial since our word was reduced. Then by Theorem 6.0.3 and induction on n, it follows that

$$\sum_{\substack{S \in \mathcal{S}_H^n \\ [\mathrm{Stab}(S)]_H}} \mathrm{rr}(\mathrm{Stab}(S)) = \left(\sum_{[A_n] \in \bar{\mathcal{A}}_n^\psi} \mathrm{rr}(A_n) \right) + \left(\sum_{[A_n] \in \bar{\mathcal{A}}_n^{\psi^{-1}}} \mathrm{rr}(A_n) \right)$$
$$= 2 \sum_{[A_n] \in \bar{\mathcal{A}}_n^\psi} \mathrm{rr}(A_n) \ .$$

$\square$

The following lemma tells us that essentially, each element in $\bar{\mathcal{A}}_n^\psi$ corresponds to a unique segment in $\mathcal{S}_H^n$, directed towards the vertex H.

Lemma 6.1.4. *Let $[A_n] \in \bar{\mathcal{A}}_n^\psi$. Then one of the following holds:*

1. if $[A_n] \in \bar{\mathcal{A}}_n^{\psi^{-1}}$, there are precisely two H-orbits of elements $S \in \mathcal{S}_H^n$ such that $[\mathrm{Stab}(S)]_H = [A_n]_H$,

2. if $[A_n] \notin \bar{\mathcal{A}}_n^{\psi^{-1}}$, there is precisely one H-orbit of elements $S \in \mathcal{S}_H^n$ such that $[\mathrm{Stab}(S)]_H = [A_n]_H$.

Proof. Suppose $S, S' \in \mathcal{S}_H^n$ are in distinct H-orbits and that $\mathrm{Stab}(S) = \mathrm{Stab}(S') = A_n$. Let $g, h \in G$ be elements such that the endpoints of S and S' are gH and hH respectively. Then g and h are equal to reduced words:

$$g = g_0 t^{\epsilon_1} g_2 ... t^{\epsilon_n} g_n,$$
$$h = h_0 t^{\eta_1} h_2 ... t^{\eta_n} h_n.$$

By Lemma 6.1.3, we have that $\epsilon_i = 1$ or $\epsilon_i = -1$ for all i. Similarly for η_i. If $\epsilon_i = \eta_i$ for all i, then $g^{-1}h \in H$ or $\mathrm{rr}(H \cap H^{h^{-1}g}) = 0$. The former implies that S and S' were in the same H-orbit and the latter implies that $\mathrm{rr}(A_n) = 0$ as A_n stabilises the geodesic connecting gH and hH. Hence, we must have $\epsilon_i = -\eta_i$ for all i. Then, by definition, we have $[A_n] \in \bar{\mathcal{A}}_n^{\psi^{-1}}$. $\square$

The following proposition is key in our proof of Corollary 6.3.4, where we show that one-relator groups with negative immersions have stable hierarchies.

Proposition 6.1.5. *If $s(\psi) = \infty$, then there are $1 \leqslant n \leqslant \mathrm{rr}(A)$ many H-orbits of biinfinite geodesics $S \subset T$ such that the following holds:*

1. *S contains the vertex H,*

2. *every finite subset of S has non-cyclic, non-trivial stabiliser subgroup.*

Moreover, for every such S, the following holds:

1. *every edge in S is directed in the same direction,*

2. *there exists an element $g \in G$ acting by translation on S.*

Proof. The fact that every such geodesic must consist of edges directed in the same direction is Lemma 6.1.3. The fact that there are $1 \leqslant n \leqslant \mathrm{rr}(A)$ many such geodesics follows from Lemma 6.1.4 and by definition of $s(\psi)$. So now let $S_1, ..., S_n$ be the collection of such biinfinite geodesics in T. Identify each vertex of S_i with an integer so that H is the vertex associated with 0. Let $g_{i,j} \in G$ be an element such that $g_{i,j}H$ is the jth vertex in S_i. Then $g_{i,j}^{-1} \cdot S_i$ must be in the same H-orbit of some S_m. But then by the pigeonhole principle, $S_i, g_{i,1}^{-1} \cdot S_i, ..., g_{i,\mathrm{rr}(A)}^{-1} \cdot S_i$ must contain two

biinfinite geodesics in the same H-orbit. Suppose that $h_1 g_{i,k}^{-1} \cdot S_i = h_2 g_{i,l}^{-1} \cdot S_i = S_l$ for some $h_1, h_2 \in H$. Then we have

$$S_i = h_2 g_{i,l}(h_1 g_{i,k})^{-1} \cdot S_i \ ,$$

where $h_2 g_{i,l}(h_1 g_{i,k})^{-1}$ acts by translation on S_i. $\qquad\square$

Remark 6.1.6. The main consequence of Proposition 6.1.5 is that if $s(\psi) = \infty$, then there exists some element $g \in G$ acting hyperbolically on T, and such that $\mathrm{rr}(H \cap H^{g^n}) \neq 0$ for all $n \in \mathbb{Z}$.

We conclude this subsection with an algorithm to compute generating sets for conjugacy class representatives of elements in $\bar{\mathcal{A}}_n^{\psi}$.

Proposition 6.1.7. *There is an algorithm that, given as input the following:*

1. *a one-relator presentation $H \cong \langle \Sigma \mid r \rangle$,*

2. *a pair of subsets $X, Y \subset \Sigma$ that generate a pair of Magnus subgroups of H,*

3. *a tuple of words $\{a_1, ..., a_k\} \subset F(X)$ that form a free basis of the subgroup $A < H$,*

4. *a tuple of words $\{b_1, ..., b_k\} \subset F(Y)$ that form a free basis of the subgroup $B < H$,*

5. *an integer $n \geqslant 1$,*

computes generating sets in $F(Y)$ for representatives of each conjugacy class in $\bar{\mathcal{A}}_n^{\psi}$ where $\psi : A \to B$ is given by $\psi(a_i) = b_i$.

Proof. By [How05, Theorem E], we may compute generating sets $\{c_1, ..., c_m\} \subset F(X)$ and $\{d_1, ..., d_m\} \subset F(Y)$ for the subgroup $\langle X \rangle \cap \langle Y \rangle$. We may assume that $c_i = d_i$ as elements in H. The proof is by induction on n. The base case of $n = 1$ is immediate. So now suppose that $n \geqslant 2$ and that we have computed $\bar{\mathcal{A}}_{n-1}^{\psi}$. Let

$$\{\{y_{1,1}, y_{1,2}, ..., y_{1,m_1}\}, ..., \{y_{k,1}, y_{k,2}, ..., y_{k,m_k}\}\}$$

be generating sets in $F(Y)$ for conjugacy class representatives of elements in $\bar{\mathcal{A}}_{n-1}^{\psi}$. Denote by $Y_i = \langle y_{i,1}, ..., y_{i,m_i} \rangle < F(Y)$. By our discussion of the results in [Sta83] in Section 2.1.3, we may compute generating sets for conjugacy class representatives of each $[D \cap Y_i^y]_{F(Y)}$ where $D = \langle d_1, ..., d_m \rangle$ and $y \in F(Y)$. By substituting each

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appearance of d_i with c_i and then $\psi(c_i)$, we obtain generating sets in $F(Y)$ for conjugacy class representatives of each conjugacy class in the set

$$\{[\psi(D^x \cap Y_i^y)]\}_{x \in \langle X \rangle, y \in \langle Y \rangle, 1 \leqslant i \leqslant k} .$$

Then by Theorem 6.0.3, after removing each generating set of a cyclic group, we are done. $\qquad \square$

Remark 6.1.8. Proposition 6.1.7 provides us with a partial algorithm to decide stability of inertial one-relator extensions. The existence of a full algorithm is not known.

Example 6.1.9. Consider the following one-relator group:

$$\langle a, b \mid b^2 a^2 b^{-1} a b a^2 b^{-2} a^{-2} \rangle .$$

This group appears as $BBABaBBAAbbaa$ in the database [Cas21]. We see that it has one-relator splitting:

$$H *_{\psi} \cong \langle x, y, z \mid z^2 y z^2 x^{-2} \rangle *_{\psi} ,$$

where ψ is given by $\psi(x) = y$, $\psi(y) = z$. Since $y = z^{-2} x^2 z^{-2}$, we see that $\langle x, y, z \rangle$ is a free group freely generated by x and z. Thus we have:

$$\begin{aligned}
\bar{\mathcal{A}}_0^{\psi} &= \{[\langle x, z \rangle]\} , \\
\bar{\mathcal{A}}_1^{\psi} &= \{[\langle x^2, z \rangle]\} , \\
\bar{\mathcal{A}}_2^{\psi} &= \{[\langle (z^{-2} x^2 z^{-2})^2, z \rangle]\} , \\
\bar{\mathcal{A}}_3^{\psi} &= \varnothing .
\end{aligned}$$

Hence, $s(\psi) = 3$ and $H *_{\psi}$ is stable.

6.2 Finding Baumslag–Solitar subgroups

6.2.1 The graph of cyclic stabilisers

We now introduce the graph of cyclic stabilisers. This is a graph of cyclic groups $\mathcal{G}$, associated to an inertial one-relator extension $H *_{\psi}$, that encodes relations between the cyclic stabilisers of segments leading out of H in the Bass-Serre tree. An important consequence of the construction is that certain Baumslag–Solitar subgroups of $H *_{\psi}$ can be read off from $\pi_1(\mathcal{G})$.

Fix a non-cyclic one-relator group H and an isomorphism $\psi : A \to B$ such that $H *_\psi$ is an inertial one-relator extension. Denote by $\mathcal{A}$ the set of all the A-conjugacy classes of maximal cyclic subgroups of A with the following property: for any $S \in \mathcal{S}_H$ with $\mathrm{Stab}(S)$ cyclic, either $\mathrm{Stab}(S)$ is H-conjugate into some representative in $\mathcal{A}$, or it is not H-conjugate into A. We similarly define $\mathcal{B}$, replacing A with B. Since A and B are free groups, every element is contained in a unique maximal cyclic subgroup and so these sets are well defined. If $S \in \mathcal{S}_H$ is any segment with $\mathrm{Stab}(S)$ cyclic, then $\mathrm{Stab}(S)$ is H-conjugate into some representative in $\mathcal{A}$ or $\mathcal{B}$.

We now define *the graph of cyclic stabilisers* $\mathcal{G} = (\Gamma, \{\langle c_v \rangle\}, \{\langle c_e \rangle\}, \{\partial_e^\pm\})$ as follows. Identify $V(\Gamma)$ with the disjoint union $\mathcal{A} \sqcup \mathcal{B}$. Choose any map $\nu : V(\Gamma) \to H$ sending an element $[\langle a \rangle]_A \in \mathcal{A}$ to a generator of a conjugacy class representative $a \in A$ and $[\langle b \rangle]_B \in \mathcal{B}$ to a generator of a representative $b \in B$. There are two types of edges: *H-edges* and *t-edges*.

For each pair of vertices $v, w \in V(\Gamma)$ and each double coset $\langle \nu(v) \rangle h \langle \nu(w) \rangle$, such that $h \in H$ or $h \in At^{-1}B \sqcup BtA$ and

$$\langle \nu(v) \rangle \cap \langle \nu(w) \rangle^h \neq 1,$$

there is an edge e connecting v and w. If $v, w \in \mathcal{A}$ or $v, w \in \mathcal{B}$, then we assume that $h \notin \langle \nu(v) \rangle \langle \nu(w) \rangle$. If $h \in At^{-1}B$, then $v \in \mathcal{A}$ and $w \in \mathcal{B}$. If $h \in BtA$, then $v \in \mathcal{B}$ and $w \in \mathcal{A}$.

The boundary maps $\partial_e^\pm$ are induced by the monomorphisms

$$\langle \nu(v) \rangle \cap \langle \nu(w) \rangle^h \hookrightarrow \langle \nu(v) \rangle \,,$$
$$\langle \nu(v) \rangle^{h^{-1}} \cap \langle \nu(w) \rangle \hookrightarrow \langle \nu(w) \rangle \,.$$

If $h \in H$, then we say that e is an *H-edge* and if $h \in At^{-1}B \sqcup BtA$, we say that e is a *t-edge*. All edges between vertices in $\mathcal{A}$ and $\mathcal{B}$ are oriented towards $\mathcal{B}$ and a choice of orientation is made for the remaining edges.

Note that our construction of $\mathcal{G}$ did not depend on the choice of map ν. We record this with the following lemma.

Lemma 6.2.1. *The isomorphism class of $\mathcal{G}$ as a graph of groups depends only on ψ.*

Choose any map $\xi : E(\Gamma) \to G$ such that $\xi(e) \in \langle \nu(o(e)) \rangle h \langle \nu(t(e)) \rangle$ where $\langle \nu(o(e)) \rangle h \langle \nu(t(e)) \rangle$ is the double coset associated with e. For each $v \in V(\Gamma)$, we may define a group homomorphism, induced by the choices ν, ξ, from the fundamental

group of the graph of cyclic stabilisers

$$\mu : \pi_1(\mathcal{G}, v) \to G \ ,$$

as follows.

Each element of $\pi_1(\mathcal{G}, v)$ can be represented by words of the form

$$c_{v_0}^{i_0} e_1^{\epsilon_1} c_{v_1}^{i_1} ... e_n^{\epsilon_n} c_{v_n}^{i_n} \ ,$$

where $\epsilon_j = \pm 1$, $v_0, v_n = v$, $e_j \in E(\Gamma)$, $v_j = t(e_j)$ and $v_j = o(e_{j+1})$. Then we define

$$\mu(c_{v_0}^{i_0} e_1^{\epsilon_1} c_{v_1}^{i_1} ... e_n^{\epsilon_n} c_{v_n}^{i_n}) = \nu(v_0)^{i_0} \xi(e_1)^{\epsilon_1} \nu(v_1)^{i_1} ... \xi(e_n)^{\epsilon_n} \nu(v_n)^{i_n} \ .$$

One can check that this is well defined and depends only on ν, ξ. We will call μ *the homomorphism induced by ν, ξ.*

A path $\gamma : I \to \Gamma$ is *alternating* if it does not traverse two H-edges or t-edges in a row. An H-*path* is a path $\gamma : I \to \Gamma$ that only traverses H-edges. We say a word $e_1^{\epsilon_1} c_{v_1}^{i_1} ... e_n^{\epsilon_n} c_{v_n}^{i_n}$ is an *alternating word* if $e_1^{\epsilon_1} * e_2^{\epsilon_2} * ... * e_n^{\epsilon_n}$ is an alternating path. Recall that a word $c_{v_0}^{i_0} e_1^{\epsilon_1} c_{v_1}^{i_1} ... e_n^{\epsilon_n} c_{v_n}^{i_n}$ is *cyclically reduced* if all of its cyclic permutations $e_j^{\epsilon_j} c_{v_j}^{i_j} ... e_n^{\epsilon_n} c_{v_n}^{i_n + i_0} ... e_{j-1}^{\epsilon_{j-1}} c_{v_{j-1}}^{i_{j-1}}$ are also reduced. A word is *cyclically alternating* if all of its cyclic permutations are also alternating.

A *generalised Baumslag–Solitar group* is a group that splits as the fundamental group of a graph of groups with infinite cyclic vertex and edge groups. A non-cyclic generalised Baumslag–Solitar group always contains a Baumslag–Solitar subgroup.

Lemma 6.2.2. *If $c = e_1^{\epsilon_1} c_{v_1}^{i_1} ... e_n^{\epsilon_n} c_{v_n}^{i_n}$ is a cyclically reduced and cyclically alternating word, then $\mu(c)$ is a cyclically reduced word. In particular, if $n \geqslant 2$, then G contains a Baumslag–Solitar subgroup that is not conjugate into H.*

Proof. If $n = 1$, then the result is clear. So from now on assume that $n \geqslant 2$. We may also assume that e_1 is a t-edge. Taking indices modulo n, we have that $\mu(c)$ is not cyclically reduced if and only if $\epsilon_{2j-1} = -\epsilon_{2j+1} = 1$ for some j and

$$\nu(v_{2j-1})^{i_{2j-1}} \xi(e_{2j})^{\epsilon_{2j}} \nu(v_{2j})^{i_{2j}} \in A \ ,$$

or $\epsilon_{2j-1} = -\epsilon_{2j+1} = -1$ for some j and

$$\nu(v_{2j-1})^{i_{2j-1}} \xi(e_{2j})^{\epsilon_{2j}} \nu(v_{2j})^{i_{2j}} \in B \ .$$

In the first case, $\nu(v_{2j-1})^{i_{2j-1}} \xi(e_{2j})^{\epsilon_{2j}} \nu(v_{2j})^{i_{2j}} \in A$ if and only if $\xi(e_{2j})^{\epsilon_{2j}} \in A$. But

this would then imply that $v_{2j-1} = v_{2j}$. Since A is free and $\nu(v_{2j})$ generates a maximal cyclic subgroup, it follows that $\xi(e_{2j})^{\epsilon_{2j}} \in \langle \nu(v_{2j}) \rangle$ which is not possible by construction. The same argument is valid for the other case and so $\mu(c)$ is cyclically reduced. In particular, since $n \geqslant 2$, we have that $\mu(c)$ acts hyperbolically on T.

Now, by definition of $\mathcal{G}$, there are integers k, l, such that

$$\mu(c)^{-1}\mu(c_{v_n})^k\mu(c) = \mu(c_{v_n})^l.$$

We may assume that k and l are minimal possible. If $k = \pm 1$ or $l = \pm 1$, then $\langle \mu(c), \mu(c_{v_n}) \rangle \cong \mathrm{BS}(1, \pm l)$ or $\mathrm{BS}(1, \pm k)$ respectively. Since $\mu(c)$ acts hyperbolically on T, it follows that this Baumslag–Solitar subgroup is not conjugate into H. If $k, l \neq \pm 1$, since $\mu(c)$ is cyclically reduced, we have that $[\mu(c), \mu(c_{v_n})]$ is also cyclically reduced. Thus, $\langle \mu(c_{v_n})^l, [\mu(c), \mu(c_{v_n})] \rangle \cong \mathbb{Z}^2$. As before, this copy of $\mathbb{Z}^2$ cannot be conjugate into H. $\qquad\square$

We will denote by $\mathcal{G}_k$ the full subgraph of groups of $\mathcal{G}$ on the vertices corresponding to the maximal cyclic subgroups containing some $\mathrm{Stab}(S)$ where $S \in \mathcal{S}_H^i$ and $i \leqslant k$. Under certain conditions, we may compute $\mathcal{G}_k$.

Theorem 6.2.3. *There is an algorithm that, given as input:*

1. *a one-relator presentation $H \cong \langle \Sigma \mid r \rangle$, where H is hyperbolic and non-cyclic,*

2. *a pair of subsets $X, Y \subset \Sigma$ that generate a pair of Magnus subgroups of H,*

3. *a tuple of words $\{a_1, ..., a_m\} \subset F(X)$ that form a free basis of a subgroup A, strongly inert in $F(X)$ and quasi-convex in H,*

4. *a tuple of words $\{b_1, ..., b_m\} \subset F(X)$ that form a free basis of a subgroup B, strongly inert in $F(Y)$ and quasi-convex in H,*

5. *an isomorphism $\psi : A \to B$ given by $\psi(a_i) = b_i$,*

6. *an integer $k \geqslant 0$,*

computes $\mathcal{G}_k$. Furthermore, if $s(\psi) < \infty$, then $\mathcal{G}_{s(\psi)} = \mathcal{G}$.

Proof. By [KMW17, Proposition 6.7], for any pair of vertices $v, w \in V(\mathcal{G})$, there are at most finitely many edges connecting them. By [KMW17, Corollary 6.10], given a choice of representatives $\nu : V(\Gamma) \to H$, for any given pair of vertices $v, w \in V(\mathcal{G})$, we may decide if v and w are connected by an edge. Moreover, we may find double coset representatives for each such edge. Hence, in order to compute $\mathcal{G}_k$, it suffices to find representatives $\nu_k : V(\Gamma_k) \to H$, where Γ_k is the underlying graph of $\mathcal{G}_k$.

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Note that $\Gamma_0 = \varnothing$ since H is non-cyclic. If A and B are cyclic, then $\mathcal{A} = \{[A]\}$ and $\mathcal{B} = \{[B]\}$. If not, then $\mathcal{S}_H^1$ only contains segments with non-cyclic stabilisers and so $\Gamma_1 = \varnothing$. Thus, representatives ν_0 and ν_1 are computable.

By [KMW17, Proposition 6.7 and Corollary 6.10], there exist finite computable sets R_{AB}, R_{BA}, R_{AA} and R_{BB}, satisfying the following:

1. R_{AB} is a complete set of A, B double coset representatives with the property that $A^g \cap B \neq 1$ for all $g \in R_{AB}$,

2. R_{BA} is a complete set of B, A double coset representatives with the property that $A \cap B^g \neq 1$ for all $g \in R_{BA}$,

3. R_{AA} is a complete set of A, A double coset representatives with the property that $A^g \cap A \neq 1$ for all $g \in R_{AA}$,

4. R_{BB} is a complete set of B, B double coset representatives with the property that $B^g \cap B \neq 1$ for all $g \in R_{BB}$.

Let $S \in \mathcal{S}_H$. We have $\operatorname{Stab}(S) = H \cap H^c$ where c is equal to a reduced word

$$t^{\epsilon_1} c_1 d_1 t^{\epsilon_2} c_2 d_2 ... t^{\epsilon_n} c_n \ ,$$

with $\epsilon_i = \pm 1$, $c_n \in H$ and such that the following hold:

1. if $\epsilon_i = \epsilon_{i+1} = 1$, then $c_i \in R_{AB}$ for some j and $d_i \in B$,

2. if $\epsilon_i = \epsilon_{i+1} = -1$, then $c_i \in R_{BA}$ for some j and $d_i \in A$,

3. if $\epsilon_i = 1$, $\epsilon_{i+1} = -1$, then $c_i \in R_{AA}$ for some j and $d_i \in A$,

4. if $\epsilon_i = -1$, $\epsilon_{i+1} = 1$, then $c_i \in R_{BB}$ for some j and $d_i \in B$.

Denote by $c_{i,j} = t^{\epsilon_i} c_i d_i ... t^{\epsilon_j} c_j d_j$ and $A' = \langle X \rangle$ and $B' = \langle Y \rangle$. If for some i we have $\epsilon_i = -\epsilon_{i+1}$, then
$$H \cap H^{c_{1,i} t^{\epsilon_{i+1}}} < H \cap H^{t^{\epsilon_i} c_i d_i t^{\epsilon_{i+1}}} \cong \mathbb{Z}$$

by Theorem 6.0.1. Similarly, if $\epsilon_i = \epsilon_{i+1} = 1$ and $c_i \notin A'B'$, or $\epsilon_i = \epsilon_{i+1} = -1$ and $c_i \notin B'A'$, then
$$H \cap H^{c_{1,i} t^{\epsilon_{i+1}}} < H \cap H^{t^{\epsilon_i} c_i d_i t^{\epsilon_{i+1}}} \cong \mathbb{Z} \ .$$

But since $H \cap H^{t^{\epsilon_i} c_i d_i t^{\epsilon_{i+1}}} = \operatorname{Stab}(S)$ for some $S \in \mathcal{S}_H^2$, it suffices to only consider

the words of the form:

$$ta_1 b_1 t a_2 b_2 ... t a_n b_n \ ,$$
$$t^{-1} h t a_2 b_2 ... t a_n b_n \ ,$$
$$t^{-1} b_1 a_1 t^{-1} b_2 a_1 ... t^{-1} b_n a_n \ ,$$
$$tgt^{-1} b_2 a_2 ... t^{-1} b_n a_n \ ,$$

where $h \in R_{BB}$, $g \in R_{AA}$ and $a_i \in A'$, $b_i \in B'$. The first two cases are the same as the seccond two up to symmetry.

Denote by $\mathcal{W}_n$ the collection of words of the first type where t appears n times. If $C < A'$ is a finitely generated subgroup, then there are finitely many B'-conjugacy classes of subgroups $B \cap C^{ab} = (B^{b^{-1}} \cap C^a)^b$ where $a \in A'$ and $b \in B'$ (see Section 2.1.3). Since B is a finitely generated subgroup of the free group B', there are also finitely many B-conjugacy classes of such subgroups. We may apply the same argument, replacing B with A. This implies that the sets

$$\{[H \cap H^c]_A, [H \cap H^c]_B\}_{c \in \mathcal{W}_2} \ ,$$
$$\{[H \cap H^{t^{-1} hc}]_A, [H \cap H^{t^{-1} hc}]_B\}_{h \in R_{BB}, c \in \mathcal{W}_2}$$

are finite. Hence, by induction on n, the sets

$$\{[H \cap H^c]_A, [H \cap H^c]_B\}_{c \in \mathcal{W}_n} \ ,$$
$$\{[H \cap H^{t^{-1} hc}]_A, [H \cap H^{t^{-1} hc}]_B\}_{h \in R_{BB}, c \in \mathcal{W}_n}$$

are also finite. The symmetric argument covers the remaining cases and we see that $V(\Gamma_k)$ must be finite for all $k \geqslant 0$. By [How05, Theorem E], we may compute a free basis for the free group $A' \cap B'$ in the generators for A' or B'. We may use Stallings graphs (again, see Section 2.1.3 and [Sta83]) to find double coset representatives and compute all the other intersections in the same way as in the proof of Proposition 6.1.7. Thus, a representative $\nu_k : V(\Gamma_k) \to H$ is computable.

Now suppose that $k \geqslant s(\psi)$. By definition, if $c \in \mathcal{W}_k$, then we have $H \cap H^c \cong \mathbb{Z}$ or $H \cap H^c = 1$. Since $\mathrm{Stab}(S_1) < \mathrm{Stab}(S_2)$ for any pair of segments $S_2 \subset S_1 \subset T$, we get that $\mathcal{G}_k = \mathcal{G}_{s(\psi)}$ and thus $\mathcal{G} = \mathcal{G}_{s(\psi)}$. $\qquad \square$

Example 6.2.4. Let us consider the one-relator group

$$\langle a, b, t \mid t a^2 t^{-1} b a b^{-1} t a t^{-1} b a b^{-1} \rangle \ ,$$
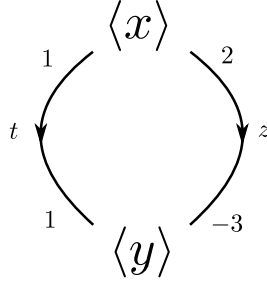
Figure 6.1: The graph of cyclic stabilisers $\mathcal{G}$

appearing in the one-relator group database [Cas21] as $BBABaBABa$. It has a one-relator splitting:

$$\langle a, b, t \mid ta^2t^{-1}bab^{-1}tat^{-1}bab^{-1} \rangle \cong \langle x, y, z, t \mid txt^{-1} = y, y^2zxz^{-1}yzxz^{-1} \rangle$$
$$\cong \langle x, y, z \mid y^2zxz^{-1}yzxz^{-1} \rangle *_\psi \ .$$

This one-relator splitting is also a one-relator hierarchy of length one since the element $y^2zxz^{-1}yzxz^{-1}$ is primitive in $F(x, y, z)$. One can check that this HNN-extension is stable and so it satisfies the hypotheses of Theorem 6.2.5.

Since the edge groups of this splitting are cyclic, it follows that $\mathcal{G}$ has two vertices, one corresponding to $\langle x \rangle$ and the other to $\langle y \rangle$, with a t-edge connecting them. Since $y^2zxz^{-1}yzxz^{-1} = \mathrm{pr}_{3/2}(y, zxz^{-1})$, we see that $zx^2z^{-1} = y^{-3}$. Hence, there is an edge connecting $\langle x \rangle$ with $\langle y \rangle$ corresponding to conjugation by z, where the edge monomorphisms are given by multiplication by 2 and -3 respectively. See Figure 6.1 for the graph of cyclic stabilisers $\mathcal{G}$.

We see that $\pi_1(\mathcal{G})$ has a cyclically alternating and cyclically reduced word zt^{-1} and so our one-relator group contains a Baumslag–Solitar subgroup by Lemma 6.2.2. More explicitly, we have

$$\langle x, zt^{-1} \rangle = \langle a, bt^{-1} \rangle \cong BS(2, -3) \ .$$

6.2.2 A criterion

Lemma 6.2.2 told us that certain Baumslag–Solitar subgroups of $H*_\psi$ could be read off from its graph of cyclic stabilisers. Although we may not find all such subgroups in this way, we now show that under certain conditions, if $H*_\psi$ does contain Baumslag–Solitar subgroups, then Lemma 6.2.2 will always produce a witness.

Theorem 6.2.5. *Let $G = H*_\psi$ be an inertial one-relator extension where H is*

non-cyclic and ψ identifies A and B. Suppose that $s(\psi) < \infty$, H is hyperbolic and that A and B are quasi-convex in H. The following are equivalent:

1. *G acts acylindrically on T,*

2. *G does not contain any Baumslag–Solitar subgroups,*

3. *$\pi_1(\mathcal{G})$ does not admit any cyclically alternating and cyclically reduced word.*

Proof. We prove that (3) implies (1), that (1) implies (2) and that (2) implies (3) by proving the contrapositive statements.

Let $\mathcal{G}$ be the graph of cyclic stabilisers of ψ and let $\nu : V(\Gamma) \to H$ be a choice of representatives. Suppose that G does not act acylindrically on T. Then, for any $n \geqslant 1$, there exists a sequence of geodesic segments $S_1 \subset S_2 \subset ... \subset S_n$ with $S_i \in \mathcal{S}_H^i$, each with infinite stabiliser. Let $g_i \in G$ be an element such that the endpoints of S_i are H and g_iH. For all $i \geqslant s(\psi)$, by Lemma 6.1.3, $\mathrm{Stab}(S_i)$ is cyclic. By Theorem 6.2.3, each subgroup $\mathrm{Stab}(g_i^{-1} \cdot S_i)$ is H-conjugate into some $\langle \nu(v) \rangle$ where $v \in V(\Gamma)$. By the pigeonhole principle, there are three integers $s(\psi) < i < j < k$ such that $\mathrm{Stab}(g_i^{-1}S_i)$, $\mathrm{Stab}(g_j^{-1}S_j)$ and $\mathrm{Stab}(g_k^{-1}S_k)$ are H-conjugate into some $\langle \nu(v) \rangle$. Thus, since $\mathrm{Stab}(S_k) < \mathrm{Stab}(S_j) < \mathrm{Stab}(S_i) < \langle h \rangle$, there are elements h_1, h_2, h_3, h_4 such that:

$$\langle h \rangle^{g_j h_2} \cap \langle h \rangle^{g_i h_1} \neq 1 \ ,$$
$$\langle h \rangle^{g_k h_4} \cap \langle h \rangle^{g_i h_3} \neq 1 \ .$$

In particular, if $f = g_i h_1 h_2^{-1} g_j^{-1}$ and $g = g_i h_3 h_4^{-1} g_k^{-1}$, then

$$\langle h \rangle \cap \langle h \rangle^f \neq 1 \ ,$$
$$\langle h \rangle \cap \langle h \rangle^g \neq 1 \ ,$$
$$\langle h \rangle \cap \langle h \rangle^{fg} \neq 1 \ .$$

Suppose first that both f and g act elliptically on T. If they do not both fix a common vertex, then fg acts hyperbolically on T. So suppose that they both fix a common vertex u. Now the geodesic connecting u with S_k must meet S_k at the midpoint between g_iH and g_jH and the midpoint between g_iH and g_kH. Since $j < k$, this is not possible and so we may assume that one of f, g or fg acts hyperbolically on T. The cyclic reduction of f, g or fg provides us with a cyclically alternating and cylically reduced word in $\pi_1(\mathcal{G})$ by construction.

Now suppose that G contains a Baumslag–Solitar subgroup $J < G$. Since H is hyperbolic, it cannot contain a Baumslag–Solitar subgroup. This implies that J

cannot act elliptically on T. It follows from [MO15, Theorem 2.1] that J cannot act acylindrically on T. Hence G does not act acylindrically on T and (1) implies (2).

Finally, Lemma 6.2.2 shows that (2) implies (3). □

Corollary 6.2.6. *Let $G = H *_\psi$ be an inertial one-relator extension where ψ identifies A and B. Suppose that $s(\psi) < \infty$, that H is hyperbolic and that A and B are quasiconvex in H. Then there is an algorithm that decides if G contains a Baumslag–Solitar subgroup.*

Example 6.2.7. Consider the one-relator group:

$$\langle a, b \mid b^2 a^2 b^{-1} a b a^2 b^{-2} a^{-2} \rangle$$

from Example 6.1.9. It has one-relator splitting

$$H *_\psi = \langle x, y, z \mid z^2 y z^2 x^{-2} \rangle *_\psi \ ,$$

where $\psi : A \to B$ is given by $\psi(x) = y$, $\psi(y) = z$. In Example 6.1.9, we showed that $H *_\psi$ is stable and that the vertex group was freely generated by x, z. We see that the edge groups are generated by $x, z^{-2} x^2 z^{-2}$ and x^2, z respectively. Every subgroup of the form $A \cap B^g$, $A \cap A^g$ or $B \cap B^g$ is either trivial, or conjugate to A, B or one of the following:

$$\langle x, z^{-2} x^2 z^{-2} \rangle \cap \langle x^2, z \rangle = \langle x^2, z^{-2} x^2 z^{-2} \rangle \ ,$$
$$\langle x, z^{-2} x^2 z^{-2} \rangle \cap \langle x^2, z \rangle^x = \langle x^2 \rangle \ ,$$
$$\langle x^2, z \rangle \cap \langle x^2, z \rangle^x = \langle x^2 \rangle \ .$$

By applying ψ to x and ψ^{-1} to $x^2 = z^2 y z^2$, we see that the vertices in Γ corresponding to stabilisers of elements in $\mathcal{S}_H^2$ are the $\mathcal{B}$-vertex $[\langle y \rangle]$ and the $\mathcal{A}$-vertices $[\langle y \rangle]$ and $[\langle y^2 x y^2 \rangle]$. The graph of cyclic stabilisers can be seen in Figure 6.2. By Theorem 6.2.5, since $\mathcal{G}$ does not admit any cyclically alternating and cyclically reduced words, we see that $H *_\psi$ does not contain any Baumslag–Solitar subgroups.

6.3 Stable one-relator splittings

6.3.1 One-relator groups with torsion

The following lemma allows one to easily establish stability of a one-relator splitting in certain cases: if the edge groups of a one-relator splitting have no exceptional intersection, then the splitting is stable.
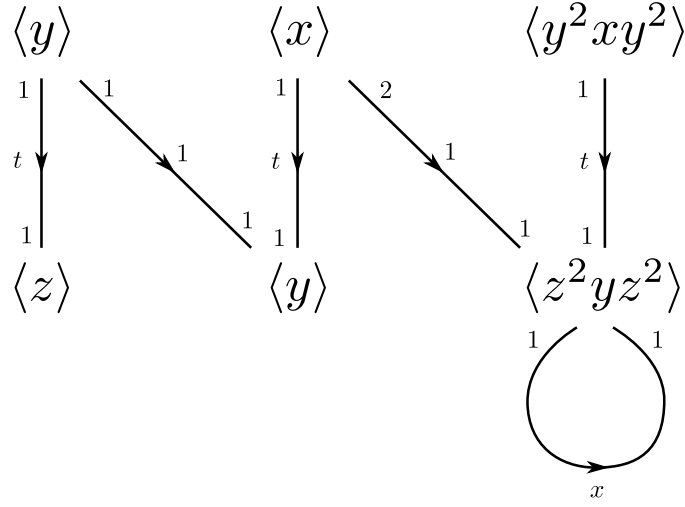
$$\langle y \rangle \qquad \langle x \rangle \qquad \langle y^2 x y^2 \rangle$$

Figure 6.2: The graph of cyclic stabilisers $\mathcal{G}$

Lemma 6.3.1. *Let X be a finite one-relator complex and let $\pi_1(X) \cong \pi_1(Z)*_\psi$ be a one-relator splitting where $A, B \subset Z$ are the associated Magnus subcomplexes. Let $A', B' \subset Z$ be Magnus subcomplexes such that $A \subset A'$, $B \subset B'$ and $A' \cap B'$ is connected. If $\pi_1(A') \cap \pi_1(B') = \pi_1(A' \cap B')$, then $s(\psi) < \infty$.*

Proof. Let $Y \to X$ be the cyclic cover containing Z. Let $\theta : \Theta \looparrowright A$ be a graph immersion with $\chi(\Theta) \leqslant -1$. By assumption and Theorem 6.0.1, θ is homotopic in Z to a graph immersion $\theta_1 : \Theta_1 \looparrowright B = 1 \cdot A$, if and and only if θ is homotopic in A to a graph immersion $\theta_1 : \Theta_1 \looparrowright A \cap B$. If $s(\psi) \geqslant 2$, there are graph immersions θ, θ_1 as a above, such that $\theta_1 : \Theta_1 \looparrowright A \cap B \hookrightarrow 1 \cdot A$ is homotopic in $1 \cdot Z$ to a graph immersion $\theta_2 : \Theta_2 \looparrowright A \cap 1 \cdot A \cap 2 \cdot A \hookrightarrow 1 \cdot (A \cap B)$. Carrying on this argument, we see that if $s(\psi) \geqslant n$, then there exists a graph immersion $\theta : \Theta \looparrowright A$ such that $\chi(\Theta) \leqslant -1$ and such that θ factors through $A \cap 1 \cdot A \cap ... \cap n \cdot A$. However, for large enough n we have that $A \cap 1 \cdot A \cap ... \cap n \cdot A = \varnothing$. $\qquad \square$

The condition from Lemma 6.3.1 always holds for one-relator splittings of one-relator groups with torsion by Remark 6.1.2 and Corollary 5.0.3, yielding the following.

Corollary 6.3.2. *All one-relator splittings of one-relator groups with torsion are stable.*

6.3.2 One-relator groups with negative immersions

The aim of this subsection is to show that every one-relator splitting of a one-relator group with negative immersions is stable. In order to do this, we show that certain

constraints on the subgroups of $H*_\psi$ that are not conjugate into H imply stability.

Theorem 6.3.3. *If every descending chain of non-cyclic, freely indecomposable proper subgroups of bounded rank*

$$G = H_0 > H_1 > ... > H_k > ...$$

is either finite or eventually conjugate into H, then $s(\psi) < \infty$.

Proof. Suppose for a contradiction that $s(\psi) = \infty$. Denote by $\phi : G \to G/\langle\langle H \rangle\rangle \cong \mathbb{Z}$. Then, by Proposition 6.1.5, there is some biinfinite geodesic S in the Bass-Serre tree for $H*_\psi$, containing the vertex H, such that there exists some element $g \in G$ acting by translations on S. Moreover, we have that $\phi(g) \neq 0$. Hence, $H \cap H^{g^n} \neq 1$ for all n.

Consider the descending chain of subgroups

$$G = \langle H, g \rangle > \langle H, g^2 \rangle > ... > \langle H, g^{2^k} \rangle > ...$$

and denote by $H_k = \langle H, g^{2^k} \rangle$. Since $\text{rk}(H_k) \leqslant \text{rk}(H) + 1$, the ranks in this chain are bounded. The chain must be proper since $\phi(H_i) = \langle \phi(g)2^i \rangle \neq \langle \phi(g)2^j \rangle = \phi(H_j)$ for all $i \neq j$.

Each H_i has a Grushko decomposition

$$H_i = F(X_i) * J_{i,1} * ... * J_{i,k_i}$$

that is unique up to permutation and conjugation of factors and where each $J_{i,j}$ is non-cyclic and freely indecomposable. By the Kurosh subgroup theorem, each $J_{i,j}$ is a conjugate of a subgroup of some $J_{i-1,l}$. Furthermore, we have that $|X_i| + 2k_i \leqslant \text{rk}(H) + 1$.

Now the claim is that for some integer $m \geqslant 0$, each $J_{i,j}$ is either a conjugate of some $J_{i-1,l}$, or is conjugate into H, for all $i > m$. If this was not the case, then there would be infinite sequences of integers $i_0, i_1, i_2, ...$ and $j_0, j_1, j_2, ...$ and elements $g_1, g_2, ... \in G$, such that

$$J_{i_0,j_0} > J_{i_1,j_1}^{g_1} > J_{i_2,j_2}^{g_2} > ...$$

where the inclusions are all proper and no J_{i_k,j_k} is conjugate into H. But this contradicts the hypothesis and so the claim is proven.

So now set $J_i = J_{i,1} * ... * J_{i,k_i}$. For all $i \geqslant m$, we have that $\phi(J_i) = \langle l \rangle$ for some fixed l. But since $\phi(H_i) = \langle \phi(g)2^i \rangle$, this forces $J_i < \ker(\phi)$. Thus, the induced homomorphism $\phi \mid H_m$ factors through the projection $H_m \to H_m/\langle\langle J_m \rangle\rangle \cong$

$F(X_m)$. But now $\phi(g^{2^m})$ is a generator of $\phi(H_m)$, so there is some primitive element $x \in F(X_m)$ that maps to a generator of $\phi(H_m)$ and such that $x = kg^{2^m}$ for some $k \in \ker(\phi \mid H_m)$. Since x is primitive in $F(X_m)$, we have:

$$H_m = \langle x \rangle * K = \langle kg^{2^m} \rangle * K \ ,$$

for some subgroup $K < H_m$ such that $\langle\langle K \rangle\rangle = \ker(\phi)$.

Now let T_m be the Bass-Serre tree associated with the HNN-extension with trivial edge groups, $\langle x \rangle * K$. Each vertex is stabilised by a conjugate of K and each edge has trivial stabiliser. Since $H < H_m$, $H < \ker(\phi)$ and H is finitely generated, it follows that H is a subgroup of a free product of finitely many conjugates of K. Since g^{2^m} acts hyperbolically on T_m and T_m has trivial edge stabilisers, it follows that

$$H \cap H^{g^{2^{km}}} = 1 \ ,$$

for all k sufficiently large. But then this contradicts the assumption that $s(\psi) = \infty$. □

Corollary 6.3.4. *All one-relator splittings of one-relator groups with negative immersions are stable.*

Proof. Let X be a one-relator complex with negative immersions. Let $\pi_1(X) \cong \pi_1(Z)*_\psi$ be a one-relator splitting where $A, B \subset Z$ are the Magnus subcomplexes identified. By Remark 6.1.2, we may assume that $\pi_1(A)$ and $\pi_1(B)$ are strongly inert subgroups of Magnus subgroups of $\pi_1(Z)$ for some one-relator presentation. This is because adding a free factor preserves stability. Now the result follows from Theorems 2.3.6 and 6.3.3. □

6.3.3 The general case

A one-relator group that does not have torsion or negative immersions may not have any stable one-relator splittings. However, there is still something to be said in the general case: let G be a one-relator group and $G = H*_\psi$ a one-relator splitting. Either $H*_\psi$ is stable or H is a one-relator group with exceptional intersection by Lemma 6.3.1. In particular, by Theorem 5.2.8, one of the following hold:

1. $H*_\psi$ is stable,

2. H has negative immersions,

3. H contains a Baumslag–Solitar subgroup.

Chapter 7

Main results and further questions

7.1 Hierarchies

We are finally ready to prove our main results. Let us first define stable hierarchies.

Definition 7.1.1. A one-relator tower (hierarchy) $X_N \looparrowright ... \looparrowright X_1 \looparrowright X_0$ is a *stable one-relator tower (hierarchy)* if each associated one-relator splitting $\pi_1(X_{i+1})*_{\psi_i}$ is stable.

The most important tool established in this thesis is the following one-relator hierarchy equivalence theorem.

Theorem 7.1.2. *Let X be a one-relator complex and $X_N \looparrowright ... \looparrowright X_1 \looparrowright X_0 = X$ a one-relator hierarchy. The following are equivalent:*

1. *$X_N \looparrowright ... \looparrowright X_1 \looparrowright X_0 = X$ is a quasi-convex hierarchy and $\pi_1(X)$ is hyperbolic,*

2. *$X_N \looparrowright ... \looparrowright X_1 \looparrowright X_0 = X$ is an acylindrical hierarchy,*

3. *$X_N \looparrowright ... \looparrowright X_1 \looparrowright X_0 = X$ is a stable hierarchy and $\pi_1(X)$ contains no Baumslag–Solitar subgroups.*

Moreover, if any of the above is satisfied, then $\pi_1(X)$ is virtually special and $\pi_1(A)$ is quasi-convex in $\pi_1(X)$ for any subcomplex $A \subset X_i$.

Proof. The equivalence of (1) and (2) is Theorem 4.4.1.

We prove that (2) and (3) are equivalent by induction on hierarchy length. The base case is clear. Suppose that the equivalence holds for all one-relator hierarchies

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$X_N \looparrowright ... \looparrowright X_1$. If $X_N \looparrowright ... \looparrowright X_1 \looparrowright X_0 = X$ is an acylindrical hierarchy, then clearly it is a stable hierarchy. By Theorem 4.4.1, $\pi_1(X)$ is hyperbolic and so contains no Baumslag–Solitar subgroups; this shows that (2) implies (3). By Theorem 6.2.5 and induction, we see that (2) and (3) are equivalent.

If any of (1), (2) or (3) hold, then the remaining assertions hold by Theorem 4.4.1. $\qquad\square$

We now prove a stronger form of [LW22, Conjecture 1.9].

Theorem 7.1.3. *Let X be a one-relator complex with negative immersions. Then $\pi_1(X)$ is hyperbolic, virtually special and all of its one-relator hierarchies $X_N \looparrowright ... \looparrowright X_1 \looparrowright X_0 = X$ are quasi-convex hierarchies.*

Proof. Since $\pi_1(X)$ does not contain any Baumslag–Solitar subgroups by Theorem 2.3.5, the result follows from Corollary 6.3.4 and Theorem 7.1.2. $\qquad\square$

Corollary 7.1.4. *Let X be a one-relator complex with negative immersions. Then every finitely generated subgroup of $\pi_1(X)$ is hyperbolic.*

Proof. Follows from Theorem 7.1.3, [Ger96, Corollary 7.8] and Theorem 2.3.6. $\qquad\square$

Corollary 7.1.5. *An exceptional intersection group is hyperbolic (and virtually special) if and only if it does not contain a Baumslag–Solitar subgroup.*

Proof. Follows from Theorems 5.2.8 and 7.1.3. $\qquad\square$

Corollary 7.1.6. *Parafree one-relator groups are hyperbolic and virtually special. In particular, their isomorphism problem is decidable.*

Proof. By [Bau69, Theorem 4.2] and Theorem 2.4.5, parafree one-relator groups have negative immersions. Now the result follows from Theorem 7.1.3 and [DG11, Theorem 1]. $\qquad\square$

In [MUW11, Problem 1.2], it is asked whether it is decidable if a one-relator group contains a Baumslag–Solitar subgroup. We show that for one-relator groups with stable hierarchies, this is indeed the case.

Theorem 7.1.7. *There is an algorithm that, given as input a one-relator complex that has a stable one-relator hierarchy, decides whether $\pi_1(X)$ contains a Baumslag–Solitar subgroup or not.*

Proof. We may enumerate hierarchies $X_N \looparrowright ... \looparrowright X_1 \looparrowright X_0 = X$ and by Proposition 6.1.7, check if the induced one-relator splittings are stable. Since X has a stable hierarchy, we will eventually find a hierarchy $X_N \looparrowright ... \looparrowright X_1 \looparrowright X_0 = X$ that is stable. Now the result follows from Theorem 7.1.2, Corollary 6.2.6 and by induction on N. $\qquad\square$

7.2 Gersten's conjecture

In [Ger92b], Gersten conjectured that a one-relator group is hyperbolic if and only if it does not contain a Baumslag–Solitar subgroup. Theorem 7.1.3 tells us that if a counterexample exists to Gersten's conjecture, then it must be a one-relator group that contains a non-cyclic two-generator one-relator subgroup. In this section, we obtain a further restriction.

If $i < j \in \mathbb{Z}$ are integers, denote by

$$A_{i,j} = \{a^{t^i}, a^{t^{i+1}}, ..., a^{t^j}\} \subset F(a, t) .$$

For each rational $p/q \in \mathbb{Q}_{>0}$, we define two new families of one-relator groups. The first family is parametrised by two words

$$x \in \langle A_{0,k-1} \rangle - \langle A_{1,k-1} \rangle ,$$
$$y \in \langle A_{1,k} \rangle - \langle A_{1,k-1} \rangle ,$$

such that $\mathrm{pr}_{p/q}(x, y)$ is a primitive exceptional intersection word of the first type. We then define:

$$E_{p/q}(x, y) = \langle a, t \mid \mathrm{pr}_{p/q}(x, y) \rangle . \tag{7.1}$$

We call this a *primitive extension group of the first type.*

The second family is parametrised by three words

$$x \in \langle A_{0,k-1} \rangle - \langle A_{1,k-1} \rangle ,$$
$$y \in \langle A_{1,k} \rangle - \langle A_{1,k-1} \rangle ,$$
$$z \in \langle A_{1,k-1} \rangle - 1 ,$$

such that $\mathrm{pr}_{p/q}(xy, z)$ is a primitive exceptional intersection word of the second type. We then define:

$$F_{p/q}(x, y, z) = \langle a, t \mid \mathrm{pr}_{p/q}(xy, z) \rangle . \tag{7.2}$$

We call this a *primitive extension group of the second type.*

Definition 7.2.1. A group G is a *primitive extension group* if it is a primitive extension group of the first or second type.

We now show that Gersten's conjecture is true if and only if it is true for primitive extension groups. In order to do so, a more refined hierarchy result is required.

Theorem 7.2.2. *Let $X = (\Gamma, \lambda)$ be a one-relator complex such that $\pi_1(X)$ does not contain Baumslag–Solitar subgroups. There exists a stable one-relator tower:*

$$X_N \looparrowright ... \looparrowright X_1 \looparrowright X_0 = X \ ,$$

such that one of the following holds:

1. *$\pi_1(X_N)$ is finite,*

2. *$\pi_1(X_N)$ is a primitive extension group.*

Moreover, if $\pi_1(X_N)$ is hyperbolic, then the tower is acylindrical and $\pi_1(X)$ is hyperbolic.

Proof. By Theorem 7.1.3, we may assume that X does not have negative immersions. We may also assume that λ is primitive. By Theorem 3.3.16, there is a one-relator tower:

$$Q = X_K \looparrowright ... \looparrowright X_1 \looparrowright X_0 = X \ ,$$

such that $\pi_1(Q)$ is the 2-generator w-subgroup. Thus, by Theorem 2.3.7, every two-generator subgroup of $\pi_1(X)$ is either free or conjugate into $\pi_1(Q)$. Since $\pi_1(X)$ cannot contain Baumslag–Solitar subgroups, we have that no $\pi_1(X_i)$ can have exceptional intersection by Theorem 5.2.8. Thus, by Lemma 6.3.1, each splitting is stable.

Now let $p : Y \to Q$ be any cyclic cover and $Z \in \mathrm{TD}(p)$ a minimal tree domain. Since $\chi(Q) = 0$, there is a pair of generators a, t for $\pi_1(Q)$ such that

$$\pi_1(Z) \cong \left\langle a, a^t, ..., a^{t^k} \ \middle| \ w\left(a, a^t, ..., a^{t^k}\right) \right\rangle$$

and both a and a^{t^k} are mentioned in w. Moreover, $\pi_1(Q) = \pi_1(Z)*_\psi$ where ψ maps $\langle a, a^t, ..., a^{t^{k-1}} \rangle$ to $\langle a^t, a^{t^2}, ..., a^{t^k} \rangle$. If these Magnus subgroups have exceptional intersection, then by Theorem 5.2.8, $\pi_1(Z)$ must be a primitive exceptional intersection group. Then by definition, $\pi_1(Q)$ must be a primitive extension group. If the

Magnus subgroups do not have exceptional intersection, since $c(Z) < c(X)$, we may use induction to complete the tower to a stable tower:

$$X_N \looparrowright ... \looparrowright X_1 \looparrowright X_0 = X \ ,$$

where $\pi_1(X_N)$ is either finite or is a primitive extension group.

All that is left to prove is that this tower is acylindrical when $\pi_1(X_N)$ is hyperbolic. If $\pi_1(X_N)$ is finite, by Theorem 7.1.2, we have that the tower is acylindrical. Suppose then that $\pi_1(X_N)$ is not finite. Since $\chi(X_N) = 0$ and $\pi_1(X_N)$ is hyperbolic, each Magnus subgroup is cyclic and so must be quasi-convex. Then, by Theorems 4.3.1 and 6.2.5 and induction, we see that the tower must be acylindrical.
$\square$

Theorem 7.2.3. *A one-relator group is hyperbolic if and only if its primitive extension subgroups are hyperbolic.*

Proof. If G is a one-relator group containing a finitely presented non-hyperbolic subgroup, then G cannot be hyperbolic by [Ger96, Corollary 7.8]. So now suppose that G is a one-relator group not containing any non-hyperbolic primitive extension subgroup. Since $BS(1, n)$ is a primitive extension group for all $n \neq 0$, it follows that G does not contain any Baumslag–Solitar subgroups. Now the result follows from Theorem 7.2.2.
$\square$

Corollary 7.2.4. *Gersten's conjecture is true if and only if it is true for primitive extension groups.*

7.3 Further questions

Extending results in this thesis to all one-relator groups would require a better understanding of one-relator hierarchies that are not stable. Therefore, we pose the following problems.

Problem 7.3.1. Characterise one-relator hierarchies that are not stable.

Problem 7.3.2. Find an algorithm to decide stability for one-relator splittings.

Any one-relator group satisfying the hypothesis of Brown's criterion, either has a one-relator splitting that is not stable, or is isomorphic to a Baumslag–Solitar group $BS(1, n)$. More generally, if $\pi_1(X) \cong \pi_1(Z) *_\psi$ is a one-relator splitting and there is some $g \in \pi_1(X)$ acting hyperbolically on its Bass-Serre tree and some $A_n \in [A_n] \in \bar{\mathcal{A}}_n^\psi$

such that $A_n^g < A_n$, then $\langle A_n, g \rangle$ splits as an ascending HNN-extension of a finitely generated free group. We ask whether this is the only situation that can occur.

Question 7.3.3. Let X be a finite one-relator complex and let $\pi_1(X) \cong \pi_1(Z)*_\psi$ be a one-relator splitting that is not stable. Is there some $n \geqslant 0$ such that one of the following holds

$$\bar{\mathcal{A}}_n^\psi = \bar{\mathcal{A}}_{n+i}^\psi \ ,$$
$$\bar{\mathcal{A}}_n^{\psi^{-1}} = \bar{\mathcal{A}}_{n+i}^{\psi^{-1}} \ ,$$

for all $i \geqslant 0$?

If so, does there exist an immersion of one-relator complexes $Q \looparrowright X$ that does not factor through $Z \looparrowright X$ and such that $\pi_1(Q)$ splits as an ascending HNN-extension of a non-trivial, finitely generated free group?

In light of the results in [LW22] and this thesis, we generalise a question asked by Moldavanskii [KM18, Question 11.63].

Question 7.3.4. Let G be a one-relator group and let $[P_1], ..., [P_n]$ be the conjugacy classes of w-subgroups of G. If $H < G$ is a subgroup such that $H \cap \langle\langle P_1, ..., P_n \rangle\rangle = 1$, is H free?

By [Sel95] and [DG11], the isomorphism problem for one-relator groups with torsion and negative immersions is solvable. Before then, it was known by [Pri77] that the isomorphism problem was decidable for 2-generator one-relator groups with torsion. There, Pride proves a stronger result: that freely indecomposable 2-generator one-relator groups with torsion have precisely one generating set, up to Nielsen equivalence. To conclude, we ask if this property holds also for one-relator groups with negative immersions.

Question 7.3.5. Do one-relator groups with negative immersions have only one generating set, up to Nielsen equivalence?

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