Iwasawa theory and p-adic families of cohomology classes

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Contents

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Declaration

The results of Chapter 2 appear in the paper [Roc22]. The remaining chapters are, to the best of my knowledge and unless otherwise indicated, my original work.

This thesis has not been submitted for a degree at any other university.

Abstract

This thesis consists of four papers written during the course of my PhD.

The first concerns an approach to non-ordinary Iwasawa theory and generalises the ‘plus/minus’ approach of Pollack.

The subsequent three chapters all concern p-adic families of cohomology classes. The first considers interpolation of Euler system classes for GSp_4 in ordinary families. The second generalises work of Hida and Tilouine–Urban by proving control theorems for a large class of reductive groups. The third and final paper concerns a construction for varying Euler system classes in non-ordinary families.

1 Introduction

Hello. This thesis concerns the theory of Euler systems, p-adic L-functions and their variation in p-adic families. The (mostly conjectural) interplay between these two objects has important consequences for Iwasawa theory and the theory of special values of L-functions, which we will expounded upon below, before giving a more detailed run down of the contents of this thesis.

1.1 Mathematical context

1.1.1 Some Big conjectures

Our story begins, as it so often does, with the Birch–Swinnerton-Dyer conjecture. Fix a prime p and let $A/\mathbb{Q}$ be a d-dimensional abelian variety. Its rational points $A(\mathbb{Q})$ admit the structure of a finitely generated abelian group; we write r_{alg} for its rank. We define the *p-adic Tate module*

$$T_pA = \varprojlim_n A[p^n]$$

then $V_pA := T_pA \otimes \mathbb{Q}_p$ is a $2d$-dimensional $\mathbb{Q}_p$-linear $G_{\mathbb{Q}} := \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$-representation. Let $\ell \neq p$ be a prime, then if A has good reduction at ℓ then V_pA is unramified at ℓ. We define the L-function of A to be

$$L(A, s) = \prod_{\substack{\ell \neq p \\ A \text{ good reduction at } \ell}} P_\ell(V_pA, \ell^{-s})^{-1}$$

for $s \in \mathbb{C}$, $s \gg 0$ where

$$P_\ell(T_pA, t) := \det(1 - \mathrm{Frob}_\ell^{-1} t | V_pA)$$

and Frob_ℓ is the arithmetic Frobenius. The product converges for $\mathfrak{Re}(s) > 3/2$ and moreover when $A = E$ is an elliptic curve, $L(E, s)$ extends analytically to the entire complex plane by a reasonably well-known result of Wiles [Wil95], Taylor-Wiles [TW95] and Breuil-Conrad-Diamond-Taylor [BCDT01]. Such a meromorphic continuation is expected to hold in general. We write r_{an} for the order of vanishing of $L(A, s)$ at $s = 1$. We are now in a position state the conjecture of Birch–Swinnerton-Dyer:

Conjecture 1.1.1. *For A as above, we have*

$$r_{\mathrm{an}} = r_{\mathrm{alg}}.$$

This conjecture is remarkable as it compares two seemingly disparate objects of fundamentally different natures: the L-function and the Mordell–Weil group of the abelian variety, one analytic, one algebraic.

Remark 1.1.2. There is another statement, sometimes referred to as the full Birch–Swinnerton-Dyer conjecture, which gives a precise formula for the r_{an}-th derivative of $L(A, s)$ at $s = 1$ in terms of a number of invariants of A including the order of the Tate–Shafarevich group

$$Ш(A) = \ker\left(H^1(\mathbb{Q}, A(\mathbb{Q})) \to \prod_{\ell \text{ prime}} H^1(\mathbb{Q}_\ell, A(\mathbb{Q}_\ell))\right)$$

a mysterious group which is not even known to be finite. In the case that $A = E$ is an elliptic curve, finiteness of $Ш(E)$ implies the existence of an effective algorithm for computing the rank of the Mordell–Weil group. This finiteness is expected but far from proven. It is, however, implied by the full Birch–Swinnerton-Dyer conjecture.

The Birch–Swinnerton-Dyer conjecture is (modulo a few fickle grains) part of a vast web of conjectures known collectively as the Bloch–Kato conjectures. Let $K/\mathbb{Q}_p$ be a finite extension of fields with ring of integers $\mathcal{O}$ and let V be a K-linear continuous $G_{\mathbb{Q}}$-representation unramified at almost all $\ell \neq p$ and de Rham at p with $T \subset V$ a $G_{\mathbb{Q}}$-invariant $\mathcal{O}$-lattice. Such a representation is called *geometric*. The nomenclature is justified by the fact that all Galois representations arising as subquotients of the étale cohomology of smooth projective varieties are geometric. Furthermore, the Fontaine–Mazure conjecture posits that all irreducible geometric Galois representations arise in this way. Much like the Birch–Swinnerton-Dyer conjecture, the Bloch-Kato conjecture concerns a comparison of algebraic and analytic invariants coming from V. On the algebraic side we have the *Bloch–Kato Selmer group* $H^1_f(\mathbb{Q}, T)$:

Definition 1.1.3. Define subgroups of $H^1(\mathbb{Q}_\ell, T)$ by

$$H^1_f(\mathbb{Q}_\ell, T) := \begin{cases} \ker\left(H^1(\mathbb{Q}_\ell, T) \to H^1(\mathbb{Q}_\ell, T \otimes \mathbb{B}_{\mathrm{cris}})\right) & \text{if } \ell = p \\ \ker\left(H^1(\mathbb{Q}_\ell, T) \to H^1(G_{\mathbb{Q}_\ell}/I_\ell, T^{I_\ell})\right) & \text{otherwise,} \end{cases}$$

where $\mathbb{B}_{\mathrm{cris}}$ is Fontaine's ring of crystalline periods, and define the *Bloch–Kato Selmer group*:

$$H^1_f(\mathbb{Q}, T) := \ker\left(H^1(\mathbb{Q}, T) \to \prod_\ell \frac{H^1(\mathbb{Q}_\ell, T)}{H^1_f(\mathbb{Q}_\ell, T)}\right).$$

On the analytic side we construct the L-function of V:

$$L(V, s) = \prod_{\ell : V \text{ unramified at } \ell} \det(1 - \mathrm{Frob}_\ell^{-1} \ell^{-s} | V)^{-1}.$$

The Bloch–Kato conjecture is as follows:

Conjecture 1.1.4. *For V as above we have*

$$\dim_K H^1_f(\mathbb{Q}, V^*(1)) - \dim_K H^0(\mathbb{Q}, V^*(1)) = \mathrm{ord}_{s=0} L(V, s)$$

where $V^*(1)$ is the Tate dual of V.

In the case that $V = V_pA$, the p-adic Tate module of an abelian variety $A/\mathbb{Q}$, the Kummer map gives an injection

$$A(\mathbb{Q}) \otimes \mathbb{Q}_p \hookrightarrow H^1_f(\mathbb{Q}, V_pA)$$

the cokernel of which has dimension equal to the rank of the p-part of $Ш(A)$.

When $r_{\mathrm{an}} \in \{0, 1\}$, the Birch–Swinnerton-Dyer conjecture for elliptic curves has been proved by Kolyvagin, building off work of Gross–Zagier, using the (anticyclotomic) Euler system of Heegner points. Later Kato [Kat04] was able to prove Bloch–Kato in analytic rank 0 for modular forms of weight $k \geq 2$, recovering the $r_{\mathrm{an}} = 0$ elliptic curve case as a special case. We give an overview of Kato's method as the main motivation for much of our work will be based around a generalisation of this method.

1.1.2 Kato's method and generalisations

Just what is an Euler system?

Definition 1.1.5. Let V be as above and let $T \subset V$ be a $G_{\mathbb{Q}}$-stable lattice and Σ a finite set of primes. An Euler system for (T, Σ) is a collection of classes $\{c_m\}$ for $m \geq 0$

$$c_m \in H^1(\mathbb{Q}(\zeta_m), T)$$

satisfying the following *norm relations*:

$$\mathrm{cores}^{\mathbb{Q}(\zeta_{m\ell})}_{\mathbb{Q}(\zeta_m)}(c_{m\ell}) = \begin{cases} c_m & \text{if } \ell \in \Sigma \text{ or } \ell | m \\ P_\ell(V^*(1), \sigma_\ell^{-1}) \cdot c_m & \text{otherwise} \end{cases},$$

where σ_ℓ is the image of Frob_ℓ in $\mathrm{Gal}(\mathbb{Q}(\zeta_m)/\mathbb{Q})$.

The primary utility of an Euler system is that if one can prove its non-vanishing (and certain local conditions) then one obtains bounds on Selmer groups.

In [Kat04], Kato constructs an Euler system $\{z_m^{\mathrm{Kato}}\}$ associated to an elliptic curve E (or, more generally, a modular form) by pushing forward classes in the étale cohomology of modular curves.

If we fix m and set $H^1_{\mathrm{Iw}}(\mathbb{Q}(\zeta_{mp^\infty}), V_pE) = \left(\varprojlim_n H^1(\mathbb{Q}(\zeta_{mp^n}), T_pE)\right) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ then the system $\{z^{\mathrm{Kato}}_{mp^n}\}_{n\geq 1}$ defines an element $z^{\mathrm{Kato}}_{mp^\infty} \in H^1_{\mathrm{Iw}}(\mathbb{Q}(\zeta_{mp^\infty}), V_pE)$. When p is a prime of good reduction for E, Perrin-Riou has constructed a map

$$\mathcal{L} : H^1_{\mathrm{Iw}}(\mathbb{Q}(\zeta_{mp^\infty}), V_pE) \to \mathcal{H} \otimes \mathbb{D}_{\mathrm{cris}}(V_pE)$$

interpolating the Bloch-Kato logarithm, where $\mathcal{H}$ is the ring of rigid analytic functions on the rigid space $\mathcal{W}$ parameterising characters of $\mathbb{Z}_p^\times$. When α is a root of the polynomial

$$X^2P(V_pE, X^{-1}) = X^2 + a_pX + p$$

of p-slope < 1, Kato proves that the projection of $\mathcal{L}(z^{\mathrm{Kato}}_{mp^\infty})$ to the α-eigenspace of $\mathbb{D}_{\mathrm{cris}}(V_pE)$ is a rigid analytic function with prescribed growth depending on the slope of α and whose specialisations at finite order characters χ of $\mathbb{Z}_p^\times$ are, up to an explicit non-zero factor, equal to the value at $s = 1$ of the twisted L-function $L(E, \chi, s)$. We call such a function a p-adic L-function for E, and a result linking the Bloch-Kato logarithm of (the bottom class of) an Euler system with L-values is called an *explicit reciprocity law.*

In summary, Kato showed that

$$L(E, 1) \neq 0 \implies z_0^{\mathrm{Kato}} \neq 0 \implies H^1_f(\mathbb{Q}, V_pE) = 0,$$

where the last implication is from the bounds on Selmer groups given by a non-vanishing Euler system. Since $H^0(\mathbb{Q}, V_pE) = 0$ we have proved one implication of Bloch–Kato when $r_{\mathrm{an}} = 0$.

1.1.3 Automorphic forms and p-adic variation

Applying Kato's method to elliptic curves uses as crucial input the modularity theorem of Wiles et.al. which implies that the Tate module of a rational elliptic curve occurs in the étale cohomology of a modular curve. The following folklore conjecture is a vast generalisation of the modularity theorem (see, for example, [BG10, Conjecture 3.2.1]).

Conjecture 1.1.6. *Any irreducible geometric p-adic representation*

$$\rho : G_{\mathbb{Q}} \to \mathrm{GL}_n(\bar{\mathbb{Q}}_p)$$

is automorphic in the sense that there is an algebraic (in the sense of Clozel [Clo]) automorphic representation π *of* $\mathrm{GL}_n(\mathbb{A}_{\mathbb{Q}})$ *such that*

$$\mathrm{WD}_p(\rho) = \mathrm{rec}(\pi_p \otimes |\det|^{\frac{1-n}{2}})$$

where rec *is the local Langlands correspondence and* $\mathrm{WD}_p(\rho)$ *is the Frobenius semisimple Weil–Deligne representation associated to* ρ *at p.*

Since automorphic L-functions are generally more amenable to analysis (for example, the functional equation and meromorphic extension of automorphic L-functions for GL_n are known) it behooves us to study the automorphic side of things. In the special case of abelian surfaces, the above conjecture is given by the paramodular conjecture:

Conjecture 1.1.7. *Let* $A/\mathbb{Q}$ *be an abelian variety satisfying* $\mathrm{End}_{\mathbb{Q}}(A) = \mathbb{Z}$ *and of conductor* N*. Then there is a Siegel modular form* f_A *of weight* $(2, 2)$ *and paramodular level* N *whose associated p-adic (spin)* $G_{\mathbb{Q}}$*-representations coincides with the p-adic Tate module of* A*.*

The Galois representations associated to Siegel forms of weight $(2, 2)$ do not occur in the étale cohomology of Siegel modular threefolds. In order to work with these Galois representations (and indeed, in order to construct them) we need to work with p-adic families of Siegel modular forms.

In [Ser73], Serre defined p-adic modular forms as p-adic limits of classical modular forms and used his Eisenstein family to give a new construction of the p-adic L-function of Kubota–Leopoldt interpolating the values of the Riemann zeta function at odd negative integers. Serre's space of p-adic modular forms comes equipped with an action of the Hecke operators $\{T_\ell\}_{\ell\neq p}, U_p$ but is rather unwieldy and, in particular, the operator U_p has an extremely large continuous spectrum.

Serre's p-adic modular forms were refined by Katz [Kat73] who defined a subspace of p-adic modular forms, called overconvergent modular forms. Whereas p-adic modular forms arise as global sections over the ordinary locus [1] of the modular curve, Katz defines overconvergent forms as sections which 'overconverge' to small neighbourhoods of the ordinary locus. The space of overconvergent modular forms of fixed radius of overconvergence and tame level N is a Banach space admitting an action of the Hecke operators $\{T_\ell\}_{\ell \neq p}, U_p$, the latter of which acts compactly, allowing for a rich spectral theory. Using Serre's Eisenstein family, Coleman was able to define overconvergent modular forms of an arbitrary p-adic weight and showed that they vary in families, eventually culminating in the construction of the eigencurve by Coleman–Mazur [CM98], a rigid analytic curve whose points parameterise eigensystems of overconvergent eigenforms with finite U_p-slope. The construction of the eigencurve has been generalised to automorphic forms over more general groups by work of Buzzard [Buz07], Urban [Urb11], Johansson–Newton [JN19], Ash-Stevens [AS08], Andreatta–Iovita–Pilloni [AIP15] among others. The resulting rigid spaces are known as eigenvarieties, and their points parameterise Hecke eigensystems of finite-slope automorphic forms.

A different approach to p-adic modular forms was given by Hida, who realised one could get a good, and in particular *integral*, theory of p-adic modular forms by focusing only on those that are ordinary; their U_p eigenvalue is a p-adic unit. Hida shows that ordinary modular forms vary p-adically in families known as *Hida families*. Unlike the theory of overconvergent forms which has a predominantly p-adic analytic flavour, the theory of Hida families takes on a more algebraic shape; the interpolating spaces considered in Hida theory are projective limits of classical spaces and are projective over the Iwasawa algebra. The space of ordinary p-adic forms can be obtained directly from the space of p-adic modular forms via an ordinary idempotent which can be defined algebraically and in particular does not require any Banach structure.

Central to the theories of both Coleman and Hida are *control theorems* which allow one to isolate classical forms within a p-adic family using the action of the U_p-operator. In Section 4 we generalise control theorems of Hida and Tilouine–Urban to the setting of reductive groups which are quasi-split at p.

1.1.4 Overconvergent cohomology

Let $\Gamma_1(N) \subset \mathrm{SL}_2(\mathbb{Z})$ be the usual congruence subgroup of level N coprime to p. For $k \geq 2$ and L a field of characteristic 0 the classical Eichler–Shimura isomorphism identifies group cohomology with spaces of classical modular forms:

$$H^1(\Gamma_1(N), \mathrm{Sym}^{k-2}L^2) \cong S_k(\Gamma_1(N), L) \oplus \overline{S_k(\Gamma_1(N), L)} \oplus \mathcal{E}_k(\Gamma_1(N), L),$$

where $S_k(\Gamma_1(N), L)$ is the space of weight k cusp forms for $\Gamma_1(N)$ with Fourier coefficients in L and $\mathcal{E}_k(\Gamma, L)$ is the space of weight k complex Eisenstein series. The isomorphism is equivariant for the action of the Hecke operators on both sides and canonical when $L = \mathbb{C}$. This isomorphism has been vastly generalised by work of Franke [Fra98].

The above isomorphisms motivate the use of one of our main tools, the *overconvergent cohomology* of Ash–Stevens. Ash–Stevens define large modules $\mathbb{D}$ of p-adic distributions equipped with an action of a monoid Σ containing $\Gamma_1(N)$. The cohomology groups $H^1(\Gamma_1(N) \cap \Gamma_0(p), \mathbb{D})$ are p-adic Banach spaces admitting an action of the Hecke operators T_ℓ for $\ell \nmid Np$ and U_p with U_p acting compactly. One can use these cohomology groups to construct a rigid analytic space parameterising the Hecke eigensystems occurring in $H^1(\Gamma_1(N), \mathrm{Sym}^{k-2}\mathbb{Q}_p^2)$ with finite U_p-slope. By a theorem of Chenevier [Che05], this 'cohomological' eigencurve is isomorphic to the Coleman–Mazur eigencurve. The study of overconvergent cohomology has both pros and cons compared to Coleman's families of overconvergent modular forms. For our applications we restrict ourselves to noting that, for a neat congruence subgroup $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$, since the complex upper half plane $\mathcal{H}$ is the universal cover of $Y(\Gamma) := \mathcal{H}/\Gamma$ with fundamental group Γ, there is an isomorphism of cohomology groups

$$H^1(\Gamma, \mathrm{Sym}^{k-2}L^2) \cong H^1_B(Y(\Gamma)(\mathbb{C}), \mathrm{Sym}^{k-2}L^2)$$

where the right hand side is Betti cohomology of $Y(\Gamma)(\mathbb{C})$ with coefficients in the local system induced by $\mathrm{Sym}^{k-2}L^2$. The Betti-étale comparison isomorphism then identifies

$$H^1(\Gamma, \mathrm{Sym}^{k-2}\mathbb{Q}_p^2) \cong H^1_{\acute{e}t}(Y(\Gamma)_{\bar{\mathbb{Q}}}, \mathrm{Sym}^{k-2}\mathbb{Q}_p^2),$$

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[1] We remove the points corresponding to elliptic curves which are supersingular at p.

where the right hand side is the absolute étale cohomology of the $\mathbb{Q}$-curve $Y(\Gamma)_{\mathbb{Q}}$ with coefficients in the lisse étale sheaf associated to $\mathrm{Sym}^{k-2}\mathbb{Q}_p^2$. The cohomology classes we will be working with will naturally live in Betti and étale cohomology and so the above isomorphisms provide us an avenue to constructing families of classes over subspaces of the eigencurve.

There is a general theory of overconvergent cohomology for the class of reductive groups G admitting discrete series representations. This theory was developed in this generality by Urban [Urb11] who used it to construct eigenvarieties in this setting. In general the representation $\mathrm{Sym}^{k-2}L^2$ will be replaced by an L-linear irreducible algebraic representation V_λ associated to a character $\lambda \in X^\bullet(T_G)$ of a maximal torus T_G. For applications to the Birch–Swinnerton-Dyer conjectures we want to work with $G = \mathrm{GSp}_{2g}$ for $g \geq 1$. Automorphic forms for G include the classical genus g Siegel modular forms. We will focus on how to generalise Kato's argument for $g = 1$ to the case $g = 2$ which is where the necessity of working with p-adic families first appears.

Recall that the paramodular conjecture associates to an abelian surface a weight $(2, 2)$ Siegel modular form of paramodular level. Much like weight 1 modular forms, the eigensystems of these forms do not occur in classical cohomology (their weight is not *cohomological*). The Hecke eigensystems of these forms do however exist as a p-adic limit of forms of cohomological weight, giving a point on the eigenvariety $\mathcal{E}$ for G. One can define a Galois representation associated to these forms by p-adic interpolation. Work of Loeffler–Skinner–Zerbes [LSZ21] associates an Euler system, the *Lemma–Flach* Euler system, to cohomological weight Siegel modular forms by pushing forward canonical classes in the étale cohomology of Siegel threefolds. By varying these Euler system classes in a family over a subspace of $\mathcal{E}$ we hope to construct an Euler system for weight $(2, 2)$ forms by interpolation, giving us access to the first step in our adaptation of Kato's argument. In recent work, Loeffler–Zerbes [LZ21] have succeeded in constructing an Euler system for *ordinary* abelian surfaces using the above method and have applied Kato's method to prove new cases of Birch–Swinnerton-Dyer. In Section 5 of this thesis we show how one can interpolate the Lemma–Flach Euler system in *non-ordinary* families, taking the first step in applying Kato's argument to a class of non-ordinary abelian surfaces.

1.1.5 Loeffler's machine

Whereas most of the discussion above has been focused on the classical cases of GSp_{2g} for $g \geq 1$, the constructions of this thesis will often be applied in much greater generality. The jumping off point is Loeffler's machine for constructing norm compatible elements [Loe21] in the cohomology of locally symmetric spaces. The generality of this material makes any exposition rather cumbersome, so the author hopes you will forgive him some vagueities,[2]. Fundamental in this construction is the theory of *spherical varieties*:

Definition 1.1.8. Let $H \hookrightarrow G$ be an embedding of connected reductive group schemes. The pair (G, H) is called a *spherical pair* if H has an open orbit on the flag variety $\mathcal{F}_G := \bar{Q}_G \backslash G$ where Q_G is a choice of parabolic subgroup of G and $\bar{Q}_G$ is its conjugate under the long Weyl element of G.

Remark 1.1.9. The definition of a spherical pair is usually reserved for the case that Q_G is a Borel subgroup of G, so if we were being super fastidious we might call the above pairs Q_G-spherical to emphasise the dependency on the parabolic Q_G but we aren't and we won't.

In practice we want finer control over the open orbit. Let Q_H be a parabolic subgroup of H with Levi decomposition $Q_H = L_H \ltimes N_H$ and suppose there is a normal algebraic subgroup $L_H^0 \subset L_H$ such that for $Q_H^0 := L_G^0 \ltimes N_H \subset Q_H$ (such a subgroup is a called a *mirabolic* subgroup) we have

- An element $u \in G(\mathbb{Z}_p)$ mapping to $[u] \in \mathcal{F}_G$ such the Q_H^0-orbit of $[u]$ is Zariski open.
- A subgroup $Q_G^0 = L_G^0 \times N_G \subset Q_G$ such that $uQ_H^0u^{-1} \cap \bar{Q}_G \subset Q_G^0$.

Obviously the second point can be made to be trivial, but taking smaller Q_G^0 allows us to work with a greater range of weights.

For open compact subgroups $K_H \subset H(\mathbb{Z}_p), K_G \subset G(\mathbb{Z}_p)$ let $Y_H(K_H), Y_G(K_G)$ be the locally symmetric spaces for H, G level K_H, K_G at p and some fixed tame level, chosen so that we have a

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[2] The dictionary defies coldly my assertion that vagueities is a real English word, but I will wash my hands in the cauldron of Hell before I concede.

closed immersion

$$\iota : Y_H(K_H) \hookrightarrow Y_G(K_G).$$

Let V_H, V_G be irreducible algebraic representations of H, G respectively and by abuse of notation use the same notation for their induced local-systems/lisse étale sheaves as appropriate.

Given a map

$$\iota^{\#} : V_H \to \iota^* V_G$$

we obtain a pushforward map on cohomology groups

$$\iota_* : H^i(Y_H(K_H), V_H) \to H^{i+c}(Y_G(K_G), V_G),$$

where c is essentially the real codimension of $Y_H(K_H)$ in $Y_G(K_G)$ modulo a central irritation. Define

$$H^i_{\mathrm{Iw}}(Q^0_H, V_H) = \varprojlim_{K_H \supset Q^0_H} H^i(Y_H(K_H), V_H)$$

for a suitable[3] cohomology theory H^i with appropriate coefficients V_H.

Theorem 1.1.10. *Let H, G, Q^0_H, Q^0_G be as above and let $\xi_H \in H^i_{\mathrm{Iw}}(Q^0_H, V_H)$ be a system of norm compatible elements for H. Then there is an element*

$$\xi_G := ([u]_* \circ \iota_*)(\xi_H) \in H^{i+c}_{\mathrm{Iw}}(Q^0_G, V_G)^{\mathrm{fs}}$$

where the superscript refers to the finite-slope part of cohomology for a choice Hecke operator at p depending on the parabolic Q_G.

When ξ_H arises as the realisation of some canonical classes in motivic cohomology we expect ξ_G to be related to motivic L-functions i.e. the p-part of an Euler system or related to a p-adic L-function.

Example 1.1.11. When $H = \mathrm{GL}_2 \times_{\mathrm{GL}_1} \mathrm{GL}_2$, $G = \mathrm{GSp}_4$ then the subgroup

$$Q^0_H = \{ \left(\begin{smallmatrix} x & * \\ & 1 \end{smallmatrix}\right) \times \left(\begin{smallmatrix} x & * \\ & 1 \end{smallmatrix}\right) \}$$

has an open orbit on the flag variety $\bar{Q}_G \backslash G$ associated to the Siegel parabolic

$$Q_G = \left(\begin{smallmatrix} * & * & * & * \\ * & * & * & * \\ & & * & * \\ & & * & * \end{smallmatrix}\right) \cap \mathrm{GSp}_4.$$

Loeffler's machine applied to cup products of Beilinson's Eisenstein classes provides a construction of the Lemma-Flach Euler system of [LSZ21] along with the proof of the p-direction norm relations.

Given this general p-adic theory of Euler systems, one can ask whether there is a general theory of variation in families. This is answered in the affirmative for ordinary (or 'Hida') families by Loeffler–Zerbes and the author in [LRZ21], the origins of which can be found in this thesis. The situation of Coleman (non-ordinary) families is considered in the final part of this thesis.

Remark 1.1.12. There is (in some sense) an automorphic counterpart to this theory due to work of Sakellaridis–Venkatesh on the *relative Langlands programme*. Let G, H be as above and recall that a cuspidal automorphic representation Π of $G(\mathbb{A})$ is called H-distinguished if there is $\varphi \in \Pi$ such that

$$\mathcal{P}(\varphi) = \int_{[H]} \varphi(h) dh \neq 0.$$

Associated to a spherical pair (G, H) we have a *spherical variety* $X = G/H$. Suppose for simplicity that G is split. We can associate to X a split reductive group G_X and a general principle states (roughly) that a cuspidal automorphic representation Π of G is H-distinguished if and only if Π is a functorial transfer from $G_X(\mathbb{A})$ plus an additional condition featuring L-functions. An additional principle states that the functorial transfer condition is essentially equivalent to the existence (for almost all primes v) of Π'_v in the local Vogan L-packet of Π_v which is H-distinguished (there are

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[3] 'suitable' refers to Loeffler's theory of *cohomology functors* [Loe21, Section 2]. In practice these will be Betti or étale cohomology depending on whether we want to construct Euler systems or p-adic L-functions.

various technical reasons why this principle is probably wrong as stated, hence its status as a principle and not a conjecture).

The Ichino–Ikeda conjecture gives a formula for the square of $\mathcal{P}(\varphi)$ in terms of the L-function of Π multiplied by some local data and thus when Π is H-distinguished we expect the integral to give us information about the L-values of Π, and the above principle suggests that we should be able to find Π' in the (Vogan) L-packet of Π such that $\mathcal{P}(\varphi) \neq 0$ for some $\varphi \in \Pi'$.

The interpolation of period integrals such as $\mathcal{P}(\varphi)$ plays an important role in constructing p-adic L-functions; see, for example, [LPSZ19], [BSDW21] and so this consonance of the above theory of cohomological classes with the theory of Sakellaridis–Venkatesh is of explicit utility in the pursuit of automorphic Iwasawa theory.

1.2 Contents of this thesis

We give an overview of the work contained in this thesis.

1.2.1 Plus/Minus p-adic L-functions for GL_{2n}

This work has appeared, modulo minuscule revisions, in *Annales mathématiques du Québec* [Roc22].

As the material in this chapter is somewhat disjoint from subsequent chapters, we give a brief motivation for the theory.

Let V be a geometric p-adic $G_{\mathbb{Q}}$-representation with integral $G_{\mathbb{Q}}$-lattice T and let $d_-(T) = \mathrm{rank} V^{c=-1}$ be the the rank of the -1 eigenspace for complex conjugation. Assume for simplicity that V is crystalline at p. Set

$$r(V) := \max\{0, d_-(T) - \mathrm{Fil}^0\mathbb{D}_{\mathrm{dR}}(V)\}$$

We say that V satisfies the rank r Panchishkin condition at p if $r(V) = r, r(V^*(1)) = 0$ and there is a subspace $V^+ \subset V$ satisfying

- V^+ is stable under $G_{\mathbb{Q}_p}$,
- V^+ has all Hodge-Tate weights ≥ 0,
- V/V^+ has all Hodge-Tate weights ≤ 0.

Such a local subrepresentation is called a *Panchishkin subrepresentation.* The existence of a Panchishkin subrepresentations is related to the notion of *ordinarity.*

Example 1.2.1. Let $\mathcal{F}$ be a genus 2 weight 3 Siegel modular form with associated $G_{\mathbb{Q}}$-representation $V_{\mathcal{F}}$. If we set $V = V_{\mathcal{F}}^*$, then:

- V satisfies $r(V_{\mathcal{F}}^*) = 1, r(V_{\mathcal{F}}(1)) = 0$ and satisfies the rank 1 Panchishkin condition if and only if $\mathcal{F}$ is ordinary for the Hecke operator

$$T(p) = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & p & \\ & & & p \end{pmatrix}.$$

- $V(-1)$ satisfies $r(V(-1)) = 0, r(V^*(2)) = 0$ and satisfies the rank 0 Panchishkin condition if and only if $\mathcal{F}$ is ordinary for the Hecke operator

$$T_1(p^2) = \begin{pmatrix} 1 & & & \\ & p & & \\ & & p & \\ & & & p^2 \end{pmatrix}.$$

Let Λ be the $\mathbb{Z}_p^\times$ Iwasawa algebra, which can be embedded as the bounded-by-1 sections over the weight space $\mathcal{W}$. The rank 0 Panchishkin condition is thought to be the 'correct' condition for a bounded p-adic L-function $L_p(V) \in \Lambda[1/p]$ to exist for V. In this case the Iwasawa main conjecture for V says that the characteristic ideal of a 'Greenberg-Iwasawa' Selmer group $\widetilde{H}^1(\mathbb{Q}, T \otimes \Lambda)$ is generated by $L_p(V)$. When the rank 0 Panchishkin condition is not satisfied we can often still construct *unbounded* p-adic L-functions living in $\mathcal{H}$ whose growth is determined by a choice of eigenvalue of the Frobenius φ acting on $\mathbb{D}_{\mathrm{cris}}(V)$. A natural question to ask is if there is an analogous Iwasawa theory for these unbounded p-adic L-functions.

Assume for the rest of this section (for ease of notation) that $p > 2$. One approach to 'non-ordinary' Iwasawa theory has been initiated by Pollack [Pol03] and Kobayashi [Kob03]. Given a prime to p level modular newform f of weight k and nebentype χ one can construct a p-adic L-function $L_{p,\alpha}(f)$ for each root α of the Hecke polynomial

$$X^2 - a_p(f)X + \chi(p)p^{k-1} \tag{1}$$

of growth $v_p(\alpha)$. Suppose further that f satisfies $a_p = 0$. In this case both roots of (1) satisfy $v_p(\alpha) > 0$ and as a result neither of our p-adic L-functions are bounded. In this setting Pollack [Pol03] constructs bounded '$\pm$ p-adic L-functions' $L_p^{\pm}(f)$ satisfying

$$L_{p,\alpha}(f) = \log_k^+ L_p^+(f) + \log_k^- L_p^-(f),$$

where $\log_k^{\pm}$ are weight k 'half-logarithms' and are explicitly defined with prescribed zeroes. Using these 'plus/minus' L-functions, Kobayashi [Kob03] (for elliptic curves) and Lei [Lei11] (weight ≥ 2 modular forms) define $\pm$-Selmer groups and formulate analogues of the Iwasawa main conjecture for these objects. Using Kato's Euler system, the respective authors were able to prove one inclusion in their $\pm$-main conjectures. The full conjecture for GL_2 has been proved by Xin Wan [Wan16] in many cases and as a corollary used to prove full BSD for an infinite family of elliptic curves without complex multiplication.

In the paper [Roc22] we generalise the construction of plus/minus p-adic L-functions to certain cuspidal automorphic representations of GL_{2n}.

To be precise, let Π be a cuspidal automorphic representation of GL_{2n} unramified at p and admitting a Shalika model [4]. Denote the Satake parameters at p by $\alpha_1, \dots, \alpha_{2n}$, ordered by increasing p-adic valuation (after fixing an isomorphism $\mathbb{C} \cong \bar{\mathbb{Q}}_p$). Suppose that

$$\alpha_n + \alpha_{n+1} = 0,$$

and that Π has sufficiently small slope at p in a precise sense. Given a p-stabilisation of Π satisfying certain technical conditions, Barrera–Dimitrov–Williams [BSDW21] have constructed a p-adic L-function $L_p(\Pi) \in \mathcal{O}(\mathcal{W})$ interpolating the critical values of the complex L-function $L(\Pi)$. In this paper the following theorem has been proved under some further constraints on Π:

Theorem 1.2.2. *Suppose the parameters $\alpha_1, \dots, \alpha_{n-1}$ are such that the product $\prod_{i=1}^{n-1} \alpha_i$ has the smallest possible valuation. Then there are bounded rigid functions $L_p^{\pm}(\Pi) \in \mathcal{O}(\mathcal{W})^{\circ}$ satisfying*

$$L_p(\Pi) = \log_p^+ L_p^+(\Pi) + \log_p^- L_p^-(\Pi),$$

where the locus of zeroes of $\log^+(\Pi) \in \mathcal{O}(\mathcal{W})$ (resp $\log_p^-(\Pi)$) is given precisely by characters of $\mathbb{Z}_p^\times$ of the form $\theta \cdot x^j$ where j is a critical integer for Π and θ is a finite-order character of even (resp. odd) p-power order.

This generalises work of Pollack [Pol03] in the case of GL_2. As a novel application of this construction, we use the fact that bounded functions on $\mathcal{W}$ have finitely many zeroes to prove the following theorem:

Theorem 1.2.3. *Under some technical assumptions on the complex L-function $L(\Pi)$, for infinitely many p-power Dirichlet characters we have*

$$L(\Pi \otimes \chi, \frac{\omega + 1}{2}) \neq 0,$$

where ω is the purity weight of Π.

---

[4] An automorphic representation Π of $\mathrm{GL}_n(\mathbb{A}_f)$ admits a Shalika model if it is globally distinguished for a particular subgroup known as the *Shalika subgroup*. A special case of a the general theory of Sakellaridis–Venkatesh discussed in Remark 1.1.12 shows that this is equivalent to Π being the Langlands transfer of an automorphic representation of GSp_4.

This theorem extends work of Dimitrov–Januszewski–Raghuram who work under the assumption of Borel-ordinarity [DJR20].

Remark 1.2.4. We note that while we have labelled this under the banner of 'non-ordinary' Iwasawa theory, for $n > 1$ we are still operating under an ordinarity hypothesis, namely *Siegel ordinarity.* This corresponds to the existence of an $(n-1)$-dimensional subrepresentation of the $G_{\mathbb{Q}_p}$-representation $\mathrm{rec}(\Pi_p \otimes |\det|_p^{\frac{1-n}{2}})$. For $n = 2$ this is precisely the condition under which we expect the existence of a rank 1 Euler system, an expectation which has been realised in the construction of the Lemma-Flach Euler system of Loeffler–Skinner–Zerbes [LSZ21]. Note that this does give us a construction of bounded measures in a case where we do not have the rank 0 Panchishkin condition.

The construction of signed p-adic L-functions for GL_{2n} has been generalised by Lei–Ray [LR20] who have managed to relax the condition $\alpha_n + \alpha_{n+1} = 0$. They formulate signed main conjectures using their signed p-adic L-functions.

1.2.2 Interpolating Iwahori level Lemma-Flach classes

Let $H = \mathrm{GL}_2 \times_{\mathrm{GL}_1} \mathrm{GL}_2$, $G = \mathrm{GSp}_4$ and let $a, b, m \geq 0$ and $a \geq q \geq 0, b \geq r \geq 0$ be integers. In [LSZ21] Loeffler–Skinner–Zerbes construct classes

$${}_{c_1,c_2}z^{[a,b,q,r]}_{M,m,n} \in H^4_{\acute{e}t}(Y_G(M, p^m, p^n), \mathscr{D}^{a,b}(-q))$$

where the notation is as in *op.cit.* The tuple $[a, b, q, r]$ parameterises the *branching law* describing how the algebraic representation $\mathscr{D}^{a,b}(-q)$ breaks up into irreducible representations after restriction to H. The (Siegel) ordinary part of these classes satisfy the Euler system norm relations after projecting to Galois cohomology. These classes can be constructed using Loeffler's machine (Theorem 1.1.10) by noting that the mirabolic subgroup

$$Q^0_H = \{\left(\begin{smallmatrix} x & * \\ & 1 \end{smallmatrix}\right) \times \left(\begin{smallmatrix} x & * \\ & 1 \end{smallmatrix}\right)\}$$

of the Siegel parabolic $Q_S = \{\left(\begin{smallmatrix} A & X \\ & B \end{smallmatrix}\right) \in G : A, B \in \mathrm{GL}_2, X \in M_2\}$ has an open orbit on the Siegel flag variety $\mathcal{F}_S = \bar{Q}_S \backslash G$.

In Section 9 of *op.cit* the authors construct Iwasawa cohomology classes

$${}_{c_1,c_2}z_{\acute{e}t} \in H^4_{\mathrm{Iw}}(N_S(\mathbb{Z}_p), \mathbb{Z}_p)$$

where N_S is the unipotent radical of Q_S, and *moment maps*

$$\mathrm{mom}^{[a,b,q,r]}_{m,n} : H^4_{\mathrm{Iw}}(N_S(\mathbb{Z}_p), \mathbb{Z}_p) \to H^4_{\acute{e}t}(Y_G(M, p^m, p^n), \mathscr{D}^{a,b}(-q))$$

satisfying

$$\mathrm{mom}^{[a,b,q,r]}_{m,n}({}_{c_1,c_2}z_{\acute{e}t}) = {}_{c_1,c_2}z^{[a,b,q,r]}_{M,m,n}.$$

In [LZ20b] these classes are used to construct Galois cohomology classes

$${}_{c_1,c_2}z^{[\underline{\Pi},r]}_{\mathrm{Iw}} \in H^1_{\mathrm{Iw}}(\mathbb{Q}(\zeta_{Mp^\infty}), W_{\underline{\Pi}}),$$

where $\underline{\Pi}$ is a one-parameter family of Siegel ordinary Hecke eigensystems, $W_{\underline{\Pi}}$ is the Λ-adic Galois representation associated to $\underline{\Pi}$ and $r \geq 0$ is fixed. These classes interpolate the Lemma–Flach Euler system classes

$${}_{c_1,c_2}z^{[\Pi(n),q,r]}_{m} \in H^1(\mathbb{Q}(\zeta_{Mp^m}), W_{\Pi(n)})$$

where $\Pi(n)$ is a classical specialisation of $\underline{\Pi}$ at an integer n.

What we are doing here is varying the variables (a, q) occurring in the branching law. We would like to construct a class

$$z^{[\underline{\Pi}']}_{\mathrm{Iw}} \in H^1_{\mathrm{Iw}}(\mathbb{Q}(\zeta_{Mp^\infty}), W_{\underline{\Pi}'} \otimes \Lambda)$$

for $\underline{\Pi}'$ a family of ordinary eigensystems varying in both weight variables a, b and $W_{\underline{\Pi}'}$ is a family of Galois representations interpolating the Galois representations of classical specialisations of $\underline{\Pi}'$, interpolating the classes

$${}_{c_1,c_2}z^{[\underline{\Pi}'(a,b),q,r]}_{m} \in H^1(\mathbb{Q}(\zeta_{Mp^m}), W_{\Pi(a,b)})$$

for all a, b, q, r occurring in the branching law. Siegel-ordinarity is too weak of a condition to allow for such variation; the quotient of the Siegel Levi subgroup by its derived subgroup is a rank 2 torus so we can only ever hope to vary two variables in this setting.

In Chapter 3 we consider the modified groups $\tilde{G} = G \times \mathrm{GL}_1$ and $\tilde{H} = H \times \mathrm{GL}_1$. The natural extension $B_{\tilde{G}} := B_G \times \mathrm{GL}_1$ of the Borel subgroup of G to $\tilde{G}$ has unchanged flag variety and moreover the subgroup

$$Q^0_{\tilde{H}}(R) := \{ \left(\begin{smallmatrix} x & * \\ & y \end{smallmatrix}\right) \times \left(\begin{smallmatrix} xy & * \\ & 1 \end{smallmatrix}\right) \times (y) : x, y \in R^\times \}$$

has an open orbit on the flag variety for $B_{\tilde{G}}$. Using Loeffler's machine we obtain classes

$${}_{c_1,c_2}\tilde{z}^{[a,b,q,r]}_{\mathrm{Iw}} \in H^4_{\mathrm{Iw}}(B_{\tilde{G}}(\mathbb{Z}_p), \tilde{\mathscr{D}}^{a,b} \otimes \mu^q \sigma^{r-q})^{\mathrm{ord}}$$

where $(\cdot)^{\mathrm{ord}}$ denotes the ordinary subspace for the Hecke operator at p associated to the Borel subgroup, $\tilde{\mathscr{D}}^{a,b}$ is an explicit twist of $\mathscr{D}^{a,b}$, μ is the similitude character on G and σ is the projection of $\tilde{G}$ to its GL_1 factor. We are then able to construct moment maps

$$\mathrm{mom}^{[a,b,q,r]} : H^4_{\mathrm{Iw}}(B_{\tilde{G}}(\mathbb{Z}_p), \mathbb{Z}_p(3)) \to H^4_{\mathrm{Iw}}(B_{\tilde{G}}(\mathbb{Z}_p), \tilde{\mathscr{D}}^{a,b} \otimes \mu^q \sigma^{r-q})$$

satisfying

$$\mathrm{mom}^{[a,b,q,r]}({}_{c_1,c_2}\tilde{z}^{[0,0,0,0]}_{\mathrm{Iw}}) = {}_{c_1,c_2}\tilde{z}^{[a,b,q,r]}_{\mathrm{Iw}}.$$

Pushing forward to Galois cohomology gives us the required class varying in a 4-parameter family. This fulfils the promise of [LZ20b, Remark 17.3.10], rectifying that papers contemptible cowardice.

1.2.3 Derived control theorems for reductive groups

The classical control theorem of Hida gives us precise information as to when the specialisation of a Λ-adic eigensystem is classical. To be precise, for integers $k \geq 0, r \geq 1$ let $I_{k,r}$ be the ideal of $\Lambda = \mathbb{Z}_p[[\mathbb{Z}_p^\times]]$ generated by the elements $[1+p^r] - (1+p^r)^k$, where square-brackets refer to group-like elements. Note that this is the image of the kernel of the homomorphism

$$\begin{aligned} \Lambda_r := \mathbb{Z}_p[[1 + p^r\mathbb{Z}_p]] &\to \mathbb{Z}_p^\times \\ [x] &\mapsto x^k \end{aligned}$$

under the natural inclusion $\Lambda_r \to \Lambda$. If we define

$$H^1_{\mathrm{ord}}(\Gamma_1(p^\infty), \mathbb{Z}_p) = \varprojlim_r H^1_{\mathrm{ord}}(\Gamma_1(p^r), \mathbb{Z}_p),$$

where ord refers to the subspace on which U_p acts invertibly, then this is a projective Λ-module and Hida's control theorem (in this particular case due to Ohta [Oht99]) gives an isomorphism

$$H^1_{\mathrm{ord}}(\Gamma_1(p^\infty), \mathbb{Z}_p)/I_{k,r} \cong H^1(\Gamma_1(p^r), \mathrm{Sym}^{k-2}\mathbb{Z}_p^2).$$

In particular one sees that one can construct families of ordinary Hecke eigensystems in 'infinite p-level' cohomology groups whose specialisations at classical points give classical Hecke eigensystems by classical Eichler–Shimura theory. This approach was utilised by Tilouine–Urban [TU99] in order to construct several variable families of Hecke eigensystems specialising to Hecke eigensystems of classical Siegel–Hilbert cusp forms.

We generalise previous work of Hida [Hid95] for SL_n and Tilouine-Urban [TU99] for GSp_4 and prove control theorems for the Betti cohomology of locally symmetric spaces associated to a large class of reductive groups. As in the above cases, the Betti cohomology groups carry a natural Hecke action and our theorems give precise information about when one can lift an ordinary integral Hecke eigensystem to a family of eigensystems taking values in an Iwasawa algebra. Unlike in the case of SL_2 we will in general have non-vanishing infinite p-level ordinary cohomology groups outside of the middle degree. This suggests that the correct generalisation of Hida's control theorem should use the language of derived categories.

More precisely, let G be a connected reductive $\mathbb{Q}$-group which is quasi-split at a prime p. Let $Q = M_Q N_Q \subset G$ be a parabolic subgroup with Levi M_Q and unipotent radical N_Q, and let

$K = K_p K^p \subset G(\mathbb{A}_f)$ be an open compact subgroup of $G(\mathbb{A}_f)$. We consider the locally symmetric space

$$S_K = G(\mathbb{Q})\backslash G(\mathbb{A})/K \cdot K_\infty$$

where K_∞ is the product of a maximal compact subgroup for $G^{\mathrm{der}}(\mathbb{R})$ and the real points of the centre of G. Let $K_0(p^n) \subset G(\mathbb{A})$ be the depth n parahoric subgroup associated to Q and let $K_1(p^n)$ be the points of $K_0(p)$ which are in $N_Q(\mathbb{Z}/p^n\mathbb{Z})$ mod p^n. Suppose $L/\mathbb{Q}_p$ is a finite extension over which G splits with ring of integers $\mathcal{O}_L$. Given an irreducible $\mathcal{O}_L$-linear representation V_λ of $G_{/\mathcal{O}_L}$ of highest weight $\lambda \in X^\bullet(M_Q)$ we can construct a locally constant sheaf $\mathcal{V}_\lambda$ on S_K. The Betti cohomology $R\Gamma(S_{K^p K_?(p)}, \mathcal{V}_\lambda)$ carries a natural action of the Hecke operator U_Q associated to Q. Define the Q-ordinary cohomology of $S_{K^p K_?(p^n)}$

$$R\Gamma_{Q-\mathrm{ord}}(S_{K^p K_?(p^n)}, \mathcal{V}_\lambda) := e_Q R\Gamma(S_{K^p K_?(p^n)}, \mathcal{V}_\lambda),$$

where $e_Q := \lim_n U_Q^{n!}$ is the ordinary idempotent associated to Q. Denote by $\Lambda = \Lambda(\mathfrak{S}_G)$ the Iwasawa algebra $\mathcal{O}_L[[\mathfrak{S}_G]]$, where $\mathfrak{S}_G = M_Q(\mathbb{Z}_p)/M_Q^{\mathrm{der}}(\mathbb{Z}_p)$ and for $n \geq 1$ let Λ_n be the Iwasawa algebra of $\mathfrak{S}_n = \{s \in \mathfrak{S}_G : s \equiv 1 \bmod p^n\}$.

Theorem 1.2.5. *For all λ as above there is a perfect complex $M_\lambda^\bullet \in \mathscr{D}(\Lambda)$ concentrated in degrees $[0, \nu]$ satisfying*

$$H^i(M_\lambda^\bullet) = \varprojlim_n H^i_{Q-\mathrm{ord}}(K_1(p^n), \mathcal{V}_\lambda/p^n)$$

and for all χ as above there is a quasi-isomorphism

$$M_\lambda^\bullet \otimes^L_{\Lambda_n} \mathcal{O}_L^{(\chi)} \sim R\Gamma_{Q-\mathrm{ord}}(K_1(p^n), \mathcal{V}_{\lambda+\chi}),$$

for $n \geq 1$ and a quasi-isomorphism

$$M_\lambda^\bullet \otimes^L_{\Lambda} \mathcal{O}_L^{(\chi)} \sim R\Gamma_{Q-\mathrm{ord}}(K_0(p), \mathcal{V}_{\lambda+\chi}).$$

We also construct pairings on ordinary cohomology in the case that $Q = B$ is a Borel subgroup and give criteria for localisations of the cohomology of $M_\lambda^\bullet$ to vanish outside the middle degree d, at least after taking invariants under the prime-to-p part Δ of $\mathfrak{S}_G$, in which case $H^d_{Q-\mathrm{ord}}(M_\lambda^\bullet)^\Delta$ is a projective Λ-module.

The techniques used Chapter 4 are analogous to those used in the previously cited papers of Hida and Tilouine–Urban, however those papers do not work in the derived setting, as we do. Furthermore, this paper is intended to be used as a toolkit for those wanting to work with Euler systems in families. The papers of Hida and Tilouine–Urban work with $\mathbb{Q}_p/\mathbb{Z}_p$- coefficients, and thus are not directly applicable to Euler systems constructed using the methods of Loeffler-Zerbes et.al. which exist in étale cohomology with $\mathbb{Z}_p$-linear coefficients. The results of Chapter 4 have already found applications in a general construction of Euler systems in ordinary families in the work of Loeffler–Zerbes and the author [LRZ21].

1.2.4 Spherical varieties and non-ordinary families of cohomology classes

Previously we gave an example of how interpolating cohomology classes is an indispensable tool for work on the Bloch–Kato conjectures. In Chapter 5 we construct a family of Euler systems varying over a family of Borel-ordinary Siegel forms. This construction has been massively generalised by Loeffler–Zerbes [LRZ21] and the author to include classes constructed using Loeffler's machine i.e. spherical pairs of reductive groups (G, H). In these constructions we always make an *ordinarity* assumption with respect to some parabolic subgroup Q_G of our reductive group G. An algebraic automorphic representation unramified at p is Q_G-ordinary if a certain parahoric Hecke operator U_Q (determined by Q_G) at p has an eigenvalue which is a p-adic unit (for some embedding of the field of definition into $\overline{\mathbb{Q}}_p$).

Example 1.2.6. In the case of GL_2, weight 2 modular forms with rational Fourier coefficients which are ordinary at p correspond to elliptic curves which are ordinary at p, so by focusing on ordinary forms we miss out on curves which are supersingular at p.

In this paper we relax the ordinarity assumption of [LRZ21] and construct cohomology classes varying in p-adic families with only a finite slope assumption; the Hecke operator U_Q has a non-zero eigenvalue. Under an additional *non-critical slope* condition, requiring the Hecke operator U_Q to instead have an eigenvalue of slope less than a prescribed value determined by the weight, we show that these classes can be pushed forward into Galois cohomology. We show that in the case of GSp_4 this construction can be used to give Galois cohomology classes interpolating the Lemma–Flach Euler system of Loeffler–Skinner–Zerbes [LSZ21] with full variation in the weight variables. One expects that the image of this class under the 'overconvergent Perrin-Riou regulator' should be related to a multi-variable p-adic L-function interpolating the p-adic L-functions of a family of Siegel modular forms as they vary in a Coleman family.

We assume the setting of Loeffler's machine i.e. we have a spherical pair of reductive groups (G, H) and we have constructed norm-compatible cohomology classes

$$z_r^{[\lambda]} \in H^i(I_r, V_\lambda)^{\mathrm{fs}}$$

for[5] $r \geq 0$ and $\lambda \in X^\bullet(S_G)$ where $S_G = M_G/M_G^{\mathrm{der}}$ is the maximal torus quotient of the Levi of Q_G and I_r is the depth r Iwahori subgroup. In this paper we construct 'large' cohomology classes interpolating the classes $z_0^{[\lambda]}$ as λ varies over the weight space $\mathcal{W}_G$ parameterising continuous characters of $S_G(\mathbb{Z}_p)/S_G^0(\mathbb{Z}_p)$, where S_G^0 is the image of Q_G^0 in S_G. We show that one can use this construction to interpolate the Lemma-Flach Euler system classes in Coleman families of Siegel modular forms.

This construction generalises previous constructions of large Euler system classes in the case of $G = \mathrm{GL}_2 \times_{\mathrm{GL}_1} \mathrm{GL}_2, H = \mathrm{GL}_2$ (Beilinson–Flach case [LZ16]), $G = GL_2, H = \mathrm{Res}_{E/\mathbb{Q}}(\mathrm{GL}_1)$ for E an imaginary quadratic field (Heenger point case [JLZ21]), $G = \mathrm{GL}_2 \times \mathrm{GL}_2 \times \mathrm{GL}_2, H = \mathrm{GL}_2$ (diagonal cycle case [BSV20]). The method is inspired by the last of these papers and differs greatly from the first two. In the Beilinson–Flach and Heegner point cases the authors work with modules of p-adic distributions and treat interpolation in the weight and cyclotomic variables separately. In the diagonal cycle case the authors use modules of p-adic analytic functions (of which the spaces of distributions are their duals). Using these modules of analytic functions it is straightforward to interpolate the algebraic branching maps and include the cyclotomic variable as part of the whole package. We develop the method of [BSV20] in the following novel ways:

- The methods of [BSV20] involves pushing forward the trivial class in the 0-degree cohomology of the trivial representation. Our method interpolates branching laws corresponding to any irreducible H-representation of an irreducible G-representation and expands the family of classes one can pushforward to a much wider class including Beilinson's Eisenstein classes.
- Our methods require no compatibility between the parabolics chosen for H and G.
- We develop a theory of *locally Iwasawa* functions. The modules of these functions sit between the modules of locally analytic functions of differing analytic radii and are profinite, giving us greater control over the étale cohomology groups utilised in the construction of Galois cohomology classes.

We show how, in the étale case, we can project these classes into $H^1(\mathbb{Q}, W_{\underline{\Pi}})$ for a family of Galois representations $W_{\underline{\Pi}}$ varying in a Coleman family $\underline{\Pi}$ passing through a small-slope classical point.

In [LZ21] the authors crucially utilise the interpolation results of [LRZ21] to prove cases of Birch–Swinnerton-Dyer for abelian surfaces satisfying an ordinarity condition at p. The construction of the large Lemma–Flach class in this paper is expected to be the first step in extending these results to the non-ordinary setting.

2 Plus/Minus p-adic L-functions for GL_{2n}

2.1 Introduction

Let $f = \sum_{n=0}^{\infty} a_n q^n$ be a normalized cuspidal newform of weight k and level N with character ε, and let p be a prime such that $p \nmid N$. Let α be a root of the Hecke polynomial $X^2 - a_p X + p^{k-1}\varepsilon(p)$

---

[5] the norm relation at $r = 0$ does not follow immediately from Loeffler's machine and will be given by some Euler factor.

which, after fixing an isomorphism $\bar{\mathbb{Q}}_p \cong \mathbb{C}$, satisfies $r := v_p(a_p) < k-1$, where v_p is the p-adic valuation on $\mathbb{C}_p$ normalized so that $v_p(p) = 1$. From this data we can construct an order r locally analytic distribution $L_p^{(\alpha)}$ on $\mathbb{Z}_p^\times$ whose values at special characters interpolate the critical values of the complex L-function of f and its twists. The arithmetic of $L_p^{(\alpha)}$ is well understood in the case that f is ordinary at p i.e. when $r = 0$, but is more mysterious in the non-ordinary case, since the unbounded growth of $L_p^{(\alpha)}$ means that it does not lie in the Iwasawa algebra, and hence cannot be the characteristic element of an Iwasawa module.

In [Pol03] Pollack provides a solution to this problem in the case that $a_p = 0$ by constructing bounded distributions L_p^+, L_p^- each of which interpolate half the values of the complex L-function of f and its twists. Kobayashi [Kob03] and Lei [Lei11] have formulated Iwasawa main conjectures using these 'plus/minus p-adic L-functions', shown them to be equivalent to Kato's main conjecture and proved one inclusion in these conjectures using Kato's Euler system. The converse inclusion has been proved in many cases by Wan [Wan16].

Now let Π be a cuspidal automorphic representation of $\mathrm{GL}_{2n}(\mathbb{A}_\mathbb{Q})$. Suppose that Π is cohomological with respect to some pure dominant integral weight μ, and that it is the transfer of a globally generic cuspidal automorphic representation of $\mathrm{GSpin}_{2n+1}(\mathbb{A}_\mathbb{Q})$. Let p be a prime at which Π is unramified, and let $\alpha_1, \ldots, \alpha_{2n}$ be the Satake parameters at p. We call a choice of $\alpha = \prod_{i=1}^n \alpha_{j_i}$ for $\{j_1, \ldots, j_n\} \subset \{1, \ldots, 2n\}$ a p-stabilisation of Π. When a p-stabilisation α is *non-critical* and under some further auxiliary technical assumptions Dimitrov, Januszewski and Raghuram [DJR20] (ordinary case) and Barrera, Dimitrov and Williams [BSDW21] construct a locally analytic distribution $L_p^{(\alpha)}$ on $\mathbb{Z}_p^\times$ interpolating the L-values of Π. If we assume α satisfies a non-critical slope condition then this p-stabilisation is non-critical, although this is a stronger condition. We show that that there are at most two choices of α satisfying the non-critical slope condition and thus at most two non-critical slope $L_p^{(\alpha)}$ can be constructed from a given Π.

There is an increasing filtration $\mathscr{D}^r(\mathbb{Z}_p^\times, \mathbb{C}_p)$ on the space $\mathscr{D}(\mathbb{Z}_p^\times, \mathbb{C}_p)$ of $\mathbb{C}_p$-valued distributions on $\mathbb{Z}_p^\times$ which measures the 'growth' of the distribution in a precise way (Definition 2.2.15). The 0th part of this filtration is the space of measures on $\mathbb{Z}_p^\times$. The construction of [BSDW21] shows that

$$L_p^{(\alpha)} \in \mathscr{D}^{v_p(\alpha)}(\mathbb{Z}_p^\times, \mathbb{C}_p).$$

Suppose we have two non-critical slope p-adic L-functions for a given Π and suppose the following condition, which we dub the 'Pollack condition', holds:

$$\textbf{Pollack condition: } \alpha_n + \alpha_{n+1} = 0. \tag{2}$$

We prove the following theorem, stated for an odd prime p:

Theorem 2.1.1. *Let α be a p-stabilisation satisfying the non-critical slope condition and let* $\mathrm{Crit}(\Pi)$ *be the set of critical integers for Π defined in Definition 2.2.20. There exist a pair of distributions $L_p^\pm \in \mathscr{D}^{v_p(\alpha) - \#\mathrm{Crit}(\Pi)/2}(\mathbb{Z}_p^\times, \mathbb{C}_p)$ satisfying*

$$L_p^{(\alpha)} = \log_\Pi^+ L_p^+ + \log_\Pi^- L_p^-,$$

where $\log_\Pi^\pm \in \mathscr{D}^{\#\mathrm{Crit}(\Pi)/2}(\mathbb{Z}_p^\times, \mathbb{C}_p)$ are distributions depending only on $\mathrm{Crit}(\Pi)$ *of order* $\#\mathrm{Crit}(\Pi)/2$. *If the valuation of $\prod_{i=1}^{n-1} \alpha_i$ is minimal (see Proposition 2.2.22) the distributions $L_p^\pm$ are contained in $\mathscr{D}^0(\mathbb{Z}_p^\times, \mathbb{C}_p)$. These distributions satisfy the following interpolation property for $j \in$* $\mathrm{Crit}(\Pi)$*:*

$$\int_{\mathbb{Z}_p^\times} x^j \theta(x) L_p^+(x) = (*) \frac{L(\Pi \otimes \theta, j + 1/2)}{\log_\Pi^+(x^j \theta)}$$

for θ a Dirichlet character of conductor an even power of p, and

$$\int_{\mathbb{Z}_p^\times} x^j \theta(x) L_p^-(x) = (*) \frac{L(\Pi \otimes \theta, j + 1/2)}{\log_\Pi^-(x^j \theta)}$$

for θ a Dirichlet character of conductor an odd power of p, where the $(*)$ are non-zero constants.

When $p = 2$ the result holds with the signs of the distributions $\log_\Pi^\pm$ swapped.

Remark 2.1.2. Since we assume that μ is pure, the condition that $\prod_{i=1}^{n-1} \alpha_i$ be minimal is equivalent to the statement that this p-stabilisation is $\mathscr{P}$-ordinary (See [Hid98, Section 6.2]) where $\mathscr{P} \subset \mathrm{GL}_{2n}$ is the parabolic subgroup given by the partition $2n = (n-1) + 2 + (n-1)$.

As an application we prove the following extension of the main result of [DJR20]:

Theorem 2.1.3. *In the case that $L_p^{\pm}$ are bounded distributions, the purity weight w is even, and* $\mathrm{Crit}(\Pi) \neq \{w/2\}$, *we have*

$$L(\Pi \otimes \theta, (w+1)/2) \neq 0$$

for all but finitely many characters θ of p-power conductor.

Remark 2.1.4. The assumption on the purity weight is to ensure that the central L-value is critical.

Relation to other work: Since this paper first appeared in preprint form, Lei and Ray [LR20] have used the results of this paper to formulate an Iwasawa main conjecture for Π, relating the signed p-adic L-functions of Theorem 1.0.1 to signed Selmer groups. They have also generalised the construction of the signed p-adic L-functions to allow certain cases with $\alpha_n + \alpha_{n+1} \neq 0$, using the theory of Wach modules.

2.2 Preliminaries

2.2.1 p-adic distribution spaces

We recall the relevant theory of continuous functions on $\mathbb{Z}_p^{\times}$. The main reference for this section is [Col10, Section I.5].

Let L be a complete extension of $\mathbb{Q}_p$.

Definition 2.2.1. Define

$$\mathscr{C}(\mathbb{Z}_p, L) := \{f : \mathbb{Z}_p \to L : f \text{ continuous}\},$$

the space of continuous L-valued functions on $\mathbb{Z}_p$.

If we equip $\mathscr{C}(\mathbb{Z}_p, L)$ the infimum valuation it becomes an L-Banach space in the sense of [Col10, Section I.1].

Definition 2.2.2. For $a \in \mathbb{C}_p$, $h \in \mathbb{R}$ define

$$\bar{\mathbb{B}}(a, h) = \{z \in \mathbb{C}_p : v(z-a) \geq h\}.$$

Write $\mathcal{O}_{\bar{\mathbb{B}}(a,h),L}$ for the space of L-valued rigid functions on $\bar{\mathbb{B}}(a, h)$.

We have an isomorphism

$$\mathcal{O}_{\bar{\mathbb{B}}(a,h),L} \cong L\langle X - a\rangle.$$

Definition 2.2.3. Define

$$\mathrm{LA}(\mathbb{Z}_p, L) = \{f : \mathbb{Z}_p \to L : \forall a \in \mathbb{Z}_p, \exists n \in \mathbb{Z}_{\geq 0}, F_{a,n} \in \mathcal{O}_{\bar{B}(a,h),L} \text{ s.t. } \forall z \in a + p^n\mathbb{Z}_p, f(z) = F_{a,n}(z)\},$$

the space of L-valued locally analytic functions on $\mathbb{Z}_p$. This is the space of functions locally described by a convergent power series.

Since $\mathbb{Z}_p$ is compact, for any $f \in \mathrm{LA}(\mathbb{Z}_p, L)$ there exists (non-unique) $n \in \mathbb{Z}_{\geq 0}$, called the *radius of analyticity*, such that the restriction of f to $a + p^n\mathbb{Z}_p$ is described by a power series for all $a \in \mathbb{Z}_p$.

Definition 2.2.4. Define for $h \in \mathbb{Z}_{\geq 0}$ a filtration

$$\mathrm{LA}_h(\mathbb{Z}_p, L) = \{f \in \mathrm{LA}(\mathbb{Z}_p, L) : f \text{ has radius of analyticity } h\}.$$

We call these locally h-analytic functions.

We give the spaces $\mathrm{LA}_h(\mathbb{Z}_p, L)$ a valuation v_{LA_h} in the following way: Let $u_h = (p^h(1-p))^{-1}$ and let $v_{\mathbb{B}(a,u_h)}$ be the valuation on $\mathcal{O}_{\mathbb{B}(a,u_h)}$ given by

$$v_{\mathbb{B}(a,u_h)}(f) = \inf_m\{v_p(a_m) + nu_h : f(X) = \sum_{i=0}^{\infty} a_i(X-a)^i\}.$$

An element $f \in \mathrm{LA}_h(\mathbb{Z}_p, L)$ locally extends to such a power series and we define

$$v_{\mathrm{LA}_h}(f) = \inf_{a\in\mathbb{Z}_p} v_{\mathbb{B}(a,u_h)}(f).$$

This gives $\mathrm{LA}(\mathbb{Z}_p, L)$ the structure of a Fréchet space.

Definition 2.2.5. Let $r \in \mathbb{R}_{\geq 0}$. Let $f \in \mathscr{C}(\mathbb{Z}_p, L)$. We say that f is of *order* r if there are functions $f^{(i)} : \mathbb{Z}_p \to L$ such that if we define

$$\varepsilon_h(f) = \inf_{\substack{x\in\mathbb{Z}_p\\ y\in p^h\mathbb{Z}_p}} v_p\left(f(x+y) - \sum_{i=0}^{\lfloor r\rfloor} f^{(i)}(x)y^i/i!\right),$$

then

$$\varepsilon_h(f) - rh \to \infty \text{ as } h \to \infty.$$

We denote the set of such functions by $\mathscr{C}^r(\mathbb{Z}_p, L)$.

The space $\mathscr{C}^r(\mathbb{Z}_p, L)$ is a Banach space with valuation given by

$$v_{\mathscr{C}^r}(f) = \inf\left(\inf_{0\leq j\leq\lfloor r\rfloor, x\in\mathbb{Z}_p}\left(\frac{f^{(i)}(x)}{i!}\right), \inf_{x,y\in\mathbb{Z}_p}\left(\varepsilon_n(f) - rv_p(y)\right)\right).$$

For any r we have a continuous inclusion

$$\mathrm{LA}(\mathbb{Z}_p, L) \hookrightarrow \mathscr{C}^r(\mathbb{Z}_p, L)$$

with dense image, and $\mathscr{C}^0(\mathbb{Z}_p, L) = \mathscr{C}(\mathbb{Z}_p, L)$

Definition 2.2.6. Define

$$\mathscr{D}(\mathbb{Z}_p, L) = \mathrm{Hom}_{\mathrm{cont}}(\mathrm{LA}(\mathbb{Z}_p, L), L),$$

the space of *locally analytic distributions* on $\mathbb{Z}_p$.

The space $\mathscr{D}(\mathbb{Z}_p, L)$ admits the structure of a Fréchet space via the family of valuations given by restricting to $LA_h(\mathbb{Z}_p, L)$ and taking the dual of v_{LA_h}.

Definition 2.2.7. Let $r \in \mathbb{R}_{\geq 0}$. Define

$$\mathscr{D}^r(\mathbb{Z}_p, L) = \mathrm{Hom}_{\mathrm{cont}}(\mathscr{C}^r(\mathbb{Z}_p, L), L).$$

The space $\mathscr{D}^r(\mathbb{Z}_p, L)$ embeds as a subspace of $\mathscr{D}(\mathbb{Z}_p, L)$.

Remark 2.2.8. The space $\mathscr{D}^0(\mathbb{Z}_p, L)$ of bounded distributions is often referred to as the space of *measures* on $\mathbb{Z}_p$.

We equip each $\mathscr{D}^r(\mathbb{Z}_p, L)$ with the valuation

$$v_{\mathscr{D}^r}(\mu) = \inf_{f\in\mathscr{C}^r(\mathbb{Z}_p,L)\setminus\{0\}}\left(v_p(\mu(f)) - v_{\mathscr{C}_r}(f)\right).$$

For $\mu \in \mathscr{D}^r(\mathbb{Z}_p, L), f \in \mathscr{C}^r(\mathbb{Z}_p, L)$ we write

$$\mu(f) =: \int_{\mathbb{Z}_p} f(x)\mu(x).$$

We give the space $\mathscr{D}(\mathbb{Z}_p, L)$ the structure of an L-algebra via convolution of distributions:

$$\int_{\mathbb{Z}_p} f(x)(\mu * \lambda)(x) := \int_{\mathbb{Z}_p}\left(\int_{\mathbb{Z}_p} f(x+y)\mu(x)\right)\lambda(y).$$

2.2.2 Integral transforms

We recall the theory of p-adic integral transforms, allowing us to identify the distribution modules $\mathscr{D}^r(\mathbb{Z}_p, L)$ with certain spaces of rigid analytic functions.

Definition 2.2.9. For $x \in \mathbb{C}_p, a \in \mathbb{R}$, let $\mathbb{B}(x,a) = \{y \in \mathbb{C}_p : v_p(y-x) > a\}$. We define

$$\mathscr{R}^+ = \left\{ f = \sum_{n=0}^{\infty} a_n X^n \in L[[X]] : f \text{ converges on } \mathbb{B}(0,0) \right\}$$

We give $\mathscr{R}^+$ the structure of a Fréchet space via the family of valuations $v_{\mathbb{B}(0,u_h)}$.

Definition 2.2.10. Let $\ell(n) = \inf\{m : n < p^m\}$, and for $r \in \mathbb{R}_{\geq 0}$ define

$$\mathscr{R}_r^+ = \{f = \sum_{n=0}^{\infty} a_n X^n \in L[[X]] : v_p(a_n) + r\ell(n) \text{ is bounded below as } n \to \infty\}.$$

We can put a valuation on these spaces

$$v_r(f) = \inf{}_h b_h + r\ell(h),$$

where $\ell(h)$ is the smallest integer satisfying $p^{\ell(h)} > h$. However, a different valuation will be useful for our purposes.

Lemma 2.2.11. *A power series $f \in L[[X]]$ is in $\mathscr{R}_r^+$ if and only if $\inf_{h \in \mathbb{Z}_{\geq 0}}(v_{\mathbb{B}(0,u_h)}(f) + rh) \neq -\infty$. Furthermore, the spaces $\mathscr{R}_r^+$ are Banach spaces when equipped with the valuation*

$$v_r(f) = \inf{}_{h \in \mathbb{Z}_{\geq 0}}(v_{\mathbb{B}(0,u_h)}(f) + rh).$$

Moreover, $v_r(f)$ is equivalent to $v_r'(f)$.

Proof. [Col10, Lemme II.1.1]. □

Lemma 2.2.12. *If $f \in \mathscr{R}_r^+$, $g \in \mathscr{R}_s^+$, then $fg \in \mathscr{R}_{r+s}^+$.*

Proof. [Col10, Corollaire II.1.2]. □

Theorem 2.2.13. *Define the Amice transform:*

$$\begin{aligned} \mathscr{A} : \mathscr{D}(\mathbb{Z}_p, L) &\cong \mathscr{R}^+ \\ \mu &\mapsto \int_{\mathbb{Z}_p} (1+X)^x \mu(x). \end{aligned}$$

The Amice transform is an isomorphism of L-algebras under which the spaces $\mathscr{D}^r(\mathbb{Z}_p, L)$ and $\mathscr{R}_r^+$ are identified isometrically with respect to the valuations $v_{\mathscr{D}^r}$ and v_r'.

Proof. [Col10, Théorème II.2.2] and [Col10, Proposition II.3.1]. □

We now consider the multiplicative topological group $\mathbb{Z}_p^\times$. Let

$$q = \begin{cases} p & \text{if } p \text{ odd} \\ 4 & \text{otherwise.} \end{cases}$$

We have the well-known isomorphism

$$\mathbb{Z}_p^\times \cong (\mathbb{Z}/q\mathbb{Z})^\times \times 1 + q\mathbb{Z}_p,$$

the second factor of which is topologically cyclic. Let γ be a topological generator of $1+q\mathbb{Z}_p$. Such a choice allows us to write any $x \in 1+q\mathbb{Z}_p$ in the form $x = \gamma^s$ for a unique $s \in \mathbb{Z}_p$, giving us an isomorphism of topological groups

$$\begin{aligned} 1+q\mathbb{Z}_p &\cong \mathbb{Z}_p \\ \gamma^s &\mapsto s. \end{aligned}$$

Thus $\mathbb{Z}_p^\times$ is homeomorphic to $p-1$ (resp. 2 when $p=2$) copies of $\mathbb{Z}_p$, and we can use the above theory of $\mathbb{Z}_p$ in this context, defining $\mathrm{LA}(\mathbb{Z}_p^\times, L)$, $\mathscr{D}(\mathbb{Z}_p^\times, L)$ in the obvious way; each space decomposes as a direct sum over their restrictions to each $\mathbb{Z}_p$ component and we take the infimum of the valuations on each summand.

Definition 2.2.14. Define *weight space* to be the rigid analytic space $\mathcal{W}$ over $\mathbb{Q}_p$ representing

$$L \mapsto \mathrm{Hom}_{\mathrm{cont}}(\mathbb{Z}_p^\times, L^\times).$$

Integrating characters gives a canonical identification

$$\mathscr{D}(\mathbb{Z}_p^\times, \mathbb{Q}_p) = H^0(\mathcal{W}, \mathcal{O}_{\mathcal{W}}),$$

where $\mathcal{O}_{\mathcal{W}}$ is the structure sheaf of $\mathcal{W}$. This is isomorphism commutes with base change in the sense that for a finite extension $L/\mathbb{Q}_p$ we have

$$\mathscr{D}(\mathbb{Z}_p^\times, L) = \mathscr{D}(\mathbb{Z}_p^\times, \mathbb{Q}_p)\hat{\otimes}_{\mathbb{Q}_p} L = H^0(\mathcal{W}_L, \mathcal{O}_{\mathcal{W}_L}),$$

where $\mathcal{W}_L = \mathcal{W} \times_{\mathbb{Q}_p} \mathrm{Sp}(L)$ and $\mathrm{Sp}(L)$ is the affinoid space associated to L. We identify $\mathcal{W}(\mathbb{C}_p)$ with the set $\sqcup_\psi \mathbb{B}_\psi$, where $\mathbb{B}_\psi = \mathbb{B}(0,0)$ and the disjoint union runs over characters of $\mathbb{Z}_p^\times$ which factor through $(\mathbb{Z}/q\mathbb{Z})^\times$. We can thus identify $\mathscr{D}(\mathbb{Z}_p^\times, L)$ with functions on $\sqcup_\psi \mathbb{B}_\psi$ which are described by elements of $\mathscr{R}^+$ on each $\mathbb{B}_\psi$. Given a distribution $\mu \in \mathscr{D}(\mathbb{Z}_p^\times, L)$ we write the corresponding rigid function on $\mathcal{W}$ as $\mathscr{M}(\mu)$.

On each $\mathbb{B}_\psi$ the global sections $\mathcal{O}_{\mathcal{W}}(\mathbb{B}_\psi)$ are given (after choosing a coordinate X) by precisely $\mathscr{R}^+$. As these are quasi-Stein spaces, the topology on $\mathcal{O}_{\mathcal{W}}(\mathbb{B}_\psi)$ is that of a Fréchet space induced by an increasing chain of affinoids

$$Y_1 \subset Y_2 \subset \ldots,$$

which we can choose to be the closed discs of radius u_h, whence the topology as global sections over a rigid space coincides with the topology on $\mathscr{R}^+$ given by the family of valuations $v_{\overline{\mathbb{B}}(0,u_h)}$ (see [Pot13, 1C]).

Definition 2.2.15. For $r \in \mathbb{R}_{\geq 0}$ we define a subspace $\mathscr{D}^r(\mathbb{Z}_p^\times, L) \subset \mathscr{D}(\mathbb{Z}_p^\times, L)$ by

$$\mathscr{D}^r(\mathbb{Z}_p^\times, L) = \{\mu \in \mathscr{D}(\mathbb{Z}_p^\times, L) : \mathscr{M}(\mu)|_{\mathbb{B}_\psi} \in \mathscr{R}_r^+ \text{ for all } \psi\}.$$

These spaces decompose as a direct sum

$$\mathscr{D}^r(\mathbb{Z}_p^\times, L) = \oplus_\psi \mathscr{D}^r(\mathbb{Z}_p, L)$$

and we equip it with the valuation given by

$$v_{\mathscr{D}_r}(\mu) := \inf{}_\psi v_{\mathscr{D}^r}(\mu_\psi)$$

where μ_ψ is the projection of μ to the ψ component.

2.2.3 Automorphic representations

Fix $n \geq 1$ and set $G = \mathrm{GL}_{2n}$. Let Π be a cuspidal automorphic representation of $G(\mathbb{A}_\mathbb{Q})$. Let $T \subset G$ be the maximal diagonal torus and let

$$\mu = (\mu_1, \ldots, \mu_{2n}) \in \mathbb{Z}^{2g}$$

be an integral weight. We say μ is *dominant* if $\mu_1 \geq \cdots \geq \mu_{2n}$, and we say μ is *pure* if there is $\omega \in \mathbb{Z}$, the *purity weight* of μ, such that

$$\mu_i + \mu_{2n+1-i} = \omega$$

for all $i = 1, \ldots, n$.

Definition 2.2.16. We say that Π is *cohomological* with respect to a dominant integral weight μ if the $(\mathfrak{g}_\infty, K_\infty)$-cohomology

$$H^q(\mathfrak{g}_\infty, K_\infty, \Pi \otimes V_{\mathbb{C}}^\mu)$$

is non-vanishing for some q. Here $\mathfrak{g}_\infty$ is the Lie algebra of $G(\mathbb{R})$, $K_\infty^\circ \subset G(\mathbb{R})$ is the identity component of the maximal open compact subgroup and $V_{\mathbb{C}}^\mu$ is the irreducible $\mathbb{C}$-linear G-representation of highest weight μ.

Cohomological representations occur in the Betti cohomology of locally symmetric spaces for G. Purity of μ is a necessary condition for Π to be cohomological.

The complex dual group of GSpin_{2n+1} is given by $\mathrm{GSp}_{2n}(\mathbb{C})$. Let

$$\iota : \mathrm{GSp}_{2n}(\mathbb{C}) \to \mathrm{GL}_{2n}(\mathbb{C})$$

by the natural inclusion.

Definition 2.2.17. We say that Π is the transfer of a globally generic cuspidal automorphic representation π of $\mathrm{GSpin}_{2n+1}(\mathbb{A}_\mathbb{Q})$ if for each unramified place ℓ such that π_ℓ corresponds to semi-simple conjugacy class $[t_\ell]$ in $\mathrm{GSp}_{2n}(\mathbb{C})$, the local representation Π_ℓ is the unique irreducible unramified admissible representation corresponding to $\iota([t_\ell])$ under the Satake isomorphism.

Remark 2.2.18.

- For a given globally generic automorphic representation π of $\mathrm{GSpin}_{2n+1}(\mathbb{A})$ the existence of such a transfer was proved by Asgari-Shahidi [AS06, Theorem 1.1].
- A necessary and sufficient condition for Π to be the transfer of a globally generic cuspidal automorphic representation of GSpin_{2n+1} is that it admits a *Shalika model* which realises Π in a certain space of functions $W : G(\mathbb{A}_\mathbb{Q}) \to \mathbb{C}$, see [BSDW21, Section 2.6] for details.

2.2.4 p-stabilisations

Let Π be a cuspidal automorphic representation of $G(\mathbb{A}_\mathbb{Q})$ which is cohomological with respect to a pure dominant integral weight μ and suppose that Π is the transfer of a globally generic cuspidal automorphic representation of $\mathrm{GSpin}_{2n+1}(\mathbb{A}_\mathbb{Q})$. Let B denote the upper triangular Borel subgroup of G.

Given a prime p at which Π is unramified, define the *Hodge-Tate weights* of Π at p to be the integers

$$h_i = \mu_i + 2n - i, \ i = 1, \ldots, 2n. \tag{3}$$

Remark 2.2.19. These weights coincide with the Hodge-Tate weights of the Galois representation associated to Π when the Hodge-Tate weight of the cyclotomic character is taken to be 1.

Definition 2.2.20. Define a set

$$\mathrm{Crit}(\Pi) = \{j \in \mathbb{Z} : \mu_n \geq j \geq \mu_{n+1}\}.$$

Remark 2.2.21. It is shown in [GR14, Proposition 6.1.1] that the half integers $j + 1/2$ for $j \in \mathrm{Crit}(\Pi)$ are precisely the critical points of the L-function $L(s, \Pi)$ in the sense of Deligne [Del79, Definition 1.3].

Let p be a prime at which Π is unramified. There is an unramified character

$$\lambda_p : T(\mathbb{Q}_p) \to \mathbb{C}^\times$$

such that Π_p is isomorphic to the normalised parabolic induction module $\mathrm{Ind}_{B(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)}(|\cdot|^{\frac{2n-1}{2}}\lambda_p)$. We define the Satake parameters at p to be the values $\alpha_i = \lambda_{p,i}(p)$, where $\lambda_{p,i}$ denotes the projection to the ith diagonal entry. After choosing an isomorphism $\overline{\mathbb{Q}}_p \cong \mathbb{C}$, we reorder the α_i so that they are ordered with respect to decreasing p-adic valuation and such that $\alpha_i\alpha_{2n+1-i} = \lambda$ for a fixed λ with p-adic valuation $2n - 1 + w$. That we can do this is a result of the transfer from GSpin_{2n+1}, see [AS06, (64)].

We define the Hodge polygon of Π to be the piecewise linear curve joining the following points in $\mathbb{R}^n$:

$$\left\{(0,0),(j,\sum_{i=1}^{j} h_{2n+1-i}) : j = 1,\ldots,2n\right\}$$

and define the Newton polygon on Π at p to be the piecewise linear curve joining the points

$$\left\{(0,0),(j,\sum_{i=1}^{j} v_p(\alpha_{2n+1-i})) : j = 1,\ldots,2n\right\}.$$

The following result is due in this form to Hida [Hid98, Theorem 8.1].

Proposition 2.2.22. *The Newton polygon lies on or above the Hodge polygon and the end points coincide.*

Definition 2.2.23. Let $I = (i_1,\ldots,i_n) \subset \mathbb{Z}^n$ satisfy $1 \leq i_1 < \ldots < i_n \leq 2n$, and set

$$\alpha_I := \alpha_{i_1}\cdots\alpha_{i_n}.$$

We call α_I *p-stabilisation data* for Π.

Let $Q \subset \mathrm{GL}_{2n}$ be the parabolic subgroup given by the partition $2n = n + n$. The following conditions are translations of the conditions of the same name given in [BSDW21, Section 2.7].

Definition 2.2.24. Let I be as above.

- We say that the product α_I is of *Shalika type* if I contains precisely one element of each pair $\{i, 2n+1-i\}$ for $i = 1,\ldots,n$, see [DJR20, Definition 3.5(ii)].
- We say that α_I is *Q-regular* if it is of Shalika type and if for any other choice of $J \subset \mathbb{Z}^n$ satisfying the above properties $a_J \neq a_I$. This amounts to choosing a simple Hecke eigenvalue for the U_p-operator associated to Q acting on the Q-parahoric invariants of Π_p, see [DJR20, Definition 3.5(i)] and [BSDW21, Section 2.7].
- Set $r_I = v_p(\alpha_I) - \sum_{i=n+1}^{2n} h_i$. We say that α_I is *non-critical slope* if it satisfies

$$r_I < \#\mathrm{Crit}(\Pi).$$

- We say that α_I is *minimal slope* if

$$r_I = \#\mathrm{Crit}(\Pi)/2.$$

Remark 2.2.25. The conditions in Definition 2.2.24 are used to control a certain local twisted integral at p, attached to a choice of parahoric-invariant vector W in the Shalika model. In [DJR20, Proposition 3.4], the authors show that this local zeta integral is an explicit multiple of $W(1)$. In [DJR20, Lem. 3.6], they use the Shalika-type and Q-regular conditions to exhibit an explicit vector W in the Shalika model attached to α_I with $W(1) = 1$, and hence deduce non-vanishing of the local zeta integral. We note that Π_p always admits p-stabilisations of Shalika type. The terminology is justified by [BSDW21, Remark 2.5], which explains that the refinements of Shalika type are exactly those that arise from refinements of GSpin_{2n+1}. Finally, if the Satake parameter of Π_p is regular semisimple, then all stabilisations of Π_p are Q-regular.

In [BSDW21] the authors construct [6] a locally analytic distribution $L_p^{(\alpha_I)} \in \mathscr{D}(\mathbb{Z}_p^\times, \mathbb{C}_p^\times)$ with respect to a choice of non-critical slope Q-regular p-stabilization data α_I. The distribution $L_p^{(\alpha_I)}$ is of order r_I and by [Vis76, Lemma 2.10] is uniquely defined by the following interpolation property: Let $\theta : \mathbb{Z}_p^\times \to \bar{\mathbb{Q}}_p$ be a finite-order character of conductor p^m, then for $m \geq 1$ we have

$$\int_{\mathbb{Z}_p^\times} x^j\theta(x)L_p^{(\alpha_I)}(x) = \xi_{\infty,j}\frac{c_{x^j\theta}}{\alpha_I^m}L(\Pi\otimes\theta, j+1/2),\ j \in \mathrm{Crit}(\Pi), \tag{4}$$

---

[6] The authors actually construct p-adic L-functions for the wider class of *non-critical* p-stabilisations, but we only work with non-critical slope p-stabilisations in this paper.

where $c_{x^j\theta}$ is a constant depending only on $x^j\theta$ and the infinite factor $\xi_{\infty,j}$ is the product of a choice of period and a zeta integral at infinity. We call such a $L_p^{(\alpha_I)}$ a 'non-critical slope p-adic L-function'.

2.3 Plus/Minus p-adic L-functions

We construct the titular plus/minus L-functions. We first note that the condition of non-critial slope imposes strong restrictions on the number of p-adic L-functions we can construct.

Theorem 2.3.1. *There are at most two choices of p-stabilization α_I for which $L_p^{(\alpha_I)}$ is non-critical slope.*

Proof. Without loss of generality we may assume that $\mu_{2n} = 0$, forcing $w = \mu_1$. The end points of the Newton and Hodge polygons coinciding implies that

$$v_p(\lambda) = h_i + h_{2n+1-i}, \ i = 1, \dots, n. \tag{\dagger}$$

The 'non-critical slope' condition for $I = (i_1, \dots, i_n)$ is equivalent to

$$v_p(\alpha_I) - \sum_{i=n+1}^{2n} h_i < h_n - h_{n+1}.$$

We observe that any I that includes a 2-tuple of integers the form $(i, 2n+1-i)$ is not non-critical slope. Indeed, we can find an explicit I containing some $(i, 2n+1-i)$ with minimal valuation, namely $(n, n+1, n+3, \dots, 2n)$, amongst all I containing some $(i, 2n+1-i)$. For such an I we have

$$\begin{aligned} v_p(\alpha_I) - (h_{n+1} + h_{n+2} \dots + h_{2n}) &\geq h_n + h_{n+1} + h_{n+3} + \cdots + h_{2n} - (h_{n+1} + \dots + h_{2n}) \\ &= h_n - h_{n+2} \\ &> h_n - h_{n+1}, \end{aligned}$$

where the first inequality is a consequence of the Newton polygon lying above the Hodge polygon and (†), and the strict inequality is due to dominance. Thus any I containing a pair of integers (i, j) with $i < j$ and $j \leq 2n+1-i$ cannot be non-critical slope, since any such I has greater valuation than $(n, n+1, n+3, \dots, 2n)$. This leaves us with two choices of potential non-critical slope n-tuples:

$$I_{n+1} = (n+1, n+2, \dots, 2n),$$

and

$$I_n = (n, n+2, \dots, 2n).$$

□

In light of Theorem 6.3.3 it is clear that the only two choices of p-stabilization data which can give a non-critical slope distribution are

$$\alpha = \alpha_{n+1}\alpha_{n+2} \dots \alpha_{2n}, \ \beta = \alpha_n \alpha_{n+2} \dots \alpha_{2n}.$$

Definition 2.3.2. We say that Π satisfies the 'Pollack condition' if

$$\alpha_n + \alpha_{n+1} = 0.$$

Corollary 2.3.3. *For a non-critical slope p-stabilisation α_I we have the following:*

- *The p-stabilisation α_I is of Shalika type.*
- *If we assume the Pollack condition and that at least one of $\alpha_n, \alpha_{n+1} \neq 0$, then α_I is Q-regular.*

Proof. The first part is immediate from Theorem 6.3.3.

For the second claim recall that a p-stabilisation α_I is Q-regular if

$$\alpha_I \neq \alpha_J$$

for all $I \neq J$. Suppose α_I is a non-critical slope p-stabilisation. Then by Theorem 6.3.3

$$\alpha_I = \alpha_n \alpha_{n+2} \cdots \alpha_{2n} \text{ or } \alpha_{n+1}\alpha_{n+2}\cdots\alpha_{n+1},$$

and by the Pollack condition these are clearly not equal. Finally, for a critical slope p-stabilisation α_J we must have $v_p(\alpha_J) > v_p(\alpha_I)$ so $\alpha_J \neq \alpha_I$. □

2.3.1 Pollack ±-L-functions

Let Π be as in the previous section. Set

$$\alpha = \alpha_{n+1}\alpha_{n+2}\ldots\alpha_{2n}, \;\; \beta = \alpha_n\alpha_{n+2}\ldots\alpha_{2n},$$

and let $r = v_p(\alpha) - \sum_{i=n+1}^{2n} h_i = v_p(\beta) - \sum_{i=n+1}^{2n} h_i$. The Pollack condition forces

$$r \geq \#\mathrm{Crit}(\Pi)/2$$

since

$$\begin{aligned} r = v_p(\alpha) - \sum_{i=n+1}^{2n} h_i &\geq v_p(\alpha_{n+1}) - h_{n+1} \\ &= \frac{h_n + h_{n+1}}{2} - h_{n+1} \\ &= \frac{h_n - h_{n+1}}{2} \\ &= \#\mathrm{Crit}(\Pi)/2, \end{aligned}$$

where the first inequality comes from Newton-above-Hodge, and the lower bound given is tight, with the bound being achieved when the end point of the segment of the Newton polygon corresponding to $\alpha_{n+2}\cdots\alpha_{2n}$ touches the Hodge polygon. This justifies the use of the term 'minimal slope' in Definition 2.2.24.

We assume that

$$r < \#\mathrm{Crit}(\Pi)$$

so that we can construct precisely two non-critical slope p-adic L-functions $L_p^{(\alpha)}, L_p^{(\beta)} \in \mathscr{D}^r(\mathbb{Z}_p^\times, \mathbb{C}_p)$.

Remark 2.3.4. Unlike in the case of GL_2, for $n > 1$ the non-critical slope condition for α, β is *a priori* implied by the Pollack condition. Indeed, suppose one has a cuspidal automorphic representation Π of $\mathrm{GL}_{2n}(\mathbb{A}_\mathbb{Q})$ satisfying the Pollack condition at a prime p for which $v_p(\alpha_i) = v_p(\alpha_j)$ for $1 \leq i, j \leq 2n$. The value r is then the same for any choice of p-stabilization, so there are either $\binom{2n}{n}$ non-critical slope p-adic L-functions or there are none. But Theorem 6.3.3 says there can be at most two choices of non-critical slope p-stabilization.

Following Pollack, we define

$$G^\pm = \frac{L_p^{(\alpha)} \pm L_p^{(\beta)}}{2},$$

so that

$$\begin{aligned} L_p^{(\alpha)} &= G^+ + G^- \\ L_p^{(\beta)} &= G^+ - G^-. \end{aligned}$$

We note that in the case of $L_p^{(\beta)}$, the interpolation formula is given by

$$\int_{\mathbb{Z}_p^\times} x^j \theta(x) L_p^{(\beta)}(x) = (-1)^m \xi_{\infty,j} \frac{c_{x^j\theta}}{\alpha^m} L(\Pi \otimes \theta, j + 1/2), \; j \in \mathrm{Crit}(\Pi),$$

from which it follows that

$$\int_{\mathbb{Z}_p^\times} x^j\theta(x)G^+(x) = 0, \text{ if the conductor of } \theta \text{ is } p^m,\ m \text{ odd}$$
$$\int_{\mathbb{Z}_p^\times} x^j\theta(x)G^-(x) = 0, \text{ if the conductor of } \theta \text{ is } p^m,\ m \text{ even.}$$

Equivalently (noting that characters of conductor p^m correspond to $(m-1)$th roots of unity), if ζ_{p^m} is any p^mth root of unity and p is odd,

$$\mathscr{M}(G^+)(\gamma^j\zeta_{p^m}-1) = 0 \text{ for } m \text{ even}$$
$$\mathscr{M}(G^-)(\gamma^j\zeta_{p^m}-1) = 0 \text{ for } m \text{ odd}$$

on each of the connected components[7] of $\mathcal{W}(\mathbb{C}_p)$ (which we recall we are identifying with $p-1$ copies of $\mathbb{B}(0,0)$). When $p=2$ the sign flips and the above vanishing is equivalent to.

$$\mathscr{M}(G^+)(\gamma^j\zeta_{p^m}-1) = 0 \text{ for } m \text{ odd}$$
$$\mathscr{M}(G^-)(\gamma^j\zeta_{p^m}-1) = 0 \text{ for } m \text{ even}$$

For any $j \in \mathbb{Z}$, Pollack defines the following power series

$$\begin{aligned}\log^+_{p,j}(X) &:= \frac{1}{p}\prod_{m=1}^{\infty}\frac{\Phi_{2m}(\gamma^{-j}(1+X))}{p}\\ \log^-_{p,j}(X) &:= \frac{1}{p}\prod_{m=1}^{\infty}\frac{\Phi_{2m-1}(\gamma^{-j}(1+X))}{p},\end{aligned}$$

in $\mathbb{Q}_p[[X]]$, where Φ_m is the p^mth cyclotomic polynomial.

Lemma 2.3.5. *The power series $\log^+_{p,j}(X)$ (resp. $\log^-_{p,j}(X)$) is contained in $\mathscr{R}^+_{1/2}$ and vanishes at precisely the points $\gamma^j\zeta_{p^m}-1$ for every p^mth root of unity ζ_{p^m} with m even (resp. odd).*

Proof. The statements in the lemma are proved in [Pol03, Lemma 4.1 4.5]. We reprove that $\log^+_{p,j}$ is contained in $\mathscr{R}^+_{1/2}$ using the setup of Section 2.2, the result for $\log^-_{p,j}$ being similar.

An analysis of the Newton copolygon of the Eisenstein polynomial Φ_n gives us that

$$v_{\mathbb{B}(0,u_h)}(\Phi_n((\gamma^{-j}(1+X))/p) = \begin{cases}0 & \text{if } h \leq n-1\\ p^{n-h-1}-1 & \text{otherwise,}\end{cases}$$

and thus

$$\begin{aligned}v_{\mathbb{B}(0,u_h)}(\log^+_{p,j}) &= \sum_{m=1}^{\frac{h+1}{2}}\left(p^{2m-h-1}-1\right)\\ &= \frac{p^{-(h+1)}-1}{1-p^2}-\frac{1}{2}-\frac{h}{2},\end{aligned}$$

whence

$$\inf_h\left(v_{\mathbb{B}(0,u_h)}(\log^+_{p,j})+\frac{h}{2}\right) = \frac{1}{p^2-1}-\frac{1}{2} < \infty,$$

so $\log^+_{p,j}(X) \in \mathscr{R}^+_{1/2}$ by Lemma 2.2.11. □

---

[7] We are referring to the connected components as a rigid space as opposed to those of the topology on $\mathbb{C}_p$ induced by v_p.

We define

$$\log_{\Pi}^{\pm}(X) = \prod_{j \in \#\mathrm{Crit}(\Pi)} \log_{p,j}^{\pm}(X) \in \mathscr{R}^{+}_{\mathrm{Crit}(\Pi)/2}.$$

By abuse of notation we will write $\log_{\Pi}^{\pm}(X)$ for the element of $\mathcal{O}_{\mathcal{W}}(\mathcal{W})$ given by $\log_{\Pi}^{\pm}(X)$ on each connected component of $\mathcal{W}$.

Lemma 2.3.6. *We have*

$$\limsup_{h} \left(v_{\mathbb{B}(0,u_h)}(\log_{\Pi}^{\pm}) + \frac{\#\mathrm{Crit}(\Pi)}{2} h \right) < \infty.$$

Proof. It follows from the proof of Lemma 2.3.5 and the multiplicativity of $v_{\mathbb{B}(0,u_h)}$ that

$$v_{\mathbb{B}(0,u_h)}(\log_{\Pi}^{+}) + \frac{\#\mathrm{Crit}(\Pi)}{2} h = \#\mathrm{Crit}(\Pi) \left(\frac{p^{-(h+1)} - 1}{1 - p^2} - \frac{1}{2} \right).$$

The right side converges as $h \to \infty$ so the lim sup is finite. A similar argument works for $\log_{\Pi}^{-}$. ☐

It follows from the above discussion and [Laz62, 4.7] that for odd p the rigid function $\log_{\Pi}^{\pm}(X)$ divides $\mathscr{M}(G^{\pm})$ in $\mathcal{O}_{\mathcal{W}}(\mathcal{W})$, and for $p = 2$ we have that $\log_{\Pi}^{\mp}(X)$ divides $\mathscr{M}(G^{\pm})$ in $\mathcal{O}_{\mathcal{W}}(\mathcal{W})$. Define plus/minus p-adic L-functions $L_p^{\pm}(X)$ to be the elements of $\mathcal{O}_{\mathcal{W}}(\mathcal{W})$ satisfying

$$\mathscr{M}(G^{\pm}) = \log_{\Pi}^{\pm}(X) \cdot L_p^{\pm}(X)$$

for p odd, and

$$\mathscr{M}(G^{\pm}) = \log_{\Pi}^{\mp}(X) \cdot L_p^{\pm}(X)$$

for $p = 2$. We write $L_p^{\pm}$ for the distribution $\mathscr{M}^{-1}(L_p^{\pm}(X))$.

Proposition 2.3.7. *We have*

$$L_p^{\pm} \in \mathscr{D}^{r - \#\mathrm{Crit}(\Pi)/2}(\mathbb{Z}_p^{\times}, \mathbb{C}_p).$$

Proof. We note that

$$\liminf_{h} \left(-v_{\mathbb{B}(0,u_h)}(\log_{\Pi}^{\pm}) - \tfrac{\#\mathrm{Crit}(\Pi)}{2} h \right) = -\limsup_{h} \left(v_{\mathbb{B}(0,u_h)}(\log_{\Pi}^{\pm}) + \tfrac{\#\mathrm{Crit}(\Pi)}{2} h \right) > -\infty.$$

By the additivity of $v_{\mathbb{B}(0,u_h)}$ ([Col10, Proposition I.4.2]) we have

$$v_{\mathbb{B}(0,u_h)}(L_p^{\pm}) + (r - \tfrac{\#\mathrm{Crit}(\Pi)}{2})h = v_{\mathbb{B}(0,u_h)}(G^{\pm}) + rh - v_{\mathbb{B}(0,u_h)}(\log_{\Pi}^{\pm}) - \tfrac{\#\mathrm{Crit}(\Pi)}{2} h,$$

and so since $G^{\pm} \in \mathscr{D}^{r}(\mathbb{Z}_p^{\times}, \mathbb{C}_p)$ (and thus $\liminf \big(v_{\mathbb{B}(0,u_h)}(G^{\pm}) + rh \big) > -\infty$) we have

$$\liminf_{h} \left(v_{\mathbb{B}(0,u_h)}(L_p^{\pm}) + (r - \tfrac{\#\mathrm{Crit}(\Pi)}{2})h \right) > -\infty$$

and so by Lemma 2.2.11 we are done. ☐

In particular, in the minimal slope case $r = \frac{\#\mathrm{Crit}(\Pi)}{2}$ we get two bounded distributions.

Remark 2.3.8.

- One might ask if there is an analogue of the plus/minus theory for p-adic L-functions for GL_{2n+1}. Beyond the exact methods used in the present paper, there is an immediate stumbling block: in general, for $n \geq 3$ odd, the usual theory of p-adic L-functions is very poorly developed. For non-ordinary Π on GL_{2n+1}, the only constructions of p-adic L-functions are for $n = 1$ and Π a symmetric square lift from GL_2; in this case, a study of signed Iwasawa theory has been considered in [BLV18].
- The proofs above show that if we relax the minimal slope hypothesis we still obtain a pair of plus/minus L-functions which are unfortunately not bounded. Since the subsets of weight space on which these functions interpolate L-values are disjoint it seems that there is no hope in attempting a similar construction for these functions.

- In the case that we have a $\mathrm{GL}_{2n}(\mathbb{A}_{\mathbb{Q}})$ representation admitting p-stabilisations which are critical but not non-critical slope Theorem 6.3.3 no longer holds. As a result, for each such p-stabilisation we can construct a p-adic L-function, giving us at most $\binom{2n}{n}$ p-adic L-functions. It's possible that one could generalise the methods of this paper and utilise all of these p-adic L-functions to construct bounded functions analogous to $L_p^{\pm}$, but this is not something we have explored.

2.3.2 An example of a $\mathrm{GL}_4(\mathbb{A}_{\mathbb{Q}})$ representation satisfying the Pollack condition

We give an example of a cuspidal automorphic representation of $\mathrm{GL}_4(\mathbb{A}_{\mathbb{Q}})$ satsifying the Pollack condition and having minimal slope using the theory of twisted Yoshida lifts (see [LZ20a, Section 6] for an overview).

Let $F = \mathbb{Q}(\sqrt{5})$ and let $\sigma_i : F \to \mathbb{R}$, $i = 1, 2$ be the embeddings of F into $\mathbb{R}$. The prime 41 splits in F and we write M for one of its prime factors. Using Magma we see that there is a weight $(4, 2)$ cuspidal Hilbert newform f over F of level $N = M^2$ and of trivial character and with complex multiplication by the unique extension E/F such that $E/\mathbb{Q}$ is not Galois and in which M ramifies. Thus there is a Hecke character ψ over E with infinity type $z \mapsto \varepsilon_1(z)^3\varepsilon_2(z)^2\bar{\varepsilon}_2(z)$, where $\varepsilon_i, \bar{\varepsilon}_i : E \to \mathbb{C}$, $i = 1, 2$, are the pairs of conjugate embeddings of E and such that the Hecke eigenvalue of f at a prime $\wp$ of F is given by

$$a_{\wp} = \begin{cases} \psi(\mathfrak{q}_1) + \psi(\mathfrak{q}_2) & \text{if } \wp = \mathfrak{q}_1\mathfrak{q}_2 \text{ in } E \\ \psi(\mathfrak{q}) & \text{if } \wp = \mathfrak{q}^2 \text{ in } E \\ 0 & \text{otherwise.} \end{cases}$$

Let L be the number field generated by the Hecke eigenvalues of f. Since the weight is not parallel we have $F \subset L$. Fix a rational prime $p \nmid N$ and let L_v be the completion of L at a prime v over p. Suppose now that p splits in F and write $p = \wp_1 \cdot \wp_2$, labelled such that $\sigma_i(\wp_i)$ is below v.

Let $\psi_{\mathrm{Gal},v} : G_E \to L_v$ be the v-adic character of $G_F = \mathrm{Gal}(\bar{F}/F)$ associated to ψ by class field theory, so that $V_{f,v} := \mathrm{Ind}_{G_E}^{G_F}\psi_{\mathrm{Gal},v}$ is the G_F-representation associated to f. This representation is crystalline at primes not dividing N. The Hodge-Tate weights of $V_{f,v}$ are given by $(0, 3)$ at σ_1 and $(1, 2)$ at σ_2.

The restriction of $V_{f,v}$ to G_E splits as a direct sum of characters:

$$V_{f,v}|_{G_E} = \psi_{\mathrm{Gal},v} \oplus \psi^c_{\mathrm{Gal},v},$$

where $c \in \mathrm{Gal}(E/F)$ is the non-trivial element. If a prime $\wp$ of F above p splits in E then a decomposition group $D_{\wp} \subset G_F$ at a prime over $\wp$ is contained in G_E and we have $\mathbb{D}_{\mathrm{cris}}(V_{f,v}|_{D_{\wp}}) = \mathbb{D}_{\mathrm{cris}}(\psi_{\mathrm{Gal},v}|_{D_{\wp}}) \oplus \mathbb{D}_{\mathrm{cris}}(\psi^c_{\mathrm{Gal},v}|_{D_{\wp}})$ whence f is ordinary at $\wp$ as the image of the functor $\mathbb{D}_{\mathrm{cris}}$ is weakly admissible. Taking v to be above $\sigma_1(\wp)$ and writing the prime decomposition of $\wp$ in E as $\wp = \mathfrak{q}_1\mathfrak{q}_2$ we have

$$v_v(\psi(\mathfrak{q}_1)) = 0$$
$$v_v(\psi(\mathfrak{q}_2)) = 3$$

up to reordering of the $\mathfrak{q}_i$.

Theorem 2.3.9. *Suppose π is a cuspidal automorphic representation generated by a holomorphic Hilbert modular form of weight (k_1, k_2) over a totally real field F. Suppose that:*

- *For $1 \neq \theta \in \mathrm{Gal}(F/\mathbb{Q})$ we have $\pi^\theta \not\cong \pi$,*
- *There is a Hecke character ε over $\mathbb{Q}$ such that the central character ω_π of π satisfies*

$$\omega_\pi = \varepsilon \circ \mathrm{Norm}_{F/\mathbb{Q}}.$$

Then there is a unique globally generic cuspidal automorphic representation $\Theta(\pi, \omega_\pi)$ of $\mathrm{GSp}_4(\mathbb{A}_{\mathbb{Q}})$ (a twisted Yoshida lift) of weight $(\frac{k_1+k_2}{2}, \frac{|k_1-k_2|}{2} - 2)$ with central character ε satisfying

$$L(\Pi, s) = L(\pi, s + \frac{\max\{k_1, k_2\} - 1}{2}).$$

Proof. See [LZ20a, Theorem 6.1.1 and Proposition 6.1.4]. ☐

Let π be the cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_E)$ generated by f. Since f has non-parallel weight we see that $\pi \not\approx \pi^\theta$ for non-trivial $\theta \in \mathrm{Gal}(F/\mathbb{Q})$.

Set $\Pi = \Theta(\pi, 1)$. Then Π a cuspidal automorphic representation of $\mathrm{GSp}_4(\mathbb{A}_{\mathbb{Q}})$ of weight $(3, 3)$ with trivial central character. This weight lies in the cohomological range and thus Π is cohomological. The Hodge-Tate weights of Π are $(0, 1, 2, 3)$.

Recall that we have a rational prime p such that p splits in F:

$$p\mathcal{O}_F = \wp_1\wp_2.$$

We assume further that $\wp_2$ is inert in E and $\wp_1$ splits, and we write the factorisation of $\wp_1$ as

$$\wp_1 = \mathfrak{P}_1\mathfrak{P}_2.$$

Primes satisfying the above conditions are not uncommon, for example, the primes 11 and 19 admit this splitting phenomena in the tower $E/F/\mathbb{Q}$. We remark that a necessary condition for such a splitting is that E is non-Galois over $\mathbb{Q}$. The local L-factor of Π at such a p is given by

$$L_p(\Pi, s)^{-1} = (1 - \psi(\mathfrak{P}_1)p^{-s})(1 - \psi(\mathfrak{P}_2)p^{-s})(1 - \psi(\wp_2)p^{-2s}).$$

Choosing a prime v of L lying above p such that v lies above $\sigma_i(\wp_i)$, we deduce that Π satisfies the Pollack condition and has two minimal slope p-stabilisations. By Corollary 2.3.3 these p-stabilisations are Q-regular and Shalika. There is an exceptional isomorphism $\mathrm{GSp}_4 \cong \mathrm{GSpin}_5$ and so we are done by applying the functorial lift from GSpin_5 to GL_4.

2.4 Non-vanishing of twists

We use $L_p^\pm$ to show non-vanishing of the complex L-function of Π at the central value, extending work of Dimitrov, Januszewski, Raghuram [DJR20] to a non-ordinary setting.

Proposition 2.4.1. *In the case that* $\mathrm{Crit}(\Pi) \neq \{w/2\}$, *we have*

$$L_p^\pm \neq 0.$$

Proof. We consider L_p^+, the case of L_p^- being essentially identical and we further assume p is odd for brevity of notation (although the same argument works in this case). Note that a character $\theta : \mathbb{Z}_p^\times \to \mathbb{C}_p^\times$ of conductor p^{m+1} corresponds to a choice of primitive p^mth root of unity ζ_θ in a disc $\mathcal{W}$ determined by the restriction ψ_θ of θ to $(\mathbb{Z}/p\mathbb{Z})^\times$, which gives us the identification

$$\int_{\mathbb{Z}_p^\times} x^j\theta(x)\log_\Pi^+(x) = \log_\Pi^+(\psi_\theta, \gamma^j\zeta_\theta - 1),$$

where on the left hand side we use the description of $\log_\Pi^+$ as a distribution and on the right hand side $\log_\Pi^+(\psi_\theta, -)$ is the restriction of $\log_\Pi^+$ to the disc in $\mathcal{W}$ corresponding to ψ_θ. We adopt the analogous notation for L_p. It follows from Lemma 2.3.5 that $\log_\Pi^+(\psi_\theta, \gamma^j\zeta_\theta - 1) \neq 0$ if m is odd (resp. even for $p = 2$). Thus for characters θ of odd p-power conductor we have the interpolation property

$$L_p(\psi_\theta, \gamma^j\zeta_\theta - 1) \sim \frac{L(\Pi \otimes \theta, j + 1/2)}{\log_\Pi^+(\psi_\theta, \gamma^j\zeta_\theta - 1)},\ j \in \mathrm{Crit}(\Pi),$$

where $\sim$ is used here to mean 'up to non-zero constant'. By Jacquet-Shalika [JS76, 1.3] we have

$$L(\Pi \otimes \theta, s) \neq 0 \text{ for } \mathfrak{Re}(s) \geq w/2 + 1$$

for finite order characters θ, and by applying the functional equation we get non-vanishing for $\mathfrak{Re}(s) \leq w/2$. Since $\mathrm{Crit}(\Pi)$ contains an integer k not equal to $w/2$, the above discussion gives us

$$L(\Pi, k + 1/2) \neq 0,$$

and thus $L_p^+ \neq 0$. ☐

Remark 2.4.2. Proposition 2.4.1 actually proves the stronger result that the power series $\mathscr{M}(L_p^\pm)|_{\mathbb{B}_\psi}$ is non-zero for each choice of ψ.

We can turn this back on itself and use $L_p^\pm$ to say something about nonvanishing of $L(\Pi \otimes \theta, (\omega + 1)/2)$ in the case when $L_p^\pm \in \mathscr{D}^0(\mathbb{Z}_p^\times, \mathbb{C}_p)$.

Theorem 2.4.3. *In the case that* $L_p^\pm$ *are bounded distributions,* w *is even, and* $\mathrm{Crit}(\Pi) \neq \{w/2\}$, *we have*

$$L(\Pi \otimes \theta, (w+1)/2) \neq 0$$

for all but finitely many characters θ *of* p*-power conductor.*

Proof. Assume p odd for brevity, again noting that the argument works fine for $p = 2$. For any character ψ of $(\mathbb{Z}/p\mathbb{Z})^\times$ we can write $\mathscr{M}(L_p^\pm)|_{\mathbb{B}_\psi} = L_p^\pm(\psi, T) \in \mathcal{O}_L[[T]] \otimes_{\mathcal{O}_L} L$ for some finite extension $L/\mathbb{Q}_p$. We note that $\mathscr{M}(L_p^\pm) = \sum_\psi \mathbb{1}_{\mathbb{B}_\psi} L_p^\pm(\psi)$ where $\mathbb{1}_{\mathbb{B}_\psi}$ denotes the indicator function on $\mathbb{B}_\psi$. This power series is non-zero by Proposition 2.4.1 and Remark 2.4.2, and so Weierstrass preparation tells us that each $L_p^\pm(\psi, T)$, and thus $L_p^\pm$, has only finitely many zeroes. Given any character θ of p-power conductor, we have

$$\int_{\mathbb{Z}_p^\times} x^{w/2}\theta(x) L_p^?(x) \sim L(\Pi \otimes \theta, (w+1)/2),$$

where

$$? = \begin{cases} + & \text{if the conductor of } \theta \text{ is odd } p\text{-power} \\ - & \text{otherwise} \end{cases}.$$

Thus, for all but finitely many θ, we have

$$L(\Pi \otimes \theta, (w+1)/2) \neq 0.$$

□

3 Lemma–Flach classes in Hida families

3.1 Introduction

We construct classes in Galois cohomology interpolating the Lemma–Flach Euler system of [LSZ21] as it varies in a 4-parameter Hida family. This generalises [LSZ21, Section 9] in which variation in a two parameter family was considered.

The novelty in this construction is in the application of Loeffler's machine; in order to obtain classes varying in a 4-parameter family we must increase the number of variables in our Iwasawa algebra by working with Shimura varieties for the group $\tilde{G} := \mathrm{GSp}_4 \times \mathrm{GL}_1$. In this setting we can find a subgroup

$$Q^0_{\tilde{H}} \subset \tilde{H} = \mathrm{GL}_2 \times_{\mathrm{GL}_1} \times \mathrm{GL}_2 \times \mathrm{GL}_1$$

which has an open orbit on the flag variety $B_{\tilde{G}} \backslash \tilde{G}$ and thus gives classes in the Borel-ordinary Iwasawa cohomology of $\tilde{G}$, a module over a rank 4 Iwasawa algebra. This gives the required variation and an analogous argument to that in [LSZ21] in the 2-variable case gives classes in the Galois cohomology of the families of Borel-ordinary Galois representations constructed in [TU99].

3.2 Prerequisites

3.2.1 Algebraic groups

All algebraic groups will be treated as group schemes over $\mathbb{Z}$. Let $G = \mathrm{GSp}_4$ be the group scheme defined for a $\mathbb{Z}$-algebra R by

$$G(R) = \{(X, \mu) \in \mathrm{GL}_4(R) \times R^\times : X^T J X = \mu J\},$$

where $J := \begin{pmatrix} & & & 1 \\ & & 1 & \\ & -1 & & \\ -1 & & & \end{pmatrix}$. We write $\mu : G \to G/[G,G] \cong \mathbb{G}_m$ for the similitude character sending (X, μ) to μ. Set

$$H = \mathrm{GL}_2 \times_{\mathrm{GL}_1} \mathrm{GL}_2,$$

where product is fibred over the determinant. There is a natural inclusion $\iota : H \hookrightarrow G$ given by

$$\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \times \left(\begin{smallmatrix} a' & b' \\ c' & d' \end{smallmatrix}\right) \mapsto \begin{pmatrix} a & & & b \\ & a' & b' & \\ & c' & d' & \\ c & & & d \end{pmatrix} \times (\det). \tag{5}$$

Let $B_G = T_G \times N_G \subset G$ be the Borel subgroup of upper triangular matrices, $T_G \subset B$ the maximal torus of diagonal matrices and N_G the unipotent radical of B_G. Define a Borel subgroup $B_H = B_G \cap H$ of H with Levi decomposition $T_H \ltimes N_H$, where $T_H = T_G \cap H, N_H = N_G \cap H$.

For an algebraic group $\mathcal{G}$ write

$$\tilde{\mathcal{G}} = \mathcal{G} \times \mathrm{GL}_1.$$

The above inclusion of H into G extends naturally to an inclusion

$$\iota : \tilde{H} \hookrightarrow \tilde{G},$$

by extending to the identity on the GL_1 factor. We have a Borel subgroup $B_{\tilde{G}} = \tilde{B}_G$ with Levi decomposition $B_{\tilde{G}} = T_{\tilde{G}} \ltimes N_G$ where $T_{\tilde{G}} = \tilde{T}_G$ and similarly for H.

3.2.2 Algebraic representations

Let $\chi_i \in X^\bullet(T_G), i = 1, 2$ be the projection to the first and second diagonal entries of T_G, so that $\{\chi_1, \chi_2, \mu\}$ forms a basis for $X^\bullet(T_G)$. These elements extend trivially to the maximal torus $\tilde{T}_G$ of $\tilde{G}$. Writing σ for the character given by projection to the GL_1 factor in the direct product, the set $\{\chi_1, \chi_2, \mu, \sigma\}$ forms a basis for $X^\bullet(\tilde{T}_G)$. For $a, b \geq 0$, we write $V^{a,b}$ for the $\mathbb{Q}$-linear representation of $\tilde{G}$ of highest weight $(a + b)\chi_1 + a\chi_2$. In general, given any dominant integral weight $\lambda \in X^\bullet(T_G)$, let $V_\lambda = (V_\lambda, v_\lambda)$ be the irreducible representation of highest weight λ with a choice of highest weight vector v_λ. Let $V_{\lambda,\mathbb{Z}}$ be the *maximal* integral lattice with respect to v_λ in the sense of [LSZ21, Definition 4.2].

Proposition 3.2.1. *Let λ, λ' be dominant integral weights. The Cartan product is the unique $\tilde{G}$-equivariant homomorphism*

$$V_{\lambda,\mathbb{Z}} \otimes V_{\lambda',\mathbb{Z}} \to V_{\lambda+\lambda',\mathbb{Z}}$$

satisfying $v_\lambda \cdot v_{\lambda'} = v_{\lambda+\lambda'}$. For any non-zero $v \in V_{\lambda,\mathbb{Z}}$, $v' \in V_{\lambda',\mathbb{Z}}$ we have $v \cdot v' \neq 0$.

Proof. [LSZ21, Proposition 4.2.1]. ☐

Remark 3.2.2. The Borel–Weil theorem realises algebraic representations as subrepresentations of the coordinate ring of $\tilde{G}$. Via this optic, the Cartan product is nothing but multiplication of regular functions.

For a representation V of $\tilde{G}$ write $\iota^* V$ for the restriction of the representation to $\tilde{H}$ via ι. The following branching law describes how $V^{a,b}$ decomposes as a representation of H.

Proposition 3.2.3. *We have*

$$\iota^* V^{a,b} = \bigoplus_{0 \leq q \leq a} \bigoplus_{0 \leq r \leq b} W^{a-q+b-r,a-q+r} \otimes \det{}^q,$$

as H-representations, where $W^{c,d} = \mathrm{Sym}^c(\mathbb{Q}^2) \boxtimes \mathrm{Sym}^d(\mathbb{Q}^2)$ is the irreducible H-representation of highest weight (c, d) with respect to B_H.

As in [LSZ21, Section 4.3] we fix choices $v^{a,b,q,r} \in V^{a,b}$ of highest weight vectors for $\tilde{H}$ in each $W^{a-q+b-r,a-q+r} \otimes \det^q$ factor. As $W^{c,d}$ has a canonical choice of highest weight vector $e_1^c \boxtimes f_1^d$ where $\{e_1, e_2\}, \{f_1, f_2\}$ are standard bases for the standard representation of GL_2, we can define canonical branching maps

$$\mathrm{br}^{[a,b,q,r]} : W^{a-q+b-r,a-q+r} \otimes \det{}^q \to \iota^* V^{a,b}$$

by sending $e_1^c \boxtimes f_1^d$ to $v^{a,b,q,r}$. By [LSZ21, Proposition 4.3.5], these branching maps restrict to inclusions of admissible lattices:

$$\mathrm{br}^{[a,b,q,r]} : W_{\mathbb{Z}}^{a-q+b-r,a-q+r} \otimes \det{}^q \to \iota^* V_{\mathbb{Z}}^{a,b}, \tag{6}$$

where $W_{\mathbb{Z}}^{a-q+b-r,a-q+r}$ is the *minimal* admissible lattice in $W^{a-q+b-r,a-q+r}$.

Remark 3.2.4. The minimal admissible lattice in $W_{\mathbb{Z}}^{c,d}$ is given by $\mathrm{TSym}^c(\mathbb{Z}^2) \boxtimes \mathrm{TSym}^d(\mathbb{Z}^2)$ where TSym^r is the space of degree r symmetric tensors. The maximal admissible lattice is given by $\mathrm{Sym}^c(\mathbb{Z}^2) \boxtimes \mathrm{Sym}^d(\mathbb{Z}^2)$.

3.2.3 A (slightly) different Lie-theoretic computation

Define a cocharacter $T_B \in X_\bullet(\tilde{T}_G)$ by $T_B(x) = \begin{pmatrix} x^3 & & & \\ & x^2 & & \\ & & x & \\ & & & 1 \end{pmatrix} \times (x)$ and set $u = \begin{pmatrix} 1 & 1 & 1 & \\ & 1 & & 1 \\ & & 1 & -1 \\ & & & 1 \end{pmatrix}$. We will need the natural analogue of 'a Lie-theoretic computation' [LSZ21, Section 4.4]:

Lemma 3.2.5. *Let $v^{a,b,q,r} \in V^{a,b}$ be as above. Then the projection of $u^h v^{a,b,q,r}$ to the highest T_B weight subspace of $V^{a,b}$ is given by $(2h)^q h^r v^{a,b,0,0}$.*

Proof. Let $V = V^{0,1}$ be the standard representation of GSp_4 and note that we can identify $V^{1,0}$ with the irreducible direct summand of $\bigwedge^2 V^{0,1}$ spanned by $e_1 \wedge e_2$. The vectors $v^{a,b,q,r}$ are defined in [LSZ21, Section 4.3] as the Cartan product $v^{a,b,q,r} = v^{b-r} \cdot (v')^r \cdot w^{a-q} \cdot (w')^q$ where $v = e_1 \in V^{0,1}$, $v' = e_2 \in V^{0,1}$, $w = e_1 \wedge e_2 \in V^{1,0}$ and $w' = e_1 \wedge e_4 - e_2 \wedge e_3 \in V^{1,0}$. Notice that v, w are highest weight vectors for T_B (and are thus fixed by u) whereas v', w' are not. By the equivariance of the Cartan product it suffices to compute the projections of $u^h v'$ and $u^h w'$ to the highest T_B weight subspaces to get the result. □

3.2.4 Shimura varieties

Set $\mathbb{S} = \mathrm{Res}_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_m)$ and define

$$\begin{aligned} h_{\tilde{H}} &: \mathbb{S} \to \tilde{H}_{\mathbb{R}} \\ a + ib &\mapsto \left(\frac{1}{a^2+b^2}\begin{pmatrix} a & b \\ -b & a \end{pmatrix}\right)^2 \times (1). \end{aligned}$$

This defines a Shimura datum $(\tilde{H}, h_{\tilde{H}})$ for $\tilde{H}$. The inclusion $\iota : \tilde{H} \hookrightarrow \tilde{G}$ induces a compatible Shimura datum $(\tilde{G}, h_{\tilde{G}})$, where $h_{\tilde{G}} = \iota \circ h_{\tilde{H}}$.

Definition 3.2.6. Let $R \in \{G, H, \tilde{G}, \tilde{H}\}$. We say $U \subset R(\mathbb{A}_f)$ is *sufficiently small* if it acts without fixed points on the set

$$R(\mathbb{Q})\backslash R(\mathbb{A}) \times \mathcal{H}_R,$$

where

$$\mathcal{H}_R = \begin{cases} \mathcal{H} \times \mathcal{H} & \text{if } R = H, \tilde{H} \\ \mathcal{H}_2 & \text{if } R = G, \tilde{G} \end{cases}$$

and $\mathcal{H}$ is the complex upper half plane and $\mathcal{H}_2$ is the genus 2 Siegel space.

Given open compact subgroups $U \subset \tilde{H}(\mathbb{A}_f), K \subset \tilde{G}(\mathbb{A}_f)$ the above Shimura data induce smooth quasiprojective varieties $Y_{\tilde{H}}(U), Y_{\tilde{G}}(K)$ which are canonically defined over $\mathbb{Q}$ and whose $\mathbb{C}$-points are given by

$$\begin{aligned} Y_{\tilde{H}}(U)(\mathbb{C}) &= \tilde{H}(\mathbb{Q})\backslash\tilde{H}(\mathbb{A}_f) \times \mathcal{H}^2/U, \\ Y_{\tilde{G}}(K)(\mathbb{C}) &= \tilde{H}(\mathbb{Q})\backslash\tilde{G}(\mathbb{A}_f) \times \mathcal{H}_2/K. \end{aligned}$$

For open compact subgroups $U \subset U'$ of $\tilde{H}(\mathbb{A}_f)$ there is a natural finite etale 'projection' morphism

$$\mathrm{pr}^U_{U'} : Y_{\tilde{H}}(U) \to Y_{\tilde{H}}(U'),$$

and similarly for $\tilde{G}$. Define pro-varieties

$$\begin{aligned} Y_{\tilde{H}} &:= \varprojlim_U Y_{\tilde{H}}(U) \\ Y_{\tilde{G}} &:= \varprojlim_K Y_{\tilde{H}}(K), \end{aligned}$$

where the limit is taken over all open compact subgroups ordered by inclusion. These carry natural actions of $\tilde{H}(\mathbb{A}_f)$ and $\tilde{G}(\mathbb{A}_f)$ induced by the conjugation isomorphisms

$$Y_*(U) \xrightarrow{[g]} Y_*(g^{-1}Ug)$$

which are given on $\mathbb{C}$-points by right-translation. We have

$$Y_{\tilde{H}}(\mathbb{C}) = \tilde{H}(\mathbb{Q})\backslash\tilde{H}(\mathbb{A}_f) \times \mathcal{H} \times \mathcal{H}$$
$$Y_{\tilde{G}}(\mathbb{C}) = \tilde{H}(\mathbb{Q})\backslash\tilde{G}(\mathbb{A}_f) \times \mathcal{H}_2.$$

The group homomorphism $\iota : \tilde{H} \hookrightarrow \tilde{G}$ induces an injective map

$$\iota : Y_{\tilde{H}} \hookrightarrow Y_{\tilde{G}}$$

which descends to a map

$$\iota_K^U : Y_{\tilde{H}}(U) \to Y_{\tilde{G}}(K)$$

whenever $\iota^{-1}(U) \subset K$. This map is not necessarily injective, with a criteria for injectivity being given by [LSZ21, Proposition 5.3.1].

We can, in a similar fashion, define compatible Shimura data for the groups H and G which give varieties Y_H, Y_G such that the natural projection $q : \tilde{G} \to G$ induces a morphism of Shimura varieties

$$q : Y_{\tilde{G}} \to Y_G$$

compatible with the maps induced by ι.

We will also briefly need the Shimura variety $Y_{\mathrm{GL}_2\times\mathrm{GL}_1}$ defined by the Shimura datum

$$h_{\mathrm{GL}_2\times\mathrm{GL}_1} : a+ib \mapsto (\frac{1}{a^2+b^2}\left(\begin{smallmatrix} a & b \\ -b & a\end{smallmatrix}\right)) \times (1).$$

This Shimura datum is compatible with the GL_2 Shimura datum

$$h_{\mathrm{GL}_2} : a+ib \mapsto (\frac{1}{a^2+b^2}\left(\begin{smallmatrix} a & b \\ -b & a\end{smallmatrix}\right))$$

under the natural projection homomorphism

$$q : \mathrm{GL}_2 \times \mathrm{GL}_1 \to \mathrm{GL}_2$$

and thus we get a morphism of Shimura varieties

$$Y_{\mathrm{GL}_2\times\mathrm{GL}_1} \xrightarrow{q} Y_{\mathrm{GL}_2}.$$

3.2.5 Level groups

We give a reference for the level groups we will be using in this section. We first define normal subgroup $Q^\circ_H \subset Q_{\tilde{H}}$ which will be the key input into construction of the branching maps: For a $\mathbb{Z}_p$-algebra R define

$$Q^\circ_{\tilde{H}}(R) := \left\{\begin{pmatrix} x & * \\ & 1\end{pmatrix} \times \begin{pmatrix} xy & * \\ & y^{-1}\end{pmatrix} \times (y) : x, y \in R^\times\right\}.$$

Definition 3.2.7. We define open compact subgroups of $\tilde{H}(\mathbb{Q}_p)$ and $\tilde{G}(\mathbb{Q}_p)$:

- $Q^\circ_{\tilde{H}_p}(p^n) = \{g \in Q_{\tilde{H}}(\mathbb{Z}_p) : g \in Q^\circ_{\tilde{H}} \bmod p^n\}$, $n \geq 0$,
- $Q^\circ_{\tilde{H}_p}(p^m, p^n) = \{h \in Q^\circ_{\tilde{H}_p}(p^n), \mu, \sigma \equiv 1 \bmod p^m\}$, $n, m \geq 0$,
- $Q^\circ_{\tilde{G}_p}(p^m, p^n) = \{g \in \tilde{G}(\mathbb{Z}_p) : g \in N_{\tilde{G}}(\mathbb{Z}_p) \bmod p^n, \mu, \sigma \equiv 1 \bmod p^m\}$, $n, m \geq 0$,

recalling that μ, σ are our basis for $X^\bullet(\tilde{G})$ given by the similitude character and projection to the direct GL_1-factor respectively.

Fix a finite set of primes S not including p. We define level groups

$$Q^{\circ}_{\tilde{G}}(p^m,p^n) = Q^{\circ}_{\tilde{G}_p}(p^m,p^n) \times K_S \times \prod_{\ell\notin S\cup\{p\}} \tilde{G}(\mathbb{Z}_\ell) \subset \tilde{G}(\mathbb{A}_f)$$
$$Q^{\circ}_{\tilde{H}}(p^m,p^n) = Q^{\circ}_{\tilde{H}_p}(p^m,p^n) \times U_S \times \prod_{\ell\notin S\cup\{p\}} \tilde{H}(\mathbb{Z}_\ell) \subset \tilde{H}(\mathbb{A}_f),$$

where $K_S \subset \tilde{G}(\mathbb{Q}_S), U_S \subset \tilde{H}(\mathbb{Q}_S)$ are chosen so that the above groups are sufficiently small (this may require enlarging S by finitely many primes). Let $M > 0$ be a square-free integer coprime to p. For an open compact subgroup $K \subset \tilde{G}(\hat{\mathbb{Z}})$ we write $K(M) = \{k \in K : \mu, \sigma(k) \equiv 1 \bmod M\}$, for example $Q_{\tilde{G}}(M,p^m,p^n)$.

We define the non-tilde'd groups $Q^{\circ}_G(M,p^m,p^n)$ to be the image of the tilde'd group under the natural projection map $q : \tilde{G} \to G$.

3.2.6 Coefficient sheaves

For $\mathcal{G} \in \{H, G, \tilde{H}, \tilde{G}\}$, given a sufficiently small open compact subgroup $U \subset \mathcal{G}(\mathbb{A}_f)$ and a free $\mathbb{Z}_p$-module V of finite rank, equipped with a continuous left action of U, we can assign to V a locally constant U-equivariant étale sheaf $\mathscr{V}$ on $Y_{\mathcal{G}}(U)$ such that for any $U' \lhd U$ open we have

$$\mathscr{V}(Y_{\mathcal{G}}(U')) = \mathscr{V}^{U'}$$

and such that the pullback action of $u \in U/U'$ on $H^0_{et}(Y_{\mathcal{G}}(U), \mathscr{V})$ is given by the natural action of u on $\mathscr{V}^U$. If the action of U on V extends to some monoid $\mathcal{M} \subset \mathcal{G}(\mathbb{A}_f)$ then the sheaf $\mathscr{V}$ becomes $\mathcal{M}$-equivariant and we get an action of the Hecke algebra $\mathcal{H}(U\backslash\mathcal{M}/U)$ on the cohomology groups $H^*_{et}(Y_{\mathcal{G}}(U), \mathscr{V})$.

Set

$$\begin{aligned}
\tilde{D}^{a,b}_{\mathbb{Z}_p} &= V^{a,b}_{\mathbb{Z}_p} \otimes \mu^{-(2a+b)} \times \sigma^a \\
\tilde{H}^{c,d}_{\mathbb{Z}_p} &= W^{c,d}_{\mathbb{Z}_p} \otimes \mu^{-(c+d)} \times \sigma^d \\
D^{a,b}_{\mathbb{Z}_p} &= V^{a,b}_{\mathbb{Z}_p} \otimes \mu^{-(2a+b)} \\
H^{c,d}_{\mathbb{Z}_p} &= W^{c,d}_{\mathbb{Z}_p} \otimes \mu^{-(c+d)},
\end{aligned}$$

and let $\tilde{\mathscr{D}}^{a,b}_{\mathbb{Z}_p}, \tilde{\mathscr{H}}^{c,d}_{\mathbb{Z}_p}, \mathscr{D}^{a,b}_{\mathbb{Z}_p}, \mathscr{H}^{c,d}_{\mathbb{Z}_p}$ be their respective associated etale sheaves. By abuse of notation we will write $V \otimes \mu^x \times \sigma^y$ for the etale sheaf associated to a $\tilde{G}$-representation $V \otimes \mu^x \times \sigma^y$.

Our chosen normalizations ensure that the highest weight space for the cocharacter T_B is exactly 0. Moreover, this also holds for the cocharacters

$$T_S : x \mapsto \begin{pmatrix} x &&& \\ & x && \\ && 1 & \\ &&& 1 \end{pmatrix} \times (1), T_{\mathcal{K}} : x \mapsto \begin{pmatrix} x^2 &&& \\ & x && \\ && x & \\ &&& 1 \end{pmatrix} \times (x).$$

This will be key when constructing Hecke operators that preserve integrality.

For $c = a + b - q - r, d = a - q + r.$ the branching maps (6) induce morphisms of sheaves

$$\mathrm{br}^{[a,b,q,r]} : \mathscr{H}^{c,d} \to \iota^* \tilde{\mathscr{D}}^{a,b}_{\mathbb{Z}_p} \otimes \mu^q \sigma^{r-q}.$$

We note that if $f^{c,d} \in \tilde{H}^{c,d}_{\mathbb{Z}_p}$ is a highest weight vector then $Q^0_{\tilde{H}} \cdot f^{c,d} = f^{c,d}$ and thus

$$\mathrm{br}^{[a,b,q,r]}(f^{c,d}) \in (\tilde{D}^{a,b}_{\mathbb{Z}_p} \otimes \mu^q \sigma^{r-q})^{Q^0_{\tilde{H}}}$$

and this space is one-dimensional.

3.2.7 Hecke operators

Definition 3.2.8. For $? \in \{B, S, \mathcal{K}\}$, define

$$\tilde{\alpha}_? = T_?(p) \in \tilde{T}_G(\mathbb{Q}_p)$$

and

$$\alpha_? := q(\alpha_?).$$

Remark 3.2.9. The letters $B, S, \mathcal{K}$ refer to Borel, Siegel and Klingen and refer to the fact that the cocharacters $T_B, T_S, T_{\mathcal{K}}$ are *strictly dominant* with respect to the (natural extensions to $\tilde{G}$ of the) parabolic subgroups $P_B = B_G, P_S, P_{\mathcal{K}}$ of G which bear these names. Specifically, this means that for $? \in \{B, S, \mathcal{K}\}$

$$\langle T_?, \alpha \rangle > 0$$

for all relative roots α of G with respect to $P_?$.

For $? \in \{B, S, \mathcal{K}\}$, $m \geq 1$ the element $\tilde{\alpha}_?^{-m} \in \tilde{G}(\mathbb{Q}_p)$ has a well defined action on the integral representation $\tilde{D}^{a,b}_{\mathbb{Z}_p}$.

For $? \in \{S, \mathcal{K}, B\}$ we define morphisms of sheaves

$$[\tilde{\alpha}_?^{-1}]_\# : \tilde{\mathscr{D}}^{a,b}_{\mathbb{Z}_p} \to [\alpha_?^{-1}]^* \tilde{\mathscr{D}}^{a,b}_{\mathbb{Z}_p}$$

via the action of $\tilde{\alpha}_?^{-m}$ on $\tilde{D}^{a,b}_{\mathbb{Z}_p}$ and another morphism of sheaves

$$p^X [\alpha_?^{-1}]_\# : \mathscr{D}^{a,b}_{\mathbb{Z}_p} \to [q(\alpha_?^{-1})]^* \mathscr{D}^{a,b}_{\mathbb{Z}_p}$$

via the action of $p^{X_?} \alpha_?^{-1}$ on $D^{a,b}_{\mathbb{Z}_p}$ where

$$X_? = \begin{cases} 0 & \text{if } ? = S \\ a & \text{otherwise.} \end{cases}$$

We write

$$\begin{aligned} [\tilde{\alpha}_?]_* &= ([\tilde{\alpha}_?], [\tilde{\alpha}_?^{-1}]_\#), \\ [\alpha_?]_* &= ([\alpha_?], p^X [\alpha_?^{-1}]_\#). \end{aligned}$$

For any open compact subgroup $K \subset \tilde{G}(\mathbb{A}_f)$, consider the diagram of Shimura varieties

$$\begin{tikzcd} Y_{\tilde{G}}(\tilde{\alpha}_? K \tilde{\alpha}_?^{-1} \cap K) \arrow[r, "{[\tilde{\alpha}_?]}"] \arrow[d, "\mathrm{pr}"] & Y_{\tilde{G}}(K \cap \tilde{\alpha}_?^{-1} K \tilde{\alpha}_?) \arrow[d, "\mathrm{pr}"] \\ Y_{\tilde{G}}(K) & Y_{\tilde{G}}(K) \end{tikzcd}$$

Definition 3.2.10. For K as above and $? \in \{B, S, \mathcal{K}\}$, we define Hecke operators $\tilde{\mathcal{U}}_?, \mathcal{U}_?$, on $H^*_{\mathrm{et}}(Y_{\tilde{G}}(K), \tilde{\mathscr{D}}^{a,b}_{\mathbb{Z}_p} \otimes \mu^q \sigma^{r-q})$ and $H^*_{\mathrm{et}}(Y_G(q(K)), \mathscr{D}^{a,b}_{\mathbb{Z}_p} \otimes \mu^q)$ respectively by

$$\begin{aligned} \tilde{\mathcal{U}}_? &= (\mathrm{pr}_* \circ [\tilde{\alpha}_?]_* \circ \mathrm{pr}^*) \\ \mathcal{U}_? &= (\mathrm{pr}_* \circ [\alpha_?]_* \circ \mathrm{pr}^*) \end{aligned}$$

where the projection morphisms are as in the above diagram and the sheaf morphisms $[\tilde{\alpha}_?]_\#$, $[\alpha_?]_\#$ ignore the twists by $\mu^q \sigma^{r-q}$ and μ^q respectively.

3.2.8 Cohomology functors

We recall the formalism of cohomology functors from [Loe21]. For this section only, let G be a locally profinite topological group and $\Sigma \subset G$ an open submonoid.

Definition 3.2.11. Let $\mathcal{P}(G, \Sigma)$ be the category whose objects consist of the open compact subgroups of G contained in Σ, and whose morphisms are given by

$$\mathrm{Hom}_{\mathcal{P}(G,\Sigma)}(U, V) = \{g \in U\backslash\Sigma/V : g^{-1}Ug \subset V\}.$$

We write $[g]$ for the corresponding double coset, with composition defined to be $[g] \circ [h] = [hg]$. If $U \subset V$ we write $\mathrm{pr}_V^U = [1]$, or often just pr.

Definition 3.2.12. Define a *cohomology functor* for (G, Σ) with coefficients in a commutative ring A to be a pair of functors (M_*, M^*), where

$$M_* : \mathcal{P}(G, \Sigma^{-1}) \to A - \mathrm{mod}, M^* : \mathcal{P}(G, \Sigma)^{\mathrm{op}} \to A - \mathrm{mod},$$

such that

1. $M_*(U) = M^*(U) =: M(U)$ for all $U \in \mathcal{P}(G, \sigma)$
2. If for $g : V \to U$ we write $[g]_* = M_*([g])$ and similarly for M^*, then

   $$[g]_* = [g^{-1}]^* \in \mathrm{Hom}_A(M(V), M(U))$$

   whenever this makes sense.

A morphism of cohomology functors $M \to N$ is a pair of natural transformations

$$\begin{aligned} M_* &\implies N_*, \\ M^* &\implies N^*. \end{aligned}$$

We will be working with cohomology functors satisfying the following useful axiom.

Definition 3.2.13. We say a cohomology functor M is *Cartesian* if for any open compact subgroup $V \subset \Sigma$ and any open compact subgroups $U, U' \subset V$ there is a commutative diagram

$$\begin{tikzcd} \bigoplus_\gamma M(U_\gamma) \arrow[r, "\sum \mathrm{pr}_*"] & M(U') \\ M(U) \arrow[u, "\sum [\gamma]_*"] \arrow[r, "\mathrm{pr}_*"] & M(V) \arrow[u, "\mathrm{pr}^*"'] \end{tikzcd}$$

where the sum runs over a set of representatives for the double quotient $U\backslash V/U'$, and $U_\gamma := U \cap \gamma U' \gamma^{-1}$.

Definition 3.2.14. For a cohomology functor M define the *Iwasawa cohomology* $M_{\mathrm{Iw}}(K)$ for any compact subgroup $K \subset G$ to be

$$M_{\mathrm{Iw}}(K) = \varprojlim_{U \supset K} M(U).$$

Given any triple (g, K, K') such that $g^{-1}Kg \subset K'$ we can define pushforward maps $[g]_* : M_{\mathrm{Iw}}(K) \to M_{\mathrm{Iw}}(K')$ and furthermore if M is Cartesian and $K \subset K'$ has finite index, we can define a pullback map

$$M_{\mathrm{Iw}}(K') \to M_{\mathrm{Iw}}(K).$$

Let M_G be a Cartesian cohomology functor for (G, Σ). Suppose H is another locally profinite topological group, and that we have an injective group homomorphism

$$\iota : H \hookrightarrow G$$

onto a closed subgroup of G.

Definition 3.2.15. Given Cartesian cohomology functors M_H, M_G for H and G respectively, a *pushforward* $\iota_* : M_H \to M_G$ is a collection of maps $M_H(V) \to M_G(K)$ for each pair of open compacts $U \subset \Sigma_H, K \subset \Sigma_G$, satisfying $\iota(V) \subset K$ which are compatible with pushforward maps $[h]_*$ for $h \in H \cap \Sigma^{-1}$ and satisfying the following diagram

$$\begin{tikzcd}
\bigoplus_\gamma M_H(V \cap \gamma\iota^{-1}(U)\gamma^{-1}) \arrow[r, "\sum[\gamma]_*"] & M_G(U) \\
M_H(V) \arrow[u, "\sum \mathrm{pr}^*"] \arrow[r, "\iota_*"] & M_G(K) \arrow[u, "\mathrm{pr}_*"]
\end{tikzcd}$$

where the sum runs over a fixed set of representatives γ for the double quotient $V\backslash K/U$, and $[\gamma]_*$ is the composition

$$M_H(V \cap \gamma\iota^{-1}(U)\gamma^{-1}) \xrightarrow{\iota_*} M_G(\gamma U \gamma^{-1}) \xrightarrow{[\gamma]_*} M_G(U).$$

Example 3.2.16. Let $G = \mathcal{G}(\mathbb{Q}_p)$ be the $\mathbb{Q}_p$-points of a connected reductive group $\mathcal{G}$ defined over $\mathbb{Q}$. Suppose $\mathcal{G}$ admits a Shimura datum $(\mathcal{G}, h)$ which satisfies the axiom 'SV5' (see [Mil05]). Let Y_G be the associated Shimura variety defined over its reflex field E. Choose an open compact subgroup $U^p \subset \mathcal{G}(\mathbb{A}_f^{(p)})$ such that for any open compact $U \subset G$ the product $U^pU \subset \mathcal{G}(\mathbb{A}_f)$ is neat, and let $\mathscr{V}$ be an étale sheaf on $Y_G(U^pU)$ induced from a $\mathbb{Z}_p$-linear algebraic representation V of $\mathcal{G}$. If we set $\Sigma = \{g \in G : gV \subset V\}$, and for any integers i, n, consider the functor

$$\begin{aligned} \mathcal{P}(G, \Sigma) &\to \mathbb{Z}_p\text{-mod} \\ U &\mapsto H^i_{et}(Y_G(U^pU), \mathscr{V}(n)), \end{aligned}$$

where for $g^{-1}Vg \subset U$, the pullback $[g]^*$ is defined as the composition

$$H^i_{et}(Y_G(U^pV), \mathscr{V}(n)) \to H^i_{et}(Y_G(U^pV), [g]^*\mathscr{V}(n)) \to H^i_{et}(Y_G(U^pU), \mathscr{V}(n)).$$

The above functor is then a Cartesian cohomology functor for (G, Σ) with coefficients in $\mathscr{V}$.

Given a morphism of Shimura data

$$(\mathcal{H}, h_H) \to (\mathcal{G}, h_G)$$

induced by a closed immersion of reductive groups $\mathcal{H} \hookrightarrow \mathcal{G}$, both satisfying Milne's axiom SV5 [8], there is a closed immersion of Shimura varieties $Y_{\mathcal{H}} \overset{\iota}{\hookrightarrow} Y_{\mathcal{G}}$, say of algebraic codimension d, which induces a pushforward map

$$\iota_* : H^i_{et}(Y_{\mathcal{H}}, \iota^*\mathscr{V}) \to H^{i+2d}_{et}(Y_{\mathcal{G}}, \mathscr{V}(d))$$

3.2.9 Eisenstein classes and Lemma-Flach elements

Let $\mathcal{S}_0(\mathbb{A}_f^2, \mathbb{Q})$ denote the space of Schwartz functions ϕ on $\mathbb{A}_f^2$ which satisfy $\phi(0,0) = 0$. This space has a right action of $\mathrm{GL}_2(\mathbb{A}_f)$ given by $A \cdot \phi(x, y) = \phi((x, y) \cdot A)$. Let $Y_{\mathrm{GL}_2} = \varprojlim_U Y_{\mathrm{GL}_2}(U)$ be the infinite level modular curve, let $\mathcal{H}^k_{\mathbb{Q}}$ be the motivic sheaf over Y associated to the highest weight representation $\mathrm{Sym}^k\mathbb{Q}^2$ of GL_2 and let $\mathscr{H}^k_{\mathbb{Z}_p}$ be the Lisse étale sheaf induced by $\mathrm{Sym}^k(\mathbb{Z}_p^2)$.

Theorem 3.2.17 (Beilinson)**.** *There is a* $\mathrm{GL}_2(\mathbb{A}_f)$*-equivariant map*

$$\mathcal{S}_0(\mathbb{A}_f^2, \mathbb{Q}) \to H^1_{\mathrm{mot}}(Y_{\mathrm{GL}_2}, \mathcal{H}^k_{\mathbb{Q}}(1))$$

written as $\phi \mapsto \mathrm{Eis}^k_{\mathrm{mot},\phi}$*, the motivic Eisenstein symbol.*

We write $\mathrm{Eis}^k_{\acute{e}t,\phi} = r_{\acute{e}t}(\mathrm{Eis}^k_{\mathrm{mot},\phi}) \in H^1_{\acute{e}t}(Y_{\mathrm{GL}_2}, \mathscr{H}^k_{\mathbb{Q}_p}(1))$, where $r_{\acute{e}t}$ is the *étale regulator*. We will need integral versions of the these classes. For an integer $c > 1$, let ${}_c\mathcal{S}((\mathbb{A}_f^{(p)} \times \mathbb{Z}_p)^2, \mathbb{Z}) \subset \mathcal{S}_0((\mathbb{A}_f)^2, \mathbb{Q})$ be the subspace of functions of the form $\phi^{(c)} \cdot \mathrm{ch}(\mathbb{Z}_c^2)$, where $\phi^{(c)}$ is a $\mathbb{Z}_p$-valued Schwartz function on $(\mathbb{A}_f^{(c)})^2$ such that $\phi_c(0,0) = 0$.

---

[8] '*G* has no $\mathbb{R}$-split torus in the centre which is not $\mathbb{Q}$-split'

Theorem 3.2.18 (Kings). *For a sufficiently small open compact subgroup $U \subset \mathrm{GL}_2(\mathbb{A}_f^{(pc)} \times \mathbb{Z}_p)$, and any c coprime to $6p$ there is a $\mathrm{GL}_2(\mathbb{A}_f^{(pc)} \times \mathbb{Z}_p)$-equivariant map*

$$ {}_c\mathcal{S}((\mathbb{A}_f^{(p)} \times \mathbb{Z}_p)^2, \mathbb{Z}_p)^U \to H^1_{ét}(Y(U), \mathscr{H}^k_{\mathbb{Z}_p}(1)), $$

written $\phi \mapsto {}_c\mathrm{Eis}^k_{ét,\phi}$, such that

$$ {}_c\mathrm{Eis}^k_{ét,\phi} = (c^2 - c^{-k}\left(\begin{smallmatrix} c & \\ & c \end{smallmatrix}\right)^{-1})\mathrm{Eis}^k_{ét,\phi}. $$

Recall we defined Shimura varieties $Y_{\mathrm{GL}_2 \times \mathrm{GL}_1}$ and Y_{GL_2} with a natural projection morphism

$$ Y_{\mathrm{GL}_2 \times \mathrm{GL}_1} \xrightarrow{q} Y_{\mathrm{GL}_2}. $$

Define a morphism of sheaves

$$ q^{\#} : \tilde{\mathscr{H}}^k_{\mathbb{Z}_p} \to q^* \mathscr{H}^k_{\mathbb{Z}_p} $$

to be the identity on $\mathscr{H}^k_{\mathbb{Z}_p}$ i.e. we ignore the GL_1-twist.

Definition 3.2.19. Define

$$ \tilde{\mathrm{Eis}}^k_{\mathrm{mot},\phi} = q^*(\mathrm{Eis}^k_{\mathrm{mot},\phi}) \in H^1_{\mathrm{mot}}(Y_{\mathrm{GL}_2 \times \mathrm{GL}_1}, \tilde{\mathscr{H}}^k_{\mathbb{Q}_p}(1)). $$

Definition 3.2.20. By taking the cup product of an Eisenstein symbol $\mathrm{Eis}^c_{\mathrm{mot},\phi_1}$ with $\tilde{\mathrm{Eis}}^d_{\mathrm{mot},\phi_2}$ there is an $\tilde{H}(\mathbb{A}_f)$-equivariant map

$$ \mathcal{S}_0(\mathbb{A}_f^2, \mathbb{Q})^{\otimes 2} \to H^2_{\mathrm{mot}}(Y_{\tilde{H}}, \mathscr{H}^{c,d}_{\mathbb{Q}_p}(2)). $$

Proposition 3.2.21. *For an appropriate choice of integers $c_1, c_2 > 1$ and a sufficiently small open compact subgroup $U \subset \tilde{H}(\mathbb{A}_f^{(pc)} \times \mathbb{Z}_{pc})$, there is a $\tilde{H}(\mathbb{A}_f^{(pc)} \times \mathbb{Z}_{pc})$-equivariant map*

$$ ({}_{c_1}\mathcal{S}((\mathbb{A}_f^{(p)} \times \mathbb{Z}_p)^2, \mathbb{Z}_p) \otimes {}_{c_2}\mathcal{S}((\mathbb{A}_f^{(p)} \times \mathbb{Z}_p)^2, \mathbb{Z}_p))^U \to H^2_{ét}(Y_{\tilde{H}}(U), \tilde{\mathscr{H}}^{i,j}_{\mathbb{Z}_p}(2)) $$

written $\underline{\phi} \mapsto {}_{c_1,c_2}\tilde{\mathrm{Eis}}^{i,j}_{ét,\underline{\phi}}$, where

$$ \left(c_1 - c_2^{-i}(\left(\begin{smallmatrix} c_1 & \\ & c_1 \end{smallmatrix}\right), \mathrm{id})^{-1}\right) \left(c_2 - c_1^{-j}(\mathrm{id}, \left(\begin{smallmatrix} c_2 & \\ & c_2 \end{smallmatrix}\right))^{-1}\right) \tilde{\mathrm{Eis}}^{i,j}_{ét,\underline{\phi}} = r_{et}({}_{c_1,c_2}\tilde{\mathrm{Eis}}^{i,j}_{\mathrm{mot},\underline{\phi}}). $$

We introduce level groups which will be useful in proving the norm relations at p.

Definition 3.2.22. Define level groups $U_m, V_m \subset \tilde{G}(\mathbb{A}_f)$ to be equal to $Q_{\tilde{G}}(M, p^m, p^n)$ away from p (suppressing the dependence on M) and at p let:

$$ \begin{aligned} (U_m)_p &= \{g \in \tilde{G}(\mathbb{Z}_p) : g \in \bar{Q}^0_{\tilde{G}} \bmod p^m, \alpha_B^{-m} g \alpha_B^m \in \tilde{G}(\mathbb{Z}_p)\} \\ (V_m)_p &= \alpha_B^m U_m \alpha_B^{-m}. \end{aligned} $$

As in [LSZ21] we choose the following integral test data

- An element $\xi_{U_m} \in \mathcal{H}(\tilde{G}(\mathbb{A}_f), \mathbb{Z})$ fixed by the right translation action of U_m,
- A subgroup $W \subset \tilde{H}(\mathbb{A}_f)$ such that for all x in the support of ξ_{U_m} we have $W \subset \tilde{H}(\mathbb{A}_f) \cap xU_m x^{-1}$,
- An element
$$ \phi_{U_m} \in ({}_{c_1}\mathcal{S}((\mathbb{A}_f^{(p)} \times \mathbb{Z}_p)^2, \mathbb{Z}_p) \otimes {}_{c_2}\mathcal{S}((\mathbb{A}_f^{(p)} \times \mathbb{Z}_p)^2, \mathbb{Z}_p)) $$
invariant under W.

Recall that S is the finite set of primes away from p at which $Q_{\tilde{G}}(M, p^m, p^n)$ is ramified. We define the above data as products

$$\xi_{U_m} = \operatorname{ch}(K_S) \otimes \bigotimes_{\ell \notin S} \xi_\ell, \;\; W = W_S \otimes \prod_{\ell} W_\ell, \;\; \phi_{U_m} = \phi_S \otimes \bigotimes_{\ell \notin S} \phi_\ell.$$

The data at primes in S or primes dividing M is not so important for our current applications and is chosen as in [LSZ21, Section 8.4.4]. At p we choose the following data:

- Set $\xi_p = \operatorname{ch}(uU_m)$
- Set $W_p := Q^{\circ}_{\tilde{H}_p}(M, p^m, p^m)$.
- Set ϕ_p to be the characteristic function of $(0,1) \cdot W_p$.

We choose our pair of integers $c_1, c_2 > 1$ such that the following conditions are satisfied:

- The c_i are coprime to $6p \prod_{\ell \in S} \ell$,
- Our chosen vector ϕ_S is preserved by the action of the elements $\left(\left(\begin{smallmatrix} c_i & \\ & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} c_i & \\ & 1 \end{smallmatrix}\right)^{-1}\right) \in (\mathrm{GL}_2 \times \mathrm{GL}_2)(\mathbb{Q}_S)$ (note that these elements are not in H),
- For each $\ell \in S$ the subgroup K_S is normalised by the elements $\begin{pmatrix} 1 & & & \\ & c_1 & & \\ & & 1 & \\ & & & c_1 \end{pmatrix}$ and $\begin{pmatrix} c_2 & & & \\ & 1 & & \\ & & c_2 & \\ & & & 1 \end{pmatrix}$.

Definition 3.2.23. Define a class

$${}_{c_1,c_2}\tilde{\mathscr{Z}}^{[a,b,q,r]}_{U_m} \in H^4_{\acute{e}t}(Y_{\tilde{G}}(U_m), \tilde{\mathscr{D}}^{a,b}_{\mathbb{Z}_p} \otimes \mu^q \sigma^{r-q})$$

in the following way: Write ξ_{U_m} as a finite $\mathbb{Z}$-linear sum of characteristic functions $\operatorname{ch}(x_i U_m)$. Then $W \subset \tilde{H}(\mathbb{Z}_p) \cap x_i U_m x_i^{-1}$ by definition and we define ${}_{c_1,c_2}\tilde{\mathscr{Z}}^{[a,b,q,r]}_{U_m}$ as the sum over the images of ${}_{c_1,c_2}\tilde{\mathrm{Eis}}^{c,d}_{\acute{e}t,\phi_{U_m}}$ under the pushforwards

$$\begin{aligned} H^2_{\acute{e}t}(Y_{\tilde{H}}(W), \tilde{\mathscr{H}}^{c,d}_{\mathbb{Z}_p}(2)) &\to H^2_{\acute{e}t}(Y_{\tilde{H}}(\tilde{H}(\mathbb{Z}_p) \cap x_i U_m x_i^{-1}), \tilde{\mathscr{H}}^{c,d}_{\mathbb{Z}_p}(2)) \\ &\xrightarrow{\mathrm{br}^{[a,b,q,r]}} H^2_{\acute{e}t}(Y_{\tilde{H}}(\tilde{H}(\mathbb{Z}_p) \cap x_i U_m x_i^{-1}), \iota^* \tilde{\mathscr{D}}^{a,b}_{\mathbb{Z}_p} \otimes \mu^q \sigma^{r-q}(2)) \\ &\xrightarrow{\iota_*} H^4_{\acute{e}t}(Y_{\tilde{G}}(x_i U_m x_i^{-1}), \tilde{\mathscr{D}}^{a,b}_{\mathbb{Z}_p} \otimes \mu^q \sigma^{r-q}(3)) \\ &\xrightarrow{[x_i]_*} H^4_{\acute{e}t}(Y_{\tilde{G}}(U_m), \tilde{\mathscr{D}}^{a,b}_{\mathbb{Z}_p} \otimes \mu^q \sigma^{r-q}(3)) \end{aligned}$$

for each x_i.

3.3 Norm relations in the p-direction

3.3.1 Norm relations for $m \geq 1$

We prove the p-direction norm relations for $m \geq 1$ using the theory of [Loe21].

As usual we define an open compact subgroup $U^p \subset \tilde{G}(\mathbb{A}_f^{(p)})$ by

$$U^p = U_S \times \prod_{\ell \notin S \cup \{p\}} \tilde{G}(\mathbb{Z}_\ell),$$

where the subgroup $U_S \subset \tilde{G}(\mathbb{Q}_\ell)$ is chosen such that the product $U^p U$ is sufficiently small. For K, U open compact subgroups of $\tilde{H}(\mathbb{Q}_p)$ and $\tilde{G}(\mathbb{Q}_p)$ respectively, we set

$$\begin{aligned} M_{\tilde{H}}(K) &:= H^2_{\acute{e}t}(Y_{\tilde{H}}(\iota^{-1}(U^p)K), \tilde{\mathscr{H}}^{c,d}_{\mathbb{Z}_p}) \\ M_{\tilde{G}}(U) &:= H^4_{\acute{e}t}(Y_{\tilde{G}}(U^p U), \tilde{\mathscr{D}}^{a,b}_{\mathbb{Z}_p} \otimes \mu^q \sigma^{r-q}). \end{aligned}$$

Let $\tilde{B}^{\mathrm{op}}_G$ denote the conjugate of $\tilde{B}_G$ under the long Weyl element. The mirabolic subgroup $Q^{\circ}_{\tilde{H}} \subset \tilde{H}$ can be easily seen to have dimension 4, equal to the dimension of the flag variety $\mathcal{F} := \tilde{G}/\tilde{B}^{\mathrm{op}}_G \cong G/B^{\mathrm{op}}_G$. In order the apply the machinery of [Loe21] we verify the following conditions hold:

1. There is an element $u \in \tilde{G}$ such that the stabiliser $\mathrm{Stab}_{Q^{\circ}_{\tilde{H}}}([u])$ is contained in $Q^{\circ}_{\tilde{H}} \cap u\tilde{T}_G u^{-1}$.
2. The $Q^{\circ}_{\tilde{H}}$ orbit of $[u]$ in $\mathcal{F}$ is open i.e. $(\tilde{G}, Q^{\circ}_{\tilde{H}})$ is a *spherical pair.*

Lemma 3.3.1. *The image of the element*

$$u = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \times (1) \in \tilde{G}(\mathbb{Z}_p)$$

in $\mathcal{F}(\mathbb{Z}_p)$ *has trivial* $Q^{\circ}_{\tilde{H}}$*-stabiliser and satisfies conditions 1 and 2 above.*

Proof. We have $Q^{\circ}_{\tilde{H}} \cap u\tilde{T}_G u^{-1} = \{1\}$ and that $\mathrm{Stab}_{Q^{\circ}_{\tilde{H}}}([u])$ is trivial follows from a routine computation. By [DA70, Théorème 10.1.2] the orbit map

$$\begin{aligned} Q^{0}_{\tilde{H}} &\to \mathcal{F} \\ q &\mapsto qu \end{aligned}$$

is a monomorphism and its set theoretic image is dense (for dimension reasons) and constructible and therefore open in $\mathcal{F}$. By comparing structure sheaves we see that this is an open immersion. ☐

Since $Q^{\circ}_{\tilde{H}} \cap u\tilde{T}_G u^{-1} = \{1\}$ we take $\bar{Q}^{0}_{\tilde{G}} = \bar{N}_{\tilde{G}}$.

Definition 3.3.2. Define a map

$$s_{m,*} = (s_m, s_m^{\#}) : H^4_{\acute{e}t}(Y_{\tilde{G}}(U_m), \tilde{\mathscr{D}}^{a,b}_{\mathbb{Z}_p} \otimes \mu^q \sigma^{r-q}) \to H^4_{\acute{e}t}(Y_{\tilde{G}}(V_m), \tilde{\mathscr{D}}^{a,b}_{\mathbb{Z}_p} \otimes \mu^q \sigma^{r-q})$$

to be the map induced by

$$s_m : Y_{\tilde{G}}(U_m) \xrightarrow{\alpha_B^m} Y_{\tilde{G}}(V_m)$$

and the morphism of sheaves

$$s_m^{\#} : \tilde{\mathscr{D}}^{a,b}_{\mathbb{Z}_p} \otimes \mu^q \sigma^{r-q} \to s_m^* \tilde{\mathscr{D}}^{a,b}_{\mathbb{Z}_p} \otimes \mu^q \sigma^{r-q}$$

induced by the action of α_B^{-m} on $\tilde{D}^{a,b}_{\mathbb{Z}_p}$ (independent of r, q).

Define

$$\tilde{\mathscr{Z}}^{[a,b,q,r]}_{V_m} := s_{m,*}\left(\tilde{\mathscr{Z}}^{[a,b,q,r]}_{U_m}\right)$$

Proposition 3.3.3. *The classes* $\tilde{\mathscr{Z}}^{[a,b,q,r]}_{V_n}$ *satisfy the following norm compatibility:*

$$\mathrm{pr}^{V_{m+1}}_{V_m}(\tilde{\mathscr{Z}}^{[a,b,q,r]}_{V_{m+1}}) = \tilde{\mathcal{U}}_B \cdot \tilde{\mathscr{Z}}^{[a,b,q,r]}_{V_m}$$

Proof. There are finitely many $k \in \mathbb{Z}$ and elements $x \in \tilde{G}(\mathbb{A}_f)$ such that

$${}_{c_1,c_2}\tilde{\mathscr{Z}}^{[a,b,q,r]}_{U_m} = \sum_k k([x]_* \circ \iota_* \circ \mathrm{br}^{[a,b,q,r]})({}_{c_1,c_2}\tilde{\mathrm{Eis}}^{c,d}_{\acute{e}t,\phi_{U_m}}).$$

The elements x all have p-part u and thus we can apply [Loe21, Theorem 4.5.4] to get the desired compatibility after applying $s_{m,*}$ ☐

For $n \leq m$ we have inclusions

$$V_m \subset Q_{\tilde{G}}(M, p^m, p^m) \subset Q_{\tilde{G}}(M, p^m, p^n). \tag{7}$$

Definition 3.3.4. For $n \leq m$ define classes

$$\tilde{z}^{[a,b,q,r]}_{\acute{e}t,M,m,n} \in H^4_{\acute{e}t}(Y_{\tilde{G}}(M, p^m, p^n), \tilde{\mathscr{D}}^{a,b}_{\mathbb{Z}_p} \otimes \mu\sigma^{r-q})$$

by pushing forward the classes $\tilde{\mathscr{Z}}^{[a,b,q,r]}_{V_m}$ along the inclusion (7).

Finally, we define classes

$$\tilde{z}^{[a,b,q,r]}_{\acute{e}t,M,m,n} \in H^4_{\acute{e}t}(Y_{\tilde{G}}(M,p^m,p^n), \mathscr{D}^{a,b}_{\mathbb{Z}_p} \otimes \mu\sigma^{r-q})$$

for all $m, n \geq 0$ by taking $n' \geq m$ and pushing forward along

$$Q_{\tilde{G}}(M,p^m,p^{n'}) \subset Q_{\tilde{G}}(M,p^m,p^n).$$

Proposition 3.3.5. *The classes* $\tilde{z}^{[a,b,q,r]}_{et,M,m,n}$ *satisfy the following norm relations:*

1. $(\mathrm{pr}^{n+1}_n)_*(\tilde{z}^{[a,b,q,r]}_{et,M,m,n+1}) = \tilde{z}^{[a,b,q,r]}_{et,M,m,n}$.
2. $(\mathrm{pr}^{m+1}_m)_*(\tilde{z}^{[a,b,q,r]}_{et,M,m+1,n}) = \tilde{\mathcal{U}}_B \tilde{z}^{[a,b,q,r]}_{et,M,m,n}, m \geq 1$.

Proof. The first relation is obvious and the second follows from the fact that for $m \geq 1$ both V_m and $Q_{\tilde{G}}(M,p^m,p^n)$ have Iwahori decompositions and so the coset representatives defining $\tilde{\mathcal{U}}_B$ are the same at both levels allowing us to apply the Cartesian axiom for cohomology functors. ☐

Remark 3.3.6. One can show that

$$(\mathrm{pr}^1_0)_*(\tilde{z}^{[a,b,q,r]}_{et,M,1,n}) = (\tilde{\mathcal{U}}_{\mathcal{K}} - p^r\{p\}\tilde{\mathcal{U}}_S)(\tilde{\mathcal{U}}_S - p^q)(\tilde{z}^{[a,b,q,r]}_{M,0,n})$$

by a careful analysis of many diagrams.

Let $\Delta^{(r)}_m = (\mathbb{Z}/Mp^m\mathbb{Z})^\times$ then there is an isomorphism of $G_{\mathbb{Q}}$-modules

$$H^i_{\acute{e}t}(Y_{\tilde{G}}(M,p^m,p^n), \tilde{\mathscr{D}}^{a,b} \otimes \mu^q\sigma^r) \cong H^i_{\acute{e}t}(Y_G(M,p^m,p^n), \mathscr{D}^{a,b}(-q)) \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[\Delta^{(r)}_m]$$

where the $G_{\mathbb{Q}}$-action on the group ring $\mathbb{Z}_p[\Delta^{(r)}_m]$ is trivial and the action of $(t,z) \in T_{\tilde{G}}(\mathbb{Z}_p) = T_G(\mathbb{Z}_p) \times \mathbb{Z}_p^\times$ is given for $[g] \in \Delta^{(r)}_m$ by

$$(t,z)[g] = z^{r+a}[zg].$$

We obtain classes

$$\tilde{z}^{[a,b,q,r]}_{\acute{e}t,M,m,n} \in H^4_{\acute{e}t}(Y_G(M,p^m,p^n), \mathscr{D}^{a,b}(-q)) \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[\Delta^{(r)}_m]$$

satisfying norm-compatibility in the p-direction.

3.4 Moment maps and p-adic interpolation

3.4.1 Fractured moment maps for $\tilde{H}$ and the über-branch

We rejig the moment maps in [LSZ21] to work for norm compatible classes in the Iwahori tower. Let Σ be a finite set of primes such that the prime to p level is unramified outside Σ, so that our Shimura varieties all have models over $\mathbb{Z}[\Sigma^{-1}]$. For $n \geq m \geq 0$ we introduce auxiliary level groups

$$Q'_{\tilde{G}}(M,p^m,p^n) = \{g \in U_m : g \equiv 1 \bmod p^n\},$$

writing $Y'_{\tilde{G}}(M,p^m,p^n)$ for the $\tilde{G}$-Shimura variety of level $Q'_{\tilde{G}}(M,p^m,p^n)$, and setting

$$\tilde{\mathscr{Z}}^{[a,b,q,r]}_{M,m,n} := \tilde{\mathscr{Z}}^{[a,b,q,r]}_{Q'_{\tilde{G}}(M,p^m,p^n)}.$$

We recall that our Euler system elements have the form

$$\tilde{\mathscr{Z}}^{[a,b,q,r]}_{M,m,n} = \sum a([x] \circ \iota_* \circ \mathrm{br}^{[a,b,q,r]})(\tilde{\mathrm{Eis}}^{c,d})$$

where all of the x have p-part u. For brevity reasons we prove our results assuming that

$$\tilde{\mathscr{Z}}^{[a,b,q,r]}_{m,n} = ([u] \circ \iota_* \circ \mathrm{br}^{[a,b,q,r]})(\tilde{\mathrm{Eis}}^{c,d}).$$

It will be reasonably easy to see that in the general case we can apply the results to each summand independently, making sure to choose the appropriate prime-to-p $\tilde{H}$-level group for each summand.

Lemma 3.4.1. *For any c, d, the mod p^n reduction of the vectors*

$$e^c \boxtimes f^d \in \tilde{H}^{c,d}_{\mathbb{Z}_p}$$

are invariant under $Q^{\circ}_{\tilde{H}}(p^m, p^n)$ for any m, n.

Proof. For all n, the vectors $e^c_n \boxtimes f^d_n$ are highest weight vectors for the maximal torus of H and so are invariant under $N_H(\mathbb{Z}_p)$. Since $Q^{\circ}_{\tilde{H}}(p^m, p^n)$ is upper triangular mod p^n it suffices to compute the action of the torus

$$\left(\begin{smallmatrix} x & \\ & 1 \end{smallmatrix}\right) \times \begin{pmatrix} xy & \\ & y^{-1} \end{pmatrix} \times (y^{-1})$$

which can easily be seen to fix the vector in question. $\square$

We thus obtain a canonical section $e^c_n \boxtimes f^d_n \in H^0_{et}(Y_{\tilde{H}}(p^m, p^n)_{\Sigma}, \tilde{\mathscr{H}}^{c,d}_n)$, where $\tilde{\mathscr{H}}^{c,d}_n$ is the mod p^n reduction of the sheaf $\tilde{\mathscr{H}}^{c,d}_{\mathbb{Z}_p}$.

Definition 3.4.2. Define moment maps

$$\mathrm{mom}^{c,d}_n : H^2_{\mathrm{Iw}}(Y_{\tilde{H}}(p^m, p^{\infty})_{\Sigma}, \mathbb{Z}_p(2)) \to H^2_{et}(Y_{\tilde{H}}(p^m, p^n), \tilde{\mathscr{H}}^{c,d}_{\mathbb{Z}_p}(2))$$

by restricting the image of

$$(z_s)_s \mapsto (\mathrm{pr}^s_n)_*(z_s \cup e^c_s \boxtimes f^d_s)_{s \geq n} \in \varprojlim_s H^2_{et}(Y_{\tilde{H}}(p^m, p^n)_{\Sigma}, \tilde{\mathscr{H}}^{c,d}_s) = H^2_{et}(Y_{\tilde{H}}(p^m, p^n)_{\Sigma}, \tilde{\mathscr{H}}^{c,d}_{\mathbb{Z}_p})$$

to the generic fibre.

Theorem 3.4.3. *There is a class*

$$\tilde{\mathcal{EI}}_m \in H^2_{\mathrm{Iw}}(Y_{\tilde{H}}(p^m, p^{\infty})_{\Sigma}, \mathbb{Z}_p(2)),$$

depending on some suppressed parameters, which satisfies

$$\mathrm{mom}^{c,d}_n(\tilde{\mathcal{EI}}_m) = \tilde{\mathrm{Eis}}^{c,d}_{m,n}.$$

Proof. Define

$$\tilde{\mathcal{EI}}_m = (\tilde{\mathrm{Eis}}^{0,0}_{m,s})_{s \geq 1}.$$

The result follows from [LSZ21, Theorem 9.1.4] and fact that the following diagram commutes for $s \geq n$

$$\begin{tikzcd}
H^1_{et}(Y_{\tilde{H}}(p^s), \mathscr{H}^{c,d}_s) \arrow[r, "\mathrm{pr}_*"] & H^1_{et}(Y_{\tilde{H}}(p^n), \mathscr{H}^{c,d}_s) \\
H^1_{et}(Y_H(p^s), \mathscr{H}^{c,d}_s) \arrow[r, "\mathrm{pr}_*"] \arrow[u, "q^*"] & H^1_{et}(Y_H(p^n), \mathscr{H}^{c,d}_s) \arrow[u, "q^*"']
\end{tikzcd}$$

which can be deduced from the Cartesian axiom of cohomology functors. $\square$

In order to interpolate in the r variable we will need an auxiliary moment map.

Definition 3.4.4. We define 'fractured moment maps'

$$\mathfrak{mom}^{[c,d,r]}_n : H^2_{\mathrm{Iw}}(Y_{\tilde{H}}(p^m, p^{\infty})_{\Sigma}, \mathscr{H}^{0,r}_{\mathbb{Z}_p}(2)) \to H^2_{et}(Y_{\tilde{H}}(p^m, p^n), \mathscr{H}^{c,d}_{\mathbb{Z}_p}(2))$$

to be the map given by restricting

$$(z_s)_s \mapsto (\mathrm{pr}^s_n)_*((z_s \cup e^c_s \boxtimes f^{d-r}_s)_{s \geq n}$$

to the generic fibre.

Lemma 3.4.5. *The map* $\mathfrak{mom}_n^{[c,d,r]}$ *fits into a diagram*

$$\begin{tikzcd}
H^2_{\mathrm{Iw}}(Y_{\tilde{H}}(p^m, p^\infty), \mathbb{Z}_p(2)) \arrow[r, "\ddot{U}_r"] \arrow[dr, "\mathrm{mom}_n^{c,d}"] & H^2_{\mathrm{Iw}}(Y_{\tilde{H}}(p^m, p^\infty), \mathscr{H}^{0,r}_{\mathbb{Z}_p}(2)) \arrow[d, "\mathfrak{mom}_n^{[c,d,r]}"] \\
 & H^2_{ét}(Y_{\tilde{H}}(p^m, p^n), \mathscr{H}^{c,d}_{\mathbb{Z}_p}(2))
\end{tikzcd}$$

where the top map $\ddot{U}_r$ *is given by taking the limit over* $z_s \mapsto z_s \cup \mathbb{1} \boxtimes f_s^r$.

Proof. This follows from the definition of the Cartan product. ☐

3.4.2 Moment maps for $\tilde{G}$

Lemma 3.4.6. *For all* a, b, *the mod* p^n *reduction of the vectors* $d_n^{[a,b,0,0]}$ *are invariant under* $Q^\circ_{\tilde{G}}(p^m, p^n)$ *for all* m, n.

Proof. We observe that $d^{[a,b,0,0]}$ are highest weight vectors for $\tilde{D}^{a,b}_{\mathbb{Z}_p}$ and thus are invariant under the unipotent radical N_G of the Borel $B_{\tilde{G}}$. Since $Q^\circ_{\tilde{G}}(p^m, p^n)$ is contained in $N_{\tilde{G}}(\mathbb{Z}_p)$ mod p^n we are done. ☐

Definition 3.4.7. For $0 \le q \le a, 0 \le r \le b$, define

$$\mathrm{mom}_n^{[a,b,q,r]} : H^4_{\mathrm{Iw}}(Y_{\tilde{G}}(p^m, p^\infty), \mathscr{D}^{q,r}_{\mathbb{Z}_p}(3) \otimes \mu^q \sigma^{r-q}) \to H^4_{ét}(Y_{\tilde{G}}(p^m, p^n), \mathscr{D}^{a,b}_{\mathbb{Z}_p}(3) \otimes \mu^q \sigma^{r-q})$$

by cup product with $d_s^{[a-q,b-r,0,0]}$ modulo p^s, projection to the nth level of the projective limit and lifting to the generic fibre.

Lemma 3.4.8. *The following diagram commutes:*

$$\begin{tikzcd}
H^2_{\mathrm{Iw}}(Y_{\tilde{H}}(p^m, p^\infty)_\Sigma, \mathscr{H}^{0,r}_{\mathbb{Z}_p}) \arrow[r, "{[u]_* \circ \iota_\infty \circ \mathrm{br}^{[q,r,q,r]}}"] \arrow[d, "\mathfrak{mom}_n^{[c,d,r]}"] & H^4_{\mathrm{Iw}}(Y_{\tilde{G}}(p^m, p^\infty)_\Sigma, \mathscr{D}^{q,r}_{\mathbb{Z}_p}(3) \otimes \mu^q \sigma^{r-q}) \arrow[d, "\mathrm{mom}_n^{[a,b,q,r]}"] \\
H^2_{et}(Y_{\tilde{H}}(p^m, p^n), \mathscr{H}^{c,d}_{\mathbb{Z}_p}) \arrow[r, "{[u]_* \circ \iota_{n,*} \circ \mathrm{br}^{[a,b,0,r]}}"] & H^4_{ét}(Y_{\tilde{G}}(p^m, p^n), \mathscr{D}^{a,b}_{\mathbb{Z}_p}(3) \otimes \mu^q \sigma^r).
\end{tikzcd}$$

Proof. This follows from the definition of the branching maps and the fact that the $d^{[a,b,q,r]}$ are compatible under the Cartan product. ☐

The above definitions and diagrams carry over perfectly well to the case of the dashed level groups $Q'_{\tilde{G}}(p^m, p^n)$. We define a class

$$\tilde{\mathscr{Z}}^{q,r}_m := \Big(u_* \circ \iota_{\infty,*} \circ \mathrm{br}^{[q,r,q,r]} \circ \ddot{U}_r(\tilde{\mathcal{EI}}_m)\Big) \in H^4_{\mathrm{Iw}}(Y'_{\tilde{G}}(p^m, p^\infty), \mathscr{D}^{q,r}_{\mathbb{Z}_p}(3) \otimes \mu^q \sigma^{r-q}).$$

Remark 3.4.9. In the general case this should of course be replaced by the sum

$$\tilde{\mathscr{Z}}^{q,r}_m = \sum a([x]_* \circ \iota_{\infty,*} \circ \mathrm{br}^{[q,r,q,r]} \circ \ddot{U}_r)(\tilde{\mathcal{EI}}_m)$$

for some $a \in \mathbb{Z}$.

Proposition 3.4.10. *The moment maps* $\mathrm{mom}_n^{[a,b,q,r]}$ *satisfy the following interpolation property:*

$$\mathrm{mom}_n^{[a,b,q,r]}(\tilde{\mathscr{Z}}^{q,r}_m) = \tilde{\mathscr{Z}}^{[a,b,q,r]}_{m,n}$$

for all $n \ge \max\{m, 1\}$.

Proof. Using the above lemma and [LSZ21, Proposition 9.3.1] we get

$$\begin{aligned}\mathrm{mom}_n^{[a,b,q,r]}(u_* \circ \iota_{\infty,*} \circ \mathrm{br}^{[q,r,q,r]}((\mathcal{EI}_{m,n} \cup f_n^r)_n)) &= u_* \circ \iota_{n,*} \circ \mathrm{br}^{[a,b,q,r]} \circ \mathfrak{mom}^{[c,d,r]}((\mathcal{EI}_{m,n} \cup f_n^r)_n) \\ &= u_* \circ \iota_{n,*} \circ \mathrm{br}^{[a,b,q,r]} \circ \mathrm{mom}_n^{c,d}(\mathcal{EI}_m) \\ &= \tilde{\mathscr{Z}}_{m,n}^{[a,b,q,r]},\end{aligned}$$

where we can pull out $[u]_*$ because it fixes the highest weight vectors $d^{[a-q,b-r,0,0]}$ and thus commutes with the moment maps. ☐

Lemma 3.4.11. *For $0 \leq q \leq a, 0 \leq r \leq b$, $d_m^{[q,r,q,r]}$ is invariant under $Q'_{\tilde{G}}(p^m, p^\infty)$.*

Proof. $Q'_{\tilde{G}}(p^m, p^\infty)$ is contained in the principal congruence subgroup of level p^m and is thus trivial mod p^m. ☐

Proposition 3.4.12. *We have*

$$\tilde{\mathscr{Z}}_m^{q,r} \equiv \tilde{\mathscr{Z}}_m^{0,0} \cup \left(u_* d_m^{[q,r,q,r]} \otimes \zeta_m^q \otimes \tau_m^{r-q}\right) \mod p^m,$$

where ζ_m, τ_m are choices of (multiplicative) basis elements for the mod p^m representations defined respectively by μ and σ.

Proof. Clear from the definitions. ☐

We now prepare ourselves to move back to our non-dashed level groups.

Definition 3.4.13. Define

$$\tilde{z}_m^{q,r} = s_{m,*}(\tilde{\mathscr{Z}}_m^{q,r}) \in H^4_{\mathrm{Iw}}(Y_{\tilde{G}}(p^m, p^\infty), \mathscr{D}_{\mathbb{Z}_p}^{q,r}(3) \otimes \mu^q \sigma^{r-q}).$$

Corollary 3.4.14. *We have the following interpolation property:*

$$\mathrm{mom}_n^{[a,b,q,r]}(\tilde{z}_m^{q,r}) = \tilde{z}_{\acute{e}t,m,n}^{[a,b,q,r]}$$

for all $a \geq q, b \geq r$.

Proof. This follows from the fact that α_B fixes the vectors $d^{[a-q,b-r,0,0]}$ and is identical to [LSZ21, Corollary 9.4.3]. ☐

3.4.3 Interpolating in q and r

We need the following technical lemma which is analagous to [LSZ21, Proposition 9.5.1]:

Lemma 3.4.15.

$$\tilde{z}_{\mathrm{Iw},m}^{q,r} = (-1)^{q+r} 2^q \tilde{z}_{\mathrm{Iw},m}^0 \cup (d_m^{[q,r,0,0]} \otimes \zeta_m^q \otimes \tau_m^{r-q}) \mod p^m.$$

Proof. Recall that, by definition, the morphisms of sheaves $s_m^\#$ ignores twists of $\mathscr{D}_{\mathbb{Z}_p}^{a,b}$. By the previous section we have

$$\tilde{\mathscr{Z}}_m^{q,r} = \tilde{\mathscr{Z}}_m^0 \cup (u_* d_m^{[q,r,q,r]} \otimes \zeta_m^q \otimes \tau_m^{r-q}) \mod p^m.$$

Since the torus $T''(x)$ has all weights ≤ 0 then α_B^{-m} acts through positive integral powers of p^m, which kill all weight subspaces of $\mathscr{D}_m^{q,r}$ except the 0-weight subspace. Thus it suffices to compute the projection of $u_* d^{[q,r,q,r]}$ to the weight 0 subspace, which is precisely the content of Lemma 3.2.5 when $h = -1$. ☐

Set

$$H^4_{\mathrm{Iw}}(Y_{\tilde{G}}(p^\infty, p^\infty), \mathbb{Z}_p(3)) = \varprojlim_m H^4_{\mathrm{Iw}}(Y_{\tilde{G}}(p^m, p^\infty), \mathbb{Z}_p(3))$$

and define the *Borel-ordinary projector*

$$\tilde{e}_{\mathrm{ord}} = \lim_{k\to\infty} \tilde{\mathcal{U}}_B^{k!},$$

which is well-defined on the above projective limit by Tilouine-Urban [TU99].

Definition 3.4.16. Set

$$\tilde{z}_{\mathrm{Iw}} := (\tilde{\mathcal{U}}_B^{-m} \cdot \tilde{e}_{\mathrm{ord}}(\tilde{z}^0_{\mathrm{Iw},m}))_{m\geq 1}$$

Definition 3.4.17. We define

$$\mathrm{mom}^{[a,b,q,r]}_{m,n} : H^4_{\mathrm{Iw}}(Y_{\tilde{G}}(p^\infty, p^\infty)_\Sigma, \mathbb{Z}_p(3)) \to H^4_{et}(Y_{\tilde{G}}(p^m, p^n), \mathscr{D}^{a,b}_{\mathbb{Z}_p}(3) \otimes \mu^q\sigma^{r-q})$$

by cup product with $d^{[a,b,0,0]} \otimes \zeta^q \otimes \tau^{r-q} \in H^0_{\mathrm{Iw}}(Y_{\tilde{G}}(p^\infty, p^\infty)_\Sigma, \mathscr{D}^{a,b}_{\mathbb{Z}_p} \otimes \mu^q\sigma^{r-q})$

Theorem 3.4.18. *For $m \geq 0, n \geq 1, 0 \leq q \leq a, 0 \leq r \leq b$ we have*

$$\mathrm{mom}^{[a,b,q,r]}_{m,n}(\tilde{z}_{\mathrm{Iw}}) = \frac{1}{(-1)^{q+r}2^q} \begin{cases} \tilde{\mathcal{U}}_B^{-m} \cdot \tilde{e}_{\mathrm{ord}}(\tilde{z}^{[a,b,q,r]}_{et,m,n}) & m \geq 1 \\ (1 - \frac{p^r\{p\}\tilde{\mathcal{U}}_S}{\tilde{\mathcal{U}}_{\mathcal{K}}})(1 - \frac{p^q}{\tilde{\mathcal{U}}_S})\tilde{e}_{\mathrm{ord}}(\tilde{z}^{[a,b,q,r]}_{et,m,n}) & m = 0. \end{cases}$$

Proof. Essentially identical to [LSZ21, Theorem 9.6.4]. □

We have an isomorphism of $\mathbb{Z}_p[[T_{\tilde{G}}(\mathbb{Z}_p)]][G_{\mathbb{Q}}]$-modules

$$H^i_{\mathrm{Iw}}(Y_{\tilde{G}}(M, p^\infty, p^\infty)_{\bar{\mathbb{Q}}}, \mathbb{Z}_p(3)) \cong H^4(Y_G(M, p^\infty, p^\infty)_{\bar{\mathbb{Q}}}, \mathbb{Z}_p(3)) \otimes \mathbb{Z}_p[[\Delta_\infty]]$$

where $\Delta_\infty \cong \mathbb{Z}_p^\times \times (\mathbb{Z}/M\mathbb{Z})^\times$ equipped with the trivial Galois action. Similarly there is an isomorphism of $\mathbb{Z}_p[[T_{\tilde{G}}(\mathbb{Z}_p)]]$-modules in integral étale cohomology:

$$H^i_{\mathrm{Iw}}(Y_{\tilde{G}}(M, p^\infty, p^\infty)_\Sigma, \mathbb{Z}_p(3)) \cong H^4(Y_G(M, p^\infty, p^\infty)_\Sigma, \mathbb{Z}_p(3)) \otimes \mathbb{Z}_p[[\Delta_\infty]].$$

This gives an interpretation of $\tilde{z}_{\mathrm{Iw}}$ as a measure on Δ_∞ valued in $H^4(Y_G(M, p^\infty, p^\infty)_\Sigma, \mathbb{Z}_p(3))$, where the moment map $\mathrm{mom}^{[a,b,q,r]}_m$ is given on $\mathbb{Z}_p[\Delta_\infty]$ for $\delta \in \Delta_\infty$ by

$$[\delta] \mapsto \delta^{r+a}[\delta \bmod p^m].$$

Remark 3.4.19. When $m = 0, n = 1$ the above map becomes $[\delta] \mapsto \delta^{r+a}$ which can be interpreted for $\mu \in \mathbb{Z}_p[[\Delta_\infty]]$ as an integral

$$\int_{\mathbb{Z}_p^\times} z^{r+a}\mu(z).$$

In this case the moment maps interpolate Iwahori level classes for G.

3.4.4 Classes in Galois cohomology

Consider the map

$$H^4_{\acute{e}t}(Y_G(M, p^m, p^n)_\Sigma, \mathscr{D}^{a,b}(3-q)) \otimes \mathbb{Z}_p[\Delta_m] \to H^0(\mathbb{Z}[\Sigma^{-1}], H^4(Y_G(M, p^m, p^n)_{\bar{\mathbb{Q}}}, \mathscr{D}^{a,b}(3-q)) \otimes \mathbb{Z}_p[\Delta_m]) \tag{8}$$

induced by the base change map $Y_{\tilde{G},\Sigma} \to Y_{\tilde{G},\bar{\mathbb{Q}}}$, where the Galois cohomology on the right is unramified outside of Σ. We call classes in the kernel of this map *cohomologically trivial* classes. As in the Siegel ordinary case [LSZ21, Corollary 9.6.6], we see that the Borel-ordinary classes are cohomologically trivial:

Lemma 3.4.20. *For $m \geq 1$ or $q, r \geq 1$ the classes $\tilde{e}_{\mathrm{ord}}({}_{c_1,c_2}\tilde{z}^{[a,b,q,r]}_{\acute{e}t,M,m,n})$ are cohomologically trivial.*

Proof. An application of Deligne reciprocity and Shaprio's lemma gives us

$$H^0(\mathbb{Q}, H^4_{\acute{e}t}(Y_{\tilde{G}}(M,p^m,p^n)_{\bar{\mathbb{Q}}}, \mathscr{D}^{a,b} \otimes \mu^q\sigma^{r-q})) = H^0(\mathbb{Q}(\zeta_{Mp^m}), H^4_{\acute{e}t}(Y_{\tilde{G}}(p^n)_{\bar{\mathbb{Q}}}, \tilde{\mathscr{D}}^{a,b} \otimes \mu^q\sigma^{r-q}))$$

and taking the inverse limit over m on gives $H^0(\mathbb{Q}, H^4_{\acute{e}t}(Y_{\tilde{G}}(M,p^\infty,p^n)_{\bar{\mathbb{Q}}}, \tilde{\mathscr{D}}^{a,b} \otimes \mu^q\sigma^{r-q})) = 0$ by [Nek06, Proposition 8.3.5] (using crucially that $H^4_{\acute{e}t}(Y_{\tilde{G}}(p^n)_{\bar{\mathbb{Q}}}, \tilde{\mathscr{D}}^{a,b} \otimes \mu^q\sigma^{r-q})$ is a finitely generated $\mathbb{Z}_p$-module). We thus immediately see from Proposition 3.3.5 that for $m \geq 1$ the class $\tilde{e}^{\mathrm{ord}}({}_{c_1,c_2}\tilde{z}^{[a,b,q,r]}_{\acute{e}t,M,m,n})$ maps to zero under the edge map (8) and if $m = 0$ and $q, r \geq 1$ the Euler factor $(1 - \frac{p^r\{p\}\tilde{\mathcal{U}}_S}{\tilde{\mathcal{U}}_{\mathcal{K}}})(1 - \frac{p^q}{\tilde{\mathcal{U}}_S})$ is invertible and the result follows. ☐

If we consider the Hochschild Serre spectral sequence for $Y_{\tilde{G}}$

$$\begin{aligned} &H^i(\mathbb{Z}[\Sigma^{-1}], H^j_{\acute{e}t}(Y_G(M,p^m,p^n)_{\bar{\mathbb{Q}}}, \mathscr{D}^{a,b}(3-q)) \otimes \mathbb{Z}_p[\Delta_m]) \\ &\qquad\qquad\qquad \Longrightarrow H^{i+j}_{\acute{e}t}(Y_G(M,p^m,p^n)_\Sigma, \tilde{\mathscr{D}}^{a,b}(3-q)) \otimes \mathbb{Z}_p[\Delta_m], \end{aligned}$$

we see that the cohomologically trivial classes in $H^4_{\acute{e}t}(Y_G(M,p^m,p^n)_\Sigma, \mathscr{D}^{a,b}(3-q)) \otimes \mathbb{Z}_p[\Delta_m]$ map into the Galois cohomology group $H^1(\mathbb{Z}[\Sigma^{-1}], H^3_{\acute{e}t}(Y_G(M,p^m,p^n)_{\bar{\mathbb{Q}}}, \mathscr{D}^{a,b}(3-q)) \otimes \mathbb{Z}_p[\Delta_m])$.

Remark 3.4.21. Since Iwasawa cohomology is automatically unramified outside p, the results of Lemma 3.4.20 hold for Galois cohomology with restricted ramification.

Let Π be a non-endoscopic, non-CAP cuspidal automorphic representation of $G(\mathbb{A}_{\mathbb{Q}})$, discrete series at infinity and of (cohomological) weight (k_1, k_2) with $k_1 \geq k_2 \geq 3$ and $(a, b) = (k_2 - 3, k_1 - k_2)$. Suppose further that Π_p is unramified and has a Borel-ordinary p-stabilisation. As in [LSZ21, Section 10], a choice of Iwahori-invariant $\mathcal{U}_B$ eigenvector for Π_p gives us a map

$$H^3_{\acute{e}t}(Y_{\tilde{G}}(M,p^m,p^n)_{\bar{\mathbb{Q}}}, \mathscr{D}^{a,b}_{\mathbb{Z}_p}(3) \otimes \mu^q\sigma^{r-q}) \to W_\Pi(-q) \otimes \mathbb{Z}_p[\Delta^{(r)}_m]$$

where W_Π is the Galois representation associated to Π by Taylor [Tay89] and Weissauer [Wei05].

Definition 3.4.22. We define

$$z^\Pi_m \in H^1(\mathbb{Q}(\zeta_{Mp^m}), W_\Pi(-q) \otimes \mathbb{Z}_p[\Delta^{(r)}_m])$$

to be the pushforward of $\tilde{e}_{\mathrm{ord}}({}_{c_1,c_2}z^{[a,b,q,r]}_{\acute{e}t,M,m,n})$.

We recall some results of Tilouine–Urban [TU99] on families of Siegel modular forms.

Definition 3.4.23. Write $\mathcal{W}$ for the rigid analytic *weight space* parameterising characters of $(\mathbb{Z}_p^\times)^2$.

The following definition is the Borel analogue of [LZ20b, Definition 17.1.2]

Definition 3.4.24. Let $\mathcal{U} \subset \mathcal{W}$ be an affinoid subspace containing 0. We define a *Borel-type Hida family* $\underline{\Pi}$ passing through (a, b) to be the data of:

- For each pair of non-negative integers $(m, n) \in \mathcal{U}$ a globally generic cuspidal automorphic representation Π', cohomological at infinity with coefficients in $V^{a+m,b+n}$ such that $\underline{\Pi}(m, n) = \Pi'$.
- An embedding of the coefficient field of $\Pi(m, n)$ into $\bar{\mathbb{Q}}_p$ with respect to which $\Pi(m, n)$ is Borel-ordinary.
- Rigid analytic functions $t_{i,\ell} \in \mathcal{O}(\mathcal{U})$ for $i = 1, 2$, $\ell \neq p$ such that for each $(m, n) \in \mathcal{U} \cap \mathbb{Z}^2_{\geq 0}$ the values of $t_{1,\ell}, t_{2,\ell}$ at (m, n) are given respectively by the eigenvalues of the spherical Hecke operators $\mathrm{diag}(\ell, \ell, 1, 1)$, $\ell^{-(a+m+b+n)}\mathrm{diag}(\ell^2, \ell, \ell, 1)$ on the 'arithmetic twist' $\underline{\Pi}(m, n)^{\mathrm{arith}} := \underline{\Pi}(m, n) \otimes \|\cdot\|^{-\frac{2a+b}{2}}$.
- Rigid analytic functions $u_{1,p}, u_{2,p}$ with $u_{1,p}u_{2,p}$ taking p-adic unit values such that for all $(m, n) \in \mathcal{U} \cap \mathbb{Z}^2_{\geq 0}$ we can write the Hecke-parameters of $\underline{\Pi}(m, n)^{\mathrm{arith}}$ as $(\alpha_{m,n}, \beta_{m,n}, \gamma_{m,n}, \delta_{m,n})$ with
$$u_{1,p}(m, n) = \alpha_{m,n}, u_{2,p}(m, n) = \frac{\beta_{m,n} + \gamma_{m,n}}{p^{a+m+b+n+1}}.$$

Theorem 3.4.25 (Tilouine–Urban). *Let* Π *be as above.*

- *There is a disc* $\mathcal{U} \subset \mathcal{W}$ *and a Borel-type Hida family* $\underline{\Pi}$ *over* $\mathcal{U}$ *such that* $\underline{\Pi}(0,0) = \Pi$.
- *After possibly shrinking* $\mathcal{U}$ *there is a free rank 4* $\mathcal{O}(\mathcal{U})$*-module* $W_{\underline{\Pi}}$ *whose fibre at* $(m,n) \in \mathcal{U} \cap \mathbb{Z}^2_{\geq 0}$ *gives the Galois representation* $W_{\Pi(m,n)}$ *associated to* $\Pi(m,n)$.

The Galois representation $W_{\underline{\Pi}}$ occurs as a direct summand in $H^3_{\acute{e}t}(Y_G(M,p^\infty)_{\bar{\mathbb{Q}}}, \mathbb{Z}_p(3))$ and so we can pushforward the classes $\tilde{z}_{\mathrm{Iw}}$ to get a class

$$\tilde{z}^{\Pi}_{\mathrm{Iw}} \in H^1_{\mathrm{Iw}}(\mathbb{Q}(\zeta_{Mp^\infty}), W_{\underline{\Pi}} \otimes \mathbb{Z}_p[[\Delta_\infty]])$$

interpolating the classes $\tilde{e}_{\mathrm{ord}}({}_{c_1,c_2}\tilde{z}_m^{\Pi(a,b),q,r})$ for $(a,b) \in \mathcal{U}$, $0 \leq q \leq a, 0 \leq r \leq b$ via the moment maps $\mathrm{mom}^{[a,b,q,r]}_{m,n}$.

4 Derived control theorems for reductive groups

4.1 Introduction

Fix a prime p. Since the 1980s, starting with the seminal work of Hida [Hid86], p-adic families of Hecke eigensystems have been an indispensible tool in arithmetic geometry. Control theorems allow us to isolate classical eigensystems using the action of the Hecke algebra. We prove control theorems for the ordinary arithmetic cohomology associated to a large class of reductive groups.

To be precise, let $\mathcal{G}$ be a connected reductive algebraic group over $\mathbb{Q}$ unramified over $\mathbb{Q}_p$ with Borel subgroup B_G, splitting field $K/\mathbb{Q}_p$ and reductive model G over $\mathbb{Z}_p$. Let Q_G be a parabolic subgroup of G with Levi decomposition $Q_G = L_G \times N_G$, where N_G is the unipotent radical of Q_G and L_G is the Levi subgroup. Let T_G be a maximal torus contained in Q_G and let $S_G = L_G^{\mathrm{der}} \backslash L_G$. Write $S_n(\mathbb{Z}_p) \subset S_G(\mathbb{Z}_p)$ for the subgroup of points which reduce to the identity mod p^n. Let $\chi \in X^\bullet(L_G)$, $\lambda \in X^\bullet(T_G)$ be characters such that λ is dominant for B_{L_G} and $\lambda + \chi$ is dominant for B_G and write $V_{\lambda+\chi}$ for the K-linear irreducible representation of G of highest weight λ and W_λ for the K-linear irreducible representation of L_G of highest weight λ. Write $V_{\lambda,\mathcal{O}_K}$ for the minimal admissible $\mathcal{O}_K$-lattice in $V_{\lambda+\chi}$ and $W_{\lambda,\mathcal{O}_K}$ for the minimal admissible lattice in W_λ. Let $\Gamma \subset G(\mathbb{Q}) \cap G(\hat{\mathbb{Z}})$ be a congruence subgroup of level prime to p, let $\Gamma_0(p^n)$ be the subgroup of points which reduce to $Q_G(\mathbb{Z}/p^n\mathbb{Z}) \mod p^n$ and let $\Gamma_1(p^n)$ be the subgroup of points which reduce to $N_G(\mathbb{Z}/p^n\mathbb{Z}) \mod p^n$.

We prove the following theorem:

Theorem 4.1.1. *For all* λ *as above there is a perfect complex* $M^\bullet_\lambda \in \mathscr{D}(\mathcal{O}_K[[S_G(\mathbb{Z}_p)]])$ *concentrated in degrees* $[0,\nu]$ *satisfying*

$$H^i(M^\bullet_\lambda) = \varprojlim_n H^i(\Gamma_1(p^n), W_{\lambda,\mathcal{O}_K}/p^n)^{\mathrm{ord}}$$

and for all χ *as above there is a quasi-isomorphism*

$$M^\bullet_\lambda \otimes^L_{\mathcal{O}_K[[S_n(\mathbb{Z}_p)]]} \mathcal{O}_K^{(\chi)} \sim R\Gamma(\Gamma_1(p^n), V_{\lambda+\chi,\mathcal{O}_K})^{\mathrm{ord}},$$

for $n \geq 1$ *and a quasi-isomorphism*

$$M^\bullet_\lambda \otimes^L_{\mathcal{O}_K[[S_G(\mathbb{Z}_p)]]} \mathcal{O}_K^{(\chi)} \sim R\Gamma(\Gamma_0(p^n), V_{\lambda+\chi,\mathcal{O}_K})^{\mathrm{ord}}$$

for $n = 0$.

The aim of this work is to provide a toolbox for those working with Euler systems varying in Hida families, such as those constructed in [KLZ17] and [LSZ21], and to act as a companion piece to forthcoming work of Loeffler–Zerbes in which they construct such interpolating classes for the same broad class of reductive groups with which we work.

We remark that there are many similar results in the literature, for example the work of Hida [Hid95] for SL_n and Tilouine-Urban [TU99] for GSp_4, albeit not in the derived setting. Indeed many of their proofs generalise readily to the general setting with only minor tweaks in order to work with complexes instead of cohomology groups and to account for changes in convention. The conventions in the aforementioned papers tend to differ greatly from those occurring in the literature on Euler systems and so we think it valuable, even in the existing cases, to have statements of these results with our conventions.

The layout of the paper is as follows:

- In Section 4.2 we fix the notations and conventions we will use for reductive groups, highest weight representations and interpolating modules.
- In Section 4.3 we prove the derived control theorem.
- In Section 4.4 we prove p-stabilisation and duality results.
- In Section 4.5 we deduce control results for 'adèlic cohomology'. We prove compatibility with the Hecke algebra $\mathbb{T}_{S,p}$ generated by the anemic Hecke algebra $\mathbb{T}_S$ and the U_p-operator and use this to prove a vanishing result for Iwasawa cohomology under the assumption that the Iwahori-level cohomology vanishes outside of the middle degree when localised at some maximal ideal of $\mathbb{T}_{S,p}$.

4.2 Notation

4.2.1 Algebraic groups and Iwasawa algebras

The setting:

- $\mathcal{G}$ is a connected reductive algebraic $\mathbb{Q}$-group, unramified over $\mathbb{Q}_p$ and satisfying Milne's axiom (SV5) i.e. the centre contains no $\mathbb{R}$-split torus which is not $\mathbb{Q}$-split. The group-scheme $\mathcal{G}/\mathbb{Q}_p$ thus splits over a finite unramified extension K of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}$ and admits a reductive group-scheme model G over $\mathbb{Z}_p$.
- Fix a choice of Borel subgroup and maximal torus $B_G \supset T_G$ defined over $\mathbb{Z}_p$.
- Fix a choice Q_G of standard parabolic subgroup of G with Levi factor L_G and unipotent radical N_G. Let L_G^{der} denote the derived subgroup of L_G. We write $\bar{Q}_G$ for the image of Q_G under the longest Weyl element.
- Let $S_G = L_G^{\mathrm{der}} \backslash L_G$ and let $\mathfrak{S}_G = L_G^{\mathrm{der}}(\mathbb{Z}_p) \backslash L_G(\mathbb{Z}_p) \subset S_G(\mathbb{Z}_p)$.

Let $\eta : \mathbb{G}_{m/\mathbb{Z}_p} \to Z(L_G)$ be a cocharacter which is strictly dominant with respect to Q_G in the sense that $\langle \eta, \Phi \rangle > 0$ for all relative roots Φ. Set $\tau = \eta(p)$. We then have

$$\bigcap_i \tau^{-i} \bar{N}_G(\mathbb{Z}_p) \tau^i = \{1\}.$$

Define

$$\begin{aligned} N_r &= \tau^r N_G(\mathbb{Z}_p) \tau^{-r}, \\ \bar{N}_r &= \tau^{-r} N_G(\mathbb{Z}_p) \tau^r, \\ L_r^{\mathrm{der}} &= \{\ell \in L_G(\mathbb{Z}_p) : \ell \bmod p^r \in L_G^{\mathrm{der}}(\mathbb{Z}/p^r\mathbb{Z})\} \end{aligned}$$

for $r \geq 1$ and set $L_0^{\mathrm{der}} = L_G$ for future notational convenience. Define open-compact subgroups

$$\begin{aligned} V_{0,r} &= \bar{N}_r L_G N_0 \\ V_{1,r} &= \bar{N}_r L_r^{\mathrm{der}} N_0. \end{aligned}$$

Fix a prime-to-p congruence subgroup $\Gamma \subset G(\hat{\mathbb{Z}})$ and let

$$\Gamma_{?,r} = V_{?,r} \cap \Gamma,$$

for $? \in \{0, 1\}$.

Define

$$\Lambda_0 := \mathcal{O}[[\mathfrak{S}_G]].$$

Let $\mathfrak{S}_r = \{s \in \mathfrak{S}_G : s \equiv 1 \bmod p^r\} = L_G^{\mathrm{der}}(\mathbb{Z}_p)\backslash L_r^{\mathrm{der}}$. This is a free $\mathbb{Z}_p$-module. Set

$$\Lambda_r := \mathcal{O}[[\mathfrak{S}_r]].$$

The ring Λ_0 decomposes into a direct sum

$$\Lambda_0 = \bigoplus_{\psi} \Lambda_1^{(\psi)},$$

where the sum runs over characters $\psi : \mathfrak{S}_1\backslash\mathfrak{S}_G \to \mathcal{O}^\times$ and $\Lambda_1^{(\psi)} = \Lambda_1$ with the action of $\mathfrak{S}_1\backslash\mathfrak{S}_G$ given by ψ.

Lemma 4.2.1. *Let M be a Λ_0-module. Suppose M is free as a Λ_1-module under the inclusion $\Lambda_1 \hookrightarrow \Lambda_0$. Then M is projective as a Λ_0-module.*

Proof. It suffices to prove that Λ_1 is a projective Λ_0-module, which is clear from the above decomposition. □

Given a character $\chi : \mathfrak{S}_r \to \mathcal{O}^\times$ we write

$$\chi^\dagger : \Lambda_r \to \mathcal{O}$$

for the induced homomorphism.

4.2.2 Chain complexes and Hecke algebras

Let R be a ring. For an arithmetic subgroup $\Gamma \subset G(\mathbb{Z})$ we can find a resolution $\mathscr{C}_\bullet(\Gamma)$ of R by finite free $R[\Gamma]$-modules [Urb11, Lemma 4.2.2]. Given a left $R[\Gamma]$-module M, we define a complex

$$\mathscr{C}^\bullet(\Gamma, M) := \mathrm{Hom}_{R[\Gamma]}(\mathscr{C}_\bullet(\Gamma), M),$$

satisfying $H^i(\mathscr{C}^\bullet(\Gamma, M)) = H^i(\Gamma, M)$. This of course depends on the choice of $\mathscr{C}_\bullet(\Gamma)$ but its image $R\Gamma(\Gamma, M)$ in the derived category does not.

Given groups Γ, Δ, a Γ-module M and Δ-module N, it is a standard fact from group cohomology (see e.g. [Urb11, 4.2.5]) that a pair (ϕ, f) consisting of a group homomorphism $\phi : \Gamma \to \Delta$ and a map of abelian groups $f : N \to M$ satisfying

$$f(\phi(\gamma)m) = \gamma f(m)$$

for all $n \in N$ and $\gamma \in \Gamma$ induce a natural map

$$\mathscr{C}^\bullet(\Delta, M) \to \mathscr{C}^\bullet(\Gamma, N)$$

Example 4.2.2.

- If $\iota : \Gamma \hookrightarrow \Delta$ and $M = N$ then we have the restriction map

$$\mathrm{res}_\Gamma^\Delta = (\iota, \mathrm{id})$$

- If $\alpha \in G(\mathbb{Q})$ acts on M, then define

$$[\alpha] = (\alpha(\cdot)\alpha^{-1}, \alpha(\cdot))$$

If $\Gamma' \subset \Gamma$ is a finite index subgroup, then $\mathscr{C}_\bullet(\Gamma)$ is also a resolution of R by finite free Γ'-modules so there is a unique homotopy equivalence $\delta : \mathscr{C}_\bullet(\Gamma) \to \mathscr{C}_\bullet(\Gamma')$ extending the identity. We define the corestriction map

$$\mathrm{cores}_{\Gamma'}^\Gamma : \mathrm{Hom}_{\Gamma'}(\mathscr{C}_\bullet(\Gamma'), M) \xrightarrow{\circ\delta} \mathrm{Hom}_{\Gamma'}(\mathscr{C}_\bullet(\Gamma), M) \xrightarrow{\sum \gamma_i} \mathrm{Hom}_{\Gamma}(\mathscr{C}_\bullet(\Gamma), M),$$

where γ_i is a full set of coset representatives for $\Gamma'\backslash\Gamma$.

Now suppose we have arithmetic groups $\Gamma, \Gamma' \subset \Gamma''$ and that M is an $R[\Gamma'']$-module with a compatible action of $\alpha \in G(\mathbb{Q})$. The double coset $\Gamma\alpha\Gamma'$ defines a map

$$\mathscr{C}^\bullet(\Gamma, M) \to \mathscr{C}^\bullet(\Gamma', M)$$

via

$$[\Gamma\alpha\Gamma'] = \mathrm{cores}^{\Gamma'}_{\Gamma'\cap\alpha^{-1}\Gamma\alpha} \circ [\alpha] \circ \mathrm{res}^{\Gamma}_{\alpha\Gamma'\alpha^{-1}\cap\Gamma}, \tag{9}$$

where we suppress the dependence on the homotopy δ.

Definition 4.2.3. Define

$$\mathcal{T} := [\Gamma\tau^{-1}\Gamma].$$

4.2.3 Ordinary subspaces

We perform a somewhat ad-hoc construction of the ordinary subspace. Let $(R, \mathfrak{m})$ be a local ring complete with respect to the $\mathfrak{m}$-adic topology. Suppose M is a topological $R[\Gamma]$-module with a compatible action of τ^{-1} and such that the action of $\mathcal{T}$ on $\mathscr{C}^\bullet(\Gamma, M)$ is continuous for the induced product topology. Suppose further that M is compact so that $\mathscr{C}^\bullet(\Gamma, M)$ is also compact. Then we can make sense of the ordinary part of $\mathscr{C}^\bullet(\Gamma, M)$:

$$\mathscr{C}^i(\Gamma, M)^{\mathrm{ord}} := \bigcap_{n\geq 0} \mathcal{T}^n\mathscr{C}^i(\Gamma, M).$$

All of the coefficient modules that we consider will satisfy these conditions.

Suppose we know that the quotients $\mathscr{C}^i(\Gamma, M)/\mathfrak{m}^n$ are finite $R/\mathfrak{m}^n$-modules. Then by results of Pilloni [Pil20] there is an idempotent $e = \lim_n \mathcal{T}^{n!}$ such that

$$e\mathscr{C}^i(\Gamma, M) = \mathscr{C}^i(\Gamma, M)^{\mathrm{ord}}.$$

4.2.4 Algebraic representations

Let $\lambda \in X^\bullet(T_G)$ be dominant with respect to the Borel $B_{L_G} := B_G \cap L_G$ of L_G. Define

$$\mathscr{C}^{L_G}_{\mathrm{alg}}(\lambda) := \{f \in \mathcal{O}[L_{G/\mathcal{O}}] : f(bx) = (-\lambda)(b)f(x) \ \forall\ b \in B_{L_G/\mathcal{O}}\},$$

an admissible $\mathcal{O}$-lattice (in the sense of [LSZ21, 4.2]) in the K-linear irreducible representation of $L_{G/K}$ of lowest weight $-\lambda$ with left L_G action given by right translation. Suppose $\chi \in X^\bullet(L_G)$ is such that $\lambda + \chi$ is dominant for G, and write

$$\begin{aligned}\mathscr{C}^G_{\mathrm{alg}}(\lambda+\chi) &= \{f \in \mathcal{O}[G_{/\mathcal{O}}] \otimes \mathscr{C}^{L_G}_{\mathrm{alg}}(\lambda) : f(qg) = (-\chi(q))qf(g) \ \forall\ q \in Q_{G/\mathcal{O}}\} \\ &\cong \{f \in \mathcal{O}[G_{/\mathcal{O}}] : f(bg) = (-\lambda-\chi)(b)f(g) \ \forall\ b \in B_{G/\mathcal{O}}\}\end{aligned}$$

an admissible $\mathcal{O}$-lattice in the irreducible representation of $G_{/K}$ of lowest weight $-(\lambda+\chi)$. The above isomorphism is given by mapping the $\mathscr{C}^{L_G}_{\mathrm{alg}}(\lambda)$ factor to K under the 'evaluation at 1' map.

Definition 4.2.4. Define

$$\begin{aligned}W_{\lambda,\mathcal{O}} &= \mathrm{Hom}_{\mathcal{O}}(\mathscr{C}^{L_G}_{\mathrm{alg}}(\lambda), \mathcal{O}) \\ V_{\lambda+\chi,\mathcal{O}} &= \mathrm{Hom}_{\mathcal{O}}(\mathscr{C}^{G}_{\mathrm{alg}}(\lambda+\chi), \mathcal{O}).\end{aligned}$$

given the structure of $L_{G/\mathcal{O}}$ (resp. $G_{/\mathcal{O}}$) representations via the contragredient representation. We note that $W_{\lambda,\mathcal{O}}$ is an admissible lattice in the K-linear representation of L_G of highest weight λ and $V_{\lambda+\chi,\mathcal{O}}$ is an admissible lattice in the K-linear highest weight representation of highest weight $\lambda+\chi$.

We define an action of τ on $V_{\lambda,\mathcal{O}}$ as follows: τ gives a well-defined map $V_{\lambda,\mathcal{O}} \otimes K \to V_{\lambda,\mathcal{O}} \otimes K$ and if we set $h_\lambda = \langle\eta, \lambda\rangle$, then $p^{h_\lambda}\tau^{-1}$ preserves the lattice $V_{\lambda,\mathcal{O}}$, so we let

$$\tau^{-1} * v = p^{h_\lambda}\tau^{-1}v.$$

Remark 4.2.5. This action corresponds to the action of τ on $\mathscr{C}^G_{\mathrm{alg}}(\lambda)$ given by restricting to the big Bruhat cell $\bar{N}_G L_G N_G$ and setting $(\tau * f)(\bar{n}\ell n) = f(\tau^{-1}\bar{n}\tau\ell n)$.

4.2.5 Distribution modules

We define the distribution modules which will serve as the coefficients for our interpolating complexes.

Definition 4.2.6. Define spaces

$$\begin{aligned} Y_r &:= L_r^{\mathrm{der}} N_0 \backslash V_{0,1} \cong \mathfrak{S}_r \backslash \mathfrak{S}_G \times \bar{N}_1, \\ Y_{\mathrm{univ}} &:= L_G^{\mathrm{der}} N_0 \backslash V_{0,1} \cong \mathfrak{S}_G \times \bar{N}_1. \end{aligned}$$

We extend the natural right action of $V_{0,1}$ on Y_r to an action of the monoid generated by $V_{0,1}$ and τ by letting

$$(\ell, n) * \tau = (\ell, \tau^{-1} n \tau).$$

Definition 4.2.7. Given a character $\lambda \in X^\bullet(T_G)$ dominant with respect to B_{L_G}, let

$$\begin{aligned} \mathscr{C}_r(\lambda) &= \{\text{Continuous } f : N_0 \backslash V_{0,1} \to \mathscr{C}_{\mathrm{alg}}^{L_G}(\lambda) : f(\ell x) = \ell f(x) \ \forall\ \ell \in L_r^{\mathrm{der}}\}, \\ \mathscr{C}_{\mathrm{univ}}(\lambda) &= \{\text{Continuous } f : N_0 \backslash V_{0,1} \to \mathscr{C}_{\mathrm{alg}}^{L_G}(\lambda) : f(\ell x) = \ell f(x) \ \forall\ \ell \in L_G^{\mathrm{der}}\} \end{aligned}$$

where the functions are continuous for the p-adic topologies on the source and target. These spaces are isomorphic to the spaces of continuous $\mathscr{C}_{\mathrm{alg}}^{L_G}(\lambda)$-valued functions on Y_r, Y_{univ} respectively via the map

$$\phi : f \mapsto \left(\ell \bar{n} \mapsto \ell^{-1} f(\ell \bar{n})\right). \tag{10}$$

We endow these spaces with an action of $V_{0,1}$ by right translation:

$$(g \cdot f)(x) = f(xg).$$

We define a twisted action of $\mathfrak{S}_G$ on $\mathscr{C}_{\mathrm{univ}}(\lambda)$:

$$(\ell * f)(x) = \ell^{-1} f(\ell x). \tag{11}$$

The isomorphism (10) is equivariant for this action if we give the target the natural left translation action of $\mathfrak{S}_G$.

Definition 4.2.8. Define modules of distributions

$$\begin{aligned} \mathbb{D}_r(\lambda) &= \mathrm{Hom}_{\mathcal{O},\mathrm{cont}}(\mathscr{C}_r(\lambda), \mathcal{O}), \\ \mathbb{D}_{\mathrm{univ}}(\lambda) &= \mathrm{Hom}_{\mathcal{O},\mathrm{cont}}(\mathscr{C}_{\mathrm{univ}}(\lambda), \mathcal{O}). \end{aligned}$$

Using the identification (10) we have isomorphisms of $\mathcal{O}$-modules:

$$\mathbb{D}_r(\lambda) \cong \mathcal{O}[[Y_r]] \otimes W_{\lambda,\mathcal{O}}, \tag{12}$$

$$\mathbb{D}_{\mathrm{univ}}(\lambda) \cong \mathcal{O}[[Y_{\mathrm{univ}}]] \otimes W_{\lambda,\mathcal{O}}, \tag{13}$$

from which we see $\mathbb{D}_{\mathrm{univ}}$ obtains a natural action of Λ_0 corresponding to the $*$-action (11) in the sense that for $\mu \in \mathbb{D}_{\mathrm{univ}}(\lambda)$, $f \in \mathcal{C}_{\mathrm{univ}}(\lambda)$ and $[s] \in \Lambda_0$ corresponding to $s \in \mathfrak{S}_G$:

$$\int_{N_0 \backslash V_{0,1}} f(x)([s] \cdot \mu)(x) = \int_{N_0 \backslash V_{0,1}} (s * f)(x) \mu(x).$$

Let $\chi \in X^\bullet(S_G)$ be a character such that $\lambda + \chi$ is dominant for G. There is a natural map

$$\mathbb{D}_{\mathrm{univ}}(\lambda) \to \mathbb{D}_r(\lambda + \chi)$$

factoring through $\mathbb{D}_{\mathrm{univ}}(\lambda) \otimes \mathcal{O}^{(\chi)}$, given by dualising the inclusion

$$\mathscr{C}_r(\lambda + \chi) \hookrightarrow \mathscr{C}_{\mathrm{univ}}(\lambda).$$

In our proof of the control theorem we will need a few finite modules.

Definition 4.2.9. Set $\mathcal{O}_s := \mathcal{O}/p^s$ and $\mathscr{C}^{L_G}_{\mathrm{alg}}(\lambda; p^s) := \mathscr{C}^{L_G}_{\mathrm{alg}}(\lambda) \otimes \mathcal{O}_s$. For $s \geq r$ we define the following $\mathcal{O}_s$-modules:

$$\begin{aligned}
\mathscr{C}_r(\lambda; p^s) &:= \{f : N_G(\mathbb{Z}/p^s\mathbb{Z})\backslash V_{0,1}(p^s) \to \mathscr{C}^{L_G}_{\mathrm{alg}}(\lambda; p^s) : f(\ell x) = \ell f(x)\ \forall \ell \in L_r^{\mathrm{der}}\} \\
\mathbb{D}_r(\lambda; p^s) &:= \mathrm{Hom}_{\mathcal{O}_s}(\mathscr{C}_r(\lambda; p^s), \mathcal{O}_s) \\
\tilde{\mathscr{C}}_r(\lambda; p^s) &:= \{f : L_G(\mathbb{Z}/p^s\mathbb{Z}) \to \mathscr{C}^{L_G}_{\mathrm{alg}}(\lambda; p^s) : f(\ell x) = \ell f(x)\ \forall \ell \in L_r^{\mathrm{der}}\} \\
\tilde{\mathbb{D}}_r(\lambda; p^s) &:= \mathrm{Hom}_{\mathcal{O}_s}(\tilde{\mathscr{C}}_r(\lambda; p^s), \mathcal{O}_s),
\end{aligned}$$

where $V_{0,1}(p^s) \subset G(\mathbb{Z}/p^s\mathbb{Z})$ is the mod p^s reduction of $V_{0,1}$. Note that we endow $\mathbb{D}_r(\lambda; p^s)$ with the action of $\Gamma_{0,1}$ corresponding to right translation of functions and give $\tilde{\mathbb{D}}_r(\lambda; p^s)$ an analogous action of $\Gamma_{0,s}$.

The utility of these modules is given by the fact that

$$\tilde{\mathbb{D}}_r(\lambda; p^s) = \mathrm{Ind}^{\Gamma_{0,s}}_{\Gamma_{1,r} \cap \Gamma_{0,s}} W_{\lambda,s},$$

and

$$\varprojlim_s \mathbb{D}_r(\lambda; p^s) = \mathbb{D}_r(\lambda).$$

4.3 Derived control

Let ν be the virtual cohomological dimension of G. For a commutative ring R let $\mathscr{D}(R)$ denote the derived category of R-modules.

Definition 4.3.1. A bounded complex of R-modules is called *perfect* if it consists of finite projective R-modules. We call an object $M \in \mathscr{D}(R)$ perfect if it can be lifted to a perfect complex of R-modules.

Write $\mathcal{O}^{(\chi)}$ for $\mathcal{O}$ with Λ_0-module structure given by $\chi^\dagger$. We prove the following theorem.

Theorem 4.3.2. *For each $\lambda \in X^\bullet(T_G)$ dominant for B_{L_G} there is a perfect complex $M^\bullet_\lambda \in \mathscr{D}(\Lambda_0)$ concentrated in degrees $[0, \nu]$ satisfying*

$$H^i(M^\bullet_\lambda) = \varprojlim_r H^i(\Gamma_{1,r}, W_{\lambda,\mathcal{O}} \otimes \mathcal{O}_r)^{\mathrm{ord}}$$

and for all $\chi \in X^\bullet(S_G)$ such that $\lambda + \chi$ is dominant for G there are quasi-isomorphisms

$$M^\bullet_\lambda \otimes^L_{\Lambda_r} \mathcal{O}^{(\chi)} \sim R\Gamma(\Gamma_{1,r}, V_{\lambda+\chi,\mathcal{O}})^{\mathrm{ord}}$$

for $r \geq 1$ and

$$M^\bullet_\lambda \otimes^L_{\Lambda_0} \mathcal{O}^{(\chi)} \sim R\Gamma(\Gamma_{0,1}, V_{\lambda+\chi,\mathcal{O}})^{\mathrm{ord}}$$

for $r = 0$.

Remark 4.3.3. This result corrects an error in the main result of [AS97] in which a small indexing mistake in the proof of Lemma 1.1 hides the contribution of some inscrutable Tor groups to the kernel of the specialisation map. Explicitly, it is stated in *op.cit* that for $G = \mathrm{GL}_n$ there is an injective map

$$H^i(\Gamma_{0,1}, \mathbb{D}_{\mathrm{univ}})^{\mathrm{ord}}/I_0^{(\lambda)} \hookrightarrow H^i(\Gamma_{0,1}, V_{\lambda,\mathcal{O}})^{\mathrm{ord}}$$

for all i. Our analysis shows that for this to happen it is necessary that the image of the Tor group $\mathrm{Tor}^{\Lambda_0}_{-2}(H^{i-1}(\Gamma_{0,1}, \mathbb{D}_{\mathrm{univ}})^{\mathrm{ord}}, \mathcal{O}^{(\lambda)})$ in $H^i(\Gamma_{0,1}, \mathbb{D}_{\mathrm{univ}})^{\mathrm{ord}}/I_0^{(\lambda)}$ vanishes. It seems to us that there is no *a priori* reason that this should be the case- it is not purely formal from the numerology. We describe in Section 4.5 some additional hypothesis that force the vanishing of the Tor groups which constitute the obstruction to the statement in *op. cit.*

We prove that there is a sequence of quasi-isomorphisms for $r \geq 0$:

$$\mathscr{C}^\bullet(\Gamma_{0,1}, \mathbb{D}_{\mathrm{univ}}(\lambda))^{\mathrm{ord}} \otimes_{\Lambda_r} \mathcal{O}^{(\chi)} \sim \mathscr{C}^\bullet(\Gamma_{0,1}, \mathbb{D}_r(\lambda + \chi))^{\mathrm{ord}} \sim \mathscr{C}^\bullet(\Gamma_{1,r}, V_{\lambda+\chi,\mathcal{O}})^{\mathrm{ord}}.$$

Definition 4.3.4. Let $\{s_i^{(r)}\}_i$ be a $\mathbb{Z}_p$-basis for $T_r(\mathbb{Z}_p)$. Given an algebraic character $\chi : S_G(\mathbb{Z}_p) \to \mathcal{O}^\times$ we write $I_r^{(\chi)}$ for the kernel of the induced homomorphism

$$\chi^\dagger : \Lambda_r \to \mathcal{O}.$$

It is generated by the regular sequence $([s_i^{(r)}] - \chi(s_i^{(r)}))_i$.

The first of the above sequence of quasi-isomorphisms is an immediate consequence of the following lemma.

Lemma 4.3.5. *The kernel of the map*

$$r_\chi^\lambda : \mathbb{D}_{\mathrm{univ}}(\lambda) \to \mathbb{D}_r(\lambda + \chi) \tag{14}$$

dualising the inclusion

$$\iota_\chi^\lambda : \mathcal{C}_r(\lambda + \chi) \to \mathcal{C}_{\mathrm{univ}}(\lambda) \tag{15}$$

is given by $I_r^{(\chi)}\mathbb{D}_{\mathrm{univ}}(\lambda)$.

Proof. For simplicity we assume that λ is trivial. In particular this means that the action (11) is just left translation and the isomorphism (10) is essentially trivial. The image of $\mathscr{C}_r(\chi)$ under (15) is given by the subspace of $\mathscr{C}_{\mathrm{univ}}(1)$ satisfying $(\ell \cdot f)(x) = \chi(\ell) f(x)$ for all $\ell \in L_r^{\mathrm{der}}$.

Note that $I_r^{(\chi)}$ is the kernel of the map

$$\mu \mapsto \int_{S_r} \chi(x)\mu(x),$$

which is merely a distribution-theoretic way of writing $\chi^\dagger$. There is a finite set $\Phi^r = \{\gamma_1, \dots, \gamma_m\}$ such that

$$\mathfrak{S}_G = \Phi^r \times \mathfrak{S}_r.$$

Suppose $\mu = \sum_{i \geq 0} \mu_{1,i} \otimes \mu_{2,i} \in \mathbb{D}_{\mathrm{univ}}(1) \cong \mathcal{O}[[\mathfrak{S}_G]] \hat{\otimes} \mathcal{O}[[\bar{N}_1]]$ and suppose without loss of generality that the set $\{\mu_{2,i}\}_{i \geq 0}$ is independent over $\mathcal{O}$ in the sense that if there are $c_i \in \mathcal{O}$ such that $\sum c_i \mu_{2,i} = 0$ then $c_i = 0$ for all i. Write

$$a_i := \int_{\mathfrak{S}_r} \chi(s)\mu_{1,i}(s)$$

Suppose $\mu \in \ker(r_\chi^1)$. We then have for $f \in \mathcal{C}_r(\chi)$:

$$\begin{aligned}
\int_{\mathfrak{S}_G \times \bar{N}_1} f(s\bar{n})\mu(s\bar{n}) &= \int_{\mathfrak{S}_r \times \Phi^r \times \bar{N}_1} f(s\phi\bar{n})\mu(s\phi\bar{n}) \\
&= \sum_j \int_{\mathfrak{S}_r \times \bar{N}_1} f(s\gamma_j\bar{n})\mu(s\bar{n}) \\
&= \sum_j \int_{\mathfrak{S}_r \times \bar{N}_1} \chi(s) f(\gamma_j\bar{n})\mu(s, \bar{n}) \\
&= \sum_i \sum_j \int_{\mathfrak{S}_r} \chi(s)\mu_{1,i}(s) \int_{\bar{N}_1} f(\gamma_j\bar{n})\mu_{2,i}(\bar{n}) \\
&= \sum_i a_i \int_{\Phi^r \times \bar{N}_1} f(x)\mu_{2,i}(x) \\
&= \int_{\Phi^r \times \bar{N}_1} f(x) \left(\sum_i a_i \mu_{2,i} \right)(x) \\
&= 0
\end{aligned}$$

Since restriction of continuous functions in $\mathcal{C}_r(\chi)$ to $\Phi^r \times \bar{N}_1$ is an isomorphism, this implies that $\sum_i a_i \mu_{2,i} = 0$ and by our assumption on the independence of the distributions $\mu_{2,i}$ we get that

$$a_i = \int_{\mathfrak{S}_r} \chi(s)\mu_{1,i}(s) = 0$$

for all i as required.

$\square$

The second quasi-isomorphism follows from the following few lemmas. The next result is a variation of a lemma which appears frequently in papers on Hida theory, for example [Hid95, Proposition 4.1], [TU99, Lemma 3.1], and shall be henceforth known as the 'Hida lemma'.

Lemma 4.3.6. *For $s \geq r$ let M be a compact $\Gamma_{1,r}$-module with a compatible action of τ^{-1}. The following diagram commutes on cohomology*

$$\begin{tikzcd}
\mathscr{C}^\bullet(\Gamma_{1,r} \cap \Gamma_{0,s}, M) \arrow[r, "\text{cores}"] \arrow[d, "\mathcal{T}^{s-r}"] & \mathscr{C}^\bullet(\Gamma_{1,r}, M) \arrow[d, "\mathcal{T}^{s-r}"] \arrow[dl, "\tau^{-(s-r)}"'] \\
\mathscr{C}^\bullet(\Gamma_{1,r} \cap \Gamma_{0,s}, M) \arrow[r, "\text{cores}"'] & \mathscr{C}^\bullet(\Gamma_{1,r}, M)
\end{tikzcd}$$

Proof. We do the proof for the top triangle, the bottom triangle being similar. We first note that

$$(\Gamma_{1,r} \cap \Gamma_{0,s}) \cap \tau^{-(s-r)}(\Gamma_{1,r} \cap \Gamma_{0,s})\tau^{s-r} = \bar{N}_{2s-r} L_r N_0 \cap G(\mathbb{Q})$$

so that when computing $\mathcal{T}^{s-r}$ our corestriction will sum over representatives for $\bar{N}_s/\bar{N}_{2s-r}$. Then

$$\Gamma_{1,r}/\Gamma_{1,r} \cap \Gamma_{0,s} = \bar{N}_r/\bar{N}_s,$$

and so if γ_i' is a set of representatives for $\Gamma_{1,r}/\Gamma_{1,r} \cap \Gamma_{0,s}$, then $\gamma_i := \tau^{-(s-r)}\gamma_i'\tau^{s-r}$ is a set of representatives for $\bar{N}_s/\bar{N}_{2s-r}$.

The Hecke operator $\mathcal{T}$ is given on complexes by the composition of pairs

$$(\delta_1, \sum_i \gamma_i) \circ (\tau^{-1}(\cdot)\tau, \tau^{-1}) \circ (\iota, \mathrm{id}) = (\tau^{-1}\delta_1(\cdot)\tau, \sum \gamma_i \tau^{-1})$$

where δ_1 is the canonical homotopy equivalence

$$\mathscr{C}_\bullet((\Gamma_{1,r} \cap \Gamma_{0,s}) \cap \tau(\Gamma_{1,r} \cap \Gamma_{0,s})\tau^{-1}) \to \mathscr{C}_\bullet(\Gamma_{1,r} \cap \Gamma_{0,s})$$

extending the identity and we abuse notation by writing corestriction as a compatible pair, and $\tau^{-1} \circ$ cores is given by

$$\begin{aligned}(\tau^{-1}(\cdot)\tau, \tau^{-1}) \circ (\delta_2, \sum \gamma_i') &= (\delta_2(\tau^{-1}(\cdot)\tau), \sum \tau^{-1}\gamma_i') \\ &= (\delta_2(\tau^{-1}(\cdot)\tau), \sum \gamma_i \tau^{-1}),\end{aligned}$$

where δ_2 is the canonical homotopy equivalence of $\Gamma_{1,r} \cap \Gamma_{0,s}$-complexes

$$\mathscr{C}_\bullet(\Gamma_{1,r} \cap \Gamma_{0,s}) \to \mathscr{C}_\bullet(\Gamma_{1,r}).$$

The maps δ_i induce the identity on cohomology so the result follows. $\square$

The upshot of this lemma is that the restriction of the above corestriction map to the ordinary subspace is a quasi-isomorphism, so we have a quasi-isomorphism

$$\mathscr{C}^\bullet(\Gamma_{1,r} \cap \Gamma_{0,s}, M)^{\mathrm{ord}} \cong \mathscr{C}^\bullet(\Gamma_{1,r}, M)^{\mathrm{ord}}.$$

The following proposition will allow us to attack our problem using the Hida lemma in conjunction with Shapiro's lemma by reducing to the case of twisting by characters. Let $V_{\lambda+\chi,s} := V_{\lambda+\chi,\mathcal{O}} \otimes \mathcal{O}_s$, $W_{\lambda,s} := W_{\lambda,\mathcal{O}} \otimes \mathcal{O}_s$.

Proposition 4.3.7. *Let $s \geq r$. There are isomorphisms*

$$\mathscr{C}^\bullet(\Gamma_{1,r} \cap \Gamma_{0,s}, V_{\lambda+\chi,s})^{\mathrm{ord}} \cong \mathscr{C}^\bullet(\Gamma_{1,r} \cap \Gamma_{0,s}, W_{\lambda,s}(\chi))^{\mathrm{ord}}$$
$$\mathscr{C}^\bullet(\Gamma_{0,s}, \mathbb{D}_r(\lambda; p^s))^{\mathrm{ord}} \cong \mathscr{C}^\bullet(\Gamma_{0,s}, \tilde{\mathbb{D}}_r(\lambda; p^s))^{\mathrm{ord}}$$

Proof. We prove the first isomorphism, the second being not dissimilar. Consider the map

$$\mathscr{C}^\bullet(\Gamma_{1,r}\cap\Gamma_{0,s}, V_{\lambda+\chi,s})^{\mathrm{ord}} \to \mathscr{C}^\bullet(\Gamma_{1,r}\cap\Gamma_{0,s}, W_{\lambda,s}(\chi))^{\mathrm{ord}}$$

induced by the dual of the inclusion of L_G-modules

$$\mathscr{C}^{L_G}_{\mathrm{alg}}(\lambda;p^s)(\chi^{-1}) \hookrightarrow \mathscr{C}^{G}_{\mathrm{alg}}(\lambda+\chi;p^s).$$

The image of $\mathscr{C}^{L_G}_{\mathrm{alg}}(\lambda;p^s)(\chi^{-1})$ in $\mathscr{C}^{G}_{\mathrm{alg}}(\lambda+\chi;p^s)$ under the above inclusion is given by $\mathscr{C}^{G}_{\mathrm{alg}}(\lambda+\chi;p^s)^{\bar{N}_G(\mathbb{Z}_p)}$. When L_G is a maximal torus this is just the inclusion of the lowest weight subspace. The kernel $\mathcal{K}_s \subset V_{\lambda+\chi,s}$ of the dual map is given by functionals

$$\mathscr{C}^{G}_{\mathrm{alg}}(\lambda+\chi;p^s) \to \mathcal{O}_s$$

whose restriction to $\mathscr{C}^{L_G}_{\mathrm{alg}}(\lambda;p^s)(\chi^{-1})$ is zero. We want to show that

$$\mathscr{C}^i(\Gamma_{1,r}\cap\Gamma_{0,s},\mathcal{K}_s)^{\mathrm{ord}} = 0.$$

Let $z \in \mathscr{C}^i(\Gamma_{1,r}\cap\Gamma_{0,s},\mathcal{K}_s)$, then for $c \in \mathscr{C}_i(\Gamma)$ there are functionals $\phi_i^{(c)} \in \mathcal{K}_s$ such that

$$(\mathcal{T}^k z)(c) = \sum \gamma_i \tau^{-k}\cdot \phi_i^{(c)},$$

so via the contragredient representation it suffices to show that if $f \in \mathscr{C}^{G}_{\mathrm{alg}}(\lambda+\chi;p^s)$ and $k \gg 0$ then $\tau^k \cdot f$ is $\bar{N}_G(\mathbb{Z}_p)$-invariant. Since $\tau^{-k}\bar{N}_0\tau^k = \bar{N}_k$, then for $k \geq s$

$$\tau^{-k}\bar{N}_0\tau^k \equiv 1 \bmod p^s,$$

whence we are done. ☐

Lemma 4.3.8. *For* $r \geq 1$ *there is a quasi-isomorphism*

$$\mathscr{C}^\bullet(\Gamma_{0,1},\mathbb{D}_r(\lambda+\chi))^{\mathrm{ord}} \cong \mathscr{C}^\bullet(\Gamma_{1,r}, V_{\lambda+\chi,\mathcal{O}})^{\mathrm{ord}}.$$

Proof. We have the following chain of quasi-isomorphism

$$\begin{aligned}
\mathscr{C}^\bullet(\Gamma_{0,1},\mathbb{D}_r(\lambda+\chi))^{\mathrm{ord}} &\cong \varprojlim_s \mathscr{C}^\bullet(\Gamma_{0,1},\mathbb{D}_r(\lambda+\chi;p^s))^{\mathrm{ord}}\\
&\cong \varprojlim_s \mathscr{C}^\bullet(\Gamma_{0,s},\mathbb{D}_r(\lambda+\chi;p^s))^{\mathrm{ord}}\\
&\cong \varprojlim_s \mathscr{C}^\bullet(\Gamma_{0,s},\tilde{\mathbb{D}}_r(\lambda+\chi;p^s))^{\mathrm{ord}}\\
&\cong \varprojlim_s \mathscr{C}^\bullet(\Gamma_{1,r}\cap\Gamma_{0,s},W_{\lambda,s}(\chi))^{\mathrm{ord}}\\
&\cong \varprojlim_s \mathscr{C}^\bullet(\Gamma_{1,r}\cap\Gamma_{0,s},V_{\lambda+\chi,s})^{\mathrm{ord}}\\
&\cong \varprojlim_s \mathscr{C}^\bullet(\Gamma_{1,r},V_{\lambda+\chi,s})^{\mathrm{ord}}\\
&\cong \mathscr{C}^\bullet(\Gamma_{1,r},V_{\lambda+\chi,\mathcal{O}})^{\mathrm{ord}},
\end{aligned}$$

where the second line is the Hida lemma, the fourth line is Shapiro's lemma and the sixth line is the Hida lemma again. ☐

The case $r = 0$ has the same proof with the Shapiro's lemma step being trivial.

Corollary 4.3.9. *There is a quasi-isomorphism*

$$\mathscr{C}^\bullet(\Gamma_{0,1},\mathbb{D}_{\mathrm{univ}}(\lambda))^{\mathrm{ord}} \cong \varprojlim_r \mathscr{C}^\bullet(\Gamma_{1,r},W_{\lambda,r})^{\mathrm{ord}}.$$

Proof. Note that $\mathbb{D}_{\mathrm{univ}}(\lambda) = \varprojlim_r \mathbb{D}_r(\lambda) = \varprojlim_{r,s} \mathbb{D}_r(\lambda; p^s)$. The previous lemma gives us that

$$\mathscr{C}^\bullet(\Gamma_{0,1}, \mathbb{D}_r(\lambda; p^s))^{\mathrm{ord}} = \mathscr{C}^\bullet(\Gamma_{1,r} \cap \Gamma_{0,s}, W_{\lambda,s})^{\mathrm{ord}}$$

and the result follows from setting $s = r$ and taking the inverse limit over r on both sides. $\square$

We write

$$\mathscr{C}^i(\Gamma_{1,\infty}, W_{\lambda,\mathcal{O}})^{\mathrm{ord}} := \varprojlim_r \mathscr{C}^i(\Gamma_{1,r}, W_{\lambda,r})^{\mathrm{ord}}$$

and similarly for cohomology.

4.3.1 Perfection

Lemma 4.3.10. *Let N be a Λ_r-module with trivial $\Gamma_{0,1}$-action. Then*

$$\mathscr{C}^\bullet(\Gamma_{0,1}, \mathbb{D}_{\mathrm{univ}}(\lambda))^{\mathrm{ord}} \otimes_{\Lambda_r} N \cong \mathscr{C}^\bullet(\Gamma_{0,1}, \mathbb{D}_{\mathrm{univ}}(\lambda) \otimes_{\Lambda_r} N)^{\mathrm{ord}}$$

Proof. There's an integer n such that

$$\mathscr{C}^\bullet(\Gamma_{0,1}, \mathbb{D}_{\mathrm{univ}}(\lambda)) \cong \oplus_{i=1}^n \mathbb{D}_{\mathrm{univ}}(\lambda)$$

as $\Lambda_r[\Gamma_{0,1}]$-modules. The result follows since the Λ_r-action is continuous and commutes with the Hecke action. $\square$

Lemma 4.3.11. *The Λ_r-regular sequence $([s_i^{(r)}] - \chi(s_i^{(r)}))_i$ generating $I_r^{(\chi)}$ is $\mathbb{D}_{\mathrm{univ}}(\lambda)$-regular for any λ.*

Proof. We reduce, using the isomorphism (13), to showing that the given sequence is regular in Λ_0, which is visibly the case. $\square$

Proposition 4.3.12. *For any choice of $\mathscr{C}_\bullet(\Gamma_{0,1})$, the modules $\mathscr{C}^\bullet(\Gamma_{0,1}, \mathbb{D}_{\mathrm{univ}}(\lambda))^{\mathrm{ord}}$ are finite flat Λ_1-modules.*

Proof. Let $\mathfrak{m}_1$ denote the maximal ideal of Λ_1. There is an exact sequence

$$0 \to \mathfrak{m}_1\mathscr{C}^\bullet(\Gamma_{0,1}, \mathbb{D}_{\mathrm{univ}}(\lambda))^{\mathrm{ord}} \to \mathscr{C}^\bullet(\Gamma_{0,1}, \mathbb{D}_{\mathrm{univ}}(\lambda))^{\mathrm{ord}} \to \mathscr{C}^\bullet(\Gamma_{0,1}, \mathbb{D}_1(\lambda)/p)^{\mathrm{ord}} \to 0.$$

In a similar way to Proposition 4.3.7 we can show that

$$\mathscr{C}^\bullet(\Gamma_{0,1}, \mathbb{D}_1(\lambda)/p)^{\mathrm{ord}} \cong \mathscr{C}^\bullet(\Gamma_{0,1}, (\mathcal{O}/p)[\mathfrak{S}_1 \backslash \mathfrak{S}_G])^{\mathrm{ord}}$$

and thus that it is finite. Thus we can apply Nakayama's lemma to conclude that $\mathscr{C}^\bullet(\Gamma_{0,1}, \mathbb{D}_{\mathrm{univ}}(\lambda))^{\mathrm{ord}}$ is a complex of finitely generated Λ_1-modules. The ideal $\mathfrak{m}_1$ is generated by a regular sequence $(p, x_1, \ldots, x_n)$, and $\Lambda_1/\mathfrak{m}_1 \cong \mathbb{F}_{p^k}$ for $k = [K : \mathbb{Q}_p]$. By the local criterion for flatness as stated in [Eis13, theorem 6.8], it suffices to show that

$$\mathrm{Tor}_1^{\Lambda_1}(\mathscr{C}^*(\Gamma_{0,1}, \mathbb{D}_{\mathrm{univ}}(\lambda))^{\mathrm{ord}}, \mathbb{F}_{p^k}) = 0.$$

This group is computed by the Koszul complex for $(p, x_1, \ldots, x_m)$ tensored with $\mathscr{C}^*(\Gamma_{0,1}, \mathbb{D}_{\mathrm{univ}}(\lambda))^{\mathrm{ord}}$, and thus it suffices to prove that the above sequence is $\mathscr{C}^*(\Gamma_{0,1}, \mathbb{D}_{\mathrm{univ}}(\lambda))^{\mathrm{ord}}$-regular. This follows from the fact that this sequence is $\mathbb{D}_{\mathrm{univ}}(\lambda)$-regular and Lemma 4.3.10. $\square$

Corollary 4.3.13. *The complex $\mathscr{C}^\bullet(\Gamma_{0,1}, \mathbb{D}_{\mathrm{univ}}(\lambda))^{\mathrm{ord}}$ is a perfect complex of Λ_0-modules concentrated in degrees $[0, \nu]$.*

Proof. By the above lemmas the modules $\mathscr{C}^i(\Gamma_{0,1}, \mathbb{D}_{\mathrm{univ}}(\lambda))^{\mathrm{ord}}$ are finite flat over the local ring Λ_1 and are thus free. We are done by Lemma 4.2.1. $\square$

Definition 4.3.14. Write $M_\lambda^\bullet$ for the image of $\mathscr{C}^\bullet(\Gamma_{0,1}, \mathbb{D}_{\mathrm{univ}}(\lambda))^{\mathrm{ord}}$ in $\mathcal{D}(\Lambda_0)$, the derived category of Λ_0-modules.

Proof. (of Theorem 4.3.2) By Proposition 4.3.12 we have

$$M_\lambda^\bullet \otimes^L_{\Lambda_r} \mathcal{O}^{(x)} \sim \mathscr{C}^\bullet(\Gamma_{0,1}, \mathbb{D}_{\mathrm{univ}}(\lambda))^{\mathrm{ord}} \otimes_{\Lambda_r} \mathcal{O}^{(x)}$$

by Lemma 4.3.5 we have

$$M_\lambda^\bullet \otimes^L_{\Lambda_r} \mathcal{O}^{(x)} \sim \mathscr{C}^\bullet(\Gamma_{0,1}, \mathbb{D}_r(\lambda))^{\mathrm{ord}}$$

and by Lemma 4.3.8

$$M_\lambda^\bullet \otimes^L_{\Lambda_r} \mathcal{O}^{(x)} \sim \mathscr{C}^\bullet(\Gamma_{1,r}, V_{\lambda,\mathcal{O}})^{\mathrm{ord}}.$$

□

4.4 p-stabilisation and duality

4.4.1 p-stabilisation

Let $\mathcal{K} = G(\mathbb{Z}_p)$ and $J = \tau^{-1}\mathcal{K}\tau \cap \mathcal{K}$. Set

$$\Gamma = K^p \times \mathcal{K} \cap G(\mathbb{Q})$$

for a fixed open-compact subgroup $K^p \subset G(\hat{\mathbb{Z}}^{(p)})$. Define a Hecke operator $\mathcal{T}_0 = [\Gamma\tau^{-1}\Gamma]$. Let W be the Weyl group of G, W_L the Weyl group of L_G and w_0 the long Weyl element in W. For any W-set X, $x \in X$, $w \in W$ write $x_w := w \cdot x$. Let I be the parahoric subgroup associated to Q_G.

Lemma 4.4.1.

$$\mathcal{K}\tau^{-1}\mathcal{K} = \sqcup_{w\in W/W_{L_G}} \sqcup_{u\in (w_0w)^{-1}Jw_0w\cap I\backslash I}\, uw\tau^{-1}\mathcal{K} \tag{16}$$

Proof. The proof given for the GSp_4 case in [TU99] carries over verbatim. □

We will call a weight $\lambda \in X^\bullet(T)$ *very regular* if it has trivial stabiliser in the Weyl group. An equivalent formulation is that it does not lie in the wall of one of the Weyl chambers. The method of proof of the following theorem is, for the most part, the same as that used in [TU99, Proposition 3.2] and [Hid95, Lemma 7.2].

Theorem 4.4.2. *For λ very regular and dominant for B_G there is a quasi-isomorphism*

$$\mathscr{C}^\bullet(\Gamma, V_{\lambda,\mathcal{O}})^{\mathcal{T}_0-\mathrm{ord}} \cong \mathscr{C}^\bullet(\Gamma_{0,1}, V_{\lambda,\mathcal{O}})^{\mathrm{ord}}$$

given by restricting $e \circ \mathrm{res}$ to the ordinary subspace defined by $\mathcal{T}_0$.

Proof. Consider the restriction map

$$\mathrm{res} : \mathscr{C}^\bullet(\Gamma, V_{\lambda,\mathcal{O}}) \to \mathscr{C}^\bullet(\Gamma_{0,1}, V_{\lambda,\mathcal{O}}).$$

and its mod p^r reductions

$$\mathrm{res}_r : \mathscr{C}^\bullet(\Gamma, V_{\lambda,r}) \to \mathscr{C}^\bullet(\Gamma_{0,1}, V_{\lambda,r}).$$

Write ℓ, ℓ' respectively for the restrictions of the maps $e \circ \mathrm{res}$, $e_0 \circ \mathrm{cores}$ to the ordinary subspaces on their respective sources, and write ℓ_r for the analagous maps

$$\mathscr{C}^\bullet(\Gamma, V_{\lambda,r})^{\mathrm{ord}} \longleftrightarrow \mathscr{C}^\bullet(\Gamma_{0,1}, V_{\lambda,r})^{\mathrm{ord}}.$$

We claim that

$$\mathcal{T} \circ \mathrm{res}_1 = \mathrm{res}_1 \circ \mathcal{T}_0$$

on cohomology.

It suffices to show that the strata in the decomposition (16) corresponding to non-trivial elements of W/W_L give elements that are divisible by p when applied $\mathscr{C}^i(\Gamma, V_{\lambda,\mathcal{O}})$, since the stratum corresponding to the trivial element is precisely $V_{0,1}\tau^{-1}V_{0,1}$.

Fix representatives $w \in G(\mathcal{O})$ of W. None of the following will depend on this choice. Let $f \in \mathscr{C}^i(\Gamma, V_{\lambda,\mathcal{O}})$, then for $z \in \mathscr{C}^i(\Gamma)$ there are $v_{uw}^{(z)} \in V_{\lambda,\mathcal{O}}$ such that

$$(\mathcal{T}f)(z) = \sum_{u,w} uw\tau^{-1}v_{uw}^{(z)}.$$

Recall that we can consider the $v_i^{(z)}$ as functions $G(\mathcal{O}) \to W_{\lambda,\mathcal{O}}$. In this optic we have

$$\begin{aligned}(w\tau^{-1} \cdot v_i^{(z)})(g) &= p^{h_\lambda} v_{uw}^{(z)}(g\tau^{-1}w) \\ &= p^{h_\lambda} v_{uw}^{(z)}(gw\tau_w^{-1}) \\ &= p^{h_\lambda - h_{\lambda_{w^{-1}}}} (\tau_w^{-1} \cdot v_{uw}^{(z)})(gw).\end{aligned}$$

We claim that $h_\lambda = h_{\lambda_{w^{-1}}}$ if and only if $w \in W_L$. We note that if $w \in W_L$ then

$$\begin{aligned}h_{\lambda_{w^{-1}}} &:= \langle \eta_w, \lambda \rangle \\ &= \langle \eta, \lambda \rangle \\ &= h_\lambda,\end{aligned}$$

because η takes values in $Z(L_G)$, thus we descend to an action of W/W_{L_G} on $\langle \eta, \lambda \rangle$.

Letting Φ_L be the set of simple roots of L_G corresponding to B_L we note that

$$Z(L_G) = \bigcap_{\alpha \in \Phi_L} \ker(\alpha),$$

and thus that $\langle \eta, \alpha \rangle = 0$ for $\alpha \in \Phi_L$. Thus for any $w \in W$, if we write $\Phi'_L = \Phi_G \backslash \Phi_L$ for the set of relative simple roots of L_G, then as λ is dominant there are non-negative integers n_α such that

$$\lambda - \lambda_w = \sum_{\alpha \in \Phi_L} n_\alpha \alpha + \sum_{\alpha \in \Phi'_L} n_\alpha \alpha$$

whence it is clear that

$$h_\lambda \geq h_{\lambda_w^{-1}}$$

with equality holding if

$$\lambda - \lambda_{w^{-1}} = \sum_{\alpha \in \Phi_L} n_\alpha \alpha$$

Note that if $w \in W, w' \in W_L$ then if $\lambda_w = \lambda_{w'}$ then $\lambda_{w'w^{-1}} = 1$ and so by very regularity $w = w'$, and in particular $w \in W_L$. This says that for $w \in W$ the character λ_w occurs in W_λ if and only if $w \in W_L$ or, in other words

$$\lambda - \lambda_{w^{-1}} = \sum_{\alpha \in \Phi_L} n_\alpha \alpha \iff w \in W_L,$$

which is what we want. We conclude that

$$w \notin W_L \implies uw\tau^{-1} v_{uw}^{(z)} \equiv 0 \bmod p$$

and thus $\mathcal{T}_0 \equiv \mathcal{T} \bmod p$.

In particular we have

$$\mathcal{T} \circ \mathrm{res}_1 = \mathrm{res}_1 \circ \mathcal{T}_0,$$

which implies that res maps between the ordinary subspaces for the respective Hecke operators. Recall the definition of the ordinary idempotents

$$e_? = \lim_{n \to \infty} \mathcal{T}_?^{n!}$$

for $? \in \{\emptyset, 0\}$.

The above result shows that

$$e \circ \mathrm{res}_1 = \mathrm{res}_1 \circ e_0.$$

To show the map ℓ'_1 is surjective we note that

$$\ell'_1 \circ \ell_1 = [\Gamma : \Gamma_{0,1}].$$

This index is prime to p so injectivity follows.

Proving surjectivity for ℓ_1 is a little trickier. Define maps

$$\begin{aligned}\mathscr{P} &: H^i(\Gamma_{0,1}, V_{\lambda,1}) \to H^i(\Gamma_{0,1}, W_{\lambda,1})\\ \iota &: H^i(\Gamma_{0,1}, W_{\lambda_{w_0},1}) \to H^i(\Gamma_{0,1}, V_{\lambda,1})\end{aligned}$$

via the natural maps

$$\begin{aligned}V_{\lambda,1} &\to W_{\lambda,1}\\ W_{\lambda,1} &\to V_{\lambda,1}\end{aligned}$$

given by dualising evaluation at 1 and inclusion respectively. These maps are intertwined by the map

$$[\mathcal{W}] : H^i(\Gamma_{0,1}, W_{\lambda,1}) \to H^i(\Gamma_{0,1}, W_{\lambda_{w_0},1})$$

induced by $\mathcal{W} = w_0\tau^{-1} \in \Gamma\tau^{-1}$. Recall that $\mathscr{P}$ restricts to an isomorphism on $\mathcal{T}$-ordinary subspaces. We will show that the map

$$\mathscr{P} \circ \mathrm{res}^{\Gamma}_{\Gamma_{0,1}} \circ \mathrm{cores}^{\Gamma}_{\Gamma_{0,1}} \circ \iota \circ [\mathcal{W}] : H^i(\Gamma_{0,1}, W_{\lambda,1}) \to H^i(\Gamma_{0,1}, W_{\lambda,1})$$

is a bijection on $\mathcal{T}$-ordinary subspaces, whence we can conclude that the restriction map is surjective. Note that

$$\begin{aligned}\Gamma &= \sqcup_{u,w} uw(\tau\Gamma\tau^{-1} \cap \Gamma)\\ &= \sqcup_{u,w} uw(w_0\Gamma_{0,1}w_0^{-1})\\ &= \sqcup_{u,w} uww_0\Gamma_{0,1},\end{aligned}$$

where u, w are as in (16), and the last line is obtained by multiplying both sides on the right by w_0 which can be assumed to be in Γ by weak approximation. Let $f \in \mathscr{C}^i(\Gamma_{0,1}, W_{\lambda,1})$, then for $z \in \mathscr{C}_i(\Gamma_{0,1})$ we have

$$(\mathrm{res} \circ \mathrm{cores} \circ \iota)(\mathcal{W} \cdot f)(z) = \sum_{u,w} uww_0(\mathcal{W} \cdot f)(w_0^{-1}w^{-1}uz).$$

Since $(\mathcal{W} \cdot f)$ takes values in $W_{\lambda_{w_0},1}$ we see that only the terms for $w = 1$ will be non-zero under $\mathscr{P}$, so

$$(\mathscr{P} \circ \mathrm{res} \circ \mathrm{cores} \circ \iota)(\mathcal{W} \cdot f)(z) = \sum_{u} uw_0(\mathcal{W} \cdot f)(w_0^{-1}w^{-1}uz)$$

but

$$uw_0(\mathcal{W} \cdot f)(w_0^{-1}w^{-1}uz) = u\tau^{-1}f(\tau u^{-1}z)$$

so the sum is just the expression for the Hecke operator $\mathcal{T}$ and this is invertible on the $\mathcal{T}$-ordinary subspace by definition.

Now that we know that the maps

$$\ell_1, \ell_1'$$

are both surjective, we see from the diagram

$$\begin{tikzcd}
& H^i(\Gamma, V_{\lambda,r})^{\mathrm{ord}} \arrow[r, "\ell_r"'] \arrow[dl] \arrow[d] & H^i(\Gamma_{0,1}, V_{\lambda,r})^{\mathrm{ord}} \arrow[l, bend right, "\ell_r'"'] \arrow[d] \arrow[dr] & \\
H^i(\Gamma, V_{\lambda,r})^{\mathrm{ord}}/p \arrow[r, hook] & H^i(\Gamma, V_{\lambda,1})^{\mathrm{ord}} \arrow[r, "\ell_1"'] & H^i(\Gamma_{0,1}, V_{\lambda,1})^{\mathrm{ord}} \arrow[l, bend right, "\ell_1'"'] & H^i(\Gamma_{0,1}, V_{\lambda,r})^{\mathrm{ord}}/p \arrow[l, hook']
\end{tikzcd}$$

that we can use Nakayama's lemma to deduce that the maps ℓ_r, ℓ_r' are also both surjective. Since these are finite sets we can deduce that ℓ_r is a bijection for all r and by taking the inverse limit we get that ℓ is an isomorphism. Therefore we get the desired quasi-isomorphism

$$\mathscr{C}^\bullet(\Gamma, V_{\lambda,\mathcal{O}})^{\mathcal{T}_0-\mathrm{ord}} \sim \mathscr{C}^\bullet(\Gamma_{0,1}, V_{\lambda,\mathcal{O}})^{\mathrm{ord}}.$$

□

Let F be a number field over which G splits. By various results of Borel and Tits there is a finite set of primes S such that G has a split reductive model over the ring of S-integers $\mathcal{O}_{F,S}$. We note that this implies that for every $p \notin S$ the group G is unramified at p. For $\lambda \in X^\bullet(T_G)$ dominant for B_{L_G} and $\chi \in X^\bullet(L_G)$ such that $\lambda + \chi$ is dominant for G, define

$$\begin{aligned} W_{\lambda,F,S} &= \{f \in \mathcal{O}_{F,S}[L_G] : f(\bar{b}x) = \lambda(\bar{b})f(x) \; \forall \; \bar{b} \in \bar{B}_{L_G}\} \\ V_{\lambda,F,S} &= \{f \in \mathcal{O}_{F,S}[G] \otimes W_{\lambda,F,S} : f(\bar{q}g) = \chi(\bar{q})\bar{q}f(g) \; \forall \; \bar{q} \in \bar{Q}_G\}. \end{aligned}$$

Note that these $\mathcal{O}_{F,S}$-modules are independent of p. It is clear from our previous definition of the action of τ^{-1} that it preserves these modules and thus we have an action of $\mathcal{T}_0$ on the cohomology groups $H^i(\Gamma, V_{\lambda,F,S})$. For a prime p write F_p for a p-adic completion of F and $\mathcal{O}_{F_p}$ for its ring of integers.

Corollary 4.4.3. *There is a finite set of primes S_Γ containing S such that for primes $p \notin S_\Gamma$ the cohomology groups $H^i(\Gamma_{0,1}, V_{\lambda,\mathcal{O}_{F_p}})^{\mathrm{ord}}$ are torsion-free as $\mathcal{O}$-modules.*

Proof. The cohomology group $e_0H^i(\Gamma, V_{\lambda,F,S})$ gives an $\mathcal{O}_{F,S}$-lattice in $e_0H^i(\Gamma, V_{\lambda,\mathcal{O}_{F_p}})$ and thus, by the previous theorem, in $eH^i(\Gamma_{0,1}, V_{\lambda,\mathcal{O}_{F_p}})$. Since this lattice is independent of p and finitely generated if we let

$$S_\Gamma = S \cup \{\text{torsion primes in } e_0H^i(\Gamma, V_{\lambda,F,S})\}$$

then S_Γ is finite and for all $p \notin S_\Gamma$ the $\mathcal{O}_{F_p}$-module $eH^i(\Gamma_{0,1}, V_{\lambda,\mathcal{O}_{F_p}}) = e_0H^i(\Gamma, V_{\lambda,F,S}) \otimes \mathcal{O}_{F_p}$ is torsion free. $\square$

4.4.2 Duality results

For this section only set $Q_G = B_G$. We show how to obtain a derived control result for compactly supported group cohomology. We show that the $\mathcal{O}$-linear Poincaré duality pairing at level $\Gamma_{0,1}$ is non-degenerate outside of a finite set of primes. Throughout this section λ is dominant for B_G.

Definition 4.4.4. For any left $\Gamma_{0,1}$-module M define a complex

$$\mathscr{C}_c^\bullet(\Gamma_{0,1}, M) = \mathscr{C}_{2d-\bullet}(\Gamma_{0,1}) \otimes_{\Gamma_{0,1}} M,$$

where $2d := \dim_{\mathbb{R}} G(\mathbb{R})/K_\infty$ for a maximal open compact subgroup $K_\infty \subset G(\mathbb{R})$.

This complex satisfies

$$H^i(\mathscr{C}_c^\bullet(\Gamma_{0,1}, M)) = H_c^i(\Gamma_{0,1}, M)$$

We can define a Hecke operator $\mathcal{T}$ on $\mathscr{C}_c^\bullet(\Gamma_{0,1}, M)$ in the same way as for $\mathscr{C}^\bullet(\Gamma_{0,1}, M)$ and whence define the ordinary part of the complex, denoted $\mathscr{C}_c^\bullet(\Gamma_{0,1}, M)^{\mathrm{ord}}$ and uniquely defined up to homotopy equivalence.

Remark 4.4.5. As modules we have

$$\mathscr{C}_c^i(\Gamma_{0,1}, M) = \mathscr{C}^i(\Gamma_{0,1}, M).$$

In particular $\mathscr{C}_c^\bullet(\Gamma_{0,1}, \mathbb{D}_{\mathrm{univ}}(\lambda))^{\mathrm{ord}}$ is a perfect complex of Λ_0-modules.

Definition 4.4.6. We define a pairing of $\mathcal{O}$-modules

$$(-,-)_r : H_c^i(\Gamma_{1,r}, V_{\lambda,\mathcal{O}}) \otimes_{\mathcal{O}} H^{2d-i}(\Gamma_{1,r}, (V_{\lambda,\mathcal{O}})^\vee) \to \mathcal{O}.$$

by

$$(x,y)_r = \varphi_\lambda(x \cup \mathcal{W}(y))$$

where φ_λ is the natural map $V_{\lambda,\mathcal{O}} \otimes (V_{\lambda,\mathcal{O}})^\vee \to \mathcal{O}$.

Let $\lambda^\vee := -\omega_0\lambda$ where ω_0 is the long Weyl element. We note that in general $(V_{\lambda,\mathcal{O}})^\vee \neq V_{\lambda^\vee,\mathcal{O}}$; we can at most say that

$$V_{\lambda^\vee,\mathcal{O}} \subset (V_{\lambda,\mathcal{O}})^\vee$$

since the former is the minimal lattice and the latter is the maximal admissible lattice.

Lemma 4.4.7. *Let $V_{\lambda,\mathcal{O}}^{\min}$ be the maximal admissible lattice in V_λ. There is an isomorphism*

$$\mathscr{C}^i(\Gamma_{0,1}, V_{\lambda,\mathcal{O}}^{\min})^{\mathrm{ord}} \cong \mathscr{C}^i(\Gamma_{0,1}, V_{\lambda,\mathcal{O}})^{\mathrm{ord}}.$$

Proof. The strict dominance requirement for η forces

$$h_\lambda - h_\mu > 0 \tag{17}$$

for all characters $\mu \neq \lambda$ of T_G appearing in V_λ. We recall that, given a choice of highest weight vector v_λ, an admissible lattice in V_λ is the direct sum of its intersections with the weight spaces for T_G, and the highest weight spaces are the $\mathcal{O}$-linear span of v_λ. By construction of the $*$ action, τ^{-1} fixes λ and for $\mu \neq \lambda$ we have that $\tau^{-r} * (V_{\lambda,\mathcal{O}}^{\max})_\mu = p^{r(h_\lambda - h_\mu)}(V_{\lambda,\mathcal{O}}^{\max})_\mu$. By (17) we see that

$$\lim_r \tau^{-r} * V_{\lambda,\mathcal{O}}^{\max} = \mathcal{O} \cdot v_\lambda.$$

In particular, as $V_{\lambda,\mathcal{O}}$ is an open subgroup of $V_{\lambda,\mathcal{O}}^{\max}$ we see that for $r \gg 0$:

$$\tau^{-r} * V_{\lambda,\mathcal{O}}^{\max} \subset V_{\lambda,\mathcal{O}}.$$

It's easy to see that this implies the result. ☐

We now discuss the Hecke action on $(-,-)_r$. We define an 'adjoint' Hecke operator $\mathcal{T}^*$ defined by

$$\mathrm{cores}^{\Gamma'}_{\Gamma' \cap \tau^{-1}\Gamma\tau} \circ [\tau] \circ \mathrm{res}^{\Gamma}_{\tau\Gamma'\tau^{-1} \cap \Gamma}$$

and with action on $V_\lambda^\vee$ given by $\tau * v = p^{-h_\lambda}\tau v$. This satisfies

$$\mathcal{T}x \cup y = x \cup \mathcal{T}^* y.$$

Lemma 4.4.8. *The ordinary idempotent e_{ord} is self-adjoint for the pairing $(-,-)_r$.*

Proof. Note that $\mathcal{W}$ normalizes V_r. Thus

$$\begin{aligned}\mathcal{W}^{-1}\Gamma\tau\Gamma\mathcal{W} &= \tau w_0^{-1}\Gamma\tau\Gamma w_0 \tau^{-1} \\ &= \Gamma\tau w_0^{-1}\tau w_0 \tau^{-1}\Gamma \\ &= \Gamma w_0^{-1}\tau w_0 \Gamma.\end{aligned}$$

The element $w_0^{-1}\tau w_0 \in T(\mathbb{Q}_p)$ is the image of p under the cocharacter $\eta_{w_0}^{-1}$ which satsfies $\langle \eta_{w_0}^{-1}, \alpha \rangle > 0$ for all positive roots α. Thus $\Gamma w_0^{-1}\tau w_0 \Gamma$ defines the same idempotent e_{ord} as $\Gamma\tau^{-1}\Gamma$ so we are done. ☐

Proposition 4.4.9. *There is a cofinite set of primes for which the pairing*

$$(-,-)_r : H_c^i(\Gamma_{1,r}, V_{\lambda,\mathcal{O}})^{\mathrm{ord}} \otimes H^{2d-i}(\Gamma_{1,r}, (V_{\lambda,\mathcal{O}})^\vee)^{\mathrm{ord}} \to \mathcal{O}$$

is non-degenerate.

Proof. Note that $\mathcal{W}$ induces an isomorphism

$$H^j(\Gamma_{1,r}, (V_{\lambda,\mathcal{O}})^\vee)^{\mathrm{ord}} \cong H^j(\Gamma_{1,r}, (V_{\lambda,\mathcal{O}})^\vee)^{\mathcal{T}^*-\mathrm{ord}}$$

where the latter is the ordinary subspace for the adjoint Hecke operator $\mathcal{T}^*$. Thus non-degeneracy of the pairing is implied by non-degeneracy of the standard Poincaré duality pairing because this restricts to a pairing between the ordinary and $\mathcal{T}^*$-ordinary subspaces. This pairing is well-known to descend to a non-degenerate pairing

$$H_c^i(\Gamma_{1,r}, V_{\lambda,\mathcal{O}})/(\mathrm{tors}) \otimes H^{2d-i}(\Gamma_{1,r}, V_{\lambda,\mathcal{O}})/(\mathrm{tors}) \to \mathcal{O}.$$

We know from Corollary 4.4.3 (and the analogous version for compactly supported cohomology) that the groups $H_c^j(\Gamma_{1,r}, V_{\lambda,\mathcal{O}})^{\mathrm{ord}}, H^j(\Gamma_{1,r}, (V_{\lambda,\mathcal{O}})^\vee)^{\mathrm{ord}}$ are free $\mathcal{O}$-modules for a cofinite set of primes and thus the result follows. ☐

Thus the pairing $(-,-)_r$ descends to a pairing

$$H^i_c(\Gamma_{1,r},V_{\lambda,\mathcal{O}})^{\mathrm{ord}}\otimes H^{2d-i}(\Gamma_{1,r},(V_{\lambda,\mathcal{O}})^\vee)^{\mathrm{ord}}\to\mathcal{O}$$

which is non-degenerate outside of a finite set of primes S_Γ.

Remark 4.4.10. Although one can extend the proof of Lemma 4.4.7 to the situation that λ is a character of L_G for a general parabolic Q_G, a generalisation of the duality pairing to general parabolics is hampered by the fact that the 'Atkin-Lehner' style operator $\mathcal{W}$ does not necessarily preserve Levi subgroups.

4.5 Adèlic cohomology and Hecke algebras

4.5.1 Adèlic cohmology and localisations

For $U=U_{?,r}\subset G(\mathbb{A}_f)$, let $\mathbb{T}(U)$ be the Hecke algebra of locally constant U-biinvariant functions on $G(\mathbb{A}_f)$. It is generated by indicator functions on double cosets $U\alpha U$, $\alpha\in G(\mathbb{A}_f)$. We write $\mathbb{T}^S(U)$ for the Hecke algebra restricted to places at which U is hyperspecial and let

$$\mathbb{T}^S_p(U)=\mathbb{T}^S(U)\otimes\mathcal{T}\subset\mathbb{T}(U).$$

This subalgebra is well known to be commutative. We will often drop the open compact U from the notation. Assuming G^{der} sastisfies strong approximation (which we do from now on), there is a finite set $\{t_1,\dots,t_n\}$ such that

$$G(\mathbb{A}_f)=\sqcup_i G(\mathbb{Q})\times t_iU,$$

and if we define

$$\Gamma(t_i,U):=t_iUt_i^{-1}\cap G(\mathbb{Q})$$

then for any $\mathcal{O}[U]$-module M with a compatible τ^{-1} action, $\mathbb{T}^S_p(U)$ acts naturally on the complex

$$R\Gamma(U,M):=\bigoplus_i\mathscr{C}^\bullet(\Gamma(t_i,U),M).$$

It can be shown that the image of this complex in the homotopy category does not depend on the choice of t_i. We can define the double coset action (and thus the action of $\mathbb{T}^S_p(U)$) in terms of non-adèlicised Hecke operators (defined in Section 4.2.2) on summands in the following way:

Let $x\in G(\mathbb{A}_f)$ have p-part τ^{-1} and let $xt_i=\gamma_{x,i}t_{j_i}k\in G(\mathbb{Q})t_{j_i}U$. The action is then given by:

$$[UxU]=\bigoplus_i[\Gamma(t_i,U)\gamma_{x,i}^{-1}\Gamma(t_{j_i},U)].$$

As $\mathbb{T}^S_p(U)$ is generated by double coset operators, this description suffices to define the action of $\mathbb{T}^S_p(U)$.

We assume now that M is finitely-generated as an $\mathcal{O}$-module. The definition of these coset operators as given by (9) in terms of functorial morphisms shows that the double cosets $[V_{1,r}xV_{1,r}]$ for $r\geq 1$ commute with corestriction maps and thus define Hecke operators on the inverse limit

$$R\Gamma(V_{1,\infty},M):=R\varprojlim_r R\Gamma(V_{1,r},M/p^r)=\varprojlim_r R\Gamma(V_{1,r},M/p^r).$$

In particular we get an action of $\mathcal{T}$ and can use this to define the ordinary complex $R\Gamma(V_{1,\infty},M)^{\mathrm{ord}}$ defined uniquely up to homotopy and satisfying

$$R\Gamma(V_{1,\infty},M)^{\mathrm{ord}}=\varprojlim_r R\Gamma(V_{1,r},M)^{\mathrm{ord}}$$

in the homotopy category.

Since by weak approximation we can choose $t_i\in G(\hat{\mathbb{Z}})$ to be trivial at p, by looking on individual summands we obtain derived control theorems for this 'adèlic cohomology'.

For $? \in \{\emptyset, p\}$, write $\mathbb{T}^S_?(U, M)$ for the image of $\mathbb{T}^S_? \otimes R$ in $\mathrm{End}(R\Gamma(U, M))$. Write $\mathbb{T}^S_\lambda := \mathbb{T}^S(V_{0,1}, \mathbb{D}_{\mathrm{univ}})^{\mathrm{ord}}$, where the superscript ord refers to restricting the ordinary subspace. This is well-defined as $\mathbb{T}^S_p$ is commutative and acts continuously. We prove a variant of the derived control theorem for localisations by ideals of $\mathbb{T}^S_\lambda$. The proof of the following theorem is pretty much identical to [APS08, Theorem 5.1] except that we work with complexes of Hecke modules.

Theorem 4.5.1. *There is a natural bijection (not an isomorphism of schemes)*

$$\mathrm{Spec}\ \mathbb{T}^S_\lambda / I_r^{(\chi)} \leftrightarrow \mathrm{Spec}\ \mathbb{T}^S(V_{1,r}, V_{\lambda+\chi,\mathcal{O}})^{\mathrm{ord}}.$$

Proof. We note that since the elements of $\mathbb{T}^S_\lambda$ are Λ_0-linear there is a natural map

$$q : \mathbb{T}^S_\lambda \to \mathbb{T}^S(V_{1,r}, V_{\lambda+\chi,\mathcal{O}})^{\mathrm{ord}}$$

which lifts into a diagram

$$\begin{tikzcd} \mathbb{T}^S \arrow[d] \arrow[dr] & \\ \mathbb{T}^S_\lambda \arrow[r, "q"] & \mathbb{T}^S(V_{1,r}, V_{\lambda+\chi,\mathcal{O}})^{\mathrm{ord}}. \end{tikzcd}$$

The two vertical maps are surjective and thus so is q.

To prove the theorem it suffices to show that $\ker(q) \subset \mathrm{Rad}(I_r^{(\chi)}\mathbb{T}^S_\lambda)$. The ideal $I_r^{(\chi)} \subset \Lambda_r$ is generated by a regular sequence $(x_1, \ldots, x_m)$. We proceed by induction on m. Suppose $I_r^{(\chi)} = (x)$ and let $T \in \ker(q)$. Then $T(\mathscr{C}^\bullet(V_{0,1}, \mathbb{D}_{\mathrm{univ}}(\lambda))^{\mathrm{ord}}) \subset x\mathscr{C}^\bullet(V_{0,1}, \mathbb{D}_{\mathrm{univ}}(\lambda))^{\mathrm{ord}}$. We know that the modules $\mathscr{C}^i(V_{0,1}, \mathbb{D}_{\mathrm{univ}}(\lambda))^{\mathrm{ord}}$ have no x-torsion, so

$$T' := x^{-1}T \in \mathrm{End}_{\Lambda_0}(M^\bullet_\lambda)$$

is well-defined. Define

$$\mathscr{X} = \{\phi \in \mathrm{End}_{\Lambda_0}(M^\bullet_\lambda) : \exists k \text{ such that } x^k\phi \in \mathbb{T}^S_\lambda\}.$$

Then $\mathscr{X}$ is a finitely generated Λ_0-module and $T' \in \mathscr{X}$. Since $\mathscr{X}$ is finitely generated there is an integer N such that $x^N\mathscr{X} \subset \mathbb{T}^S_\lambda$. Thus

$$T^{N+1} = x(x^N T^{N+1}) \in \mathbb{T}^S_\lambda,$$

so $T \in \mathrm{Rad}(I_r^{(\chi)}\mathbb{T}^S_\lambda)$.

Now let $m \geq 1$ and suppose the induction hypothesis holds for regular sequences of length m. Suppose $I_r^{(\chi)} = (x_1, \ldots, x_{m+1})$ then consider

$$\psi : \mathbb{T}^S_\lambda \xrightarrow{\varphi} \mathbb{T}^S(M^\bullet_\lambda/(x_1, \ldots, x_m)) \to \mathbb{T}^S(V_{1,r}, V_{\lambda+\chi,\mathcal{O}})^{\mathrm{ord}}.$$

We note that the proof of the $m = 1$ case holds perfectly well for the second map in the above sequence and so if $T \in \ker(\psi)$ then $\varphi(T) \in \mathrm{Rad}(x_{m+1}\mathbb{T}^S(M^\bullet_\lambda/(x_1, \ldots, x_m)))$ so that there is N such that $\varphi(T^N)$ is in this radical and thus there is an element $z \in \mathbb{T}^S_\lambda$ such that $\varphi(T^N) = x_{m+1}\varphi(z)$. By the induction hypothesis

$$T^N - \alpha z \in \mathrm{Rad}((x_1, \ldots, x_m)\mathbb{T}^S_\lambda)$$

so there is M such that

$$(T^N - \alpha z)^M \in (x_1, \ldots, x_m)\mathbb{T}^S_\lambda$$

whence it is clear that

$$T^{NM} \in I_r^{(\chi)}\mathbb{T}^S_\lambda.$$

□

Corollary 4.5.2. *Let $\wp \in \mathrm{Spec}\ \mathbb{T}^S_\lambda/I_r^{(\chi)}$, then*

$$(M^\bullet_\lambda)_\wp \otimes^L_{\Lambda_r} \mathcal{O}^{(\chi)} \sim R\Gamma(V_{1,r}, V_{\lambda+\chi,\mathcal{O}})^{\mathrm{ord}}_\wp$$

Proof. The control theorem tells us precisely how prime ideals move between the big and small Hecke algebras and localisation is exact. ☐

Let $V = K^p \times G(\mathbb{Z}_p)$.

Theorem 4.5.3. *Suppose there is a maximal ideal* $\mathfrak{m} \in \operatorname{Spec}\mathbb{T}^S$ *such that*

$$H^\bullet_?(V_{0,1}, V_{\lambda,\mathcal{O}})_\mathfrak{m} = H^d_?(V_{0,1}, V_{\lambda,\mathcal{O}})_\mathfrak{m}$$

for $? \in \{\emptyset, c\}$ *and* $H^d_?(V_{0,1}, V_{\lambda,\mathcal{O}})_\mathfrak{m}$ *is finite free as an* $\mathcal{O}$*-module. Let* Δ^p *be the prime-to-*p *part of* $S_G(\mathbb{Z}_p)$*. Then*

$$H^\bullet_?(V_{1,\infty}, V_{\lambda,\mathcal{O}})^{\mathrm{ord},\Delta^p}_\mathfrak{m} = H^d_?(V_{1,\infty}, V_{\lambda,\mathcal{O}})^{\mathrm{ord},\Delta^p}_\mathfrak{m},$$

where $(\cdot)^{\Delta^p}$ *refers to* Δ^p*-invariants. Furthermore,* $H^d_?(V_{1,\infty}, V_{\lambda,\mathcal{O}})^{\mathrm{ord},\Delta^p}_\mathfrak{m}$ *is a free* Λ_1*-module.*

Proof. We note that as Δ^p has prime-to-p order, taking Δ^p invariants is an exact functor on $\mathcal{O}$-modules. Let $\mathfrak{m}_r$ be the maximal ideal of Λ_r generated by p and $I_r^{(\mathbb{1})}$. An easy corollary of derived control is the following 'mod p' variant:

$$M^\bullet_{\lambda,?} \otimes^{\mathbb{L}}_{\Lambda_r} \Lambda_r/\mathfrak{m}_r \sim R\Gamma_?(V_{1,r}, V_{\lambda,1})^{\mathrm{ord}}.$$

We note that since $H^i_?(V_{0,1}, V_{\lambda,\mathcal{O}})^{\mathrm{ord}}_\mathfrak{m}$ is a free $\mathcal{O}$-module for all i, the short exact sequence

$$0 \to V_{\lambda,\mathcal{O}} \xrightarrow{p} V_{\lambda,\mathcal{O}} \to V_{\lambda,1} \to 0$$

gives us that $H^i_?(V_{0,1}, V_{\lambda,\mathcal{O}})^{\mathrm{ord}}_\mathfrak{m}/p = H^i_?(V_{0,1}, V_{\lambda,1})^{\mathrm{ord}}_\mathfrak{m}$ and so

$$H^i_?(V_{0,1}, V_{\lambda,1})^{\mathrm{ord}}_\mathfrak{m} = 0$$

for all i. We consider the spectral sequence

$$E_2^{i,j} : \operatorname{Tor}^{\Lambda_1}_{-i}(H^j_?(V_{1,\infty}, V_{\lambda,\mathcal{O}})^{\mathrm{ord}}, \Lambda_1/\mathfrak{m}_1) \implies H^{i+j}_?(V_{1,1}, V_{\lambda,1})^{\mathrm{ord}},$$

which satisfies $E_2^{i,j} = 0$ for $j > \nu$. We can apply the exact functor $(\cdot)^{\Delta^p}$ to the spectral sequence to get

$$(E_2^{i,j})^{\Delta^p} : \operatorname{Tor}^{\Lambda_1}_{-i}(H^j_?(V_{1,\infty}, V_{\lambda,\mathcal{O}})^{\mathrm{ord}}, \Lambda/\mathfrak{m}_1)^{\Delta^p} \implies H^{i+j}_?(V_{1,1}, V_{\lambda,1})^{\mathrm{ord},\Delta^p} = H^{i+j}_?(V_{0,1}, V_{\lambda,1})^{\mathrm{ord}}.$$

Moreover, we can show that

$$\operatorname{Tor}^{\Lambda_1}_{-i}(H^j_?(V_{1,\infty}, V_{\lambda,\mathcal{O}})^{\mathrm{ord}}, \Lambda_1/\mathfrak{m}_1)^{\Delta^p} = \operatorname{Tor}^{\Lambda_1}_{-i}(H^j_?(V_{1,\infty}, V_{\lambda,\mathcal{O}})^{\mathrm{ord},\Delta^p}, \Lambda_1/\mathfrak{m}_1),$$

(see Lemma 4.5.5) so the spectral sequence becomes

$$(E_2^{i,j})^{\Delta^p} : \operatorname{Tor}^{\Lambda_1}_{-i}(H^j_?(V_{1,\infty}, V_{\lambda,\mathcal{O}})^{\mathrm{ord},\Delta^p}, \Lambda_1/\mathfrak{m}_1) \implies H^{i+j}_?(V_{0,1}, V_{\lambda,1})^{\mathrm{ord}}.$$

We can then read off that

$$H^\nu_?(V_{1,\infty}, V_{\lambda,\mathcal{O}})^{\mathrm{ord},\Delta^p}_\mathfrak{m}/\mathfrak{m}_1 \cong H^\nu_?(V_{0,1}, V_{\lambda,1})^{\mathrm{ord}}_\mathfrak{m} = 0$$

and thus Nakayama allows us to conclude that

$$H^\nu_?(V_{1,\infty}, V_{\lambda,\mathcal{O}})^{\mathrm{ord},\Delta^p}_\mathfrak{m} = 0.$$

Now $(E_2^{i,j})^{\Delta^p} = 0$ for $j > \nu - 1$ and so we perform this step inductively to get $H^j_?(V_{1,\infty}, V_{\lambda,\mathcal{O}})^{\mathrm{ord},\Delta^p}_\mathfrak{m} = 0$ for $j > d$, so that $(E_2^{i,j})^{\Delta^p} = 0$ for $j > d$. We can now read off that there is a surjection

$$H^{d-1}_?(V_{0,1}, V_{\lambda,1})^{\mathrm{ord},\Delta^p}_\mathfrak{m} = H^{d-1}_?(V_{0,1}, V_{\lambda,1})^{\mathrm{ord}}_\mathfrak{m} \twoheadrightarrow \operatorname{Tor}^{\Lambda}_1(H^d_?(V_{1,\infty}, V_{\lambda,\mathcal{O}})^{\mathrm{ord},\Delta^p}_\mathfrak{m}, \Lambda_1/\mathfrak{m}_1),$$

and since $H^{d-1}_?(V_{0,1}, V_{\lambda,1})^{\mathrm{ord}}_\mathfrak{m} = 0$ the Tor group vanishes and so by the local criterion for flatness $H^d_?(V_{1,\infty}, V_{\lambda,\mathcal{O}})^{\mathrm{ord},\Delta^p}_\mathfrak{m}$ is a flat Λ-module. Since

$$H^d_?(V_{1,\infty}, V_{\lambda,1})^{\mathrm{ord},\Delta^p}_\mathfrak{m}/\mathfrak{m}_1 \cong H^d_?(V_{0,1}, V_{\lambda,1})^{\mathrm{ord}}_\mathfrak{m}$$

Nakayama gives us that $H^d_?(V_{1,\infty}, V_{\lambda,1})^{\mathrm{ord},\Delta^p}_\mathfrak{m}$ is finitely generated and thus it is free. We're left to showing

$$H^j_?(V_{1,\infty}, V_{\lambda,1})^{\mathrm{ord},\Delta^p}_\mathfrak{m} = 0$$

for $j < d$, but since $H^d_?(V_{1,\infty}, V_{\lambda,1})^{\mathrm{ord},\Delta^p}_\mathfrak{m}$ is free we have $(E_2^{i,d})^{\Delta^p} = 0$ for $i < 0$ so we can proceed much as we did for $j > d$. ☐

Remark 4.5.4. Examples of our assumption holding include the work of Mokrane-Tilouine [MT02], where they prove that the assumptions hold for $G = \mathrm{GSp}_{2g}$ and suitably nice maximal ideals of $\mathbb{T}^S$. Such results will generally be proved at prime-to-p level and can be deduced at level $V_{0,1}$ via Theorem 4.4.2.

Lemma 4.5.5. *Let M be a Λ_0-module, then*

$$(M \otimes_\Lambda \Lambda/\mathfrak{m}_1)^{\Delta^p} = M^{\Delta^p} \otimes_\Lambda \Lambda/\mathfrak{m}_1.$$

Proof. We can decompose M into a direct sum of ψ-eigenspaces $M^{(\psi)}$ where ψ runs over characters $\psi : \Delta^p \to \mathcal{O}^\times$. Thus

$$M \otimes_\Lambda \Lambda/\mathfrak{m}_1 = \oplus_\psi (M^{(\psi)} \otimes_\Lambda \Lambda/\mathfrak{m}_1).$$

Since Δ^p acts on the tensor product through M and preserves the eigenspaces by definition, we need only to show that

$$\psi(\Delta^p) \equiv 1 \bmod p \implies \psi \equiv \mathbb{1}$$

but Δ^p is finite so ψ takes values in $(\mathcal{O}/p)^\times \hookrightarrow \mathcal{O}^\times$, so the implication holds. □

5 Spherical varieties and non-ordinary families of cohomology classes

5.1 Introduction

Let p be a prime. In this paper we give a construction of non-ordinary p-adic families of cohomology classes interpolating classes constructed by pushing forward classes from a 'small' reductive group to a larger one. The construction of such classes has been automated via the 'spherical varieties' machine of Loeffler [Loe21]. These cohomology classes arise naturally in the study of Iwasawa theory as Euler systems and in the construction of p-adic L-functions. Our construction provides a vast generalisation of previous works such as [LZ16] and [BSV20] and acts a sequel to [LRZ21] in which ordinary families of cohomology classes were considered.

The non-ordinary setting requires fundamentally different methods than in the ordinary case. We use a method inspired by work of Greenberg–Seveso [GS20] and Bertolini–Seveso–Venerucci [BSV20] on balanced diagonal classes. Our method utilises modules of analytic functions in place of the usual modules of analytic distributions to interpolate branching maps between algebraic representations.

Suppose we have an inclusion of reductive groups $H \hookrightarrow G$ satisfying the conditions set out in [LRZ21, 2.1]. Let μ, λ be dominant algebraic weights of H, G and let V_μ^H, V_λ^G be the respective irreducible representations. For a wide-open disc $\mathcal{U} \subset \mathcal{W}_G$ containing λ, where $\mathcal{W}_G$ is a suitable weight space, we define modules of locally analytic functions $A_\lambda^{\mathrm{an}}, A_{\mathcal{U}}^{\mathrm{an}}$ which are modules for a parahoric subgroup J_G. There is a natural inclusion

$$V_\lambda^G \hookrightarrow A_\lambda^{\mathrm{an}}$$

and a specialisation map

$$A_{\mathcal{U}}^{\mathrm{an}} \to A_\lambda^{\mathrm{an}}.$$

Let $Y_H(J_H), Y_G(J_G)$ denote the parahoric level locally symmetric spaces (of some suitably small tame level) associated to H, G. Suppose λ is Q_H^0-admissible in the sense of Definition 5.2.3 and let $V_\mu^H \to V_\lambda^G$ be the resulting H-inclusion. As a key ingredient in his construction of norm-compatible classes, Loeffler [Loe21] constructs a map

$$H^i(Y_H(J_H), V_\mu^H) \to H^{i+c}(Y_G(J_G), V_\lambda^G)$$

for $i \geq 0$ and $c = \dim_{\mathbb{R}} Y_G - \dim_{\mathbb{R}} Y_H - \mathrm{rk}_{\mathbb{R}}(\frac{Z_H}{Z_G \cap Z_H})$. We can lift this map to a map

$$H^i(Y_H(J_H), V_\mu^H) \to H^{i+c}(Y_G(J_G), A_\lambda^{\mathrm{an}}). \tag{18}$$

In Section 6 we construct what we call a 'big branching map' map in Betti cohomology:

$$H^i(Y_H(J_H), D_{\mathcal{U}}^H) \to H^{i+c}(Y_G(J_G), A_{\mathcal{U}}^{\mathrm{an}}), \tag{19}$$

where $D^H_{\mathcal{U}}$ is a module of distributions over $\mathcal{U}$ specialising to V^H_μ at λ. Our main result is the following:

Theorem 5.1.1. *There is a commutative diagram of Betti cohomology groups*

$$\begin{tikzcd}
H^i(Y_H(J_H), D^H_{\mathcal{U}}) \arrow[r] \arrow[d] & H^{i+c}(Y_G(J_G), A^{\mathrm{an}}_{\mathcal{U}}) \arrow[d] \\
H^i(Y_H(J_H), V^H_\mu) \arrow[r] & H^{i+c}(Y_G(J_G), A^{\mathrm{an}}_{\lambda}),
\end{tikzcd} \tag{20}$$

where the top map is (19) *and the bottom is* (18).

When Y_H, Y_G admit compatible Shimura data we construct an analagous map in étale cohomology. This requires a construction of new objects: profinite modules $A^{\mathrm{Iw}}_{\mathcal{U}}, A^{\mathrm{Iw}}_{\lambda}$ which we dub 'Iwasawa analytic functions'. Roughly speaking, these consist of functions on J_G which extend to a wide-open rigid analytic neighbourhood.

Theorem 5.1.2. *There is a commutative diagram of étale cohomology groups*

$$\begin{tikzcd}
H^i_{\acute{e}t}(Y_H(J_H)_{\bar{\mathbb{Q}}}, D^H_{\mathcal{U}}(j)) \arrow[r] \arrow[d] & H^{i+2e}_{\acute{e}t}(Y_G(J_G)_{\bar{\mathbb{Q}}}, A^{\mathrm{Iw}}_{\mathcal{U}}(j+e)) \arrow[d] \\
H^i_{\acute{e}t}(Y_H(J_H)_{\bar{\mathbb{Q}}}, V^H_\mu(j)) \arrow[r] & H^{i+2e}_{\acute{e}t}(Y_G(J_G)_{\bar{\mathbb{Q}}}, A^{\mathrm{Iw}}_{\lambda}(j+e))
\end{tikzcd} \tag{21}$$

where $j \in \mathbb{Z}$ and $c = 2e$.

Let Σ be a finite set of primes containing p and the primes at which the level of G ramifies and suppose $Y_H(J_H), Y_G(J_G)$ admit Σ-integral models $Y_G(J_H)_\Sigma, Y_H(J_G)_\Sigma$. The diagram of Theorem 5.1.2 then also holds for the étale cohomology of these integral models. Let $q = \dim Y_H(J_G)$. In Section 6.4 we show how, under certain hypotheses (small slope, vanishing outside the middle degree after localisation at a maximal ideal of the Hecke algebra) we can push-forward classes in $H^{q+1}(Y_G(J_G)_\Sigma, A^{\mathrm{Iw}}_{\mathcal{U}})$ to obtain classes in Galois cohomology.

Theorem 5.1.3. *Suppose $\mathfrak{m}$ is a 'nice' (in a precise sense) maximal ideal of the unramified Hecke algebra acting on $H^\bullet(Y_G(J_G)_{\bar{\mathbb{Q}}}, V^G_\lambda)$. Then there is an affinoid $\mathcal{V} \subset \mathcal{W}_G$ containing λ and an 'Abel–Jacobi' map*

$$AJ_{\mathcal{U}} : H^{q+1}(Y_G(J_G)_\Sigma, A^{\mathrm{Iw}}_{\mathcal{U},m}) \to H^1(\mathcal{O}_{E,\Sigma}, W_{\mathcal{U}})$$

where E is the reflex field of Y_G, $\mathcal{O}_{E,\Sigma}$ is its ring of Σ-integers, and $W_{\mathcal{U}}$ is a family of Galois representations over $\mathcal{U}$ specialising to $H^q(Y_G(J_G)_{\bar{\mathbb{Q}}}, V^G_\lambda)_{\mathfrak{m}}$ at λ.

The proof of this theorem uses in an essential way the profiniteness of the modules $A^{\mathrm{Iw}}_{\mathcal{U},m}$. We can similarly construct Abel–Jacobi maps at finite level and the above commutative diagram in étale cohomology shows that the image of classes constructed using Loeffler's machine under this Abel–Jacobi map interpolate in families over $\mathcal{U}$.

Example 5.1.4. In Section 6.6 we consider the case of $G = \mathrm{GSp}_4$ and $H = \mathrm{GL}_2 \times_{\mathrm{GL}_1} \mathrm{GL}_2$. By considering auxiliary groups $\tilde{G} = G \times \mathrm{GL}_1$ and $\tilde{H} = H \times \mathrm{GL}_1$ we obtain a spherical pair for the Borel subgroup of $\tilde{G}$ and our construction gives a Galois cohomology class $z_{\mathcal{U}} \in H^1(\mathbb{Q}, W_{\mathcal{U}})$ interpolating the Lemma–Flach classes constructed in [LSZ21] (or, more precisely, their Iwahori-variants constructed in [LRZ21, Section 7]). We expect this class to play a role in proving new cases of Bloch–Kato and Birch–Swinnerton-Dyer, much as their ordinary counterparts have in [LZ21].

This work contains several novel developments of the theory initiated with [GS20] and [BSV20]:

- We work in the full generality of $\mathbb{Q}$-groups which are unramified at p (modulo an easily removed condition on the centre).
- The methods of *op.cit.* involve pushing forward the trivial class in $H^0(Y_H(J_H), \mathbb{Z}_p)$ along a J_G-map $\mathbb{Z}_p \to A^{\mathrm{an}}_{\mathcal{U},m}$ interpolating H-maps $\mathbb{Z}_p \to V^G_\lambda$ for $\lambda \in \mathcal{U}$. Our use of the distribution modules $D^H_{\mathcal{U},m}$ over Y_H allows us to interpolate a wider class of branching maps and expands

the canonical classes we can consider. In particular we can now consider push-forwards of families of Beilinson's Eisenstein classes (generalising Siegel units on modular curves), see Section 6.2.2.

- Our construction and use of the profinite modules $A^{\mathrm{Iw}}_{\mathcal{U},m}$, which are essential in showing cohomological triviality of both interpolating classes and 'finite-level' classes satisfying a small slope condition (as opposed to an ordinarity condition where the proof is simpler and does not generalise, relying on the vanishing of Iwasawa cohomology in degree 0).
- Our construction allows for interpolation of 'admissible' weights λ in the disc $\mathcal{U}$ for whom V_λ^G has non-zero invariants under a mirabolic subgroup $Q_H^0 \subset H$. In Section 6.5.1 we show that there is a torus S such that if one considers the pair

  $$(\tilde{G}, \tilde{H}) = (G \times S, H \times S),$$

  then every weight of λ is invariant under a mirabolic subgroup $Q^0_{\tilde{H}} \subset \tilde{H}$ after twisting by a character of S acting trivially on parahoric level cohomology, essentially removing the admissibility condition.

5.2 Setup

We fix the notation used throughout the paper. Note that here G is an arbitrary group, whereas later we will fix groups G and H both of which satisfy the conditions set forth below.

Fix a prime p. We recall the following setting from [LRZ21]:

- Let $\mathcal{G}$ be a connected reductive group over $\mathbb{Q}$ satisfying Milne's axiom (SV5), that is, the centre contains no $\mathbb{R}$-split torus which is not $\mathbb{Q}$-split [9] .
- G is a reductive group scheme over $\mathbb{Z}_p$ whose base-extension to $\mathbb{Q}_p$ coincides with that of $\mathcal{G}$.
- Q_G is a choice parabolic subgroup of G and $\bar{Q}_G$ is the opposite parabolic, so that $L_G = Q_G \cap \bar{Q}_G$ is a Levi subgroup of Q_G and the big Bruhat cell $U^G_{\mathrm{Bru}} = \bar{N}_G \times L_G \times N_G$ is an open subscheme of G over $\mathbb{Z}_p$, where N_G is the unipotent radical of Q_G and $\bar{N}_G$ is its opposite.
- Let S_G denote the torus L_G/L_G^{der} with character lattice $X^\bullet(S_G)$ and let $X^\bullet_+(S_G)$ denote the Q_G-dominant weights. Let $C_G = G/G^{\mathrm{der}}$ denote the maximal torus quotient of G.
- Let $J_G \subset G(\mathbb{Z}_p)$ be the parahoric subgroup associated to Q_G. This group admits an Iwahori decomposition

  $$J_G = (J_G \cap \bar{N}_G(\mathbb{Z}_p)) \times L_G(\mathbb{Z}_p) \times N_G(\mathbb{Z}_p)$$

- Let A be the maximal $\mathbb{Q}_p$-split torus in the centre of L_G.

We choose a subtorus $S_G^0 \subset S_G$ and let L_G^0 and Q_G^0 be its preimages under the quotient maps $L_G \to S_G$ and $Q_G \to S_G$ respectively. Let Φ_G denote the roots of G, Δ_G the simple roots and Φ_G^+ the positive roots. Set $d_G = \mathrm{dim} N_G$.

5.2.1 Algebraic representations

Let $K/\mathbb{Q}_p$ be a finite unramified extension over which G splits and let $\mathcal{O}$ be the ring of integers of K. For $\lambda \in X^\bullet_+(S_G)$ we define

$$V_\lambda = \{f \in K[G] : f(\bar{n}\ell g) = \lambda(\ell) f(g) \ \forall \bar{n} \in \bar{N}_G, \ell \in L_G, g \in G\}$$

with g acting by right-translation. By the Borel–Weil–Bott theorem this is the irreducible G-representation of highest weight λ with respect to a Borel subgroup $B_G \subset Q_G$. We let f_λ denote the unique choice of highest weight vector satisfying $f(\bar{n}\ell n) = \lambda(\ell)$ for $\bar{n}\ell n \in U^G_{\mathrm{Bru}}$. We also write

$$\mathcal{P}^G_\lambda = \{f \in K[G] : f(n\ell g) = \lambda^{-1}(\ell) f(g) \ \forall n \in N_G, \ell \in L_G, g \in G\},$$

so that $(\mathcal{P}^G_\lambda)^\vee \cong V_\lambda$ as G-representations. A canonical choice of highest weight vector of $(\mathcal{P}^G_\lambda)^\vee$ with respect to B_G is given by the functional $\delta_1 : f \mapsto f(1)$.

---

[9] As noted in [LRZ21] this condition is for convenience and can be relaxed with some fastidious bookkeeping.

5.2.2 Integral lattices

Definition 5.2.1. An *admissible lattice* in V_λ is an $\mathcal{O}$-lattice $\mathcal{L} \subset V_\lambda$ invariant under $G_{/\mathcal{O}}$ and whose intersection with the highest weight subspace is $\mathcal{O} \cdot f_\lambda$.

We refer to [LRZ21, Section 2.3] for properties of admissible lattices. Let $V_{\lambda,\mathcal{O}}$ denote the maximal admissible lattice in V_λ, given in the Borel–Weil–Bott presentation as

$$V_{\lambda,\mathcal{O}} = \{f \in \mathcal{O}[G] : f(\bar{n}\ell g) = \lambda(\ell) f(g) \ \forall \ \bar{n} \in \bar{N}_G, \ell \in L_G, g \in G\}.$$

Similarly defining $\mathcal{P}^G_{\lambda,\mathcal{O}}$, then $(\mathcal{P}^G_{\lambda,\mathcal{O}})^\vee$ is isomorphic to the *minimal* admissible lattice in V_λ.

5.2.3 Cohomology of locally symmetric spaces

As in [LRZ21] we fix a neat prime-to-p level group K^p and for an open compact subgroup $U \subset G(\mathbb{Z}_p)$ let $Y_G(U)$ denote the locally symmetric space of level K^pU. We let $H^i(Y_G(U), \mathscr{F})$ refer to one of the following cohomology theories on $Y_G(U)$:

- Betti cohomology of the locally symmetric space $Y_G(U)$ viewed as a real manifold, with coefficients in a locally constant sheaf $\mathscr{F}$.

Suppose now that G admits a Shimura datum with reflex field E.

- Étale cohomology of $\bar{Y}_G(U) := Y_G(U)_{\bar{\mathbb{Q}}}$ with coefficients in a lisse étale sheaf $\mathscr{F}$.
- Let Σ be a sufficiently large finite set of primes containing those dividing p. We consider the étale cohomology of an integral model $Y_G(U)_\Sigma$ of $Y_G(U)$ defined over $\mathcal{O}_E[\Sigma^{-1}]$ with coefficients in $\mathscr{F}$. Here we assume the Shimura datum is of Hodge type.

5.2.4 Branching laws for algebraic representations

We now work in the situation of [Loe21] and [LRZ21] and consider an embedding $\mathcal{H} \to \mathcal{G}$ of reductive $\mathbb{Q}$-groups, extending to an embedding $H \to G$ of reductive group schemes over $\mathbb{Z}_p$. We assume we have choices of data as in Section 5.2 for both H and G. We in particular note that we require no compatibility between the choices of parabolic subgroups Q_H, Q_G other than those stated below.

Denote by $\mathcal{F} := \bar{Q}_G \backslash G$ the flag variety associated to the parabolic $\bar{Q}_G$. We assume that there is $u \in \mathcal{F}(\mathbb{Z}_p)$ satisfying:

(A) The Q^0_H-orbit of u is Zariski open in $\mathcal{F}$,

(B) The image of $\bar{Q}_G \cap uQ^0_H u^{-1}$ under the projection $\overline{Q}_G \to S_G$ is contained in S^0_G.

Under the above assumptions, the space $U_{\mathrm{Sph}} = \bar{Q}_G u Q^0_H$ is a Zariski open subset of G. We refer to it (and its image in the flag variety) as the *spherical cell.*

Remark 5.2.2. Note that since the flag variety is connected, $U^G_{\mathrm{Bru}} \cap U_{\mathrm{sph}} \neq \emptyset$ so we can always take $u \in U_{\mathrm{Bru}}(\mathbb{Z}_p)$ and in particular we can take $u \in N_G(\mathbb{Z}_p)$, which we assume from now on.

By [LRZ21, Proposition 3.2.1], for $\lambda \in X^\bullet_+(S_G)$ the space $V_\lambda^{Q^0_H}$ has dimension ≤ 1.

Definition 5.2.3. We call weights satisfying $\dim V_\lambda^{Q^0_H} = 1$ Q^0_H-*admissible weights* and denote the cone of such weights by $X^\bullet_+(S_G)^{Q^0_H}$.

For $\lambda \in X^\bullet_+(S_G)^{Q^0_H}$ the space $V_\lambda^{Q^0_H}$ is spanned by the polynomial function $f^{\mathrm{sph}}_\lambda \in K[G]$ uniquely defined by setting $f^{\mathrm{sph}}_\lambda(u) = 1$. The H-orbit of f^{sph}_λ generates an irreducible representation of H of some highest weight $\mu \in X^\bullet_+(S_H)$ with respect to a Borel $B_H \subset Q_H$. By [LRZ21, Proposition 3.2.6] $f^{\mathrm{sph}}_\lambda \in V_{\lambda,\mathcal{O}}$.

Definition 5.2.4. Define the *twig* $\phi \in \mathcal{O}(G \times H)$ by

$$\phi(g, h) = f^{\mathrm{sph}}_\lambda(gh^{-1}).$$

Lemma 5.2.5. *The twig ϕ has the following properties:*

- *For fixed $g \in G$, the function $h \mapsto \phi(g, h)$ is in $\mathcal{P}^H_{\mu,\mathcal{O}}$.*
- *For fixed $h \in H$, the function $g \mapsto \phi(g, h)$ is in $V_{\lambda,\mathcal{O}}$.*

Proof. The function ϕ is clearly algebraic in both variables. Fix $g \in G$, then for $\ell_h \in L_H$

$$\phi(g, \ell_h h) = f^{\mathrm{sph}}_\lambda(gh^{-1}\ell_h^{-1}) = \mu(\ell_h)^{-1}\phi(g, h),$$

Now fix $h \in H$. Let $\ell_g \in L_G$, then

$$\phi(\ell_g g, h) = f^{\mathrm{sph}}_\lambda(\ell_g gh^{-1}) = \lambda(\ell_g)\phi(g, h),$$

whence we are done. $\square$

Proposition 5.2.6. *Let $\lambda \in X^\bullet_+(S_G)^{Q^0_H}$ and let $\mu \in X^\bullet_+(S_H)$ be the associated weight of H. Then there is a unique H-equivariant map*

$$\mathrm{br}^\lambda_\mu : (\mathcal{P}^H_{\mu,\mathcal{O}})^\vee \to V_{\lambda,\mathcal{O}}$$

sending δ_1 to f^{sph}_λ. Under this map a functional $\varepsilon \in (\mathcal{P}^H_{\mu,\mathcal{O}})^\vee$ is mapped to the function on $G_{/\mathcal{O}}$ given by

$$g \mapsto \langle \varepsilon, \phi(g, -)\rangle.$$

5.2.5 Weight spaces

Write $\mathfrak{S}_G = L_G(\mathbb{Z}_p)/L^0_G(\mathbb{Z}_p) = S_G(\mathbb{Z}_p)/S^0_G(\mathbb{Z}_p) \subset (S_G/S^0_G)(\mathbb{Z}_p)$. The torus $\mathfrak{S}_G$ splits into a direct product $\mathfrak{S}_G = \mathfrak{S}^{\mathrm{tor}}_G \times \mathfrak{S}_{G,1}$ with the logarithm map identifying $\mathfrak{S}_{G,1} \cong \mathbb{Z}_p^{n_G}$ for some integer n_G and $\mathfrak{S}^{\mathrm{tor}}_G$ of finite order.

Definition 5.2.7. Write $\mathcal{W}_G$ for the rigid analytic space over $\mathbb{Q}_p$ parameterising continuous characters of $\mathfrak{S}_G$. The space $\mathcal{W}_G$ admits a formal model $\mathfrak{W}_G := \mathrm{Spf}\Lambda(\mathfrak{S}_G)$ over $\mathbb{Z}_p$, where $\Lambda(\mathfrak{S}_G)$ is the Iwasawa algebra associated to $\mathfrak{S}_G$.

Definition 5.2.8. For $i = 1, \dots, n$ let $s_i \in \mathfrak{S}_{G,1}$ be a $\mathbb{Z}_p$-basis. For an integer $m \geq 0$ we denote by $\mathcal{W}_m \subset \mathcal{W}_G$ the wide-open subspace consisting of weights λ satisfying

$$v_p(\lambda(s_i) - 1) > \frac{1}{p^m(p-1)}$$

for all i.

Fix a finite extension $E/\mathbb{Q}_p$ and for some $m \geq 0$ let $\mathcal{U} \subset \mathcal{W}_m$ be a wide-open disc defined over E. Let $\Lambda_G(\mathcal{U}) \cong \mathcal{O}_E[[t_1, \dots, t_n]]$ denote the $\mathcal{O}_E$-algebra of bounded-by-one rigid functions on $\mathcal{U}$ with $\mathfrak{m}_{\mathcal{U}}$ its maximal ideal.

Let

$$S^{\mathrm{stab}}_H = \frac{Q_H \cap u^{-1}\bar{Q}_G u}{Q^0_H \cap u^{-1}\bar{Q}_G u},$$

the quotient of the stabilisers of $[u] \in \mathcal{F}(\mathbb{Z}_p)$ in Q_H and Q^0_H. As remarked in [LRZ21, Remark 3.2.4] the natural map $S^{\mathrm{stab}}_H \to Q_H/Q^0_H$ is an isomorphism. It turns out more is true:

Lemma 5.2.9. *The natural inclusion*

$$(Q_H \cap u^{-1}\bar{Q}_G u)(\mathbb{Z}_p) \to Q_H(\mathbb{Z}_p)$$

induces an isomorphism

$$\frac{(Q_H \cap u^{-1}\bar{Q}_G u)(\mathbb{Z}_p)}{(Q^0_H \cap u^{-1}\bar{Q}_G u)(\mathbb{Z}_p)} \cong Q_H(\mathbb{Z}_p)/Q^0_H(\mathbb{Z}_p).$$

Proof. For $g \in Q_H(\mathbb{Z}_p)$, write $U_g := U_{\mathrm{sph}} g$. Each U_g is Zariski open and thus $U_g \cap U_1 \neq \emptyset$ for all g. Thus for any $g \in Q_H(\mathbb{Z}_p)$ there is $r_1, r_g \in \bar{Q}_G(\mathbb{Z}_p), q_1, q_g \in Q_H^0(\mathbb{Z}_p)$ such that

$$r_1 u q_1 = r_g u g q_g$$

and so $g q_g q_1^{-1} = u^{-1} r_g^{-1} r_1 u \in (u^{-1}\bar{Q}_G u \cap Q_H)(\mathbb{Z}_p)$. In particular, for any $g \in Q_H(\mathbb{Z}_p)$ there is $q \in Q_H^0(\mathbb{Z}_p)$ such that $gq \in (u^{-1}\bar{Q}_G u)(\mathbb{Z}_p)$, whence the claim follows. ☐

Definition 5.2.10. We define a group scheme homomorphism

$$\omega : S_H/S_H^0 \to Q_H/Q_H^0 \to S_H^{\mathrm{stab}} \to S_G/S_G^0$$

where the last map is given by conjugation by u and projection to S_G. This is well defined by assumption (B).

Given a Q_H^0-admissible weight λ the associated H-weight μ is given by $\lambda \circ \omega$. By Lemma 5.2.9, ω induces a homomorphism

$$\mathfrak{S}_H \to \mathfrak{S}_G$$

of profinite abelian groups.

Lemma 5.2.11. *There is a morphism of affine formal schemes:*

$$\Omega : \mathfrak{W}_G \to \mathfrak{W}_H$$

given on points by $\Omega(\lambda)(s) = \lambda(\omega(s))$.

Proof. The map

$$\Lambda(\mathfrak{S}_H) \to \Lambda(\mathfrak{S}_G)$$

given by linearly extending $[s] \mapsto [\omega(s)]$ is a map of topological rings since ω is algebraic and therefore continuous. ☐

Define the universal character for the torus $\mathfrak{S}_G$

$$\begin{aligned} k_{\mathrm{univ}}^G : \mathfrak{S}_G &\to \Lambda(\mathfrak{S}_G)^\times \\ s &\mapsto [s] \end{aligned}$$

where $\Lambda(\mathfrak{S}_G)$ is the Iwasawa algebra of $\mathfrak{S}_G$ and is canonically isomorphic to the bounded-by-1 global sections of $\mathcal{W}_G$. Note that $k_{\mathrm{univ}}^G \in \mathfrak{W}_G(\Lambda_G(\mathcal{U}))$.

Define

$$k_{\mathrm{univ}}^H : \mathfrak{S}_H \to \Lambda(\mathfrak{S}_G)^\times$$

by $k_{\mathrm{univ}}^H = \Omega(k_{\mathrm{univ}}^G)$. Let $\mathcal{U} \subset \mathcal{W}_G$ be a wide open disc. Define

$$k_{\mathcal{U}}^G : \mathfrak{S}_G \to \Lambda_G(\mathcal{U})^\times$$

to be the character given by composing k_{univ}^G with restriction to $\mathcal{U}$. Define

$$k_{\mathcal{U}}^H = \Omega(k_{\mathcal{U}}^G) : \mathfrak{S}_H \to \Lambda_G(\mathcal{U})^\times.$$

This character has the useful property that for $\lambda \in \mathcal{U}$

$$\lambda \circ k_{\mathcal{U}}^H = \Omega(\lambda).$$

Lemma 5.2.12. *If $\mathcal{U} \subset \mathcal{W}_m$ then the characters $k_{\mathcal{U}}^G, k_{\mathcal{U}}^H$ are m-analytic on $\mathfrak{S}_G$, (resp. $\mathfrak{S}_H$), viewed as a disjoint union of copies of $\mathbb{Z}_p^{n_G}$ (resp. $\mathbb{Z}_p^{n_H}$) indexed by $\mathfrak{S}_G^{\mathrm{tor}}$ (resp. $\mathfrak{S}_H^{\mathrm{tor}}$).*

Proof. Since $k_{\mathcal{U}}^H$ is the pullback of $k_{\mathcal{U}}^G$ by an algebraic map, it suffices to prove the lemma for the latter character. Moreover, since $k_{\mathcal{U}}^G$ is a character it suffices to show m-analyticity on $\mathfrak{S}_{G,1}$. If we let $\{s_i\}_i$ be a basis for $\mathfrak{S}_{G,1} \cong (1 + p\mathbb{Z}_p)^n$ then we need to show that for any $z \in \mathbb{Z}_p$ $z \mapsto k_{\mathcal{U}}^G(s_i^z)$ is m-analytic on $\mathbb{Z}_p$. We can then proceed much as in [LZ16, Lemma 4.1.5]. ☐

5.2.6 Hecke algebras

Let $K^p \subset G(\mathbb{A}_f^{(p)})$ be a prime-to-p level group and let S be a finite set of primes containing those at which K^p is ramified and not containing p. Define

$$\mathbb{T}_S := \mathcal{C}_c^\infty(K^p \backslash G(\mathbb{A}_f^{S\cup\{p\}})/K^p, \mathbb{Z}_p)$$

the space of $\mathbb{Z}_p$-valued compactly supported locally constant K^p-biinvariant functions on $G(\mathbb{A}_f^{S\cup\{p\}})$. This is a commutative $\mathbb{Z}_p$-algebra.

Set

$$A^- = \{a \in A : v_p(\alpha(a)) \geq 0 \ \forall \ \alpha \in \Delta_G \backslash \Delta_L\}$$

and define $A^{--} \subset A^-$ with a strict inequality. We then define the double coset algebra

$$\mathfrak{U}_p^- := \mathbb{Z}_p[J_G a J_G : a \in A^-],$$

and similarly $\mathfrak{U}_p^{--}$. There is a $\mathbb{Z}_p$-algebra isomorphism

$$\mathbb{Z}_p[A^-/A(\mathbb{Z}_p)] \cong \mathfrak{U}_p^-.$$

In particular, $\mathfrak{U}_p^-$ is commutative.

Definition 5.2.13. Define the unramified Q_G-parahoric Hecke algebra:

$$\mathbb{T}_{S,p}^- := \mathbb{T}_S \otimes \mathfrak{U}_p^-.$$

5.2.7 Locally analytic function spaces

Let B be one of the local rings $\mathcal{O}$ or $\Lambda_G(\mathcal{U})$ for a wide-open disc $\mathcal{U} \subset \mathcal{W}_G$ and let $\mathfrak{m}_B$ be its maximal ideal and a character $\kappa : T_G(\mathbb{Z}_p) \to B^\times$ such that

$$\kappa = \begin{cases} \lambda & \text{if } B = \mathcal{O} \\ k_{\mathcal{U}}^G & \text{if } B = \Lambda_G(\mathcal{U}) \end{cases}$$

for some $\lambda \in X^\bullet(T_G)$.

Definition 5.2.14. For $m \geq 0$ define

$$\mathrm{LA}_m(\mathbb{Z}_p^d, B) = \{f : \mathbb{Z}_p^d \to B : \forall \ \underline{a} \in \mathbb{Z}_p^d, \exists f_{\underline{a}} \in B\langle T_1 \dots, T_d\rangle \text{ s.t. } f(\underline{a} + p^m \underline{x}) = f_{\underline{a}}(\underline{x}) \ \forall \underline{x} \in \mathbb{Z}_p^d\}.$$

This space is isomorphic to $\prod_{\underline{a}} B\langle p^{-m}T_1, \dots, p^{-m}T_d\rangle$ as a B-module, where $\underline{a}$ runs over $(\mathbb{Z}/p^m\mathbb{Z})^d$.

The exponential map gives an isomorphism $N_G(\mathbb{Z}_p) \cong (\mathrm{Lie}N_G)(\mathbb{Z}_p) \cong \mathbb{Z}_p^{d_G}$, where for the second isomorphism we choose a basis such that $N_G(p\mathbb{Z}_p) = p\mathbb{Z}_p^{d_G}$.

Definition 5.2.15. For $m \geq 0$, define

$$A_{\kappa,m}^{\mathrm{an}} = \{f : U_{\mathrm{Bru}}^G(\mathbb{Z}_p) \to B : f(\bar{n}tn) = \kappa(t)f(n), \text{ and } f|_{N_G(\mathbb{Z}_p)} \in \mathrm{LA}_m(\mathbb{Z}_p^{d_G}, B)\}$$

equipped with the $\mathfrak{m}_B$-adic topology. If $B = \Lambda_G(\mathcal{U})$ for a wide open disc $\mathcal{U} \subset \mathcal{W}_m$ and $\kappa = k_{\mathcal{U}}^G$, we write $A_{\mathcal{U},m}^{\mathrm{an}} := A_{k_{\mathcal{U}}^G,m}^{\mathrm{an}}$.

Restriction to $N_G(\mathbb{Z}_p)$ gives a B-module isomorphism $A_{\kappa,m}^{\mathrm{an}} \cong \mathrm{LA}_m(\mathbb{Z}_p^{d_G}, B)$ with inverse $f \mapsto (\bar{n}tn \mapsto \kappa(t)f(n))$. We give these spaces an action of $a \in A^-$ via

$$(a \cdot f)(\bar{n}\ell n) = f(\bar{n}\ell a n a^{-1}).$$

The following proposition is well-known and the proof is very similar to that of Proposition 5.2.26 below (which is less well-known) so we omit it.

Proposition 5.2.16. *Suppose $\kappa : \mathfrak{S}_G(\mathbb{Z}_p) \to B^\times$ is an m-analytic character. The modules $A_{\kappa,m}^{\mathrm{an}}$ are preserved by both the right translation action of J_G and the action of A^- described above.*

Adapting the proof of [Urb11, 3.2.8] to our situation we see that for $a \in A^{--}$

$$aA^{\mathrm{an}}_{\kappa,m+1} \subset A^{\mathrm{an}}_{\kappa,m} \tag{22}$$

and thus the action of A^{--} is by compact operators since the inclusions $A^{\mathrm{an}}_{\kappa,m} \subset A^{\mathrm{an}}_{\kappa,m+1}$ are compact.

If $\lambda \in X^{\bullet}_{+}(S_G)$ then there is a natural J_G-equivariant inclusion

$$V^G_\lambda \hookrightarrow A^{\mathrm{an}}_{\lambda,m},$$

which also preserves the A^- action.

Definition 5.2.17. If $\lambda \in \mathcal{U} \subset \mathcal{W}_m$ is the restriction to $\mathfrak{S}_G(\mathbb{Z}_p)$ of an algebraic character of $(S_G/S^0_G)(\mathbb{Q}_p)$, then there is a natural specialisation map

$$\rho_\lambda : A^{\mathrm{an}}_{\mathcal{U},m} \to A^{\mathrm{an}}_{\lambda,m}$$

given by post-composing with the map

$$\Lambda_G(\mathcal{U}) \to \Lambda_G(\mathcal{U}) \otimes_\lambda \mathcal{O}_E \cong \mathcal{O}_E.$$

The space $A^{\mathrm{an}}_{\mathcal{U},m}$ is not the unit ball in a Banach algebra, but we can define a basis $\{e_i\}_{i\in I}$ for a countable indexing set I such that for any $f \in A^{\mathrm{an}}_{\mathcal{U},m}$ there are $a_i \in \Lambda_G(\mathcal{U})$ such that

- $a_i \to 0$ in the cofinite filtration on I.
- We have $f = \sum_{i\in I} a_i e_i$.

This can be seen from the identification of $A^{\mathrm{an}}_{\mathcal{U},m}$ with a product of Tate algebras. This basis is sufficient to define Fredholm determinants of compact operators.

Definition 5.2.18. Let $\mathcal{V} \subset \mathcal{U} \subset \mathcal{W}_G$ be an affinoid contained in a wide open disc $\mathcal{U}$. Define

$$A^{\mathrm{an}}_{\mathcal{V},m} := A^{\mathrm{an}}_{\mathcal{U},m} \hat{\otimes} \mathcal{O}(\mathcal{V})^\circ$$

where $\mathcal{O}(\mathcal{V})^\circ$ are the bounded-by-1 global sections of $\mathcal{V}$.

The space $A^{\mathrm{an}}_{\mathcal{V},m}[1/p] = A^{\mathrm{an}}_{\mathcal{U},m} \hat{\otimes} \mathcal{O}(\mathcal{V})$ is an orthonormalisable Banach $\mathcal{O}(\mathcal{V})$-module with unit ball $A^{\mathrm{an}}_{\mathcal{V},m}$.

Let $H \hookrightarrow G$ be an embedding and let $\bar{N}_H(p\mathbb{Z}_p) \cong \mathbb{Z}_p^{d_H}$ be the algebraic isomorphism given by the logarithm map.

Definition 5.2.19. For $\kappa \in \{k^H_{\mathcal{U}}, \mu\}$ where $\mu = \lambda \circ \omega$ for some $\lambda \in X^{\bullet}_{+}(S_G)$, define

$$\mathcal{P}^{H,\mathrm{an}}_{\kappa,m} := \{f : J_H \to B : f(nt\bar{n}) = \kappa(t)^{-1} f(\bar{n}), \text{ and } f|_{\bar{N}_H(p\mathbb{Z}_p)} \in \mathrm{LA}_m(\mathbb{Z}_p^{d_H}, B)\}.$$

These modules clearly satisfy all the properties of $A^{\mathrm{an}}_{\kappa,m}$ and admit a natural J_H-equivariant inclusion

$$\mathcal{P}^H_\lambda \hookrightarrow \mathcal{P}^{H,\mathrm{an}}_{\lambda,m}.$$

5.2.8 Locally analytic distribution modules

Suppose we have an embedding of reductive $\mathbb{Q}$-groups $H \hookrightarrow G$ satisfying the conditions outlined in the introduction.

Definition 5.2.20. For $\kappa \in \{k^H_{\mathcal{U}}, \mu\}$ where $\mu = \lambda \circ \omega$ for some $\lambda \in X^{\bullet}_{+}(S_G)$, define

$$D^H_{\kappa,m} = \mathrm{Hom}_{\mathrm{cont},B}(\mathcal{P}^{H,an}_{\kappa,m}, B),$$

given the weak topology i.e. the weakest topology making the evaluation maps continuous.

These spaces are topologically generated as a $B[J_H]$-modules by evaluation-at-1 distribution δ_1. For $m \geq 1$ let $X^{(i)}_{\kappa,m}$ be the image of $D^H_{\kappa,m}$ in $D^H_{\kappa,m-1}/p^i$.

Definition 5.2.21. Define

$$\mathrm{Fil}^n D^H_{\kappa,m} \subset D^H_{\kappa,m}$$

to be the kernel of the composition

$$D^H_{\kappa,m} \to X^{(i)}_{\kappa,m} \to X^{(i)}_{\kappa,m} \otimes_{B/p^i} B/\mathfrak{m}_B.$$

By generalising the argument of Hansen [Han15] we see that $\{\mathrm{Fil}^n D^H_{\kappa,m}\}_{n\geq 0}$ is a decreasing filtration invariant under J_H such that $D^H_{\kappa,m}/\mathrm{Fil}^n D^H_{\kappa,m}$ is finite for all $n \geq 0$ and

$$D^H_{\kappa,m} = \varprojlim_n D^H_{\kappa,m}/\mathrm{Fil}^n_{\kappa,m},$$

which allows us to define a lisse étale sheaf $\mathscr{D}^H_{\kappa,m}$ over our symmetric spaces for H.

Definition 5.2.22. Define a $\Lambda_G(\mathcal{U})$-linear specialisation map

$$\mathrm{sp}_\mu : D^H_{\mathcal{U},m} \to (\mathcal{P}^H_{\mu,\mathcal{O}})^\vee.$$

as the map uniquely characterised by preserving the evaluation-at-1 map δ_1.

5.2.9 Locally Iwasawa functions

Let B, d_G be as in the previous section.

Definition 5.2.23. For $m \geq 0$ define

$$\mathrm{LI}_{m+1}(\mathbb{Z}_p^{d_G}, B) = \{f : \mathbb{Z}_p^{d_G} \to B : \forall\, \underline{a} \in \mathbb{Z}_p^d, \exists f_{\underline{a}} \in B[[T_1 \dots, T_{d_G}]] \text{ s.t. } f(\underline{a}+p^{m+1}\underline{x}) = f_{\underline{a}}(p\underline{x})\ \forall \underline{x} \in \mathbb{Z}_p^{d_G}\}.$$

This space admits a natural structure as a $B[[T_1, \dots, T_{d_G}]]$-module and is isomorphic to $\prod_{\underline{a}} B[[p^{-m}T_1 \dots, p^{-m}T_{d_G}]]$.

Definition 5.2.24. Let $\mathfrak{n}_B$ be the maximal ideal of $B[[T_1, \dots, T_{d_G}]]$. Define a filtration Fil^n_{m+1} on $\mathrm{LI}_{m+1}(\mathbb{Z}_p^{d_G}, B)$ by

$$\mathrm{Fil}^n_{m+1} := \mathfrak{n}_B^n \mathrm{LI}_{m+1}(\mathbb{Z}_p^{d_G}, B).$$

We see that

$$\mathrm{LI}_{m+1}(\mathbb{Z}_p^{d_G}, B) \cong \varprojlim_n \mathrm{LI}_{m+1}(\mathbb{Z}_p^{d_G}, B)/\mathrm{Fil}^n_{m+1}.$$

The modules $\mathrm{LI}_{m+1}(\mathbb{Z}_p^{d_G}, B)/\mathrm{Fil}^n_{m+1}$ are finite and thus the modules $\mathrm{LI}_{m+1}(\mathbb{Z}_p^{d_G}, B)$ are profinite and in particular they are $\mathfrak{n}_B$-adically complete and separated.

For $m \geq 0$ there is a chain of inclusions

$$\mathrm{LA}_m(\mathbb{Z}_p^{d_G}, B) \subset \mathrm{LI}_{m+1}(\mathbb{Z}_p^{d_G}, B) \subset \mathrm{LA}_{m+1}(\mathbb{Z}_p^{d_G}, B) \tag{23}$$

given by restriction. This corresponds to

$$\begin{aligned}\prod_{\underline{a} \mod p^m} B\langle p^{-m}T_1, \dots, p^{-m}T_{d_G}\rangle \hookrightarrow \prod_{\underline{a} \mod p^m} & B[[p^{-m}T_1, \dots, p^{-m}T_{d_G}]] \\ \hookrightarrow \prod_{\underline{a} \mod p^{m+1}} & B\langle p^{-(1+m)}T_1, \dots, p^{-(1+m)}T_{d_G}\rangle.\end{aligned}$$

Write $\varinjlim_m \mathrm{LA}_m(\mathbb{Z}_p^{d_G}, B) = \mathrm{LA}(\mathbb{Z}_p^{d_G}, B)$ for the space of all B-valued locally analytic functions on $\mathbb{Z}_p^{d_G}$. The inductive systems $\{\mathrm{LA}_m(\mathbb{Z}_p^d, B)\}_{m\geq 0}$ and $\{\mathrm{LI}_{m+1}(\mathbb{Z}_p^{d_G}, B)\}_{m\geq 0}$ are cofinal in the inverse system

$$\dots \to \mathrm{LA}_m(\mathbb{Z}_p^{d_G}, B) \to \mathrm{LI}_{m+1}(\mathbb{Z}_p^{d_G}, B) \to \mathrm{LA}_{m+1}(\mathbb{Z}_p^{d_G}, B) \to \dots,$$

with the arrows given by the inclusions (23). Thus

$$\varinjlim_m \mathrm{LI}_m(\mathbb{Z}_p^{d_G}, B) = \mathrm{LA}(\mathbb{Z}_p^{d_G}, B).$$

Let $\kappa : S_G(\mathbb{Z}_p) \to B$ be an m-analytic character.

Definition 5.2.25. For $m \geq 0$ define

$$A^{\mathrm{Iw}}_{\kappa,m+1} := \{f : U^G_{\mathrm{Bru}}(\mathbb{Z}_p) \to B : f(\bar{n}tn) = \kappa(t)f(n), \text{ and } f|_{N_G(\mathbb{Z}_p)} \in \mathrm{LI}_{m+1}(\mathbb{Z}_p^{d_G}, B)\},$$

given the $\mathfrak{m}_B$-adic topology. As in the locally analytic case we write $A^{\mathrm{Iw}}_{\mathcal{U},m} := A^{\mathrm{Iw}}_{k_{\mathcal{U}},m}$ when $\mathcal{U} \subset \mathcal{W}_m$.

These spaces are isomorphic to $\mathrm{LI}_m(\mathbb{Z}_p^{d_G}, B)$ as B-modules and the filtration on $\mathrm{LI}_m(\mathbb{Z}_p^{d_G}, B)$ defines a filtration $\mathrm{Fil}^n_{\kappa,m} A^{\mathrm{Iw}}_{\kappa,m} \subset A^{\mathrm{Iw}}_{\kappa,m}$.

When no danger of ambiguity exists we will write $\mathrm{Fil}^n_{\kappa,m}$ for $\mathrm{Fil}^n_{\kappa,m} A^{\mathrm{Iw}}_{\kappa,m}$.

Proposition 5.2.26. *for $n \geq 1$ the modules $A^{\mathrm{Iw}}_{\kappa,m}, \mathrm{Fil}^n_{\kappa,m}$ are preserved by the actions of J_G and A^- inherited from those on $A^{\mathrm{an}}_{\kappa,m+1}$.*

Proof. For $m \in \mathbb{Q}_{\geq 0}$, let $\mathcal{G}_m$ be the $\mathbb{Q}_p$-rigid space of points $g \in G$ such that

$$|g - 1| \leq p^m$$

and set $\mathcal{G}^\circ_m = \cup_{m' > m} \mathcal{G}_{m'} \subset \mathcal{G}_0$. Define a rigid analytic group $\mathcal{J}^\circ_{G,m} = \mathcal{G}^\circ_m \cdot J_G$ (the group generated by J_G and $\mathcal{G}^\circ_m$ in $\mathcal{G}_0$). This group admits an Iwahori decomposition

$$\mathcal{J}^\circ_{G,m} = (\bar{\mathcal{N}}^\circ_m \cdot \bar{N}_G(\mathbb{Z}_p)) \times (\mathcal{L}^\circ_m \cdot L_G(\mathbb{Z}_p)) \times (\mathcal{N}^\circ_m \cdot N_G(\mathbb{Z}_p)).$$

Choosing a set Γ of representatives for J_G mod p^{m+1} then

$$\mathcal{J}^\circ_{G,m} = \sqcup_{\gamma \in \Gamma} \mathcal{G}^\circ_m \gamma$$

The space $A^{\mathrm{Iw}}_{\kappa,m}$ is identified with the module of B-valued bounded-by-1 rigid functions F on $\mathcal{J}^\circ_{G,m}$ such that for $\bar{n}\ell \in \bar{Q}_G(\mathbb{Z}_p) \cdot \mathcal{G}^\circ_m$, $j \in \mathcal{J}^\circ_{G,m}$ we have

$$F(\bar{n}\ell j) = \kappa(\ell) F(j).$$

Viewed via this optic, it is clear that $A^{\mathrm{Iw}}_{\kappa,m}$ is stable under right translation by J_G.

Let $\mathbb{1}$ be the trivial character. The space $A^{\mathrm{Iw}}_{\mathbb{1},m}$ is a subring of $\mathcal{O}(\mathcal{J}^\circ_{G,m})_B := \mathcal{O}(\mathcal{J}^\circ_{G,m})^{\leq 1} \hat{\otimes}_{\mathbb{Z}_p} B$ and acts on $A^{\mathrm{Iw}}_{\kappa,m}$ via multiplication of functions in $\mathcal{O}(\mathcal{J}^\circ_{G,m})_B$. Then $\mathrm{Fil}^n_{\mathbb{1},m}$ is an ideal of $A^{\mathrm{Iw}}_{\mathbb{1},m}$ and $\mathrm{Fil}^n_{\mathbb{1},m} = (\mathrm{Fil}^1_{\mathbb{1},m})^n$. Furthermore,

$$\mathrm{Fil}^n_{\kappa,m} = \mathrm{Fil}^n_{\mathbb{1},m} A^{\mathrm{Iw}}_{\kappa,m}.$$

Since the action of J_G respects the ring structure of $\mathcal{O}(\mathcal{J}^\circ_{G,m})_B$ it suffices to show that $\mathrm{Fil}^1_{\kappa,m}$ is preserved by J_G.

We note that

$$\mathrm{Fil}^1_{\kappa,m} = \{F \in A^{\mathrm{Iw}}_{\kappa,m} : F(\gamma) \equiv 0 \bmod p \ \forall \ \gamma \in \Gamma\},$$

so taking $F \in \mathrm{Fil}^1, j \in J_G$ then for $\gamma \in \Gamma$ there is $\gamma_j \in \Gamma$ and $\varepsilon \equiv \ 1 \bmod p^{m+1}$ such that $\varepsilon\gamma_j = \gamma j$ and thus

$$(j \cdot F)(\gamma) = F(\varepsilon\gamma_j) \equiv 0 \bmod p.$$

The action of A^- on $\mathcal{J}^\circ_{G,m}$ is by

$$a * \mathcal{J}^\circ_{G,m} = (\bar{\mathcal{N}}^\circ_m \cdot \bar{N}_G(\mathbb{Z}_p)) \times (\mathcal{L}^\circ_m \cdot L_G(\mathbb{Z}_p)) \times a(\mathcal{N}^\circ_m \cdot N_G(\mathbb{Z}_p))a^{-1}$$

We can see that this is well-defined noting that for a set of representatives Γ' of $N_G(\mathbb{Z}_p)$ mod p^{m+1} we have

$$\mathcal{N}^\circ_m \cdot N_G(\mathbb{Z}_p) = \sqcup_{n \in \Gamma'} \mathcal{N}^\circ_m \cdot n$$

and each $\mathcal{N}^\circ_m \cdot n$ is isomorphic to a direct product of balls $\mathcal{U}_{\alpha,m}$ of radius p^{-m} and centre 0 contained in the root spaces $\mathcal{U}_\alpha$:

$$\mathcal{N}^\circ_m \cdot n = \prod_{\alpha \in \Phi^L} \mathcal{U}_{\alpha,m},$$

where $\Phi_L = \Phi_G \backslash \Phi_L$. Thus we have an isomorphism

$$\mathcal{N}_m^\circ \cdot N_G(\mathbb{Z}_p) = \prod_{\alpha \in \Phi^L} \mathcal{U}_{\alpha,m} \times N_G(\mathbb{Z}/p^{m+1}\mathbb{Z})$$

with the action of $a \in A^-$ given by

$$\begin{aligned} a\mathcal{N}_m^\circ \cdot N_G(\mathbb{Z}_p)a^{-1} &= \prod_{\alpha \in \Phi^L} p^{v_p(\alpha(a))}\mathcal{U}_{\alpha,m} \times aN_G(\mathbb{Z}/p^{m+1}\mathbb{Z})a^{-1} \\ &= \prod_{\alpha \in \Phi^L} \mathcal{U}_{\alpha,m+v_p(\alpha(a))} \times aN_G(\mathbb{Z}/p^{m+1}\mathbb{Z})a^{-1} \end{aligned}$$

and $\mathcal{U}_{\alpha,m+v_p(\alpha(a))} \subset \mathcal{U}_{\alpha,m}$ since $v_p(\alpha(a)) \geq 0$ by definition of A^-, so $a\mathcal{N}_m^\circ \cdot N_G(\mathbb{Z}_p)a^{-1} \subset \mathcal{N}_m^\circ \cdot N_G(\mathbb{Z}_p)$. We can prove the filtration $\mathrm{Fil}^n_{\kappa,m}$ is invariant under the action of A^- in a similar way to that of J_G. ☐

Corollary 5.2.27. *Let $K^p \subset G(\mathbb{A}^p)$ be a neat open-compact subgroup. The modules $A^{\mathrm{Iw}}_{\kappa,m}$ induce lisse étale sheaves $\mathscr{A}^{\mathrm{Iw}}_{\kappa,m}$ over $Y_G(K^pJ_G)$.*

6 Locally analytic branching laws

Let $\mathcal{U} \subset \mathcal{W}_G$ be a wide-open disc and $m \geq 0$. We construct branching maps

$$D^H_{\mathcal{U},m} \to A^{\mathrm{an}}_{\mathcal{U},m}$$

interpolating the algebraic branching maps of Section 5.2.4.

6.1 The Big Twig

We consider functions on the set $(U_{\mathrm{sph}} \cdot Q_H)(\mathbb{Z}_p) = (\bar{Q}_G u Q_H)(\mathbb{Z}_p)$.

Lemma 6.1.1. *The function*

$$\begin{aligned} f_{\mathcal{U}} : (U_{\mathrm{sph}} \cdot Q_H)(\mathbb{Z}_p) &\to \Lambda_G(\mathcal{U}) \\ \bar{n}\ell u q g &\mapsto k^H_{\mathcal{U}}(g)k^G_{\mathcal{U}}(\ell) \end{aligned}$$

for $\bar{n}\ell \in \bar{Q}_G(\mathbb{Z}_p), q \in Q^0_H(\mathbb{Z}_p), g \in Q_H(\mathbb{Z}_p)$ is well-defined.

Proof. For $i = 1, 2$, let $g_i \in Q_H(\mathbb{Z}_p), q_i \in Q^0_H(\mathbb{Z}_p)$ and $\bar{n}_i\ell_i \in \bar{Q}_G(\mathbb{Z}_p)$, and suppose (noting that Q_H normalises Q^0_H so without loss of generality we can swap g_i and q_i in our presentation) $\bar{n}_1\ell_1 u g_1 q_1 = \bar{n}_2\ell_2 u g_2 q_2$, then rearranging we get

$$(\bar{n}_2\ell_2 u g_2)^{-1}\bar{n}_1\ell_1 u g_1 \in Q^0_H(\mathbb{Z}_p), \tag{†}$$

so the map ω of Definition 5.2.10 applied to the left hand side is trivial. Since $Q^0_H(\mathbb{Z}_p) \subset Q_H(\mathbb{Z}_p)$ we have that

$$u^{-1}\ell_2^{-1}\bar{n}_2^{-1}\bar{n}_1\ell_1 u \in Q_H(\mathbb{Z}_p) \cap u^{-1}\bar{Q}_G(\mathbb{Z}_p)u,$$

whose image under

$$Q_H(\mathbb{Z}_p) \cap u^{-1}\bar{Q}_G(\mathbb{Z}_p)u \to S^{\mathrm{stab}}_H(\mathbb{Z}_p) \to (S_G/S^0_G)(\mathbb{Z}_p)$$

is equal to the image of $\ell_2^{-1}\ell_1$ under the projection $L_G(\mathbb{Z}_p) \to (S_G/S^0_G)(\mathbb{Z}_p)$. Applying ω to (†), we get

$$1 = \omega((\bar{n}_2\ell_2 u g_2)^{-1}\bar{n}_1\ell_1 u g_1) = \omega(g_2)^{-1}\omega(g_1)\omega(u^{-1}\ell_2^{-1}\bar{n}_2^{-1}\bar{n}_1\ell_1 u) = \ell_2^{-1}\ell_1\omega(g_2)^{-1}\omega(g_1),$$

so

$$\ell_2\omega(g_2) = \ell_1\omega(g_1) \bmod S^0_G(\mathbb{Z}_p)$$

and applying $k^G_{\mathcal{U}}$ allows us to conclude. ☐

Remark 6.1.2. The above proof shows that we can view $f_{\mathcal{U}}$ as the pullback of $k_{\mathcal{U}}^{G}$ via the map

$$\begin{aligned}(U_{\mathrm{sph}} \cdot Q_H)(\mathbb{Z}_p) &\to \mathfrak{S}_G, \\ \bar{n}\ell u q &\mapsto \ell\omega(q) \bmod L_G^0(\mathbb{Z}_p).\end{aligned}$$

Definition 6.1.3. Define a $*$-action of A^- on $U_{\mathrm{Bru}}(\mathbb{Z}_p)$ by

$$a * \bar{n}\ell n = \bar{n}\ell a n a^{-1}.$$

Lemma 6.1.4. *For any $a \in A^{--}$ we have*

$$(a * U_{\mathrm{Bru}}(\mathbb{Z}_p))\, u \subset U_{\mathrm{Sph}}(\mathbb{Z}_p).$$

Proof. It suffices to check this on $\mathcal{F}(\mathbb{Z}_p)$. Furthermore, as U_{Sph} is Zariski open in $\mathcal{F}_G$, it suffices to show that the inclusion holds mod p. We compute

$$\begin{aligned}(a * U_{\mathrm{Bru}}(\mathbb{Z}_p))\, u &= \bar{Q}_G(\mathbb{Z}_p) a N_G(\mathbb{Z}_p) a^{-1} u \\ &\equiv \bar{Q}_G(\mathbb{F}_p) u && \bmod p \\ &\in U_{\mathrm{Sph}}(\mathbb{F}_p),\end{aligned}$$

where the second equality follows from the fact that $aN_G(\mathbb{Z}_p)a^{-1} \equiv 1 \bmod p$. □

Fix $\tau \in A^{--}$. We want to study locally analytic functions on U_{sph}. Let $[U_{\mathrm{sph}}]$ denote the image of U_{Sph} in $\mathcal{F}_G$ (alternatively the orbit of the image of u under Q_H^0). For any r there is an injection

$$\begin{aligned}\phi : \mathbb{Z}_p^d \cong \tau^r N_G(\mathbb{Z}_p)\tau^{-r} &\hookrightarrow [U_{\mathrm{Sph}}] \\ \tau^r n \tau^{-r} &\mapsto [u\tau^r n \tau^{-r}]\end{aligned}$$

which defines an open compact chart around $[u] \in U_{\mathrm{Sph}}(\mathbb{Z}_p) \subset \mathcal{F}_G(\mathbb{Z}_p)$. Let $V_u^{(r)}$ be the image of this map, an open compact neighbourhood of $[u]$. By Q_H^0-homogeneity this defines an atlas $\{\phi_q : \mathbb{Z}_p^d \to V_u^{(r)} q\}_{q \in Q_H^0}$, giving $[U_{\mathrm{Sph}}]$ the structure of a p-adic manifold.

Definition 6.1.5. For a character $\kappa : \mathfrak{S}_G \to B_\kappa^\times$, define

$$A_{\kappa,\mathrm{Sph}}^{an} := \{f : U_{\mathrm{Sph}} \to B : f(sg) = \kappa(s) f(g)\ \forall\, g \in U_{\mathrm{Sph}}, s \in S_G^0(\mathbb{Z}_p), f|_{V_u^{(r)} q} \text{ locally analytic } \forall\, q \in Q_H^0\}$$

Lemma 6.1.6. *We have*

$$f_{\mathcal{U}} \in A_{\kappa,\mathrm{Sph}}^{an}.$$

Proof. By Lemma 6.1.1 we can identify

$$A_{\kappa,\mathrm{Sph}}^{an} = \{f : \mathfrak{S}_G \times \{u\} \times \left(Q_H^0 \cap u^{-1}\bar{Q}_G u\right) \backslash Q_H^0(\mathbb{Z}_p) \to \Lambda_G(\mathcal{U}) : f(s,u,q) = \kappa(s) f(u,q), f|_{\left(Q_H^0 \cap u^{-1}\bar{Q}_G u\right) \backslash Q_H^0} \text{ locall}$$

where $\left(Q_H^0 \cap u^{-1}\bar{Q}_G u\right) \backslash Q_H^0 \cong [U_{\mathrm{Sph}}]$ and the image of the identity has an open compact-neightbourhood $V^{(r)}$ given by the image of $\tau^r N_G(\mathbb{Z}_p)\tau^r$. Since $f_{\mathcal{U}}$ is Q_H^0-invariant and the Q_H^0-translates of $V^{(r)}$ cover $\left(Q_H^0 \cap u^{-1}\bar{Q}_G u\right) \backslash Q_H^0(\mathbb{Z}_p)$ it suffices to show that $f_{\mathcal{U}}$ is locally analytic when restricted to $V^{(r)} = \{1\} \times \{u\} \times \tau^r N_G(\mathbb{Z}_p)\tau^{-r}$, but $f(1, u, \tau^r n \tau^{-r}) = 1$ so we are done. □

Definition 6.1.7. For $r \geq 0$, write

$$N_r := \tau^r N_G(\mathbb{Z}_p)\tau^{-r}, \bar{N}_r := \tau^r \bar{N}_G(\mathbb{Z}_p)\tau^{-r}, L_r = \{\ell \in L_G(\mathbb{Z}_p) : \ell \in L_G^0 \bmod p^r\}.$$

We define the following open-compact subgroups of $G(\mathbb{Z}_p)$

$$W_r := \bar{N}_r \times L_r \times N_r, V_r = \bar{N}_0 \times L_r \times N_r.$$

We note that $\tau^{-r} W_r \tau^r \subset V_r$.

Let $\{\gamma_1, \dots, \gamma_n\}$ be a set of representatives for $N_r \backslash N_0$, so that

$$U_{\mathrm{Bru}} = \sqcup_{i=1}^{n} \bar{Q}_G N_r \gamma_i.$$

Let $A^{(r)}_{\kappa,m} \subset A^{\mathrm{an}}_{\kappa,m}$ denote the subspace of locally analytic functions supported on $\bar{Q}_G N_r$.

Lemma 6.1.8. *For every $r \geq 0$ we have a $u^{-1}W_r u$-equivariant decomposition*

$$A^{\mathrm{an}}_{\kappa,m} = A^{(r)}_{\kappa,m} u \oplus A_{\kappa,m,(r)} u,$$

where $A_{\kappa,m,(r)}u$ are the functions supported $\sqcup_{i=2}^{n} \bar{Q}_G N_r \gamma_i u$.

Proof. That there is such a decomposition of B-modules follows from the fact that each $\bar{Q}_G N_r \gamma_i$ is open-compact in U_{Bru}. The equivariance condition follows from the fact that W_r preserves $\bar{Q}_G N_r$. ☐

We have a natural $J_H \cap u^{-1}W_r u$-equivariant restriction map

$$\psi^0_{m,r} : A^{\mathrm{an}}_{\kappa,\mathrm{sph},m} \to A^{(r)}_{\kappa,m} u$$

which we extend to

$$\psi_{m,r} : A^{\mathrm{an}}_{\kappa,\mathrm{sph},m} \to A^{\mathrm{an}}_{\kappa,m}$$

via the above decomposition.

Definition 6.1.9. Define the level p^r *Big Twig* $\Phi_{\mathcal{U},r} : U_{\mathrm{Bru}} \times J_H \to \Lambda_G(\mathcal{U})$ by

$$\Phi_{\mathcal{U}}(g,j) = \psi_{m,r}\left(\left(j^{-1} \cdot f_{\mathcal{U}}\right)\right)(g).$$

The following lemma is then clear from the above discussion:

Lemma 6.1.10. *Suppose $\mathcal{U} \subset \mathcal{W}_m$. The Big Twig satisfies the following properties:*

- *For fixed $j \in J_H$*
$$g \mapsto \Phi_{\mathcal{U},r}(g,j) \in A^{\mathrm{an}}_{\mathcal{U},\mathrm{sph},m}$$
- *For fixed $g \in U_{\mathrm{Bru}}$*
$$j \mapsto \Phi_{\mathcal{U},r}(g,j) \in \mathcal{P}^{H,\mathrm{an}}_{\mathcal{U},m}.$$

6.2 The Big Branch

Definition 6.2.1. For $m \geq 0$, define the level p^r, m-analytic, *Big Branch*

$$\mathscr{BR}_{m,r} : D^H_{\mathcal{U},m} \to A^{\mathrm{an}}_{\mathcal{U},m}$$

defined for $\varepsilon \in D^H_{\mathcal{U},m}, g \in U_{Bru}$ by

$$\mathscr{BR}_{m,r}(\varepsilon)(g) = \langle \varepsilon, \Phi_{\mathcal{U},r}(g,\cdot)\rangle.$$

These maps are equivariant for the action of $J_H \cap u^{-1}W_r u$ on both sides.

Proposition 6.2.2. *Let $\lambda \in X^\bullet_+(S^0_G)^{Q^0_H}$ and set $\mu = \Omega(\lambda)$. Let $\mathcal{U} \subset \mathcal{W}_G$ be a wide-open disc containing λ. The diagram*

$$\begin{tikzcd}
D^H_{\mathcal{U},m} \arrow[r, "\mathscr{BR}_{m,r}"] \arrow[d, "\mathrm{sp}_\mu"] & A^{\mathrm{an}}_{\mathcal{U},m} \arrow[r, "\tau^{-r}u^{-1}"] & A^{\mathrm{an}}_{\mathcal{U},m} \arrow[d, "\rho_\lambda"] \\
\mathcal{P}^H_\mu)^\vee \arrow[r, "\mathrm{br}_\lambda"] & A^{\mathrm{an}}_{\lambda,m} \arrow[r, "\tau^{-r}u^{-1}"] & A^{\mathrm{an}}_{\lambda,m}
\end{tikzcd}$$

Proof. By density of Dirac measures it suffices to show, for $j \in J_H$, commutativity on the distributions δ_j defined by

$$\int_{J_H} f(x)\delta_j(x) = f(j).$$

Note that $(j \cdot f_\lambda^{\mathrm{sph}})$ agrees with $\rho_\lambda(\psi_{m,r}(j \cdot f_{\mathcal{U}}))$ after restriction to $\bar{Q}_G N_r u$. Write $f_\lambda^{\mathrm{sph}} = f_{\lambda,0}^{\mathrm{sph}} + F$, where $f_{\lambda,0}^{\mathrm{sph}}$ is the function supported on f_λ^{sph} obtained by restricting f_λ^{sph} to $\bar{Q}_G N_r u$ and extending by zero. Clearly F vanishes identically on $\bar{Q}_G N_r$, so in particular $[\tau^{-r}u^{-1}]F \equiv 0$, so $[\tau^{-r}u^{-1}]$ factors through restriction to $\bar{Q}_G N_r u$ whence the result follows. ☐

Theorem 6.2.3. *For each $r \geq 1$, there is a commutative diagram of Betti cohomology groups*

$$\begin{tikzcd}
H^i(Y_H(J_H \cap u^{-1}W_r u), D^H_{\mathcal{U},m+1}) \arrow[r, "{[u\tau]_* \circ \iota_* \circ \mathscr{BR}_{m,r}}"] \arrow[d] & H^{i+c}(Y_G(W_r), A^{\mathrm{an}}_{\mathcal{U},m+1}) \arrow[d] \\
H^i(Y_H(J_H \cap u^{-1}W_r u), (\mathcal{P}^H_\mu)^\vee) \arrow[r, "{[u\tau]_* \circ \iota_* \circ \mathrm{br}^\lambda_\mu}"] & H^{i+c}(Y_G(V_r), A^{\mathrm{an}}_{\lambda,m+1}).
\end{tikzcd} \tag{24}$$

Proof. This follows from the above Proposition, using the fact that $\tau^{-r}W_r\tau^r \subset V_r$ ☐

Corollary 6.2.4. *Suppose we have elements $z^H_{\mathcal{U},r} \in H^i(Y_H(J_H \cap u^{-1}W_r u), D^H_{\mathcal{U},m+1})$ which are compatible under the natural projections*

$$\mathrm{pr}_{r+1} : H^i(Y_H(J_H \cap u^{-1}W_{r+1}u), D^H_{\mathcal{U},m+1}) \to H^i(Y_H(J_H \cap u^{-1}W_r u), D^H_{\mathcal{U},m+1})$$

for $r \geq 1$. Then, writing

$$z^G_{\mathcal{U},r} = [u\tau]_* \circ \iota_* \circ \mathscr{BR}_{m,r}\left(z^H_{\mathcal{U},r}\right) \in H^{i+c}(Y_G(W_r), A^{\mathrm{an}}_{\mathcal{U},m+1}),$$

we have

$$\mathrm{pr}_{r+1}\left(z^G_{\mathcal{U},r+1}\right) = \mathcal{U}_p \cdot z^G_{\mathcal{U},r}.$$

Proof. This is (essentially) the main result of [Loe21]. In *op. cit.* the author uses the level groups $U_r := \bar{N}_0 \times L_r \times N_r$ where we use W_r. ☐

Example 6.2.5. We compute the Big Twig in some familiar situations, using the techniques of [BSV20]. Let $H = \mathrm{GL}_2$ and suppose for simplicity that $p \neq 2$. In this case $\mathcal{F}_H \cong \mathbb{P}^1_{\mathbb{Z}_p}$ and for an integer $k \geq 0$ we can identify the representation V_k of highest weight k with the global sections of the sheaf $\mathcal{O}_{\mathbb{P}^1}(k)$, the space degree k homogeneous polynomials on $\mathbb{Z}_p^2$.

We consider the setting of [LZ16], given by taking $G = \mathrm{GL}_2 \times_{\mathrm{GL}_1} \mathrm{GL}_2$ and $H = \mathrm{GL}_2$ embedded diagonally. Taking $u = \left(\begin{smallmatrix}1 & \\ & 1\end{smallmatrix}\right) \times \left(\begin{smallmatrix}1 & 1 \\ & 1\end{smallmatrix}\right)$ and $Q^0_H = \{\left(\begin{smallmatrix}x & y \\ & 1\end{smallmatrix}\right)\}$ then Q^0_H has an open orbit U_{Sph} on $\mathcal{F}_G = (\mathbb{P}^1)^2$ given for a $\mathbb{Z}_p$-algebra R by

$$U_{\mathrm{Sph}}(R) = \{[x_1 : x_2] \times [y_1 : y_2] \in (\mathbb{P}^1 \times \mathbb{P}^1)(R) : x_1, y_1, x_2 - y_2 \in R^\times\}.$$

We have $S_G = T_G$, the standard diagonal torus, and $S^0_G = \{1\}$ so the weight space $\mathcal{W}_G$ parameterises characters of T_G. The cocharacters

$$\lambda_1^\vee : x \mapsto \begin{pmatrix} x & \\ & x^{-1} \end{pmatrix} \times \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}$$

$$\lambda_2^\vee : x \mapsto \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} \times \begin{pmatrix} x & \\ & x^{-1} \end{pmatrix}$$

$$\lambda_3^\vee : x \mapsto \begin{pmatrix} 1 & \\ & x \end{pmatrix} \times \begin{pmatrix} 1 & \\ & x \end{pmatrix}$$

determine a decomposition

$$\mathcal{W}_G = \mathcal{W}_{\mathrm{GL}_1} \times \mathcal{W}_{\mathrm{GL}_1} \times \mathcal{W}_{\mathrm{GL}_1}$$

where $\mathcal{W}_{\mathrm{GL}_1}$ is the standard weight space parameterising characters of $\mathbb{Z}_p^\times$. Write $k_{\mathcal{U},i} = k_{\mathcal{U}}^G \circ \lambda_i^\vee$ and further define

$$\begin{aligned} k_{\mathcal{U},1}^* &= k_{\mathcal{U},1} - k_{\mathcal{U},3} \\ k_{\mathcal{U},2}^* &= k_{\mathcal{U},2} - k_{\mathcal{U},3}. \end{aligned}$$

Let $k, k' \geq 0$ be integers, then for $0 \leq j \leq \min\{k, k'\}$ there is an H-equivariant map

$$V_{k+k'-2j} \otimes \det{}^j \to V_k \otimes V_{k'}.$$

Define a section

$$\begin{aligned} F_{k,k',j} &\in \mathcal{O}(\mathbb{P}^1)(k) \otimes \mathcal{O}(\mathbb{P}^1)(k') \\ F_{k,k',j}(x_1, x_2, y_1, y_2) &= x_1^{k-j} y_1^{k'-j} \det\left(\begin{smallmatrix} x_1 & x_2 \\ y_1 & y_2 \end{smallmatrix}\right)^j, \end{aligned}$$

then $F_{k,k',j} \in (V_k \otimes V_{k'} \otimes \det^{j-k-k'})^{Q_H^0}$ and $F_{k,k',j}(u) = 1$ so $F_{k,k',j}$ is a highest weight vector for the action of H of highest weight $\left(\begin{smallmatrix} x & \\ & x^{-1}\det \end{smallmatrix}\right) \mapsto x^{k+k'-2j}\det^j$. Let $\mathcal{U} \subset \mathcal{W}_G$ be a wide-open disc with universal character $k_{\mathcal{U}}^G : T_G(\mathbb{Z}_p) \to \Lambda_G(\mathcal{U})^\times$. Then $F_{k,k',j}$ restricted to $U_{\mathrm{sph}}(\mathbb{Z}_p)$ takes values in $\mathbb{Z}_p^\times$ so the function

$$\begin{aligned} F_{\mathcal{U}} &: U_{\mathrm{Sph}}(\mathbb{Z}_p) \to \Lambda_G(\mathcal{U})^\times \\ F_{\mathcal{U}}(x_1, x_2, y_1, y_2) &= x_1^{k_{\mathcal{U},1}^*} y_1^{k_{\mathcal{U},2}^*} \det \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix}^{k_{\mathcal{U},3}} \end{aligned}$$

is well defined and is homogeneous of weight $k_{\mathcal{U}}^G$. Setting $\tau = \left(\begin{smallmatrix} p & \\ & 1 \end{smallmatrix}\right) \times \left(\begin{smallmatrix} p & \\ & 1 \end{smallmatrix}\right)$ then for $n = \left(\begin{smallmatrix} 1 & z_1 \\ & 1 \end{smallmatrix}\right) \times \left(\begin{smallmatrix} 1 & z_2 \\ & 1 \end{smallmatrix}\right) \in N_G(\mathbb{Z}_p)$ and $i^{-1} = \left(\begin{smallmatrix} i_1 & i_2 \\ pi_3 & i_4 \end{smallmatrix}\right) \in J_H$ the Big twig is given by

$$\Phi_{\mathcal{U}}(\tau n \tau^{-1} u, i) = (i_1 + p^2 i_3 z_1)^{k_{\mathcal{U},1}^*} (i_1 + p i_3 (1 + p z_2))^{k_{\mathcal{U},2}^*} (\det(i)^{-1}(1 + p(z_2 - z_1)))^{k_{\mathcal{U},3}}.$$

Example 6.2.6. We consider the situation of [GS20] and [BSV20]. Let H be as in the previous exapmle and set $G = \mathrm{GL}_2 \times_{\mathrm{GL}_1} \mathrm{GL}_2 \times_{\mathrm{GL}_1} \mathrm{GL}_2$, $Q_G = B_G = T_G \times N_G$ the upper triangular Borel subgroup, and $S_G = T_G$. We consider the diagonal embedding of H into G. Taking $u = \left(\begin{smallmatrix} 1 & \\ & 1 \end{smallmatrix}\right) \times \left(\begin{smallmatrix} 1 & -1 \\ & 1 \end{smallmatrix}\right) \times \left(\begin{smallmatrix} 1 & 1 \\ & 1 \end{smallmatrix}\right)$ we see that H has an open orbit on $\mathcal{F}_G \cong (\mathbb{P}^1)^3$. For a triple of non-negative integers $\underline{r} = (r_1, r_2, r_3)$ let $V_{\underline{r}} = V_{r_1} \boxtimes V_{r_2} \boxtimes V_{r_3}$. We have $uHu^{-1} \cap \bar{B}_G = Z_H = H \cap u^{-1}\bar{B}_G u = S_G^0$ from which it follows that a necessary condition for $V_{\underline{r}}$ to have an H-invariant element is for there to exist an integer r such that $r_1 + r_2 + r_3 = 2r$. If we further suppose that for each permutation $\sigma \in S_3$ we have $r_{\sigma(1)} + r_{\sigma(2)} \geq r_{\sigma(3)}$ then the function $\mathrm{Det}_r \in V_{\underline{r}}$ given by

$$(x_1, x_2) \times (y_1, y_2) \times (z_1, z_2) \mapsto 2^{-r} \det \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix}^{r-r_3} \det \begin{pmatrix} x_1 & x_2 \\ z_1 & z_2 \end{pmatrix}^{r-r_2} \det \begin{pmatrix} y_1 & y_2 \\ z_1 & z_2 \end{pmatrix}^{r-r_1},$$

is well defined and $\mathrm{Det}_r \in (V_{\underline{r}} \otimes \det^{-r})^H$. Writing, for $g \in G$, $\mathrm{Det}_r(g) := \mathrm{Det}_r((1,0)^3 g)$, we see that $\mathrm{Det}_r(u) = 1$ (this is the reason for the factor of 2^{-r}) and thus for $\bar{n}t \in \bar{B}, h \in H$ that:

$$\mathrm{Det}_r(\bar{n}tuh) = \det(h)^r t^{\lambda_{\underline{r}}},$$

where $\lambda_{\underline{r}}$ is the highest weight of $V_{\underline{r}}$, using additive notation for characters.

Recall that $\mathcal{W}_G$ parameterises weights of the rank 3 torus $\mathfrak{S}_G(\mathbb{Z}_p) = (T_G/Z_H)(\mathbb{Z}_p)$. For $i = 1, 2, 3$ write $\lambda_i^\vee \in X_\bullet(S_G/S_G^0)$ for the cocharacter given by composing

$$x \mapsto \begin{pmatrix} x & \\ & x^{-1} \end{pmatrix}$$

with the inclusion into the ith GL_2-component of S_G and reduction modulo S_G^0. These cocharacters determine a decomposition of $\mathcal{W}_G$:

$$\mathcal{W}_G = \mathcal{W}_{\mathrm{GL}_1} \times \mathcal{W}_{\mathrm{GL}_1} \times \mathcal{W}_{\mathrm{GL}_1} \tag{25}$$

and we define $\mathcal{U} = \mathcal{U}_1 \times \mathcal{U}_2 \times \mathcal{U}_3 \subset \mathcal{W}_G$ as a product of wide-open discs $\mathcal{U}_i \subset \mathcal{W}_{\mathrm{GL}_1}$ contained in each factor of the decomposition (25). Suppose that each $\mathcal{U}_i$ is centred on an integer $k_i \in \mathbb{Z}$; the condition that our weights be trivial on Z_H forces $k_1 + k_2 + k_3$ to be even. We define

$$k_{\mathcal{U},i} = k^G_{\mathcal{U}} \circ \lambda_i^\vee.$$

These characters take the form $k_{\mathcal{U},i}(z) = \omega(z)^{k_1+k_2+k_3} \langle z \rangle^{k_{\mathcal{U},i}}$ where $\omega(z)$ is the Teichmüller representative of $z \in \mathbb{Z}_p^\times$. We further define

$$k^*_{\mathcal{U},1}(z) = \omega(z)^{\frac{k_1+k_2-k_3}{2}} \langle z \rangle^{\frac{k_{\mathcal{U},1}+k_{\mathcal{U},2}-k_{\mathcal{U},3}}{2}}$$

and similarly for $i = 2, 3$ to obtain characters $k^*_{\mathcal{U},i}$ satisfying $k^*_{\mathcal{U},\sigma(1)} + k^*_{\mathcal{U},\sigma(2)} = k_{\mathcal{U},\sigma(3)}$ for all permutations $\sigma \in S_3$ and write

$$k^*_{\mathcal{U},123} = k^*_{\mathcal{U},1} + k^*_{\mathcal{U},2} + k^*_{\mathcal{U},3}.$$

Set $\tau = \left(\begin{smallmatrix} p & \\ & 1 \end{smallmatrix}\right) \times \left(\begin{smallmatrix} p & \\ & 1 \end{smallmatrix}\right) \times \left(\begin{smallmatrix} p & \\ & 1 \end{smallmatrix}\right)$. Retaining additive notation for characters, we then have for $\bar{n}t \in \bar{B}_G(\mathbb{Z}_p)$, $n = \left(\begin{smallmatrix} 1 & z_1 \\ & 1 \end{smallmatrix}\right) \times \left(\begin{smallmatrix} 1 & z_2 \\ & 1 \end{smallmatrix}\right) \times \left(\begin{smallmatrix} 1 & z_3 \\ & 1 \end{smallmatrix}\right) \in N_G(\mathbb{Z}_p)$:

$$\begin{aligned} \Phi_{\mathcal{U}}(\bar{n}t\tau n\tau^{-1}u, h) = {} & (2\det(h))^{-k^*_{\mathcal{U},123}} \times \\ & t^{k^G_{\mathcal{U}}} \det \begin{pmatrix} 1 & pz_1 \\ 1 & -1+pz_2 \end{pmatrix}^{k^*_{\mathcal{U},3}} \det \begin{pmatrix} 1 & pz_1 \\ 1 & -1+pz_2 \end{pmatrix}^{k^*_{\mathcal{U},2}} \det \begin{pmatrix} 1 & -1+pz_2 \\ 1 & 1+pz_3 \end{pmatrix}^{k^*_{\mathcal{U},1}} \end{aligned}$$

defining an element of $A^{\mathrm{an}}_{\mathcal{U},m} \otimes (k^*_{\mathcal{U},123} \circ \det)^{-1}$ for suitably large m.

6.2.1 Étale cohomology

Lemma 6.2.7. *For $\mathcal{U} \subset \mathcal{W}_{m+1}$ there is a commutative diagram*

$$\begin{tikzcd} D^H_{\mathcal{U},m} \arrow[r] \arrow[d] & A^{\mathrm{Iw}}_{\mathcal{U},m+1} \arrow[d] \\ (\mathcal{P}^H_\mu)^\vee \arrow[r] & A^{\mathrm{Iw}}_{\lambda,m+1} \end{tikzcd}$$

Proof. Just compose the diagram (6.2.3) with the inclusions

$$A^3_{\kappa,m} \cong A^{\mathrm{an}}_{\kappa,m} \hookrightarrow A^{\mathrm{Iw}}_{\kappa,m+1}$$

for appropriate κ. ☐

Lemma 6.2.8. *Suppose $\mathcal{U} \subset \mathcal{W}_m$, $m \geq 0$. The Big Branch $\mathscr{BR}_{m+1}$ is continuous for the profinite topologies on $D^H_{\mathcal{U},m+1}$ and $A^{\mathrm{Iw}}_{\mathcal{U},m+1}$.*

Proof. The assumption on $\mathcal{U}$ means that $h \mapsto \Phi(g, h)$ is an element of $\mathcal{P}^{H,\mathrm{an}}_{\mathcal{U},m}$ so the $(m+1)$-analytic Big Branch $\mathscr{BR}_{m+1}$ factors through restriction to this module and lands in $A^{\mathrm{Iw}}_{\mathcal{U},m+1}$. Recalling notation from Section 5.2.8, we have a commutative diagram

$$\begin{tikzcd} D^H_{\mathcal{U},m+1} \arrow[r, "\mathscr{BR}_{m+1}"] \arrow[d] & A^{\mathrm{Iw}}_{\mathcal{U},m+1} \arrow[dr] & \\ X^{(i)}_{\mathcal{U},m+1} \arrow[r] \arrow[d] & X^{(i)}_{\mathcal{U},m+1} \otimes \Lambda_{\mathcal{U}}/\mathfrak{m}^i_{\mathcal{U}} \arrow[r, "\mathscr{BR}_m"] & A^{\mathrm{Iw}}_{\mathcal{U},m+1} \otimes \Lambda_{\mathcal{U}}/\mathfrak{m}^i_{\mathcal{U}} \\ D^H_{\mathcal{U},m}/p^i \arrow[urr, "\mathscr{BR}_m"] & & \end{tikzcd}$$

for $i \geq 0$. We need to show that for each $n \geq 0$ there is $i = i(n)$ such that $\mathscr{BR}_{m+1}(\mathrm{Fil}^i D^H_{\mathcal{U},m+1}) \subset \mathrm{Fil}^n A^{\mathrm{Iw}}_{\mathcal{U},m+1}$. From the diagram it suffices to show that there is $i = i(n)$ such that $\mathfrak{m}^i_{\mathcal{U}} A^{\mathrm{Iw}}_{\mathcal{U},m} \subset \mathrm{Fil}^n A^{\mathrm{Iw}}_{\mathcal{U},m+1}$ but this is easily verifiable. ☐

Corollary 6.2.9. *Suppose Y_H, Y_G admit compatible Shimura data. In this case there is $e \in \mathbb{Z}$ such that $2e = c$ For any integer j there is a commutative diagram of étale cohomology groups*

$$\begin{tikzcd}
H^i(Y_H(J_H \cap u^{-1}U_1u), \mathscr{D}^H_{\mathcal{U},m+1}(j)) \arrow[r, "\mathrm{Tr} \circ [u\tau]_* \circ \mathscr{B}\mathscr{R}"] \arrow[d] & H^{i+2e}(Y_G(J_G), \mathscr{A}^{\mathrm{Iw}}_{\mathcal{U},m+1}(j+e)) \arrow[d] \\
H^i(Y_H(J_H \cap u^{-1}U_1u), (\mathcal{P}^H_\mu)^\vee(j)) \arrow[r, "\mathrm{Tr} \circ [u\tau]_* \circ \mathrm{br}^\lambda_\mu"] & H^{i+2e}(Y_G(J_G), \mathscr{A}^{\mathrm{Iw}}_{\lambda,m+1}(j+e)),
\end{tikzcd}$$

where (j) denotes a cyclotomic twist.

6.2.2 Eisenstein classes

Let $H = \mathrm{GL}_2$, $Q_H = B_H$ and as in Example 6.2.5 let V_k denote the irreducible H-rep of highest weight $k \geq 0$. A large proportion of examples of Euler systems arise as push-forwards of Eisenstein classes in the cohomology of modular curves, for example [KLZ17], [LSZ21]. We briefly discuss how these classes fit into our framework.

Let $\mathcal{S}_0((\mathbb{A}_f^{(p)} \times \mathbb{Z}_p)^2, \mathbb{Z}_p)$ be the space of $\mathbb{Z}_p$-valued Schwartz functions ϕ on $(\mathbb{A}_f^{(p)} \times \mathbb{Z}_p)^2$ satisfying $\phi(0,0) = 0$ and for an integer c coprime to $6p$, let ${}_c\mathcal{S}_0((\mathbb{A}_f^{(p)} \times \mathbb{Z}_p)^2, \mathbb{Z}_p)$ denote the subspace of $\mathcal{S}_0((\mathbb{A}_f^{(p)} \times \mathbb{Z}_p)^2, \mathbb{Z}_p)$ of the form $\phi^{(c)} \otimes \mathrm{ch}(\mathbb{Z}_c^2)$, where ϕ is a $\mathbb{Z}$-valued Schwartz function on $(\mathbb{A}_f^{(c)})^2$ and $\mathbb{Z}_c = \prod_{\ell | c} \mathbb{Z}_\ell$. We equip these spaces with the natural right translation action of H.

Proposition 6.2.10. *Let $k \geq 0$. For c as above and $U \subset H(\mathbb{A}_f^{(pc)} \times \mathbb{Z}_{pc})$ a neat open compact subgroup there is a map*

$$\begin{gathered} {}_c\mathcal{S}_0((\mathbb{A}_f^{(p)} \times \mathbb{Z}_p)^2, \mathbb{Z}_p) \to H^1_{\acute{e}t}(Y_H(U)_\Sigma, V_{k,\mathbb{Z}_p}(1)) \\ \phi \mapsto_c \mathrm{Eis}^k_{\acute{e}t,\phi} \end{gathered}$$

whose image in $H^1(Y_H(U)_\Sigma, V_k)$ satisfies

$${}_c\mathrm{Eis}^k_{\acute{e}t,\phi} = \left(c^2 - c^{-k} \begin{pmatrix} c & \\ & c \end{pmatrix}^{-1} \right) r_{\acute{e}t}(\mathrm{Eis}^k_{\mathrm{mot},\phi}),$$

where $\mathrm{Eis}^k_{\mathrm{mot},\phi}$ is Beilinson's motivic Eisenstein class, and $r_{\acute{e}t}$ is the étale regulator.

Let $K^p \subset H(\mathbb{A}_f^{(pc)} \times \mathbb{Z}_c)$ be a choice of tame subgroup such that K^pJ_H is neat and let ϕ be a $\mathbb{Z}_p$-valued Schwartz function on $(\mathbb{A}_f^{(p)})^2$ invariant under K^p. By [KLZ17, Section 4] there is an *Eisenstein–Iwasawa class*

$${}_c\mathcal{EI}_\phi \in H^2(Y_H(J_H)_\Sigma, \Lambda(N_H(\mathbb{Z}_p) \backslash J_H)),$$

where $\Lambda(N_H(\mathbb{Z}_p) \backslash J_H)$ is the étale sheaf associated to the space of $\mathbb{Z}_p$-valued measures on $N_H(\mathbb{Z}_p) \backslash J_H$. for all $k \geq 0$ There is a *kth moment map*

$$\mathrm{mom}^k : H^1(Y_H(J_H)_\Sigma, \Lambda(N_H(\mathbb{Z}_p) \backslash J_H)(1)) \to H^1(Y_H(J_H)_\Sigma, V^\vee_{k,\mathbb{Z}_p}(1))$$

satisfying

$$\mathrm{mom}^k({}_c\mathcal{EI}_\phi) = {}_c\mathrm{Eis}^k_{\acute{e}t,\phi_1},$$

where $\phi_1 = \phi \otimes \mathrm{ch}(p\mathbb{Z}_p \times \mathbb{Z}_p^\times)$.

Remark 6.2.11. The space $V^\vee_{k,\mathbb{Z}_p}$ is H-isomorphic to the space $\mathrm{TSym}^k(\mathbb{Z}_p^2)$ of weight k symmetric tensors on $\mathbb{Z}_p^2$.

Let $\mathcal{U} \subset \mathcal{W}_m$. We define a map

$$\psi_{\mathcal{U}} : \Lambda(N_H(\mathbb{Z}_p) \backslash J_H) \to D^H_{\mathcal{U},m}$$

given, for $j \in N_H(\mathbb{Z}_p)\backslash J_H$, by sending the Dirac measure δ_j to the 'evaluation-at-j' distribution. The kth moment map factors through $\psi_{\mathcal{U}}$ as

$$\mathrm{mom}^k : \Lambda(N_H(\mathbb{Z}_p)\backslash J_H) \xrightarrow{\psi_{\mathcal{U}}} D^H_{\mathcal{U}} \xrightarrow{\rho_k} V^\vee_{k,\mathbb{Z}_p},$$

whenever $k \in \mathcal{U}$, and so we define

$${}_c\mathcal{EI}^{\mathcal{U}}_{\acute{e}t,\phi} := \psi_{\mathcal{U},*}({}_c\mathcal{EI}_\phi) \in H^1(Y_H(J_H), \mathscr{D}^H_{\mathcal{U}}(1)).$$

Proposition 6.2.12. *For $k \in \mathbb{Z}_{\geq 0} \cap \mathcal{U}$, the class ${}_c\mathcal{EI}^{\mathcal{U}}_{\acute{e}t,\phi}$ satisfies*

$$\rho_k({}_c\mathcal{EI}^{\mathcal{U}}_{\acute{e}t,\phi}) = {}_c\mathrm{Eis}^k_{\acute{e}t,\phi_1}.$$

Proof. Clear from the above discussion. □

We will use these classes in Section 6.6 to interpolate the Lemma–Flach Euler system of [LSZ21] in Coleman families.

6.3 Complexes of Banach modules

6.3.1 Slope decompositions

Definition 6.3.1. Let $F \in A\{\{T\}\}$ and let $h \in \mathbb{R}_{\geq 0}$. We say that F has a slope $\leq h$ factorisation if we have a factorisation

$$F = Q \cdot S$$

where Q is a polynomial and S is Fredholm, such that

1. Every slope of Q is $\leq h$
2. S has slope $> h$
3. p^h is in the interval of convergence of S.

Such a factorisation is unique if it exists.

Definition 6.3.2. Let M be an R-module equipped with an R-linear endomorphism u. For $h \in \mathbb{Q}$ we say that M has a $\leq h$*-slope decomposition* if it decomposes as a direct sum

$$M = M^{u \leq h} \oplus M^{u > h}$$

such that

- Both summands are u-stable.
- $M^{u\leq h}$ is finitely generated over A.
- For every $m \in M^{u\leq h}$ there is a polynomial $Q \in R[t]$ of slope $\leq h$ with $Q^*(0)$ a multiplicative unit, such that $Q^*(u)m = 0$.
- For any polynomial $Q \in R[t]$ of slope $\leq h$ with $Q^*(0)$ a multiplicative unit, the map

  $$Q^*(u) : M^{u>h} \to M^{u>h}$$

  is an isomorphism.

If such a decomposition exists it is unique and u acts invertibly on $M^{u\leq h}$. Let R be a Banach $\mathbb{Q}_p$-algebra and let M be a Banach R-module with an action of $\mathfrak{U}_p^{--}$ by compact operators. For $u \in \mathfrak{U}_p^{--}$ let $F_u \in R\{\{t\}\}$ denote the Fredholm determinant of u acting on M. The following theorem follows directly from results of Coleman, Serre and Buzzard (see [Urb11, Theorem 2.3.8], for example):

Theorem 6.3.3. *Let R, M be as above and suppose that M is projective as a Banach module with R-linear compact operator u.*

If we have a prime decomposition $F_u(T) = Q(T)S(T)$ in $R\{\{T\}\}$ with Q a polynomial such that $Q(0) = 1$ and $Q^*(T)$ invertible in R then there exists $R_Q(T) \in TR\{\{T\}\}$ whose coefficients are polynomials in the coefficients of Q and S and we have a decomposition of M:

$$M = N_u(Q) \oplus F_u(Q)$$

of closed R submodules satisfying

- *The projector on $N_u(Q)$ is given by $R_Q(u)$.*
- *$Q^*(u)$ annihilates $N_u(Q)$.*
- *$Q^*(u)$ is invertible on $F_u(Q)$.*

If A is Noetherian then $N_u(Q)$ is projective of finite rank and

$$\det(1 - tu \mid N_u(Q)) = Q(t).$$

When the decomposition $F_u = QS$ is a slope $\leq h$ factorisation then the decomposition in the above theorem is a slope $\leq h$ factorisation and $M^{u\leq h} = N_u(Q)$

6.3.2 Slope decompositions on cohomology

Let $K = K_pK^p \subset G(\mathbb{A}_f)$ be a neat open compact subgroup with $K_p \subset J_G$ admitting an Iwahori decomposition and let R be a $\mathbb{Q}_p$ Banach algebra. From now on we assume that $\mathcal{G}, \mathcal{H}$ admit compatible Shimura data and set $q = \dim_{\mathbb{C}} Y_G$.

Definition 6.3.4. For an $R[K]$-module M let

$$\mathscr{C}^\bullet(\bar{Y}_G(K), M)$$

be the 'Borel–Serre' complex defined in [Han17, Section 2.1] whose cohomology computes $H^\bullet(\bar{Y}_G(K), M)$ (as R-modules). We let $R\Gamma(\bar{Y}_G(K), M)$ be the image of the above complex in the derived category of Banach R-modules.

Remark 6.3.5. We won't always have defined an étale sheaf associated to M so by abuse of notation we let $H^\bullet(\bar{Y}_G(K), M)$ denote the Betti cohomology of the locally constant sheaf of R-modules induced by M in this case, noting that this is isomorphic as an R-module to the étale cohomology when M has an associated étale sheaf (we might think of this as bequeathing the Betti cohomology with a Galois action).

Suppose M is an orthonormalisable Banach R-module and with a continuous action of A^-. Then we can define an action of the Hecke algebra $\mathbb{T}^-_{S,p}$ on the complex $\mathscr{C}^\bullet(\bar{Y}_G(K), M)$ via its interpretation as an algebra of double coset operators. Suppose further that A^{--} acts compactly on M. Then the action of $\mathfrak{U}_p^{--}$ on $\mathscr{C}^\bullet(\bar{Y}_G(K), M)$ acts compactly on the total complex $\oplus_i \mathscr{C}^i(\bar{Y}_G(K), M)$. We refer to the following proposition from [Han17, 2.3.3]:

Proposition 6.3.6. *Let R be an affinoid algebra. If $C^\bullet$ is a complex of projective Banach R-algebras equipped with an R-linear compact operator u, then for any $x \in \mathrm{Sp}(R)$ and $h \in \mathbb{Q}_{\geq 0}$ there is an affinoid subdomain $\mathrm{Sp}(R') \subset \mathrm{Sp}(R)$ such that $x \in \mathrm{Sp}(R')$ and such that the complex $C^\bullet \hat{\otimes}_R R'$ admits a slope $\leq h$ decomposition for u and $(C^\bullet \hat{\otimes}_R R')^{u\leq h}$ is a complex of finite flat R'-modules.*

We will also need the following easy technical lemma.

Lemma 6.3.7. *Let $N \subset M$ be an inclusion of projective Banach R-modules with R-linear compact operator u such that*

$$uM \subset N$$

Suppose further that N, M admit slope $\leq h$ decompositions, then for $h \in \mathbb{Q}_{\geq 0}$ we have

$$M^{u\leq h} = N^{u\leq h}.$$

Moreover, if $e_u^{\leq h}$ is the slope $\leq h$ idempotent on M associated to u by Theorem 6.3.3 then

$$e_u^{\leq h} N = N^{u \leq h}$$

i.e. $e_u^{\leq h}$ is a slope $\leq h$ idempotent for N.

Proof. By [AS08, Theorem 4.1.2(c)&(d)] we have a slope $\leq h$ decomposition on M/N such that

$$0 \to N^{u\leq h} \to M^{u\leq h} \to (M/N)^{u\leq h} \to 0$$

is an exact sequence. However, $u(M/N) = 0$ by our hypothesis so u has infinite slope on M/N and thus $(M/N)^{u\leq h} = 0$ for any $h \in \mathbb{Q}_{\geq 0}$ (as u must be invertible on finite slope parts) whence $N^{u\leq h} = M^{u\leq h}$.

For the statement involving the idempotent it's easy to see that for any idempotent operator ϕ on M preserving N we have $\phi(N) = \phi(M)$ and that $e_u^{\leq h}$ is immediate since

$$e_u^{\leq h} M = M^{u\leq h} = N^{u\leq h}.$$

$\square$

By Proposition 6.3.6 we can shrink $\mathcal{V}$ to an affinoid $\mathcal{V}'$ also containing x_0 and such that the complex admits a slope $\leq h$ decomposition for any $u \in \mathfrak{U}_p^{--}, h \in \mathbb{Q}_{\geq 0}$ with projector $e^{\leq h} \in \mathcal{O}(\mathcal{V}')\{\{u\}\}$. In this case by Lemma 6.3.7 we have

$$\mathscr{C}^\bullet(\bar{Y}_G(K), A^{\mathrm{an}}_{\mathcal{V}',m})^{u\leq h} \cong \mathscr{C}^\bullet(\bar{Y}_G(K), A^{\mathrm{an}}_{\mathcal{V}',m+1})^{u\leq h}.$$

We will need slope decompositions for coefficents defined over a wide-open disc in weight space.

Lemma 6.3.8. *Let $\mathcal{U} \subset \mathcal{W}_G$ be a wide open disc and let $x_0 \in \mathcal{U}$. There is a wide open disc $x_0 \in \mathcal{U}' \subset \mathcal{U}$ such that the complex $\mathscr{C}^\bullet(\bar{Y}_G(K), A^{\mathrm{an}}_{\mathcal{U},m})$ admits a slope $\leq h$ decomposition.*

Proof. As explained above, there is an affinoid $\mathcal{V} \subset \mathcal{U}$ containing x_0 over which $\mathscr{C}^\bullet(\bar{Y}_G(K), A^{\mathrm{an}}_{\mathcal{V},m})$ admits a slope $\leq h$ decomposition. By shrinking we can assume that $\mathcal{V}$ is a closed disc centered on x_0. Let $\mathcal{U}'$ be the wide-open disc given by taking the 'interior' wide-open disc of this affinoid disc. Since an orthonormal basis of $A^{\mathrm{an}}_{\mathcal{V},m}$ gives an orthonormal basis of $A^{\mathrm{an}}_{\mathcal{U}',m}$ and the Banach norm on $\Lambda_G(\mathcal{U}')[1/p]$ restricts to the Gauss norm on $\mathcal{O}(\mathcal{V})$ we get a slope $\leq h$ decomposition on $\mathscr{C}^\bullet(\bar{Y}_G(K), A^{\mathrm{an}}_{\mathcal{U}',m})$ and all the above results hold in this case. Note in particular that the projector $e^{\leq h}$ is still in $\mathcal{O}(\mathcal{V})\{\{u\}\}$ when computing the decomposition for $\mathcal{U}'$. $\square$

Lemma 6.3.9. *Let $h \in \mathbb{Q}_{\geq 0}$ and let $\mathcal{U}$ be a wide-open disc such that $\mathscr{C}^\bullet(\bar{Y}_G(K), A^{\mathrm{an}}_{\mathcal{U},m})$ admits a slope $\leq h$ decomposition. Then the slope $\leq h$ total cohomology $H^\bullet(\bar{Y}_G(K), A^{\mathrm{an}}_{\mathcal{U},m+1})^{u\leq h}$ is a Galois-stable direct summand of $H^\bullet(\bar{Y}_G(K), \mathscr{A}^{\mathrm{Iw}}_{\mathcal{U},m+1})$ as a $\Lambda_G(\mathcal{U})$-module.*

Proof. Write

$$\begin{aligned} M_m &:= \mathscr{C}^\bullet(\bar{Y}_G(K), A^{\mathrm{an}}_{\mathcal{U},m}) \\ I_m &:= \mathscr{C}^\bullet(\bar{Y}_G(K), A^{\mathrm{Iw}}_{\mathcal{U},m}), \end{aligned}$$

so that we have natural inclusions

$$M_m \subset I_{m+1} \subset M_{m+1}. \tag{26}$$

The key point is that the natural action of u on M_m, I_{m+1} is the restriction of the action of u on M_{m+1}, so we can apply Lemma 6.3.7. Since $uM_{m+1} \subset M_m$ it follows from (26) that

$$uI_{m+1} \subset M_m.$$

By Theorem 6.3.3 there is an idempotent $e_m^{\leq h} \in uR\{\{u\}\}$ such that

$$e_m^{\leq h} M_{m+1} = I_{m+1}^{u\leq h} = M_m^{u\leq h}$$

where the second equality is Lemma 6.3.7. Moreover, by Lemma 6.3.7 we have that

$$e_m^{\leq h} M_m = M_m^{u \leq h}$$

(i.e. $e_m^{\leq h}$ does not depend on m) from which we can infer that

$$M_m^{u \leq h} = e^{\leq h} I_{m+1}$$

is a direct summand of I_{m+1} and thus $H^\bullet(M_m)^{u \leq h}$ is a direct summand of the total cohomology $H^\bullet(I_m)$ by functoriality.

To show Galois-stability we note that for each i

$$H^i(\bar{Y}_G(K), A^{\mathrm{Iw}}_{\mathcal{U},m}) = \varprojlim_n H^i(\bar{Y}_G(K), A^{\mathrm{Iw}}_{\mathcal{U},m}/\mathrm{Fil}^n)$$

as Galois modules, with each $H^\bullet(\bar{Y}_G(K), A^{\mathrm{Iw}}_{\mathcal{U},m}/\mathrm{Fil}^n)$ finite. Since $e^{\leq h}$ can be represented by a polynomial in u mod p^n and since the Hecke operators commute with the Galois action we see that for $g \in G_{\mathbb{Q}}$

$$g \cdot e^{\leq h} = e^{\leq h} \cdot g \bmod p^n$$

for all n and by taking the limit over n we get the result. □

6.3.3 Refined slope decompositions and classicality

As in [SW21, Section 3.5] we consider a more refined slope decomposition. For $i = 1, \ldots, n$ let $Q_{G,i}^{\max}$ denote the maximal parabolic subgroups of G containing Q_G. These correspond to a subset $\{\alpha_1, \ldots, \alpha_n\}$ of the simple roots of G and by taking $a_i \in A^-$ such that $v(\alpha_i(a_i)) > 0$ and $v(\alpha_j(a_i)) = 0$ for $j \neq i$ we can associate Hecke operators $U_i \in \mathfrak{U}_p^-$ as the image of a_i under the isomorphism $\mathbb{Z}_p[A^-/A(\mathbb{Z}_p)] \cong \mathfrak{U}_p^-$.

Definition 6.3.10. Set $h_i^{\mathrm{crit}} := -(\langle \lambda, \alpha_i \rangle + 1) v(\alpha(a_i))$.

Let $\mathbf{h} := (h_1, \ldots, h_n) \in \mathbb{Q}^n_{\geq 0}$. Suppose a Banach R module M admits a slope $\leq h_{\mathrm{aux}}$ decomposition with respect to the operator $U_0 := U_1 \cdots U_n \in \mathfrak{U}_p^{--}$ for some $h_{\mathrm{aux}} > \prod_i h_i$, so that $M^{\leq h_{\mathrm{aux}}}$ is a finite projective Banach R-module. In particular, the whole of A^- acts compactly on $M^{\leq h_{\mathrm{aux}}}$ and supposing that the Fredholm series F_i admit slope h_i decompositions for each i then we can define

$$M^{\leq \mathbf{h}} = \cap_i (M^{\leq h_{\mathrm{aux}}})^{\leq h_i}.$$

Lemma 6.3.11. *For $0 \leq i \leq n$ let $\boldsymbol{h}^{(i)} = (h_1, \ldots, h_i)$ and suppose the U_i act compactly on M. Set*

$$M^{\leq \boldsymbol{h}^{(i+1)}} = (M^{\leq \boldsymbol{h}^{(i)}})^{U_{i+1} \leq \boldsymbol{h}_{i+1}}.$$

Then

$$M^{\leq \boldsymbol{h}^{(n)}} = M^{\leq \boldsymbol{h}}.$$

Proof. It suffices to prove the following statement: Suppose M is a Banach module equipped with two compact operators U_1, U_2 whose Fredholm determinants F_1, F_2 admit slope h_1, h_2 factorisations respectively. Then

$$M^{U_1 \leq h_1} \cap M^{U_2 \leq h_2} = (M^{U_1 \leq h_1})^{U_2 \leq h_2}$$

Suppose $F_2 = Q_2 S_2$ is the slope $\leq h_2$ factorisation and $\tilde{F}_2 = \tilde{Q}_2 \tilde{S}_2$ is a slope factorisation of the Fredholm determinant $\tilde{F}_2$ of U_2 restricted to $M^{U_1 \leq h_1}$. Then $\tilde{Q}_2$ divides Q_2 so for $m \in (M^{U_1 \leq h_1})^{U_2 \leq h_2}$ we have $Q_2^* m = 0$ and thus $m \in M^{U_1 \leq h_1} \cap M^{U_2 \leq h_2}$.

Conversely suppose $m \in M^{U_1 \leq h_1} \cap M^{U_2 \leq h_2}$. Then in particular $m \in M^{U_1 \leq h_1}$ and so we can write $m = m_2 + n$ where $m_2 \in (M^{U_1 \leq h_1})^{U_2 \leq h_2}$ and n is in the complement $(M^{U_1 \leq h_1})^{U_2 > h_2}$. As $m \in M^{U_2 \leq h_2}$ we have $Q_2^*(U_2) m = 0$ but also $Q_2^*(U_2) m_2 = 0$ by the same argument as in the first inclusion, so $Q_2^*(U_2) n = 0$ and since Q_2^* is a slope $\leq h_2$ polynomial and $Q_2^*(0)$ is a multiplicative unit then $n = 0$ by Definition 6.3.2. □

Corollary 6.3.12. *Let M be a projective Banach R-module with an action of an R-linear compact operator u. Then the module $M^{\leq \boldsymbol{h}}$ is a finite projective R-module and a direct summand of M with projector $e^{\leq \boldsymbol{h}} \in R\{\{U_1, \ldots, U_n\}\}$.*

Proof. The above lemma states that we can obtain the refined slope decomposition $M^{\leq \mathbf{h}}$ on $M^{\leq h_{\text{aux}}}$ by recursively taking a finite number of slope decompositions so the result follows from Theorem 6.3.3. ☐

We say $\mathbf{h}$ is *non-critical* if for all $i = 1, \ldots, n$ we have $h_i < h_i^{\text{crit}}$.

Theorem 6.3.13. *For $\boldsymbol{h} \in \mathbb{Q}^n_{\geq 0}$ non-critical and $\lambda \in X^{\bullet}_{+}(S_G)$ there is a quasi-isomorphism:*

$$R\Gamma(\bar{Y}_G(K), \mathscr{A}^{\text{an}}_{\lambda,m})^{\leq \boldsymbol{h}} \cong R\Gamma(\bar{Y}_G(K), V^G_{\lambda,m})^{\leq \boldsymbol{h}}.$$

Proof. This is proved for compactly supported cohomology with coefficients in modules of distributions in [SW21, Theorem 4.4] using the dual of a parabolic locally analytic BGG complex. The same proof (sans dualising) can easily be adapted to our setting using this complex. ☐

We end with a variation of Proposition 6.3.6

Lemma 6.3.14. *Let R be an affinoid algebra and let $C^{\bullet}$ be a complex of projective Banach R-algebras equipped with a continuous R-linear action of $\mathfrak{U}_p^{-}$ such that $\mathfrak{U}_p^{--}$ acts via compact operators. Then for any $x \in \mathrm{Sp}(R)$ and $\boldsymbol{h} \in \mathbb{Q}^n_{\geq 0}$ there is an affinoid subdomain $\mathrm{Sp}(R') \subset \mathrm{Sp}(R)$ such that $x \in \mathrm{Sp}(R')$ and such that the complex $C^{\bullet}\hat{\otimes}_R R'$ admits a slope $\leq \boldsymbol{h}$ decomposition and $(C^{\bullet}\hat{\otimes}_R R')^{\leq \boldsymbol{h}}$ is a complex of finite flat R'-modules.*

Proof. We know that by Proposition 6.3.6 there is an affinoid $\mathrm{Sp}(R_0)$ containing x and such that U_0 admits a slope $\leq h_{\text{aux}}$ decomposition on $\oplus_i C^i$ and affinoids $\mathrm{Sp}(R_i)$ such that U_i admits a slope decomposition on $(\oplus_i C^i)^{\leq h_{\text{aux}}}$ and $x \in \mathrm{Sp}(R_i)$ for each i. Taking the intersection of these subsets gives us the required affinoid. The finite flatness follows from the above corollary. ☐

Definition 6.3.15. If $\mathcal{U}$ is a wide-open disc such that $H^{\bullet}(\bar{Y}_G(K), A^{\text{an}}_{\mathcal{U},m})$ admits a slope $\leq \mathbf{h}$ decomposition we say that $\mathcal{U}$ is *$\boldsymbol{h}$-adapted.*

6.3.4 Control theorem

We prove control results for the cohomology of locally symmetric spaces.

Lemma 6.3.16. *Let $\lambda \in \mathcal{W}_m$ with residue field L and $\mathcal{U} \subset \mathcal{W}_m$ a wide-open disc containing λ, then there is a quasi-isomorphism*

$$R\Gamma(\bar{Y}_G(K), A^{\text{an}}_{\mathcal{U},m})[1/p] \otimes^L_{\Lambda_G(\mathcal{U})[1/p]} L \sim R\Gamma(\bar{Y}_G(K), A^{\text{an}}_{\lambda,m} \otimes_{\mathbb{Q}_p} L).$$

Proof. This follows from the fact that

$$A^{\text{an}}_{\mathcal{U},m}[1/p] \otimes_{\Lambda_G(\mathcal{U})[1/p]} L = A^{\text{an}}_{\lambda,m}[1/p] \otimes_{\mathbb{Q}_p} L. \quad \square$$

Corollary 6.3.17. *For $\boldsymbol{h} \in \mathbb{Q}^n_{\leq 0}$ non-critical, an $\boldsymbol{h}$ adapted wide-open disc $\mathcal{U} \subset \mathcal{W}_m$ and algebraic $\lambda \in \mathcal{U}$, there is a quasi-isomomorphism*

$$R\Gamma(\bar{Y}_G(K), A^{\text{an}}_{\mathcal{U},m})[1/p]^{\leq \boldsymbol{h}} \otimes^L_{\Lambda_G(\mathcal{U})[1/p]} L \sim R\Gamma(\bar{Y}_G(K), V_{\lambda,\mathcal{O}})[1/p]^{\leq \boldsymbol{h}} \otimes_{\mathbb{Q}_p} L.$$

Proof. This is an immediate corollary of Theorem 6.3.13 and the previous lemma. ☐

6.3.5 Vanishing results

Let $\mathcal{U}$ be an $\mathbf{h}$-adapted wide-open disc.

Definition 6.3.18. For $\mathbf{h} \in \mathbb{Q}^n_{\geq 0}$ set

$$\mathcal{T}_{\mathcal{U},\mathbf{h}} = \operatorname{im}\left(\mathbb{T}^-_{S,p} \to \operatorname{End}_{\Lambda_G(\mathcal{U})[1/p]}(R\Gamma(\bar{Y}_G(K), A^{\mathrm{an}}_{\mathcal{U},m})[1/p]^{\leq \mathbf{h}})\right)$$
$$\mathcal{T}_{\lambda,\mathbf{h}} = \operatorname{im}\left(\mathbb{T}^-_{S,p} \to \operatorname{End}_{\Lambda_G(\mathcal{U})[1/p]}(R\Gamma(\bar{Y}_G(K), V_{\lambda,\mathcal{O}})[1/p]^{\leq \mathbf{h}})\right)$$

Lemma 6.3.19. *The natural map*

$$r_\lambda : \mathcal{T}_{\mathcal{U},\boldsymbol{h}} \to \mathcal{T}_{\lambda,\boldsymbol{h}}$$

induces a bijection

$$\operatorname{Spec}(\mathcal{T}_{\lambda,\boldsymbol{h}}) \to \operatorname{Spec}(\mathcal{T}_{\mathcal{U},\boldsymbol{h}}/\ker(r_\lambda)) \tag{27}$$

Proof. This is [AS08, Theorem 6.2.1(ii)]. ☐

Lemma 6.3.20. *Suppose $\lambda \in \mathcal{U}$ and $\mathfrak{m}_\lambda \subset \mathcal{T}_{\lambda,\boldsymbol{h}}$ is a maximal ideal such that*

$$R\Gamma(\bar{Y}_G(K), V_{\lambda,\mathcal{O}})[1/p]^{\leq \boldsymbol{h}}_{\mathfrak{m}_\lambda}$$

is quasi-isomorphic to a complex concentrated in degree $q = \dim_{\mathbb{C}} Y_G$. Then if $\mathfrak{M}_{\mathcal{U}}$ is the image of $\mathfrak{m}_\lambda$ under the identification (27) *then*

$$R\Gamma(\bar{Y}_G(K), A^{\mathrm{an}}_{\mathcal{U},m})^{\leq \boldsymbol{h}}_{\mathfrak{M}_{\mathcal{U}}}$$

is quasi-isomorphic to a complex of projective $(\mathcal{T}_{\mathcal{U},\boldsymbol{h}})_{\mathfrak{M}_{\mathcal{U}}}$ modules concentrated in degree q.

Proof. This follows from the Lemma 2.9 of [BDJ21]. ☐

6.4 Classes in Galois cohomology

We give a recipe for mapping étale classes into Galois cohomology.

6.4.1 Bits of eigenvarieties and families of Galois representations

Let $\mathbf{h} \in \mathbb{Q}^n_{\geq 0}$. Consider the total étale cohomology $H^\bullet(\bar{Y}_G(K), \mathscr{A}_{\mathcal{U},m})[1/p]^{\leq \mathbf{h}}$ for an $\mathbf{h}$-adapted wide-open disc $\mathcal{U}$.

Definition 6.4.1. For $\mathcal{U}, \mathbf{h}$ as above define $\mathcal{E}_{\mathcal{U},\mathbf{h}}$ to be the quasi-Stein rigid space defined by

$$\mathcal{O}(\mathcal{E}_{\mathcal{U},\mathbf{h}}) := \mathcal{T}_{\mathcal{U},\mathbf{h}} \otimes_{\Lambda_G(\mathcal{U})[1/p]} \mathcal{O}(\mathcal{U}).$$

The structure morphism

$$\mathbf{w} : \mathcal{E}_{\mathcal{U},\mathbf{h}} \to \mathcal{W}_G$$

is finite and we refer to it as the *weight map.*

For $L/\mathbb{Q}_p$ a point $x \in \mathcal{E}_{\mathcal{U},\mathbf{h}}(L)$ corresponds to an eigensystem of $\mathbb{T}^-_{S,p}$ acting on $H^\bullet(\bar{Y}_G(K), \mathscr{A}_{\mathbf{w}(x)})[1/p]^{\leq \mathbf{h}} \otimes L$.

Definition 6.4.2. We call a point $x \in \mathcal{E}_{\mathcal{U},\mathbf{h}}(L)$ *classical* if $\mathbf{w}(x)$ is the restriction of a dominant algebraic character and the associated eigensystem occurs in $H^\bullet(\bar{Y}_G(K), V_{\mathbf{w}(x)}) \otimes_{\mathbb{Q}_p} L$.

Remark 6.4.3. Theorem 6.3.13 says that non-critical slope eigensystems of classical weight are classical.

Definition 6.4.4. We say a classical point $x \in \mathcal{E}_{\mathcal{U},\mathbf{h}}(L)$ is *really nice* if

$$\left(H^\bullet(\bar{Y}_G(K), V_{\mathbf{w}(x)})[1/p] \otimes_{\mathbb{Q}_p} L\right)_x = \left(H^q(\bar{Y}_G(K), V_{\mathbf{w}(x)})[1/p] \otimes_{\mathbb{Q}_p} L\right)_x$$

is a free $\mathcal{O}(\mathcal{E}_{\mathcal{U},\mathbf{h}})_x$-module of rank 1 and the weight map is étale at x.

Definition 6.4.5. Define a complex of coherent sheaves $M^\bullet_{\mathcal{U},h}$ over $\mathcal{E}_{\mathcal{U},\mathbf{h}}$ as that induced by the complex of $\mathcal{O}(\mathcal{E}_{\mathcal{U},\mathbf{h}})$-modules

$$\mathscr{C}^\bullet(\bar{Y}_G(K), \mathscr{A}_{\mathcal{U},m})[1/p]^{\leq \mathbf{h}} \otimes_{\Lambda_G(\mathcal{U})[1/p]} \mathcal{O}(\mathcal{U}).$$

Proposition 6.4.6. *Let $x \in \mathcal{E}_{\mathcal{U},\boldsymbol{h}}$ be a really nice point. Then there is an affinoid neighbourhood $x \in \mathcal{V} \subset \mathcal{E}_{\mathcal{U},\boldsymbol{h}}$ such that for non-critical $\boldsymbol{h}$ the complex of sheaves*

$$M^\bullet_{\mathcal{U},\boldsymbol{h}}|_{\mathcal{V}}$$

is quasi-isomorphic to a complex of locally free sheaves concentrated in degree q.

Proof. By Lemma 6.3.20 the stalk of $M^\bullet_{\mathcal{U},\mathbf{h}}$ at x is quasi-isomorphic to a complex concentrated in degree q. By coherence we can find an affinoid $\mathcal{V} \subset \mathcal{U}$ containing x such that

$$M^\bullet_{\mathcal{U},\mathbf{h}}|_{\mathcal{V}}$$

is quasi-isomorphic to a complex concentrated in degree q. ☐

Definition 6.4.7. We say an affinoid $\mathcal{V} \subset \mathcal{E}_{\mathcal{U},\mathbf{h}}$ is *righteous* if the restriction of the weight map to $\mathcal{V}$ is an isomorphism onto its image and $M^\bullet_{\mathcal{U},\mathbf{h}}|_{\mathcal{V}}$ is quasi-isomorphic to a complex of locally free sheaves concentrated in degree q.

Clearly a subaffinoid of a righteous affinoid is also righteous.

Lemma 6.4.8. *If $x \in \mathcal{E}_{\mathcal{U},\boldsymbol{h}}$ is really nice then it has an affinoid neighbourhood $\mathcal{V}$ which is righteous.*

Proof. This follows immediately from the weight map being étale at really nice points and Proposition 6.4.6. ☐

Suppose now that $x \in \mathcal{E}_{\mathcal{U},\mathbf{h}}$ is really nice with righteous neighbourhood $\mathcal{V}$. Take a wide-open disc $\mathcal{U}' \subset \mathcal{V}$ containing x so that $\mathbf{w}^{-1}(\mathbf{w}(\mathcal{U}'))$ is a finite disjoint union of spaces isomorphic to $\mathbf{w}(\mathcal{U}')$. Let $\tilde{f} \in \mathcal{T}_{\mathbf{w}(\mathcal{U}'),\mathbf{h}}$ be an idempotent satisfying

$$\tilde{f} \cdot \mathcal{T}_{\mathbf{w}(\mathcal{U}'),\mathbf{h}} \cong \mathcal{O}(\mathcal{U}')$$

and $\tilde{f} \cdot H^\bullet(\bar{Y}_G(K), \mathscr{A}_{\mathbf{w}(\mathcal{U}')})[1/p]^{\leq \mathbf{h}} = H^q(\bar{Y}_G(K), \mathscr{A}_{\mathbf{w}(\mathcal{U}')})[1/p]^{\leq \mathbf{h}}|_{\mathcal{U}'}$ and let $f \in \mathbb{T}^-_{S,p} \hat{\otimes} \mathcal{O}(\mathcal{U}')$ be a lift of $\tilde{f}$.

Definition 6.4.9. Define an $\mathcal{O}(\mathcal{U}')$-linear Galois representation

$$W_{\mathcal{U}'} := H^q(\bar{Y}_G(K), \mathscr{A}_{\mathbf{w}(\mathcal{U}')})[1/p]^{\leq \mathbf{h}} \hat{\otimes}_{\mathcal{T}_{\mathbf{w}(\mathcal{U}'),h}} \mathcal{O}(\mathcal{U}') = f \cdot H^q(\bar{Y}_G(K), \mathscr{A}_{\mathbf{w}(\mathcal{U}')})[1/p]^{\leq \mathbf{h}}.$$

This Galois representation is a direct summand of $H^q(\bar{Y}_G(K), \mathscr{A}_{\mathbf{w}(\mathcal{U}')})[1/p]^{\leq \mathbf{h}}$ with projector f.

Lemma 6.4.10. *For $x \in \mathcal{E}_{\mathcal{U},\boldsymbol{h}}$ really nice with residue field $L(x)$ and $\mathcal{U}'$ as above, we have*

$$W_{\mathcal{U}'} \otimes_{\mathcal{O}(\boldsymbol{w}(\mathcal{U}'))} L(x) \cong f \cdot H^q(\bar{Y}_G(K), V_{\boldsymbol{w}(x)})_x =: W_x.$$

Proof. The left hand side is isomorphic to the stalk of $H^q(\bar{Y}_G(K), \mathscr{A}_{\mathcal{U}})[1/p]^{\leq \mathbf{h}}$ at x and this is equal to the right hand side by standard control results. ☐

6.4.2 Abel–Jacobi maps

Let Σ be a set of primes of the reflex field E such that we have an integral model $Y_{G,\Sigma}$ of Y_G over $\mathcal{O}_E[\Sigma^{-1}]$ and let $K = K^pK_p \subset G(\mathbb{A}_f)$ be a neat open compact subgroup with $K_p \subset J_G$ and admitting an Iwahori decomposition. In this section we construct a *weight* κ *Abel–Jacobi map*

$$AJ_\kappa^{\leq \mathbf{h}} : (f \cdot e^{\leq \mathbf{h}})H^{q+1}_{\acute{e}t}(Y_G(K)_\Sigma, \mathscr{A}^{\mathrm{Iw}}_{\kappa,m})[1/p] \to H^1(\mathcal{O}_E[\Sigma^{-1}], W_\kappa)$$

for weights $\kappa : \mathfrak{S}_G \to B^\times$, where W_κ is a Galois representation defined below.

By the Hochschild–Serre spectral sequence there is an Abel–Jacobi map

$$\mathrm{AJ} : H^{q+1}_{\acute{e}t}(Y_G(K)_\Sigma, \mathscr{A}^{\mathrm{Iw}}_{\kappa,m})[1/p]_0 \to H^1(\mathcal{O}_E[\Sigma^{-1}], H^q_{\acute{e}t}(\bar{Y}_G(K), \mathscr{A}^{\mathrm{Iw}}_{\kappa,m})[1/p])$$

where $H^{q+1}_{\acute{e}t}(Y_G(K)_\Sigma, \mathscr{A}^{\mathrm{Iw}}_{\kappa,m})[1/p]_0$ is the kernel of the base-change map

$$H^{q+1}_{\acute{e}t}(Y_G(K)_\Sigma, \mathscr{A}^{\mathrm{Iw}}_{\kappa,m}) \to H^{q+1}_{\acute{e}t}(\bar{Y}_G(K), \mathscr{A}^{\mathrm{Iw}}_{\kappa,m}).$$

Let $\mathbf{h} \in \mathbb{Q}^n_{\geq 0}$.

Lemma 6.4.11. *Let* $e^{\leq \boldsymbol{h}} \in B\{\{\mathfrak{U}_p^-\}\}$ *be the slope* $\leq \boldsymbol{h}$ *projector on* $\mathscr{C}^\bullet(\bar{Y}_G(K), A^{\mathrm{Iw}}_{\kappa,m})[1/p]$*, then*

$$e^{\leq \boldsymbol{h}} H^\bullet_{\acute{e}t}(\bar{Y}_G(K), \mathscr{A}^{\mathrm{Iw}}_{\kappa,m})[1/p] = H^\bullet_{\acute{e}t}(\bar{Y}_G(K), \mathscr{A}^{\mathrm{Iw}}_{\kappa,m})[1/p]^{\leq \boldsymbol{h}}.$$

Proof. Since the differentials d are continuous and $\mathfrak{U}_p$-equivariant and since $e^{\leq \mathbf{h}}$ converges on each $\mathscr{C}^i(\bar{Y}_G(K), A^{\mathrm{Iw}}_{\kappa,m})$ then $d(e^{\leq \mathbf{h}}x) = e^{\leq \mathbf{h}}d(x)$ and so $e^{\leq \mathbf{h}}$ preserves cocycles and coboundaries and by Lemma 6.3.7

$$e^{\leq \mathbf{h}}Z(\bar{Y}_G(K), A^{\mathrm{Iw}}_{\kappa,m})[1/p] = Z(\bar{Y}_G(K), A^{\mathrm{Iw}}_{\kappa,m})[1/p]^{\leq \mathbf{h}}$$

and

$$e^{\leq \mathbf{h}}B(\bar{Y}_G(K), A^{\mathrm{Iw}}_{\kappa,m})[1/p] = B(\bar{Y}_G(K), A^{\mathrm{Iw}}_{\kappa,m})[1/p]^{\leq \mathbf{h}}$$

and the result follows by comparing Betti and étale cohomologies. ☐

We note that for $B = \mathcal{O}, \Lambda_G(\mathcal{U})$, the series $e^{\leq \mathbf{h}}$ is a p-adic limit of polynomials. This is clear for $B = \mathcal{O}$ and also holds when $B = \Lambda_G(\mathcal{U})$. Indeed, by construction $e^{\leq \mathbf{h}}$ converges on an affinoid $\mathcal{V} \times \mathbb{A}^n_{\mathrm{rig}}$ containing $\mathcal{U} \times \mathbb{A}^n_{\mathrm{rig}}$ and thus we can write $e^{\leq \mathbf{h}}$ as a p-adic limit of $\mathcal{O}(\mathcal{V})$-coefficient polynomials in the operators $U_1, \ldots, U_n$. We can assume without loss of generality that $e^{\leq \mathbf{h}}$ is optimally integrally normalised in the sense that $e^{\leq \mathbf{h}} \in \mathcal{O}(\mathcal{V})^\circ\{\{U_1, \ldots, U_n\}\}$ and $e^{\leq \mathbf{h}} \not\equiv 0 \bmod p$. Since p^r vanishes on $H^i_{\acute{e}t}(\bar{Y}_G(K), \mathscr{A}^{\mathrm{Iw}}_{\kappa,m}/\mathrm{Fil}^r)$ we have a well-defined action of $e^{\leq \mathbf{h}}$ which mod p^r is represented by a polynomial $e_r^{\leq \mathbf{h}} \in B[U_1, \ldots, U_n]$ and this sequence satisfies $e^{\leq \mathbf{h}} = \lim_r e_r^{\leq \mathbf{h}}$. We can arrange it such that

$$e_{r+1}^{\leq \mathbf{h}} \equiv e_r^{\leq \mathbf{h}} \bmod p^r.$$

Since $\mathfrak{U}_p$ acts on $H^\bullet_{\acute{e}t}(Y_G(K)_\Sigma, \mathscr{A}^{\mathrm{Iw}}_{\kappa,m}/\mathrm{Fil}^r)$ for all r the collection of elements $\{e_r^{\leq \mathbf{h}}\}_{r \geq 0}$ map to a compatible system of endomorphisms of the inverse system $H^\bullet_{\acute{e}t}(Y_G(K)_\Sigma, \mathscr{A}^{\mathrm{Iw}}_{\kappa,m}/\mathrm{Fil}^r)$ and thus we get an action of $e^{\leq \mathbf{h}}$ on $\varprojlim H^\bullet_{\acute{e}t}(Y_G(K)_\Sigma, \mathscr{A}^{\mathrm{Iw}}_{\kappa,m}/\mathrm{Fil}^r) = H^\bullet_{\acute{e}t}(Y_G(K)_\Sigma, \mathscr{A}^{\mathrm{Iw}}_{\kappa,m})$. Moreover, since the base-change map

$$BC_n : H^{q+1}_{\acute{e}t}(Y_G(K)_\Sigma, \mathscr{A}^{\mathrm{Iw}}_{\kappa,m}/\mathrm{Fil}^r) \to H^{q+1}_{\acute{e}t}(\bar{Y}_G(K), \mathscr{A}^{\mathrm{Iw}}_{\kappa,m}/\mathrm{Fil}^r)$$

is $\mathfrak{U}_p$-equivariant for $0 \leq r \leq \infty$ and p-adically continuous it commutes with $e^{\leq \mathbf{h}}$ and thus induces a map

$$e^{\leq \mathbf{h}}H^{q+1}_{\acute{e}t}(Y_G(K)_\Sigma, \mathscr{A}^{\mathrm{Iw}}_{\kappa,m}) \to H^{q+1}_{\acute{e}t}(\bar{Y}_G(K), \mathscr{A}^{\mathrm{Iw}}_{\kappa,m})[1/p]^{\leq \mathbf{h}}.$$

Remark 6.4.12. There is no reason for us to believe that the image of $e^{\leq \mathbf{h}}$ in $\mathrm{End}(H^{q+1}_{\acute{e}t}(Y_G(K)_\Sigma, \mathscr{A}^{\mathrm{Iw}}_{\kappa,m}))$ is an idempotent.

Let $\mathbf{h}$ be non-critical, $\mathcal{V} \subset \mathcal{E}_{\mathcal{U},\mathbf{h}}$ a righteous affinoid containing a very nice point x, and let $\mathcal{U}', f$ be as in Section 6.4.1. We then have

$$H^{q+1}_{\acute{e}t}(\bar{Y}_G(K), \mathscr{A}^{\mathrm{Iw}}_{\mathcal{U},m})[1/p]^{\leq \mathbf{h}} \otimes_{\mathcal{T}_{\mathcal{U},\mathbf{h}}} \mathcal{O}(\mathcal{U}') = f \cdot H^{q+1}_{\acute{e}t}(\bar{Y}_G(K), \mathscr{A}^{\mathrm{Iw}}_{\mathcal{U},m})[1/p]^{\leq \mathbf{h}} = 0$$

and

$$H^{q+1}_{\acute{e}t}(\bar{Y}_G(K), \mathscr{A}^{\mathrm{Iw}}_{\lambda,m})[1/p]^{\leq \mathbf{h}}_x = 0$$

We thus have for $\kappa \in \{k^G_{\mathcal{U}}, \lambda\}$

$$(f \cdot e^{\leq \mathbf{h}}) H^{q+1}_{\acute{e}t}(Y_G(K)_\Sigma, \mathscr{A}^{\mathrm{Iw}}_{\kappa,m})[1/p] \subset H^{q+1}_{\acute{e}t}(Y_G(K)_\Sigma, \mathscr{A}^{\mathrm{Iw}}_{\kappa,m})[1/p]_0.$$

Definition 6.4.13. We define the weight κ *slope* $\leq \boldsymbol{h}$ *Abel–Jacobi map*

$$AJ^{\leq \mathbf{h}}_{\kappa} : (f \cdot e^{\leq \mathbf{h}}) H^{q+1}_{\acute{e}t}(Y_G(K)_\Sigma, \mathscr{A}^{\mathrm{Iw}}_{\kappa,m})[1/p] \to H^1(\mathcal{O}_E[\Sigma^{-1}], W_\kappa)$$

to be the restriction of AJ to $(f \cdot e^{\leq \mathbf{h}}) H^{q+1}_{\acute{e}t}(Y_G(K)_\Sigma, \mathscr{A}^{\mathrm{Iw}}_{\kappa,m})[1/p]$.

6.5 Criterion for Q^0_H-admissibility and superfluous variables

6.5.1 Criterion for Q^0_H-admissibility

Let $\mu \in X^\bullet_+(S_H), \lambda \in X^\bullet_+(S_G)$ and suppose there is an injective H-map

$$V^H_\mu \to V^G_\lambda.$$

Is there a character $\chi \in X^\bullet(G)$ such that $\mu + \chi \in X^\bullet_+(S_H)^{Q^0_H}$?

Lemma 6.5.1. *Suppose μ is trivial on $Q^0_H \cap G^{\mathrm{der}}$ and the maximal torus quotient C_G of G is split. Then there is $\chi_\mu \in X^\bullet(G)$ such that $(V_\lambda \otimes \chi_\mu^{-1})^{Q^0_H} \neq \{0\}$.*

Proof. We have an injection

$$Q^0_H/(Q^0_H \cap G^{\mathrm{der}}) \to C_G$$

and thus $Q^0_H/(Q^0_H \cap G^{\mathrm{der}})$ is a split torus. The restriction of μ to Q^0_H lifts (non-uniquely) to a character χ_μ of G whence the result follows. ☐

The assumptions in Lemma 6.5.1 won't hold in every case, so we describe a process to widen the number of Q^0_H admissible weights by substituting the pair (G, H) for a slightly modified pair $(\tilde{G}, \tilde{H})$.

Assume S^0_H satisfies Milne's assumption (SV5). This is equivalent to the real points of the subgroup

$$(S^0_H)^a = \cap_{\chi \in X^\bullet(S_H)} \mathrm{Ker}(\chi)$$

being compact [10]. For an algebraic group M define $\tilde{M} := M \times S^0_H$ and let

$$Q^0_{\tilde{H}} := \{(h, \bar{h}) \in Q^0_H \times S^0_H\}$$

where $\bar{h}$ denotes the image of h in S^0_H. Since this is the kernel of the map $\tilde{Q}_H \to S^0_H$ given by $(q, s) \mapsto \bar{q}s^{-1}$ it is a mirabolic subgroup of $\tilde{Q}_H$. We have that $\tilde{\mathcal{F}} := \tilde{\bar{Q}}_G \backslash \tilde{G} \cong \mathcal{F}$ and its easy to see that $Q^0_{\tilde{H}}$ has an open orbit on $\tilde{\mathcal{F}}$. A character $\mu \in X^\bullet(S_H)$ induces a character $\tilde{\mu} \in X^\bullet(Q^0_{\tilde{H}})$ given by

$$(h, \bar{h}) \mapsto \mu(\bar{h})$$

which corresponds to μ under the isomorphism $Q^0_H \cong Q^0_{\tilde{H}}$ sending h to $(h, \bar{h})$. What's more, $\tilde{\mu}$ admits an extension to a character $\chi_\mu \in X^\bullet(\tilde{G})$ by simply taking for $(g, s) \in \tilde{G}$

$$\chi_\mu(g, s) = \mu(s).$$

Thus

$$(V^G_\lambda \otimes \chi_\mu^{-1})^{Q^0_{\tilde{H}}} \neq \{0\}$$

i.e. the weight $\lambda - \chi_\mu$ is $Q^0_{\tilde{H}}$-admissible.

---

[10] We emphasise again that we are only imposing this assumption for convenience.

6.5.2 Superfluous variables

For this section we suppose that there is a central torus $T_Z \subset Z_G$ such that

$$S_G/S_G^0 = T_Z/(S_G^0 \cap T_Z) \times S^Z = S_Z \times S^Z \tag{28}$$

for some complementary torus S^Z.

Remark 6.5.2. This will notably occur when the group itself is of the form $G \times T_Z$ as in, for example, the construction given in Section 6.5.1.

In this case we get a product decomposition

$$\mathcal{W}_G = \mathcal{W}_Z \times \mathcal{W}^Z$$

where the components correspond to those in the decomposition (28).

Lemma 6.5.3. *Let $K_G = K^p J_G \subset G(\mathbb{A}_f)$ be a neat open compact subgroup. Let $\mathcal{U}_Z \subset \mathcal{W}_Z, \mathcal{U}^Z \subset \mathcal{W}^Z$ be wide-open discs and set $\mathcal{U} = \mathcal{U}_Z \times \mathcal{U}^Z$. For any $\boldsymbol{h} \in \mathbb{Q}^n_{\geq 0}$ there is an isomorphism of $\Lambda_{\mathcal{U}} = \Lambda_{\mathcal{U}^Z} \hat{\otimes}_{\mathcal{O}} \Lambda_{\mathcal{U}_Z}$-modules*

$$H^\bullet(\bar{Y}_G(K_G), A^{\mathrm{Iw}}_{\mathcal{U},m})^{\leq h} \hat{\otimes}_{\mathcal{O}} \Lambda_{\mathcal{U}_Z}$$

Proof. There are finitely many $g \in G(\mathbb{A}_f)$ and arithmetic subgroups $\Gamma_g \subset G(\mathbb{Q})$ such that

$$\mathscr{C}^\bullet(\bar{Y}_G(K_G), A^{\mathrm{Iw}}_{\mathcal{U},m}) = \oplus \mathscr{C}^\bullet(\Gamma_g, A^{\mathrm{Iw}}_{\mathcal{U},m})$$

as $\Lambda_{\mathcal{U}}$-modules. Moreover, since K_G is assumed neat, the groups Γ_g have trivial centre so in particular for any g we have a Γ_g-module isomorphism

$$A^{\mathrm{Iw}}_{\mathcal{U},m} \cong A^{\mathrm{Iw}}_{\mathcal{U}^Z,m} \hat{\otimes} \Lambda_{\mathcal{U}_Z}$$

and

$$\mathscr{C}^\bullet(\Gamma_g, A^{\mathrm{Iw}}_{\mathcal{U},m}) = \mathscr{C}^\bullet(\Gamma_g, A^{\mathrm{Iw}}_{\mathcal{U}^Z,m}) \hat{\otimes} \Lambda_{\mathcal{U}_Z}.$$

Recall from Section 6.3.3 that $a_i \in A^-$ are elements defining the Hecke operators U_i used in our slope decompositions. Taking the image of a_i in S_G/S_G^0 we have a decomposition $a_i = a_{i,Z} \times a_i^Z$ corresponding to (28) and it's easy to see that $a_{i,Z}$ acts trivially on $A^{\mathrm{Iw}}_{\mathcal{U},m}$ from which we can infer that for $\mathbf{h} \in \mathbb{Q}^n_{\geq 0}$:

$$\mathscr{C}^\bullet(\Gamma_g, A^{\mathrm{Iw}}_{\mathcal{U},m})^{\leq \mathbf{h}} \cong \mathscr{C}^\bullet(\Gamma_g, A^{\mathrm{Iw}}_{\mathcal{U}^Z,m})^{\leq \mathbf{h}} \hat{\otimes} \Lambda_{\mathcal{U}_Z}$$

and thus

$$\mathscr{C}^\bullet(\bar{Y}_G(K_G), A^{\mathrm{Iw}}_{\mathcal{U},m}) \cong \mathscr{C}^\bullet(\bar{Y}_G(K_G), A^{\mathrm{Iw}}_{\mathcal{U}^Z,m})^{\leq \mathbf{h}} \hat{\otimes} \Lambda_{\mathcal{U}_Z}.$$

Since $\Lambda_{\mathcal{U}_Z}$ is a flat $\mathcal{O}$-module we deduce the result. ☐

The lemma has a global geometric interpretation:

Lemma 6.5.4. *Let $\mathcal{U} = \mathcal{U}^Z \times \mathcal{W}_Z$, then*

$$\mathcal{E}^{\leq \boldsymbol{h}}_{\mathcal{U}} \cong \mathcal{E}^{\leq \boldsymbol{h}}_{\mathcal{U}^Z} \times \mathcal{W}_Z.$$

Proof. This follows from the fact that

$$\mathrm{End}_{\Lambda_{\mathcal{U}^Z} \hat{\otimes} \mathcal{O}_{\mathcal{W}_Z}}(\mathscr{C}^\bullet(\bar{Y}_G(K_G), A^{\mathrm{Iw}}_{\mathcal{U}^Z,m})^{\leq \mathbf{h}} \hat{\otimes} \mathcal{O}_{\mathcal{W}_Z}) = \mathrm{End}_{\Lambda_{\mathcal{U}^Z}}(\mathscr{C}^\bullet(\bar{Y}_G(K_G), A^{\mathrm{Iw}}_{\mathcal{U}^Z,m})^{\leq \mathbf{h}}) \hat{\otimes} \mathcal{O}_{\mathcal{W}_Z}.$$

☐

Throughout this paper there have been a number of times where we have had to shrink our subspace $\mathcal{U} \subset \mathcal{W}_G$. The upshot of the above discussion is that if $\mathcal{U}$ decomposes as a product over the decomposition (28) then we only need to shrink the $\mathcal{W}^Z$-component. If $D^{\ell a}(S_Z, M)$ denotes the space of locally analytic distributions on S_Z with values in a Banach-module M then there is a canonical isomorphism

$$D^{\ell a}(S_Z, M) \cong \mathcal{O}_{\mathcal{W}_Z} \hat{\otimes} M.$$

For $\mathcal{U} = \mathcal{U}_Z \times \mathcal{W}^Z$ the Galois representation of Definition 6.4.9 looks like

$$W_{\mathcal{U}} \cong D^{\ell a}(S_H, W_{\mathcal{U}^Z})$$

echoing Theorem A [LZ16].

Example 6.5.5. In Section 6.5.1 we showed that we could modify a spherical pair (G, H) to get a pair $(\tilde{G}, \tilde{H})$ such that any weight $\lambda \in X_+^\bullet(S_G)$ is Q_H^0-admissible up to a twist by a character χ_λ of $\tilde{G}$. Recall that $\tilde{G} = G \times S_H^0$ so there is an obvious decomposition $S_{\tilde{G}}/S_{\tilde{G}}^0 = S_G/S_G^0 \times S_H^0$ of the form (28) and thus a decomposition

$$\mathcal{W}_{\tilde{G}} = \mathcal{W}_G \times \mathcal{W}_Z.$$

Moreover, as showed in Section 6.5.1, the character χ_λ can be chosen to factor through the S_H^0 component, i.e. $\chi \in \mathcal{W}_Z$. Therefore, using the results of this section, taking $\tilde{\mathcal{U}} = \mathcal{U} \times \mathcal{W}_Z$ with $\mathcal{U} \subset \mathcal{W}_G$ we can construct a class $f_{\tilde{\mathcal{U}}}^{\mathrm{sph}} \in A_{\tilde{\mathcal{U}},m}^{\mathrm{Iw}}$ such that for *any* $\lambda \in \mathcal{U}$, $f_{\tilde{\mathcal{U}}}^{\mathrm{sph}}$ interpolates the Q_H^0-invariant vectors $f_{\lambda-\chi_\lambda}^{\mathrm{sph}} \in A_{\lambda,m}^{\mathrm{Iw}}(-\chi_\lambda)$.

6.6 Example: $(\mathrm{GSp}_4, \mathrm{GL}_2 \times_{\mathrm{GL}_1} \mathrm{GL}_2)$

We show how the above theory can be used to construct a class interpolating non-ordinary variants of the Lemma–Flach Euler system constructed in [LSZ21]. Let $G = \mathrm{GSp}_4$ and $H = \mathrm{GL}_2 \times_{\mathrm{GL}_1} \mathrm{GL}_2$. These groups admit a natural embedding

$$H \hookrightarrow G.$$

as used in [LSZ21]. As in [LRZ21, Section 7.1] to get the full weight variation we need to modify G and H. Set

- $\tilde{G} = G \times \mathrm{GL}_1 \times \mathrm{GL}_1$, $\tilde{H} = (\mathrm{GL}_2 \times_{\mathrm{GL}_1} \mathrm{GL}_2) \times \mathrm{GL}_1 \times \mathrm{GL}_1$,
- $Q_{\tilde{G}} = B_{\tilde{G}} = B_G \times \mathrm{GL}_1 \times \mathrm{GL}_1$, $Q_{\tilde{H}} = B_{\tilde{H}} \times \mathrm{GL}_1 \times \mathrm{GL}_1$, where $B_G = T_G \times N_G, B_H = T_H \times N_H$ are the respective upper triangular Borels.
- For a $\mathbb{Z}_p$-algebra R define $Q_{\tilde{H}}^0(R) = \{\left(\begin{smallmatrix} x & * \\ & 1 \end{smallmatrix}\right) \times \begin{pmatrix} xy & * \\ & y^{-1} \end{pmatrix} \times (y) \times (x) : x, y \in R^\times \}$.
- Set $L_{\tilde{G}} = T_{\tilde{G}} = T_G \times \mathrm{GL}_1 \times \mathrm{GL}_1$ and $L_H = T_{\tilde{H}} = T_H \times \mathrm{GL}_1 \times \mathrm{GL}_1$.

There is a natural embedding

$$\tilde{H} \hookrightarrow \tilde{G}$$

extending the embedding of H into G and $Q_{\tilde{H}}^0$ has an open orbit on the flag variety $\mathcal{F}$ with trivial stabiliser.

Let $0 \leq q \leq a$, $0 \leq r \leq b$. Consider the following dominant character of $T_{\tilde{G}}$

$$\lambda^{[a,b,q,r]} : \begin{pmatrix} x_1 & & & \\ & x_2 & & \\ & & x_2^{-1}x_3 & \\ & & & x_1^{-1}x_3 \end{pmatrix} \times (x_4) \times (x_5) \to x_1^{a+b} x_2^a x_3^{-2a-b} x_4^{r-q+a} x_5^q$$

and write $V^{[a,b,q,r]}$ for the irreducible $\tilde{G}$-representation of highest weight $\lambda^{[a,b,q,r]}$ with maximal admissible lattice $V_{\mathbb{Z}_p}^{[a,b,q,r]}$. Note that $V^{[a,b,0,-a]} = D^{a,b}$ in the notation of [LSZ21]. An easy computation using the branching law for $H \hookrightarrow G$ [LSZ21, Proposition 4.3.1] shows that

$$(V^{[a,b,q,r]})^{Q_{\tilde{H}}^0} \neq 0$$

and thus that there is an $\tilde{H}$-map

$$\left(\mathcal{P}_{\mathbb{Z}_p}^{[c,d]}\right)^\vee \to V_{\mathbb{Z}_p}^{[a,b,q,r]}$$

where $\mathcal{P}_{\mathbb{Z}_p}^{[c,d]} := \mathcal{P}_{\lambda^{[c,d]},\mathbb{Z}_p}^{\tilde{H}}$ for the weight

$$\lambda^{[c,d]} : \begin{pmatrix} y_1 & \\ & y_1^{-1}y_3 \end{pmatrix} \times \begin{pmatrix} y_3 & \\ & y_2^{-1}y_3 \end{pmatrix} \times (y_4) \times (y_5) \mapsto y_1^c y_2^d y_3^{-(c+d)} y_4^{-d},$$

and $c = a + b - q - r, d = a - q + r$.

Fix prime-to-p open compact subgroups $K_G^p \subset G(\mathbb{A}_f^{(p)})$ and $K_G^p = K_G^p \cap H$ such that $K_G^p J_G$ and $K_H^p J_H$ are neat. Define $K_G^p = K_G^p \times \mathrm{GL}_1(\mathbb{A}_f^{(p)})^2$ and $K_{\tilde{H}}^p = K_{\tilde{G}}^p \cap \tilde{H}(\mathbb{A}_f^{(p)})$. We see from [LRZ21, Section 7.1] (but using parahoric test data at p) that there is a class

$$_{c_1,c_2}z_{\acute{e}t}^{[a,b,q,r]} \in H^4(Y_G(J_G)_\Sigma, V_{\mathbb{Z}_p}^{[a,b,q,r]}(3))$$

obtained by pushing forward a cup-product of Eisenstein classes

$$_{c_1,c_2}\mathrm{Eis}_\phi^{c,d} = {}_{c_1}\mathrm{Eis}_{\acute{e}t,\phi_1}^c \sqcup {}_{c_2}\mathrm{Eis}_{\acute{e}t,\phi_2}^d \in H^2(Y_H(J_H)_\Sigma, (\mathcal{P}_{\mathbb{Z}_p}^{[c,d]})^\vee(2))$$

where the auxiliary values c_1, c_2 are chosen to ensure integrality of the classes as in Section 6.2.2 and $\phi = \phi_1 \otimes \phi_2$.

By the results of Section 6.4, if Π is a cohomological cuspidal automorphic representation of G of weight (a, b) and non-critical slope $\leq \mathbf{h}$ giving a really nice point on the G eigenvariety, then there is an Abel–Jacobi map

$$AJ_\Pi : (f \cdot e^{\leq \mathbf{h}})H^4(Y_{\tilde{G}}(J_G)_\Sigma, V_{\mathbb{Z}_p}^{[a,b,q,r]}(3))_0 \to H^1(\mathbb{Q}, W_\Pi)$$

where W_Π is the 4-dimensional Galois representation constructed by Taylor and Weissauer [Tay91], [Wei05].

Theorem 6.6.1. *Let $\mathcal{U} \subset \mathcal{W}_m$ be a wide-open disc. There is a class*

$$_{c_1,c_2}z_{\mathcal{U},m} \in D^{\ell a}((\mathbb{Z}_p^\times)^2, H^4(Y_G(J_G)_\Sigma, \mathscr{A}_{\mathcal{U},m}^{\mathrm{Iw}}))$$

such that for any cohomological cuspidal automorphic representation Π of weight (a, b) and non-critical slope $\leq \boldsymbol{h}$ at p giving a really nice point on the GSp_4 *eigenvariety and for any $0 \leq q \leq a, 0 \leq r \leq b$, up to shrinking $\mathcal{U}$, we have*

$$\rho^{[a,b,q,r]}\left(AJ_{\mathcal{U}}^{\leq \boldsymbol{h}}({}_{c_1,c_2}z_{\mathcal{U},m})\right) = \left(1 - \frac{p^q}{\alpha}\right)\left(1 - \frac{p^{1+a+r}\chi_2(p)}{\beta}\right) AJ_\Pi({}_{c_1,c_2}z_{\acute{e}t}^{[a,b,q,r]})$$

where α is the eigenvalues for the Siegel operator U_S, $\alpha\beta/p^{1+a}$ is the eigenvalue for the Klingen operator $U_{\mathcal{K}}$ and χ_2 is a prime-to-p Dirichlet character depending on the choice of Schwartz function away from p.

Proof. The weight space $\mathcal{W}_{\tilde{G}}$ decomposes as

$$\mathcal{W}_{\tilde{G}} = \mathcal{W}_G \times \mathcal{W}_{\mathrm{GL}_1} \times \mathcal{W}_{\mathrm{GL}_1}.$$

Let $\tilde{\mathcal{U}} = \mathcal{U} \times \mathcal{W}_{\mathrm{GL}_1} \times \mathcal{W}_{\mathrm{GL}_1}$

Define

$$_{c_1,c_2}\tilde{\mathcal{EI}}_{\phi^{(p)}} \in H^2(Y_{\tilde{H}}(J_{\tilde{H}}), \Lambda(N_{\tilde{H}}(\mathbb{Z}_p)\backslash J_{\tilde{H}})(2)) \cong H^2(Y_H(J_H), \Lambda(N_H(\mathbb{Z}_p)\backslash J_H)(2)) \otimes \Lambda(\mathbb{Z}_p^\times) \otimes \Lambda(\mathbb{Z}_p^\times)$$

for prime-to-p Schwartz functions $\phi_i^{(p)}$, $\phi^{(p)} = \phi_1^{(p)} \otimes \phi_2^{(p)}$ as the image of ${}_{c_1}\mathcal{EI}_{\phi_1^{(p)}} \sqcup {}_{c_2}\mathcal{EI}_{\phi_2^{(p)}}$. These classes interpolate the classes ${}_{c_1,c_2}\mathrm{Eis}_{\acute{e}t,\phi}^{c,d}$ for varying c, d. Similar to Section 6.2.2, define

$$_{c_1,c_2}\tilde{\mathcal{EI}}_{\phi^{(p)}}^{\mathcal{U}} \in H^2(Y_{\tilde{H}}(K_H)_\Sigma, \mathscr{D}_{\tilde{\mathcal{U}},m}^{\tilde{H}}(2))$$

by pushing forward ${}_{c_1,c_2}\tilde{\mathcal{EI}}_{\phi^{(p)}}$ along the natural map

$$\Lambda(N_{\tilde{H}}(\mathbb{Z}_p)\backslash J_{\tilde{H}}) \to D_{\tilde{\mathcal{U}},m}^{\tilde{H}}.$$

We have an isomorphism $H^4(Y_{\tilde{G}}(J_{\tilde{G}}), \mathscr{A}_{\tilde{\mathcal{U}},m}^{\mathrm{Iw}}) = H^4(Y_G(J_G), \mathscr{A}_{\mathcal{U},m}^{\mathrm{Iw}}) \otimes \Lambda(\mathbb{Z}_p^\times) \otimes \Lambda(\mathbb{Z}_p^\times)$.

Applying the machinery of this paper to ${}_{c_1,c_2}\tilde{\mathcal{EI}}_{\mathcal{U}}$ we obtain

$$ {}_{c_1,c_2}z_{\mathcal{U}}^{J_G} \in H^4(Y_G(J_G)_\Sigma, \mathscr{A}_{\mathcal{U},m}^{\mathrm{Iw}}) \otimes \Lambda(\mathbb{Z}_p^\times) \otimes \Lambda(\mathbb{Z}_p^\times). $$

Shrinking $\mathcal{U}$ if necessary, the slope $\leq \mathbf{h}$ Abel–Jacobi map $AJ_{\mathcal{U}}^{\leq \mathbf{h}}$ is well-defined (after inverting p), and so for admissible weights $\lambda^{[a,b,q,r]}$ we have an equality of classes

$$ \rho^{[a,b,q,r]}(AJ_{\mathcal{U}}^{\leq \mathbf{h}}({}_{c_1,c_2}z_{\mathcal{U}}^{J_G})) = \left(1 - \frac{p^q}{\alpha}\right)\left(1 - \frac{p^{1+a+r}\chi_2(p)}{\beta}\right) AJ_{\Pi}^{\leq \mathbf{h}}({}_{c_1,c_2}z_{\acute{e}t}^{[a,b,q,r]}) \in H^1(\mathbb{Q}, W_\Pi), $$

where the Euler factor is computed by a zeta integral computation (due to Loeffler) comparing classes at V_1 and Iwahori level (see Remark **??**). ☐

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