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A compound double pendulum with friction

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ABSTRACT

We study a version of the two-degree-of-freedom double pendulum in which the two point masses are replaced by rigid bodies of irregular shape and nonconservative forces are permitted. We derive the equations of motion by analysing the forces involved in the framework of screw theory. This distinguishes the work from similar studies in the literature, which typically consider a double pendulum composed with rods and assume equations of motion without derivation. The equations of motion are solved numerically using the fourth-order Runge-Kutta method to show that decreasing the friction of the axles can cause the trajectory of one of the pendulums to become aperiodic. The stability of steady state solutions is also analysed.

1. Introduction

The simple double pendulum is one of the most famous dynamical systems in classical mechanics, consisting of two point masses joined by two massless rods which are free to rotate [1,2]. The system is one of the simplest to exhibit features typical of chaotic motion (motion which is sensitive to small changes in the initial conditions) [3]. It is also a common pedagogical example of a nonlinear dynamical system [4]. Many variants of the double pendulum have been considered, including a simple asymmetric double pendulum [5]. Various types of compound distributed-mass pendulum have also been studied. The most typical examples are the double bar pendulum and the double square pendulum [6]. An interesting variation on the compound pendulum was considered by Amer et al., who considered the nonlinear oscillations of a rigid body constrained to the plane and attached at a point to one end of a massless damped spring, with stiffness k, whose other end is attached to another point [7]. This type of system is known to be useful in modelling ship motions with nonlinear coupling between pitching and rolling motions and is an example of a harmonically excited dynamical system with three degrees of freedom [8]. Furthermore, the system has intricate dynamics marked by the presence of both bifurcations and chaotic motions [9]. A similar two-degree-of-freedom system with non-linear damping was studied by Zhu et al., who found that the system can exhibit periodic, quasiperiodic and chaotic motions [10]. Their results also showed that the amplitude could be reduced by carefully adjusting the parameters of the system.

More recently, El-Sabaa et al. considered the planar motion of a two-degree-of-freedom double rigid body pendulum, where the pivot point is constrained to move in a Lissajous curve [11]. The resonances were classified for this system and solvability conditions were identified for the steady-state solutions. The equations of motion were solved numerically using the fourth-order Runge-Kutta method. This work builds on previous investigations of chaotic responses of a harmonically excited spring pendulum system using the method of multiple scales [12]. Using this method, the nonautonomous system is reduced to an approximate autonomous system of amplitude and phase variables which is show to have Hopf bifurcation and a sequence of period-doubling bifurcations leading to chaotic motions. Eissa et al. used this method to solve the equations of motion up to the fourth order approximation and extracted all primary resonance cases to this order [13].

The stability of the system was also investigated using both frequency response equations and the phase-plane method. Addelhfeez et al. considered the planar motion of a three-degree-of-freedom triple pendulum, consisting of a double rigid pendulum attached to an unstretched rod, whose suspension point is fixed [14]. The resonance conditions have been analysed for harmonically excited three-degrees-of-freedom pendulum using the asymptotic method of multiple scales [15–17]. Finally, both regular and chaotic behaviour has been studied for a two-degree-of-freedom double pendulum with repulsive permanent magnets embedded in each body, with the first magnet installed at the end of the second pendulum and the second one attached to the body of the experimental rig. It was found that the motion is governed by a pair of coupled non-linear second-order ODEs which include friction and magnetic interaction torques [18]. This study was subsequently extended to a two-degree-of-freedom system

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consisting of a double pendulum with magnets embedded in a variable magnetic field, where the pivots of the two pendulums are coupled by an elastic element [19].

In this article, we study the planar motion of a two-degree-of-freedom rigid body pendulum, where the upper body is attached to a stationary axle. Similarly to [18], the mathematical model which we use allows for the inclusion of nonconservative forces. The equations of motion are derived using screw theory formalism and analysis of forces. It is then easy to include the effects of friction at both axles. This is to be compared with [18], where both the form of the equations of motion and the governing equations for friction torques are assumed a priori to fit with experimental data. Since both the equations of motion and the equations for the torques have a form which is somewhat long and complicated, the mathematical model ends up having 15 parameters and has very long transitional processes. This can be compared with our model, which allows for the inclusion of friction and other nonconservative forces in a simple way. It is anticipated that the model which we study could be useful when studying locomotion of a robot, where the robot has a leg composed of two links of identical irregular shape and friction occurs at both axles. In Section 2, we describe the dynamical system which we study and the physical assumptions which we employ in our mathematical model. In Section 3, we give a brief introduction to screw theory and the notation of screws. This notation is then used to derive the equations of motion. In Section 4, we solve the equations numerically and demonstrate an interesting point that decreasing friction at the axles of the two bodies can cause them to stop oscillating in phase. In Section 5, we study stability of steady state solutions for the system, the case which would likely be of most interest for applications in robot walking. In Section 6, we finish with conclusions and future directions.

2. The compound double pendulum

We will begin by setting up the mechanical system which we would like to study. Consider a double pendulum composed of two identical rigid bodies with equal mass m and some irregular shape (see Fig. 1 for an example). Note that the two bodies do not necessarily need to be rods or squares. The upper body rotates around a fixed axle at P in the x-direction and the lower body rotates about an axle at Q which passes through the upper body. The two axles are assumed to be massless and the axles are situated at the same points within the geometries of their respective plates. It is also assumed that the two bodies have uniform mass density distributions and that both bodies have the same moments of inertia about their axles, where the moment of inertia is defined to be the second moment of the mass density. Our setting is generalized insofar as we do not neglect the effects of friction or other nonconservative forces at the axles. A coordinate system with origin at P is defined as shown in Fig. 1 and the centers of mass for the upper and lower bodies are located at positions (x_1, y_1) and (x_2, y_2). The centers of mass of the two bodies form angles θ_1 and θ_2 with respect to the y axis. The distance between the two axles is defined to be d and the distance between each axle and the respective center of mass at (x_1, y_1) or (x_2, y_2) is defined to be c. The notation for all the quantities used in given in Table 1.

To begin deriving the equations of motion, note that the orientation of each plate can be described by a rotation matrix $\mathbf{R}(\theta_i)$ from the group of special orthogonal matrices SO(2):

$$\mathbf{R}(\theta_i) = \begin{pmatrix} \cos\theta_i & -\sin\theta_i \\ \sin\theta_i & \cos\theta_i \end{pmatrix}, \tag{1}$$

for $i = 1, 2$. Such a transformation rotates points in the xy-plane through an angle θ_i with respect to the x-axis about the origin of a Cartesian coordinate system. The aim is to arrive at a system of equations

$$\boldsymbol{\tau} = \mathbf{A}\ddot{\boldsymbol{\theta}} + \mathbf{b}, \tag{2}$$

where $\boldsymbol{\tau}$ is a vector of torques, $\ddot{\theta}$ denotes an angular acceleration, and $\mathbf{A}$ and $\mathbf{b}$ are defined in terms of angular positions θ and angular speeds $\dot{\theta}$. As we have two plates, we must ultimately get rid of all the reaction forces and produce a system of two coupled scalar equations.

Table 1
Parameters of the system.

Quantity	Dimension	Description
m	kg	Mass of first and second pendulum
θ_1	Dimensionless	Angle which first pendulum makes with y axis
θ_2	Dimensionless	Angle which second pendulum makes with y axis
d	m	Distance between the two axles
I	kg m^2	Moment of inertia
c	m	Distance from an axle to the centre of mass of pendulum
γ_1	Dimensionless	Friction coefficient for first axle
γ_2	Dimensionless	Friction coefficient for second axle
g	m s^{-2}	Acceleration due to gravity

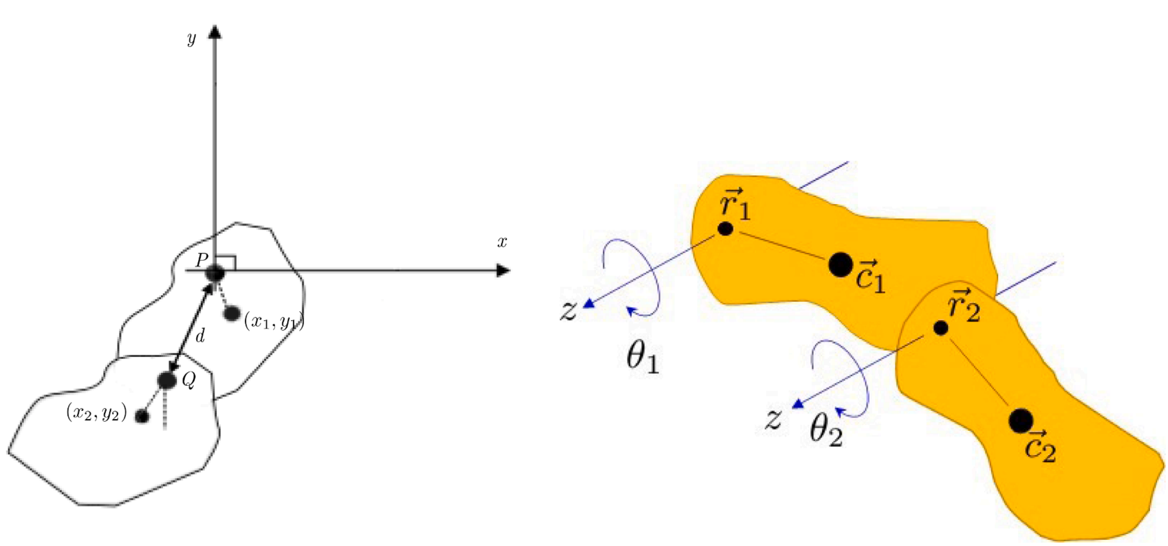

Fig. 1. Left: a diagram of the rigid body double pendulum showing the locations of the axles and the centres of mass in the xy-plane. The first axle is located at the point P and the second is located at the point Q. Right: the rigid body pendulum from an angle. Both axles are shown to go through the z-axis. The angles θ_1 and θ_2 of the centres of mass with respect to the y-axis are also shown.

The position vector for the center of mass $\mathbf{c}_1$ of the first plate is a function of the angle θ_1 which we can define as

$$\mathbf{c}_1 = \mathbf{R}(\theta_1)\begin{pmatrix} 0 \\ -c \end{pmatrix} = \begin{pmatrix} c\sin\theta_1 \\ -c\cos\theta_1 \end{pmatrix}. \tag{3}$$

The position vector for the second center of mass $\mathbf{c}_2$ is found by adding the relative position vector for the second axle $\mathbf{r}_2$:

$$\mathbf{c}_2 = \mathbf{R}(\theta_1)\begin{pmatrix} 0 \\ -d \end{pmatrix} + \mathbf{R}(\theta_2)\begin{pmatrix} 0 \\ -c \end{pmatrix} = \begin{pmatrix} d\sin\theta_1 + c\sin\theta_2 \\ -d\cos\theta_1 - c\cos\theta_2 \end{pmatrix}. \tag{4}$$

We then need to derive the rotational velocities and accelerations for both bodies and the linear velocities and accelerations of the points P and Q. The angular velocities of the two plates are simply

$$\boldsymbol{\omega}_1 = \dot{\theta}_1\mathbf{k}, \qquad \boldsymbol{\omega}_2 = \dot{\theta}_2\mathbf{k}, \tag{5}$$

whereas the angular accelerations are given by

$$\dot{\boldsymbol{\omega}}_1 = \ddot{\theta}_1\mathbf{k}, \qquad \dot{\boldsymbol{\omega}}_2 = \ddot{\theta}_2\mathbf{k}. \tag{6}$$

The first axle is fixed, so we have for the linear velocities

$$\mathbf{v}_1 = \boldsymbol{0}, \qquad \mathbf{v}_2 = (\mathbf{k}\times\mathbf{r}_2)\dot{\theta}_1. \tag{7}$$

For the linear accelerations of the two plates, we have

$$\mathbf{a}_1 = \boldsymbol{0}, \qquad \mathbf{a}_2 = (\mathbf{k}\times\mathbf{r}_2)\ddot{\theta}_1 - \mathbf{r}_2\dot{\theta}_1^2. \tag{8}$$

Similarly, the linear and angular accelerations of the two centers of mass are given by

$$\dot{\mathbf{c}}_1 = (\mathbf{k}\times\mathbf{c}_1)\dot{\theta}_1, \qquad \dot{\mathbf{c}}_2 = (\mathbf{k}\times\mathbf{c}_2)\dot{\theta}_2, \qquad \ddot{\mathbf{c}}_1 = (\mathbf{k}\times\mathbf{c}_1)\ddot{\theta}_1 - \mathbf{c}_1\dot{\theta}_1^2, \qquad \ddot{\mathbf{c}}_2 = (\mathbf{k}\times\mathbf{c}_2)\ddot{\theta}_2 - \mathbf{c}_2\dot{\theta}_2^2. \tag{9}$$

We now need to balance the forces and torques involved in the problem. The equation for one single rigid body is given by Newton's second law. For the first plate, we obtain

$$\mathbf{R}_1 - \mathbf{R}_2 - mg\mathbf{j} = m\ddot{\mathbf{c}}_1, \tag{10}$$

where $\mathbf{R}_i$ denotes the reaction force at the i-th axle and mg is the restoring force due to gravity which tries to move the plate back to its equilibrium position. For the second plate, we have

$$\mathbf{R}_2 - mg\mathbf{j} = m\left(\ddot{\mathbf{r}}_2 + \ddot{\mathbf{c}}_2\right). \tag{11}$$

Substituting equation (11) into (10) eliminates the reaction force $\mathbf{R}_2$. However, $\mathbf{R}_2$ appears again in the torque balance equations. These are written down using the fact that the net torque about a joint is zero, which gives us

$$\tau_1 - \tau_2 = I\ddot{\theta}_1 + \mathbf{k}\cdot(\mathbf{r}_2\times\mathbf{R}_2), \tag{12a}$$
$$\tau_2 = I\ddot{\theta}_2 + \mathbf{k}\cdot\left(\mathbf{c}_2\times m\ddot{\mathbf{r}}_2\right). \tag{12b}$$

The hope is that we can then get rid of the reaction forces by back-substituting and plugging the result back into equations (12a) and (12b). This ends up giving extremely long and cumbersome expressions which would need to be processed with a computer algebra system. Instead of proceeding in this way, it is easier for our purposes to introduce an alternative type of notation called screw notation.

3. Basics of screw notation

Screw notation pairs up two vectors into one object with six components (a screw being a pair of three-dimensional real vectors). Note that this switch in notation changes the usual vector calculus operations ($\times$ is no longer a vector cross-product because one is working with 6×1 vectors). The physical intuition behind the use of screws is that any spatial displacement of a rigid body can be defined using a translation along a line, followed by a rotation around an axis parallel to that line (a fact known as Chasles' theorem). The line along which the body is translated is known as a screw axis, by analogy with the fact that the pitch of a screw relates a rotation around an axis to a translation along that axis. A screw is a six-dimensional vector constructed from a pair of three-dimensional vectors. The key idea from the computational point of view is that the algebraic manipulations can be simplified using the fact that in screw notation the magnitude of a vector along a line is coupled to the moment about this line, which removes the necessity of dealing with the forces and the torques separately.

The screw which is formed from a force vector and a torque vector is known in the literature as a wrench:

$$\mathscr{W} = \begin{pmatrix} \mathbf{r}\times\mathbf{F} \\ \mathbf{F} \end{pmatrix}. \tag{13}$$

For example, the pure gravity wrench $\mathscr{G}_i$ for both plates is a 6-component vector defined to be

$$\mathscr{G}_i = \begin{pmatrix} -mg\mathbf{c}_i\times\mathbf{j} \\ -mg\mathbf{j} \end{pmatrix}, \tag{14}$$

where the direction of gravity is taken to be in the negative y direction, g is the acceleration due to gravity, $\mathbf{c}_i$ is the position vector for the center of mass of the plate, and $\mathbf{j}$ is the unit vector in the y direction. On the other hand, the gravity screw is defined to be

$$\mathbf{g} = \begin{pmatrix} \boldsymbol{0} \\ -g\mathbf{j} \end{pmatrix}, \tag{15}$$

where $\boldsymbol{0}$ denotes the three-dimensional vector whose components are all zero. Finally, a twist is a screw formed from the angular velocity of a rigid body around an axis and the linear velocity along that axis.

The dynamics of a robot can be simplified further by directly writing equations for the angular accelerations and angular speeds of the bodies about the axles. Since we are dealing with two links, we will write these for a pair of torques:

$$\tau_1 = A_{1j}\ddot{\theta}_j + B_{1jk}\dot{\theta}_j\dot{\theta}_k + C_1, \tag{16a}$$
$$\tau_2 = A_{2j}\ddot{\theta}_j + B_{2jk}\dot{\theta}_j\dot{\theta}_k + C_2, \tag{16b}$$

where repeated indices are summed over. As long as we know the forms for $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$, this then directly provides the dependencies of the torques on the angular speeds and accelerations which would have to be obtained at great length using the usual vector notation. $\mathbf{A}$ is a 6×6 matrix defined to be

$$A_{ij} = \begin{cases} \mathbf{s}_i^T(N_1+N_2)\mathbf{s}_j & \text{if } i\ge j \\ \mathbf{s}_j^T(N_1+N_2)\mathbf{s}_i & \text{if } i<j \end{cases} \tag{17}$$

and the $\mathbf{B}$ and $\mathbf{C}$ terms are Coriolis and gravity terms, respectively, which are defined in terms of $\mathbf{s}$, $\mathbf{N}$, and the gravity screw $\mathbf{g}$. The usual choice for the tensor $\mathbf{B}$ is given by

$$B_{ijk} = \frac{1}{2}\left(\mathbf{s}_j^T(N_x+\ldots+N_2)[\mathbf{s}_i,\mathbf{s}_k] + \mathbf{s}_k^T(N_x+\ldots+N_2)\left[\mathbf{s}_i,\mathbf{s}_j\right] - \mathbf{s}_i^T(N_x+\ldots+N_2)\left[\mathbf{s}_k,\mathbf{s}_j\right]\right). \tag{18}$$

The gravity terms $\mathbf{C}$ are defined by

$$C_i = \mathbf{s}_i^T(N_i+\ldots+N_2)\mathbf{g}. \tag{19}$$

$\mathbf{s}_i$ denotes an eigenvector of the inertia matrix, where i refers to the i-th principal axis (also known as a principal screw of inertia). Substituting in the necessary quantities, we arrive at two coupled scalar equations for the torques:

$$\tau_1 = 2md^2\ddot{\theta}_1 + I\ddot{\theta}_1 + md^2\ddot{\theta}_2\cos(\theta_1 - \theta_2) + md^2\dot{\theta}_2^2\sin(\theta_1 - \theta_2) + 2mgd\sin\theta_1, \quad (20a)$$

$$\tau_2 = md^2\ddot{\theta}_2 + I\ddot{\theta}_2 + md^2\ddot{\theta}_1\cos(\theta_1 - \theta_2) - md^2\dot{\theta}_1^2\sin(\theta_1 - \theta_2) + mgd\sin\theta_2. \quad (20b)$$

The equations of motion in Eq. (20) form a pair of coupled nonlinear differential equations which can only be solved numerically. We will here assume that the instantaneous joint torque τ_i at a moment of time may be taken to be proportional to the friction of the axle, so that we have

$$\tau_1 = -\gamma_1\dot{\theta}_1 - \gamma_2(\dot{\theta}_1 - \dot{\theta}_2), \qquad \tau_2 = \gamma_2(\dot{\theta}_1 - \dot{\theta}_2), \quad (21)$$

where γ_1 and γ_2 are coefficients of friction at the joints. This then leads to the following equations of motion:

$$-\gamma_1\dot{\theta}_1 - \gamma_2(\dot{\theta}_1 - \dot{\theta}_2) = 2md^2\ddot{\theta}_1 + I\ddot{\theta}_1 + md^2\ddot{\theta}_2\cos(\theta_1 - \theta_2) + md^2\dot{\theta}_2^2\sin(\theta_1 - \theta_2) + 2mgd\sin\theta_1, \quad (22a)$$

$$\gamma_2(\dot{\theta}_1 - \dot{\theta}_2) = md^2\ddot{\theta}_2 + I\ddot{\theta}_2 + md^2\ddot{\theta}_1\cos(\theta_1 - \theta_2) - md^2\dot{\theta}_1^2\sin(\theta_1 - \theta_2) + mgd\sin\theta_2. \quad (22b)$$

By way of verification of these equations, we note that they match with equations of motion which were assumed by Tian et al. [20] for a two degree-of-freedom hinged double rod pendulum which impacts with a rigid wall. We note that with some modifications the method which we have used here could also be used to rigorously derive the equations of motion used in [20]. The method which we have used can also be generalised to any type of torque apart from friction torques.

4. Numerical solution

To make an initial investigation of the dynamics of the pendulum the equations of motion are solved numerically using the fourth-order RungeKutta method. For this purpose, the ode45 solver was implemented in MATLAB. This solver uses an adaptive time step and cannot be used with a fixed time step. In the numerical solution, it is also assumed that the two pendulums can only swing at a small angle in the same vertical plane. For the solver, the absolute and relative error tolerances are both set to 1×10^{-6}. Typically in the literature on double pendulums, one notes that if the two pendulums are released from rest at the same initial angle, they will oscillate in phase [9]. Chaotic motion appears if the pendulums are released from a larger initial angle, which causes the two trajectories to diverge so that the in-phase oscillations are broken. It is interesting to ask whether the presence of friction or other non-conservative forces could have an influence on whether the motion can become unpredictable.

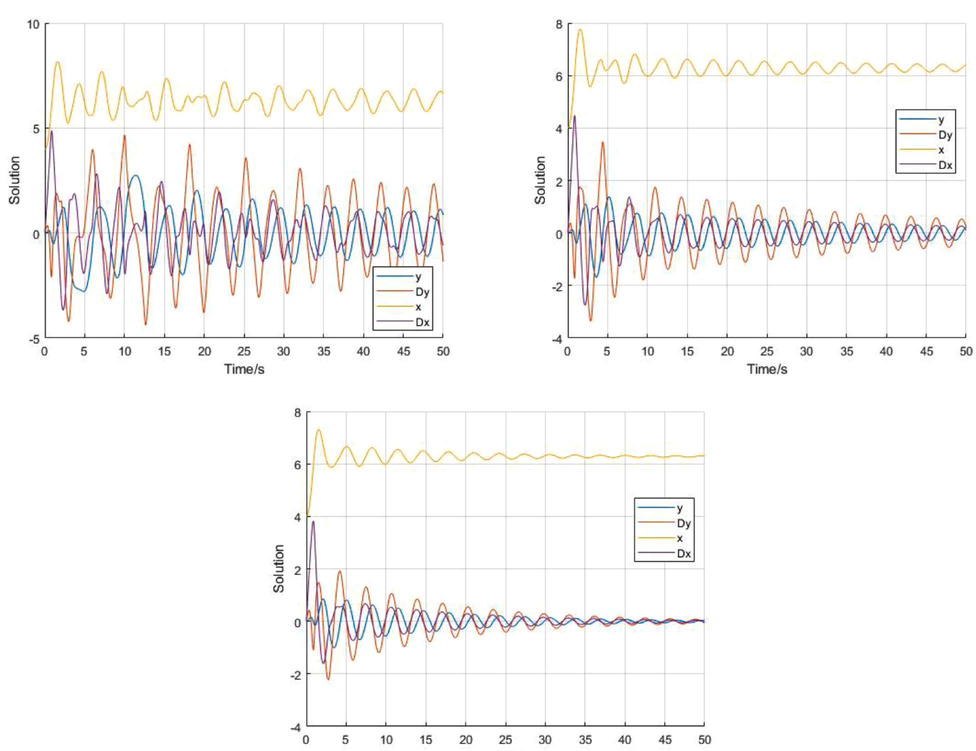

Fig. 2. θ_1 θ_2 $\dot{\theta}_1$ and $\dot{\theta}_2$ against time for $\gamma_1 = \gamma_2 = 1, 0.4, 0.1$.

To solve the equations numerically, we set all the parameters to unity and g to 9.8 ms^{-1}. As example initial conditions, we use $\theta_1 = 0^\circ$, $\theta_2 = 4^\circ$, and $\dot{\theta}_1 = \dot{\theta}_2 = 0$. In Fig. 2, we plot θ_1, θ_2, $\dot{\theta}_1$ and $\dot{\theta}_2$ against time with $\gamma_1 = \gamma_2 = 0.1, 0.4, 1$, respectively. As one might expect, the presence of friction has a strong damping effect, but interestingly, the motion can become somewhat aperiodic as the friction is lowered, showing typical modes for the first pendulum which are out of phase with the second pendulum. We attribute the intuitive reason for this as the fact that stronger friction at the joints stops the bodies from turning as easily and so restricts them into following relatively regular trajectories, whereas without friction it becomes easier for the upper pendulum to be jerked back and forth such that it no longer follows a normal mode.

5. Stability

We will comment here on the formal stability of solutions at the steady state. A possible application of the system which we have discussed is to model the leg of a locomoting robot in a fashion which is slightly more realistic than that of a double rod pendulum, since both the upper and lower limb are allowed to have an irregular shape. In this case, it is sufficient to make a theoretical study of the system for motions with small angles. In this approximation, one has

$$\cos(\theta_1-\theta_2)\approx 1, \quad \sin(\theta_1-\theta_2)\approx\theta_1-\theta_2, \quad \cos\theta_i\approx 1, \quad \sin\theta_i \approx\theta_i, \quad \dot{\theta}_i^2\approx 0. \tag{23}$$

Using this approximation, the equations of motion become

$$\gamma_1\dot{\theta}_1-\gamma_2\left(\dot{\theta}_1-\dot{\theta}_y\right)=2md^2\ddot{\theta}_1+I\ddot{\theta}_1+md^2\ddot{\theta}_2+2mdg\theta_1, \tag{24a}$$
$$\gamma_2(\dot{\theta}_1-\dot{\theta}_2)=md^2\ddot{\theta}_2+I\ddot{\theta}_2+md^2\ddot{\theta}_1+dmg\theta_2. \tag{24b}$$

To reduce this system from second order to first order, we let $\theta_1 = x_1$, $\dot{x}_1 = x_2$, $\theta_2 = y_1$, $\dot{y}_1 = y_2$ and solve for $\dot{x}_1$, $\dot{x}_2$, $\dot{y}_1$ and $\dot{y}_2$:

$$\dot{x}_1=x_2, \tag{25a}$$
$$\dot{x}_2=\frac{d^3\ gm^2y_1-2dgx_1m\left(d^2m+I\right)-\gamma_2x_2\left(2d^2m+I\right)+\gamma_2y_2\left(2d^2m+I\right)}{d^4m^2-\gamma_1\left(d^2m+I\right)+3d^2Im+I^2}, \tag{25b}$$
$$\dot{y}_1=y_2, \tag{25c}$$
$$\dot{y}_2=\frac{2d^3\ gm^3x_1-dgm_2y_1\left(2d^2m-\gamma_1+I\right)+\gamma_2x_2\left(3d^2m-\gamma_1+I\right)-\gamma_2y_2\left(3d^2m-\gamma_1+I\right)}{d^4m^2-\gamma_1\left(d^2m+I\right)+3d^2Im+I^2}. \tag{25d}$$

Calculating the Jacobian $\mathbf{J}(\mathbf{x})$, we obtain

$$\mathbf{J}(\mathbf{x})=\begin{pmatrix} 0 & 1 & 0 & 0 \\ -\frac{2dgm^2\left(d^2m+I\right)}{(*)} & -\frac{\gamma_2\left(2d^2m+I\right)}{(*)} & \frac{d^3\ gm^2}{(*)} & \frac{\gamma_2\left(2d^2m+I\right)}{(*)} \\ 0 & 0 & 0 & 1 \\ \frac{2d^3\ gm^2}{(*)} & \frac{\gamma_2\left(3md^2-\gamma_1+I\right)}{(*)} & \frac{-dgm\left(2md^2-\gamma_1+I\right)}{(*)} & -\frac{\gamma_2\left(3md^2-\gamma_1+I\right)}{(*)} \end{pmatrix}, \tag{26}$$

where $(*)$ denotes $d^4m^2-\gamma_1(d^2m+I)+3d^2Im+I^2$. Numerically evaluating the determinant equation of the Jacobian to determine its eigenvalues, we find that $\mathbf{J}$ always has four eigenvalues with negative real part (ie. solutions are stable) unless $a\lesssim\gamma_1$. In this case, there is one eigenvalue with positive real part, implying instability. It follows that steady state solutions are stable as long as this condition is avoided.

6. Conclusions

The problem of the nonlinear dynamical motion of a 2-DOF compound double pendulum with bodies of irregular shape has been investigated. We have considered a fixed pivot point for the first pendulum, but an interesting generalisation might be extend to the case the pivot point moves in an elliptical trajectory with a steady angular velocity, similarly to [17]. The analysis could also be easily extended to include two bodies with differing masses or moments of inertia. The governing equations of motion were derived using screw theory formalism and solve numerically using the fourth-order Runge-Kutta method in order to examine the influence of nonconservative forces on a typical trajectory. In this work, we considered friction torques in the equations of motion, but our analysis could easily be extended to include more general types of nonconservative force. Our method also allows for a rigorous, transparent derivation of similar equations of motion which were stated without justification for a hinged double pendulum in [20]. We have also analysed the stability of solutions at the steady state. The main applications of our model are likely in rotor dynamics and in walking analysis for humans or robots. The model may be particularly useful for robot walking, since although a human leg could be modelled by a double pendulum where each pendulum is a rod, a robot leg could have an upper and lower limb of irregular shape (for example, where both limbs are armour-plated and have mass distributions which can be encoded by the moment of inertia).

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

Data will be made available on request.

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