POST-CRITICALLY FINITE MAPS ON $\mathbb{P}^n$ FOR $n \geq 2$ ARE SPARSE

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ABSTRACT. Let $f : \mathbb{P}^n \to \mathbb{P}^n$ be a morphism of degree $d \geq 2$. The map f is said to be *post-critically finite* (PCF) if there exist integers $k \geq 1$ and $\ell \geq 0$ such that the critical locus Crit_f satisfies $f^{k+\ell}(\mathrm{Crit}_f) \subseteq f^{\ell}(\mathrm{Crit}_f)$. The smallest such ℓ is called the *tail-length*. We prove that for $d \geq 3$ and $n \geq 2$, the set of PCF maps f with tail-length at most 2 is not Zariski dense in the the parameter space of all such maps. In particular, maps with periodic critical loci, i.e., with $\ell = 0$, are not Zariski dense.

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1. Introduction

A rational map $f : \mathbb{P}^1 \to \mathbb{P}^1$ of degree $d \geq 2$ is said to be *post-critically finite* (PCF) if all of its critical points have finite forward orbits. PCF maps play a fundamental role in the study of one-dimensional dynamics; see Remark 6 for a brief history. In particular, PCF maps are ubiquitous in the sense that they are Zariski dense in the parameter space of all degree d rational maps of $\mathbb{P}^1$, and the same is true of the smaller collection of post-critically periodic (PCP) maps, which are the maps whose critical points are periodic; see [6, Theorem A].

Fornæss and Sibony [8] introduced an analogue of PCF maps on $\mathbb{P}^n$ for $n \geq 2$, and a number of authors have constructed examples of such maps and studied their properties; see [1, 2, 5, 13, 14, 15, 21, 25] for examples in complex dynamics, and [3, 13] for some arithmetic results. Our aim in this paper is to explain why it is likely that the set of such maps is much sparser than in the one-dimensional case, and to prove a result which quantifies this statement for PCF maps having small tail length. We set the notation

$$\mathrm{End}_d^n := \left\{ \begin{matrix} \text{morphisms } f : \mathbb{P}^n \to \mathbb{P}^n \text{ of algebraic} \\ \text{degree } d, \text{ i.e., } f^*\mathcal{O}_{\mathbb{P}}(1) = \mathcal{O}_{\mathbb{P}}(d) \end{matrix} \right\}.$$

We note that End_d^n is naturally identified with a Zariski open subset of $\mathbb{P}^N$, where $N = (n+1)\binom{d+n}{n} - 1$. More precisely, the variety End_d^n is the complement of the hypersurface in $\mathbb{P}^N$ defined by the vanishing of the Macaulay resultant. See [23, Chapter 1] for details.

In this paper we always work over[1]

$$\mathbb{F} := \text{an algebraically closed field of characteristic } 0.$$

Definition 1. The *critical locus* of a map $f = [f_0, \ldots, f_n] \in \mathrm{End}_d^n$ given by homogeneous polynomials $f_i(x_0, \ldots, x_n)$ is the variety

$$\mathcal{C}_f := \left\{ \det\left(\frac{\partial f_i}{\partial x_j} \right) = 0 \right\} \subset \mathbb{P}^n.$$

The *branch locus* of f is the image of the critical locus, taken with the reduced scheme structure and denoted by

$$\mathcal{B}_f := f(\mathcal{C}_f).$$

Definition 2. A map $f \in \mathrm{End}_d^n$ is *post-critically finite* (PCF) if there exist $k \geq 1$ and $\ell \geq 0$ such that

$$f^{k+\ell}(\mathcal{C}_f) \subseteq f^\ell(\mathcal{C}_f).$$

[1] Some parts of this paper remain true over infinite fields of characteristic p, but to avoid separability complications, we restrict to the case of characteristic 0.

If k and ℓ are chosen minimally, we say that f is *PCF of Type* (k, ℓ), where k is the *period* and ℓ is the *tail-length*. A PCF map with tail length 0 is said to be *post-critically periodic* (PCP).

Our main theorem says that in dimension greater than one, post-critically periodic maps are comparatively rare, and more generally the same is true for post-critically finite maps whose tail-length is at most 2.

Theorem 3. *Let $d \geq 3$ and $n \geq 2$. Fix some $\ell \leq 2$. Then*

$$\{f \in \mathrm{End}_d^n : f \textit{ is post-critically finite of Type } (k, \ell) \textit{ for some } k \in \mathbb{N}\}$$

is contained in a proper Zariski closed subset of End_d^n.

We conjecture that Theorem 3 is true for any fixed tail-length, and we ask whether it remains valid for the union over all tail-lengths.

Conjecture 4. *Let $d \geq 3$ and $n \geq 2$. Then for all $\ell \geq 1$,*

$$\{f \in \mathrm{End}_d^n : f \textit{ is post-critically finite of Type } (k, \ell) \textit{ for some } k \in \mathbb{N}\}$$

is contained in a proper Zariski closed subset of End_d^n.

Question 5. *Let $d \geq 3$ and $n \geq 2$. Is the set*

$$\{f \in \mathrm{End}_d^n : f \textit{ is post-critically finite}\}$$

contained in a proper Zariski closed subset of End_d^n*?*

Remark 6. One motivation for studying PCF endomorphisms in higher dimensions comes from work of Nekrashevych [18], in which he studies the Julia set of a PCF map $f : \mathbb{P}_{\mathbb{C}}^N \to \mathbb{P}_{\mathbb{C}}^N$ using an associated iterated monodromy group. In [1], Belk and Koch explicitly compute the iterated monodromy group associated to a particular example, which in fact turns out to be post-critically periodic. We also mention that the algebraic analogue of the partial self-covering property is exploited in [3] to show that extensions of number fields obtained by adjoining backward orbits of points relative to PCF endomorphisms of any smooth, projective variety are finitely ramified.

Remark 7. For ease of exposition, we work in the parameter space End_d^n, but we note that since the PCF property is invariant under PGL_{n+1}-conjugation, Theorem 3 could equally well be formulated for the dynamical moduli space $\mathcal{M}_d^n := \mathrm{End}_d^n /\!/ \mathrm{PGL}_{n+1}$ constructed via GIT in [17, 20]. And similarly for Conjecture 4 and Question 5.

Remark 8. The property of being PCF as given in Definition 2 admits two other equivalent characterizations that are sometimes useful. First, a map $f \in \operatorname{End}_d^n$ is PCF if and only if the post-critical locus

$$\operatorname{PostCrit}(f) := \bigcup_{m \geq 1} f^m(\mathcal{C}_f)$$

is algebraic, that is, if $\operatorname{PostCrit}(f)$ consists of a finite union of algebraic hypersurfaces. This equivalence follows immediately from the fact that for each m, the image $f^m(\mathcal{C}_f)$ is a finite union of algebraic hypersurfaces. Second, a map f is PCF if and only if there exists a Zariski-open subset $U \subseteq \mathbb{P}^N$ such that $f^{-1}(U) \subseteq U$ and such that $f : U \to \mathbb{P}^N$ is unramified; specifically, if such a U exists, then its complement is algebraic and contains the post-critical locus.

We briefly summarize the contents of this paper. In Section 2 we give various constructions of PCF maps and non-PCF maps, and in particular show that for all d and n, every period and tail length can occur. In Section 3 we prove that there is a Zariski dense set of $f \in \operatorname{End}_d^n$ such that $\mathcal{C}_f$ is a variety of general type. (We thank Jason Starr for showing us this proof.) We use this in Section 4 to show that the set of PCP maps, i.e., the set of maps f of PCF Type $(k, 0)$, is not Zariski dense. Section 5 contains two multiplicity lemmas. In Section 6 we construct maps whose branch locus has a minimally branched point and use this map to show that the set of f of PCF Type $(k, 1)$ is not Zariski dense. Section 7 gives a general method for proving, for any fixed ℓ, that the set of f of PCF Type (k, ℓ) is not Zariski dense. This method requires showing that there exists a single map having certain properties. In Section 8 we construct such a map for $\ell = 2$, thereby completing the proof that the set of f of PCF Type $(k, 2)$ is not Zariski dense.

2. Examples of PCF maps

Before proving our main results on higher dimensional PCF maps, we pause in this section to give a number of examples. We remark that in all of these examples, the critical locus $\mathcal{C}_f$ is reducible, and indeed it is generally a union of rational hypersurfaces, the multiplicity $\operatorname{Mult}_{\mathcal{C}_f}(f)$ is strictly greater than 2 and generally equal to $\deg(f)$, and the restriction $f|_{\mathcal{C}_f} : \mathcal{C}_f \to \mathcal{B}_f$ is generally not 1-to-1. This highlights the difficulty of constructing maps whose critical and branch loci are sufficiently generic, and the existence of such maps is the key to proving results such as Theorem 3.

Example 9. The most obvious PCF map is the d-power map

$$f = [x_0^d, \ldots, x_n^d] \text{ with critical locus } \mathcal{C}_f = \{x_0 x_1 \cdots x_n = 0\}$$

consisting of the coordinate hyperplanes. Thus $f(\mathcal{C}_f) = \mathcal{C}_f$, so f is PCF of Type $(1, 0)$.

Example 10 (Symmetric powers of PCF maps on $\mathbb{P}^1$). Let $f : \mathbb{P}^1 \to \mathbb{P}^1$ be a map of degree d. Then the n-fold product map, which we denote by

$$F_n := f \times f \times \cdots \times f : (\mathbb{P}^1)^n \longrightarrow (\mathbb{P}^1)^n,$$

descends to a map $\tilde{F}_n$ on the symmetric product $(\mathbb{P}^1)^n/\mathcal{S}_n$. Using the standard isomorphism $\mathbb{P}^n \cong (\mathbb{P}^1)^n/\mathcal{S}_n$, we obtain a map $\tilde{F}_n$ on $\mathbb{P}^n$ such that the diagram in Figure 1 commutes.

$$\begin{array}{ccc} (\mathbb{P}^1)^n & \xrightarrow{F_n} & (\mathbb{P}^1)^n \\ \pi \downarrow & & \downarrow \pi \\ \mathbb{P}^n & \xrightarrow{\tilde{F}_n} & \mathbb{P}^n \end{array}$$

FIGURE 1. The symmetric power of a PCF map

The commutative diagram above can be used to relate the dynamical properties of $\tilde{F}_n$ to those of f; see [28] and [27] for a systematic study of symmetric power maps. Firstly, we observe that $\tilde{F}_n$ and f have the same algebraic degree, and an explicit chain-rule calculation (whose details we omit here) can be used to relate the critical locus of $\tilde{F}_n$ to that of f. Secondly, the branch locus of $\tilde{F}_n$ is reducible, with each irreducible component rational. Also, $\tilde{F}_n$ is PCF if and only if f is PCF. Now, suppose that f is PCF. Given $p \in \mathcal{C}_f$, p is pre-periodic under f; we denote the tail-length and period of p by ℓ_p and k_p, respectively. A straightforward diagram chase and chain-rule computation can be used to show that $\tilde{F}_n$ is PCF of Type (k, ℓ) with

$$k = \operatorname*{lcm}_{p \in \mathcal{C}_f} k_p \quad \text{and} \quad \ell = \max\bigl(1, \max_{p \in \mathcal{C}_f}\{\ell_p\}\bigr). \tag{1}$$

As a special case of Example 10, we obtain the following result.

Proposition 11. *For all $n \geq 1$, all $d \geq 2$, all $k \geq 1$ and all $\ell \geq 1$, there exists a PCF map of Type (k, ℓ) in* End_d^n.

Proof. It is known that for all $d \geq 2$, all $k \geq 1$ and all $\ell \geq 1$, there exists a PCF map f of Type (k, ℓ) in End_d^1 such that f has exactly two critical points, one fixed, and one pre-periodic of Type (k, ℓ). More precisely, one can take $f(x) = x^d + c$ for an appropriate choice of c. It follows from (1) that $\tilde{F}_n \in \mathrm{End}_d^n$ is PCF of Type (k, ℓ). $\square$

Example 12. Koch [15] has used Teichmüller theory and Thurston's topological characterization of PCF maps on $\mathbb{P}^1$ (presented in [7] by Douady and Hubbard) to construct interesting PCF maps in all dimensions and degrees. We omit the details of the construction but note that for every PCF map on $\mathbb{P}^n$ arising from Koch's construction, the post-critical locus is contained in the union of hyperplanes

$$\Delta := \bigcup_{0 \le i \le n} \{x_i = 0\} \cup \bigcup_{0 \le i < j \le n} \{x_i = x_j\}.$$

It follows from counting the number of hyperplanes in Δ that if a map arising from Koch's construction is PCF Type (k, ℓ) then $1 \le k, \ell \le (n + 1 + \binom{n+1}{2})$. (In fact, the postcritical portraits of these maps can be completely described; see [15, Propositions 6.1 and 6.2].)

Example 13. Although our contention is that PCF maps are rare, it is perhaps not obvious that there exist any maps that are not PCF. One non-constructive way to see that there exist PCF maps defined over $\mathbb{C}$ is to apply Fakhruddin's result [29] that a very general endomorphism of $\mathbb{P}^n$ does not have any positive-dimensional periodic subvarieties other than $\mathbb{P}^n$, hence cannot be PCF if $n \ge 2$. We take the time here to construct examples of non-PCF maps in End_d^n, defined over $\mathbb{Q}$, for all $d \ge 2$ and all $n \ge 1$. We consider the family of morphisms

$$f_t : \mathbb{P}^n \longrightarrow \mathbb{P}^n, \quad f_t(X_0, \ldots, X_n) := [X_0^d + tX_1^d, X_1^d, X_2^d, \ldots, X_n^d].$$

The support of the critical locus of f_t is the union of the coordinate hyperplanes $\{X_i = 0\}$, each of which is fixed by f_t except $\{X_0 = 0\}$. Set

$$H_\alpha := \{X_0 = \alpha X_1\} \subset \mathbb{P}^n,$$

and note that $(f_t)_* H_\alpha = H_{\alpha^d + t}$. It follows that f_t is PCF if and only if the univariate polynomial $z^d + t$ is, and so f_1 in particular is not PCF.

We observe as an immediate consequence that for any fixed k and ℓ, the set

$$\{f \in \mathrm{End}_d^n : f \text{ is post-critically finite of type } (k, \ell)\} \tag{2}$$

is not Zariski dense in End_d^n. This follows, since elimination theory says that the set (2) is Zariski closed, and our example says that the complement of (2) is non-empty. Of course, the fact that (2) is not Zariski dense for each fixed pair (k, ℓ) is much weaker than Theorem 3, which implies that if $\ell \le 2$, then the union of (2) over all k is still not Zariski dense.

3. Determinental varieties are of general type

A key tool in the proof that PCP maps are sparse is the following result, whose proof was shown to us by Jason Starr.

Theorem 14. *Let $n \geq 2$ and $d \geq 3$. Then the set*

$$\{f \in \mathrm{End}_d^n : \mathcal{C}_f \textit{ is an irreducible variety of general type}\}$$

is a non-empty Zariski open subset of End_d^n.

Proof. The generic determinantal variety

$$\mathcal{D} = \left\{M \in \mathrm{Mat}_{(n+1)\times(n+1)}(\mathbb{F}) \cong \mathbb{F}^{(n+1)^2} : \det(M) = 0\right\}$$

is singular, but its singularities are relatively mild. More precisely, the generic determinantal variety $\mathcal{D}$ is canonical, and thus all global sections of (positive powers of) the dualizing sheaf on the singular determinantal variety lift to global sections (rather than to rational/meromorphic sections) of (positive powers of) the dualizing sheaf on any desingularization. This follows from results of Vainsencher [26], who describes an explicit desingularization $\tilde{\mathcal{D}}$ of $\mathcal{D}$ as the space of complete linear collineations. The result is also stated explicitly and proven in the preprint of Starr [24, Corollary 3.14].

Since $\mathcal{D}$ has canonical singularities, and since the total space of the incidence correspondence is smooth over the parameter space of matrices, it follows that the inverse image of $\mathcal{D}$ in this total space also has canonical singularities. Thus when we project this total space to the parameter space of $(n+1)$-tuples of homogeneous degree d polyomials, the (geometric) generic fiber has canonical singularities. Hence the open set of the parameter space consisting of fibers that have canonical singularities is dense.

Since these fibers have canonical singularities, they are of general type once the dualizing sheaf is ample. But for degree d maps $\mathbb{P}^n \to \mathbb{P}^n$, the dualizing sheaf of the critical locus is the restriction of

$$\mathcal{O}_{\mathbb{P}^n}\big((n+1)(d-2)\big).$$

Hence if $d \geq 3$, then a general $(n+1)$-tuple of degree d homogeneous polynomials has a critical locus whose desingularization is of general type. ∎

Theorem 14 covers maps of degree $d \geq 3$ for all dimensions n. For dimension 2 we can prove something stronger that includes quadratic maps.

Theorem 15. *We consider the set of maps*

$$\mathcal{E}_d^{\text{sm-irr}} := \{f \in \mathrm{End}_d^2 : \mathcal{C}_f \textit{ is smooth and irreducible}\}.$$

(a) *Let $d \ge 1$. Then $\mathcal{E}_d^{\text{sm-irr}}$ is a non-empty Zariski open subset of* End_d^2.
(b) *Let $d \ge 2$. Then $\mathcal{E}_d^{\text{sm-irr}}$ does not contain any PCP maps.*
(c) *For all $d \ge 2$, the set*

$$\{f \in \operatorname{End}_d^2 : f \text{ is PCP}\}$$

is not Zariski dense in End_d^2.

Proof. (a) The set $\mathcal{E}_d^{\text{sm-irr}}$ is clearly Zariski open, so the only question is whether it's empty. To prove that $\mathcal{E}_d^{\text{sm-irr}}$ is not empty, we use [4, Theorem 1], which says that for any smooth irreducible surface $S \subset \mathbb{P}^r$, the set of linear projections $\pi : \mathbb{P}^r \to \mathbb{P}^2$ such that the critical locus of $\pi|_S$ is smooth and irreducible is a non-empty Zariski open subset of the space of linear projections. (The special case that S is a Veronese embedding of $\mathbb{P}^2$ is proven in [16].) Taking S to be the image of the d-uple embedding $\rho_d : \mathbb{P}^2 \hookrightarrow \mathbb{P}^r$ [11, Exercise I.2.12], we see that compositions with linear projections $\pi \circ \rho_d$ correspond exactly to degree d rational maps $\mathbb{P}^2 \to \mathbb{P}^2$. So the desired result is the special case of [4] in which $S = \rho_d(\mathbb{P}^2)$.
(b) Let $f \in \mathcal{E}_d^{\text{sm-irr}}$. Then $\mathcal{C}_f$ is a smooth irreducible curve of degree $3(d-1)$ in $\mathbb{P}^2$, so it has genus $g(\mathcal{C}_f) = \frac{1}{2}(3d-4)(3d-5) \ge 1$ for all $d \ge 2$.[2] Suppose now that f is PCP, so $f^k(\mathcal{C}_f) = \mathcal{C}_f$ for some $k \ge 1$. (Note that we must have equality, since $\mathcal{C}_f$ is irreducible.) Thus $\mathcal{C}_f$ is an irreducible curve that is (forward) invariant for the map f^k. Further, since

$$\mathcal{C}_{f^k} = \mathcal{C}_f + f^*\mathcal{C}_f + \cdots + f^{(k-1)*}\mathcal{C}_f,$$

we see that $\mathcal{C}_f$ is also critical for f^k. We now apply [2, Theorem 4.1], which says that an irreducible curve in $\mathbb{P}^2$ that is forward invariant and critical for a non-linear morphism $\mathbb{P}^2 \to \mathbb{P}^2$ is necessarily a rational curve, i.e., has genus 0. This contradicts $g(\mathcal{C}_f) \ge 1$, which completes the proof that the set $\mathcal{E}_d^{\text{sm-irr}}$ does not contain any PCP maps.
(c) This is immediate from (a) and (b), since (a) gives a non-empty Zariski open subset of End_d^2, and (b) says that this open set contains no PCP maps. □

4. Proof that post-critically periodic maps are sparse

In this section we prove the tail length 0 part of Theorem 3, i.e., we prove the following result:

[2]For $d \ge 3$, the genus satisfies $g(\mathcal{C}_f) \ge 10$, so in particular $\mathcal{C}_f$ is of general type, as predicted by Theorem 14; but for $d = 2$ we see that $\mathcal{C}_f$ is not of general type. This shows that Theorem 14 cannot be extended to $d = 2$.

Theorem 16. *Let $d \geq 3$ and $n \geq 2$. Then*

$$\{f \in \mathrm{End}_d^n : f \text{ is post-critically periodic}\}$$

is contained in a proper Zariski closed subset of End_d^n.

Proof. For notational convenience we let

$$\mathrm{PCP}_d^n := \{f \in \mathrm{End}_d^n : f \text{ is post-critically periodic}\}.$$

We assume that $\mathrm{PCP}_d^n(\mathbb{F})$ is a Zariski dense subset of $\mathrm{End}_d^n(\mathbb{F})$ and derive a contradiction.

Step 1: Theorem 14 tells us that

$$\{f \in \mathrm{End}_d^n(\mathbb{F}) : \mathcal{C}_f \text{ is irreducible and of general type}\} \tag{3}$$

is a non-empty Zariski open subset of $\mathrm{End}_d^n(\mathbb{F})$. Under our assumption that $\mathrm{PCP}_d^n(\mathbb{F})$ is a Zariski dense subset of $\mathrm{End}_d^n(\mathbb{F})$, it follows that the intersection of $\mathrm{PCP}_d^n(\mathbb{F})$ with (3), i.e., the set

$$\{f \in \mathrm{PCP}_d^n(\mathbb{F}) : \mathcal{C}_f \text{ is irreducible and of general type}\},$$

is also a Zariski dense subset of $\mathrm{End}_d^n(\mathbb{F})$.

Step 2: We next show that for every map f in the set

$$\{f \in \mathrm{PCP}_d^n(\mathbb{F}) : \mathcal{C}_f \text{ is irreducible and of general type}\},$$

there is an integer $m(f) \geq 1$ such that

$$\mathcal{C}_f \subseteq \mathrm{Fix}(f^{m(f)}),$$

i.e., there is an iterate of f that fixes every point in $\mathcal{C}_f$. To see this, we use the definition of PCP to find some $k \geq 1$ such that $f^k(\mathcal{C}_f) \subseteq \mathcal{C}_f$. But f is a morphism, so for any irreducible subvariety $V \subset \mathbb{P}^n$ we have $\dim f(V) = \dim V$. Hence $\dim f^k(\mathcal{C}_f) = \dim \mathcal{C}_f$, and the irreducibility of $\mathcal{C}_f$ implies that $f^k(\mathcal{C}_f) = \mathcal{C}_f$. In other words, the map $f^k|_{\mathcal{C}_f}$ is a surjective endomorphism of $\mathcal{C}_f$. But $\mathcal{C}_f$ is of general type, and it is known that for varieties of general type, every surjective endomorphism is an automorphism; see [9, Lemma 3.4] or [12, Proposition 10.10]. Further, the automorphism group of a variety of general type is finite; see [10] for a recent strong upper bound on its order.[3] Hence there exists an r such that f^{kr} fixes every point of $\mathcal{C}_f$, and we take $m(f) = kr$.

Step 3: We note that endomorphisms of $\mathbb{P}^n$ fix no positive-dimensional subvarieties.

Lemma 17. *Let $f \in \mathrm{End}_d^n(\mathbb{F})$ with $d \geq 2$. Then* $\dim \mathrm{Fix}(f) = 0$.

[3]The quintessential example is that of a curve of genus $g \geq 2$, whose automorphism group has order at most $84(g-1)$.

Proof. Note that $\operatorname{Fix}(f)$ is certainly Zariski closed, and suppose that $Y \subseteq \operatorname{Fix}(f)$ is an irreducible subvariety of positive dimension. Setting $L = \mathcal{O}(1)|_Y$, we see that $f = \mathrm{id}$ on Y implies $f^*L = L$. On the other hand, since $f^*\mathcal{O}(1) \cong \mathcal{O}(1)^{\otimes d}$, we must also have $L = f^*L \cong L^{\otimes d}$. As $d \geq 2$, this contradicts L being ample. $\square$

Step 4: We resume the proof of Theorem 16. Let f be an element of the set

$$\{f \in \mathrm{PCP}_d^n(\mathbb{F}) : \mathcal{C}_f \text{ is irreducible and of general type}\}. \tag{4}$$

Applying Step 2, we find an integer $m = m(f) \geq 1$ so that

$$\mathcal{C}_f \subseteq \operatorname{Fix}(f^m).$$

The map f^m is in $\mathrm{End}_{d^m}^n(\mathbb{F})$, so applying Lemma 17 to the map f^m tells us that $\dim \operatorname{Fix}(f^m) = 0$. Hence

$$n - 1 = \dim \mathcal{C}_f \leq \dim \operatorname{Fix}(f^m) = 0,$$

contradicting our assumption that $n \geq 2$. $\square$

5. Two multiplicity lemmas

In this section we prove two multiplicity lemmas that will be used to deal with PCF maps of tail length 1.

Definition 18. We use Mult to denote multiplicity in various contexts. Thus if s is a local parameter cutting out $\mathcal{C}_f$ near p and t is a local parameter cutting out $\mathcal{B}_f$ near $f(p)$, then

$$f^{\#}(t) = (\text{unit in the local ring at } p) \cdot s^k \quad \text{with} \quad \mathrm{Mult}_{\mathcal{C}_f}(f) = k.$$

And if Z is a zero-dimensional scheme and $p \in Z$, then $\mathrm{Mult}_Z(p)$ is the scheme-theoretic multiplicity of Z at p.

Lemma 19. *Let X and Y be projective varieties of dimension n, let $f : X \to Y$ be a morphism, and let $p \in \mathcal{C}_f$ be a point satisfying*:

- *p is a smooth point of $\mathcal{C}_f$.*
- *p is a smooth point of X.*
- *$f(p)$ is a smooth point of Y.*
- *The restriction $f|_{\mathcal{C}_f}$ is an immersion near p.*

Then we have:

(a) *The point p is an isolated point of $f^{-1}(f(p))$.*
(b) *The multiplicity of p in this set equals the multiplicity of f along its critical locus,*

$$\mathrm{Mult}_{f^{-1}(f(p))}(p) = \mathrm{Mult}_{\mathcal{C}_f}(f).$$

Proof. We let
$$k = \mathrm{Mult}_{\mathcal{C}_f}(f).$$
We first note that since p is a smooth point of $\mathcal{C}_f$ and $f|_{\mathcal{C}_f}$ is an immersion near p, it follows that $f(p)$ is a smooth point of $\mathcal{B}_f$. We work in the completions of the local rings at p and $f(p)$, so we can pick local equations s cutting out $\mathcal{C}_f$ at p and t cutting out $\mathcal{B}_f$ at $f(p)$ such that $f^\#(t) = s^k$. We complete s and t respectively to local coordinates $(x_1, \ldots, x_{n-1}, x_n) = s$ for X at p and $(y_1, \ldots, y_{n-1}, y_n) = t$ for Y at $f(p)$ in such a way that $(x_1, \ldots, x_{n-1})$ restrict to local coordinates for $\mathcal{C}_f$ at p, and $(y_1, \ldots, y_{n-1})$ restrict to local parameters for $\mathcal{B}_f$ at $f(p)$, and further so that in these coordinates, the map induced by $f_{\mathcal{C}_f}$ from the completion of the local ring of $\mathcal{B}_f$ at $f(p)$ to the completion of the local ring of $\mathcal{C}_f$ at p is
$$\begin{aligned} f^\#_{\mathcal{C}_f} : \mathbb{F}[\![y_1, \ldots, y_{n-1}]\!] &\to \mathbb{F}[\![x_1, \ldots, x_{n-1}]\!], \\ y_i &\mapsto x_i, \qquad i = 1, \ldots, n-1. \end{aligned}$$
Then in these coordinates, the map induced by f from the completion of the local ring of Y at $f(p)$ to the completion of the local ring of X at p is
$$\begin{aligned} f^\# : \mathbb{F}[\![y_1, \ldots, y_{n-1}, y_n]\!] &\to \mathbb{F}[\![x_1, \ldots, x_{n-1}, x_n]\!], \\ y_i &\mapsto \begin{cases} x_i + f_i(x_n) & \text{for } i = 1, \ldots, n-1, \\ x_n^k & \text{for } i = n, \end{cases} \end{aligned}$$
where each f_i is a power series in x_n whose constant term is zero.

Claim. The set $\{1, x_n, x_n^2, \ldots, x_n^{k-1}\}$ is an $\mathbb{F}$-basis for the vector sapce
$$\frac{\mathbb{F}[\![x_1, \ldots, x_n]\!]}{\big(x_1 + f_1(x_n), \ldots, x_{n-1} + f_{n-1}(x_n), x_n^k\big)}.$$

Proof of Claim. Both spanning and linear independence can easily be shown directly. □

We conclude that
$$\begin{aligned} &\frac{\mathbb{F}[\![x_1, \ldots, x_n]\!]}{\big(\text{pullback of maximal ideal of } f(p)\big)} \\ &\qquad = \frac{\mathbb{F}[\![x_1, \ldots, x_n]\!]}{\big(x_1 + f_1(x_n), \ldots, x_{n-1} + f_{n-1}(x_n), x_n^k\big)} \end{aligned}$$
has dimension k over $\mathbb{F}$, so p is an isolated point of multiplicity k in $f^{-1}(f(p))$. □

Lemma 20. *Let X and Y be projective varieties of dimension n, let $f : X \to Y$ be a morphism, and let $p \in X$ and $q \in Y$ be smooth points such that p is an isolated point of multiplicity k in $f^{-1}(q)$. Let $(x_1, \dots x_n)$ be coordinates at p, so the completion of the local ring to X at p is $\mathbb{F}[\![x_1, \dots, x_n]\!]$, and let $(z_1, \dots z_n)$ be coordinates at q, so the completion of local ring to Y at q is $\mathbb{F}[\![z_1, \dots, z_n]\!]$, and suppose that in these coordinates we have $z_i = f_i(x_1, \dots, x_n)$. Denote the maximal ideals of the completions of the local rings at p and q by $\mathfrak{m}$ and $\mathfrak{n}$ respectively.*

(1) *The following are equivalent*:
  (A) $k = 1$.
  (B) $f_1, \dots, f_n$ *generate* $\mathfrak{m}$.
  (C) $\{f_1, \dots, f_n\} \bmod \mathfrak{m}^2$ *is an $\mathbb{F}$-basis for* $\mathfrak{m}/\mathfrak{m}^2$.
  (D) $p \notin \mathcal{C}_f$.
(2) *If $k = 2$, then the following are true*:
  (a) *p is a smooth point of $\mathcal{C}_f$.*
  (b) *$f|_{\mathcal{C}_f}$ is an immersion near p.*
  (c) *f has multiplicity 2 along $\mathcal{C}_f$ near p.*

Proof. Recall that
$$k = \dim_{\mathbb{F}} \frac{\mathbb{F}[\![x_1, \dots, x_n]\!]}{(f_1, \dots, f_n)}.$$
This implies the equivalence of (A) and (B). Nakayama's lemma implies the equivalence of (B) and (C). By definition, $p \in \mathcal{C}_f$ if and only if the Jacobian of f, i.e., the induced map on tangent spaces, drops rank at p. The Jacobian at p is dual to the induced map from $\mathfrak{n}/\mathfrak{n}^2$ to $\mathfrak{m}/\mathfrak{m}^2$. In turn, the map from $\mathfrak{n}/\mathfrak{n}^2$ to $\mathfrak{m}/\mathfrak{m}^2$ sends the basis $\{z_1 \dots, z_n\}$ to $\{f_1, \dots, f_n\} \bmod \mathfrak{m}^2$. Thus the Jacobian at p is full rank if and only if $\{f_1, \dots, f_n\} \bmod \mathfrak{m}^2$ is an $\mathbb{F}$-basis for $\mathfrak{m}/\mathfrak{m}^2$, proving the equivalence of (C) and (D). This completes the proof of Part (1) of Lemma 20.

For Part (2) we suppose that $k = 2$. By the preceding discussion, the set $\{f_1, \dots, f_n\} \bmod \mathfrak{m}^2$ does not generate $\mathfrak{m}/\mathfrak{m}^2$. Let $g_1, \dots, g_s$ be functions whose reductions modulo $\mathfrak{m}^2$ form a basis for
$$\frac{\mathfrak{m}/\mathfrak{m}^2}{\operatorname{Span}\bigl(\{f_1, \dots, f_n\} \bmod \mathfrak{m}^2\bigr)}.$$
Note that we have that $s \geq 1$. Also
$$1, g_1, \dots, g_s \text{ are linearly independent in } \frac{\mathbb{F}[\![x_1, \dots, x_n]\!]}{(f_1, \dots, f_n)}.$$
But
$$\dim_{\mathbb{F}} \frac{\mathbb{F}[\![x_1, \dots, x_n]\!]}{(f_1, \dots, f_n)} = 2,$$

which implies that $s = 1$, and hence that $1, g_1$ form a basis. We conclude that $\{f_1, \dots, f_n\}$ mod $\mathfrak{m}^2$ span an $(n-1)$-dimensional subspace, so without loss of generality we may assume that $\{f_1, \dots, f_{n-1}\}$ mod $\mathfrak{m}^2$ are linearly independent, and that $\{f_1, \dots, f_{n-1}, g_1\}$ mod $\mathfrak{m}^2$ is a basis for $\mathfrak{m}/\mathfrak{m}^2$. By Nakayama's lemma again,

$$\{y_1, \dots, y_n\} := \{f_1, \dots, f_{n-1}, g_1\}$$

generate $\mathfrak{m}$ and form an alternate system of coordinates at p. With respect to these new coordinates, f_n is a power series f_n' in $y_1, \dots, y_n$. We expand f_n' with respect to the last coordinate y_n,

$$f_n'(y_1, \dots, y_n) = c_0 + c_1 y_n + c_2 y_n^2 + \cdots ,$$

where each c_i is a power series in $y_1, \dots, y_{n-1}$. Also

$$\frac{\partial f_n'}{\partial y_n}(y_1, \dots, y_n) = c_1 + 2c_2 y_n + 3c_3 y_n^2 + \cdots .$$

We know that $1, y_n$ forms a basis for

$$\frac{\mathbb{F}[\![y_1, \dots, y_n]\!]}{(y_1, \dots, y_{n-1}, f_n')} \cong \frac{\mathbb{F}[\![y_n]\!]}{c_0(0, \dots, 0) + c_1(0, \dots, 0) y_n + c_2(0, \dots, 0) y_n^2 + \cdots},$$

so we must have

$$c_0(0, \dots, 0) = c_1(0, \dots, 0) = 0 \quad \text{and} \quad c_{2,0} := c_2(0, \dots, 0) \neq 0.$$

Let

$$c_1 = c_{1,1} y_1 + \cdots + c_{1,n-1} y_{n-1} + (\text{higher order terms in } \mathfrak{m}^2),$$

where each $c_{1,i} \in \mathbb{F}$. Then

$$\frac{\partial f_n'}{\partial y_n} = c_{1,1} y_1 + \cdots c_{1,n-1} y_{n-1} + 2c_{2,0} y_n + (\text{something in } \mathfrak{m}^2).$$

We want to re-write f in coordinates $y_1, \dots y_n$ at p and $z_1, \dots z_n$ at q. We have the induced map on the completions of local rings,

$$\begin{aligned} f^{\#} : \mathbb{F}[\![z_1, \dots, z_n]\!] &\to \mathbb{F}[\![y_1, \dots, y_n]\!] \\ z_i &\mapsto \begin{cases} y_i & \text{for } i = 1, \dots, n-1, \\ f_n' & \text{for } i = n. \end{cases} \end{aligned}$$

In these coordinates, the Jacobian matrix J_f is of the form

$$J_f = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & \frac{\partial f_n'}{\partial y_1} \\ 0 & 1 & 0 & \cdots & 0 & \frac{\partial f_n'}{\partial y_2} \\ 0 & 0 & 1 & \cdots & 0 & \frac{\partial f_n'}{\partial y_3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & \frac{\partial f_n'}{\partial y_{n-1}} \\ 0 & 0 & 0 & \cdots & 0 & \frac{\partial f_n'}{\partial y_n} \end{bmatrix}.$$

The critical locus $\mathcal{C}_f$ is locally cut out by the determinant of J_f, which in these coordinates is

$$\det(J_f) = \frac{df_n'}{dy_n} = c_{1,1}y_1 + \cdots c_{1,n-1}y_{n-1} + 2c_{2,0}y_n + (\text{something in } \mathfrak{m}^2).$$

Since $c_{2,0}$ is non-zero in $\mathbb{F}$, we see that $\det(J_f)$ is non-zero in $\mathfrak{m}/\mathfrak{m}^2$, which implies that $\mathcal{C}_f$ is smooth at p.

The tangent space to $\mathcal{C}_f$ at p is cut out by the equation

$$y_n = -\frac{c_{1,1}}{2c_{2,0}}y_1 - \cdots - \frac{c_{1,n-1}}{2c_{2,0}}y_{n-1},$$

so $y_1, \ldots, y_{n-1}$ restrict to give local coordinates (a basis) for the cotangent space to $\mathcal{C}_f$ at p. The map $f|_{\mathcal{C}_f} : \mathcal{C}_f \to Y$ induces the following map of completions of local rings at p and q:

$$f_{\mathcal{C}_f}^{\#} : \mathbb{F}[\![z_1, \ldots, z_n]\!] \to \frac{\mathbb{F}[\![y_1, \ldots, y_n]\!]}{(\det(J_f))} \cong \mathbb{F}[\![y_1, \ldots, y_{n-1}]\!]$$
$$z_i \mapsto \begin{cases} y_i & \text{for } i = 1, \ldots, n-1, \\ f_n' \bmod \det(J_f) & \text{for } i = n. \end{cases}$$

In these coordinates, it is clear that the map on cotangent spaces is surjective, so the map on tangent spaces is injective. Thus the map $f|_{\mathcal{C}_f} : \mathcal{C}_f \to Y$ is an immersion near p, as desired. Finally, a direct application of Lemma 19 tells us that f has multiplicity 2 along $\mathcal{C}_f$ near p. □

6. A map with a minimally branched point

In this section we construct a map f whose branch locus contains a point that is minimally branched. We call this the "hyperplance construction" because the coordinates of the map f that we construct vanish along hyperplanes.

Proposition 21 (Hyperplane Construction). *Let $n \geq 1$ and $d \geq 2$.*

(a) *There exists a morphism $f : \mathbb{P}^n \to \mathbb{P}^n$ of degree d containing a branch point $q \in \mathcal{B}_f$ with the property that*

$$f^*(q) = 2p + \underbrace{p_1 + p_2 + \cdots + p_{d^n-2}}_{\text{distinct points different from } p}. \tag{5}$$

(b) *Let f be a map as in* (a) *with a point q satisfying* (5)*. Then the following are true*:

(1) *The point q is a smooth point of $\mathcal{B}_f$, and thus lies on exactly one irreducible component B of $\mathcal{B}_f$.*

(2) *There exists a unique irreducible component C of $\mathcal{C}_f$ mapping to B.*

(3) *The map $f|_C : C \to B$ is generically* 1*-to-*1*.*

(4) *The map f has multiplicity* 2 *along C.*

Proof. (a) We take

$$q = [0 : 0 : \cdots : 0 : 1] \in \mathbb{P}^n,$$

and we use $\boldsymbol{X} = [X_1 : \cdots : X_{n+1}]$ as homogeneous coordinates on $\mathbb{P}^n$. We are going to create a map

$$f(\boldsymbol{X}) = [f_1 : \cdots : f_{n+1}] \quad \text{with} \quad f_i(\boldsymbol{X}) = \prod_{j=1}^{d} L_{i,j}(\boldsymbol{X}),$$

where the $L_{i,j}(\boldsymbol{X})$ are linear forms that will be constructed inductively.

We note that

$$f(P) = q \quad \Longleftrightarrow \quad \begin{pmatrix} \text{for all } 1 \le i \le n \text{ there is some index} \\ 1 \le \sigma(i) \le d \text{ such that } L_{i,\sigma(i)}(P) = 0. \end{pmatrix}$$

In other words, the solutions to $f(P) = q$ are parameterized by the d^n functions

$$\sigma : \{1, 2, \ldots, n\} \longrightarrow \{1, 2, \ldots, d\},$$

where a given σ corresponds to the solution(s) P_σ to the system of linear equations

$$L_{1,\sigma(1)}(P) = L_{2,\sigma(2)}(P) = \cdots = L_{n,\sigma(n)}(P) = 0. \tag{6}$$

To ease notation, we denote this set of index maps by

$$[n : d] := \bigl(\text{collection of maps } \sigma : \{1, \ldots, n\} \to \{1, \ldots, d\}\bigr).$$

We start our construction by setting

$$L_{n+1,j}(\boldsymbol{X}) = X_{n+1} \text{ for all } 1 \le j \le d,$$

i.e., we take

$$f_{n+1}(\boldsymbol{X}) := X_{n+1}^d.$$

This allows us to dehomogenize $X_{n+1} = 1$, and then by abuse of notation, we write $f = (f_1, \dots, f_n)$ for the affine map $f : \mathbb{A}^n \to \mathbb{A}^n$ having affine coordinates $(X_1, \dots, X_n)$, and $q = (0, 0, \dots, 0)$.

We next assign the initial linear form in each f_i to be X_i, i.e.,

$$L_{1,1} = X_1,\ L_{2,1} = X_2, \dots, L_{n,1} = X_n, \text{ and thus } f_i = X_i L_{i,2} L_{i,3} \cdots L_{i,d}.$$

The next step is to select the second linear form in f_1, which we do by setting

$$L_{1,2} = X_1 - X_2. \quad \text{Thus} \quad f_1 = X_1(X_1 - X_2) L_{1,3} \cdots L_{1,d}.$$

This allows us to determine the solution P_σ to (6) for the following two particular index maps σ_1 and σ_2 in $[n : d]$:

$$\sigma_1 \in [n : d] \text{ is defined by } \sigma_1(i) = 1 \text{ for all } 1 \le i \le n.$$

$$\sigma_2 \in [n : d] \text{ is defined by } \sigma_2(i) = \begin{cases} 2 & \text{if } i = 1, \\ 1 & \text{for } 2 \le i \le n. \end{cases}$$

For these index maps we have

$$\begin{aligned} P_{\sigma_1} &= \{X_1 = X_2 = X_3 = \cdots = X_n = 0\} = q, \\ P_{\sigma_2} &= \{X_1 - X_2 = X_2 = X_3 = \cdots = X_n = 0\} = q. \end{aligned}$$

Now suppose that for a given $k_1, \dots, k_n \in \{1, \dots, d\}$, we have constructed linear forms

$$\begin{gathered} L_{1,1}, \dots, L_{1,k_1}, \\ L_{2,1}, \dots, L_{2,k_2}, \\ \vdots \\ L_{n,1}, \dots, L_{n,k_n}, \end{gathered}$$

such that for every

$$\sigma \in [n : d] \quad \text{satisfying } \sigma(i) \le k_i \text{ for all } 1 \le i \le n, \tag{7}$$

the following hold:

- There is a solution P_σ to (6).
- The solutions P_σ corresponding to the σ satisfying (7) are distinct except for the duplicate value $P_{\sigma_1} = P_{\sigma_2} = q$ noted earlier.

Suppose that

$$k_t < d \text{ for some } 1 \le t \le n.$$

Then we choose a linear form L_{t,k_t+1} such that

$$L_{t,k_t+1}(P_\sigma) \ne 0 \text{ for all } \sigma \text{ satisfying (7)},$$

i.e., we want L_{t,k_t+1} to not vanish at all of the previously selected points. We can find such a linear form by choosing a point in the dual space $\check{\mathbb{P}}^n$ that is not on any of the hyperplanes defined by the previously selected P_σ. (This is where we use the assumption that our field $\mathbb{F}$ is infinite, since it ensures that $(\check{\mathbb{P}}^n)(\mathbb{F})$ is not covered by finitely many hyperplanes.)

Note that it also follows that for all σ satisfying (7), the hyperplane $L_{t,k_t+1} = 0$ does not contain the line

$$\bigcap_{i\ne t} L_{i,\sigma(i)},$$

since if it did, then the form L_{t,k_t+1} would vanish at all points on this line, including P_σ. Hence for every σ satisfying

$$\sigma(i) \le k_i \text{ for } i \ne t \quad \text{and} \quad \sigma(t) = k_t + 1,$$

the hyperplanes

$$L_{1,\sigma(1)},\ L_{2,\sigma(2)}, \ldots, L_{n,\sigma(n)}$$

intersect properly at a point P_σ that cannot equal any of the previously constructed points.

Continuing this process, we end up with linear forms

$$L_{i,j} \text{ for all } 1 \le i \le n \text{ and all } 1 \le j \le d$$

such that for $\sigma, \tau \in [n : d]$, we have

$$P_\sigma = P_\tau \iff \sigma = \tau \text{ or } \{\sigma,\tau\} = \{\sigma_1,\sigma_2\},$$

where σ_1 and σ_2 are the maps defined earlier. It follows that the map

$$f(\boldsymbol{X}) := \left[\prod_{j=1}^{d} L_{1,j}(\boldsymbol{X}) : \cdots : \prod_{j=1}^{d} L_{n,j}(\boldsymbol{X}) : X_{n+1}^d\right],$$

satisfies

$$f^*(q) = 2q + p_1 + p_2 + \cdots + p_{d^n-2},$$

where the points $q, p_1, \ldots, p_{d^n-2}$ are distinct. This completes the proof of Proposition 21(a).

(b) Lemma 20 tells us that:

- p is the only point on $\mathcal{C}_f$ that maps to q.
- p is a smooth point of $\mathcal{C}_f$.
- The map $f|_{\mathcal{C}_f} : \mathcal{C}_f \to \mathbb{P}^n$ is an immersion near p.

This implies that $\mathcal{B}_f$ is smooth at q, so q lies on a unique irreducible component of $\mathcal{B}_f$, as desired. We know already that $q \in B$ has exactly one pre-image point in $\mathcal{C}_f$, and that that pre-image point p is a smooth point of $\mathcal{C}_f$, which implies that the unique irreducible component C of $\mathcal{C}_f$ containing p is the only irreducible component of $\mathcal{C}_f$ mapping to B. Since $f|_C : C \to \mathbb{P}^n$ is an immersion near p, it is generically

1-to-1. Finally, Lemma 20 also tells us that f has order 2 along C, which completes the proof of Proposition 21(b). □

Remark 22. With minor modifications, the proof of Proposition 21(a) can be modified to construct a map satisfying $f^*(q) = ep + p_1 + p_2 + \cdots + p_{d^n - e}$ for any $e \geq 2$. To do this, in the proof we simply start by choosing $L_{1,2}, \ldots, L_{1,e}$ to be linear forms defining hyperplanes in general position.

7. PCF maps with fixed tail length

In this section, we prove a number of results about PCF maps with fixed tail length ℓ. An immediate consequence will be a proof that PCF maps with tail length 1 are sparse, and the methods that we develop will then be used in Section 8 to show that PCF maps with tail length at most 2 are sparse.

We recall that in Section 4 we proved that a map f whose critical locus is irreducible and of general type cannot be PCP. The key to the proof is the fact that these assumptions imply that some iterate f^m is an endomorphism of $\mathcal{C}_f$, and hence is an automorphism of finite order, since varieties of general type have finite automorphism groups.

More generally, suppose that f is PCF of type (k, ℓ). Then f^k restricts to an endomorphism of $f^\ell(\mathcal{C}_f)$, but if $f^\ell(\mathcal{C}_f)$ is not general type, then it may admit endomorphisms that are not of finite order. On the other hand, by Theorem 14, we know that for most maps f, the critical locus $\mathcal{C}_f$ is of general type. Our next proposition lays out a roadmap for proving that PCF maps with fixed tail length ℓ are sparse. It says, roughly, that such maps are sparse provided that we can find even a single map f with the property that $f^\ell(\mathcal{C}_f)$ is of general type. Using this proposition, we will easily be able to handle the case $\ell = 1$, and with significantly more work as described in Section 8, the case $\ell = 2$.

Proposition 23. *Let $n \geq 2$ and $d \geq 3$ and $\ell \geq 1$. Suppose that there exists at least one endomorphism $f_0 \in \operatorname{End}_d^n$ such that $f_0^\ell(\mathcal{C}_{f_0})$ has an irreducible component B with the following properties*:

(1) *There is exactly one irreducible component C of $\mathcal{C}_{f_0}$ satisfying*
$$f_0^\ell(C) = B.$$
(2) *None of the images $f_0(C), \ldots, f_0^{\ell-1}(C)$ is contained in $\mathcal{C}_{f_0}$.*
(3) *The map $f_0^\ell|_C : C \to B$ is generically* 1*-to-*1*.*
(4) *The map f_0^ℓ has multiplicity* 2 *along C.*

Then the following are true:

(a) *There is a non-empty Zariski open subset $U^n_{d,\ell} \subset \mathrm{End}^n_d$ such that for all $f \in U^n_{d,\ell}$:*
- *$\mathcal{C}_f$ is irreducible and of general type.*
- *The map $f^\ell|_{\mathcal{C}_f} : \mathcal{C}_f \to f^\ell(\mathcal{C}_f)$ is generically 1-to-1.*
- *The map f is **not** PCF with tail-length ℓ.*

(b) *The set of PCF maps with exact tail length ℓ is not Zariski dense in End^n_d.*

We start with some preliminary results.

Lemma 24. *Let $n \geq 3$ and $d \geq 3$ and $\ell \geq 1$. There exists a positive integer $r^n_{d,\ell}$ and a non-empty Zariski-open set $U^n_{d,\ell} \subset \mathrm{End}^n_d$ such that every $f \in U^n_{d,\ell}$ has the following properties:*

(1) *The critical locus $\mathcal{C}_f$ is irreducible and of general type.*
(2) *The map $f^\ell|_{\mathcal{C}_f} : \mathcal{C}_f \to f^\ell(\mathcal{C}_f)$ is generically $r^n_{d,\ell}$-to-1.*

Proof. We first observe that there is a non-empty Zariski open set $(U^n_{d,\ell})_1 \subset \mathrm{End}^n_d$ such that for all $f \in (U^n_{d,\ell})_1$:

(1) The critical locus $\mathcal{C}_f$ is irreducible and of general type. The fact that this is a non-empty open condition follows from Theorem 14.
(2′) The maps $f, f^2, \dots, f^\ell$ have no non-trivial automorphisms.[4] The fact that this is a non-empty open condition follows from [17].

Then over $(U^n_{d,\ell})_1$ there is a universal family

$$\mathcal{F} : \mathbb{P}^n \times (U^n_{d,\ell})_1 \to \mathbb{P}^n \times (U^n_{d,\ell})_1,$$

with universal critical locus $\hat{\mathcal{C}} \to (U^n_{d,\ell})_1$. We denote by $\hat{\mathcal{B}}_\ell$ the underlying reduced variety of the image $\mathcal{F}^\ell(\hat{\mathcal{C}})$ of the universal critical locus under the ℓth iterate of $\mathcal{F}$.

The restriction $\mathcal{F}^\ell|_{\hat{\mathcal{C}}} : \hat{\mathcal{C}} \to \hat{\mathcal{B}}_\ell$ is generically finite, so has some generic degree $r^n_{d,\ell}$. There is an open set $(U^n_{d,\ell})_2 \subset (U^n_{d,\ell})_1$ over which $\pi_{\hat{\mathcal{B}}_\ell}$ is flat, as well as an open set $\hat{\mathcal{B}}^\circ_\ell \subset \pi^{-1}_{\hat{\mathcal{B}}_\ell}\big((U^n_{d,\ell})_2\big)$ over which $\mathcal{F}^\ell|_{\hat{\mathcal{C}}}$ is étale of degree exactly $r^n_{d,\ell}$. Since a flat map of finite type of Noetherian schemes is open, the set

$$U^n_{d,\ell} := \pi_{\hat{\mathcal{B}}_\ell}(\hat{\mathcal{B}}^\circ_\ell) \subset (U^n_{d,\ell})_2$$

is open, and over $U^n_{d,\ell}$, the map

$$\mathcal{F}^\ell|_{\hat{\mathcal{C}}} : \hat{\mathcal{C}} \to \hat{\mathcal{B}}_\ell$$

has generic degree $r^n_{d,\ell}$ by construction. □

[4]In general, the automorphism group of a dynamical system $f : \mathbb{P}^n \to \mathbb{P}^n$ is $\mathrm{Aut}(f) := \{\alpha \in \mathrm{PGL}_{n+1} : \alpha \circ f = f \circ \alpha\}$. It is proven in [17] that if f is a morphism and $d \geq 2$, then $\mathrm{Aut}(f)$ is finite, and that the set of $f \in \mathrm{End}^n_d$ with $\mathrm{Aut}(f) \neq 1$ is a Zariski closed set.

Lemma 25. *Suppose that there is an endomorphism $f_0 \in \mathrm{End}_d^n$ satisfying the hypothesis* (1)–(4) *of Proposition* 23. *Then the degree $r_{d,\ell}^n$ described in Lemma* 24 *satisfies $r_{d,\ell}^n = 1$.*

Proof. We are given a map f_0 that satisfies the four hypotheses of Proposition 23. Since f_0 is in the closure of $U_{d,\ell}^n$, we can find a map

$$F : \mathrm{Spec}\bigl(\mathbb{F}[\![t]\!]\bigr) \to \mathrm{End}_d^n$$

such that the generic point $\mathrm{Spec}\bigl(\mathbb{F}(\!(t)\!)\bigr)$ maps to $U_{d,\ell}^n$ and the special point at $t = 0$ maps to f_0. Taking a ramified base change if necessary, we obtain from F a family of degree d morphisms over $\mathrm{Spec}\bigl(\mathbb{F}[\![t]\!]\bigr)$,

$$\mathcal{F} : \mathrm{Spec}\bigl(\mathbb{F}[\![t]\!]\bigr) \times \mathbb{P}^n \to \mathrm{Spec}\bigl(\mathbb{F}[\![t]\!]\bigr) \times \mathbb{P}^n.$$

Denote by $\hat{\mathcal{C}}$ the underlying reduced scheme of the critical locus of $\mathcal{F}$. It has pure codimension one. Denote by $\hat{\mathcal{B}}_\ell$ the underlying reduced variety of the image $\mathcal{F}^\ell(\hat{\mathcal{C}})$ of the universal critical locus under the ℓth iterate of $\mathcal{F}$. Denote by $\mathcal{F}_\eta$ the restriction of $\mathcal{F}$ to the generic fiber $\mathrm{Spec}\bigl(\mathbb{F}(\!(t)\!)\bigr) \times \mathbb{P}^n$ and by $\mathcal{F}_0$ the restriction of $\mathcal{F}$ to the special fiber $\mathbb{P}_{\mathbb{F}}^n$. By construction, we have:

- $\mathcal{F}_0 = f_0$
- $\hat{\mathcal{C}}_\eta$ is irreducible general type.
- $\mathcal{F}_\eta^\ell|_{\mathcal{C}_\eta}$ has degree $r_{d,\ell}^n$.

Further, since $\deg(\mathcal{F}_0) = \deg(\mathcal{F}) = d$, the map $\mathcal{F}$ is not ramified along the special fiber. We conclude that $\hat{\mathcal{C}}$ is the Zariski closure of $\hat{\mathcal{C}}_\eta$ and that $\hat{\mathcal{B}}_\ell$ is the Zariski closure of $(\hat{\mathcal{B}}_\ell)_\eta$.

Let $p \in C$ be a smooth point such that:

- The points $f_0(p), f_0^2(p), \ldots, f_0^\ell(p) = q$ are not in the critical locus $\mathcal{C}_{f_0}$.
- The point p is not in the critical loci of any of the restrictions

$$(f_0)|_C, (f_0^2)|_C, \ldots (f_0^\ell)|_C.$$

Then p and $q = f_0^\ell(p)$ satisfy the conditions in Proposition 21(a) with respect to f_0^ℓ, that is, the divisor $(f_0^\ell)^*(q)$ is the sum of $2p$ and $(d^\ell)^n - 2$ points having multiplicity 1.

On the one hand, $(\mathcal{F}_0^\ell)^{-1}(q)$ is a subscheme of $(\mathcal{F}^\ell)^{-1}(q)$, while on the other hand, both schemes have degree $(d^\ell)^n$ over $\mathbb{F}$. Therefore $(\mathcal{F}_0^\ell)^{-1}(q) = (\mathcal{F}^\ell)^{-1}(q)$. This means that p has multiplicity exactly 2 in $(\mathcal{F}^\ell)^{-1}(q)$. Since the proof of Lemma 20 was local, we conclude that $\hat{\mathcal{C}}$ is smooth at p, and that $\hat{\mathcal{B}}_\ell$ is smooth at q. We also have that $(\hat{\mathcal{B}}_\ell)_0$ is smooth at q.

Claim. The following are true:

- The map $\pi_{\hat{\mathcal{C}}} : \hat{\mathcal{C}} \to \operatorname{Spec}\big(\mathbb{F}[\![t]\!]\big)$ is smooth at p.
- The map $\pi_{\hat{\mathcal{B}}_\ell} : \hat{\mathcal{B}}_\ell \to \operatorname{Spec}\big(\mathbb{F}[\![t]\!]\big)$ is smooth at q.

Proof of Claim. We follow the proof of Lemma 20. Let $(t, x_1, \dots x_n)$ be coordinates at p, so the completion of the local ring to $\operatorname{Spec}\big(\mathbb{F}[\![t]\!]\big) \times \mathbb{P}^n$ at p is $\mathbb{F}[\![t, x_1, \dots, x_n]\!]$, and let $(z_1, \dots z_n)$ be coordinates at q, so the completion of the local ring to $\operatorname{Spec}\big(\mathbb{F}[\![t]\!]\big) \times \mathbb{P}^n$ at q is $\mathbb{F}[\![t, z_1, \dots, z_n]\!]$. Using these coordinates, we suppose that $\mathcal{F}^\ell$ is given by

$$z_i = f_i(t, x_1, \dots, x_n).$$

Without loss of generality, we may assume that

$$z_i = f_i(t, x_1, \dots, x_n) = x_i \quad \text{for } i = 1, \dots, n-1.$$

As in the proof of Lemma 20, we conclude that $(t, x_1, \dots, x_{n-1})$ restrict to local coordinates on $\hat{\mathcal{C}}$, and that $(t, z_1, \dots, z_{n-1})$ restrict to local coordinates on $\hat{\mathcal{B}}_\ell$. In these coordinates, the maps $\pi_{\hat{\mathcal{C}}}$ and $\pi_{\hat{\mathcal{B}}_\ell}$ are obtained, respectively, by forgetting all of the x_i and z_i coordinates, and thus they are smooth maps. This completes the proof of the claim. $\square$

We resume the proof of Lemma 25. The claim implies that there exists a section

$$P : \operatorname{Spec}\big(\mathbb{F}[\![t]\!]\big) \to \hat{\mathcal{C}}$$

with $P(0) = p$. Then $Q := \mathcal{F}^\ell \circ P$ is a section of $\hat{\mathcal{B}}$. Since $P_\eta \in \hat{\mathcal{C}}_\eta$, we see that P_η appears in $(\mathcal{F}_\eta^\ell)^{-1}(Q_\eta)$ with multiplicity at least 2. On the other hand, by construction we know that $(\mathcal{F}^\ell)^{-1}(Q)|_{t=0}$ has $d^n - 1$ distinct $\mathbb{F}$-points, and that $(d^\ell)^n - 2$ of them appear with multiplicity exactly 1. Hence $(\mathcal{F}_\eta^\ell)^{-1}(Q_\eta)$ must have at least $(d^\ell)^n - 2$ distinct $\mathbb{F}(\!(t)\!)$-points appearing with multiplicity exactly 1. Therefore $(\mathcal{F}_\eta^\ell)^{-1}(Q_\eta)$ must have exactly $(d^\ell)^n - 1$ distinct $\mathbb{F}(\!(t)\!)$-points, with exactly one of them, P_η, appearing with multiplicity 2. Proposition 21(b) implies that $(\mathcal{F}_\eta^\ell)|_{\hat{\mathcal{C}}_\eta}$ has degree 1, so $r_{d,\ell}^n = 1$, as desired. $\square$

We can now finish the proof of Proposition 23.

Proof of Proposition 23. Suppose that, for some fixed ℓ, the hypotheses of Proposition 23 are satisfied. Then, by Lemmas 24 and 25, there is a non-empty Zariski open subset $U_{d,\ell}^n \subset \operatorname{End}_d^n$ such that for all $f \in U_{d,\ell}^n$:

- $\mathcal{C}_f$ is irreducible and of general type.
- The map $f^\ell|_{\mathcal{C}_f} : \mathcal{C}_f \to f^\ell(\mathcal{C}_f)$ is generically 1-to-1.

It remains to show that if $f \in U_{d,\ell}^n$, then f is not PCF of tail-length ℓ. Suppose we have some $f \in U_{d,\ell}^n$. Then $\mathcal{C}_f$ is irreducible and of general type, and since $f^\ell|_{\mathcal{C}_f} : \mathcal{C}_f \to f^\ell(\mathcal{C}_f)$ is generically 1-to-1, we know that $f^\ell(\mathcal{C}_f)$ is birational to $\mathcal{C}_f$, and hence $f^\ell(\mathcal{C}_f)$ is irreducible and of general type. Assume for contradiction that f is PCF of tail-length ℓ and some period $k > 0$. Then f^k defines an endomorphism of $f^\ell(\mathcal{C}_f)$. As in Step 2 of Theorem 16, we conclude that $f^k|_{f^\ell(\mathcal{C}_f)}$ is a finite-order automorphism. Thus there exists some $r > 0$ such that $f^{kr}|_{f^\ell(\mathcal{C}_f)}$ is the identity, i.e., such that $\mathcal{C}_f \subseteq \operatorname{Fix}(f^{kr})$. But $\mathcal{C}_f$ is a hypersurface, so it has dimension $n-1 \geq 1$, while Lemma 17 tells us that $\operatorname{Fix}(f^{kr})$ has dimension 0. The contradiction completes the proof of Proposition 23. $\square$

It is now a simple matter to prove that PCF maps with tail length $\ell = 1$ are sparse.

Theorem 26. *Let $n \geq 3$ and $d \geq 3$. Then*

$$\{f \in \operatorname{End}_d^n : f^k(\mathcal{C}_f) \subseteq f(\mathcal{C}_f) \text{ for some } k \geq 2\}$$

is contained in a proper closed subvariety of End_d^n.

Proof. The map constructed in Proposition 21 satisfies the hypotheses of Proposition 23 for $\ell = 1$. We conclude that there is a non-empty Zariski open subset $U_{d,1}^n \subset \operatorname{End}_d^n$ such that for all $f \in U_{d,1}^n$, the map f is not PCF of tail-length 1. $\square$

8. PCF maps with tail-length 2 are sparse

The main result of this section is as stated in the title. As in the previous section, we begin with a number of preliminary results.

Lemma 27. *Let $n \geq 2$, and let $f : \mathbb{P}^n \to \mathbb{P}^n$ be a morphism of degree $d \geq 2$. Suppose that $H \subset \mathbb{P}^n$ is an irreducible hypersurface satisfying:*

- *$f(H)$ is not contained in $\mathcal{B}_f$.*
- *$f|_H$ is generically r-to-1 for some $r \geq 2$.*

Then there exists an automorphism $\alpha \in \mathrm{PGL}_{n+1}(\mathbb{F})$ such that:

(1) *$f(\alpha(H))$ is not contained in $\mathcal{B}_f$, and*
(2) *$f|_{\alpha(H)}$ is generically s-to-1, for some $s < r$.*

Remark 28. Applying Lemma 27 repeatedly, we see that there exists an $\alpha \in \mathrm{PGL}_{n+1}(\mathbb{F})$ such that $f|_{\alpha(H)}$ is generically 1-to-1.

Proof of Lemma 27. First, we note that the conditions (1) and (2) on $\alpha \in \mathrm{PGL}_{n+1}(\mathbb{F})$ are both Zariski-open, and so it suffices to show that the sets of $\alpha \in \mathrm{PGL}_{n+1}(\mathbb{F})$ satisfying (1) and (2) are both non-empty. Next, we note that, by assumption, $\mathrm{Id} \in \mathrm{PGL}_{n+1}(\mathbb{F})$ satisfies condition (1), so the set of $\alpha \in \mathrm{PGL}_{n+1}(\mathbb{F})$ satisfying (1) is non-empty. So it remains only to show that the set of $\alpha \in \mathrm{PGL}_{n+1}(\mathbb{F})$ satisfying (2) is non-empty. For this, set $e = \deg(H)$. Then $f_*([H]) = r[f(H)]$ is $d^{n-1}e$ times the class of a hyperplane, so

$$f(H) \text{ is a hypersurface of degree } D := \frac{d^{n-1}e}{r},$$

where for notational convenience we let D denote the frequently appearing quantity $D = D(d,n,e,r) := d^{n-1}e/r$.

We pick a line L such that the intersection $L \cap f(H)$ has the following properties:

- L and $f(H)$ intersect transversally.
- The intersection consists of exactly D smooth points of $f(H)$, say
$$L \cap f(H) = \{q_1, \dots, q_D\}.$$
- $L \cap f(H) \cap \mathcal{B}_f = \emptyset$, i.e., $q_i \notin \mathcal{B}_f$ for all $1 \le i \le D$.
- $L \cap f(H) \cap f(\text{singular locus of } H) = \emptyset$.

It is possible to find such a line L because the "bad locus" that we must avoid has codimension at least 2 in $\mathbb{P}^n$.

By construction, L is not contained in $\mathcal{B}_f$, so $f^{-1}(L)$ is a curve C of degree d^{n-1}. Also, the intersection $C \cap H$ is transversal, consisting of exactly $d^{n-1}e = rD$ smooth points of H, which we label as

$$C \cap H = \{p_{i,j} : 1 \le i \le D,\ 1 \le j \le r\}$$

so that:

$$\begin{array}{rcl} p_{1,1}, \dots, p_{1,r} & \text{map to} & q_1 \\ & \vdots & \\ p_{i,1}, \dots, p_{i,r} & \text{map to} & q_i \\ & \vdots & \\ p_{D,1}, \dots, p_{D,r} & \text{map to} & q_D. \end{array}$$

Without loss of generality, we may assume that

$$p_{1,1} = [1 : 0 : \dots : 0] \quad \text{and} \quad p_{1,2} = [0 : 1 : 0 \cdots : 0].$$

For all i and j, the point $p_{i,j}$ is not in the branch locus of f, so f induces isomorphisms of completions of local rings of $\mathbb{P}^n$. Writing $\mathcal{R}_p$ for the completion of the local ring at p, we have

$$f_{i,j} : \mathcal{R}_{p_{i,j}} \longrightarrow \mathcal{R}_{q_i},$$
$$f_{i,j_1,j_2} := f_{i,j_2}^{-1} \circ f_{i,j_1} : \mathcal{R}_{p_{i,j_1}} \longrightarrow \mathcal{R}_{p_{i,j_2}}.$$

We pick a local parametrization of C near $p_{1,1}$, i.e., we fix a map

$$P_{1,1} : \mathrm{Spec}\big(\mathbb{F}[\![t]\!]\big) \to C \quad \text{with} \quad P_{1,1}(0) = p_{1,1}$$

that induces an isomorphism between $\mathbb{F}[\![t]\!]$ and the completion of the local ring of C at $p_{1,1}$. We then obtain a local parametrization of C near $p_{1,2}$ as follows: First we pre-compose $P_{1,1}$ with a specified involution of $\operatorname{Spec}\bigl(\mathbb{F}[\![t]\!]\bigr)$, then we apply $f_{1,1,2}$. Specifically, we set

$$P_{1,2}(t) = f_{1,1,2}(P_{1,1}(-t)), \tag{8}$$

and then

$$\begin{aligned}
&(f \circ P_{1,2})(t) \\
&\quad = (f \circ f_{1,1,2} \circ P_{1,1})(-t) \quad \text{from (8)}, \\
&\quad = \bigl(f \circ (f_{1,2})^{-1} \circ f_{1,1} \circ P_{1,1}\bigr)(-t) \quad \text{since } f_{1,1,2} := (f_{1,2})^{-1} \circ f_{1,1}, \\
&\quad = (f_{1,1} \circ P_{1,1})(-t) \quad \text{since } f \circ (f_{1,2})^{-1} = \mathrm{Id} \text{ on } U_{1,2}, \\
&\quad = (f \circ P_{1,1})(-t) \quad \text{since } f_{1,1} = f \text{ on } U_{1,1}.
\end{aligned} \tag{9}$$

We note that $\frac{d}{dt} f(P_{1,2}(t))\big|_{t=0} \ne 0$, so taking derivatives of (9) and evaluating at $t = 0$ yields

$$0 \ne \frac{d}{dt}(f \circ P_{1,2})(t)\Big|_{t=0} = \frac{d}{dt}(f \circ P_{1,1})(-t))\Big|_{t=0} = -\frac{d}{dt}(f \circ P_{1,1})(t))\Big|_{t=0}.$$

The condition on t that the points

$$P_{1,1}(t),\ P_{1,2}(t),\ [0:0:1:0\cdots:0],\ \ldots,\ [0:\cdots:0:1],\ [1:1\cdots:1]$$

are in general position is an open condition that is satisfied at $t = 0$, and thus it is satisfied over $\operatorname{Spec}\bigl(\mathbb{F}[\![t]\!]\bigr)$.

There is thus a unique element $\alpha_t \in \mathrm{PGL}_{n+1}\bigl(\mathbb{F}[\![t]\!]\bigr)$ satisfying

$$\begin{aligned}
\alpha_t([1:0:0:0:\cdots:0:0]) &= P_{1,1} \\
\alpha_t([0:1:0:0:\cdots:0:0]) &= P_{1,2} \\
\alpha_t([0:0:1:0:\cdots:0:0]) &= [0:0:1:0:\cdots:0:0] \\
\vdots \quad &\quad \vdots \\
\alpha_t([0:0:0:0:\cdots:0:1]) &= [0:0:0:0:\cdots:0:1] \\
\alpha_t([1:1:1:1:\cdots:1:1]) &= [1:1:1:1:\cdots:1:1].
\end{aligned}$$

We note that α has the following properties:

$$\alpha_0 = \mathrm{Id} \in \mathrm{PGL}_{n+1}(\mathbb{F}). \tag{10}$$

$$\alpha_t(p_{1,1}) \in C\bigl(\mathbb{F}[\![t]\!]\bigr) \quad \text{and} \quad \alpha_t(p_{1,2}) \in C\bigl(\mathbb{F}[\![t]\!]\bigr). \tag{11}$$

$$0 \ne \frac{d}{dt} f(\alpha_t(p_{1,1}))\Big|_{t=0} = -\frac{d}{dt}(f(\alpha_t(p_{1,2}))\Big|_{t=0}. \tag{12}$$

Condition (12) implies that for $t \neq 0$, i.e., over the generic point $\operatorname{Spec}\big(\mathbb{F}(\!(t)\!)\big)$, we have

$$f(\alpha_t(p_{1,1})) \neq f(\alpha_t(p_{1,2})).$$

We conclude that $f(\alpha_t(p_{1,1}))$ and $f(\alpha_t(p_{1,2}))$ restrict to distinct points of $L\big(\mathbb{F}(\!(t)\!)\big)$.

We can parametrize the intersection points of $\big(\alpha_t(H) \cap C\big)\big(\mathbb{F}[\![t]\!]\big)$, i.e., we can find maps

$$P_{i,j} : \operatorname{Spec}\big(\mathbb{F}[\![t]\!]\big) \to \alpha_t(H) \cap C$$

such that $P_{i,j}(0) = p_{i,j}$ for all i, j. We have that

$$\begin{aligned} &P_{1,1}(t) = \alpha_t(p_{1,1}), \\ &P_{1,2}(t) = \alpha_t(p_{1,2}), \\ &f \circ P_{i,j} \in (f(\alpha_t(H)) \cap L)(\operatorname{Spec}\big(\mathbb{F}[\![t]\!]\big)). \end{aligned}$$

The conditions on t that

$$f \circ P_{i,1} \neq f \circ P_{1,1} \quad \text{and} \quad f \circ P_{i,1} \neq f \circ P_{1,2} \quad \text{for all } 2 \leq i \leq D$$

are open conditions satisfied at $t = 0$, and thus are satisfied over $\mathbb{F}[\![t]\!]$. On the other hand, for $t \neq 0$, i.e., over $\operatorname{Spec}\big(\mathbb{F}(\!(t)\!)\big)$, we have

$$f \circ P_{1,1} = f(\alpha_t(p_{1,1})) \neq f(\alpha_t(p_{1,2})) = f \circ P_{1,2}.$$

Thus $(f(\alpha_t(H)) \cap L)(\operatorname{Spec}\big(\mathbb{F}(\!(t)\!)\big))$ contains at least $D + 1$ distinct points, specifically

$$f \circ P_{1,1}, f \circ P_{2,1}, \ldots, f \circ P_{D,1}, f \circ P_{1,2} \in \big(f(\alpha_t(H)) \cap L\big)\big(\operatorname{Spec}\big(\mathbb{F}(\!(t)\!)\big)\big).$$

Thus over $\mathbb{F}(\!(t)\!)$ we have

$$\frac{d^{n-1}e}{\deg(f|_{\alpha_t(H)})} = \deg(f(\alpha_t(H))) \geq |f(\alpha_t(H)) \cap L| \geq D + 1 > D.$$

Since $D = d^{n-1}e/r$, this gives a strict inequality

$$\deg(f|_{\alpha_t(H)}) < r,$$

showing that the set of $\alpha \in \mathrm{PGL}_{n+1}(\mathbb{F})$ satisfying (2) is non-empty. This completes the proof of Lemma 27 over the algebraically closed characteristic 0 field $\mathbb{F}$. $\square$

Lemma 29. *Let $n \geq 3$ and $d \geq 3$ and $\ell = 2$. Then there exists an $f_0 \in \mathrm{End}_d^n$ that satisfies Conditions* (1)–(4) *of Proposition* 23.

Proof. By Proposition 21 and Theorem 14, there exists $f \in \mathrm{End}_d^n$ such that

- $\mathcal{C}_f$ is irreducible and of general type.

- f is not PCF with tail length 1, i.e.,
$$f(\mathcal{B}_f) = f^2(\mathcal{C}_f) \not\subset f(\mathcal{C}_f) = \mathcal{B}_f\,.$$
- $f : \mathcal{C}_f \to \mathcal{B}_f$ is generically 1-to-1.
- f has multiplicity 2 along $\mathcal{C}_f$

Thus f satisfies conditions (1), (2) and (4) of the hypotheses of Proposition 23. If $f|_{\mathcal{B}_f}$ is generically 1-to-1, then f also satisfies condition (3) so we are done. If not, we use Lemma 27 to find an $\alpha \in \mathrm{PGL}_{n+1}$ such that $f|_{\alpha(\mathcal{B}_f)}$ is generically 1-to-1. Set $f_0 = \alpha \circ f$. Then

$$\mathcal{C}_{f_0} = \mathcal{C}_f, \quad \mathcal{B}_{f_0} = \alpha(\mathcal{B}_f), \quad \text{and} \quad (f_0)|_{\mathcal{B}_{f_0}} = (f_0)|_{\alpha(\mathcal{B}_f)} = (\alpha \circ f)|_{\alpha(\mathcal{B}_f)}.$$

This last map $(f_0)|_{\mathcal{B}_{f_0}}$ is generically 1-to-1 because $f|_{\alpha(\mathcal{B}_f)}$ is generically 1-to-1 and α is everywhere 1-to-1. Finally the multiplicity of f_0 equals the multiplicity of f along $\mathcal{C}_{f_0} = \mathcal{C}_f$, thus is 2. Thus f_0 satisfies the hypotheses of Proposition 23 for $\ell = 2$. $\square$

We now have the tools to prove the main result of this section, which is that PCF maps with tail length at most 2 are sparse.

Theorem 30. *Let $n \geq 3$ and $d \geq 3$. Then*
$$\{f \in \mathrm{End}_d^n : f^k(\mathcal{C}_f) \subseteq f^2(\mathcal{C}_f) \textit{ for some } k \geq 2\}$$
is contained in a proper closed subvariety of End_d^n.

Proof. By Lemma 29, there exists a map $f_0 \in \mathrm{End}_d^n$ satisfying the hypotheses of Proposition 23 for $\ell = 2$. Thus we can use Proposition 23 to conclude that
$$\{f \in \mathrm{End}_d^n : f^k(\mathcal{C}_f) \subseteq f^2(\mathcal{C}_f) \text{ for some } k \geq 2\}$$
is contained in a proper closed subvariety of End_d^n. $\square$

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References

[1] J. Belk and S. Koch. Iterated monodromy for a two-dimensional map. In *In the tradition of Ahlfors-Bers. V*, volume 510 of *Contemp. Math.*, pages 1–11. Amer. Math. Soc., Providence, RI, 2010.

[2] A. M. Bonifant and M. Dabija. Self-maps of $\mathbb{P}^2$ with invariant elliptic curves. In *Complex manifolds and hyperbolic geometry (Guanajuato, 2001)*, volume 311 of *Contemp. Math.*, pages 1–25. Amer. Math. Soc., Providence, RI, 2002.

[3] A. Bridy, P. Ingram, R. Jones, J. Juul, A. Levy, M. Manes, S. Rubinstein-Salzedo, and J. H. Silverman. Finite ramification for preimage fields of post-critically finite morphisms. *Math. Res. Lett.*, 24(6):1633–1647, 2017.

[4] C. Ciliberto and F. Flamini. On the branch curve of a general projection of a surface to a plane. *Trans. Amer. Math. Soc.*, 363(7):3457–3471, 2011.

[5] M. Dabija and M. Jonsson. Algebraic webs invariant under endomorphisms. *Publ. Mat.*, 54(1):137–148, 2010.

[6] L. De Marco. Dynamical moduli spaces and elliptic curves. *Ann. Fac. Sci. Toulouse Math. (6)*, 27(2):389–420, 2018. KAWA 2015 Pisa.

[7] A. Douady and J. H. Hubbard. A proof of Thurston's topological characterization of rational functions. *Acta Math.*, 171(2):263–297, 1993.

[8] J. E. Fornæ ss and N. Sibony. Critically finite rational maps on $\mathbf{P}^2$. In *The Madison Symposium on Complex Analysis (Madison, WI, 1991)*, volume 137 of *Contemp. Math.*, pages 245–260. Amer. Math. Soc., Providence, RI, 1992.

[9] Y. Fujimoto and N. Nakayama. Complex projective manifolds which admit non-isomorphic surjective endomorphisms. In *Higher dimensional algebraic varieties and vector bundles*, RIMS Kôkyûroku Bessatsu, B9, pages 51–79. Res. Inst. Math. Sci. (RIMS), Kyoto, 2008.

[10] C. D. Hacon, J. McKernan, and C. Xu. On the birational automorphisms of varieties of general type. *Ann. of Math. (2)*, 177(3):1077–1111, 2013.

[11] R. Hartshorne. *Algebraic Geometry*, volume 52 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1977.

[12] S. Iitaka. *Algebraic geometry*, volume 76 of *Graduate Texts in Mathematics*. Springer-Verlag, New York-Berlin, 1982. An introduction to birational geometry of algebraic varieties, North-Holland Mathematical Library, 24.

[13] P. Ingram. Rigidity and height bounds for certain post-critically finite endomorphisms of $\mathbb{P}^N$. *Canad. J. Math.*, 68(3):625–654, 2016.

[14] M. Jonsson. Some properties of 2-critically finite holomorphic maps of $\mathbf{P}^2$. *Ergodic Theory Dynam. Systems*, 18(1):171–187, 1998.

[15] S. Koch. Teichmüller theory and critically finite endomorphisms. *Adv. Math.*, 248:573–617, 2013.

[16] V. S. Kulikov and V. S. Kulikov. Generic coverings of the plane with *A*-*D*-*E*-singularities. *Izv. Ross. Akad. Nauk Ser. Mat.*, 64(6):65–106, 2000.

[17] A. Levy. The space of morphisms on projective space. *Acta Arith.*, 146(1):13–31, 2011.

[18] V. Nekrashevych. Combinatorial models of expanding dynamical systems. *Ergodic Theory Dynam. Systems*, 34(3):938–985, 2014.

[19] D. G. Northcott. Periodic points on an algebraic variety. *Ann. of Math. (2)*, 51:167–177, 1950.

[20] C. Petsche, L. Szpiro, and M. Tepper. Isotriviality is equivalent to potential good reduction for endomorphisms of $\mathbb{P}^N$ over function fields. *J. Algebra*, 322(9):3345–3365, 2009.
[21] F. Rong. The Fatou set for critically finite maps. *Proc. Amer. Math. Soc.*, 136(10):3621–3625, 2008.
[22] J. H. Silverman. *The Arithmetic of Dynamical Systems*, volume 241 of *Graduate Texts in Mathematics*. Springer, New York, 2007.
[23] J. H. Silverman. *Moduli Spaces and Arithmetic Dynamics*, volume 30 of *CRM Monograph Series*. American Mathematical Society, Providence, RI, 2012.
[24] J. Starr. The Kodaira dimension of spaces of rational curves on low degree hypersurfaces, 2003. arxiv.org/abs/math/0305432.
[25] T. Ueda. Critically finite maps on projective spaces. Number 1087, pages 132–138. 1999. Research on complex dynamical systems: current state and prospects (Japanese) (Kyoto, 1998).
[26] I. Vainsencher. Complete collineations and blowing up determinantal ideals. *Math. Ann.*, 267(3):417–432, 1984.
[27] T. Gauthier, B. Hutz, and S. Kaschner. Symmetrization of Rational Maps: Arithmetic Properties and Families of Lattés Maps of $\mathbb{P}^k$. arxiv.org/abs/math/1603.04887.
[28] T. Dinh and N. Sibony. Dynamics in several complex variables: endomorphisms of projective spaces and polynomial-like mappings. Holomorphic dynamical systems, 165–294, Springer, 2010.
[29] N. Fakhruddin. The algebraic dynamics of generic endomorphisms of $\mathbb{P}^n$. *Algebra & Number Theory*, 8(3): 587–608, Mathematical Sciences Publishers, 2014

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