# Some Applications of Optimal Stopping and Control in Finance and Economics

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Thesis submitted for the degree of *Doctor of Philosophy*

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October 2022

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# Acknowledgements

Firstly, I would like to thank my supervisors, Saul D. Jacka and Vicky Henderson, for their tremendous support and guidance during my research journey.

Secondly, I am very grateful to the examiners Aleksandar Mijatovic and Mihail Zervos for many helpful suggestions and insightful comments.

Next, I wish to thank Martin Herdegen for motivating me to pursue a doctoral degree and organising an excellent reading group.

My gratitude goes to Nazem Khan, Urvashi Behary Paray and Xin Zhi for their encouragement and friendship. I would like to thank Justin Gwee for our stimulating discussions and Haodong Sun for the valuable advice on my research and career.

I would like to acknowledge the financial support from the Statistics Department at the University of Warwick.

Last but not least, my special thanks are extended to my parents and grandparents.

# Declaration

This thesis is submitted to the University of Warwick in support of my application for the degree of Doctor of Philosophy. It has been composed by myself and has not been submitted in any previous application for any degree. Work based on collaborative research is declared as follows:

Chapter 2 is based on a joint work with Saul D. Jacka and Vicky Henderson, 'The Support and Resistance Line Method: An Analysis via Optimal Stopping', arXiv preprint, arXiv:2103.02331, 2021.

Chapter 3 is based on a new section in the revised version of the joint work with Saul D. Jacka and Vicky Henderson, 'The Support and Resistance Line Method: An Analysis via Optimal Stopping', following a review of Finance and Stochastics.

Chapter 4 is based on a joint work with Saul D. Jacka and Vicky Henderson.

# Abstract

In this thesis, we consider some applications of optimal stopping and control problems in specific scenarios. In Chapter 1, a review of the established general results is provided. In Chapter 2, we study a mathematical model capturing the support/resistance line method (a technique in technical analysis) where the underlying stock price transitions between two states of nature in a path-dependent manner. For optimal stopping problems with respect to a general class of reward functions and dynamics, using probabilistic methods, we show that the value function is $C^1$ and solves a general free boundary problem. Moreover, for a wide range of utilities, we prove that the best time to buy and sell the stock is obtained by solving free boundary problems corresponding to two linked optimal stopping problems. We use this to numerically compute optimal trading strategies and compare the strategies with the standard trading rule to investigate the viability of this form of technical analysis. In Chapter 3, the model studied in Chapter 2 is extended by adding a partial reflection boundary and an additional regime (the 0 regime). In Chapter 4, we study a two dimensional continuous-time infinite horizon singular control problem related with the optimal management of inventory and production. The primary source of production is modeled as an uncontrolled one-dimensional diffusion process with general dynamics. By controlling the accumulated secondary source of production and output, which are both finite variation processes, we aim to optimise the inventory process under a general concave running reward function and maximise the profit generated from the production. By solving the associated Dynkin game, we obtain two non-intersecting bounded and monotone free-boundaries where one is directly computable and the other is characterised by a free-boundary problem with smooth-pasting conditions. By restricting the volatility term of the diffusion to linear functions with no intercepts, desired smoothness of the value function is obtained by utilising its viscosity property. This leads to the verification of the proposed candidate optimal control that keeps the state process within the inaction set by reflecting the inventory process at the free-boundaries with the minimum effort.

# 1

# Introduction

Many problems in finance and economics are, in essence, stochastic optimization problems. Modern probability theory enables the construction of continuous-time stochastic processes and the mathematical formulation of the optimization problem as the maximisation of anticipated gain over a range of random actions. If we want to choose a stopping time to maximise the expected reward at that time, this problem is called an *optimal stopping problem*. Moreover, if we are allowed to choose a process which influences the underlying process and the expected reward, then we have a *stochastic control problem*. There exists an enormous literature on these two types of problems (see Pham [57] and references therein).

In this thesis, we aim to study some specific applications of optimal stopping and stochastic control problems in finance and energy. In Chapter 2 and 3, we introduce a novel stock price model inspired by Technical Analysis and formulate two related optimal stopping problems for seeking the best trading time. In Chapter 4, we study the optimal singular control of the inventory process under a mixture of controllable and stochastic production. The main focus of our research is to find methods that can compute solutions or optimal strategies in some explicit form.

In this chapter, we begin with reviewing some general theory on optimal stopping and singular control problems, which forms the basis of our analysis in later chapters, and then provide a more detailed outline of the rest of the thesis.

## 1.1 Optimal stopping problems

Let $G$ be an adapted càdàg process defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ which satisfies the usual conditions (i.e. $(\mathcal{F}_t)_{t\geq 0}$ is right continuous and $\mathbb{P}$-complete). The (infinite horizon) optimal stopping problem admits the following mathematical formulation:

$$v_0 := \sup_{\tau} \mathbb{E}[G_\tau], \tag{1.1.1}$$

where the supremum is taken over stopping times $\tau$. Imagining at time $t$ that we know the maximised expected gain for stopping the process in the future given the current information, then if it is the same as the gain at time $t$, it would be optimal to stop. In other words, define the process

$$v_t := \operatorname*{ess\,sup}_{\tau \geq t} \mathbb{E}[G_\tau], \tag{1.1.2}$$

then we expect the stopping time:

$$\tau_* := \inf\{t \geq 0 : v_t = G_t\} \tag{1.1.3}$$

is the minimal optimal stopping time, i.e. $v_o = \mathbb{E}[G_{\tau_*}]$. It is well-known that, under the assumption that $\mathbb{E}[\sup_t |G_t|] < \infty$ and $\tau_* < \infty$ $\mathbb{P}$-a.s., our claim is correct. Moreover, $v_t$ is the minimal supermartingale dominating $G$, and $v_{t\wedge\tau_*}$ is a martingale. We often refer $v_t$ as the *Snell envelope* of $G$ and this way of studying optimal stopping problems as the martingale approach. For proofs of these results, we refer to Peskir and Shiryaev [56] Theorem 2.2.

Although the martingale approach benefits from its generality, to derive explicit solutions, the so-called Markovian approach is popular among applications of optimal stopping. Let $X$ be a Feller process (for its definition, see Revuz and Yor [59] Chapter III.2) defined on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}_x)$ taking values in $(E, \mathcal{E})$ where $E$ is LCCB (locally compact with countable base) and $\mathcal{E}$ is the $\sigma$-algebra of Borel sets in $E$. Then, given a continuous gain function $g : \mathbb{E} \to \mathbb{R}$ and a positive constant $r$, we consider the following optimal stopping problem:

$$V(x) = \sup_{\tau} \mathbb{E}_x[e^{-r\tau} g(X_\tau)]. \tag{1.1.4}$$

Then, we can define the stopping set $D$ and continuation set $C$ as:

$$D := \{x \in E : V(x) = g(x)\}; \tag{1.1.5}$$

$$C := \{x \in E : V(x) > g(x)\}. \tag{1.1.6}$$

The classical result (Shiryaev [65] Theorem 3.3) tells us, assuming $\mathbb{E}[\sup_t |e^{-rt} g(X_t)|] < \infty$, the first entry time to set $D$, denoted by $\tau_D$ (i.e. $\tau_D := \inf\{t \geq 0 : X_t \in D\}$) is the minimal optimal stopping time if $\tau_D < \infty$ $\mathbb{P}_x$-a.s.

To connect these two approaches together, comparing with (1.1.1), we see $G_t = e^{-rt}g(X_t)$ ($X$ is càdàg by Revuz and Yor [59] Chapter 3 Theorem 2.7). In fact, by Theorem 3.4 in El Karoui et al.[23], the Snell envelope of $G$ is $e^{-rt}V(X_t)$, i.e. $v_t = e^{-rt}V(X_t)$ $\mathbb{P}_x$-a.s., and hence $\tau_* = \tau_D$ $\mathbb{P}_x$-a.s..

The main advantage of the Markovian framework is that we split the state space into two disjoint sets $C$ and $D$. In $D$, we already know the value of $V$. So the remaining job is to compute $V$ on $C$ and also $C$ itself. We know that, $v_{t\wedge\tau_D}$ is a martingale, and suppose $V \in \mathcal{D}(\mathcal{L})$ where $\mathcal{L}$ is the infinitesimal generator of $X$ and $\mathcal{D}(\mathcal{L})$ is the domain of $\mathcal{L}$ (for more details, see Revuz and Yor [59] Chapter VII.1), then

$$e^{-rt\wedge\tau_D}V(X_t \wedge \tau_D) - V(x) - \int_0^{t\wedge\tau_D} e^{-rs}(\mathcal{L} - r)V(X_s)ds, \quad x \in C, \tag{1.1.7}$$

is also a martingale. Thus, we obtain the following *free-boundary problem*:

$$\mathcal{L}V(x) - rV(x) = 0, \quad x \in C; \tag{1.1.8}$$

$$V(x) = g(x), \quad x \in D. \tag{1.1.9}$$

This provides a nice characterization for $V$. However, to derive $C$ together with $V$, we often need one more condition, which is to check $\mathbf{V}$ satisfies the *smooth-pasting* (or smooth-fitting) principle which states that

$$\frac{\partial V}{\partial x}(y) = \frac{\partial g}{\partial x}(y), \quad y \in \partial C \cap \partial D. \tag{1.1.10}$$

In other words, assuming $g$ is smooth, then $\mathbf{V}$ is also smooth on the intersected boundary of $C$ and $D$. This is often true for diffusion processes, but can fail in general. Once the smooth pasting principle is verified, $\mathbf{V}$ is characterised as a solution to the free-boundary problem defined by (1.1.8)-(1.1.10). If we can also prove $\mathbf{V}$ is the unique solution, then the numerical solution to the free-boundary problem may be used as an estimate of $\mathbf{V}$ and $C$.

## 1.2 Singular stochastic control problems

Consider a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ which satisfies the usual conditions and support a $d$-dimensional Brownian motion $W$. The $n$-dimensional *controlled diffusion processes* is the solution to the following stochastic differential equation (SDE):

$$dX_t = \mu(X_t, \alpha_t)dt + \sigma(X_t, \alpha_t)dW_t, \quad X_0 = x, \tag{1.2.1}$$

where $\alpha$ is a progressively measurable process taking values in $A \subset \mathbb{R}^m$. Under appropriate Lipschitz and $L^2$ boundedness conditions, the existence and uniqueness of the solution are ensured (Yong and Zhou [70] Chapter 1 Theorem 6.16). Let $f : \mathbb{R}^n \times A$ be a measurable function. Then, the payoff function $G$ is defined by

$$G(x;\alpha) := \mathbb{E}\Big[\int_0^\infty e^{-rt} f(X_t, \alpha_t)dt\Big]. \tag{1.2.2}$$

This leads to a (infinite horizon) stochastic control problem given by

$$v(x) := \sup_{\alpha\in\mathcal{A}} G(x;\alpha), \tag{1.2.3}$$

where $\mathcal{A}$ denotes the set of progressively measurable $\alpha$ taking values in $A \subset \mathbb{R}^m$ such that (1.2.1) admits a unique strong solution and $f(X,\alpha) \in L^1(d\mathbb{P} \otimes e^{-rt}dt)$. Then, $\mathcal{A}$ is called the set of *admissible controls*.

Similar to the optimal stopping problem, the key principle is that optimising the control at each time $t$ would result in a global optimisation over all time. Mathematically, the *dynamic programming principle* (DPP) states,

$$v(x) = \sup_{\alpha\in\mathcal{A}} \sup_{\rho\in\mathcal{T}} \mathbb{E}\Big[\int_0^\rho e^{-rt} f(X_t, \alpha_t)dt + e^{-r\rho} v(X_\rho)\Big], \tag{1.2.4}$$

$$= \sup_{\alpha\in\mathcal{A}} \inf_{\rho\in\mathcal{T}} \mathbb{E}\Big[\int_0^\rho e^{-rt} f(X_t, \alpha_t)dt + e^{-r\rho} v(X_\rho)\Big], \tag{1.2.5}$$

where $\mathcal{T}$ denotes the set of stopping times. By Pham [57] Theorem 3.3.1, we know our set sup implies $v$ satisfies DPP. Assuming $v$ is smooth and applying Itô's formula to $e^{-rt}v(X_t)$, by DPP, we can derive the following *Hamilton-Jacobi-Bellman* (HJB) equation of the value function:

$$rv(x) - \sup_{a\in A}\{\mathcal{L}^a v + f(x,a)\} = 0, \tag{1.2.6}$$

where $\mathcal{L}^a v := \mu(x,a)D_x v + \frac{1}{2}\operatorname{Tr}(\sigma\sigma^T(x,a)D_x^2 v)$.

Define the *Hamiltonian* for $(x,p,M) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathcal{S}_n$,

$$H(x,p,M) := \sup_{a\in A}\{\mu(x,a)p + \frac{1}{2}\operatorname{Tr}(\sigma\sigma^T(x,a)M) + f(x,a)\}, \tag{1.2.7}$$

where $\mathcal{S}_n$ denotes the set of symmetric $n \times n$ matrices.

Now, let us consider a special case. Suppose $f(x,a) = u(x) + ca$, $\mu(x,a) = \mu(x) + Ba$, and $\sigma(x,a) = \sigma(x)$, where $u : \mathbb{R}^n \to \mathbb{R}$, $c \in \mathbb{R}^m$ and $B \in \mathbb{R}^{n\times m}$. Then,

$$H(x,p,M) = \mu(x)p + \frac{1}{2}\operatorname{Tr}(\sigma\sigma^T(x)M) + u(x) + \sup_{a\in A} a^T\{B^T p - c\}. \tag{1.2.8}$$

Set $G(p) := c - B^T p$. By convention, we say $G(p) \geq 0$ if $G(p)_i \geq 0$ for all $i = 1, 2, ..., m$. Assume the set $A$ is unbounded above but bounded below by 0. We can conclude that

$$G(p) \geq 0 \Longleftrightarrow H(x, p, M) < \infty. \tag{1.2.9}$$

In this case, we call it a *singular control problem*[1] since the control $a$ can be infinity. Moreover, by considering DPP when $G(p) > 0$, we obtain a variational inequality

$$\min\{rv(x) - H(x, D_x v(x), D_x^2 v(x)), G(D_x v(x))\} = 0. \tag{1.2.10}$$

Probabilistically, consider the integral control process $\xi_t := \int_0^t \alpha_s ds$. If $\alpha$ is bounded, then $\xi_t$ is well-defined. However, as $\alpha$ shall be allowed to be infinity, we need enlarge the space of the control such that $\xi$ is a càdlàg increasing[2] process. This leads to a reformulation of the original problem as follows:

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t + Bd\xi_t, \quad X_0 = x, \tag{1.2.11}$$

$$G(x; \xi) = \mathbb{E}\Big[\int_0^\infty e^{-rt}\{u(X_t)dt + cd\xi_t\}\Big], \tag{1.2.12}$$

$$v(x) = \sup_{\xi \in \mathcal{A}} G(x; \xi), \tag{1.2.13}$$

where $\mathcal{A}$ contains all adapted càdlàg increasing processes such that $u(X) + c\xi \in L^1(d\mathbb{P} \otimes e^{-rt}dt)$. Of course, one can generalize $c$ and $B$ to be random processes instead of being deterministic, but let us stay with this relatively simpler set-up.

One approach to compute $v$ is to show (1.2.10) admits a $C^2$ solution and standard verification theorem (e.g. Fleming and Soner [30] Chapter VIII Theorem 4.1) implies it must be equal to $v$ under suitable conditions. However, it is often hard to independently verify the existence of smooth solution to (1.2.10) and sometimes $v$ is indeed non-smooth (e.g. Pham [57] Chapter 3.3.7). To overcome this difficulty, the concept of viscosity solution is introduced by Lions [46], which provides a weaker notation of solutions to nonlinear second-order partial differential equations (PDE). Classical results (e.g. Fleming and Soner [30] Chapter VIII Theorem 5.1) state that under DPP and continuity, $v$ is a solution to (1.2.10) in the viscosity sense.

To explicitly determine the optimal control $\xi^*$, it is often required to prove $v$ is smooth. A result of Karatzas and Wang [42] illustrates a method to connect $D_x v$ with the value function of an *zero sum optimal stopping game* (Dynkin game), denoted by $w$, when $n = 1$, $m = 2$ and $u$ is concave, in the way that if the associated optimal stopping game has a Nash equilibrium, then $v'(x) = w(x)$ for all $x$. This

[1] In general $G$ can be dependent on $x$ and $M$ as well under different set-up.

[2] We say a process $X$ is increasing if it is non-decreasing, and *strictly* increasing if $X_t > X_s$ for all $t > s \geq 0$.

probabilistic characterisation enables us to use well-known results of Dynkin game to study the continuity and smoothness of $w$ (i.e. of $v'$). Similarly to optimal stopping problems, $w$ can be characterised as a solution to a free-boundary problem. Note the free-boundary is equal to the boundary of the set $C_a := \{G(v'(x)) > 0\}$. We call $C_a$ the *inaction set* because in many cases $\xi^*$ stays constant when $X$ is in $C_a$. Moreover, when $X$ hits the boundary of $C_a$, very often the optimal control takes the least effort to keep $X$ within $\bar{C}_a$. Thus, if the free-boundary can be computed, the optimal control $\xi^*$ is also obtained.

## 1.3 Outline of the thesis

Broadly speaking, Technical Analysis are a class of trading strategies which are based on historical patterns of the stock prices. In Chapter 2, we study a novel path-dependent regime-switching model given by

$$dS_t = \mu_{F_t}(S_t)dt + \sigma_{F_t}(S_t)dW_t, \tag{1.3.1}$$

where the flag process $F$, valued in $\{+,-\}$, transitions according to the downcrossing time of $L$ or upcrossing time of $H$ for constants $L < H$, so that

$$F_t = \begin{cases} + & \text{if } F_{t-} = -, \text{ and } S_t = H; \\ - & \text{if } F_{t-} = +, \text{ and } S_t = L; \\ F_{t-} & \text{otherwise.} \end{cases} \tag{1.3.2}$$

This model adheres to the notion of support/resistance lines, a technique utilised frequently by technical traders, where we assume there exists an unspecified price level $R$ between $L$ and $H$ and that a significant deviation from $R$, i.e. reaching $L$ or $H$, would result in a regime-change. We then consider the following two optimal stopping problems:

$$\mathbf{V}_s(x,f) := \sup_{\tau} \mathbb{E}^{x,f}[e^{-r\tau}u(S_\tau)], \tag{1.3.3}$$

$$\mathbf{V}_p(x,f) := \sup_{\tau} \mathbb{E}^{x,f}[e^{-r\tau}\{\mathbf{V}_s(S_\tau, F_\tau) - u(S_\tau)\}], \tag{1.3.4}$$

where $\mathbf{V}_s$ represents the problem of best selling time and $\mathbf{V}_p$ represents the problem of best purchasing time. After showing $(S,F)$ is a Feller process, under additional assumptions on the sign of $\mathcal{L}u - ru$, where $\mathcal{L}$ denotes the infinitesimal generator of $(S,F)$, both $\mathbf{V}_s$ and $\mathbf{V}_p$ can be characterised as solutions to free-boundary problems with smoothing-pasting conditions. Furthermore, uniqueness of solution is also proved, allowing for numerical estimation of the optimal trading boundaries and a discussion of the optimality of Technical Analysis trading prescriptions.

Chapter 3 extends the model for $(S, F)$ by setting $R$ to be a partial reflection boundary for $S$, which directly captures the key behavior assumed by support/resistance line. Moreover, an additional regime 0 is introduced, into which the flag process $F$ jumps at exponentially distributed random times if the process is in the + regime. Under this generalized model, we can still characterise $V_s$ by a free-boundary problem up until the lower boundary of the stopping set in the + regime. If the technicality is reduced by switching off the 0 regime, then precisely analogous results to those seen in Chapter 2 can be obtained.

In many applications, such as the energy industry, inventory management requires controlling the inflow and outflow of products to optimise the inventory level and profit generation. In Chapter 4, we investigate an inventory control problem where the inflow (production) and outflow (output) are modeled as increasing càdlàg processes and the inventory receives an additional uncontrolled inflow (e.g. wind power in the case of energy generation) with the rate modeled by an Itô diffusion. After choosing appropriate $\mu$, $\sigma$, $B$, and $c$, the control problem fits into the formulation given by (1.2.11)-(1.2.13) where we denote the value function by $\mathbf{V}(i, k)$. The associated Dynkin game discussed in the previous section allows us to obtain the existence and the continuity of $\mathbf{V}_i$. An in-depth analysis of the Dyknin game reveals the fact that its continuation set (which is equal to $C_a$) is determined by two free-boundaries, and one of them can be easily computed (which only depends on $u$ and $c$). We further solve the Dynkin game by characterising its value function and the other unknown free-boundary as the unique solution to a free-boundary problem with smooth-pasting conditions. With an application of the viscosity approach, under the restriction to linear volatility and other regularity assumptions on $u$ and $\mu$, it follows that $\mathbf{V} \in C^{1,2}$ in the closure of the inaction set $C_a$. Finally, we propose that the candidate optimal control is the process that only increases when the state process reaches the free-boundaries after a potential initial jump at time 0 and keeps the state process within $\bar{C}_a$ with the minimum effort. Its optimality is verified by using the smoothness of $\mathbf{V}$.

# 2

# Optimal trading strategy under the support and resistance line method

## 2.1 Introduction

Technical analysis (TA) is a method to identify trading opportunities by analysing historical market data and price patterns. Some traders believe that by observing key market indicators and charts (i.e. graphs of price data) they can predict future price movement, and construct profitable trading strategies. TA is extremely popular among investors. In a survey of 678 fund managers Menkhoff [51] found that 86% of fund managers rely on TA as one of their investment tools. Hoffmann and Shefrin [35] analyze survey responses from individual investors and report that 32% use TA.

Through the continuous development of TA, numerous trading rules have been introduced. For example, traders may generate buy/sell signals by comparisons of short and long-term moving-averages; from breakthroughs of market support and resistance levels; from so-called Bollinger bands, and from directional indicators.

In this chapter we study the support/resistance line method. Under this method, traders usually buy (sell) an asset if its price goes below (above) a support (resistance) level, or simply, "buy at low" (BL) and "sell at high" (SH). This is the so-called *standard trading rule*. The support (resistance) line is viewed as a local minimum (maximum) of the asset price over a period of time. However, when the price goes *substantially* below (above) the support (resistance) line, it is said to have *broken-through* the line and it is widely accepted that the support (resistance) line will become the new resistance (support) line because of the negative (positive) outlook for the asset resulting from such a price movement.

Despite the richness of technical trading strategies, many of them have been criticised for being subjective and lacking mathematical justification. Furthermore, TA is also contentious due to the perception of conflict between its claimed predictive power and the Efficient Markets Hypothesis (see Section 2 of Park and Irwin [53] for more details). A key motivation for our analysis is to answer the question: if we make reasonable assumptions about utility and stock price dynamics, can the model make trading prescriptions in line with those of TA?

The vast majority of studies of TA devote their efforts to finding empirical evidence for the profitability of technical trading rules by examining historical data. For example, Brock, Lakonishok, and LeBaron [7] tested moving-average-type trading rules and the support/resistance line method on the Dow Jones Industrial Average on a time scale of 90 years. This study suggested that the technical trading strategies considered there were significantly profitable. Based on a similar approach but with the data taken from Asian markets, Bessembinder and Chan [4] further confirmed the forecasting power of trading rules based on TA. Lo, Mamaysky, and Wang [47] implemented an automatic trading algorithm based on more sophisticated pattern-based trading rules (such as triangle, rectangle, and head-and-shoulders) by using kernel regressions, and a significant profit was observed. Park and Irwin [53] provided a comprehensive review of the literature on the profitability of TA and concluded that more than half showed positive evidence, though many of them had imperfections in their test procedures (for example, some ignored transaction costs). Ebert and Hilpert [21] demonstrated that the market timing of technical trading rules induced skewed trading profits. Popular rules were studied by a combination of simple models, simulations and analysis of empirical data. They argued that investors' preference for positive skewness partially explained the popularity of TA.

Tremendous effort has also been made to build algorithms which implement technical analysis-based trading strategies fast and accurately. For instance, Sezer, Ozbayoglu, and Dogdu [64] designed a trading system based on a neural network constructed by using technical trading rules (based on the simple moving average and the relative strength index), and they showed the optimised system did outperform a buy-and-hold strategy.

In contrast, very little research has been done on the mathematical modelling side. Blanchet-Scalliet et al. [5] derived the optimal expected portfolio wealth at some terminal time $T$ where the underlying price process was assumed to have a mis-specified drift from time 0 to an exponentially distributed random time $\tau$, and (using Monte Carlo methods) they numerically compared it with the expected portfolio wealth resulted from a simple moving-average trading strategy. Lorig, Zhou, and Bin [48] studied a logarithmic utility maximization problem when trading strategies are based on exponential moving averages of the price of an underlying risky asset. De Angelis and Peskir [16] determined the optimal stopping time that minimised the expected absolute distance between the stock price and the unknown support/resistance line which was assumed to be a random variable independent of the price. Under an unrealistic constraint with only linear utility, Jacka and Maeda [49] solved two linked optimal stopping problems with a model for the stock price inspired by the support/resistance line method.

Nevertheless, this literature either focused on particular dynamics (e.g. De Angelis and Peskir [16]) or a specific utility function (e.g. Blanchet-Scalliet et al. [5] and Lorig, Zhou, and Bin [48], Jacka and Maeda [49]).

Our aim in this chapter is to study the modelling of the support/resistance line method and provide very general results for a wide class of dynamics and utility functions. We will rigorously study its mathematical properties with a broad range of reward functions with the aid of (what we like to think are elegant) probabilistic arguments. We will show, under mild assumptions on reward functions and dynamics, the $C^1$ smoothness of the value function. Hence, we will prove the value function is the solution to a generalized free boundary problem. Using these results, we show how solutions of two relevant linked optimal stopping problems are found by solving two free boundary problems whose solutions are easily (numerically) computed. Then, the resulting optimal trading strategy derived from various plausible choices for price dynamics and the trader's utility function will be contrasted with the standard trading rule. To the best of our knowledge, no literature to date attempts to address these issues.

The stock price process may be described as follows. We assume there are two regimes for the stock price process, termed the *positive* and *negative* regime respectively. The dynamics of the stock price process are dependent on its current regime. We further assume that there is an unobserved fixed price level located in some known interval $[L, H]$, and this price level is the support line if the stock is in the positive regime and the resistance line if it is in the negative regime. The regime changes from the negative (positive) to the positive (negative) regime if the stock price crosses $H(L)$ from below (above). So, when a transition of regimes occurs, there is a reversal of the role of the resistance and support level in line with what traders would expect. Note that the stock price process can be in either regime on the interval $(L, H)$, which provides the flexibility to move around the support/resistance line without changing regimes. In an extended version of our model presented in the next chapter, we explore two additional features - a partial reflection boundary and an additional regime (the 0 regime) such that the the regime transitions to 0 from the positive regime after an exponential waiting time, and from 0, can only transition to the negative regime by hitting $L$ from above.

We emphasise that, in contrast to standard regime-switching models, the regime transition in our model is path-dependent and not specified by an exogenous Markov chain. The path-dependent regime-changing can be viewed as a novel method of introducing a market signalling effect into the price process (see Lehalle and Neuman [44] for a different approach).

By making mild assumptions on dynamics, no-arbitrage can be ensured in our main model (for a market consisting of a bond paying the risk-free rate and the stock). A potential criticism of our class of models might be that the exogenous stock price process reduces the economic credibility of our results and that a multiple-agent-based model with an endogenous price for the stock would be more desirable. We argue that, since endogenous specification of prices in a dynamic context usually requires that agents, although they may differ as to their utilities, agree on the probabilistic specification of the world, modelling a world where *only some* agents believe in TA renders this approach non-viable and the best that can be achieved is a price specification which is arbitrage-free and hence in dynamic equilibrium.

We stress that the optimal stopping problems presented here are not standard since the stock price process is *not* a diffusion and the regime process on its own is *not* Markovian.

The rest of this chapter proceeds as follows. In Section 2.2, we provide definitions for key ingredients of the model and establish important mathematical properties. In Section 2.3, we give some general results regarding the optimal stopping problem. In Section 2.4, we describe and solve *the seller's problem* and obtain the optimal selling boundaries. In Section 2.5, we define and solve *the buyer's problem*, which provides the optimal buying boundaries. Finally, in Section 2.6, we analyse numerically the influence of the degree of relative risk aversion to optimal trading strategies under different types of dynamics, and we will discuss the optimality of the standard trading rule within the context of our modelling. Section 2.7 contains some supplementary results and proofs.

## 2.2 A path-dependent regime-switching model

Suppose for the stock price, there are two price levels $L$ and $H$ such that $0 < L < H$ and two regimes: positive (denoted $+$) and negative (denoted $-$). We assume there is a price level $R$ located in $(L, H)$ which is a support line if the stock price is in the positive regime and becomes a resistance line if the stock transitions into the negative regime. We emphasise that, at the point, the model does not dependent on $R$ explicitly (and the value of $R$ is not specified). We will extend this setting by including $R$ in our model directly in Chapter 3.

We are given a stochastic process $(S, F)$ where $S$ is the stock price process taking values in $\mathbb{R}_+ = [0, \infty)$ and $F$ is a flag-process taking values in $\{-, +\}$. Then, the law of $S$ is given by the following (regime-dependent) SDE

$$dS_t = \mu_{F_t}(S_t)dt + \sigma_{F_t}(S_t)dW_t, \tag{2.2.1}$$

for suitable Borel measurable real functions $\mu_+$, $\mu_-$, $\sigma_+$, and $\sigma_-$ (hereafter referred to collectively as *the dynamics*) defined on $\mathbb{R}_+$, and $F$ is a piecewise constant process which satisfies

$$F_t = \begin{cases} + & \text{if } F_{t-} = +, \text{ and } S_t > L \\ + & \text{if } F_{t-} = -, \text{ and } S_t = H \\ - & \text{if } F_{t-} = -, \text{ and } S_t < H \\ - & \text{if } F_{t-} = +, \text{ and } S_t = L, \end{cases} \tag{2.2.2}$$

where $W$ denotes a one dimensional Brownian motion and $t \in \mathbb{R}_+$.

Thus the regime transitions happen when $S$ hits $L$ from above when $S$ is in the positive regime and when $S$ hits $H$ from below when $S$ is in the positive regime. It follows that the regime switching times are a sequence of stopping times which depend on the path of $S$. Clearly, such a set-up gives a state space $(E, \mathcal{B})$, where $E = [0, H) \times \{-\} \cup (L, \infty] \times \{+\}$ and $\mathcal{B} = \mathcal{B}(\mathbb{R}_+) \otimes 2^{\{+,-\}}$.

To ensure the (weak) existence and uniqueness of $(S, F)$, we make the following assumptions on the dynamics.

**Assumption 2.2.1.** Define $M := \{(x, f) \in E : \sigma_f(x) = 0\}$ and $N := \{(x, f) \in E : \int_{N_x} \sigma_f^{-2}(y)dy = \infty, \text{ for any open set } N_x \text{ in } \mathbb{R}_+ \text{ containing } x \}$. Then,

$$M = N = \{(0, -)\}, \tag{2.2.3}$$

$$\{(x, f) \in E : \int_{N_x} \frac{\mu_f(y)}{\sigma_f^2(y)}dy < \infty, \text{ for any open sets } N_x \text{ containing } x \} = E \setminus N. \tag{2.2.4}$$

It is useful to separately define the two 'component' diffusion processes $S^+$ and $S^-$ which are solutions of the following SDEs,

$$\begin{cases} dS_t^+ = \mu_+(S_t^+)dt + \sigma_+(S_t^+)dW_t, \\ dS_t^- = \mu_-(S_t^-)dt + \sigma_-(S_t^-)dW_t. \end{cases} \tag{2.2.5}$$

Note that, by Theorem 4.53 (2) of Engelbert and Schmidt [24], under Assumption 2.2.1, there exist solutions of (2.2.5) which are unique in law. 'To construct $(S, F)$ we need versions of $S^+$ and $S^-$, killed on hitting $L$ and $H$. We denote the infinitesimal generators of the killed processes by $\mathcal{L}+$ and $\mathcal{L}^-$ respectively.

Denoting the scale functions and speed measures in the two regimes by $s_\pm$ and $m_\pm$ respectively, we assume:

**Assumption 2.2.2.** Khasminskii's condition holds in the positive regime:

$$\int_1^\infty s'_+(x)dx \int_1^x m'_+(y)dy = \infty. \tag{2.2.6}$$

This implies the process $S^+$ does not explode in finite time (see Rogers and Williams [62] p.297), and hence $S$ inherits this property. We will assume Assumptions 2.2.1 and 2.2.2 are in force in the rest of this chapter.

From Assumption 2.2.1. the Itô diffusions $S^+$ and $S^-$ are *regular* except at 0 (i.e. $\mathbb{P}_x(S^f \text{ hits } y) > 0$, for all $x > 0$ and $y \geq 0$, for each $f \in \{+,-\}$).

**Theorem 2.2.3.** *Under Assumptions 2.2.1 and 2.2.2, a (time-homogeneous) Markov process $(S,F)$ satisfying (2.2.1) and 2.2.2) exists and is unique in law. It has infinitesimal generator $\mathcal{L}$ given by*

$$\mathcal{L}g(x,f) = \mathcal{L}^f g(x,f),$$

*with $\mathcal{D}(\mathcal{L}) = \{g :\ g_\epsilon(\cdot,f) \in \mathcal{D}(\mathcal{L}^f) \text{ for } f = \pm\}$. The process $(S,F)$ is Feller, and thus has the strong Markov property, and is regular at all points in $\mathbb{E}$ except $(0,-)$. It follows from (2.2.1) that $S$ is a continuous semimartingale. Moreover $(S,F)$ is càdlàg.*

The central idea of the proof of Theorem 2.2.3 is that we can glue the laws of $S^+$ and $S^-$ together at the stopping times corresponding to regime transitions. We present the proof of Theorem 2.2.3 in Section 2.7.2.

From now on, we will work with a process $(S,F)$ which is defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = \{\mathcal{F}_t\}_{t\in\mathbb{R}_+\cup\{\infty\}}, \mathbb{P}_{x,f})$, satisfying the usual conditions, which supports a Brownian motion $W$. We stress that, by Assumption 2.2.1, either $(0,-)$ is inaccessible or is absorbing for $(S,F)$. We will consider both cases and distinguish the results where they are different. In the case where $(0,-)$ is inaccessible (e.g. $S^-$ is a geometric Brownian motion), the state space $E$ may exclude $(0,-)$, but we trust that there is no prospect of confusion in still using $E$ to denote the state space.

Note that, for $A \in \mathcal{F}$, we say $A$ a.s. (or $\mathbb{P}$-a.s.), if $\mathbb{P}_{x,f}(A) = 1$ for each $(x,f) \in E$.

For any $A \in \mathcal{B}$, we define the hitting time of $A$ by $\tau_A := \inf\{t \geq 0 :\ (S_t,F_t) \in A\}$. In the case of $A \in \mathcal{B}(\mathbb{R}_+)$, we denote by $\tau_A^+$ and $\tau_A^-$ the first time $S$ enters $A$ while in the positive regime and negative regime respectively, e.g., $\tau_A^+ := \inf\{t \geq 0 :$

$S_t \in A, F_t = +\}$. If $A = \{a\}$ for some $a \in \mathbb{R}_+$, we simply use $\tau_a^f$ to denote $\tau_A^f$, $f \in \{+,-\}$. If $A \in \mathcal{B}(\mathbb{R}_+)$, we set $\tau_A := \tau_A^+ \wedge \tau_A^-$. According to Kallenberg [39], since $(S,F)$ is a right-continuous adapted process, it is progressively measurable (i.e., $(S,F)$ restricted to $\Omega \times [0,t]$ is $\mathcal{F}_t \otimes \mathcal{B}[0,t]$-measurable for every $t \geq 0$) and Theorem 7.7 in [39] ensures $\tau_A$ is a Markov time (a stopping time if $\tau_A$ is finite a.s.) for any $A \in \mathcal{B}$.

Let $\sigma_A^+$ and $\sigma_A^-$ denote the first hitting time by $S^+$ and $S^-$ of a Borel measurable set $A$ respectively. We note that $S$ has the same law as $S^f$ until the first time that the regime changes.

Theorem 2.2.3 identifies $\mathcal{L}$, the infinitesimal generator of $(S,F)$. More generally, let $\mathbb{L}$ denote the *martingale generator* of $(S,F)$, i.e. for a measurable function $h$, if there is a measurable function $g$ such that, $\int_0^t |g(S_s,F_s)|ds < \infty$ a.s. and for each $(x,f) \in E$,

$$M_t := h(S_t,F_t) - h(x) + \int_0^t g(S_s,F_s)ds \tag{2.2.7}$$

is a local martingale under each $\mathbb{P}_{x,f}$, then we say $\mathbb{L}h = g$ and $h \in \mathcal{D}(\mathbb{L})$. Similarly, for $S^f$, we denote its martingale generator by $\mathbb{L}^f$. Note that if $h \in C^1$ with absolutely continuous first derivative, then $\mathbb{L}^f h = \mathcal{L}^f h$ and that $\mathcal{D}(\mathcal{L}) \subset \mathcal{D}(\mathbb{L})$ by Proposition 1.7 in Chapter VII of [59].

Fix a positive constant $r$, which shall be understood as the "interest rate" in later sections. Then, we define *fundamental solutions,* which we denote by $\phi^+$ and $\psi^+$ for (2.2.9), and $\phi^-$ and $\psi^-$ for (2.2.10) as follow:

$$\psi^f(x) = \begin{cases} \mathbb{E}^x[e^{-r\sigma_{c^f}}] & \text{if } x \leq c^f \\ \frac{1}{\mathbb{E}^{c^f}[e^{-r\sigma_x}]} & \text{if } x > c^f, \end{cases} \qquad \phi^f(x) = \begin{cases} \frac{1}{\mathbb{E}^{c^f}[e^{-r\sigma_x}]} & \text{if } x \leq c^f \\ \mathbb{E}^x[e^{-r\sigma_{c^f}}] & \text{if } x > c^f, \end{cases} \tag{2.2.8}$$

where $c^+ = H$ and $c^- = L$. Note that $\psi^f$ is increasing and $\phi^f$ is decreasing. Since both the speed measures and scale functions are absolutely continuous with respect to Lebesgue measure, fundamental solutions are solutions to the following ODEs:

$$\mathcal{L}^+ v - rv = 0, \tag{2.2.9}$$

$$\mathcal{L}^- v - rv = 0. \tag{2.2.10}$$

## 2.3 The optimal stopping problems

We study two problems in this chapter. The first is called the *seller's problem.* In this problem, a trader initially holds the stock and seeks a selling time which gives the maximum gains (utility in this chapter). The second one we term the *buyer's problem* : here a trader wants to maximise expected utility (gain) by first purchasing a stock and then selling it later. Both problems are formulated as optimal stopping problems, and we will present some general results here.

We assume we have a *gains function* $h : E \to \mathbb{R}_+$ and a discount rate $r \geq 0$. We introduce the following assumptions.

**Assumption 2.3.1.** $h \in \mathcal{D}(\mathbb{L}) \cap C(E)$, i.e. $h$ is continuous and in the domain of the martingale generator of $(S, F)$.

*Remark* 2.3.2. Note that if $h : \mathbb{R}_+ \to \mathbb{R}$ and we extend its domain to $E$ by setting $h_E(x, f) = h(x)$ then $h_E$ satisfies Assumption 2.3.1 if $h \in \mathcal{D}(\mathbb{L}^+) \cap \mathcal{D}(\mathbb{L}^-) \cap C(\mathbb{R}_+)$.

Henceforward, we believe there is no confusion caused by using the notation $h$ interchangeably with $h_E$ and we will assume that $h$ is of this form, i.e. is independent of $F$.

As we will see, the following assumption guarantees the finiteness of the value function for our optimal stopping problems.

**Assumption 2.3.3.** $\mathbb{E}^{x,f}\Big[\sup_{t\geq 0} |e^{-rt}h(S_t)|\Big] < \infty$.

*Remark* 2.3.4. Assumption 2.3.1 holds for any $h \in C^2$ by applying Itô's formula for semimartingales, and it holds even for $h \in C^1$ where the first derivative is absolutely continuous following an extended version of Itô's formula (e.g. (45.9) Lemma in [62] p.105). Conversely, if a function $h$ can be written as the difference of two convex functions , then $h \in \mathcal{D}(\mathbb{L})$ would imply $h \in C^1$ by the Itô-Tanaka formula. We will see that Assumption 2.3.1 is satisfied by the set-up of the seller/buyer's problem.

The *optimal stopping problem* is defined by

$$\mathbf{V}(x, f) := \sup_{\tau} \mathbb{E}^{x,f}[e^{-r\tau}h(S_\tau)], \tag{P}$$

where the supremum is taken over all stopping times, and we call $\mathbf{V}(x, f)$ the *value function.* We also look for the optimal stopping time $\tau^*$ making

$$\mathbf{V}(x, f) = \mathbb{E}^{x,f}[e^{-r\tau^*}h(S_{\tau^*})]. \tag{2.3.1}$$

The following lemmas are required for the proof of Theorem 2.3.7.

**Lemma 2.3.5.** *Under Assumption 2.3.3,* $\mathbf{V}(x, f) < \infty$ *for any* $(x, f) \in E$.

*Proof.* By definition,

$$\mathbf{V}(x, f) = \sup_{\tau} \mathbb{E}^{x,f}[e^{-r\tau}h(S_\tau)] \leq \mathbb{E}^{x,f}\Big[\sup_{\tau} e^{-r\tau}h(S_\tau)\Big] = \mathbb{E}^{x,f}\Big[\sup_{t} e^{-rt}h(S_t)\Big] \tag{2.3.2}$$

Hence, by Assumption 2.3.3, $\mathbf{V}(x, f) < \infty$. ♢

**Lemma 2.3.6.** *Under Assumption 2.3.1 and 2.3.3, the value function* $\mathbf{V}(x, f)$ *is lower semicontinuous*
$\big(i.e.\ \liminf_{y \to x} \mathbf{V}(y, f) \geq \mathbf{V}(x, f)\big)$.

*Proof.* By Assumption 2.3.3, we can apply Theorem 1 in Chapter 3 of [65] to see $\mathbf{V}$ is the smallest excessive majorant of the gains function h. Then by Lemma 4 in Chapter 3 of [65], since $(S, F)$ is a Feller process and $h$ is bounded below by 0, $\mathbf{V}$ is lower semicontinuous. $\diamondsuit$

Define the *stopping set* $D$ and *continuation set* $C$ by

$$D = \{(x, f) \in [0, \infty) \times \{+, -\} | \mathbf{V}(x, f) = h(x)\}, \tag{2.3.3}$$

$$C = \{(x, f) \in [0, \infty) \times \{+, -\} | \mathbf{V}(x, f) > h(x)\}. \tag{2.3.4}$$

As $\mathbf{V}$ is lower semicontinuous, $D$ is closed and $C$ is open. The following theorem follows immediately from Shiryaev [65] Chapter 3, Theorem 3.

**Theorem 2.3.7.** *For any gains function* $h$ *satisfying Assumptions 2.3.1 and 2.3.3, if* $\tau_D < \infty$ *a.s. for every* $(x, f) \in E$*, the stopping time* $\tau_D$ *is optimal in the sense that equation (2.3.1) holds.*

By a well-known result (see Jacka and Norgilas [38] Theorem 2.10), $e^{-rt}\mathbf{V}(S_t, F_t)$ is the *Snell envelope* of the gains process $e^{-rt}h(S_t)$ under Assumption 2.3.3, i.e. $e^{-rt}\mathbf{V}(S_t, F_t) = \operatorname{ess\,sup}_{\tau \geq t} \mathbb{E}[e^{-r\tau}h(S_\tau)|\mathcal{F}_t]$, a.s. Moreover, standard theory of optimal stopping (e.g. Theorem 2.2 of [56]) tells us that $e^{-rt}\mathbf{V}(S_t, F_t)$ is a supermartingale and the stopped process $e^{-rt\wedge\tau_D}\mathbf{V}(S_{t\wedge\tau_D}, F_{t\wedge\tau_D})$ is a martingale. The theorem below is a direct consequence of Theorem 3.11 in Jacka and Norgilas [38].

**Theorem 2.3.8.** *Recall that* $\mathbb{L}$ *denotes the martingale generator of* $(S, F)$*. Under Assumptions 2.3.1 and 2.3.3,* $\mathbb{L}\mathbf{V}(x, f) - r\mathbf{V}(x, f) = 0$ *on* $C$ *almost everywhere.*

*Proof.* Fix $(x, f) \in C$. Theorem 3.11 in Jacka and Norgilas [38] states that, under Assumption 2.3.3, $h \in \mathcal{D}(\mathbb{L})$ implies that $\mathbf{V} \in \mathcal{D}(\mathbb{L})$. Thus

$$e^{-rt\wedge\tau_D}\mathbf{V}(S_{t\wedge\tau_D}, F_{t\wedge\tau_D}) = \mathbf{V}(x, f) + \int_0^{t\wedge\tau_D} e^{-rs}dM_s + \int_0^{t\wedge\tau_D} e^{-rs}(\mathbb{L}-r)\mathbf{V}(S_s, F_s)ds. \tag{2.3.5}$$

Since $e^{-rt\wedge\tau_D}\mathbf{V}(S_{t\wedge\tau_D}, F_{t\wedge\tau_D})$ is a martingale, we must have, for all $t \geq 0$,

$$\int_0^t e^{-rs}(\mathbb{L} - r)\mathbf{V}(S_s, F_s)\mathbb{1}_{(S_s, F_s)\in C}ds = 0 \ a.s. \tag{2.3.6}$$

Moreover, by the Doob-Meyer decomposition (e.g. Theorem 2.4 of [38]), $\int_0^t e^{-rs}(\mathbb{L}-r)\mathbf{V}(S_s,F_s)ds$ is the unique decreasing integrable variation process in the decomposition of $e^{-rt}\mathbf{V}(S_t,F_t)$. This implies $\mathbb{L}\mathbf{V}(x,f)-r\mathbf{V}(x,f)\le 0$ a.e. on $E$, because otherwise the compensator would be increasing on a set in which $(S,F)$ spends positive time with positive probability. If $\mathbb{L}\mathbf{V}(x,f)-r\mathbf{V}(x,f)<0$ on a subset of $C$ with positive measure, then with positive probability, there is a $t$ such that

$$\int_0^t e^{-rs}(\mathbb{L}-r)\mathbf{V}(S_s,F_s)\mathbb{1}_{(S_s,F_s)\in C}ds<0. \tag{2.3.7}$$

This leads to a contradiction. Thus, we have $\mathbb{L}\mathbf{V}(x,f)-r\mathbf{V}(x,f)=0$ on $C$ almost everywhere. We are left to verify the remaining assumption of Theorem 3.11 in Jacka and Norgilas [38] — that $(S,F)$ is a right process, which is satisfied since $(S,F)$ is Feller.

$\diamondsuit$

If a process $X$ starts at $x$ in the boundary of the continuation region, $C$ and enters $\mathrm{int}(D)$ immediately with positive probability, then the smooth pasting principle is often valid at $x$ (see Section 9 in Peskir and Shiryaev [56]). The smooth pasting principle is well established for one dimensional Itô diffusion processes (see e.g., Jacka and Norgilas [38]), but not in greater generality. Nevertheless, since the process $S$ is an Ito diffusion before regime transitions, the smooth pasting principle indeed holds:

**Theorem 2.3.9.** *Under Assumption 2.3.1 and 2.3.3,* $\mathbf{V}\in C^1(E\setminus\{(0,-)\})$*. In particular,* $\mathbf{V}$ *is continuously differentiable (in $x$) at the boundary of $C$, $\partial C$, apart from at* $(0,-)$.

*Proof.* The strategy is similar to the proof of Theorem 4.9 of Jacka and Norgilas [38].

Fix $f=+$ and $x>L$. Let $\tilde{s}(x):=\psi_+(x)/\phi_+(x)$, then $\tilde{s}$ is continuous and increasing. Pick an arbitrary interval $x\in[a,b]$ and $a>L$. Set $\tau:=\inf\{t\ge 0: S=a \text{ or } S=b\}$.

By following the same argument as (4.14) - (4.15) in Jacka and Norgilas [38] p.1884-p.1885, $J(x):=\mathbf{V}(x,+)/\phi_+(x)$ is $\tilde{s}$-concave. Let $K:[\tilde{s}(a),\tilde{s}(b)]\to\mathbb{R}_+$ be the function defined by $K(x):=J\circ\tilde{s}^{-1}(x)$. Then, $K$ is concave and $K(\tilde{s}(x))\phi_+(x)=\mathbf{V}(x,+)$. Further define $Y_t=\tilde{s}(S_t)$, and we have

$$e^{-rt\wedge\tau}\mathbf{V}(S_{t\wedge\tau},+)=e^{-rt\wedge\tau}\phi_+(S_{t\wedge\tau})K(Y_{t\wedge\tau}).$$

Set $N_t:=e^{-rt}\phi_+(S_t)$, which implies that $N_{t\wedge\tau}$ is a local martingale.[1] Applying the

[1] Since changing the reference point $\phi$ will only replace it by fixed multiples (see, e.g. (50.5) p.293

generalised Itô formula for convex/concave functions (see Revuz and Yor [59]) , we have

$$K(Y_{t\wedge\tau}) = K(Y_0) + \int_0^{t\wedge\tau} K'_-(Y_s)dY_s - \int_{\tilde{s}(a)}^{\tilde{s}(b)} L^z_{t\wedge\tau}\nu(dz), \tag{2.3.8}$$

where $L^z_t$ is the local time of $Y_t$ at $z$, and $\nu$ is the measure corresponding to the second derivative of $-K$ in the sense of distribution. Therefore,

$$\begin{aligned} e^{-rt\wedge\tau}\mathbf{V}(S_{t\wedge\tau},+) = N_0K(Y_0) + \int_0^{t\wedge\tau} K(Y_s)dN_s \\ + \int_0^{t\wedge\tau} K'_-(Y_s)(N_sdY_s + d[N,Y]_s) + \int_0^{t\wedge\tau} N_sdA_s, \end{aligned} \tag{2.3.9}$$

where $A_t := \int_{\tilde{s}(a)}^{\tilde{s}(b)} L^z_t\nu(dz)$ is a continuous and increasing process. Note that $N_{t\wedge\tau}Y_{t\wedge\tau} = e^{-rt\wedge\tau}\psi_+(S_{t\wedge\tau})$ is a local martingale. Hence,

$$\int_0^{t\wedge\tau} N_sdY_s + [N,Y]_{t\wedge\tau} = e^{-rt\wedge\tau}\psi_+(S_{t\wedge\tau}) - \int_0^{t\wedge\tau} Y_sdN_s, \tag{2.3.10}$$

which again is a local martingale. Furthermore,

$$\int_0^t N_sdA_s = N_tA_t - \int_0^t A_sdN_s. \tag{2.3.11}$$

So, we get

$$e^{-rt\wedge\tau}\mathbf{V}(S_{t\wedge\tau},+) = \mathbf{V}(x,+) + M_{t\wedge\tau} - \int_{\tilde{s}(a)}^{\tilde{s}(b)} N_{t\wedge\tau}L^z_{t\wedge\tau}\nu(dz) \tag{2.3.12}$$

for some local martingale $M$.

On the other hand, we know $\mathbf{V} \in \mathcal{D}(\mathbb{L})$, thus,

$$e^{-rt\wedge\tau}\mathbf{V}(S_{t\wedge\tau},+) = \mathbf{V}(x,+) + M^V_{t\wedge\tau} + \int_0^{t\wedge\tau} e^{-rs}(\mathbb{L}^+ - r)\mathbf{V}(S_s,+)ds, \tag{2.3.13}$$

where $M^V$ is a local martingale. By the uniqueness of the Doob-Meyer decomposition, the finite variation terms of (2.3.12) and (2.3.13) must agree. Hence,

$$\int_0^{t\wedge\tau} e^{-rs}(\mathbb{L}^+ - r)\mathbf{V}(S_s,+)ds = -\int_{\tilde{s}(a)}^{\tilde{s}(b)} e^{-rt\wedge\tau}\phi_+(S_{t\wedge\tau})L^z_{t\wedge\tau}\nu(dz) \tag{2.3.14}$$

of [62]), WLOG, we can assume $b < c^+$. Then, on $\{t < \tau\}$, $e^{-rt}\phi_+(S_t) = \mathbb{E}^x[e^{-r\sigma_{c^+}}|\mathcal{F}_t]$, which is a martingale. We can repeat the same argument to show $e^{-rt\wedge\tau}\psi_+(S_{t\wedge\tau})$ is a martingale as well.

Suppose $\nu(\{\tilde{s}(x)\}) > 0$ (i.e. $K$ is not differentiable at $\tilde{s}(x)$). Then, (2.3.14) becomes

$$\int_0^{t\wedge\tau} e^{-rs}(\mathbb{L}^+ - r)\mathbf{V}(S_s,+)ds$$
$$= -e^{-rt\wedge\tau}\phi_+(S_{t\wedge\tau})\Big\{L^{\tilde{s}(x)}_{t\wedge\tau}\nu(\{\tilde{s}(x)\}) + \int_{\tilde{s}(a)}^{\tilde{s}(b)} \mathbb{1}_{z\neq\tilde{s}(x)}L^z_{t\wedge\tau}\nu(dz)\Big\}. \quad (2.3.15)$$

We see the left hand side of (2.3.15) is absolutely continuous with respect to Lebesgue measure. However, the right hand side is not absolutely continuous since the measure induced by $L^{\tilde{s}(x)}$ is singular with respect to Lebesgue measure.[2] Hence, we obtain a contradiction, which implies $\mathbf{V}(x,+)$ is differentiable at $x$ since $\tilde{s} \in C^1$. Therefore, as $x$ is arbitrary and the left and right derivative of $\mathbf{V}(x,+)$ exist, we conclude $\mathbf{V}(x,+) \in C^1$. The proof for $\mathbf{V}(x,-)$ is identical. $\diamondsuit$

*Remark* 2.3.10. We might extend the dynamics of $S$ to have multiple regimes on overlapping intervals in $\mathbb{R}+$. The proof of Theorem 2.3.9 can be easily extended to show that the $C^1$ smoothness of the value function holds in this more general set-up.

We look for a measurable function $v : E \to \mathbb{R}$ and a set $\tilde{D}$ such that $v \in \mathcal{D}(\mathbb{L})$ and

$$\mathbb{L}v - rv = 0 \text{ in } \tilde{C}, \quad (2.3.16)$$
$$v|_{\tilde{D}} = h|_{\tilde{D}}, \quad (2.3.17)$$
$$\frac{\partial v}{\partial x}|_{\partial\tilde{C}} = \frac{\partial h}{\partial x}|_{\partial\tilde{C}}, \quad (2.3.18)$$

where $\tilde{C} := \tilde{D}^c$. By Theorem 2.3.8 and 2.3.9, the value function $\mathbf{V}$ and stopping set $D$ is a solution to a *free boundary problem* . Moreover, if the drifts and volatilities are sufficiently smooth (so that fundamental solutions are $C^2$), the usual argument (e.g. Section 7.1 in Peskir and Shiryaev [56]) allows us to replace condition (2.3.16) by ODEs (2.2.9) and (2.2.10).

We want to show that conversely, for our two problems the value function is the unique solution to the free boundary problem. This is done in Section 2.4.3 for the seller's problem and in Section 2.5.2 for the buyer's problem.

[2] For Brownian Motion, this is a standard result (see e.g. Theorem 9.6 in [40]). For a weak solution of an SDE with zero drift, by considering it as a time-changed BM (with an absolutely continuous time change) and applying Tanaka's formula, we can prove its local time is a time-changed BM local time and hence singular w.r.t to Lebesgue measure as well. Finally, applying the relevant scale function to the weak solution of a general SDE leads to our claim.

## 2.4 The seller's problem

Let $u : [0, \infty) \to \mathbb{R}_+$ denote a positive increasing function. A typical example of $u$ is a utility function, e.g. $x^\gamma$ for $\gamma \in (0, 1)$. We assume

**Assumption 2.4.1.** $u$ is $C^2$ in $(0, \infty)$, so satisfies Assumption 2.3.1,

Moreover, we further assume $u$ satisfies Assumption 2.3.3. Replacing the gains function $h$ in (P) by $u$, the *seller's problem* is defined by (SP).

$$\mathbf{V}(x, f) := \sup_\tau \mathbb{E}^{x,f}[e^{-r\tau} u(S_\tau)]. \tag{SP}$$

It is important to note that there is a large class of utility functions which satisfy these assumptions, but the following results hold even without assuming concavity.

By Theorem 2.3.7, an arg max of (SP) is given by $\tau_D$. Our next step is to consider what additional assumptions are needed on $u$ to determine the shape of the stopping set $D$.

### 2.4.1 Assumptions and their motivations

Our preliminary work, [49], fixed the gains function to be $h : x \mapsto x$, and in this case, by assuming $\mu_- < r < \mu_+$, it is clear that the sign of $\mathcal{L}^f h - rh$ is $f$.

We now discuss some reasonable assumptions to make on the sign of $\mathcal{L}^f u - ru$. Consonant with the description 'negative regime', we assume that the sign of $\mathcal{L}^- u - ru$ is $-$. Moreover, since we are primarily interested in the case where $u$ is a utility function, it is reasonable to posit that, for large $x$, the sign of $\mathcal{L}^+ u - ru$ is $-$. So, we assume there is a constant $A$ such that the sign of $\mathcal{L}^+ u - ru$ is $+$ for $x < A$ and $-$ for $x > A$. In short, we have:

**Assumption 2.4.2.**

$$\mathcal{L}^- u - ru < 0 \quad \text{in} \quad (0, H]. \tag{2.4.1}$$

$$\mathcal{L}^+ u - ru > 0 \quad \text{in} \quad [L, A). \tag{2.4.2}$$

$$\mathcal{L}^+ u - ru < 0 \quad \text{in} \quad (A, \infty) \tag{2.4.3}$$

for some $A \in (L, \infty)$.

We will solve the seller's and buyer's problem under these three assumptions on the seller's utility.

*Remark* 2.4.3. For some common choices of dynamics and utility functions (e.g. a geometric Brownian motion with a power utility function), $\mathcal{L}^+ u - ru$ can only have one sign. In this case, the optimal stopping time (in the positive regime) can be proven to be either 0 or $\infty$, which is neither very interesting nor realistic. We observe that our assumptions *are* satisfied for a wide class of realistic dynamics and utility functions.

*Remark* 2.4.4. For the geometric Brownian motion, there is a wide range of utility functions which satisfies Assumption 2.4.2. For example, consider an exponential utility function $u(x) = \frac{1-e^{-ax}}{a}$ for $a > 0$. Then,

$$\mathcal{L}^f u - ru = e^{-ax}(-\frac{1}{2}\sigma_f x^2 a + \mu_f x + \frac{r}{a}) - r. \tag{2.4.4}$$

For $a \geq 1$, we can choose $\sigma_f$ and $\mu_f$ such that Assumption 2.4.2 holds. Furthermore, for power utility functions, we can overcome the problem of $A$ being infinite by imposing a selling boundary $M$ (i.e. the seller is forced to clear the position when the price reaches $M$), and only consider $\tau \leq \tau_M$ in (SP). By applying an analogous argument to that seen in Section 2.4.2, we can show that the stopping set is $\{M\}$ in the positive regime and $(0, m]$ in the negative regime. Then, we can propose a candidate solution satisfying (2.3.16) - (2.3.18) with the martingale generator replaced by the differential operator, which leads to a free boundary problem. By mimicking the proof of Theorem 2.4.10, it is not hard to verify the candidate solution is the value function. A more complete discussion can be found in [49]. We will not study this case further since we would like to focus our attention on the more interesting case where $A < \infty$.

*Remark* 2.4.5. Note that neither does $u$ being a utility function imply Assumption 2.4.2 nor the reverse.

**Theorem 2.4.6.** *Suppose that, in addition, $u$ satisfies*

$$\limsup_{x\to\infty} \left( \frac{\sigma_+(x)u'(x)}{u(x)} \right)^2 < \infty, \tag{2.4.5}$$

*and*

$$\text{there exists } \epsilon > 0 \text{ such that } \limsup_{x\to\infty} \mathcal{L}^+ u(x) - (r - \epsilon)u(x) < 0 \tag{2.4.6}$$

*then $u$ satisfies Assumption 2.3.3.*

See Section 2.7 2.7.1 for the proof.

### 2.4.2 The shape of the stopping set, $D$

We define $D^+$, $C^+$ , $D^-$ and $C^-$ by

$$\begin{aligned}
D^+ &= \{x \in (L,\infty) | \mathbf{V}(x,+) = u(x)\},\\
C^+ &= \{x \in (L,\infty) | \mathbf{V}(x,+) > u(x)\},\\
D^- &= \{x \in [0,H) | \mathbf{V}(x,-) = u(x)\},\\
C^- &= \{x \in [0,H) | \mathbf{V}(x,-) > u(x)\}.
\end{aligned}$$

It is clear that $D = D^+ \times \{+\} \cup D^- \times \{-\}$ and $C = C^+ \times \{+\} \cup C^- \times \{-\}$.

**Theorem 2.4.7.** *Under Assumptions 2.3.1, 2.3.3, and 2.4.2, $D^+ = [B,\infty)$ for some $B \in [A,\infty)$, and $D^- = [0,m]$ for some $m \in [0,H)$, or $D^- = (0,m]$ for some $m \in [0,H)$ in the case where $0$ is inaccessible (with the convention that $(0,0] = \emptyset$).*

*Proof.* Before we start, recall $D^+$ and $D^-$ are both closed as $\mathbf{V}$ is lower semicontinuous.
If $\exists y \in (L,A)$ such that $y \in D^+$, there is $\epsilon > 0$ making $L+\epsilon < y < A-\epsilon$. Define $\tau = \tau^+_{L+\epsilon} \wedge \tau^+_{A-\epsilon}$. So $S_{t\wedge\tau}$ is an Itô diffusion staring at $y$ with absorbing states $L+\epsilon$ and $A-\epsilon$. Then, by Ito's Lemma,

$$\mathbb{E}^{(y,+)}[e^{-r\tau}u(S_\tau)] = u(y) + \mathbb{E}^{(y,+)}\Big[\int_0^\tau e^{-rt}(\mathcal{L}^+u(S_t) - ru(S_t))dt\Big] > u(y) = \mathbf{V}(y,+), \tag{2.4.7}$$

where the inequality follows from Assumption 2.4.2 (more specifically (2.4.2)). But this contradicts the definition of $\mathbf{V}$. Thus, $D^+ \cap (L,A) = \emptyset$.

Now, we need to show there are no 'gaps' in $D^+$. Since $C^+$ is open, it can be written as a countable unions of (disjoint) open intervals. Let $(y_1,y_2)$ be such an interval with $y_1,y_2 \in D^+$, and hence $y_2 > y_1 \geq A$. Take any $y \in (y_1,y_2)$ and define $\tau = \tau^+_{y_1} \wedge \tau^+_{y_2}$. It is obvious that

$$\mathbf{V}(y,+) = \mathbb{E}^{(y,+)}[e^{-r\tau}u(S_\tau)] = u(y) + \mathbb{E}^{(y,+)}\Big[\int_0^\tau e^{-rt}(\mathcal{L}^+u(S_t) - ru(S_t))dt\Big] \leq u(y), \tag{2.4.8}$$

which contradicts the assumption that $y \in (y_1,y_2) \subset C^+$. This implies that $D^+$ must be an interval. Since $D^+$ is closed, we complete our claim apart from the special case where $D^+$ is empty.
To prove $D^-$ is an interval, suppose there are $y_1,y_2 \in D^-$ such that $H > y_2 > y_1 > 0$ and $(y_1,y_2) \subset C^-$. Take any $y \in (y_1,y_2)$ and define $\tau = \tau^-_{y_1} \wedge \tau^-_{y_2}$. It is obvious that

$$\mathbf{V}(y,-) = \mathbb{E}^{(y,-)}[e^{-r\tau}u(S_\tau)] = u(y) + \mathbb{E}^{(y,-)}\Big[\int_0^\tau e^{-rt}(\mathcal{L}^-u(S_t) - ru(S_t))dt\Big] \leq u(y), \tag{2.4.9}$$

but this contradicts the inequality $\mathbf{V}(y,-) > u(y)$ because $y \in C^-$. Therefore, $D^-$ is a closed interval.
Moreover,if $(0,-) \in \mathbb{E}$, we must have $e^{-r\tau}u(S_\tau) = e^{-r\tau}u(0) \le u(0)$ for all stopping time $\tau$. Hence, $\mathbb{E}^{(0,-)}[\sup_\tau e^{-r\tau}u(S_\tau)] \le u(0)$, which implies $0 \in D^-$.
So either $D^- = [0,m]$ for some $m \in [0,H)$ or $D^- = [0,H)$. To rule out the latter possibility, assume that it is true. Then $\forall \epsilon > 0$, $\mathbf{V}(H-\epsilon,-) = u(H-\epsilon)$, which implies $\lim_{\epsilon\to 0}\mathbf{V}(H-\epsilon,-) = u(H)$. However, because

$$\mathbf{V}(H-\epsilon,-) \ge \mathbf{V}(H,+)\mathbb{E}^{(H-\epsilon,-)}[e^{-r\tau_H}\mathbb{1}_{\tau_H<\tau_{H/2}}]+u(H/2)\mathbb{E}^{(H-\epsilon,-)}[e^{-r\tau_{H/2}}\mathbb{1}_{\tau_{H/2}<\tau_H}], \tag{2.4.10}$$

taking $\epsilon$ to 0, we can see that

$$\lim_{\epsilon\to 0}\mathbf{V}(H-\epsilon,-) \ge \mathbf{V}(H,+) > u(H),$$

as $\mathbb{E}^{(H-\epsilon,-)}[e^{-r\tau_H}\mathbb{1}_{\tau_H<\tau_{H/2}}]$ converges to 1 and $\mathbb{E}^{(H-\epsilon,-)}[e^{-r\tau_{H/2}}\mathbb{1}_{\tau_{H/2}<\tau_H}]$ converges to 0 by continuity of $\phi_-$ and $\psi_-$. Therefore, by contradiction, $D^- \neq [0,H)$.
Next, suppose $(0,-) \notin E$. This implies $D^- = (0,m]$ or $D^- = [n,m]$ for some $n > 0$. Assume $D^- = [n,m]$ and fix $x \in (0,n)$. Then, $V(x,-) > u(x)$ since $x \in C^-$. However, since $\mathcal{L}^-u - ru < 0$ on $(0,n)$, by Itô's lemma,

$$\begin{aligned}\mathbf{V}(x,-) &= \mathbb{E}^{(x,-)}[e^{-r\tau_n}u(S_{\tau_n})]\\ &= u(x) + \mathbb{E}^{(x,-)}\Big[\int_0^{\tau_n} e^{-rt}(\mathcal{L}^-u(S_t) - ru(S_t))dt\Big] \le u(x),\end{aligned} \tag{2.4.11}$$

which leads to a contradiction. Thus, we must have $D^- = (0,m]$.
Finally, we shall rule out the case where $D^+$ is empty. Suppose $D^+ = \emptyset$. Denote $D_\epsilon := \{(x,f) \in E; \mathbf{V}(x,f) \le u(x)+\epsilon\}$. Since $D^-$ takes the form $[0,m]$, we know $D_\epsilon$ decreases to $D^- \times \{-\}$ as $\epsilon$ tends to 0. Hence, $\tau_{D_\epsilon}$ converges to $\tau_{D^-}$ a.s.. Therefore, by the Dominated Convergence Theorem,

$$\lim_{\epsilon\downarrow 0}\mathbb{E}^{x,f}[e^{-r\tau_{D_\epsilon}}u(S_{\tau_{D_\epsilon}})] = \mathbb{E}^{x,f}[e^{-r\tau_{D^-}}u(S_{\tau_{D^-}})] \le u(m). \tag{2.4.12}$$

On the other hand,

$$\begin{aligned}\mathbb{E}^{x,f}[e^{-r\tau_{D_\epsilon}}u(S_{\tau_{D_\epsilon}})] &\ge \mathbb{E}^{x,f}[e^{-r\tau_{D_\epsilon}}(\mathbf{V}(S_{\tau_{D_\epsilon}},F_{\tau_{D_\epsilon}}) - \epsilon)]\\ &\ge \mathbb{E}^{x,f}[e^{-r\tau_{D_\epsilon}}\mathbf{V}(S_{\tau_{D_\epsilon}},F_{\tau_{D_\epsilon}})] - \epsilon = \mathbf{V}(x,f) - \epsilon,\end{aligned}$$

where the first inequality follows from the definition of $D_\epsilon$ and the last equality follows from Theorem 2 in Chapter 3 of Shiryaev [65]. Letting $\epsilon$ tend to 0, we find that

$$\lim_{\epsilon \downarrow 0} \mathbb{E}^{x,f}[e^{-r\tau_{D_\epsilon}} u(S_{\tau_{D_\epsilon}})] \geq \mathbf{V}(x, f). \tag{2.4.13}$$

Therefore, as the left hand side of (2.4.12) and (2.4.13) are the same, we have $u(m) \geq \mathbf{V}(x, f)$ for any $(x, f) \in E$. However, since $u$ is increasing, $\mathbf{V}(x, f) \geq u(x) > u(m)$ for any $(x, f) \in E \setminus ([0, m] \times \{-\})$, which results in a contradiction. $\diamondsuit$

### 2.4.3 The value function

Recalling the definition of the free boundary problem (2.3.16)-(2.3.18), we have shown that $\mathbf{V}$ is a solution to the free boundary problem (suitably modified when $m = 0$, where the smooth pasting condition (2.3.18) only holds at $B$) *provided we interpret the ODEs as relating to the martingale generator.*

Now we want to show that $\mathbf{V}$ is the unique *classical* solution of the free boundary problem.

From now on, we assume

**Assumption 2.4.8.** $\mu_\pm$ and $\sigma_\pm$ are $\alpha$-Hölder continuous (for some $\alpha > 0$).

We seek a pair of functions $v(\cdot, f) \in C^2$, constants $\hat{B} \geq A$ and $\hat{m} < H$, such that

$$\mathcal{L}^+ v(\cdot, +) - r v(\cdot, +) = 0, \text{ in } (\mathrm{L}, \hat{B}) \tag{2.4.14}$$

$$\mathcal{L}^- v(\cdot, -) - r v(\cdot, -) = 0, \text{ in } (\hat{m}, \mathrm{H}) \tag{2.4.15}$$

$$v(\hat{B}, +) = u(\hat{B}), \quad v(L, +) = v(L, -)\mathbb{1}_{\hat{m}<L} + u(L)\mathbb{1}_{\hat{m} \geq L}, \tag{2.4.16}$$

$$v(H, -) = v(H, +), \quad v(\hat{m}, -) = u(\hat{m}), \tag{2.4.17}$$

$$\frac{\partial v}{\partial x}(x, +; \hat{B}, \hat{m})\Big|_{x=\hat{B}} = u'(\hat{B}), \tag{2.4.18}$$

$$\frac{\partial v}{\partial x}(x, -; \hat{B}, \hat{m})\Big|_{x=\hat{m}} = u'(\hat{m}) \text{ if } \hat{m} > 0. \tag{2.4.19}$$

$$\frac{\partial v}{\partial x}(x, -; \hat{B}, \hat{m})\Big|_{x=0_+} \geq u'(0_+) \text{ if } \hat{m} = 0. \tag{2.4.20}$$

*Remark* 2.4.9. If $\hat{m} = 0$, the boundary condition (2.4.17) is to be interpreted as:

$$v(H, -) = v(H, +), \quad v(0_+, -) = u(0_+). \tag{2.4.21}$$

The following theorem establishes that the value function has the desired property.

**Theorem 2.4.10.** *Under Assumptions 2.3.1, 2.3.3, 2.4.2, and 2.4.8, a (classical) solution to the free boundary problem (2.4.14)-(2.4.19) exists. Let $(B^*, m^*)$ and*

$v(x, f; B^*, m^*)$ *denote the solution. Define* $V : E \to \mathbb{R}$ *by:*

$$V(x,f) = \begin{cases} v(x,f;B^*,m^*) & \text{if } x \in (L, B^*),\ f = + \text{ or } x \in (m^*, H),\ f = -; \\ u(x) & \text{if } x \in [B^*, \infty),\ f = + \text{ or } x \in [0, m^*],\ f = -. \end{cases} \tag{2.4.22}$$

*Then,* $V = \mathbf{V}$ *on* $E$ *and* $(B^*, m^*) = (B, m)$*.*

The following lemma provides sufficient conditions for a function to be the value function.

**Lemma 2.4.11.** *Let* $V(x, f)$ *denote a function on* $E$*. Define* $N_t := e^{-rt}V(S_t, F_t)$*. If* $N$ *satisfies P1,P2 and P3 defined as follows:*

*P1.* $N_t$ *is a class D supermartingale,*

*P2.* $\exists \tau < \infty$ *a.s. such that* $N_0 = \mathbb{E}^{x,f}[e^{-r\tau}u(S_\tau)]$*,*

*P3.* $N_t \geq e^{-rt}u(S_t)$ *for all* $t \geq 0$*,*

*then,* $V(x, f) = \mathbf{V}(x, f)$*.*

*Proof.* By the Optional Sampling Theorem for class D supermartingales (see Rogers and Williams [61] pp.189), for any stopping time $\tau$,

$$V(x,f) = N_0 \geq \mathbb{E}^{x,f}[e^{-r\tau}V(S_\tau, F_\tau)] \geq \mathbb{E}^{x,f}[e^{-r\tau}u(S_\tau)], \tag{2.4.23}$$

where the last inequality follows from P3. Since (2.4.23) holds for any $\tau$, we get $V(x, f) \geq \mathbf{V}(x, f)$. On the other hand, by P2, for some $\tau$,

$$V(x,f) = N_0 = \mathbb{E}^{x,f}[e^{-r\tau}u(S_\tau)],$$

and hence $V(x, f) \leq \mathbf{V}(x, f)$. ♢

We can now give the proof of Theorem 2.4.10.

*Proof of Theorem 2.4.10 .* We first prove that $\mathbf{V}$ solves the free boundary problem.

Suppose $m > 0$. Since the dynamics are Hölder-continuous and $\mathcal{L}^\pm$ are uniformly elliptic, by the continuity of $\mathbf{V}$, we can apply usual argument to prove that $\mathbf{V}$ solves the Dirichlet problem (and gives a classical solution) on the continuation set by fixing $\hat{m} = m$ and $\hat{B} = B$. Moreover, Theorem 2.3.9 implies that the (2.4.18) and (2.4.19) hold.

Suppose $m = 0$. Then it is possible that $\mathcal{L}^-$ is elliptic on $(0, H)$ but not uniformly. In this case, some general results (e.g. Theorem 6.25 of Gilbarg and Trudinger [34]) can be adapted to show that $\mathbf{V}$ still solves the Dirichlet problem in the classical sense. Therefore, this proves the existence of a solution to the free boundary problem.

We move on to show every solution to the free boundary problem equals to the value function.

Let $v$ denote a solution to the free boundary problem with boundaries $B^*$ and $m^*$. Define $N_t := e^{-rt}V(S_t, F_t)$ where $V$ is defined by (2.4.22). According to Lemma 2.4.11, to show $V$ is the value function, it is sufficient to prove $N$ satisfies P1-P3. Firstly, it is obvious that $V(0,-) = u(0) = \mathbf{V}(0,-)$. Now, take an arbitrary initial position $(x, f) \in E \setminus \{(0,-)\}$.
(P1) As $v(x,+)$ and $v(x,-)$ are $C^2$ on compact domains, they are bounded by some finite constant $M$. So,

$$|N_t| = e^{-rt}|V(S_t, F_t)| \leq e^{-rt}(|M| \vee |u(S_t)|) \leq |M| \vee e^{-rt}|u(S_t)|. \tag{2.4.24}$$

Hence,

$$\mathbb{E}^{x,f}[\sup_{\tau} |N_\tau|] \leq \mathbb{E}^{x,f}[|M| \vee \sup_{\tau} e^{-r\tau}|u(S_\tau)|] \leq |M| + \mathbb{E}^{x,f}[\sup_{\tau} e^{-r\tau}|u(S_\tau)|] < \infty, \tag{2.4.25}$$

which implies that $N_t$ is of class D.
Now, we are going to show $N_t$ is a supermartingale. Using Peskir's change-of-variable formula with local time (see [54]), as $S_t$ is a continuous semimartingale and $V(x, f)$ is a piecewise $C^2$ function of $x$ given $f$, it follows that

$$\begin{aligned} dN_t =& e^{-rt}\Big[\big(-rV(S_t,+) + \mathcal{L}^+V(S_t,+)\big)\mathbb{1}_{\{F_t=+, S_t\neq B^*\}}dt \\ &+ \big(-rV(S_t,-) + \mathcal{L}^-V(S_t,-)\big)\mathbb{1}_{\{F_t=-, S_t\neq m^*\}}dt \\ &+ \frac{\partial V}{\partial x}(S_t, F_t)\sigma_{F_t}(S_t)\mathbb{1}_{(S_t,F_t)\notin\{(B^*,+),(m^*,-)\}}dW_t \\ &+ \frac{1}{2}\big(\frac{\partial V}{\partial x}(m^{*+}, -; m^*, B^*) - \frac{\partial V}{\partial x}(m^{*-}, -; m^*, B^*)\big)\mathbb{1}_{\{F_t=-, m^*\neq 0\}}dl_t^{m^*}(S) \\ &+ \frac{1}{2}\big(\frac{\partial V}{\partial x}(B^{*+}, +; m^*, B^*) - \frac{\partial V}{\partial x}(B^{*-}, +; m^*, B^*)\big)\mathbb{1}_{F_t=+}dl_t^{B^*}(S)\Big], \end{aligned} \tag{2.4.26}$$

where $dl_t^a(S)$ is the measure induced by the local time of $S$ at the point $a$. By the smooth pasting conditions (2.4.18) and (2.4.19), the local time terms in (2.4.26) disappear. Recall $v(x, \pm)$ satisfies ODEs (2.4.14) and (2.4.15) respectively. By the construction of $V$, equation (2.4.26) becomes

$$\begin{aligned} dN_t =& e^{-rt}\Big[\big(\mathcal{L}^- u(S_t) - ru(S_t)\big)\mathbb{1}_{(F_t=-,S_t<m^*)}dt + \big(\mathcal{L}^+ u(S_t) - ru(S_t)\big)\mathbb{1}_{(F_t=+,S_t>B^*)}dt \\ &+ \frac{\partial V}{\partial x}(S_t,F_t)\sigma_{F_t}(S_t)\mathbb{1}_{(S_t,F_t)\notin\{(B^*,+),(m^*,-)\}}dW_t\Big]. \end{aligned}$$

Since $\mathcal{L}^- u - ru \leq 0$ on $[0, m^*]$ and $\mathcal{L}^+ u - ru \leq 0$ on $[B^*, \infty)$, we can conclude that the drift terms are non-positive. Moreover, we can find a localising sequence for the $dW_t$ term such that the stopped process is a martingale. Thus, we conclude that $N_t$ is a local supermartingale. Since it is also class D, $N_t$ is therefore a supermartingale.
(P2) Define $\tau_{B^*}^{m^*} := \inf\{t \geq 0; S_t \leq m^* \text{ if } F_t = - \text{ or } S_t \geq B^* \text{ if } F_t = +\}$. Suppose for the starting position $(x, f)$ we have that $x \geq m^*$ if $f = -$ and $x \leq B^*$ if $f = +$. Applying Itô's formula to $N_{t\wedge\tau_{B^*}^{m^*}}$, we obtain:

$$\begin{aligned} dN_{t\wedge\tau_{B^*}^{m^*}} =& \mathbb{1}_{t<\tau_{B^*}^{m^*}} e^{-rt}\Big[\big(\mathcal{L}^- v(S_t,-) - rv(S_t,-)\big)\mathbb{1}_{(F_t=-)}dt \\ &+ \big(\mathcal{L}^+ v(S_t,+) - rv(S_t,+)\big)\mathbb{1}_{(F_t=+)}dt + \frac{\partial v}{\partial x}(S_t,F_t)\sigma_{F_t}(S_t)dW_t\Big]. \end{aligned} \tag{2.4.27}$$

Again, by (2.4.14) and (2.4.15), the drift terms vanish, which implies $\mathbb{E}^{x,f}[N_{t\wedge\tau_{B^*}^{m^*}}] = N_0$. Thus, $\mathbb{E}^{x,f}[N_{\tau_{B^*}^{m^*}}] = N_0$ by the Dominated Convergence Theorem.
(P3) It is sufficient to show $v(x,+) \geq u(x)$ on $[L, B^*]$ and $v(x,-) \geq u(x)$ on $[m^*, H]$. We start by proving the second inequality. Define $g(x) := v(x,-) - u(x)$. Now we further define $g_\epsilon(x) := g(x) + \epsilon u(x)$ for $\epsilon < 1$. Hence,

$$\mathcal{L}^- g_\epsilon - rg_\epsilon = (1-\epsilon)(ru - \mathcal{L}^- u) \geq 0.$$

Therefore, $\mathcal{L}^- g_\epsilon - rg_\epsilon \geq 0$ on $(m^*, H]$. Moreover, we have $g_\epsilon(m^*_+) = \epsilon u(m^*_+) \geq 0$, and $\frac{\partial g_\epsilon}{\partial x}\Big|_{x=m^*_+} = v'(m^*_+) - u'(m^*_+) + \epsilon u(m^*_+) \geq \epsilon u(m^*_+) > 0$. By the strong maximum principle (see, e.g. Theorem 6.2.3 in [32]), $g_\epsilon$ has no positive maximum on $(m^*, H)$. So $g_\epsilon$ must be increasing and hence positive. Let $\epsilon$ tend to 0 to see that $g \geq 0$. Therefore, $v(x,-) \geq u(x)$.
To show $v(x,+) \geq u(x)$ on $[L, B^*]$, we can now define $g$ by $g(x) := v(x,+) - u(x)$ so $g$ satisfies $\mathcal{L}^+ g - rg \leq 0$ on $(L, A]$ and $\mathcal{L}^+ g - rg \geq 0$ on $[A, B^*]$.
For $x \in [A, B^*]$, define $g_\epsilon(x) = g(x) + \epsilon\phi_+(x)$. Hence,

$$\mathcal{L}^+ g_\epsilon - rg_\epsilon = ru - \mathcal{L}^+ u \geq 0.$$

Therefore, $\mathcal{L}^+ g_\epsilon - rg_\epsilon$ is non-negative on $[A, B^*]$. Moreover, we have $g_\epsilon(B^*) = \epsilon\phi_+(B^*) > 0$, and $\frac{\partial g_\epsilon}{\partial x}\Big|_{x=B^*} = \phi'_-(B^*) < 0$. By the strong maximum principle, there

is no positive maximum of $g_\epsilon$ on $(A, B^*)$. So $g_\epsilon$ must be strictly decreasing and hence positive. Let $\epsilon$ tend to 0 to see $g \geq 0$ on $[A, B^*]$ .
Finally, for $x \in [L, A]$, we know $\mathcal{L}^+ g - rg \leq 0$ and $g(A) \geq 0$. We further notice that $v(L,+) = u(L)\mathbb{1}_{m^* \geq L} + v(L,-)\mathbb{1}_{m^*<L} \geq u(L)$ since we have shown that $v(x,-) \geq u(x)$. Thus, by the strong minimum principle, there is no negative minimum of $g$ on $(L, A)$, which ensures $g \geq 0$. So we conclude that $v(x,+) \geq u(x)$ on $[L, A]$. $\diamondsuit$

### 2.4.4 An example

Here we present a simple example where we calculate all quantities in closed form. Section 2.6 will present a numerical approach to treat more realistic examples.

**Example 2.4.12.** *Let $u(x) = x^\gamma$, $\gamma \in (0, 1)$. We take $\mu_-(x) = \mu_- x$, $\sigma_-(x) = \sigma_- x$, $\mu_+(x) = \mu_+(x + 1)$, $\sigma_+(x) = \sigma_+ x$, where $\mu_-$, $\mu_+$, $r$, $\sigma_-$ and $\sigma_+$ are all positive constants. It is not hard to see that Assumption 2.3.1, 2.3.3, 2.4.2 and 2.4.8 all hold. To have a closed form solution, we further assume $\mu_+ = \sigma_+^2 = r = c$ (later in Section 6 we will relax this assumption to obtain numerical solutions).*

*Thus, we can find the value function by solving the free boundary problem.*

*The ODE*

$$\mathcal{L}^- v - rv = \frac{1}{2}\sigma_-^2 x^2 v''(x) + \mu_- x v'(x) - rv(x) = 0,$$

*admits a general solution of the form $v(x,-) = C_3 x^\alpha + C_4 x^\beta$ where $\alpha$ and $\beta$ are the roots of $\frac{1}{2}\sigma_-^2 x^2 + (\mu_- - \frac{1}{2}\sigma_-^2)x - r = 0$. The ODE*

$$\mathcal{L}^+ v - rv = \frac{1}{2} r x^2 v''(x) + r(x + 1)v'(x) - rv(x) = 0$$

*has general solution $v(x,+) = C_1(x-1)e^{\frac{2}{x}} + C_2(x+1)$. Now, set $\gamma = 0.8$, $\mu_- = 1/30$, $\sigma_-^2 = 1/30$, $r = 0.1$, $L = 1$, and $H = 2$. After some calculations, one obtains $A = 20/7$ and $v(x,-) = C_3 x^{-3} + C_4 x^2$. Assume $m^* \geq 1$, we compute the value of $B^*$ by condition (2.4.16) and (2.4.18). Numerical approximation gives $B^* = 3.839282$ and $v_1(x,+) = 0.1075171(x-1)e^{\frac{2}{x}} + 0.5(x+1)$. Therefore by (2.4.17) and (2.4.19), we compute $m^* = 1.775502$ and $v_1(x,-) = 2.126333x^{-3} + 0.3816175x^2$. Thus, by Theorem 2.4.10, we derive the value function $\mathbf{V}(x, f)$:*

$$\mathbf{V}(x, f) = \begin{cases} 0.1075171(x-1)e^{\frac{2}{x}} + 0.5(x+1) & \textit{if } x \in (1, 3.839282),\ f = + \\ 2.126333x^{-3} + 0.3816175x^2 & \textit{if } x \in (1.775502, 2),\ f = - \\ x^{0.8} & \textit{otherwise.} \end{cases} \tag{2.4.28}$$

*The optimal strategy is to sell the stock when its price is higher than* 3.839282 *in the* *positive regime or lower than* 1.775502 *in the negative regime.*

## 2.5 The buyer's problem

If traders want to find the best time to purchase a stock and sell it later to maximise their *incremental* expected utility, they will try to solve the following double optimal stopping problem:

$$\mathbf{V}_p(x,f) := \sup_{\tau_1<\tau_2} \mathbb{E}^{x,f}[e^{-r\tau_2}u(S_{\tau_2}) - e^{-r\tau_1}u(S_{\tau_1})]. \tag{2.5.1}$$

In other words, we would like to maximize the marginal difference of the utility between buying and selling, where $\tau_1$ (resp. $\tau_2$) is interpreted as the buying (resp. selling) time. We call this the *buyer's problem.* By Lemma 2.7.1, the buyer's problem admits an equivalent formulation given by

$$\mathbf{V}_p(x,f) := \sup_{\tau} \mathbb{E}^{x,f}[e^{-r\tau}\{\mathbf{V}(S_\tau,F_\tau) - u(S_\tau)\}], \tag{BP}$$

where $\mathbf{V}$ is the value function from the seller's problem (SP). From now on, we work on the equivalent formulation (BP) .

Let $g(x,f) := \mathbf{V}(x,f) - u(x)$. Thus $g(x,f)$ is the gains function and for fixed $f$, it is a $C^1$ and piecewise $C^2$ function on $\mathbb{R}_+$, which has value 0 on $[0,m]\times\{-\}\cup[B,\infty)\times\{+\}$ and is positive elsewhere. We can see that $g$ satisfies Assumptions 2.3.1, 2.3.3 (see Lemma 2.7.2). Moreover, by Assumption 2.4.2, $g$ satisfies the following conditions

$$\mathcal{L}^+g(x,+) - rg(x,+) < 0 \quad \text{for} \quad x\in(L,A). \tag{2.5.2}$$

$$\mathcal{L}^+g(x,+) - rg(x,+) > 0 \quad \text{for} \quad x\in(A,B). \tag{2.5.3}$$

$$\mathcal{L}^-g(x,-) - rg(x,-) > 0 \quad \text{for} \quad x\in(m,H). \tag{2.5.4}$$

Let $D_p$ and $C_p$ denote the stopping set and continuation set for the buyer's problem. By Theorem 2.3.7, since Assumptions 2.3.1 and 2.3.3 hold, $\tau_{D_p}$ is an optimal stopping time.

### 2.5.1 The shape of the stopping set $D_p$

Analogously to the seller's problem, we define the stopping sets $D_p^+$ and $D_p^-$.

**Theorem 2.5.1.** *Under Assumptions 2.3.1, 2.3.3, and 2.4.2,* $D_p^+ = [a,b]$ *for some* $a,b$ *with* $L<a<b\le A$*. Moreover,* $D_p^- = \{0\}$ *if* 0 *is absorbing and* $D_p^- = \emptyset$ *if* 0 *is inaccessible.*

*Proof.* Suppose 0 is absorbing. If $S_0 = 0$, then $g(S_t, F_t) = 0$ for any $t$ and hence $V_p(0, -) = 0 = g(0, -)$, which implies $0 \in D_p^-$. If $x \in (0, m]$, set $\tau := \tau^-_{\frac{m+H}{2}} \wedge \tau_0^-$. Then, $\mathbb{E}[e^{-r\tau} g(S_\tau, F_\tau)] > 0 = g(x, -)$. So $D_p^- \cap (0, m] = \emptyset$. Moreover, for $x \in (m, H)$, we can define $\tau := \tau_m^- \wedge \tau_H^-$. Then by (2.5.4),

$$\mathbb{E}^{x,-}[e^{-r\tau} g(S_\tau, F_\tau)] = g(x, -) + \mathbb{E}^x[\int_0^\tau \mathcal{L}^- g(S_t^-, -) - rg(S_t^-, -)dt] > g(x, -). \tag{2.5.5}$$

Thus, $D_p^- \cap (m, H) = \emptyset$, and this shows $D_p^- = \{0\}$ if 0 is absorbing and $D_p^- = \emptyset$ if 0 is inaccessible.

For $D_p^+$, suppose it is empty. Then, $\tau_{D_p} = \tau_0$, which implies $\mathbf{V} = 0$ on $\mathbb{E}$. However, this contradicts the definition of $C_p^+$ because $g(x, +) > 0 = \mathbf{V}(x, +)$ for $x < B$. Thus, $\mathcal{D}_p^+ \neq \emptyset$.

Since $g(x, +) = 0$ in $[B, \infty)$, it is clear that $D_p^+ \cap [B, \infty) = \emptyset$. By (2.5.3), we can make a very similar argument to show $D_p^+ \cap (A, B) = \emptyset$. Now we claim $D^+$ is an interval. If it is not, then there exist $L < y_1 < y < y_2 \leq A$ such that $y_1, y_2 \in D_p^+$ and $y \in (y_1, y_2) \subset C_p^+$. Let $\tau := \tau_{y_1}^+ \wedge \tau_{y_2}^+$. It is clear that, by Theorem 2.3.7 and (2.5.2), we have

$$\begin{aligned} \mathbf{V}_p(y, +) &= E^{y,+}[e^{-r\tau} g(S_\tau, F_\tau)] \\ &= g(y, -) + \mathbb{E}^x[\int_0^\tau \mathcal{L}^+ g(S_t^+, +) - rg(S_t^+, +)dt] < g(y, -). \end{aligned} \tag{2.5.6}$$

By contradiction, $D_p^+$ must be an interval.
Now we claim that $L$ is not an endpoint of $D_p^+$. Suppose that $D_p^+$ is of the form $(L, b]$ for some $b \in (L, A]$ (we know $D_p^+$ is closed). Then $\lim_{x \to L} \mathbf{V}_p(x, +) = \lim_{x \to L} g(x, +) = g(L, -)$. However we also have

$$\mathbf{V}_p(x, +) \geq \mathbf{V}_p(L, -)\mathbb{E}^x[e^{-r\tau_L^+} \mathbb{1}_{\tau_L^+ < t_A^+}] + g(A, +)\mathbb{E}^x[e^{-r\tau_A^+} \mathbb{1}_{\tau_L^+ < t_A^+}]. \tag{2.5.7}$$

Letting $x$ decrease to $L$ in (2.5.7), we can conclude $\lim_{x \downarrow L} \mathbf{V}_p(x, +) \geq \mathbf{V}(L, -) > g(L, -)$ (since $L \notin \mathcal{D}_p^-$), which gives a contradiction. Therefore $D_p^+$ has to be of the form $[a, b]$, where $L < a < b \leq A$. $\diamondsuit$

Thus, the trader should only buy the stock when it is in the positive regime and its price falls into the interval $[a, b]$.

Note that $b > L$. If the initial position $x$ is greater than $b$ and the stock is in the positive regime, then we will stop before $S$ hits $L$, which means no regime transition can occur. This contradicts the TA principle of "buy low" and makes the value of $\mathbf{V}_p$ on $[b, \infty)$ easy to compute.

**Lemma 2.5.2.** $\mathbf{V}_p(x,+) = \frac{g(b,+)}{\phi_+(b)}\phi_+(x)$ *for* $x \in [b,\infty)$.

*Proof.* Let $x \in [b,\infty)$. Then, $\mathbf{V}_p(x,+) = \mathbb{E}^{x,+}[e^{-r\tau_b} g(S_{\tau_b}, F_{\tau_b})] = \mathbb{E}^x[e^{-r\sigma_b} g(S^+_{\sigma_b},+)]$ $= g(b,+)\mathbb{E}^x[e^{-r\sigma_b}]$. By the strong Markov property, $\phi_+(x) = \mathbb{E}^x[e^{-r\sigma_b}]\phi_+(b)$. Thus, $\mathbf{V}_p(x,+) = \frac{g(b,+)}{\phi_+(b)}\phi_+(x)$. ♢

### 2.5.2 The value function

Analogously to the seller's problem, the free boundary problem corresponding to the buyer's problem is as follows. We look for a pair of functions $v(x,f) \in C^2$, constants $\hat{a} \geq L$ and $\hat{b} \in (\hat{a}, A]$, such that

$$\mathcal{L}^+ v_p(\cdot,+) - r v_p(\cdot,+) = 0, \text{ in } (L,\hat{a}) \tag{2.5.8}$$

$$\mathcal{L}^- v_p(\cdot,-) - r v_p(\cdot,-) = 0, \text{ in } (0,H) \tag{2.5.9}$$

$$v_p(\hat{a},+) = g(\hat{a},+), \quad v_p(L,+) = v_p(L,-) \tag{2.5.10}$$

$$v_p(H,-) = k(\hat{a},\hat{b}), \quad v_p(0,-) = 0, \tag{2.5.11}$$

$$\frac{\partial v_p}{\partial x}(x,+)\Big|_{x=\hat{a}} = g'(\hat{a},+), \tag{2.5.12}$$

$$\frac{\partial}{\partial x}\frac{g(\hat{b},+)}{\phi_+(\hat{b})}\phi_+(x)\Big|_{x=\hat{b}} = g'(\hat{b},+), \tag{2.5.13}$$

where $k(\hat{a},\hat{b}) := v_p(H,+)\mathbb{1}_{\{H\leq\hat{a}\}} + g(H,+)\mathbb{1}_{\{\hat{a}<H\leq\hat{b}\}} + \frac{g(\hat{b},+)}{\phi_+(\hat{b})}\phi_+(H)\mathbb{1}_{\{\hat{b}<H\}}$.

We are now ready to present the main result in this section: the solution to the buyer's problem.

**Theorem 2.5.3.** *Under Assumption 2.3.1, 2.3.3, 2.4.2, and 2.4.8, the free boundary problem defined via conditions (2.5.8) to (2.5.13) admits a solution* $v_p$, $a^*$, *and* $b^*$. *Moreover,* $V_p(x,f)$ *defined by (2.5.14) is equal to the value function* $\mathbf{V}_p(x,f)$ *and* $\tau := \tau^+_{[a^*,b^*]}$ *is the optimal stopping time in (BP).*

$$V_p(x,f) = \begin{cases} v_p(x,f;a^*,b^*) & \text{if } x \in (L,a^*],\ f=+ \text{ or } x \in [0,H),\ f=-, \\ g(x,+) & \text{if } x \in (a^*,b^*],\ f=+, \\ \frac{g(b^*,+)}{\phi_+(b^*)}\phi_+(x) & \text{if } x \in (b^*,\infty),\ f=+. \end{cases} \tag{2.5.14}$$

*Proof.* By Theorem 2.3.8, 2.3.9, and 2.5.1, $\mathbf{V}_p$ is indeed a solution to the free boundary problem.

Conversely, let $(x,f) \in E\setminus\{(0,-)\}$ and assume we have a solution denote by $v_p$, $a^*$, and $b^*$ to the free boundary problem. Define $N_t := e^{-rt}V_p(S_t,F_t)$. According to Lemma 2.4.11, to show $V_p$ is the value function, it is sufficient to prove $N$ satisfies P1-P3.

(P1) $|V_p|$ is bounded by some constant $K$. So $\mathbb{E}^{x,f}[\sup_\tau N_\tau] \leq K$, which implies class D. Using Peskir's change-of-variable formula with local time (see [54]), as $S_t$ is a continuous semimartingale and $V_p(x, f)$ is a piecewise $C^2$ function of $x$ given f, it follows That

$$\begin{aligned} dN_t =& e^{-rt}\Big[\big(-rV_p(S_t,+)+\mathcal{L}^+V_p(S_t,+)\big)\mathbb{1}_{\{F_t=+,S_t\neq a^*\}}\mathbb{1}_{\{F_t=+,S_t\neq b^*\}}dt \\ &+\big(-rV_p(S_t,-)+\mathcal{L}^-V_p(S_t,-)\big)\mathbb{1}_{\{F_t=-\}}dt \\ &+\frac{\partial V_p}{\partial x}(S_t,F_t)\sigma_{F_t}(S_t)\mathbb{1}_{(S_t,F_t)\notin\{(a^*,+),(b^*,+)\}}dW_t \\ &+\frac{1}{2}\big(\frac{\partial V_p}{\partial x}(a^{*+},+;a^*,b^*)-\frac{\partial V_p}{\partial x}(a^{*-},+;a^*,b^*)\big)\mathbb{1}_{F_t=-}dl_t^{a^*}(S) \\ &+\frac{1}{2}\big(\frac{\partial V_p}{\partial x}(b^{*+},+;a^*,b^*)-\frac{\partial V_p}{\partial x}(b^{*-},+;a^*,b^*)\big)\mathbb{1}_{F_t=+}dl_t^{b^*}(S)\Big]. \end{aligned} \tag{2.5.15}$$

By the smooth pasting principle (2.5.12) and (2.5.13), the local time terms in (2.5.15) disappear. Recall $v_p$ satisfies (2.5.8) and (2.5.13). By the construction of $V_p$, equation (2.5.15) becomes

$$\begin{aligned} dN_t^i = e^{-rt}\Big[&\big(\mathcal{L}^+g(S_t,+)-rg(S_t,+)\big)\mathbb{1}_{(F_t=+,a^*<S_t<b^*)}dt \\ &+\frac{\partial V_p}{\partial x}(S_t,F_t)\sigma_{F_t}(S_t)\mathbb{1}_{(S_t,F_t)\notin\{(a^*,+),(b^*,+)\}}dW_t\Big]. \end{aligned} \tag{2.5.16}$$

Since $\mathcal{L}^+g(x,+) - rg(x,+) < 0$ on $[a^*, b^*]$, we can conclude the drift terms are non-positive. Moreover, by smoothness of $v_p$ and $g$, we have $\frac{\partial V_p}{\partial x}(S_t, F_t)\mathbb{1}_{(S_t,F_t)\notin\{(a^*,+),(b^*,+)\}}$ is locally bounded. We also have $\sigma_\pm$ is bounded locally. These together imply that the $dW_t$ term is a local martingale. Thus, we conclude that $N_t$ is a local supermartingale. Since $N$ is class D, it is also a supermartingale.

(P2) Define $\tau_{b^*}^{a^*} := \inf\{t \geq 0; S_t \leq a^* \text{ if } F_t = + \text{ or } S_t \geq b^* \text{ if } F_t = +\}$. Suppose for the initial position $(x, f)$ we have that $x \leq a^*$ or $x \geq b^*$ when $f = +$. Applying Ito's formula to $N_{t\wedge\tau_{b^*}^{a^*}}$, we can get

$$\begin{aligned} dN_{t\wedge\tau_{b^*}^{a^*}} =& \mathbb{1}_{t<\tau_{b^*}^{a^*}}e^{-rt}\Big[\big(\mathcal{L}^-v_p(S_t,+)-rv_p(S_t,+)\big)\mathbb{1}_{\{F_t=+,S_t\leq a\}}dt \\ &+\big(\mathcal{L}^-v_p(S_t,-)-rv_p(S_t,-)\big)\mathbb{1}_{\{F_t=-\}}dt \\ &+\big(\mathcal{L}^+\frac{g(b^*,+)}{\phi_+(b^*)}\phi_+(S_t)-r\frac{g(b^*,+)}{\phi_+(b^*)}\phi_+(S_t)\big)\mathbb{1}_{\{F_t=+,S_t\geq b\}}dt \\ &+\big(\frac{\partial v_p}{\partial x}(S_t,F_t)\sigma_{F_t}(S_t)\mathbb{1}_{S_t\leq a}+\frac{g(b^*,+)}{\phi_+(b^*)}\phi_+'(S_t)\mathbb{1}_{S_t\geq b}\big)dW_t\Big]. \end{aligned}$$

Again by (2.5.8) and (2.5.9), the drift terms vanish, which implies $\mathbb{E}^{x,f}[N_{t\wedge\tau_{b^*}^{a^*}}] = N_0$ as the expectation of the $dW_t$ term should be 0. Let $t$ tend to infinity to see that $\mathbb{E}^{x,f}[N_{\tau_{b^*}^{a^*}}] = N_0$ by dominated convergence theorem. For other initial positions

$(x, f)$, it is trivial that $N_{\tau_{b^*}^{a^*}} = N_0$, which also leads to $\mathbb{E}^{x,f}[N_{\tau_{b^*}^{a^*}}] = N_0$.
(P3) It is sufficient to show $v_p(x,-) \geq g(x,-)$ on $[m, H]$, $v_p(x,+) \geq g(x,+)$ on $[L, a^*]$, and $\frac{g(b^*,+)}{\phi_+(b^*)}\phi_+(x) \geq g(x,+)$ on $[b^*, B]$. We start with proving the third inequality. Define $h(x) := \frac{g(b^*,+)}{\phi_+(b^*)}\phi_+(x) - g(x,+)$, and we have $\mathcal{L}^+h - rh \geq 0$ on $[b^*, A]$ and $\mathcal{L}^+h - rh \leq 0$ on $[A, B]$. Now we further define $h_\epsilon(x) := h(x) + \epsilon\psi_+(x)$. Hence

$$\mathcal{L}^+h_\epsilon - rh_\epsilon = \mathcal{L}^+h - rh \geq 0 \text{ on } [b^*, A].$$

By the strong maximum principle, there is no positive maximum on $(b^*, A)$. Because $h_\epsilon(b) > 0$ and $h'_\epsilon(b) > 0$, $h_\epsilon$ must be strictly increasing and hence positive. Let $\epsilon$ tend to 0 to see $h \geq 0$. Therefore, $\frac{g(b^*,+)}{\phi_+(b^*)}\phi_+(x) \geq g(x)$. Moreover, as $\mathcal{L}^+h - rh \leq 0$ on $[A, B]$, by the strong minimum principle, there is no negative minimum on $(A, B)$. As $h(A) \geq 0$ and $h(B) > 0$, we can see $h(x) \geq 0$ on $[A, B]$.
To show $v_p(x,+) \geq g(x,+)$ on $[L, a^*]$, we first define $h(x) := v_p(x,+) - g(x,+)$. Then, further define $h_\epsilon(x) := h(x) + \epsilon\phi_+(x)$. Hence we have

$$\mathcal{L}^+h_\epsilon - rh_\epsilon = \mathcal{L}^+h - rh \geq 0 \;\text{ on } [L, a^*]$$

By the strong maximum principle, there is no positive maximum on $(b^*, A)$. Moreover, $h_\epsilon(a) > 0$ and $h'_\epsilon(a) < 0$, we must have $h_\epsilon$ being strictly increasing and hence positive. Let $\epsilon$ tend to 0 to see $h \geq 0$.
Finally, we can define $h(x) := v_p(x,-) - g(x,-)$. Then, $\mathcal{L}^-h_\epsilon - rh_\epsilon \leq 0$ on $[m, H]$. By strong minimum principle, there is no negative minimum on $(m, H)$. However, since $h(H) = v_p(H,+) - g(H,+) \geq 0$ and $h(m) \geq v_p(m,-) > 0$, $h$ must stay non-negative, i.e. $h(x) \geq 0$ on $[m, H]$.

$\diamondsuit$

### 2.5.3 Example 2.4.12 revisited

**Example 2.5.4.** *Recall Example 2.4.12. We now solve the purchase problem. By Theorem 2.5.3, we can compute $a^* = 1.1632$ and $b^* = 2.1686$. Thus, the value function is given below.*

$$\mathbf{V}_p(x, f) = \begin{cases} 0.0408(x-1)e^{\frac{2}{x}} + 0.0138(x+1) & \text{if } x \in (1, 1.1632),\ f = + \\ 0.1075(x-1)e^{\frac{2}{x}} + 0.5(x+1) - x^{0.8} & \text{if } x \in [1.1632, 2.1686],\ f = + \\ -0.1858(x-1)e^{\frac{2}{x}} + 0.1858(x+1) & \text{if } x \in (2.1686, \infty),\ f = + \\ 0.0277x^2 & \text{if } x \in (0, 2),\ f = - \end{cases} \tag{2.5.17}$$

## 2.6 Optimal trading strategies and degrees of relative risk aversion

In the preceding we identified four price levels, namely $B, m, b, a$, which together determine the optimal trading strategies. In this section, we explore the relation between these price levels and degrees of relative risk aversion. To do this, we first need to implement an algorithm which allows us to estimate these stopping boundaries numerically.

### 2.6.1 The numerical algorithm

The algorithm needs to numerically solve two free boundary problems defined via (2.4.14)-(2.4.19) and (2.5.8)-(2.5.13). Essentially, for each free boundary problem, we need to solve two linear second order ODEs that are linked via boundary conditions where the boundaries are estimated simultaneously. The numerical methods for solving an ODE are well established. For each iteration with a different boundary value, we numerically solve the corresponding boundary value problem (BVP) and check the smooth pasting condition. In the case where the boundary condition is given by the solution of the other BVP (e.g. $v(L,+) = v(L,-)$), we have to loop through different values of the boundary, and in every iteration, we numerically solve the linked BVPs with the aid of the smooth pasting conditions and check the boundary condition.

Figure 1 provides more details about the algorithm implemented for the seller's problem. Essentially, we begin with the assumption that $m \geq L$ so that the BVP for the positive regime can be solved in isolation, which gives $v(x,+)$. Then, we compute $v(x,-)$ and check whether $m \geq L$. If not, then we have to set initially the boundary condition $v(L,\pm) = u(L)$ and continue the computation as Figure 1 indicates until the boundary condition $v(H,+) = v(H,-)$ is (approximately) satisfied. Whenever "compute" appears in Figure 1, we mean numerically solve the related BVPs.

In the following sections, we take the utility function to be a power function of the form $u(x) = x^\gamma$ and the dynamics in the negative regime are of the form $\mu_-(x) = \mu_- x$ and $\sigma_-(x) = \sigma_- x$.

### 2.6.2 Affine drift with linear volatility

Recall the example studied in Section 2.4.4 and 2.5.3 where we have parameters $\mu_\pm, \sigma_\pm$ and $c$. Table 1 summarises the values of all parameters. Note the assumption $\mu_+ = \sigma_+^2 = r = c$ is relaxed. We vary $\gamma$ between 0.7 and 0.95. Then, we can compute $B, m, b, a$ accordingly. The results are plotted in Figure 2 where the solid horizontal line is the reference line representing $L = 1$.

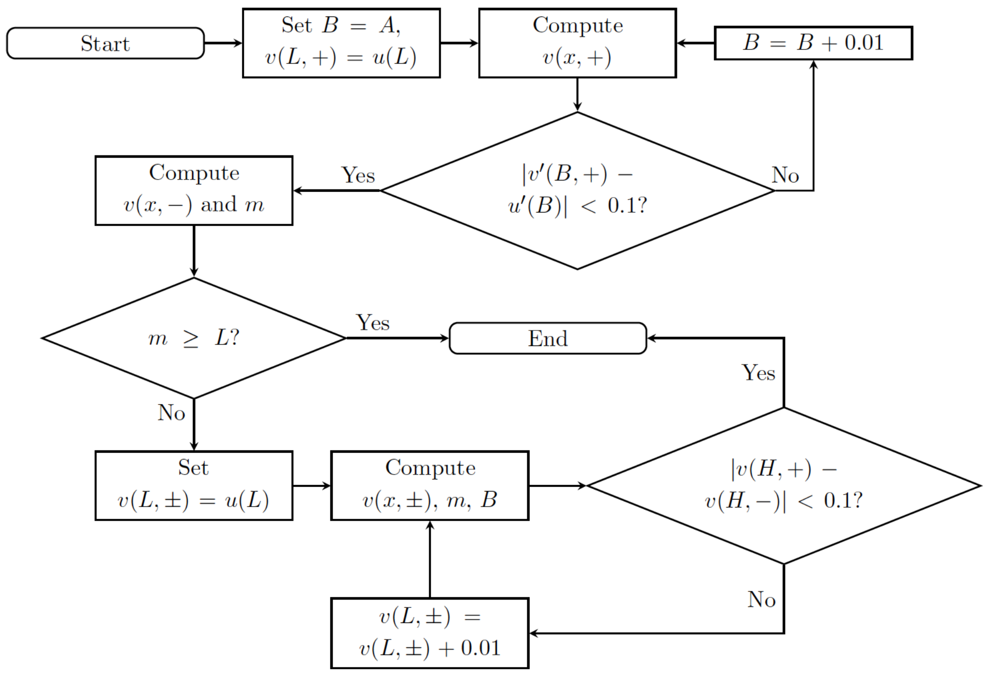


Figure 1: Flow chart of the algorithm used to solve the seller's problem.

Table 1: Parameter values.

| $\mu_+$ | $\mu_-$ | $\sigma_+^2$ | $\sigma_-^2$ | $c$ | $r$ | $L$ | $H$ |
|---|---|---|---|---|---|---|---|
| 0.15 | 1/30 | 0.1 | 1/30 | 0.16 | 0.15 | 1 | 2 |

We observe some patterns in Figure 2. For the seller's problem, as $\gamma$ increases, $B$ increases exponentially fast, and $m$ decreases roughly linearly and never drops below $L$. Moreover, for the buyer's problem, as $\gamma$ increases, $b$ increases exponentially fast and $a$ increases approximately linearly. These patterns can be explained qualitatively. Since $1-\gamma$ is equal to the relative risk aversion for the power utility, as $\gamma$ increases, the degree of risk aversion decreases. Therefore, as $\gamma$ increases, traders are content to take more risk, which implies optimally traders shall sell at a higher profit-taking boundary $B$, or a lower stop-loss boundary $m$. Similarly, the increase of $b$ and decrease of $a$ are easily understood.

From Figure 2, we can clearly see how the magnitude of the trader's risk preference influences the optimal trading strategies. When a trader is risk-neutral, as Jacka and Maeda [38] observe, the profit-taking boundary $B$ is equal to infinity. Then, as the trader becomes more risk-averse, $B$ quickly decreases to a comparably small level . On the other hand, the stop-loss boundary $m$ is relatively stable with

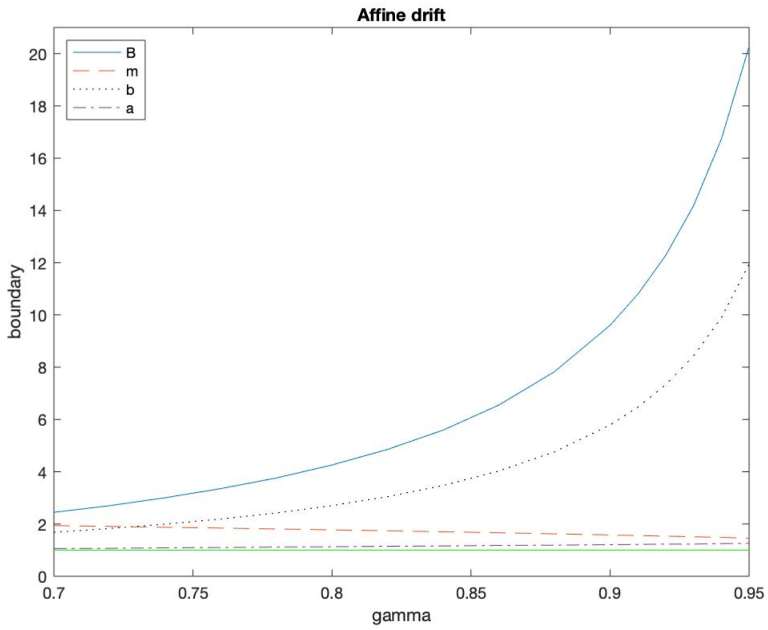


Figure 2: Values of $B, m, b, a$ against $\gamma$.

regard to the change of risk aversion.

We are now at a good point to explain the relation between the standard trading rule (recall trading strategies BL and SH from Section 1) and the optimal solution derived in our model. If traders follow BL (SH), they should buy (sell) at a price between 1 and 2 in the positive (negative) regime. Assuming that the trader's $\gamma$ is below 0.75, we can see that, since $a$ is close to 1 and $b$ is close to 2, BL would result in the trader behaving approximately optimally. However, in the negative regime, the optimal trading strategy looks very different from SH, because the trader optimally sells at any price below $m$ (instead of only at points between 1 and 2) and does not sell at price levels above $m$. Hence, there is disagreement between the standard trading rule and the optimal trading strategy under this model.

How should we understand this? Technical analysts think of a resistance line as a *reflecting boundary*, i.e. the price *must* go down once it reaches this price level. This naturally leads to arbitrage opportunities — there is no EMM for such a model— it would clearly be optimal to sell the stock at the resistance line since the price is guaranteed to go no lower and may rise. However, our model does not permit arbitrage. (In Chapter 3 we will extend the model to include such a partial reflection boundary). Our stock dynamics model the idea of the resistance line being *broken-through* (the old resistance line becomes the new support after it is broken from below), which is compatible with technical analysis. This explains why sell-at-high

Table 2: Parameter values.

| $\mu_+$ | $\mu_-$ | $\sigma_+^2$ | $\sigma_-^2$ | $c$ | $r$ | $L$ | $H$ |
|---|---|---|---|---|---|---|---|
| 0.1 | 1/30 | 0.1 | 1/30 | 0.7 | 0.1 | 1 | 2 |

is not the best trading strategy in our model. Indeed, no matter how strong the downwards drift around the resistance line, Theorem 2.4.10 tells us that it is optimal not to sell when the price is near $H$ in the negative regime.

We can provide another viewpoint on the inconsistency raised from the negative regime. Since traders know the possibility of a break-through in our model, the optimal strategy takes this into account. When the stock price goes above $m$ in the negative regime, it is likely that there will be a break-through. Hence, the current resistance can be thought of as the future support, which means "not selling" by SH because the trader now believes the stock is soon going to enter the positive regime.

Our analysis provides some evidence that, under the possibility of a break-through, the standard trading rule is not optimal. So, when contemplating the use of technical analysis, the standard trading rule should be used with caution.

### 2.6.3 Mean-reverting models: Vasicek and CIR

We would like to investigate how our findings in the previous section change when the drift becomes mean-reverting. In this section we consider the Vasicek [66] and Cox, Ingersoll and Ross [12] (CIR) models. The drifts of Vasicek and CIR are both of the form $c - \mu_+ x$, and the volatilities are of the form $\sigma_+$ and $\sigma_+\sqrt{x}$ respectively, for some positive constants $\mu_+, \sigma_+$ and $c$. We can check that (2.5.2)-(2.5.4) have all been met for our choice of parameters listed in Table 2. We will allow the value of $\gamma$ to be greater than 1 (i.e. traders become risk-seeking). We can do this because concavity is not needed for Theorem 2.4.10 and 2.5.3 to be valid (cf. Remark 2.4.5). Figure 3 and 4 present the results for the Vasicek and CIR models, respectively, and vary $\gamma$ between 0.5 and 1.5.

Comparing to Figure 2, the main pattern (e.g. the signs of the slopes of boundaries) is preserved, which makes our previous analysis regarding the standard trading rule more evident. It is also worth noticing that Figure 3 and 4 from the two mean-reverting models are very similar.

On the other hand, it is clear that there are also a few differences. Firstly, the slopes of $B$ and $b$ decrease as $\gamma$ increases. Intuitively, since the mean-reverting drift would push the stock price down with increasing force as the stock price increases, which means the risk associated with waiting for a higher selling boundary $B$ (less

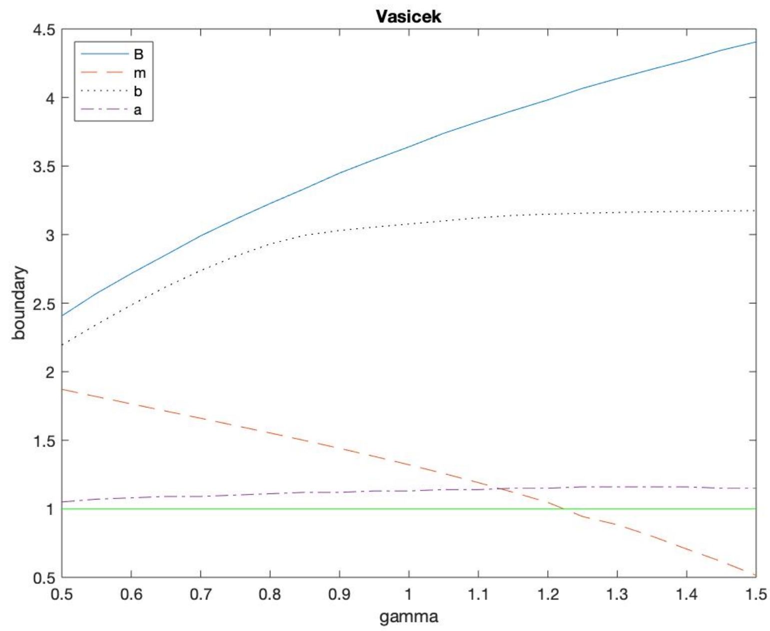


Figure 3: Values of $B, m, b, a$ against $\gamma$ for the Vasicek [66] model.

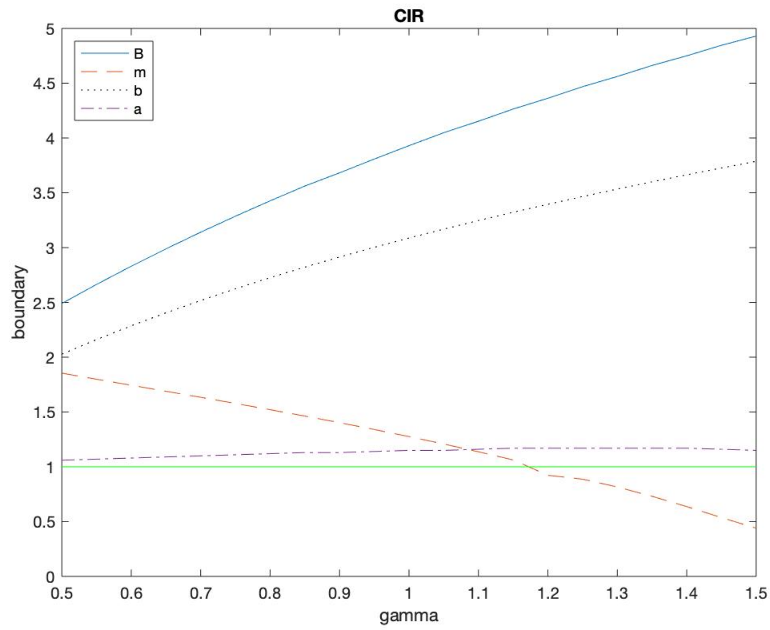


Figure 4: Values of $B, m, b, a$ against $\gamma$ for the CIR [12] model.

chance of getting there) or buying at a higher price $b$ (greater chance of making a loss) is much greater than in the model with affine drift. This makes the trader increasingly less willing to increase $B$ or $b$ for each smaller (and eventually negative) degree of relative risk aversion, which results in the concavity seen in Figures 3 and 4.

Secondly, in contrast to the affine drift model, the values of $B$ or $b$ do not tend to infinity, even for $\gamma$ close to, or greater than one. Mathematically, Theorem 2.4.7 and 2.5.1 together show, under the existence of $A < \infty$, we must have $b \leq A \leq B < \infty$ (the finiteness of $B$ is proved in Theorem 2.4.7).

Finally, the value of $m$ drops below reference line $L = 1$ at $\gamma \geq 1.25$. This is not observed in Figure 2 for the affine drift model. Moreover, there is a kink for $m$ for $\gamma$ around 1.25. This is because the boundary condition changes substantially for $m < L$ as Figure 1 shows. Essentially, when $m < L$, the trader would continue to hold the stock when the price process transitions from the positive to the negative regime. From Figure 3 and 4, this happens only in the case where traders are risk-seeking (i.e. $\gamma > 1$), which suggests (at least under our modelling and specifications) waiting for a break-through from the negative to the positive regime is a very risky strategy and should be avoided by risk-averse traders.

## 2.7 Proofs and additional results

### 2.7.1 Additional lemmas

**Lemma 2.7.1.** $\mathbf{V}_p$ *defined by (2.5.1) has an equivalent formulation as*

$$\mathbf{V}_p(x, f) := \sup_{\tau} \mathbb{E}^{x,f}[e^{-r\tau}\{\mathbf{V}(S_\tau, F_\tau) - u(S_\tau)\}], \tag{2.7.1}$$

*where* $\mathbf{V}$ *is given by (SP).*

*Proof.* Note that

$$e^{-r\tau}\mathbf{V}(S_\tau, F_\tau) = \operatorname*{ess\,sup}_{\tau_0 \geq \tau} \mathbb{E}[e^{-r\tau_0}u(S_{\tau_0})|\mathcal{F}_\tau].$$

Let $Z^\tau = \mathbb{E}[e^{-r\tau}u(S_\tau)|\mathcal{F}_{\tau_1}]$ and $Z^* := \operatorname{ess\,sup}_{\tau \geq \tau_1} \mathbb{E}[e^{-r\tau}u(S_\tau)|\mathcal{F}_{\tau_1}]$. It is sufficient to show that

$$\sup_{\tau_1 \leq \tau_2} \mathbb{E}[Z^{\tau_2}] \geq \sup_{\tau_1} \mathbb{E}[Z^*]. \tag{2.7.2}$$

For arbitrary stopping times $\tau \geq \tau_1$ and $\sigma \geq \tau_1$, define stopping time $\tau^0$ by

$$\tau^0 = \tau \mathbb{1}_{Z^\tau \geq Z^\sigma} + \sigma \mathbb{1}_{Z^\tau < Z^\sigma}. \tag{2.7.3}$$

Hence, $Z^{\tau^0} = \mathbb{E}[e^{-r\tau}u(S_\tau)\mathbb{1}_{Z^\tau \geq Z^\sigma} + e^{-r\sigma}u(S_\sigma)\sigma\mathbb{1}_{Z^\tau < Z^\sigma}|\mathcal{F}_{\tau_1}] \geq \max\{Z^\tau, Z^\sigma\}$. Thus, there is an sequence of stopping times $\sigma_n$ such that $Z^{\sigma_n}$ increases to $Z^*$. Moreover, since

$$\mathbb{E}[|Z^*|] \leq \mathbb{E}[\operatorname*{ess\,sup}_{\tau \geq \tau_1} \mathbb{E}[|e^{-r\tau}u(S_\tau)||\mathcal{F}_{\tau_1}]]] \leq \mathbb{E}[\sup_t |e^{-rt}u(S_t)|] < \infty, \tag{2.7.4}$$

by monotone convergence theorem, we conclude

$$\sup_{\tau_1} \mathbb{E}[Z^*] = \sup_{\tau_1} \lim_{n\to\infty} \mathbb{E}[Z^{\sigma_n}] \leq \sup_{\tau_1} \sup_{\tau_2 \geq \tau_1} \mathbb{E}[Z^{\tau_2}] = \sup_{\tau_1 \leq \tau_2} \mathbb{E}[Z^{\tau_2}]. \tag{2.7.5}$$

$\diamondsuit$

**Lemma 2.7.2.** *$g(x, f) := \mathbf{V}(x, f) - u(x)$ satisfies Assumptions 2.3.1, 2.3.3.*

*Proof.* $u$ is of $C^2$ on $(0, \infty)$. Moreover, by Theorem 2.4.10, $\mathbf{V}$ is $C^2$ on $C \cup int(D)$ and $C^1$ on $E \setminus (0, -)$. Therefore, Assumption 2.3.1 holds for $g$.

Since $\mathbf{V} - u$ is continuous with a compact support, $g$ is bounded by some constant $K$. Hence,

$$0 \leq \mathbb{E}^{x,f}\left[\sup_{t\geq 0} e^{-rt}g(S_t, F_t)\right] \leq \mathbb{E}^{x,f}\left[\sup_{t\geq 0} e^{-rt}K\right] \leq K < \infty. \tag{2.7.6}$$

$\diamondsuit$

### 2.7.2 Proof of Theorem 2.2.3

*Proof of Theorem 2.2.3.* The proof constructs the resolvent for $(S, F)$ via the obvious iteration scheme and then deduces the other properties from those of the resolvent.

For ease of notation, we denote the unique element of $(+, -) \setminus \{f\}$ by $-f$ and temporarily relabel $L$ and $H$ by $F^-$, $F^+$ respectively.

Denote the resolvent of $S^f$ (killed at $F^{-f}$) by $R^f_\lambda$ and define $R^{f,n}_\lambda$ (acting on $C_b(\mathbb{R} \cup \{\partial\}, \mathbb{R})$) inductively by

$$R^{f,1}_\lambda = R^f_\lambda; \;\; R^{f,n+1}_\lambda g(x) = R^f_\lambda g(x) + (1 - \lambda R^f_\lambda 1(x))R^{-f,n}_\lambda g(F^{-f}), \tag{2.7.7}$$

where $1(x) := I_{(x \neq \partial)}$. It should be clear that $R^{f,n}$ corresponds to a process which looks like the desired $S$ but dies on the $n$th regime switch.

Now we take limits in $n$ in equation (2.7.7)

$$\tilde{R}^f := \lim_{n\to\infty} R^{f,n}.$$

The limit is guaranteed to exist since $0 \leq (1 - \lambda R^f_\lambda 1(F^{-f})) < 1$

Now define $\bar{R}$ by

$$\bar{R}g(\cdot, f) = \tilde{R}^f g(\cdot, f) \text{ for } f = \pm.$$

It is easy to check from this definition that $\bar{R}$ is a contraction resolvent on $C_b(\mathbb{E}, \mathbb{R})$ (see [68] III.4) and is the unique contraction resolvent $T$ on $C_b(\mathbb{E}, \mathbb{R})$ (bounded continuous functions from $\mathbb{E}$ to $\mathbb{R}$) satisfying

$$T_\lambda g(x, f) = R^f_\lambda g(x, f) + (1 - \lambda R^f_\lambda 1(x)) T_\lambda g(F^f, -f). \tag{2.7.8}$$

It follows from the definition that

$$\bar{R}_\lambda 1(F^f) = R^f_\lambda 1(F^f) + \big(1 - \lambda R^f_\lambda 1(F^f)\big)\Big(R^{-f}_\lambda 1(F^{-f}) + (1 - \lambda R^{-f}_\lambda 1(F^{-f}))\bar{R}_\lambda 1(F^f)\Big), \tag{2.7.9}$$

and substituting into (2.7.8) we see that $\lambda \bar{R}_\lambda 1(F^f) = 1$. It follows from (2.7.8) that

$$\lambda \bar{R} 1_E(x, f) = 1 \text{ for all } (x, f) \in \mathbb{E},$$

and so $\bar{R}$ is conservative.

To show that $\bar{R}$ is the resolvent of a conservative transition semigroup $(P_t)_{t \geq 0}$, it remains to show that $\lambda \bar{R}$ is positive and a contraction on $C_b(\mathbb{E}, \mathbb{R})$ (equipped with the sup-norm, $|| \cdot ||_\infty$). Positivity follows immediately from the positivity of $\mathbb{R}^f$ and the iteration (2.7.7). Contractivity follows from the contractivity of $R^f$ by induction and the fact that $||R^f g||_\infty \leq ||g||_\infty ||R^f 1||_\infty$ (which follows from positivity).

Using the identity

$$\Gamma g = \lim_{\lambda \to \infty} \lambda(\lambda R_\lambda - I) g,$$

where $R$ is a resolvent corresponding to the infinitesimal generator $\Gamma$ and $g \in \mathcal{D}(\Gamma)$ (see (4.12) p111 in [68]), it is easy to deduce from (2.7.9) that the infinitesimal generator of the semigroup is $\mathcal{L}$ and hence that $(S, F)$ satisfies (2.2.1) and (2.2.2).

To show that $(P_t)$ is Feller, it remains (see p 166 of [26]) to show that the transition semigroup is strongly continuous, but this follows from the strong continuity of the semigroups for $S^f$, $f = \pm$.

Theorem 2.7 of Ch.4 of [26] now tells us that $(S, f)$ is strong Markov.

The càdlàg property follows from the fact that $S^f$ is continuous up to its death time (the first hitting time of $F^{-f}$), and so the time for $F_t$ to jump is always strictly positive unless $(S_{t-}, F_{t-}) = (F^{-f}, f)$, while these times cannot cluster since $(P_t)$ is conservative.

Finally, the desired regularity follows from that of $S^f$, $f = \pm$ $\diamondsuit$

### 2.7.3 Proof of Theorem 2.4.6

We shall appeal to the following result which is mentioned in [37] but for which no proof was given.

**Theorem 2.7.3.** *Suppose that $Z$ is a continuous, positive process adapted to $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$. Define $S$ to be the running maximum of $Z$ so that*

$$S_t := \sup_{s \leq t} Z_s$$

*Further suppose that there is a $p > 1$ and a sequence of stopping times $T_n \uparrow \infty$ a.s.such that*

$$\lim_n \sup_{\tau \leq T_n} \mathbb{E}[Z_\tau^p] < \infty, \tag{2.7.10}$$

*then*

$$\mathbb{E}[S_\infty] < \infty.$$

*Proof.* For each $x \in (0, \infty)$ define

$$\tau_x^n = \min(\inf\{t \leq T_n : \quad Z_t \geq x\}, T_n).$$

Clearly $(S_{T_n} \geq x) = (Z_{\tau_x^n} \geq x)$.

Take $C :\ \sup_{\tau \leq T_n} \mathbb{E}[Z_\tau^p] \leq C$ for all $n$ (which we can, since the limit in (2.7.10) is a monotone one) then, by Markov's inequality,

$$\mathbb{P}(S_{T_n} \geq x) = \mathbb{P}(Z_{\tau_x^n} \geq x) \leq \frac{\mathbb{E}[Z_{\tau_x^n}^p]}{x^p} \leq \frac{C}{x^p}. \tag{2.7.11}$$

Now, using the standard result that, if $X$ is a non-negative random variable,

$$\mathbb{E}[X] = \int_0^\infty \mathbb{P}(X \geq x) dx,$$

we deduce from (2.7.11 that

$$\mathbb{E}[S_{T_n}] \leq 1 + \int_1^\infty \frac{C}{x^p} dx = 1 + \frac{C}{p-1}.$$

Since $S$ is an increasing process, we see, by the Monotone Convergence Theorem, that

$$\mathbb{E}[S_\infty] \leq 1 + \frac{C}{p-1}$$

$\diamondsuit$

*Remark* 2.7.4. Continuity is not actually required in Theorem 2.7.3. The argument only needs small modifications if $Z$ is just predictable, on appealing to the Predictable Section Theorem.

*Proof of Theorem 2.4.6.* We define a positive, continuous process $Z$ by

$$Z_t = e^{-rt}u(S_t),$$

and set $Q = Z^p$. Then, from Ito's Lemma we see that

$$dQ_t = e^{-prt}pu(S_t)^{p-1}\left((\mathcal{L}^{F_t} - r)u(S_t) + \frac{1}{2}(p-1)\frac{(\sigma_{F_t}u'(S_t))^2}{u(S_t)}\right)dt + dN_t, \quad (2.7.12)$$

where $N$ is a local martingale. Since $u$ is $C^2$ by assumption, denoting $\{(x,f) \in E^- \times \{-\} \cup (L,M] \times \{+\}\}$ by $E^M$,

$$\sup_{E^M}\left((\mathcal{L}^{F_t} - r)u(S_t) + \frac{1}{2}(p-1)\frac{(\sigma_{F_t}u'(S_t))^2}{u(S_t)}\right) = \kappa_M < \infty,$$

for any $M \geq A$. Now, thanks to (2.4.5) and (2.4.6), we may take $M$ such that $\sup_{x\geq M}\left(\frac{\sigma_+(x)u'(x)}{u(x)}\right)^2 = D$ and $\big(\mathcal{L}^+u(x) - ru(x)\big) \leq -\epsilon u(x)$ for all $x \geq M$.

Then denoting $\sup_{E^M} u(x)$ by $d_M$,

$$dQ_t \leq dN_t + e^{-prt}p\left(d_M^{p-1}\kappa_M - u(S_t)^p[\epsilon - \frac{1}{2}(p-1)D]1_{(S_t\geq M)}\right)dt. \quad (2.7.13)$$

Setting $p = 1 + \frac{\epsilon}{D}$ in (2.7.13), and taking a localising sequence $T_n$ for the local martingale $N$ we see that $E[Q_\tau] \leq \frac{d_M^{p-1}\kappa_M}{r}$, for $\tau$ any stopping time bounded by $T_n$. Since the bound is independent of $n$, we conclude that $Q$ satisfies (2.7.10) and thus, by Theorem 2.7.3, Assumption 2.3.3 holds. $\diamondsuit$

# 3

# Partially reflected support and resistance line model with random switching

## 3.1 An extended stock price process

We now extend the model studied in Chapter 2 by adding two more features: (i) an additional regime, denoted by 0, such that $F$ transitions to 0 from the + regime after an exponential waiting time, and from 0, $F$ can only transition to the negative regime by hitting $L$ from above; (ii) the support/resistance line $R$ becomes a partial reflecting boundary for the process, with parameters $p_+ \in (0.5, 1)$, $p_- \in (0, 0.5)$ and $p_0 = 0.5$ (we also use $q_f$ to denote $1 - p_f$).

More formally, define $\tau_n$ to be the $n$th transition time of $F$ from $-$ to $+$ with the convention $\tau_0 = 0$ if $F_0 = +$ and $\tau_0 = \infty$ if $F_0 = -$, and let $(J_n)$ be a sequence of i.i.d. Exponential($\lambda$) random variables independent of $W$. Then, the process $(S, F)$ is specified by

$$dS_t = \mu_{F_t}(S_t)dt + \sigma_{F_t}(S_t)dW_t + (p_{F_t} - q_{F_t})dl_t^R, \tag{3.1.1}$$

$$F_t = \begin{cases} + & \text{if } F_{t-} = +, \text{ and } S_t > L \\ + & \text{if } F_{t-} = -, \text{ and } S_t = H \\ 0 & \text{if } F_{t-} = 0, \text{ and } S_t > L \\ 0 & \text{if } F_{t-} = +, \text{ and } t - \tau_n = J_n \text{ for some } n > 0 \\ - & \text{if } F_{t-} = -, \text{ and } S_t < H \\ - & \text{if } F_{t-} = +, \text{ and } S_t = L \\ - & \text{if } F_{t-} = 0, \text{ and } S_t = L, \end{cases} \tag{3.1.2}$$

where $l_t^R$ is the symmetric local time process of $S$ at $R$. Note the notation $S^{x,f}$ will be used if we wish to emphasize the initial position. Where there is no fear of confusion, we will use $J$ (instead of $J_n$) to represent the jump time of $F$ from $+$ to 0. We assume $\lambda \geq 0$, and in particular when $\lambda = 0$, it means $F$ never transitions from $+$ to 0 and hence we view $F$ only takes value in $\{+,-\}$.

With this set-up, the bouncing-back effect of $R$ is directly introduced into the model. The effect of the additional 0 regime is to prevent the process from remaining in the $+$ regime for a long time. The resulting state space is $E = \bigcup_{f\in\{+,-,0\}} E^f \times \{f\}$ where $E^+ = E^0 = (L,\infty)$ and $E^- = [0,H)$.

*Remark* 3.1.1. Although the process dynamics have been significantly extended, the majority of results in previous sections can still be verified by very similar arguments. Hence, the proofs contained in the section will be omitted or abbreviated unless there is significant divergence from the previous proof.

Insofar as the results from previous sections are not necessarily repeated we note that our results still show that the prescriptions of TA remain only partially valid. In particular, it is still optimum to "sell at low" under certain circumstances.

As before, it is useful to define the diffusions $S^f$ for $f \in \{+,0,-\}$:

$$dS^f_t = \mu_f(S^f_t)dt + \sigma_f(S^f_t)dW_t + (p_f - q_f)dl^R_t. \tag{3.1.3}$$

**Assumption 3.1.2.**

1. There exist unique (weak) solutions to SDEs (3.1.3), and they live on a completed filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = \{\mathcal{F}_t\}_{t\in\mathbb{R}_+\cup\{\infty\}}, \mathbb{P}_{x,f})$ which supports a Brownian motion $W$.

2. For any $f \in \{+,0,-\}$, $\mu_f$ and $\sigma_f$ are $\alpha$-Hölder continuous (for some $\alpha > 0$) and $\sigma_f(x) > 0$ for $x > 0$.

3. 0 is an absorbing or inaccessible state for $S^-$.

From now on, once an assumption is stated, it is assumed to hold until the end of this section. Under Assumption 3.1.2, we obtain the following theorem.

**Theorem 3.1.3.** *There is a solution $(S,F)$ to (3.1.1) and (3.1.2). The solution is unique in law. Moreover, $(S,F)$ is regular apart from at $(0,-)$, and $(S,F)$ is also a Feller process.*

*Proof.* The proof follows exactly the same lines as that of Theorem 2.2.3. ♢

The generator of $(S,F)$ (restricted to functions independent of $f$) is given by

$$\begin{aligned}\mathcal{L}h(x,f) = \frac{1}{2}\sigma_f^2(x)h''(x,f) + \mu_f(x)h'(x,f)&\\ + \mathbb{1}_{f=+}\lambda\{h(x,0) - h(x,+)\} \quad &\text{for } x \neq R,\end{aligned} \tag{3.1.4}$$

for $h : E \to \mathbb{R}$ in the set $\mathcal{Z} = \{h$ is $C^2$ in direction $x$ except at (R,f) and $p_f h'(R_+, f)$ $= q_f h'(R_-, f)\}$, where $h'$ and $h''$ denote the first and second partial derivatives of $h$ in direction $x$.

For $g : \mathbb{R}_+ \to \mathbb{R}$, we define $\mathcal{L}^f$ by

$$\mathcal{L}^f g(x) = \frac{1}{2}\sigma_f^2(x)g''(x) + \mu_f(x)g'(x). \tag{3.1.5}$$

## 3.2 Seller's Problem

We restrict our attention to the class of gains function of the form $u : \mathbb{R}_+ \to \mathbb{R}_+$ such that

**Assumption 3.2.1.**

1. $u \in C(\mathbb{R}_+) \cap C^2(0, \infty)$.
2. $u$ is strictly increasing and positive.
3. $\mathbb{E}^{x,f}\Big[\sup_{t\geq 0} |e^{-rt}u(S_t)|\Big] < \infty$.

The optimal stopping problem is, as before,

$$\mathbf{V}(x, f) := \sup_{\tau} \mathbb{E}^{x,f}[e^{-r\tau}u(S_\tau)], \tag{SP}$$

We denote the stopping set by $D$ and continuation set by $C$ as before. If $\tau_D < \infty$ a.s., we conclude that $\tau_D$ is optimal by Shiryaev [65] Chapter 3 Theorem 3, and once again $e^{-rt}\mathbf{V}(S_t, F_t)$ is the Snell envelope of $e^{-rt}u(S_t)$.

The following generalises Assumption 2.4.2:

**Assumption 3.2.2.** There is constant $A > H$ such that

$$\mathcal{L}^- u - ru < 0 \quad \text{in} \quad (0, H]. \tag{3.2.1}$$

$$\mathcal{L}^+ u - ru > 0 \quad \text{in} \quad [L, A). \tag{3.2.2}$$

$$\mathcal{L}^+ u - ru < 0 \quad \text{in} \quad (A, \infty). \tag{3.2.3}$$

$$\mathcal{L}^0 u - ru < 0 \quad \text{in} \quad [L, \infty). \tag{3.2.4}$$

### Boundaries of stopping sets

Define $D^f$, $C^f$ as in previous sections. We see that

**Theorem 3.2.3.**

*1. $B := \inf\{x \in D^+\} \geq A$.*

2. *$D^- = [0, m]$ (if 0 is absorbing) or $(0, m]$ (if 0 is inaccessible) for some $m \in [0, H)$.*

3. *If $m \geq L$, then $D^0 = (L, \infty)$. If $m < L$, then there exists constant $c \in (L, \infty)$ such that $D^0 = [c, \infty)$.*

4. *If $B \geq c$ or $\lambda = 0$, then $D^+ = [B, \infty)$.*

*Sketch of proof.* First, let us start with item 1. Since $\mathcal{L}^+ u - ru > 0$ on $(L, A)$, we can argue as in Theorem 2.4.7 that if $x \in (L, A)$ then $V(x, +) > u(x)$.

Let us prove items 2 and 3. To show $D^0$ is an interval, we follow a similar argument to that in Theorem 2.4.7. For $D^-$, its shape specified by item 2 can be proved using the same argument as in Theorem 2.4.7, with a little extra to allow for the negative local time term.

Assuming that $m \geq L$ but $D^0 = [c, \infty)$ leads to a contradiction by observing that in this case $e^{-rt} u(S^{x,0}_{t \wedge \tau_D})$ is a supermartingale and so $V(x, 0) = u(x)$.

Next, let us prove $D^+$ is connected if $c \leq B$. Suppose (to seek a contradiction) there exist $y_1, y_2 \in D^+$, such that $(y_1, y_2) \subset C^+$ (here $y_1 \geq B \geq A$ and $y_2$ is allowed to be infinity). However, let $\tau = \tau_{y_1} \wedge \tau_{y_2} \wedge J$, we see for $y \in (y_1, y_2)$,

$$\begin{aligned}
\mathbf{V}(y, f) &= \mathbb{E}^{(y,f)}[e^{-r\tau} u(S_\tau)] \\
&= u(y) + \mathbb{E}^{(y,f)}\Big[\int_0^\tau e^{-rt}(\mathcal{L}^+ u(S_t) - ru(S_t))dt\Big] \\
&< u(y)
\end{aligned} \tag{3.2.5}$$

where the equality follows from the fact the $\tau$ is the optimal stopping time and by time $\tau$ the process has not hit the partial reflection boundary $R$.

Finally, to show $D^+$ is connected if $\lambda = 0$, we notice (3.2.5) still holds with $\tau$ changed to $\tau = \tau_{y_1} \wedge \tau_{y_2}$. So we can argue analogously. This completes the proof. $\diamondsuit$

**Corollary 3.2.4.** If $V(x, +) \geq V(x, 0)$ then $D^+ = [B, \infty)$

*Proof.* Since $u(B) = V(B, +) \geq V(x, 0)$ we see that $c \leq B$ and the result follows. $\diamondsuit$

We shall see that if the free boundary points, $c$ and $m$, are not equal to $R$, then the smooth pasting conditions still hold because heuristically the smoothness at the free boundary is a local property.

**Theorem 3.2.5.**

1. *$\mathbf{V}_x(B, +) = u'(B)$.*

2. *If $m < L$ and $c \neq R$ then $\mathbf{V}_x(c, 0) = u'(c)$.*

*3. If $m > 0$ and $m \neq R$, $\mathbf{V}_x(m,-) = u'(m)$.*

*Sketch of proof.* We will only prove item 1 as the proof of item 2 and item 3 are analogous. Fix $\epsilon > 0$, abbreviate $\tau_\epsilon := \tau_{B-\epsilon} \wedge \tau_{B+\epsilon}$. Since $\mathbf{V}(x,0) \geq 0$, we obtain

$$\begin{aligned}
\mathbf{V}(B,+) &\geq \mathbb{E}^{B,+}[e^{-r\tau_\epsilon \wedge J}\mathbf{V}(S_{\tau_\epsilon \wedge J}, F_{\tau_\epsilon \wedge J})] \\
&\geq \mathbb{E}^{B}[e^{-r\tau_\epsilon}\mathbf{V}(S^+_{\tau_\epsilon}, +)\mathbb{1}_{\tau_\epsilon < J}] \\
&= \mathbb{E}^{B}\big[\mathbb{E}[e^{-r\tau_\epsilon}\mathbf{V}(S^+_{\tau_\epsilon}, +)\mathbb{1}_{\tau_\epsilon < J}|\mathcal{F}_{\tau_\epsilon}]\big] \\
&= \mathbb{E}^{B}\big[e^{-r\tau_\epsilon}\mathbf{V}(S^+_{\tau_\epsilon}, +)\mathbb{E}[\mathbb{1}_{\tau_\epsilon < J}|\mathcal{F}_{\tau_\epsilon}]\big] \\
&= \mathbb{E}^{B}[e^{-(r+\lambda)\tau_\epsilon}\mathbf{V}(S^+_{\tau_\epsilon}, +)] \\
&= \mathbf{V}(B-\epsilon)\mathbb{E}^{B}[e^{-(r+\lambda)\tau_{B-\epsilon}}\mathbb{1}_{\tau_{B-\epsilon} < \tau_{B+\epsilon}}] + \mathbf{V}(B+\epsilon)\mathbb{E}^{B}[e^{-(r+\lambda)\tau_{B+\epsilon}}\mathbb{1}_{\tau_{B+\epsilon} < \tau_{B-\epsilon}}]
\end{aligned} \tag{3.2.6}$$

where the first inequality follows from the fact that the Snell envelope is a supermartingale and the second inequality follows the definition of a killed process.
Let $\phi$ and $\psi$ denote the decreasing and increasing fundamental solutions to

$$\mathcal{L}^+ f - (r+\lambda)f = 0. \tag{3.2.7}$$

As in [63], we can deduce that $V(x,+)/\phi(x)$ is $\tilde{s}$-concave locally around $B$ where $\tilde{s} := \psi/\phi$, which allows us to apply the arguments from [63] to show item 1. $\diamondsuit$

**Solution to the seller's problem**

We are now ready to propose the candidate solution via the following free boundary problem, and we will prove the candidate solution is indeed the value function (restricted to the interval $(L,B)$ in the case $f = +$).

Let $\tilde{D} := \bigcup_{f\in\{+,-,0\}} \tilde{D}^f$ and $\tilde{C} := \bigcup_{f\in\{+,-,0\}} \tilde{C}^f = E \setminus \tilde{D}$, where $\tilde{D}^f$ and $\tilde{C}^f$ are assumed to have the same structure as $D^f$ and $C^f$ specified in Theorem 3.2.3. Let us use $B', m', c'$ to denote the corresponding free boundaries in $\partial\tilde{C}$. Furthermore, we use $\tilde{C}_R$ to denote $\tilde{C} \setminus \{(R,-),(R,+)\}$, and $\tilde{C}^\circ_R$ is the interior of $\tilde{C}_R$.

Let $v : E \to \mathbb{R}$. We say $(v, B', m', c')$ is a *solution to the free boundary problem*

(3.2.8) if $v \in C(E) \cap C^1(\tilde{C}_R) \cap C^2(\tilde{C}_R^\circ)$ such that

$$\begin{cases} (\mathcal{L}^f - r)v(x,f) + \lambda\big(v(x,0) - v(x,+)\big)\mathbb{1}_{\{f=+\}} = 0, & \text{in } \tilde{C}_R^\circ \\ v(x,f) = u(x), & \text{in } \tilde{D} \\ v(L,f) = v(L,-), & f \in \{+,0\} \\ v(H,-) = v(H,+), & \\ p_+ v_x(R+,+) = q_+ v_x(R-,+), & \\ p_- v_x(R+,-) = q_- v_x(R-,-), & \text{if } R \geq m \\ v_x(B',+) = u'(B'), & \\ v_x(m',-) = u'(m), & \text{if } m' > 0 \text{ and } m' \neq R \\ v_x(c',0) = u'(c), & \text{if } m' < L,\ c' \neq R, \text{ and } c' \neq L \\ v(x,+) \geq u(x), & \text{on } [A,c'] \text{ if } c' > A \\ A \leq B', L \leq c' \leq B', \text{ and } 0 \leq m' < H & \end{cases} \tag{3.2.8}$$

**Theorem 3.2.6.** *Assuming $C \leq B$, the quadruplet $(\mathbf{V}, B, m, c)$ is a solution to the free boundary problem (3.2.8).*

*Sketch of proof.* Fix $f$. Take an interval $I := (y,z)$ such that $I \subset C^f \setminus \{R\}$. Suppose there exists a solution $v(x,f) \in C^2(I) \cap C(\bar{I})$ to the following ODE (in the case $f = +$ we implicitly assume $v(x,0)$ is already known):

$$\begin{cases} (\mathcal{L}^f - r)v(x,f) + \lambda\big(v(x,0) - v(x,+)\big)\mathbb{1}_{\{f=+\}} = 0, & \text{in } (y,z) \\ v(y,f) = \mathbf{V}(y,f), \quad v(z,f) = \mathbf{V}(z,f). & \end{cases} \tag{3.2.9}$$

Let $\tau := \tau_y \wedge \tau_z \wedge \sigma$ where $\sigma := \inf\{t \geq 0 : F_t \neq f\}$. Then, if $f \neq +$, by Dynkin's formula, we obtain

$$v(x,f) = \mathbb{E}^{x,f}[e^{-r\tau} v(S_\tau, F_\tau)] - \mathbb{E}^{x,f}\Big[\int_0^\tau e^{-rt}(\mathcal{L}^f - r)v(S_t, F_t)dt\Big]. \tag{3.2.10}$$

We see $v(S_\tau, F_\tau) = \mathbf{V}(S_\tau, F_\tau)$ $P$-a.s. since $\sigma \geq \tau_y \wedge \tau_z$, and $(\mathcal{L}^f - r)v(S_t, F_t) = 0$ on $[0,\tau)$. This leads to

$$v(x,f) = \mathbb{E}^{x,f}[e^{-r\tau}\mathbf{V}(S_\tau, F_\tau)] = \mathbf{V}(x,f), \tag{3.2.11}$$

where the second inequality holds since $e^{-rt\wedge\tau}\mathbf{V}(S_{t\wedge\tau}, F_{t\wedge\tau})$ is a martingale. If $f = +$, we need to define $h(x,l) := v(x,l)\mathbb{1}_{l=+} + \mathbf{V}(x,0)\mathbb{1}_{l=0}$. For any $\epsilon > 0$, we can still apply Dynkin's formula for the stopping time $\tau_\epsilon := (\tau - \epsilon)^+$ and get

$$h(x) = \mathbb{E}^{x,f}[e^{-r\tau_\epsilon} h(S_{\tau_\epsilon}, F_{\tau_\epsilon})] \\ - \mathbb{E}^{x,f}\Big[\int_0^{\tau_\epsilon} e^{-rt}\Big\{(\mathcal{L}^f - r)v(S_t, F_t) + \lambda\big(v(S_t, 0) - v(S_t, +)\big)\Big\}dt\Big].$$

Since the integrand of the $dt$ term is 0 and $h$ is bounded on $I$, we can apply dominated convergence and take $\epsilon$ to 0 to show

$$v(x, f) = h(x) = \mathbb{E}^{x,f}[e^{-r\tau} h(S_\tau, F_\tau)] = \mathbb{E}^{x,f}[e^{-r\tau}\mathbf{V}(S_\tau, F_\tau)] = \mathbf{V}(x, f). \quad (3.2.12)$$

The existence of a classical solution to ODE (3.2.9) in the case $f \neq +$ simply follows from Theorem 6.2.4 of [32]. If $f = +$, since we have shown $v(x, 0) \in C^2(I)$ and $v(x, 0) = \mathbf{V}(x, 0)$, Theorem 6.2.4 of [32] can still be applied.
Next, if we can show $p_f \mathbf{V}_x(R+, f) = q_f \mathbf{V}_x(R-, f)$, the proof is completed as the rest of assertions in (3.2.8) are straightforward to verify. By now we know $v(x, f)$ is a piecewise $C^2$ function for fixed $f$, and hence it can be written as the difference of two convex functions. Thus, we can apply the symmetric Meyer-Tanaka's formula, which shows, on $\{t \leq \tau_D\}$,

$$\begin{aligned} de^{-rt}\mathbf{V}(S_t, F_t) &= e^{-rt}\Big\{(\mathcal{L} - r)\mathbf{V}(S_t, F_t)\mathbb{1}_{S_t \neq R}dt + (p_{F_t}\mathbf{V}_x(R+, F_t) \\ &\quad - q_{F_t}\mathbf{V}_x(R-, F_t))dl_t^R\Big\} + dM_t \\ &= e^{-rt}(p_{F_t}\mathbf{V}_x(R+, F_t) - q_{F_t}\mathbf{V}_x(R-, F_t))dl_t^R + dM_t, \end{aligned} \quad (3.2.13)$$

where $M$ is a martingale. Since $e^{-rt\wedge\tau_D}\mathbf{V}(S_{t\wedge\tau_D}, F_{t\wedge\tau_D})$ is a martingale, it follows that $\int[p_{F_t}\mathbf{V}_x(R+, F_t) - q_{F_t}\mathbf{V}_x(R-, F_t)]dl_t^R = 0$ and hence that $p_f\mathbf{V}_x(R+, f) - q_f\mathbf{V}_x(R-, f) = 0$.

$\diamondsuit$

**Theorem 3.2.7.** *Let $(v, B', m', c')$ be a solution to (3.2.8) with $c' \leq B'$. Then, $v = \mathbf{V}$, and $\tilde{D} = \mathcal{D}$.*

*Sketch of proof.* Define $N_t := e^{-rt}v(S_t, F_t)$. It is sufficient to prove $N$ satisfies P1-P3 stated in Lemma 2.4.11. As there is no risk of confusion, we use $m, c, B$ to denote $m', c', B'$ respectively.
(P1) Since $v(x, f)$ is a continuous function for each fixed $f$, there is a constant $M$ such that $v(x, f) <= M$ if $x <= B$. Hence, we can argue analogously as in Theorem 2.4.10 (P1) to prove $N_t$ is of class D. In order to use the change-of-variable formula

as before, we first need to consider a sequence of stopping times $J_n$ where $J_n$ is the $n^{th}$ time that $F$ jumps from $+$ to 0 and $J_0 = 0$. Then, defining

$$A_t := \sum_{n=0}^{\infty} \mathbb{1}_{t \geq J_n} \triangle N_{J_n}, \tag{3.2.14}$$

$$N_t = N_0 + \sum_{n=0}^{\infty} \int \mathbb{1}_{t \in [J_n, J_{n+1})} dN_t + A_t. \tag{3.2.15}$$

Since $|v(x,+) - v(x,0)| \leq \sup_{x \in [L,B]} \{v(x,+) + v(x,0)\} < \infty$ for all $x \geq L$, we see that the jump of $A_t$ can be bounded by some constant denoted by $k$ . Therefore the variation process $|A_t|$ is bounded by $k\Lambda_t$ where $\Lambda_t$ denotes a Poisson process with intensity $\lambda$. Let $A^0$ be the compensator of $A$ . It can be shown easily that

$$dA_t^0 = e^{-rt} \lambda (v(S_t, 0) - v(S_t, +)) \mathbb{1}_{\{F_t = +\}} dt. \tag{3.2.16}$$

Therefore, by adding and subtracting $A_t^0$ in equality (3.2.15), applying the symmetric Meyer-Tanaka's formula, it is evident that

$$\begin{aligned} dN_t =& e^{-rt} \Big[ \big( - rv(S_t,+) + \mathcal{L}^+ v(S_t,+) + \lambda \big[ v(S_t,0) - v(S_t,+) \big] \big) \mathbb{1}_{\{F_t=+, S_t \neq B, S_t \neq R\}} dt \\ & + \big( - rv(S_t,0) + \mathcal{L}^+ v(S_t,0) \big) \mathbb{1}_{\{F_t=0, S_t \neq c\}} dt \\ & + \big( - rv(S_t,-) + \mathcal{L}^+ v(S_t,-) \big) \mathbb{1}_{\{F_t=-, S_t \neq m, S_t \neq R\}} dt \\ & + \big( p_{F_t} v_x(R+, F_t) - q_{F_t} v_x(R-, F_t) \big) \mathbb{1}_{F_t \neq 0} dl_t^R \\ & + (v_x(B+,+) - v_x(B-,+)) \mathbb{1}_{F_t=+} dl_t^B \\ & + (v_x(c+,0) - v_x(c-,0)) \mathbb{1}_{\{F_t=+, c>L\}} dl_t^c \\ & + (v_x(m+,0) - v_x(m-,0)) \mathbb{1}_{\{F_t=+, m \neq 0\}} dl_t^c \Big] + dM_t, \end{aligned} \tag{3.2.17}$$

where $M_t$ is a local martingale. By (3.2.8), all local time terms vanish and the $dt$ terms are all non-positive. Thus, $N_t$ is a class D supermartingale.
(P2) By setting $\tau := \tau_{\tilde{D}}$, it is not hard to see from (3.2.17) that $N_{t \wedge \tau}$ is a martingale. Hence, (P2) holds.
(P3) First, we show $v(x,-) \geq u(x)$ on $[m, H)$. Set $\epsilon > 0$. Define $g(x) := v(x,-) - u(x)$ and $g_\epsilon(x) := g(x) + \epsilon \psi_-(x)$ where $\psi_-(x)$ is the increasing fundamental solution of $\mathcal{L}^- f - rf = 0$. If $m \geq R$, we can argue exactly as in P3 of the proof of Theorem 2.4.10. Note that the case $m = R$ does not present a problem since in that case $v_x(m+,-) \geq u'(R)$.

Next, suppose $m < R$. On $(m, H) \setminus \{R\}$, we have

$$\mathcal{L}^- g_\epsilon - r g_\epsilon = ru - \mathcal{L}^- u > 0. \tag{3.2.18}$$

Since $g_\epsilon(m) = \epsilon\psi_-(m) > 0$ and $g'_\epsilon(m_+) = \epsilon\psi'_-(m) > 0$, by strong maximum principle, $g_\epsilon$ is increasing on $(m, R)$. Thus, letting $\epsilon$ tend to 0, we find $g$ is an increasing function on $(m, R)$, and hence $v(x, -) \geq u(x)$.
Moreover, from the monotonicity of $g$, we get $v_x(R-, -) \geq u'(R) \geq 0$. Therefore,

$$g'_\epsilon(R+) = \frac{q_-}{p_-} v_x(R-, -) - u'(R) + \epsilon\psi'_-(R) \geq v_x(R-, -) - u'(R) + \epsilon\psi'_-(R) > 0. \tag{3.2.19}$$

Thus, we can apply the maximum principle to $g_\epsilon$ on $(R, H)$ to see that $g_\epsilon$ is increasing, which completes the proof on taking $\epsilon$ to 0.
The proof that $v(x, 0) \geq u(x)$ follows the same argument as in P3 of the proof of Theorem 2.4.10 and is omitted.
Finally, to prove $v(x, +) \geq u(x)$, define $g(x) := v(x, +) - u(x)$. Recall that $\psi$ and $\phi$ denote the increasing and decreasing fundamental solutions respectively to the ODE:

$$\mathcal{L}^+ w(x) - (r + \lambda) w(x) = 0. \tag{3.2.20}$$

Depending on the value of $c$, there are two cases, namely $c \leq A$, and $c > A$. Let us split the discussion into these two cases.
*Case 1.* $c \leq A$. For $\epsilon > 0$, set $g_\epsilon(x) := g(x) + \epsilon\phi(x)$. Note that on $(A, B)$, by Assumption 3.2.2 and $c \leq A$, it holds

$$\mathcal{L}^+ g_\epsilon(x) - (r + \lambda) g_\epsilon(x) \geq \lambda u(x) - \lambda v(x, 0) = 0. \tag{3.2.21}$$

Moreover, $g_\epsilon(B) = \epsilon\phi(B) > 0$ and $g'_\epsilon(B) = \epsilon\phi'(B) < 0$. By the strong maximum principle,$g_\epsilon$ must be decreasing on $[A, B]$ and hence stay positive. Thus, taking $\epsilon \to 0$, we see $g \geq 0$ on $[A, B]$, i.e. $v(x, +) \geq u(x)$.
Notice on $(L, R) \cup (R, A)$, by Assumption 3.2.2 and the fact $v(x, 0) \geq u(x)$, it holds

$$\mathcal{L}^+ g_\epsilon(x) - (r + \lambda) g_\epsilon(x) \leq \lambda u(x) - \lambda v(x, 0) \leq 0. \tag{3.2.22}$$

Thus, if $g_\epsilon(R) \geq 0$ for all $\epsilon > 0$, since $g_\epsilon(A) \geq 0$ and $g_\epsilon(L) = v(L, -) - u(L) + \epsilon\hat{\phi}(x) \geq 0$, we can use strong minimum principle to show $g_\epsilon \geq 0$ on $[L, A]$, and taking $\epsilon \to 0$ implies $g \geq 0$. Suppose on the contrary, there exists $\epsilon > 0$ such that $g_\epsilon(R) < 0$. Then, by the strong minimum principle, we must have $g'_\epsilon(R+) \geq 0$ and $g'_\epsilon(R-) \leq 0$.

However, this leads to

$$0 \le p_+ g'_\epsilon(R+) - q_+ g'_\epsilon(R-) = (q_+ - p_+)(u'(R) - \epsilon\hat{\phi}'(R)) < 0. \tag{3.2.23}$$

This leads to a contradiction.
*Case 2.* $c > A$. We start by proving $g \ge 0$ on $[c, B]$ following the same argument as in Case 1. Moreover, in this case, the free-boundary problem assumes $g \ge 0$ on $[A, c]$. On $(L, R) \cup (R, A)$, we observe (3.2.22) still holds. Thus, argue analogously as in *Case 1*, we can show $g \ge 0$ on $[L, A]$. ♢

*Remark* 3.2.8. In order to prove uniqueness of the solution to the free boundary problem we have had to assume that $c \le B$ and constrain the solution to dominate $u$ for $(x, f) \in (A, c) \times \{+\}$.

## 3.3 Buyer's Problem

The buyer's problem is given by

$$\mathbf{W}(x, f) := \sup_\tau \mathbb{E}^{x,f}[e^{-r\tau} g(S_\tau, F_\tau)], \tag{BP}$$

where $g(x, f) := \mathbf{V}(x, f) - u(x)$ and $\mathbf{V}$ is the value function of the seller's problem (SP). We assume $u$ satisfy Assumption 3.2.1 and 3.2.2. This leads to

$$\begin{cases} \mathcal{L}^- g - rg > 0, & \text{in } (m, H) \setminus \{R\} \\ \mathcal{L}^+ g - rg < 0, & \text{in } (L, A) \setminus \{R\} \\ \mathcal{L}^+ g - rg > 0, & \text{in } (A, B). \end{cases} \tag{3.3.1}$$

To tackle the increasing technicality, we now make the following restrictive assumption.

**Assumption 3.3.1.** $\lambda = 0$, i.e. the regime 0 is inaccessible for $F$.

There are a few direct consequences based on the formulation. Since $g$ is positive and bounded, so is $\mathbf{W}$. Define the stopping set $\hat{D}$ and continuation set $\hat{C}$ for (BP). If $\tau_{\hat{D}} < \infty$ $\mathbb{P}$-a.s., we conclude that $\tau_{\hat{D}}$ is optimal by Shiryaev [65] Chapter 3 Theorem 3, and $e^{-rt}\mathbf{W}(S_t, F_t)$ is the Snell envelope of $e^{-rt} g(S_t, F_t)$.

Further define $\hat{D}^f$ and $\hat{C}^f$ analogously for $f \in \{+, -\}$. We find

**Theorem 3.3.2.**

1. *$\hat{D}^+ = [a, b]$ for some $a, b \in (L, A]$ and $a < b$.*

*2. $\hat{D}^- = \emptyset$ if $0$ is inaccessible or $\hat{D}^- = \{(0,-)\}$ if $0$ is absorbing.*

*Sketch of proof.* We begin with proving item 2. It is equivalent to show $\hat{C}^- = (0, H)$. Fix $x > 0$ and $f = -$. If $x \neq R$, , we can argue analogously as Theorem 2.5.1 to show $x \in \hat{C}^-$. If $x = R$, set $\tau = \tau_{R+\epsilon} \wedge \tau_{R-\epsilon}$ for $\epsilon$ sufficiently small. Then,

$$\begin{aligned}
\mathbb{E}^{R,-}[e^{-r\tau} g(S_\tau, -)] &= g(R,-) + \mathbb{E}^{R,-}[\int_0^\tau e^{-rt}\{\mathcal{L}^- - r\} g(S_t, -) \mathbb{1}_{S_t \neq R} dt] \\
&+ (p_- g(R+,-) - q_- g(R-,-)) \mathbb{E}^{R,-}[\int_0^\tau e^{-rt} dl_t^R] \\
&= g(R,-) + \mathbb{E}^{R,-}[\int_0^\tau e^{-rt}\{\mathcal{L}^- - r\} g(S_t, F_t) \mathbb{1}_{S_t \neq R} dt] \\
&+ (q_- - p_-) u(R) \mathbb{E}^{R,-}[\int_0^\tau e^{-rt} dl_t^R] \\
&> g(R,-).
\end{aligned} \tag{3.3.2}$$

Thus, $R \in \hat{C}^-$ and item 2 is proved (by observing $\mathbf{W}(0,-) = g(0,-) = 0$ if $0$ is absorbing).

In a similar way, we can show $\hat{D}^+ \cap (A, \infty) = \emptyset$. Moreover, as in Theorem 2.5.1, we can show $\lim_{x \to L} \mathbf{W}(x,+) = \mathbf{W}(L,-) > g(L,-) = g(L,+)$, which ensures $\inf \hat{D}^+ > L$. To complete the proof of item 1, suppose $\hat{D}^+$ is not connected. Then there exists an interval $(y_1, y_2) \subset \hat{C}^+$ such that $y_1, y_2 \in \hat{D}^+$. For $y \in (y_1, y_2)$, it is optimal to stop by $\tau := \tau_{y_1} \wedge \tau_{y_2}$. Therefore, by (3.3.1),

$$\begin{aligned}
\mathbf{W}(y,+) &= \mathbb{E}^{y,+}[e^{-r\tau} g(S_\tau, +)] \\
&= g(y,+) + \mathbb{E}^{y,+}[\int_0^\tau e^{-rt}\{\mathcal{L}^+ - r\} g(S_t, F_t) \mathbb{1}_{S_t \neq R} dt] \\
&+ (q_+ - p_+) u(R) \mathbb{E}^{y,+}[\int_0^\tau e^{-rt} dl_t^R] \\
&< g(y,+).
\end{aligned} \tag{3.3.3}$$

Hence, we obtain a contradiction. Therefore, $\hat{D}^+$ is an interval, and item 1 is proved. ♢

The smooth pasting conditions still hold at the free boundaries $a$ and $b$.

**Theorem 3.3.3.** *$\mathbf{W}_x(a,+) = g_x(a,+)$ and $\mathbf{W}_x(b,+) = g_x(b,+)$ if $a, b$ are not equal to $R$.*

*Sketch of proof.* To follow analogously the proof of Theorem 3.2.5, we need $g(x,+) \geq 0$ on a neighbourhood of $a$ and $b$ respectively, which is ensured by the assumption that $g \geq 0$ on $[0, A]$. To avoid repetition, the rest of the proof is omitted. ♢

Define $\tilde{D}$ and $\tilde{C}$ analogously as in the seller's problem. Let $w : E \to \mathbb{R}$. We call $(w, a, b)$ is a solution to the free boundary problem if $w \in C(E) \cap C^1(\tilde{C}_R \cup \tilde{D}) \cap C^2(\tilde{C}_R^\circ \cup \tilde{D}^\circ)$ such that

$$\begin{cases} (\mathcal{L}^f - r)w(x,f) = 0, & \text{in } \tilde{C}_R^\circ \\ w(x,f) = g(x), & \text{in } \tilde{D} \\ w(L+,+) = w(L,-), \quad \lim_{x\to\infty} w(x,+) = 0, & \\ w(H-,-) = w(H,+), \quad w(0,-) = 0, & \\ p_f w_x(R+,f) = q_f w_x(R-,f), & f \in \{-,+\} \text{ and } (R,f) \notin \tilde{D}^\circ \\ w_x(a,+) = g_x(a,+), & \text{if } a \neq R \\ w_x(b,-) = g_x(b,+), & \text{if } b \neq R \\ w(x,f) \geq 0, & \text{in } E \end{cases} \tag{3.3.4}$$

**Theorem 3.3.4.** *The triplet $(\mathbf{W}, a, b)$ is a solution to the free boundary problem (3.3.4).*

*Sketch of proof.* The smoothness of $\mathbf{W}$ can be proved via usual arguments (cf. proof of Theorem 3.2.6). Boundary conditions are easy to show. Finally, using symmetric Meyer-Tanaka's formula, we can argue as the proof of Theorem 3.2.6 to show $p_f w_x(R+,f) = q_f w_x(R-,f)$ if $R \in \hat{C}^f$. ◇

**Theorem 3.3.5.** *Let $(w, a, b)$ be a solution to (3.2.8). Then, $w = \mathbf{W}$, and $\tilde{D} = \hat{D}$.*

*Sketch of proof.* By Lemma 2.4.11, it is sufficient to prove P1-P3. P1 and P2 can be proved very similarly to Theorem 3.2.7, and their proofs are omitted. To show P3, let $h(x) := w(x,+) - g(x,+)$. We have to prove $h \geq 0$ on $(L, \infty)$. Let us assume $R \geq b$ first.

Mimicking the proof of Theorem 2.5.3, we can prove $h(x) \geq 0$ on $(L, a)$. Moreover, define $h_\epsilon(x) := h(x) + \epsilon\psi_+(x)$ for an arbitrary $\epsilon > 0$. Then, $\mathcal{L}^+ h_\epsilon - r h_\epsilon \geq 0$ on $(b, R) \cup (R, A)$. Since $h_\epsilon(b) > 0$ and $h'_\epsilon(b) > 0$, by the strong maximum principle, $h'_\epsilon(x) \geq 0$ for all $x \in (b, R)$ and $h'_\epsilon(R-) \geq 0$. Furthermore, as

$$p_+ h'_\epsilon(R+) - q_+ h'_\epsilon(R-) = (p_+ - q_+)(u'(R) + \epsilon\psi'_+(R)) > 0, \tag{3.3.5}$$

we have $h'_\epsilon(R+) > 0$. Combining with the fact that $h_\epsilon$ is continuous at $R$, by the strong maximum principle, $h'_\epsilon(x) \geq 0$ for all $x \in (R, A)$. Therefore $h_\epsilon$ is increasing on $[b, A]$, which implies $h_\epsilon(x) \geq 0$ on $[b, A]$. Taking $\epsilon$ to 0, we see $h(x) \geq 0$ on $[b, A]$. Now we know $h(A) \geq 0$. Moreover, since $\mathcal{L}^+ h - rh \leq 0$ on $[A, B]$ and $h(B) = 0$, the

strong minimum principle implies $h \geq 0$ on $[A, B]$.
Next, let us show $w(x,-) \geq g(x,-)$. Set $h(x) := w(x,-) - g(x,-)$ and $h_\epsilon(x) := h(x) + \epsilon\psi_+(x)$ for an arbitrary $\epsilon > 0$. We know $h(x) = 0$ on $[0, m]$. Moreover, by the boundary condition, $h(H) = w(H,+) - g(H,+) \geq 0$. Note $\mathcal{L}^- h - rh < 0$ on $(m, H) \setminus \{R\}$. Thus, if $R \leq m$, by the strong minimum principle, we conclude $h \geq 0$ on $[m, H]$.
Now assume $R > m$. Notice that $h_\epsilon(m) \geq 0$, $h_\epsilon(H) \geq 0$, and $\mathcal{L}^- h - rh < 0$ on $(m, H) \setminus \{R\}$. If $h_\epsilon(R) \geq 0$ for all $\epsilon > 0$, by the strong minimum principle, we get $h_\epsilon \geq 0$ on $[m, H]$. Taking $\epsilon$ to 0, we must have $h \geq 0$ on $[m, H]$, which completes the proof. Let us suppose, on the contrary, there exists $\epsilon > 0$ such that $h_\epsilon(R) < 0$. By the strong minimum principle, it is necessary that $h'_\epsilon(R-) \leq 0$ and $h'_\epsilon(R+) \geq 0$ because otherwise there would be a negative minimum on $(m, R)$ or $(R, H)$. From the smoothness conditions at $R$, it follows that

$$q_- h_\epsilon(R-) - p_- h_\epsilon(R_+) = (q_- - p_-)(u'(R) + \epsilon\psi'_+(R)) > 0, \tag{3.3.6}$$

which implies $h'_\epsilon(R-) > h'_\epsilon(R_+)$. This leads to a contradiction. $\diamondsuit$

If we compare the new models developed in this chapter to those in Chapter 2, then, in the absence of the 0 regime, we observe the structure of the stopping sets is preserved. Even with the introduction of the 0 regime, the stopping set in the negative regime for the seller's problem retains the same structure. The inconsistency between the standard trading rule (of TA) and the trading strategy from our model remains, even under the assumption of the support/resistance line having (partial) bouncing-back effect. Similarly to the examples we have seen in Section 2.6, in the negative regime, if $m < R$ and the initial price $x < R$, then following a sell-at-high strategy and selling the stock when the price hits $R$ is not optimal.

We emphasise that with the introduction of a partially-reflecting boundary, arbitrage opportunities are introduced, i.e. there is no EMM. This makes the model unsuitable for pricing derivatives. Nevertheless, it directly reflects the technical traders' belief about stock price movements, which makes it useful for comparing the output of our model to the standard trading rules from TA .

# 4

# Singular control of inventory and hybrid production

## 4.1 Introduction

In this chapter, we would like to consider the following economic problem. A producer is capable of making products from two sources. One is called the primary source, and the other is called the secondary source. The key difference is that the production from the primary source is completely stochastic, whereas the production from the secondary source is entirely controllable. We further assume that, regardless of the origin of the sources, once the product is produced, it would enter into storage equipment and be stored there. Simultaneously, the producer supplies the products from the inventory to the market directly. From such an operation process, the producer receives positive rewards through supplying their products. Moreover, we assume there is no cost of production from the primary source, but there is a cost associated with the secondary source and we assume it exceeds the reward. Furthermore, depending on the set-up, the producer could obtain positive or negative benefits from holding the inventory, but the marginal gain must be decreasing, which reflects the nature of their risk preference. Finally, the main problem faced by this producer is to determine, with regard to the uncertainty of the primary source production, the optimal production plan of the secondary source and supply strategy which maximise the expected storage and output rewards after deducting the cost of secondary source production.

This problem is inspired by the concept of energy production. Recent decades have seen a rapid development of green energy while the world faces challenges from global warming. Wind power, as one of the main sources of renewable energy, has had its production capacity increase exponentially in the last 20 years [52]. However, the main distinction between wind power and conventional power is that the energy generation of wind power is exposed to substantial uncertainties and fluctuations. To stabilize the energy supply, it is a common practice to install some forms of energy storage equipment (e.g. batteries). Therefore, it naturally raises the question of how to optimally manage the inventory and output from the energy producer's point of view. In this context, we consider winder power as the primary source and thermal power as the secondary source. We emphasise that the modeling studied in this chapter is more generalised but still can be seen as an approximation to the energy management problem proposed above.

To properly formulate such a problem, it is crucial to consider what method should be used to model the stochastic production from the primary source. In terms of energy production, there are various methods to choose from with regard to the prediction of wind speed and power. For example, statistical and data-driven approaches that focus on spot forecasting are proved to be popular, which include time series models [25] and artificial neural networks [1, 11, 45]. On the other hand, a wide range of probabilistic models has been explored in the literature, where the use of distributions such as Weibull, Gamma, and truncated normal have been extensively studied by Carta et al. [10]. Moreover, the idea of using continuous-time stochastic processes for wind power prediction has emerged. For instance, geometric Brownian motion (Wang et al. [67]) and mean-reverting models (e.g., Arenas-López and Badaoui [2] for Ornstein-Uhlenbeck process and Bensoussan and Brouste [2] for Cox-Ingersoll-Ross model), which have traditionally arisen from financial applications, have been used in modeling wind speed and/or power.

In this study, we begin with modeling the rate of production from the primary source by the strong solution of the following SDE:

$$dK_t = \mu(K_t)dt + \sigma(K_t)dW_t, \quad t \geq 0.$$

Of course, more regularity assumptions on $\mu$ and $\sigma$ are needed for the well-posedness of the problem. In particular, as we will see, results in Section 4.5 and 4.6 rely on the assumption that the volatility term takes the form $\sigma x$.

The producer can control the production from the secondary source and the output, but instead of controlling their rates, we assume the producer directly controls the corresponding accumulated processes denoted by $B$ and $A$ respectively (which are increasing and hence of finite variation). Such a formulation allows jumps of the accumulated processes, which corresponds to their rates being infinity at the time of jumps. Of course, in reality, neither the production nor the output rates can be infinitely large, but our results show that, optimally $A$ is always continuous except for a possible jump at time 0 and $B$ stays constant after some first hitting time (of the inventory process), which guarantees the applicability of our modeling. The inventory process is then defined as follows:

$$I_t = i + \int_0^t K_s ds + B_t - A_t, \quad t \geq 0.$$

This enables us to introduce a two-dimensional infinite-time horizon singular stochastic control problem that is given by

$$\mathbf{V}(i,k) := \sup_{A,B} \mathbb{E}_{i,k}\Big[\int_0^\infty e^{-rt}\{u(I_t)dt + pdA_t - qdB_t\}\Big],$$

where $u$ is a concave running reward function, $p$ is the (long-term) price of the product per unit, and $q$ is the (long-term) cost of production (from the secondary source) per unit.

The control problem stated above can be considered as a specific example in the field of production-inventory problems. Traditionally a problem of this kind assumes that the producer aims at minimizing the expected cost of inventory under stochastic demand by controlling the production. In many studies (e.g. Fleming et al. [31]), the cost function associated with the inventory is assumed to be convex, and a quadratic function is a common choice as in [3, 8, 19, 69]. Compared to the classical set-up, our modeling is non-standard as we assume the production (from the primary source) is purely random and the output (which can be understood as the demand) is controllable. We would like to stress that our reformulation has substantially distinguished itself from the traditional problem by restricting the output rate to be positive only (whereas classically the demand is modeled as a general diffusion process and allowed to be negative) and introducing two distinct sources of production.

There is rich literature on the application of singular control in optimal production and inventory problems (e.g. [13, 14] and references therein). From the mathematical perspective, our modeling is closely related to the stochastic reversible investment problem where the investor manages the cash injection and withdrawal of their investment (exposed to the uncertainty from self-evolution or a separate demand process) to maximise the accumulated investment profit or minimise the total management cost. Under our modeling, the investment process is regarded as the inventory process, and the injection and withdrawal are thought of as production from the secondary and primary source respectively. Federico and Pham [27] explicitly derived the optimal control of a reversible investment problem with respect to a bivariate cost function that links the investment and demand processes together. Later Federico et al. [28] generalized this model by allowing a direct interaction of the investment process with the demand process. A different model can be found in the paper by De Angelis and Ferrari [15] where the reversible investment problem is defined on a finite-time horizon and a stochastic self-evolving investment process. A multidimensional setup is investigated by Dianetti and Ferrari [18], where the 'investment' process is still one-dimensional and evolving autonomously, but it enters into the dynamics of diffusion processes in other dimensions. A recent work by Federico et al. [29] brought such a formulation into the context of production-inventory management, and they investigated this problem under partial information on the inventory process. Our modeling, though similar to those papers in many ways, has a combination of two distinct mathematical features which, to the best of our knowledge, do not occur in the current literature: (a) our model allows direct interaction of the diffusion process with the inventory process and hence the inventory process is not autonomous (c.f. [18, 27]); (b) our model permits a general class of dynamics and reward functions (c.f. [8, 15, 28, 29]).

To thoroughly study our problem, a combination of analytical and probabilistic arguments are applied. More specifically, we start with proving a polynomial growth condition of $\mathbf{V}$ and the concavity of $\mathbf{V}(\cdot, k)$ by elementary analysis, which leads to the existence and uniqueness of the optimal control. Secondly, we define a zero-sum optimal stopping game (Dynkin game) and prove that its value function $\mathbf{W}$ is connected with $\mathbf{V}$ via $\mathbf{W} = \mathbf{V}_i$, which also implies the continuity and monotonicity of $\mathbf{V}_i$. Thirdly, depending on the value of $\mathbf{V}_i$, the state space is split into three disjoint sets, namely $C$, $D_p$ and $D_q$, where the intersecting boundary of $C$ with $D_p$ is denoted by $a_p(k)$ (a function of $k$) and the other intersecting boundary of $C$ with $D_q$ is denoted by $a_q(k)$. Then, using standard results of Dynkin games for Markov processes (see [22, 55]), we see that $\mathbf{W}$ is a solution to a free-boundary problem. From this, similarly to De Angelis and Ferrari [15], we can further derive various properties of the free-boundaries $a_p$ and $a_q$, such as monotonicity, (left/right) continuity, and boundedness. In particular, we show that $a_p$ is equal to a constant whose value is directly computable. Moreover, an in-depth probabilistic analysis reveals the fact that a desirable smooth-pasting principle holds along these two boundaries, which enables us to characterise $\mathbf{W}$ and the inverse of $a_q$ as the unique solution to a free-boundary problem with smooth-pasting conditions. Furthermore, by assuming a geometric Brownian motion style volatility term and either a concave drift term or a locally semiconvex reward function, the existence and continuity of $\mathbf{V}_k$ are proved through exploiting the viscosity property of $\mathbf{V}$. By usual arguments based on the $C^{1,1}$ smoothness and the viscosity property of $\mathbf{V}$, it follows that $\mathbf{V}$ is also $C^{1,2}$ in the closure of $C$. Finally, we propose a candidate optimal control which reflects $I$ when $(I, K)$ hits the boundaries $a_p$ or $a_q$ in such a way to keep $(I, K)$

inside the closure of $C$ with the minimum effort[1], and the proven $C^{1,2}$ smoothness of $\mathbf{V}$ allows us to establish a verification theorem for the candidate optimal control.

The rest of this chapter is organised as follows. In Section 4.2, we rigorously formulate the singular control problem and prove some fundamental results. In Section 4.3, we formally define the Dynkin game and prove its connection to the control problem. In Section 4.4, we explore various properties of the Dynkin game including establishing the smooth-pasting principle and undertaking a free-boundary analysis which leads to their characterisations. In Section 4.5, we derive the desired smoothness property of the value function via the viscosity approach. In Section 4.6, we propose the candidate optimal control and obtain its optimality through a verification theorem. Finally, in Section 4.7, we provide some economic interpretations of the optimal control based on our modeling, and Section 4.8 contains some supplementary results and proofs.

## 4.2 Problem formulation and preliminary results

### 4.2.1 Model setup

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a filtered probability space satisfying the usual conditions, which supports a one dimensional Brownian motion $W$. The production rate of the primary source $K$ is modeled by the following SDE:

$$\begin{cases} dK_t = \mu(K_t)dt + \sigma(K_t)dW_t, \quad t \geq 0 \\ K_0 = k. \end{cases} \tag{4.2.1}$$

Let $\mathbb{R} := (-\infty, \infty)$, $\mathbb{R}_+ := [0, \infty)$ and $\mathbb{R}_+^\circ := (0, \infty)$. We make the following assumptions on $K$ and its dynamics.

**Assumption 4.2.1.**

(i) $\mu$ and $\sigma$ are in $C^1(\mathbb{R}_+^\circ)$, and $\mu'$ and $\sigma'$ are locally Hölder continuous.

(ii) $\sigma(x) > 0$ for $x > 0$.

(iii) There exists a constant $N$ such that for all $x \in \mathbb{R}_+^\circ$, $|\mu(x)| \vee |\sigma(x)| \leq N(1+|x|)$.

(iv) $\mu$ and $\sigma$ are Lipschitz continuous with the Lipschitz constant $L$, i.e. there exists a constant $L > 0$ such that for all $x, y \in \mathbb{R}_+^\circ$,

$$|\mu(x) - \mu(y)| \vee |\sigma(x) - \sigma(y)| \leq L|x - y|.$$

[1] Such a control is often referred as a Skorokhod reflection-type policy (see e.g. [8, 15, 27, 28, 29]).

(v) 0 is inaccessible and $K$ is non-explosive, i.e. $\mathbb{P}(\sup_{s\le t} K_s = \infty$ or $K_t = 0) = 0$, $\forall t \ge 0$.

*Remark* 4.2.2. By standard theory (e.g. [40], Theorem 5.2.5 and 5.2.9), Assumption 4.2.1 (ii), (iii), and (iv) guarantee that for any $k \in \mathbb{R}_+^\circ$, there is a unique strong solution $K^k$ taking values in $\mathbb{R}_+^\circ$ with continuous paths. Moreover, according to Feller's test ([40], Theorem 5.5.29), to ensure $K$ is non-explosive and 0 is inaccessible, it is sufficient to assume:

$$\int_1^\infty s'(y)\int_1^y \frac{2dz}{s'(z)\sigma^2(z)}dy = \infty \quad \text{and} \quad \int_1^0 s'(y)\int_1^y \frac{2dz}{s'(z)\sigma^2(z)}dy = \infty,$$

where $s'(y) := \exp\Big\{-2\int_1^y \frac{\mu(x)}{\sigma^2(x)}dx\Big\}$.

By Proposition 5.2.18 in [40], Assumption 4.2.1 (ii), (iii), and (iv) ensure that the following comparison result holds.

**Lemma 4.2.3.** *If $k_1 \le k_2$, then $K_t^{k_1} \le K_t^{k_2}$, for all $t \ge 0$, $\mathbb{P}$-almost surely.*

It will be useful to define $\hat{K}_t := \int_0^t K_s ds$, which represents the accumulated production from the primary source.

To proceed, let us state some standard results of estimates of moments. For fixed $\nu \ge 2$, set $\eta := 2N + (\nu-1)N^2$, by Theorem 4.1 in [50] Chapter 2, Assumption 4.2.1 implies, that for all $k, k' \in \mathbb{R}_+^\circ$ and $t \ge 0$,

$$\mathbb{E}[|K_t^k|^\nu] \le 2^{\frac{\nu-2}{2}} e^{\nu\eta t}(1+|k|^\nu). \tag{4.2.2}$$

By Hölder's inequality, we obtain

$$\mathbb{E}[|\hat{K}_t^k|^\nu] \le 2^{\frac{\nu-2}{2}} e^{\nu\eta t} t^{\nu-1}(1+|k|^\nu). \tag{4.2.3}$$

By Jensen's inequality, we also have estimates for $0 < \nu < 2$, which are given by

$$\mathbb{E}[|K_t^k|^\nu] \le e^{\nu(2N+N^2)t}(1+k^2)^{\frac{\nu}{2}}, \tag{4.2.4}$$

$$\mathbb{E}[|\hat{K}_t^k|^\nu] \le e^{\nu(2N+N^2)t} t^{\frac{\nu}{2}}(1+k^2)^{\frac{\nu}{2}}. \tag{4.2.5}$$

Furthermore, we also need the following stability result which states that as the initial position converges, the corresponding diffusion processes also converge in probability and $L^2$.

**Lemma 4.2.4.** *For any sequence $k_n \to k$,*

$$\sup_{0\le s\le t} |K_s^{k_n} - K_s^k| \xrightarrow[p]{L^2} 0, \qquad \textit{for all } t \ge 0. \tag{4.2.6}$$

*Moreover, there exists a subsequence, still denoted by $k_n$, such that*

$$\mathbb{P}(\lim_{n\to\infty} K_t^{k_n} = K_t^k, \text{ for all } t \geq 0) = 1. \tag{4.2.7}$$

*Proof.* Standard estimates ([57], Theorem 1.3.16 ) show that there exits a constant $c$ such that for all $t \geq 0$ and $x, y \in \mathbb{R}_+^\circ$,

$$\mathbb{E}[\sup_{0\leq s\leq t} |K_s^x - K_s^y|^2] \leq e^{ct}|x-y|^2. \tag{4.2.8}$$

Taking a sequence $k_n \to k$, by (4.2.8), we obtain

$$\lim_{n\to\infty} \mathbb{E}[\sup_{0\leq s\leq t} |K_s^{k_n} - K_s^k|^2] = 0, \tag{4.2.9}$$

and (4.2.6) follows. Thus, for each $t \geq 0$, there exits a subsequence $k_n$ such that

$$\mathbb{P}(\lim_{n\to\infty} K_s^{k_n} = K_s^k, \ s \leq t) = 1. \tag{4.2.10}$$

Take $t_m$ a sequence in $\mathbb{Q}$ such that $t_m \to \infty$. Then, using diagonalization, there exists a subsequence $k_n$ such that

$$\mathbb{P}(\lim_{n\to\infty} K_t^{k_n} = K_t^k, \forall t \geq 0) = \lim_{m\to\infty} \mathbb{P}(\lim_{n\to\infty} K_s^{k_n} = K_s^k, \ s \leq t_m) = 1. \tag{4.2.11}$$

$\diamondsuit$

We use $A$ to denote the *accumulated* output and $B$ to denote the *accumulated* secondary production. We say $(A, B)$ is *admissible* if $A$ and $B$ are adapted, right continuous with left limit (or equivalently càdlàg), and non-decreasing.

We set $A_{0^-} = B_{0^-} = 0$ [2]. Note that by definition, $A$ and $B$ are finite variation process and non-negative. It is also important to note that, unlike $\hat{K}$, we allow the control to have jumps. The inventory process $I$ satisfies

$$I_t = i + \hat{K}_t + B_t - A_t, \quad i \in \mathbb{R}, \tag{4.2.12}$$

and it will be useful to set $\hat{I}_t := i + \hat{K}_t$, i.e. $\hat{I}$ is the process $I^{A,B}$ where $A = B = 0$. We say $(I, K)$ is the state process and define the state space $S := \mathbb{R} \times \mathbb{R}_+^\circ$.

*Remark* 4.2.5. We allow $I$ to take negative values. Considering that $I$ represents the inventory, it might seem that this assumption is unrealistic. However, this is not

[2] In our setting, for a finite variation process $R$, on finite interval $[s, t]$, the Lebesgue-Stieltjes integral $\int_s^t dR_u := R_t - R_{s^-}$, i.e. the jump at time $s$ is also included. We use $\int_{s+}^t dR_u$ or $\int_s^{t^-} dR_u$ to exclude the jump at either end points.

a major disadvantage of our modeling because our results show that, it is very easy to choose $u$ such that the optimised inventory level is non-negative all the time for all $i \geq 0$. More details are provided in Section 4.7.

Given a control process $(A, B)$, the *payoff function* $G$ is given by

$$G(i, k; A, B) := \mathbb{E}\Big[\int_0^\infty e^{-rt}\{u(I_t)dt + pdA_t - qdB_t\}\Big], \tag{4.2.13}$$

where $u : \mathbb{R} \to \mathbb{R}$ is the running profit/cost function of the inventory, $p > 0$ is the unit price of output, $q > 0$ is the unit cost of secondary production, and $r > 0$ is the discount factor.

To avoid an infinite payoff obtained by simply scaling up $A$ and $B$, we assume $p < q$. So there is no incentive to produce from the secondary source and sell the products immediately. Note that this also ensures $A$ and $B$ are the minimal decompositions of a finite variation process.

To ensure the problem is well-posed, we make some assumptions on $u$, which are listed below.

**Assumption 4.2.6.**

(i) $u \in C^2(\mathbb{R})$.

(ii) There exist constants $Z > 0$ and $\gamma \geq 2$, such that, $|u(x)|+|u'(x)| \leq Z(1+|x|^\gamma)$.

(iii) $\lim_{x\to-\infty} u'(x) = \infty$ and $\lim_{x\to\infty} u'(x) \leq 0$.

(iv) $u$ is strictly concave.

Define two important constants:

$$i_p := (u')^{-1}(rp) \quad \text{and} \quad i_q := (u')^{-1}(rq). \tag{4.2.14}$$

Or equivalently, they satisfy $u'(i_p) = rp$ and $u'(i_q) = rq$. Observe they are well defined since we have assumed $u'(x) \to \infty$ as $x \to -\infty$. They will be used frequently in Section 4.4.

Furthermore, we also need the discount factor $r$ to be large enough.

**Assumption 4.2.7.** $r > \iota := \gamma(2N + (\gamma - 1)N^2)$, where $\gamma$ is given by Assumption 4.2.6 (ii).

*Remark* 4.2.8. Based on Assumption 4.2.7, since $r > 2N + N^2$, we see from (4.2.5)

$$\mathbb{E}[e^{-rt}|K_t^k|] \leq e^{(2N+N^2-r)t}(1 + k^2)^{\frac{1}{2}} \tag{4.2.15}$$

$$\mathbb{E}[e^{-rt}|\hat{K}_t^k|] \leq e^{(2N+N^2-r)t}t^{\frac{1}{2}}(1 + k^2)^{\frac{1}{2}}, \tag{4.2.16}$$

which implies that there exists a constant $c > 0$, such that, for all $k > 0$,

$$\mathbb{E}\Big[\int_0^\infty e^{-rt} K_t^k dt\Big] \leq c(1+k^2)^{\frac{1}{2}}, \tag{4.2.17}$$

$$\mathbb{E}\Big[\int_0^\infty e^{-rt} \hat{K}_t^k dt\Big] \leq c(1+k^2)^{\frac{1}{2}}. \tag{4.2.18}$$

Moreover, since $d(e^{-rt}\hat{K}_t) = e^{-rt}\big(K_t - r\hat{K}_t\big)dt \leq e^{-rt}K_t dt$, we get

$$\mathbb{E}[\sup_{t\geq 0} e^{-rt}\hat{K}_t^k] \leq \mathbb{E}\Big[\int_0^\infty e^{-rt} K_t^k dt\Big] \leq c(1+k^2)^{\frac{1}{2}}. \tag{4.2.19}$$

So, $\sup_{t\geq 0} e^{-rt}\hat{K}_t^k$ is in $L^1(\mathbb{P})$.

Let $\bar{\mathcal{A}}$ denote a subset of admissible controls in which

$$\mathbb{E}\big[\sup_{T\geq 0} e^{-rT} A_T\big] < \infty. \tag{4.2.20}$$

We are interested in solving the following stochastic singular control problem.

$$\overline{\mathbf{V}}(i,k) := \sup_{(A,B)\in\bar{\mathcal{A}}} G(i,k;A,B). \tag{CP}$$

We call $\overline{\mathbf{V}}(i,k)$ the *value function* of the control problem CP.

### 4.2.2 Restricted control problem

Our analysis of the control problem will rely on the existence of the optimal control. However, this result is not easy to obtain in $\bar{\mathcal{A}}$ (for more details, see Remark 4.2.16). To tackle this difficulty, we have to further restrict the set of admissible controls, but we will see in Section 4.6 that this restriction can be relaxed.

Let $\upsilon \in L^1_+(\mathbb{P})$ where $L^1_+(\mathbb{P})$ denotes the space of non-negative integrable random variables. Then, we define $\mathcal{A}(\upsilon)$ to be a subset of $\bar{\mathcal{A}}$ in the following way:

$$\mathcal{A}(\upsilon) := \{(A,B)\in\bar{\mathcal{A}} : \sup_{t\geq 0} e^{-rt}A_t \leq \sup_{t\geq 0} e^{-rt}\hat{K}_t + (i-i_p)^+ + \upsilon,\ \mathbb{P}\text{-a.s. } \forall (i,k)\in S\}.$$

Define the restricted control problem by

$$\mathbf{V}^\upsilon(i,k) := \sup_{(A,B)\in\mathcal{A}(\upsilon)} G(i,k;A,B), \tag{RCP}$$

where $\mathbf{V}^\upsilon$ is the value function of the restricted control problem (RCP).

$\overline{\mathbf{V}}$ and $\mathbf{V}^\upsilon$ are connected via the following proposition.

**Proposition 4.2.9.** *For $(i,k) \in S$, we have*

$$\overline{\mathbf{V}}(i,k) = \sup_{\upsilon \in L^1_+(\mathbb{P})} \mathbf{V}^{\upsilon}(i,k). \tag{4.2.21}$$

*Therefore, if there exists $(A^*, B^*) \in \mathcal{A}(0)$ such that*

$$\mathbf{V}^{\upsilon}(i,k) = G(i,k;A^*,B^*), \quad \forall \upsilon \in L^1_+(\mathbb{P}), \tag{4.2.22}$$

*then $\overline{\mathbf{V}}(i,k) = G(i,k;A^*,B^*)$.*

*Proof.* On one hand, it is obvious that $\mathbf{V}^{\upsilon}(i,k) \le \overline{\mathbf{V}}(i,k)$ for all $\upsilon \in L^1_+(\mathbb{P})$ since $\mathcal{A}(\upsilon) \subset \bar{\mathcal{A}}$. On the other hand, for $\epsilon > 0$, let $(A_\epsilon, B_\epsilon)$ be an $\epsilon$-optimal control [3] of $\overline{\mathbf{V}}$, i.e.

$$G(i,k;A_\epsilon,B_\epsilon) \ge \overline{\mathbf{V}}(i,k) - \epsilon.$$

Since $(A_\epsilon, B_\epsilon) \in \bar{\mathcal{A}}$, there must exist $\upsilon_\epsilon \in L^1_+(\mathbb{P})$ such that $(A_\epsilon, B_\epsilon) \in \mathcal{A}(\upsilon_\epsilon)$. As $(A_\epsilon, B_\epsilon)$ is not necessarily optimal for $\mathbf{V}^{\upsilon_\epsilon}$, we observe that

$$\sup_{\upsilon \in L^1_+(\mathbb{P})} \mathbf{V}^{\upsilon}(i,k) \ge \mathbf{V}^{\upsilon_\epsilon}(i,k) \ge G(i,k;A_\epsilon,B_\epsilon) \ge \overline{\mathbf{V}}(i,k) - \epsilon. \tag{4.2.23}$$

Since $\epsilon$ can be arbitrarily small, we conclude that (4.2.21) holds. Finally, $\overline{\mathbf{V}}(i,k) = G(i,k;A^*,B^*)$ is a direct consequence of (4.2.21) and (4.2.22). $\diamondsuit$

*Remark* 4.2.10. In Section 4.6, we will propose a candidate optimal control $(A^*, B^*)$ belongs to $\mathcal{A}(0)$ and prove it is optimal for $\mathbf{V}^{\upsilon}$. As $\upsilon$ is arbitrary, by Proposition 4.2.9, $(A^*, B^*)$ must be optimal for $\overline{\mathbf{V}}$ as well. Thus, in this way, we relax the restriction from $\mathcal{A}(\upsilon)$ to $\bar{\mathcal{A}}$. For more details, see Theorem 4.6.4

From now on, fix $\upsilon \in L^1_+(\mathbb{P})$. For ease of notation, let us drop the superscript $\upsilon$ in $\mathbf{V}^{\upsilon}$ and use $\mathcal{A}$ to denote $\mathcal{A}(\upsilon)$.

### 4.2.3 Existence and uniqueness of the optimal control

The first step of our study is to obtain some integrability results which are very useful for later analysis.

**Proposition 4.2.11.** *There exists a constant $\alpha \ge 0$, such that, for all $i \in \mathbb{R}$, and $k \in \mathbb{R}^{\circ}_+$,*

$$\mathbb{E}\Big[\int_0^\infty e^{-rt}\{|u(\hat{I}_t)| + |u'(\hat{I}_t)|\}dt\Big] \le \alpha(1 + |i|^\gamma + |k|^\gamma) \tag{4.2.24}$$

[3] The existence of the $\epsilon$-optimal control follows from the finteness of $\overline{\mathbf{V}}$ which can be proved by (almost) the same argument as in Proposition 4.2.12. We omit the detailed proof.

*Proof.* By Assumption 4.2.6 (ii) and the standard estimate (4.2.3), we have

$$\begin{aligned}
&\mathbb{E}\Big[\int_0^\infty e^{-rt}\{|u(\hat{I}_t)| + |u'(\hat{I}_t)|\}dt\Big] \\
&\leq \mathbb{E}\Big[\int_0^\infty e^{-rt} Z(1 + |i + \hat{K}_t|^\gamma)dt\Big] \\
&\leq \mathbb{E}\Big[\int_0^\infty e^{-rt}\alpha(1 + |i|^\gamma + |\hat{K}_t|^\gamma)dt\Big] \\
&\leq \alpha(1 + |i|^\gamma) + \alpha\mathbb{E}\Big[\int_0^\infty e^{-rt}|\hat{K}_t|^\gamma dt\Big] \\
&\leq \alpha(1 + |i|^\gamma) + \alpha(1 + |k|^\gamma)\int_0^\infty e^{-(r-\iota)t} t^{(\gamma-1)}dt \\
&\leq \alpha(1 + |i|^\gamma + |k|^\gamma),
\end{aligned} \tag{4.2.25}$$

where $\alpha$ is a constant that may vary from line to line. $\diamondsuit$

The following proposition provides a polynomial growth condition for $\mathbf{V}$.

**Proposition 4.2.12.** *There a constant $\alpha$ and such that for all $i \in \mathbb{R}$ and $k \in \mathbb{R}_+^\circ$*

$$-\alpha(1 + |i|^\gamma + |k|^\gamma) \leq \mathbf{V}(i,k) \leq \alpha(1 + |i|^\gamma + |k|^\gamma). \tag{4.2.26}$$

*Proof.* Taking $A = B = 0$, $-\alpha(1 + |i|^\gamma + |k|^\gamma) \leq \mathbf{V}(i,k)$ follows directly from Proposition 4.2.11.
By Assumption 4.2.6(iii), for any $\epsilon > 0$, there exists $M$ such that $u(x) \leq M + \epsilon x$. Therefore, fixing $T > 0$,

$$\begin{aligned}
&\int_0^T e^{-rt}\{u(I_t)dt + pdA_t - qdB_t\} \\
\leq &\int_0^T e^{-rt}\{(M + \epsilon i + \epsilon\hat{K}_t + \epsilon B_t - \epsilon A_t)dt + pdA_t - qdB_t\} \\
\leq &(M + \epsilon i)\frac{1}{r} + \epsilon\int_0^\infty e^{-rt}\hat{K}_t dt + \int_0^T e^{-rt}\{pdA_t - \epsilon A_t dt\} + \int_0^T e^{-rt}\{\epsilon B_t dt - qdB_t\}
\end{aligned} \tag{4.2.27}$$

We now apply the integration by parts formula for a finite variation process to obtain

$$\begin{aligned}
\int_0^T e^{-rt}\{u(I_t)dt + pdA_t - qdB_t\} \leq (M + \epsilon i)\frac{1}{r} + \epsilon\int_0^\infty e^{-rt}\hat{K}_t dt + pe^{-rT}A_T \\
+ (rp - \epsilon)\int_0^T e^{-rt}A_t dt - qe^{-rT}B_T + (\epsilon - rq)\int_0^T e^{-rt}B_t dt.
\end{aligned} \tag{4.2.28}$$

Let $\epsilon = rp$. Since $p < q$, we get

$$\begin{aligned} &\int_0^\infty e^{-rt}\{u(I_t)dt + pdA_t - qdB_t\} \\ &\leq \limsup_{T\to\infty} \int_0^T e^{-rt}\{u(I_t)dt + pdA_t - qdB_t\} \\ &\leq (M + rpi)\frac{1}{r} + rp\int_0^\infty e^{-rt}\hat{K}_t dt + p\sup_{T\geq 0} e^{-rT}A_T. \end{aligned} \tag{4.2.29}$$

Taking expectation on both sides, by the definition of $\mathcal{A}$ (note we have dropped $\upsilon$) and Remark 4.2.8, we can find a constant $\alpha$ such that

$$G(i,k;A,B) \leq \alpha(1 + |i| + (1+k^2)^{\frac{1}{2}} + |i|^\gamma + |k|^\gamma), \text{ for all } (A,B) \in \mathcal{A}. \tag{4.2.30}$$

Since $\gamma \geq 2$, we can always choose $\alpha$ big enough such that $G(i,k;A,B) \leq \alpha(1 + |i|^\gamma + |k|^\gamma)$. As $(A,B)$ is arbitrary and $\alpha$ is independent of $(A,B)$ and $(i,k)$, this shows $V(i,k) \leq \alpha(1 + |i|^\gamma + |k|^\gamma)$ for all $(i,k) \in S$. ♢

We can further prove the concavity of $G$ and $\mathbf{V}$ for fixed $k$ as the proposition below states. This is a standard result based on the concavity of $u$.

**Proposition 4.2.13.** *The payoff function $G(i,k;A,B)$ is strictly concave in $i$ and over the space of admissible control $\mathcal{A}$. Moreover, the value function $\mathbf{V}(i,k)$ is concave in $i$. Hence, the left and right partial derivatives $\mathbf{V}_i^\pm(i,k)$ exist, and $\mathbf{V}_i^+ \leq \mathbf{V}_i^-$.*

*Proof.* Fix $k > 0$. Take $(i,k),(i',k) \in S$ and $(A,B),(A',B') \in \mathcal{A}$. Pick $\lambda \in (0,1)$. Define $i^\lambda = \lambda i + (1-\lambda)i'$, $A^\lambda = \lambda A + (1-\lambda)A'$, and $B^\lambda = \lambda B + (1-\lambda)B'$.

Hence, by the definition of $G$ and strict concavity of $u$, we have

$$\begin{aligned} &G(i^\lambda, k; A^\lambda, B^\lambda) \\ &= \mathbb{E}\Big[\int_0^\infty e^{-rt}\{u(i^l + \hat{K}_t^k + A_t^\lambda + B_t^\lambda)dt + pdA_t^\lambda - qdB_t^\lambda\}\Big] \\ &> \mathbb{E}\Big[\int_0^\infty e^{-rt}\{\lambda u(I_t^{i,k,A,B})dt + (1-\lambda)u(I_t^{i',k,A',B'})dt + pdA_t^\lambda - qdB_t^\lambda\}\Big] \\ &= \lambda G(i,k;A,B) + (1-\lambda)G(i',k;A',B'). \end{aligned} \tag{4.2.31}$$

Thus, $G(\cdot,k;\cdot,\cdot)$ is strictly concave. Hence, by the defintion of $\mathbf{V}$,

$$\begin{aligned}\mathbf{V}(i^\lambda,k) &\geq \sup_{(A,B),(A',B')\in\mathcal{A}} G(i^l,k;A^\lambda,B^\lambda)\\ &\geq \sup_{(A,B),(A',B')\in\mathcal{A}} \big\{\lambda G(i,k;A,B)+(1-\lambda)G(i',k;A',B')\big\}\\ &= \lambda \sup_{(A,B)\in\mathcal{A}} G(i,k;A,B)+(1-\lambda)\sup_{(A',B')\in\mathcal{A}} G(i',k;A',B')\\ &= \lambda\mathbf{V}(i,k)+(1-\lambda)\mathbf{V}(i',k),\end{aligned} \tag{4.2.32}$$

which proves the concavity of $\mathbf{V}(\cdot,k)$. ♢

*Remark* 4.2.14. It is easy to see from the proof of Proposition 4.2.12 and 4.2.13 that analogous results hold for $\overline{\mathbf{V}}$ as well.

Now we are in a good position to prove that there exists a unique pair of optimal controls for the problem (RCP). This forms the basis of the results in Section 4.3.

**Theorem 4.2.15.** *There exists, up to a.s., a unique admissible control* $(A^*,B^*)$ *which is optimal for problem (RCP), i.e.* $G(i,k;A^*,B^*)=\mathbf{V}(i,k)$.

*Proof. Existence.* Let $(i,k)\in S$ and take a maximising sequence $(A^n,B^n)$ such that $G(i,k;A^n,B^n)\to\mathbf{V}(i,k)$. WLOG, let's assume $G(i,k;A^n,B^n)\geq\mathbf{V}(i,k)-1$. Substituting $\epsilon\in(p,q)$ in (4.2.28) and then taking $T\to\infty$, we obtain

$$\begin{aligned}\int_0^\infty e^{-rt}\{u(I_t)dt+pdA_t^n-qdB_t^n\}+r(\epsilon-p)\int_0^\infty e^{-rt}A_t^ndt+r(q-\epsilon)\int_0^\infty e^{-rt}B_t^ndt\\ \leq (M+rpi)\frac{1}{r}+rp\int_0^\infty e^{-rt}\hat{K}_tdt+p\sup_{T\geq 0}e^{-rT}A_T^n.\end{aligned} \tag{4.2.33}$$

Take expectation on both sides of the above inequality. By our choice of $(A^n,B^n)$, we further deduce

$$\begin{aligned}r(\epsilon-p)\mathbb{E}\Big[\int_0^\infty e^{-rt}A_t^ndt\Big]+r(q-\epsilon)\mathbb{E}\Big[\int_0^\infty e^{-rt}B_t^ndt\Big]\\ \leq 1-\mathbf{V}(i,k)+\alpha(1+|i|^\gamma+|k|^\gamma),\end{aligned} \tag{4.2.34}$$

where $\alpha$ is a constant. By the lower bound of $\mathbf{V}$ developed in Proposition 4.2.12, we see $\mathbb{E}\Big[\int_0^\infty e^{-rt}A_t^ndt\Big]<\infty$ and $\mathbb{E}\Big[\int_0^\infty e^{-rt}B_t^ndt\Big]<\infty$, i.e. both $A^n$ and $B^n$ are bounded in the space $L^1(\Omega\times\mathbb{R}_+,\mathbb{P}\otimes e^{-rt}dt)$. By a theorem of Komlós [43], there exist subsequences (still labelled by $A^n$ and $B^n$) and a pair of measurable processes

$A^*$ and $B^*$ such that the Cesáro sequences of processes

$$A^{*n} := \frac{1}{n}\sum_{j=0}^{n} A^j \quad \text{and} \quad B^{*n} := \frac{1}{n}\sum_{j=0}^{n} B^j \tag{4.2.35}$$

converge to $A^*$ and $B^*$ $\mathbb{P}\otimes e^{-rt}dt$-a.e. respectively. Notice that it is easy to check $(A^{*n}, B^{*n}) \in \mathcal{A}$. Moreover,

$$\begin{aligned}\mathbb{E}[\sup_{t\geq 0} e^{-rt}\mathcal{A}^*_t] &= \mathbb{E}[\sup_{t\geq 0}\liminf_{n\to\infty} e^{-rt}\frac{1}{n}\sum_{j=0}^{n} A^j_t] \\ &\leq \mathbb{E}[\liminf_{n\to\infty}\sup_{t\geq 0} e^{-rt}\frac{1}{n}\sum_{j=0}^{n} A^j_t].\end{aligned} \tag{4.2.36}$$

By Fatou's Lemma, since $(A^n, B^n) \in \mathcal{A}$, we have

$$\mathbb{E}[\sup_{t\geq 0} e^{-rt}\mathcal{A}^*_t] \leq \liminf_{n\to\infty}\frac{1}{n}\sum_{j=0}^{n}\mathbb{E}[\sup_{t\geq 0} e^{-rt}A^j_t] \leq \Lambda(1+|i|^\gamma+|k|^\gamma). \tag{4.2.37}$$

Similarly,

$$\sup_{t\geq 0} e^{-rt}\mathcal{A}^*_t \leq \lim_{n\to\infty}\frac{1}{n}\sum_{j=0}^{n}\sup_{t\geq 0} e^{-rt}A^j_t \leq \sup_{t\geq 0} e^{-rt}\hat{K}_t + (i-i_p)^+ + \upsilon. \tag{4.2.38}$$

Therefore, we see that the integrability and dominance requirements specified in $\mathcal{A}$ are satisfied. Arguing as Lemma 4.5-4.7 in [41], we see that there are modifications of $(A^*, B^*)$ which are adapted, càdlàg, and increasing. Hence, $(A^*, B^*) \in \mathcal{A}$.
By concavity of $G$ over $\mathcal{A}$, $(A^{*n}, B^{*n})$ is still a maximising sequence. Hence, if we can use the reverse Fatou's Lemma, we would have

$$\mathbf{V}(i,k) = \limsup_{n\to\infty} G(i,k;A^{*n},B^{*n}) \leq G(i,k;A^*,B^*). \tag{4.2.39}$$

To show that the reverse Fatou's Lemma can be applied, we use (4.2.28) again to get

$$\begin{aligned}&\int_0^\infty e^{-rt}\{u(I_t)dt + pdA^{*n}_t - qdB^{*n}_t\} \\ &\leq \alpha + pi + rp\int_0^\infty e^{-rt}K_t dt + \sup_{t\geq 0} e^{-rt}A^{*n}_t \\ &\leq \alpha + p|i| + rp\int_0^\infty e^{-rt}K_t dt + \sup_{t\geq 0} e^{-rt}\hat{K}_t + (i-i_p)^+ + \upsilon =: Y,\end{aligned} \tag{4.2.40}$$

where $\alpha \geq 0$ is a constant and the last inequality follows from the definition of $\mathcal{A}$.

Note $\int_0^\infty e^{-rt} K_t dt$ and $\sup_{t\geq 0} e^{-rt}\hat{K}_t$ are integrable (cf. Assumption 4.2.7, Remark 4.2.8 and standard estimate (4.2.4)), which ensures $Y \in L^1(\mathbb{P})$. Moreover, $Y$ is non-negative and independent the choice of $(A^{*n}, B^{*n})$. Therefore, Fatou's Lemma can be applied.

*Uniqueness.* Suppose there are two optimal controls $(A_1, B_1)$ and $(A_2, B_2)$. Take $\lambda \in (0,1)$. Let $(A^\lambda, B^\lambda)$denote the convex combination of $(A_1, B_1)$ and $(A_2, B_2)$. As $G$ is strictly concave in the control, it follows

$$G(i,k; A^\lambda, B^\lambda) > \lambda G(i,k; A_1, B_1) + (1-\lambda) G(i,k; A_2, B_2) = \mathbf{V}(i,k), \qquad (4.2.41)$$

unless $(A_1, B_1) = (A_2, B_2)$. ♢

*Remark* 4.2.16. The difficulty in proving the existence of the optimal control for $\bar{\mathcal{A}}$ lies in the application of the reverse Fatou's lemma. To obtain (4.2.39), it is necessary to find a non-negative random variable such as $Y$ which makes (4.2.40) hold. However, without the dominance property assumed in $\mathcal{A}(v)$, it is not trivial how to construct such a random variable for controls in $\bar{\mathcal{A}}$.

### 4.2.4 Variational HJB and heuristic discussion

The classical approach to the derivation of the value function is through solving the HJB equation. Let $\mathcal{L}$ denote the infinitesimal generator of $K$, i.e.

$$\mathcal{L} := \frac{1}{2}\sigma^2(k)\frac{\partial^2}{\partial k^2} + \mu(k)\frac{\partial}{\partial k}. \qquad (4.2.42)$$

Following the standard arguments based on dynamic programming principle, the variational *Hamilton–Jacobi–Bellman* equation (HJB) associated with control problem (RCP) (and problem (CP)) is given by

$$\min\{rv(i,k) - \mathcal{L}v(i,k) - kv_i(i,k) - u(i), v_i(i,k) - p, q - v_i(i,k)\} = 0, \qquad \text{(HJB)}$$

for some function $v : S \to \mathbb{R}$.

We can make a few observations from (HJB). If $\mathbf{V}$ indeed solves (HJB) in the classical sense, then $\mathbf{V}_i$ is bounded between $p$ and $q$. Moreover, when $\mathbf{V}_i$ is strictly between $p$ and $q$, $\mathbf{V}$ must solve the following PDE:

$$rv(i,k) - \mathcal{L}v(i,k) - kv_i(i,k) - u(i) = 0. \qquad (4.2.43)$$

Based on relevant research on singular control problem (e.g. [15, 27, 28, 29]), we expect the optimal control is to take no action when the state process $(I, K)$

is in $\{p < \mathbf{V}_i < q\}$, and the control takes the least effort to keep $(I, K)$ within $\{p \leq \mathbf{V}_i \leq q\}$ when $(I, K)$ is in $\{\mathbf{V}_i = p\} \cup \{\mathbf{V}_i = q\}$,.

If we can obtain a candidate value function $v$ which is $C^{1,2}$, then via the classical verification approach, it is possible to verify that $v = \mathbf{V}$, and obtain the optimal control from there. However, this is not feasible in our case because the closed-form solution to (4.2.43) is unknown.

To this end, we will directly show that $\mathbf{V}$ is a classical $C^{1,2}$ solution to (4.2.43) in the closure of $\{p < \mathbf{V}_i < q\}$ and prove the verification theorem of the candidate optimal control. We achieve this in the following steps:

**Step 1.** Given the existence of the optimal control, we show $\mathbf{V}_i$ is equal to the value function of a zero-sum optimal stopping game, which gives its existence and continuity. This is done in Section 4.3;

**Step 2.** In Section 4.4, we characterise $\mathbf{V}_i$ and the boundary of $\{p < \mathbf{V}_i < q\}$ by a free-boundary problem through an in-depth analysis of the zero-sum optimal stopping game;

**Step 3.** By assuming a geometric Brownian motion type of volatility term and other regularity conditions on $\mu$ and $u$, we show that $\mathbf{V}_k$ is well-defined and continuous through utilising the viscosity property of $\mathbf{V}$. This leads to the $C^{1,2}$ smoothness of $\mathbf{V}$ in the closure of $\{p < \mathbf{V}_i < q\}$. This step is completed in Section 4.5;

**Step 4.** Finally, in Section 4.6, we propose a candidate optimal control based on the characterisation developed in Step 2 and verify that it is the unique solution to the restricted control problem (RCP), which also leads to its optimality for the unrestricted control problem (CP).

## 4.3 Existence of $\mathbf{V}_i$ and the related zero-sum optimal stopping game

The problem (RCP) belongs to the class of two dimensional singular control problem. One crucial result in this area is that, in many cases (e.g. [15, 27, 28, 29]), $\mathbf{V}_i$ can be characterised by the value function of a zero-sum optimal stopping game (Dynkin game) defined as the following. Let $J(i, k; \tau, \sigma)$ be a payoff function for the initial position $(i, k)$ given a pair of stopping times $(\tau, \sigma)$, which is explicitly given by

$$J(i, k; \tau, \sigma) = \mathbb{E}\Big[\int_0^{\tau\wedge\sigma} e^{-rt} u'(\hat{I}_t)dt + qe^{-r\sigma}\mathbb{1}_{\sigma<\tau} + pe^{-r\tau}\mathbb{1}_{\tau\leq\sigma}\Big]. \tag{4.3.1}$$

There are two players in the game and each of them chooses a stopping time. The player who controls $\tau$ wants to maximise the payoff $J$, and is called the *sup-player*, whereas the player who controls $\sigma$ wants to minimise the payoff $J$, and is called the *inf-player*.

The value function of the optimal stopping game, denoted by $\mathbf{W}(i,k)$, exists if

$$\mathbf{W}(i,k) = \sup_{\tau} \inf_{\sigma} J(i,k;\tau,\sigma) = \inf_{\sigma} \sup_{\tau} J(i,k;\tau,\sigma). \tag{SP}$$

We say a pair of stopping times $(\tau_*, \sigma_*)$ is a *Nash equilibrium* (or saddle point) if for all stopping times $\sigma, \tau$,

$$J(i,k;\tau,\sigma_*) \leq J(i,k;\tau_*,\sigma_*) \leq J(i,k;\tau_*,\sigma). \tag{4.3.2}$$

In other words, any deviation from the strategy $(\tau_*, \sigma_*)$ would result in a worse outcome for each player. It is clear that if such $(\tau_*, \sigma_*)$ exists, then $\mathbf{W}$ is well defined in the sense that (SP) holds.

The next theorem establishes the existence of Nash equilibrium and $\mathbf{W} = \mathbf{V}_i$. Karatzas and Shreve [42] proved a similar result on a more general set-up, but it does not cover our case. We provide the proof in Section 4.8.1. The proof mainly mimics the proof in [42].

**Theorem 4.3.1.** *Suppose $(A^*, B^*)$ is the optimal control of the control problem (RCP). Define*

$$\sigma_* := \inf\{t \geq 0 : B_t^* > 0\}, \quad \tau_* := \inf\{t \geq 0 : A_t^* > 0\}. \tag{4.3.3}$$

*Then, for any stopping times $\sigma$ and $\tau$,*

$$J(i,k;\tau,\sigma_*) \leq \mathbf{V}_i^+(i,k), \quad \mathbf{V}_i^-(i,k) \leq J(i,k;\tau_*,\sigma), \tag{4.3.4}$$

*which implies $(\tau_*, \sigma_*)$ is a Nash equilibrium. Moreover, $\mathbf{W}(i,k) = J(i,k;\tau_*,\sigma_*) = \mathbf{V}_i(i,k)$.*

By the above theorem, $\mathbf{V}_i$ is well-defined and equals to $\mathbf{W}$. From now on, we do not have to distinguish $\mathbf{V}_i$ and $\mathbf{W}$, though we will use $\mathbf{W}$ in most cases. In the next proposition, we want to show some basic properties for $\mathbf{W}$.

**Proposition 4.3.2.**

1. *$\mathbf{W}$ is continuous in $S$*

2. *$\mathbf{W}(i,\cdot)$ and $\mathbf{W}(\cdot,k)$ are decreasing for all $(i,k) \in S$.*

*3.* $p \leq \mathbf{W} \leq q$.

*Proof.* 1. For fixed $(i,k) \in S$, take a sequence $(i_n, k_n) \to (i,k)$. WLOG, $|i - i_n| + |k - k_n| \leq 1$. Let $(\tau, \sigma)$ and $(\tau_n, \sigma_n)$ denote the optimal stopping times with respect to $(i,k)$ and $(i_n, k_n)$ respectively. Then,

$$\begin{aligned}\mathbf{W}(i,k) - \mathbf{W}(i_n,k_n) &\leq J(i,k;\tau,\sigma_n) - J(i_n,k_n;\tau,\sigma_n) \\ &\leq \mathbb{E}\Big[\int_0^\infty e^{-rt}|u'(\hat{I}_t^{i,k}) - u'(\hat{I}_t^{i_n,k_n})|dt\Big].\end{aligned} \tag{4.3.5}$$

By Assumption 4.2.6 (ii) and the standard estimate (4.2.3), it is evident that

$$|u'(\hat{I}_t^{i,k}) - u'(\hat{I}_t^{i_n,k_n})| \leq \alpha(1 + |i|^\gamma + |\hat{K}_t^k|^\gamma + |\hat{K}_t^{k+1}|^\gamma + |\hat{K}_t^{k-1}|^\gamma) \in L^1(d\mathbb{P} \otimes e^{-rt}dt).$$

Moreover, by Lemma 4.2.4, $\hat{I}_t^{i_n,k_n} \to \hat{I}_t^{i,k}$ in probability for all $t \geq 0$. Therefore, taking $n$ to $\infty$, by the Dominated Convergence Theorem, we have $\mathbf{W}(i,k) \leq \liminf_{n\to\infty} \mathbf{W}(i_n,k_n)$. In a similar way, we can show $\mathbf{W}(i,k) \geq \limsup_{n\to\infty} \mathbf{W}(i_n,k_n)$.
2. Since $u$ is concave, $u'$ is decreasing. By Lemma 4.2.3, for all $i_1 \geq i_2$ and $k_1 \geq k_2$,

$$J(i_1,k_1;\tau,\sigma) \leq J(i_2,k_2;\tau,\sigma). \tag{4.3.6}$$

Hence, $\mathbf{W}(i_1,k_1) \leq \mathbf{W}(i_2,k_2)$.
3. $J(i,k;0,\sigma) = p$ and $J(i,k;\tau,0) \leq q$ for all $\sigma \geq 0$ and $\tau \geq 0$. Therefore, $p \leq \mathbf{W}(i,k) \leq q$. ♢

## 4.4 Regularity of W and free-boundaries

Let us begin by introducing the following notations. Define three sets $C$, $D_p$, and $D_q$ by

$$C = \{(i,k) \in S : p < \mathbf{W}(i,k) < q\}, \tag{4.4.1}$$

$$D_p = \{(i,k) \in S : \mathbf{W}(i,k) = p\}, \tag{4.4.2}$$

$$D_q = \{(i,k) \in S : \mathbf{W}(i,k) = q\}. \tag{4.4.3}$$

In the language of optimal stopping problems, we call $C$ the *continuation set* and $D_p$ or $D_q$ the *stopping sets.*

Furthermore, define two *free-boundaries* $a_p(k)$ and $a_q(k)$ which map from $\mathbb{R}_+^\circ$ to

$\mathbb{R}$ such that

$$a_p(k) := \inf\{i \in \mathbb{R} : \mathbf{W}(i,k) = p\}, \tag{4.4.4}$$

$$a_q(k) := \sup\{i \in \mathbb{R} : \mathbf{W}(i,k) = q\}. \tag{4.4.5}$$

As $\mathbf{W}$ is decreasing with respect to $i$, it is easy to see that $a_q(k) \le a_p(k)$. Furthermore, since $\mathbf{W}$ is continuous, it follows $\mathbf{W}(a_p(k),k) = p$ and $\mathbf{W}(a_q(k),k) = q$. Moreover, since $\mathbf{W}$ is decreasing, we can rewrite $C$, $D_p$, and $D_q$ by

$$C := \{(i,k) \in S : a_q(k) < i < a_p(k)\}, \tag{4.4.6}$$

$$D_p := \{(i,k) \in S : a_q(k) \ge i\}, \tag{4.4.7}$$

$$D_q := \{(i,k) \in S : i \ge a_p(k)\}. \tag{4.4.8}$$

The previous section established the result that $\mathbf{V}_i = \mathbf{W}$ and we proved some properties of $\mathbf{W}$. In this section, we will further explore the connection between $\mathbf{W}$ and free-boundary problems. In particular, we are interested in finding an approach to characterise its free-boundaries $a_p(k)$ and $a_q(k)$ because as we have discussed at the end of Section 4.2, we expect the optimal control of $\mathbf{V}$ is determined by them.

### 4.4.1 Free-boundary problem

It turns out, in a similar fashion to the optimal stopping problems, the first hitting time of $D_p$ and $D_q$ of the process $(\hat{I}, K)$ form a Nash equilibrium to the zero-sum optimal stopping game (SP).

**Theorem 4.4.1.** *Set $\sigma_* = \inf\{t \ge 0 : (\hat{I}_t, K_t) \in D_q\}$ and $\tau_* = \inf\{t \ge 0 : (\hat{I}_t, K_t) \in D_p\}$. Then, $\mathbf{W}(i,k) = J(i,k;\tau_*,\sigma_*)$.*

*Proof.* Let $Y_t := \int_0^t e^{-rs} u'(\hat{I}_s) ds$. Then, by Lemma 2.12 in [38], $(I, K, Y)$ is still a continuous Feller process. Moreover, by Proposition 4.2.11,

$$\mathbb{E}[\sup_t |Y_t|] \le \mathbb{E}[\lim_{t\to\infty} |Y_t|] < \infty. \tag{4.4.9}$$

Thus, we can apply Theorem 2.1 in [22], which shows $(\tau_*, \sigma_*)$ is indeed optimal. $\diamondsuit$

Moreover, it can be shown that $\mathbf{W}$ is a smooth solution to the following free-boundary problem

$$
\begin{cases}
\mathcal{L}w(i,k) + kw_i(i,k) - rw(i,k) = -u'(i), & (i,k) \in C; \\
\mathcal{L}w(i,k) + kw_i(i,k) - rw(i,k) \geq -u'(i), & (i,k) \in D_q^o; \\
\mathcal{L}w(i,k) + kw_i(i,k) - rw(i,k) \leq -u'(i), & (i,k) \in D_p^o; \\
p \leq w(i,k) \leq q, & (i,k) \in S; \\
w(i,k) = p, & i \geq a_p(k); \\
w(i,k) = q, & i \leq a_q(k),
\end{cases}
\tag{4.4.10}
$$

where $D_q^o$ and $D_p^o$ are the interior of $D_q$ and $D_p$ respectively.

**Proposition 4.4.2.** $\mathbf{W} \in C^{1,2}(C)$ *and satisfies all conditions listed by (4.4.10).*

*Proof.* Proceeding as in the proof of Theorem 4.4.1, we can show that it is suitable to apply Theorem 2.1 in [55], which ensures

$$e^{-rt\wedge\tau}\mathbf{W}(\hat{I}_{t\wedge\tau}, K_{t\wedge\tau}) + \int_0^{t\wedge\tau} e^{-rs}u'(\hat{I}_s)ds, \tag{4.4.11}$$

$$e^{-rt\wedge\sigma}\mathbf{W}(\hat{I}_{t\wedge\sigma}, K_{t\wedge\sigma}) + \int_0^{t\wedge\sigma} e^{-rs}u'(\hat{I}_s)ds, \tag{4.4.12}$$

$$e^{-rt\wedge\tau\wedge\sigma}\mathbf{W}(\hat{I}_{t\wedge\tau\wedge\sigma}, K_{t\wedge\tau\wedge\sigma}) + \int_0^{t\wedge\tau\wedge\sigma} e^{-rs}u'(\hat{I}_s)ds, \tag{4.4.13}$$

are continuous submartingale, supermartingale, and martingale respectively, where $\tau$ and $\sigma$ are the Nash equilibrium defined in Theorem 4.4.1. If we can verify $\mathbf{W} \in C^{1,2}(C)$, all conditions in (4.4.10) can be checked straightforwardly. To prove $\mathbf{W} \in C^{1,2}(C)$, we can use the usual argument. In other words, let $f : S \to \mathbb{R}$ be a $C^{1,2}$ solution to

$$\mathcal{L}f(i,k) + kf_i(i,k) - rf(i,k) = -u'(i) \text{ in } R, \tag{4.4.14}$$

with boundary condition $f|_{\partial R} = \mathbf{W}|_{\partial R}$, where $R \subset C$ is an open rectangle set. As $\sigma, \mu, u'$ are Lipschitz continuous in the closure of $R$ and $\mathbf{W}$ is continuous, the existence and uniqueness of $f$ is ensured by Theorem 6.3.6 in [32]. Take $(i,k) \in R$ and $\rho := \inf\{t \geq 0 : (\hat{I}_t, K_t) \in \partial R\}$. Following the proof of Theorem 6.5.1 in [32], for $(i,k) \in R$, we can argue that, by appying Ito's formula to $e^{-rt}f(\hat{I}_t, K_t)$, it follows

$$\mathbb{E}[e^{-r\rho}f(\hat{I}_\rho, K_\rho)] = f(i,k) + \mathbb{E}\Big[\int_0^\rho e^{-rs}\{\mathcal{L}f(\hat{I}_s, K_s) + kf_i(\hat{I}_s, K_s) - rf(\hat{I}_s, K_s)\}ds\Big]. \tag{4.4.15}$$

Therefore,

$$\begin{aligned} f(i,k) &= \mathbb{E}\Big[e^{-r\rho} f(\hat{I}_\rho, K_\rho) + \int_0^\rho e^{-rs} u'(\hat{I}_s) ds\Big] \\ &= \mathbb{E}\Big[e^{-r\rho} \mathbf{W}(\hat{I}_\rho, K_\rho) + \int_0^\rho e^{-rs} u'(\hat{I}_s) ds\Big] = \mathbf{W}(i,k), \end{aligned} \tag{4.4.16}$$

where the last equality holds since (4.4.13) is a martingale, and $\rho \le \tau \wedge \sigma$. ♢

Since (4.4.11) is a submartingale and (4.4.12) is a supermartingale, we get the following inequalities directly.

**Lemma 4.4.3.** *For all $(i,k) \in S$ and the optimal stopping times $\sigma$ and $\tau$ defined in Theorem 4.4.1, let $\rho$ be an arbitrary stopping time, then we have*

$$\mathbf{W}(i,k) \le \mathbb{E}\Big[e^{-r\rho\wedge\tau} \mathbf{W}(\hat{I}_{\rho\wedge\tau}, K_{\rho\wedge\tau}) + \int_0^{\rho\wedge\tau} e^{-rt} u'(\hat{I}_t) dt\Big], \tag{4.4.17}$$

$$\mathbf{W}(i,k) \ge \mathbb{E}\Big[e^{-r\rho\wedge\sigma} \mathbf{W}(\hat{I}_{\rho\wedge\sigma}, K_{\rho\wedge\sigma}) + \int_0^{\rho\wedge\sigma} e^{-rt} u'(\hat{I}_t) dt\Big]. \tag{4.4.18}$$

### 4.4.2 Free-boundaries analysis

Recall the definition of two important constants:

$$i_p := (u')^{-1}(rp) \quad \text{and} \quad i_q := (u')^{-1}(rq). \tag{4.4.19}$$

Or equivalently, they satisfy $u'(i_p) = rp$ and $u'(i_q) = rq$. The next proposition lists some important properties about the free-boundaries. In particular, $a_p$ is a horizontal line.

**Proposition 4.4.4.** *For all $k > 0$,*

1. *$a_q(k) \le i_q$;*
2. *$a_q(k)$ is decreasing and left continuous;*
3. *$a_q(k) > -\infty$;*
4. *$a_p(k) = i_p$.*

*Proof.* 1. For $(i,k) \in D_q^o$, $\mathbf{W}(i,k) = q$. Thus, all partial derivatives of $\mathbf{W}$ are 0. Plugging in (4.4.10), we get $u'(i) \ge rq$, which implies $i \le i_q$. Hence, $a_q(k) \le i_q$.
2. Since $\mathbf{W}$ is decreasing, for all $0 < k_1 \le k_2$ and $i > a_q(k_1)$,

$$q = \mathbf{W}(a_q(k_1), k_1) > \mathbf{W}(i, k_1) \ge \mathbf{W}(i, k_2). \tag{4.4.20}$$

If $i = a_q(k_2)$, then the equation above would lead to a contradiction. Therefore, $a_q(k_2) \le a_q(k_1)$, i.e. $a_q$ is a decreasing function. Let $k_n$ be a sequence increasing to $k$. As a result, $a_q(k_n)$ is a decreasing sequence and $a_q(k_n) \ge a_q(k)$. Suppose $\lim_{n\to\infty} a_q(k_n) > a_q(k)$. Take $i \in (a_q(k), \lim_{n\to\infty} a_q(k_n))$. Then, $\forall n \ge 0$, $\mathbf{W}(i, k_n) = q$ and $\mathbf{W}(i, k) < q$ because $i < a_q(k_n)$ and $i > a_q(k)$. This contradicts to the continuity of $\mathbf{W}$. Therefore, $\lim_{n\to\infty} a_q(k_n) = a_q(k)$ and $a_q$ is left continuous.
3. Suppose there exits $k$ such that $a_q(k) = -\infty$. By monotonicity of $a_q$, $\forall k' > k$, $a_q(k') = -\infty$. Take $k_0, k_1 \in (k, \infty)$, $k' \in (k_0, k_1)$, and $i' \in \mathbb{R}$. Moreover, define

$$\tau := \inf\{t \ge 0 : K_t = k_0 \text{ or } K_t = k_1\}, \quad \sigma := \inf\{t \ge 0 : a_q(K_t) \ge \hat{I}_t\}. \tag{4.4.21}$$

Then, $\sigma$ is optimal by Theorem 4.4.1 and (4.4.6), and $\tau \le \sigma$. Therefore, for any $T > 0$,

$$\begin{aligned} \mathbf{W}(i', k') &\ge \mathbb{E}\Big[\int_0^{\tau\wedge T\wedge\sigma} e^{-rt} u'(\hat{I}_t)dt + qe^{-r\sigma}\mathbb{1}_{\sigma<\tau} + pe^{-r\tau}\mathbb{1}_{\tau\le\sigma}\Big] \\ &\ge \mathbb{E}\Big[\int_0^{\tau\wedge T} e^{-rt} u'(i' + \hat{K}_t)dt\Big] \\ &\ge \mathbb{E}\Big[\int_0^{\tau\wedge T} e^{-rt} u'(i' + k_1 T)dt\Big] \\ &\ge \frac{u'(i' + k_1 T)}{r}\mathbb{E}\Big[1 - e^{-r\tau\wedge T}\Big]. \end{aligned} \tag{4.4.22}$$

Letting $i'$ go to $-\infty$, by Assumption 4.2.6 (iii), the right hand side of the above inequality converges to $\infty$. This contradicts to $\mathbf{W} \le q$.
4. We first prove $a_p(k) \ge i_p$. By (4.4.10), in $D_p$, $u'(i) \le rp$, which implies $i \ge i_p$. Hence, $a_p(k) \ge i_p$. Next, we aim at showing $a_p(k) \le i_p$. Define an auxiliary optimal stopping problem $\hat{\mathbf{W}}$ by

$$\hat{\mathbf{W}}(i, k) := \sup_\tau \mathbb{E}\Big[\int_0^\tau e^{-rt} u'(\hat{I}_t)dt + pe^{-r\tau}\Big]. \tag{4.4.23}$$

Take $i \ge i_p$ and $k > 0$ arbitrarily. Notice that $\inf\{t \ge 0 : \hat{I}_t \le a_q(K_t)\} = \infty$ as $\hat{I}_t \ge i \ge i_p$ and $a_q \le i_q < i_p$. Therefore, $\mathbf{W}(i, k) = \hat{\mathbf{W}}(i, k)$. Note that

$$\mathbb{E}\Big[\int_0^\tau e^{-rt} u'(\hat{I}_t)dt + pe^{-r\tau}\Big] = p + \mathbb{E}\Big[\int_0^\tau e^{-rt}\{u'(\hat{I}_t) - rp\}dt\Big]. \tag{4.4.24}$$

Moreover, since $\hat{I}_t \ge i_p$, $u'(\hat{I}_t) - rp \le 0$ for all $t \ge 0$. Hence, for all $\tau$,

$$\mathbb{E}\Big[\int_0^\tau e^{-rt} u'(\hat{I}_t)dt + pe^{-r\tau}\Big] \le p. \tag{4.4.25}$$

Thus, $\hat{\mathbf{W}}(i,k) = p$, which implies $\mathbf{W} = p$ in $[i_p, \infty) \times \mathbb{R}_+^\circ$. So $a_p(k) \le i_p$. $\diamondsuit$

We can define the inverse of $a_p$ and $a_q$, denoted by $d_p$ ans $d_p$, as follows:

$$b_p(i) = \inf\{k \in \mathbb{R}_+^\circ : a_p(k) \le i\}, \quad b_q(i) = \sup\{k \in \mathbb{R}_+^\circ : a_q(k) \ge i\}. \tag{4.4.26}$$

By Proposition 4.4.4 and the convention that $\inf \emptyset = \infty$ and $\sup \emptyset = 0$, we have

$$b_p(i) = \begin{cases} \infty, & i < i_p; \\ 0, & i \ge i_p; \end{cases} \tag{4.4.27}$$

and $b_q(i) = 0$ if $i \ge i_q$. Moreover, we can show $b_q$ is continuous.

**Proposition 4.4.5.** *$b_q$ is decreasing and continuous on $\mathbb{R}$.*

*Proof.* For $i \le i'$,

$$\{k \in \mathbb{R}_+^\circ : a_q(k) \ge i'\} \subset \{k \in \mathbb{R}_+^\circ : a_q(k) \ge i\}. \tag{4.4.28}$$

Hence, $b_q(i) \ge b_q(i')$, which shows $b_q$ is decreasing. For $i > i_q$, it is obvious $b_q$ is continuous (as $b_q = 0$). Moreover, $d_q$ admits an equivalent definition given by

$$b_q(i) = \sup\{k \in \mathbb{R}_+^\circ : \mathbf{W}(i,k) = q\}. \tag{4.4.29}$$

Argue analogously as in Proposition 4.4.4, we can show $b_q$ is left continuous.
To complete the proof, we need to show $b_q$ is continuous on $(-\infty, i_q]$. Suppose there exists $i_0 \le i_q$ where $b_q$ is not continuous at $i_0$, or equivalently $b_q(i_0) > b_q(i_0+)$. Let $f(i,k) := q - \mathbf{W}(i,k)$. Choose a rectangle set $R = [i_0, i_1] \times [k_0, k_1]$ such that $b_q(i) > k_1 > k_0 > b_q(i_0+)$. As $b_q$ is decreasing, we know the interior of $R$, denoted by $R^o$, is contained by the continuation set $C$. Then, we know $f$ solves

$$\begin{cases} (\mathcal{L} - r)f(i,k) + kf_i(i,k) = u'(i) - rq, & \text{on } R^o; \\ f(i_0, k) = 0, & k \in [k_0, k_1]. \end{cases} \tag{4.4.30}$$

Since $b_q \ge 0$, we can divide both sides of the equation by $k$ and get

$$\frac{1}{k}\mathcal{L}f(i,k) - \frac{r}{k}f(i,k) + f_i(i,k) = \frac{u'(i) - rq}{k}, \quad \text{on } R^o. \tag{4.4.31}$$

By Assumption 4.2.1 (i), Theorem 3.10 in [33] implies $f_{kkk}$ and $f_{ik}$ exist and are Lipschitz continuous. Then, after differentiating both sides of (4.4.31), there exists an second order differentiable operator $\hat{\mathcal{L}}$ such that on $R^o$

$$\hat{\mathcal{L}}\hat{f}(i,k)+\hat{f}_i(i,k)+\frac{r}{k^2}f(i,k)=-\frac{u'(i)-rq}{k^2}, \tag{4.4.32}$$

where $\hat{f}:=f_k$. Noticing $f(i,k)\geq 0$, we obtain

$$\hat{\mathcal{L}}\hat{f}(i,k)+\hat{f}_i(i,k)\leq -\frac{u'(i)-rq}{k^2}. \tag{4.4.33}$$

Take $\psi\in C_c^\infty([k_0,k_1])$ (where $C_c^\infty$ denotes the set of infinitely continuously differentiable functions with compact support) such that $\psi\geq 0$ and $\int_{k_0}^{k_1}\psi(x)dx=1$. Further define

$$F_\psi(i):=\int_{k_0}^{k_1}\hat{f}_i(i,k)\psi(k)dk. \tag{4.4.34}$$

We denote by $\hat{\mathcal{L}}^*$ the adjoint operator of $\hat{\mathcal{L}}$. So we use integration by parts repeatedly to get

$$\begin{aligned} F_\psi(i) &\leq \int_{k_0}^{k_1}\{-\frac{u'(i)-rq}{k^2}\psi(k)-\hat{\mathcal{L}}\hat{f}(i,k)\psi(k)\}dk \\ &= \int_{k_0}^{k_1}\{-\frac{u'(i)-rq}{k^2}\psi(k)-\hat{f}(i,k)\hat{\mathcal{L}}^*\psi(k)\}dk \\ &= \int_{k_0}^{k_1}\{-\frac{u'(i)-rq}{k^2}\psi(k)+f(i,k)\frac{\partial}{\partial k}(\hat{\mathcal{L}}^*\psi(k))\}dk. \end{aligned} \tag{4.4.35}$$

Let $i\downarrow i_0$, by dominated convergence, we obtain

$$F_\psi(i_0+)\leq \int_{k_0}^{k_1}-\frac{u'(i_0)-rq}{k^2}\psi(k)dk<0. \tag{4.4.36}$$

Hence, there exists $l,\epsilon>0$ such that $F_\psi(i)\leq -l$ on $(i_0,i_0+\epsilon]$. Therefore, for any $\delta\in(0,\epsilon)$, using Fubini's theorem, we have

$$\begin{aligned} -l(\epsilon-\delta) &\geq \int_\delta^\epsilon F_\psi(i_0+i)di \\ &= \int_{k_0}^{k_1}(\hat{f}(i_0+\epsilon,k)-\hat{f}(i_0+\delta,k))\psi(k)dk \\ &= \int_{k_0}^{k_1}f_k(i_0+\epsilon,k)\psi(k)dk+\int_{k_0}^{k_1}f(i_0+\delta,k)\psi'(k)dk \end{aligned} \tag{4.4.37}$$

Let $\delta\downarrow 0$, by dominated convergence

$$0>-l\epsilon\geq \int_{k_0}^{k_1}f_k(i_0+\epsilon,k)\psi(k)dk. \tag{4.4.38}$$

However, since $\mathbf{W}$ is decreasing with respect to $k$, we must have $f_k \geq 0$, which leads to a contradiction. $\diamondsuit$

*Remark* 4.4.6. The continuity of $b_q$ implies $a_q$ is a strictly decreasing function.

By Proposition 4.4.5, there exists a constant, denoted by $i'_q$ such that $b_q(i) > 0$ for all $i < i'_q$ and $b_q(i'_q) = 0$. It is obvious that $i'_q \leq i_q$.

Figure 5 provides a possible shape of $C$, $D_p$ and $D_q$ based on our previous analysis.

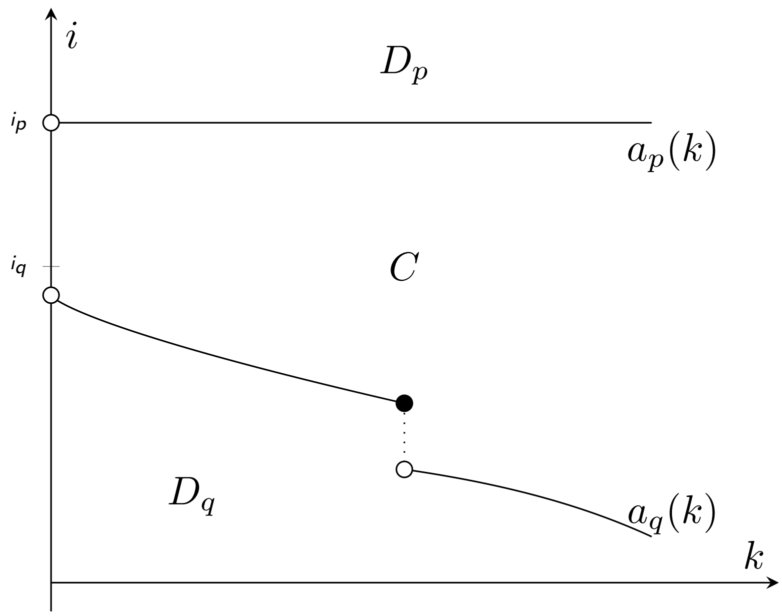


Figure 5: An illustration of the continuation set, stopping sets, and free boundaries.

### 4.4.3 Smooth-pasting principle

When studying optimal stopping problems, it is often important to verify that the value function is differentiable on the free-boundaries. Such property is called the *smooth-pasting principle*. In the sequel, we will show the existence of $\mathbf{W}_k$ along $b_q$ [4] and $\mathbf{W}$ is $C^{1,1}$ across $a_p$.

**Proposition 4.4.7.** *For all $i \in \mathbb{R}$, if $b_q(i) > 0$, then $\mathbf{W}_k(i, b_q(i))$ exists and*

$$\mathbf{W}_k(i, b_q(i)) = 0. \tag{4.4.39}$$

[4] Of course, it is equivalent to prove this along $a_q$.

*Proof.* Take $k > 0$ and $i < i_p$. Let $(\tau, \sigma)$ denote the optimal stopping times for $(i,k)$. For any stopping time $\rho$, by Lemma 4.4.3 and the fact that $\hat{I}_t \geq i$ for all $t \geq 0$ $\mathbb{P}$-a.s., we get

$$\begin{aligned}
\mathbf{W}(i,k) - \frac{u'(i)}{r} &\leq \mathbb{E}\Big[e^{-r\rho\wedge\tau}\mathbf{W}(\hat{I}_{\rho\wedge\tau}, K_{\rho\wedge\tau}) + \int_0^{\rho\wedge\tau} e^{-rt}u'(\hat{I}_t)dt\Big] - \frac{u'(i)}{r} \\
&\leq \mathbb{E}\Big[e^{-r\rho\wedge\tau}\mathbf{W}(\hat{I}_{\rho\wedge\tau}, K_{\rho\wedge\tau}) + \int_0^{\rho\wedge\tau} e^{-rt}u'(i)dt\Big] - \frac{u'(i)}{r} \\
&= \mathbb{E}\Big[e^{-r\rho\wedge\tau}\big\{\mathbf{W}(\hat{I}_{\rho\wedge\tau}, K_{\rho\wedge\tau}) - \frac{u'(i)}{r}\big\}\Big]
\end{aligned} \tag{4.4.40}$$

Take $a < k < b$ and let $\tau_a, \tau_b$ denote the hitting time of $K$ at $a$ and $b$ respectively. Substituting $\rho = \tau_a \wedge \tau_b$ into the above inequality, we find

$$\begin{aligned}
\mathbf{W}(i,k) - \frac{u'(i)}{r} \leq\; & \mathbb{E}\Big[e^{-r\tau}\big\{\mathbf{W}(\hat{I}_\tau, K_\tau) - \frac{u'(i)}{r}\big\}\mathbb{1}_{\tau<\tau_a\wedge\tau_b}\Big] \\
&+ \mathbb{E}\Big[e^{-r\tau_a}\big\{\mathbf{W}(\hat{I}_{\tau_a}, a) - \frac{u'(i)}{r}\big\}\mathbb{1}_{\tau>\tau_a\wedge\tau_b}\mathbb{1}_{\tau_a<\tau_b}\Big] \\
&+ \mathbb{E}\Big[e^{-r\tau_b}\big\{\mathbf{W}(\hat{I}_{\tau_b}, b) - \frac{u'(i)}{r}\big\}\mathbb{1}_{\tau>\tau_a\wedge\tau_b}\mathbb{1}_{\tau_b<\tau_a}\Big].
\end{aligned} \tag{4.4.41}$$

Note that $\mathbf{W}(\hat{I}_\tau, K_\tau) - \frac{u'(i)}{r} = p - \frac{u'(i)}{r} \leq 0$ as $u'(i)/r > p$. Since $\mathbf{W}$ is decreasing with respect to $i$, and $W(i,b) > p$ and $W(i,a) > p$, we can derive

$$\begin{aligned}
&\mathbf{W}(i,k) - \frac{u'(i)}{r} \\
\leq& \mathbb{E}\Big[e^{-r\tau_a}\big\{p - \frac{u'(i)}{r}\big\}\mathbb{1}_{\tau<\tau_a\wedge\tau_b}\mathbb{1}_{\tau_a<\tau_b}\Big] + \mathbb{E}\Big[e^{-r\tau_b}\big\{p - \frac{u'(i)}{r}\big\}\mathbb{1}_{\tau<\tau_a\wedge\tau_b}\mathbb{1}_{\tau_b<\tau_a}\Big] \\
&+ \mathbb{E}\Big[e^{-r\tau_a}\big\{\mathbf{W}(i,a) - \frac{u'(i)}{r}\big\}\mathbb{1}_{\tau>\tau_a\wedge\tau_b}\mathbb{1}_{\tau_a<\tau_b}\Big] \\
&+ \mathbb{E}\Big[e^{-r\tau_b}\big\{\mathbf{W}(i,b) - \frac{u'(i)}{r}\big\}\mathbb{1}_{\tau>\tau_a\wedge\tau_b}\mathbb{1}_{\tau_b<\tau_a}\Big] \\
\leq& \mathbb{E}\Big[e^{-r\tau_a}\big\{\mathbf{W}(i,a) - \frac{u'(i)}{r}\big\}\mathbb{1}_{\tau_a<\tau_b}\Big] + \mathbb{E}\Big[e^{-r\tau_b}\big\{\mathbf{W}(i,b) - \frac{u'(i)}{r}\big\}\mathbb{1}_{\tau_b<\tau_a}\Big].
\end{aligned} \tag{4.4.42}$$

Set $f(k;i) := \mathbf{W}(i,k) - \frac{u'(i)}{r}$. We have shown, for $i < i_p$ and $k \in (a,b)$,

$$f(k;i) \leq f(a;i)\mathbb{E}[e^{-r\tau_a}\mathbb{1}_{\tau_a<\tau_b}] + f(b;i)\mathbb{E}[e^{-r\tau_b}\mathbb{1}_{\tau_b<\tau_a}]. \tag{4.4.43}$$

Let $\phi$ and $\psi$ are the unique decreasing/increasing fundamental solutions to $\mathcal{L}f - rf = 0$. Define $h(k;i) := \frac{f(k;i)}{\phi(k)}$ and $\tilde{s} := \psi/\phi$. Following (2.9)-(2.13) in [63], we can prove that $h$ is $\tilde{s}$-convex. Hence, we have

$$x \mapsto \frac{h(x;i) - h(k;i)}{\tilde{s}(x) - \tilde{s}(k)} \text{ is increasing on } x \neq k. \tag{4.4.44}$$

Now choose $i$ such that $b_q(i) > 0$ and set $k = b_q(i)$. Since $\mathbf{W} \leq q$ and $\mathbf{W}(i,k) = q$, for $\epsilon > 0$ sufficiently small

$$\begin{aligned}\frac{\frac{q-u'(i)/r}{\phi(k-\epsilon)} - \frac{q-u'(i)/r}{\phi(k)}}{\tilde{s}(k-\epsilon) - \tilde{s}(k)} &\leq \frac{h(k-\epsilon;i) - h(k;i)}{\tilde{s}(k-\epsilon) - \tilde{s}(k)} \\ &\leq \frac{h(k+\epsilon;i) - h(k;i)}{\tilde{s}(k+\epsilon) - \tilde{s}(k)} \leq \frac{\frac{q-u'(i)/r}{\phi(k+\epsilon)} - \frac{q-u'(i)/r}{\phi(k)}}{\tilde{s}(k+\epsilon) - \tilde{s}(k)}. \end{aligned} \tag{4.4.45}$$

Note that under Assumption 4.2.1, $\phi$ and $\psi$ are both $C^1$. Therefore, by mimicking Theorem 2.3 in [63], taking $\epsilon \to 0$ gives $W_k(i, b_q(i)) = f'(b_q(i);i) = 0$. $\diamondsuit$

Along the boundary $a_p$, we can show the existence and continuity of $\mathbf{W}_i$ and $\mathbf{W}_k$, which ensures the differentiability of $\mathbf{W}$. To do so, we need the following lemma.

**Lemma 4.4.8.** *Let $\tilde{C}$ denote the union of $C$ and boundary $a_p$, i.e. $\tilde{C} := C \cup \{(i_p, k) : k > 0\}$. Take an arbitrary sequence $(i_n, k_n) \subset C$ which converges to $(i,k) \in \tilde{C}$. Denote the hitting time of $\hat{I}^{i_n,k_n}$ at $i_p$ by $\tau_n$ and the hitting time of $\hat{I}^{i,k}$ at $i_p$ by $\tau$. Then, $\tau_n \to \tau$ $\mathbb{P}$-a.s..*

*Proof.* Fix $\omega \in \Omega$. By the definition of $\tau_n$, for all $n \in \mathbb{N}$,

$$i_n + \int_0^\infty K_t^{k_n}(\omega) \mathbb{1}_{t<\tau_n(\omega)} dt = i_p. \tag{4.4.46}$$

Set $k^* := \max k_n$ and $k_* := \min k_n$. Denote the corresponding hitting time of $\hat{I}$ at $i_p$ by $\tau^*$ and $\tau_*$ respectively. Then, by Lemma 4.2.3, $\tau^*(\omega) \leq \tau_n(\omega) \leq \tau_*(\omega)$ for every $n$. Since $\tau_*$ is (almost surely) finite it follows $\tau_n(\omega)$ is a bounded sequence. Thus, it has a convergent subsequence, and for ease of notation, let us still use $\tau_n(\omega)$ to denote this convergent subsequence from now on and we denote its limit by $T$. Moreover, by Lemma 4.2.4, there exists a further subsequence, still denoted by $K^{k_n}(\omega)$ such that $K_t^{k_n}(\omega) \to K_t^k(\omega)$ for each $t \geq 0$, which implies

$$\lim_{n\to\infty} K_t^{k_n}(\omega) \mathbb{1}_{t<\tau_n(\omega)} = K_t^k(\omega) \mathbb{1}_{t\leq T}. \tag{4.4.47}$$

Moreover, by Lemma 4.2.3, we conclude, for all $n \in \mathbb{N}$ and $t \geq 0$,

$$|K_t^{k_n}(\omega) \mathbb{1}_{t<\tau_n(\omega)}| \leq K_t^{k^*}(\omega) \mathbb{1}_{t\leq\tau_*(\omega)}. \tag{4.4.48}$$

Since $\tau_*$ is (almost surely) finite and $K^{k^*}$ does not explode, it follows,

$$\int_0^\infty K_t^{k^*}(\omega)\mathbb{1}_{t\le\tau_*(\omega)}dt < \infty. \tag{4.4.49}$$

Therefore, we can apply Lebesgue's dominated convergence theorem to (4.4.46) and obtain

$$\begin{aligned} i_p &= \lim_{n\to\infty}\left\{i_n + \int_0^\infty K_t^{k_n}(\omega)\mathbb{1}_{t<\tau_n(\omega)}dt\right\} \\ &= i + \int_0^\infty K_t^k(\omega)\mathbb{1}_{t\le T}dt = i + \int_0^\infty K_t^k(\omega)\mathbb{1}_{t<T}dt. \end{aligned} \tag{4.4.50}$$

On the other hand, by the definition of $\tau$,

$$i + \int_0^\infty K_t^k(\omega)\mathbb{1}_{t<\tau(\omega)}dt = i_p. \tag{4.4.51}$$

Since $K_t^k(\omega) > 0$ for all $t \ge 0$, we must have $\tau(\omega) = T$ otherwise it is easy to find a contradiction. Thus, we have shown that every convergent subsequence of the bounded sequence $\tau_n(\omega)$ converges to the same limit $\tau(\omega)$. Thus, $\tau_n(\omega)$ must also converge to $\tau(\omega)$. As we can prove this except for $\omega$ in a $\mathbb{P}$-null set, it follows $\tau_n \to \tau$ $\mathbb{P}$-a.s.. $\diamondsuit$

The concept of *stochastic flow* will play an important role in the proof of the smooth-pasting principle. Define a function $\phi(k) : k \to K_t^k(\omega)$ for fixed $(t,\omega)$. If there exists a universal set $\mathcal{Z} \subset \Omega$ with $\mathbb{P}(\mathcal{Z}) = 0$ such that $\phi$ is continuously differentiable on $\mathbb{R}_+^\circ$ for every $\omega \in \Omega\setminus\mathcal{Z}$ and $t \ge 0$, then we say $K$ has a continuously differentiable stochastic flow. The first derivative of $\phi$ is denoted by $\partial K_t(\omega)$.

Under Assumption 4.2.1, by Theorem 39 in Protter [58] p.305, we know $K$ generates continuously differentiable flows, and $\partial K$ solves the following SDE:

$$\partial K_t = 1 + \int_0^t \mu'(K_s)\partial K_s ds + \int_0^t \sigma'(K_s)\partial K_s dW_s. \tag{4.4.52}$$

Moreover, if we apply Theorem 39 in [58] again to the joint process $(\hat{K}, K)$, then we see that $\hat{K}$ also generates continuously differentiable flows (in variable $k$) and the first derivative $\partial\hat{K}$ (in the $k$ direction) satisfies

$$\partial\hat{K}_t = \int_0^t \partial K_s ds. \tag{4.4.53}$$

Our proof of smooth-pasting principle relies on the following assumption.

**Assumption 4.4.9.** $u'$ is global Lipschitz with the Lipschitz constant $L_u$, and when $(i_p - i)^+$ is sufficiently small, for each $k > 0$, there exists a constant $R > 0$ such that

$$\mathbb{E}\Big[\sup_{y\in B(k,R)}\int_0^{\tau^y}\sup_{x\in B(k,R)} e^{-rt}|\partial\hat{K}_t^x|dt\Big] < \infty, \tag{4.4.54}$$

where $B(k,R)$ denotes the ball centered at $k$ with radius $R$ and $\tau^y := \inf\{t \geq 0 : \hat{I}_t^{i,y} = i_p\}$.

*Remark* 4.4.10. When $K$ is a geometric Brownian motion, it satisfies (4.4.54) because $\partial\hat{K}_t = \int_0^t K_s^1 ds$. Then, by Assumption 4.2.7, it follows

$$\mathbb{E}\Big[\sup_{y\in B(k,R)}\int_0^{\tau^y}\sup_{x\in B(k,R)} e^{-rt}|\partial\hat{K}_t^x|dt\Big] \leq \int_0^\infty e^{-rt}\mathbb{E}\Big[\partial\hat{K}_t\Big]dt < \infty. \tag{4.4.55}$$

**Proposition 4.4.11.** *For every sequence $(i_n, k_n) \subset C$ which converges to $(i_p, k)$ where $k > 0$, it follows*

$$\lim_{n\to\infty}\mathbf{W}_i(i_n,k_n) = \lim_{n\to\infty}\mathbf{W}_k(i_n,k_n) = 0. \tag{4.4.56}$$

*Proof.* This proof largely mimics the proof of Theorem 8 in [17]. Take $k > 0$ and suppose $(i_n, k_n) \subset C$ converges to $(i_p, k)$. We will only prove $\lim_{n\to\infty}\mathbf{W}_k(i_n,k_n) = 0$ as the proof for $\mathbf{W}_i$ is analogous.
First note that, as $\mathbf{W}$ is decreasing with respect to $k$, we must have

$$\limsup_{n\to\infty}\mathbf{W}_k(i_n,k_n) \leq 0.$$

Next, fix a sequence $\epsilon_n \downarrow 0$ as $n \to \infty$. Without loss of generality, let us assume that

$$\liminf_{n\to\infty}\mathbf{W}_k(i_n,k_n) = \lim_{n\to\infty}\frac{\mathbf{W}(i_n,k_n+\epsilon_n) - \mathbf{W}(i_n,k_n)}{\epsilon_n}. \tag{4.4.57}$$

Let $\tau_n^\epsilon$ (resp. $\tau_n$) and $\sigma_n^\epsilon$ (resp. $\sigma_n$) denote the optimal stopping times for the sup and inf players corresponding to the initial position $(i_n, k_n + \epsilon_n)$ (resp. $(i_n, k_n)$). By the definition of $\mathbf{W}$, we get

$$\begin{aligned}&\mathbf{W}(i_n,k_n+\epsilon_n) - \mathbf{W}(i_n,k_n)\\ &\geq J(i_n,k_n+\epsilon_n;\tau_n,\sigma_n^\epsilon) - J(i_n,k_n;\tau_n,\sigma_n^\epsilon)\\ &= \mathbb{E}\Big[\int_0^{\tau_n\wedge\sigma_n^\epsilon} e^{-rt}\big\{u'(\hat{I}_t^{i_n,k_n+\epsilon_n}) - u'(\hat{I}_t^{i_n,k_n})\big\}dt\Big].\end{aligned} \tag{4.4.58}$$

By the global Lipschitz continuity of $u'$ and the Mean Value Theorem, for each time

$t \geq 0$, there exists a constant $\eta_t^n$ valued in $(k_n, k_n + \epsilon_n)$ such that

$$|u'(\hat{I}_t^{i_n, k_n+\epsilon_n}) - u'(\hat{I}_t^{i_n, k_n})| \leq L_u \big|\hat{K}_t^{k_n+\epsilon_n} - \hat{K}_t^{k_n}\big| = L_u |\partial \hat{K}_t^{\eta_t^n}| \epsilon_n. \tag{4.4.59}$$

Therefore, after dividing both sides of (4.4.58) by $\epsilon_n$ and taking $n \to \infty$, we obtain

$$\liminf_{n\to\infty} \mathbf{W}_k(i_n, k_n) \geq -L_u \lim_{n\to\infty} \mathbb{E}\Big[\int_0^{\tau_n \wedge \sigma_n^\epsilon} e^{-rt} |\partial \hat{K}_t^{\eta_t^n}| dt\Big] = 0, \tag{4.4.60}$$

where the last equality follows from the fact that $\tau_n$ converges to 0 $\mathbb{P}$-a.s. by Lemma 4.4.8 and the Dominated Convergence Theorem which is applicable thanks to Assumption 4.4.9. $\diamondsuit$

We are ready to show the $C^{1,2}$ smoothness of $\mathbf{W}$ in $C \cup D_p$.

**Theorem 4.4.12.** *In $C \cup D_p$, $\mathbf{W} \in C^{1,2}$.*

*Proof.* We have already proved $\mathbf{W} \in C^{1,2}(C)$ and $\mathbf{W} = p$ in $D_p$. By Proposition 4.4.11, we see that $\mathbf{W}_k$ and $\mathbf{W}_i$ exist along $a_p(k)$ for every $k > 0$ and they are continuous. Therefore, $\mathbf{W} \in C^{1,1}(C \cup D_p)$. Moreover,

$$\frac{1}{2}\sigma^2(k)\mathbf{W}_{kk}(i,k) + \mu(k)\mathbf{W}_k(i,k) + k\mathbf{W}_i(i,k) - r\mathbf{W}(i,k) = -u'(i), \quad (i,k) \in C. \tag{4.4.61}$$

Taking $(i, k)$ to $(i_p, k')$, we find for all $k' > 0$,

$$\lim_{(i,k)\to(i_p,k')} \frac{1}{2}\sigma^2(k)\mathbf{W}_{kk}(i,k) - rp = -u'(i_p) = -rp. \tag{4.4.62}$$

Since $\sigma^2(k) > 0$, we conclude that $\lim_{(i,k)\to(i_p,k')} \mathbf{W}_{kk}(i,k) = 0$, i.e. $\mathbf{W} \in C^{1,2}(C \cup D_p)$. $\diamondsuit$

### 4.4.4 Characterisation of W and $b_q$

Recall the free-boundary problem (4.4.10). Given the smooth-pasting principle and our knowledge of $b_q$, the free-boundary problem can be updated as follows.

Consider $w : S \to \mathbb{R}_+$ and $\beta : \mathbb{R} \to \mathbb{R}_+$. For ease of notation, let us introduce $C^\beta$, and $D_q^\beta$ and recall $D_p$, which are defined by

$$C^\beta := \{(i,k) \in S : \beta(i) < k \text{ and } i < i_p\}; \tag{4.4.63}$$

$$D_q^\beta := \{(i,k) \in S : k \leq \beta(i)\}; \tag{4.4.64}$$

$$D_p := \{(i,k) \in S : i \geq i_p\}. \tag{4.4.65}$$

Let $\mathcal{B}$ denote a set of functions that is given by

$$\mathcal{B} := \left\{ \begin{array}{c} f : \mathbb{R} \to \mathbb{R}_+ \text{ such that } f \text{ is continuous,} \\ \text{decreasing, surjective, and } f(i_q) = 0 \end{array} \right\}. \tag{4.4.66}$$

Then, we say $(w, \beta)$ is a solution to the free-boundary problem (4.4.67) defined below if $w \in C(S)$ and $\beta \in \mathcal{B}$, and all conditions specified in (4.4.67) are satisfied.

$$\begin{cases} \mathcal{L}w(i,k) + kw_i(i,k) - rw(i,k) = -u'(i), & (i,k) \in C^\beta; \\ w(i,k) = p, & (i,k) \in D_p; \\ w(i,k) = q, & (i,k) \in D_q^\beta; \\ w \in C^{1,2}(C^\beta \cup D_p); & \\ w_k(i, \beta(i)) = 0, & i < \beta^{-1}(0); \\ p \le w(i,k) \le q, & (i,k) \in S. \end{cases} \tag{4.4.67}$$

The main result of this subsection is given by the following theorem.

**Theorem 4.4.13.** $(\mathbf{W}, b_q)$ *is the unique solution to the free-boundary problem (4.4.67).*

It is easy to see that $(\mathbf{W}, b_q)$ solves (4.4.67) by Proposition 4.4.2, 4.4.7 and Theorem 4.4.12. Therefore, the rest of this section is devoted to the proof of uniqueness.

Let $(w, \beta)$ be an arbitrary solution of (4.4.67). Then, we can show there exists a probabilistic representation of $w$ that depends on $\beta$.

**Proposition 4.4.14.** *For all* $(i,k) \in S$*,* $w(i,k)$ *can be written as*

$$\begin{aligned} w(i,k) = \mathbb{E}_{i,k}\Big[\int_0^\infty e^{-rs} u'(\hat{I}_s) \mathbb{1}_{\beta(\hat{I}_s) < K_s} \mathbb{1}_{\hat{I}_s < i_p} ds\Big] \\ + \mathbb{E}_{i,k}\Big[\int_0^\infty e^{-rs} \big\{ rp \mathbb{1}_{\hat{I}_s \ge i_p} ds + rq \mathbb{1}_{K_s \le \beta(\hat{I}_s)} \big\} ds\Big], \end{aligned} \tag{4.4.68}$$

*where the initial position of* $(\hat{I}, K)$ *is emphasized with subscript* $\mathbb{E}_{i,k}$*. Moreover,* $\beta(i)$ *is a solution to the following integral equation.*

$$\begin{aligned} q = \int_0^\infty e^{-rs} \mathbb{E}_{i,\beta(i)} \Big[u'(\hat{I}_s) \mathbb{1}_{\beta(\hat{I}_s) < K_s} \mathbb{1}_{\hat{I}_s < i_p}\Big] ds \\ + \int_0^\infty e^{-rs} \big\{ rp \mathbb{P}_{i,\beta(i)}(\hat{I}_s \ge i_p) + rq \mathbb{P}_{i,\beta(i)}(K_s \le \beta(\hat{I}_s) \big\} ds. \end{aligned} \tag{4.4.69}$$

*Proof.* The main idea is to apply a general version of Itô's rule to $e^{-rt} w(\hat{I}_t, K_t)$. Despite that the $C^{1,2}$ smoothness of $w$ along $b_q$ is unknown, as $\hat{I}$ is a finite variation process, we are in the setting of the change-of-variable formula derived by Du Toit

[20] Theorem 5.1. Following Remark 5.4 in [20], to use this result, it is sufficient to check the following conditions:

$\beta$ and $w$ are continuous functions;

$b_t := \beta(\hat{I}_t)$ has paths of bounded variation;

$w_k(i, \beta(i)\pm)$ exist for all $i$ such that $\beta(i) > 0$;

$t \mapsto w_k(\hat{I}_t, b_t+) - w_k(\hat{I}_t, b_t-)$ is continuous on $\{t < \rho\}$ where $\rho := \inf\{s : b_s = 0\}$;

$u(i)\mathbb{1}_{(i,k)\in C^\beta} + rp\mathbb{1}_{(i,k)\in D_p^\beta} + rq\mathbb{1}_{(i,k)\in D_q^\beta}$ is locally bounded.

It turns out these conditions are satisfied simply because $(w, \beta)$ is a solution to the free-boundary problem (4.4.67). Therefore, for any fixed time $t \geq 0$, we can apply the change-of variable-formula to $e^{-rt}w(\hat{I}_t, K_t)$ and obtain

$$\begin{aligned}
w(i,k) &= -\int_0^t e^{-rs}\{\mathcal{L}w(\hat{I}_s, K_s-) - rw(\hat{I}_s, K_s) + kw_i(\hat{I}_s, K_s-)\}ds \\
&\quad + e^{-rt}w(\hat{I}_t, K_t) - M_t \\
&= -\int_0^t e^{-rs}\{\mathcal{L}w(\hat{I}_s, K_s) - rw(\hat{I}_s, K_s) + kw_i(\hat{I}_s, K_s)\}\mathbb{1}_{\beta(\hat{I}_s)<K_s}\mathbb{1}_{\hat{I}_s<i_p}ds \\
&\quad + \int_0^t e^{-rs}rp\mathbb{1}_{\hat{I}_s\geq i_p}ds + \int_0^t e^{-rs}rq\mathbb{1}_{K_s\leq\beta(\hat{I}_s)}ds + e^{-rt}w(\hat{I}_t, K_t) - M_t. \\
&= \int_0^t e^{-rs}u'(\hat{I}_s)\mathbb{1}_{\beta(\hat{I}_s)<K_s}\mathbb{1}_{\hat{I}_s<i_p}ds + \int_0^t e^{-rs}rp\mathbb{1}_{\hat{I}_s\geq i_p}ds \\
&\quad + \int_0^t e^{-rs}rq\mathbb{1}_{K_s\leq\beta(\hat{I}_s)}ds + e^{-rt}w(\hat{I}_t, K_t) - M_t
\end{aligned} \tag{4.4.70}$$

where $M_t = \int_0^t e^{-rs}w_k(\hat{I}_s, K_s-)dW_s$. Take $T_n := \inf\{t \geq 0 : |\hat{I}_t| \vee K_t \geq n \text{ or } K_t \leq 1/n\}$. Then $M_{t\wedge T_n}$ is a martingale with initial value 0. Thus,

$$\begin{aligned}
w(i,k) &= \mathbb{E}_{i,k}\Big[\int_0^{t\wedge\mathcal{T}_n} e^{-rs}u'(\hat{I}_s)\mathbb{1}_{\beta(\hat{I}_s)<K_s}\mathbb{1}_{\hat{I}_s<i_p}ds\Big] \\
&\quad + \mathbb{E}_{i,k}\Big[\int_0^{t\wedge\mathcal{T}_n} e^{-rs}\big\{rp\mathbb{1}_{\hat{I}_s\geq i_p} + rq\mathbb{1}_{K_s\leq\beta(\hat{I}_s)}\big\}ds\Big] + \mathbb{E}_{i,k}\Big[e^{-rt\wedge\mathcal{T}_n}w(\hat{I}_{t\wedge\mathcal{T}_n}, K_{t\wedge\mathcal{T}_n})\Big]
\end{aligned} \tag{4.4.71}$$

By the boundedness of $w$ and Proposition 4.2.11, we can take $t$ and $n$ to infinity and use the Dominated Convergence Theorem, which gives (4.4.68). Finally, (4.4.69) is a direct consequence of (4.4.68) by substituting $(i, \beta(i))$ on the left hand side of (4.4.68) and using Fubini's theorem. $\diamondsuit$

We define a progressively measurable process $h_t$ by

$$h_t := e^{-rt}w(I_t, K_t) + \int_0^t e^{-rs}\big\{u'(\hat{I}_s)\mathbb{1}_{\beta(\hat{I}_s)<K_s}\mathbb{1}_{\hat{I}_s<i_p} + rp\mathbb{1}_{\hat{I}_s\geq i_p} + rq\mathbb{1}_{K_s\leq\beta(\hat{I}_s)}\big\}ds. \tag{4.4.72}$$

Then, it is not hard to show

**Lemma 4.4.15.** *$h_t$ is a U.I. martingale.*

*Proof.* Arbitrarily choose a stopping time $\tau$. Because $(I, K)$ is a Feller (or Feller-Dynkin) process, hence a strong Markov process, it follows from the probabilistic representation of $w$ (4.4.68) that

$$e^{-r\tau}w(I_\tau, K_\tau) = \mathbb{E}\Big[\int_\tau^\infty e^{-rs}\big\{u'(\hat{I}_s)\mathbb{1}_{\beta(\hat{I}_s)<K_s}\mathbb{1}_{\hat{I}_s<i_p}+rp\mathbb{1}_{\hat{I}_s\geq i_p}+rq\mathbb{1}_{K_s\leq\beta(\hat{I}_s)}\big\}ds\big|F_\tau\Big]. \tag{4.4.73}$$

Therefore, we see that $\mathbb{E}[h_\tau] = w(i, k)$. Moreover, it is also easy to check that $|h_t|$ is a bounded process since $w$ is a bounded function. Thus, $h_t$ is a U.I. martingale (see e.g. Theorem II.77.6 in [61] p.190). ♢

We can now prove Theorem 4.4.13.

*Proof of Theorem 4.4.13. Existence.* By Proposition 4.3.2, $\mathbf{W}$ is continuous. By the definition of $b_q$ and Proposition 4.4.5, it is obvious that $b_q \in \mathcal{B}$. All conditions in (4.4.67) follow from Proposition 4.4.2, Proposition 4.4.7, and Theorem 4.4.12.

*Uniqueness.* Let $(w, \beta)$ be an arbitrary solution to (4.4.67). By the probabilistic representation (4.4.68), it is sufficient to prove $\beta \geq b_q$ and $\beta \leq b_q$.
To begin with, suppose there exists $i_0 < i_q$ such that $b_q(i_0) < \beta(i_0)$. Take $k_0 \in (b_q(i_0), \beta(i_0))$. Set $\tau$ and $\sigma$ be the optimal stopping times for $\mathbf{W}$ as in Theorem 4.4.1. Then, by Lemma 4.4.15 and the definitions of $\tau$ and $\sigma$, we get

$$\begin{aligned} w(i_0, k_0) &= \mathbb{E}_{i_0,k_0}\Big[e^{-r\tau\wedge\sigma}w(I_{\tau\wedge\sigma}, K_{\tau\wedge\sigma}) \\ &\qquad + \int_0^{\tau\wedge\sigma} e^{-rs}\big\{u'(\hat{I}_s)\mathbb{1}_{\beta(\hat{I}_s)<K_s}\mathbb{1}_{\hat{I}_s<i_p} + rp\mathbb{1}_{\hat{I}_s\geq i_p} + rq\mathbb{1}_{K_s\leq\beta(\hat{I}_s)}\big\}ds\Big] \\ &\leq \mathbb{E}_{i_0,k_0}\Big[e^{-r\tau}p\mathbb{1}_{\tau\leq\sigma} + e^{-r\sigma}q\mathbb{1}_{\sigma<\tau} + \\ &\qquad + \int_0^{\tau\wedge\sigma} e^{-rs}\big\{u'(\hat{I}_s)\mathbb{1}_{\beta(\hat{I}_s)<K_s}\mathbb{1}_{\hat{I}_s<i_p} + rq\mathbb{1}_{K_s\leq\beta(\hat{I}_s)}\big\}ds\Big] \\ &< \mathbb{E}_{i_0,k_0}\Big[e^{-r\tau}p\mathbb{1}_{\tau\leq\sigma} + e^{-r\sigma}q\mathbb{1}_{\sigma<\tau} + \int_0^{\tau\wedge\sigma} e^{-rs}u'(\hat{I}_s)ds\Big] \\ &= J(i_0, k_0; \tau, \sigma) = \mathbf{W}(i_0, k_0), \end{aligned} \tag{4.4.74}$$

where the first inequality holds since $w \leq q$ and the second inequality follows from the fact that on $\{K_s \leq \beta(\hat{I}_s)\}$, we have $rq < u'(\hat{I}_s)$. However, since $(i_0, k_0) \in D_q^{\beta}$, it follows $w(i_0, k_0) = q$ and hence $\mathbf{W}(i_0, k_0) > q$, which leads to a contradiction. Therefore, $\beta \geq b_q$.

On the other hand, suppose there exists $i_0 < i_q$ such that $\beta(i_0) < b_q(i_0)$. Take $k_0 \in (\beta(i_0), b_q(i_0))$. Let $\tau$ still be as in Theorem 4.4.1, i.e. the optimal stopping time for the sup-player. Define $\sigma_\beta := \inf\{t \geq 0 : K_t \leq \beta(\hat{I}_t)\}$. Then, by Lemma 4.4.15, we get

$$\begin{aligned}
\mathbf{W}(i_0, k_0) &= \mathbb{E}_{i_0,k_0}\Big[e^{-r\tau\wedge\sigma_\beta}\mathbf{W}(\hat{I}_{\tau\wedge\sigma_\beta}, K_{\tau\wedge\sigma_\beta}) \\
&\qquad + \int_0^{\tau\wedge\sigma_\beta} e^{-rs}\big\{u'(\hat{I}_s)\mathbb{1}_{b_q(\hat{I}_s)<K_s}\mathbb{1}_{\hat{I}_s<i_p} + rp\mathbb{1}_{\hat{I}_s\geq i_p} + rq\mathbb{1}_{K_s\leq b_q(\hat{I}_s)}\big\}ds\Big] \\
&\leq \mathbb{E}_{i_0,k_0}\Big[e^{-r\tau}p\mathbb{1}_{\tau\leq\sigma_\beta} + e^{-r\sigma_\beta}q\mathbb{1}_{\sigma_\beta<\tau} \\
&\qquad + \int_0^{\tau\wedge\sigma_\beta} e^{-rs}\big\{u'(\hat{I}_s)\mathbb{1}_{b_q(\hat{I}_s)<K_s}\mathbb{1}_{\hat{I}_s<i_p} + rq\mathbb{1}_{K_s\leq b_q(\hat{I}_s)}\big\}ds\Big] \\
&< \mathbb{E}_{i_0,k_0}\Big[e^{-r\tau\wedge\sigma_\beta}w(\hat{I}_{\tau\wedge\sigma_\beta}, K_{\tau\wedge\sigma_\beta}) + \int_0^{\tau\wedge\sigma_\beta} e^{-rs}u'(\hat{I}_s)ds\Big] \\
&= \mathbb{E}_{i_0,k_0}[h_{\tau\wedge\sigma_\beta}] = w(i_0, k_0),
\end{aligned} \tag{4.4.75}$$

where the first inequality holds since $\mathbf{W} \leq q$ and the second inequality follows from the fact that $rq < u'(\hat{I}_s)$ on $\{K_s \leq b_q(\hat{I}_s)\}$. However, since $(i_0, k_0) \in D_p$, we know $\mathbf{W}(i_0, k_0) = q$, which contradicts to the condition that $w(i_0, k_0) \leq q$ . Therefore, $\beta \geq b_q$. ♢

*Remark* 4.4.16. In some cases (e.g. [15, 28]), one can prove that $\beta_q$ is the unique solution to the integral equation (4.4.69) over a space of functions like $\mathcal{B}$. However, that normally requires the knowledge of the probability density function of the space variable. In our case, $\hat{I}$ is an integral process whose density is unknown, which makes it nontrivial to obtain uniqueness. Given this technical difficulty, we leave the discussion of the uniqueness of the solution to (4.4.69) to future research.

## 4.5 Viscosity solution and regularity of $\mathbf{V}_k$

For the value function $\mathbf{V}$ defined by (RCP), we have shown $\mathbf{V}_i$ is well defined and continuous on $S$. We would like to further explore the smoothness of $\mathbf{V}$, in particular in the $k$-direction. The smoothness of $\mathbf{V}$ will play a key role in the verification of the candidate optimal control in Section 4.6. To achieve this, we require Assumption 4.5.1 which is stated below. We stress that all results proved in Section 4.2, 4.3 and

4.4 do *not* rely on this assumption.

**Assumption 4.5.1.** Either of the three conditions is satisfied:

1. $\sigma(k) = \sigma k$ and $\mu(k) = \mu k$, i.e. $K$ is a geometric Brownian motion.
2. $\sigma(k) = \sigma k$, $\mu(k)$ is concave, and $u$ is an increasing function.
3. $\sigma(k) = \sigma k$ and $\mu'(k)$ is globally Lipschitz with Lipschitz constant $L$ (recall Assumption 4.2.1 (iv)). Moreover, there exist constants $c_1, c_2, M_1, M_2, \kappa, \theta \in \mathbb{R}_+^\circ$ such that:
   (a) $c_2 \in [rp, rq]$, $\kappa \in (0, 1)$, and $\theta \geq 1$;
   (b) $u(x) \leq c_1(1 - |x|^\theta) + c_2 x$;
   (c) $|u(x) - u(y)| \leq M_1(1 + |x|^\theta + |y|^\theta)^\kappa |x - y|$ for all $x, y \in \mathbb{R}_+^\circ$;
   (d) For every $x_1, x_2 \in \mathbb{R}$, and $\lambda \in [0, 1]$,
   $$u(x^\lambda) - \lambda u(x_1) - (1 - \lambda)u(x_2) \leq M_2(1 + |x|^\theta + |y|^\theta)^\kappa \lambda(1 - \lambda)|x_1 - x_2|^2,$$
   where $x^\lambda := \lambda x_1 + (1 - \lambda)x_2$;
   (e) Set $p_0 = \frac{1}{\kappa}$ and $q_0 := \frac{1}{1-\kappa}$. We assume $r \geq \xi := 2q_0 L - q_0\sigma^2 + 2\sigma^2 q_0^2$.

*Remark* 4.5.2. Under Assumption 4.5.1 Condition 3, we can see that $u$ is locally semiconvex, i.e. for any $R > 0$, there exists a constant $c_R > 0$ such that in the ball with radius $R$, $u(x) + c_R|x|^2$ is convex.

*Remark* 4.5.3. An example of a reward function that satisfies Condition 3 in Assumption 4.5.1 is $u(x) = -(x - c)^2$ where $c$ is a constant. Let $\theta = 2$ and $\kappa = 0.5$. As $u$ is locally Lipschitz with a linear derivative, we see that (c) holds. $u$ is semiconvex, and hence (d) is satisfied. It is easy to see, since $u$ is a quadratic function, (b) holds for suitable $c_1$ and $c_2$.

On one hand, Assumption 4.5.1 Condition 1 or 2 enables us to improve the concavity property of $\mathbf{V}$ to the whole state space $S$.

**Proposition 4.5.4.** *Under Assumption 4.5.1 Condition 1 or 2, the payoff function $G(i, k; A, B)$ is strictly concave over $S \times \mathcal{A}$. Moreover, the value function $\mathbf{V}(i, k)$ is concave in $S$. As a result of concavity, $\mathbf{V}$ is a continuous function, and the left and right partial derivatives $\mathbf{V}_k^\pm(i, k)$ exist, and $\mathbf{V}_k^+ \leq \mathbf{V}_k^-$.*

*Proof.* Take $(i, k), (i', k') \in S$ and $(A, B), (A', B') \in \mathcal{A}$. Pick $\lambda \in (0, 1)$. Define $i^\lambda = \lambda i + (1-\lambda)i'$, $k^\lambda = \lambda k + (1-\lambda)k'$, $A^\lambda = \lambda A + (1-\lambda)A'$, and $B^\lambda = \lambda B + (1-\lambda)B'$.

Under Condition 1, we notice that

$$K_t^{k^\lambda} = \lambda K_t^k + (1-\lambda) K_t^{k'}, \quad t \geq 0. \tag{4.5.1}$$

Hence, by strict concavity of $u$, we have

$$\begin{aligned}
&G(i^\lambda, k^\lambda; A^\lambda, B^\lambda) \\
&= \mathbb{E}\Big[\int_0^\infty e^{-rt}\{u(I_t^{i^l,k^l,A^\lambda,B^\lambda})dt + pdA_t^\lambda - qdB_t^\lambda\}\Big] \\
&> \mathbb{E}\Big[\int_0^\infty e^{-rt}\{\lambda u(I_t^{i,k,A,B})dt + (1-\lambda)u(I_t^{i',k',A',B'})dt + pdA_t^\lambda - qdB_t^\lambda\}\Big] \\
&= \lambda G(i,k;A,B) + (1-\lambda)G(i',k';A',B').
\end{aligned} \tag{4.5.2}$$

Arguing analougouly as in Proposition 4.2.13, the concavity of $\mathbf{V}$ follows from the concavity of $G$.

Under Condition 2, suppose we have shown

$$K_t^{k^\lambda} \geq \lambda K_t^k + (1-\lambda) K_t^{k'}, \quad t \geq 0. \tag{4.5.3}$$

Since $u$ is increasing, we find

$$u(I_t^{i^l,k^l,A^\lambda,B^\lambda}) \geq \lambda u(I_t^{i,k,A,B}) + (1-\lambda)u(I_t^{i',k',A',B'}), \tag{4.5.4}$$

which is sufficient to prove the concavity of $G$ by following (4.5.2). So, the remaining task is to prove (4.5.3). For ease of notation, define $K^\lambda := \lambda K^k + (1-\lambda)K^{k'}$. As $K$ solves an SDE, we exploit the concavity of $\mu$ and see that

$$\begin{aligned}
K_t^\lambda &= k^\lambda + \int_0^t \{\lambda\mu(K_s^k) + (1-\lambda)\mu(K_s^{k'})\}ds + \int_0^t \sigma K_s^\lambda dW_s \\
&\leq k^\lambda + \int_0^t \mu(K_t^\lambda)ds + \int_0^t \sigma K_s^\lambda dW_s.
\end{aligned} \tag{4.5.5}$$

Since we also know that

$$K_t^{k^\lambda} = k^\lambda + \int_0^t \mu(K_t^{k^\lambda})ds + \int_0^t \sigma K_s^{k^\lambda} dW_s,$$

employing a comparison principle (see [36] Theorem 1.1), we immediately obtain (4.5.3). $\diamondsuit$

On the other hand, Assumption 4.5.1 Condition 3 is sufficient to pass the local semiconvex property of $u$ on to $\mathbf{V}$.

**Proposition 4.5.5.** *Under Assumption 4.5.1 Condition 3,* $\mathbf{V}$ *is locally semiconvex. In particular, by Theorem 2.17 in [9],* $\mathbf{V}$ *is locally Lipschitz continuous.*

Observe since $\mathbf{V}(i,k) \geq -\alpha(1+|i|^\gamma+|k|^\gamma)$ by Proposition 4.2.12, without loss of generality, we can restrict attention to the following class of control $\mathcal{A}_0$ given by

$$\mathcal{A}_0 := \{(A,B) \in \mathcal{A} : G(i,k;A,B) \geq -\alpha(1+|i|^\gamma+|k|^\gamma), \forall (i,k) \in S\}. \tag{4.5.6}$$

Before we give a proof of Proposition 4.5.5, we first derive an estimate for $|I|^\theta$.

**Lemma 4.5.6.** *Take* $R>0$ *and* $B_R$ *an open ball with radius* $R$*. Under Assumption 4.5.1 Condition 3, there exists a constant* $c_R$*, depending only on* $R$*, such that, for every* $(i,k) \in B_R \cap S$*,*

$$\sup_{(A,B)\in\mathcal{A}_0} \mathbb{E}\Big[\int_0^\infty e^{-rt}|I_t^{i,k}|^\theta dt\Big] \leq c_R. \tag{4.5.7}$$

*Proof.* Fix $(i,k) \in B_R \cap S$ and $(A,B) \in \mathcal{A}_0$. Using Condition 3 (3b), $rp \leq c_2 \leq rq$ and integration by parts, it follows that

$$\begin{aligned}
&\int_0^T e^{-rt}\{u(I_t)dt + pdA_t - qdB_t\} \\
&\leq \int_0^T e^{-rt}\{c_1 - c_1|I_t|^\theta + c_2 I_t\}dt + \int_0^T e^{-rt}\{pdA_t - qdB_t\} \\
&\leq \frac{1}{r}(c_1 + c_2|i|) - c_1\int_0^T e^{-rt}|I_t|^\theta dt + c_2\int_0^T e^{-rt}\hat{K}_t dt \\
&\quad + \int_0^T e^{-rt}\{pdA_t - c_2A_t dt\} + \int_0^T e^{-rt}\{-qdB_t + c_2B_t dt\} \\
&\leq \frac{1}{r}(c_1 + c_2|i|) - c_1\int_0^T e^{-rt}|I_t|^\theta dt + c_2\int_0^T e^{-rt}\hat{K}_t dt + pe^{-rT}A_T - qe^{-rT}B_T \\
&\leq \frac{1}{r}(c_1 + c_2|i|) - c_1\int_0^T e^{-rt}|I_t|^\theta dt + c_2\int_0^\infty e^{-rt}\hat{K}_t dt + \sup_{T\geq 0} pe^{-rT}A_T.
\end{aligned} \tag{4.5.8}$$

Therefore, taking $T \to \infty$ and expectation on both sides of the inequality, by Remark 4.2.8 and the definition of $\mathcal{A}$, we see there exists a constant $c' > 0$ such that

$$G(i,k;A,B) \leq \frac{1}{r}(c_1+c_2|i|) - c_1\mathbb{E}\Big[\int_0^\infty e^{-rt}|I_t|^\theta dt\Big] + c'(1+k^2)^{1/2} + \Lambda(1+|i|^\gamma+|k|^\gamma). \tag{4.5.9}$$

As $(A, B) \in \mathcal{A}_0$, we further obtain that

$$\mathbb{E}\Big[\int_0^\infty e^{-rt}|I_t|^\theta dt\Big] \le \frac{1}{c_1}\big\{\frac{1}{r}(c_1 + c_2|i|) + c'(1+k^2)^{1/2} + (\alpha+\Lambda)(1+|i|^\gamma + |k|^\gamma)\big\}. \tag{4.5.10}$$

Observe that the right-hand side of (4.5.10) is bounded uniformly by a constant $c_R$ as $(i, k) \in B_R$. Therefore, the proof is completed by taking the supremum over control $A_0$ on the left-hand side of (4.5.10). $\diamondsuit$

*Proof of Proposition 4.5.5.* Let $R > 0$ and $B_R$ be a ball with radius $R$. Take $(i, k), (i', k') \in B_R \cap S$ and $\lambda \in [0, 1]$. Using an equivalent definition of semiconvexity (see Proposition 4.4 in [70], p. 187), we aim at proving there exists a constant $M_R > 0$, depending only on $R$, such that

$$\mathbf{V}(i^\lambda, k^\lambda) - \lambda\mathbf{V}(i, k) - (1-\lambda)\mathbf{V}(i', k') \le M\lambda(1-\lambda)(|i-i'|^2 + |k-k'|^2), \tag{4.5.11}$$

where $i^\lambda = \lambda i + (1-\lambda)i'$ and $k^\lambda = \lambda k + (1-\lambda)k'$. Set $I := i + \hat{K}^k + B - A$ and $I' := i' + \hat{K}^{k'} + B - A$. For $(A, B) \in \mathcal{A}_0$, further define

$$K^\lambda := \lambda K^k + (1-\lambda)K^{k'}, \tag{4.5.12}$$

$$\hat{K}^\lambda := \lambda \hat{K}^k + (1-\lambda)\hat{K}^{k'}, \tag{4.5.13}$$

$$I^\lambda := \lambda I + (1-\lambda)I', \tag{4.5.14}$$

$$\tilde{I}^\lambda := i^\lambda + \hat{K}^{k^\lambda} + B - A. \tag{4.5.15}$$

We complete the proof in three steps.

*Step 1.* First, we derive a stochastic upper-bound for $|\hat{K}^{k^\lambda} - \hat{K}^\lambda|$ and $|K^k - K^{k'}|^2$. By Assumption4.5.1 Condition 3 and in particular the fact that $\mu'$ is Lipschitz, we obtain

$$\begin{aligned}
&|\mu(k^\lambda) - \lambda\mu(k) - (1-\lambda)\mu(k')| \\
&= \Big|\lambda(1-\lambda)(k-k')\int_0^1 \mu'(k^\lambda + y(1-\lambda)(k-k'))dy \\
&\quad - \lambda(1-\lambda)(k-k')\int_0^1 \mu'(k^\lambda + y\lambda(k'-k))dy\Big| \\
&= \lambda(1-\lambda)|k-k'|\Big|\int_0^1 \{\mu'(k^\lambda + y(1-\lambda)(k-k')) - \mu'(k^\lambda + y\lambda(k'-k))\}dy\Big| \\
&\le L\lambda(1-\lambda)|k-k'|^2.
\end{aligned}$$

Next, set $X := K^{k^\lambda} - K^\lambda$, then we see

$$\begin{aligned} X_t &= \int_0^t \big\{\mu(K_s^{k^\lambda}) - \mu(K_s^\lambda) + \mu(K_s^\lambda) - \lambda\mu(K_s^k) - (1-\lambda)\mu(K_s^{k'})\big\}dt + \int_0^t \sigma X_s dW_s \\ &\le \int_0^t \big\{L|X_t| + L\lambda(1-\lambda)|K_s^k - K_s^{k'}|^2\big\}dt + \int_0^t \sigma X_s dW_s. \end{aligned} \tag{4.5.16}$$

Define $E_t := \exp\big((L - \frac{1}{2}\sigma^2)t + \sigma W_t\big)$. Let $Y_t := L\lambda(1-\lambda)E_t \int_0^t \frac{|K_s^k - K_s^{k'}|^2}{E_s} ds$. Applying integration by parts, we see $Y$ solves the following SDE:

$$dY_t = \big\{L|Y_t| + L\lambda(1-\lambda)|K_s^k - K_s^{k'}|^2\big\}dt + \sigma Y_t dW_t. \tag{4.5.17}$$

By a comparison principle (see [36] Theorem 1.1), we see that $X_t \le Y_t$. By symmetry of the argument, we conclude $|X_t| \le Y_t$. Analogously we can apply the comparison principle again to $|K_t^k - K_t^{k'}|$ and $E_t$. In summary, we have

$$|K_t^k - K_t^{k'}|^2 \le |k - k'|E_t^2, \tag{4.5.18}$$

$$|K_t^{k^\lambda} - K_t^\lambda| \le L\lambda(1-\lambda)|k-k'|^2 E_t F_t, \tag{4.5.19}$$

where $F_t := \int_0^t E_s ds$. By Hölder's inequality, the upper-bound is inherited by $\hat{K}$, which leads to the inequalities

$$|\hat{K}_t^k - \hat{K}_t^{k'}|^2 \le |k-k'|^2 t\zeta_t; \tag{4.5.20}$$

$$|\hat{K}_t^{k^\lambda} - \hat{K}_t^\lambda| \le L\lambda(1-\lambda)|k-k'|^2 \varrho_t. \tag{4.5.21}$$

where $\zeta_t := \int_0^t E_s^2 ds$ and $\varrho_t := \int_0^t E_s F_s ds$.

*Step 2.* Recall the definition of $q_0 = \frac{1}{1-k}$ and $\xi := 2q_0 L - q_0\sigma^2 + 2\sigma^2 q_0^2$. In this step, we derive estimates for $\mathbb{E}[\zeta_t^{q_0}]$ and $\mathbb{E}[\varrho_t^{q_0}]$. By the definition of $\zeta_t$ and Hölder's inequality, we get

$$\begin{aligned} \mathbb{E}[\zeta_t^{q_0}] = \mathbb{E}\Big[\Big(\int_0^t E_s^2 ds\Big)^{q_0}\Big] &\le t^{\kappa q_0}\int_0^t \mathbb{E}[E_s^{2q_0}]ds \\ &= t^{\kappa q_0}\int_0^t \mathbb{E}\big[\exp(q_0(2L-\sigma^2)s + 2q_0\sigma W_s)\big]ds \\ &= t^{\kappa q_0}\int_0^t \exp(\xi s)ds \le \frac{t^{\kappa q_0}}{\xi}\exp(\xi t). \end{aligned} \tag{4.5.22}$$

Analogously, it is straightforward to derive the following estimate for $\mathbb{E}[\varrho_t^{q_0}]$.

$$\begin{aligned}\mathbb{E}[\varrho_t^{q_0}] &= \mathbb{E}\Big[\Big(\int_0^t E_s F_s ds\Big)^{q_0}\Big] \le t^{kq_0}\int_0^t \mathbb{E}[E_s^{q_0}F_s^{q_0}]ds \\ &\le t^{kq_0}\int_0^t \mathbb{E}[E_s^{2q_0}]^{\frac{1}{2}}\mathbb{E}[F_s^{2q_0}]^{\frac{1}{2}}ds \\ &\le t^{kq_0}\int_0^t \exp(\frac{1}{2}\xi s)s^{\frac{(k+1)q_0}{2}}\Big(\int_0^s \mathbb{E}[E_u^{2q_0}]du\Big)^{\frac{1}{2}}ds \\ &\le \frac{t^{\kappa q_0}}{\sqrt{\xi}}\int_0^t s^{\frac{(k+1)q_0}{2}}\exp(\xi s)ds \le \frac{t^{\frac{(3k+1)q_0}{2}+1}}{\sqrt{\xi}(\frac{(k+1)q_0}{2}+1)}\exp(\xi t).\end{aligned} \tag{4.5.23}$$

*Step 3.* This is the main step in the derivation of (4.5.11).
After restricting to $\mathcal{A}_0$, by the definition of $\mathbf{V}$ and the local Lipschitz and semiconvex property of $u$ specified in Condition 3, employing the inequality $\sup(f+g)\le\sup f+\sup g$ and (4.5.20), it follows that

$$\begin{aligned}&\mathbf{V}(i^\lambda,k^\lambda)-\lambda\mathbf{V}(i,k)-(1-\lambda)\mathbf{V}(i',k') \\ \le&\sup_{(A,B)\in\mathcal{A}_0}\Big\{G(i^\lambda,k^\lambda;A,B)-\lambda G(i,k;A,B)-(1-\lambda)G(i',k';A,B)\Big\} \\ \le&\sup_{(A,B)\in\mathcal{A}_0}\Big\{\mathbb{E}\Big[\int_0^\infty e^{-rt}\{u(\tilde{I}_t^\lambda)-\lambda u(I_t)-(1-\lambda)u(I_t')\}dt\Big]\Big\} \\ =&\sup_{(A,B)\in\mathcal{A}_0}\Big\{\mathbb{E}\Big[\int_0^\infty e^{-rt}\{u(\tilde{I}_t^\lambda)-u(I_t^\lambda)+u(I_t^\lambda)-\lambda u(I_t)-(1-\lambda)u(I_t')\}dt\Big]\Big\} \\ \le&\sup_{(A,B)\in\mathcal{A}_0}\Big\{M_1\mathbb{E}\Big[\int_0^\infty e^{-rt}\big(1+|\tilde{I}_t^\lambda|^\theta+|I_t^\lambda|^\theta\big)^\kappa|\hat{K}_t^{k^\lambda}-\hat{K}_t^\lambda|dt\Big]\Big\} \\ +&\sup_{(A,B)\in\mathcal{A}_0}\Big\{M_2\lambda(1-\lambda)\mathbb{E}\Big[\int_0^\infty e^{-rt}\big(1+|I_t|^\theta+|I_t'|^\theta\big)^\kappa\{|i-i'|^2+|\hat{K}_t^k-\hat{K}_t^{k'}|^2\}dt\Big]\Big\}.\end{aligned} \tag{4.5.24}$$

Then, we can use the stochastic upper-bound (4.5.20), which shows

$$\begin{aligned}
&\mathbf{V}(i^\lambda, k^\lambda) - \lambda \mathbf{V}(i,k) - (1-\lambda)\mathbf{V}(i',k') \\
&\le M_1 L \lambda(1-\lambda)|k-k'|^2 \underbrace{\sup_{(A,B)\in\mathcal{A}_0} \left\{ \mathbb{E}\Big[ \int_0^\infty e^{-rt} \big(1 + |\tilde{I}_t^\lambda|^\theta + |I_t^\lambda|^\theta\big)^\kappa \varrho_t dt \Big] \right\}}_{N_1} \\
&+ M_2 \lambda(1-\lambda)|i-i'|^2 \underbrace{\sup_{(A,B)\in\mathcal{A}_0} \left\{ \mathbb{E}\Big[ \int_0^\infty e^{-rt} \big(1 + |I_t|^\theta + |I_t'|^\theta\big)^\kappa dt \Big] \right\}}_{N_2} \\
&+ M_2 \lambda(1-\lambda)|k-k'|^2 \underbrace{\sup_{(A,B)\in\mathcal{A}_0} \left\{ \mathbb{E}\Big[ \int_0^\infty e^{-rt} \big(1 + |I_t|^\theta + |I_t'|^\theta\big)^\kappa t\zeta_t dt \Big] \right\}}_{N_3}.
\end{aligned} \tag{4.5.25}$$

The remaining job is for us to compute upper bounds for $N_i$, $i = 1,2,3$. By utilising Hölder's inequality on the product measure $d\mathbb{P} \otimes e^{-rt}dt$, it follows that

$$N_1 \le \sup_{(A,B)\in\mathcal{A}_0} \left\{ \mathbb{E}\Big[ \int_0^\infty e^{-rt} \big(1 + |\tilde{I}_t^\lambda|^\theta + |I_t^\lambda|^\theta\big) dt \Big]^\kappa \right\} \mathbb{E}\Big[ \int_0^\infty e^{-rt} \varrho_t^{q_0} dt \Big]^{1-\kappa}. \tag{4.5.26}$$

We then apply the convexity of $|x|^\theta$ and Lemma (4.5.6) to the first term in the product, which gives

$$\begin{aligned}
&\sup_{(A,B)\in\mathcal{A}_0} \left\{ \mathbb{E}\Big[ \int_0^\infty e^{-rt} \big(1 + |\tilde{I}_t^\lambda|^\theta + |I_t^\lambda|^\theta\big) dt \Big]^\kappa \right\} \\
&\le \sup_{(A,B)\in\mathcal{A}_0} \left\{ \mathbb{E}\Big[ \int_0^\infty e^{-rt} \big(1 + |\tilde{I}_t^\lambda|^\theta + \lambda|I_t|^\theta + (1-\lambda)|I_t'|^\theta\big) dt \Big]^\kappa \right\} \\
&\le (\frac{1}{r} + 2c_R)^\kappa.
\end{aligned} \tag{4.5.27}$$

Using the estimate (4.5.23) and the assumption $r > \xi$, there must exist a constant $C_1$ such that

$$\mathbb{E}\Big[ \int_0^\infty e^{-rt} \varrho_t^{q_0} dt \Big]^{1-\kappa} \le \Big( \int_0^\infty e^{-rt} \mathbb{E}[\varrho_t^{q_0}] dt \Big)^{1-\kappa} \le C_1. \tag{4.5.28}$$

Thus, we obtain $N_1 \le (\frac{1}{r} + 2c_R)^\kappa C_1$. Similarly, it is easy to check there exists a constant $C_2$ such that $N_2 \le (\frac{1}{r} + 2c_R)^\kappa (\frac{1}{r})^{1-\kappa} \le C_2$. Finally, for $N_3$, by the estimate (4.5.22), there must exist a constant $C_3$ such that

$$N_3 \le (\frac{1}{r} + 2c_R)^\kappa \Big( \int_0^\infty e^{-rt} t^{q_0} \mathbb{E}[\zeta_t^{q_0}] dt \Big)^{1-\kappa} \le (\frac{1}{r} + 2c_R)^\kappa C_3. \tag{4.5.29}$$

Therefore, there exists a constant $M$, depending only on $R$, such that (4.5.11) holds, i.e. $\mathbf{V}$ is locally semiconvex. $\diamondsuit$

Next, let us recall the definition of a viscosity solution. We first recall the HJB equation below:

$$\min\{rv(i,k) - \mathcal{L}v(i,k) - kv_i(i,k) - u(i), v_i(i,k) - p, q - v_i(i,k)\} = 0. \tag{HJB}$$

Though it is unknown whether $\mathbf{V}$ is a $C^{1,2}$ solution to the HJB equation, it is often possible to prove it is a solution in the viscosity sense. We say $v \in C(S)$ is a *viscosity subsolution* to (HJB) if, for every $(i,k) \in S$ and every $\psi \in C^2(S)$ such that $v(i,k) = \psi(i,k)$ and $v - \psi$ attains a local maximum at $(i,k)$, the following holds:

$$\min\{r\psi(i,k) - \mathcal{L}\psi(i,k) - k\psi_i(i,k) - u(i), \psi_i(i,k) - p, q - \psi_i(i,k)\} \leq 0. \tag{4.5.30}$$

We say $v \in C(S)$ is a *viscosity supersolution* to (HJB) if, for every $(i,k) \in S$ and every $\psi \in C^2(S)$ such that $v(i,k) = \psi(i,k)$ and $v - \psi$ attains a local minimum at $(i,k)$, the following holds:

$$\min\{r\psi(i,k) - \mathcal{L}\psi(i,k) - k\psi_i(i,k) - u(i), \psi_i(i,k) - p, q - \psi_i(i,k)\} \geq 0. \tag{4.5.31}$$

If $v \in C(S)$ is both a viscosity subsolution and supersolution, then we call $v$ a *viscosity solution.*

**Theorem 4.5.7.** *The value function* $\mathbf{V}$ *is a viscosity solution to (HJB).*

*Proof.* By Proposition 4.5.4 or 4.5.5, we know $\mathbf{V}$ is continuous. Applying the weak dynamic programming principle proved by Bouchard and Touzi [6] (also see Remark 3.10 and 3.11 therein) and combining with continuity of $\mathbf{V}$, we obtain a classical dynamic programming principle. Then, we can argue analogously as in Theorem 5.1 Section VIII.5 of [30] to prove $\mathbf{V}$ is a viscosity solution to (HJB). $\diamondsuit$

Based on the viscosity supersolution property, we will show in the following proposition that $\mathbf{V}_k$ is well-defined. Together with the continuity of $\mathbf{V}_i$, we also see that $\mathbf{V} \in C^{1,1}(S)$.

**Proposition 4.5.8.** *$V_k$ is well defined and continuous on $S$. Hence,* $\mathbf{V} \in C^1(S)$.

*Proof.* We begin by assuming Condition 3 of Assumption 4.5.1 is satisfied. Fix $R > 0$ and $(i_0, k_0) \in S \cap B_{\mathbb{R}}$. Define $\hat{v}(i,k) := \mathbf{V}(i,k) + M|(i,k) - (i_0,k_0)|^2$ where $M$ is a constant specified by Lemma 4.8.1 which ensures $\hat{v}$ is a convex function.

Therefore, the left and right partial derivative in the $k$ direction, denoted by $\hat{v}_k^+$ and $\hat{v}_k^-$ respectively, always exist, and

$$\hat{v}_k^-(i,k) \le \hat{v}_k^+(i,k) \quad \text{for all } (i,k) \in S.$$

If $\hat{v}_k^-(i_0,k_0) = \hat{v}_k^+(i_0,k_0)$, then $\hat{v}_k(i_0,k_0)$ exists, which ensures $\mathbf{V}_k(i_0,k_0)$ is well defined. Suppose on the contrary that $\hat{v}_k^-(i_0,k_0) < \hat{v}_k^+(i_0,k_0)$. Then, there exist two constants $a < b$ and

$$(\hat{v}_i(i_0,k_0), a) \in \partial\hat{v}(i_0,k_0) \text{ and } (\hat{v}_i(i_0,k_0), b) \in \partial\hat{v}(i_0,k_0), \tag{4.5.32}$$

where $\partial\hat{v}(i_0,k_0)$ denotes the *subdifferential* of $\hat{v}$ at $(i_0,k_0)$ (for more details, see [60], ch.23). Now, we define a sequence of functions $f^n$ in the following way:

$$f^n(i,k) := \hat{v}(i_0,k_0) + \hat{v}_i(i_0,k_0)(i-i_0) + \frac{1}{2}(a+b)(k-k_0) + \frac{n}{2}(k-k_0)^2. \tag{4.5.33}$$

By the convexity of $\hat{v}$, there must exist a neighbourhood around $(i_0,k_0)$, denoted by $N_{(i_0,k_0)}$, such that, for all $n \in \mathbb{N}$

$$f^n(i,k) \le \hat{v}(i,k), \quad \forall (i,k) \in N_{(i_0,k_0)}. \tag{4.5.34}$$

Further define $g^n(i,k) := f^n(i,k) - M|(i,k)-(i_0,k_0)|^2$. Then, for all $n \in \mathbb{N}$,

$$g^n(i,k) \le \mathbf{V}(i,k), \quad \forall (i,k) \in N_{(i_0,k_0)}. \tag{4.5.35}$$

Meanwhile, it is not hard to see that for all $n \in \mathbb{N}$,

$$\begin{cases} g^n(i_0,k_0) = f^n(i_0,k_0) = \hat{v}(i_0,k_0) = \mathbf{V}(i_0,k_0), \\ g^n \in C^2(S), \\ g_i^n(i_0,k_0) = f_i^n(i_0,k_0) = \hat{v}_i(i_0,k_0) = \mathbf{V}_i(i_0,k_0), \\ g_k^n(i_0,k_0) = \frac{1}{2}(a+b) \quad \text{and} \quad g_{kk}^n(i_0,k_0) = n - 2M. \end{cases} \tag{4.5.36}$$

Therefore, $\mathbf{V}-g^n$ achieves a local minimum at $(i_0,k_0)$. By the viscosity supersolution property of $\mathbf{V}$, it follows, for all $n \in \mathbb{N}$,

$$r\mathbf{V}(i_0,k_0) - \frac{1}{2}\sigma^2(k_0)(n-2M) - \mu(k_0)\frac{1}{2}(a+b) - k\mathbf{V}_i(i_0,k_0) - u(i_0) \ge 0. \tag{4.5.37}$$

However, the above inequality cannot hold for sufficiently large $n$, which leads to a contradiction.

Moreover, under Condtion 1 or 2, since $\mathbf{V}$ is concave, we derive analogously (by

exploiting the subsolution property of $\mathbf{V}$) a contradiction if $\mathbf{V}_k^+(i_0,k_0) < \mathbf{V}_k^-(i_0,k_0)$. The detailed proof is omitted.
Finally, under Condition 3 (resp. Condtion 1 or 2), by Theorem 25.2 in [60], since $(\hat{v}_i, \hat{v}_k)$ (resp. $(\mathbf{V}_i, \mathbf{V}_k)$) is always well-defined for any $(i,k) \in S$, it follows $\hat{v}$ (resp. $\mathbf{V}$) is differentiable on $S$. Furthermore, Theorem 25.5 in [60] allows us to conclude that the gradient of $\hat{v}$ (resp. $\mathbf{V}$) is continuous on $S$, which implies that $\mathbf{V}_k$ is continuous on $S$. $\diamondsuit$

Finally, we can show $\mathbf{V}$ is actually $C^{1,2}$ on the closure of $C$. This helps us to apply Itô's rule later in the proof of the verification theorem.

**Proposition 4.5.9.** *Let $\bar{C}$ denote the closure of $C$ in the set $S$. Then, $\mathbf{V}_{kk}$ admits a continuous extension on $\bar{C}$, and the extended $\mathbf{V}$ is a classical $C^{1,2}$ solution to*

$$\mathcal{L}v(i,k) - rv(i,k) + kv_i(i,k) + u(i) = 0, \quad (i,k) \in \bar{C}. \tag{4.5.38}$$

*Proof.* Let us first prove $\mathbf{V}$ is a viscosity solution to (4.5.38) in $C$. since $\mathbf{V} \in C^1(S)$, for any test function $\psi \in C^2(S)$, if $\mathbf{V} - \psi$ attains a local maximum or minimum at $(i_0,k_0) \in C$, then $\mathbf{V}_i(i_0,k_0) = \psi_i(i_0,k_0)$. As $(i_0,k_0) \in C$, we get $p < \psi_i(i_0,k_0) < q$. Thus, by the viscosity solution property of $\mathbf{V}$ to (HJB), it follows

$$\mathcal{L}\psi(i_0,k_0) - r\psi(i_0,k_0) + k\psi_i(i_0,k_0) + u(i) = 0. \tag{4.5.39}$$

Therefore, $\mathbf{V}$ is a viscosity solution to (4.5.38) in $C$.

Next, we show $\mathbf{V}$ is a $C^{1,2}$ solution to (4.5.38) on $C$. Take a rectangle set $[a,b] \times [c,d] \subset C$. Let us set up an initial-boundary value problem:

$$\begin{cases} \mathcal{L}v(i,k) - rv(i,k) + kv_i(i,k) + u(i) = 0, & (i,k) \in (a,b] \times [c,d]; \\ v(i,c) = \mathbf{V}(i,c), \ v(i,d) = \mathbf{V}(i,d) & i \in (a,b); \\ v(a,k) = \mathbf{V}(a,k), & k \in (c,d). \end{cases} \tag{4.5.40}$$

We observe that $\mathcal{L}$ is uniformly elliptic and that $\sigma, \mu, u, \mathbf{V}$ are Lipschitz continuous. By Theorem 6.3.6 in [32], there exists a unique classical solution $\hat{v}$ to (4.5.40). We also know that $\mathbf{V}$ is a viscosity solution to (4.5.40). Therefore, we can use the strong comparison principle (e.g. Theorem 4.4.5 of [57]) to show $\hat{v}(\cdot) = \mathbf{V}(i,\cdot)$ on $[a,b] \times [c,d]$. Thus, $\mathbf{V} \in C^{1,2}(C)$.
Moreover, for all $(i,k) \in C$, as $\sigma(k) > 0$ for all $k > 0$, it follows that

$$\mathbf{V}_{kk}(i,k) = \frac{2}{\sigma^2(k)}\Big\{ -\mu(k)\mathbf{V}_k(i,k) + r\mathbf{V}(i,k) - k\mathbf{V}_i(i,k) - u(i)\Big\}. \tag{4.5.41}$$

As $\mathbf{V} \in C^1(S)$, we see the right hand side of the equation is continuous in $S$. Thus, taking any sequence $(i_n, k_n) \to (i, k) \in \partial C$, it follows that $\mathbf{V}_{kk}(i_n, k_n)$ converges. Therefore, $\mathbf{V}_{kk}$ admits a continuous extension on $\bar{C}$. $\diamondsuit$

## 4.6 Candidate optimal control and verification theorem

We can now formally propose a candidate optimal control. Fix $(i, k) \in \mathcal{S}$. Let us specify a *Skorokhod reflection-type* policy, denoted by $(A^*, B^*)$ which satisfies

$$\begin{cases} a_q(K_t) \leq I_t^{A^*,B^*} \leq i_p, \quad \mathbb{P} \otimes dt - \text{a.e.}; \\ I_t^{A^*,B^*} = i + \hat{K}_t + B_t^* - A_t^*, \quad \mathbb{P} \otimes dt - \text{a.e.}; \\ dA^* \text{ has support on } \{t \geq 0 : \mathbf{V}_i(I_t^{A^*,B^*}, K_t) = p\}; \\ dB^* \text{ has support on } \{t \geq 0 : \mathbf{V}_i(I_t^{A^*,B^*}, K_t) = q\}. \end{cases} \tag{4.6.1}$$

In other words, according to this policy, there is no action taken when the state process is in set $C$, and $(A^*, B^*)$ takes the least effort to keep $(I, K)$ staying inside $\bar{C}$.

We can explicitly construct such a $(A^*, B^*)$. Let us give a brief description here. Define $\rho_p := \inf\{t \geq 0 : I_t \geq i_p\}$ and $\rho_q := \inf\{t \geq 0 : I_t \geq i_q\}$. By convention, we set $\rho_p = 0$ if $I_{0-} = i > i_p$. Then, for $t \geq 0$, $(A_t^*, B_t^*)$ is defined by

$$A_t^* := (i - i_p)^+ + \int_{\rho_p \wedge t}^{t} K_s ds, \tag{4.6.2}$$

$$B_t^* := \max\big(0, -\inf_{0 \leq s \leq t}\{i + \hat{K}_s - a_q(K_s)\}\big). \tag{4.6.3}$$

Indeed, by this construction, we see that $(A^*, B^*)$ is adapted, càdlàg, non-negative and increasing. It is easy to see that $A_t^* = (i - i_p)^+$ when $t < \rho_p$. Moreover, it follows from the definition of $B^*$ that $i + \hat{K}_t + B_t^* \geq a_q(K_t)$ for $t < \rho_p$. So, we see that $a_q(K_t) \leq I_t^{A^*,B^*} \leq i_p$ in $\{t < \rho_p\}$ [5].

Notice that, since $a_q(k) \leq i_q$ and $\hat{K}$ is increasing, for any $t \geq \rho_q$, we get

$$B_{\rho_q}^* \geq i_q - i - \hat{K}_{\rho_q} \geq a_q(K_t) - i - \hat{K}_t. \tag{4.6.4}$$

Therefore, for all $t \geq \rho_q$, $B_t^* = B_{\rho_q}^*$, i.e. $B^*$ will stay constant after $I$ exceeds $i_q$, and hence for $t \geq \rho_p$ (note that $\rho_p \geq \rho_q$), we get

$$I_t^{A^*,B^*} = I_{\rho_p-}^{A^*,B^*} + \int_{\rho_p}^{t} K_s ds - A_t^* = i_p, \tag{4.6.5}$$

[5] $\{t < \rho_p\} := \{(\omega, t) : t < \rho_p(\omega)\}$

which implies $a_q(K_t) \le I_t^{A^*,B^*} \le i_p$ in $\{t \ge \rho_p\}$. The rest of conditions in (4.6.1) can be checked directly.

By the above construction, the following lemma is sufficient to prove $(A^*, B^*) \in \mathcal{A}$.

**Lemma 4.6.1.** $\mathbb{P}$ *almost surely,* $\sup_{t\ge 0} e^{-rt} A_t^* \le (i - i_p)^+ + \sup_{t\ge 0} e^{-rt} \hat{K}_t$. *Moreover, there exists* $\Lambda > 0$ *such that* $\mathbb{E}\big[\sup_{T\ge 0} e^{-rT} A_T^*\big] \le \Lambda(1 + |i|^\gamma + |k|^\gamma)$.

*Proof.* Take $(i, k) \in S$. By the definition of $A^*$, for all $t \ge 0$,

$$A_t^* = (i - i_p)^+ + \int_{\rho_p \wedge t}^{t} K_s ds. \tag{4.6.6}$$

In this way, we observe that $A_t^* \le (i - i_p)^+ + \hat{K}_t$ for all $t \ge 0$, which implies that $\sup_{t\ge 0} e^{-rt} A_t^* \le (i - i_p)^+ + \sup_{t\ge 0} e^{-rt} \hat{K}_t$. Moreover, by Remark 4.2.8, we know that $\sup_{t\ge 0} e^{-rt} \hat{K}_t \in L^1(\mathbb{P})$ and there exists a constant $\Lambda_0$ such that

$$\mathbb{E}\big[\sup_{T\ge 0} e^{-rT} A_T^*\big] \le (i - i_p)^+ + \Lambda_0(1 + k^2)^{\frac{1}{2}}. \tag{4.6.7}$$

Since $\gamma \ge 2$, we can choose $\Lambda$ large enough so that

$$\mathbb{E}\big[\sup_{T\ge 0} e^{-rT} A_T^*\big] \le \Lambda(1 + |i|^\gamma + |k|^\gamma). \tag{4.6.8}$$

$\diamondsuit$

Before we present a verification theorem, it is necessary to make an additional assumption:

**Assumption 4.6.2.** $\lim_{n\to\infty} \frac{n^\gamma}{\psi(n)} = 0$ where $\psi$ is the unique (up to multiplicity) increasing fundamental solution to $\mathcal{L}f - rf = 0$.

Finally, the following verification theorem confirms the optimality of $(A^*, B^*)$.

**Theorem 4.6.3.** *The control process* $(A^*, B^*)$ *specified by (4.6.1) is the unique solution of the restricted control problem (RCP).*

*Proof.* Fix $(i, k) \in S$. All we need to do is to show

$$G(i, k : A^*, B^*) \ge \mathbf{V}(i, k). \tag{4.6.9}$$

Abbreviate $I^{A^*,B^*}$ by $I$. We want to apply a general version of Itô's formula (e.g., Theorem 33 [58], p.81) to $e^{-rt}V(I_t, K_t)$. Normally Itô's formula requires $C^2$ smoothness. However, thanks to Proposition A.2 of [27], if $\tau$ is a bounded stopping time such that $(I_t, K_t)$ is contained in a compact subset of $S$, then for $t \leq \tau$, we have

$$\mathbf{V}(i,k) = \mathbb{E}[e^{-r\tau}\mathbf{V}(I_\tau, K_\tau)] - \mathbb{E}\Big[\int_0^\tau e^{-rt}\{\mathcal{L}\mathbf{V}(I_t,K_t) - r\mathbf{V}(I_t,K_t) + K\mathbf{V}_i(I_t,K_t)\}dt\Big]$$
$$- \mathbb{E}\Big[\int_0^\tau e^{-rt}\mathbf{V}_i(I_t,K_t)d(B_t^{*c} - A_t^{*c})\Big] - \mathbb{E}\Big[\sum_{0\leq t\leq\tau} e^{-rt}\{\mathbf{V}(I_t,K_t) - \mathbf{V}(I_{t^-},K_t)\}\Big], \tag{4.6.10}$$

where $B^{*c}$ (resp. $A^{*c}$) denotes the continuous part of $B^*$ (resp. $A^*$). Now define $\tau_n := \rho_n \wedge n$ where $\rho_n := \inf\{t \geq 0 : |(I_t, K_t)| \geq n\}$ and substitute $\tau_n$ into (4.6.10). Set $\Delta A_t^* := A_t^* - A_{t^-}^*$ and $\Delta B_t^* := B_t^* - B_{t^-}^*$. According to (4.6.1), it is not hard to see

$$\begin{aligned}\sum_{0\leq t\leq\tau_n} e^{-rt}\{\mathbf{V}(I_t,K_t) - \mathbf{V}(I_{t^-},K_t)\} &= \sum_{0\leq t\leq\tau_n} e^{-rt}\int_0^{\Delta B_t^*}\mathbf{V}_i(I_{t^-}+x,K_t)dx \\ &\quad - \sum_{0\leq t\leq\tau_n} e^{-rt}\int_0^{\Delta A_t^*}\mathbf{V}_i(I_{t^-}+x,K_t)dx \\ &= \sum_{0\leq t\leq\tau_n} e^{-rt}\{q\Delta B_t^* - p\Delta A_t^*\}.\end{aligned} \tag{4.6.11}$$

In the same way, we get

$$\int_0^{\tau_n} e^{-rt}\mathbf{V}_i(I_t,K_t)d(B_t^{*c} - A_t^{*c}) = \int_0^{\tau_n} e^{-rt}q dB_t^{*c} - \int_0^{\tau_n} e^{-rt}p dA_t^{*c}. \tag{4.6.12}$$

Next, as $(I,K) \in \bar{C}$ $\mathbb{P}$-a.s., it follows

$$\mathcal{L}\mathbf{V}(I_t,K_t) - r\mathbf{V}(I_t,K_t) + K\mathbf{V}_i(I_t,K_t) = -u(I_t) \quad \text{for all } t \geq 0. \tag{4.6.13}$$

Therefore, equality (4.6.10) becomes

$$\mathbf{V}(i,k) = \mathbb{E}[e^{-r\tau_n}\mathbf{V}(I_{\tau_n},K_{\tau_n})] + \mathbb{E}\Big[\int_0^{\tau_n} e^{-rt}\{u(I_t)dt + p dA_t^* - q dB_t^*\}\Big]. \tag{4.6.14}$$

The final step is to pass $n$ to infinity.

To do so, firstly, let us check $\lim_{n\to\infty}\mathbb{E}[e^{-r\tau_n}\mathbf{V}(I_{\tau_n},K_{\tau_n})] = 0$. Note $|\mathbf{V}(i,k)| \leq$

$\alpha(1+|i|^\gamma+|k|^\gamma)$ by Proposition 4.2.12. Since $i \le I_t \le i_p$, we see that

$$\begin{aligned}
&\mathbb{E}[e^{-r\tau_n}|\mathbf{V}(I_{\tau_n},K_{\tau_n})|] \\
&\le \mathbb{E}[e^{-r\tau_n}\alpha(1+|I_{\tau_n}|^\gamma+|K_{\tau_n}|^\gamma)] \\
&\le \alpha(1+|i|^\gamma \vee |i_p|^\gamma)\mathbb{E}[e^{-r\tau_n}]+\alpha\mathbb{E}[e^{-r\tau_n}|K_{\tau_n}|^\gamma] \\
&\le \alpha(1+|i|^\gamma \vee |i_p|^\gamma)\mathbb{E}[e^{-r\tau_n}]+\alpha\mathbb{E}[e^{-rn}|K_n|^\gamma]+\alpha\mathbb{E}[e^{-r\rho_n}|K_{\rho_n}|^\gamma] \\
&\le \alpha(1+|i|^\gamma \vee |i_p|^\gamma)\mathbb{E}[e^{-r\tau_n}]+\alpha\mathbb{E}[e^{-rn}|K_n|^\gamma]+\alpha|n|^\gamma\mathbb{E}[e^{-r\rho_n}],
\end{aligned} \tag{4.6.15}$$

where the last inequality follows from the definition of $\rho_n$.

It is easy to check that $\lim_{n\to\infty}\mathbb{E}[e^{-r\tau_n}]=0$ by the Dominated Convergence Theorem. Next, it follows from the standard estimate (4.2.1) that there exists a constant $\tilde{c}$ such that

$$\mathbb{E}[e^{-rn}|K_n|^\gamma] \le \tilde{c}e^{-(r-\iota)n}(1+|k|^\gamma), \tag{4.6.16}$$

where $\iota$ is defined in Assumption 4.2.7. Since $r>\iota$ by Assumption 4.2.7, we obtain $\lim_{n\to\infty}\mathbb{E}[e^{-rn}|K_n|^\gamma]=0$. Lastly, without loss of generality, let us assume $k<n$ and $|i|\vee|i_p|<n$. Then, $\rho_n=\inf\{t\ge 0: K_t\ge n\}$, and it is well known that $\mathbb{E}[e^{-r\rho_n}]=\frac{\psi(k)}{\psi(n)}$ where $\psi$ is the increasing fundamental solution to $\mathcal{L}f-rf=0$. By Assumption 4.6.2, we get

$$\lim_{n\to\infty}|n|^\gamma\mathbb{E}[e^{-r\rho_n}]=\psi(k)\lim_{n\to\infty}\frac{n^\gamma}{\psi(n)}=0. \tag{4.6.17}$$

Taking $n\to\infty$ in (4.6.15), we see that all expectation terms in the last line converge to 0, and hence

$$\lim_{n\to\infty}\mathbb{E}[e^{-r\tau_n}|\mathbf{V}(I_{\tau_n},K_{\tau_n})|]=0, \tag{4.6.18}$$

which implies $\lim_{n\to\infty}|\mathbb{E}[e^{-r\tau_n}\mathbf{V}(I_{\tau_n},K_{\tau_n})]|=0$.

Secondly, set $\sigma:=\inf\{t\ge 0: I_t=i_p\}$. We see there exists a positive constant $M$ such that

$$\begin{aligned}
&\left|\int_0^{\tau_n} e^{-rt}\{u(I_t)dt+pdA_t^*-qdB_t^*\}\right| \\
&\le \int_0^{\tau_n} e^{-rt}\max_{x\in[i_q,i_p]}|u(x)|dt+p(i-i_p)^+ + \int_{\sigma\wedge\tau_n}^{\tau_n} e^{-rt}pK_tdt+q|i-i_q| \\
&\le M+p\int_0^\infty e^{-rt}K_tdt.
\end{aligned} \tag{4.6.19}$$

As we know $\int_0^\infty e^{-rt}K_tdt$ is integrable by Remark 4.2.8, we can apply the Dominated

Convergence Theorem and get

$$\lim_{n\to\infty}\mathbb{E}\Big[\int_0^{\tau_n} e^{-rt}\{u(I_t)dt+pdA_t^*-qdB_t^*\}\Big]=\mathbb{E}\Big[\int_0^{\infty} e^{-rt}\{u(I_t)dt+pdA_t^*-qdB_t^*\}\Big].$$

Finally, taking $n\to\infty$ in (4.6.14), we conclude that

$$\begin{aligned}\mathbf{V}(i,k)=&\lim_{n\to\infty}\mathbb{E}[e^{-r\tau_n}\mathbf{V}(I_{\tau_n},K_{\tau_n})]\\&+\lim_{n\to\infty}\mathbb{E}\Big[\int_0^{\tau_n} e^{-rt}\{u(I_t)dt+pdA_t^*-qdB_t^*\}\Big]=G(i,k;A^*,B^*).\end{aligned}$$

$\diamondsuit$

Before we conclude this section, let us come back to the unrestricted control problem (CP) and prove the optimality of $(A^*,B^*)$.

**Theorem 4.6.4.** $(A^*,B^*)$ *is the unique optimal control of the control problem (CP).*

*Proof.* Recall the definition of $\mathcal{A}(\upsilon)$ and $\bar{\mathcal{A}}$ from Section 4.2. Note that $(A^*,B^*)\in\mathcal{A}(0)$ by Lemma 4.6.1. In our study of the restricted control problem (RCP), the random variable $\upsilon$ is arbitrarily chosen, and Theorem 4.6.3 shows that $(A^*,B^*)$ is optimal for $\mathbf{V}^\upsilon$. Therefore, by Proposition 4.2.9, we obtain $\overline{\mathbf{V}}(i,k)=G(i,k;A^*,B^*)$. As $(A^*,B^*)\in\bar{\mathcal{A}}$, we see $(A^*,B^*)$ is the optimal control. The uniqueness easily follows from the strict concavity of $G(i,k;\cdot,\cdot)$, which can be proved analogously as in the uniqueness part of the proof of Theorem 4.2.15. $\diamondsuit$

*Remark* 4.6.5. According to the definition of $(A^*,B^*)$, $A^*$ and $B^*$ stay constant when $(I,K)$ is in $C$. Thus, $I$ is an increasing process in $C$. It shows once $I$ exceeds $i_q$ (or the smallest $i'_q<i_q$ such that $b_q(i'_q)=0$), $(I,K)$ cannot hit $a_q$ anymore, i.e. $B^*$ will stay constant forever. Then we wait until $I$ hits $i_p$ and from that time onward, $I$ stays at $i_p$.

## 4.7 Concluding remarks

Under a general concave preference on the level of inventory, constant price, constant cost of production from the secondary source, and zero cost of production from the primary source, our model tells us that, qualitatively:

- It is optimal to not output any products until the inventory has accumulated to a certain level which is not stochastic (i.e. $i_p$). Once the inventory reaches that level, we shall sell all primary production and keep the inventory at that level;

- In some cases, when the initial inventory is not large enough or the rate of production from the primary source is low, optimally we have to make extra production from the secondary source to boost the inventory;
- When we produce from the secondary source, it is optimal to produce as little as possible while still keeping the inventory just on or above a level which depends on the rate of production of the primary source (i.e. $a_q(k)$);
- There exists another fixed inventory level (i.e. $i_q$) such that after the inventory accumulates higher than that, it is optimal to not produce from the secondary source whatever the production rate of the primary source.

The current assumptions of the model allow the inventory $I$ to be negative. However, assuming $i_p \geq 0$, it is not hard to see that for every $i \geq 0$ and $k > 0$, $I_t^{A^*,B^*} \geq 0$ for all $t \geq 0$. Therefore, even if we add an extra constraint that $I \geq 0$ to the set of admissible control $\bar{\mathcal{A}}$, $(A^*, B^*)$ defined by (4.6.1) would still be admissible and hence optimal. Thus, if we want to use the result of our model with the restriction on the inventory being non-negative, the only additional requirement is to choose $u$ such that $i_p \geq 0$.

## 4.8 Proofs and additional results

### 4.8.1 Proof of Theorem 4.3.1

*Proof.* If (4.3.4) holds, then by Proposition 4.2.13 it is obvious that

$$\begin{aligned}\inf_\sigma \sup_\tau J(i,k;\tau,\sigma) &\leq \sup_\tau J(i,k;\tau,\sigma_*) \leq \mathbf{V}_i^+(i,k)\\ &\leq \mathbf{V}_i^-(i,k) \leq \inf_\sigma J(i,k;\tau_*,\sigma) \leq \sup_\tau \inf_\sigma J(i,k;\tau,\sigma). \quad (4.8.1)\end{aligned}$$

Since $\sup_\tau \inf_\sigma J(i,k;\tau,\sigma) \leq \inf_\sigma \sup_\tau J(i,k;\tau,\sigma)$, we must have $\mathbf{W}(i,k) = J(i,k;\tau_*,\sigma_*) = \mathbf{V}_i(i,k)$. So to complete the proof, it is sufficient to show (4.3.4). We will only prove $v_i^+(i,k) \geq J(i,k;\tau,\sigma_*)$ for every stopping time $\tau$. The other assertion can be proved analogously.
Fix $(i,k) \in S$. Suppose $(A^*, B^*)$ is the optimal control w.r.t $(i,k)$. Take a stopping time $t$. For any $\epsilon > 0$, define

$$\sigma_\epsilon = \inf\{t \geq 0 : B_t^* \geq \epsilon\} \quad (4.8.2)$$

$$A_t^\epsilon = \begin{cases} A_t^*, & \text{if } t < \tau \wedge \sigma_\epsilon \\ A_t^*. & \text{if } \sigma_\epsilon < \tau,\ t \geq \sigma_\epsilon \\ A_t^* + (\epsilon - B_\tau^*), & \text{if } \tau \leq \sigma_\epsilon,\ t \geq \tau \end{cases} \tag{4.8.3}$$

$$B_t^\epsilon = \begin{cases} 0, & \text{if } t < \tau \wedge \sigma_\epsilon \\ B_t^* - \epsilon, & \text{if } \sigma_\epsilon < \tau,\ t \geq \sigma_\epsilon \\ B_t^* - B_\tau^*, & \text{if } \tau \leq \sigma_\epsilon,\ t \geq \tau \end{cases} \tag{4.8.4}$$

Therefore,

$$B_t^\epsilon - A_t^\epsilon = \begin{cases} -A_t^*, & \text{if } t < \tau \wedge \sigma_\epsilon \\ B_t^* - A_t^* - \epsilon, & \text{if } t \geq \tau \wedge \sigma_\epsilon \end{cases} \tag{4.8.5}$$

It follows

$$\begin{aligned} G(i+\epsilon, k; A^\epsilon, B^\epsilon) = \mathbb{E}\Big[ &\int_0^{\tau\wedge\sigma_*} e^{-rt} u(i + \epsilon + \hat{K}_t - A_t^*) dt \\ &+ \int_{\tau\wedge\sigma_*}^{\tau\wedge\sigma_\epsilon} e^{-rt} u(i + \epsilon + \hat{K}_t - A_t^*) dt \\ &+ \int_{\tau\wedge\sigma_\epsilon}^{\infty} e^{-rt} u(i + \hat{K}_t + B_t^* - A_t^*) dt \\ &+ \mathbb{1}_{\sigma_\epsilon < \tau} \left\{ \int_0^\infty e^{-rt} p dA_t^* - e^{-r\sigma_\epsilon} q (B_{\sigma_\epsilon}^* - \epsilon) - \int_{\sigma_\epsilon{}^+}^\infty e^{-rt} q dB_t^* \right\} \\ &+ \mathbb{1}_{\sigma_* < \tau \leq \sigma_\epsilon} \left\{ \int_0^\infty e^{-rt} p dA_t^* + e^{-r\tau} p (\epsilon - B_\tau^*) - \int_{\tau^+}^\infty e^{-rt} q dB_t^* \right\} \\ &+ \mathbb{1}_{\tau \leq \sigma_*} \left\{ \int_0^\infty e^{-rt} p dA_t^* + e^{-r\tau} p\epsilon - \int_{\sigma_*}^\infty e^{-rt} q dB_t^* \right\} \Big], \end{aligned} \tag{4.8.6}$$

and

$$
\begin{aligned}
G(i,k;A^*,B^*) = \mathbb{E}\Big[ & \int_0^{\tau\wedge\sigma_*} e^{-rt}u(i+\hat{K}_t+B_t^*-A_t^*)dt \\
& + \int_{\tau\wedge\sigma_*}^{\tau\wedge\sigma_\epsilon} e^{-rt}u(i+\hat{K}_t+B_t^*-A_t^*)dt \\
& + \int_{\tau\wedge\sigma_\epsilon}^{\infty} e^{-rt}u(i+\hat{K}_t+B_t^*-A_t^*)dt \\
& + \mathbb{1}_{\sigma_\epsilon<\tau}\Big\{ \int_0^\infty e^{-rt}pdA_t^* - e^{-r\sigma_\epsilon}q(B^*_{\sigma_\epsilon}-B^*_{\sigma_\epsilon -}) \\
& - \int_{\sigma_*}^{\sigma_\epsilon -} e^{-rt}qdB_t^* - \int_{\sigma_\epsilon +}^{\infty} e^{-rt}qdB_t^* \Big\} \\
& + \mathbb{1}_{\sigma_*<\tau\le\sigma_\epsilon}\Big\{ \int_0^\infty e^{-rt}pdA_t^* - \int_{\sigma_*}^{\infty} e^{-rt}qdB_t^* \Big\} \\
& + \mathbb{1}_{\tau\le\sigma_*}\Big\{ \int_0^\infty e^{-rt}pdA_t^* - \int_{\sigma_*}^{\infty} e^{-rt}qdB_t^* \Big\}\Big].
\end{aligned}
\tag{4.8.7}
$$

Thus, we have

$$
\begin{aligned}
\mathbf{V}(i+\epsilon,k) - \mathbf{V}(i,k) &\ge G(i+\epsilon,k;A^\epsilon,B^\epsilon) - G(i,k;A^*,B^*) \\
&= \mathbb{E}\Big[ \int_0^{\tau\wedge\sigma_*} e^{-rt}\{\underbrace{u(i+\epsilon+\hat{K}_t-A_t^*) - u(i+\hat{K}_t+B_t^*-A_t^*)}_{(I)}\}dt \\
&+ \int_{\tau\wedge\sigma_*}^{\tau\wedge\sigma_\epsilon} e^{-rt}\{\underbrace{u(i+\epsilon+\hat{K}_t-A_t^*) - u(i+\hat{K}_t+B_t^*-A_t^*)}_{(II)}\}dt \\
&+ \underbrace{\mathbb{1}_{\sigma_\epsilon<\tau}\Big\{ -e^{r\sigma_\epsilon}q(B^*_{\sigma_\epsilon -}-\epsilon) + \int_{\sigma_*}^{\sigma_\epsilon -} e^{-rt}qdB_t^* \Big\}}_{(III)} \\
&+ \underbrace{\mathbb{1}_{\sigma_*<\tau\le\sigma_\epsilon}\Big\{ e^{-r\tau}p(\epsilon-B_\tau^*) + \int_{\sigma_*}^{\tau} e^{-rt}qdB_t^* \Big\}}_{(IV)} + \mathbb{1}_{\tau\le\sigma_*}e^{-r\tau}p\epsilon\Big].
\end{aligned}
\tag{4.8.8}
$$

Since $u$ is concave, on $[0,\tau\wedge\sigma_*]$, $(I)\geq \epsilon u'(i+\hat{K}_t)$, and on $[\tau\wedge\sigma_*,\tau\wedge\sigma_\epsilon]$, $(II)\geq (\epsilon-B^*_t)u'(i+\epsilon+\hat{K}_t)$. Moreover,

$$\begin{aligned}(III)\geq\ & \mathbb{1}_{\sigma_\epsilon<\tau}\epsilon q e^{-r\sigma_\epsilon}-\mathbb{1}_{\sigma_*<\tau}\epsilon q e^{-r\sigma_*}+\mathbb{1}_{\sigma_*<\tau}\epsilon q e^{-r\sigma_*}\\ & -\mathbb{1}_{\sigma_\epsilon<\tau}e^{-r\sigma_\epsilon}qB^*_{\sigma_\epsilon-}+\mathbb{1}_{\sigma_\epsilon<\tau}\int_{\sigma_*}^{\sigma_\epsilon-}e^{-rt}q\,dB^*_t\\ \geq\ & \mathbb{1}_{\sigma_*<\tau}\epsilon q e^{-r\sigma_*}+\epsilon q\{\mathbb{1}_{\sigma_\epsilon<\tau}e^{-r\sigma_\epsilon}-\mathbb{1}_{\sigma_*<\tau}e^{-r\sigma_*}\}\\ & -\mathbb{1}_{\sigma_\epsilon<\tau}\Big\{\int_{\sigma_*}^{\sigma_\epsilon-}q\{e^{-r\sigma_\epsilon}-e^{-rt}\}dB^*_t\Big\}\\ \geq\ & \mathbb{1}_{\sigma_*<\tau}\epsilon q e^{-r\sigma_*}+\epsilon q\{\mathbb{1}_{\sigma_\epsilon<\tau}e^{-r\sigma_\epsilon}-\mathbb{1}_{\sigma_*<\tau}e^{-r\sigma_*}\}-\mathbb{1}_{\sigma_\epsilon<\tau}qB^*_{\sigma_\epsilon-}|e^{-r\sigma_*}-e^{-r\sigma_\epsilon}|,\end{aligned}\tag{4.8.9}$$

where the second inequality follows from the fact that $B^*_{\sigma_*-}=0$ and the $dB^*_t$ integral in our definition includes the jump at the lower boundary time. Furthermore,

$$(IV)\geq \mathbb{1}_{\sigma_*<\tau\leq\sigma_\epsilon}\left\{e^{-r\tau}p(\epsilon-B^*_\tau)+e^{-r\tau}qB^*_\tau\right\}\geq \mathbb{1}_{\sigma_*<\tau\leq\sigma_\epsilon}e^{-r\tau}p\epsilon. \tag{4.8.10}$$

Thus,

$$\begin{aligned}\frac{\mathbf{V}(i+\epsilon,k)-\mathbf{V}(i,k)}{\epsilon}\geq\ & \mathbb{E}\Big[\int_0^{\tau\wedge\sigma_*}e^{-rt}u'(i+\hat{K}_t)dt+qe^{-r\sigma_*}\mathbb{1}_{\sigma_*<\tau}+pe^{-r\tau}\mathbb{1}_{\tau\leq\sigma_*}\\ & +\frac{1}{\epsilon}\int_{\tau\wedge\sigma_*}^{\tau\wedge\sigma_\epsilon}e^{-rt}(\epsilon-B^*_t)u'(i+\epsilon+\hat{K}_t)dt\\ & +q\{\mathbb{1}_{\sigma_\epsilon<\tau}e^{-r\sigma_\epsilon}-\mathbb{1}_{\sigma_*<\tau}e^{-r\sigma_*}\}-\mathbb{1}_{\sigma_\epsilon<\tau}\frac{B^*_{\sigma_\epsilon-}}{\epsilon}q|e^{-r\sigma_*}-e^{-r\sigma_\epsilon}|\\ & -\mathbb{1}_{\sigma_*<\tau\leq\sigma_\epsilon}e^{-r\tau}p\Big].\end{aligned}\tag{4.8.11}$$

Finally, we notice

$$\begin{aligned}&\frac{1}{\epsilon}\int_{\tau\wedge\sigma_*}^{\tau\wedge\sigma_\epsilon}e^{-rt}(\epsilon-B^*_t)u'(i+\epsilon+\hat{K}_t)dt\geq 0,\\ &\mathbb{1}_{\sigma_\epsilon<\tau}e^{-r\sigma_\epsilon}-\mathbb{1}_{\sigma_*<\tau}e^{-r\sigma_*}\overset{\epsilon\to 0}{\to}0,\\ &-\mathbb{1}_{\sigma_\epsilon<\tau}\frac{B^*_{\sigma_\epsilon-}}{\epsilon}q|e^{-r\sigma_*}-e^{-r\sigma_\epsilon}|\geq -\mathbb{1}_{\sigma_\epsilon<\tau}q|e^{-r\sigma_*}-e^{-r\sigma_\epsilon}|\overset{\epsilon\to 0}{\to}0,\\ &\mathbb{1}_{\sigma_*<\tau\leq\sigma_\epsilon}\overset{\epsilon\to 0}{\to}0.\end{aligned}\tag{4.8.12}$$

Taking $\epsilon\to 0$, since all terms in (4.8.11) are either bounded or dominated by some integrable random variables(cf. Proposition 4.2.11) , we can apply the Dominated Convergence Theorem to (4.8.11) and get $\mathbf{V}^+_i(i,k)\geq J(i,k;\tau,\sigma_*)$. ♢

### 4.8.2 Auxiliary lemmas

**Lemma 4.8.1.** *If a function $f : \mathbb{R}^n \to \mathbb{R}$ is semiconvex, then for suitable $M$, $g(x) := f(x) + M|x-c|^2$ is concave for all $c \in \mathbb{R}^n$.*

*Proof.* Take $x, y \in \mathbb{R}^n$ and $\lambda \in [0,1]$. Then, by semiconvexity of $f$, there is constant $M$ such that

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y) + M\lambda(1-\lambda)|x-y|^2. \tag{4.8.13}$$

Set $g(x) = f(x) + M|x-c|^2$. It is easy to check

$$\lambda(1-\lambda)|x-y|^2 + |\lambda x + (1-\lambda)y - c|^2 = \lambda|x-c|^2 + (1-\lambda)|y-c|^2. \tag{4.8.14}$$

Therefore, we obtain

$$\begin{aligned} &g(\lambda x + (1-\lambda)y) \\ &\leq \lambda f(x) + (1-\lambda)f(y) + M\lambda(1-\lambda)|x-y|^2 + M|\lambda x + (1-\lambda)y - c|^2 \\ &= \lambda\big(f(x) + M|x-c|^2\big) + (1-\lambda)\big(f(y) + M|y-c|^2\big) = \lambda g(x) + (1-\lambda)g(y). \end{aligned} \tag{4.8.15}$$

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