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Estimates for the Heat Flow in Optimal Spaces of Unbounded Initial Data in $\mathbb{R}^d$ and Applications to the Ornstein–Uhlenbeck Semigroup

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Abstract. In this paper, we improve some estimates from [7] for solutions of the heat equation with initial conditions that are large 'at infinity' and employ them to obtain suitable estimates on the solutions of the Ornstein–Uhlenbeck semigroup [solutions of $u_t - \Delta u = \alpha x \cdot \nabla u$] for the same class of unbounded data.

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1. Introduction

In two previous papers [6, 7], we considered solutions of the heat equation

$$u_t - \Delta u = 0, \ x \in \mathbb{R}^d, \ t > 0, \quad u(x, 0) = u_0(x) \tag{1.1}$$

given by

$$u(x,t) = S(t)u_0(x) := \frac{1}{(4\pi t)^{d/2}} \int_{\mathbb{R}^d} \mathrm{e}^{-\frac{|x-y|^2}{4t}} u_0(y)\,\mathrm{d}y, \quad x \in \mathbb{R}^d, \ t > 0, \tag{1.2}$$

with initial data in the weighted spaces of Radon measures

$$\mathcal{M}_\varepsilon(\mathbb{R}^d) := \left\{ \mu \in \mathcal{M}_{\mathrm{loc}}(\mathbb{R}^d) : \ \int_{\mathbb{R}^d} \rho_\varepsilon(x)\,\mathrm{d}|\mu|(x) < \infty \right\};$$

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or of locally integrable functions

$$L^p_\varepsilon(\mathbb{R}^d) := \left\{ f \in L^p_{\mathrm{loc}}(\mathbb{R}^d) : \int_{\mathbb{R}^d} \rho_\varepsilon(x)|f(x)|^p \, \mathrm{d}x < \infty \right\}$$

for $1 \leq p < \infty$ with the norm

$$\|f\|_{L^p_\varepsilon(\mathbb{R}^d)} := \left(\int_{\mathbb{R}^d} \rho_\varepsilon(x)|f(x)|^p \, \mathrm{d}x \right)^{\frac{1}{p}}; \tag{1.3}$$

and for $p = \infty$

$$L^\infty_\varepsilon(\mathbb{R}^d) := \left\{ f \in L^\infty_{\mathrm{loc}}(\mathbb{R}^d) : \sup_{x \in \mathbb{R}^d} \rho_\varepsilon(x)|f(x)| < \infty \right\}$$

with the norm $\|f\|_{L^\infty_\varepsilon(\mathbb{R}^d)} := \sup_{x\in\mathbb{R}^d} \rho_\varepsilon(x)|f(x)|$. Here, the weight functions ρ_ε, $\varepsilon > 0$, are given by

$$\rho_\varepsilon(x) = \left(\frac{\varepsilon}{\pi}\right)^{d/2} \mathrm{e}^{-\varepsilon|x|^2}, \qquad \int_{\mathbb{R}^d} \rho_\varepsilon(x) \, \mathrm{d}x = 1.$$

These spaces satisfy

$$L^p_\varepsilon(\mathbb{R}^d) \subset L^q_\varepsilon(\mathbb{R}^d) \subset L^1_\varepsilon(\mathbb{R}^d) \subset \mathcal{M}_\varepsilon(\mathbb{R}^d), \qquad 1 \leq q \leq p < \infty,$$

and the last inclusion is an isometry.

We showed in [6] that since these spaces allow initial data which may be very large at infinity, these solutions may not exist for all time, by a mechanism of *mass moving from infinity*. Indeed, for a non-negative initial data in $\mathcal{M}_\varepsilon(\mathbb{R}^d)$, the solution may only exist up to $T(\varepsilon) = \frac{1}{4\varepsilon}$. We also characterized in [6] those initial data and the points at which the solution blows up. As a consequence of the results above, the heat equation defines a global semigroup in the Fréchet spaces

$$L^1_0(\mathbb{R}^d) = \bigcap_{\varepsilon>0} L^1_\varepsilon(\mathbb{R}^d), \qquad \mathcal{M}_0(\mathbb{R}^d) = \bigcap_{\varepsilon>0} \mathcal{M}_\varepsilon(\mathbb{R}^d),$$
$$L^p_0(\mathbb{R}^d) = \bigcap_{\varepsilon>0} L^p_\varepsilon(\mathbb{R}^d), \qquad \text{and} \qquad L^\infty_0(\mathbb{R}^d) = \bigcap_{\varepsilon>0} L^\infty_\varepsilon(\mathbb{R}^d)$$

which satisfy

$$L^p_0(\mathbb{R}^d) \subset L^q_0(\mathbb{R}^d) \subset L^1_0(\mathbb{R}^d) \subset \mathcal{M}_0(\mathbb{R}^d), \qquad 1 \leq q \leq p < \infty.$$

This semigroup is shown in [7] to be analytic and to have suitable smoothing estimates.

In particular, in [7], we obtain quantitative smoothing estimates for the semigroup of the type

$$\|u(t)\|_{L^q_{q\delta}(\mathbb{R}^d)} \leq c_{p,q} \frac{(1+4\delta t)^{\frac{d}{2}(1-\frac{1}{q})}}{(4\delta t)^{\frac{d}{2}(\frac{1}{p}-\frac{1}{q})}} \|u_0\|_{L^p_{p\delta(t)}(\mathbb{R}^d)} \tag{1.4}$$

or $1 \leq p \leq q < \infty$ with any $\delta > 0$ and $\delta(t) = \frac{\delta}{1+4\delta t}$.

One of our main goals here is to provide an improved set of estimates

$$\|u(t)\|_{L^q_{q\delta}(\mathbb{R}^d)} \leq c_{p,q} \left(\frac{1+4p\delta t}{4p\delta t}\right)^{\frac{d}{2}(\frac{1}{p}-\frac{1}{q})} \|u_0\|_{L^p_{\delta_p(t)}(\mathbb{R}^d)}, \quad \delta_p(t) = \frac{p\delta}{1+4p\delta t} \tag{1.5}$$

for $1 \leq p \leq q < \infty$.

Both (1.4) and (1.5) share some common features. First, to estimate the solution at time t in a fixed space $L^q_{q\delta}(\mathbb{R}^d)$, it is necessary to measure the initial data in the space $L^p_{p\delta(t)}(\mathbb{R}^d)$ or $L^p_{\delta_p(t)}(\mathbb{R}^d)$, respectively, which changes with time, requiring better and better integrability of u_0 as t increases, since $p\delta(t)$ and $\delta_p(t)$ are decreasing. Second, besides the typical term $t^{-\frac{d}{2}(\frac{1}{p}-\frac{1}{q})}$ that describes the time singularity of the solution for short times, an algebraically increasing term $(1+4\delta t)^{\frac{d}{2}(1-\frac{1}{q})}$ or $(1+4\delta t)^{\frac{d}{2}(\frac{1}{p}-\frac{1}{q})}$, respectively, appears, reflecting the fact that we are dealing with functions that are very large at infinity. This term reflects the mechanism of *mass moving from infinity* that was shown in [6] to be able to produce nonlinear-like behaviour in the solutions of (1.1).

On the one hand, since $p\delta(t) > \delta_p(t)$, $L^p_{\delta_p(t)}(\mathbb{R}^d) \subset L^p_{p\delta(t)}(\mathbb{R}^d)$ with an inclusion constant that is bounded in $t > 0$ and then (1.4) implies an estimate like (1.5) but with a worse time-dependent constant. On the other hand, (1.5) does not imply (1.4). Notice that (1.5) requires stronger integrability conditions on the initial data than (1.4), but the time-dependent constant is bounded in the former and unbounded in the latter as $t \to \infty$ provided $p > 1$.

In the next section, we prove (1.5) along with new versions of the other estimates in [7] that can be improved in a similar way. In particular, we refine the conditions on the initial data required to guarantee that the solution remains bounded or decays to zero as $t \to \infty$; see Proposition 2.6.

These improved estimates will have an impact in the forthcoming analysis in [8] for the non-autonomous linear problem and applications to elliptic equations. In this paper, we demonstrate the utility of (1.5) by applying these estimates to the Ornstein–Uhlenbeck semigroup in Sect. 3

$$u_t - \Delta u = \alpha x \cdot \nabla u, \quad x \in \mathbb{R}^d,\ t > 0, \qquad u(x,0) = u_0(x); \tag{1.6}$$

see [1,3–5]. Here, we deal simultaneously with the cases $\alpha > 0$ and $\alpha < 0$, and our results are consequences of the improved estimates above: they cannot be obtained using the earlier estimates (1.4). In particular, we show optimal results for global existence or blow-up in finite time. For solutions that exist for all times, we obtain a complete set of smoothing estimates, including derivatives. Those estimates use suitable Fréchet spaces when $\alpha > 0$ and extend and recover some known estimates in the "invariant measure spaces" when $\alpha < 0$; see the references above. We also obtain results on the asymptotic behaviour and decay of solutions, including the complexity of behaviour for $\alpha > 0$.

2. Improved $L^p_\varepsilon(\mathbb{R}^d)$ Estimates

In this section, we improve several estimates obtained in [7]; in particular, we will obtain (1.5). For this, a crucial tool is a result from [7] that characterizes $L^p_{p\varepsilon}(\mathbb{R}^d)$, for $1 < p < \infty$, as the complex interpolation spaces between $L^1_\varepsilon(\mathbb{R}^d)$

and $L^\infty_\varepsilon(\mathbb{R}^d)$, that is

$$L^p_{p\varepsilon}(\mathbb{R}^d) = [L^1_\varepsilon(\mathbb{R}^d), L^\infty_\varepsilon(\mathbb{R}^d)]_{\frac{1}{p'}} \tag{2.1}$$

with norm

$$|||f|||_{p,\varepsilon} := \|f\|_{[L^1_\varepsilon(\mathbb{R}^d), L^\infty_\varepsilon(\mathbb{R}^d)]_{\frac{1}{p'}}} = \|f\rho_\varepsilon\|_{L^p(\mathbb{R}^d)} = \left(\frac{\varepsilon}{\pi}\right)^{\frac{d}{2}(1-\frac{1}{p})} \frac{1}{p^{d/2p}} \|f\|_{L^p_{p\varepsilon}(\mathbb{R}^d)}; \tag{2.2}$$

see [2]. Note that the natural norm on the interpolation space differs from the norm on $L^p_{p\varepsilon}(\mathbb{R}^d)$ by a factor that depends on both p and ε.

The following basic (end point) estimates were proved in [6] and will be used further below.

Proposition 2.1. *If $u_0 \in \mathcal{M}_0(\mathbb{R}^d)$, then $u(t) = S(t)u_0$ given by (1.2) satisfies the following estimates for any $\delta > 0$, $t > 0$ and $\delta(t) = \frac{\delta}{1+4\delta t}$.*

(i)
$$\|u(t)\|_{L^1_\delta(\mathbb{R}^d)} \le \|u_0\|_{\mathcal{M}_{\delta(t)}(\mathbb{R}^d)}, \quad t > 0 \tag{2.3}$$
with an equality if $u_0 \ge 0$.

(ii) *For all $t > 0$, $u(t) \in L^\infty_0(\mathbb{R}^d)$ and*
$$\|u(t)\|_{L^\infty_\delta(\mathbb{R}^d)} \le \left(\frac{1+4\delta t}{4\pi t}\right)^{d/2} \|u_0\|_{\mathcal{M}_{\delta(t)}(\mathbb{R}^d)}. \tag{2.4}$$

(iii) *For any $1 \le q < \infty$*
$$\|u(t)\|_{L^q_{q\delta}(\mathbb{R}^d)} \le q^{d/2q} \left(\frac{1+4\delta t}{4\delta t}\right)^{\frac{d}{2}(1-\frac{1}{q})} \|u_0\|_{\mathcal{M}_{\delta(t)}(\mathbb{R}^d)}. \tag{2.5}$$

(iv) *If $u_0 \in L^\infty_0(\mathbb{R}^d)$, then $u(t) \in L^\infty_0(\mathbb{R}^d)$ for all $t > 0$ and*
$$\|u(t)\|_{L^\infty_\delta(\mathbb{R}^d)} \le (1+4\delta t)^d \|u_0\|_{L^\infty_{\delta(t)}(\mathbb{R}^d)}. \tag{2.6}$$

Now, we prove the improved regularity estimates (1.5). This result improves Theorem 4.6 in [7].

Theorem 2.2. *If $u_0 \in L^p_0(\mathbb{R}^d)$ with $1 \le p < \infty$, then $u(t) \in L^q_0(\mathbb{R}^d)$ for any $1 \le p \le q < \infty$ and $t > 0$. Moreover, for any $\delta > 0$, $t > 0$, we have*

$$\|u(t)\|_{L^q_{q\delta}(\mathbb{R}^d)} \le \left(\frac{q}{p}\right)^{d/2q} \left(\frac{1+4p\delta t}{4p\delta t}\right)^{\frac{d}{2}(\frac{1}{p}-\frac{1}{q})} \|u_0\|_{L^p_{\delta_p(t)}(\mathbb{R}^d)}, \tag{2.7}$$

where

$$\delta_p(t) = \frac{p\delta}{1+4p\delta t}.$$

In particular, setting $q = p$, we have

$$\|u(t)\|_{L^p_{p\delta}(\mathbb{R}^d)} \le \|u_0\|_{L^p_{\delta_p(t)}(\mathbb{R}^d)}. \tag{2.8}$$

For $q = \infty$, we have

$$\|u(t)\|_{L^\infty_\delta(\mathbb{R}^d)} \le \left(\frac{\delta}{\pi}\right)^{\frac{d}{2}} \left(\frac{1+4p\delta t}{4p\delta t}\right)^{\frac{d}{2p}} \|u_0\|_{L^p_{\delta_p(t)}(\mathbb{R}^d)}. \tag{2.9}$$

Proof. (i) We first prove (2.7) for $q = p$. For this, setting $0 < \delta = \frac{1}{4\tau}$, from (1.2), we have

$$e^{-\delta|x|^2}|u(x,t)| \leq \frac{1}{(4\pi t)^{d/2}} \int_{\mathbb{R}^d} e^{-|x|^2/4\tau} e^{-|x-z|^2/4t} |u_0(z)| \, dz \tag{2.10}$$

with $u_0 \in L^1_0(\mathbb{R}^d)$, $1 < p < \infty$; by Jensen's inequality, we get

$$e^{-p\delta|x|^2}|u(x,t)|^p \leq \frac{1}{(4\pi t)^{d/2}} \int_{\mathbb{R}^d} e^{-p|x|^2/4\tau} |u_0(z)|^p e^{-|x-z|^2/4t} \, dz. \tag{2.11}$$

From this, we obtain

$$\int_{\mathbb{R}^d} e^{-p\delta|x|^2}|u(x,t)|^p \, dx \leq \frac{1}{(4\pi t)^{d/2}} \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} e^{-p|x|^2/4\tau} e^{-|x-z|^2/4t} \, dx \right) |u_0(z)|^p \, dz$$

and rearranging squares as

$$\frac{|x|^2}{\frac{\tau}{p}} + \frac{|x-z|^2}{t} = \frac{t + \frac{\tau}{p}}{t\frac{\tau}{p}} \left| x - \frac{\frac{\tau}{p}}{t + \frac{\tau}{p}} z \right|^2 + \frac{|z|^2}{t + \frac{\tau}{p}}$$

and using

$$\int_{\mathbb{R}^d} \exp\left(-\frac{t + \frac{\tau}{p}}{4t\frac{\tau}{p}} \left| x - \frac{\frac{\tau}{p}}{t + \frac{\tau}{p}} z \right|^2 \right) dx = \int_{\mathbb{R}^d} \exp\left(-\frac{t + \frac{\tau}{p}}{4t\frac{\tau}{p}} |x|^2 \right) dx = \left(\frac{4\pi t \frac{\tau}{p}}{t + \frac{\tau}{p}} \right)^{\frac{d}{2}} = \left(\frac{4\pi t \tau}{pt + \tau} \right)^{\frac{d}{2}}$$

and $0 < \delta = \frac{1}{4\tau}$, we have

$$\int_{\mathbb{R}^d} e^{-p\delta|x|^2}|u(x,t)|^p \, dx \leq \frac{1}{(4\pi t)^{d/2}} \left(\frac{4\pi t}{1 + 4p\delta t} \right)^{\frac{d}{2}} \int_{\mathbb{R}^d} \exp\left(-\frac{|z|^2}{4(t + \frac{\tau}{p})} \right) |u_0(z)|^p \, dz = \frac{1}{(1 + 4p\delta t)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{-\delta_p(t)|z|^2} |u_0(z)|^p \, dz.$$

Therefore

$$\|u(t)\|^p_{L^p_{p\delta}(\mathbb{R}^d)} \leq \left(\frac{p\delta}{\pi} \right)^{\frac{d}{2}} \frac{1}{(1 + 4p\delta t)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{-\delta_p(t)|z|^2} |u_0(z)|^p \, dz = \left(\frac{p\delta}{\pi} \right)^{\frac{d}{2}} \frac{1}{(1 + 4p\delta t)^{\frac{d}{2}}} \left(\frac{\pi}{\delta_p(t)} \right)^{\frac{d}{2}} \|u_0\|^p_{L^p_{\delta_p(t)}(\mathbb{R}^d)} = \|u_0\|^p_{L^p_{\delta_p(t)}(\mathbb{R}^d)}.$$

(ii) Now, we prove (2.9). For this, from (2.11)

$$e^{-\delta|x|^2}|u(x,t)| \leq \frac{1}{(4\pi t)^{d/2p}} \left(\int_{\mathbb{R}^d} e^{-p|x|^2/4\tau} |u_0(z)|^p e^{-|x-z|^2/4t} \, dz \right)^{\frac{1}{p}}$$

and rearranging the squares as above, we get

$e^{-\delta|x|^2}|u(x,t)|$

$$\le \frac{1}{(4\pi t)^{d/2p}}\left(\int_{\mathbb{R}^d}\exp\left(-\frac{t+\frac{\tau}{p}}{4t\frac{\tau}{p}}\left|x-\frac{\frac{\tau}{p}}{t+\frac{\tau}{p}}z\right|^2\right)\exp\left(-\frac{|z|^2}{4(t+\frac{\tau}{p})}\right)|u_0(z)|^p\,\mathrm{d}z\right)^{\frac{1}{p}}$$
$$\le \frac{1}{(4\pi t)^{d/2p}}\left(\int_{\mathbb{R}^d}\exp\left(-\frac{|z|^2}{4(t+\frac{\tau}{p})}\right)|u_0(z)|^p\,\mathrm{d}z\right)^{\frac{1}{p}}$$
$$= \frac{1}{(4\pi t)^{d/2p}}\left(\int_{\mathbb{R}^d}\mathrm{e}^{-\delta_p(t)|z|^2}|u_0(z)|^p\,\mathrm{d}z\right)^{\frac{1}{p}},$$

which leads to

$$\sup_{x\in\mathbb{R}^d}\mathrm{e}^{-\delta|x|^2}|u(x,t)| \le \frac{1}{(4\pi t)^{d/2p}}\left(\int_{\mathbb{R}^d}\mathrm{e}^{-\delta_p(t)|z|^2}|u_0(z)|^p\,\mathrm{d}z\right)^{\frac{1}{p}}.$$

Then, we get

$$\|u(t)\|_{L^\infty_\delta(\mathbb{R}^d)} \le \left(\frac{\delta}{\pi}\right)^{\frac{d}{2}}\frac{1}{(4\pi t)^{d/2p}}\left(\frac{\pi}{\delta_p(t)}\right)^{\frac{d}{2p}}\|u_0\|_{L^p_{\delta_p(t)}(\mathbb{R}^d)}$$
$$= \left(\frac{\delta}{\pi}\right)^{\frac{d}{2}}\left(\frac{1+4p\delta t}{4p\delta t}\right)^{\frac{d}{2p}}\|u_0\|_{L^p_{\delta_p(t)}(\mathbb{R}^d)}$$

and (2.9) follows.

(iii) Using (2.2), we rewrite (2.8) as

$$|||u(t)|||_{p,\delta} \le \left(\frac{\delta}{\pi}\right)^{\frac{d}{2}(1-\frac{1}{p})}\frac{1}{p^{d/2p}}\|u_0\|_{L^p_{\delta_p(t)}(\mathbb{R}^d)}.$$

Interpolating this and (2.9), we get for any $0<\theta<1$

$$\|u(t)\|_{[L^p_{p\delta}(\mathbb{R}^d),L^\infty_\delta(\mathbb{R}^d)]_\theta}$$
$$\le \left(\frac{\delta}{\pi}\right)^{\frac{\theta d}{2}}\left(\frac{1+4p\delta t}{4p\delta t}\right)^{\frac{\theta d}{2p}}\left(\frac{\delta}{\pi}\right)^{\frac{d}{2}(1-\frac{1}{p})(1-\theta)}\frac{1}{p^{d(1-\theta)/2p}}\|u_0\|_{L^p_{\delta_p(t)}(\mathbb{R}^d)}.$$

By reiteration of the interpolation in (2.1), we have that for $\theta = 1-\frac{p}{q}$, with $q>p$, we get

$$[L^p_{p\delta}(\mathbb{R}^d),L^\infty_\delta(\mathbb{R}^d)]_\theta = L^q_{q\delta}(\mathbb{R}^d).$$

Hence, taking $\theta = 1-\frac{p}{q}$, this leads to the estimate

$$|||u(t)|||_{q,\delta} \le \left(\frac{\delta}{\pi}\right)^{\frac{d}{2}(1-\frac{p}{q})}\left(\frac{1+4p\delta t}{4p\delta t}\right)^{\frac{d}{2}(\frac{1}{p}-\frac{1}{q})}\left(\frac{\delta}{\pi}\right)^{\frac{d}{2}(\frac{p}{q}-\frac{1}{q})}\frac{1}{p^{d/2q}}\|u_0\|_{L^p_{\delta_p(t)}(\mathbb{R}^d)}. \tag{2.12}$$

From (2.2), we get

$$\left(\frac{\delta}{\pi}\right)^{\frac{d}{2}(1-\frac{1}{q})}\frac{1}{q^{d/2q}}\|u(t)\|_{L^q_{q\delta}(\mathbb{R}^d)}$$
$$\le \left(\frac{\delta}{\pi}\right)^{\frac{d}{2}(1-\frac{p}{q})}\left(\frac{1+4p\delta t}{4p\delta t}\right)^{\frac{d}{2}(\frac{1}{p}-\frac{1}{q})}\left(\frac{\delta}{\pi}\right)^{\frac{d}{2}(\frac{p}{q}-\frac{1}{q})}\frac{1}{p^{d/2q}}\|u_0\|_{L^p_{\delta_p(t)}(\mathbb{R}^d)},$$

which leads to (2.7). □

We can also obtain the following bounds on the derivatives of the solutions. This result improves Proposition 4.7 in [7].

Proposition 2.3. *If* $u_0 \in L^p_0(\mathbb{R}^d)$ *with* $1 \leq p \leq \infty$, *take any multi-index* $\alpha \in \mathbb{N}^d$, *any* $1 \leq p \leq q < \infty$, *any* $\delta > 0$, $\gamma > 1$ *and define* $\tilde{\delta}_p(t) = \frac{p\delta}{1+4p\gamma\delta t}$ *for* $t > 0$. *Then*

$$\|D^\alpha_x u(t)\|_{L^q_{q\delta}(\mathbb{R}^d)} \leq \frac{c_{\alpha,p,q,\delta,\gamma}}{t^{\frac{|\alpha|}{2}}} \left(\frac{1+4p\gamma\delta t}{t}\right)^{\frac{d}{2}(\frac{1}{p}-\frac{1}{q})} \|u_0\|_{L^p_{\tilde{\delta}_p(t)}(\mathbb{R}^d)}$$

and

$$\|D^\alpha_x u(t)\|_{L^\infty_\delta(\mathbb{R}^d)} \leq \frac{c_{\alpha,p,\delta,\gamma}}{t^{\frac{|\alpha|}{2}}} \left(\frac{1+4p\gamma\delta t}{t}\right)^{\frac{d}{2p}} \|u_0\|_{L^p_{\tilde{\delta}_p(t)}(\mathbb{R}^d)}.$$

Proof. Using, for $\gamma > 1$

$$|D^\alpha_x u(x,t)| \leq \frac{c_{\alpha,\gamma}}{t^{d/2+|\alpha|/2}} \int_{\mathbb{R}^d} \mathrm{e}^{-\frac{1}{\gamma}\frac{|x-y|^2}{4t}} \, \mathrm{d}|u_0|(y), \qquad (2.13)$$

the right-hand side is a multiple of the solution of the heat equation with initial data $|u_0|$ in a time $t' = \gamma t$, with the extra factor $\frac{1}{t^{\frac{|\alpha|}{2}}}$.

Hence, we can proceed as in Theorem 2.2 and we can obtain all estimates there with the right-hand side in terms of $t' = \gamma t$ and with $\tilde{\delta}_p(t) = \frac{p\delta}{1+4p\delta t'}$. Writing everything in terms of t, we get analogously to (2.9)

$$\|D^\alpha_x u(t)\|_{L^\infty_\delta(\mathbb{R}^d)} \leq c_\alpha \left(\frac{1+4p\gamma\delta t}{4p\gamma\delta t}\right)^{\frac{d}{2p}} \|u_0\|_{L^p_{\tilde{\delta}_p(t)}(\mathbb{R}^d)}$$

and analogously to (2.12)

$$|||D^\alpha_x u(t)|||_{q,\delta} \leq \frac{c_\alpha}{t^{\frac{|\alpha|}{2}}} \left(\frac{1+4p\gamma\delta t}{4p\gamma\delta t}\right)^{\frac{d}{2}(\frac{1}{p}-\frac{1}{q})} \|u_0\|_{L^p_{\tilde{\delta}_p(t)}(\mathbb{R}^d)}.$$

Again, using the norms (1.3) and (2.2), we get the result. □

Notice that all the estimates we have obtained so far in this section have been for initial data in the spaces $L^p_0(\mathbb{R}^d)$, $1 \leq p \leq \infty$. Now, we will use the fact that for $1 \leq p < \infty$, the space $L^p_0(\mathbb{R}^d)$ is dense in all spaces $L^p_\varepsilon(\mathbb{R}^d)$ for $\varepsilon > 0$; see [7]. Given this, a density argument allows us to extend the estimates above to initial data in the spaces $L^p_\varepsilon(\mathbb{R}^d)$, where solutions of the heat equation may exist only for finite time; see [6]. This result improves Proposition 4.10 in [7].

Proposition 2.4. *Assume that* $\varepsilon > 0$ *and set* $T(\varepsilon) = \frac{1}{4\varepsilon}$. *For every* $\delta > \varepsilon$, *define* $\delta_p(t) = \frac{p\delta}{1+4p\delta t}$. *For any* $u_0 \in L^p_{p\varepsilon}(\mathbb{R}^d)$ *with* $1 \leq p < \infty$ *and for any* $1 \leq p \leq q < \infty$, *we have*

$$\|u(t)\|_{L^q_{q\delta}(\mathbb{R}^d)} \leq c_{p,q} \left(\frac{1+4p\delta t}{4p\delta t}\right)^{\frac{d}{2}(\frac{1}{p}-\frac{1}{q})} \|u_0\|_{L^p_{\delta_p(t)}(\mathbb{R}^d)}$$

$$\|u(t)\|_{L^\infty_\delta(\mathbb{R}^d)} \leq c_{p,\infty} \left(\frac{1+4p\delta t}{4p\delta t}\right)^{\frac{d}{2p}} \|u_0\|_{L^p_{\delta_p(t)}(\mathbb{R}^d)},$$

which hold for $0 < t \leq \frac{1}{p}\big(T(\varepsilon) - T(\delta)\big)$.

Proof. For $u_0 \in L^p_{p\varepsilon}(\mathbb{R}^d) \subset L^1_{p\varepsilon}(\mathbb{R}^d)$, (2.7) and (2.9) remain valid as long as $0 < t < T(p\varepsilon) = \frac{1}{p}T(\varepsilon)$ and $\delta_p(t) \geq p\varepsilon$, i.e., $0 < t \leq \frac{1}{p}\big(T(\varepsilon) - T(\delta)\big)$. □

We now rephrase this result, estimating the solution in a norm that depends on time, but the initial space is fixed. This result improves Proposition 4.11 in [7].

Proposition 2.5. *Assume that* $\varepsilon > 0$ *and* $0 < t < T(\varepsilon) = \frac{1}{4\varepsilon}$. *For any* $u_0 \in L^p_{p\varepsilon}(\mathbb{R}^d)$ *with* $1 \leq p < \infty$ *and for any* $1 \leq p \leq q < \infty$, *we have for* $0 < t < T(p\varepsilon) = \frac{1}{p}T(\varepsilon)$ *and* $\varepsilon_p(t) = \frac{\varepsilon}{1-4p\varepsilon t}$

$$\|u(t)\|_{L^q_{q\varepsilon_p(t)}(\mathbb{R}^d)} \leq c_{p,q}\frac{1}{(4p\delta t)^{\frac{d}{2}(\frac{1}{p}-\frac{1}{q})}}\frac{1}{(1-4p\varepsilon t)^{\frac{d}{2}(\frac{1}{p}-\frac{1}{q})}}\|u_0\|_{L^p_{p\varepsilon}(\mathbb{R}^d)}$$

$$\|u(t)\|_{L^\infty_{\varepsilon_p(t)}(\mathbb{R}^d)} \leq c_{p,\infty}\frac{1}{(4p\delta t)^{\frac{d}{2p}}}\frac{1}{(1-4p\varepsilon t)^{\frac{d}{2p}}}\|u_0\|_{L^p_{p\varepsilon}(\mathbb{R}^d)}.$$

Proof. We set $\delta_p(t) = p\varepsilon$, that is, $1 + 4p\delta t = \frac{\delta}{\varepsilon}$ and $\delta = \varepsilon_p(t) = \frac{\varepsilon}{1-4p\varepsilon t}$ for $0 < t < \frac{1}{p}T(\varepsilon)$. Then, Proposition 2.4 gives the result. □

Notice that estimates on derivatives in Proposition 2.3 can be extended along the same lines.

As a consequence of the results above, we obtain the following characterization of the initial data for which the solution of the heat equation stays bounded or decays to zero in the Fréchet spaces $L^q_0(\mathbb{R}^d)$. This result improves parts (ii) and (iii) in Proposition 5.1 in [7].

Proposition 2.6. *If* $u_0 \in L^p_0(\mathbb{R}^d)$, $1 \leq p < \infty$, *then* $u(t) = S(t)u_0$ *is bounded in* $L^q_0(\mathbb{R}^d)$, $p \leq q \leq \infty$ *if*

$$\|u_0\|_{L^p_\varepsilon(\mathbb{R}^d)} \leq C, \quad \varepsilon \to 0,$$

while it decays to zero in $L^q_0(\mathbb{R}^d)$ *provided that*

$$\|u_0\|_{L^p_\varepsilon(\mathbb{R}^d)} \to 0, \quad \varepsilon \to 0.$$

Proof. This follows from (2.7) and (2.9) where we set $\varepsilon = \delta_p(t) = \frac{p\delta}{1+4p\delta t} \to 0$ as $t \to \infty$, and then, for large t, the factor $\frac{1+4p\delta t}{4p\delta t}$ is bounded. □

Observe that an analogous result can be obtained for the derivatives $D^\alpha u(x,t)$, from (2.13) by adding a factor $\varepsilon^{|\alpha|/2}$ to the estimates on u_0 above.

3. The Ornstein–Uhlenbeck Semigroup

In this section, we consider the Ornstein–Uhlenbeck semigroup defined by the solutions of the problem

$$u_t - \Delta u = \alpha x \cdot \nabla u, \quad x \in \mathbb{R}^d,\ t > 0, \qquad u(x,0) = u_0(x) \tag{3.1}$$

with $\alpha \in \mathbb{R}$, $\alpha \neq 0$. We will derive estimates on the solutions by means of the results in the previous section. In more general, Ornstein–Uhlenbeck equations have been thoroughly studied in [1,3–5] and references therein.

Observe that in the absence of diffusion, integrating along characteristics leads to

$$u(x,t) = u_0(x\mathrm{e}^{\alpha t}), \quad x \in \mathbb{R}^d,\ t > 0.$$

Hence, we look for solutions of (3.1) of the form

$$u(x,t) = w(x\mathrm{e}^{\alpha t}, s(t)), \quad s(0) = 0.$$

Theorem 3.1. *For* $\varepsilon > 0$ *and* $u_0 \in \mathcal{M}_\varepsilon(\mathbb{R}^d)$*, the function*

$$u(x,t) = w(x\mathrm{e}^{\alpha t}, s(t)), \quad s(t) = \frac{\mathrm{e}^{2\alpha t}}{2\alpha} - \frac{1}{2\alpha}, \quad x \in \mathbb{R}^d,\ t > 0$$

with

$$w(y,s) = \frac{1}{(4\pi s)^{d/2}} \int_{\mathbb{R}^d} \mathrm{e}^{-\frac{|y-z|^2}{4s}}\, \mathrm{d}u_0(z), \quad y \in \mathbb{R}^d,\ s > 0$$

is a classical solution of (3.1) *for* $0 < t < T_{OU}(\varepsilon)$*, where*

(i) *if* $\alpha > -2\varepsilon$ *and* $\alpha \neq 0$ *then* $T_{OU}(\varepsilon)$ *is the unique solution of* $s(t) = \frac{1}{4\varepsilon}$*;*
(ii) *if* $\alpha \leq -2\varepsilon$*, then* $T_{OU}(\varepsilon) = \infty$*.*

Moreover, for every $\varphi \in C_c(\mathbb{R}^d)$ *and* $0 \leq t < T_{OU}(\varepsilon)$

$$\int_{\mathbb{R}^d} \varphi\, u(t) \to \int_{\mathbb{R}^d} \varphi\, \mathrm{d}u_0, \quad t \to 0^+,$$

i.e., $u(t) \to u_0$ *as* $t \to 0^+$ *in the sense of measures.*

Also, if we define $\tilde{\varepsilon}(t) := \frac{\varepsilon \mathrm{e}^{2\alpha t}}{(1-4\varepsilon s(t))}$*, then we have*

$$\|u(t)\|_{L^1_{\tilde{\varepsilon}(t)}(\mathbb{R}^d)} \leq \|u_0\|_{\mathcal{M}_\varepsilon(\mathbb{R}^d)}, \qquad 0 < t < T_{OU}(\varepsilon)$$

with equality if $u_0 \geq 0$*.*

Finally, if $u_0 \geq 0$ *and is non-trivial, then* $u(x,t) > 0$ *for all* (x,t)*.*

Proof. Looking for a solution of (3.1) of the form $u(x,t) = w(x\mathrm{e}^{\alpha t}, s(t))$, $s(0) = 0$, gives

$$w_s \frac{s'(t)}{\mathrm{e}^{2\alpha t}} - \Delta w = 0.$$

Then, we take $s'(t) = \mathrm{e}^{2\alpha t}$ which, thanks to the results in [6] for $w(y,s)$, means that $u(x,t)$ is a classical solution of (3.1) as w is defined for $0 < s < \frac{1}{4\varepsilon}$. Hence, u is defined as long as $s(t) < \frac{1}{4\varepsilon}$, that is, for $0 < t < T_{OU}(\varepsilon)$. Since $s(t)$ is increasing and as $t \to \infty$

$$s(t) = \frac{\mathrm{e}^{2\alpha t}}{2\alpha} - \frac{1}{2\alpha} \to \begin{cases} \frac{1}{2|\alpha|} & \text{if } \alpha < 0 \\ \infty & \text{if } \alpha > 0, \end{cases}$$

we get the description of $T_{OU}(\varepsilon)$ in the statement.

Also, for every $\varphi \in C_c(\mathbb{R}^d)$ and $0 < t < T_{OU}(\varepsilon)$

$$\int_{\mathbb{R}^d} \varphi\, u(t) = \mathrm{e}^{d\alpha t} \int_{\mathbb{R}^d} \varphi(y\mathrm{e}^{-\alpha t})\, w(y, s(t))\, \mathrm{d}y = \mathrm{e}^{d\alpha t} \int_{\mathbb{R}^d} S(s(t))\big(\varphi(\cdot \mathrm{e}^{-\alpha t})\big)\, \mathrm{d}u_0,$$

where we have used the results in [6], and for $\eta \in C_c(\mathbb{R}^d)$, we denote by $S(s)\eta$ the solution of the heat equation. Observe then that since $|\varphi(x)| \leq A\mathrm{e}^{-\gamma|x|^2}$,

$x \in \mathbb{R}^d$, with $\gamma > 2\varepsilon$, $|\varphi(x\mathrm{e}^{-\alpha t})| \le A\mathrm{e}^{-\tilde{\gamma}|x|^2}$ with $\tilde{\gamma} > \varepsilon$ for all t sufficiently close to 0. Hence, by Corollary 3.4 in [6], we have $|S(s)\varphi(\cdot\mathrm{e}^{-\alpha t})|(x) \le C\mathrm{e}^{-\varepsilon|x|^2} \in L^1(\mathrm{d}|u_0|)$. Also, $S(s(t))\varphi(\cdot\mathrm{e}^{-\alpha t})(x) \to \varphi(x)$ for $x \in \mathbb{R}^d$ as $t \to 0$, see, e.g., Lemma 3.3 in [6], and then, Lebesgue's theorem gives

$$\int_{\mathbb{R}^d} S(s(t))\big(\varphi(\cdot\mathrm{e}^{-\alpha t})\big)\,\mathrm{d}u_0 \to \int_{\mathbb{R}^d} \varphi\,\mathrm{d}u_0,$$

as $t \to 0$ and we get the result.

Now, observe that for any $t > 0$ and $\delta > 0$, we have

$$\begin{aligned}\|w(\cdot\mathrm{e}^{\alpha t})\|_{L^1_\delta(\mathbb{R}^d)} &= \left(\frac{\delta}{\pi}\right)^{d/2}\int_{\mathbb{R}^d} \mathrm{e}^{-\delta|x|^2}|w(x\mathrm{e}^{\alpha t})|\,\mathrm{d}x\\ &= \left(\frac{\delta\mathrm{e}^{-2\alpha t}}{\pi}\right)^{d/2}\int_{\mathbb{R}^d} \mathrm{e}^{-\delta\mathrm{e}^{-2\alpha t}|y|^2}|w(y)|\,\mathrm{d}y = \|w\|_{L^1_{\delta\mathrm{e}^{-2\alpha t}}(\mathbb{R}^d)}.\end{aligned}$$

Also, from the results in [6], for $0 < s < T(\varepsilon)$ and for any $\delta \ge \frac{1}{4(T(\varepsilon)-s)} > \varepsilon$, we have $w(s) \in L^1_\delta(\mathbb{R}^d)$, and if we set $\varepsilon(s) := \frac{1}{4(T(\varepsilon)-s)} = \frac{\varepsilon}{(1-4\varepsilon s)}$, then $\|w(s)\|_{L^1_{\varepsilon(s)}(\mathbb{R}^d)} \le \|u_0\|_{\mathcal{M}_\varepsilon(\mathbb{R}^d)}$, with equality if $u_0 \ge 0$.

Therefore

$$\|u(t)\|_{L^1_\delta(\mathbb{R}^d)} = \|w(s(t))\|_{L^1_{\delta\mathrm{e}^{-2\alpha t}}(\mathbb{R}^d)},$$

so we choose $\delta\mathrm{e}^{-2\alpha t} = \varepsilon(s(t))$, that is, $\delta = \frac{\mathrm{e}^{2\alpha t}}{4(T(\varepsilon)-s(t))} = \frac{\varepsilon\mathrm{e}^{2\alpha t}}{(1-4\varepsilon s(t))} := \tilde{\varepsilon}(t)$, and we get

$$\|u(t)\|_{L^1_{\tilde{\varepsilon}(t)}(\mathbb{R}^d)} \le \|u_0\|_{\mathcal{M}_\varepsilon(\mathbb{R}^d)}, \quad 0 < t < T_{OU}(\varepsilon).$$

3.1. Blow-Up in Finite Time

For a non-negative $0 \le u_0 \in \mathcal{M}_{\mathrm{loc}}(\mathbb{R}^d)$, define its 'optimal index' as

$$\varepsilon_0(u_0) := \inf\{\varepsilon :\ u_0 \in \mathcal{M}_\varepsilon(\mathbb{R}^d)\} = \sup\{\varepsilon :\ u_0 \notin \mathcal{M}_\varepsilon(\mathbb{R}^d)\} \le \infty. \qquad (3.2)$$

Consider now the case $\alpha > 0$ or $\alpha < 0$ and $|\alpha| < 2\varepsilon_0$ and define then $T_{OU}(\varepsilon_0)$ is the unique solution of $s(t) = \frac{1}{4\varepsilon_0}$.

Theorem 3.2. *Assume that $0 \le u_0 \in \mathcal{M}_{\mathrm{loc}}(\mathbb{R}^d)$ with optimal index, such that $0 < \varepsilon_0 < \infty$ and either $\alpha > 0$ or $\alpha < 0$ and $|\alpha| < 2\varepsilon_0$.*

Then, the solution u of the Ornstein–Uhlenbeck equation given in Theorem 3.1 is not defined at any point $x \in \mathbb{R}^d$ beyond $T = T_{OU}(\varepsilon_0) < \infty$.

Furthermore, there exists a convex set $K \subset \mathbb{R}^d$, such that for $x \in K$ the limit $\lim_{t\to T} u(x,t)$ exists and is finite and for $x \notin K$, $\lim_{t\to T} u(x,t) = \infty$.

Conversely, for each convex and closed set $K \subset \mathbb{R}^d$ and for any ε_0, there exits $0 \le u_0 \in \mathcal{M}_{\mathrm{loc}}(\mathbb{R}^d)$ with optimal index ε_0, such that for $x \in K$, the limit $\lim_{t\to T} u(x,t)$ exists and is finite, and for $x \notin K$, $\lim_{t\to T} u(x,t) = \infty$.

Proof. From [6], we have that $w(y,s)$ in Theorem 3.1 is not defined at any point in $\mathbb{R}^d$ beyond time $s = \hat{T} = \frac{1}{4\varepsilon_0}$. Also, there exists a convex set $\hat{K}$, such that for $y \in K$, the limit $\lim_{s\to\hat{T}} w(y,s)$ exists and is finite and for $y \notin K$, $\lim_{s\to\hat{T}} w(y,s) = \infty$. Then, we obtain the result for $u(x,t) = w(x\mathrm{e}^{\alpha t}, s(t))$ as $t \to T$, since $s(T) = \hat{T}$ and for $x \in K = \mathrm{e}^{-\alpha T}\hat{K}$.

Conversely, for each convex and closed set $\hat{K} \subset \mathbb{R}^d$ and for any ε_0, there exits $0 \le u_0 \in \mathcal{M}_{\mathrm{loc}}(\mathbb{R}^d)$ with optimal index ε_0, such that for $y \in \hat{K}$, the limit $\lim_{s\to\hat{T}} w(y,s)$ exists and is finite and for $y \notin \hat{K}$, $\lim_{s\to\hat{T}} w(y,s) = \infty$, where $\hat{T} = \frac{1}{4\varepsilon_0}$. Again, we obtain the result for $u(x,t) = w(x\mathrm{e}^{\alpha t}, s(t))$ as $t \to T$, since $s(T) = \hat{T}$ and for $x \in K = \mathrm{e}^{-\alpha T}\hat{K}$. □

3.2. Estimates for All Times

Hereafter, we will set

$$\mathcal{M}_{OU}(\mathbb{R}^d) = \begin{cases} \mathcal{M}_0(\mathbb{R}^d) = \bigcap_{\varepsilon>0} \mathcal{M}_\varepsilon(\mathbb{R}^d), & \text{if } \alpha > 0 \\ \mathcal{M}_{\frac{|\alpha|}{2}}(\mathbb{R}^d) & \text{if } \alpha < 0 \end{cases}$$

the set of Radon measures, such that the solution of (3.1) given in Theorem 3.1 exists for all times. Analogously, we denote for $1 \le p \le \infty$

$$L^p_{OU}(\mathbb{R}^d) = \begin{cases} L^p_0(\mathbb{R}^d) = \bigcap_{\varepsilon>0} L^p_\varepsilon(\mathbb{R}^d), & \text{if } \alpha > 0 \\ L^p_{\frac{|\alpha|}{2}}(\mathbb{R}^d) & \text{if } \alpha < 0 \end{cases}$$

and $L^p_{OU}(\mathbb{R}^d) \subset \mathcal{M}_{OU}(\mathbb{R}^d)$.

Now, we derive estimates in Lebesgue spaces for solutions of (3.1).

Proposition 3.3. *For $1 \le p \le q \le \infty$ and $u_0 \in L^p(\mathbb{R}^d)$*

$$\|u(t)\|_{L^q(\mathbb{R}^d)} \le \mathrm{e}^{\frac{-d\alpha t}{q}} \left(\frac{\alpha}{2\pi(\mathrm{e}^{2\alpha t}-1)}\right)^{\frac{d}{2}(\frac{1}{p}-\frac{1}{q})} \|u_0\|_{L^p(\mathbb{R}^d)}, \quad t > 0.$$

If $u_0 \ge 0$

$$\|u(t)\|_{L^1(\mathbb{R}^d)} = \mathrm{e}^{-d\alpha t}\|u_0\|_{L^1(\mathbb{R}^d)}, \quad t > 0.$$

Proof. The result follows from the fact that:

$$\|w(\cdot\mathrm{e}^{\alpha t})\|_{L^q(\mathbb{R}^d)} = \mathrm{e}^{\frac{-d\alpha t}{q}}\|w(\cdot)\|_{L^q(\mathbb{R}^d)},$$

the standard estimates for the solutions of the heat equation

$$\|w(s)\|_{L^q(\mathbb{R}^d)} \le (4\pi s)^{-\frac{d}{2}(\frac{1}{p}-\frac{1}{q})}\|u_0\|_{L^p(\mathbb{R}^d)}, \quad \text{for every } s > 0 \text{ and } 1 \le p \le q \le \infty,$$

and then

$$\|u(t)\|_{L^q(\mathbb{R}^d)} = \mathrm{e}^{\frac{-d\alpha t}{q}}\|w(s(t))\|_{L^q(\mathbb{R}^d)} \le \mathrm{e}^{\frac{-d\alpha t}{q}}(4\pi s(t))^{-\frac{d}{2}(\frac{1}{p}-\frac{1}{q})}\|u_0\|_{L^p(\mathbb{R}^d)}.$$

Now, we use the estimates in Proposition 2.1 on the heat equation to derive the following estimates for the solutions of the Ornstein–Uhlenbeck equation.

Proposition 3.4. *For $u_0 \in \mathcal{M}_{OU}(\mathbb{R}^d)$ and $\delta > 0$ and $\tilde{\delta}(t) = \frac{\delta\mathrm{e}^{-2\alpha t}}{1+4\delta\mathrm{e}^{-2\alpha t}s(t)}$ for $t > 0$*

(i)

$$\|u(t)\|_{L^1_{\tilde{\delta}}(\mathbb{R}^d)} \le \|u_0\|_{\mathcal{M}_{\tilde{\delta}(t)}(\mathbb{R}^d)}$$

with equality if $u_0 \ge 0$.

(ii)

$$\|u(t)\|_{L^{\infty}_{\delta}(\mathbb{R}^d)} \leq \mathrm{e}^{d\alpha t}\left(\frac{1+4\delta\mathrm{e}^{-2\alpha t}s(t)}{4\pi s(t)}\right)^{d/2}\|u_0\|_{\mathcal{M}_{\tilde{\delta}(t)}(\mathbb{R}^d)}.$$

(iii)

$$\|u(t)\|_{L^{q}_{q\delta}(\mathbb{R}^d)} \leq q^{d/2q}\left(\frac{1+4\delta\mathrm{e}^{-2\alpha t}s(t)}{4\delta\mathrm{e}^{-2\alpha t}s(t)}\right)^{\frac{d}{2}(1-\frac{1}{q})}\|u_0\|_{\mathcal{M}_{\tilde{\delta}(t)}(\mathbb{R}^d)}.$$

(iv)

$$\|u(t)\|_{L^{\infty}_{\delta}(\mathbb{R}^d)} \leq \mathrm{e}^{d\alpha t}(1+4\delta\mathrm{e}^{-2\alpha t}s(t))^{d}\|u_0\|_{L^{\infty}_{\tilde{\delta}(t)}(\mathbb{R}^d)}.$$

Proof. Observe that for $\delta > 0$, if $1 \leq q < \infty$

$$\begin{aligned}\|w(\cdot\mathrm{e}^{\alpha t})\|^q_{L^q_{q\delta}(\mathbb{R}^d)} &= \left(\frac{q\delta}{\pi}\right)^{d/2}\int_{\mathbb{R}^d}\mathrm{e}^{-q\delta|x|^2}|w(x\mathrm{e}^{\alpha t})|^q\,\mathrm{d}x\\ &= \left(\frac{q\delta\mathrm{e}^{-2\alpha t}}{\pi}\right)^{d/2}\int_{\mathbb{R}^d}\mathrm{e}^{-q\delta\mathrm{e}^{-2\alpha t}|y|^2}|w(y)|^q\,\mathrm{d}y = \|w\|^q_{L^q_{q\delta\mathrm{e}^{-2\alpha t}}(\mathbb{R}^d)},\end{aligned} \tag{3.3}$$

while for $q = \infty$

$$\begin{aligned}\|w(\cdot\mathrm{e}^{\alpha t})\|_{L^{\infty}_{\delta}(\mathbb{R}^d)} &= \sup_{x\in\mathbb{R}^d}\left(\frac{\delta}{\pi}\right)^{d/2}\mathrm{e}^{-\delta|x|^2}|w(x\mathrm{e}^{\alpha t})|\\ &= \mathrm{e}^{d\alpha t}\sup_{y\in\mathbb{R}^d}\left(\frac{\delta\mathrm{e}^{-2\alpha t}}{\pi}\right)^{d/2}\mathrm{e}^{-\delta\mathrm{e}^{-2\alpha t}|y|^2}|w(y)| = \mathrm{e}^{d\alpha t}\|w\|_{L^{\infty}_{\delta\mathrm{e}^{-2\alpha t}}(\mathbb{R}^d)}.\end{aligned} \tag{3.4}$$

Then, (3.3) combined with (2.3) gives

$$\|u(t)\|_{L^1_{\delta}(\mathbb{R}^d)} = \|w(s(t))\|_{L^1_{\delta\mathrm{e}^{-2\alpha t}}(\mathbb{R}^d)} \leq \|u_0\|_{\mathcal{M}_{\tilde{\delta}(t)}(\mathbb{R}^d)}$$

with $\tilde{\delta}(t) = \frac{\delta\mathrm{e}^{-2\alpha t}}{1+4\delta\mathrm{e}^{-2\alpha t}s(t)}$, with equality if $u_0 \geq 0$.

In the same way, from (2.4)

$$\begin{aligned}\|u(t)\|_{L^{\infty}_{\delta}(\mathbb{R}^d)} &= \mathrm{e}^{d\alpha t}\|w(s(t))\|_{L^{\infty}_{\delta\mathrm{e}^{-2\alpha t}}(\mathbb{R}^d)}\\ &\leq \mathrm{e}^{d\alpha t}\left(\frac{1+4\delta\mathrm{e}^{-2\alpha t}s(t)}{4\pi s(t)}\right)^{d/2}\|u_0\|_{\mathcal{M}_{\tilde{\delta}(t)}(\mathbb{R}^d)}.\end{aligned}$$

From (2.5)

$$\begin{aligned}\|u(t)\|_{L^q_{q\delta}(\mathbb{R}^d)} &= \|w(s(t))\|_{L^q_{q\delta\mathrm{e}^{-2\alpha t}}(\mathbb{R}^d)}\\ &\leq q^{d/2q}\left(\frac{1+4\delta\mathrm{e}^{-2\alpha t}s(t)}{4\delta\mathrm{e}^{-2\alpha t}s(t)}\right)^{\frac{d}{2}(1-\frac{1}{q})}\|u_0\|_{\mathcal{M}_{\tilde{\delta}(t)}(\mathbb{R}^d)}.\end{aligned}$$

Finally, from (2.6)

$$\|u(t)\|_{L^{\infty}_{\delta}(\mathbb{R}^d)} = \mathrm{e}^{d\alpha t}\|w(s(t))\|_{L^{\infty}_{\delta\mathrm{e}^{-2\alpha t}}(\mathbb{R}^d)} \leq \mathrm{e}^{d\alpha t}(1+4\delta\mathrm{e}^{-2\alpha t}s(t))^{d}\|u_0\|_{L^{\infty}_{\tilde{\delta}(t)}(\mathbb{R}^d)}.$$

□

Observe that in the proof above that if $\alpha > 0$, then $\tilde{\delta}(t) \leq \delta e^{-2\alpha t} \leq \delta$ and $\tilde{\delta}(t) \to 0$ as $t \to \infty$. Therefore, as in Sect. 2 for the heat equation, to estimate the solution of (3.1) for large times, we need to estimate the norm of the initial data in stronger norms.

On the other hand, if $\alpha < 0$ and $u_0 \in \mathcal{M}_\delta(\mathbb{R}^d)$ with $\delta \leq \frac{|\alpha|}{2}$, then $\delta \leq \tilde{\delta}(t) = \frac{\delta e^{-2\alpha t}}{1+4\delta e^{-2\alpha t} s(t)} \leq \frac{|\alpha|}{2}$ for $t > 0$ and $\tilde{\delta}(t) \to \frac{|\alpha|}{2}$ as $t \to \infty$. On the other hand, if $u_0 \in \mathcal{M}_{OU}(\mathbb{R}^d) \subset \mathcal{M}_\delta(\mathbb{R}^d)$ with $\frac{|\alpha|}{2} < \delta$, then $\frac{|\alpha|}{2} < \tilde{\delta}(t) < \delta$ for $t > 0$ and $\tilde{\delta}(t) \to \frac{|\alpha|}{2}$ as $t \to \infty$. Therefore, in this case, the estimates on the solution for large times are given essentially in terms of the norm of $\mathcal{M}_{\frac{|\alpha|}{2}}(\mathbb{R}^d)$.

In particular, we get the following result.

Corollary 3.5. *Assume* $\alpha < 0$. *Then, the space* $\mathcal{M}_{OU}(\mathbb{R}^d) = \mathcal{M}_{\frac{|\alpha|}{2}}(\mathbb{R}^d)$ *is invariant and the Ornstein–Uhlenbeck flow is a contraction in this space. Moreover, we have the following smoothing estimates:*

$$\|u(t)\|_{L^\infty_{\frac{|\alpha|}{2}}(\mathbb{R}^d)} \leq \left(\frac{|\alpha|}{2\pi}\right)^{d/2} \left(\frac{1}{1-e^{-2|\alpha|t}}\right)^{\frac{d}{2}} \|u_0\|_{\mathcal{M}_{\frac{|\alpha|}{2}}(\mathbb{R}^d)}.$$

and for $1 \leq q < \infty$

$$\|u(t)\|_{L^q_{q\frac{|\alpha|}{2}}(\mathbb{R}^d)} \leq q^{d/2q} \left(\frac{1}{1-e^{-2|\alpha|t}}\right)^{\frac{d}{2}(1-\frac{1}{q})} \|u_0\|_{\mathcal{M}_{\frac{|\alpha|}{2}}(\mathbb{R}^d)}.$$

The space $L^\infty_{\frac{|\alpha|}{2}}(\mathbb{R}^d)$ *is also invariant and*

$$\|u(t)\|_{L^\infty_{\frac{|\alpha|}{2}}(\mathbb{R}^d)} \leq e^{d|\alpha|t} \|u_0\|_{L^\infty_{\frac{|\alpha|}{2}}(\mathbb{R}^d)}.$$

Proof. From part (i) Proposition 3.4, observe that $\tilde{\delta}(t) = \frac{\delta e^{-2\alpha t}}{1+4\delta e^{-2\alpha t} s(t)} = \delta$ if and only if $\delta = \frac{|\alpha|}{2}$ and we get that the Ornstein–Uhlenbeck flow is a contraction in $\mathcal{M}_{OU}(\mathbb{R}^d)$.

For the smoothing estimates, using $\tilde{\delta}(t) = \frac{\delta e^{-2\alpha t}}{1+4\delta e^{-2\alpha t} s(t)} = \delta$, parts (ii) and (iii) in Proposition 3.4 lead to the results.

For $L^\infty_{\frac{|\alpha|}{2}}(\mathbb{R}^d)$, the result follows from part (iv) in Proposition 3.4. □

Now, the estimates in Theorem 2.2 give us the following.

Theorem 3.6. *For* $1 \leq p \leq q < \infty$ *and* $u_0 \in L^p_{OU}(\mathbb{R}^d)$, *we have*

$$\|u(t)\|_{L^q_{q\delta}(\mathbb{R}^d)} \leq \Phi_{p,q}(t) \|u_0\|_{L^p_{\tilde{\delta}_p(t)}(\mathbb{R}^d)} \quad t > 0$$

with $\tilde{\delta}_p(t) = \frac{p\delta e^{-2\alpha t}}{1+4p\delta e^{-2\alpha t} s(t)}$ *and*

$$\Phi_{p,q}(t) = \left(\frac{q}{p}\right)^{d/2q} \left(\frac{1+4p\delta e^{-2\alpha t} s(t))}{4p\delta e^{-2\alpha t} s(t)}\right)^{\frac{d}{2}(\frac{1}{p}-\frac{1}{q})} \qquad t > 0.$$

Also, for $q = \infty$, $1 \leq p \leq \infty$

$$\|u(t)\|_{L^\infty_\delta(\mathbb{R}^d)} \leq \Phi_{p,\infty}(t) \|u_0\|_{L^p_{\tilde{\delta}_p(t)}(\mathbb{R}^d)} \quad t > 0$$

with $\tilde{\delta}_\infty(t) = \tilde{\delta}_1(t)$ *and*

$$\Phi_{p,\infty}(t) = \left(\frac{\delta}{\pi}\right)^{\frac{d}{2}} \left(\frac{1 + 4p\delta \mathrm{e}^{-2\alpha t} s(t)}{4p\delta \mathrm{e}^{-2\alpha t} s(t)}\right)^{\frac{d}{2p}}.$$

In particular, if $p = q < \infty$

$$\|u(t)\|_{L^p_{p\delta}(\mathbb{R}^d)} \leq \|u_0\|_{L^p_{\tilde{\delta}_p(t)}(\mathbb{R}^d)} \qquad t > 0.$$

Proof. Observe that (3.3), (3.4) combined with (2.7) give for $1 \leq q < \infty$

$$\begin{aligned}\|u(t)\|_{L^q_{q\delta}(\mathbb{R}^d)} &= \|w(s(t))\|_{L^q_{q\delta \mathrm{e}^{-2\alpha t}}(\mathbb{R}^d)} \\ &\leq \left(\frac{q}{p}\right)^{d/2q} \left(\frac{1 + 4p\delta \mathrm{e}^{-2\alpha t} s(t))}{4p\delta \mathrm{e}^{-2\alpha t} s(t)}\right)^{\frac{d}{2}(\frac{1}{p} - \frac{1}{q})} \|u_0\|_{L^p_{\tilde{\delta}_p(t)}(\mathbb{R}^d)}\end{aligned}$$

with $\tilde{\delta}_p(t) = \frac{p\delta \mathrm{e}^{-2\alpha t}}{1+4p\delta \mathrm{e}^{-2\alpha t} s(t)}$. The case $p = q$ follows from this or from (2.8).

Also, for $q = \infty$, (2.9) gives

$$\begin{aligned}\|u(t)\|_{L^\infty_\delta(\mathbb{R}^d)} &= \mathrm{e}^{d\alpha t} \|w(s(t))\|_{L^\infty_{\delta \mathrm{e}^{-2\alpha t}}(\mathbb{R}^d)} \\ &\leq \mathrm{e}^{d\alpha t} \left(\frac{\delta \mathrm{e}^{-2\alpha t}}{\pi}\right)^{\frac{d}{2}} \left(\frac{1 + 4p\delta \mathrm{e}^{-2\alpha t} s(t)}{4p\delta \mathrm{e}^{-2\alpha t} s(t)}\right)^{\frac{d}{2p}} \|u_0\|_{L^p_{\tilde{\delta}_p(t)}(\mathbb{R}^d)} \\ &= \left(\frac{\delta}{\pi}\right)^{\frac{d}{2}} \left(\frac{1 + 4p\delta \mathrm{e}^{-2\alpha t} s(t)}{4p\delta \mathrm{e}^{-2\alpha t} s(t)}\right)^{\frac{d}{2p}} \|u_0\|_{L^p_{\tilde{\delta}_p(t)}(\mathbb{R}^d)}.\end{aligned}$$

As observed before Corollary 3.5, if $\alpha > 0$, then $\tilde{\delta}_p(t) \leq p\delta e^{-2\alpha t} \leq p\delta$ and $\tilde{\delta}_p(t) \to 0$ as $t \to \infty$. Again, estimates for large times require estimates of the initial data in stronger norms.

On the other hand, if $\alpha < 0$ and $u_0 \in L^p_\delta(\mathbb{R}^d)$ with $\delta \leq \frac{|\alpha|}{2}$, then $p\delta \leq \tilde{\delta}_p(t) = \frac{p\delta \mathrm{e}^{-2\alpha t}}{1+4p\delta \mathrm{e}^{-2\alpha t} s(t)} \leq \frac{|\alpha|}{2}$ for $t > 0$ and $\tilde{\delta}_p(t) \to \frac{|\alpha|}{2}$ as $t \to \infty$. On the other hand, if $u_0 \in L^p_{OU}(\mathbb{R}^d) \subset L^p_\delta(\mathbb{R}^d)$ with $\frac{|\alpha|}{2} < p\delta$, then $\frac{|\alpha|}{2} < \tilde{\delta}_p(t) < \delta$ for $t > 0$ and $\tilde{\delta}_p(t) \to \frac{|\alpha|}{2}$ as $t \to \infty$.

In particular, we get the following result.

Corollary 3.7. *Assume* $\alpha < 0$. *Then, the spaces* $L^p_{\frac{|\alpha|}{2}}(\mathbb{R}^d)$, $1 \leq p < \infty$ *are invariant and the Ornstein–Uhlenbeck flow is a contraction in them.*

Also

$$\|u(t)\|_{L^\infty_{\frac{|\alpha|}{2p}}(\mathbb{R}^d)} \leq \left(\frac{\delta}{\pi}\right)^{\frac{d}{2}} \left(\frac{1}{1 - \mathrm{e}^{-2|\alpha| t}}\right)^{\frac{d}{2p}} \|u_0\|_{L^p_{\frac{|\alpha|}{2}}(\mathbb{R}^d)},$$

and for $1 \leq p \leq q < \infty$

$$\|u(t)\|_{L^q_{\frac{q|\alpha|}{2p}}(\mathbb{R}^d)} \leq \left(\frac{q}{p}\right)^{d/2q} \left(\frac{1}{1 - \mathrm{e}^{-2|\alpha| t}}\right)^{\frac{d}{2}(\frac{1}{p} - \frac{1}{q})} \|u_0\|_{L^p_{\frac{|\alpha|}{2}}(\mathbb{R}^d)} \quad t > 0.$$

Proof. From Theorem 3.6 with $p = q$, observe that $\tilde{\delta}_p(t) = \frac{p\delta \mathrm{e}^{-2\alpha t}}{1+4p\delta \mathrm{e}^{-2\alpha t} s(t)} = p\delta$ if and only if $\delta = \frac{|\alpha|}{2p}$ and we get a contraction in $L^p_{\frac{|\alpha|}{2}}(\mathbb{R}^d)$, $1 \leq p < \infty$.

Also, using $\tilde{\delta}_p(t) = \frac{p\delta e^{-2\alpha t}}{1+4p\delta e^{-2\alpha t}s(t)} = p\delta = \frac{|\alpha|}{2}$, Theorem 3.6 leads to

$$\Phi_{p,q}(t) = \left(\frac{q}{p}\right)^{d/2q} \left(\frac{1}{1-e^{-2|\alpha|t}}\right)^{\frac{d}{2}(\frac{1}{p}-\frac{1}{q})} \qquad t > 0$$

for $1 \leq p \leq q < \infty$ and $\Phi_{p,\infty}(t) = \left(\frac{\delta}{\pi}\right)^{\frac{d}{2}} \left(\frac{1}{1-e^{-2|\alpha|t}}\right)^{\frac{d}{2p}}$. □

For derivatives, we have the following estimates.

Proposition 3.8. *Assume $\alpha < 0$. If $u_0 \in \mathcal{M}_{OU}(\mathbb{R}^d)$, then we have the following estimates.*

(i) *For any multi-index $\beta \in \mathbb{N}^d$ and $1 \leq q < \infty$, for any $\delta > 0$, $t > 0$, $\gamma > 1$ and $\tilde{\delta}^\gamma(t) = \frac{\delta e^{-2\alpha t}}{1+4\delta e^{-2\alpha t}\gamma s(t)}$*

$$\|D_x^\beta u(t)\|_{L^q_{q\delta}(\mathbb{R}^d)} \leq c_{\beta,\gamma} q^{d/2q} \frac{e^{\alpha|\beta|t}}{s(t)^{\frac{|\beta|}{2}}} \left(\frac{1+4\delta e^{-2\alpha t}\gamma s(t)}{4\delta e^{-2\alpha t}\gamma s(t)}\right)^{\frac{d}{2}(1-\frac{1}{q})} \|u_0\|_{\mathcal{M}_{\tilde{\delta}^\gamma(t)}(\mathbb{R}^d)}$$

and

$$\|D_x^\beta u(t)\|_{L^\infty_\delta(\mathbb{R}^d)} \leq c_{\beta,\gamma} \frac{e^{(d+|\beta|)\alpha t}}{s(t)^{\frac{|\beta|}{2}}} \left(\frac{1+4\delta e^{-2\alpha t}\gamma s(t)}{4\pi\gamma s(t)}\right)^{\frac{d}{2}} \|u_0\|_{\mathcal{M}_{\tilde{\delta}^\gamma(t)}(\mathbb{R}^d)}.$$

(ii) *If $u_0 \in L^p_{OU}(\mathbb{R}^d)$ with $1 \leq p \leq \infty$ for any multi-index $\beta \in \mathbb{N}^d$ and for any $1 \leq p \leq q \leq \infty$, then for any $\delta > 0$, $t > 0$, $\gamma > 1$ and $\tilde{\delta}_p^\gamma(t) = \frac{p\delta e^{-2\alpha t}}{1+4p\delta e^{-2\alpha t}\gamma s(t)}$*

$$\|D_x^\beta u(t)\|_{L^q_{q\delta}(\mathbb{R}^d)} \leq c_{\beta,\gamma} \left(\frac{q}{p}\right)^{d/2q} \frac{e^{\alpha|\beta|t}}{s(t)^{\frac{|\beta|}{2}}} \left(\frac{1+4p\delta e^{-2\alpha t}\gamma s(t)}{4p\delta e^{-2\alpha t}\gamma s(t)}\right)^{\frac{d}{2}(\frac{1}{p}-\frac{1}{q})} \|u_0\|_{L^p_{\tilde{\delta}_p^\gamma(t)}(\mathbb{R}^d)}$$

and

$$\|D_x^\beta u(t)\|_{L^\infty_\delta(\mathbb{R}^d)} \leq c_{\beta,\gamma} \left(\frac{\delta}{\pi}\right)^{\frac{d}{2}} \frac{e^{(d+|\beta|)\alpha t}}{s(t)^{\frac{|\beta|}{2}}} \left(\frac{1+4p\delta e^{-2\alpha t}\gamma s(t)}{4p\delta e^{-2\alpha t}\gamma s(t)}\right)^{\frac{d}{2p}} \|u_0\|_{L^p_{\tilde{\delta}_p^\gamma(t)}(\mathbb{R}^d)}$$

$$\|D_x^\beta u(t)\|_{L^\infty_\delta(\mathbb{R}^d)} \leq c_{\beta,\gamma} \frac{e^{(d+|\beta|)\alpha t}}{s(t)^{\frac{|\beta|}{2}}} (1+4\delta e^{-2\alpha t}\gamma s(t))^d \|u_0\|_{L^\infty_{\tilde{\delta}_\infty^\gamma(t)}(\mathbb{R}^d)}$$

with $\tilde{\delta}_\infty^\gamma(t) = \tilde{\delta}_1^\gamma(t)$.

Proof. Observe that for any multi-index $\beta \in \mathbb{N}^d$, $D_x^\beta u(x,t) = e^{\alpha|\beta|t} D_y^\beta w(e^{\alpha t}x, s(t))$ and from (2.13), for any $\gamma > 1$

$$|D_y^\beta w(y,s)| \leq \frac{c_{\beta,\gamma}}{s^{d/2+|\beta|/2}} \int_{\mathbb{R}^d} e^{-\frac{1}{\gamma}\frac{|y-z|^2}{4s}} \, d|u_0|(z)$$

and the integral term above is a multiple of the solution of the heat equation with initial data $|u_0|$ with the extra factor $\frac{1}{s^{\frac{|\beta|}{2}}}$ and at time $s' = \gamma s$.

(i) Hence, from (2.5)

$$\|D_x^\beta u(t)\|_{L^q_{q\delta}(\mathbb{R}^d)} = e^{\alpha|\beta|t} \|D_y^\beta w(s(t))\|_{L^q_{q\delta e^{-2\alpha t}}(\mathbb{R}^d)}$$

$$\leq \frac{e^{\alpha|\beta|t}}{s(t)^{\frac{|\beta|}{2}}} c_{\beta,\gamma} q^{d/2q} \left(\frac{1 + 4\delta e^{-2\alpha t}\gamma s(t)}{4\delta e^{-2\alpha t}\gamma s(t)} \right)^{\frac{d}{2}(1-\frac{1}{q})} \|u_0\|_{\mathcal{M}_{\tilde{\delta}^{\gamma}(t)}(\mathbb{R}^d)}.$$

Analogously from (2.4)

$$\|D_x^\beta u(t)\|_{L^\infty_\delta(\mathbb{R}^d)} = e^{(d+|\beta|)\alpha t} \|D_y^\beta w(s(t))\|_{L^\infty_{\delta e^{-2\alpha t}}(\mathbb{R}^d)}$$
$$\leq \frac{e^{(d+|\beta|)\alpha t}}{s(t)^{\frac{|\beta|}{2}}} c_{\beta,\gamma} \left(\frac{1 + 4\delta e^{-2\alpha t}\gamma s(t)}{4\pi\gamma s(t)} \right)^{\frac{d}{2}} \|u_0\|_{\mathcal{M}_{\tilde{\delta}^{\gamma}(t)}(\mathbb{R}^d)}.$$

(ii) From (2.7)

$$\|D_x^\beta u(t)\|_{L^q_{q\delta}(\mathbb{R}^d)} = e^{\alpha|\beta|t} \|D_y^\beta w(s(t))\|_{L^q_{q\delta e^{-2\alpha t}}(\mathbb{R}^d)}$$
$$\leq \frac{e^{\alpha|\beta|t}}{s(t)^{\frac{|\beta|}{2}}} c_{\beta,\gamma} \left(\frac{q}{p}\right)^{d/2q} \left(\frac{1 + 4p\delta e^{-2\alpha t}\gamma s(t)}{4p\delta e^{-2\alpha t}\gamma s(t)} \right)^{\frac{d}{2}(\frac{1}{p}-\frac{1}{q})} \|u_0\|_{L^p_{\tilde{\delta}^{\gamma}_p(t)}(\mathbb{R}^d)}.$$

Also, from (2.9)

$$\|D_x^\beta u(t)\|_{L^\infty_\delta(\mathbb{R}^d)} = e^{(d+|\beta|)\alpha t} \|D_y^\beta w(s(t))\|_{L^\infty_{\delta e^{-2\alpha t}}(\mathbb{R}^d)}$$
$$\leq \frac{e^{(d+|\beta|)\alpha t}}{s(t)^{\frac{|\beta|}{2}}} c_{\beta,\gamma} \left(\frac{\delta}{\pi}\right)^{\frac{d}{2}} \left(\frac{1 + 4p\delta e^{-2\alpha t}\gamma s(t)}{4p\delta e^{-2\alpha t}\gamma s(t)} \right)^{\frac{d}{2p}} \|u_0\|_{L^p_{\tilde{\delta}^{\gamma}_p(t)}(\mathbb{R}^d)}.$$

Finally, from (2.6)

$$\|D_x^\beta u(t)\|_{L^\infty_\delta(\mathbb{R}^d)} = e^{(d+|\beta|)\alpha t} \|D_y^\beta w(s(t))\|_{L^\infty_{\delta e^{-2\alpha t}}(\mathbb{R}^d)}$$
$$\leq \frac{e^{(d+|\beta|)\alpha t}}{s(t)^{\frac{|\beta|}{2}}} c_{\beta,\gamma} (1 + 4\delta e^{-2\alpha t}\gamma s(t))^d \|u_0\|_{L^\infty_{\tilde{\delta}^{\gamma}_1(t)}(\mathbb{R}^d)}.$$

As in Corollaries 3.5 and 3.7, we get the following particular estimates.

Corollary 3.9. *Assume* $\alpha < 0$ *and* $u_0 \in \mathcal{M}_{OU}(\mathbb{R}^d) = \mathcal{M}_{\frac{|\alpha|}{2}}(\mathbb{R}^d)$. *Then*

(i) *For any multi-index* $\beta \in \mathbb{N}^d$ *and* $1 \leq q < \infty$

$$\|D_x^\beta u(t)\|_{L^q_{q\frac{|\alpha|}{2}}(\mathbb{R}^d)} \leq c_{\beta,q,\alpha} e^{\alpha|\beta|t} \left(\frac{1}{1 - e^{-2|\alpha|t}} \right)^{\frac{d}{2}(1-\frac{1}{q})+\frac{|\beta|}{2}} \|u_0\|_{\mathcal{M}_{\frac{|\alpha|}{2}}(\mathbb{R}^d)}$$

with $c_{\beta,q,\alpha} = c_\beta q^{d/2q} |\alpha|^{\frac{|\beta|}{2}}$ *and*

$$\|D_x^\beta u(t)\|_{L^\infty_{\frac{|\alpha|}{2}}(\mathbb{R}^d)} \leq c_{\beta,\alpha} e^{\alpha|\beta|t} \left(\frac{1}{1 - e^{-2|\alpha|t}} \right)^{\frac{d}{2}+\frac{|\beta|}{2}} \|u_0\|_{\mathcal{M}_{\frac{|\alpha|}{2}}(\mathbb{R}^d)}$$

with $c_{\beta,\alpha} = c_\beta |\alpha|^{\frac{d}{2}+\frac{|\beta|}{2}}$.

(ii) *If* $u_0 \in L^p_{\frac{|\alpha|}{2}}(\mathbb{R}^d)$ *with* $1 \leq p \leq \infty$ *for any multi-index* $\beta \in \mathbb{N}^d$ *and for any* $1 \leq p \leq q \leq \infty$

$$\|D_x^\beta u(t)\|_{L^q_{\frac{q|\alpha|}{2p}}(\mathbb{R}^d)} \leq c_{\beta,p,q,\alpha} e^{\alpha|\beta|t} \left(\frac{1}{1 - e^{-2|\alpha|t}} \right)^{\frac{d}{2}(\frac{1}{p}-\frac{1}{q})+\frac{|\beta|}{2}} \|u_0\|_{L^p_{\frac{|\alpha|}{2}}(\mathbb{R}^d)}$$

with $c_{\beta,p,q,\alpha} = c_\beta \left(\frac{q}{p}\right)^{d/2q} |\alpha|^{\frac{|\beta|}{2}}$ *and*

$$\|D_x^\beta u(t)\|_{L^\infty_{\frac{|\alpha|}{2p}}(\mathbb{R}^d)} \leq c_{\beta,p,\alpha} \mathrm{e}^{(d+|\beta|)\alpha t} \left(\frac{1}{1-\mathrm{e}^{-2|\alpha|t}}\right)^{\frac{d}{2p}+\frac{|\beta|}{2}} \|u_0\|_{L^p_{\frac{|\alpha|}{2}}(\mathbb{R}^d)}$$

with $c_{\beta,p,\alpha} = c_\beta \left(\frac{|\alpha|}{2\pi p}\right)^{\frac{d}{2}} |\alpha|^{\frac{|\beta|}{2}}$ *and*

$$\|D_x^\beta u(t)\|_{L^\infty_{\frac{|\alpha|}{2}}(\mathbb{R}^d)} \leq c_{\beta,\alpha} \mathrm{e}^{(|\beta|-d)\alpha t} \left(\frac{1}{1-\mathrm{e}^{-2|\alpha|t}}\right)^{\frac{|\beta|}{2}} \|u_0\|_{L^\infty_{\frac{|\alpha|}{2}}(\mathbb{R}^d)}$$

with $c_{\beta,\alpha} = c_\beta |\alpha|^{\frac{|\beta|}{2}}$.

Proof. Notice that $\tilde{\delta}^\gamma(t) = \frac{1}{\gamma}\frac{\delta\gamma \mathrm{e}^{-2\alpha t}}{1+4\delta\gamma\mathrm{e}^{-2\alpha t}s(t)}$ and then as in Corollary 3.5, $\tilde{\delta}^\gamma(t) = \delta$ if and only if $\delta = \frac{|\alpha|}{2}$. In an analogous way, $\tilde{\delta}_p^\gamma(t) = \frac{1}{\gamma}\frac{p\delta\gamma\mathrm{e}^{-2\alpha t}}{1+4p\delta\gamma\mathrm{e}^{-2\alpha t}s(t)}$ and as in Corollary 3.7, $\tilde{\delta}_p^\gamma(t) = p\delta$ if and only if $\delta = \frac{|\alpha|}{2p}$.

In both cases, Proposition 3.8 gives the results. For example, for the first estimate, since $\gamma > 1$

$$\left(\frac{1+4\delta\mathrm{e}^{-2\alpha t}\gamma s(t)}{4\delta\mathrm{e}^{-2\alpha t}\gamma s(t)}\right)^{\frac{d}{2}(1-\frac{1}{q})} = \left(\frac{1}{\gamma\frac{|\alpha|}{2}s(t)}\right)^{\frac{d}{2}(1-\frac{1}{q})} \leq \left(\frac{1}{1-\mathrm{e}^{-2|\alpha|t}}\right)^{\frac{d}{2}(1-\frac{1}{q})}.$$

For case (ii), observe for example that

$$\left(\frac{1+4p\delta\mathrm{e}^{-2\alpha t}\gamma s(t)}{4p\delta\mathrm{e}^{-2\alpha t}\gamma s(t)}\right)^{\frac{d}{2}(\frac{1}{p}-\frac{1}{q})} = \left(\frac{1}{\gamma\frac{|\alpha|}{2}s(t)}\right)^{\frac{d}{2}(\frac{1}{p}-\frac{1}{q})} \leq \left(\frac{1}{1-\mathrm{e}^{-2|\alpha|t}}\right)^{\frac{d}{2}(\frac{1}{p}-\frac{1}{q})}.$$

As in Proposition 2.4 and arguing as in Theorem 3.6, for solutions that may not exist for all times, we get the following results.

Proposition 3.10. *Assume* $\alpha > 0$ *and* $\varepsilon > 0$ *or* $\alpha < 0$ *and* $|\alpha| < 2\varepsilon$ *and define* $T_{OU}(\varepsilon)$ *as the unique solution of* $s(t) = \frac{1}{4\varepsilon}$.

For every $\delta > \varepsilon$, *define* $\tilde{\delta}_p(t) = \frac{p\delta\mathrm{e}^{-2\alpha t}}{1+4p\delta\mathrm{e}^{-2\alpha t}s(t)}$. *For any* $u_0 \in L^p_{p\varepsilon}(\mathbb{R}^d)$ *with* $1 \leq p < \infty$ *and for any* $1 \leq p \leq q < \infty$, *we have*

$$\|u(t)\|_{L^q_{q\delta}(\mathbb{R}^d)} \leq c_{p,q}\left(\frac{1+4p\delta\mathrm{e}^{-2\alpha t}s(t)}{4p\delta\mathrm{e}^{-2\alpha t}s(t)}\right)^{\frac{d}{2}(\frac{1}{p}-\frac{1}{q})} \|u_0\|_{L^p_{\tilde{\delta}_p(t)}(\mathbb{R}^d)}$$

$$\|u(t)\|_{L^\infty_{\delta}(\mathbb{R}^d)} \leq c_{p,\infty}\left(\frac{1+4p\delta\mathrm{e}^{-2\alpha t}s(t)}{4p\delta\mathrm{e}^{-2\alpha t}s(t)}\right)^{\frac{d}{2p}} \|u_0\|_{L^p_{\tilde{\delta}_p(t)}(\mathbb{R}^d)}$$

which hold for $0 < t \leq \big(T_{OU}(p\varepsilon) - T_{OU}(p\delta)\big)$.

3.3. Asymptotic Behaviour

We start with the case $\alpha < 0$. The next result states, in particular, that solutions converge to a constant. Observe that Corollary 3.9 shows that the derivatives converge to zero as $t \to \infty$.

Proposition 3.11. *Assume* $\alpha < 0$.

(i) *If* $u_0 \in \mathcal{M}_{\frac{|\alpha|}{2}}(\mathbb{R}^d)$, *then*

$$\lim_{t\to\infty} u(x,t) = \int_{\mathbb{R}^d} \rho_{\frac{|\alpha|}{2}}(z)\,\mathrm{d}u_0(z)$$

uniformly for $(x,t) \in \mathbb{R}^{d+1}$ *in sets, such that* $|x|e^{\alpha t} \to 0$ *uniformly as* $t \to \infty$ *(in particular for* x *in bounded sets of* $\mathbb{R}^d$*). Also*

$$\lim_{t\to\infty} u(t) = \int_{\mathbb{R}^d} \rho_{\frac{|\alpha|}{2}}(z)\,\mathrm{d}u_0(z) \quad \textit{in } L^p_{\frac{|\alpha|}{2}}(\mathbb{R}^d) \textit{ for all } 1 \le p < \infty.$$

(ii) *On the other hand, if* $u_0 \notin \mathcal{M}_{\frac{|\alpha|}{2}}(\mathbb{R}^d)$ *but in* (3.2), *we have* $\varepsilon_0(u_0) = \frac{|\alpha|}{2}$, *then for all* $x \in \mathbb{R}^d$

$$\lim_{t\to\infty} u(x,t) = \infty.$$

Proof. (i) With the notations in (3.2), since $\varepsilon_0(u_0) \le \frac{|\alpha|}{2}$, the function $w(y,s)$ is Theorem 3.1 is defined for $0 < s < \hat{T} = \frac{1}{4\varepsilon_0} \ge \frac{1}{2|\alpha|}$, and then, the solution of (3.1) is defined for $0 < t < \infty$ and for all $(x,t) \in \mathbb{R}^{d+1}$ as in the statement

$$u(x,t) = w(x\mathrm{e}^{\alpha t}, s(t)) \to w\left(0, \frac{1}{2|\alpha|}\right) = \int_{\mathbb{R}^d} \rho_{\frac{|\alpha|}{2}}(z)\,\mathrm{d}u_0(z) := \mathbb{E}(u_0) \quad \text{as } t \to \infty.$$

Assume now $u_0 \ge 0$. Then, Fatou's Lemma gives, using $\rho_{\frac{|\alpha|}{2}}$ is a probability density

$$\mathbb{E}(u_0) \le \liminf_{t\to\infty} \|u(t)\|_{L^1_{\frac{|\alpha|}{2}}(\mathbb{R}^d)} \le \limsup_{t\to\infty} \|u(t)\|_{L^1_{\frac{|\alpha|}{2}}(\mathbb{R}^d)} = \|u_0\|_{\mathcal{M}_{\frac{|\alpha|}{2}}(\mathbb{R}^d)} = \mathbb{E}(u_0),$$

and this implies $u(t) \to \mathbb{E}(u_0)$ in $L^1_{\frac{|\alpha|}{2}}(\mathbb{R}^d)$ as $t \to \infty$. This and the uniform bounds in $L^\infty_{\frac{|\alpha|}{2}}(\mathbb{R}^d)$ for $t \ge 1$ in Corollary 3.5 imply convergence in $L^p_{\frac{|\alpha|}{2}}(\mathbb{R}^d)$ for all $1 \le p < \infty$.

For non-negative u_0, we split it into positive and negative parts and then $u(t,u_0) = u(t,u_0^+) - u(t,u_0^-)$ and apply the argument above to each term, since $\mathbb{E}(u_0^+) - \mathbb{E}(u_0^-) = \mathbb{E}(u_0)$.

(ii) On the other hand, if $u_0 \notin \mathcal{M}_{\frac{|\alpha|}{2}}(\mathbb{R}^d)$ but $\varepsilon_0(u_0) = \frac{|\alpha|}{2}$, then $w(y,s)$ is defined up to time $\hat{T} = \frac{1}{4\varepsilon_0} = \frac{1}{2|\alpha|}$ and then $u(x,t)$ is defined for $0 < t < \infty$ and for all $x \in \mathbb{R}^d$

$$u(x,t) = w(x\mathrm{e}^{\alpha t}, s(t)) \to w\left(0, \frac{1}{2|\alpha|}\right) = \infty \quad \text{as } t \to \infty,$$

since $u_0 \notin \mathcal{M}_{\frac{|\alpha|}{2}}(\mathbb{R}^d)$. □

We now consider the case $\alpha > 0$.

Proposition 3.12. *Assume* $\alpha > 0$ *and* $u_0 \in \mathcal{M}_0(\mathbb{R}^d)$. *Then, the solution of* (3.1) *in Theorem* 3.1 *is bounded for all* $t > 0$ *in* $L^q_0(\mathbb{R}^d)$, $1 \le q \le \infty$

$$\|u_0\|_{\mathcal{M}_\varepsilon(\mathbb{R}^d)} \le C, \quad \varepsilon \to 0,$$

while it decays to zero in $L^q_0(\mathbb{R}^d)$ *provided that*

$$\|u_0\|_{\mathcal{M}_\varepsilon(\mathbb{R}^d)} \to 0, \quad \varepsilon \to 0.$$

Proof. Since $\alpha > 0$, $\tilde{\delta}(t) \leq \delta e^{-2\alpha t} \leq \delta$ and, in fact, $\tilde{\delta}(t) = O(e^{-2\alpha t}) \to 0$ as $t \to \infty$. Also, $s(t) = O(e^{2\alpha t})$ as $t \to \infty$.

Then, in Proposition 3.4, observe that in parts (ii) and (iii), the factors $\mathrm{e}^{d\alpha t}\left(\frac{1+4\delta \mathrm{e}^{-2\alpha t}s(t)}{4\pi s(t)}\right)^{d/2}$ and $\left(\frac{1+4\delta \mathrm{e}^{-2\alpha t}s(t)}{4\delta \mathrm{e}^{-2\alpha t}s(t)}\right)^{\frac{d}{2}(1-\frac{1}{q})}$ are bounded for large times.

Hence, by parts (i), (ii), and (iii) in Proposition 3.4, the solution remains bounded or converges to zero provided $\|u_0\|_{\mathcal{M}_{\tilde{\delta}(t)}(\mathbb{R}^d)}$ is bounded or converges to 0 for $t \to \infty$, respectively. □

Recall that for $\varepsilon > 0$ and w as in Theorem 3.1

$$\|u_0\|_{\mathcal{M}_\varepsilon(\mathbb{R}^d)} = w\left(0, \frac{1}{4\varepsilon}, |u_0|\right)$$

and that for $u_0 \in \mathcal{M}_0(\mathbb{R}^d)$, $u(0,t,u_0) = w(0, s(t), u_0)$ and then

$$|u(0,t,u_0)| \leq u(0,t,|u_0|) = w(0,s(t),|u_0|) = \|u_0\|_{\mathcal{M}_{\frac{1}{4s(t)}}(\mathbb{R}^d)}$$

and $s(t) \to \infty$ as $t \to \infty$. Hence, the behaviour of the solution of (3.1) at $x = 0$ controls the behaviour of the norms of the solution.

Then, the results in [6] on the heat equation immediately lead to the following results. The first one shows how the distribution of mass of the initial data controls the behaviour of $u(0,t)$ for large times.

Theorem 3.13. *Suppose that $\alpha > 0$ and $u_0 \in \mathcal{M}_0(\mathbb{R}^d)$.*

(i) *If* $\sup_{R>0} \dfrac{1}{R^d} \displaystyle\int_{R/2 \leq |x| \leq R} \mathrm{d}|u_0|(x) \leq M$, *then* $u(0,\cdot,|u_0|) \in L^\infty(0,\infty)$.

(ii) *If* $\lim_{R\to\infty} \dfrac{1}{R^d} \displaystyle\int_{R/2 \leq |x| \leq R} \mathrm{d}|u_0|(x) = 0$, *then* $u(0,t,|u_0|) \to 0$ *as* $t \to \infty$.

Assume, in addition, that $u_0 \geq 0$. Then

(iii) *If* $\liminf_{R\to\infty} \dfrac{1}{R^d} \displaystyle\int_{|x| \leq R} \mathrm{d}u_0(x) > 0$, *then* $\liminf_{t\to\infty} u(0,t,u_0) > 0$.

(iv) *If* $\lim_{R\to\infty} \dfrac{1}{R^d} \displaystyle\int_{|x| \leq R} \mathrm{d}u_0(x) = \infty$, *then* $u(0,t,u_0) \to \infty$ *as* $t \to \infty$.

The second one shows that typically the behaviour of $u(0,t)$ can be very complex.

Theorem 3.14. *Assume $\alpha > 0$ and fix an arbitrary sequence of non-negative numbers $\{\alpha_k\}_k$. Then*

(i) *There exists a non-negative $u_0 \in L^1_0(\mathbb{R}^d)$ and a sequence $t_n \to \infty$, such that for every k, there exists a subsequence $t_{k,j}$ with $u(0,t_{k,j}) \to \alpha_k$ as $j \to \infty$.*

(ii) *The set of non-negative $u_0 \in L^1_0(\mathbb{R}^d)$ satisfying (i) above, $\mathscr{O}_\alpha$, is dense in the subset of non-negative functions in $L^1_0(\mathbb{R}^d)$.*

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