WARWICK
THE UNIVERSITY OF WARWICK

Manuscript version: Author's Accepted Manuscript
The version presented in WRAP is the author's accepted manuscript and may differ from the published version or Version of Record.

Persistent WRAP URL:
http://wrap.warwick.ac.uk/174324

How to cite:
Please refer to published version for the most recent bibliographic citation information. If a published version is known of, the repository item page linked to above, will contain details on accessing it.

Copyright and reuse:
The Warwick Research Archive Portal (WRAP) makes this work by researchers of the University of Warwick available open access under the following conditions.

Copyright © and all moral rights to the version of the paper presented here belong to the individual author(s) and/or other copyright owners. To the extent reasonable and practicable the material made available in WRAP has been checked for eligibility before being made available.

Copies of full items can be used for personal research or study, educational, or not-for-profit purposes without prior permission or charge. Provided that the authors, title and full bibliographic details are credited, a hyperlink and/or URL is given for the original metadata page and the content is not changed in any way.

Publisher's statement:
Please refer to the repository item page, publisher's statement section, for further information.

For more information, please contact the WRAP Team at: wrap@warwick.ac.uk.

warwick.ac.uk/lib-publications

KNESER GRAPHS ARE HAMILTONIAN

ARTURO MERINO, TORSTEN MÜTZE, AND NAMRATA

Abstract. For integers $k \geq 1$ and $n \geq 2k+1$, the Kneser graph $K(n,k)$ has as vertices all k-element subsets of an n-element ground set, and an edge between any two disjoint sets. It has been conjectured since the 1970s that all Kneser graphs admit a Hamilton cycle, with one notable exception, namely the Petersen graph $K(5,2)$. This problem received considerable attention in the literature, including a recent solution for the sparsest case $n = 2k+1$. The main contribution of this paper is to prove the conjecture in full generality. We also extend this Hamiltonicity result to all connected generalized Johnson graphs (except the Petersen graph). The generalized Johnson graph $J(n,k,s)$ has as vertices all k-element subsets of an n-element ground set, and an edge between any two sets whose intersection has size exactly s. Clearly, we have $K(n,k) = J(n,k,0)$, i.e., generalized Johnson graph include Kneser graphs as a special case. Our results imply that all known families of vertex-transitive graphs defined by intersecting set systems have a Hamilton cycle, which settles an interesting special case of Lovász' conjecture on Hamilton cycles in vertex-transitive graphs from 1970. Our main technical innovation is to study cycles in Kneser graphs by a kinetic system of multiple gliders that move at different speeds and that interact over time, reminiscent of the gliders in Conway's Game of Life, and to analyze this system combinatorially and via linear algebra.

1. Introduction

For integers $k \geq 1$ and $n \geq 2k+1$, the *Kneser graph* $K(n,k)$ has as vertices all k-element subsets of $[n] := \{1, 2, \ldots, n\}$, and an edge between any two sets A and B that are disjoint, i.e., $A \cap B = \emptyset$. Kneser graphs were introduced by Lovász [Lov78] in his celebrated proof of Kneser's conjecture. Using the Borsuk-Ulam theorem, he proved that the chromatic number of $K(n,k)$ equals $n-2k+2$, and his proof gave rise to the field of topological combinatorics. We proceed to list a few other important properties of Kneser graphs. The maximum independent set in $K(n,k)$ has size $\binom{n-1}{k-1}$ by the famous Erdős-Ko-Rado [EKR61] theorem. Furthermore, the graph $K(n,k)$ is vertex-transitive, i.e., it 'looks the same' from the point of view of any vertex, and all vertices have degree $\binom{n-k}{k}$. Lastly, note that when $n < ck$, the Kneser graph $K(n,k)$ does not contain cliques of size c, whereas it does contain such cliques when $n \geq ck$. Many other properties of Kneser graphs have been studied, for example their diameter [VPV05], treewidth [HW14], boxicity [CL21], and removal lemmas [FR18]. $K(n,k)$

1.1. Hamilton cycles in Kneser graphs.

In this work we investigate Hamilton cycles in Kneser graphs, i.e., cycles that visit every vertex exactly once. Kneser graphs have long been conjectured

(Arturo Merino) Department of Mathematics, TU Berlin, Germany

(Torsten Mütze) Department of Computer Science, University of Warwick, United Kingdom & Department of Theoretical Computer Science and Mathematical Logic, Charles University, Prague, Czech Republic

(Namrata) Department of Computer Science, University of Warwick, United Kingdom

E-mail addresses: merino@math.tu-berlin.de, torsten.mutze@warwick.ac.uk, namrata@warwick.ac.uk.

This work was supported by Czech Science Foundation grant GA 22-15272S..

An extended abstract of this work has been accepted for presentation that the 55th Annual ACM Symposium on the Theory of Computing (STOC 2023).

to have a Hamilton cycle, with one notable exception, the Petersen graph $K(5,2)$ (see Figure 2), which only admits a Hamilton path. This conjecture goes back to the 1970s, and in the following we give a detailed account of this long history. As Kneser graphs are vertex-transitive, this is a special case of Lovász' famous conjecture [Lov70], which asserts that every vertex-transitive graph admits a Hamilton path. A stronger form of the conjecture asserts that every vertex-transitive graph admits a Hamilton cycle, apart from five exceptional graphs, one of them being the Petersen graph. So far, the conjecture for Hamilton cycles in Kneser graphs has been tackled from two angles, namely for sufficiently dense Kneser graphs, and for the sparsest Kneser graphs. From the aforementioned results about the degree and cliques in $K(n,k)$, we see that $K(n,k)$ is relatively dense when n is large w.r.t. k, and relatively sparse otherwise. The sparsest case is when $n = 2k+1$, and the graphs $O_k := K(2k+1,k)$ are also known as *odd graphs*. Intuitively, proving Hamiltonicity should be easier for the dense cases, and harder for the sparse cases. O_k

We first recap the known results for dense Kneser graphs. Heinrich and Wallis [HW78] showed that $K(n,k)$ has a Hamilton cycle if $n \geq 2k + k/(\sqrt[k]{2}-1) = (1+o(1))k^2/\ln 2$. This was improved by B. Chen and Lih [CL87], whose results imply that $K(n,k)$ has a Hamilton cycle if $n \geq (1+o(1))k^2/\log k$; see [CI96]. In another breakthrough, Y. Chen [Che00] showed that $K(n,k)$ is Hamiltonian when $n \geq 3k$. A particularly nice and clean proof for the cases where $n = ck$, $c \in \{3,4,\ldots\}$, was obtained by Y. Chen and Füredi [CF02]. Their proof uses Baranyai's well-known partition theorem for complete hypergraphs [Bar75] to partition the vertices of $K(ck,k)$ into cliques of size c. This proof method was extended by Bellmann and Schülke to any $n \geq 4k$ [BS21]. The asymptotically best result known to date, again due to Y. Chen [Che03], is that $K(n,k)$ has a Hamilton cycle if $n \geq (3k+1+\sqrt{5k^2-2k+1})/2 = (1+o(1))2.618\ldots\cdot k$. With the help of computers, Shields and Savage [SS04] found Hamilton cycles in $K(n,k)$ for all $n \leq 27$ (except for the Petersen graph).

We now briefly summarize the Hamiltonicity story of the sparsest Kneser graphs, namely the odd graphs. Note that $O_k = K(2k+1,k)$ has degree $k+1$, which is only logarithmic in the number of vertices. The conjecture that O_k has a Hamilton cycle for all $k \geq 3$ originated in the 1970s, in papers by Meredith and Lloyd [ML72, ML73] and by Biggs [Big79]. Already Balaban [Bal72] exhibited a Hamilton cycle for the cases $k = 3$ and $k = 4$, and Meredith and Lloyd described one for $k = 5$ and $k = 6$. Later, Mather [Mat76] solved the case $k = 7$. Mütze, Nummenpalo and Walczak [MNW21] finally settled the problem for all odd graphs, proving that O_k has a Hamilton cycle for every $k \geq 3$. In fact, they even proved that O_k admits double-exponentially (in k) many distinct Hamilton cycles. Already much earlier, Johnson [Joh11] provided an inductive argument that establishes Hamiltonicity of $K(n,k)$ provided that the existence of Hamilton cycles is known for several smaller Kneser graphs. Combining his result with the unconditional results from [MNW21] yields that $K(2k+2^a,k)$ has a Hamilton cycle for all $k \geq 3$ and $a \geq 0$. These results still leave infinitely many open cases, the sparsest one of which is the family $K(2k+3,k)$ for $k \geq 1$.

Another line of attack towards proving Hamiltonicity is to find long cycles in $K(n,k)$. To this end, Johnson [Joh04] showed that there exists a constant $c > 0$ such that the odd graph O_k has a cycle that visits at least a $(1-c/\sqrt{k})$-fraction of all vertices, which is almost all vertices as k tends to infinity. This was generalized and improved in [MS17], where it was shown that $K(n,k)$ has a cycle visiting a $2k/n$-fraction of all vertices. For $n = 2k+1$ this fraction is $(1-1/(2k+1))$, and more generally for $n = 2k+o(k)$ it is $1-o(1)$.

The main contribution of this paper is to settle the conjecture on Hamilton cycles in Kneser graphs affirmatively in full generality.

Theorem 1. *For all $k \geq 1$ and $n \geq 2k+1$, the Kneser graph $K(n,k)$ has a Hamilton cycle, unless it is the Petersen graph, i.e., $(n,k) = (5,2)$.*

In the following we present generalizations of this result that we establish in this paper, and we discuss how they extend previously known Hamiltonicity results. The relations between these results for different families of vertex-transitive graphs are illustrated in Figure 1. In fact, our proof of Theorem 1 enables us to settle all known instances of Lovász' conjecture for vertex-transitive graphs defined by intersecting set systems. As we shall see, Kneser graphs are the hardest cases among them to prove. Indeed, the more general families of graphs can be settled easily once Hamiltonicity is established for Kneser graphs.

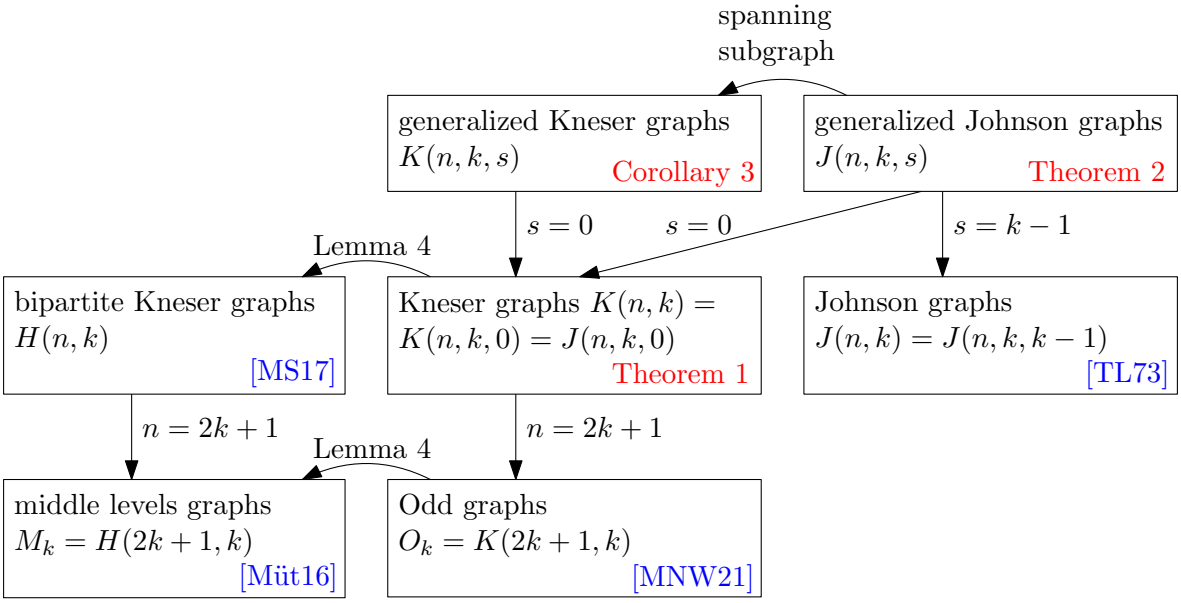


FIGURE 1. Relation between Hamiltonicity results established in this paper and previous papers. Arrows indicate implications.

1.2. **Generalized Johnson graphs.** The *generalized Johnson graph* $J(n,k,s)$ has as vertices all k-element subsets of $[n]$, and an edge between any two sets A and B that satisfy $|A \cap B| = s$, i.e., the intersection of A and B has size exactly s. To ensure that the graph is connected, we assume that $s < k$ and $n \geq 2k - s + \mathbf{1}_{[s=0]}$, where $\mathbf{1}_{[s=0]}$ denotes the indicator function that equals 1 if $s = 0$ and 0 otherwise. Generalized Johnson graphs are sometimes called 'uniform subset graphs' in the literature, and they are also vertex-transitive. Furthermore, by taking complements, we see that $J(n,k,s)$ is isomorphic to $J(n, n-k, n-2k+s)$. Clearly, Kneser graphs are special generalized Johnson graphs obtained for $s = 0$. On the other hand, the graphs obtained for $s = k-1$ are known as (ordinary) *Johnson graphs* $J(n,k) := J(n,k,k-1)$. $J(n,k,s)$ $J(n,k)$

Chen and Lih [CL87] conjectured that all graphs $J(n,k,s)$ admit a Hamilton cycle except the Petersen graph $J(5,2,0) = J(5,3,1)$, and this problem was reiterated in Gould's survey [Gou91]. In their original paper, Chen and Lih settled the cases $s \in \{k-1, k-2, k-3\}$. It is known that a Hamilton cycle in the Johnson graph $J(n,k) = J(n,k,k-1)$ can be obtained by restricting the binary reflected Gray code for bitstrings of length n to those strings with Hamming weight k [TL73]. In fact, for Johnson graphs $J(n,k)$ much stronger Hamiltonicity properties are known [JR94, Kno94]. Other properties of generalized Johnson graphs were investigated in [CW08a, AAC+18, Zak20, KZ22].

We generalize Theorem 1 further, by showing that all connected generalized Johnson graphs admit a Hamilton cycle. This resolves Chen and Lih's conjecture affirmatively in full generality.

Theorem 2. *For all $k \geq 1$, $0 \leq s < k$, and $n \geq 2k - s + \mathbf{1}_{[s=0]}$ the generalized Johnson graph $J(n,k,s)$ has a Hamilton cycle, unless it is the Petersen graph, i.e., $(n,k,s) \in \{(5,2,0),(5,3,1)\}$.*

1.3. **Generalized Kneser graphs.** The *generalized Kneser graph* $K(n,k,s)$ has as vertices all k-element subsets of $[n]$, and an edge between any two sets A and B that satisfy $|A \cap B| \leq s$, i.e., the intersection of A and B has size at most s. The definition is very similar to generalized Johnson graphs, only the equality condition on the size of the set intersection is replaced by an inequality. As a consequence, we clearly have $K(n,k,s) = \bigcup_{t \leq s} J(n,k,t)$, i.e., $K(n,k,s)$ has the same vertex set as $J(n,k,s)$, but more edges. In other words, $J(n,k,s)$ is a spanning subgraph of $K(n,k,s)$. Generalized Kneser graphs are also vertex-transitive, and they have been studied heavily in the literature; see e.g. [Fra85, Den97, CW08b, BCK19, JM20, GMKM21, LCL22, Met22]. $K(n,k,s)$

As $J(n,k,s)$ is a spanning subgraph of $K(n,k,s)$, Theorem 2 yields the following immediate corollary.

Corollary 3. *For all $k \geq 1$, $0 \leq s < k$, and $n \geq 2k - s + \mathbf{1}_{[s=0]}$ the generalized Kneser graph $K(n,k,s)$ has a Hamilton cycle, unless it is the Petersen graph, i.e., $(n,k,s) \in \{(5,2,0),(5,3,1)\}$.*

1.4. **Bipartite Kneser graphs and the middle levels problem.** For integers $k \geq 1$ and $n \geq 2k+1$, the *bipartite Kneser graph* $H(n,k)$ has as vertices all k-element and $(n-k)$-element subsets of $[n]$, and an edge between any two sets A and B that satisfy $A \subseteq B$. It is easy to see that bipartite Kneser graphs are also vertex-transitive. The following simple lemma shows that Hamiltonicity of $K(n,k)$ is harder than the Hamiltonicity of $H(n,k)$. $H(n,k)$

Lemma 4. *If $K(n,k)$ admits a Hamilton cycle, then $H(n,k)$ admits a Hamilton cycle or path.*

Proof. Given a Hamilton cycle $C = (x_1, x_2, \ldots, x_N)$ in $K(n,k)$, the sequences $P := (x_1, \overline{x_2}, x_3, \overline{x_4}, \ldots)$ and $P' := (\overline{x_1}, x_2, \overline{x_3}, x_4, \ldots)$, where $\overline{x_i} := [n] \setminus x_i$, are two spanning paths in $H(n,k)$. Consequently, if $N = \binom{n}{k}$ is odd, then the concatenation PP' is a Hamilton cycle in $H(n,k)$, and if N is even, then P and P' are two disjoint cycles that together span the graph and that can be joined to a Hamilton path. □

The sparsest bipartite Kneser graphs $M_k := H(2k+1,k)$ are known as *middle levels graphs*, as they are isomorphic to the subgraph of the $(2k+1)$-dimensional hypercube induced by the middle two levels. The well-known *middle levels conjecture* asserts that M_k has a Hamilton cycle for all $k \geq 1$. This conjecture was raised in the 1980s, settled affirmatively in [Müt16], and a short proof was given in [GMN18]. More generally, all bipartite Kneser graphs $H(n,k)$ were shown to have a Hamilton cycle in [MS17], via a short argument that uses the sparsest case M_k as a basis for induction. These papers completed a long line of previous partial results on these problems; see the papers for more references and historical remarks. Via Lemma 4 and its proof shown before, our Theorem 1 thus also yields a new alternative proof for the Hamiltonicity of bipartite Kneser graphs. Consequently, our results in this paper settle Lovász' conjecture for all known families of vertex-transitive graphs that are defined by intersecting set systems. M_k

1.5. **Algorithmic considerations.** A *combinatorial Gray code* [Sav97, Müt22] is an algorithm that computes a listing of combinatorial objects such that any two consecutive objects in the list satisfy a certain adjacency condition. Many such algorithms are covered in depths in Knuth's book 'The Art of Computer Programming Vol. 4A' [Knu11], and several of them correspond to

computing a Hamilton cycle in a vertex-transitive graph, thus algorithmically solving one special case of Lovász' conjecture. For example, the classical binary reflected Gray code computes a Hamilton cycle in the n-dimensional hypercube, which can be seen as the Cayley graph of $\mathbb{Z}_2^n$ given by the standard generators. Another example is the well-known Steinhaus-Johnson-Trotter algorithm, which computes a Hamilton cycle in the Cayley graph of the symmetric group when the generators are adjacent transpositions. Similarly, the recent solution [SW20] of Nijenhuis and Wilf's sigma-tau problem [NW75, Ex. 6] computes a Hamilton cycle in the Cayley (di)graph of the symmetric group with the two generators being cyclic left-shift or transposition of the first two elements. Similar Gray code algorithms have been discovered for the symmetric group with other generators, such as prefix reversals [Ord67, Zak84], prefix shifts [Cor92, CW93, RW10], and for other groups such as the alternating group [GR87, Hol17].

Subsets of size k of an n-element ground set are known as (n,k)-*combinations* in the Gray code literature. Many different algorithms have been devised for generating (n,k)-combinations by element exchanges, i.e., any two consecutive combinations differ in removing one element from the subset and adding another one [TL73, EM84, Cha89, BW84, EHR84, Rus88]. This is equivalent to saying that any two consecutive sets intersect in exactly $k-1$ elements, i.e., such a Gray code computes a Hamilton cycle in the Johnson graph $J(n,k)$.

Computing a Hamilton cycle in the Kneser graph $K(n,k)$ thus corresponds to computing a Gray code for (n,k)-combinations where the adjacency condition is disjointness. Our proof of the existence of a Hamilton cycle in $K(n,k)$ is constructive, and it translates straightforwardly into an algorithm for computing the cycle whose running time is polynomial in the size $N := \binom{n}{k}$ of the Kneser graph. It remains open whether there exists a more efficient algorithm, i.e., one with running time that is polynomial in n and k per generated combination (note that N is exponential in k), similarly to the previously mentioned combination generation algorithms; see also the discussion at the end of this paper.

1.6. **Proof ideas.** In Section 1.7 below we demonstrate how Theorem 1 can be used to establish Theorem 2 by a simple inductive construction. Consequently, the main work in this paper is to prove Theorem 1.

As mentioned before, Mütze, Nummenpalo and Walczak [MNW21] proved that $K(n,k)$ has a Hamilton cycle for $n = 2k+1$ and all $k \geq 3$. Combining this result with Johnson's construction [Joh11] shows that $K(n,k)$ has a Hamilton cycle for $n = 2k+2^a$ and all $k \geq 3$ and $a \geq 0$, in particular for $n = 2k+2$. The techniques developed in this paper work whenever $n \geq 2k+3$, and thus they settle all remaining cases of Theorem 1. It should be noted that our proof does not work in the cases $n = 2k+1$ and $n = 2k+2$, so the two earlier constructions do not become obsolete.

We follow a two-step approach to construct a Hamilton cycle in $K(n,k)$ for $n \geq 2k+3$. In the first step, we construct a *cycle factor* in the graph, i.e., a collection of disjoint cycles that together visit all vertices. In the second step, we join the cycles of the factor to a single cycle. In the following we discuss both of these steps in more detail, outlining the main obstacles and novel ingredients to overcome them. This outline reflects the structure of the remainder of this paper.

1.6.1. *Cycle factor construction.* The starting point is to consider the characteristic vectors of the vertices of $K(n,k)$. For every k-element subset of $[n]$, this is a bitstring of length n with exactly k many 1s at the positions corresponding to the elements of the set. For example, the vertex $\{1,7,9\}$ of $K(9,3)$ is represented by the bitstring 100000101; see also Figure 2. In this figure and the following ones, 1s are often represented by black squares, and 0s by white squares.

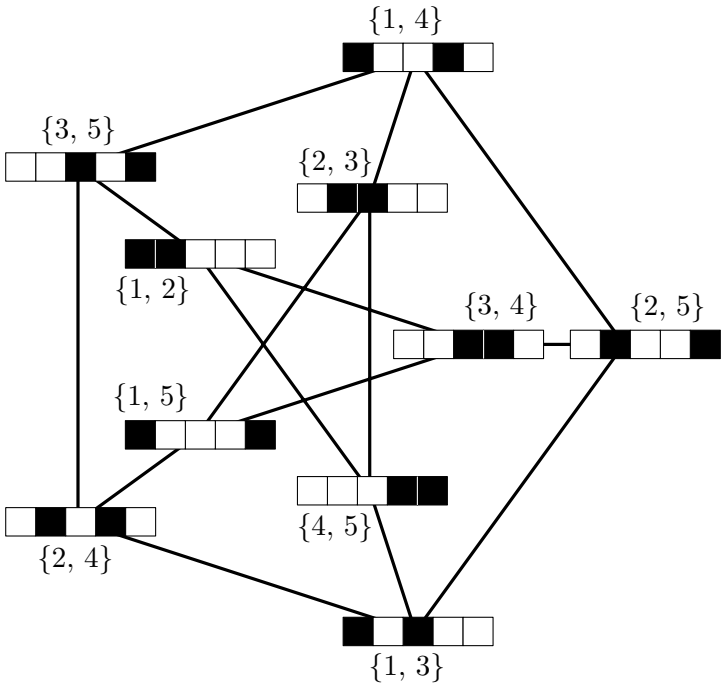


FIGURE 2. The Petersen graph $K(5,2)$. The vertices are all 2-elements subsets of $[5] = \{1,2,3,4,5\}$, and in the corresponding bitstrings, 1s are represented by black squares and 0s by white squares.

Clearly, two sets A and B that are vertices of $K(n,k)$ are disjoint if and only if the corresponding bitstrings have no 1s at the same positions.

Our construction of a cycle factor in the Kneser graph $K(n,k)$ uses the following simple rule based on parenthesis matching, which is a technique pioneered by Greene and Kleitman [GK76] (in a completely different context): Given a vertex represented by a bitstring x, we interpret the 1s in x as opening brackets and the 0s as closing brackets, and we match closest pairs of opening and closing brackets in the natural way, which will leave some 0s unmatched. This matching is done *cyclically* across the boundary of x, i.e., x is considered as a cyclic string. We write $f(x)$ for the vertex obtained from x by complementing all matched bits, leaving the unmatched bits unchanged. For example, $x = 100000101$ is interpreted as $x =))))))(=))\text{---}(($, where each - denotes an unmatched closing bracket, and then complementing matched bits (the first three and last three in this case) yields the vertex $f(x) = 011000010$. Repeatedly applying f to every vertex partitions the vertices of the Kneser graph into cycles, and we write $C(x) := (x, f(x), f^2(x), \ldots)$ for the cycle containing x. For example, for x from before we obtain $C(x) = (100000101, 011000010, 000110001, 100001100, 010000011, \ldots, 000011010)$. Figure 3 shows several more examples of cycles generated by this parenthesis matching rule. The reason that this rule indeed generates disjoint cycles is that f is invertible and that $f(x) \neq x$ and $f^2(x) \neq x$. Indeed, x is obtained from $f(x)$ by applying the same parenthesis matching procedure as before, but with interpreting the 1s as closing brackets and the 0s are opening brackets instead.

1.6.2. *Analysis via gliders.* The next key step is to understand the structure of the cycles generated by f, as this is important for joining the cycles to a single Hamilton cycle. Unfortunately, the number of cycles and their lengths in our factor are governed by intricate number-theoretic phenomena, which we are unable to understand fully. Instead, we describe the evolution of a

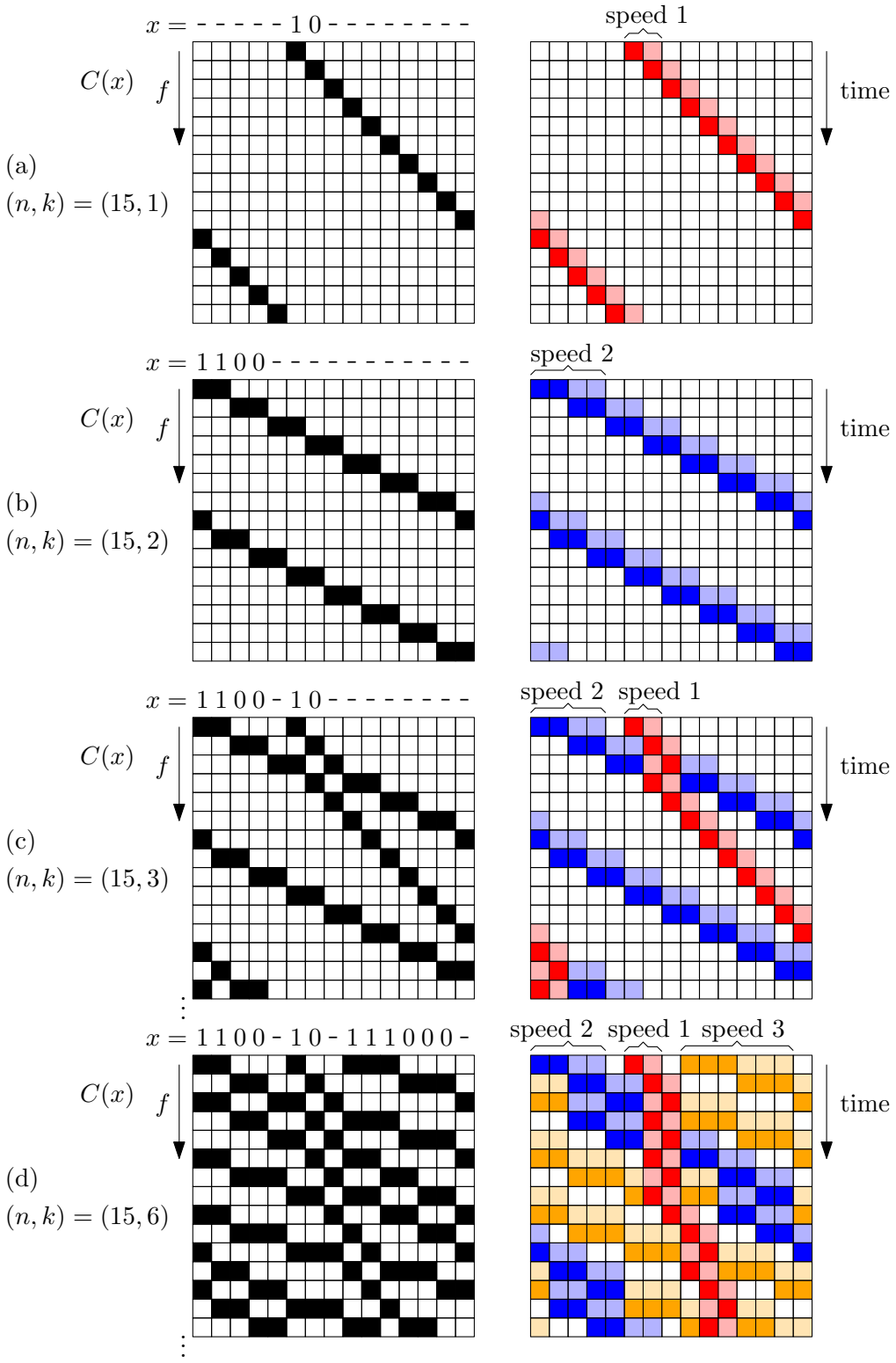


FIGURE 3. Cycles of the factor $\mathcal{C}_{n,k}$ in several different Kneser graphs $K(n,k)$. The cycles in (a) and (b) are shown completely, whereas in (c) and (d) only the first 15 vertices are shown. The right hand side shows the interpretation of certain groups of bits as gliders, and their movement over time. Matched bits belonging to the same glider are colored in the same color, with the opaque filling given to 1-bits, and the transparent filling given to 0-bits. (a) one glider of speed 1; (b) one glider of speed 2; (c) two gliders with speeds 1 and 2 that participate in an overtaking; (d) three gliders of speeds 1, 2 and 3 that participate in multiple overtakings. Animations of these examples are available at [Müt23].

bitstring x under repeated applications of f combinatorially, which enables us to extract some important cycle properties and invariants (other than the number of cycles and the cycle lengths). Specifically, we describe this evolution by a kinetic system of multiple gliders that move at different speeds and that interact over time, reminiscent of the gliders in Conway's Game of Life. This physical interpretation and its analysis are one of the main innovations of this paper. Specifically, we view each application of f as one unit of time moving forward. Furthermore, we partition the matched bits of x into groups, and each of these groups is called a *glider*. A glider has a *speed* associated to it, which is given by the number of 1s in its group. As a consequence of this definition, the sum of speeds of all gliders equals k. For example, in the cycle shown in Figure 3 (a), there is a single matched 1 and the corresponding matched 0, and together these two bits form a glider of speed 1 that moves one step to the right in every time step. Applying f means going down to the next row in the picture, so the time axis points downwards. Similarly, in Figure 3 (b), there are two matched 1s and the corresponding two matched 0s, and together these four bits form a glider of speed 2 that moves two steps to the right in every time step. As we see from these examples, a single glider of speed v simply moves uniformly, following the basic physics law

$$s(t) = s(0) + v \cdot t,$$

where t is the time (i.e., the number of applications of f) and $s(t)$ is the position of the glider in the bitstring as a function of time. The position $s(t)$ has to be considered modulo n, as bitstrings are considered as cyclic strings and the gliders hence wrap around the boundary. The situation gets more interesting and complicated when gliders of different speeds interact with each other. For example, in Figure 3 (c), there is one glider of speed 2 and one glider of speed 1. As long as these groups of bits are separated, each glider moves uniformly as before. However, when the speed 2 glider catches up with the speed 1 glider, an overtaking occurs. During an overtaking, the faster glider receives a boost, whereas the slower glider is delayed. This can be captured by augmenting the corresponding equations of motion by introducing additional terms, making them non-uniform. In the simplest case of two gliders of different speeds, the equations become

$$\begin{aligned} s_1(t) &= s_1(0) + v_1 \cdot t - 2v_1c_{1,2}, \\ s_2(t) &= s_2(0) + v_2 \cdot t + 2v_1c_{1,2}, \end{aligned}$$

where the subscript 1 stands for the slower glider and the subscript 2 stands for the faster glider, and the additional variable $c_{1,2}$ counts the number of overtakings. Note that the terms $2v_1c_{1,2}$ occur with opposite signs in both equations, capturing the fact that the faster glider is boosted by the same amount that the slower glider is delayed. This can be seen as 'energy conservation' in the system of gliders. Overall, the slower glider stands still for two time steps during an overtaking, as $v_1 \cdot 2 - 2v_1 \cdot 1 = 0$, and the faster glider's position changes by an additional amount of $2v_1$ (compared to its movement without overtaking). For more than two gliders, the equations of motion can be generalized accordingly, by introducing additional overtaking counters between any pair of gliders (see Proposition 28). Nevertheless, as the reader may appreciate from Figure 3 (d), in general it is highly nontrivial to recognize from an arbitrary bitstring x which of its matched bits belong to which glider, and consequently which glider is currently overtaking which other glider. Note that in general the gliders will not be nicely separated, but will be involved in simultaneous interactions, so that the groups of bits forming the gliders will be interleaved in complicated ways. Our general rule that achieves the glider partition is based on a recursion that uses an interpretation of x as a Motzkin path, where every matched 1 becomes an $\nearrow$-step in the Motzkin path, every matched 0 becomes a $\searrow$-step, and every unmatched 0 becomes a $\rightarrow$-step (see Section 3.4).

One important property that we extract from the aforementioned physics interpretation is that the number of gliders and their speeds are invariant along each cycle (see Lemma 23). For example, in Figure 3 (d), every bitstring along this cycle has three gliders of speeds 1, 2 and 3. Note in this example that the speeds do not necessarily correspond to the lengths of maximal sequences of consecutive 1s in the bitstrings, due to the interleaving of gliders. We also use the equations of motion to derive a seemingly innocent, but very crucial property, namely that no glider stands still forever, but will move eventually (see Lemma 30). Note that the speed 1 glider in Figure 3 (d) stands still between time steps 2–8, as during those steps it is overtaken once by the speed 2 glider, and twice by the speed 3 glider (wrapping around the boundary). We establish this fact by linear algebra, by showing that the determinant of the linear systems of equations that governs the gliders' movements is non-singular (see Lemma 29).

For the reader's entertainment, we programmed an interactive animation of gliders over time, and we encourage experimentation with this code, which can be found at [Müt23]. In particular, this link contains animations of many examples used in figures from our paper, which greatly improves their educational value.

The cycle factor construction discussed before and our analysis via gliders actually work for all $n \geq 2k+1$, not just for $n \geq 2k+3$. The assumption $n \geq 2k+3$ will become crucial in the next step, though.

1.6.3. *Gluing the cycles together.* To join the cycles of our factor to a single Hamilton cycle, we consider a 4-cycle D that shares two opposite edges with two cycles C, C' from our factor. Clearly, the symmetric difference of the edge sets $(C \cup C')\Delta D$ yields a single cycle on the same vertex set as $C \cup C'$. We may repeatedly apply such gluing operations, each time reducing the number of cycles in the factor by one, until the resulting factor has a single cycle, namely a Hamilton cycle. It turns out that the cycle factor defined by f admits a lot of such gluing 4-cycles. Note that $K(n,k)$ does not have any 4-cycles for $n = 2k+1$, so the assumption $n \geq 2k+2$ is needed here.

The two main technical obstacles we have to overcome are the following: (a) All of the 4-cycles used for the gluing must be edge-disjoint, so that none of the gluings interfere with each other. (b) We must use sufficiently many gluings to achieve connectivity, i.e., every cycle must be connected to every other cycle via a sequence of gluings. These two objectives are somewhat conflicting with each other, so satisfying both at the same time is nontrivial. The final gluings that we use and that satisfy both conditions are described by a set of nine intricate regular expressions (see (53)).

The 4-cycles that we use for the gluings are based on local modifications of two bitstrings x and y that satisfy certain conditions and that lie on two different cycles $C(x)$ and $C(y)$ from our factor, by considering the gliders in x and y. Specifically, this local modification changes the speed sets of the gliders in x and y in a controllable way. Recall that the speeds of gliders are invariant along each cycle, so these speeds will only change along the gluing 4-cycles. To control the gluing, we consider the speeds of gliders in a bitstring x in non-increasing order. Recall that the sum of speeds equals k, so such a sorted sequence forms a number partition of k. To establish (b) we choose gluings that guarantee a lexicographic increase in those number partitions. This ensures that every cycle is joined, via a sequence of gluings, to a cycle that has the lexicographically largest number partition, namely the number k itself. This corresponds to a single glider of maximum speed k, i.e., to a bitstring x in which all 1s are consecutive.

For example, consider the two cycles $C(x)$ and $C(y)$ shown in Figure 4, which can be glued together using the 4-cycle $C_4(x,y) := (x, f(x), y, f(y))$. Note that in $C(x)$, there are two gliders of speed 1 and one glider of speed 3, whereas in $C(y)$ there is one glider of speed 2 and one of

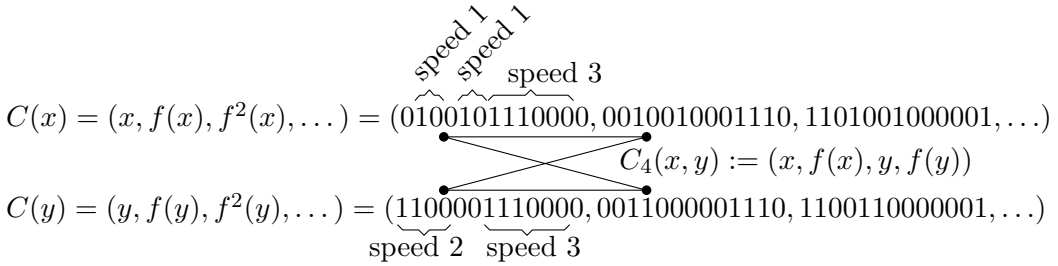


FIGURE 4. Gluing of two cycles from the factor via a 4-cycle in $K(13, 5)$.

speed 3. Consequently, via the gluing we have moved from the number partition $(3, 1, 1)$ to the lexicographically larger partition $(3, 2)$.

The general idea for choosing the gluings $C_4(x, y)$, which can already be seen in this example, is such that in x we decrease the speed of a glider of minimum speed by 1, and instead we increase the speed of any other glider by 1, which ensures that the number partition associated with y is lexicographically larger than that of x. Unfortunately, it is not always possible to use gluings that guarantee such immediate lexicographic improvement. In some cases we have to use gluings where a small lexicographic decrease occurs. It then has to be argued that subsequent gluings compensate for this defect such that the overall effect of the resulting sequence of gluings is again a lexicographic improvement. For example, from a vertex with associated number partition $(4, 4)$, the first gluing may lead to a vertex with number partition $(4, 3, 1)$, and the next gluing may lead to $(5, 3)$. While the step $(4, 4) \to (4, 3, 1)$ is a lexicographic decrease instead of an increase, overall $(4, 4) \to (4, 3, 1) \to (5, 3)$ is a lexicographic increase. In this step of the proof the assumption $n \geq 2k + 3$ finally enters the picture, as it gives us the necessary flexibility in choosing gluings that are guaranteed to achieve this improvement in all cases.

The arguments so far show that every cycle is connected, via a sequence of gluings, to a cycle in which all 1s are consecutive. Note however, that there may be several such cycles, depending on the values of n and k. Specifically, there are exactly $\gcd(n, k)$ such cycles. To join those, we observe that the subgraph of $K(n, k)$ induced by those special cycles is isomorphic to a Cayley graph of $\mathbb{Z}/n\mathbb{Z}$, which admits many gluing 4-cycles to join them (see Lemma 34).

1.7. **Proof of Theorem 2.** We first show how Theorem 1 can be used to establish the more general Theorem 2 quite easily. Chen and Lih showed the following about generalized Johnson graphs.

Lemma 5 ([CL87, Thm. 1])**.** *If $J(n-1, k-1, s-1)$ and $J(n-1, k, s)$ have a Hamilton cycle, then $J(n, k, s)$ also has a Hamilton cycle.*

The proof of Lemma 5 given in [CL87] is based on a straightforward partitioning of the graph $J(n, k, s)$ into two subgraphs that are isomorphic to $J(n-1, k-1, s-1)$ and $J(n-1, k, s)$. Specifically, this partition is obtained by considering all vertices (=sets) that contain a fixed element, n say, and those that do not contain it. One can then join the cycles in the two subgraphs to one, by taking the symmetric difference with a 4-cycle that has one edge in each of the two subgraphs, using the fact that Johnson graphs are edge-transitive, i.e., we can force each of the cycles in the two subgraphs to use this edge from the 4-cycle. All that is needed now for the proof of Theorem 2 is the following simple observation.

Lemma 6. *If $J(n, k, s)$ is a generalized Johnson graph, then either it is a Kneser graph or $J(n-1, k-1, s-1)$ and $J(n-1, k, s)$ are both generalized Johnson graphs.*

In the proof we will use the aforementioned observation that $J(n,k,s)$ is isomorphic to $J(n,n-k,n-2k+s)$.

Proof. Let $k \geq 1$, $0 \leq s < k$ and $n \geq 2k-s+\mathbf{1}_{[s=0]}$. If $s=0$, then $J(n,k,s) = J(n,k,0) = K(n,k)$ is a Kneser graph. This happens in particular if $k=1$. If $s>0$ and $n=2k-s$, then $J(n,k,s) = J(n,n-k,n-2k+s) = J(n,k-s,0) = K(n,k-s)$ is also a Kneser graph. Otherwise, we have $k>1$, $s>0$ and $n>2k-s$, and we consider the graphs $H_1 := J(n-1,k-1,s-1)$ and $H_0 := J(n-1,k,s)$. From $k>1$ we obtain $k-1 \geq 1$, and from $s>0$ and $s<k$ we obtain that $0 \leq s-1 < k-1$. Furthermore, the inequality $n>2k-s$ is equivalent to $n-1 > 2(k-1)-(s-1)$, which implies $n-1 \geq 2(k-1)-(s-1)+1$. Combining these observations shows that the graph H_1 is indeed a valid generalized Johnson graph. Similarly, the inequality $n>2k-s$ implies that $n-1 \geq 2k-s = 2k-s+\mathbf{1}_{[s=0]}$ (since $s>0$), and consequently the graph H_0 is also a valid generalized Johnson graph. □

Proof of Theorem 2. Combine Lemmas 5 and 6, and use Theorem 1 and induction. Because of the exceptional cases $J(5,2,0) = J(5,3,1)$, in a few base cases the existence of a Hamilton cycle in $J(n,k,s)$ has to be checked directly, namely for $(n,k,s) \in \{(3,1,0),(4,1,0),(4,2,1),(5,1,0),(5,2,1),(6,1,0),(6,2,0),(6,2,1),(6,3,1),(6,3,2)\}$. Using that $J(n,k,s) = J(n,n-k,n-2k+s)$ this settles all cases with $n \leq 6$. □

2. Cycle factor construction

In this section we describe in detail the construction of a cycle factor in the Kneser graph $K(n,k)$ outlined in Section 1.6.1. This construction is valid for the entire range of values $n \geq 2k+1$.

2.1. **Preliminaries.** We let $X_{n,k}$ denote the set of all bitstrings of length n with exactly k many 1s. We interpret the vertices of the Kneser graph $K(n,k)$ as bitstrings in $X_{n,k}$, by considering the corresponding characteristic vectors. Every pair of disjoint sets, which is an edge in the Kneser graph, corresponds to a pair of bitstrings that have no 1s at the same positions. These definitions are illustrated in Figure 2. $X_{n,k}$

We write ε for the empty string. Moreover, for any bitstring x, we write $\overline{x}$ for the bitstring obtained from x by complementing every bit. We also write $x\,y$ for the concatenation of the bitstrings x and y, and x^a for the a-fold repetition of x. $\varepsilon, \overline{x}$ $x\,y, x^a$

For integers $a \leq b$ we define $[a,b] := \{a, a+1, \ldots, b\}$, and we refer to this set of integers as an *interval*. $[a,b]$

Throughout this paper, important terminology and symbols are printed on the page boundaries at the place where they are first defined, to facilitate going back and looking up the definitions.

2.2. **Cycle factor construction.** We consider every bitstring $x \in X_{n,k}$, and we apply *parenthesis matching* to it, which is a technique developed by Greene and Kleitman [GK76] in the context of symmetric chain partitions of posets. For this we interpret the 1s in x as opening brackets and the 0s as closing brackets, and we match closest pairs of opening and closing brackets in the natural way. This matching is done *cyclically* across the boundary of x, i.e., x is considered as a cyclic string; see Figure 5. In particular, in the following we will consider indices in x modulo n, with $1, \ldots, n$ as representatives of the equivalence classes. As $n \geq 2k+1$, there are more 0s than 1s in x, and consequently every 1 is matched to some 0, but not every 0 is matched to a 1. Whenever we want to emphasize that we consider a bitstring x with parenthesis matching applied to it, we write every unmatched 0 in x as -. For example, we write $x = 001100001 = 0\text{-}1100\text{--}1$.

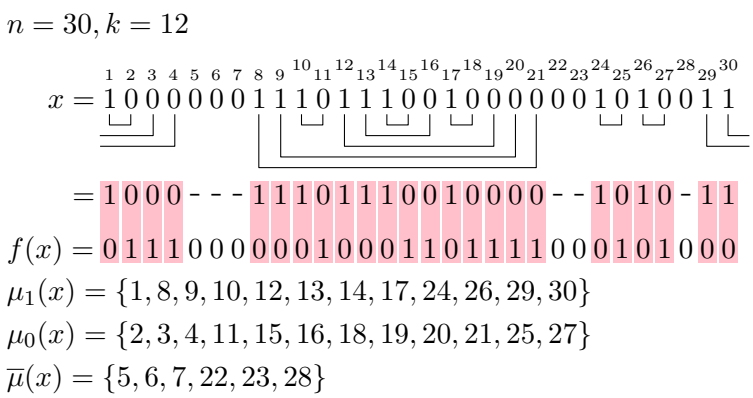


FIGURE 5. Parenthesis matching for a bitstring $x \in X_{30,12}$. Matched pairs of bits are indicated by square brackets, and bits that are complemented to obtain $f(x)$ are highlighted.

The parenthesis matching procedure can be described equivalently as follows: For every 1-bit in x, we consider the shortest (cyclic) substring starting from this bit to the right that contains the same number of 0s as 1s, and we match it to the last 0-bit of this substring.

We let $\mu(x) \subseteq [n]$ denote the set of positions of bits that are matched in x, and we write $\overline{\mu}(x) := [n] \setminus \mu(x)$ for the positions of unmatched bits. Moreover, we partition $\mu(x)$ into the sets $\mu_1(x)$ and $\mu_0(x)$ of the positions of matched 1s and 0s, respectively. By these definitions, the sets $\mu_1(x)$, $\mu_0(x)$ and $\overline{\mu}(x)$ are all disjoint, their union is $[n]$, the union of $\mu_1(x)$ and $\mu_0(x)$ is $\mu(x)$, and the sizes of the four sets $\mu_0(x)$, $\mu_1(x)$, $\mu(x)$, and $\overline{\mu}(x)$ are k, k, $2k$, and $n-2k$, respectively. Recall that every 1-bit is matched, or equivalently, all unmatched bits are 0s. Consequently, $\mu_1(x)$ is the set of positions of *all* 1s in x. $\mu(x), \overline{\mu}(x)$ $\mu_1(x), \mu_0(x)$

For any $x \in X_{n,k}$, we let $f(x) \in X_{n,k}$ denote the bitstring obtained from x by complementing all matched bits, i.e., the bits at all positions in $\mu(x)$; see Figure 5. $f(x)$

By this definition we have $\mu_1(f(x)) = \mu_0(x)$. Consequently, x and $f(x)$ have no 1s at the same positions, i.e., they are characteristic vectors of disjoint sets. It follows that $(x, f(x))$ is an edge in the Kneser graph $K(n,k)$.

For any $x \in X_{n,k}$ we define a sequence of vertices in $K(n,k)$ by $C(x)$

$$C(x) := (x, f(x), f^2(x), \dots), \tag{1a}$$

i.e., we repeatedly apply f to x until we obtain x again.

Lemma 7. *Let $n \geq 2k+1$. For any $x \in X_{n,k}$, the sequence $C(x)$ defined in* (1a) *describes a cycle of length at least 3 in the Kneser graph $K(n,k)$.*

Proof. We first argue that the mapping f is invertible. Specifically, we can obtain x from $f(x)$ by cyclically matching 0s and 1s (instead of 1s and 0s) and complementing matched bits. It follows that there are no two distinct bitstrings $x, x' \in X_{n,k}$ with $f(x) = f(x')$. As the set $X_{n,k}$ is finite, we conclude that the first duplicate bitstring in the sequence $C(x)$ is the first string x, so the sequence $C(x)$ is indeed cyclic.

Now consider three consecutive bitstrings x, $f(x)$ and $f^2(x)$ in the sequence $C(x)$. To complete the proof of the lemma, we show that $x \neq f(x)$ and $x \neq f^2(x)$. For this we identify a position $i \in [n]$ such that $i \in \overline{\mu}(x)$, $i \in \mu_0(f(x))$ and $i \in \mu_1(f^2(x))$. Specifically, consider a maximal sequence of unmatched 0s in x, and let i be the position of the first such bit in the sequence. Note that the set $\overline{\mu}(x)$ is nonempty by the assumption $n \geq 2k+1$. By definition we clearly have $i \in \overline{\mu}(x)$. Moreover, we also have $i-1 \in \mu(x)$, specifically $i-1 \in \mu_0(x)$, and consequently

$i-1 \in \mu_1(f(x))$, which implies that the 0-bit at position i in $f(x)$ is matched to the 1-bit to its left, so $i \in \mu_0(f(x))$. It follows that $i \in \mu_1(f^2(x))$, as claimed. □

Using Lemma 7, we may define a *cycle factor* in $K(n,k)$ by $\mathcal{C}_{n,k}$

$$\mathcal{C}_{n,k} := \{C(x) \mid x \in X_{n,k}\}. \tag{1b}$$

For example, for the Petersen graph $K(5,2)$, for $x := 10100$ and $x' := 11000$ we get

$$C(x) = (1010\text{-}, \text{-}1010, 0\text{-}101, 10\text{-}10, 010\text{-}1),$$
$$C(x') = (1100\text{-}, 0\text{-}110, 100\text{-}1, \text{-}1100, 00\text{-}11),$$

and $\mathcal{C}_{5,2} = \{C(x), C(x')\}$. More examples are shown in Figure 3.

3. Analysis via gliders

In this section we analyze various properties and invariants of the cycles of the factor $\mathcal{C}_{n,k}$ defined in the previous section via a system of interacting gliders, as sketched in Section 1.6.2. All of these results hold for the entire range of values $n \geq 2k+1$.

3.1. **Motzkin paths and Dyck paths.** We identify any bitstring $x \in X_{n,k}$ with a Motzkin path in the integer lattice $\mathbb{Z}^2$ in the following way; see Figure 6. We apply parenthesis matching to x and we read the bits of x from left to right. Every matched 1 is drawn as an $\nearrow$-step, every matched 0 as a $\searrow$-step, and every unmatched 0 as a $\rightarrow$-step, and these steps change the current coordinate by $(+1,+1)$, $(+1,-1)$, or $(+1,0)$, respectively. In other words, the lattice path corresponding to x has $\nearrow$-steps at the positions $\mu_1(x)$, $\searrow$-steps at the positions $\mu_0(x)$, and $\rightarrow$-steps at the positions $\overline{\mu}(x)$. To define the absolute position of this lattice path, it suffices to specify the coordinate of one point on it. Specifically, if i is the position of the first unmatched 0-bit in x, then this $\rightarrow$-step starts at the coordinate $(i-1,0)$. Motzkin path

In every substring of x for which every prefix has at least as many 1s as 0s, every 0-bit is matched to some 1-bit to its left in this substring. As a consequence, the Motzkin path x never moves below the abscissa and all $\rightarrow$-steps of x lie on the abscissa. It follows that in the above definition, we can choose i as the position of *any* unmatched 0-bit in x, not necessarily the first one.

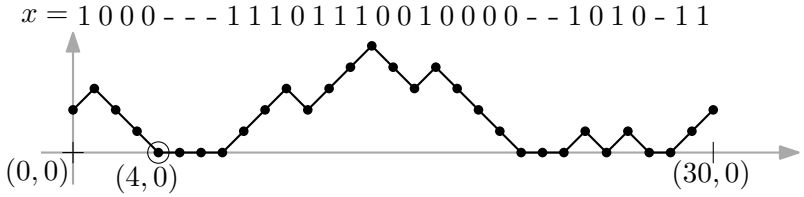


FIGURE 6. The Motzkin path corresponding to the bitstring x from Figure 5. The starting point $(4,0)$ of the first $\rightarrow$-step of x is marked.

For any integer $\ell \geq 0$, we write D_ℓ for the set of Dyck words of length 2ℓ, i.e., bitstrings with ℓ many 1s and 0s that have at least as many 1s as 0s in every prefix. Note that we have $D_0 = \{\varepsilon\}$ and $D_\ell = \{1\,u\,0\,v \mid u \in D_i, v \in D_{\ell-i-1}, \text{ and } i = 0, \ldots, \ell-1\}$ for $\ell > 0$. Moreover, we write D for the set of Dyck words of arbitrary length, i.e., $D := \bigcup_{\ell \geq 0} D_\ell$. Lastly, we use $D' \subseteq D$ to denote the set of bitstrings that have strictly more 1s than 0s in every proper prefix, i.e., we have $D' = \{1\,u\,0 \mid u \in D\}$. Dyck word D, D'

Observe that a substring y of x with $y \in D$ corresponds to a subpath of the Motzkin path x that has the same number of $\nearrow$-steps and $\searrow$-steps, does not contain any $\rightarrow$-steps, and that never moves below the height of the starting point, i.e., it is a Dyck subpath.

3.2. **The infinite string $\widehat{x}$.** We let $\widehat{x}$ be the bitstring obtained by extending x by copies of itself infinitely in both directions. We can think of $\widehat{x}$ as the string obtained by unrolling the cyclic string x. As the parenthesis matching procedure interprets x as a cyclic string, the notion of matched and unmatched bits extends in the natural way from x to $\widehat{x}$. Consequently, the Motzkin path corresponding to $\widehat{x}$ is obtained by concatenating infinitely many copies of the Motzkin path of x. In this way, the indices of bits in $\widehat{x}$ or of steps on the corresponding Motzkin path continue beyond the indices $1, \dots, n$ used in x, i.e., these indices continue with $0, -1, -2, \dots$, to the left, and $n+1, n+2, \dots$ to the right, and in $\widehat{x}$ we see the same bits/steps in each equivalence class of indices modulo n. We can translate the indices in $\widehat{x}$ back to x simply by considering them modulo n, with $1, \dots, n$ as representatives of the equivalence classes. $\widehat{x}$

The motivation for considering the infinite Motzkin path $\widehat{x}$ in addition to the finite path x is that we aim to partition certain steps of $\widehat{x}$ into groups, which we will call *gliders*. We do this for every vertex along the cycle $C(x)$, with the goal of tracking the movement of gliders along $C(x)$. By (1a), moving one step along the cycle $C(x)$ corresponds to one application of f, which we interpret as time moving forward by one unit; recall Figure 3. As $\widehat{x}$ has periodicity n, each glider repeats periodically every n steps along $\widehat{x}$; see Figure 9. However, for formulating continuous equations of motions for the gliders, it is crucial to treat these periodic copies as separate entities that continually move towards $+\infty$ along the cycle $C(x)$, and not to treat them as a single entity that moves and wraps around the boundary of x. So in the infinite string $\widehat{x}$, an infinite periodic set of gliders continually moves to the right over time, and we consider them through the finite 'window' x, in which they appear to wrap around the boundary.

3.3. **Hills and valleys.** We refer to any Dyck subpath $y \in D$ of $\widehat{x}$ as a *hill*, and to any complemented Dyck subpath y with $\overline{y} \in D$ as a *valley*. If a hill y of $\widehat{x}$ starts and ends at the abscissa, we refer to it as a *base hill*. We define the *height* of a hill y in x as the difference between the ordinates of its highest point and its starting point. Similarly, the *depth* of a valley y in x is defined as the difference between the ordinates of its starting point and its lowest point. hill, valley, base hill, height, depth

We consider a hill $y \in D'$ in $\widehat{x}$ of height h, and we let p denote the leftmost highest point of y on the Motzkin path. The hill y can be decomposed uniquely as

$$y = \begin{cases} 1\,0 & \text{if } h = 1, \\ 1\,u_1\,1\,u_2\,\cdots\,1\,u_{h-2}\,1\,1\,v_0\,0\,v_1\,0\,\cdots\,0\,v_{h-2}\,0\,0, & \\ \qquad \text{with } u_1, \dots, u_{h-2}, \overline{v_0}, \dots, \overline{v_{h-2}} \in D, & \text{if } h \geq 2, \end{cases} \tag{2}$$

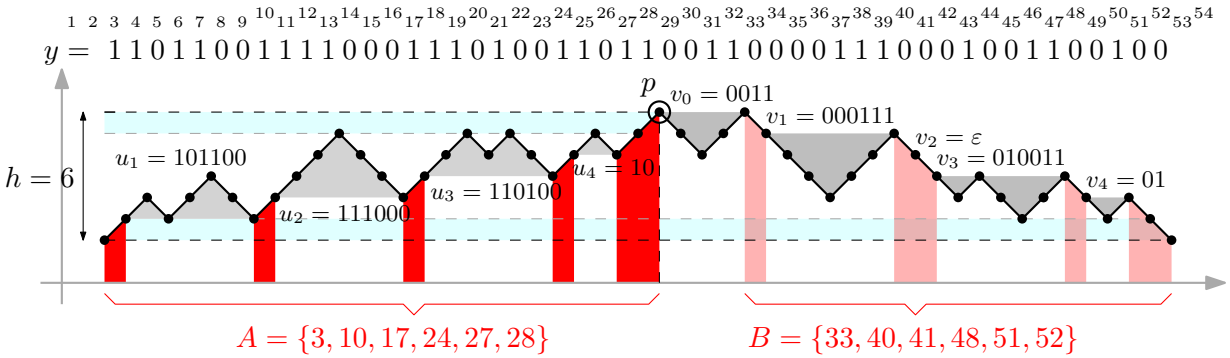


FIGURE 7. Decomposition of a hill $y \in D'$ of height $h = 6$ contained in some larger Motzkin path $\widehat{x}$ into bulges u_i and dents v_i. The leftmost highest point p of y is marked.

i.e., the u_i are maximal hills in y to the left of the point p, and the v_i are maximal valleys in y to the right of the point p; see Figure 7. We refer to the hill u_i as the *ith bulge of* y and to the valley v_i as the *ith dent of* y. If $h = 1$ then $y = 1\,0$ has neither bulges nor dents. From (2) we obtain the following lemma. bulge dent

Lemma 8. *For any hill $y \in D'$ of height h in $\widehat{x}$, the ith bulge of y has height at most $h-1-i$ for all $i = 1, \dots, h-2$, and the ith dent of y has depth at most $h-1-i$ for all $i = 0, \dots, h-2$.*

3.4. **Glider partition.** In the following we define a function $\Gamma(x)$ that recursively computes a partition of the set of indices of all $\nearrow$-steps and $\searrow$-steps in the infinite Motzkin path $\widehat{x}$. This corresponds to the set of all indices of matched bits in the bitstring $\widehat{x}$, i.e., to the set $\mu(\widehat{x}) \subseteq \mathbb{Z}$. Moreover, the sets of the partition are grouped into pairs (A, B) with $|A| = |B|$ and $\max A < \min B$, and either A contains positions of $\nearrow$-steps of $\widehat{x}$ and B contains positions of $\searrow$-steps, or vice versa. We will refer to every such pair (A, B) as a *glider* (the formal definition is given below), and we think of the steps of $\widehat{x}$ at the positions in A and B as the steps of the Motzkin path that 'belong' to the glider (A, B). Every $\nearrow$-step and $\searrow$-step of $\widehat{x}$ belongs to precisely one glider, whereas $\rightarrow$-steps do not belong to any glider; see Figure 8.

We first define $\Gamma(x)$

$$\Gamma(x) := \bigcup_{y \text{ base hill in } \widehat{x}} \Gamma(y). \tag{3a}$$

For any hill $y \in D \setminus D'$ in $\widehat{x}$, if $y = \varepsilon$ we define

$$\Gamma(y) := \emptyset, \tag{3b}$$

and if $y \neq \varepsilon$ we consider the partition $y = y_1 \cdots y_\ell$ with $y_1, \dots, y_\ell \in D'$ and define

$$\Gamma(y) := \bigcup_{i=1}^{\ell} \Gamma(y_i). \tag{3c}$$

The interesting step of the recursion happens for hills $y \in D'$ in $\widehat{x}$, for which we partition y uniquely as in (2), and define

$$\Gamma(y) := \Big(\bigcup_{i=1}^{h-2} \Gamma(u_i) \cup \bigcup_{i=0}^{h-2} \Gamma(\overline{v_i}) \Big) \cup \{(A, B)\}, \tag{3d}$$

where A and B are the set of indices of the 1s and 0s of y, respectively, that do not belong to any of the bulges u_i or the dents v_i in the decomposition (2); see Figure 7. Recall that if $h = 1$, then $y = 1\,0$ has no bulges or dents, so in this case the unions in (3d) are empty and then A and B are 1-element sets containing the positions of the 1 and the 0 in y, respectively. In general, the indices in A and B are absolute with respect to the Motzkin path $\widehat{x}$ that contains y as a subpath. We refer to any pair $(A, B) \in \Gamma(x)$ computed in (3d) as a *glider*. glider

The complemented strings $\overline{v_i}$ on the right hand side of (3d) need further explanation. By definition we have $\overline{v_i} \in D$, i.e., the subpath v_i in $\widehat{x}$ is a valley. Therefore, to compute $\Gamma(\overline{v_i})$ we apply Γ to the hill $\overline{v_i}$, obtained by complementing the valley v_i, without changing any other steps of $\widehat{x}$, as they are irrelevant for the computation of $\Gamma(\overline{v_i})$; see Figure 8. Note that the vertical positions of the steps of $\widehat{x}$ or its subpaths are irrelevant for the definition of Γ, but what matters are their indices on the horizontal axis (as they enter the sets A and B in (3d)), which are not modified by the complementation. However, complementation changes the roles of 0s and 1s, so for any glider $(A, B) \in \Gamma(x)$, A and B are either indices of $\nearrow$-steps and $\searrow$-steps, respectively, of some hill in the original Motzkin path $\widehat{x}$, or indices of $\searrow$-steps and $\nearrow$-steps, respectively, of some valley in the original path $\widehat{x}$. What is important is that $\max A < \min B$, i.e., all steps in A are to the left of all steps in B.

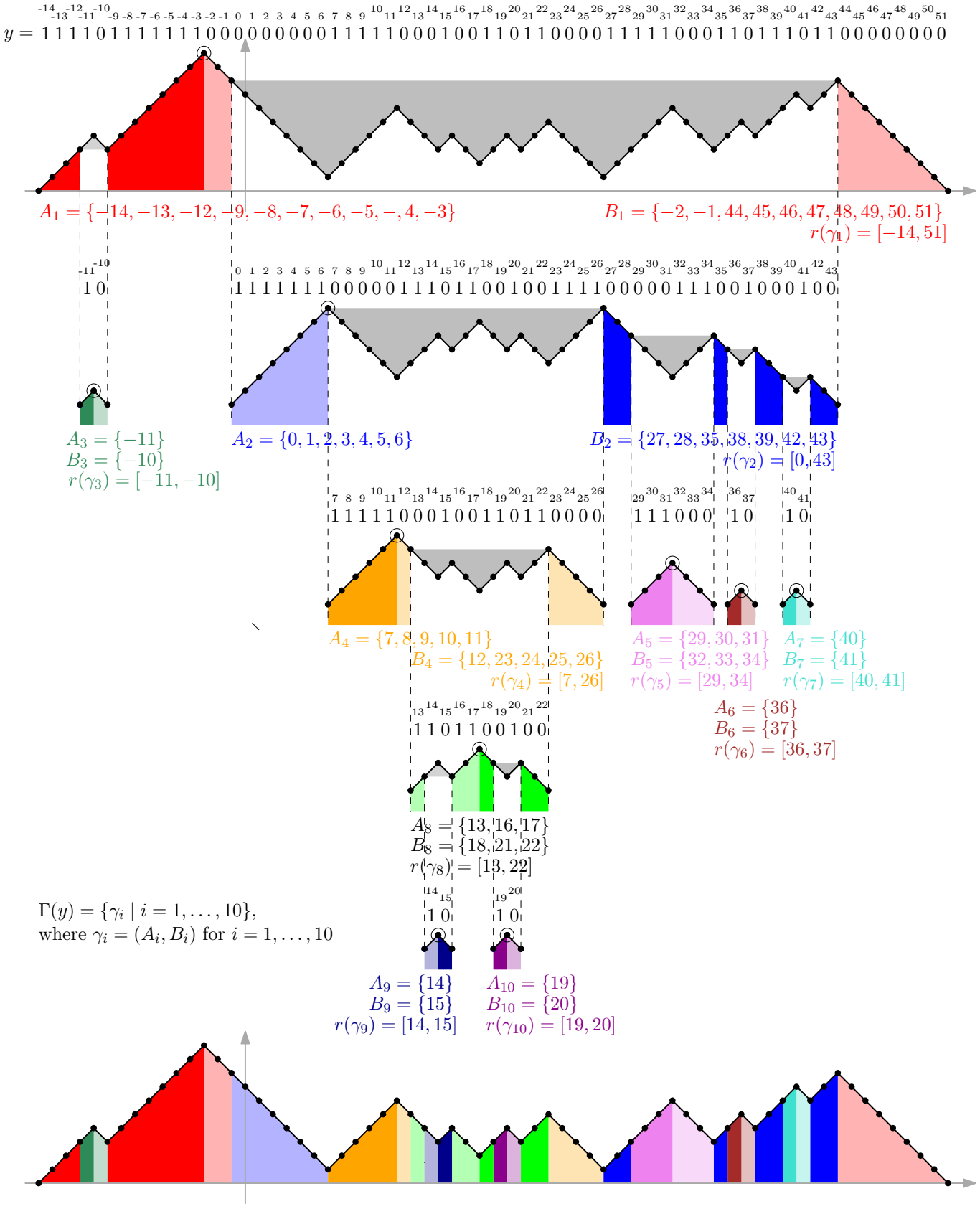


FIGURE 8. Glider partitioning in a hill $y \in D'$ contained in some larger Motzkin path $\widehat{x}$.

3.5. **Range of gliders and equivalence classes.** We define the *range* of a glider $\gamma = (A, B) \in \Gamma(x)$ as the interval

range $r(\gamma)$

$$r(\gamma) := [\min A, \max B] = \{\min A, \min A + 1, \dots, \max B\}.$$

Moreover, we write $\widehat{x}_{r(\gamma)}$ for the subpath of $\widehat{x}$ on the interval $r(\gamma)$. This subpath contains all steps from A and B, plus possibly steps of other gliders, and it starts with a step from A and ends with a step from B. We refer to a glider $\gamma \in \Gamma(x)$ as *non-inverted* if $\widehat{x}_{r(\gamma)}$ is a hill in $\widehat{x}$, and as *inverted* if $\widehat{x}_{r(\gamma)}$ is a valley. Non-inverted gliders $\gamma = (A, B)$ have $\nearrow$-steps of $\widehat{x}$ at the positions in A and $\searrow$-steps at the positions in B, and for inverted gliders the situation is reversed. For example, in Figure 8, the gliders $\gamma_1, \gamma_3, \gamma_4, \gamma_5, \gamma_6, \gamma_7, \gamma_{10}$ are non-inverted, whereas $\gamma_2, \gamma_8, \gamma_9$ are inverted.

inverted

Clearly, as $\widehat{x}$ is an infinite string containing infinitely many hills, the set $\Gamma(x)$ is an infinite set. However, as $\widehat{x}$ has periodicity n, we can partition the set $\Gamma(x)$ into finitely many equivalence classes of gliders. Specifically, for any two gliders $(A, B), (A', B') \in \Gamma(x)$, we write $(A, B) \sim (A', B')$ if these sets are the same modulo n. For any glider $\gamma \in \Gamma(x)$, its equivalence class is denoted by $[\gamma]$ and the set of all equivalence classes by

$\Gamma(x)/\!\sim$

$$\Gamma(x)/\!\sim := \{[\gamma] \mid \gamma \in \Gamma(x)\}; \tag{4}$$

see Figure 9. We also define

$\nu(x)$

$$\nu(x) := |\Gamma(x)/\!\sim|, \tag{5}$$

and we simply refer to this quantity as the *number of gliders*. Of course, $\Gamma(x)$ is an infinite set, but there are only finitely many equivalence classes with respect to $\sim$.

3.6. **Position and speed of gliders.** For any glider $\gamma = (A, B) \in \Gamma(x)$ we define

$s_1(\gamma), s_2(\gamma)$

$$s_1(\gamma) := \max A \quad \text{and} \quad s_2(\gamma) := \max B. \tag{6a}$$

Using these, the *position of* γ is defined as

position $s(\gamma)$

$$s(\gamma) := \frac{1}{2}(s_1(\gamma) + s_2(\gamma)) = \frac{1}{2}(\max A + \max B). \tag{6b}$$

In words, the position $s(\gamma)$ is the average position of the rightmost $\nearrow$-step and the rightmost $\searrow$-step that belong to γ, which is true regardless of whether γ is inverted or not. Clearly, the numbers $s(\gamma)$, $\gamma \in \Gamma(x)$, are either integers or half-integers; see Figure 9. Also note from the example in the figure that two distinct gliders may have the same position. For technical reasons we also define

$s_0(\gamma)$

$$s_0(\gamma) := \min A,$$

so we have $r(\gamma) = [s_0(\gamma), s_2(\gamma)]$.

The *speed* of $\gamma = (A, B) \in \Gamma(x)$ is defined as

speed $v(\gamma)$

$$v(\gamma) := |A| = |B|. \tag{7}$$

Note that the speed equals the height of the hill $\widehat{x}_{r(\gamma)}$ if γ is non-inverted, or the depth of the valley $\widehat{x}_{r(\gamma)}$ if γ is inverted. We also define the *speed set*

speed set $V(x)$

$$V(x) := \{v(\gamma) \mid [\gamma] \in \Gamma(x)/\!\sim\}, \tag{8}$$

which we define as a multiset of size $\nu(x)$, as gliders from distinct equivalence classes may have the same speed. Note that

$$\sum_{v \in V(x)} v = k, \tag{9}$$

as the sum of speeds equals the number of 1s in x. In other words, the speed set is a number partition of k.

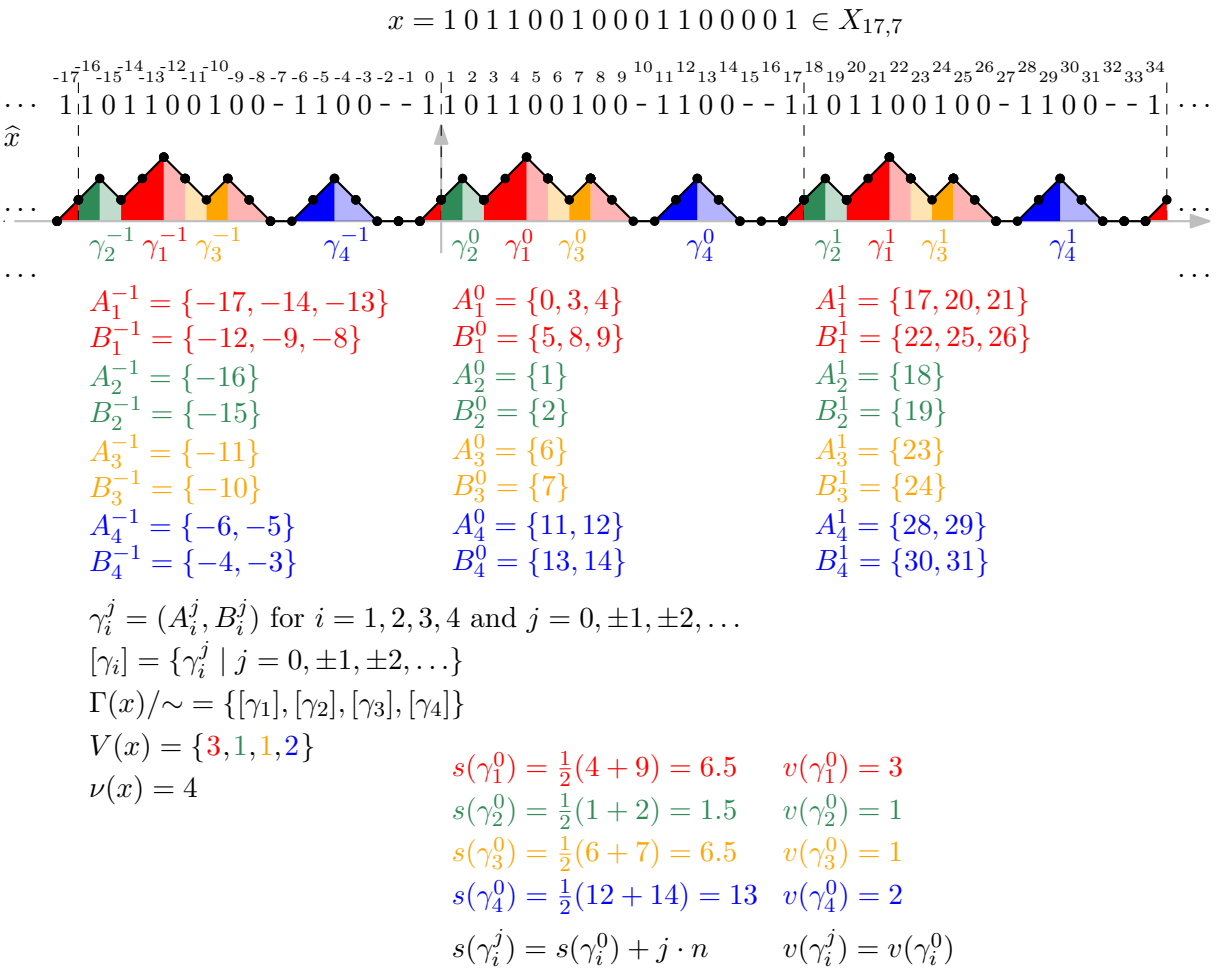


FIGURE 9. Position and speed of gliders in $\widehat{x}$.

3.7. **Properties of the speed set.** Before proceeding with the description of the movement of gliders over time, we establish a few combinatorial properties about the speed set $V(x)$. Specifically, Lemma 9 below asserts that the speed set can be computed more directly, without invoking the glider partitioning recursion described in Section 3.4. Furthermore, Lemma 10 asserts that the cardinality of $V(x)$, which equals the number of gliders $\nu(x)$, is given by the number of descents in x.

Let X be a nonempty multiset of positive integers. We write $X \oplus 1$ for the multiset obtained by incrementing one of the largest elements of X by 1. For example, for $X = \{4, 4, 4, 2, 1, 1\}$ we have $X \oplus 1 = \{5, 4, 4, 2, 1, 1\}$. $X \oplus 1$

In a bitstring $x \in X_{n,k}$ under parenthesis matching, we refer to any maximal (cyclic) substring of matched bits as a *block*. We define a multiset of integers $W(x)$ as follows; see Figure 10. We first define block $W(x)$

$$W(x) := \bigcup_{y \text{ block in } x} W(y). \tag{10a}$$

For any $y \in D \setminus D'$, $y \neq \varepsilon$, we consider the partition $y = y_1 \cdots y_\ell$ with $y_1, \ldots, y_\ell \in D'$ and define

$$W(y) := \bigcup_{i=1}^{\ell} W(y_i). \tag{10b}$$

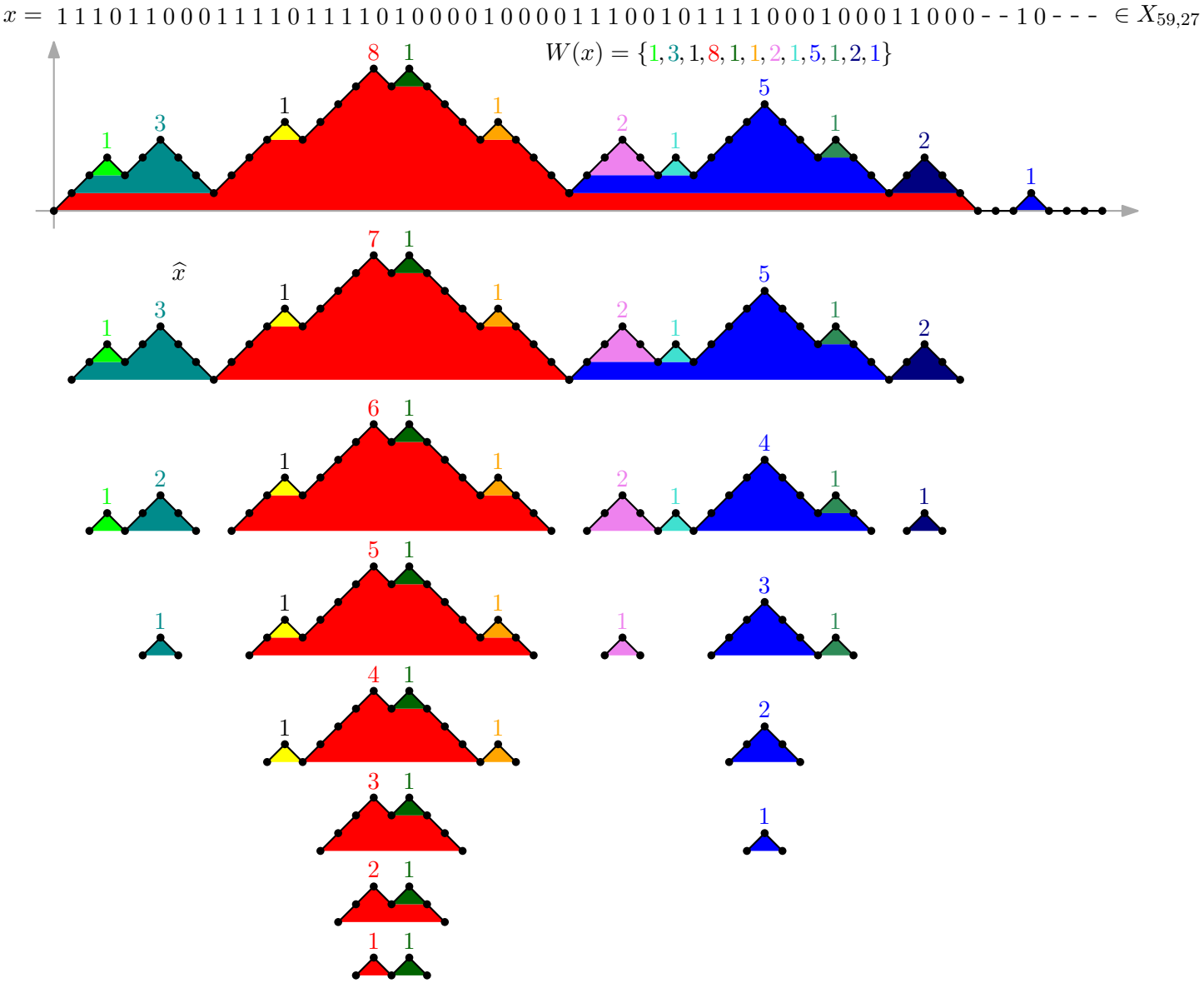


FIGURE 10. Illustration of the recursive definition of the multiset $W(x)$.

Lastly, for $y \in D'$ we have $y = 1\,u\,0$ with $u \in D$ and define

$$W(y) := \begin{cases} \{1\} & \text{if } u = \varepsilon, \\ W(u) \oplus 1 & \text{if } u \neq \varepsilon. \end{cases} \tag{10c}$$

Lemma 9. *For any $x \in X_{n,k}$, the multisets $V(x)$ and $W(x)$ defined in* (8) *or* (10)*, respectively, are the same, i.e., we have $V(x) = W(x)$.*

Proof. Consider a block $y \in D$ in x and the corresponding Dyck path, and let j by the position of one of its $\nearrow$-steps. A *staircase* $S(j)$ is the maximum size set $\{j = j_1 < \cdots < j_h\}$ of positions of $\nearrow$-steps in y such that the subpath u_i of y between positions j_i and j_{i+1} is a hill, i.e., $u_i \in D$, of height at most $h - 1 - i$, for all $i = 1, \ldots, h-1$. In particular, u_{h-1} has height at most 0, i.e., $u_{h-1} = \varepsilon$.

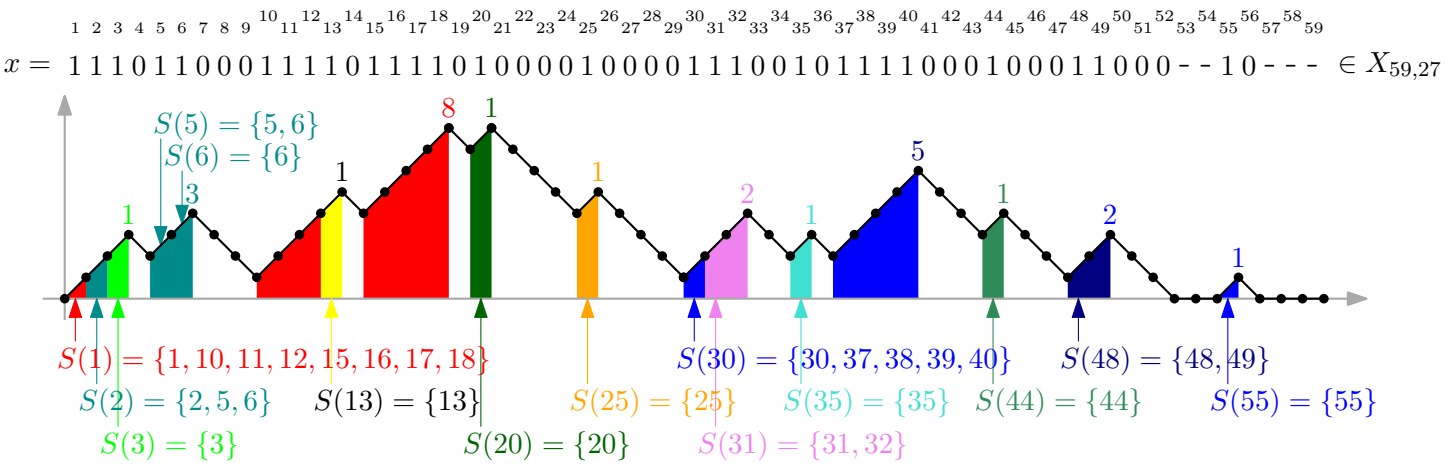


FIGURE 11. Illustration of staircases. All shown staircases are full, except $S(5)$ and $S(6)$. This continues the example from Figure 10.

Note that any two staircases are either disjoint or one is a contained in the other (as a suffix when the elements are ordered increasingly). A staircase is *full* if it is not contained in another staircase. For example, in Figure 11 we have $S(2) \supseteq S(5) \supseteq S(6)$, so $S(5)$ and $S(6)$ are not full, but $S(2)$ is full. The set of full staircases partitions the set of positions of all $\nearrow$-steps in y; see Figure 11. We write $S(x)$ for the union of all full staircases over all blocks $y \in D$ in x. We also define the multiset $T(x) := \{|S| \mid S \in S(x)\}$.

We first show that

$$T(x) = W(x). \tag{11}$$

To prove (11) we consider the recursive definition of $W(x)$, and we consider a substring $y \in D$, $y \neq \varepsilon$, of x that arises in this computation. If $y = 1\,0 \in D'$ then by the first case in (10c) we have $W(y) = \{1\}$. Furthermore, we clearly have $T(y) = \{1\}$, so indeed $T(y) = W(y)$. If $y = 1\,u\,0 \in D'$ with $u \neq \varepsilon$ then by the second case in (10c) we have $W(y) = W(u) \oplus 1$. Furthermore, the leftmost full staircase in $S(u)$ is extended by the leading 1 in y to a full staircase in $S(y)$, implying that $T(y) = T(u) \oplus 1$. By induction we know $T(u) = W(u)$ and consequently $T(y) = W(y)$. If $y \in D \setminus D'$, $y \neq \varepsilon$, then we have $y = y_1 \cdots y_\ell$ for $y_1, \ldots, y_\ell \in D'$ and by (10b) we have $W(y) = \bigcup_{i=1}^{\ell} W(y_i)$. Furthermore, the staircases in y are simply obtained as the union of staircases in all substrings y_i, implying that $T(y) = \bigcup_{i=1}^{\ell} T(y_i)$. By induction we know $T(y_i) = W(y_i)$ for all $i = 1, \ldots, \ell$ and consequently $T(y) = W(y)$.

We now show that

$$T(x) = V(x). \tag{12}$$

We claim that for any glider $\gamma = (A, B) \in \Gamma(x)$, the set of positions of its $\nearrow$-steps is a full staircase S in the base hill of $\widehat{x}$ containing this glider. Specifically, if γ is non-inverted, then $S = A$, whereas if γ is inverted, then $S = B$. In both cases we obtain $v(\gamma) = |A| = |B| = |S|$ (recall (7)), i.e., there is a bijection between the multisets $T(x)$ and $V(x)$, which proves (12). To prove our claim we distinguish whether γ is non-inverted or inverted. The former case splits into three subcases, and the latter case into two subcases, as follows; see Figure 12 (a)–(c2):

- Case (a): $\widehat{x}_{r(\gamma)}$ is a base hill.
- Case (b1): $y\,\widehat{x}_{r(\gamma)}\,y'$ with $y, y' \in D$ is a bulge of $\widehat{x}_{r(\gamma')}$ for some non-inverted glider $\gamma' \in \Gamma(x)$.

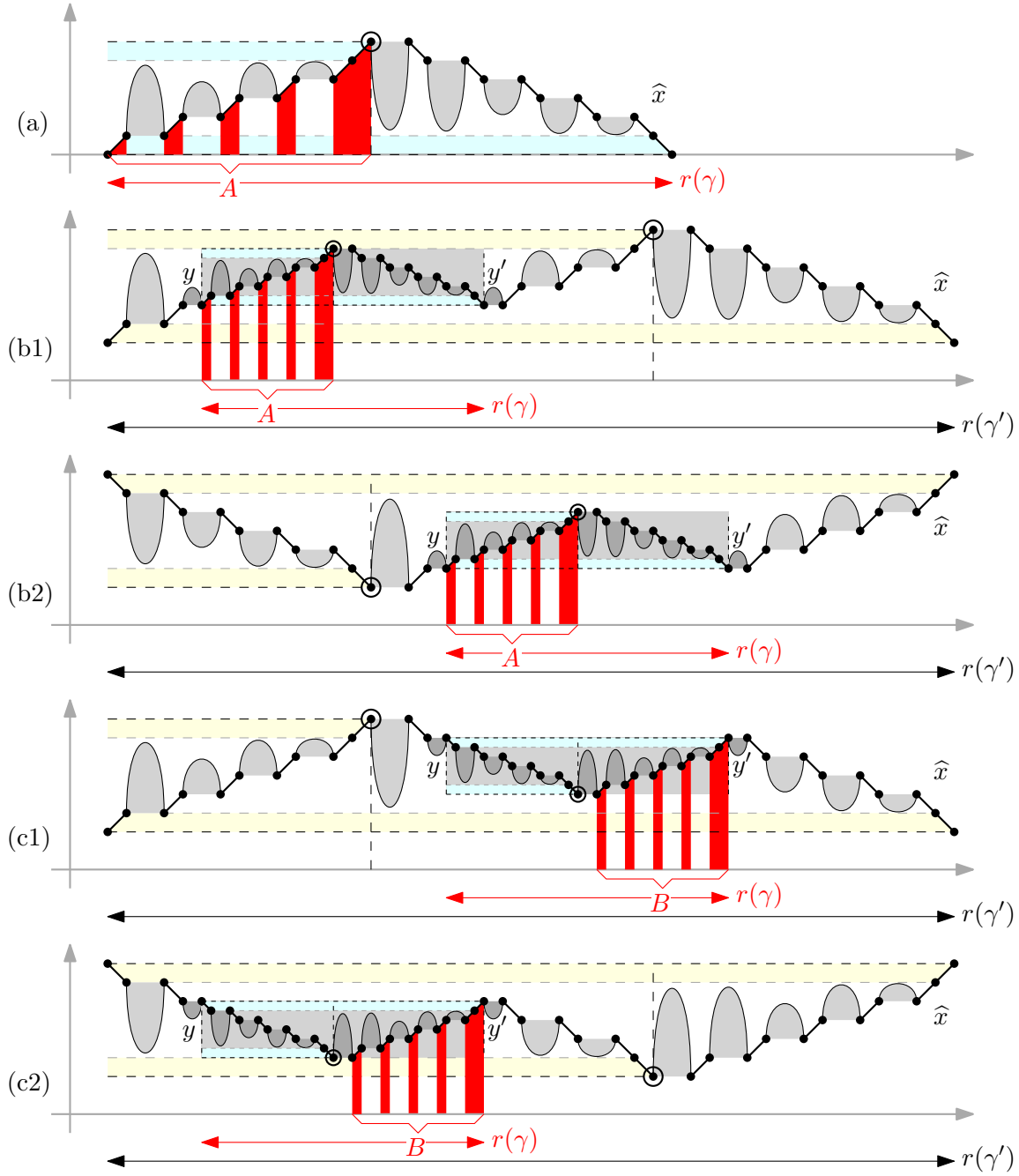


FIGURE 12. Illustration of the five cases in the proof of Lemma 9. The gray-shaded regions represent bulges and dents.

- Case (b2): $\overline{y\,\widehat{x}_{r(\gamma)}\,y}$ with $y, y' \in D$ is a dent of $\overline{\widehat{x}_{r(\gamma')}}$ for some inverted glider $\gamma' \in \Gamma(x)$.
- Case (c1): $y\,\widehat{x}_{r(\gamma)}\,y'$ with $\overline{y}, \overline{y'} \in D$ is a dent of $\widehat{x}_{r(\gamma')}$ for some non-inverted glider $\gamma' \in \Gamma(x)$.
- Case (c2): $\overline{y\,\widehat{x}_{r(\gamma)}\,y'}$ with $\overline{y}, \overline{y'} \in D$ is a bulge of $\overline{\widehat{x}_{r(\gamma')}}$ for some inverted glider $\gamma' \in \Gamma(x)$.

Note that the glider γ' in the latter four cases satisfies $v(\gamma') > v(\gamma)$. In all five cases the claim can be verified directly from the definition of full staircases and the definition of $\Gamma(x)$ given in (3).

To complete the proof we combine (11) and (12). □

We emphasize that in the first part of the proof that establishes (11) the glider partitions for $y = 1\,u\,0$ and u, i.e., the sets $\Gamma(y)$ and $\Gamma(u)$, may be considerably different; see Figure 27. More specifically, $\Gamma(y)$ may differ from $\Gamma(u)$ by more than simply adding the positions of the first and last step of y to a fastest glider from $\Gamma(u)$.

descents $d(x)$

We write $d(x)$ for the number of descents in the (cyclic) string x, i.e., the number of occurrences of the substring 10. Clearly, this is equal to the number of ascents, i.e., the number of occurrences of the substring 01. Furthermore, this is the same as the number of maximal substrings of 1s, or the number of maximal substrings of 0s. For example, $x = 001100010001$ has three descents, one of them across the boundary, i.e., $d(x) = 3$. As this quantity has nothing to do with parenthesis matching, we did not distinguish matched or unmatched 0s in x in this example. Nonetheless, the next lemma asserts that $d(x)$ equals the number of gliders $\nu(x)$.

Lemma 10. *For any $x \in X_{n,k}$ we have $d(x) = \nu(x)$.*

Proof. We prove inductively that $d(x) = |W(x)|$, and then the claim follows with the help of Lemma 9 and the definitions (5) and (8). To show that $d(x) = |W(x)|$ we consider the recursive definition of $W(x)$, and we consider a substring $y \in D$, $y \neq \varepsilon$, of x that arises in this computation. If $y = 1\,0 \in D'$ then by the first case in (10c) we have $|W(y)| = 1$. Furthermore, we clearly have $d(y) = 1$, so indeed $d(y) = |W(y)|$. If $y = 1\,u\,0 \in D'$ with $u \neq \varepsilon$ then by the second case in (10c) we have $|W(y)| = |W(u) \oplus 1| = |W(u)|$. Furthermore, as the first and last bit of u are 1 and 0, respectively, the first 1 and last 0 in y do not create additional occurrences of 10, implying that $d(y) = d(u)$. By induction we know $d(u) = |W(u)|$ and consequently $d(y) = |W(y)|$. If $y \in D \setminus D'$, $y \neq \varepsilon$, then we have $y = y_1 \cdots y_\ell$ for $y_1, \ldots, y_\ell \in D'$ and by (10b) we have $|W(y)| = \sum_{i=1}^{\ell} |W(y_i)|$. Furthermore, the number of occurrences of 10 in y is simply the sum of the number of such occurrences in each of the substrings y_i, implying that $d(y) = \sum_{i=1}^{\ell} d(y_i)$. By induction we know $d(y_i) = |W(y_i)|$ for all $i = 1, \ldots, \ell$ and consequently $d(y) = |W(y)|$. □

forest $\Lambda(x)$

3.8. **Free and trapped gliders.** We associate the recursive computation of $\Gamma(x)$ in (3) with a forest $\Lambda(x)$ of rooted unordered trees as follows; see Figure 13. The vertex set of $\Lambda(x)$ is the set of gliders $\Gamma(x)$. Moreover, for any glider $\gamma \in \Gamma(x)$ we consider the recursion step (3d) in which $\gamma = (A, B)$ is added to the set $\Gamma(x)$, and we consider the corresponding hill $y \in D'$ with bulges u_i and dents v_i as defined by (2). The set of all descendants (direct or indirect) of γ in the tree of $\Lambda(x)$ is the set $\bigcup_{i=1}^{h-2} \Gamma(u_i) \cup \bigcup_{i=0}^{h-2} \Gamma(\overline{v_i})$. Consequently, the direct descendants, i.e., the children, of γ are exactly the gliders computed in the *next* recursion step for the u_i and $\overline{v_i}$. Moreover, every tree in $\Lambda(x)$ corresponds to a base hill $y \in D'$ in $\widehat{x}$ (recall (3a)), specifically the root γ of this tree satisfies $y = \widehat{x}_{r(\gamma)}$.

Lemma 11. *For any inverted glider $\gamma \in \Gamma(x)$, the step at position $s_2(\gamma) + 1$ in $\widehat{x}$ is a $\searrow$-step, i.e., the corresponding bit in x is a matched 0.*

As γ is inverted, its last step at position $s_2(\gamma)$ is an $\nearrow$-step, and the $\searrow$-step at position $s_2(\gamma)+1$ is directly to its right.

Proof. As γ is inverted, it cannot be the root of a tree in $\Lambda(x)$. Let γ' be the parent of γ in the forest $\Lambda(x)$. If γ' is non-inverted, then $y := \widehat{x}_{r(\gamma)}$ belongs to a dent of $y' := \widehat{x}_{r(\gamma')}$ and y starts and ends on the same height as the starting point and endpoint of this dent. From (2) it follows that y must be followed by a $\searrow$-step of y', which either belongs to the same dent or to γ'. If γ' is inverted, then $\overline{y}$ belongs to a bulge of $\overline{y'}$ and $\overline{y}$ starts and ends on the same height as the starting point and endpoint of this bulge. From (2) it follows that $\overline{y}$ must be followed by a $\nearrow$-step of $\overline{y'}$, which either belongs to the same bulge or to γ' (in the latter case this is a $\searrow$-step of γ', however). In both cases y is followed by a $\searrow$-step in $\widehat{x}$. □

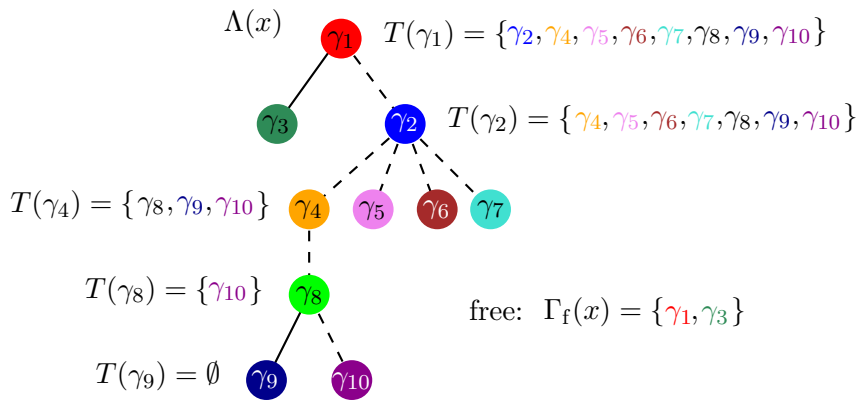


FIGURE 13. Tree of free and trapped gliders present in the hill $y \in D'$ of Figure 8.

For a glider γ as before in the definition of $\Lambda(x)$, we define a set $T(\gamma) \subseteq \Gamma(x)$ as $T(\gamma)$

$$T(\gamma) := \bigcup_{i=0}^{h-2} \Gamma(\overline{v_i}), \tag{13}$$

and we refer to any glider $\gamma' \in T(\gamma)$ as *trapped by* γ. In the forest $\Lambda(x)$, all descendants of γ in the subtrees on the gliders in $\Gamma(\overline{v_i})$ are trapped by γ. In Figure 13, children that are trapped by their parent are connected by dashed lines. A glider that is not trapped by any of its predecessors in the forest $\Lambda(x)$ is called *free*. In particular, the gliders at the roots of the trees in $\Lambda(x)$ are all free. We write $\Gamma_f(x)$ for the set of all free gliders. Note that free gliders are always non-inverted, whereas trapped gliders can be inverted or non-inverted. Specifically, a trapped glider is inverted if and only if it is trapped by an odd number of predecessors in its tree in $\Lambda(x)$. trapped free $\Gamma_f(x)$

The following lemma asserts that the speed of gliders decreases along parent-child pairs in the forest $\Lambda(x)$; cf. Figure 13.

Lemma 12. *Let $\gamma, \gamma' \in \Gamma(x)$. If γ' is a child of γ in the forest $\Lambda(x)$, then $v(\gamma) > v(\gamma')$.*

Proof. We assume w.l.o.g. that $y := \widehat{x}_{r(\gamma)}$ is a hill in $\widehat{x}$. Let h denote the height of y, and recall that $v(\gamma) = h$. From Lemma 8 we obtain that the height of the ith bulge u_i of y is at most $h - 1 - i \leq h - 2$ and the depth of the ith dent v_i of y is at most $h - 1 - i \leq h - 1$. It follows that $v(\gamma) > v(\gamma')$ for all $\gamma' \in \bigcup_{i=1}^{h-2} \Gamma(u_i) \cup \bigcup_{i=0}^{h-2} \Gamma(\overline{v_i})$, which are precisely the children of γ in the forest $\Lambda(x)$. □

Lemma 13. *For any base hill y in $\widehat{x}$ of the form $y = 1\,u\,0$, $u \in D$, the first and last step of y belong to the glider of maximum speed in $\Gamma(y)$.*

Proof. From (3) we see that the first and last step of y belong to the same glider $\gamma \in \Gamma(x)$. Furthermore, γ is the root of a tree in $\Lambda(x)$ and every other glider in $\Gamma(y)$ is a descendant of γ in this tree. The statement hence follows from Lemma 12. □

We say that a glider $\gamma \in \Gamma(x)$ is *clean*, if $\widehat{x}_{r(\gamma)} = 1^{v(\gamma)}0^{v(\gamma)}$ or $\widehat{x}_{r(\gamma)} = 0^{v(\gamma)}1^{v(\gamma)}$; see Figure 14. In other words, in the infinite Motzkin path $\widehat{x}$, the steps belonging to γ are all consecutive, i.e., all steps of $\widehat{x}_{r(\gamma)}$ belong to γ. These are $2v(\gamma)$ steps in total, $v(\gamma)$ many $\nearrow$-steps followed by $v(\gamma)$ many $\searrow$-steps if γ is non-inverted, and vice versa if γ is inverted. clean

Lemma 14. *If $\gamma \in \Gamma(x)$ has minimum speed $v(\gamma) = \min V(x)$, then it is clean.*

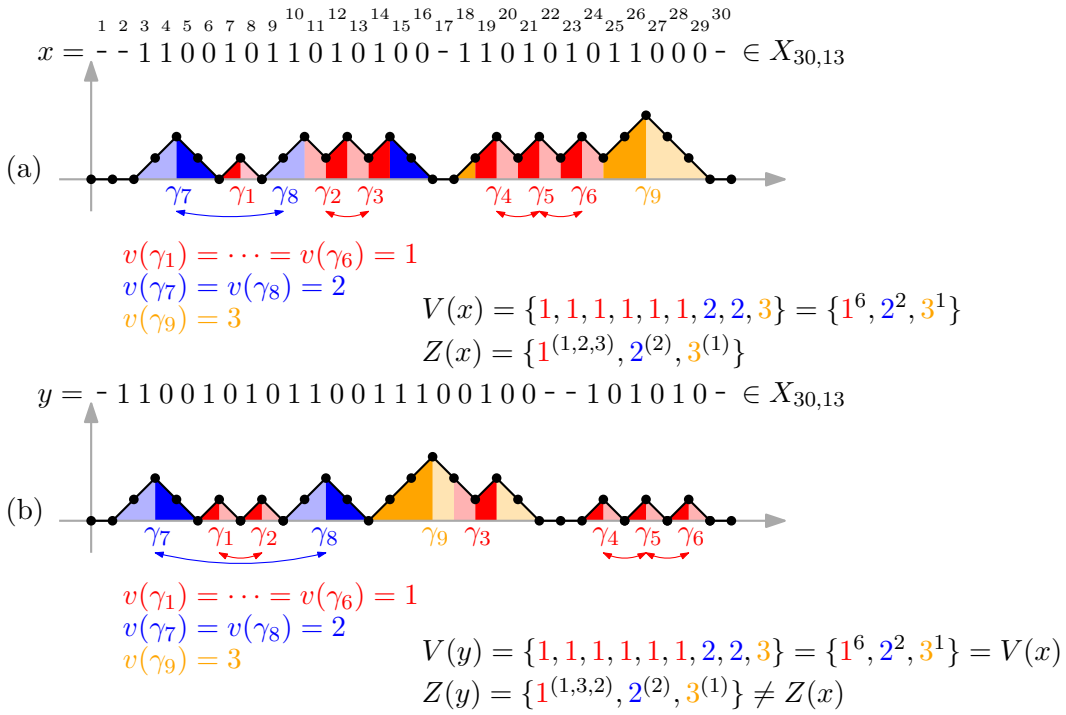


FIGURE 14. Illustration of clean and coupled gliders, and of trains. In (a), all gliders except γ_8 and γ_9 are clean, and in (b), all gliders except γ_9 are clean. In both figures, pairs of coupled gliders are connected by double arrows.

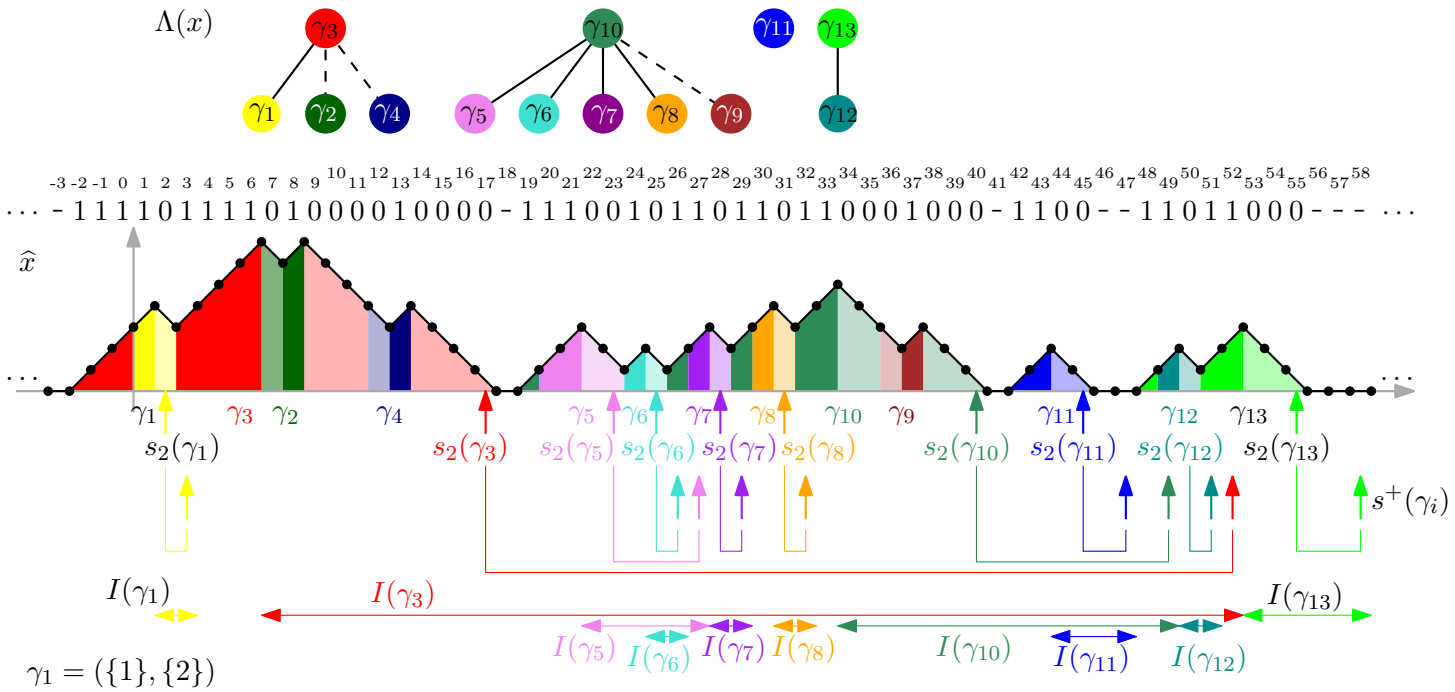
Proof. As γ has minimum speed, it must be a leaf in the tree $\Lambda(x)$ by Lemma 12. Consequently, in the corresponding hill $y \in D'$ whose support is $r(\gamma)$, all bulges and dents are empty, i.e., we have $u_1 = \cdots = u_{h-2} = v_0 = \cdots = v_{h-2} = \varepsilon$, $h = v(\gamma)$, in the decomposition (2). □

We say that two gliders $\gamma', \gamma'' \in \Gamma(x)$ of the same speed $v(\gamma') = v(\gamma'')$ are *coupled*, if all steps between the last step of γ' and the first step of γ'' belong to gliders of strictly smaller speed; see Figure 14. In particular, no $\rightarrow$-steps are allowed between γ' and γ''. Note that if $v(\gamma') = v(\gamma'') = \min V(x)$, then the last step of γ' must be directly next to the first step of γ''. coupled

Lemma 15. *Let $\gamma, \gamma', \gamma'' \in \Gamma(x)$ such that γ' and γ'' are coupled. Then either both γ' and γ'' are trapped by γ or none of them.*

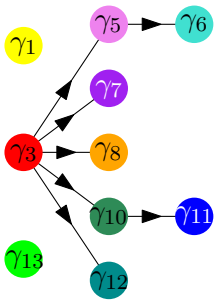
Proof. Suppose for the sake of contradiction that only one of γ', γ'' is trapped by γ, but not the other one. Then clearly we have $v(\gamma) > v(\gamma') = v(\gamma'')$ by Lemma 12. However, note that in the decomposition (2) of the hill $y \in D'$ whose support is $r(\gamma)$, all dents are separated by the $\nearrow$-step at position $s_1(\gamma)$ and the $\searrow$-step at position $s_2(\gamma)$ from the rest of the Motzkin path $\widehat{x}$, so one of these steps is between γ' and γ'', a contradiction. □

3.9. **Capturing relation.** In the previous sections, we have partitioned certain steps of $\widehat{x}$ into gliders, and defined various properties for them, such as position, speed etc. All of these definitions were 'static', i.e., for one particular vertex $x \in X_{n,k}$. We now aim to investigate the behavior of gliders as we move from x to $f(x)$ along the cycle $C(x)$, i.e., we aim to understand the 'dynamic' behavior of gliders over one time step. As long as a glider does not interact with other gliders, this motion is uniform with the corresponding speed; recall Figure 3 (a)+(b). However, two gliders of different speeds interact with each other. Specifically, they participate



$\gamma_1 = (\{1\},\{2\})$
$\gamma_2 = (\{7\},\{8\})$
$\gamma_3 = (\{-2,-1,0,3,4,5,6\},\{9,10,11,14,15,16,17\})$
$\gamma_4 = (\{12\},\{13\})$
$\gamma_5 = (\{20,21\},\{22,23\})$
$\gamma_6 = (\{24\},\{25\})$
$\gamma_7 = (\{27\},\{28\})$
$\gamma_8 = (\{30\},\{31\})$
$\gamma_9 = (\{36\},\{37\})$
$\gamma_{10} = (\{19,26,29,32,33\},\{34,35,38,39,40\})$
$\gamma_{11} = (\{42,43\},\{44,45\})$
$\gamma_{12} = (\{49\},\{50\})$
$\gamma_{13} = (\{48,51,52\},\{53,54,55\})$

capturing relation $\succ$



$\Gamma_f(x) = \{\gamma_1,\dots,\gamma_{13}\} \setminus \{\gamma_2,\gamma_4,\gamma_9\}$

$C(\gamma_3) = \{\gamma_5,\gamma_6,\gamma_7,\gamma_8,\gamma_{10},\gamma_{11},\gamma_{12}\}$
$C(\gamma_5) = \{\gamma_6\}$
$C(\gamma_{10}) = \{\gamma_{11}\}$

$S(\gamma_3) = \{18,41,46,47,48,51,52\}$
$S(\gamma_5) = \{26,27\}$
$S(\gamma_{10}) = \{41,46,47,48,49\}$

$M(x) = \{\gamma_1,\gamma_3,\gamma_{13}\}$

$C(\gamma_3)_0 = \emptyset$
$C(\gamma_3)_1 = \{\gamma_5,\gamma_6,\gamma_7,\gamma_8,\gamma_{10}\}$
$C(\gamma_3)_2 = \{\gamma_{11}\}$
$C(\gamma_3)_3 = C(\gamma_3)_4 = \emptyset$
$C(\gamma_3)_5 = \{\gamma_{12}\}$

$C(\gamma_5)_0 = \{\gamma_6\}$

$C(\gamma_{10})_0 = \emptyset$
$C(\gamma_{10})_1 = \{\gamma_{11}\}$
$C(\gamma_{10})_2 = C(\gamma_{10})_3 = \emptyset$

FIGURE 15. Illustration of the capturing relation and related definitions.

in an overtaking, which boosts the faster glider that is overtaking, and delays the slower glider that is being overtaken; see Figure 3 (c)+(d). During an overtaking, the glider being overtaken is trapped by the overtaking glider and does not change its position. However, its bits are complemented, i.e., if it was non-inverted, it becomes inverted, and vice versa. In the following, we determine which free gliders participate in overtakings in the next time step (trapped gliders do not move, so they are ignored).

The following definitions are illustrated in Figure 15. We first define a binary relation $\succ$ on the set of free gliders $\Gamma_f(x)$ as follows. For any $i \in \mathbb{Z}$ we define $w(i), w(I)$

$$w(i) = \begin{cases} 1 & \text{if step } i \text{ of } \widehat{x} \text{ is a } \rightarrow\text{-step or } \nearrow\text{-step,} \\ -1 & \text{if step } i \text{ of } \widehat{x} \text{ is a } \searrow\text{-step,} \end{cases}$$

and for any interval I we define

$$w(I) := \sum_{i \in I} w(i).$$

For a free glider $\gamma \in \Gamma_{\mathrm{f}}(x)$, we consider the subpath of $\widehat{x}$ strictly to the right of position $s_2(\gamma)$, i.e., starting from $a := s_2(\gamma) + 1$, and we consider the minimal index b such that $w([a, b]) = v(\gamma)$. Formally, we define $s^+(\gamma)$

$$s^+(\gamma) := \min\{b \mid w([s_2(\gamma) + 1, b]) = v(\gamma)\}. \tag{14}$$

From this definition we see that $w([s_2(\gamma) + 1, s^+(\gamma)]) = v(\gamma)$. Intuitively, $s^+(\gamma)$ is the rightmost position that the glider γ would occupy after its movement with speed $v(\gamma)$ in one time step, assuming that it moves independently of all other free gliders (i.e., if there are no faster gliders that overtake γ in this time step). For two free gliders $\gamma, \gamma' \in \Gamma_{\mathrm{f}}(x)$ such that $s_1(\gamma) < s_1(\gamma')$, we say that γ *captures* γ', which we denote by $\gamma \succ \gamma'$, if $s^+(\gamma) > s^+(\gamma')$; see Figure 15. It follows directly from this definition that the capturing relation $\succ$ is transitive, i.e., $\gamma \succ \gamma'$ and $\gamma' \succ \gamma''$ implies that $\gamma \succ \gamma''$. capture $\succ$

The next lemma describes a transitivity property for coupled gliders.

Lemma 16. *Let $\gamma, \gamma', \gamma'' \in \Gamma_{\mathrm{f}}(x)$ such that γ' and γ'' are coupled. If $\gamma \succ \gamma'$, then we have $\gamma \succ \gamma''$.*

Proof. This proof is illustrated in Figure 16. We define $a := s_2(\gamma')+1$, $b := s_0(\gamma'')-1$, $c := s_1(\gamma'')$, and $d := s_2(\gamma'')$. As γ' and γ'' are free, they are non-inverted. Furthermore, as γ' and γ'' are coupled, all steps of $\widehat{x}_{[a,b]}$ belong to gliders of strictly smaller speed, and all steps of those gliders are in the interval $[a, b]$ (recall Lemma 12). Consequently, $\widehat{x}_{[a,b]}$ is a hill of height less than $v(\gamma') = v(\gamma'')$, i.e., we have $w([a, p]) < v(\gamma')$ for all $p \in [a, c-1]$, and $w([a, c]) = v(\gamma')$, and therefore $s^+(\gamma') = c$. From the assumption $\gamma \succ \gamma'$ we know that $s^+(\gamma) > s^+(\gamma') = c$ and therefore $v(\gamma) > w([s_2(\gamma) + 1, c])$. As $w([c + 1, d]) = -v(\gamma'')$, we obtain $v(\gamma) - v(\gamma'') > w([s_2(\gamma), d])$. Using the definition of $s^+(\gamma)$ and $s^+(\gamma'')$ from (14), this inequality implies $s^+(\gamma) > s^+(\gamma'')$, and therefore $\gamma \succ \gamma''$, as claimed. □

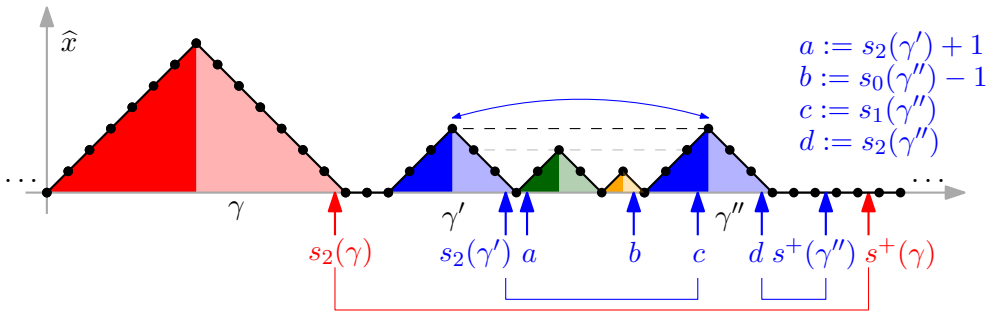


FIGURE 16. Illustration of the proof of Lemma 16.

The following lemma describes another transitivity property, namely that when a glider γ captures another glider γ', then γ also captures the free descendants of γ' in $\Lambda(x)$.

Lemma 17. *Let $\gamma, \gamma', \gamma'' \in \Gamma_{\mathrm{f}}(x)$. If $\gamma \succ \gamma'$ and γ'' is a child of γ' in $\Lambda(x)$, then we have $\gamma \succ \gamma''$.*

Proof. This proof is illustrated in Figure 17. Consider two gliders $\gamma', \gamma'' \in \Gamma_{\mathrm{f}}(x)$ such that γ'' is a child of γ' in $\Lambda(x)$. We have that $\widehat{x}_{r(\gamma'')}$ is a bulge of the hill $\widehat{x}_{r(\gamma')}$, and hence by Lemma 8 the

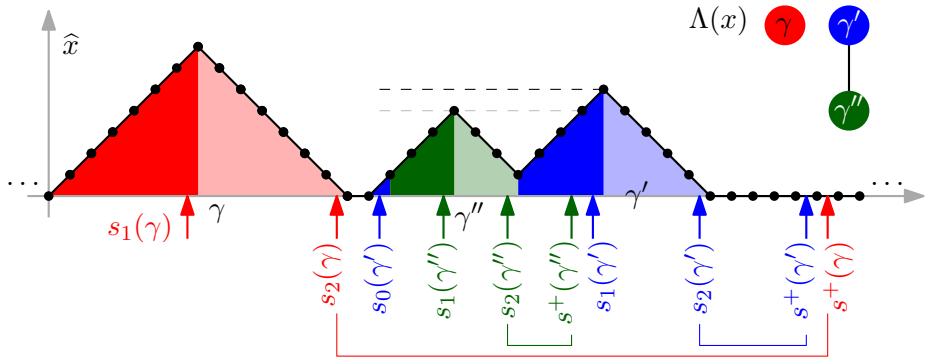


FIGURE 17. Illustration of the proof of Lemma 17.

highest point of the hill $\widehat{x}_{r(\gamma'')}$ is strictly lower than the highest point of the hill $\widehat{x}_{r(\gamma')}$. From (14) we see that the height of the hill $\widehat{x}_{r(\gamma'')}$ equals $v(\gamma'') = w([s_2(\gamma'')+1, s^+(\gamma'')])$. Combining these observations shows that $s^+(\gamma'') < s_1(\gamma')$, in particular $\gamma'' \not\succ \gamma'$.

Now let $\gamma, \gamma', \gamma'' \in \Gamma_{\mathrm{f}}(x)$ be as in the lemma. From the arguments before we obtain that $s^+(\gamma'') < s_1(\gamma') < s^+(\gamma')$. Furthermore, using that $\gamma \succ \gamma'$ we also obtain that γ is not a descendant of γ' in $\Lambda(x)$, i.e., $s_1(\gamma) < s_2(\gamma) < s_0(\gamma') < s_1(\gamma'')$ and that $s^+(\gamma) > s^+(\gamma')$. Combining these observations shows that $s^+(\gamma) > s^+(\gamma'')$, yielding $\gamma \succ \gamma''$, as claimed. □

We introduce a few more definitions, illustrated in Figure 15.

For any free glider $\gamma \in \Gamma_{\mathrm{f}}(x)$, we define a set $C(\gamma) \subseteq \Gamma_{\mathrm{f}}(x)$ as $C(\gamma)$

$$C(\gamma) := \{\gamma' \in \Gamma_{\mathrm{f}}(x) \mid \gamma \succ \gamma'\}, \tag{15}$$

and we refer to any glider $\gamma' \in C(\gamma)$ as *captured by* γ. By Lemma 17, $C(\gamma)$ is a downset of vertices of the subforest of $\Lambda(x)$ induced by free gliders. captured

For any glider $\gamma \in \Gamma_{\mathrm{f}}(x)$, we let $S(\gamma)$ be the set of positions in the interval $[s_2(\gamma)+1, s^+(\gamma)]$ not belonging to the range of any glider in $C(\gamma)$. Formally, we define $S(\gamma)$

$$S(\gamma) := [s_2(\gamma)+1, s^+(\gamma)] \setminus \bigcup_{\gamma' \in C(\gamma)} r(\gamma'). \tag{16}$$

Furthermore, we define a partition of the set $C(\gamma)$ of captured gliders as follows: We let $C(\gamma)_0$ $C(\gamma)_i$ be the subset of gliders γ' from $C(\gamma)$ for which $r(\gamma')$ is smaller than the minimum of $S(\gamma)$. Furthermore, we let $C(\gamma)_i$, $i \geq 1$, be the subset of gliders γ' from $C(\gamma)$ for which $r(\gamma')$ is between the ith and $(i+1)$th smallest element of $S(\gamma)$. Note that $s^+(\gamma) \in S(\gamma)$ and therefore no gliders in $C(\gamma)$ are to the right of the largest element of $S(\gamma)$. Lastly, for any glider $\gamma \in \Gamma_{\mathrm{f}}(x)$ we define $I(\gamma)$

$$I(\gamma) := [s_1(\gamma)+1, s^+(\gamma)]. \tag{17}$$

The next lemma describes a number of crucial properties of the capturing relation and the corresponding concepts defined before.

Lemma 18. *Any free glider $\gamma \in \Gamma_{\mathrm{f}}(x)$ has the following properties:*

(i) The steps of $\widehat{x}$ whose support is the set $S(\gamma)$ defined in (16) form a sequence of 0 or more $\rightarrow$-steps followed by 0 or more $\nearrow$-steps.

(ii) We have $|S(\gamma)| = v(\gamma)$, i.e., the cardinality of $S(\gamma)$ equals the speed of γ.

(iii) For any glider γ' in the set $C(\gamma)_i$ defined after (16), we have $v(\gamma') < v(\gamma) - i$. In particular, $v(\gamma') < v(\gamma)$, i.e., a glider can only capture slower gliders, and a glider can only be captured by faster gliders.

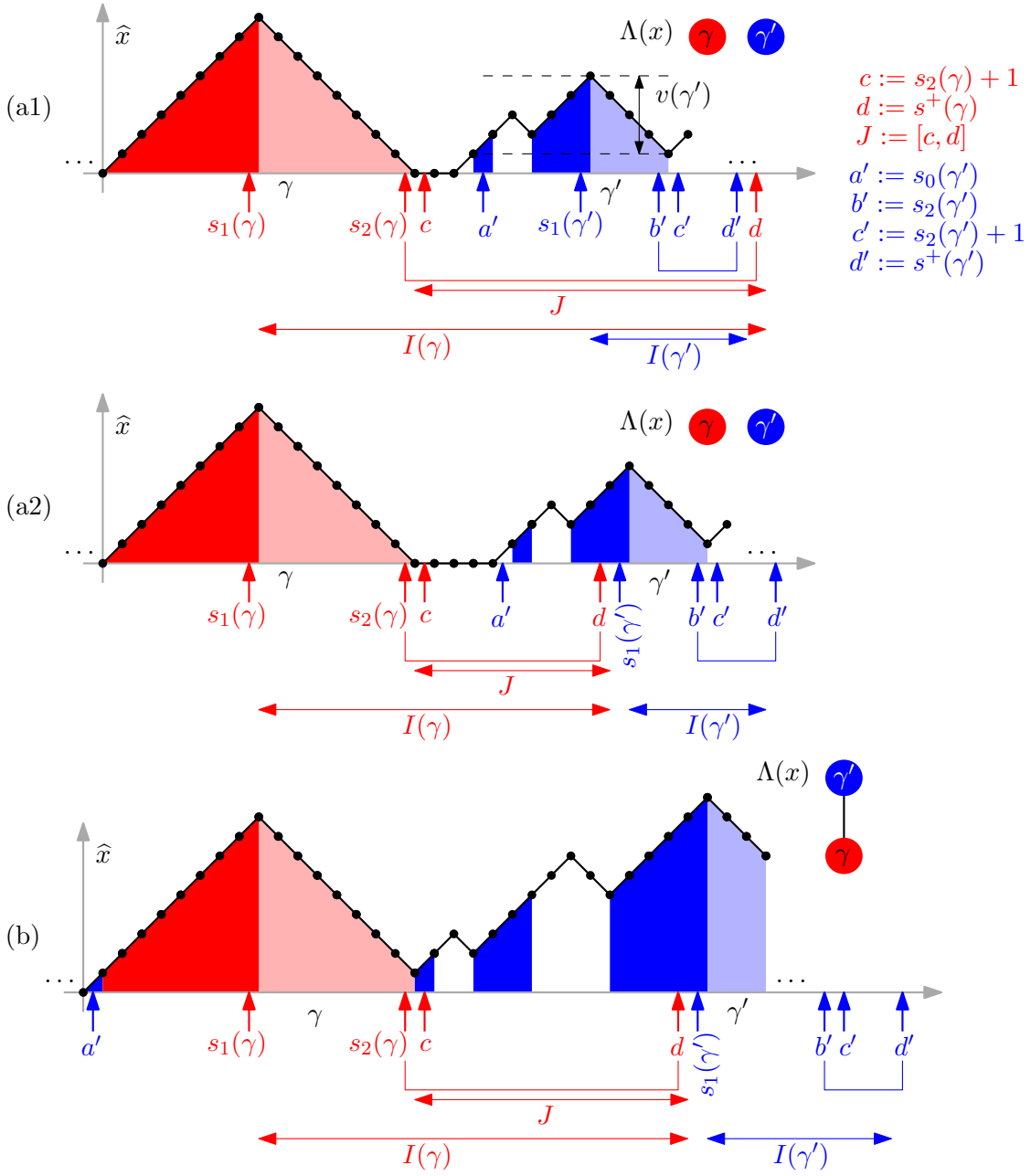


FIGURE 18. Illustration of the three cases in the proof of Lemma 18.

(iv) For any glider $\gamma' \in \Gamma_{\mathrm{f}}(x)$ such that $s_1(\gamma) < s_1(\gamma')$, we have $I(\gamma') \subseteq I(\gamma)$ if $\gamma \succ \gamma'$, and $I(\gamma') \cap I(\gamma) = \emptyset$ if $\gamma \not\succ \gamma'$, i.e., any two of the intervals defined in (17) are either disjoint or nested.

Proof. The definitions used in this proof are illustrated in Figure 18.

Let $c := s_2(\gamma) + 1$ and $d := s^+(\gamma)$, and consider the subpath of $\widehat{x}$ on the interval $J := [c, d]$. From (14) we obtain the following *Observation H:* For any hill $y = \widehat{x}_{[a,b]}$ of height h with $a \geq c$ and with the $\nearrow$-step at position $p \in [a, b]$ leading to the leftmost highest point of y, if $d > p$ then we have $d > b$ and

$$w([c, b]) < v(\gamma) - h. \tag{18}$$

Now consider any glider $\gamma' \in \Gamma_{\mathrm{f}}(x)$ with $r(\gamma') \cap J \neq \emptyset$, and define $[a', b'] := r(\gamma') = [s_0(\gamma'), s_2(\gamma')]$, $c' := s_2(\gamma') + 1 = b' + 1$, and $d' := s^+(\gamma')$. From (14) we know that

$$w([c', d']) = v(\gamma'). \tag{19}$$

Note that γ' is not a descendant of γ in $\Lambda(x)$, otherwise we would have $r(\gamma') \cap J = \emptyset$. It remains to consider two possible cases:

Case (a): γ is not a descendant of γ' in $\Lambda(x)$. In this case, we must have $a' \in J$.

Case (a1): If $b' \in J$, then we apply Observation H. Specifically, using (18) with $h = v(\gamma')$ we obtain

$$w([c, b']) < v(\gamma) - v(\gamma'), \tag{20}$$

which combined with (19) and $c' = b' + 1$ yields

$$v(\gamma) > w([c, b']) + w([c', d']) = w([c, d']). \tag{21}$$

Combining (14) and (21) shows that $d > d'$, i.e., we have $s^+(\gamma) > s^+(\gamma')$, which implies $\gamma \succ \gamma'$. We also see from (17) that $I(\gamma') \subseteq I(\gamma)$. By the definition (15), the glider γ' is contained in the set $C(\gamma)$, and hence none of the positions in $r(\gamma') = [a', b']$ are contained in the set $S(\gamma)$ by (16).

Case (a2): If $b' \notin J$, then by Observation H we have $d \leq s_1(\gamma')$. It follows that $\gamma \not\succ \gamma'$, and therefore $\gamma' \notin C(\gamma)$. We also see that $I(\gamma') \cap I(\gamma) = \emptyset$. Furthermore, all steps of γ' at positions in $S(\gamma)$ are $\nearrow$-steps, and as there are no $\rightarrow$-steps at positions in $r(\gamma')$, these positions in $S(\gamma)$ are not interleaved or followed by positions of any $\rightarrow$-steps.

Case (b): γ is a descendant of γ' in $\Lambda(x)$. By Lemma 8 the highest point of the hill $\widehat{x}_{r(\gamma)}$ is strictly lower than the highest point of the hill $\widehat{x}_{r(\gamma')}$. From (14) we conclude that $s^+(\gamma) < s_1(\gamma')$, in particular $\gamma \not\succ \gamma'$ and therefore $\gamma' \notin C(\gamma)$. We also see that $I(\gamma') \cap I(\gamma) = \emptyset$. Furthermore, all steps of γ' at positions in $S(\gamma)$ are $\nearrow$-steps, and as there are no $\rightarrow$-steps at positions in $r(\gamma')$, the set $S(\gamma)$ contains no positions of any $\rightarrow$-steps at all.

Combining the observations resulting from cases (a1), (a2) and (b), we are now in position to prove (i)-(iv).

We first obtain that for any $i \in S(\gamma)$ we have $\widehat{x}_i \in \{\nearrow, \rightarrow\}$ and the positions of any $\nearrow$-steps must be preceded by the positions of any $\rightarrow$-steps, which proves (i).

For any $\gamma' \in C(\gamma)$ we have

$$w(r(\gamma')) = 0, \tag{22}$$

and so $w([c, d]) = w([s_2(\gamma) + 1, s^+(\gamma)]) = v(\gamma)$ equals the cardinality of $S(\gamma)$ by (16), which proves (ii).

Consider a glider $\gamma' \in C(\gamma)$ and define $[a', b'] := r(\gamma') = [s_0(\gamma'), s_2(\gamma')]$. From (20) and (22) we obtain $w([c, a' - 1]) = w([s_2(\gamma) + 1, s_0(\gamma') - 1]) < v(\gamma) - v(\gamma')$. By (16) and (22), if the number of elements in $S(\gamma)$ smaller than $r(\gamma')$ equals i, then we have $i = w([s_2(\gamma)+1, s_0(\gamma')-1])$. Combining these observations yields $i < v(\gamma)-v(\gamma')$, which proves (iii). This argument also works in the special case $i = 0$, when the interval $[c, a' - 1]$ is empty and therefore $w([c, a' - 1]) = 0$.

To prove (iv), note that any glider $\gamma' \in \Gamma_{\mathrm{f}}(x)$ as in case (a1) satisfies $\gamma \succ \gamma'$ and $I(\gamma') \subseteq I(\gamma)$, whereas any glider $\gamma' \in \Gamma_{\mathrm{f}}(x)$ as in case (a2) or case (b) satisfies $\gamma \not\succ \gamma'$ and $I(\gamma') \cap I(\gamma) = \emptyset$.

This completes the proof of the lemma. □

3.10. **Movement of gliders in one time step.** We now define a bijection g between gliders in the sets $\Gamma(x)$ and $\Gamma(f(x))$, so that repeatedly applying this bijection will enable us to track the movement of each glider over time. Whereas the mapping f defined in Section 2.2 describes the changes from x to $f(x)$ on the level of 0s and 1s, the bijection g describes the changes on the level of gliders.

To define the bijection $g : \Gamma(x) \to \Gamma(f(x))$, we let $M(x) \subseteq \Gamma_{\mathrm{f}}(x)$ be the set of free gliders that are not captured by any other glider, i.e., $M(x)$

$$M(x) := \{\gamma \in \Gamma_{\mathrm{f}}(x) \mid \text{there is no } \gamma' \in \Gamma_{\mathrm{f}}(x) \text{ such that } \gamma' \succ \gamma\}. \tag{23a}$$

For any glider $\gamma =: (A, B) \in M(x)$ we consider the set $S(\gamma)$ defined in (16) and define $g(\gamma)$

$$g(\gamma) := (B, S(\gamma)), \tag{23b}$$

whereas for any other glider $\gamma \in \Gamma(x) \setminus M(x)$ we define

$$g(\gamma) := \gamma. \tag{23c}$$

In words, the gliders in $M(x)$ move forward according to (23b), i.e., the downsteps of γ in $\widehat{x}$, which are at the positions in B, become the upsteps of $g(\gamma)$ in $\widehat{f(x)}$, and the downsteps of $g(\gamma)$ in $\widehat{f(x)}$ are at the positions in $S(\gamma)$. Moreover, by (23c) none of the gliders in $\Gamma(x) \setminus M(x)$ changes its position. However, we will see that each of these gliders changes from being non-inverted to inverted, or vice versa. See Figure 19 for an example illustrating these definitions.

We need to show that the bijection g defined in (23) is well-defined, i.e., that g maps a glider from $\Gamma(x)$ to a glider in $\Gamma(f(x))$. This is established in Lemma 20 below.

To prove Lemma 20, we need the following auxiliary construction and lemma. Specifically, given the Motzkin path $\widehat{x}$, we first define another infinite Motzkin path $\varphi(\widehat{x})$. Lemma 19 below shows that actually $\varphi(\widehat{x}) = \widehat{f(x)}$, and the lemma establishes additional properties about gliders in the Motzkin paths $\widehat{x}$ and $\widehat{f(x)}$; see Figure 19. All the following definitions and computations rely on the definitions provided in Section 3.9.

The Motzkin path $\varphi(\widehat{x})$ is defined as follows: For all steps $i \in \bigcup_{\gamma \in M(x)} I(\gamma)$ we define $\varphi(\widehat{x})$

$$\varphi(\widehat{x})_i := \begin{cases} \nearrow & \text{if } \widehat{x}_i = \ \searrow, \\ \searrow & \text{if } \widehat{x}_i \in \{\rightarrow, \nearrow\}, \end{cases} \tag{24a}$$

and otherwise we define

$$\varphi(\widehat{x})_i := \ \rightarrow . \tag{24b}$$

Recall from Lemma 18 (iv) that for any two gliders $\gamma, \gamma' \in M(x)$, the intervals $I(\gamma)$ and $I(\gamma')$ are disjoint.

Lemma 19. *For any $x \in X_{n,k}$, the Motzkin path $\varphi(\widehat{x})$ defined in (24) satisfies $\varphi(\widehat{x}) = \widehat{f(x)}$. Moreover, for any glider $\gamma =: (A, B) \in M(x)$ with $v(\gamma) = h$, we have $\varphi(\widehat{x})_{I(\gamma)} = z\,y$, where $z \in D$ and $y \in D'$ are base hills in $\varphi(\widehat{x})$ and y has height h, satisfying the following conditions:*

(i) z is the complement of the 0th dent of $\widehat{x}_{r(\gamma)}$;
(ii) the ith bulge of y is the complement of the ith dent of $\widehat{x}_{r(\gamma)}$, for all $i = 1, \dots, h-2$;
(iii) the ith dent of y is the complement of $\widehat{x}_{R_i}$, where $R_i := \bigcup_{\gamma' \in C(\gamma)_i} r(\gamma')$, for all $i = 0, \dots, h-2$;
(iv) the $\nearrow$-steps and $\searrow$-steps of y not belonging to any of its bulges or dents are at the positions B or $S(\gamma)$, respectively.

In the proof of Lemma 19 we denote steps of the Motzkin path $\varphi(\widehat{x})$ by $\nearrow$, $\searrow$, or $\rightarrow$, as given by the definition (24). Once we have established that $\varphi(\widehat{x}) = \widehat{f(x)}$, it is clear that these correspond to matched 1s, matched 0s, or unmatched 0s (-) in $f(x)$, respectively.

Proof. Let $\gamma =: (A, B) \in M(x)$. We split the interval $I(\gamma) = [s_1(\gamma) + 1, s^+(\gamma)]$ into the two smaller intervals $[a, b] := [s_1(\gamma) + 1, s_2(\gamma)]$ and $[c, d] := [s_2(\gamma) + 1, s^+(\gamma)]$.

We first consider the steps of $\varphi(\widehat{x})$ in the interval $[a, b]$. If $h = 1$, then we have $\widehat{x}_{[a,b]} = \ \searrow$ and consequently

$$\varphi(\widehat{x})_{[a,b]} = \ \nearrow \tag{25}$$

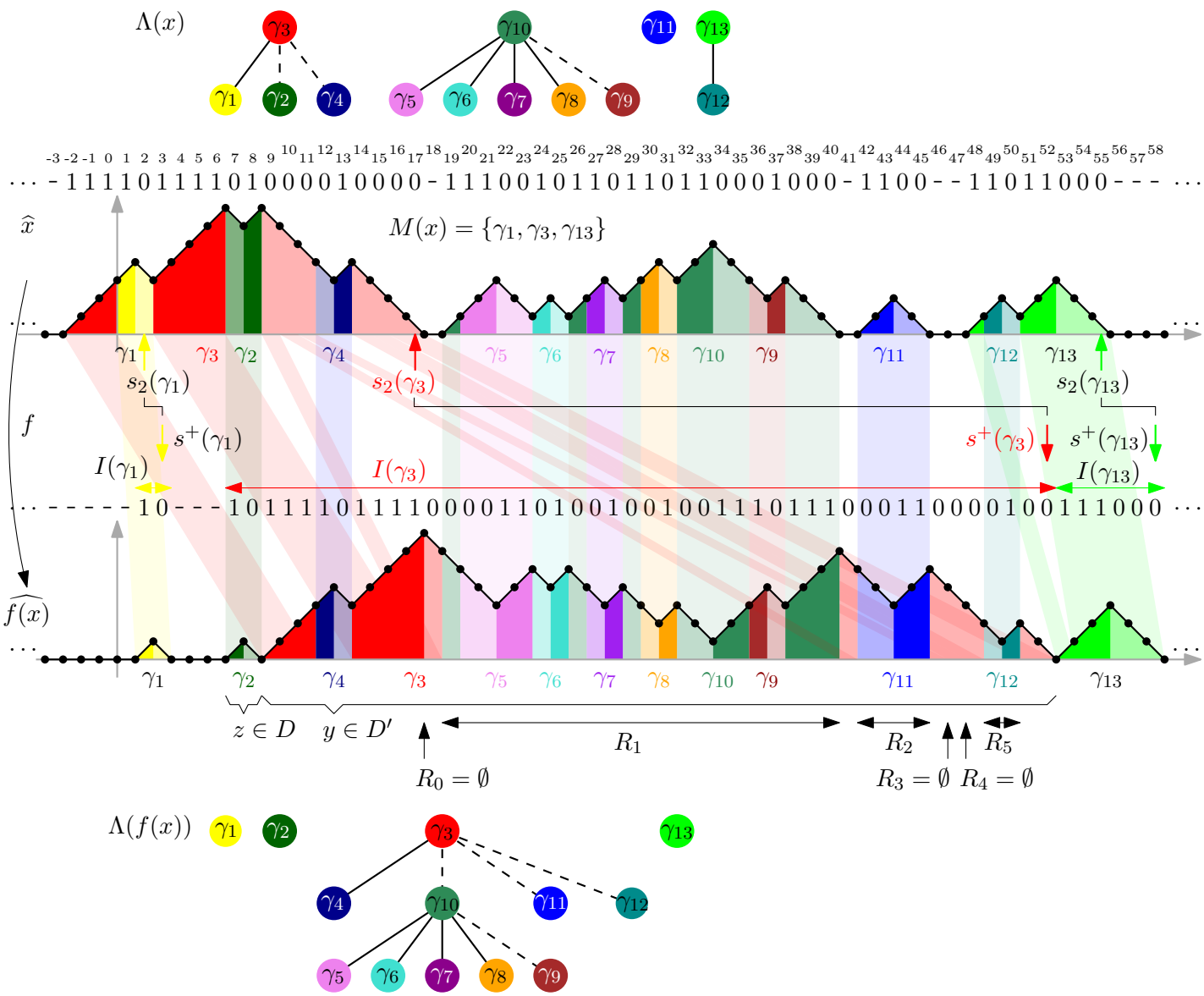


FIGURE 19. One application of f and corresponding movement of gliders, captured by the bijection g. The pairs of gliders mapped to each other under g are denoted by the same index γ_i in $\widehat{x}$ and $\widehat{f(x)}$, and connected by shaded trapezoids. Only the three gliders in $M(x) = \{\gamma_1, \gamma_3, \gamma_{13}\}$ move forward in this step, whereas all others do not change position. This figure continues the example from Figure 15.

by (24a). If $h \geq 2$, then let v'_i, $i = 0, \ldots, h-2$, be the ith dent of $\widehat{x}_{r(\gamma)}$, and let I_i be its support. As v'_0 is a valley in $\widehat{x}$, we obtain from (24a) that $z := \varphi(\widehat{x})_{I_0} = \overline{v'_0}$ is a hill, i.e., $z \in D$. Similarly, for $i = 1, \ldots, h-2$ we have that v'_i is a valley of depth at most $h-1-i$ in $\widehat{x}$ by Lemma 8, and consequently $u_i := \varphi(\widehat{x})_{I_i} = \overline{v'_i}$ is a hill of height at most $h-1-i$ in $\varphi(\widehat{x})$. The remaining steps of $\widehat{x}$ in $[a, b]$ not belonging to any of the intervals I_i, $i = 0, \ldots, h-2$, are at the positions in B (recall (3d)), and these are $v(\gamma) = h$ many $\searrow$-steps (recall (7)), so in $\varphi(\widehat{x})$ there are h many $\nearrow$-steps at the positions in B. Combining these observations shows that

$$\varphi(\widehat{x})_{[a,b]} = z \nearrow u_1 \nearrow u_2 \cdots \nearrow u_{h-2} \nearrow\nearrow \tag{26}$$

with $z \in D$ and hills $u_i \in D$ of height at most $h-1-i$ for all $i = 1, \dots, h-2$.

We now consider the steps of $\varphi(\widehat{x})$ in the interval $[c,d]$. If $h = 1$, then we have $\widehat{x}_{[c,d]} \in \{\rightarrow, \nearrow\}$ by Lemma 18 (i) and consequently

$$\varphi(\widehat{x})_{[c,d]} = \ \searrow \tag{27}$$

by (24a). If $h \geq 2$, then let $R_i := \bigcup_{\gamma' \in C(\gamma)_i} r(\gamma')$ for $i = 0, \dots, h-2$. Note that $\widehat{x}_{R_i}$ is a hill of height at most $v(\gamma) - i - 1 = h - i - 1$ in $\widehat{x}$ by Lemma 18 (iii), and consequently $v_i := \varphi(\widehat{x}_{R_i}) = \overline{\widehat{x}_{R_i}}$ is a valley of depth at most $h-i-1$ in $\varphi(\widehat{x})$. The remaining steps of $\widehat{x}$ in $[c,d]$ not belonging to any of the intervals R_i, $i = 0, \dots, h-2$, are at the positions in $S(\gamma)$, and these are $|S(\gamma)| = v(\gamma) = h$ many $\rightarrow$-steps or $\nearrow$-steps by Lemma 18 (i)+(ii), so in $\varphi(\widehat{x})$ there are h many $\searrow$-steps at these positions in $S(\gamma)$. Combining these observations shows that

$$\varphi(\widehat{x})_{[c,d]} = v_0 \searrow v_1 \searrow \cdots v_{h-2} \searrow\searrow \tag{28}$$

with valleys v_i, i.e., $\overline{v_i} \in D$, of depth at most $h-1-i$ for all $i = 0, \dots, h-2$.

If $h = 1$, then from (25) and (27) we obtain

$$\varphi(\widehat{x})_{I(\gamma)} = \ \nearrow\searrow \ = \underbrace{1\,0}_{=:y}, \tag{29}$$

i.e., the statements in the lemma are trivially satisfied with $z := \varepsilon$. If $h \geq 2$, then from (26) and (28), using the stated constraints on the height of the hills u_i and the depths of the valleys v_i we obtain

$$\varphi(\widehat{x})_{I(\gamma)} = z\ \underbrace{1\,u_1\,1\,u_2\,\cdots\,1\,u_{h-2}\,1\,1\,v_0\,0\,v_1\,0\,\cdots\,0\,v_{h-2}\,0\,0}_{=:y}, \tag{30}$$

i.e., we indeed have $z \in D$ and $y \in D'$ (recall (2) and Lemma 8). Note that the 0s and 1s in (29) and (30) denote bits that are matched to another bit inside y. Moreover, $z = \overline{v'_0}$ is the complement of the 0th dent of $\widehat{x}_{r(\gamma)}$, the ith bulge of y is $u_i = \overline{v'_i}$, which is the complement of the ith dent of $\widehat{x}_{r(\gamma)}$, and the ith dent of y is $v_i = \overline{\widehat{x}_{R_i}}$. Furthermore, the $\nearrow$-steps or $\searrow$-steps of y not belonging to any of its bulges or dents are at the positions B or $S(\gamma)$, respectively. This proves statements (i)–(iv) in the lemma.

As every $\gamma \in M(x)$ satisfies (29) or (30), and the steps of $\varphi(\widehat{x})$ not belonging to any of the intervals $I(\gamma)$, $\gamma \in M(x)$, are $\rightarrow$-steps by (24b), we see that $\varphi(\widehat{x})$ is a Motzkin path that never moves below the abscissa and all of whose $\rightarrow$-steps are on the abscissa. We also see that $\varphi(\widehat{x})_{I(\gamma)}$ is a base hill in $\varphi(\widehat{x})$ for all $\gamma \in M(x)$. As $\varphi(\widehat{x})$ has periodicity n, it follows that $\varphi(\widehat{x}) = \widehat{x'}$ for some $x' \in X_{n,k}$. By (24), the 1s in x' (corresponding to $\nearrow$-steps on the Motzkin path $\varphi(\widehat{x})$) are precisely at the positions of the matched 0s in x (corresponding to $\searrow$-steps on the Motzkin path $\widehat{x}$), so indeed we have $x' = f(x)$, i.e., $\varphi(\widehat{x}) = \widehat{f(x)}$, as claimed. □

Lemma 20. *For any $x \in X_{n,k}$, the mapping $g : \Gamma(x) \to \Gamma(f(x))$ defined in (23) is a bijection. Furthermore, for any two gliders $\gamma, \gamma' \in \Gamma(x)$ with $\gamma \sim \gamma'$ we have $s(\gamma) - s(\gamma') = s(g(\gamma)) - s(g(\gamma'))$.*

Proof. Let $\gamma =: (A, B) \in M(x)$ be as in Lemma 19.

By Lemma 19 (iv), the Motzkin path $\widehat{f(x)}$ has a base hill $y \in D'$, and the $\nearrow$-steps and $\searrow$-steps not belonging to any of its bulges or dents are at the positions B or $S(\gamma)$, respectively. Consequently, from (3d) we see that $(B, S(\gamma))$ is indeed a glider of $\Gamma(f(x))$ and hence the definition (23b) is well-defined.

Now consider the 0th dent of $\widehat{x}_{r(\gamma)}$, and let I_0 be its support. By Lemma 19 (i), $z := \widehat{f(x)}_{I_0} = \overline{\widehat{x}_{I_0}} \in D$ is base hill of the Motzkin path $\widehat{f(x)}$, so by (3) the recursion $\Gamma(z)$ yields the same results for $\widehat{x}$ and $\widehat{f(x)}$. It follows that for gliders $\gamma' \in \Gamma(x) \setminus \Gamma_{\mathrm{f}}(x)$ with $r(\gamma') \subseteq I_0$ the definition $g(\gamma') = \gamma'$ given in (23c) is well-defined, and it is a bijection between those sets of gliders.

Now consider the ith dent of $\widehat{x}_{r(\gamma)}$ for some $i = 1, \ldots, h-2$, and let I_i be its support. By Lemma 19 (ii), $u_i := \widehat{f(x)}_{I_i} = \widehat{\widehat{x}_{I_i}} \in D$ is the ith bulge of the base hill $y \in D'$ of the Motzkin path $\widehat{f(x)}$, so by (3) the recursion $\Gamma(u_i)$ yields the same results for $\widehat{x}$ and $\widehat{f(x)}$. It follows that for gliders $\gamma' \in \Gamma(x) \setminus \Gamma_{\mathrm{f}}(x)$ with $r(\gamma') \subseteq I_i$ the definition $g(\gamma') = \gamma'$ given in (23c) is a well-defined bijection between those sets of gliders.

Now consider the interval $R_i := \bigcup_{\gamma' \in C(\gamma)_i} r(\gamma')$ for some $i = 0, \ldots, h-2$. Clearly, $\widehat{x}_{R_i} \in D$ is a hill in $\widehat{x}$, either a base hill or a bulge of some base hill (recall Lemma 18 (i)). By Lemma 19 (iii), $\widehat{f(x)}_{R_i} = \widehat{\widehat{x}_{R_i}}$ is the ith dent of the base hill $y \in D'$ of the Motzkin path $\widehat{f(x)}$, so by (3) the recursion $\Gamma(\widehat{x}_{R_i})$ yields the same results for $\widehat{x}$ and $\widehat{f(x)}$. It follows that for gliders $\gamma' \in \Gamma(x) \setminus \Gamma_{\mathrm{f}}(x)$ with $r(\gamma') \subseteq R_i$ the definition $g(\gamma') = \gamma'$ given in (23c) is a well-defined bijection between those sets of gliders.

This proves the first part of the lemma. To verify the second part, consider two gliders $\gamma, \gamma' \in \Gamma(x)$ with $\gamma \sim \gamma'$. As $\widehat{x}$ has periodicity n, the set $S(\gamma)$ is obtained from $S(\gamma')$ by adding $s(\gamma) - s(\gamma')$ to all elements. As $S(\gamma)$ and $S(\gamma')$ determine the images $g(\gamma)$ and $g(\gamma')$, we obtain the desired statement.

This completes the proof of the lemma. □

The next lemma describes how the bijection g affects the trappedness relations between gliders from $\Gamma(x)$ and $\Gamma(f(x))$; see Figure 19.

Lemma 21. *Let $T(x,\gamma)$ and $C(x,\gamma)$ be the sets $T(\gamma)$ and $C(\gamma)$ of trapped and captured gliders, respectively, defined in* (13) *and* (15) *for a given bitstring $x \in X_{n,k}$. Then we have:*

(i) For any glider $\gamma \in M(x)$, we have $g(\gamma) \in \Gamma_{\mathrm{f}}(f(x))$ and $T(f(x), g(\gamma)) = \{\gamma' \cup T(x,\gamma') \mid \gamma' \in C(x,\gamma)\}$. In words, the glider $g(\gamma)$ is free and the gliders trapped by $g(\gamma)$ are precisely the free gliders captured by γ, and the gliders trapped by them. In particular, $T(x,\gamma) \cap T(f(x), g(\gamma)) = \emptyset$, i.e., none of the gliders trapped by γ is trapped by $g(\gamma)$.

(ii) For any glider $\gamma \in \Gamma(x) \setminus M(x)$ we have $T(f(x), g(\gamma)) = T(x,\gamma)$. In words, the gliders trapped by γ and $g(\gamma)$ are the same.

Proof. We first prove (ii) and then (i).

To prove (ii), let $\gamma \in \Gamma(x) \setminus M(x)$. By (23c) we have $g(\gamma) = \gamma$. Furthermore, any glider $\gamma' \in T(x,\gamma)$ is not free and therefore $\gamma' \notin M(x)$, implying that $g(\gamma') = \gamma'$ as well.

To prove (i), let $\gamma \in M(x)$. By (23b) and Lemma 19 (iv), $y := \widehat{f(x)}_{r(g(\gamma))}$ is a base hill in $\widehat{f(x)}$, and thus $g(\gamma)$ is the root of the tree in the forest $\Lambda(f(x))$ that corresponds to the recursive computation of $\Gamma(y)$. Consequently, $g(\gamma)$ is not trapped by any other glider and therefore $g(\gamma) \in \Gamma_{\mathrm{f}}(f(x))$. Furthermore, by Lemma 19 (iii), the gliders $\gamma' \in \Gamma_{\mathrm{f}}(x)$ for which $\widehat{f(x)}_{r(g(\gamma'))}$ belongs to a dent of $\widehat{f(x)}_{r(g(\gamma))}$ are precisely the gliders $C(x,\gamma)$. By part (ii) of the lemma established before, for any glider $\gamma'' \in T(x,\gamma')$, we have $r(g(\gamma'')) = r(\gamma'') \subseteq r(\gamma') = r(g(\gamma'))$, and therefore $\widehat{f(x)}_{r(g(\gamma''))}$ belongs to the same dent of $\widehat{f(x)}_{r(g(\gamma))}$ as $\widehat{f(x)}_{r(g(\gamma'))}$. □

We say that a glider $\gamma \in \Gamma(x)$ is *open*, if in the Motzkin path $\widehat{x}$, the step to the right of γ at position $s_2(\gamma) + 1$ is a $\rightarrow$-step, i.e., the corresponding bit in x is an unmatched 0. Note that if γ is open, then it must be free and therefore non-inverted. The next lemma asserts that if a glider is open, then it has moved in the preceding time step. open

Lemma 22. *Let $\gamma \in \Gamma(x)$ be such that $g(\gamma)$ is open. Then we have $\gamma \in M(x)$.*

Note that $\gamma \in M(x)$ if and only if $g(\gamma) \neq \gamma$.

Proof. For the sake of contradiction suppose that $g(\gamma) = \gamma$. In this case γ must be inverted, and by Lemma 11 the step of $\widehat{x}$ at position $s_2(\gamma)+1$ is a $\searrow$-step. This however implies that the step of $\widehat{f(x)}$ at this position is an $\nearrow$-step, contradicting the assumption that $g(\gamma)$ is open. □

3.11. **Cycle invariants.** Clearly, the bijection g defined in (23) preserves the speeds of all gliders. Together with the 'uniformity' property of g stated in Lemma 20, we obtain that the speed set $V(x)$ defined in (8) is invariant along the cycle $C(x)$.

Lemma 23. *For any $x \in X_{n,k}$ and any vertex y on the cycle $C(x)$, we have $V(x) = V(y)$. In words, the speed set is invariant along each cycle.*

Lemma 23 shows that if $V(x) \neq V(y)$, then x and y are not on the same cycle. However, if $V(x) = V(y)$ then x and y may still be on different cycles. For example, for $x = 1010\text{--} \in X_{6,2}$ and $y = 10\text{-}10\text{-} \in X_{6,2}$ we have $V(x) = V(y) = \{1, 1\}$, but in both cycles the two gliders of speed 1 are in different relative distances to each other. Furthermore, these distances do not change along the cycles, as all gliders have the same speed.

We can derive the following somewhat refined condition from Lemma 23, stated in Lemma 24 below. For this we write $V^-(x)$ for the set obtained from the multiset $V(x)$ by removing duplicates, and for each $v \in V^-(x)$ we write $\nu_v(x)$ for the number of occurrences of v in $V(x)$, i.e., $\nu_v(x)$ is the number of gliders of speed v. We can then write $V(x)$ compactly as

$V^-(x)$ $\nu_v(x)$

$$V(x) = \{v^{\nu_v(x)} \mid v \in V^-(x)\}, \tag{31}$$

i.e., we write the elements of $V^-(x)$ with their multiplicities $\nu_v(x)$ as exponents to indicate repetition. We refer to a maximal sequence of coupled gliders as a *train*, and its *size* is the number of gliders belonging to this train. For example, in Figure 14 (a), the gliders γ_2, γ_3 form a train of size 2, the gliders $\gamma_4, \gamma_5, \gamma_6$ form a train of size 3, and the gliders γ_7, γ_8 form a train of size 2, whereas γ_1 and γ_9 are both their own train of size 1 each. The train sizes formed by all equivalence classes of gliders of speed v from left to right define a cyclic composition of $\nu_v(x)$, which we denote by $z_v(x)$, and we define

train

train composition

$$Z(x) := \{v^{z_v(x)} \mid v \in V^-(x)\},$$

which we refer to as *train composition*. Note that $Z(x)$ is obtained from (31) by replacing each exponent $\nu_v(x)$ by a cyclic composition of it, i.e., by specifying how the gliders of speed v are grouped into trains. By cyclic composition we mean a partition of $\nu_v(x)$ that is ordered up to cyclic shifts, i.e., orderings that differ only by cyclic shift are considered equivalent. For the example in Figure 14, we have $z_1(x) = (1, 2, 3) = (2, 3, 1) = (3, 1, 2)$, $z_2(x) = (2)$, $z_3(x) = (1)$, $z_1(y) = (2, 1, 3) = (1, 3, 2) = (3, 2, 1)$, $z_2(y) = (2)$, and $z_3(y) = (1)$.

$Z(x)$

Lemma 24. *For any $x \in X_{n,k}$ and any vertex y on the cycle $C(x)$, we have $Z(x) = Z(y)$. In words, the train composition is invariant along each cycle.*

Proof. By Lemma 15, all gliders belonging to the same train are trapped by the same gliders. In particular, they are either all free or all trapped. Furthermore, by Lemma 16, if one of them is captured by some glider, then all of them are captured by the same glider. Furthermore, note that for two coupled gliders $\gamma, \gamma' \in M(x)$, for $\gamma =: (A, B)$ and $\gamma' =: (A', B')$ we have $S(\gamma) = A'$ and therefore $g(\gamma) = (B, A')$ and $g(\gamma') = (B', S(\gamma'))$. Consequently, the steps in $\widehat{f(x)}$ between the last step of $g(\gamma)$ and the first step of $g(\gamma')$ are steps that belong to the 0th dent of $\widehat{x}_{r(\gamma')}$ in $\widehat{x}$, i.e., these steps belong to gliders of strictly smaller speed in $\widehat{x}$ and $\widehat{f(x)}$. It follows that $g(\gamma)$ and $g(\gamma')$ are coupled. We conclude that for all gliders in a train in $\widehat{x}$, their images under g also form a train in $\widehat{f(x)}$.

To complete the proof, note that the order of gliders of the same speed in $\widehat{x}$ is the same in $\widehat{f(x)}$ under the mapping g (recall Lemma 12), so the order of trains is also the same. □

For the bitstrings x and y shown in Figure 14 (a) and (b), respectively, we have $V(x) = V(y)$, but $Z(x) \neq Z(y)$, as the train sizes of the equivalence classes of gliders of speed 1 appear in different cyclic order, so x and y lie on different cycles.

It seems that one should be able to give a complete combinatorial interpretation of the set of cycles of the factor $\mathcal{C}_{n,k}$ defined in (1) via the speed sets of gliders, their train composition, and the relative distances of the trains, but unfortunately we do not have such an interpretation. Specifically, given two vertices x and y with the same train compositions $Z(x) = Z(y)$, in general we do not have an efficient way to decide whether x and y lie on the same cycle, other than computing $f^t(x)$ for $t = 0, 1, \ldots$ and checking if $y = f^t(x)$ holds for one of them.

We can also derive the following somewhat coarser condition from Lemma 23. It uses the total number of gliders $\nu(x)$ defined in (5), which can be computed very easily as the number $d(x)$ of descents in x by Lemma 10, without invoking the somewhat intricate glider partition introduced in Section 3.4.

Lemma 25. *For any $x \in X_{n,k}$ and any vertex y on the cycle $C(x)$, we have $\nu(x) = \nu(y)$ and consequently $d(x) = d(y)$. In words, the number of descents is invariant along each cycle.*

For example, the number of descents in all bitstrings in the four cycles shown in Figure 3 (a)–(d) is 1, 1, 2, 3 respectively. As mentioned before, the number of descents equals the number of maximal substrings of 1s (or 0s), so this quantity is also a cycle invariant. As Figure 3 (d) shows, the lengths of those maximal substrings of 1s are not invariant, however (neither for 0s).

3.12. **Movement of slowest gliders.** Gliders with minimum speed play an important role in our later arguments, and we will now analyze their movement.

The next lemma can be seen as the converse of Lemma 22 for the case of gliders of minimum speed. Specifically, it asserts that if certain gliders of minimum speed move in some step, then after this step they are open.

Lemma 26. *Let $\gamma \in M(x)$ be the rightmost glider in a train of minimum speed $v(\gamma) = \min V(x)$. Then in $\widehat{f(x)}$ the glider $g(\gamma)$ is open.*

Lemma 26 is illustrated in Figure 21. Intuitively, in the step from x to $f(x)$, the glider γ moves at minimum speed, whereas all other moving gliders either move at a strictly faster speed, or they belong to another train of the same speed, i.e., a train that is separated from γ's train by a positive number of steps. This creates a 'gap' to the right of $g(\gamma)$, i.e., a step that does not belong to any other glider.

Proof. For the reader's convenience, the notations used in this proof are illustrated in Figure 20.

As $\gamma =: (A, B)$ has minimum speed, we know that γ and $g(\gamma)$ are both clean by Lemma 14. Consequently, the set B is an interval of length $v(\gamma)$, and $\widehat{x}$ has $v(\gamma)$ many $\searrow$-steps at these positions. Similarly, the set $S(\gamma)$ is an interval of length $v(\gamma)$, which by Lemma 18 (i) consists of some number a of $\rightarrow$-steps plus some number $b := v(\gamma) - a$ of $\nearrow$-steps. Consider the glider $\gamma' \in M(x)$ for which $I(\gamma')$ is to the right of $I(\gamma) = B \cup S(\gamma)$. We claim that the $\nearrow$-step of γ' at position $s_1(\gamma')$ is not contained in the set $S(\gamma)$. From this it follows that $I(\gamma)$ is separated from $I(\gamma')$ by at least one step, and by (24b) all these steps between the two intervals are $\rightarrow$-steps in $\widehat{f(x)}$. To prove this claim we consider two cases.

Case (a): $\widehat{x}_{r(\gamma)}$ is a base hill; see Figure 20 (a). As γ' has speed $v(\gamma') \geq v(\gamma)$ and precisely $v(\gamma')$ many $\nearrow$-steps, at most $b = v(\gamma) - a \leq v(\gamma) \leq v(\gamma')$ of these $\nearrow$-steps belong to $S(\gamma)$. One

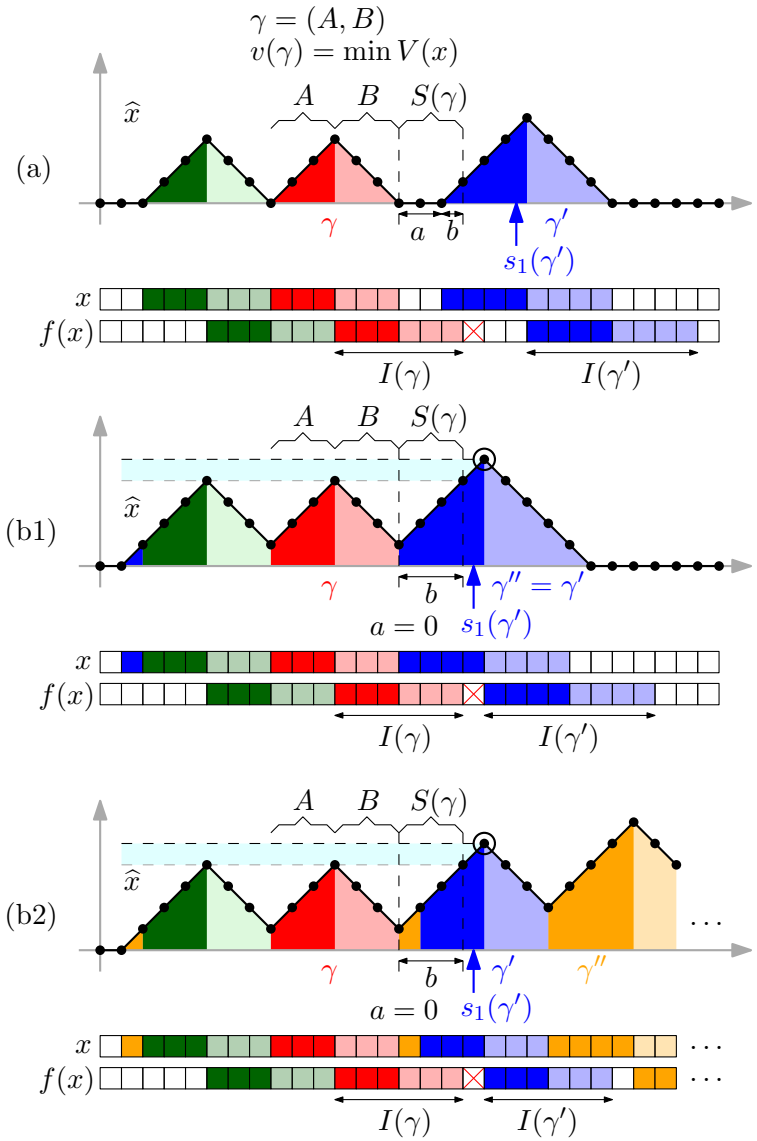


FIGURE 20. Illustration of the proof of Lemma 26.

of these inequalities is strict either because $a > 0$ or because γ is the rightmost glider in its train and therefore $v(\gamma) < v(\gamma')$. It follows that $b < v(\gamma')$ and therefore $s_1(\gamma') \notin S(\gamma)$.

Case (b): $\widehat{x}_{r(\gamma)}$ is not a base hill. Then let $\gamma'' \in \Gamma(x)$ be the parent of γ in the forest $\Lambda(x)$ and define $y := \widehat{x}_{r(\gamma'')} \in D'$. Note that $\widehat{x}_{r(\gamma)}$ belongs to a bulge of y and we have $a = 0$. If $\widehat{x}_{r(\gamma)}$ belongs to the rightmost nonempty bulge of y, then we have $\gamma'' = \gamma'$, and then the $\nearrow$-step of γ' at position $s_1(\gamma')$ is higher than the highest point of $\widehat{x}_{r(\gamma)}$, so $s_1(\gamma') \notin S(\gamma)$; see Figure 20 (b1). Otherwise $\widehat{x}_{r(\gamma')}$ is the initial part of the next nonempty bulge of y to the right of $\widehat{x}_{r(\gamma)}$; see Figure 20 (b2). By the assumption that $v(\gamma') \geq v(\gamma)$ and that γ is the rightmost glider in its train, the $\nearrow$-step of γ' at position $s_1(\gamma')$ is higher than the highest point of $\widehat{x}_{r(\gamma)}$, so again we have $s_1(\gamma') \notin S(\gamma)$. This completes the proof. □

For any $x \in X_{n,k}$ and the rightmost glider $\gamma \in \Gamma(x)$ in a train of minimum speed $v(\gamma) = \min V(x)$, we let $g' : \Gamma(f^{-1}(x)) \to \Gamma(x)$ be the bijection defined in (23) for the string $f^{-1}(x)$, and we let $\gamma' \in \Gamma(f^{-1}(x))$ be such that $g'(\gamma') = \gamma$. We define $h_\gamma(x) \in X_{n,k}$ as the bitstring obtained $h_\gamma(x)$

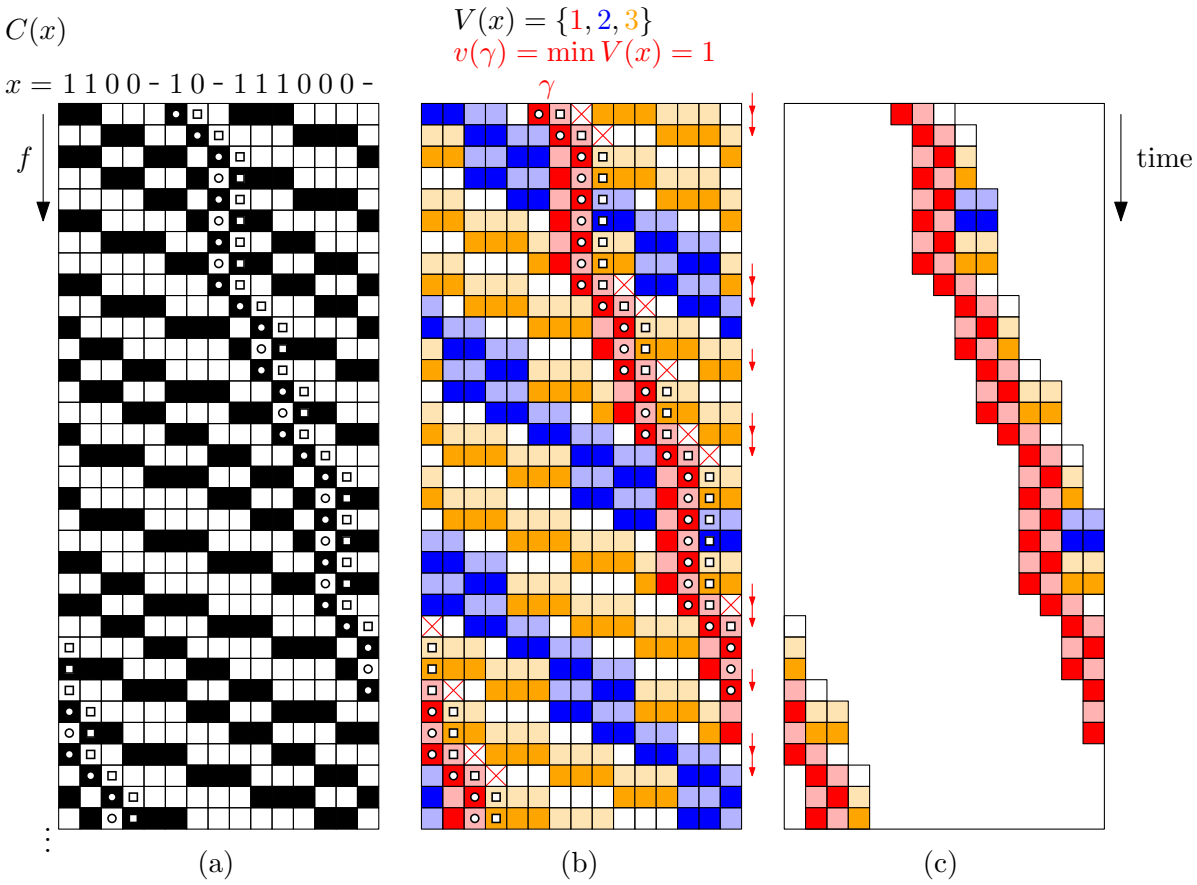


FIGURE 21. Illustration of Lemmas 26 and 31. The glider γ has minimum speed $v(\gamma) = \min V(x) = 1$. In (b), the steps in which γ moves are marked by little arrows on the right, and the corresponding unmatched 0-bit guaranteed by the lemmas is marked by a cross. Furthermore, transposing the two marked bits yields the cycle $C(h_\gamma(x))$ shown in Figure 22 for the bitstring $h_\gamma(x)$ that is obtained from x by shifting the glider γ one position to the right. Part (c) shows only the steps in (b) that are different from the ones in Figure 22 (b).

from x by transposing the two bits at positions $s_1(\gamma') + 1$ and $s_2(\gamma') + 1$; see Figures 21 and 22. We emphasize that the transposed positions are computed based on $f^{-1}(x)$ (specifically, based on $\gamma' \in \Gamma(f^{-1}(x))$), but the bits are transposed in x. In fact, we will see that the transposed bits in x are always a (matched) 1 and a matched 0. However, when applying parenthesis matching to the modified string $h_\gamma(x)$, the transposed 0 may be unmatched, and another 0-bit may instead be matched. Observe that if $\gamma' \in M(f^{-1}(x))$, then by Lemma 26 the glider γ is open in $\widehat{x}$, i.e., $\widehat{x}_{r(\gamma)} = 1^{v(\gamma)}0^{v(\gamma)}$ is a base hill followed by a $\rightarrow$-step, and then the gliders in $\Gamma(h_\gamma(x))$ are obtained from the gliders of $\Gamma(x)$ by shifting the gliders in the equivalence class $[\gamma]$ one step to the right, while leaving all other gliders unchanged. In Figure 21 (b), these steps are marked by little arrows on the right. In other cases, steps of more than one glider may change in $\Gamma(h_\gamma(x))$ compared to $\Gamma(x)$, and in particular the position of γ may change by more than $+1$; see Figures 21 (c) and 22 (c). The next lemma shows that the cycle $C(h_\gamma(x))$ is obtained from the cycle $C(x)$ by shifting the glider γ and its images under repeated applications of g one step to the right. As mentioned before, this intuition is literally true only for steps in $C(x)$ in which

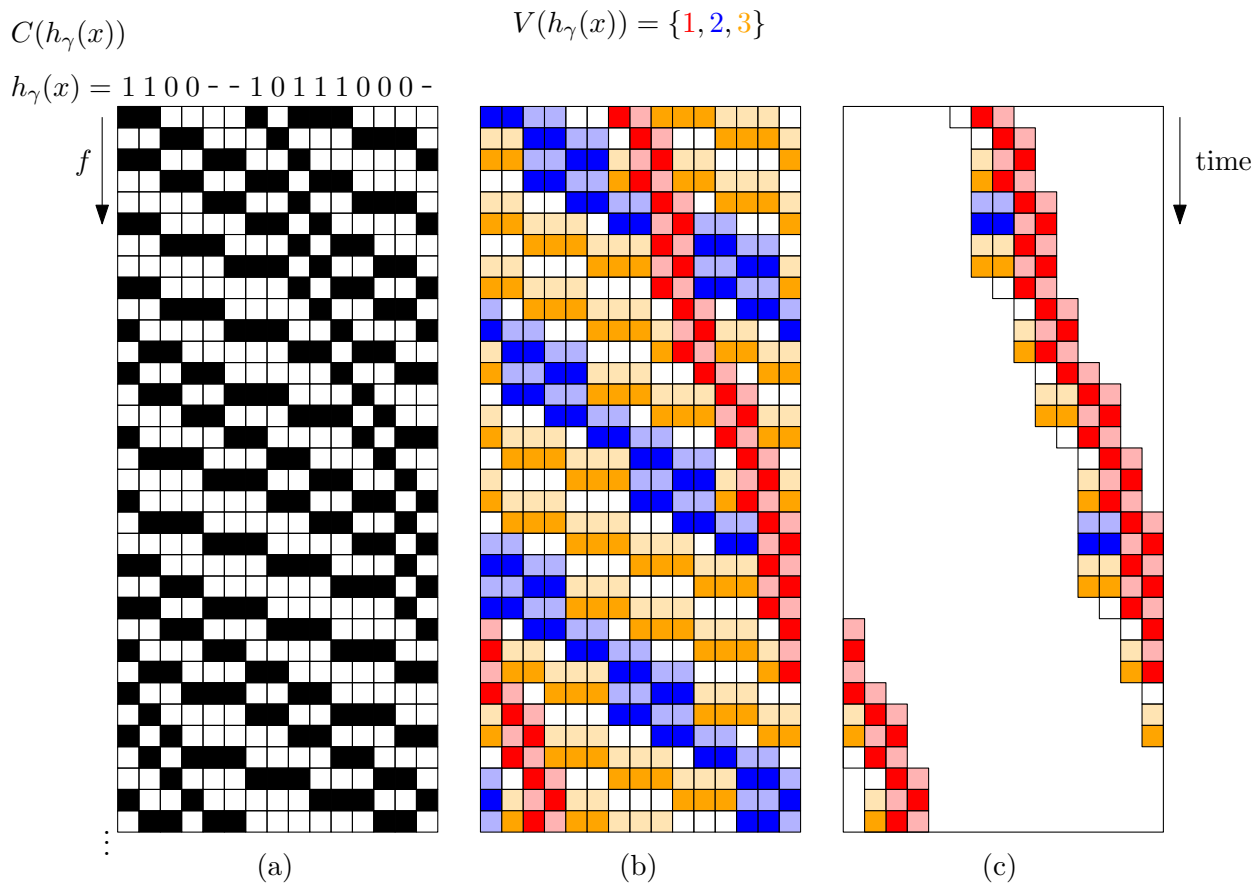


FIGURE 22. Illustration of Lemma 27. This figure continues Figure 21. Part (c) shows only the steps in (b) that are different from the ones in Figure 21 (b). To fully appreciate the two figures, the reader is encouraged to print both pages in color and alternately look at them like in a flip-book. Alternatively, the reader may want to flip back and forth between both pages on a screen reader, with fixed alignment of the page boundaries.

the glider has moved. Nonetheless, it will be the case that $V(x) = V(h_\gamma(x))$, i.e., the speed sets of gliders in the cycles $C(x)$ and $C(h_\gamma(x))$ are identical; recall Lemma 23.

Lemma 27. *We have* $f(h_\gamma(x)) = h_{g(\gamma)}(f(x))$.

Proof. In the proof we also consider the glider $\gamma' \in \Gamma(f^{-1}(x))$ defined before the lemma. Throughout this proof, we will repeatedly use that γ and its images under g are clean by Lemma 14. Let $(A, B) := \gamma$, and define the abbreviations $a := s_0(\gamma) = \min A$, $b := s_1(\gamma) + 1 = \max A + 1 = \min B$, and $c := s_2(\gamma) + 1 = \max B + 1$.

Case (a): We first consider the case that $s(\gamma) > s(\gamma')$; see Figure 23 (a). Clearly, we have

$$s_1(\gamma') + 1 = a \quad \text{and} \quad s_2(\gamma') + 1 = b.$$

Applying the definition of h_γ, we see that $h_\gamma(x)$ is obtained from x by transposing the bits at positions a and b.

In x, the 1-bits and 0-bits at the positions in A and B, respectively, are matched to each other, i.e., we have $A \subseteq \mu_1(x)$ and $B \subseteq \mu_0(x)$. Furthermore, by Lemma 26 the bit at position c

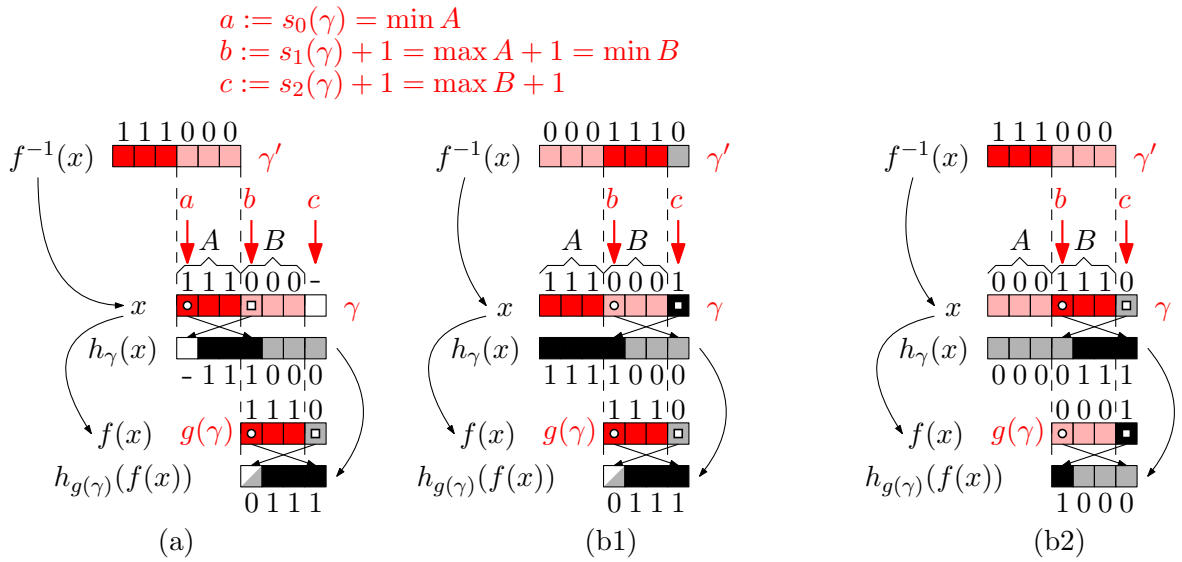


FIGURE 23. Illustration of the proof of Lemma 27. For the bits in $h_\gamma(x)$ and $h_{g(\gamma)}(f(x))$, no membership to certain gliders is specified, so they are drawn black (1-bits) and gray (matched 0-bits). The 0-bits corresponding to half-white/half-gray cells may be matched or unmatched.

is an unmatched 0, i.e., we have $c \notin \mu_0(x)$. Combining these two observations we obtain

$$\mu_0(h_\gamma(x)) = \mu_0(x) - b + c. \tag{32}$$

As $b \in \mu_0(x)$, there is a 1-bit at position b in $f(x)$. Furthermore, from $c \notin \mu_0(x)$ we see that there is a 0-bit at position c in $f(x)$. These two bits are swapped in $f(x)$ to obtain $h_{g(\gamma)}(f(x))$, and consequently we have

$$\mu_1(h_{g(\gamma)}(f(x))) = \mu_1(f(x)) - b + c. \tag{33}$$

Combining (32) and (33) proves that $f(h_\gamma(x)) = h_{g(\gamma)}(f(x))$.

Case (b): We now consider the case that $s(\gamma) = s(\gamma')$. Clearly, we have

$$s_1(\gamma') + 1 = s_1(\gamma) + 1 = b \quad \text{and} \quad s_2(\gamma') + 1 = s_2(\gamma) + 1 = c,$$

i.e., $h_\gamma(x)$ is obtained from x by transposing the bits at positions b and c.

We distinguish the subcases whether γ is inverted or non-inverted.

Case (b1): γ is non-inverted; see Figure 23 (b1). In this case γ' is inverted, and by Lemma 11 we have $c \in \mu_0(f^{-1}(x))$ and therefore $c \in \mu_1(x)$. Furthermore, in x the 1-bits and 0-bits at the positions in A and B, respectively, are matched to each other, i.e., we have $A \subseteq \mu_1(x)$ and $B \subseteq \mu_0(x)$. Combining these two observations we obtain

$$\mu_0(h_\gamma(x)) = \mu_0(x) - b + c. \tag{34}$$

As $b \in \mu_0(x)$, there is a 1-bit at position b in $f(x)$. Furthermore, as $c \in \mu_1(x)$ there is a 0-bit at position c in $f(x)$. These two bits are swapped in $f(x)$ to obtain $h_{g(\gamma)}(f(x))$, and consequently we have

$$\mu_1(h_{g(\gamma)}(f(x))) = \mu_1(f(x)) - b + c. \tag{35}$$

Combining (34) and (35) proves that $f(h_\gamma(x)) = h_{g(\gamma)}(f(x))$.

Case (b2): γ is inverted; see Figure 23 (b2). By Lemma 11 we have $c \in \mu_0(x)$. Note also that the valley $\widehat{x}_{r(\gamma)}$ does not touch the abscissa, as otherwise A and B would not belong to the same glider. Combining these two observations we obtain

$$\mu_0(h_\gamma(x)) = \mu_0(x) - c + b. \tag{36}$$

As $b \in \mu_1(x)$, there is a 0-bit at position b in $f(x)$. Furthermore, as $c \in \mu_0(x)$ there is a 1-bit at position c in $f(x)$. These two bits are swapped in $f(x)$ to obtain $h_{g(\gamma)}(f(x))$, and consequently we have

$$\mu_1\big(h_{g(\gamma)}(f(x))\big) = \mu_1(f(x)) - c + b. \tag{37}$$

Combining (36) and (37) proves that $f(h_\gamma(x)) = h_{g(\gamma)}(f(x))$.

This completes the proof of the lemma. □

3.13. **Equations of motion.** The main result of this section, Proposition 28 below, describes equations of motion that capture the movement of gliders over time, including their interactions.

For any $x \in X_{n,k}$ we define $x^t := f^t(x)$ for all $t \geq 0$, and we refer to the parameter t as *time*. x^t
We let g^t be the bijection defined in (23) between the sets $\Gamma(x^{t-1})$ and $\Gamma(x^t)$. Furthermore, for any $\gamma \in \Gamma(x) = \Gamma(x^0)$ we define $\gamma^0 := \gamma$ and $\gamma^t := g^t(\gamma^{t-1})$ for $t \geq 1$. γ^t

To describe the movement of gliders over time correctly, we need to take into account that faster gliders may overtake slower gliders; see Figure 24. An overtaking happens in two steps, namely first the slower glider is trapped by the faster glider (see the step $t-1 \to t$ in the figure), and later the slower glider is released again, i.e., it is not trapped anymore by the faster glider (see the step $t \to t+1$ in the figure). Note that these two steps need not happen directly consecutively, as in between both gliders may get trapped temporarily by an even faster glider.

Formally, we consider the equivalence classes of gliders defined in (4), namely gliders that differ by multiples of n. We say that $[\gamma']$ *gets trapped by* $[\gamma]$ *in step* t, if there are representatives $\widetilde{\gamma}' \in [\gamma']$ and $\widetilde{\gamma} \in [\gamma]$ such that $\widetilde{\gamma}'^{t-1}$ is not trapped by $\widetilde{\gamma}^{t-1}$ in x^{t-1} and $\widetilde{\gamma}'^{t}$ is trapped by $\widetilde{\gamma}^{t}$ in x^t. Similarly, we say that $[\gamma']$ *gets released by* $[\gamma]$ *in step* t, if there are representatives $\widetilde{\gamma}' \in [\gamma']$ and $\widetilde{\gamma} \in [\gamma]$ such that $\widetilde{\gamma}'^{t-1}$ is trapped by $\widetilde{\gamma}^{t-1}$ in x^{t-1} and $\widetilde{\gamma}'^{t}$ is not trapped by $\widetilde{\gamma}^{t}$ in x^t. get trapped/released

To track the trapped/released events over time we introduce half-integral parameters $\Delta c^t_{[\gamma'],[\gamma]}$ and $c^t_{[\gamma'],[\gamma]}$, defined for any two equivalence classes $[\gamma], [\gamma'] \in \Gamma(x)/\!\sim$. Specifically, we define $c^t_{[\gamma'],[\gamma]}$

$$\Delta c^t_{[\gamma'],[\gamma]} := \begin{cases} \frac{1}{2} & [\gamma'] \text{ gets trapped by } [\gamma] \text{ in step } t, \\ \frac{1}{2} & [\gamma'] \text{ gets released by } [\gamma] \text{ in step } t, \\ 0 & \text{otherwise,} \end{cases} \tag{38a}$$

for $t \geq 1$. Based on this we define

$$c^t_{[\gamma'],[\gamma]} := \begin{cases} 0 & \text{if } t = 0, \\ c^{t-1}_{[\gamma'],[\gamma]} + \Delta c^t_{[\gamma'],[\gamma]} & \text{if } t \geq 1. \end{cases} \tag{38b}$$

Figure 25 shows the evolution of these counters over several time steps for an example with four gliders.

We are now in position to formulate equations of motion that describe the movement of each glider as a function of time t. These equations involve the position $s(\gamma)$ and the speed $v(\gamma)$ of a glider γ introduced in Section 3.6, and the counters defined in (38).

Proposition 28. *For any $x \in X_{n,k}$ and $\gamma \in \Gamma(x)$, the position of the glider γ at time t is given by*

$$s(\gamma^t) = s(\gamma^0) + v(\gamma) \cdot t + \sum_{[\gamma'] \in \Gamma(x)/\!\sim} 2v(\gamma') c^t_{[\gamma'],[\gamma]} - \sum_{[\gamma'] \in \Gamma(x)/\!\sim} 2v(\gamma) c^t_{[\gamma],[\gamma']}. \tag{39}$$

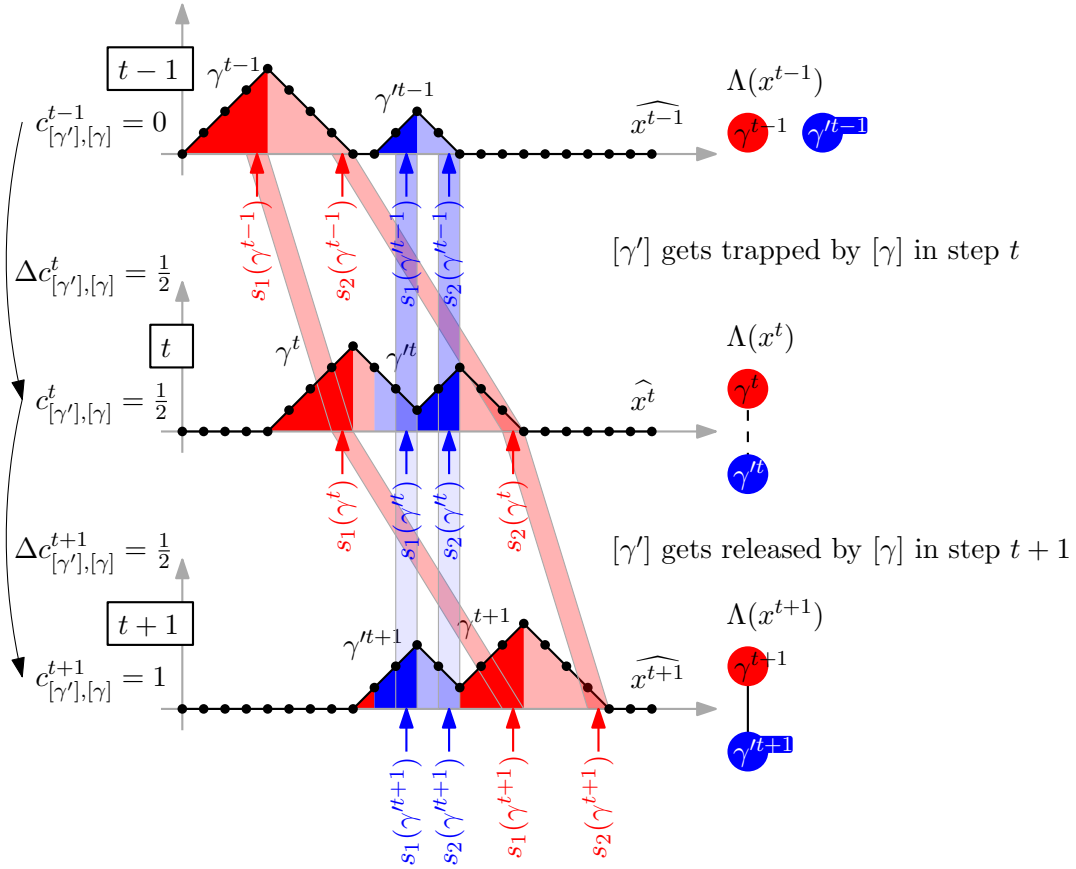


FIGURE 24. Illustration of the overtaking counter $c^t_{[\gamma'],[\gamma]}$.

Note that if γ' is trapped by γ, then $v(\gamma') < v(\gamma)$ by Lemma 12. Consequently, we have $c^t_{[\gamma'],[\gamma]} = 0$ for all $t \geq 0$ if $v(\gamma') \geq v(\gamma)$. It follows that the only nonzero contributions to the first or second sum in (39) come from gliders with $v(\gamma') < v(\gamma)$ or with $v(\gamma') > v(\gamma)$, respectively.

The equation (39) has the following physical interpretation: The first part of the equation $s(\gamma^t) = s(\gamma^0) + v(\gamma) \cdot t$ captures that the motion of γ is uniform with speed $v(\gamma)$. This is all that happens if γ never interacts with any other gliders, which occurs precisely if all gliders have the same speed. Now let us discuss the additional terms in (39) that make the motion non-uniform, in case gliders of different speeds are present. The first sum on the right hand side of (39), which has a positive sign, corresponds to a boost whenever a slower glider $[\gamma']$, i.e., one with $v(\gamma') < v(\gamma)$, gets trapped by $[\gamma]$. The second sum on the right hand side of (39), which has a negative sign, corresponds to a delay whenever $[\gamma]$ gets trapped by a faster glider $[\gamma']$, i.e., one with $v(\gamma') > v(\gamma)$. Note that when $[\gamma']$ gets trapped or released by $[\gamma]$ in step t, then we see a change of $+2v(\gamma')\Delta c^t_{[\gamma'],[\gamma]} = +2v(\gamma') \cdot \frac{1}{2} = +v(\gamma')$ (recall (38a)) in the equation for γ, and a change of $-2v(\gamma')\Delta c^t_{[\gamma'],[\gamma]} = -v(\gamma')$ in the equation for γ', i.e., two terms with the same absolute value but opposite signs. This can be seen as 'energy conservation' in the system of gliders. For the slower glider γ', the uniform change in position by $v(\gamma')$ and the delay term $-v(\gamma')$ cancel each other out, so effectively it does not change position in the two steps where it gets trapped and released; see Figure 24. On the other hand, the total boost received by the faster glider γ in these two steps is $2v(\gamma')$, which is twice the speed of the slower glider, or equivalently, the total number of its steps in the Motzkin path (number of $\nearrow$-steps plus $\searrow$-steps; recall (7)).

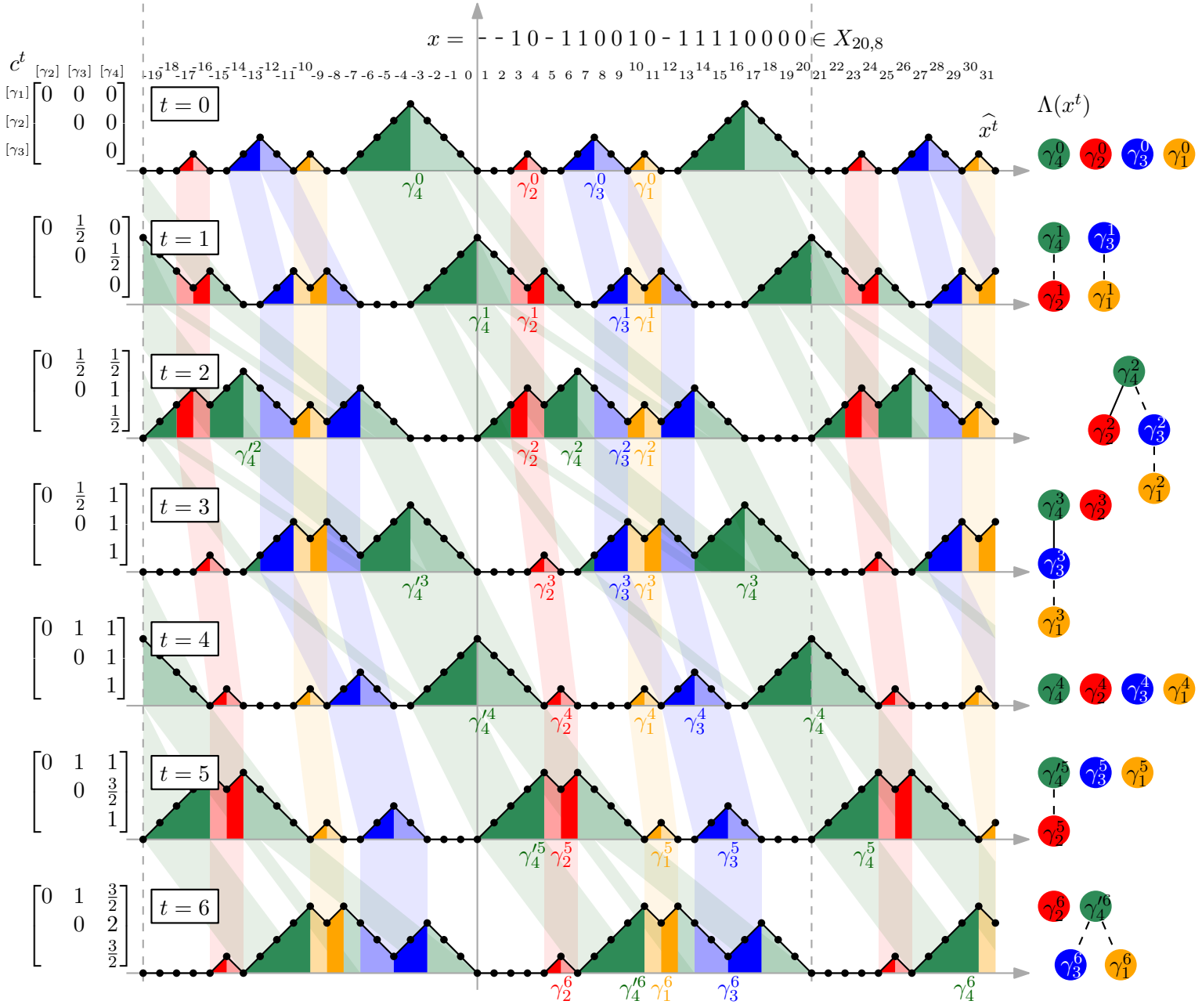


FIGURE 25. Illustration of movement of gliders over time.

Proof. We calculate the change in position of a glider in the step from $t-1$ to t. To prove (39) we will show that this change equals

$$\begin{aligned}\Delta s(\gamma^t) &:= s(\gamma^t) - s(\gamma^{t-1}) \\ &= v(\gamma) + \sum_{[\gamma']\in\Gamma(x)/\sim} 2v(\gamma')\Delta c^t_{[\gamma'],[\gamma]} - \sum_{[\gamma']\in\Gamma(x)/\sim} 2v(\gamma)\Delta c^t_{[\gamma],[\gamma']}. \end{aligned} \tag{40}$$

We first consider a glider $\gamma^{t-1} \in M(x^{t-1})$. Note that γ^{t-1} is free by assumption, and γ^t is free by Lemma 21 (i), and therefore the second sum on the right hand side of (40) equals 0.

Recall from (7) that for any glider $\gamma' = (A, B) \in \Gamma(x)$, we have $v(\gamma') = |A| = |B|$, and therefore the total number of its steps on the Motzkin path is $2v(\gamma') = 2|A| = 2|B|$.

Using this simple observation, the definitions (6a), as well as (23b), Lemma 18 (ii) and Lemma 21 (i), we get

$$\Delta s_2(\gamma^t) := s_2(\gamma^t) - s_2(\gamma^{t-1}) = v(\gamma) + \sum_{\gamma' \in T(x^t,\gamma^t)} 2v(\gamma'),$$
$$\Delta s_1(\gamma^t) := s_1(\gamma^t) - s_1(\gamma^{t-1}) = v(\gamma) + \sum_{\gamma' \in T(x^{t-1},\gamma^{t-1})} 2v(\gamma');$$

see also Figure 24. Combining these two equations via (6b) we obtain

$$\Delta s(\gamma^t) = \frac{1}{2}(\Delta s_1(\gamma^t) + \Delta s_2(\gamma^t)) = v(\gamma) + \sum_{\gamma' \in T(x^t,\gamma^t)} v(\gamma') + \sum_{\gamma' \in T(x^{t-1},\gamma^{t-1})} v(\gamma'). \tag{41}$$

We consider the first sum on the right hand side of (41). By Lemma 21 (i), $T(x^t,\gamma^t) = \{\gamma' \cup T(x^{t-1},\gamma') \mid \gamma' \in C(x^{t-1},\gamma^{t-1})\}$, and these are precisely the gliders $\gamma' \in \Gamma(x)$ for which $[\gamma']$ gets trapped by $[\gamma]$ in step t, i.e., we have $\Delta c^t_{[\gamma'],[\gamma]} = \frac{1}{2}$ by the first case in (38a) and therefore $v(\gamma') = 2v(\gamma')\Delta c^t_{[\gamma'],[\gamma]}$ for such gliders. Now consider the second sum on the right hand side of (41). By Lemma 21 (i), none of the gliders in $T(x^{t-1},\gamma^{t-1})$ are trapped by γ^t in x^t, so these are precisely the gliders $\gamma' \in \Gamma(x)$ for which $[\gamma']$ gets released by $[\gamma]$ in step t, i.e., we have $\Delta c^t_{[\gamma'],[\gamma]} = \frac{1}{2}$ by the second case in (38a) and therefore $v(\gamma') = 2v(\gamma')\Delta c^t_{[\gamma'],[\gamma]}$.

This shows that the right hand side of (41) equals (40), which completes the proof for gliders $\gamma^{t-1} \in M(x^{t-1})$.

It remains to consider a glider $\gamma^{t-1} \notin M(x^{t-1})$, i.e., we have $\gamma^t = \gamma^{t-1}$. By Lemma 21 (ii) the first sum on the right hand side of (40) equals 0. Furthermore, by Lemma 21 there is a unique glider $\gamma' \in \Gamma(x)$ such that γ^{t-1} is trapped by γ'^{t-1} and $\gamma^t = \gamma^{t-1}$ is not trapped by γ'^t, or γ^{t-1} is not trapped by γ'^{t-1} and γ^t is trapped by γ'^t. It follows that we have $\Delta c^t_{[\gamma],[\gamma']} = \frac{1}{2}$ for this unique glider γ' and therefore $\Delta s(\gamma^t) = v(\gamma) - 2v(\gamma)\Delta c^t_{[\gamma],[\gamma']} = v(\gamma) - 2v(\gamma)\frac{1}{2} = 0$, as desired. □

3.14. **Analyzing the equations.** In this section, we analyze the equations of motion derived in the previous section, by showing that the corresponding coefficient matrix is non-singular.

For any bitstring $x \in X_{n,k}$, we consider the cycle $C(x)$ defined in (1a). As the length of the cycle is finite, there is a minimum number $T > 0$, such that $x^T = x^0$ and within T time steps from x^0 to x^T every glider $\gamma \in \Gamma(x)$ has traveled an integral multiple of n many steps. In other words, after T steps, in x^T a glider γ' from the equivalence class $[\gamma^T]$ satisfies $\gamma' = \gamma^0$. Note that T is not necessarily the length of the cycle, but a multiple of it. For example, for $x = 100100 \in X_{6,2}$ we have $C(x) = (100100, 010010, 001001)$, i.e., the cycle has length 3, but we have $T = 6$, as it takes 6 time steps until the gliders of speed 1 have traveled $n = 6$ steps. T

We consider the equivalence classes of gliders $\Gamma(x)/\!\sim$, and we label one representative from each class by γ_i, $i = 1, \ldots, \nu$, $\nu = \nu(x)$, in non-decreasing order of their speeds, i.e., for $v_i := v(\gamma_i)$ we have $v_1 \le v_2 \le \cdots \le v_\nu$. Furthermore, for integers $1 \le i \le j \le \nu$ we define $c_{i,j} := c^T_{[\gamma_i],[\gamma_j]}$. γ_i, v_i $c_{i,j}$

The definition of T gives rise to the system of equations

$$s(\gamma_i^T) - s(\gamma_i^0) = (c_1 + c_{1,i})n, \quad \text{for } i = 1, \ldots, \nu, \tag{42}$$

for integers $c_1, c_{1,2}, c_{1,3}, \ldots, c_{1,\nu}$. Furthermore, by the definition of T we also have

$$c_{i,j} = c_{i,k} + c_{k,j} \tag{43}$$

for any $i \le k \le j$. Applying (43) for $i = 1$ and changing the names of indices gives in particular

$$c_{i,j} = c_{1,j} - c_{1,i}. \tag{44}$$

Combining the equations of motion (39) guaranteed by Proposition 28 and (42) yields

$$v_i T + \sum_{1\le j<i} 2v_j c_{j,i} - \sum_{i<j\le \nu} 2v_i c_{i,j} = (c_1 + c_{1,i})n, \quad \text{for } i = 1, \dots, \nu. \tag{45}$$

Eliminating from (45) all coefficients $c_{i,j}$ with $i > 1$ with the help of (44) we obtain the linear system

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} c_1 n = \underbrace{\begin{bmatrix} v_1 & -2v_1 & -2v_1 & -2v_1 & \cdots & -2v_1 \\ v_2 & V_2-2v_2-n & -2v_2 & -2v_2 & \cdots & -2v_2 \\ v_3 & -2v_2 & V_3-2v_3-n & -2v_3 & \cdots & -2v_3 \\ v_4 & -2v_2 & -2v_3 & V_4-2v_4-n & \cdots & -2v_4 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ v_\nu & -2v_2 & -2v_3 & -2v_4 & \cdots & V_\nu-2v_\nu-n \end{bmatrix}}_{=:M} \begin{bmatrix} T \\ c_{1,2} \\ c_{1,3} \\ c_{1,4} \\ \vdots \\ c_{1,\nu} \end{bmatrix}, \tag{46}$$

where

$$V_i := \sum_{j=1}^{\nu} 2v_{\min\{i,j\}}, \quad \text{for } i = 2, \dots, \nu.$$

These are ν homogeneous equations in the $\nu + 1$ integral unknowns $T, c_1, c_{1,2}, \dots, c_{1,\nu}$.

Lemma 29. *The coefficient matrix M of the linear system* (46) *satisfies*

$$\det M = (-1)^{\nu-1} v_1 \prod_{i=2}^{\nu} (n - V_i) \neq 0. \tag{47}$$

Proof. Adding twice the first column of M to every other column creates a matrix that has 0s above the diagonal, and with ith diagonal entry equal to $-(n-V_i)$ for all $i = 2, \dots, \nu$. Multiplying those values on the diagonal yields the claimed determinant. Note that $V_i \le \sum_{j=1}^{\nu} 2v_i = 2k$ (recall (9)), and therefore $n - V_i \ge n - 2k > 0$, implying that the product in (47) is nonzero. □

With similar tricks we could evaluate all cofactors of M, and thus obtain an explicit solution of the system (46). However, this is not needed for the rest of this paper, and so we omit these calculations.

3.15. **Every glider moves eventually.** As mentioned before, gliders with minimum speed play an important role in our arguments. Clearly, a glider of maximum speed moves in every time step, as it cannot be trapped by any other glider (recall Lemma 12). On the other hand, a glider of minimum speed may be trapped, and hence does not move, for several consecutive time steps by various faster gliders; see Figures 21 and 25. In fact, it is easy to construct examples where a glider is trapped for an arbitrarily long interval of time (by adding gliders of increasing speed on the left, which increases n and k). Nevertheless, the next lemma asserts that gliders of minimum speed cannot be trapped indefinitely, but eventually will move forward. This property is true more generally for gliders of any speed.

Lemma 30. *For any glider $\gamma \in \Gamma(x)$, there is a $t > 0$ such that $s(\gamma^t) - s(\gamma^0) > 0$.*

Maybe surprisingly, we do not have a purely combinatorial proof for this lemma. Instead, our proof relies on the earlier determinant computation.

Proof. It suffices to prove the lemma for a glider γ of minimum speed $v(\gamma) = \min V(x)$. If $s(\gamma^t) - s(\gamma^0) = 0$ for all $t > 0$, then in particular $s(\gamma^T) - s(\gamma^0) = 0$, i.e., we would have $c_1 = 0$. However, if $c_1 = 0$, then the left-hand side of (46) is the 0-vector, so by Lemma 29 the linear system (46) only has the trivial solution $T = 0$ and $c_{1,2} = c_{1,3} = \cdots = c_{1,\nu} = 0$. □

We consider a bitstring $x \in X_{n,k}$ under parenthesis matching, and we say that a pair of matched bits is *visible*, if there is no pair of matched bits around it. For example, the visible pairs of matched bits in the bitstring from Figure 5 are highlighted by shading in Figure 26. visible

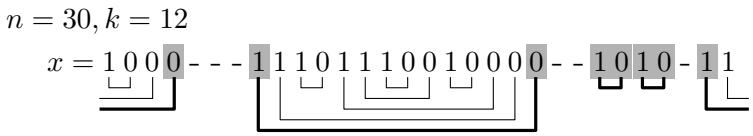


FIGURE 26. Visible pairs of matched bits highlighted in the bitstring x from Figure 5.

The next lemma asserts that for any glider $\gamma \in \Gamma(x)$ of minimum speed that is rightmost in its train, while moving along the cycle $C(x)$ this glider visits every possible position $i \in [n]$, both with its 0-bits and with its 1-bits, and moreover it is open in those steps, which implies that its outermost pair of matched bits is visible. Recall that the positions of steps of γ in the infinite Motzkin path $\widehat{x}$ translate to positions of bits in the finite string x by considering them modulo n, with $1, \dots, n$ as representatives of the equivalence classes. In this sense each step of γ corresponds to a bit in x, specifically every $\nearrow$-step corresponds to a matched 1 and every $\searrow$-step corresponds to a matched 0. This lemma is illustrated in Figure 21.

Lemma 31. *Let $x \in X_{n,k}$ and let $\gamma \in \Gamma(x)$ be the rightmost glider in a train of minimum speed $v(\gamma) = \min V(x)$. Then for $b \in \{0,1\}$ and for any position $i \in [n]$ there is a $t > 0$ such that γ^t is open and one of its b-bits is at position i in x^t.*

Proof. By Lemma 14, we know that γ^t is clean for all $t \geq 0$. If $s(\gamma^t) = s(\gamma^{t-1})$, then we have $r(\gamma^t) = r(\gamma^{t-1})$ and exactly one of γ^{t-1} and γ^t is inverted and the other is non-inverted. On the other hand, if $s(\gamma^t) > s(\gamma^{t-1})$, then $r(\gamma^t) = r(\gamma^{t-1}) + v(\gamma)$ (recall (23b)) and both γ^{t-1} and γ^t are free by Lemma 21 (i) and hence non-inverted. Moreover, by Lemma 26, γ^t is open. The claim now follows with the help of Lemma 30. □

4. Gluing the cycles together

In this section we glue the cycles from the factor $\mathcal{C}_{n,k}$ together via 4-cycles, as outlined in Section 1.6.3. The main result of this section, Theorem 35 below, asserts that this is possible without any conflicts between 4-cycles, and such that the gluing results in a single Hamilton cycle. The proof of this theorem is the first and only time in this paper that the assumption $n \geq 2k + 3$ is used. With Theorem 35 in hand, we prove Theorem 1 at the end of this section.

4.1. **Connectors.** For a bitstring $x \in X_{n,k}$ and integer $i \geq 0$, we write $\sigma^i(x)$ for the string obtained from x by cyclic right-shift by i positions. $\sigma^i(x)$

If two bitstrings $x, y \in X_{n,k}$ have the same pairs of matched bits apart from one visible pair, then we refer to $\{x, y\}$ as a *connector*. In other words, x and y have the form connector $\{x, y\}$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \sigma^i \left(\begin{bmatrix} \cdots \mathbf{1}\, u\, \mathbf{0} \cdots - w - \cdots \\ \cdots - u - \cdots \mathbf{1}\, w\, \mathbf{0} \cdots \end{bmatrix} \right) \tag{48}$$

for some $i \geq 0$, where $u, w \in D$ and the shaded pair of matched bits is visible, i.e., x and y differ by transposing a visible pair of matched bits with two unmatched 0s that have no other unmatched 0s nor the visible pair in between them. In other words, the transposed unmatched 0s are adjacent to a block other than the one containing the transposed pair of matched bits. We write $\mathcal{X}_{n,k}$ for the set of all connectors. $\mathcal{X}_{n,k}$

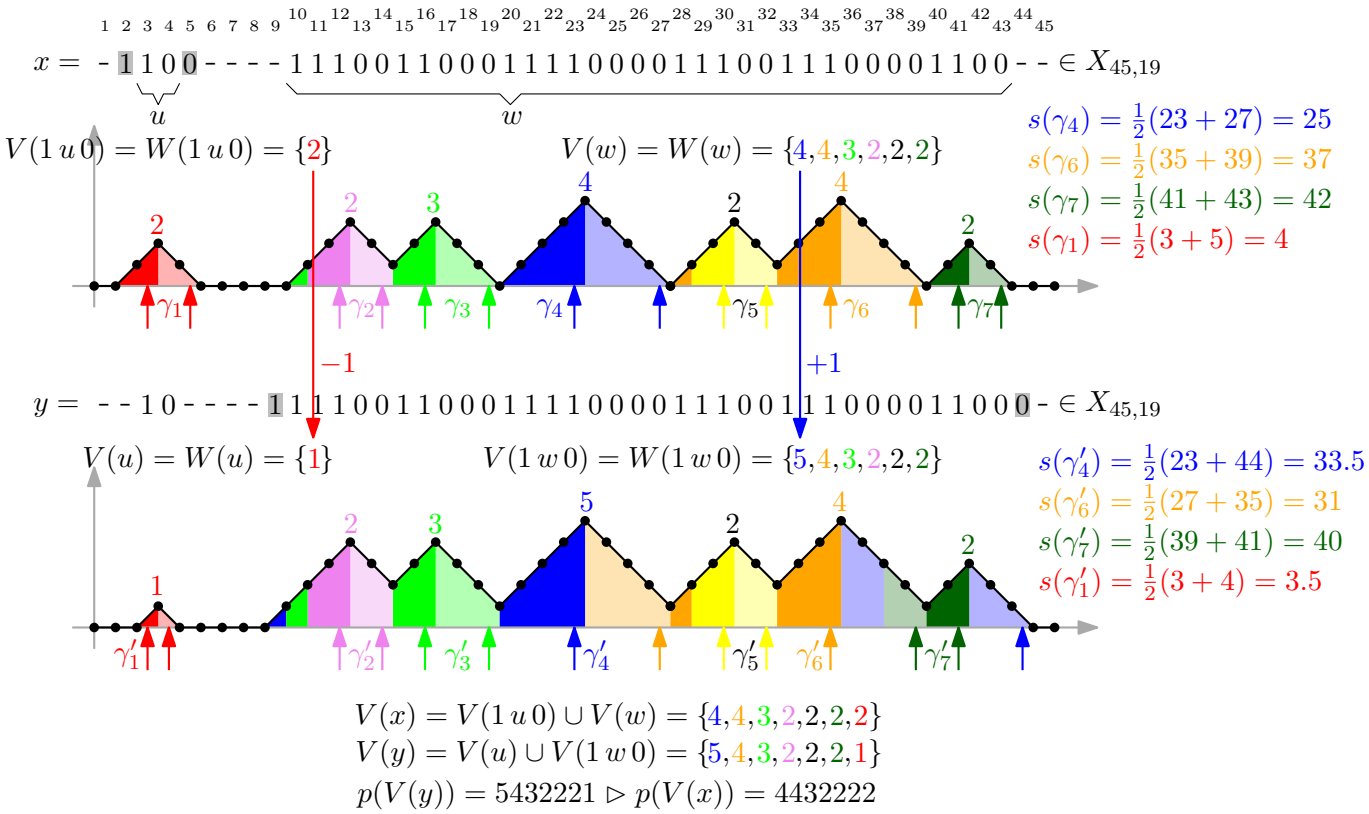


FIGURE 27. Relation between speed sets $V(x)$ and $V(y)$ for a connector $\{x, y\}$.

Lemma 32. *For any connector $\{x, y\} \in \mathcal{X}_{n,k}$, the sequence $C_4(x, y) := (x, f(x), y, f(y))$ is a 4-cycle in the Kneser graph $K(n, k)$.*

$C_4(x, y)$

Proof. Recall from Section 2.2 that f complements all matched bits. We already know that $(x, f(x))$ and $(y, f(y))$ are edges in the Kneser graph $K(n, k)$. Therefore, it remains to show that $(x, f(y))$ and $(y, f(x))$ are also edges. By symmetry, it suffices to argue that $(x, f(y))$ is an edge in $K(n, k)$. From the definition of f we know that the positions of 1s in $f(y)$ are $\mu_1(f(y)) = \mu_0(y)$. Let x and y be as in (48), and let a and b be the positions of the bits directly to the right of u and w, respectively. Clearly, we have $\mu_0(y) = \mu_0(x) - a + b$, i.e., we have $\mu_1(f(y)) = \mu_0(x) - a + b$ As $x_b =$ - is an unmatched 0, we see that the positions of 1s in $f(y)$ are where x has either matched or unmatched 0s, which proves that $(x, f(y))$ is indeed an edge in $K(n, k)$. □

Observe that if $\{x, y\} \in \mathcal{X}_{n,k}$ and $C(x)$ and $C(y)$ are two distinct cycles in the factor $\mathcal{C}_{n,k}$ defined in (1), then the symmetric difference of the edge sets $(C(x) \cup C(y)) \Delta C_4(x, y)$ is a single cycle in $K(n, k)$ on the same vertex set as $C(x) \cup C(y)$, i.e., the 4-cycle 'glues' two cycles from the factor together to a single cycle.

Given a connector $\{x, y\}$ as in (48), we now aim to understand the relation between the corresponding speed sets $V(x)$ and $V(y)$; see Figure 27. From (48) we see that

$$V(x) = V(1\,u\,0) \cup V(w) \cup S$$

for some set S (the set S contains the speeds of gliders outside of $1\,u\,0$ and w). From the definition (10) we obtain directly that $W(1\,u\,0) = W(u) \oplus 1$. Using Lemma 9 we therefore have

$$V(x) = (V(u) \oplus 1) \cup V(w) \cup S. \tag{49a}$$

With analogous arguments we obtain

$$V(y) = V(u) \cup \big(V(w) \oplus 1\big) \cup S. \tag{49b}$$

In words, $V(y)$ is obtained from $V(x)$ by decreasing the speed of the fastest glider whose bits belong to the substring $1\,u\,0$ by 1, increasing the speed of a fastest glider whose bits belong to the substring w by 1, whereas all other glider speeds remain unchanged. We emphasize that (49) is only a statement about the speed sets, not about the glider partition and the resulting glider positions in x and y, which may in fact change considerably; see Figure 27.

In the following, we write $p(V(x))$ for the sequence obtained from the multiset $V(x)$ by sorting its elements in non-increasing order. As in Section 3.14 before, we label the elements of $V(x)$ in non-decreasing order as $v_1 \le v_2 \le \cdots \le v_\nu$, $\nu = \nu(x)$, so we have $p(V(x)) = (v_\nu, v_{\nu-1}, \ldots, v_2, v_1)$. In this way we can think of the sequence $p(V(x))$ as a number partition of k (recall (9)). For two number partitions p and q of k, we write $p \rhd q$ if p is lexicographically strictly larger than q. This defines a total order on all number partitions of k. Observe that even if p and q do not have the same number of parts (i.e., different length), they will differ in a prefix of the same length in both, as they are partitions of the same number k.

$p(V(x))$

$\rhd$

Going back to (49), note that if $1\,u\,0 = 1^{v(\gamma)}0^{v(\gamma)}$ are the bits belonging to a glider $\gamma \in \Gamma(x)$ of minimum speed $v(\gamma) = \min V(x)$, then the multiset $V(y)$ is obtained from $V(x)$ by decreasing an element of minimum value $v(\gamma)$ by 1, and increasing another element by 1. When considering the corresponding Young diagrams of the associated number partitions, this corresponds to one square moving from the smallest (rightmost) stack to a stack further to the left. As a consequence, we have $p(V(y)) \rhd p(V(x))$. We will use this observation to control the process of gluing together cycles from our factor via connectors. Specifically, by using connectors that lexicographically increase the number partitions associated with the speed set of gliders, we ensure that every cycle is joined to a cycle $C(x)$ that has the lexicographically largest partition $p(V(x)) = k$, i.e., a single glider of speed k.

Furthermore, for any number partition p of k and for any integer $i \ge 0$ we write $p \boxplus i$ for the ith smallest element larger than p in the total lexicographic order $\rhd$ of all number partitions of k. Similarly, we write $p \boxminus i$ for the ith largest element smaller than p. For example, all number partition of $k = 5$ in lexicographic order are $11111, 2111, 221, 311, 32, 41, 5$, so we have $221 \boxplus 1 = 311$, and $32 \boxminus 3 = 2111$, $2111 \boxplus 5 \rhd 41$, and $11111 \boxplus 2 \rhd 32 \boxminus 3$.

We state the following simple observations for further reference.

Lemma 33. *Let $V = \{v_\nu \ge \cdots \ge v_1\}$, $\nu \ge 2$, be a multiset of positive integers.*

(i) For any $\nu \ge j > i \ge 1$ we have $p(V \setminus \{v_j, v_i\} \cup \{v_j + 1\} \cup \{v_i - 1\}) \rhd p(V)$.
(ii) If $\nu \ge 3$ and $v_3 > v_2$ then for any $\nu \ge j \ge 3$ and $i \in \{2, 1\}$ we have $p(V \setminus \{v_j, v_i\} \cup \{v_j + 1\} \cup \{v_i - 1\}) \rhd p(V) \boxplus 1$.
(iii) If $v_1 \ge 2$, then $p(V \setminus \{v_1\} \cup \{v_1 - 1, 1\}) = p(V) \boxminus 1$.
(iv) If $v_1 = 1$, then $p(V \setminus \{v_1\} \cup \{v_1 - 1, 1\}) = p(V)$.

4.2. **Gluing the single glider cycles.** Our proof will join most cycles from the factor $\mathcal{C}_{n,k}$ via the aforementioned connectors. However, the cycles which contain the n vertices

s_i

$$s_i := \sigma^i(1^k\,0^k\,\text{-}^\ell), \tag{50}$$

$\ell := n - 2k$, for $i = 0, \ldots, n-1$ will first be joined in a slightly different way. These are precisely the vertices with $V(s_i) = \{k\}$, i.e., they have only a single glider of maximum speed k. Specifically, defining $g := \gcd(n, k)$ these cycles from $\mathcal{C}_{n,k}$ are $C(s_i) = (s_{i+kj})_{j=0,\ldots,n/g-1}$ for $i = 0, \ldots, g-1$. Consequently, the n vertices s_i, $i = 0, \ldots, n-1$, are partitioned into g cycles,

g

each containing n/g of those vertices. In particular, if $g = 1$, then this is only a single cycle, and no joining is needed. We define $\mathcal{D}$

$$\mathcal{D} := \{C(s_i) \mid i = 0, \dots, g-1\}. \tag{51}$$

Note that the subgraph of $K(n,k)$ induced by the vertices s_i, $i = 0, \dots, n-1$, is isomorphic to the Cayley graph of $\mathbb{Z}/n\mathbb{Z}$ with generators $\{k+i \mid i = 0, \dots, \ell\}$. The following lemma is a simple consequence of that.

Lemma 34. *Let $k \geq 1$ and $n \geq 2k+1$. For any $r \in \{0, \dots, n-1\}$ the symmetric difference of the edge sets of the cycles $\mathcal{D}$ with the 4-cycles $C'_4(s_{r+j}, s_{r+j+k+1}) := (s_{r+j}, s_{r+j+k}, s_{r+j+2k+1}, s_{r+j+k+1})$ for $j = 0, \dots, g-2$ is a single cycle on the same vertex set.* $C'_4(x,y)$

Note that the edges that the 4-cycles $C'_4(s_{r+j}, s_{r+j+k+1})$ defined in Lemma 34 share with the cycles from $\mathcal{D}$ are precisely the edges $(s_i, f(s_i))$ with s_i in $\mathcal{S}_r$

$$\mathcal{S}_r := \{\{s_{r+j}, s_{r+j+k+1}\} \mid j = 0, \dots, g-2\}. \tag{52}$$

The pairs in $\mathcal{S}_r$ are not connectors in the sense of our definition (48). However, they serve the same purpose of describing pairs of edges on the cycle factor $\mathcal{C}_{n,k}$ that lie on a common 4-cycle (specifically, each edge is described by one endpoint x, and the other endpoint is $f(x)$). Note that the 4-cycle $C_4(x,y)$ defined in Lemma 32 for a connector $\{x,y\} \in \mathcal{X}_{n,k}$ has the form $C_4(x,y) = (x, f(x), y, f(y))$, whereas for a pair $\{x,y\} \in \mathcal{S}_r$ the 4-cycle $C'_4(x,y)$ defined in Lemma 34 has the form $C'_4(x,y) = (x, f(x), f(y), y)$.

4.3. **Auxiliary graph.** Our strategy is to join the cycles of the factor $\mathcal{C}_{n,k}$ by repeatedly gluing pairs of them together via connectors, as described before. This is modeled by the following auxiliary graph. For any set of connectors $\mathcal{U} \subseteq \mathcal{X}_{n,k}$ we define a graph $\mathcal{H}_{n,k}[\mathcal{U}]$ as follows: The $\mathcal{H}_{n,k}[\mathcal{U}]$ nodes of $\mathcal{H}_{n,k}[\mathcal{U}]$ are the cycles of the factor $\mathcal{C}_{n,k} \setminus \mathcal{D}$, plus the set $\mathcal{D}$ defined in (51), which forms its own single node. For any two distinct cycles $C, C' \in \mathcal{C}_{n,k} \setminus \mathcal{D}$ and any connector $\{x,y\} \in \mathcal{U}$ with $x \in C$ and $y \in C'$ we add an edge that connects C and C' to the graph $\mathcal{H}_{n,k}[\mathcal{U}]$. Furthermore, for any two cycles $C \in \mathcal{C}_{n,k} \setminus \mathcal{D}$ and $C' \in \mathcal{D}$ and any connector $\{x,y\} \in \mathcal{U}$ with $x \in C$ and $y \in C'$ we add an edge that connects C and $\mathcal{D}$ to the graph $\mathcal{H}_{n,k}$.

Note that every bitstring $x \in \mathcal{X}_{n,k}$ is contained in many different connectors. Specifically, let $p \geq 1$ denote the number of visible pairs of matched bits in x. There are $\ell := n-2k$ unmatched 0s in x, i.e., ℓ pairs of substrings of the form -w- with $w \in D$. Each visible pair is contained in precisely one of those substrings, and not contained in the remaining $\ell-1$ of them. Consequently, x is contained in exactly $p(\ell-1)$ connectors. For example, the bitstring in Figure 26 has $p = 4$ visible pairs and $\ell = 30 - 2 \cdot 12 = 6$ unmatched 0s, i.e., it is contained in exactly 20 connectors.

Exploiting this observation, it is relatively easy to show that the graph $\mathcal{H}_{n,k}[\mathcal{X}_{n,k}]$ is connected, i.e., that it has a spanning tree, and each of the connectors corresponding to one edge of this spanning tree glues together two cycles from the factor $\mathcal{C}_{n,k}$. However, to obtain a Hamilton cycle in $K(n,k)$, it is crucial that no two gluing operations interfere with each other, i.e., that any two of the 4-cycles used for the gluing are edge-disjoint. It is easy to see that $C_4(x,y)$ and $C_4(x',y')$ are edge-disjoint if and only if $\{x,y\} \cap \{x',y'\} = \emptyset$. We therefore require a set of *pairwise disjoint* connectors $\mathcal{U} \subseteq \mathcal{X}_{n,k}$ such that $\mathcal{H}_{n,k}[\mathcal{U}]$ is a connected graph.

Theorem 35. *For any $k \geq 1$ and $n \geq 2k+3$, there is a set $\mathcal{U} \subseteq \mathcal{X}_{n,k}$ of connectors and an $r \in \{0, \dots, n-1\}$ such that $\mathcal{U}$ and $\mathcal{S}_r$ defined in (52) satisfy the following: the sets in $\mathcal{U}$ are pairwise disjoint, the sets in $\mathcal{U}$ and $\mathcal{S}_r$ are pairwise disjoint, and $\mathcal{H}_{n,k}[\mathcal{U}]$ is a connected graph.*

Proof. We define $\ell := n - 2k$. Note that by the assumption $n \geq 2k + 3$ we have $\ell \geq 3$, and this condition will be crucial for our proof. Throughout the proof, we fix a position $p \in [n]$ arbitrarily. For any $x \in X_{n,k}$ and $i \in \mu(x)$, we write $\gamma(x, i) = (A, B) \in \Gamma(x)$ for the glider with $i \in A \cup B$.

As before, we will denote vertices from $X_{n,k}$ under parenthesis matching by strings of length n over the alphabet $\{1, 0, \text{-}\}$. We now introduce regular expressions to specify subsets of vertices from $X_{n,k}$ that satisfy certain constraints. In those expressions, the symbol $*$ is a wildcard character that represents a string of any length, possibly zero, of the symbols $\{1, 0, \text{-}\}$. Moreover, for $b \in \{1, 0, \text{-}\}$ we write b^* for any number of occurrences of the symbol b, possibly zero. For example, if $(n, k) = (7, 3)$ then $10\,\text{-}^*\,10\,*$ denotes the set of vertices $\{101010\text{-}, 1010\text{-}10, 10\text{-}1010\}$. Disjunction of two regular expressions x and y is denoted as $(x \,|\, y)$. For example, if $(n, k) = (6, 2)$ then $(1100 \,|\, \text{-}^2)*$ denotes the set of vertices $\{1100\text{--}, \text{--}1100, \text{--}1010\}$. We write $D^+ := D \setminus \{\varepsilon\}$ for the set of all nonempty Dyck words, which represent nonempty sequences of matched pairs of bits. There may be additional constraints on certain substrings in the expressions, captured by propositional formulas. For example, if $(n, k) = (11, 4)$, then $1^b\,0^b\,\text{-}^c\,w\,*$ with the constraint formula $b \geq 2 \wedge c \in \{1, 2\} \wedge w \in D^+$ represents the set of vertices $\{1100\text{-}1010\text{--}, 1100\text{-}10\text{-}10\text{-}, 1100\text{-}10\text{--}10, 1100\text{-}1100\text{--}, 1100\text{--}1010\text{-}, 1100\text{--}10\text{-}10, 1100\text{--}1100\text{-}, 111000\text{-}10\text{--}, 111000\text{--}10\text{-}\}$. Furthermore, our regular expressions contain underlined symbols to indicate strings that have to be shifted cyclically so that one of the underlined symbols is at position p. For example, if $(n, k) = (5, 2)$ and $p = 2$ then $110\underline{0\text{-}1}$ denotes the set of vertices $\{00\text{-}111, 0\text{-}1110, \text{-}11100\}$. As before, in a connector $\{x, y\}$, we shade the visible pair of matched bits in which x and y differ, which provides a visual aid to highlight the change.

We define mappings α_i, $i = 1, \ldots, 9$, whose domains and images are subsets of $X_{n,k}$ such that $\{x, \alpha_i(x)\}$ is a connector for all $x \in \mathrm{dom}(\alpha_i)$ as follows:

$$\alpha_1 : \begin{bmatrix} x \\ \alpha_1(x) \end{bmatrix} = \begin{bmatrix} u\,\underline{\text{-}}\,\mathbf{1}\,\mathbf{0}\,\text{-}^{\ell-2}\,\text{-} \\ u\,\underline{\mathbf{0}}\,\text{-}\,\text{-}\,\text{-}^{\ell-2}\,\mathbf{1} \end{bmatrix}, \qquad u \in D^+; \tag{53a}$$

$$\alpha_2 : \begin{bmatrix} x \\ \alpha_2(x) \end{bmatrix} = \begin{bmatrix} \mathbf{1}\,1^{a-1}\,\underline{0^{a-1}\,\mathbf{0}}\,\text{-}\,\text{-}\,\text{-}\,* \\ \begin{cases} \text{-}\,1^{a-1}\,\underline{0^{a-1}\,\text{-}}\,\mathbf{1}\,\mathbf{0}\,\text{-}\,* & \text{if } \tau(y,\gamma) \notin X_4 \\ \text{-}\,1^{a-1}\,\underline{0^{a-1}\,\text{-}}\,\text{-}\,\mathbf{1}\,\mathbf{0}\,* & \text{otherwise} \end{cases} \end{bmatrix}, \qquad \nu(x) \geq 2 \wedge a = v_1 \text{ even}; \tag{53b}$$

$$\alpha_3 : \begin{bmatrix} x \\ \alpha_3(x) \end{bmatrix} = \begin{bmatrix} \mathbf{1}\,1^{a-1}\,\underline{0^{a-1}\,\mathbf{0}}\,\text{-}^c\,\text{-}\,w\,\text{-}\,* \\ \text{-}\,1^{a-1}\,\underline{0^{a-1}\,\text{-}}\,\text{-}^c\,\mathbf{1}\,w\,\mathbf{0}\,* \end{bmatrix}, \qquad a = v_1 \text{ even} \wedge c \in \{0,1\} \wedge w \in D^+; \tag{53c}$$

$$\alpha_4 : \begin{bmatrix} x \\ \alpha_4(x) \end{bmatrix} = \begin{bmatrix} \underline{\mathbf{1}\,1^{a-1}}\,0^{a-1}\,\mathbf{0}\,\text{-}\,\text{-}\,\text{-}\,\text{-}^c\,* \\ \begin{cases} \underline{\text{-}\,1^{a-1}}\,0^{a-1}\,\text{-}\,\mathbf{1}\,\mathbf{0}\,\text{-}\,\text{-}^c\,* & \text{if } \tau(y,\gamma) \notin X_4 \\ \underline{\text{-}\,1^{a-1}}\,0^{a-1}\,\text{-}\,\text{-}\,\mathbf{1}\,\mathbf{0}\,\text{-}^c\,* & \text{otherwise} \end{cases} \end{bmatrix}, \quad \begin{array}{l} \nu(x) \geq 2 \wedge a = v_1 \text{ odd} \wedge \\ (a \geq 3 \vee c = \ell - 3); \end{array} \tag{53d}$$

$$\alpha_5 : \begin{bmatrix} x \\ \alpha_5(x) \end{bmatrix} = \begin{bmatrix} u\,\underline{\mathbf{1}\,1^{a-1}}\,0^{a-1}\,\mathbf{0}\,\text{-}^c\,\text{-}\,w\,\text{-}\,(*\,\text{-}\,|\,\varepsilon) \\ u\,\underline{\text{-}\,1^{a-1}}\,0^{a-1}\,\text{-}\,\text{-}^c\,\mathbf{1}\,w\,\mathbf{0}\,(*\,\text{-}\,|\,\varepsilon) \end{bmatrix},$$
$$a = v_1 \text{ odd} \wedge (a = 1 \vee c \in \{0,1\}) \wedge u \in D \wedge w \in D^+ \wedge$$
$$\neg(a = 1 \wedge c = 0 \wedge \nu(x) \geq 3 \wedge b := v_2 < v_3 \wedge u = 1^b\,0^b) \wedge$$
$$\Big(\neg(a = 1 \wedge \nu(x) \geq 3 \wedge b := v_2 < v_3 \wedge w = 1^b\,0^b) \vee (a = 1 \wedge c = 0 \wedge w = 10)\Big); \tag{53e}$$

$$\alpha_6 : \begin{bmatrix} x \\ \alpha_6(x) \end{bmatrix} = \begin{bmatrix} \mathbf{1}\,1^{b-1}\,0^{b-1}\,\mathbf{0}\,\underline{1}\,0\,\text{-}\,w\,\text{-}\,(*\,\text{-}\,|\,\varepsilon) \\ \text{-}\,1^{b-1}\,0^{b-1}\,\text{-}\,\underline{1}\,0\,\mathbf{1}\,w\,\mathbf{0}\,(*\,\text{-}\,|\,\varepsilon) \end{bmatrix}, \qquad b = v_2 < v_3 \wedge w \in D^+; \tag{53f}$$

$$\alpha_7 : \begin{bmatrix} x \\ \alpha_7(x) \end{bmatrix} = \begin{bmatrix} \underline{1}\,0\,\text{-}^*\,\text{-}\,1^b\,0^b\,\text{-}^*\,\text{-}\,w'\,\text{-}\,* \\ \underline{\ }\,\text{-}\,\text{-}^*\,\text{-}\,1^b\,0^b\,\text{-}^*\,1\,w'\,0\,* \end{bmatrix}, \qquad b = v_2 < v_3 \wedge b \geq 2 \wedge w' \in D^+; \tag{53g}$$

$$\alpha_8 : \begin{bmatrix} x \\ \alpha_8(x) \end{bmatrix} = \begin{bmatrix} u\,\underline{1}\,0\,\text{-}\,\text{-}^c\,1\,1^{b-1}\,0^{b-1}\,0\,\text{-}^*\,\text{-} \\ u\,\underline{1}\,0\,0\,\text{-}^c\,\text{-}\,1^{b-1}\,0^{b-1}\,\text{-}\,\text{-}^*\,1 \end{bmatrix}, \quad b = v_2 < v_3 \wedge (b \geq 2 \vee c = 1) \wedge u \in D^+; \tag{53h}$$

$$\alpha_9 : \begin{bmatrix} x \\ \alpha_9(x) \end{bmatrix} = \begin{bmatrix} u\,1\,0\,\text{-}^c\,\text{-}\,\text{-}\,\text{-}\,1\,0\,\text{-}^*\,\text{-} \\ u\,\underline{\ }\,\text{-}\,\text{-}^c\,1\,0\,\text{-}\,1\,0\,\text{-}^*\,\text{-} \end{bmatrix}, \qquad 1 < v_3 \wedge c \geq 0 \wedge u \in D^+. \tag{53i}$$

These specifications use the regular expressions defined before. Furthermore, preimage and image of each mapping are written as a column vector, to facilitate their positionwise comparison, which allows immediate verification that these are indeed valid connectors (recall (48)). These expressions contain several occurrences of substrings of the form $1\,1^{a-1}\,0^{a-1}\,0$, which could of course be simplified to $1^a\,0^a$, i.e., a consecutive 1s follows by a consecutive 0s, but this would make the positionwise comparison harder, as the second expression is $\text{-}\,1^{a-1}\,0^{a-1}\,\text{-}$. Note that the definitions of α_i, $i \in \{2, 4\}$, each involve a distinction between two possible cases for $\alpha_i(x)$. The condition $\tau(y, \gamma) \notin X_4$ stated in the 'if' branch of these case distinctions will be specified later at the end of this proof.

For $i = 1, \ldots, 9$ we define $X_i := \mathrm{dom}(\alpha_i)$ and $Y_i := \mathrm{img}(\alpha_i) = \alpha_i(X_i)$. Furthermore, the set of connectors $\mathcal{U} \subseteq \mathcal{X}_{n,k}$ we use to prove the theorem is defined, as one would expect at this point, by

$$\mathcal{U} := \bigcup_{i=1}^{9} \{\{x, \alpha_i(x)\} \mid x \in X_i\}. \tag{54}$$

To complete the proof, we have to show that the sets in $\mathcal{U}$ are pairwise disjoint, that the sets in $\mathcal{U}$ and $\mathcal{S}_r$ are pairwise disjoint for a suitable value of r, and that $\mathcal{H}_{n,k}[\mathcal{U}]$ is connected.

We first argue that the connectors in $\mathcal{U}$ are pairwise disjoint. As we have not yet specified the conditions in the 'if' case of the definitions of α_2 and α_4, we prove the slightly stronger statement that the connectors are disjoint for both of the two exclusive cases, in whichever combination they may occur.

For any vertex $x \in X_{n,k}$ with $\nu(x) \geq 2$ and any glider $\gamma \in \Gamma(x)$ that is the rightmost glider in a train of minimum speed $v(\gamma) = \min V(x) = v_1$, we define a vertex $\tau(x, \gamma)$ on the cycle $C(x)$ as follows. If $v(\gamma)$ is even, then let $t \geq 0$ be such that γ^t satisfies the conditions stated in Lemma 31 for $b := 0$ and $i := p$, and define $\tau(x, \gamma) := x^t$. If $v(\gamma)$ is odd, then let $t \geq 0$ be such that γ^t satisfies the conditions stated in Lemma 31 for $b := 1$ and $i := p$, and define $\tau(x, \gamma) := x^t$. In words, the vertex $\tau(x, \gamma)$ is obtained by moving along the cycle $C(x)$ starting from x and tracking the movement of the glider γ until one of its b-bits is at position p.

Clearly, the set of all vertices from $X_{n,k}$ under the mapping τ is specified by the regular expression $x = \underline{1^a\,0^a}\,\text{-}\,*$ with the condition $\nu(x) \geq 2 \wedge a = v_1$. We first argue that every such vertex x belongs to exactly one of the sets X_i, $i = 2, \ldots, 9$, and that these sets are all disjoint. This argument is given in the decision tree shown in Figure 28. Each branching distinguishes several exclusive cases, captured by logical conditions at the nodes. Furthermore, the conditions imposed on x are captured by a regular expression that gets more refined as we move towards the leaves and extra conditions are imposed. At the leaf nodes, one can check that x belongs to precisely one of the sets X_i. Note that there are two branches in the tree ending with X_4 and two branches ending with X_5. By taking the disjunction of conjunctions of logical conditions of all root-leaf paths for a particular set X_i, $i \in \{1, \ldots, 9\}$, we obtain precisely the regular expressions and corresponding logical conditions stated in the definition of α_i before.

$x = \underline{1^a}\,0^a$ - $*$ $\qquad \nu(x) \geq 2 \wedge a = v_1$

- a even
  - $x = 1^a\,\underline{0^a}$ - -^{c} $*$ $\qquad c \geq 0$
    - $c \geq 2$
      $x = 1^a\,\underline{0^a}$ - - - $*$
      $\boxed{\Rightarrow x \in X_2 = \mathrm{dom}(\alpha_2)}$
    - $c \in \{0,1\}$
      $x = 1^a\,\underline{0^a}$ -^{c} - w - $*$ $\qquad w \in D^+$
      $\boxed{\Rightarrow x \in X_3 = \mathrm{dom}(\alpha_3)}$
- a odd
  $x = \underline{1^a}\,0^a$ - $*$
  - $a \geq 3$
    $x = \underline{1^a}\,0^a$ - -^{c} $*$ $\qquad c \geq 0$
    - $c \geq 2$
      $x = \underline{1^a}\,0^a$ - - - $*$
      $\boxed{\Rightarrow x \in X_4 = \mathrm{dom}(\alpha_4)}$
    - $c \in \{0,1\}$
      $x = \underline{1^a}\,0^a$ -^{c} - w - $*$ $\qquad w \in D^+$
      $\boxed{\Rightarrow x \in X_5 = \mathrm{dom}(\alpha_5)}$
  - $a = 1$
    $x = \underline{1}\,0$ - $*$
    - x has exactly 1 block
      $x = \underline{1}\,0$ - - - -$^{\ell-3}$ $*$
      $\boxed{\Rightarrow x \in X_4 = \mathrm{dom}(\alpha_4)}$
    - x has ≥ 2 blocks
      $x = u\,\underline{1}\,0$ - -^{c} w - $(*$ - $|\,\varepsilon)$ $\qquad c \geq 0 \wedge u \in D \wedge w \in D^+$
      - $c = 0 \wedge \nu(x) \geq 3 \wedge b := v_2 < v_3 \wedge u = 1^b\,0^b$
        $x = 1^b\,0^b\,\underline{1}\,0$ - w - $(*$ - $|\,\varepsilon)$
        $\boxed{\Rightarrow x \in X_6 = \mathrm{dom}(\alpha_6)}$
      - $\nu(x) \geq 3 \wedge b := v_2 < v_3 \wedge w = 1^b\,0^b \wedge \neg(c = 0 \wedge w = 10)$
        $x = u\,\underline{1}\,0$ - -^{c} $1^b\,0^b$ - $(*$ - $|\,\varepsilon)$
        - x has ≥ 3 blocks
          $x = \underline{1}\,0$ -* - $1^b\,0^b$ -* - w' - $*$ $\qquad w' \in D^+$
          $\boxed{\Rightarrow x \in X_7 = \mathrm{dom}(\alpha_7)}$
        - x has exactly 2 blocks $\qquad (\nu(x) \geq 3 \Rightarrow u \in D^+)$
          $x = u\,\underline{1}\,0$ - -^{c} $1^b\,0^b$ -* -
          - $b \geq 2 \vee c = 1$
            $x = u\,\underline{1}\,0$ - -^{c} $1^b\,0^b$ -* -
            $\boxed{\Rightarrow x \in X_8 = \mathrm{dom}(\alpha_8)}$
          - $b = 1 \wedge c \geq 2$
            $x = u\,\underline{1}\,0$ - - - -$^{c-2}$ $1\,0$ -* -
            $\boxed{\Rightarrow x \in X_9 = \mathrm{dom}(\alpha_9)}$

          $b = 1 \wedge c = 0$ is impossible by the assumption $\neg(c = 0 \wedge w = 10)$
      - $\neg\big(c = 0 \wedge \nu(x) \geq 3 \wedge b := v_2 < v_3 \wedge u = 1^b\,0^b\big)$
        $\wedge\,\Big(\neg\big(\nu(x) \geq 3 \wedge b := v_2 < v_3 \wedge w = 1^b\,0^b\big) \vee (c = 0 \wedge w = 10)\Big)$
        $x = u\,\underline{1}\,0$ -^{c} - w - $(*$ - $|\,\varepsilon)$
        $\boxed{\Rightarrow x \in X_5 = \mathrm{dom}(\alpha_5)}$

FIGURE 28. Illustration of the proof of Theorem 35.

We have shown that X_i, $i = 2, \ldots, 9$ are pairwise disjoint sets. Note also that these sets are all disjoint from X_1, as for $x \in X_1$ we have $x_p =$ -, whereas $x'_p \in \{0, 1\}$ for all $x' \in X_2 \cup \cdots \cup X_9$. Furthermore, it is easy to check that each of the mappings α_i, $i = 1, \ldots, 9$, is an injection. Consequently, we still need to verify that $Y_i \cap X_j = \emptyset$ for all $1 \le i, j \le 9$ and $Y_i \cap Y_j = \emptyset$ for all $1 \le i < j \le 9$. We do this pinpointing, for any pair of such bitstrings, a particular property in which they differ. In the remainder of the proof we sometimes refer to the speed sets of several different bitstrings $x \in X_{n,k}$, and to avoid ambiguities we then explicitly specify the argument x for the different speed values as $V(x) = \{v_1(x), \ldots, v_{\nu(x)}(x)\}$.

Case 1: $y = \alpha_1(x) \in Y_1$. We have $y_p = 0$, whereas $x'_p \in \{1, \text{-}\}$ for $x' \in X_i, Y_i$, $i = 4, \ldots, 9$. The vertex y has exactly one visible pair of matched bits, whereas every $x' \in X_1, X_2, Y_2, X_3, Y_3$ has at least two visible pairs of matched bits.

Case 2: $y = \alpha_2(x) \in Y_2$. The vertex y has the following properties:

(i) $y_p \in \{0, \text{-}\}$.
(ii) If $y_p = 0$, then $v(\gamma(y, p))$ is odd.
(iii) If $y_p =$ -, then either $y_{p+1} = 1$ and y contains at least three blocks, or $y_{p+1} =$ -.
(iv) The first block entirely to the right of position p starts with 10.

If $x' \in X_1$, then $x'_p =$ -, $x'_{p+1} = 1$ and x' contains exactly two blocks, contradicting (iii).

If $x' \in X_2 \cup X_3$, then we have $x'_p = 0$ and $v(\gamma(x', p))$ is even, contradicting (ii). If $y' \in Y_3$, then the first block entirely to the right of position p starts with $1\,w\,0$ for some $w \in D^+$, contradicting (iv). If $x' \in X_4, X_5, X_6, X_7, X_8, X_9, Y_6, Y_8$, then we have $x'_p = 1$, contradicting (i). If $y' = \alpha_4(x') \in Y_4$, then if $y'_p = 1$ we have a contradiction to (i). Otherwise we have $y'_p =$ - and the first block entirely to the right of position p starts with $1^{a-1}\,0^{a-1}$ for some odd a, which contradicts (iv) if $a \ge 3$. Otherwise we have $a = 1$, and then the definition (53d) implies that x' has only a single block, and all - are consecutive. Then the equality $y = y'$ implies that $x = 1^{a'}\,0^{a'}\,\text{-}^{\ell}$ for some $a' \ge 2$, i.e., $\nu(x) = 1$, contradicting the condition $\nu(x) \ge 2$ in (53b). If $y' \in Y_5$, then either $y'_p = 1$, which contradicts (i), or $y'_p =$ - and the first block entirely to the right of position p is $1^{a-1}\,0^{a-1}$ for some odd a or starts with $1\,w\,0$ for $w \in D^+$, contradicting (iv). If $y' \in Y_7$, then the first block entirely to the right of position p is $1^b\,0^b$ for $b \ge 2$, contradicting (iv). If $y' \in Y_9$, then the equality $y = y'$ implies that x contains a substring -10, which means that $v_1(x) = 1$, contradicting the condition that $v_1(x)$ must be even in (53b).

Case 3: $y = \alpha_3(x) \in Y_3$. The vertex y has the following properties:

(i) $y_p \in \{0, \text{-}\}$.
(ii) If $y_p = 0$, then $v(\gamma(y, p))$ is odd.
(iii) If $y_p =$ -, then $y_{p+2} = 1$.
(iv) The first block entirely to the right of position p starts with $1\,w\,0$ for some $w \in D^+$.

If $x' \in X_1, Y_9$, then the first block entirely to the right of position p starts with 10, contradicting (iv). If $x' \in X_2, X_3$, then we have $x'_p = 0$ and $v(\gamma(x', p))$ is even, contradicting (ii). If $x' \in X_4, X_5, X_6, X_7, X_8, X_9, Y_6, Y_8$, then we have $x'_p = 1$, contradicting (i). If $y' = \alpha_4(x') \in Y_4$, then if $y'_p = 1$ there is a contradiction to (i). Otherwise we have $y' = \underline{\ }\,1^{a-1}\,0^{a-1}\,\text{-}\,\text{-}^*\,1\,0\,*$ for some odd $a \ge 1$. Consequently, if $a = 1$ then the first block in y' entirely to the right of position p starts with 10, contradicting (iv). On the other hand, if $a \ge 3$ then the equality $y = y'$ implies that x contains a substring -10, which means that $v_1(x) = 1$, contradicting the condition that $v_1(x)$ must be even in (53c). If $y' = \alpha_5(x') \in Y_5$, then if $y'_p = 1$ we have a contradiction to (i). Otherwise we have $y'_p =$ - and then a conflict $y = y'$ either implies that $y = y' = \text{-}\,1^{a-1}\,0^{a-1}\,\underline{\ }\,\text{-}\,1\,w\,0\,*$ or $y = y' = \text{-}\,1^{a-1}\,0^{a-1}\,\underline{\ }\,1\,w\,0\,\text{-}\,\text{-}^*\,1\,w'\,0\,*$ for some even $a \ge 2$ with $a = v_1(x)$ and $w, w' \in D^+$. In the first case, $x' = \text{-}\,1^{a-1}\,0^{a-1}\,\underline{1}\,0\,\text{-}\,w\,\text{-}\,*$ violates the

conditions in the second row of the conjunction in (53e), which is a contradiction. In the second case, we have $x' = \text{-}\,1^{a-1}\,0^{a-1}\,\underline{1}\,1\,w\,0\,0\,\text{-}^*\,\text{-}\,w'\,\text{-}\,*$. Note that $v_1(w) \geq a$, and consequently $v_1(1\,1\,w\,0\,0) \geq a+2 > a-1$. It follows that $v_1(x') = a-1$, and since the 1-bit in x' at position p does not belong to a glider of minimum speed by Lemma 13, we have again a contradiction to the definition (53e). If $y' \in Y_7$, then we have $y'_p = \text{-}$ and $y'_{p+2} = \text{-}$, contradicting (iii).

Case 4: $y = \alpha_4(x) \in Y_4$. The vertex y has the following properties:

(i) $y_p \in \{1, \text{-}\}$.
(ii) If $y_p = 1$ then $v(\gamma(y,p))$ is even.
(iii) $y_{[p,p+1,p+2]} \neq \text{-}\,1\,0$.
(iv) $y_{[p-1,p]} \neq 01$.
(v) The first block entirely to the right of position $p+1$ starts with 10.

If $x' \in X_1$, then we have $x'_{[p,p+1,p+2]} = \text{-}\,1\,0$, contradicting (iii). If $x' \in X_2, X_3$, then we have $x'_p = 0$, contradicting (i). If $x' \in X_4, X_5, X_6, X_7, X_8, X_9, Y_6$, then we have $x'_p = 1$ and $v(\gamma(x',p))$ is odd, contradicting (ii). If $y' \in Y_5$, then the first block entirely to the right of position $p+1$ starts with $1\,w\,0$ for some $w \in D^+$, contradicting (v). If $y' \in Y_7$, then the first block entirely to the right of position $p+1$ starts with $1^b\,0^b$ for some $b \geq 2$, contradicting (v). If $y' \in Y_8$, then we have $y'_{[p-1,p]} = 01$, contradicting (iv). If $y' \in Y_9$, then the equality $y = y'$ implies that $v_1(x) = 1$ and then the definition (53d) implies that x must have a single block, and all - are consecutive. Then y has only two blocks, whereas y' has three blocks, a contradiction.

Case 5: $y = \alpha_5(x) \in Y_5$. The vertex y has the following properties:

(i) $y_p \in \{1, \text{-}\}$.
(ii) If $y_p = 1$ then $v(\gamma(y,p))$ is even.
(iii) $y_{[p,p+1,p+2]} \neq \text{-}\,1\,0$.
(iv) $y_{p-1,p} \neq 01$.
(v) The first block entirely to the right of position $p+1$ starts with $1\,w\,0$ for some $w \in D^+$.

If $x' \in X_1$, then we have $x'_{[p,p+1,p+2]} = \text{-}\,1\,0$, contradicting (iii). If $x' \in X_2, X_3$, then we have $x'_p = 0$, contradicting (i). If $x' \in X_4, X_5, X_6, X_7, X_8, X_9, Y_6$, then we have $x'_p = 1$ and $v(\gamma(x',p))$ is odd, contradicting (ii). If $y' = \alpha_7(x') \in Y_7$, then a conflict $y = y'$ implies $y = y' = \underline{\text{-}}\,\text{-}\,\text{-}^*\,\text{-}\,1^b\,0^b\,\text{-}^*\,1\,w'\,0\,*$ for $b := v_2(x') < v_3(x')$, $b \geq 2$, and some $w' \in D^+$. This however implies $x = \underline{1}\,0\,\text{-}^*\,\text{-}\,\text{-}\,1^{b-1}\,0^{b-1}\,\text{-}\,\text{-}^*\,1\,w'\,0\,*$, so x violates the conditions in the third row of the conjunction in (53e), a contradiction. If $y' \in Y_8$, then we have $y'_{[p-1,p]} = 01$, contradicting (iv). If $y' \in Y_9$, then the first block entirely to the right of position $p+1$ starts with 10, contradicting (v).

Case 6: $y = \alpha_6(x) \in Y_6$. The vertex y has the following properties:

(i) $y_p = 1$.
(ii) The glider $\gamma(y,p)$ is not open.
(iii) $y_{p+2} = 1$.

If $x' \in X_1, X_2, X_3, Y_7, Y_9$, then we have $x'_p \in \{0, \text{-}\}$, contradicting (i). If $x' \in X_4, X_5, X_6, X_7, X_8, X_9$, then the glider $\gamma(x',p)$ is open, contradicting (ii). If $y' \in Y_8$, then we have $y'_{p+2} = 0$, contradicting (iii).

Case 7: $y = \alpha_7(x) \in Y_7$. The vertex y has the following properties:

(i) $y_p = \text{-}$.
(ii) $y_{p+1} = \text{-}$.
(iii) The first block entirely to the right of position p starts with $1^b\,0^b$ for some $b \geq 2$.

If $x' \in X_1$, then we have $x'_{p+1} = 1$, contradicting (ii). If $x' \in X_2, X_3, X_4, X_5, X_6, X_7, X_8, X_9, Y_8$, then we have $x'_p \in \{0,1\}$, contradicting (i). If $y' \in Y_9$, then the first block entirely to the right of position p starts with 10, contradicting (iii).

Case 8: $y = \alpha_8(x) \in Y_8$. If $x' \in X_1, X_2, X_3, Y_9$, then we have $x'_p \in \{0, \text{-}\}$, whereas $y_p = 1$. If $x' \in X_4, X_5, X_6, X_7, X_8, X_9$, then we have $x'_{[p-1,p+2]} \neq 0100$, whereas $y_{[p-1,p+2]} = 0100$.

Case 9: $y = \alpha_9(x) \in Y_9$. If $x' \in X_1$, then we have $x'_{p+1} = 1$, whereas $y_{p+1} = \text{-}$. If $x' \in X_2, X_3, X_4, X_5, X_6, X_7, X_8, X_9$, then we have $x'_p \in \{0, 1\}$, whereas $y_p = \text{-}$.

We now argue that the sets in $\mathcal{U}$ and $\mathcal{S}_r$ are pairwise disjoint for a suitable value of $r \in \{0, \ldots, n-1\}$. The connectors in $\mathcal{U}$ that contain a vertex s_i as defined in (50) all arise as images of α_1 and α_5. Specifically, we have $s_{p+c} = \underline{\ }^{c}\, 1^k\, 0^k\, \text{-}^{\ell-1-c} \in Y_5$ for all $c = 1, \ldots, \ell-1$ and $s_{p+\ell} = 1^k\, 0^{k-1}\, \underline{0}\, \text{-}^{\ell} \in Y_1$, i.e., all those connectors s_i satisfy

$$p + 1 \leq i \leq p + \ell. \tag{55}$$

Note that the pairs of sets $\mathcal{S}_r$ defined in (52) contain only vertices s_i with $r \leq i \leq r+(g-2)+k+1$ Consequently, if we choose $r := p + \ell + 1$, then we have

$$p + \ell + 1 \leq i \leq (p + \ell + 1) + (g - 2) + k + 1 \leq p + n = p + 1 + (n - 1), \tag{56}$$

where we used $\ell = n - 2k$ and the simple fact $g = \gcd(n, k) \leq k$ in the upper bound estimates. From (55) and (56) we see that the sets in $\mathcal{U}$ and $\mathcal{S}_r$ are pairwise disjoint, as desired.

It remains to argue that $\mathcal{H}_{n,k}[\mathcal{U}]$ is connected. To establish the connectivity, we consider the speed sets $V(x)$ of vertices $x \in X_{n,k}$, and the corresponding number partitions $p(V(x))$ of k obtained by sorting the speeds non-increasingly. Note that the speed sets and hence the number partitions are invariant along each cycle of the factor $\mathcal{C}_{n,k}$ by Lemma 23, but they can only change along connectors. Specifically, we consider the number partitions $p(V(x))$ of vertices $x \in X_i$, $i \in \{1, \ldots, 9\}$, and we argue how they change with the connectors $\{x, \alpha_i(x)\}$, in such a way that the number partitions increase lexicographically, either directly along a connector, or via a sequence of connectors. In a sequence of connectors, some intermediate steps may decrease lexicographically, but overall the change must be a lexicographic increase. This ensures that every cycle is joined, via a sequence of connectors, to a cycle in $\mathcal{D}$. This last part of the proof also specifies the conditions for the case distinctions in the definitions (53b) and (53d).

For each of the connectors $\{x, \alpha_i(x)\}$, $i = 1, \ldots, 9$, defined in (53) we first describe how the number partitions $p(V(x))$ and $p(V(\alpha_i(x)))$ compare lexicographically. For any $x \in X_1$, we obtain from (53a), (49) and Lemma 33 (i)+(ii) that

$$p(V(\alpha_1(x))) \rhd \begin{cases} p(V(x)) \boxplus 1 & \text{if } \nu(x) \geq 3 \text{ and } v_3 > v_2, \\ p(V(x)) & \text{otherwise.} \end{cases} \tag{57a}$$

Similarly, for any $x \in X_2$ the definition (53b) and Lemma 33 (iii) yield

$$p(V(\alpha_2(x))) = p(V(x)) \boxminus 1 \tag{57b}$$

(note that v_1 is assumed to be even and therefore $v_1 \geq 2$). For any $x \in X_3$ the definition (53c) and Lemma 33 (i) give

$$p(V(\alpha_3(x))) \rhd p(V(x)) \tag{57c}$$

For any $x \in X_4$, the definition (53d) and Lemma 33 (iii)+(iv) prove that

$$p(V(\alpha_4(x))) = \begin{cases} p(V(x)) \boxminus 1 & \text{if } v_1 \geq 3, \\ p(V(x)) & \text{if } v_1 = 1 \end{cases} \tag{57d}$$

(note that v_1 is assumed to be odd). For any $x \in X_5$ the definition (53e) and Lemma 33 (i)+(ii) show that

$$p(V(\alpha_5(x))) \rhd \begin{cases} p(V(x)) \boxplus 1 & \text{if } a = 1 \wedge \nu(x) \geq 3 \wedge v_3 > v_2 \wedge w \neq 10, \\ p(V(x)) & \text{otherwise.} \end{cases} \tag{57e}$$

For any $x \in X_i$, $i \in \{6,7,8\}$, the definitions (53f)–(53h) and Lemma 33 (ii) establishes

$$p(V(\alpha_i(x))) \rhd p(V(x)) \boxplus 1. \tag{57f}$$

Lastly, for any $x \in X_9$ we obtain from the definition (53i) and from Lemma 33 (iv) that

$$p(V(\alpha_9(x))) = p(V(x)). \tag{57g}$$

Let $X := \bigcup_{i=1}^{9} X_i$ and consider a vertex $x \in X$. We show that there is a sequence of connectors and cycles of the factor $\mathcal{C}_{n,k}$ in between them to reach a vertex $y \in X_{n,k}$ with $p(V(y)) \rhd p(V(x))$.

For $x \in X_1, X_3, X_5, X_6, X_7, X_8$ this follows directly from (57a), (57c), (57e), (57f), respectively. It remains to consider the cases $x \in X_2, X_4, X_9$.

Case (a): $x \in X_2$. Consider the two connectors $\{x, y\}$ and $\{x, y'\}$ where y and y' are obtained from x as described by the first and second case of the case distinction in (53b). From (57b) we see that $p(V(y)) = p(V(x)) \boxminus 1$. As $\nu(x) \geq 2$ and $v_1(x) \geq 2$ we have $\nu(y) \geq 3$, $v_1(y) = 1$ and $v_3(y) > v_2(y)$. We clearly also have $V(y') = V(y)$. Let q be the position of the 1-bit of the substring 10 in which y differs from x, and define $\gamma := \gamma(y, q) \in \Gamma(y)$ and $\hat{\gamma} := \gamma(y, q-2) \in \Gamma(y)$. By Lemma 22 and the definition of the mapping h_γ given in Section 3.12 we have $y' = h_\gamma(y)$.

We now consider the vertex $z := \tau(y, \gamma) \in X_i$, $i \in \{2, \dots, 9\}$, which satisfies $V(z) = V(y)$ by Lemma 23, and we let $t > 0$ be such that $y^t = z$. As $v_1(y) = 1$ is odd, we know that $i \notin \{2, 3\}$.

If $i \neq 4$, then we define $\alpha_2(x) := y$, and we continue the argument as follows. If $i = 6, 7, 8$, then by (57f) we have

$$p(V(\alpha_i(z))) \rhd p(V(z)) \boxplus 1 = p(V(y)) \boxplus 1 = \big(p(V(x)) \boxminus 1\big) \boxplus 1 = p(V(x)), \tag{58}$$

so we are done. If $i = 9$ then we have $z = u\,\underline{1}\,0\,\text{-}\,\text{-}^*\,1\,0\,\text{-}\,\text{-}^*$ for some $u \in D^+$. Clearly, as $v_1(x) \geq 2$ there are at most two equivalence classes of gliders in $\Gamma(y)$ and $\Gamma(z)$ that have speed 1, and there can only be exactly two if $v_1(x) = 2$. However, note that in $h_{\hat{\gamma}}(y)$ the two gliders of speed 1 form a train, and applying Lemma 27 shows that in $h_{\hat{\gamma}^t}(z)$ they do not form a train (as $u \in D^+$), a contradiction to Lemma 24. This shows that the case $i = 9$ cannot occur. If $i = 5$ then we have $z = u\,\underline{1}\,0\,\text{-}\,\text{-}^*\,w\,\text{-}\,*$. For the same reason as in the case $i = 9$, we must have $w \neq 10$, and therefore $p(V(\alpha_i(z))) \rhd p(V(z)) \boxplus 1$ by (57e), so again (58) holds.

If $i = 4$, on the other hand, then z has only a single block, i.e., we have $z = u\,\underline{1}\,0\,\text{-}^\ell$ for some $u \in D^+$. In this case we define $\alpha_2(x) := y'$. By Lemma 27 the vertex $z' := h_\gamma(z) = u\,\underline{\ }\,1\,0\,\text{-}^{\ell-1}$ lies on the cycle $C(y')$, and we have $z' \in X_1$. From (57a) we therefore obtain

$$p(V(\alpha_1(z'))) \rhd p(V(z')) \boxplus 1 = p(V(y')) \boxplus 1 = (p(V(x)) \boxminus 1) \boxplus 1 = p(V(x)),$$

so we are done.

Case (b): $x \in X_4$. Consider the two connectors $\{x, y\}$ and $\{x, y'\}$ where y and y' are obtained from x as described by the first and second case of the case distinction in (53d). If $v_1(x) \geq 3$, then the argument continues as in case (a) before. It remains to consider the case $v_1(x) = 1$. From (57d) we see that $p(V(y)) = p(V(x))$. We clearly also have $V(y') = V(y)$. Let q be the position of the 1-bit of the substring 10 in which y differs from x, and define $\gamma := \gamma(y, q) \in \Gamma(y)$.

We now consider the vertex $z := \tau(y, \gamma) \in X_i$, $i \in \{2, \dots, 9\}$. As $v_1(y) = 1$ is odd, we know that $i \notin \{2, 3\}$.

If $i \neq 4$, then we define $\alpha_4(x) := y$, and we continue the argument as follows. If $i = 5, 6, 7, 8$, then from (57e) and (57f) we have

$$p(V(\alpha_i(z))) \rhd p(V(z)) = p(V(y)) = p(V(x)).$$

If $i = 9$, then we have $z = u\,\underline{1}\,0\,\text{-}^c\,\text{-}\,\text{-}\,\text{-}\,1\,0\,\text{-}\,\text{-}^*$ for some $c \geq 0$ and $u \in D^+$. Furthermore, we have $V(\alpha_9(z)) = V(z)$ by (57g). Let $\hat{q}$ be the position of the 1-bit of the substring 10 in which $\alpha_9(z)$

differs from z, and define $\hat{\gamma} := \gamma(\alpha_9(z), \hat{q}) \in \Gamma(z)$. We observe that $\hat{z} := \tau(\alpha_9(z), \hat{\gamma})$ satisfies $\hat{z} \in X_j$ with $j \notin \{2, 3, 4, 9\}$ (apply Lemma 27 to $\hat{\gamma}$ and $\alpha_9(z)$) and therefore

$$p(V(\alpha_j(\hat{z}))) \rhd p(V(\hat{z})) = p(V(\alpha_9(z))) = p(V(z)) = p(V(y)) = p(V(x)).$$

If $i = 4$, on the other hand, then we define $\alpha_4(x) := y'$. By Lemma 27 the vertex $z' := h_\gamma(z)$ lies on the cycle $C(y')$, and we have $z' \in X_1$. Using (57a) it follows that

$$p(V(\alpha_1(z'))) \rhd p(V(z')) = p(V(y')) = p(V(x)).$$

Case (c): $x \in X_9$. This case can be settled in the same way as the subcase $i = 9$ in case (b). This completes the proof of the theorem. □

Note that the condition $\ell = n - 2k \geq 3$ is crucial in the definition (53d) for $a = 1$, as it gives us freedom to proceed among one of two possible connectors. This prevents the potential problem of an infinite cyclic sequence of connectors that yield no lexicographic change, for example by continually shifting around a speed 1 glider. This is why the aforementioned proof does not extend to the cases $n = 2k + 2$ or $n = 2k + 1$.

The reader may rightfully wonder how one would come up with the set of nine regular expressions stated in (53). In fact, they were found following a computer-guided experimental approach of trial and error. We started with a much smaller and simpler set of rules that would guarantee lexicographic improvement along the connectors. By writing a computer program that tests small cases, we figured out that those initial rules still allowed some conflicts, i.e., not all pairs of connectors were disjoint. This led to modifying the existing rules and adding new rules that would prevent those particular conflicts, sometimes creating new unforeseen conflicts, as reported by our program. We used this approach of successive refinement until eventually a conflict-free set of rules was confirmed by computer for small cases, which we then verified theoretically for all cases.

4.4. **Proof of Theorem 1.**

Proof of Theorem 1. The graphs $K(3, 1)$, $K(4, 1)$ and $K(6, 2)$ can easily be checked to admit a Hamilton cycle. On the other hand, $K(5, 2)$ is the Petersen graph, which is well known not to have a Hamilton cycle. Furthermore, it was shown in [MNW21, Thm. 1] that $K(2k + 1, k)$ has a Hamilton cycle for all $k \geq 3$. Combining this result with [Joh11, Thm. 1] proves that $K(2k + 2, k)$ has a Hamilton cycle for all $k \geq 3$. To cover the remaining cases, let $k \geq 1$ and $n \geq 2k + 3$. By Theorem 35, there is a set $\mathcal{U} \subseteq \mathcal{X}_{n,k}$ and a choice for r where $\mathcal{S}_r$ is the set defined in (52) such that the sets in $\mathcal{U}$ are pairwise disjoint, the sets in $\mathcal{U}$ and $\mathcal{S}_r$ are pairwise disjoint, and $\mathcal{H}_{n,k}[\mathcal{U}]$ is a connected graph. Let $\mathcal{T}$ be a minimal subset of $\mathcal{U}$ such that $\mathcal{H}_{n,k}[\mathcal{T}]$ is connected, i.e., this graph will be a spanning tree. By construction and by Lemma 34, the symmetric difference of the edge sets of the cycle factor $\mathcal{C}_{n,k}$ in $K(n, k)$ defined in (1) with the 4-cycles $C_4(x, y)$, $\{x, y\} \in \mathcal{T}$, defined in Lemma 32 and with the 4-cycles $C'_4(x, y)$, $\{x, y\} \in \mathcal{S}_r$, defined in Lemma 34, which are all edge-disjoint, is a Hamilton cycle in $K(n, k)$. □

5. Open questions

One interesting open problem is to develop an efficient algorithm for computing a Hamilton cycle in the Kneser graph $K(n, k)$, i.e., an algorithm whose running time is polynomial in n and k, not only polynomial in $N := \binom{n}{k}$, the size of the Kneser graph. Our construction does not give such an algorithm for fundamental reasons. Most importantly, our proof does not reveal the number of cycles in the cycle factor, so it is not even clear how many gluings via 4-cycles have to be used to join them to a Hamilton cycle. Clearly, if there are c cycles, then $c - 1$ gluings will

be needed, but we do not know how to compute c more efficiently than actually computing all the cycles of our factor, which takes time polynomial in N. Even if we knew how to compute c efficiently, we do not know how to efficiently compute a minimal set of $c-1$ connectors that joins all the cycles (our proof uses many more connectors than will eventually be needed).

Furthermore, for any of the families of vertex-transitive graphs shown in Figure 1, it would be very interesting to investigate whether they admit a Hamilton decomposition, i.e., a partition of all their edges into Hamilton cycles, plus possibly a perfect matching. This problem was raised by Meredith and Lloyd [ML73], and by Biggs [Big79] for the odd graphs $O_k = K(2k+1,k)$. Gould's survey [Gou91] mentions the analogous problem for the middle levels graphs $M_k = H(2k+1,k)$. In these two cases, we do not even know *two* edge-disjoint Hamilton cycles, so the problem seems to be very hard in general. On the other hand, for Johnson graphs $J(n,2)$ with odd n, two edge-disjoint Hamilton cycles are known [FKMS20]. To tackle this problem in general, it might be helpful to first consider decompositions of the edges of the graph into cycle factors; see [JK04].

A strengthening of the concept of containing a Hamilton cycle is to contain the rth power of a Hamilton cycle. To this end, Katona [Kat05] conjectured that the vertices of $K(n,k)$ can be ordered so that any $r+1$ consecutive vertices for $r := \lfloor n/k \rfloor - 2$ are disjoint sets, which he proved for $k=2$ using Walecki's theorem. Theorem 1 confirms this conjecture for the cases $2k+1 \leq n \leq 4k-1$ (where $r=1$). It seems plausible that Katona's conjecture holds even for $r := \lceil n/k \rceil - 2$, and our theorem confirms this stronger variant of the conjecture for the cases $2k+1 \leq n \leq 3k$ (where $r=1$).

Acknowledgements

We thank Petr Gregor and Pascal Su for several inspiring discussions about Kneser graphs, and we also thank Petr Gregor for providing feedback on the first section of this paper. We are particularly grateful to one of the STOC reviewers, whose careful reading helped eliminating several small technical errors in our proofs.

References

[AAC+18] L. A. Agong, C. Amarra, J. S. Caughman, A. J. Herman, and T. S. Terada. On the girth and diameter of generalized Johnson graphs. *Discrete Math.*, 341(1):138–142, 2018.

[Bal72] A. T. Balaban. Chemical graphs. XIII. Combinatorial patterns. *Rev. Roumain Math. Pures Appl.*, 17:3–16, 1972.

[Bar75] Zs. Baranyai. On the factorization of the complete uniform hypergraph. In *Infinite and finite sets (Colloq., Keszthely, 1973; dedicated to P. Erdős on his 60th birthday), Vol. I*, pages 91–108. Colloq. Math. Soc. János Bolyai, Vol. 10. North-Holland, Amsterdam, 1975.

[BCK19] J. Balogh, D. Cherkashin, and S. Kiselev. Coloring general Kneser graphs and hypergraphs via high-discrepancy hypergraphs. *European J. Combin.*, 79:228–236, 2019.

[Big79] N. Biggs. Some odd graph theory. In *Second International Conference on Combinatorial Mathematics (New York, 1978)*, volume 319 of *Ann. New York Acad. Sci.*, pages 71–81. New York Acad. Sci., New York, 1979.

[BS21] J. Bellmann and B. Schülke. Short proof that Kneser graphs are Hamiltonian for $n \geq 4k$. *Discrete Math.*, 344(7):Paper No. 112430, 2 pp., 2021.

[BW84] M. Buck and D. Wiedemann. Gray codes with restricted density. *Discrete Math.*, 48(2-3):163–171, 1984.

[CF02] Y. Chen and Z. Füredi. Hamiltonian Kneser graphs. *Combinatorica*, 22(1):147–149, 2002.

[Cha89] P. Chase. Combination generation and graylex ordering. *Congr. Numer.*, 69:215–242, 1989. Eighteenth Manitoba Conference on Numerical Mathematics and Computing (Winnipeg, MB, 1988).

[Che00] Y. Chen. Kneser graphs are Hamiltonian for $n \geq 3k$. *J. Combin. Theory Ser. B*, 80(1):69–79, 2000.

[Che03] Y. Chen. Triangle-free Hamiltonian Kneser graphs. *J. Combin. Theory Ser. B*, 89(1):1–16, 2003.

[CI96] W. E. Clark and M. E. H. Ismail. Binomial and Q-binomial coefficient inequalities related to the Hamiltonicity of the Kneser graphs and their Q-analogues. *J. Combin. Theory Ser. A*, 76(1):83–98, 1996.

[CL87] B. Chen and K. Lih. Hamiltonian uniform subset graphs. *J. Combin. Theory Ser. B*, 42(3):257–263, 1987.

[CL21] M. Caoduro and L. Lichev. On the boxicity of Kneser graphs and complements of line graphs. *arXiv:2105.02516*, 2021.

[Cor92] P. F. Corbett. Rotator graphs: An efficient topology for point-to-point multiprocessor networks. *IEEE Transactions on Parallel and Distributed Systems*, 3:622–626, 1992.

[CW93] R. C. Compton and S. G. Williamson. Doubly adjacent Gray codes for the symmetric group. *Linear Multilinear Algebra*, 35(3-4):237–293, 1993.

[CW08a] Y. Chen and W. Wang. Diameters of uniform subset graphs. *Discrete Math.*, 308(24):6645–6649, 2008.

[CW08b] Y. Chen and Y. Wang. On the diameter of generalized Kneser graphs. *Discrete Math.*, 308(18):4276–4279, 2008.

[Den97] T. Denley. The odd girth of the generalised Kneser graph. *European J. Combin.*, 18(6):607–611, 1997.

[EHR84] P. Eades, M. Hickey, and R. C. Read. Some Hamilton paths and a minimal change algorithm. *J. Assoc. Comput. Mach.*, 31(1):19–29, 1984.

[EKR61] P. Erdős, C. Ko, and R. Rado. Intersection theorems for systems of finite sets. *Quart. J. Math. Oxford Ser. (2)*, 12:313–320, 1961.

[EM84] P. Eades and B. McKay. An algorithm for generating subsets of fixed size with a strong minimal change property. *Inform. Process. Lett.*, 19(3):131–133, 1984.

[FKMS20] S. Felsner, L. Kleist, T. Mütze, and L. Sering. Rainbow cycles in flip graphs. *SIAM J. Discrete Math.*, 34(1):1–39, 2020.

[FR18] E. Friedgut and O. Regev. Kneser graphs are like Swiss cheese. *Discrete Anal.*, pages Paper No. 2, 18 pp., 2018.

[Fra85] P. Frankl. On the chromatic number of the general Kneser-graph. *J. Graph Theory*, 9(2):217–220, 1985.

[GK76] C. Greene and D. J. Kleitman. Strong versions of Sperner's theorem. *J. Combin. Theory Ser. A*, 20(1):80–88, 1976.

[GMKM21] I. García-Marco, K. Knauer, and L. P. Montejano. Chomp on generalized Kneser graphs and others. *Internat. J. Game Theory*, 50(3):603–621, 2021.

[GMN18] P. Gregor, T. Mütze, and J. Nummenpalo. A short proof of the middle levels theorem. *Discrete Anal.*, pages Paper No. 8, 12 pp., 2018.

[Gou91] R. J. Gould. Updating the Hamiltonian problem—a survey. *J. Graph Theory*, 15(2):121–157, 1991.

[GR87] R. J. Gould and R. Roth. Cayley digraphs and $(1, j, n)$-sequencings of the alternating groups A_n. *Discrete Math.*, 66(1–2):91–102, 1987.

[Hol17] A. E. Holroyd. Perfect snake-in-the-box codes for rank modulation. *IEEE Trans. Inform. Theory*, 63(1):104–110, 2017.

[HW78] K. Heinrich and W. D. Wallis. Hamiltonian cycles in certain graphs. *J. Austral. Math. Soc. Ser. A*, 26(1):89–98, 1978.

[HW14] D. J. Harvey and D. R. Wood. Treewidth of the Kneser graph and the Erdős-Ko-Rado theorem. *Electron. J. Combin.*, 21(1):Paper 1.48, 11 pp., 2014.

[JK04] J. R. Johnson and H. A. Kierstead. Explicit 2-factorisations of the odd graph. *Order*, 21(1):19–27, 2004.

[JM20] A. Jafari and M. J. Moghaddamzadeh. On the chromatic number of generalized Kneser graphs and Hadamard matrices. *Discrete Math.*, 343(2):111682, 3 pp., 2020.

[Joh04] J. R. Johnson. Long cycles in the middle two layers of the discrete cube. *J. Combin. Theory Ser. A*, 105(2):255–271, 2004.

[Joh11] J. R. Johnson. An inductive construction for Hamilton cycles in Kneser graphs. *Electron. J. Combin.*, 18(1):Paper 189, 12 pp., 2011.

[JR94] M. Jiang and F. Ruskey. Determining the Hamilton-connectedness of certain vertex-transitive graphs. *Discrete Math.*, 133(1-3):159–169, 1994.

[Kat05] Gy. O. H. Katona. Constructions via Hamiltonian theorems. *Discrete Math.*, 303(1-3):87–103, 2005.

[Kno94] M. Knor. Gray codes in graphs. *Math. Slovaca*, 44(4):395–412, 1994.

[Knu11] D. E. Knuth. *The Art of Computer Programming. Vol. 4A. Combinatorial Algorithms. Part 1.* Addison-Wesley, Upper Saddle River, NJ, 2011.

[KZ22] V. Kozhevnikov and M. Zhukovskii. Large cycles in generalized Johnson graphs. *arXiv:2203.03006*, 2022.

[LCL22] K. Liu, M. Cao, and M. Lu. Treewidth of the generalized Kneser graphs. *Electron. J. Combin.*, 29(1):Paper No. 1.57, 19 pp., 2022.

[Lov70] L. Lovász. Problem 11. In *Combinatorial Structures and Their Applications (Proc. Calgary Internat. Conf., Calgary, AB, 1969)*. Gordon and Breach, New York, 1970.

[Lov78] L. Lovász. Kneser's conjecture, chromatic number, and homotopy. *J. Combin. Theory Ser. A*, 25(3):319–324, 1978.

[Mat76] M. Mather. The Rugby footballers of Croam. *J. Combin. Theory Ser. B*, 20(1):62–63, 1976.

[Met22] K. Metsch. On the treewidth of generalized Kneser graphs. *arXiv:2203.14036*, 2022.

[ML72] G. H. J. Meredith and E. K. Lloyd. The Hamiltonian graphs O_4 to O_7. In *Combinatorics (Proc. Conf. Combinatorial Math., Math. Inst., Oxford, 1972)*, pages 229–236. Inst. Math. Appl., Southend-on-Sea, 1972.

[ML73] G. H. J. Meredith and E. K. Lloyd. The footballers of Croam. *J. Combin. Theory Ser. B*, 15:161–166, 1973.

[MNW21] T. Mütze, J. Nummenpalo, and B. Walczak. Sparse Kneser graphs are Hamiltonian. *J. Lond. Math. Soc. (2)*, 103(4):1253–1275, 2021.

[MS17] T. Mütze and P. Su. Bipartite Kneser graphs are Hamiltonian. *Combinatorica*, 37(6):1207–1219, 2017.

[Müt16] T. Mütze. Proof of the middle levels conjecture. *Proc. Lond. Math. Soc.*, 112(4):677–713, 2016.

[Müt22] T. Mütze. Combinatorial Gray codes—an updated survey. *arXiv:2202.01280*, 2022.

[Müt23] T. Mütze. Gliders in Kneser graphs, 2023. `http://tmuetze.de/gliders.html`.

[NW75] A. Nijenhuis and H. Wilf. *Combinatorial Algorithms*. Academic Press, New York-London, 1975. Computer Science and Applied Mathematics.

[Ord67] R. J. Ord-Smith. Algorithm 308: Generation of the permutations in pseudo-lexicographic order [G6]. *Commun. ACM*, 10(7):452, 1967.

[Rus88] F. Ruskey. Adjacent interchange generation of combinations. *J. Algorithms*, 9(2):162–180, 1988.

[RW10] F. Ruskey and A. Williams. An explicit universal cycle for the $(n-1)$-permutations of an n-set. *ACM Trans. Algorithms*, 6(3):Art. 45, 12 pp., 2010.

[Sav97] C. D. Savage. A survey of combinatorial Gray codes. *SIAM Rev.*, 39(4):605–629, 1997.

[SS04] I. Shields and C. D. Savage. A note on Hamilton cycles in Kneser graphs. *Bull. Inst. Combin. Appl.*, 40:13–22, 2004.

[SW20] J. Sawada and A. Williams. Solving the sigma-tau problem. *ACM Trans. Algorithms*, 16(1):Art. 11, 17 pp., 2020.

[TL73] D. Tang and C. Liu. Distance-2 cyclic chaining of constant-weight codes. *IEEE Trans. Comput.*, C-22:176–180, 1973.

[VPV05] M. Valencia-Pabon and J.-C. Vera. On the diameter of Kneser graphs. *Discrete Math.*, 305(1–3):383–385, 2005.

[Zak84] S. Zaks. A new algorithm for generation of permutations. *BIT*, 24(2):196–204, 1984.

[Zak20] D. Zakharov. Chromatic numbers of Kneser-type graphs. *J. Combin. Theory Ser. A*, 172:105188, 16 pp., 2020.