Singular Stochastic Partial Differential Equations On The Real Plane Under Critical Regime

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Abstract

In this thesis, the fractional incompressible Stochastic Navier-Stokes (SNS) equation on $\mathbb{R}^2$ and the fractional Anisotropic Kardar–Parisi–Zhang (AKPZ) equation on $\mathbb{R}^2$ are studied, formally defined as

$$\partial_t v = -\tfrac{1}{2}(-\Delta)^\theta v - \lambda v \cdot \nabla v + \nabla p - \nabla^\perp (-\Delta)^{\frac{\theta-1}{2}} \xi, \qquad \nabla \cdot v = 0\,, \tag{1}$$

and

$$\partial_t h = -\tfrac{1}{2}(-\Delta)^\theta h - \lambda((\partial_1 h)^2 - (\partial_2 h)^2) + (-\Delta)^{\frac{\theta+1}{2}} \xi\,, \tag{2}$$

respectively, where $\theta \in (0,1]$, ξ is the space-time white noise on $\mathbb{R}_+ \times \mathbb{R}^2$ and λ is the coupling constant. For any value of θ both equations are ill-posed due to the singularity of the noise, are critical for $\theta = 1$ and supercritical for $\theta \in (0,1)$. For $\theta = 1$, the weak coupling regime for both of the equations is shown, i.e. regularisation at scale N and coupling constant $\lambda = \hat{\lambda}/\sqrt{\log N}$, is meaningful in that the sequences $\{v^N\}_N$ of regularised solutions of SNS and the sequences $\{h^N\}_N$ of regularised solutions of AKPZ are tight and the corresponding nonlinearities do not vanish as $N \to \infty$. Instead, for $\theta \in (0,1)$ it is shown that the large scale behaviour of v and h is trivial, as the nonlinearity vanishes and v is simply converges to the solution of (1) with $\lambda = 0$, while h converges to (2) also with $\lambda = 0$. In order to further understand the limiting behaviour of AKPZ as regularisation is removed a quantity called bulk diffusivity is investigated numerically on a torus, with the aim of quantifying how different the limit is from stochastic heat equation.

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Contents

CHAPTER 1

Introduction

For a very long time mathematicians have been describing dynamics of nature via partial differential equations (PDEs), explaining propagation of heat spreading through some medium or propagation of waves, may it be sound waves, seismic waves or light waves. The equations have moved our understanding of nature forward, not just on the scale we observe on day to day basis but also on the quantum scale. However those were often describing ideal situations that do not exist in reality. In reality we have plenty of small but significant disturbances, may it be friction, wind or quantum phenomena that we have not accounted for. Sometimes the perturbations are very sharp and may render description of a deterministic equations to be lacking. Therefore it is extremely important to know how perturbations affect those models. A PDE with a stochastic noise is referred as stochastic partial differential equation, those are often harder to treat as in order to describe natures perturbation oneJacek: on->one often uses quiet irregular noise term which posses great difficulties from analytical point of view. In fact often via adding the irregular noise the equation itself becomes ill-defined these SPDEs are referred to as singular. The ramifications often passed on from physics to the mathematical realm have been a great interest to mathematicians, in recent years many approaches have been devised. A frequently used approach is to consider an analogous equation where some or all terms have higher Fourier modes artificially removed. This results in equation being smoothed out, which makes it possible to work with. Once the results are obtained for the smoothed out equation one has to remove the previously imposed Fourier cut-off in the limit to be able to describe how the original equation behaves, this often posses difficulties. Namely the equation may blow up, as such in order to obtain a meaningful limit one has to add a term to the equation which scales as the Fourier cut-off is is removed, this is referred to as renormalization. Although we are technically not considering the same equation as originally intended, under appropriate renormalization we are still describing the physical phenomena with a slight change perhaps, via different reference frame for example. A lot of commonly studied SPDEs contain a non-linear term which is often a source of many difficulties, heuristically speaking if the non-linearity matters on small scales we call the regime *super-critical*, if it doesn't we call it *sub-critical* the regime on the boundary of those two is called *critical*. One can take an SPDE which is sub-critical and change it slightly to increase the significance of the non-linearity on the solution making it of critical regime or even super-critical regime, this can be caused by say, roughening the linear part of the equation. Often we may consider an SPDE in different dimensions, this is the most common way to observe different regimes occurring naturally. In recent years a great deal of development was achieved in dealing with SPDEs in the sub-critical regime. Path-wise techniques such as rough paths [Lyo98] [Gub04] which contributed to further development such as Paracontrolled distributions [GIP15] and the Regularity Structures [Hai15b] [Hai15a] which has been built on from Rough Paths which were used in solving the famous KPZ equation [Hai13]. Since then those methods have been used countless times accelerating the development of this field of Mathematics. These methods are restricted to the locally sub-critical regime as such different approach has to be taken with SPDEs in critical regime. In critical regime often the behaviour is more subtle than that of super-critical or sub-critical, even from the Physics perspective. In this thesis two SPDEs are considered on the real plane in the critical regime, the super-critical analogous equations are also considered. The notion of bulk diffusivity is used to quantify the limit in some sense, this quantity is investigated numerically for AKPZ. Before proceeding a description of the models is given.

1.1 Stochastic Navier-Stokes

The in-compressible Navier-Stokes equation is a partial differential equation (PDE) describing the motion of an in-compressible fluid subject to an external forcing. It is given by

$$\partial_t v = \tfrac{1}{2}\Delta v - \hat{\lambda} v \cdot \nabla v + \nabla p - f\,, \qquad \nabla \cdot v = 0\,, \tag{1.1}$$

where $v = v(t,x)$ is the velocity of the fluid at $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^d$, $\hat{\lambda} \in \mathbb{R}$ is the *coupling constant* which tunes the strength of the non-linearity, p is the pressure, f the forcing and the second equation is the in-compressibility condition. For f a random noise (which will be the case throughout the paper), we will refer to the above as to the Stochastic Navier-Stokes (SNS) equation.

The SNS equation has been studied under a variety of assumptions on f. Most of the literature focuses on the case of trace-class noise, for which existence, uniqueness of solutions and ergodicity were proved (see e.g. [FG95, DPD03, HM06, FR08, RZZ14] and the references therein). The case of even rougher noises, e.g. space-time white noise and its derivatives, which is relevant in the description of motion of turbulent fluids [MR04], was first considered in $d = 2$ in [DPD02], and later, thanks to the theory of Regularity Structures [Hai14] and the paracontrolled calculus approach [GIP15], in dimension three [ZZ15].

In the present work, we focus on dimension $d = 2$ and consider the fractional stochastic Navier-Stokes equation driven by a conservative noise, which formally reads

$$\partial_t v = -\tfrac{1}{2}(-\Delta)^{\theta} v - \hat{\lambda} v \cdot \nabla v - \nabla p + \nabla^{\perp}(-\Delta)^{\frac{\theta-1}{2}} \xi\,, \qquad \nabla \cdot v = 0\,. \tag{1.2}$$

Here, θ is a strictly positive parameter, $(-\Delta)^{\theta}$ is the usual fractional Laplacian, $\nabla^{\perp} \stackrel{\text{def}}{=} (\partial_2, -\partial_1)$ and ξ is a space-time white noise on $\mathbb{R}_+ \times \mathbb{R}^2$, i.e. a Gaussian process whose covariance is given by

$$\mathbf{E}[\xi(\varphi)\xi(\psi)] = \langle \varphi, \psi \rangle_{L^2(\mathbb{R}_+ \times \mathbb{R}^2)}\,, \qquad \forall\, \varphi, \psi \in L^2(\mathbb{R}_+ \times \mathbb{R}^2)\,. \tag{1.3}$$

The choice of the forcing $f = \nabla^{\perp}(-\Delta)^{\frac{\theta-1}{2}} \xi$ in (1.2) ensures that, at least formally, the spatial white noise on $\mathbb{R}^2$, i.e. the Gaussian process whose covariance is that in (1.3) but with $\mathbb{R}^2$-valued square-integrable φ, ψ, is invariant for the dynamics.

A rigorous analysis of (1.2) has so far only been carried out for $\theta > 1$, which in the language of [Hai14, Ch. 8], corresponds to the so-called *sub-critical* regime - in [GJ13], the authors proved existence of stationary solutions while uniqueness was established in [GT20]. The goal of the present paper is instead to study the large-scale behaviour of the fractional SNS in the *critical* and *supercritical* cases, i.e. $\theta = 1$ and $\theta \in (0, 1)$ respectively. For both of those regimes the classical stochastic calculus tools does not work. In the critical regime the path-wise theories of Regularity Structures [Hai14] and paracontrolled calculus [GIP15] are not applicable, a different approach is needed.

To motivate our results, let us first consider the Vorticity formulation of (1.2). Setting

$\omega \stackrel{\text{def}}{=} \nabla^\perp \cdot v$, ω solves

$$\partial_t \omega = -\tfrac{1}{2}(-\Delta)^\theta \omega - \hat{\lambda}\,(K * \omega) \cdot \nabla \omega + (-\Delta)^{\frac{\theta+1}{2}} \xi\,, \tag{1.4}$$

where K is the Biot-Savart kernel on $\mathbb{R}^2$ given by

$$K(x) \stackrel{\text{def}}{=} \frac{1}{2\pi\iota} \int_{\mathbb{R}^2} \frac{y^\perp}{|y|^2} e^{-\iota y \cdot x}\, \mathrm{d}y\,, \tag{1.5}$$

for $y^\perp \stackrel{\text{def}}{=} (y_2, -y_1)$. Note that the neighbourhood of the origin cannot be recovered in the transition back to the velocity v due to the cut off imposed. However the rest of the space can be recovered from the Vorticity ω via $v = K * \omega$, so that (1.2) and (1.4) are equivalent when excluding the origin neighbourhood.

Due to the roughness of the noise, as written (1.4) is purely formal for any value of $\theta \in (0, 1]$. Therefore, in order to work with a well-defined object, we first regularise the equation. Let ϱ^1 be a smooth spatial mollifier, the superscript 1 representing the scale of the regularisation, and consider the regularised Vorticity equation

$$\partial_t \omega^1 = -\tfrac{1}{2}(-\Delta)^\theta \omega^1 - \hat{\lambda} \cdot \varrho^1 * \Big((K * (\varrho^1 * \omega^1)) \cdot \nabla (\varrho^1 * \omega^1) \Big) + (-\Delta)^{\frac{1+\theta}{2}} \xi\,. \tag{1.6}$$

Note in the equation above the noise is not regularised, indeed we only approximate the non-linearity. Since we are interested in the large-scale behaviour of (1.4), we rescale ω^1 according to

$$\omega^N(t,x) \stackrel{\text{def}}{=} N^2 \omega^1(tN^{2\theta}, xN)\,, \tag{1.7}$$

The rescaled function solves

$$\partial_t \omega^N = -\tfrac{1}{2}(-\Delta)^\theta \omega^N - \hat{\lambda} N^{2\theta-2} \mathcal{N}^N[\omega^N] + (-\Delta)^{\frac{1+\theta}{2}} \xi\,, \tag{1.8}$$

here the non-linearity $\mathcal{N}^N$ is defined according to

$$\mathcal{N}^N[\omega] \stackrel{\text{def}}{=} \operatorname{div} \varrho^N * \Big((K * (\varrho^N * \omega))(\varrho^N * \omega) \Big)\,. \tag{1.9}$$

where $\varrho^N(\cdot) \stackrel{\text{def}}{=} N^2 \varrho(N \cdot)$. Detailed calculations showing how the scaling is obtained can be found in the the appendix, Proposition 3.3.5. The non-linarity in (1.9) is the same as one in (1.6) except scaled (as it is present in the scaled equation whereas (1.6) is not). This can be shown to be true by using integration by parts to obtain

$$\operatorname{div}\varrho^N * \Big((K*(\varrho^N*\omega))(\varrho^N*\omega)\Big)$$
$$= \varrho^N * \Big(((\nabla\cdot K)*(\varrho^N*\omega)))(\varrho^N*\omega) + ((K*(\varrho^N*\omega))\nabla\cdot(\varrho^N*\omega)\Big)$$

second term matches the form from (1.9), meanwhile the first term is equal to 0 since by definition of Biot-Savart kernel we have

$$\nabla\cdot K = \partial_1 K_1 + \partial_2 K_2 = 0\,.$$

Note that as an effect of the scaling (1.7), the coupling constant $\hat{\lambda}$ gains an N-dependent factor which, for large N, is order 1 for $\theta = 1$, i.e. in the critical regime, while it vanishes polynomially for $\theta \in (0,1)$, which instead is the supercritical regime. The goal of the present paper is twofold. For $\theta \in (0,1)$, we will show that the non-linearity *simply goes to* 0 and that the equation trivialises, in the sense that ω^N converges to the solution of the original fractional stochastic heat equation obtained by setting $\hat{\lambda} = 0$ in (1.8). At criticality, i.e. $\theta = 1$, instead the situation is more subtle. Logarithmic corrections due to the nonlinear term are to be expected (see [WAG71, LRY05] and [CET20, CHT21] for other models in the same universality class) and need to be taken into account. In the present setting, we will do so by imposing that the coupling constant vanishes at a suitable logarithmic order (see (1.11)). We will then show that this is indeed meaningful since on the one hand sub-sequential limits for ω^N exist and on the other the nonlinear term does not vanish but is uniformly (in N) of order 1.

Before delving into the details, let us state assumptions, scalings and results more precisely. To unify notations, for $N \in \mathbb{N}$ let ω^N be the solution of

$$\partial_t\omega^N = -\tfrac{1}{2}(-\Delta)^\theta\omega^N - \lambda_N \mathcal{N}^N[\omega^N] + (-\Delta)^{\frac{1+\theta}{2}}\xi\,, \qquad \omega(0,\cdot) = \omega_0(\cdot) \tag{1.10}$$

where ω_0 is the initial condition, the value of λ_N depends on both N and θ via

$$\lambda_N \stackrel{\text{def}}{=} \begin{cases} \frac{\hat{\lambda}}{\sqrt{\log N}}\,, & \text{for } \theta = 1 \\ \hat{\lambda}N^{2\theta-2}\,, & \text{for } \theta \in (0,1), \end{cases} \tag{1.11}$$

$\mathcal{N}^N$ is defined according to (1.9) with ϱ^N satisfying the following, for all $N \in \mathbb{N}$, we require ϱ^N to be a radially symmetric smooth function such that $\|\varrho^N\|_{L^1(\mathbb{R}^2)} = 1$ and whose Fourier transform $\hat{\varrho}^N$ is compactly supported on $\{k : 1/N < |k| < N\}$. Furthermore, there exists a constant $c_\varrho > 0$ such that

$$|\hat{\varrho}^N(k)| \geq c_\varrho, \quad \forall k \in \{k : 2/N < |k| < N/2\}\,. \tag{1.12}$$

We also define $\varrho^N_y(\cdot) \stackrel{\text{def}}{=} \varrho^N(\cdot - y)$.

1.2 Anisotropic KPZ

The second model that is considered is the *Anisotropic KPZ* which is a variant of the *KPZ* equation. On $\mathbb{R}_+ \times \mathbb{R}^d$ the KPZ is formally given by

$$\partial_t h = \nu\frac{1}{2}\Delta h + \langle \nabla h, Q\nabla h\rangle + \sqrt{D}\xi\,, \tag{1.13}$$

where ξ is the space-time white noise, Q is a $d \times d$ matrix, ν and D are real constants. The equation itself describes universal phenomena of random growing interface [HZ95][BS95], with applications varying greatly. For example the KPZ model can be used to describe propagation of ink that has been dropped on a piece of paper. In addition in $d = 1$ it has connections to Gaussian unitary ensemble Tracy-Widom distribution which is interesting in itself [TW94]. The equation is sub-critical in $d = 1$, in $d = 2$ it is critical, with the super-critical regime being $d \geq 3$. The KPZ equation has been extensively studied over the years in the sub-critical regime ($d = 1$) where multiple approaches to deal with well-possednes were researched in [BG97][GJ13][GP18b] [Hai14] [GIP15] [GP17]. In super-critical regime ($d \geq 3$) the physicists predict in [KPZ86] that if one imposes restrictions on ν, D and Q then the non-linearity should not be impactful on large scales that is under appropriate re-scaling and re-normalisation (by subtracting average growth) the fluctuations should match those of the solution of the stochastic heat equation. Super-critical regime is still work in progress with some recent results for $Q = \lambda I_d$ [CCM20] [DGRZ20]. In critical regime the path-wise approaches of Paracontrolled calculus and Regularity Structures break down and hence we must work directly on the level of the equation itself. [Wol91] showed that the behaviour in this regime is subtle and depends on the determinant of Q. Two different universal behaviours occur depending on the sign of $\det Q$, the universal behaviour under $\det Q > 0$ is referred to as isotropic KPZ class, while $\det Q \leq 0$ is referred to as Anisotropic KPZ class. In the isotropic KPZ class with $Q = \lambda I_d$ and $\lambda \sim \sqrt{\frac{\hat{\lambda}}{\log N}}$ where N being the regularisation parameter multiple results are known. It has been shown there exists a phase transition for one point distribution at $\hat{\lambda} = 2\pi$ in [CSZ17], proving tightness of the sequence of approximations for $\hat{\lambda} > 0$ sufficiently small in [CD20]. This was then improved to tightness, uniqueness and characterisation of the limit for all $\hat{\lambda} \in (0, 2\pi)$ in [CSZ20]. In this thesis the Anisotropic KPZ with , $D = \nu = 1, Q = \text{diag}(1, -1)$ is investigated. To motivate our results, let us first consider the Burgers formulation of (1.13) with $d = 2$ and $Q = \text{diag}(1, -1), \nu = D = 1$. Starting with a substituting $u = (-\Delta)^{\frac{1}{2}} h$ one obtains

$$\partial_t u = -\frac{1}{2}(-\Delta)^\theta u + \lambda(-\Delta)^{\frac{1}{2}}\Big((\partial_1(-\Delta)^{-\frac{1}{2}}u)^2 - (\partial_2(-\Delta)^{-\frac{1}{2}}u)^2\Big) + (-\Delta)^{\frac{\theta}{2}}\xi\,, \tag{1.14}$$

As with Vorticity equation the roughness of the noise causes the equation (1.14) to be purely formal for any value of $\theta \in (0, 1]$. We regularise equation in the same spirit as (1.4) with the same restrictions on ϱ obtaining

$$\begin{aligned}\partial_t u^1 &= -\tfrac{1}{2}(-\Delta)^\theta u^1 + (-\Delta)^{\frac{\theta}{2}}\xi \\ &\quad - \hat{\lambda}\varrho^1 * (-\Delta)^{\frac{1}{2}}\Big((\partial_1(-\Delta)^{-\frac{1}{2}}\varrho^1 * u^1)^2 - (\partial_2(-\Delta)^{-\frac{1}{2}}\varrho^1 * u^1)^2\Big)\,.\end{aligned} \tag{1.15}$$

Since we are interested in the large-scale behaviour of (1.4), we rescale

$$u^N(t, x) \stackrel{\text{def}}{=} N^2 u^1(tN^{2\theta}, xN)\,, \tag{1.16}$$

so that u^N solves

$$\partial_t u^N = -\tfrac{1}{2}(-\Delta)^\theta u^N - \hat{\lambda}N^{2\theta-2}\tilde{\mathscr{N}}^N[u^N] + (-\Delta)^{\frac{\theta}{2}}\xi\,, \tag{1.17}$$

and the non-linearity $\tilde{\mathscr{N}}^N$ is defined according to

$$\tilde{\mathscr{N}}^N[u] \stackrel{\text{def}}{=} \varrho^N * (-\Delta)^{\frac{1}{2}}((\partial_1(-\Delta)^{-\frac{1}{2}}\varrho^N * u)^2 - (\partial_2(-\Delta)^{-\frac{1}{2}}\varrho^N * u)^2)\,. \tag{1.18}$$

Detailed calculations showing how the scaling is obtained can be found in the the appendix, Proposition 3.3.6. Before delving into the details, let us state assumptions, scalings and results more precisely. To unify notations, for $N \in \mathbb{N}$ let u^N be the solution of

$$\partial_t u^N = -\tfrac{1}{2}(-\Delta)^\theta u^N - \lambda_N N^{2\theta-2}\tilde{\mathscr{N}}^N[u^N] + (-\Delta)^{\frac{\theta}{2}}\xi\,, \qquad u(0,\cdot) = u_0(\cdot) \tag{1.19}$$

where u_0 is the initial condition, the value of λ_N depends on both N and θ via the coupling constant given in (1.11). By [DPZ14, Theorem 7.23][DPZ14, Theorem 9.20] we know there exists unique strong solution to (1.10) and to (1.19), both of which satisfy the strong Markov property. [CES21] consider KPZ with the same matrix Q on the torus and they show that space-white noise is an invariant measure for any λ_N. Moreover they have shown that

$$\left\{\int_0^t \lambda_N \tilde{\mathcal{N}}^N[u^N](s,\varphi)\,\mathrm{d}s\right\}_t$$

converges to a process with finite non-zero energy.[CET20] prove that the stationary solution to the equation (1.15) with $\theta = 1$ on $\mathbb{T}_1^2$ at stationarity is super-diffusive, with the diffusion coefficient diverging for large times as $(\log t)^\delta$ for $\delta \in (0,1)$. In this article we develop results for KPZ similar to those of [CES21] but on $\mathbb{R}^2$, expanding on the result. Furthermore similar to [CET20] the notion of bulk diffusivity is used to compare the limit of AKPZ after the mollification is removed to the stochastic heat equation corresponding to (1.13) with the non-linearity removed.

1.3 Results

The equations (1.10) and (1.19) are well-defined, we start by showing finding invariant measures for both equations, which is essential for further results and is interesting in its own right. We show that the case $\theta \in (0,1)$ is trivialises the equation to an equivalent equation without the non-linearity. Focusing on the critical case $\theta = 1$ we show the following

Theorem 1.3.1 *The spatial white noise η on $\mathbb{R}^2$ is defined by its covariance function which is given by*

$$\mathbb{E}[\eta(\varphi)\eta(\psi)] = \langle \varphi, \psi \rangle_{L^2(\mathbb{R}^2)} \quad \forall \varphi, \psi \in L^2(\mathbb{R}^2)\,.$$

Let $\mu = (-\Delta)^{\frac{1}{2}}\eta$. Then μ is invariant for the solution ω^N of (1.10)*, while η is invariant for the solution u^N of* (1.19) *.*

The question that one wants to understand is under what scaling λ_N can we expect to see a non-trivial behaviour as $N \to \infty$. That is what λ_N do we pick so that $\lambda_N \mathcal{N}^N$ gives a non-trivial limit.

Theorem 1.3.2 *Let $\mathcal{N}^N$ be the non-linearity defined in* (1.9) *and $\tilde{\mathcal{N}}^N$ be the non-linearity defined in* (1.18) *and λ_N is given in* (1.11)*. Let $\varphi \in \mathcal{S}(\mathbb{R}^2)$, ω be a solution to* (1.10) *and u be a solution to* (1.19) *with $\theta = 1$. Then*

$$\begin{aligned} \mathbf{E}\left[\sup_{s\le t} \left|\int_0^s \mathcal{N}^N[\omega_r^N](\varphi)\,\mathrm{d}r\right|^p\right]^{1/p} &\lesssim (t^{\frac{1}{2}} \vee t)\|\varphi\|_{\dot{H}^2(\mathbb{R}^2)}\,, \\ \mathbf{E}\left[\sup_{s\le t} \left|\int_0^s \tilde{\mathcal{N}}^N[u_r^N](\varphi)\,\mathrm{d}r\right|^p\right]^{1/p} &\lesssim (t^{\frac{1}{2}} \vee t)\|\varphi\|_{\dot{H}^1(\mathbb{R}^2)}\,. \end{aligned} \tag{1.20}$$

In addition for all $\kappa > 0$ we have

$$\int_0^\infty e^{-\kappa t}\mathbf{E}\left(\left[\int_0^t \lambda_N \mathcal{N}^N[\omega_s^N](\varphi)\,\mathrm{d}s\right]^2\right)\mathrm{d}t \gtrsim \frac{1}{\kappa^2}\|\varphi\|^2_{\dot{H}^2(\mathbb{R}^2)}\,,$$
$$\int_0^\infty e^{-\kappa t}\mathbf{E}\left(\left[\int_0^t \lambda_N \tilde{\mathcal{N}}^N[u_s^N](\varphi)\,\mathrm{d}s\right]^2\right)\mathrm{d}t \gtrsim \frac{1}{\kappa^2}\|\varphi\|^2_{\dot{H}^1(\mathbb{R}^2)}\,.$$

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Based on the result above tightness of $\{\omega^N\}_{N\in\mathbb{N}}$ in $C_T\mathcal{S}'(\mathbb{R}^2,\mathbb{R})$ and that of $\{u^N\}_{N\in\mathbb{N}}$ in $C_T\mathcal{S}'(\mathbb{R}^2,\mathbb{R})$ can be obtained for the same scaling factor λ_N.

Theorem 1.3.3 *Let ω^N and u^N be solutions to* (1.10) *and* (1.19) *Both sequences $\{\omega^N\}_{N\in\mathbb{N}}$ and $\{u^N\}_{N\in\mathbb{N}}$ are tight in $C_T\mathcal{S}'(\mathbb{R}^2,\mathbb{R})$ for λ_N given in* (1.11)*.*

Analogous results of the upper bound of the non-linearity and tightness are shown to be true on the torus $\mathbb{T}^2_M$ for SNS, while for AKPZ they've been shown to be true in [CES21] and are only mentioned briefly. In [CET20] a further question was presented, namely how does the limit of solution of the mollified AKPZ compare to the stochastic heat equation corresponding to simply removing the non-linearity when λ_N is independent of N. As a quantity of interest bulk diffusivity was used. It was conjectured that bulk diffusivity of the limit to the regularised AKPZ grows at a rate of $\log(t)^{\frac{1}{2}}$. Although they were able to show that bulk diffusivity of the Anisotropic KPZ equation grows as a power of the logarithm of time in a weak Tauberian sense suggesting a super-diffusive behaviour they were not able to show that this power is indeed $\frac{1}{2}$, further development has occurred in version 3 which occurred after this work was complete where they proved that the bulk diffusivity is non-trivial. A justification as to why it is believed to be $\frac{1}{2}$ has been given in [CET20] and is reviewed here along with numerical method to further support the conjecture. Having applied the method to two models there is no doubt this method is applicable to other equations in critical regime, as long as certain properties are present, such as stationarity. At last, we consider the supercritical regime $\theta \in (0,1)$. As previously anticipated, in this case the non-linearity simply converges to 0 so that ω^N trivialises.

Theorem 1.3.4 *For $N \in \mathbb{N}$ and $\theta \in (0,1)$, let ω^N be the stationary solution of* (1.10) *with λ_N defined according to* (1.11) *for $\hat{\lambda} > 0$, ϱ^N being as described in* (1.12) *and initial condition $\omega_0 = \mu$, for μ the Gaussian field with covariance as in* (1.26)*. Then, the sequence $\{\omega^N\}_N$ converges as $N \to \infty$ to the unique solution of the fractional stochastic heat equation*

$$\partial_t\omega = -\tfrac{1}{2}(-\Delta)^\theta\omega + (-\Delta)^{\frac{1+\theta}{2}}\xi\,, \qquad \omega_0 = \mu\,. \tag{1.21}$$

Analogous result follows for the stationary solution of (1.19).

1.4 Notations and function spaces

For $M \in \mathbb{N}$ let $\mathbb{T}^2_M$ be the two dimensional torus of side length $2\pi M$ and $\mathbb{Z}^2_M \stackrel{\text{def}}{=} (\mathbb{Z}_0/M)^2$ where $\mathbb{Z}_0 \stackrel{\text{def}}{=} \mathbb{Z}\backslash\{0\}$. Denote by $\{e_k\}_{k\in\mathbb{Z}^2_M}$ the usual Fourier basis, i.e. $e_k(x) \stackrel{\text{def}}{=} \frac{1}{2\pi}e^{ik\cdot x}$, which for all $j, k \in \mathbb{Z}^2$ satisfes $\langle e_k, e_j\rangle_{L^2(\mathbb{T}^2)} = 1$ and for $\varphi \in L^2(\mathbb{T}^2_M)$ let the Fourier transform of φ be

$$\mathscr{F}_M(\varphi)(k) = \hat{\varphi}(k) = \varphi_k \stackrel{\text{def}}{=} \int_{\mathbb{T}^2_M} \varphi(x)e_{-k}(x)\,\mathrm{d}x\,,$$

in particular for all $x \in \mathbb{T}^2_M$ we have

$$\varphi(x) = \frac{1}{M^2}\sum_{k\in\mathbb{Z}^2_M} \hat{\varphi}(k)e_k(x)\,,$$

note the π scalings are missing due to the chooice of the fourier basis. When the space considered is clear, the subscript M may be omitted. The previous definitions straightforwardly translate to $\mathbb{R}^2$ by replacing the integral over the torus to the full space and the Riemann-sum to an integral. For $\theta \in \mathbb{R}$ and $T = \mathbb{T}^2_M$ or $\mathbb{R}^2$, we define the fractional Laplacian $(-\Delta)^\theta$ via its Fourier transform, i.e.

$$\mathscr{F}((-\Delta)^\theta u)(z) = |z|^{2\theta}\mathscr{F}(u)(z)\,,$$

for $\varphi \in L^2(T)$ and $z \in T$ for $\theta \geq 0$ and $z \in T\backslash\{0\}$ otherwise.

We denote by $\mathcal{S}(\mathbb{R}^2)$, the classical space of Schwartz functions, i.e. infinitely differentiable functions whose derivatives of all orders decay at faster than any polynomial, formally given by

$$\mathcal{S}(\mathbb{R}^2) \stackrel{\text{def}}{=} \{\varphi \in \mathscr{C}^\infty(\mathbb{R}^2) : \|\varphi\|_{\alpha,\beta} = \sup_{x\in\mathbb{R}^2}\left|x^\alpha D^\beta\varphi(x)\right| < \infty\,\forall\alpha,\beta \in \mathbb{N}^2\}\,,$$

where we used the multi-index notation $x^\alpha = x_1^{\alpha_1}x_2^{\alpha_2}, D^\beta = \partial_1^{\beta_1}\partial_2^{\beta_2}$. Similarly to [GT20, Section 7], for $s \in \mathbb{R}$, we say $\varphi\colon (\mathbb{R}^2)^n \to \mathbb{R}$ is in the *homogeneous Sobolev space* $(\dot{H}^s(\mathbb{R}^2))^{\otimes n}$, understood as a tensor product of Hilbert spaces, if there exists a symmetric tempered distribution $\tilde{\varphi} \in \mathcal{S}'((\mathbb{R}^2)^n)$ such that

$$\|\varphi\|^2_{(\dot{H}^s(\mathbb{R}^2))^{\otimes n}} \stackrel{\text{def}}{=} \int_{(\mathbb{R}^2)^n}\left(\prod_{i=1}^n |k_i|^{2s}\right)|\hat{\tilde{\varphi}}(k_{1:n})|^2\,\mathrm{d}k_{1:n} < \infty \tag{1.22}$$

for $k_{1:n} \stackrel{\text{def}}{=} (k_1, \dots, k_n)$, and

$$\langle \varphi, \psi \rangle_{(\dot{H}^s(\mathbb{R}^2))^{\otimes n}} = \hat{\tilde{\varphi}}\left(\left(\prod_{i=1}^{n} |\cdot_i|^{2s}\right)\hat{\psi}\right), \quad \psi \in \mathcal{S}_s((\mathbb{R}^2)^n)\,, \tag{1.23}$$

here $\mathcal{S}_s$ refers to symmetric schwarz functions. Note, left hand side of (1.23) is to be intepreted as ψ tested against φ giving meaning to what it it. Clearly, for $s \geq 0$, $\tilde{\varphi}$ can be taken to be φ itself. The same conventions apply to $\dot{H}^s(\mathbb{T}_M^2)$, but in the definition of the norm the integral is replaced by a weighted Riemann-sum. For $s = 1$, which will play an important role in what follows, we point out that the norm on $(\dot{H}^1(\mathbb{R}^2))^{\otimes n}$ can be equivalently written as

$$\|\tilde{\varphi}\|^2_{(\dot{H}^1(\mathbb{R}^2))^{\otimes n}} \stackrel{\text{def}}{=} \int_{(\mathbb{R}^2)^n} |\nabla \varphi(x_{1:n})|^2 \, \mathrm{d}x_{1:n}\,.$$

here ∇ is defined as

$$\nabla \varphi(x_{1:n}) = (\nabla_{x_1}\varphi, \nabla_{x_2}\varphi, \dots, \nabla_{x_n}\varphi)(x_{1:n}) \tag{1.24}$$

where for $x_i = (x_{i_1}, x_{i_2})$ we have

$$\nabla_{x_i}\varphi = (\partial_{x_{i_1}}\varphi, \partial_{x_{i_2}}\varphi). \tag{1.25}$$

We say that a function f on $\mathbb{R}^2$ is *symmetric* if for any permutation $\sigma : \{1, \dots, n\} \to \{1, \dots, n\}$ we have

$$f(x_1, \dots, x_n) = f(x_{\sigma(1)}, \dots, x_{\sigma(n)})\,.$$

1.5 Preliminaries on Wiener space analysis

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space and H be a separable Hilbert space with scalar product $\langle \cdot, \cdot \rangle$. A stochastic process μ is called *isonormal Gaussian process* (see [Nua06, Definition 1.1.1]) if $\{\mu(h) : h \in H\}$ is a family of centred jointly Gaussian random variables with correlation $\mathbb{E}(\mu(h)\mu(g)) = \langle h, g \rangle$. Given an isonormal Gaussian process μ on H and $n \in \mathbb{N}$, we define the *n-th homogeneous Wiener chaos* $\mathcal{H}_n$ as the closed linear subspace of $L^2(\boldsymbol{\eta}) = L^2(\Omega)$ generated by the random variables $H_n(\mu(h))$, for $h \in H$ of norm 1, where H_n is the n-th Hermite polynomial defined recursively via

$$H_0(x) = 1, H_n(x) = \frac{(-1)^n}{n!} e^{x^2/2} \frac{d^n}{\mathrm{d}x^n}\left(e^{-x^2/2}\right).$$

For $m \neq n$, $\mathscr{H}_n$ and $\mathscr{H}_m$ are orthogonal and, by [Nua06, Theorem 1.1.1], $L^2(\boldsymbol{\eta}) = \oplus_n \mathscr{H}_n$.

The isonormal Gaussian process μ we will be working with in chapter 2 is such that $H = \dot{H}^1(T)$, T being either the 2-dimensional torus $\mathbb{T}^2_M$ or $\mathbb{R}^2$, and has covariance

$$\mathbb{E}[\mu(\varphi)\mu(\psi)] \stackrel{\text{def}}{=} \langle \varphi, \psi \rangle_H \,, \qquad \varphi, \, \psi \in H \,. \tag{1.26}$$

Thanks to the results in [Nua06, Chapter 1], there exists an isomorphism I between the Fock space $\Gamma L^2 \stackrel{\text{def}}{=} \oplus_{n \geq 0} \dot{H}^1_{\text{sym}}(T^n)$ and $L^2(\boldsymbol{\eta})$, where $\dot{H}^1_{\text{sym}}(T^n)$ is the space of functions in $\dot{H}^1(T^n)$ which are symmetric with respect to permutations of variables. For $n \in \mathbb{N}$, the projection I_n of the isomorphism above to $\dot{H}^1_{\text{sym}}(T^n)$ is itself an isomorphism between $\dot{H}^1_{\text{sym}}(T^n)$ and $\mathscr{H}_n$ and, by [Nua06, Theorem 1.1.2], for every $F \in L^2(\boldsymbol{\eta})$ there exists unique sequence of symmetric functions $\{f_n\}_{n \geq 0} \in \Gamma L^2$ such that $F = \sum_{n=0}^{\infty} I_n(f_n)$ and

$$\mathbb{E}[F^2] = \sum_{n=0}^{\infty} n! \|f_n\|^2_{(\dot{H}^1(T))^{\otimes n}} \,. \tag{1.27}$$

Since the Hilbert space on which μ is defined is $\dot{H}^1(T)$ (and not $L^2(T)$ as in [CES21]), the isomorphism I must be handled with care and the results in [Nua06, Ch. 1.1.2] applied accordingly. In particular, in the present context [Nua06, Proposition 1.1.3] translates as follows. Let $f \in (\dot{H}^1(T))^{\otimes n}$ and $g \in (\dot{H}^1(T))^{\otimes m}$, then

$$I_n(f) I_m(g) = \sum_{p=0}^{2} p! \binom{n}{p} \binom{m}{p} I_{m+n-2p}(f \otimes_p g) \tag{1.28}$$

where

$$f \otimes_p g(x_{1:m+n-2p}) \stackrel{\text{def}}{=} \int_{T^p} \langle \nabla_{y_{1:p}} f(x_{1:n-p}, y_{1:p}), \nabla_{y_{1:p}} g(x_{n-p+1:m+n-2p}, y_{1:p}) \rangle \, \mathrm{d}y_{1:p} \tag{1.29}$$

where $\langle \cdot, \cdot \rangle$ denotes the usual scalar product in $\mathbb{R}^p$, the gradient $\nabla_{y_{1:p}}$ is only applied to the variables $y_{1:p}$ and, as in (1.22), $x_{1:n} = (x_1, \ldots, x_n)$. In chapter 3 we will work with the same framework but with $H = L^2(T)$, η being an isonormal Gaussian process with covariance

$$\mathbb{E}[\eta(\varphi)\eta(\psi)] \stackrel{\text{def}}{=} \langle \varphi, \psi \rangle_H \,, \qquad \varphi, \, \psi \in H \,. \tag{1.30}$$

Note, throughout this thesis η and μ are the gaussian processes and not the measures with respespect to which we define our probability space. For the measures bold equivalent is used $\boldsymbol{\eta}$.

We say that $F : \mathcal{S}' \to \mathbb{R}$ is a *cylinder function* if there exist $\varphi_1, \dots, \varphi_n \in \mathcal{D}$ such that

$$F[u] = f(u(\varphi_1), \dots, u(\varphi_n)) , \tag{1.31}$$

for some function $f : \mathbb{R}^n \to \mathbb{R}$ which is smooth and whose partial derivatives grow at most polynomially at infinity. We denote the set of cylinder functions by $\mathcal{C}$. We call a random variable $F \in L^2(\boldsymbol{\eta})$ smooth if it is a cylinder function on $\mathcal{S}'$ endowed with the measure $\boldsymbol{\eta}$. The *Malliavin derivative* of a smooth random variable $F = f(\eta(\varphi_1), \dots, \eta(\varphi_n))$ is the H-valued random variable given by

$$DF \stackrel{\text{def}}{=} \sum_{i=1}^{n} \partial_i f(\eta(\varphi_1), \dots, \eta(\varphi_n))\varphi_i , \tag{1.32}$$

and we will denote by $D_x F$ the evaluation of DF at x and by $D_k F$ its Fourier transform at k. A commonly used tool in Wiener space analysis is *Gaussian integration by parts* [Nua06, Lemma 1.2.2] which states that for any two smooth random variables $F, G \in L^2(\boldsymbol{\eta})$ we have

$$\mathbf{E}[G\langle DF, h\rangle_H] = \mathbf{E}[-F\langle DG, h\rangle_H + FG\mu(h)] . \tag{1.33}$$

We will frequently work with the Fourier transform of μ which is a family of complex valued Gaussian random variables. Even though, strictly speaking, the results above do not cover this case, in [CES21, Section 2] it was shown that one can naturally extend $\dot{H}^0(\mathbb{T}^2, \mathbb{R}) = L^2(\mathbb{T}^2, \mathbb{R})$ to $L^2(\mathbb{T}^2, \mathbb{C})$. Such extension can also be performed in the present context, and we are therefore allowed to exploit (1.33) also in case of complex-valued h.

CHAPTER 2

Stochastic Navier-Stokes on the real plane under critical regime

2.1 Invariant measures of the regularised equations

The goal of this section is to construct a stationary solution to the regularised critical Navier-Stokes equation on $\mathbb{R}^2$. We will first consider the analogous equation on the torus of fixed size, where invariance is easier to obtain. Subsequently, via a compactness argument, we will scale the size of the torus to infinity and characterise the limit of the corresponding solutions via a martingale problem.

2.1.1 The regularised Vorticity equation on $\mathbb{T}_M^2$

For $\theta \in (0,1]$, we consider the periodic version on $\mathbb{T}_M^2$ of (1.10) given by

$$\partial_t \omega^{N,M} = -\tfrac{1}{2}(-\Delta)^\theta \omega^{N,M} - \lambda_N \mathcal{N}^{N,M}[\omega^{N,M}] + (-\Delta)^{\frac{\theta+1}{2}} \xi^M\,, \quad \omega^{N,M}(0,\cdot) = \omega_0^M\,, \tag{2.1}$$

where ω_0^M is the initial condition, ξ^M is a space-time white noise on $\mathbb{R} \times \mathbb{T}_M^2$ and $\mathcal{N}^{N,M}$ is the non-linearity defined in (1.9). In Fourier variables, (2.1) becomes

$$\mathrm{d}\,\hat{\omega}_k^{N,M} = -\tfrac{1}{2}|k|^{2\theta}\hat{\omega}_k^{N,M} - \lambda_N \mathcal{N}_k^{N,M}[\omega^{N,M}] + |k|^{\theta+1}\,\mathrm{d}B_k(t)\,, \qquad k \in \mathbb{Z}_M^2 \tag{2.2}$$

where the complex-valued Brownian motions B_k are defined via $B_k(t) \stackrel{\text{def}}{=} \int_0^t \hat{\xi}_k^M(\,\mathrm{d}s)$, $\hat{\xi}_k^M$ being the k-th Fourier mode of ξ^M, and $\mathcal{N}_k^{N,M}$ is the fourier transform of the non-linearity $\mathcal{N}^{N,M}$ which can be expressed in the following form

$$\mathcal{N}_k^{N,M}[\omega^{N,M}] = \frac{1}{M^2} \sum_{\ell+m=k} \mathcal{K}_{\ell,m}^N \omega_\ell^{N,M} \omega_m^{N,M}\,, \tag{2.3}$$

for

$$\mathcal{K}_{\ell,m}^N \stackrel{\text{def}}{=} \frac{1}{2\pi} \hat{\varrho}_{\ell,m}^N \frac{(\ell^\perp \cdot (\ell+m))(m \cdot (\ell+m))}{|\ell|^2 |m|^2}\,, \qquad \text{with} \quad \hat{\varrho}_{\ell,m}^N \stackrel{\text{def}}{=} \hat{\varrho}_\ell^N \hat{\varrho}_m^N \hat{\varrho}_{\ell+m}^N\,. \tag{2.4}$$

The proof of this can be found in the appendix, Proposition 3.3.3. As a first step in our analysis, we determine basic properties of the solution of (2.1). Bringing the results of the calculations above together we have

Proposition 2.1.1 *Let $M, N \in \mathbb{N}$ and $\theta \in (0,1]$. Then, for every deterministic initial condition $\omega_0^{N,M} \in \dot{H}^{-2}(\mathbb{T}_M^2)$, (2.1) has a unique strong solution $\omega^{N,M} \in C(\mathbb{R}_+, \dot{H}^{-2}(\mathbb{T}_M^2))$. Further, $\omega^{N,M}$ is a strong Markov process.*

Proof. This proof is based on [GP18b, Lemma 2.1.], we start by splitting the solution as follows

$$\omega^{N,M} = \Pi_N \omega^{N,M} + (1 - \Pi_N)\omega^{N,M}\,, p \tag{2.5}$$

here Π_N is the fourier cut-off at N that is

$$\mathcal{F}(\Pi_N f)(k) = \mathbb{1}_{k \leq N} \mathcal{F}(f)(k)$$

The second term in (2.5) is an Ornstein–Uhlenbeck process which is well-known to belong to $C(\mathbb{R}_+, \dot{H}^{-2}(\mathbb{T}_M^2))$.

Meanwhile the first term instead solves a non-linear SPDE with finite fourier modes. The non-linear part of that equation satisfies $\langle \mathcal{N}^N[\mu], \mu \rangle_{\dot{H}^{-1}(T)} = 0$ by Lemma 2.1.2, since the linear part also preserves the $\dot{H}^{-1}$ norm thus the equation as a whole preserves the $\dot{H}^{-1}$ norm. The conclusion can therefore be reached arguing as in [GJ13, Section 7] (see also [CES21, Proposition 3.4]). □

Lemma 2.1.2 *Let $T = \mathbb{T}_M^2$ or $\mathbb{R}^2$. Then for any schwarz distribution $\mu \in \mathcal{S}'(T)$ such that*

$\nabla \cdot (K * (\mu * \varrho^N)) = 0$ *we have*

$$\langle \mathcal{N}^N[\mu], \mu \rangle_{\dot{H}^{-1}(T)} = 0\,.$$

Proof. Let $\psi^N = K * (\mu * \varrho^N)$ so that $\nabla \cdot \psi^N = 0$. We will denote $\mu^N = \mu * \varrho^N$. By definition of $\mathcal{N}^N$ from (1.9) we have

$$\begin{aligned}
\langle \mathcal{N}^N[\mu], \mu \rangle_{\dot{H}^{-1}(T)} &= \langle \nabla \cdot \varrho^N * (\psi^N \mu^N), \mu \rangle_{\dot{H}^{-1}(T)} \\
&= \langle \nabla \cdot (\psi^N \mu^N), \mu^N \rangle_{\dot{H}^{-1}(T)} \\
&= \langle (\nabla \cdot \psi^N) \mu^N, \mu^N \rangle_{\dot{H}^{-1}(T)} + \langle \psi^N \cdot \nabla \mu^N, \mu^N \rangle_{\dot{H}^{-1}(T)}
\end{aligned}$$

Since $\nabla \cdot \psi^N = 0$ the first term is equal to 0, focusing on the second term

$$\begin{aligned}
\langle \psi^N \cdot \nabla \mu^N, \mu^N \rangle_{\dot{H}^{-1}(T)} &= \langle \psi_1^N \partial_1 \mu^N, \mu^N \rangle_{\dot{H}^{-1}(T)} + \langle \psi_2^N \partial_2 \mu^N, \mu^N \rangle_{\dot{H}^{-1}(T)} \\
&= \langle \psi_1^N \mu^N, \partial_1 \mu^N \rangle_{\dot{H}^{-1}(T)} + \langle \psi_2^N \mu^N, \partial_2 \mu^N \rangle_{\dot{H}^{-1}(T)} \\
&= -\langle \partial_1(\psi_1^N \mu^N), \mu^N \rangle_{\dot{H}^{-1}(T)} - \langle \partial_2(\psi_2^N \mu^N), \mu^N \rangle_{\dot{H}^{-1}(T)} \\
&= -\langle \psi^N \cdot \nabla \mu^N, \mu^N \rangle_{\dot{H}^{-1}(T)}\,,
\end{aligned}$$

And hence the second term is also 0 proving the result. □

Even though the generator $\mathscr{L}^{N,M}$ of the Markov process $\omega^{N,M}$, is a complicated operator, its action on cylinder functions F can be easily obtained by applying Itô's formula and singling out the drift term. By doing so, we deduce that for any such F, $\mathscr{L}^{N,M} F$ can be written as $\mathscr{L}^{N,M} F = \mathscr{L}_\theta^M F + \mathscr{A}^{N,M} F$. For cylinder $F[\omega] = f(\omega(\varphi_1), \dots, \omega(\varphi_n))$ the operators $\mathscr{L}_0^M$ and $\mathscr{A}^{N,M}$ are given by

$$\begin{aligned}
\mathscr{L}_\theta^M F(\omega) &\stackrel{\text{def}}{=} \frac{1}{2} \sum_{i=1}^n \omega(-(-\Delta)^\theta \varphi_i) \partial_i f + \frac{1}{2} \sum_{i,j=1}^n \langle \varphi_i, \varphi_j \rangle_{\dot{H}^{\theta+1}(\mathbb{T}_M^2)} \, \partial_{i,j}^2 f, \\
\mathscr{A}^{N,M} F(\omega) &\stackrel{\text{def}}{=} -\lambda_N \sum_{i=1}^n \mathcal{N}^{N,M}[\omega](\varphi_i) \, \partial_i f,
\end{aligned} \tag{2.6}$$

where we abbreviated $\partial_i f = \partial_i f(\omega(\varphi_1), \dots, \omega(\varphi_n))$. We are now ready to prove the following proposition.

Proposition 2.1.3 *Let μ^M be the Gaussian spatial noise on $\mathbb{T}_M^2$ with covariance given by* (1.26). *Then, for every $\theta \in (0, 1]$, μ^M is an invariant measure of the solution $\omega^{N,M}$ of* (2.1).

Proof. The proof of this statement follows the steps of [GJ13, Section 7] but we provide it here for completeness. Note the equation (2.1) on the torus is infinite dimensional, but the equation does live in a locally compact space. In such case we are able to apply Echeverrìa's criterion [Ech82]. Note this will not be possible on the real plane since there the the space on which the function lives is not locally compact. Briefly speaking the criterion states that for an equation on locally compact separable metric space for μ^M to be an invariant measure it suffices to show that

$$\mathbb{E}[\mathscr{L}^{N,M}F(\mu^M)] = 0\,,$$

for any cylinder function $F = f(\mu^M(\varphi_1), \dots, \mu^M(\varphi_n))$, where $\mathbb{E}$ is the expectation taken with respect to the law of $\boldsymbol{\eta}$. Since, throughout the proof M is fixed, we will omit it as a superscript to lighten the notation. We will use the Fourier representation of the operators $\mathscr{L}_\theta^M$ and $\mathscr{A}^{N,M}$, which can be deduced by (2.6) simply taking F depending on (finitely many) Fourier modes of μ and is

$$\mathscr{L}_\theta^M F(\mu) = \frac{1}{2M^2}\sum_k |k|^{2\theta}\Big(-\mu_{-k}D_k + |k|^2 D_{-k}D_k\Big)F(\mu)\,, \tag{2.7}$$

$$\mathscr{A}^{N,M}F(\mu) = -\frac{\lambda_N}{M^4}\sum_{i,j}\mathscr{K}^N_{i,j}\mu_i\mu_j D_{-i-j}F(\mu)\,. \tag{2.8}$$

Let us first show that $\mathbb{E}[\mathscr{L}_\theta^M F(\mu)] = 0$. Let $k \in \mathbb{Z}^2_M$. Exploiting $|k|^2 e_k = (-\Delta)e_k$ and applying Gaussian integration by parts (1.33) with $h = e_k$, $G = 1$ and $F = D_k F$, we obtain

$$\mathbb{E}[|k|^2 D_{-k}D_k F(\mu)] = \mathbb{E}[\langle D(D_k F(\mu)), e_k\rangle_{\dot{H}^1}] = \mathbb{E}[\mu^M_{-k}D_k F(\mu)]$$

which immediately implies $\mathbb{E}[\mathscr{L}_\theta F(\mu)] = 0$. We now turn to $\mathbb{E}[\mathscr{A}^{N,M}F(\mu)]$. Let $i,\ j \in \mathbb{Z}^2_M$ such that $i + j \neq 0$. We apply once more Gaussian integration by parts, this time choosing $G = \mu_i\mu_j$ and $h = e_{i+j}$, so that we have

$$\begin{aligned}
\mathbb{E}[\mu_i\mu_j D_{-i-j}F(\mu)] &= -\frac{1}{|i+j|^2}\mathbb{E}[\mu_i\mu_j\langle DF(\mu), e_{i+j}\rangle_{\dot{H}^1(\mathbb{T}^2_M)}]\\
&= -\frac{1}{|i+j|^2}\mathbb{E}[-F(\mu)\langle D(\mu_i\mu_j), e_{i+j}\rangle_{\dot{H}^1(\mathbb{T}^2_M)} + \mu_i\mu_j\mu_{-i-j}F(\mu)]\\
&= \mathbb{E}[-F(\mu)D_{-i-j}(\mu_i\mu_j)] - \mathbb{E}\left[\left(\frac{1}{|i+j|^2}\mu_i\mu_j\mu_{-i-j}\right)F(\mu)\right]
\end{aligned}$$

Now, $D_{-i-j}(\mu_i\mu_j) \neq 0$ if and only if either i or j are 0 in which case $\mathscr{K}^N_{i,j}$ in (2.4) is 0. Hence, the first summand above does not contribute to $\mathbb{E}[\mathscr{A}^{N,M}F(\mu)]$ and we obtain

$$\mathbb{E}[\mathscr{A}^{N,M}F(\mu)] = \mathbb{E}\left[\left(-\frac{\lambda_N}{M^4}\sum_{i,j}\frac{\mathscr{K}^N_{i,j}}{|i+j|^2}\mu_i\mu_j\mu_{-i-j}\right)F(\mu)\right]$$
$$= \mathbb{E}\Big[\langle \mathscr{N}^{N,M}[\mu],\mu\rangle_{\dot{H}^{-1}(\mathbb{T}^2_M)}F(\mu)\Big] = 0$$

where the last equality follows by Lemma 2.1.2 so that the proof is concluded. □

From now on, we will only work with the stationary solution of (2.1), i.e. the initial condition will always be taken to be

$$\omega_0^{N,M} \stackrel{\text{def}}{=} \mu^M \tag{2.9}$$

where μ^M is as in Proposition 2.1.3. In the following statements, we aim at obtaining estimates on the solution $\omega^{N,M}$ to (2.1) which are uniform in both N and M. A crucial tool is the so-called Itô's trick, first introduced in [GJ13]. To the reader's convenience, we now recall its statement, adapted to the present context.

Lemma 2.1.4 (Itô-Trick) *Let $\theta \in (0,1]$. Let $\mathscr{L}^{N,M}$ be the generator of the Markov process $\omega^{N,M}$, solution to* (1.10) *started from the invariant measure μ^M in* (2.9)*, and $\mathscr{L}^M_\theta$ and $\mathscr{A}^{N,M}$ be defined according to* (2.6)*. Let $T>0$ and F a cylinder function on $\mathcal{S}'(\mathbb{T}^2_M)$ as defined in* (1.31)*. Then, for every $p \geq 2$, there exists a constant $C>0$ depending only on p such that*

$$\mathbf{E}\left[\sup_{t\leq T}\left|\int_0^t \mathscr{L}^M_\theta F(\omega_s^{N,M})\,\mathrm{d}s\right|^p\right]^{1/p} \leq CT^{\frac{1}{2}}\mathbb{E}[\mathscr{E}(F)]^{1/2}\,, \tag{2.10}$$

where the energy $\mathscr{E}(F)$ is given by

$$\mathscr{E}^M(F)(\mu^M) \stackrel{\text{def}}{=} \frac{1}{M^2}\sum_{k\in\mathbb{Z}^2_M}|k|^{2+2\theta}|D_kF(\mu^M)|^2 = \int_{\mathbb{T}^2_M}|(-\Delta_x)^{\frac{1+\theta}{2}}D_xF(\mu^M)|^2\,\mathrm{d}x\,, \tag{2.11}$$

the Laplacian above clearly acting on the x variable. Here and throughout, $\mathbf{E}$ *denotes the expectation with respect to the law of the process $\{\omega_t^{N,M}\}_{t\in[0,T]}$, while $\mathbb{E}$ that with respect to the invariant measure μ^M.*

Proof. This is proof is closely following that of [GP18a][CES21]. Let F be a cylinder function on $\mathcal{S}'(\mathbb{T}^2_M)$ as defined in (1.31). Via Itô formula we get

$$F(\omega_t) = F(\omega_0) + \int_0^t(\partial s + \mathscr{L}^M_\theta + \mathscr{A}^{N,M})F(\omega^N_s)\,\mathrm{d}s + M_t(F)\,, \tag{2.12}$$

where ∂s appears as $F(\omega_s^N)$ depends on time (as well as space). Here $M_t(F) = \int_0^t (-\Delta)^{\frac{\theta+1}{2}} D_x F dB_t$ is a martingale with quadratic variation given by

$$d\langle M(F)\rangle_s = \mathscr{E}(F)(\omega_s^N)\,\mathrm{d}s\,.$$

For $\tilde{\omega}_t \stackrel{\text{def}}{=} \omega_{T-t}$ the generator is the adjoint of the generator $\mathscr{L}_\theta^M + \mathscr{A}^{N,M}$ which by (2.2.1) we know to be $\mathscr{L}_\theta^M - \mathscr{A}^{N,M}$ and so again by Itô formula we get

$$\begin{aligned}
&F(\tilde{\omega}_T) - F(\omega_t) \\
&= F(\tilde{\omega}_T) - F(\tilde{\omega}_{T-t}) \\
&= \int_{T-t}^T (\partial s + \mathscr{L}_\theta^M - \mathscr{A}^{N,M}) F(\tilde{\omega}_s)\,\mathrm{d}s + \tilde{M}_T(F) - \tilde{M}_{T-t}(F) \\
&= \int_0^t ((-\partial s + \mathscr{L}_\theta^M - \mathscr{A}^{N,M})F)(\omega_s^N)\,\mathrm{d}s + \tilde{M}_T(F) - \tilde{M}_{T-t}(F)\,,
\end{aligned} \tag{2.13}$$

where

$$\mathrm{d}\langle \tilde{M}(F)\rangle_s = \mathscr{E}(F)(\tilde{\omega}_s)\,\mathrm{d}s\,.$$

But $F(\tilde{\omega}_T) = F(\omega_0)$ hence by (2.13) and (2.12) we get

$$0 = 2\int_0^t \mathscr{L}_\theta^M F(\omega_s^N)\,\mathrm{d}s + M_t(F) + \tilde{M}_T(F) - \tilde{M}_{T-t}(F)\,.$$

By Burkholder-Davis-Gundy inequality and the equation above we obtain

$$\begin{aligned}
\mathbf{E}\bigg(\sup_{t\le T}\bigg|\int_0^t \mathscr{L}_\theta^M F(\omega_s^N)\,\mathrm{d}s\bigg|^p\bigg) &\simeq \mathbf{E}\bigg(\sup_{t\le T}|M_t(F) + \tilde{M}_T(F) - \tilde{M}_{T-t}(F)|^p\bigg) \\
&\lesssim \mathbf{E}(\langle M(F)\rangle_T^{p/2}) + \mathbf{E}(\langle \tilde{M}(F)\rangle_T^{p/2}) \\
&\simeq \mathbf{E}\bigg[\bigg(\int_0^T \mathscr{E}(F(\omega_s^N))ds\bigg)^{p/2}\bigg] \\
&\lesssim T^{p/2-1}\int_0^T \mathbf{E}\Big(\mathscr{E}(F(\omega_0))^{p/2}\Big)\,\mathrm{d}s\,.
\end{aligned}$$

Here we used Gaussian hypercontractivity [Nua06, Theorem 1.4.1] to replace the $p/2$ moment at the right hand with the square-root of the expectation of the energy. □

The Itô's trick allows us to upper-bound moments of the integral in time of certain functionals of $\omega^{N,M}$ in terms of the first moment of the energy $\mathscr{E}$ with respect to the law of $\omega^{N,M}$ at fixed time. Such a law is explicit and Gaussian making the bound particularly useful. In the following proposition, we determine suitable estimates on the non-linearity.

Proposition 2.1.5 *Let $\theta \in (0,1]$, $T > 0$ be fixed and $p \geq 2$. For $M,\ N \in \mathbb{N}$, let $\mathcal{N}^{N,M}$ be defined according to* (2.3) *and λ_N be as in* (1.11)*. Then, there exists a constant $C = C(p) > 0$, independent of $M,\ N \in \mathbb{N}$ such that for all $\varphi \in \mathcal{S}(\mathbb{T}^2_M)$ and all $t \in [0,T]$, we have*

$$\mathbf{E}\left[\sup_{s\leq t}\left|\int_0^s \omega_r^{N,M}(-(-\Delta)^\theta\varphi)\,\mathrm{d}r\right|^p\right]^{1/p} \leq Ct^{\frac{1}{2}}\|\varphi\|_{\dot{H}^{1+\theta}(\mathbb{T}^2_M)}\,, \tag{2.14}$$

$$\mathbf{E}\left[\sup_{s\leq t}\left|\lambda_N\int_0^s \mathcal{N}^{N,M}[\omega_r^{N,M}](\varphi)]\,\mathrm{d}r\right|^p\right]^{1/p} \leq CN^{\theta-1}(t\vee t^{\frac{1}{2}})\|\varphi\|_{\dot{H}^{2}(\mathbb{T}^2_M)}\,. \tag{2.15}$$

The proof of the previous proposition (and in particular of (2.15)) is based on the following lemma.

Lemma 2.1.6 *For $M,\ N \in \mathbb{N}$, $\varphi \in \mathcal{S}(\mathbb{T}^2_M)$, let $\mathcal{N}^{N,M}[\mu^M](\varphi)$ be the smooth random variable defined according to* (2.3)*, with μ^M replacing $\omega^{N,M}$. Then, $\mathcal{N}_k^{N,M}[\mu^M](\varphi)$ belongs to the second homogeneous Wiener chaos $\mathscr{H}_2$. Further, for all $\theta \in (0,1]$ the Poisson equation*

$$(1-\mathscr{L}_\theta^M)H^{N,M}[\mu^M](\varphi) = \lambda_N\mathcal{N}^{N,M}[\mu^M](\varphi) \tag{2.16}$$

has a unique solution whose energy satisfies

$$\mathbb{E}[\mathscr{E}^{N,M}(H^{N,M}[\mu^M](\varphi))] = \frac{4\lambda_N^2}{M^4}\sum_{\ell,m}|\ell|^{2+2\theta}|m|^2\frac{(\mathscr{K}^N_{\ell,m})^2}{(1+\frac{1}{2}(|\ell|^{2\theta}+|m|^{2\theta}))^2}|\varphi_{-\ell-m}|^2\,. \tag{2.17}$$

Proof. Note that, by (1.9) the non-linearity $\mathcal{N}^{N,M}$ tested against φ can be written as

$$\mathcal{N}^{N,M}[\mu^M](\varphi) = -\langle \mu^M(K*\varrho^N_\cdot)\,\mu^M(\varrho^N_\cdot)\,,\,\nabla\varphi*\varrho^N_\cdot\rangle\,, \tag{2.18}$$

the scalar product at the right hand side being the usual L^2 pairing. Now, thanks to our choice of the mollifier ϱ in (1.12), and in particular the fact that its Fourier transform is 0 in a neighbourhood of the origin, both $K*\varrho^N_\cdot$ and $\varrho^N_\cdot$ live in $\mathcal{S}(\mathbb{T}^2_M)$ so that the expectation of the right hand side of (2.18) is finite by (1.26). Hence, further using translation invariance, we have

$$\mathbb{E}[\mathcal{N}^{N,M}[\mu^M](\varphi)] = \langle\mathbb{E}[\mu^M(K*\varrho^N)\,\mu^M(\varrho^N)],\nabla\varphi*\varrho^N\rangle\rangle = \langle K*\varrho^N,\varrho^N\rangle_{\dot{H}^1(\mathbb{T}^2_M)}\langle 1,\nabla\varphi*\varrho^N\rangle\,,$$

which is zero since, by integration by parts, $\langle 1,\nabla\varphi*\varrho^N\rangle = 0$. Now, $\mathcal{N}^{N,M}[\mu](\varphi)$ is quadratic in μ and its component in the 0-th chaos is 0, hence $\mathcal{N}^N[\mu^M](\varphi) \in \mathscr{H}_2$ and $\mathcal{N}^{N,M}[\mu^M](\varphi) = I_2(\mathfrak{n}_\varphi^{N,M})$, for $\mathfrak{n}_\varphi^{N,M}$ such that

$$\hat{\mathfrak{n}}_\varphi^{N,M}(\ell,m) = \mathscr{K}^N_{\ell,m}\varphi_{-\ell-m}\,. \tag{2.19}$$

Let $\mathfrak{h}_\varphi^{N,M} \in \Gamma L^2_2$ and $H^{N,M}[\mu^M](\varphi) = I_2(\mathfrak{h}_\varphi^{N,M})$. Then by the definition of (2.6) (also shown in [GT20, Lemma 2.3]) we have

$$(1-\mathscr{L}_\theta^M)H^{N,M}[\mu^M](\varphi) = (1-\mathscr{L}_\theta^M)I_2(\mathfrak{h}_\varphi^{N,M}) = I_2\Big((1-\tfrac{1}{2}(-\Delta)^\theta)\mathfrak{h}_\varphi^{N,M}\Big)\,.$$

Equating the right hand side above and $\lambda_N I_2(\mathfrak{n}_\varphi^{N,M})$, we immediately deduce that (2.16) has a unique solution which must necessarily satisfy

$$\hat{\mathfrak{h}}_\varphi^{N,M}(\ell,m) = \lambda_N \frac{\lambda_N}{(2\pi)^2}\frac{\mathscr{K}^N_{\ell,m}}{1+\frac{1}{2}(|\ell|^{2\theta}+|m|^{2\theta})}\varphi_{-\ell-m}\,, \qquad \text{for all } \ell,m\in\mathbb{Z}^2_M. \tag{2.20}$$

In order to compute the energy of $H^{N,M}[\mu^M](\varphi)$, notice that by [Nua06, Proposition 1.2.7],

$$D_x H^{N,M}[\mu^M](\varphi) = D_x I_2(\mathfrak{h}_\varphi^{N,M}) = 2I_1(\mathfrak{h}_\varphi^{N,M}(x,\cdot))$$

which implies, by linearity of I_1,

$$\mathscr{E}^{N,M}(H^{N,M}[\mu^M](\varphi)) = 4\int_{\mathbb{T}^2_M}\Big|I_1\Big((-\Delta_x)^{\frac{1+\theta}{2}}\mathfrak{h}_\varphi^{N,M}(x,\cdot)\Big)\Big|^2\,\mathrm{d}x\,.$$

Consequently, since I_1 is an isometry from $\mathscr{H}_1$ and $\Gamma L^2_1 = \dot H^1(\mathbb{T}^2_M)$, we get

$$\mathbb{E}[\mathscr{E}^{N,M}(H^{N,M}[\mu^M](\varphi))] = 4\int_{\mathbb{T}^2_M}\|(-\Delta_x)^{\frac{1+\theta}{2}}\mathfrak{h}_\varphi^{N,M}(x,\cdot)\|^2_{\dot H^1(\mathbb{T}^2_M)}\,\mathrm{d}x$$

by Plancherel's identity and (2.20) we get

$$\begin{aligned}
&4\int_{\mathbb{T}^2_M}\|(-\Delta_x)^{\frac{1+\theta}{2}}\mathfrak{h}_\varphi^{N,M}(x,\cdot)\|^2_{\dot H^1(\mathbb{T}^2_M)}\,\mathrm{d}x\\
&= 4\int_{(\mathbb{T}^2_M)^2}(-\Delta_y)\Big((-\Delta_x)^{\frac{1+\theta}{2}}\mathfrak{h}_\varphi^{N,M}(x,\cdot)\Big)^2\,\mathrm{d}x\,\mathrm{d}y\\
&= \frac{4}{(2\pi)^2M^4}\sum_{\ell,m}|\ell|^{2+2\theta}|m|^2\mathscr{F}(\mathfrak{h}_\varphi^{N,M})(\ell,m)\,.
\end{aligned}$$

from which (2.17) follows. □

Proof of Proposition 2.1.5. For both (2.14) and (2.15), we will exploit the Itô's trick Lemma 2.1.4. Let us begin with the former. Set $K^{N,M}[\mu^M](\varphi) \stackrel{\text{def}}{=} \mu^M(\varphi)$, and notice that by (2.6), it is immediate that

$$\mathscr{L}_\theta^M K^{N,M}[\mu^M](\varphi) = \mu^M\Big(-\tfrac{1}{2}(-\Delta)^\theta\varphi\Big)\,,$$

By (2.11) we have

$$\mathscr{E}^{N,M}(K^{N,M}[\mu^M](\varphi)) = \frac{1}{M^2}\sum_k |k|^{2+2\theta}|D_k\mu^M(\varphi)|^2 = \|\varphi\|^2_{\dot H^{1+\theta}(\mathbb{T}^2_M)}\,.$$

Hence, the left hand side of (2.14) equals

$$2\mathbf{E}\Big[\sup_{s\le t}\Big|\int_0^s \mathscr{L}_\theta^M K^M[\omega_r^{N,M}](\varphi)\,\mathrm{d}r\Big|^p\Big]^{1/p} \lesssim t^{\frac{1}{2}}\|\varphi\|_{\dot H^{1+\theta}(\mathbb{T}^2_M)}\,, \tag{2.21}$$

where in the last passage we applied (2.10).

We now turn to (2.15) for which we proceed similarly to [GP18a, Proposition 3.15]. Let $H^{N,M}$ be the solution to (2.16) determined in Lemma 2.1.6. Then

$$\begin{aligned}\mathbf{E}\Big[\sup_{s\le t}\Big|\int_0^s \lambda_N\mathscr{N}^N[\omega_r^{N,M}](\varphi)\,\mathrm{d}r\Big|^p\Big]^{\frac{1}{p}} &= \mathbf{E}\Big[\sup_{s\le t}\Big|\int_0^s (1-\mathscr{L}_\theta^M)H^{N,M}[\omega_r^{N,M}](\varphi)\,\mathrm{d}r\Big|^p\Big]^{\frac{1}{p}}\\ \le \mathbf{E}\Big[\sup_{s\le t}\Big|\int_0^s H^{N,M}[\omega_r^{N,M}](\varphi)\,\mathrm{d}r\Big|^p\Big]^{\frac{1}{p}} &+ \mathbf{E}\Big[\sup_{s\le t}\Big|\int_0^s \mathscr{L}_\theta^M H^{N,M}[\omega_r^{N,M}](\varphi)\,\mathrm{d}r\Big|^p\Big]^{\frac{1}{p}}\,.\end{aligned} \tag{2.22}$$

We will separately estimate the two summands above. For the second, we apply once more (2.10), which, together with (2.17), gives

$$\begin{aligned}\mathbf{E}\Big[\sup_{s\le t}\Big|\int_0^s \mathscr{L}_\theta^M H^{N,M}[\omega_r^{N,M}](\varphi)\,\mathrm{d}r\Big|^p\Big]^{\frac{2}{p}} &\lesssim t\frac{\lambda_N^2}{M^4}\sum_{\ell,m}|\ell|^{2+2\theta}|m|^2\frac{(\mathscr{K}^N_{\ell,m})^2|\varphi_{-\ell-m}|^2}{(1+\frac{1}{2}(|\ell|^{2\theta}+|m|^{2\theta}))^2}\\ &\lesssim t\frac{1}{M^2}\sum_k |k|^4|\varphi_k|^2\frac{\lambda_N^2}{M^2}\sum_{\ell+m=k}(\hat\varrho^N_{\ell,m})^2\frac{|\ell|^{2\theta}}{(1+\frac{1}{2}(|\ell|^{2\theta}+|m|^{2\theta}))^2}\\ &\lesssim t\frac{1}{M^2}\sum_k |k|^4|\varphi_k|^2\frac{\lambda_N^2}{M^2}\sum_\ell(\hat\varrho^N_\ell)^2\frac{1}{1+\frac{1}{2}|\ell|^{2\theta}} \le t\|\varphi\|^2_{\dot H^2(\mathbb{T}^2_M)}\frac{\lambda_N^2}{M^2}\sum_{|\ell|\le N}\frac{1}{1+\frac{1}{2}|\ell|^{2\theta}}\end{aligned}$$

where we bounded $|\mathscr{K}^N_{\ell,m}| \le \hat\varrho^N_\ell|\ell+m|^2/(|\ell||m|)$ and applied a simple change of variables. Now, the remaining sum can be controlled via

$$\frac{\lambda_N^2}{M^2}\sum_{|\ell|\le N}\frac{1}{1+\frac{1}{2}|\ell|^{2\theta}} \lesssim \lambda_N^2\int_{|x|\le N}\frac{\mathrm{d}x}{1+\frac{1}{2}|x|^{2\theta}} \lesssim \begin{cases}\lambda_N^2\log N \lesssim 1\,, & \text{if } \theta=1,\\ \lambda_N^2 N^{2-2\theta}\lesssim N^{2\theta-2}\,, & \text{if } \theta\in(0,1),\end{cases}$$

the last inequality being a consequence of (1.11).

Let us turn to the first summand in (2.22). We have

$$\begin{aligned}\mathbf{E}\Big[\sup_{s\le t}\Big|\int_0^s H^{N,M}[\omega_r^{N,M}](\varphi)\,\mathrm{d}r\Big|^p\Big]^{\frac{1}{p}} &\le \mathbf{E}\Big[\Big(\int_0^t |H^{N,M}[\omega_r^{N,M}](\varphi)|\,\mathrm{d}r\Big)^p\Big]^{\frac{1}{p}}\\ &\le t^{1-\frac{1}{p}}\mathbf{E}\Big[\int_0^t |H^{N,M}[\omega_r^{N,M}](\varphi)|^p\,\mathrm{d}r\Big]^{1/p} = t\mathbb{E}[|H^{N,M}[\mu^M](\varphi)|^p]^{\frac{1}{p}}\\ &\lesssim t\mathbb{E}[|H^{N,M}[\mu^M](\varphi)|^2]^{\frac{1}{2}} \lesssim t\|\mathfrak{h}_\varphi^{N,M}\|_{\Gamma L^2_2}\end{aligned}$$

where, from the first to the second line we used Jensen's inequality, from the second to the third Gaussian hypercontractivity [Nua06, Theorem 1.4.1] and the last step is a consequence of (1.27) and the fact that, as shown in the proof of Lemma 2.1.6, $H^{N,M}[\mu^M](\varphi) = I_2(\mathfrak{h}_\varphi^{N,M})$ for $\mathfrak{h}_\varphi^{N,M}$ satisfying (2.20). In turn, the norm of $\mathfrak{h}_\varphi^{N,M}$ can be estimated via

$$\begin{aligned}\|\mathfrak{h}_\varphi^{N,M}\|^2_{\Gamma L^2_2} &= \frac{\lambda_N^2}{(2\pi)^2M^4}\sum_{\ell,m\in\mathbb{Z}^2_M}|\ell|^2|m|^2\frac{(\mathscr{K}^N_{\ell,m})^2}{(1+\frac{1}{2}(|\ell|^{2\theta}+|m|^{2\theta}))^2}|\varphi_{-\ell-m}|^2\\ &\lesssim \frac{\lambda_N^2}{M^2}\sum_{k\in\mathbb{Z}^2_M}|k|^4|\varphi_k|^2\frac{1}{M^2}\sum_{\ell+m=k}\frac{(\hat{\varrho}^N_\ell)^2}{(1+\frac{1}{2}(|\ell|^{2\theta}+|m|^{2\theta}))^2}\\ &\lesssim \frac{\lambda_N^2}{M^2}\sum_{k\in\mathbb{Z}^2_M}|k|^4|\varphi_k|^2\int_{|x|\le N}\frac{\mathrm{d}x}{(1+|x|^{2\theta})^2} \lesssim \|\varphi\|^2_{\dot{H}^2(\mathbb{T}^2_M)}\times\begin{cases}\lambda_N^2\,, & \text{if } \theta>\frac{1}{2},\\ \lambda_N^2\log N\,, & \text{if } \theta=\frac{1}{2},\\ \lambda_N^2N^{2-4\theta}\,, & \text{if } \theta<\frac{1}{2},\end{cases}\end{aligned}$$

and, for any value of $\theta\in(0,1]$ the right hand side is bounded above by $N^{2\theta-2}\|\varphi\|^2_{\dot{H}^2(\mathbb{T}^2_M)}$. □

2.1.2 The regularised Vorticity equation on $\mathbb{R}^2$

In this section, we study the regularised Vorticity equation (1.10) on the full space $\mathbb{R}^2$. Our goal is to show, on the one hand that, for $N\in\mathbb{N}$ fixed, it admits a solution and on the other that such a solution has an invariant measure μ satisfying (1.26). Let us remark that, as noted in [FQ15, Remark 3.1-(2)] in the context of the one-dimensional KPZ equation, for the latter purpose Echeverria's criterion [Ech82] is not directly applicable because the space we are working on here is not locally compact. Instead, we will follow similar methodology of that in [FQ15].

Throughout this section, $N\in\mathbb{N}$ will be fixed. For $T>0$ and $\theta\in(0,1]$, we say that

$\omega^N \in C([0,T],\mathcal{S}'(\mathbb{R}^2))$ is a *weak solution* of (1.10) starting at $\omega_0 \in \mathcal{S}'(\mathbb{R}^2)$ if for all $\varphi \in \mathcal{S}(\mathbb{R}^2)$

$$\omega_t^N(\varphi) - \omega_0(\varphi) = \frac{1}{2}\int_0^t \omega_s^N(-(-\Delta)^\theta\varphi)\,\mathrm{d}s + \lambda_N\int_0^t \mathcal{N}^N[\omega_s^N](\varphi)\,\mathrm{d}s - M_t(\varphi)\,. \tag{2.23}$$

where $\mathcal{N}^N$ is defined according to (1.9) and $M_\cdot(\cdot)$ is a continuous Gaussian process whose covariance is given by

$$\mathbf{E}[M_t(\varphi)M_s(\psi)] = (t\wedge s)\langle\varphi,\psi\rangle_{\dot{H}^{1+\theta}(\mathbb{R}^2)}\,, \qquad \varphi,\psi\in\dot{H}^{1+\theta}(\mathbb{R}^2) \tag{2.24}$$

(so that, formally, "$M_t(\varphi) = \int_0^t \xi(\,\mathrm{d}s,(-\Delta)^{\frac{1+\theta}{2}}\varphi)$" for a space-time white noise ξ on $\mathbb{R}_+\times\mathbb{R}^2$). Further, if ω_0 is distributed according to μ in (1.26), then we will say that the solution is *stationary*. Note, in Navier-Stokes community this is often referred to as *very weak* solution.

Let us introduce the operator $\mathcal{L}^N$ which is nothing but the $\mathbb{R}^2$ counterpart of $\mathcal{L}^{N,M}$ in (2.6) and formally represents the generator of (1.10).

Once again, it can be written as the sum of two operators, i.e. $\mathcal{L}^N = \mathcal{L}_\theta + \mathcal{A}^N$,

whose action of cylinder functions $F(\omega) = f(\omega(\varphi_1),\dots,\omega(\varphi_n))$ is given by

$$\mathcal{L}_\theta F(\omega) \stackrel{\text{def}}{=} \frac{1}{2}\sum_{i=1}^n \omega(-(-\Delta)^\theta\varphi_i)\partial_i f + \frac{1}{2}\sum_{i,j=1}^n \langle\varphi_i,\varphi_j\rangle_{\dot{H}^{1+\theta}(\mathbb{R}^2)}\,\partial^2_{i,j}f\,, \tag{2.25}$$

$$\mathcal{A}^N F(\omega) \stackrel{\text{def}}{=} -\lambda_N\sum_i^n \mathcal{N}^N[\omega](\varphi_i)\,\partial_i f. \tag{2.26}$$

Note that, thanks to the regularisation of the non-linearity i.e. the choice of mollifier , both $\mathcal{L}_0 F[\omega]$ and $\mathcal{A}^N F[\omega]$ are well-defined for any cylinder function F.

In the following definition, we present the martingale problem associated to $\mathcal{L}^N$.

Definition 2.1.7 Let $T>0, \Omega = C([0,T],\mathcal{S}'(\mathbb{R}^2))$ and $\mathcal{G} = \mathcal{B}(C([0,T],\mathcal{S}'(\mathbb{R}^2)))$ the canonical Borel σ-algebra on it. Let $\theta\in(0,1]$, $N\in\mathbb{N}$ and μ be a measure on $\mathcal{S}'(\mathbb{R}^2)$. We say that a probability measure $\mathbf{P}^N$ on $(\Omega,\mathcal{G})$ *solves the cylinder martingale problem for $\mathcal{L}^N$ with initial distribution μ*, if for all cylinder functions F (as defined in (1.31)) the canonical process ω^N under $\mathbf{P}^N$ is such that

$$\mathcal{M}_t(F) \stackrel{\text{def}}{=} F(\omega_t^N) - F(\mu) - \int_0^t \mathcal{L}^N F(\omega_s^N)\,\mathrm{d}s \tag{2.27}$$

is a continuous martingale.

As a first result, we determine the connection between the martingale problem in Definition 2.1.7 and weak solutions of (1.10).

Proposition 2.1.8 *Let $\theta \in (0,1]$, $N \in \mathbb{N}$ and μ be a random field on $\mathcal{S}'(\mathbb{R}^2)$. Then, $\mathbf{P}^N$ is a solution to the cylinder martingale problem for $\mathscr{L}^N$ with initial distribution μ if and only if the canonical process ω^N under $\mathbf{P}^N$ is a weak solution of* (1.10)*.*

Proof. Notice first that if ω^N is a weak solution of (1.10), then for any cylinder function F, the right hand side of (2.27) is a martingale by Itô's formula. Hence, the law of ω^N solves the martingale problem of Definition 2.1.7. In order to show that the converse also holds, we follow the strategy of [FQ15, Lemma 2.7]. Let $\mathbf{P}^N$ be a solution to the martingale problem and ω^N the canonical process with respect to $\mathbf{P}^N$. Let $\varphi \in \mathcal{S}(\mathbb{R}^2)$ and F_φ be the linear cylinder function defined as $F_\varphi(\omega^N) \stackrel{\text{def}}{=} \omega^N(\varphi)$. In view of (2.27), ω^N satisfies

$$\begin{aligned} \omega_t^N(\varphi) - \mu(\varphi) &= \int_0^t \mathscr{L}^N \omega_s(\varphi)\,\mathrm{d}s + \mathscr{M}_t(F_\varphi) \\ &= \frac{1}{2}\int_0^t \omega_s^N(-(-\Delta)^\theta \varphi)\,\mathrm{d}s + \lambda_N \int_0^t \mathcal{N}^N[\omega_s^N](\varphi)\,\mathrm{d}s + \mathscr{M}_t(F_\varphi) \end{aligned} \tag{2.28}$$

the second step being a consequence of the definition of $\mathscr{L}^N$ in (2.25) and (2.26), and where $\mathscr{M}_t(F_\varphi)$ is a continuous martingale. We are left to show that for all φ, $\mathscr{M}_t(F_\varphi)$ is Gaussian and has covariance given as in (2.24). To do so, let $\varphi,\ \psi \in \mathcal{S}(\mathbb{R}^2)$, and consider the quadratic cylinder function $F_{\varphi,\psi}(\omega^N) \stackrel{\text{def}}{=} \omega^N(\varphi)\,\omega^N(\psi)$. Exploiting (2.27) once more, we see that

$$\mathscr{M}_t(F_{\varphi,\psi}) = \omega_t^N(\varphi)\omega_t^N(\psi) - \mu(\varphi)\mu(\psi) - \int_0^t \mathscr{L}^N F_{\varphi,\psi}(\omega_s^N)\,\mathrm{d}s \tag{2.29}$$

is a martingale. Let $b_s(\varphi) \stackrel{\text{def}}{=} \mathscr{L}^N \omega_s^N(\varphi)$ and notice that (2.25) and (2.26) give

$$\mathscr{L}^N F_{\varphi,\psi}(\omega_s^N) = \omega_s^N(\varphi) b_s(\psi) + \omega_s^N(\psi) b_s(\varphi) + \langle \varphi, \psi \rangle_{\dot{H}^{1+\theta}(\mathbb{R}^2)}\,,$$

which, once plugged into (2.29), provides

$$
\begin{aligned}
\mathscr{M}_t(F_\varphi)\mathscr{M}_t(F_\psi) - t\langle\varphi,\psi\rangle_{\dot H^{1+\theta}(\mathbb{R}^2)}&\\
=&\mathscr{M}_t(F_{\varphi,\psi}) - \int_0^t \Big(b_s(\psi)\delta_{s,t}\omega^N_\cdot(\varphi) + b_s(\varphi)\delta_{s,t}\omega^N_\cdot(\psi)\Big)\,\mathrm{d}s\\
&- \mu(\varphi)\mathscr{M}_t(F_\varphi) - \mu(\psi)\mathscr{M}_t(F_\varphi) + \int_0^t\int_0^t b_s(\varphi)b_{\bar s}(\psi)\,\mathrm{d}s\,\mathrm{d}\bar s\\
=&\mathscr{M}_t(F_{\varphi,\psi}) - \int_0^t \Big(b_s(\psi)\int_s^t \mathrm{d}\mathscr{M}_{\bar s}(\varphi) + b_s(\varphi)\int_s^t \mathrm{d}\mathscr{M}_{\bar s}(\varphi)\Big)\,\mathrm{d}s\\
&- \mu(\varphi)\mathscr{M}_t(F_\varphi) - \mu(\psi)\mathscr{M}_t(F_\varphi)\\
=&\mathscr{M}_t(F_{\varphi,\psi}) - \int_0^t \Big(\int_0^{\bar s} b_s(\psi)\,\mathrm{d}s\Big)\,\mathrm{d}\mathscr{M}_{\bar s}(\varphi) + \int_0^t \Big(\int_0^{\bar s} b_s(\varphi)\,\mathrm{d}s\Big)\,\mathrm{d}\mathscr{M}_{\bar s}(\psi)\\
&- \mu(\varphi)\mathscr{M}_t(F_\varphi) - \mu(\psi)\mathscr{M}_t(F_\varphi)
\end{aligned}
\tag{2.30}
$$

where we introduced the notation $\delta_{s,t}f_\cdot \stackrel{\text{def}}{=} f(t) - f(s)$ and exploited (2.28) in the second equality. Now, all the terms at the right hand side are martingales so that, by definition, $t\langle\varphi,\psi\rangle_{\dot H^{1+\theta}(\mathbb{R}^2)}$ is the quadratic co-variation of $\mathscr{M}_t(F_\varphi)$ and $\mathscr{M}_t(F_\psi)$ and clearly (2.24) holds. For Gaussianity, taking $\psi = \varphi$ in (2.30), we deduce that $\mathscr{M}_t(F_\varphi)$ is a continuous martingale with deterministic quadratic variation which, in view of [EK09, Theorem 7.1.1], implies that, for all φ, $\mathscr{M}_t(F_\varphi)$ is Gaussian with independent increments so that the proof is concluded. □

We now show that the martingale problem of Definition 2.1.7 starting from μ as in (1.26) admits a solution. Together with the previous result, this implies the existence of a stationary weak solution to (1.10) whose invariant measure is μ thus completing the proof of Theorem 1.3.1.

Theorem 2.1.9 *Let $N \in \mathbb{N}$ be fixed, $\theta \in (0,1]$ and μ the Gaussian process with covariance given by* (1.26). *The cylinder martingale problem of Definition 2.1.7 for $\mathscr{L}^N$ with initial distribution μ has a solution $\mathbf{P}^N$. Further, the canonical process ω^N under $\mathbf{P}^N$ has invariant measure μ.*

The proof of the previous theorem exploits the Galerkin approximation $\omega^{N,M}$ of (1.4) studied in the previous section. In the next lemma, we show that the sequence is tight in M (for N fixed).

Definition 2.1.10 We say a sequence of probability distributions $\{\mu_M\}_{M\in\mathbb{N}}$ is tight in X if for any $\varepsilon > 0$ there exists $M_0 \in \mathbb{N}$ and $K_\varepsilon \subset X$ such

$$\mu_M(K_\varepsilon) > 1 - \varepsilon\,,$$

for every $M > M_0$. We say a sequence of stochastic functions are tight in space if the corresponding sequence of probability distributions are tight.

Lemma 2.1.11 *Let $N \in \mathbb{N}$ be fixed, $\theta \in (0,1]$ and $T > 0$. With a slight abuse of notation, for all $M \in \mathbb{N}$, let $\omega^{N,M}$ denote the periodically extended version of the stationary solution to* (2.1) *on $\mathbb{T}^M$. Then, the sequence $\{\omega^{N,M}\}_{M\in\mathbb{N}}$ is tight in $C([0,T], \mathcal{S}'(\mathbb{R}^2))$.*

Proof. [Mit83, Theorem 3.1.] can be applied here, in this setting it states that a sequence of continuous functions $\{\omega^{N,M}\}_{M\in\mathbb{N}}$ mapping from a closed interval ($[0,T]$ in our case) to a Frechet space, can be shown to be tight if for all $\varphi \in \mathcal{S}(\mathbb{R}^2)$ the sequence $\{t \to \omega_t^N(\varphi)\}_N$ is tight. To do so, we will exploit Kolmogorov's criterion, for which we need to prove that there exist $\alpha > 0$ and $p > 1$ such that for all $0 \le s < t \le T$ we have

$$\mathbf{E}\Big[|\omega_t^{N,M}(\varphi) - \omega_s^{N,M}(\varphi)|^p\Big]^{1/p} \lesssim_\varphi (t-s)^\alpha\,, \tag{2.31}$$

where the constant hidden into "$\lesssim$" depends on φ. Since $\omega^{N,M}$ is Markov and stationary, it is enough to show (2.31) for $s = 0$. Notice first that, by construction, the time increment of $\omega^{N,M}$ satisfies

$$\begin{aligned}&\omega_t^{N,M}(\varphi) - \mu^M(\varphi)\\ &\quad = \frac{1}{2}\int_0^t \omega_s^{N,M}(-(-\Delta)^\theta\varphi)\,\mathrm{d}s - \lambda_N\int_0^t \mathcal{N}^N[\omega_s^{N,M}](\varphi)\,\mathrm{d}s + \int_0^t \xi^M(\,\mathrm{d}s, c\varphi)\end{aligned}$$

and we will separately focus on each of the terms at the right hand side. Gaussian hypercontractivity [Nua06, Theorem 1.4.1] and the definition of ξ imply that the last term can be bounded as

$$\begin{aligned}\mathbf{E}\Big[\Big|\int_0^t \xi^M(s,(-\Delta)^{\frac{1+\theta}{2}}\varphi)\,\mathrm{d}s\Big|^p\Big]^{1/p} &\lesssim \mathbf{E}\Big[\Big|\int_0^t \xi^M(s,(-\Delta)^{\frac{1+\theta}{2}}\varphi)\,\mathrm{d}s\Big|^2\Big]^{\frac{1}{2}}\\ = \Big(\int_0^t \langle(-\Delta)^{\frac{1+\theta}{2}}\varphi,(-\Delta)^{\frac{1+\theta}{2}}\varphi\rangle_{L^2(\mathbb{T}_M^2)}\,\mathrm{d}s\Big)^{\frac{1}{2}} &= t^{\frac{1}{2}}\|\varphi\|_{\dot{H}^{1+\theta}(\mathbb{T}_M^2)} \lesssim t^{\frac{1}{2}}\|\varphi\|_{\dot{H}^{1+\theta}(\mathbb{R}^2)}\,,\end{aligned} \tag{2.32}$$

where in the last step, we simply used the fact that the $\dot{H}^{1+\theta}(\mathbb{T}_M^2)$-norm is simply a Riemann-sum approximation of the $\dot{H}^{1+\theta}(\mathbb{R}^2)$ norm. For the remaining two terms, we exploit Lemma 2.1.5 and the same argument as above. Collecting what deduced so far, we see that (2.31) holds for all φ, any $p \ge 2$ and $\alpha = 1/2$, so that, tightness of the sequence $\{\omega^N\}_N$ follows at once by Kolmogorov's criterion and [Mit83]. □

We are now ready to complete the proof of Theorem 2.1.9.

Proof of Theorem 2.1.9. Let $\mathbf{P}^{N,M}$ denote the law of the periodically extended version of the stationary solution $\omega^{N,M}$ of (2.1) on $C([0,T], \mathcal{S}'(\mathbb{R}^2))$. In order to consider $\omega^{N,M}$ on

$C([0,T],\mathcal{S}'(\mathbb{R}^2))$ we must we must extend it, we do so by periodically repeating M distance. We extend μ^M in the same fashion. Since by Lemma 2.1.11, the sequence $\{\mathbf{P}^{N,M}\}_M$ is tight, we can extract, via Prokhorov's theorem, a weakly converging subsequence that, slightly abusing the notation, we will still denote by $\{\mathbf{P}^{N,M}\}_M$. Let $\mathbf{P}^N$ be its limit. Skorokhod's representation theorem ensures that we can realise the sequence on a proper probability space in such a way that $\{\omega^{N,M}\}_M$ converges to ω^N, $\mathbf{P}^N$ almost surely in $C([0,T],\mathcal{S}'(\mathbb{R}^2))$ as $M\to\infty$. We now want to show that $\mathbf{P}^N$ is a solution to the martingale problem for $\mathcal{L}^N$, which amounts to verify that for any cylinder function F the right hand side of (2.27) is a continuous martingale.

As a preliminary step, note that since $\omega^{N,M}\to\omega^N$ almost surely in $C([0,T],\mathcal{S}'(\mathbb{R}^2))$, then, for all t, $\omega_t^{N,M}\to\omega_t^N$ almost surely in $\mathcal{S}'(\mathbb{R}^2)$. By assumption, $\omega_t^{N,M}$ is distributed according to μ^M and μ^M converges to μ. Hence ω_t^N is distributed according to μ. In other words, μ is an invariant measure for ω^N and, as μ is Gaussian, for any cylinder function G, $G(\omega_t^N)$ has finite moments of all orders.

Let $\varphi_1,\dots,\varphi_n\in\mathcal{S}(\mathbb{R}^2)$ and $F(\omega)=f(\omega(\varphi_1),\dots,\omega(\varphi_n))$ be a cylinder function on $\mathcal{S}'(\mathbb{R}^2)$. By Itô's formula, for all $t\in[0,T]$,

$$F(\omega_t^{N,M})-F(\mu^M)-\int_0^t\mathcal{L}^{N,M}F(\omega_s^{N,M})\,\mathrm{d}s\,,$$

is a square-integrable continuous martingale. Once we show that

$$\begin{aligned}&M_t^{N,M}(\omega_t^{N,M})-M_t^N(\omega^N)=\\&\Big(F(\omega_t^{N,M})-F(\mu^M)-\int_0^t\mathcal{L}^{N,M}F(\omega_s^{N,M})\,\mathrm{d}s\Big)-\Big(F(\omega_t^N)-F(\mu)-\int_0^t\mathcal{L}^NF(\omega_s^N)\,\mathrm{d}s\Big)\end{aligned}\tag{2.33}$$

goes to 0 in mean square with respect to $\mathbf{P}^N$ we can show that $M_t^N(\omega^N)$ is a martingale, because $\mathbb{E}\Big(M_t^N(\omega^N)-M_t^{N,M}(\omega^{N,M})|\mathcal{F}_s\Big)\to 0$ and so

$$\mathbb{E}\Big(M_t^N(\omega^N)|\mathcal{F}_s\Big)=\mathbb{E}\Big(M_t^{N,M}(\omega^{N,M})|\mathcal{F}_s\Big)-\mathbb{E}\Big(M_t^N(\omega^N)-M_t^{N,M}(\omega^{N,M})|\mathcal{F}_s\Big)\to_M M_s^N(\omega^N)\,.$$

We will first prove that (2.33) converges to 0 almost surely. Since $\omega^{N,M}\to\omega^N$ almost surely in $C([0,T],\mathcal{S}'(\mathbb{R}^2))$, then almost surely for all $r\in[0,T]$ and $n\in\mathbb{N}$ both

$$\begin{gathered}\partial^{(n)}f(\omega^{N,M}(\varphi_1),\dots,\omega^{N,M}(\varphi_n))\to\partial^{(n)}f(\omega_r^N(\varphi_1),\dots,\omega_r^N(\varphi_n))\,,\\ \omega_r^{N,M}(-(-\Delta)^\theta\varphi)\to\omega_r^N(-(-\Delta)^\theta\varphi)\end{gathered}\tag{2.34}$$

hold. Further, for every $i,j=1,\dots n$, $\langle\varphi_i,\varphi_j\rangle_{\dot H^{1+\theta}(\mathbb{T}_M^2)}\to\langle\varphi_i,\varphi_j\rangle_{\dot H^{1+\theta}(\mathbb{R}^2)}$ deterministically as the $\dot{H}^{1+\theta}(\mathbb{T}_M^2)$-norm is a Riemann-sum approximation of the $\dot{H}^1(\mathbb{R}^2)$-norm. Hence, by the definitions of $\mathscr{L}_0^M$ and $\mathscr{L}_0$ in (2.7) and (2.25) respectively, it follows that almost surely

$$F(\omega_r^{N,M}) \to F(\omega_r^N), \quad r \in \{0,t\} \qquad \text{and} \qquad \int_0^t \mathscr{L}_0^M F(\omega_s^{N,M})\,\mathrm{d}s \to \int_0^t \mathscr{L}_0 F(\omega_s^N)\,\mathrm{d}s\,.$$

In light of (2.34), to show that the same convergence holds for the term containing $\mathscr{A}^{N,M}F(\omega_r^{N,M})$ and $\mathscr{A}^N F(\omega_r^N)$, it suffices to argue that almost surely, for all $i = 1,\dots,n$ and $r \in [0,T]$, $\mathscr{N}^{N,M}[\omega_r^{N,M}](\varphi_i) \to \mathscr{N}^N[\omega_r^N](\varphi_i)$. This in turn is a direct consequence of the representation (2.18) and the fact that the almost sure convergence of $\omega^{N,M}$ to ω^N in $C([0,T],\mathcal{S}'(\mathbb{R}^2))$ ensures that both $\omega^{N,M}(K * \varrho_\cdot^N) \to \omega^N(K * \varrho_\cdot^N)$ and $\omega^{N,M}(\varrho_\cdot^N) \to \omega^N(\varrho_\cdot^N)$. Indeed, our choice of the mollifier guarantees that Fourier transform of ϱ^N is supported away from the origin so that $K * \varrho_\cdot^N \in \mathcal{S}(\mathbb{R}^2)$.

In conclusion, (2.33) converges to 0 almost surely. Moreover, each of its summands has finite moments of all orders as for all $r \in [0,T]$ the distribution of $\omega_r^{N,M}$ and ω_r^N is Gaussian. Therefore, by the dominated convergence theorem, (2.33) converges to 0 in mean square and the proof is concluded. □

2.2 The Vorticity equation on the real plane

Throughout this section, we will be working with a solution $\mathbf{P}^N$ of the martingale problem for $\mathscr{L}^N$ with initial distribution μ, whose canonical process ω^N is, by Proposition 2.1.8, a stationary weak solution of the fractional regularised Vorticity equation (2.23) on $\mathbb{R}^2$.

The goal is to control the behaviour of ω^N in the limit $N \to \infty$. To do so, we first need to deepen our understanding of the generator $\mathscr{L}^N$ and, in particular, determine how it acts on random variables in $L^2(\boldsymbol{\eta})$.

2.2.1 The operator $\mathscr{L}^N$

This section is devoted to the study of the properties of the operator $\mathscr{L}^N$ on $L^2(\boldsymbol{\eta})$, which is given by the sum of $\mathscr{L}_0$ and $\mathscr{A}^N$ defined in (2.25) and (2.26), respectively. Recall that, as remarked in Section 1.5, there exists an isomorphism I between $L^2(\boldsymbol{\eta})$ and the Fock space ΓL^2. With a slight abuse of notation, from here on we will denote with the same symbol any operator $\mathscr{O}$ acting on $L^2(\boldsymbol{\eta})$ and the corresponding operator acting instead on ΓL^2, where by "corresponding" we mean any operator $\mathfrak{O}$ such that $\mathscr{O} I(\varphi) = I(\mathfrak{O}\varphi)$ for all $\varphi \in \Gamma L^2$.

Proposition 2.2.1 *Let μ be the Gaussian process whose covariance function is given by* (1.26)*. Then, for any $\theta \in (0,1]$, the operator $\mathscr{L}_\theta$ is symmetric on $L^2(\boldsymbol{\eta})$, and for each n, it maps $\mathscr{H}_n$ to itself. Further, for any $f \in \Gamma L^2_n$, $\mathscr{L}_\theta f = -\frac{1}{2}(-\Delta)^\theta f$ so that the Fourier transform of the left hand side equals*

$$\mathscr{F}(\mathscr{L}_\theta f)(k_{1:n}) = -\tfrac{1}{2}|k_{1:n}|^{2\theta}\hat{f}(k_{1:n}), \qquad \textit{for all } k_{1:n} \in (\mathbb{R}^2)^n, \tag{2.35}$$

where $|k_{1:n}|^{2\theta} \stackrel{def}{=} |k_1|^{2\theta} + \cdots + |k_n|^{2\theta}$. Instead, the operator $\mathscr{A}^N$ is anti-symmetric on $L^2(\boldsymbol{\eta})$ and it can be written as the sum of two operators $\mathscr{A}^N_+$ and $\mathscr{A}^N_-$, the first mapping $\mathscr{H}_n$ to $\mathscr{H}_{n+1}$ while the second $\mathscr{H}_n$ to $\mathscr{H}_{n-1}$. Moreover, the adjoint of $\mathscr{A}^N_+$ is $-\mathscr{A}^N_-$ and for any $f \in \Gamma L^2_n$ the Fourier transform of their action on f is given by

$$\mathscr{F}(\mathscr{A}^N_+ f)(k_{1:n+1}) = \lambda_N n \mathscr{K}^N_{k_1,k_2} \hat{f}(k_1+k_2, k_{3:n+1}) \tag{2.36}$$

$$\mathscr{F}(\mathscr{A}^N_- f)(k_{1:n-1}) = 2\lambda_N n(n-1)\int_{\mathbb{R}^2} \hat{\varrho}^N_{p,k_1-p}\frac{(k_1^\perp \cdot p)(k_1\cdot(k_1-p))}{|k_1|^2}\hat{f}(p, k_1-p, k_{2:n-1})\,\mathrm{d}p \tag{2.37}$$

where $\mathscr{K}^N$ was defined in (2.4) *and $k_{1:n+1} \in (\mathbb{R}^2)^{n+1}$. Strictly speaking the functions at the right hand side need to be symmetrised with respect to all permutations of their arguments.*

Proof. The properties of $\mathscr{L}_\theta$, including (2.35) have been shown in case $\theta = 1$ in [GP18a, Lemma 3.7.], the generalisation to general θ is straightforward, we include it here for the convenience of the reader. Consider a cylinder function F of the form $F[\omega](\varphi) = H_n(\omega(\varphi))$ with φ being a Schwartz function such that $\|\varphi\|_{L^2} = 1$. Then

$$\begin{aligned}\mathscr{L}_\theta F[\omega] &= H_n'(\omega(\varphi))\omega(-\frac{1}{2}(-\Delta)^\theta\varphi) - H_n''(\omega(\varphi))\|\varphi\|_{H^{1+\theta}}\\ &= nH_{n-1}(\omega(\varphi))\omega(-\frac{1}{2}(-\Delta)^{-\theta}\varphi) - n(n-1)H_{n-2}(\omega(\varphi))\|\varphi\|_{H^{1+\theta}}\ .\end{aligned}$$

By standard properties of Hermite polynomials we have $H_k(I_1(\varphi)) = I_k(\varphi^{\otimes k})$. By (1.28) we have

$$\begin{aligned}nH_{n-1}(I_1(\varphi))I_1((-\Delta)^\theta\varphi) &= nI_{n-1}(\varphi^{\otimes(n-1)})I_1((-\Delta)^\theta\varphi)\\ &= nI_n(-\frac{1}{2}(-\Delta)^\theta\varphi^{\otimes n}) + n(n-1)H_{n-2}(I_1(\varphi))\|\varphi\|_{H^{1+\theta}}\,.\end{aligned}$$

Via polarisation we obtain $\mathscr{L}_\theta f = -\frac{1}{2}(-\Delta)^\theta f$ and hence (2.35). Concerning $\mathscr{A}^N$, let $F(\mu) =$

$f(\mu(\varphi_1),\dots,\mu(\varphi_n))$ be a generic cylinder function. By (2.26), we have

$$\begin{aligned}\mathscr{A}^N F(\mu) &= -\lambda_N \sum_i \mathcal{N}^N[\mu](\varphi_i)\partial_i f = -\lambda_N \mathcal{N}^N[\mu]\Big(\sum_i \partial_i f \varphi_i\Big)\\ &= -\lambda_N \mathcal{N}^N[\mu](DF) = -\lambda_N \int_{\mathbb{R}^2} \mathcal{N}^N[\mu](x) D_x F\,\mathrm{d}x\end{aligned} \tag{2.38}$$

where we exploited the definition of the Malliavin derivative in (1.32).

Let us first show the decomposition in $\mathscr{A}^N_+$ and $\mathscr{A}^N_-$ in (2.36) and (2.37), respectively. By polarisation it suffices to take $F(\mu) = I_n(f)$ for f of the form $\otimes^n \varphi$ and $\varphi \in \dot{H}^1(\mathbb{R}^2)$. Note that the Malliavin derivative of F satisfies

$$D_x F(\mu) = n I_{n-1}\Big(\otimes^{n-1}\varphi\Big)\varphi(x)$$

(see e.g. [CES21, proof of Lemma 3.5]). Therefore, plugging the previous into (2.38), we get

$$\mathscr{A}^N F(\mu) = -\lambda_N \int_{\mathbb{R}^2} \mathcal{N}^N[\mu](x) D_x F\,\mathrm{d}x = -n\lambda_N \mathcal{N}^N[\mu](\varphi) I_{n-1}\Big(\otimes^{n-1}\varphi\Big)\,.$$

Arguing as in the proof of Lemma 2.1.6, it is not hard to see that $\mathcal{N}^N[\mu](\varphi) \in \mathscr{H}_2$ and $\mathcal{N}^N[\mu](\varphi) = I_2(\mathfrak{n}^N_\varphi)$, the Fourier transform of $\mathfrak{n}^N_\varphi$ being given by the right hand side of (2.19) (though for $\ell, m \in \mathbb{R}^2$). Therefore,

$$\begin{aligned}\mathscr{A}^N F(\mu) = -n\lambda_N I_2(\mathfrak{n}^N_\varphi) I_{n-1}\Big(\otimes^{n-1}\varphi\Big) = &- n\lambda_N I_{n+1}(\mathfrak{n}^N_\varphi \otimes_0 \otimes^{n-1}\varphi)\\ &- 2n(n-1)\lambda_N I_{n-1}(\mathfrak{n}^N_\varphi \otimes_1 \otimes^{n-1}\varphi)\\ &- n(n-1)(n-2)\lambda_N I_{n-3}(\mathfrak{n}^N_\varphi \otimes_2 \otimes^{n-1}\varphi)\end{aligned} \tag{2.39}$$

where the last equality is a consequence of (1.28). It is not hard to see, by taking Fourier transforms and applying Plancherel's identity, that the first term indeed equals $\mathscr{A}^N_+ I_n(f)$, while the second $\mathscr{A}^N_- I_n(f)$, so that in particular $\mathscr{A}^N_+$ and $\mathscr{A}^N_-$ map $\mathscr{H}_n$ into $\mathscr{H}_{n+1}$ and $\mathscr{H}_{n-1}$ respectively. We claim that instead the last term vanishes. Indeed by (1.29), we have

$$\mathfrak{n}^N_\varphi \otimes_2 \otimes^{n-1}\varphi(x_{1:n-3}) = \prod_{i=1}^{n-3}\varphi(x_i)\int_{(\mathbb{R}^2)^2} \langle \nabla \mathfrak{n}^N_\varphi(x,y), \nabla\varphi(x)\varphi(y)\rangle\,\mathrm{d}x\,\mathrm{d}y$$

Applying Plancherel's identity and the definition of $\mathfrak{n}^N_\varphi(x,y)$, we see that the integral above equals

$$\int_{(\mathbb{R}^2)^2} |k_1|^2|k_2|^2 \hat{\mathfrak{n}}^N_\varphi(k_1,k_2)\varphi_{k_1}\varphi_{k_2}\,\mathrm{d}k_1\,\mathrm{d}k_2$$
$$= \int_{(\mathbb{R}^2)^2} |k_1|^2|k_2|^2 \mathscr{K}^N_{k_1,k_2}\varphi_{-k_1-k_2}\varphi_{k_1}\varphi_{k_2}\,\mathrm{d}k_1\,\mathrm{d}k_2 = \langle \mathcal{N}^N[(-\Delta\varphi)],(-\Delta)\varphi\rangle_{\dot H^{-1}(\mathbb{R}^2)}$$

and the right hand side is equal to 0 by Lemma 2.1.2.

We now show that $\mathscr{A}^N_+$ is the adjoint of $-\mathscr{A}^N_-$. For $F=\sum_n I_n(f_n)$ and $G=\sum_n I_n(g_n)$ we have

$$\mathbf{E}\Big[\mathscr{A}^N_+ F G\Big] = \sum_{n,m}\mathbf{E}\Big[I_{n+1}(\mathscr{A}^N_+ f_n)I_m(g_m)\Big] = \sum_n (n+1)!\langle \mathscr{A}^N_+ f_n, g_{n+1}\rangle_{\Gamma L^2_{n+1}}$$
$$\mathbf{E}\Big[F\mathscr{A}^N_- G\Big] = \sum_{n,m}\mathbf{E}\Big[I_n(f_n)I_{m-1}(\mathscr{A}^N_- g_m)\Big] = \sum_n n!\langle f_n, \mathscr{A}^N_- g_{n+1}\rangle_{\Gamma L^2_{n+1}}\,,$$

which is a consequence of orthogonality of different Wiener-chaoses. Therefore, to prove that the two right hand sides above are indeed equal, it suffices to verify that

$$(n+1)\langle \mathscr{A}^N_+ f_n, g_{n+1}\rangle_{\Gamma L^2_{n+1}} = -\langle f_n, \mathscr{A}^N_- g_{n+1}\rangle_{\Gamma L^2_n}\,.$$

By (2.36) (modulo permutations), the left hand side is given by

$$4\pi\lambda_N n(n+1)\int \Big(\Pi_{i=1}^{n+1}|k_i|^2\Big)\mathscr{K}^N_{k_1,k_2}\hat f(k_1+k_2,k_{3:n+1})\hat g_{n+1}(k_{1:n+1})\,\mathrm{d}k_{1:n+1}\,. \qquad (2.40)$$

Then, by a simple change of variables the previous integral is

$$\int \Big(\Pi_{i=1,i\neq 2}^{n+1}|k_i|^2\Big)|k_2'-k_1|^2\mathscr{K}^N_{k_1,k_2'-k_1}\hat f(k_2',k_{3:n+1})\hat g_{n+1}(k_1,k_2'-k_1,k_{3:n+1})\,\mathrm{d}k_1\,\mathrm{d}k_2'\,\mathrm{d}k_{3:n+1}$$
$$= \int \Big(\Pi_{i=1}^{n}|k_i|^2\Big)\frac{|k_1-p|^2|p|^2}{|k_1|^2}\mathscr{K}^N_{p,k_1-p}\hat f(k_{1:n})\hat g_{n+1}(p,k_1-p,k_{2:n})\,\mathrm{d}p\,\mathrm{d}k_{1:n}$$
$$= \frac{1}{2\pi}\int \Big(\Pi_{i=1}^{n}|k_i|^2\Big)\hat f(k_{1:n})\int \hat\varrho^N_{p,k_1-p}\frac{(p^\perp\cdot k_1)(k_1\cdot(k_1-p))}{|k_1|^2}\hat g_{n+1}(p,k_1-p,k_{2:n})\,\mathrm{d}p\,\mathrm{d}k_{1:n}$$
$$= -\frac{1}{2\pi}\int \Big(\Pi_{i=1}^{n}|k_i|^2\Big)\hat f(k_{1:n})\int \hat\varrho^N_{p,k_1-p}\frac{(p\cdot k_1^\perp)(k_1\cdot(k_1-p))}{|k_1|^2}\hat g_{n+1}(p,k_1-p,k_{2:n})\,\mathrm{d}p\,\mathrm{d}k_{1:n}\,,$$

from which the result follows. Further, as an immediate corollary of $(\mathscr{A}^N_+)^* = -\mathscr{A}^N_-$, we also deduce that $\mathscr{A}^N$ is anti-symmetric so that the proof of the statement is completed. □

2.2.2 Tightness and upper bound

Following techniques similar to those exploited in Section 2.1.1, we establish tightness for the sequence $\{\omega^N\}_N$ of solutions to the stationary regularised Vorticity equation under assumption (1.11). For $\theta = 1$, we also derive an order one upper bound on the integral in time of the non-linearity.

Theorem 2.2.2 *Let $\theta \in (0, 1]$. For $N \in \mathbb{N}$, let ω^N be a stationary solution to (2.23) on $\mathbb{R}^2$ with coupling constant λ_N chosen according to (1.11), started from the Gaussian process μ with covariance given by (1.26). For $\varphi \in \mathcal{S}(\mathbb{R}^2)$ and $t \geq 0$, set*

$$\mathcal{B}_t^N(\varphi) \stackrel{def}{=} \lambda_N \int_0^t \mathcal{N}_t^N[\omega_s^N](\varphi)\, \mathrm{d}s\,. \tag{2.41}$$

Then, for any $T > 0$, the couple $(\omega^N, \mathcal{B}^N)$ is tight in the space $C([0,T], \mathcal{S}'(\mathbb{R}^2))$. Moreover, for $\theta = 1$, any limit point $(\omega, \mathcal{B})$ is such that for all $p \geq 2$ there exists a constant $C = C(p)$ such that for all $\varphi \in \mathcal{S}(\mathbb{R}^2)$

$$\mathbf{E}\Big[\Big|\mathcal{B}_t(\varphi)\Big|^p\Big]^{\frac{1}{p}} \leq C(t \vee t^{\frac{1}{2}})\|\varphi\|_{\dot{H}^2(\mathbb{R}^2)}\,, \tag{2.42}$$

while, for $\theta \in (0,1)$, for all $p \geq 2$ and $\varphi \in \mathcal{S}(\mathbb{R}^2)$

$$\lim_{N\to\infty} \mathbf{E}\Big[\sup_{s\leq t}\Big|\mathcal{B}_s^N(\varphi)\Big|^p\Big]^{\frac{1}{p}} = 0\,. \tag{2.43}$$

Remark 2.2.3 For $\theta = 1$, the previous theorem proves both the tightness of the sequence $\{(\omega^N, \mathcal{B}^N)\}_N$ stated in Theorem 1.3.2 and the upper bound in (1.20). The latter can be directly verified by considering (2.42) with $p = 2$ and applying the Laplace transform at both sides.

Proof. The proof follows the same steps and computations performed in Section 2.1 for Lemma 2.1.11. More precisely, the statements of Lemma 2.1.4 (the Itô trick), Proposition 2.1.5 and Lemma 2.1.6 hold *mutatis mutandis* in the non-periodic case - it suffices to remove the superscripts M, replace every instance of $\mathbb{T}_M^2$ with $\mathbb{R}^2$ and substitute the weighted Riemann-sums with integrals. Hence, we deduce that for any $\varphi \in \mathcal{S}(\mathbb{R}^2)$ and any $p \geq 2$

$$\mathbf{E}\Big[\sup_{s\leq t}\Big|\mathcal{B}_s^N(\varphi)\Big|^p\Big]^{\frac{1}{p}} \lesssim N^{2\theta-2}(t \vee t^{\frac{1}{2}})\|\varphi\|_{\dot{H}^2(\mathbb{R}^2)}\,. \tag{2.44}$$

which implies tightness for $\mathcal{B}^N$ for $\theta \in (0,1]$ by Mitoma's and Kolmogorov's criteria, and (2.43) for $\theta \in (0,1)$ and (2.42) for $\theta = 1$. Moreover, arguing as in the proof of Lemma 2.1.11, one sees that (2.31) holds for ω^N. By invoking once more Mitoma's and Kolmogorov's criteria we conclude that tightness holds also for ω^N. □

2.2.3 Lower bound on the non-linearity for $\theta = 1$

As shown in Theorem 2.2.2, the choice of the coupling constant λ_N in (1.11) ensures tightness of the sequence $\{\omega^N\}_N$ of stationary solutions to (2.23) on $\mathbb{R}^2$ and, for $\theta = 1$, provides an upper bound on the integral in time of the non-linearity. In order to prove the lower bound the analogous statements of [CES21, Lemma 5.1] and [CES21, Lemma 5.2] will be used. For connvinience of the reader the statement and the proof for those is given below

Lemma 2.2.4 *For $N \in \mathbb{N}$, let $\mathscr{B}^N$ be defined according to* (2.41) *then for any $N \in \mathbb{N}$ we have*

$$\int_0^\infty e^{-\kappa t}\mathbf{E}\Big[\Big|\mathscr{B}_t^N(\varphi)\Big|^2\Big]\,\mathrm{d}t = \frac{2\lambda_N^2}{\kappa^2}\mathbb{E}\Big[\mathscr{N}^N[\mu](\varphi)(\kappa - \mathscr{L}^N)^{-1}\mathscr{N}^N[\mu](\varphi)\Big]\,. \tag{2.45}$$

Moreover the above equals

$$\begin{aligned}
&\frac{2\lambda_N^2}{\kappa^2}\sup_{G\in L^2(\eta)}\Big\{2\mathbb{E}[\lambda_{N,1}\mathscr{N}^N[\mu](\varphi)G] - \mathbb{E}[G(\kappa-\mathscr{L}_0)G] - \mathbb{E}[\mathscr{A}^N G(\kappa-\mathscr{L}_0)^{-1}\mathscr{A}^N G]\Big\}\\
&= \frac{2\lambda_N^2}{\kappa^2}\sup_{g\in\Gamma L^2}\Big\{2\langle\lambda_{N,1}\mathfrak{n}_\varphi^N, g\rangle_{\Gamma L^2} - \langle g, (\kappa-\mathscr{L}_0)g\rangle_{\Gamma L^2} - \langle\mathscr{A}^N g, (\kappa-\mathscr{L}_0)^{-1}\mathscr{A}^N g\rangle_{\Gamma L^2}\Big\}
\end{aligned} \tag{2.46}$$

Proof. For (2.45) we start by reformulating $\mathbf{E}\Big[\Big|\mathscr{B}_t^N(\varphi)\Big|^2\Big]$,

$$\begin{aligned}
\mathbf{E}\Big[\Big|\mathscr{B}_t^N(\varphi)\Big|^2\Big] &= 2\lambda_N^2\int_0^t \mathrm{d}s\int_0^s \mathrm{d}r\mathbf{E}[\mathscr{N}^N[\omega_r^N]\mathscr{N}^N[\omega_s^N]]\\
&= 2\lambda_N^2\int_0^t \mathrm{d}s\int_0^s \mathrm{d}r\mathbf{E}[\mathscr{N}^N[\omega_r^N]\mathbf{E}[\mathscr{N}^N[\omega_s^N]|\mathscr{G}_r]]\,,
\end{aligned}$$

here $\mathscr{G}$ denotes the natural filtration of $\{\omega_t^N\}_t$. ω^N is a Markov process which here is denoted by $\{e^{t\mathscr{L}^N}\}_{t\geq 0}$, at a fixed time it is distributed according to μ. And so the above is equal to

$$\begin{aligned}
2\lambda_N^2\int_0^t \mathrm{d}s\int_0^s \mathrm{d}r\mathbb{E}\Big[\mathscr{N}^N[\mu]\mathbf{E}^\mu\Big[\mathscr{N}^N[\omega_{s-r}^N]\Big]\Big] &= 2\lambda_N^2\int_0^t \mathrm{d}s\int_0^s \mathrm{d}r\mathbb{E}\Big[\mathscr{N}^N[\mu]e^{(s-r)\mathscr{L}^N}\mathscr{N}^N[\mu]\Big]\\
&= 2\lambda_N^2\int_0^t \mathrm{d}r(t-s)\mathbb{E}\Big[\mathscr{N}^N[\mu]e^{r\mathscr{L}^N}\mathscr{N}^N[\mu]\Big]\,,
\end{aligned}$$

here $\mathbf{E}^\mu$ denotes the expectation with respect to $\{\omega_t^N\}_t$ conditioned to start at μ. The expectation above does not depend on t, inserting this back into left hand side of (2.45) one obtains

$$2\lambda_N^2 \int_0^\infty \int_0^t (t-r) e^{-\kappa(t-r)} \mathbf{E}\Big[\mathcal{N}^N[\mu] e^{-r(\kappa - \mathscr{L}^N)} \mathcal{N}^N[\mu]\Big]\,,$$

from which right hand side of (2.45) follows as $\int_0^\infty \mathrm{d}r e^{-r(\kappa-\mathscr{L}^N)} = (\kappa - \mathscr{L}^N)^{-1}$.

To prove the above is equal to the variational problem stated in (2.46) one has to use statement from [KLO12] which is stated here in this setting (and not one of [KLO12])

Lemma 2.2.5 *Let $\mathscr{L}^N$ be the generator of the Markov process $\{\omega_t^N\}_{t\geq 0}$ and let $\mathscr{L}_0$ and $\mathscr{A}^N$ be its symmetric and antisymmetric parts with respect to the white noise measure $\boldsymbol{\eta}$. Let $F \in L(\boldsymbol{\eta})$ and denote by $\langle \cdot, \cdot \rangle_{\eta}$ the product in $\mathbf{L}^2(\boldsymbol{\eta})$, then for every $\lambda > 0$,*

$$\langle F, (\lambda - \mathscr{L}^N)^{-1} F\rangle = \sup_G \Big\{ 2\langle F, G\rangle_{\boldsymbol{\eta}} - \langle (\lambda - \mathscr{L}_0) G, G\rangle_{\boldsymbol{\eta}} - \langle \mathscr{A}^N G, (\lambda - \mathscr{L}_0)^{-1} \mathscr{A}^N G\rangle_{\boldsymbol{\eta}} \Big\}\,,$$

Here G ranges over a fixed core of $\mathscr{L}^N$.

To prove (2.46) one applies (2.45) to the above lemma twice. □

In the proposition below, we determine a matching (up to constants) lower bound on its Laplace transform thanks to which the proof of Theorem 1.3.2 is complete.

Proposition 2.2.6 *In the same setting as Theorem 2.2.2, let $\theta = 1$ and $\mathscr{B}$ be any limit point of the sequence $\mathscr{B}^N$ in (2.41). Then, there exists a constant $C > 0$ such that for all $\kappa > 0$ and $\varphi \in \mathcal{S}(\mathbb{R}^2)$ the lower bound in (1.20) holds.*

Proof. By the Lemma 2.2.4 the rest of the proof will be bounding below the following quantity

$$\begin{aligned}&\frac{2}{\kappa^2} \sup_{G \in L^2(\boldsymbol{\eta})} \Big\{ 2\mathbb{E}[\lambda_{N,1} \mathcal{N}^N[\mu](\varphi) G] - \mathbb{E}[G(\kappa - \mathscr{L}_0) G] - \mathbb{E}[\mathscr{A}^N G (\kappa - \mathscr{L}_0)^{-1} \mathscr{A}^N G] \Big\} \\ &= \frac{2}{\kappa^2} \sup_{g \in \Gamma L^2} \Big\{ 2\langle \lambda_{N,1} \mathfrak{n}_\varphi^N, g\rangle_{\Gamma L^2} - \langle g, (\kappa - \mathscr{L}_0) g\rangle_{\Gamma L^2} - \langle \mathscr{A}^N g, (\kappa - \mathscr{L}_0)^{-1} \mathscr{A}^N g\rangle_{\Gamma L^2} \Big\}\end{aligned}$$

where $\mathfrak{n}_\varphi^N$ is such that $\mathcal{N}^N[\mu](\varphi) = I_2(\mathfrak{n}_\varphi^N)$ and its Fourier transform is given by the right hand side of (2.19) (for $\ell, m \in \mathbb{R}^2$). We can further lower bound (2.46) by restricting to g to ΓL_2^2 for which, by orthogonality of different chaoses of $\mathscr{A}_+^N$ and $\mathscr{A}_-^N$ determined in Proposition 2.2.1 we have

$$\langle \mathscr{A}_+^N G, (\kappa - \mathscr{L}_0^N)^{-1} \mathscr{A}_-^N G\rangle = 0\,,$$

$$\langle \mathscr{A}_-^N G, (\kappa - \mathscr{L}_0^N)^{-1} \mathscr{A}_+^N G \rangle = 0\,.$$

hence, the last term of (2.46) equals

$$\langle \mathscr{A}^N g, (\kappa - \mathscr{L}_0)^{-1} \mathscr{A}^N g \rangle_{\Gamma L_2^2} = \langle \mathscr{A}_+^N g, (\kappa - \mathscr{L}_0)^{-1} \mathscr{A}_+^N g \rangle_{\Gamma L_3^2} + \langle \mathscr{A}_-^N g, (\kappa - \mathscr{L}_0)^{-1} \mathscr{A}_-^N g \rangle_{\Gamma L_1^2}\,.$$

Summarising, the left hand side of (2.45) is lower bounded by

$$\begin{aligned} \frac{2}{\kappa^2} \sup_{g \in \Gamma L_2^2} \Big\{ & 2\langle \lambda_{N,1} \mathfrak{n}_\varphi^N, g \rangle_{\Gamma L_2^2} - \langle g, (\kappa - \mathscr{L}_0) g \rangle_{\Gamma L_2^2} \\ & - \langle g, -\mathscr{A}_-^N (\kappa - \mathscr{L}_0)^{-1} \mathscr{A}_+^N g \rangle_{\Gamma L_2^2} - \langle g, -\mathscr{A}_+^N (\kappa - \mathscr{L}_0)^{-1} \mathscr{A}_-^N g \rangle_{\Gamma L_2^2} \Big\} \end{aligned} \tag{2.47}$$

where we further exploited that the adjoint of $\mathscr{A}_+^N$ is $-\mathscr{A}_-^N$ and vice versa. The operators $-\mathscr{A}_-^N (\kappa - \mathscr{L}_0)^{-1} \mathscr{A}_+^N$ and $-\mathscr{A}_+^N (\kappa - \mathscr{L}_0)^{-1} \mathscr{A}_-^N$, even though explicit, are difficult to handle since they are not diagonal in Fourier space, meaning that their Fourier transform cannot be expressed in terms of an explicit multiplier. Nevertheless, the following lemma, whose proof we postpone to the end of the section, ensures that they can be bounded by one.

Lemma 2.2.7 *There exists a constant $C > 0$ independent of N such that for any $g \in \Gamma L_2^2$, the following bound hold*

$$\langle g, -\mathscr{A}_-^N (\kappa - \mathscr{L}_0)^{-1} \mathscr{A}_+^N g \rangle_{\Gamma L_2^2} \vee \langle g, -\mathscr{A}_+^N (\kappa - \mathscr{L}_0)^{-1} \mathscr{A}_-^N g \rangle_{\Gamma L_2^2} \leq C \langle (-\mathscr{L}_0) g, g \rangle_{\Gamma L_2^2}\,. \tag{2.48}$$

Assuming the previous lemma holds, there exists a constant $c > 1$ independent of n such that (2.47) is bounded below by

$$\begin{aligned} & \frac{2}{\kappa^2} \sup_{g \in \Gamma L_2^2} \Big\{ 2\langle \lambda_{N,1} \mathfrak{n}_\varphi^N, g \rangle_{\Gamma L_2^2} - \langle g, (\kappa - c\mathscr{L}_0) g \rangle_{\Gamma L_2^2} \Big\} \\ & \qquad = \frac{2}{\kappa^2} \sup_{g \in \Gamma L_2^2} \Big\{ \langle \lambda_{N,1} \mathfrak{n}_\varphi^N, g \rangle_{\Gamma L_2^2} + \langle \lambda_{N,1} \mathfrak{n}_\varphi^N - (\kappa - c\mathscr{L}_0) g, g \rangle_{\Gamma L_2^2} \Big\}\,. \end{aligned} \tag{2.49}$$

Now, in order to prove (1.20), it suffices to exhibit *one* g for which the lower bound holds, and we choose it in such a way that the second scalar product in the supremum is 0, i.e. we pick $g = \mathfrak{g}$, the latter being the unique solution to

$$\lambda_{N,1} \mathfrak{n}_\varphi^N - (\kappa - c\mathscr{L}_0) \mathfrak{g} = 0\,. \tag{2.50}$$

Notice that, by (2.35), $\mathfrak{g}$ has an explicit Fourier transform which is given by

$$\hat{\mathfrak{g}}(k_{1:2}) = \lambda_{N,1}\frac{\hat{\mathfrak{n}}_\varphi(k_{1:2})}{\kappa + \frac{c}{2}|k_{1:2}|^2}\,.$$

Plugging $\mathfrak{g}$ into (2.47) we obtain a lower bound of the type

$$\begin{aligned}\frac{2}{\kappa^2}\langle \lambda_N \mathfrak{n}^N_\varphi, \mathfrak{g}\rangle_{\Gamma L^2_2} &= \frac{2\lambda^2_{N,1}}{\kappa^2}\int_{\mathbb{R}^4}|k_1|^2|k_2|^2\frac{|\hat{\mathfrak{n}}_\varphi(k_{1:2})|^2}{\kappa+\frac{c}{2}|k_{1:2}|^2}\,\mathrm{d}k_{1:2}\\ &= \frac{2}{\kappa^2}\int_{\mathbb{R}^2}\mathrm{d}k|\varphi_k|^2\Big(\lambda^2_{N,1}\int_{\mathbb{R}^2}\mathrm{d}k_2|k-k_2|^2|k_2|^2\frac{|\mathscr{K}^N_{k-k_2,k_2}|^2}{\kappa+\frac{c}{2}(|k-k_2|^2+|k_2|^2)}\Big)\end{aligned} \tag{2.51}$$

which is fully explicit and we are left to consider the inner integral. To do so, recall the definition of $\mathscr{K}^N$ in (2.4). We restrict the integral over k_2 to the sector

$$\mathscr{C}^N_k \stackrel{\text{def}}{=} \{k_2\colon \theta_{k_2}\in\theta_k + (\pi/6,\pi/3)\quad\&\quad N/3\geq|k_2|\geq(2|k|)\vee 2/N\quad\&\quad |k|\leq\sqrt{N}\}$$

where, for $j\in\mathbb{R}^2$, θ_j is the angle between the vectors j and $(1,0)$. Then we have

$$\begin{aligned}k_2\cdot k^\perp &= |k_2||k|\cos(|\theta_{k^\perp}-\theta_{k_2}|)\geq |k_2||k|\frac{\sqrt{3}}{2}\,,\\ k_2\cdot k &= |k_2||k|\cos(|\theta_k-\theta_{k_2}|)\geq|k_2||k|\frac{\sqrt{3}}{2}\,.\end{aligned} \tag{2.52}$$

Hence on $\mathscr{C}_k$, we have

$$\begin{aligned}|\mathscr{K}^N_{k-k_2,k_2}|^2 &= \frac{1}{2\pi}(\hat{\varrho}^N_{k-k_2,k_2})^2\frac{|(k-k_2)^\perp\cdot k|^2|k_2\cdot k|^2}{|k_2|^4|k-k_2|^4} = \frac{1}{2\pi}(\hat{\varrho}^N_{k-k_2,k_2})^2\frac{|k_2\cdot k^\perp|^2|k_2\cdot k|^2}{|k_2|^4|k-k_2|^4}\\ &= \frac{1}{2\pi}(\hat{\varrho}^N_{k-k_2,k_2})^2\frac{|k|^4}{|k-k_2|^4}|\cos(\theta-\theta_k)|^2|\cos(\theta-\theta_{k^\perp})|^2\geq c_\varrho\frac{|k|^4}{|k_2|^2|k-k_2|^2}\end{aligned}$$

for a constant c_ϱ depending only on ϱ but neither on k nor N. In the last step, we used that by assumption (1.12) on ϱ, $|\hat{\varrho}^N|$ is bounded below on $[2/N, N/2]$ by a constant independent of N and that on $\mathscr{C}^N_k$ we have

$$\tfrac{2}{N}\leq|k|,|k_2|,|k-k_2|\leq\tfrac{N}{2}\,,\qquad\text{and}\qquad \tfrac{3}{2}|k_2|\geq|k-k_2|\geq\tfrac{1}{2}|k_2|\,.$$

Hence, the right hand side of (2.51) is lower bounded, modulo a multiplicative constant only depending on ϱ, by

$$\frac{2}{\kappa^2}\int_{2/N\le|k|\le\sqrt{N}} \mathrm{d}k|k|^4|\varphi_k|^2\Big(\lambda_{N,1}^2\int_{\mathscr{C}_k^N}\frac{\mathrm{d}k_2}{\kappa+|k_2|^2}\Big)\,. \tag{2.53}$$

It remains to treat the quantity in parenthesis, for which we pass to polar coordinates and obtain

$$\lambda_{N,1}^2\int_{\mathscr{C}_k^N}\frac{\mathrm{d}k_2}{\kappa+|k_2|^2}\ge\lambda_{N,1}^2\int_{2\sqrt{N}}^{N/3}\frac{\varrho\,\mathrm{d}\varrho}{\kappa+\varrho^2}=\frac{\lambda}{2\log N}\log\Big(\frac{\kappa+N^2/9}{\kappa+4N}\Big)\gtrsim 1\,.$$

In conclusion, we have shown that for N large enough

$$\int_0^\infty e^{-\kappa t}\mathbf{E}\Big[\Big|\mathscr{B}_t^N(\varphi)\Big|^2\Big]\,\mathrm{d}t\gtrsim\frac{1}{\kappa^2}\int_{2/N\le|k|\le\sqrt{N}}\mathrm{d}k|k|^4|\varphi_k|^2\,, \tag{2.54}$$

and it remains to pass to the limit as $N\to\infty$. Now, thanks to (2.42) and tightness of $\mathscr{B}^N$, we can apply dominated convergence to the left hand side, while the integral at right hand side clearly converges to $\|\varphi\|^2_{\dot H^2(\mathbb{R}^2)}$, so that the proof is completed. □

Proof of Lemma 2.2.7. We will exploit the Fourier representation of the operators $\mathscr{A}_+^N$ and $\mathscr{A}_-^N$ in Proposition 2.2.1, which though still need to be symmetrised. Let $\mathfrak{a}_+^N$ be the operator defined by the right hand side of (2.36) and S_3 the set of permutations of $\{1,2,3\}$. Then,

$$\begin{aligned}\langle g,\mathscr{A}_-^N(\kappa-\mathscr{L}_0)^{-1}\mathscr{A}_+^N g\rangle_{\Gamma L_2^2}&=\langle\mathscr{A}_+^N g,(\kappa-\mathscr{L}_0)^{-1}\mathscr{A}_+^N g\rangle_{\Gamma L_3^2}\\&=\sum_{s,\bar s\in S_3}\int\frac{|k_1|^2|k_2|^2|k_3|^2}{\kappa+\frac12|k_{1:3}|^2}\mathscr{F}(\mathfrak{a}_+^N g)(k_{s(1):s(3)})\mathscr{F}(\mathfrak{a}_+^N g)(k_{\bar s(1):\bar s(3)})\,\mathrm{d}k_{1:3}\\&\lesssim\int\frac{|k_1|^2|k_2|^2|k_3|^2}{\kappa+\frac12|k_{1:3}|^2}\mathscr{F}(\mathfrak{a}_+^N g)(k_{1:3})^2\,\mathrm{d}k_{1:3}\end{aligned} \tag{2.55}$$

where in the last step we simply applied Cauchy-Schwarz inequality. Now, we bound $|\mathscr{K}_{k_1,k_2}^N|\le$

$\hat{\varrho}^N_{k_2}|k_1+k_2|^2/(|k_1||k_2|)$ so that the right hand side above can be controlled via

$$\begin{aligned}
&\lambda_{N,1}^2\int_{\mathbb{R}^6}\hat{g}(k_1+k_2,k_3)^2\hat{\varrho}_{k_2}\frac{|k_3|^2|k_1+k_2|^4}{\kappa+\frac{1}{2}|k_{1:3}|^2}\,\mathrm{d}k_{1:3}\\
&\qquad\lesssim\int_{\mathbb{R}^4}\mathrm{d}k_{1:2}\Big(\prod_{i=1}^2|k_i|^2\Big)|k_1|^2|\hat{g}(k_1,k_2)|^2\Big(\lambda_{N,1}^2\int_{\mathbb{R}^2}\frac{\hat{\varrho}_j\,\mathrm{d}j}{\kappa+|j|^2}\Big)\\
&\qquad\lesssim\int_{\mathbb{R}^4}\mathrm{d}k_{1:2}\Big(\prod_{i=1}^2|k_i|^2\Big)|k_1|^2|\hat{g}(k_1,k_2)|^2\\
&\qquad=\frac{1}{2}\int_{\mathbb{R}^4}\mathrm{d}k_{1:2}\Big(\prod_{i=1}^2|k_i|^2\Big)(|k_1|^2+|k_2|^2)|\hat{g}(k_1,k_2)|^2=\langle(-\mathscr{L}_0)g,g\rangle_{\Gamma L^2_2}
\end{aligned}\tag{2.56}$$

where the second step follows by the fact that $\hat{\varrho}_j\leq\mathbb{1}_{|j|\leq N}$ and the definition of $\lambda_{N,1}$ in (1.11), while the last by the symmetrisation of the integral.

We now turn to the other term, which is

$$\begin{aligned}
&\langle g,\mathscr{A}^N_+(\kappa-\mathscr{L}_0 0)^{-1}\mathscr{A}^N_-g\rangle_{\Gamma L^2_2}\\
&=\langle\mathscr{A}^N_-g,(\kappa-\mathscr{L}_0)^{-1}\mathscr{A}^N_-g\rangle_{\Gamma L^2_1}\lesssim\lambda_{N,1}^2\int\frac{|k|^2}{\kappa+\frac{1}{2}|k|^2}\mathscr{F}(\mathscr{A}^N_-g)(k)^2\,\mathrm{d}k\\
&=\lambda_{N,1}^2\int_{\mathbb{R}^2}\mathrm{d}k\frac{|k|^2}{\kappa+\frac{1}{2}|k|^2}\Big(\int_{\mathbb{R}^2}\hat{\varrho}^N_{p,k-p}\frac{(k^\perp\cdot p)(k\cdot(k-p))}{|k|^2}\hat{g}(p,k-p)\,\mathrm{d}p\Big)^2\\
&\lesssim\lambda_{N,1}^2\int_{\mathbb{R}^2}\mathrm{d}k\Big(\int_{\mathbb{R}^2}\hat{\varrho}^N_p|p||k-p|\hat{g}(p,k-p)\,\mathrm{d}p\Big)^2.
\end{aligned}$$

We now multiply and divide the integrand by $|p|$ and apply Cauchy-Schwarz, so that we obtain an upper bound of the form

$$\Big(\int_{\mathbb{R}^4}\mathrm{d}k_{1:2}\Big(\prod_{i=1}^2|k_i|^2\Big)|k_1|^2\hat{g}(k_1,k_2)|^2\Big)\Big(\lambda_{N,1}^2\int_{\mathbb{R}^2}(\hat{\varrho}^N_p)^2\frac{\mathrm{d}p}{|p|^2}\Big)$$

from which (2.48) follows arguing as in (2.56). □

2.2.4 Triviality of the fractional Vorticity equation for $\theta<1$

In this last section, we complete the proof of Theorem 1.3.4 and show that the re-scaled solution of the regularised fractional Vorticity equation for $\theta\in(0,1)$ converges to the fractional stochastic heat equation obtained by simply setting the coupling constant λ in (1.10) to 0.

For the proof, recall that ω is a stationary (analytically) weak solution of (1.21) if for all

$\varphi \in \mathcal{S}(\mathbb{R}^2)$, ω satisfies

$$\omega_t(\varphi) = \mu(\varphi) + \int_0^t \omega_s(-(-\Delta)^\theta \varphi)\,\mathrm{d}s + \int_0^t \xi(\,\mathrm{d}s, (-\Delta)^{\frac{1+\theta}{2}}\varphi)$$

where μ is the Gaussian process whose covariance is given by (1.26). It is not hard to see that ω admits a unique stationary weak solution. This is the only tool we need for the proof, which is then a simple corollary of Theorem 2.2.2.

Proof of Theorem 1.3.4. For $N \in \mathbb{N}$, let ω^N be a stationary weak solution to (1.10), i.e. for all $\varphi \in \mathcal{S}(\mathbb{R}^2)$ ω^N satisfies

$$\omega_t^N(\varphi) - \mu(\varphi) = \frac{1}{2}\int_0^t \omega_s^N(-(-\Delta)^\theta \varphi)\,\mathrm{d}s + \mathcal{B}_t^N(\varphi) - \int_0^t \xi(\,\mathrm{d}s, (-\Delta)^{\frac{1+\theta}{2}}\varphi)\,,$$

where $\mathcal{B}^N$ is defined according to (2.41). By Theorem 2.2.2, the sequence $(\omega^N, \mathcal{B}^N)$ is tight in the space $C([0,T], \mathcal{S}'(\mathbb{R}^2))$ and, thanks to (2.43), $\mathcal{B}^N \to 0$ as $N \to \infty$. Hence, it immediate to verify that every limit point of ω^N is a weak stationary solution of (1.21). Since the latter is unique, the result follows at once. □

CHAPTER 3

Anisotropic KPZ on the real plane under critical regime

3.1 Invariant measures of the regularised equations

Similarly to Chapter 3 the goal of this section is to construct a stationary solution to the regularised critical Anisotropic KPZ equation on $\mathbb{R}^2$. Following similar approach to that of SNS. Demonstrating this method can be applied to more than one model. Some calculations may be very similar to those from Chapter 2 and as such will be omitted.

3.1.1 The regularised Burger equation on $\mathbb{T}_M^2$

For $\theta \in (0,1]$, we consider the periodic version on $\mathbb{T}_M^2$ of (1.19) given by

$$\partial_t u^{N,M} = -\tfrac{1}{2}(-\Delta)^\theta u^{N,M} - \lambda_N \tilde{\mathcal{N}}^{N,M}[u^{N,M}] + (-\Delta)^{\frac{\theta}{2}}\xi^M\,, \quad u^{N,M}(0,\cdot) = u_0^M\,, \tag{3.1}$$

where u_0^M is the initial condition, and $\tilde{\mathcal{N}}^{N,M}$ is the non-linearity defined in (1.18). In Fourier variables, (3.1) becomes

$$\mathrm{d}\,\hat{u}_k^{N,M} = -\tfrac{1}{2}|k|^{2\theta}\hat{u}_k^{N,M} - \lambda_N \tilde{\mathcal{N}}_k^{N,M}[u^{N,M}] + |k|^\theta\,\mathrm{d}B_k(t)\,, \qquad k \in \mathbb{Z}_M^2$$

The Fourier transform of the non-linearity $\tilde{\mathcal{N}}^{N,M}$ is of the form

$$\tilde{\mathcal{N}}_k^{N,M}[u^{N,M}](x) = \frac{1}{M^2} \sum_{\substack{\ell,m \in \mathbb{Z}_M^2 \\ \ell+m=k}} \tilde{\mathcal{K}}_{\ell,m}^N u_{-\ell}^{N,M} u_{-m}^{N,M} e_{\ell+m}(x)\,, \tag{3.2}$$

for

$$\tilde{\mathcal{K}}_{\ell,m}^N \stackrel{\text{def}}{=} \frac{1}{2\pi} \hat{\varrho}_{\ell,m}^N |\ell+m| \frac{c(\ell,m)}{|\ell||m|}\,, \quad c(\ell,m) \stackrel{\text{def}}{=} m_2\ell_2 - m_1\ell_1\,, \tag{3.3}$$

and the variables ℓ and m appearing in the previous equations range over $\mathbb{Z}_M^2$. The proof of this can be found in the appendix Proposition 3.3.7. As a first step in our analysis, we determine basic properties of the solution of (3.1).

Proposition 3.1.1 *Let $M, N \in \mathbb{N}$ and $\theta \in (0,1]$. Then, for every deterministic initial condition $u_0^{N,M} \in \dot{H}^{-1}(\mathbb{T}_M^2)$, (3.1) has a unique strong solution $u^{N,M} \in C(\mathbb{R}_+, \dot{H}^{-1}(\mathbb{T}_M^2))$. Further, $u^{N,M}$ is a strong Markov process.*

Proof. The proof is the same as in Proposition 3.1.1. This is because just as in SNS the regularisation allows us to decouple the first N Fourier modes with the rest. □

Lemma 3.1.2 *Let $T = \mathbb{T}_M^2$ or $\mathbb{R}^2$ then for any distribution $\eta \in \mathcal{S}'(T)$ we have*

$$\langle \tilde{\mathcal{N}}^N[\eta], \eta \rangle_{L^2(T)} = 0\,.$$

Proof. This proof is done in [CES21, Appendix A], we include it here for completeness and convenience for the reader. Let $\psi^N = (-\Delta)^{-\frac{1}{2}} \varrho^N * \eta$ then

$$\begin{aligned}
\langle \tilde{\mathcal{N}}^N[\eta], \eta \rangle_{L^2(T)} &= \langle (-\Delta)\Big((\partial_1\psi^N)^2 - (\partial_2\psi^N)^2\Big), \psi^N \rangle_{L^2(T)} \\
&= \sum_{i=1}^2 \langle \partial_i(\partial_i\psi^N)^2, \partial_i\psi^N \rangle_{L^2(T)} + \sum_{\substack{i,j\in\{1,2\} \\ i\neq j}} (-1)^i \langle \partial_i(\partial_j\psi^N)^2, \partial_i\psi^N \rangle_{L^2(T)} \\
&= \frac{1}{3} \sum_{i=1}^2 \langle \partial_i(\partial_i\psi^N)^3, 1 \rangle_{L^2(T)} + 2 \sum_{\substack{i,j\in\{1,2\} \\ i\neq j}} (-1)^i \langle \partial_i\psi^N \partial_j\psi^N \partial_{i,j}\psi^N, 1 \rangle_{L^2(T)}\,.
\end{aligned}$$

In the first inequality we moved ϱ^N and $(-\Delta)^{-\frac{1}{2}}$ to the right hand side of the product,in the second equality we expand all the terms. In the third equality we use the fact that

$$\partial(\partial\psi^N)^3 = 3(\partial\psi^N)^2\partial\psi^N$$

In the final equation the first sum is zero by integration by parts while the second sum is made of two terms which cancel each other out giving the result. □

Even though the generator $\tilde{\mathscr{L}}^{N,M}$ of the Markov process $u^{N,M}$, is a complicated operator, its action on cylinder functions F can be easily obtained by applying Itô's formula and singling out the drift term. By doing so, we deduce that for any such F, $\tilde{\mathscr{L}}^{N,M}F$ can be written as $\tilde{\mathscr{L}}^{N,M}F = \tilde{\mathscr{L}}_\theta^M F + \tilde{\mathscr{A}}^{N,M}F$, where $\tilde{\mathscr{L}}_0^M$ and $\tilde{\mathscr{A}}^{N,M}$ are given by

$$\begin{aligned} \tilde{\mathscr{L}}_\theta^M F(u) &\stackrel{\text{def}}{=} \frac{1}{2}\sum_{i=1}^n u(-(-\Delta)^\theta\varphi_i)\partial_i f + \frac{1}{2}\sum_{i,j=1}^n \langle\varphi_i,\varphi_j\rangle_{\dot{H}^\theta(\mathbb{T}_M^2)}\,\partial_{i,j}^2 f, \\ \tilde{\mathscr{A}}^{N,M}F(u) &\stackrel{\text{def}}{=} -\lambda_N\sum_{i=1}^n \tilde{\mathscr{N}}^{N,M}[u](\varphi_i)\,\partial_i f, \end{aligned} \tag{3.4}$$

where we abbreviated $\partial_i f = \partial_i f(u(\varphi_1),\dots,u(\varphi_n))$.

Proposition 3.1.3 *Let η^M be the Gaussian spatial noise on $\mathbb{T}_M^2$ with covariance given by* (1.30)*. Then, η^M is an invariant measure of the solution $u^{N,M}$ of* (3.1)*.*

Proof. The proof for $M = 1$ is given in [CES21, Lemma 3.1] where the [Ech82] result is used, method is similar to that of Proposition 2.1.3. Generalisation of [CES21, Lemma 3.1] to general M is trivial. □

From now on, we will only work with the stationary solution of (3.1), i.e. the initial condition will always be taken to be

$$u_0^{N,M} \stackrel{\text{def}}{=} \eta^M \tag{3.5}$$

where η^M is as in Proposition 3.1.3. In the following statements, we aim at obtaining estimates on the solution $u^{N,M}$ to (3.1) which are uniform in both N and M. Even though we are considering a different equation lemma 2.1.4 still applies just the same, only difference being the energy. The energy for AKPZ is given by

$$\tilde{\mathscr{E}}^N(F)(\eta^M) \stackrel{\text{def}}{=} \frac{1}{M^2}\sum_{k\in\mathbb{Z}_M^2} |k|^{2\theta}|D_k F(\eta^M)|^2\,. \tag{3.6}$$

Proposition 3.1.4 *Let $\theta \in (0,1]$, $T > 0$ be fixed and $p \geq 2$. For $M,\ N \in \mathbb{N}$, let $\tilde{\mathcal{N}}^{N,M}$ be defined according to* (3.2) *and λ_N be as in* (1.11)*. Then, there exists a constant $C = C(p) > 0$, independent of $M,\ N \in \mathbb{N}$ such that for all $\varphi \in \mathcal{S}(\mathbb{T}^2_M)$ and all $t \in [0,T]$, we have*

$$\mathbf{E}\left[\sup_{s\leq t}\left|\int_0^s u_r^{N,M}(-(-\Delta)^\theta\varphi)\,\mathrm{d}r\right|^p\right]^{1/p} \leq Ct^{\frac{1}{2}}\|\varphi\|_{\dot{H}^\theta(\mathbb{T}^2_M)}\,, \tag{3.7}$$

$$\mathbf{E}\left[\sup_{s\leq t}\left|\lambda_N\int_0^s \tilde{\mathcal{N}}^{N,M}[u_r^{N,M}](\varphi)]\,\mathrm{d}r\right|^p\right]^{1/p} \leq CN^{\theta-1}(t\vee t^{\frac{1}{2}})\|\varphi\|_{\dot{H}^1(\mathbb{T}^2_M)}\,. \tag{3.8}$$

The proof of the previous proposition (and in particular of (2.15)) is based on the following lemma.

Lemma 3.1.5 *For $M,\ N \in \mathbb{N}$, $\varphi \in \mathcal{S}(\mathbb{T}^2_M)$, let $[\eta^M](\varphi)$ be the smooth random variable defined according to* (3.2)*, with η^M replacing u. Then, $\tilde{\mathcal{N}}_k^{N,M}[\eta^M](\varphi)$ belongs to the second homogeneous Wiener chaos $\mathscr{H}_2$.Furthermore, the Poisson equation*

$$(1-\tilde{\mathscr{L}}_0^M)\tilde{H}^{N,M}[\eta^M](\varphi) = \lambda_N\tilde{\mathcal{N}}^{N,M}[\eta^M](\varphi) \tag{3.9}$$

has a unique solution whose energy satisfies

$$\mathbb{E}[\mathscr{E}^{N,M}(\tilde{H}^{N,M}[\mu^M](\varphi))] = \frac{4\lambda_N^2}{M^4}\sum_{\ell,m}|\ell|^{2\theta}\frac{(\tilde{\mathscr{K}}^N_{\ell,m})^2}{(1+\frac{1}{2}(|\ell|^{2\theta}+|m|^{2\theta}))^2}|\varphi_{-\ell-m}|^2\,. \tag{3.10}$$

Proof. By the Fourier representation for the $\tilde{\mathcal{N}}^{N,M}$ in (3.2) we have

$$\mathbb{E}\Big(\tilde{\mathcal{N}}^{N,M}[\eta^M](\varphi)\Big) = \frac{1}{M^2}\sum_{\ell,m\in\mathbb{Z}^2_M}\tilde{\mathscr{K}}^N_{\ell,m}\mathbf{E}\Big(\eta^M_{-\ell}\eta^M_{-m}\Big)\varphi_{\ell+m} = \frac{1}{M^2}\sum_{\ell\in\mathbb{Z}^2_M}\tilde{\mathscr{K}}^N_{\ell,-\ell}\varphi_0 = 0\,,$$

the last equality is a direct consequence of the definition of $\tilde{\mathscr{K}}^N$ in (3.3). Since $\tilde{\mathcal{N}}^{N,M}[\eta^M](\varphi)$ is quadratic and its component in the 0-th chaos is 0, it follows that $\tilde{\mathcal{N}}_k^{N,M}[\eta^M](\varphi) \in \mathscr{H}_2$ and $\tilde{\mathcal{N}}^{N,M}[\eta^M](\varphi) = W_2(\tilde{\mathfrak{n}}^{N,M}_\varphi)$, for $\tilde{\mathfrak{n}}^{N,M}_\varphi$ such that

$$\hat{\tilde{\mathfrak{n}}}^N_\varphi(\ell,m) = \tilde{\mathscr{K}}^N_{\ell,m}\varphi_{\ell+m}\,. \tag{3.11}$$

Let $\tilde{\mathfrak{h}}^{N,M}_\varphi \in \dot{H}^1_{\mathrm{sym}}((\mathbb{T}^2_M)^2)$ and $\tilde{H}^{N,M}[\eta^M](\varphi) = W_2(\tilde{\mathfrak{h}}^{N,M}_\varphi)$. Then, [CES21, Lemma 3.5] implies that

$$(1-\tilde{\mathscr{L}}_\theta^M)\tilde{H}^{N,M}[\eta^M](\varphi) = W_2((1-\frac{1}{2}(-\Delta)^\theta))\tilde{\mathfrak{h}}^{N,M}_\varphi)\,.$$

from which it is immediate to see that (3.9) has a unique solution which must necessarily satisfy

$$\hat{\tilde{\mathfrak{h}}}_\varphi^{N,M}(\ell,m) = \lambda_N \frac{\tilde{\mathscr{K}}^N_{\ell,m}}{1+\frac{1}{2}(|\ell|^{2\theta}+|m|^{2\theta})}\varphi_{-\ell-m}\,, \qquad \text{for all } \ell,m\in\mathbb{Z}^2_M. \tag{3.12}$$

In order to compute the energy of $\tilde{H}^{N,M}[\eta^M](\varphi)$, let us express $W_2(\tilde{\mathfrak{h}}_\varphi^{N,M})$ in Fourier as

$$\tilde{H}^{N,M}[\eta^M](\varphi) = W_2(\tilde{\mathfrak{h}}_\varphi^{N,M}) = \frac{\lambda_N}{M^2}\sum_{\ell,m\in\mathbb{Z}^2_M}\frac{\tilde{\mathscr{K}}^N_{\ell,m}}{1+\frac{1}{2}(|\ell|^{2\theta}+|m|^{2\theta})}\eta^M_\ell\eta^M_m\varphi_{-\ell-m}\,.$$

Then, by (1.32),

$$\begin{aligned} D_k\tilde{H}^{N,M}[\eta^M](\varphi) &= \frac{\lambda_N}{M^2}\sum_{\ell,m\in\mathbb{Z}^2_M}\frac{\tilde{\mathscr{K}}^N_{\ell,m}}{1+\frac{1}{2}(|\ell|^{2\theta}+|m|^{2\theta})}D_k(\eta^M_\ell\eta^M_m)\varphi_{-\ell-m}\\ &= \frac{2\lambda_N}{M^2}\sum_{m\in\mathbb{Z}^2_M}\frac{\tilde{\mathscr{K}}^N_{k,m}}{1+\frac{1}{2}(|k|^{2\theta}+|m|^{2\theta})}\eta^M_m\varphi_{k-m}\end{aligned}$$

Hence, $\tilde{\mathscr{E}}^{N,M}(H^{N,M}[\mu^M](\varphi))$ is given by

$$\tilde{\mathscr{E}}^{N,M}(\tilde{H}^{N,M}[\mu^M](\varphi)) = \frac{4\lambda_N^2}{M^6}\sum_{k,\,m_1,m_2}|k|^{2\theta}\prod_{i=1}^2\frac{\tilde{\mathscr{K}}^N_{k,m_i}}{1+\frac{1}{2}(|k|^{2\theta}+|m_i|^{2\theta})}\mu^M_{(-1)^i m_i}\varphi_{(-1)^i(k-m_i)}\,.$$

Taking the expectation of the previous expression we get

$$\begin{aligned} &\mathbb{E}[\tilde{\mathscr{E}}^N(\tilde{H}^{N,M}[\eta^M](\varphi))]\\ &\quad= \frac{4\lambda_N^2}{M^6}\sum_{\ell,k_1,k_2}|\ell|^{2\theta}\frac{\tilde{\mathscr{K}}^N_{\ell,k_1-\ell}}{\kappa+|\ell|^{2\theta}+|k_1-\ell|^{2\theta}}\frac{\tilde{\mathscr{K}}^N_{\ell,k_2-\ell}}{\kappa+|\ell|^{2\theta}+|k_2-\ell|^{2\theta}}\mathbb{E}[\eta^M_{\ell-k_1}\eta^M_{\ell-k_2}]\varphi_{-k_1}\varphi_{-k_2}\\ &\quad= \frac{4\lambda_N^2}{M^2}\sum_\ell|\ell|^{2\theta}\frac{1}{M^2}\sum_k|\varphi_k|^2\frac{(\tilde{\mathscr{K}}^N_{\ell,k-\ell})^2}{(\kappa+|\ell|^{2\theta}+|k-\ell|^{2\theta})^2}\\ &\quad= \frac{4\lambda_N^2}{M^4}\sum_{\ell,m}|\ell|^{2\theta}\frac{(\tilde{\mathscr{K}}^N_{\ell,m})^2}{(1+\frac{1}{2}(|\ell|^{2\theta}+|m|^{2\theta}))^2}|\varphi_{-\ell-m}|^2\,.\end{aligned} \tag{3.13}$$

□

Proof of Proposition 3.1.4. For both (2.14) and (2.15), we will exploit the Itô's trick Lemma 2.1.4. Let us begin with the former. Set $\tilde{K}^{N,M}[\eta^M](\varphi) \stackrel{\text{def}}{=} \eta^M(\varphi)$, then as in SNS we have

$$\tilde{\mathscr{L}}_\theta{}^M\tilde{K}^{N,M}[\eta^M](\varphi) = \|\varphi\|^2_{\dot{H}^\theta(\mathbb{T}^2_M)}\,.$$

Hence, by (2.10)

$$2\mathbf{E}\Big[\sup_{s\le t}\Big|\int_0^s \tilde{\mathscr{L}}_0^M \tilde{K}^M[u_r^{N,M}](\varphi)\,\mathrm{d}r\Big|^p\Big]^{1/p} \lesssim t^{\frac{1}{2}}\|\varphi\|_{\dot{H}^\theta(\mathbb{T}^2_M)}\,.$$

We now turn to (3.8), just as in proof of Proposition 2.1.5 it suffices to bound two terms

$$\mathbf{E}\Big[\sup_{s\le t}\Big|\int_0^s \tilde{H}^{N,M}[\omega_r^{N,M}](\varphi)\,\mathrm{d}r\Big|^p\Big]^{\frac{1}{p}},\quad \mathbf{E}\Big[\sup_{s\le t}\Big|\int_0^s \mathscr{L}_0^M \tilde{H}^{N,M}[\omega_r^{N,M}](\varphi)\,\mathrm{d}r\Big|^p\Big]^{\frac{1}{p}} \tag{3.14}$$

For the second, we apply once more (2.10), which, together with (3.10), gives

$$\begin{aligned}
\mathbf{E}\Big[\sup_{s\le t}\Big|\int_0^s \tilde{\mathscr{L}}_0^M \tilde{H}^{N,M}[u_r^{N,M}](\varphi)\,\mathrm{d}r\Big|^p\Big]^{\frac{2}{p}} &\lesssim t\frac{\lambda_N^2}{M^4}\sum_{\ell,m}|\ell|^{2\theta}\frac{(\tilde{\mathscr{K}}^N_{\ell,m})^2}{(1+\frac{1}{2}(|\ell|^{2\theta}+|m|^{2\theta}))^2}|\varphi_{-\ell-m}|^2\\
&\lesssim t\frac{1}{M^2}\sum_k |k|^2|\varphi_k|^2\frac{\lambda_N^2}{M^2}\sum_{\ell+m=k}\hat{\varrho}^N_{\ell,m}\frac{|\ell|^{2\theta}}{(1+\frac{1}{2}(|\ell|^{2\theta}+|m|^{2\theta}))^2}\\
&\lesssim t\frac{1}{M^2}\sum_k |k|^2|\varphi_k|^2\frac{\lambda_N^2}{M^2}\sum_{\ell}\hat{\varrho}^N_{\ell}\frac{1}{1+\frac{1}{2}|\ell|^{2\theta}}\\
&= t\|\varphi\|^2_{\dot{H}^1(\mathbb{T}^2_M)}\frac{\lambda_N^2}{M^2}\sum_{\ell}\hat{\varrho}^N_{\ell}\frac{1}{1+\frac{1}{2}|\ell|^{2\theta}}\\
&\lesssim t\|\varphi\|^2_{\dot{H}^1(\mathbb{T}^2_M)}\,,
\end{aligned}$$

where we bounded $|\mathscr{K}^N_{\ell,m}|^2 \le \hat{\varrho}^N_{\ell,m}|\ell+m|^2$ and applied a simple change of variables. The last inequality being a consequence of (1.11) and Riemann approximation. Let us turn to the first summand in (3.14), in the same spirit as in SNS we have

$$\mathbf{E}\Big[\sup_{s\le t}\Big|\int_0^s \tilde{H}^{N,M}[u_r^{N,M}](\varphi)\,\mathrm{d}r\Big|^p\Big]^{\frac{1}{p}} \le \sqrt{2}t\|\tilde{\mathfrak{h}}^{N,M}_\varphi\|_{(L^2(\mathbb{T}^2_M))^{\otimes 2}}$$

which is due to Jensen's inequality and Gaussian hypercontractivity [Nua06, Theorem 1.4.1].By Lemma 3.1.5, $\tilde{H}^{N,M}[\eta^M](\varphi) = W_2(\tilde{\mathfrak{h}}^{N,M}_\varphi)$ for $\tilde{\mathfrak{h}}^{N,M}_\varphi$ satisfying (3.12). In turn, the norm of $\tilde{\mathfrak{h}}^{N,M}_\varphi$ can be estimated via

$$\begin{aligned}
\|\tilde{\mathfrak{h}}^{N,M}_\varphi\|^2_{(L^2(\mathbb{T}^2_M))^{\otimes 2}} &= \frac{\lambda_N^2}{M^4}\sum_{\ell,m\in\mathbb{Z}^2_M}\frac{(\mathscr{K}^N_{\ell,m})^2}{(1+\frac{1}{2}(|\ell|^{2\theta}+|m|^{2\theta}))^2}|\varphi_{-\ell-m}|^2\\
&\lesssim \frac{\lambda_N^2}{M^2}\sum_{k\in\mathbb{Z}^2_M}|k|^2|\varphi_k|^2\frac{1}{M^2}\sum_{\ell+m=k}\frac{1}{(1+\frac{1}{2}(|\ell|^{2\theta}+|m|^{2\theta}))^2}
\end{aligned}$$

$$\lesssim \frac{\lambda_N^2}{M^2} \sum_{k\in\mathbb{Z}_M^2} |k|^2|\varphi_k|^2 \int_{\mathbb{R}^2} \frac{\mathrm{d}x}{(1+|x|^{2\theta})^2} \lesssim \|\varphi\|_{\dot{H}^1(\mathbb{T}_M^2)}$$

where the second bound follows by $|\mathcal{K}_{\ell,m}^N| \leq |\ell+m|$ and the same change of variables as above. □

Theorem 3.1.6 *Let $N \in \mathbb{N}$ be fixed and η be a space white noise on $\mathbb{R}^2$. The cylinder martingale problem for $\tilde{\mathscr{L}}^N$ with initial distribution η in Definition 2.1.7 has a solution $\mathbb{P}$. Further, the canonical process u^N under $\mathbb{P}$ has invariant measure η.*

Proof. The proof is analogous to that of Theorem 2.1.9. Indeed one uses the Proposition 3.1.4 to obtain tightness via the use of Kolmogorov and [Mit83]. Following this a Martingale problem is formed in which almost sure convergence is trivial and is used to show mean square convergence via the fact that we know that the limiting distribution is Gaussian in nature. □

3.2 The Burger equation on the real plane

Throughout this section, we will be working with a solution $\tilde{\mathbf{P}}^N$ of the martingale problem for $\tilde{\mathscr{L}}^N$ with initial distribution η, whose canonical process u^N is, by Proposition 3.1.3, a stationary weak solution of the regularised burger equation (1.19). Similarly to SNS, the goal is to control the behaviour of u^N in the limit $N \to \infty$. We start by deepening our understanding of the generator $\tilde{\mathscr{L}}^N$ and, in particular, determine how it acts on random variables in $L^2(\boldsymbol{\eta})$.

3.2.1 The operator $\tilde{\mathscr{L}}^N$

Let us introduce the operator $\tilde{\mathscr{L}}^N$ which is nothing but the $\mathbb{R}^2$ counterpart of $\tilde{\mathscr{L}}^{N,M}$ and formally represents the generator of (1.19). Once again, it can be written as the sum of two operators, i.e. $\tilde{\mathscr{L}}^N = \mathscr{L}_\theta + \tilde{\mathscr{A}}^N$, whose action of cylinder functions $F(\omega) = f(\omega(\varphi_1), \ldots, \omega(\varphi_n))$ is given by

$$\tilde{\mathscr{L}}_\theta F(u) \stackrel{\text{def}}{=} \frac{1}{2}\sum_{i=1}^n u(-(-\Delta)^\theta \varphi_i)\partial_i f + \frac{1}{2}\sum_{i,j=1}^n \langle \varphi_i, \varphi_j \rangle_{\dot{H}^\theta(\mathbb{R}^2)} \partial_{i,j}^2 f\,, \tag{3.15}$$

$$\tilde{\mathscr{A}}^N F(u) \stackrel{\text{def}}{=} -\lambda_N \sum_i^n \tilde{\mathscr{N}}^N[u](\varphi_i)\,\partial_i f. \tag{3.16}$$

This section is devoted to the study of the properties of the operator $\tilde{\mathscr{L}}^N$ on $L^2(\boldsymbol{\eta})$, (η being the Gaussian process with covariance (1.30)) which is given by the sum of $\tilde{\mathscr{L}}_0$ and $\tilde{\mathscr{A}}^N$ defined in (3.15) and (3.16), respectively.

Proposition 3.2.1 *Let η be the Gaussian process whose covariance function is given by* (1.30)*. Then, on $L^2(\boldsymbol{\eta})$, the operator $\tilde{\mathscr{L}}_0$ is symmetric and for each n, it maps $\mathscr{H}_n$ to itself. Further, for any $f \in \Gamma L^2_n$, $\tilde{\mathscr{L}}_\theta f = -\frac{1}{2}(-\Delta)^\theta f$ so that the Fourier transform of the left hand side equals*

$$\mathscr{F}(\mathscr{L}_\theta f)(k_{1:n}) = -\tfrac{1}{2}|k_{1:n}|^{2\theta}\hat{f}(k_{1:n}) , \qquad \text{for all } k_{1:n} \in (\mathbb{R}^2)^n, \tag{3.17}$$

where $|k_{1:n}|^{2\theta} \stackrel{\text{def}}{=} |k_1|^{2\theta} + \cdots + |k_n|^{2\theta}$. Instead, the operator $\tilde{\mathscr{A}}^N$ is anti-symmetric on $L^2(\boldsymbol{\eta})$ and it can be written as the sum of two operators $\tilde{\mathscr{A}}^N_+$ and $\tilde{\mathscr{A}}^N_-$, the first mapping $\mathscr{H}_n$ to $\mathscr{H}_{n+1}$ while the second to $\mathscr{H}_{n-1}$. Moreover, the adjoint of $\tilde{\mathscr{A}}^N_+$ is $-\tilde{\mathscr{A}}^N_-$ and for any $f \in \Gamma L^2_n$ the Fourier transform of their action on f is given by

$$\mathscr{F}(\tilde{\mathscr{A}}^N_+ g)(k_{1:n+1}) = 4\pi\lambda_N \tilde{\mathscr{K}}^N_{k_{1:2}}\hat{g}(k_1+k_2, k_{3:n+1}) , \tag{3.18}$$

$$\mathscr{F}(\tilde{\mathscr{A}}^N_- g)(k_{1:n-1}) = 4\lambda_N \int |k|\hat{\varrho}^N_{k,k_1-k}\frac{c(k_1-k,k_1)}{|k_1-k||k_1|}\hat{g}(k,k_1-k,k_{2:n-1})\,\mathrm{d}k , \tag{3.19}$$

where $\tilde{\mathscr{K}}^N$ was defined in (3.3) *and $k_{1:n+1} \in (\mathbb{R}^2)^{n+1}$. Strictly speaking the functions at the right hand side need to be symmetrised with respect to all permutations of their arguments.*

Proof. The proof is analogous to the proof of Proposition 2.2.1, only difference being the noise with respect to which the Wiener chaos is used and instead of using Lemma (2.1.2) here we use Lemma (3.1.2). □

3.2.2 Tightness, upper bound and lower bound

Following techniques similar to those exploited in Section 3.1.1, we establish tightness for the sequence $\{u^N\}_N$ of solutions to the stationary regularised Vorticity equation under assumption (1.11). For $\theta = 1$, we also derive an order one upper bound on the integral in time of the non-linearity.

Theorem 3.2.2 *Let $\theta \in (0,1]$. For $N \in \mathbb{N}$, let u^N be a weak stationary solution to* (1.19) *on $\mathbb{R}^2$ with coupling constant λ_N chosen according to* (1.11)*, started from the Gaussian process η with covariance given by* (1.30)*. For $\varphi \in \mathcal{S}(\mathbb{R}^2)$ and $t \geq 0$, set*

$$\tilde{\mathscr{B}}^N_t(\varphi) \stackrel{\text{def}}{=} \lambda_N \int_0^t \tilde{\mathscr{N}}^N_t[u^N_s](\varphi)\,\mathrm{d}s . \tag{3.20}$$

Then, for any $T > 0$, the couple $(u^N, \tilde{\mathscr{B}}^N)$ is tight in the space $C([0,T], \mathcal{S}'(\mathbb{R}^2))$. Moreover, for $\theta = 1$, any limit point $(u, \tilde{\mathscr{B}})$ is such that for all $p \geq 2$ there exists a constant $C = C(p)$ such *that for all $\varphi \in \mathcal{S}(\mathbb{R}^2)$*

$$\mathbf{E}\Big[\Big|\tilde{\mathscr{B}}_t(\varphi)\Big|^p\Big]^{\frac{1}{p}} \leq C(t \vee t^{\frac{1}{2}})\|\varphi\|_{\dot{H}^1(\mathbb{R}^2)}, \tag{3.21}$$

while, for $\theta \in (0,1)$, for all $p \geq 2$ and $\varphi \in \mathcal{S}(\mathbb{R}^2)$

$$\lim_{N\to\infty} \mathbf{E}\Big[\sup_{s\leq t}\Big|\tilde{\mathscr{B}}^N_s(\varphi)\Big|^p\Big]^{\frac{1}{p}} = 0\,. \tag{3.22}$$

Proof. Just like in SNS the proof on $\mathbb{R}^2$ is very similar to the proof on $\mathbb{T}^2_M$ - it suffices to remove the superscripts M, replace every instance of $\mathbb{T}^2_M$ with $\mathbb{R}^2$ and substitute the weighted Riemann-sums with integrals. We can then apply Mitoma's and Kolmogorov's criteria to obtain (3.21) and tightness. □

We now proceed to proving the lower bound of Theorem 1.3.2 for the AKPZ, note here we take $\theta = 1$.

Proposition 3.2.3 *In the same setting as Theorem 3.2.2, let $\tilde{\mathscr{B}}$ be any limit point of the sequence $\tilde{\mathscr{B}}^N$ in (3.20). Then, there exists a constant $C > 0$ such that for all $\kappa > 0$ and $\varphi \in \mathcal{S}(\mathbb{R}^2)$, we have*

$$\int_0^\infty e^{-\kappa t}\mathbf{E}\Big[\Big|\tilde{\mathscr{B}}_t(\varphi)\Big|^2\Big]\,\mathrm{d}t \geq \frac{C}{\kappa^2}\|\varphi\|^2_{\dot{H}^1(\mathbb{R}^2)}\,. \tag{3.23}$$

Proof. For $N \in \mathbb{N}$, let $\tilde{\mathscr{B}}^N$ be defined according to (3.20), By [CES21, Lemma 5.1], for $N \in \mathbb{N}$ we have

$$\int_0^\infty e^{-\kappa t}\mathbf{E}\Big[\Big|\tilde{\mathscr{B}}^N_t(\varphi)\Big|^2\Big]\,\mathrm{d}t = \frac{2}{\kappa^2}\mathbb{E}\Big[\tilde{\mathscr{N}}^N[\eta](\varphi)(\kappa - \tilde{\mathscr{L}}^N)^{-1}\tilde{\mathscr{N}}^N[\eta](\varphi)\Big]\,. \tag{3.24}$$

Thanks to [CES21, Lemma 5.2] and the isometry W introduced in Section 1.5, the right hand side above equals

$$\begin{aligned}&\frac{2}{\kappa^2}\sup_{G\in L^2(\eta)}\Big\{2\mathbb{E}[\lambda_N\tilde{\mathscr{N}}^N[\eta](\varphi)G] - \mathbb{E}[G(\kappa-\tilde{\mathscr{L}}_0)G] - \mathbb{E}[\tilde{\mathscr{A}}^N G(\kappa-\tilde{\mathscr{L}}_0)^{-1}\tilde{\mathscr{A}}^N G]\Big\}\\ &= \frac{2}{\kappa^2}\sup_{g\in\Gamma L^2}\Big\{2\langle\lambda_N\tilde{\mathfrak{n}}^N_\varphi, g\rangle_{\Gamma L^2} - \langle g, (\kappa-\tilde{\mathscr{L}}_0)g\rangle_{\Gamma L^2} - \langle\tilde{\mathscr{A}}^N g, (\kappa-\tilde{\mathscr{L}}_0)^{-1}\tilde{\mathscr{A}}^N g\rangle_{\Gamma L^2}\Big\}\end{aligned} \tag{3.25}$$

where $\tilde{\mathfrak{n}}^N_\varphi$ is such that $\tilde{\mathscr{N}}^N[\eta](\varphi) = W_2(\tilde{\mathfrak{n}}^N_\varphi)$ and its Fourier transform is given by the right hand side of (3.11). We can further lower bound the above by restricting to g to ΓL^2_2 for which, by orthogonality of different chaoses of $\tilde{\mathscr{A}}^N_+$ and $\tilde{\mathscr{A}}^N_-$ determined in Proposition 3.2.1 we have

$$\langle\tilde{\mathscr{A}}^N_+ G, (\kappa - \tilde{\mathscr{L}}^N_0)^{-1}\tilde{\mathscr{A}}^N_- G\rangle = 0\,,$$

$$\langle \tilde{\mathscr{A}}_-^N G, (\kappa - \tilde{\mathscr{L}}_0^N)^{-1} \tilde{\mathscr{A}}_+^N G \rangle = 0\,.$$

hence, the last term of (3.25) equals

$$\langle \tilde{\mathscr{A}}^N g, (\kappa - \tilde{\mathscr{L}}_0)^{-1} \tilde{\mathscr{A}}^N g \rangle_{\Gamma L^2_2} = \langle \tilde{\mathscr{A}}_+^N g, (\kappa - \tilde{\mathscr{L}}_0)^{-1} \tilde{\mathscr{A}}_+^N g \rangle_{\Gamma L^2_3} + \langle \tilde{\mathscr{A}}_-^N g, (\kappa - \tilde{\mathscr{L}}_0)^{-1} \tilde{\mathscr{A}}_-^N g \rangle_{\Gamma L^2_1}\,.$$

Summarising, we showed that the left hand side of (3.24) is lower bounded by

$$\begin{aligned} \frac{2}{\kappa^2} \sup_{g \in \Gamma L^2_2} \Big\{ & 2 \langle \lambda_N \tilde{\mathfrak{n}}_\varphi^N, g \rangle_{\Gamma L^2_2} - \langle g, (\kappa - \tilde{\mathscr{L}}_0) g \rangle_{\Gamma L^2_2} \\ & - \langle g, -\tilde{\mathscr{A}}_-^N (\kappa - \tilde{\mathscr{L}}_0)^{-1} \tilde{\mathscr{A}}_+^N g \rangle_{\Gamma L^2_2} - \langle g, -\tilde{\mathscr{A}}_+^N (\kappa - \tilde{\mathscr{L}}_0)^{-1} \tilde{\mathscr{A}}_-^N g \rangle_{\Gamma L^2_2} \Big\} \end{aligned} \tag{3.26}$$

where we further exploited that the adjoint of $\tilde{\mathscr{A}}_+^N$ is $-\tilde{\mathscr{A}}_-^N$ and vice versa. Similarly to SNS model the operators $-\tilde{\mathscr{A}}_-^N (\kappa - \tilde{\mathscr{L}}_0)^{-1} \tilde{\mathscr{A}}_+^N$ and $-\tilde{\mathscr{A}}_+^N (\kappa - \tilde{\mathscr{L}}_0)^{-1} \tilde{\mathscr{A}}_-^N$ can be bounded by one.

Lemma 3.2.4 *There exists a constant $C > 0$ independent of N such that for any $g \in \Gamma L^2_2$, the following bound hold*

$$\langle g, -\tilde{\mathscr{A}}_-^N (\kappa - \tilde{\mathscr{L}}_0)^{-1} \tilde{\mathscr{A}}_+^N g \rangle_{\Gamma L^2_2} \vee \langle g, -\tilde{\mathscr{A}}_+^N (\kappa - \tilde{\mathscr{L}}_0)^{-1} \tilde{\mathscr{A}}_-^N g \rangle_{\Gamma L^2_2} \leq C \langle (-\tilde{\mathscr{L}}_0) g, g \rangle_{\Gamma L^2_2}\,. \tag{3.27}$$

Thanks to (3.27), there exists a constant $c > 1$ independent of n such that (3.26) is bounded below by

$$\begin{aligned} & \frac{2}{\kappa^2} \sup_{g \in \Gamma L^2_2} \Big\{ 2 \langle \lambda_N \tilde{\mathfrak{n}}_\varphi^N, g \rangle_{\Gamma L^2_2} - \langle g, (\kappa - c\tilde{\mathscr{L}}_0) g \rangle_{\Gamma L^2_2} \Big\} \\ & \qquad = \frac{2}{\kappa^2} \sup_{g \in \Gamma L^2_2} \Big\{ \langle \lambda_N \tilde{\mathfrak{n}}_\varphi^N, g \rangle_{\Gamma L^2_2} + \langle \lambda_N \tilde{\mathfrak{n}}_\varphi^N - (\kappa - c\tilde{\mathscr{L}}_0) g, g \rangle_{\Gamma L^2_2} \Big\}\,. \end{aligned} \tag{3.28}$$

Now, in order to prove (3.23), it suffices to exhibit *one* g for which it holds, and we choose it in such a way that the second scalar product in the supremum is 0, i.e. we pick the solution $\mathfrak{g}$ to

$$\lambda_N \tilde{\mathfrak{n}}_\varphi^N - (\kappa - c\tilde{\mathscr{L}}_0) \mathfrak{g} = 0\,.$$

Notice that, by (3.17), $\mathfrak{g}$ has an explicit Fourier transform which is given by

$$\hat{\mathfrak{g}}(k_{1:2}) = \lambda_N \frac{\hat{\tilde{\mathfrak{n}}}_\varphi(k_{1:2})}{\kappa + \frac{c}{2}|k_{1:2}|^2}\,.$$

Plugging $\mathfrak{g}$ into (3.26) we obtain a lower bound of the type

$$\begin{aligned}\frac{2}{\kappa^2}\langle\lambda_N\tilde{\mathfrak{n}}^N_\varphi,\mathfrak{g}\rangle_{\Gamma L^2_2} &= \frac{2\lambda_N^2}{\kappa^2}\int_{\mathbb{R}^4}\frac{|\hat{\tilde{\mathfrak{n}}}_\varphi(k_{1:2})|^2}{\kappa+\frac{c}{2}|k_{1:2}|^2}\,\mathrm{d}k_{1:2}\\ &= \frac{2}{\kappa^2}\int_{\mathbb{R}^2}\mathrm{d}k|\varphi_k|^2\Big(\lambda_N^2\int_{\mathbb{R}^2}\mathrm{d}k_2\frac{|\tilde{\mathscr{K}}^N_{k-k_2,k_2}|^2}{\kappa+\frac{c}{2}(|k-k_2|^2+|k_2|^2)}\Big)\end{aligned}\tag{3.29}$$

which is fully explicit and we are left to consider the inner integral. To do so, recall the definition of $\tilde{\mathscr{K}}^N$ in (3.3). For arbitrary $k=(k_1,k_2)\in\mathbb{R}^2$ let $k'\stackrel{\text{def}}{=}(-k_1,k_2)$ then $c(k_1,k-k_1)=k_1(k-k_1)'$. We restrict the integral over k_2 to the sector

$$\mathscr{C}^N_k\stackrel{\text{def}}{=}\{k_2\colon\theta_{k_2}\in\theta_{(k-k_2)'}+(0,\pi/6)\quad\&\quad N/3\geq|k_2|\geq(2|k|)\vee 2/N\quad\&\quad |k|\leq\sqrt{N}\}$$

where, for $j\in\mathbb{R}^2$, θ_j is the angle between the vectors j and $(1,0)$. Then we have

$$k_2\cdot(k-k_2)'=|k_2||k-k_2|\cos(|\theta_{(k-k_2)'}-\theta_{k_2}|)\geq|k_2||k|\frac{\sqrt{3}}{2}\,.$$

Hence on $\mathscr{C}_k$, we have

$$\begin{aligned}|\tilde{\mathscr{K}}^N_{k-k_2,k_2}|^2&=\frac{1}{(2\pi)^2}(\hat{\varrho}^N_{k-k_2,k_2})^2|k|^2\frac{c(k_2,k-k_2)^2}{|k_2|^2|k-k_2|^2}=\frac{1}{(2\pi)^2}(\hat{\varrho}^N_{k-k_2,k_2})^2|k|^2\frac{|k_2\cdot(k-k_2)'|^2}{|k_2|^2|k-k_2|^2}\\ &=\frac{1}{(2\pi)^2}(\hat{\varrho}^N_{k-k_2,k_2})^2|k|^2\cos(|\theta_{(k-k_2)'}-\theta_{k_2}|)^2\geq c_\varrho|k|^2\end{aligned}$$

for a constant c depending only on ϱ but neither on k nor N. In the last step, we used that by assumption 1.12 on ϱ, $|\hat{\varrho}^N|$ is bounded below on $[2/N,N/2]$ by a constant independent of N. Hence, the right hand side of (3.29) is lower bounded, modulo a multiplicative constant only depending on ϱ, by

$$\frac{2}{\kappa^2}\int_{2/N\leq|k|\leq\sqrt{N}}\mathrm{d}k|k|^2|\varphi_k|^2\Big(\lambda_N^2\int_{\mathscr{C}^N_k}\frac{\mathrm{d}k_2}{\kappa+|k_2|^2}\Big)\,.\tag{3.30}$$

The quantity in parenthesis is bounded from below by a constant as in SNS hence for N large enough

$$\int_0^\infty e^{-\kappa t}\mathbf{E}\Big[\Big|\tilde{\mathscr{B}}^N_t(\varphi)\Big|^2\Big]\,\mathrm{d}t\gtrsim\frac{1}{\kappa^2}\int_{2/N\leq|k|\leq\sqrt{N}}\mathrm{d}k|k|^2|\varphi_k|^2\,,\tag{3.31}$$

and it remains to pass to the limit as $N\to\infty$. Now, thanks to (2.42) and tightness of $\tilde{\mathscr{B}}^N$, we can apply dominated convergence to the left hand side, while the integral at right hand side clearly converges to $\|\varphi\|^2_{\dot{H}^1(\mathbb{R}^2)}$, so that the proof is completed. □

Proof of Lemma 3.2.4. We will exploit the Fourier representation of the operators $\tilde{\mathscr{A}}^N_+$ and $\tilde{\mathscr{A}}^N_-$ in Proposition 3.2.1, which though still need to be symmetrised. Let $\tilde{\mathfrak{a}}^N_+$ be the operator defined by the right hand side of (3.18) and S_3 the set of permutations of $\{1,2,3\}$. Then,

$$\begin{aligned}\langle g, \tilde{\mathscr{A}}^N_-(\kappa - \tilde{\mathscr{L}}_0)^{-1}\tilde{\mathscr{A}}^N_+ g\rangle_{\Gamma L^2_2} &= \langle \tilde{\mathscr{A}}^N_+ g, (\kappa - \tilde{\mathscr{L}}_0)^{-1}\tilde{\mathscr{A}}^N_+ g\rangle_{\Gamma L^2_3} \\ &= \sum_{s,\bar{s}\in S_3}\int \frac{1}{\kappa + \frac{1}{2}|k_{1:3}|^2}\mathscr{F}(\tilde{\mathfrak{a}}^N_+ g)(k_{s(1):s(3)})\mathscr{F}(\tilde{\mathfrak{a}}^N_+ g)(k_{\bar{s}(1):\bar{s}(3)})\,\mathrm{d}k_{1:3} \\ &\lesssim \int \frac{1}{\kappa + \frac{1}{2}|k_{1:3}|^2}\mathscr{F}(\tilde{\mathfrak{a}}^N_+ g)(k_{1:3})^2\,\mathrm{d}k_{1:3}\end{aligned} \tag{3.32}$$

where in the last step we simply applied Cauchy-Schwarz inequality. By bounding $|\tilde{\mathscr{K}}^N_{k_1,k_2}| \leq \hat{\varrho}^N_{k_2}|k_1 + k_2|$ and the bound above, one obtains

$$\langle g, \tilde{\mathscr{A}}^N_-(\kappa - \tilde{\mathscr{L}}_0)^{-1}\tilde{\mathscr{A}}^N_+ g\rangle_{\Gamma L^2_2} \lesssim \langle(-\tilde{\mathscr{L}}_0)g, g\rangle_{\Gamma L^2_2}\,. \tag{3.33}$$

We now turn to the other term, which is For the term containing $\tilde{\mathscr{A}}^N_- g$ we need to carefully consider the c function. We have

$$\begin{aligned}&\langle g, \tilde{\mathscr{A}}^N_+(\kappa - \tilde{\mathscr{L}}_0)^{-1}\tilde{\mathscr{A}}^N_- g\rangle_{\Gamma L^2_2} = \langle \tilde{\mathscr{A}}^N_- g, (\kappa - \tilde{\mathscr{L}}_0)^{-1}\tilde{\mathscr{A}}^N_- g\rangle_{L^2} \\ &\lesssim \lambda_N^2\int_{|k|<N}\frac{1}{\kappa + |k|^2}\Bigg(\int_{\substack{1/N<|k_1|<N\\ 1/N<|k-k_1|<N}} |k_1|\hat{g}(k_1, k-k_1)\frac{c(k,k-k_1)}{|k||k-k_1|}\,\mathrm{d}k_1\Bigg)^2\mathrm{d}k\,.\end{aligned}$$

By changing integration variable to $k_1' = k - k_1$ and using bi-linearity of c we have that

$$\begin{aligned}&\int |k_1|\hat{g}(k_1, k-k_1)\frac{c(k,k-k_1)}{|k||k-k_1|}\,\mathrm{d}k_1 \\ &= \frac{c(k,k)}{|k|}\int \hat{g}(k_1,k-k_1)\frac{|k_1|}{|k-k_1|}\,\mathrm{d}k_1 \\ &+ \frac{1}{|k|}\int \hat{g}(k_1,k-k_1)c(k,k_1)\bigg(\frac{|k-k_1|}{|k_1|} - \frac{|k_1|}{|k-k_1|}\bigg)\,\mathrm{d}k_1\,,\end{aligned} \tag{3.34}$$

the integrals above are over $\{k_1 : 1/N < |k_1|, |k-k_1| < N\}$ .Using $c(k,k) \leq |k|^2$ and

Cauchy-Schwarz, for the first term we have

$$\begin{aligned}
&\frac{c(k,k)}{|k|}\int \hat{g}(k_1,k-k_1)\frac{|k_1|}{|k-k_1|}\,\mathrm{d}k_1\\
&\le |k|\int \hat{g}(k_1,k-k_1)\frac{|k_1|}{|k-k_1|}\,\mathrm{d}k_1\\
&\le |k|\bigg(\int \hat{g}(k_1,k-k_1)^2|k-k_1|^2\,\mathrm{d}k_1\int\frac{1}{|k_1|^2}\,\mathrm{d}k_1\bigg)^{\frac{1}{2}}\,.
\end{aligned}\tag{3.35}$$

For the second term we use $c(k,k_1)\le |k||k_1|$ to get

$$\begin{aligned}
&\frac{1}{|k|}\int \hat{g}(k_1,k-k_1)c(k,k_1)\bigg(\frac{|k-k_1|}{|k_1|}-\frac{|k_1|}{|k-k_1|}\bigg)\,\mathrm{d}k_1\\
&\le \int \hat{g}(k_1,k-k_1)|k_1|\bigg(\frac{|k-k_1|}{|k_1|}-\frac{|k_1|}{|k-k_1|}\bigg)\,\mathrm{d}k_1\\
&= \int \hat{g}(k_1,k-k_1)\bigg(\frac{|k-k_1|^2-|k_1|^2}{|k-k_1|}\bigg)\,\mathrm{d}k_1\,.
\end{aligned}$$

Using $|k-k_1|^2-|k_1|^2\le |k|^2$ and Cauchy Schwarz we get

$$\begin{aligned}
&\int \hat{g}(k_1,k-k_1)\bigg(\frac{|k-k_1|^2-|k_1|^2}{|k-k_1|}\bigg)\,\mathrm{d}k_1\\
&\lesssim |k|^2\int \hat{g}(k_1,k-k_1)\frac{1}{|k-k_1|}\,\mathrm{d}k_1\\
&\le |k|^2\bigg(\int \hat{g}(k_1,k-k_1)^2\,\mathrm{d}k_1\int\frac{1}{|k_1|^2}\,\mathrm{d}k_1\bigg)^{\frac{1}{2}}\\
&\lesssim |k|^2(\log N)^{\frac{1}{2}}\bigg(\int \hat{g}(k_1,k-k_1)^2\,\mathrm{d}k_1\bigg)^{\frac{1}{2}}\,.
\end{aligned}$$

And so

$$\begin{aligned}
&\langle \tilde{\mathscr{A}}_-^N g,(\kappa-\tilde{\mathscr{L}}_0)^{-1}\tilde{\mathscr{A}}_-^N g\rangle_{L^2}\\
&= \langle \mathscr{F}(\tilde{\mathscr{A}}_-^N g),\mathscr{F}((\kappa-\tilde{\mathscr{L}}_0)^{-1}\tilde{\mathscr{A}}_-^N g)\rangle\\
&\lesssim \lambda_N^2\log(N)\int\bigg(\int \hat{g}(k_1,k-k_1)^2|k-k_1|^2\,\mathrm{d}k_1+|k|^2\int \hat{g}(k_1,k-k_1)^2\,\mathrm{d}k_1\bigg)\,\mathrm{d}k\\
&\lesssim \langle(-\tilde{\mathscr{L}}_0)g,g\rangle_{\Gamma L^2_2}\,.
\end{aligned}$$

□

3.3 Bulk diffusivity on the torus

It is of great interest to understand large scale properties of the solution of (1.13) with $Q = \mathrm{diag}(1,-1)$. A sensible question is to ask how does it compare with the equation where $\lambda = 0$ i.e the additive stochastic heat equation. In this section we will be comparing AKPZ with a Fourier cut-off of size 1 imposed on the non-linearity (i.e $N = 1$) at large times, that is we investigate the following equation

$$\partial_t h = \frac{1}{2}\Delta h + \lambda \Pi_1 \Big((\Pi_1 \partial_1 h)^2 - (\Pi_1 \partial_2 h)^2 \Big) + \xi\,, \tag{3.36}$$

which as before can be transformed into a Burgers equation of the following form

$$\begin{aligned} \partial_t u &= \frac{1}{2}\Delta u + \lambda \tilde{\mathcal{N}}^{1,M}[u] + (-\Delta)^{\frac{1}{2}}\xi \\ \tilde{\mathcal{N}}_t^{1,M}[u](x) &= \Pi_1 \Big((\Pi_1 \partial_1 (-\Delta)^{-\frac{1}{2}} u_t)^2 - (\Pi_1 \partial_2 (-\Delta)^{-\frac{1}{2}} u_t)^2 \Big)(x)\,, \end{aligned} \tag{3.37}$$

One way to compare stochastic heat equation corresponding to (3.36) with $\lambda = 0$ with the equation (3.36) itself is to look at the observable called *bulk diffusivity* which measures how the correlations of a process spread in space as a function of time. The classical definition of bulk diffusivity is $\int_{\mathbb{T}_M^2} |x|^2 \mathbf{E}[u_t(x)u_0(0)]$ Under mild assumptions it was shown in [CET20, Appendix A] gave a heuristic explanation as to why the classical definition above and the Green-Kubo formula are equivalent, we will be working with the Green-Kubo formulation stated below.

Definition 3.3.1 The bulk diffusivity of the process for the solution u to (3.37) is defined as

$$D_M(t) = 1 + 2\frac{\lambda^2}{t} \int_0^t \int_0^s \int_{\mathbb{T}_M^2} \mathbf{E}\Big[\tilde{\mathcal{N}}_r^{1,M}[u](x) \tilde{\mathcal{N}}_0^{1,M}[u](0) \Big] \,\mathrm{d}x\,\mathrm{d}r\,\mathrm{d}s \tag{3.38}$$

For a stochastic heat equation corresponding to removing the non-linear term it is clear that $D_M(t) \equiv 1$. Bulk diffusivity of the AKPZ equation (3.36) were extensively investigated in [CET20] where a lower and upper bounds of the Laplace transform of bulk diffusivity of solution to (3.36) were found it is stated here for reference.

Theorem 3.3.2 *Let* $\lambda > 0$ *and, for* $M \in \mathbb{N}$, D_M *be defined in* (3.38) *and* $\mathcal{D}_M$ *be the Laplace* *transform of D_M given by*

$$\mathscr{D}_M(\mu) = \mu \int_0^\infty e^{-\mu t} t D_M(t)\,\mathrm{d}t\,.$$

For lightness of notation let

$$L(x,0) \overset{def}{=} 1 + \lambda^2 \log\left(1 + \frac{1}{\mu}\right).$$

Then, there exists $\delta \in (0, \frac{1}{2}]$ and a constant $c_{bulk} \geq 1$ such that for every $M \in \mathbb{N}$ and any $\mu > 0$ sufficiently small

$$\limsup_{N\to\infty} \mathscr{D}_M(\mu) \leq \frac{(1 + c_{bulk})}{\mu} L(\mu,0)^{1-\delta}$$

and

$$\liminf_{N\to\infty} \mathscr{D}_M(\mu) \geq \frac{(1 + c_{bulk}^{-1})}{\mu} L(\mu,0)^{\delta}\,.$$

Moreover, the exponent δ is bounded away from zero for $\lambda \to 0$.

An interesting question to ask for large times t does there exist a δ such that

$$\lim_{N\to\infty} \mathscr{D}_M(\mu) = \frac{C}{\mu} L(\mu,0)^{\delta}\,,$$

for some constant C, or one might directly ask a similar question about bulk diffusivity itself namely does there exist $\delta > 0$ such that

$$\lim_{M\to\infty} D_M(t) = C \log(t)^{\delta}$$

It has been conjectured in [CET20, Appendix B] that in fact $\delta = \frac{1}{2}$. This result was the main motivation of this numerical work. Since then further development has occurred as a result an improved version of [CET20] was released where the bounds are of the form

$$\limsup_{N\to\infty} \mathscr{D}_M(\mu) \leq \frac{(c_{\text{bulk}})}{\mu} L(\mu,0)^{\frac{1}{2}} \log(L(\mu,0))^{5+\delta}$$

and

$$\liminf_{N\to\infty} \mathscr{D}_M(\mu) \geq \frac{(1+c_{\text{bulk}}^{-1})}{\mu} L(\mu,0)^{\frac{1}{2}} \log(L(\mu,0))^{-5-\delta}\,.$$

This result somewhat weakness the importance of numerical results that are to follow, nevertheless it is a nice exposition of the numerical approach and the technology used to achieve it. The rest of this subsection is structured as follows, firstly the heuristics as to why it is believed that $\delta = \frac{1}{2}$ is described, closely following [CET20, Appendix B]. Afterwards we verify this conjecture numerically using the pure spectral method.

Heuristic reason for $\delta = \frac{1}{2}$

To describe heuristic reasoning behind the $\delta = \frac{1}{2}$ conjecture the classical definition will be used= To draw a distinction between the definition of bulk diffusivity defined previously and the classical definition the (a) superscript will be used to denote the classical definition. That is

$$\begin{aligned} D^{(a)}(t) &\stackrel{\text{def}}{=} \frac{1}{2t}\int_{\mathbb{T}_M^2} |x|^2 S(t,x)\,\mathrm{d}x\,, \\ S(t,x) &\stackrel{\text{def}}{=} \mathbf{E}[u_t(x)u_0(0)]\,. \end{aligned} \tag{3.39}$$

By translation invariance we have

$$\begin{aligned} \hat{S}(t,k) &= \mathbf{E}\Big[u_0(0)\int e_{-k}(z)u_t(z)\,\mathrm{d}z\Big] \\ &= \frac{1}{(2\pi)^2}\mathbf{E}\Big[\int u_0(0)e_k(\bar z)\int e_{-k}(z+\bar z)u_t(z)\,\mathrm{d}z\,\mathrm{d}\bar z\Big] \\ &= \frac{1}{(2\pi)^2}\mathbf{E}\Big[\int u_0(\bar z)e_k(\bar z)\int e_{-k}(z+\bar z)u_t(z+\bar z)\,\mathrm{d}z\,\mathrm{d}\bar z\Big] \\ &= \frac{1}{(2\pi)^2}\mathbf{E}[\hat u_t(k)\hat u_0(-k)]\,. \end{aligned}$$

Based on (3.36) we have

$$\partial_t \hat u_t(k) + \frac{1}{2}|k|^2 \hat u_t(k) = \lambda \tilde{\mathscr{N}}_k^{1,M}[u_t] + |k|\xi_t(k)\,,$$

in particular

$$\partial_t \hat S(t,k) + \frac{1}{2}|k|^2 \hat S(t,k) = \frac{\lambda}{(2\pi)^2}\mathbf{E}\Big[\hat u_0(-k)\tilde{\mathscr{N}}_k^{1,M}[u_t]\Big] + \frac{|k|}{(2\pi)^2}\mathbf{E}[\hat u_0(-k)\xi_t(k)]$$

$$
\begin{aligned}
&= \frac{\lambda}{(2\pi)^2}\mathbf{E}\Big[\hat{u}_0(-k)\tilde{\mathcal{N}}^{1,M}_k[u_t]\Big] \\
&= \frac{\lambda}{(2\pi)^2}\mathbb{E}\Big[\hat{\eta}(-k)e^{\tilde{\mathscr{L}}t}\tilde{\mathcal{N}}^{1,M}_k[\eta]\Big]\,.
\end{aligned}
$$

In second equality the term $\frac{|k|}{(2\pi)^2}\mathbf{E}[\hat{u}_0(-k)\xi_t(k)]$ vanishes since its a product of two independent Gaussian's. As mentioned before the generator $\tilde{\mathscr{L}}$ of u can be broken down into two elements $\tilde{\mathscr{A}}$ and $\tilde{\mathscr{L}}_0$ as given in Proposition 3.2.1 (with $N = 1$). The semi-group associated to $\tilde{\mathscr{L}}$ satisfies

$$
e^{\tilde{\mathscr{L}}t} = e^{\tilde{\mathscr{L}}_0 t} + \int_0^t e^{\tilde{\mathscr{L}}_0(t-s)}e^{\tilde{\mathscr{L}}}\,\mathrm{d}s\,.
$$

$e^{\tilde{\mathscr{L}}_0 t}$ corresponds to the semi-group of Ornstein–Uhlenbeck process and $\tilde{\mathcal{N}}^{1,M}_k[\eta]$ is quadratic thus

$$
\mathbb{E}\Big[\hat{\eta}(-k)e^{\tilde{\mathscr{L}}_0 t}\tilde{\mathcal{N}}^{1,M}_k[\eta]\Big] = 0\,.
$$

For the remaining term we use the fact that adjoint of $\tilde{\mathscr{A}}$ is $-\tilde{\mathscr{A}}$ giving us

$$
-\frac{\lambda}{(2\pi)^2}\int_0^t e^{-\frac{1}{2}|k|^2(t-s)}\mathbb{E}\Big[(\tilde{\mathscr{A}}\hat{\eta})(-k)e^{\tilde{\mathscr{L}}^N s}\tilde{\mathcal{N}}^{1,M}_k[\eta]\Big]\,\mathrm{d}s\,.
$$

We now use the fact that $\tilde{\mathscr{A}}\hat{\eta}(-k) = \lambda\tilde{\mathcal{N}}^{1,M}_{-k}[\eta]$ and the Fourier representation from (3.2) to get

$$
\begin{aligned}
&-\frac{\lambda^2}{(2\pi)^2}\int_0^t e^{-\frac{1}{2}|k|^2(t-s)}\mathbb{E}\Big[\tilde{\mathcal{N}}^{1,M}_{-k}[\eta]e^{\tilde{\mathscr{L}}^N s}\tilde{\mathcal{N}}^{1,M}_k[\eta]\Big]\,\mathrm{d}s \\
&= \frac{-\lambda^2}{(2\pi)^4}|k|^2\int_0^t e^{-\frac{1}{2}|k|^2(t-s)}\int \mathrm{d}\ell\int \mathrm{d}\ell'\mathscr{K}^1_{\ell,k-\ell}\mathscr{K}^1_{\ell',k-\ell'} \\
&\quad\times \mathbf{E}[\hat{u}_s(\ell)\hat{u}_s(k-\ell)\hat{u}_0(\ell')\hat{u}_0(-k-\ell')]\,\mathrm{d}s
\end{aligned}
$$

One now uses the mode-coupling approximation which was used in [Spo14] and recently [KNSS18]. Starting with Gaussian approximation of the average of the product of four u's and then applying Wick's rule. Translation invariance gives

$$
\mathbb{E}[\hat{u}_s(\ell)\hat{u}_0(m)] = 0, \quad \text{for } \ell \neq -m
$$

Via the Wick contraction we also have

$$
|k|^2\mathbb{E}[\hat{u}_s(\ell)\hat{u}_s(k-\ell)]\mathbb{E}[\hat{u}_0(\ell')\hat{u}_0(-k-\ell')] = 0, \quad \forall k\,,
$$

All together we get

$$\begin{aligned}&\Big(\partial_t+\frac{1}{2}|k|^2\Big)\hat{S}(t,k)\\&\approx -2|k|^2\frac{\lambda^2}{(2\pi)^4}\int_0^t e^{-\frac{1}{2}|k|^2(t-s)}\int \mathrm{d}\ell(\mathscr{K}^1_{\ell,k-\ell})^2\hat{S}(s,\ell)\hat{S}(s,k-\ell)\,\mathrm{d}s\,.\end{aligned}\tag{3.40}$$

We now make an ansatz

$$\hat{S}(t,k)=\hat{S}(0,0)e^{-\frac{1}{2}|k|^2-c|k|^2t(\log t)^\delta}\,,$$

Aim is now to use the ansatz in (3.40) to find δ. The left hand side is given by

$$\Big(\partial_t+\frac{1}{2}|k|^2\Big)\hat{S}(t,k)\approx -c|k|^2(\log t)^\delta\hat{S}(0,0)\,.\tag{3.41}$$

For the right hand side we approximate $e^{-\frac{1}{2}|k|^2(t-s)}\approx 1$ and $k-\ell\approx-\ell$ so that $k\to 0$ and $t\to\infty$, this gives

$$-|k|^2\lambda^2\int_0^t \mathrm{d}s\int \mathrm{d}\ell(\mathscr{K}^1_{\ell,-\ell})^2e^{-2c|\ell|^2s(\log s)^\delta}\approx -|k|^2\lambda^2(\log t)^{1-\delta}\,,\tag{3.42}$$

comparing (3.42) with (3.41) and using $(\mathscr{K}^1_{\ell,-\ell})^2\approx 1$ one obtains $\delta=\frac{1}{2}$.

Numerical simulations of the bulk diffusivity

When aiming to approximate such quantities one resides to approximating solution to an SPDE. A most obvious way to approach this is to approximate time derivative and the Laplacian via *finite differences* and imposing Fourier cut-off via fast-Fourier transform. Although easiest in execution this approach generates errors that propagate with time as such we have no hope to obtain accurate results for large t, indeed stability of the resulting equation prevents it from behaving well for very small t as well.

Another, recently more popular method is to approach the SPDE from spectral perspective. This involves viewing (3.37) as a system of ODEs in Fourier space given by

$$\partial t\hat{u}_t(k)=\frac{1}{2}|k|^2\hat{u}_t(k)+\tilde{\mathscr{N}}^{1,M}_k[u_t]+|k|\partial_tB_t(k),\quad \hat{u}_0(k)=\hat{\eta}(k)\,\forall k\in\mathbb{Z}^2_M\tag{3.43}$$

The aim is now to approximate $\tilde{\mathscr{N}}^{1,M}_k[u_t]$ on $[0,T]$ with T large and then use the Green-Kubo formula (3.38) to obtain bulk diffusivity. Since we are going to approximate matters numerically we must discretise time, we do so uniformly between 0 and T where T is some large value. Denoting the i-th point by t_i and the distance between two points by Δ_t. In order to solve such system one considers a finite difference in time derivative

$$\partial_t \hat{u} \to \frac{1}{\Delta_t}(\hat{u}_{t_{i+1}}(k) - \hat{u}_{t_i}(k))\,.$$

The noise term is also approximated

$$\partial_t B_t(k) \overset{d}{\approx} \frac{1}{\Delta_t}\Big(B_{t_{i+1}} - B_{t_i}\Big) \overset{d}{=} \frac{1}{\Delta_t} N(0, \Delta_t) \overset{d}{=} \Delta_t^{\frac{1}{2}} N(0,1)\,.$$

From now on we denote the $N(0,1)$ variable approximation of the noise at (t_i, k) by $\Xi_{t_i}(k)$. The two approximations result in the following system of ODEs

$$\begin{aligned} \hat{u}_{t_{i+1}}(k) &= (1 - \frac{1}{2}\Delta_t |k|^2)\hat{u}_{t_i}(k) + \tilde{\mathcal{N}}_k^{1,M}[u_{t_i}] + \Delta_t^{-0.5}|k|\Xi_{t_i}(k), \\ \hat{u}_0(k) &= \hat{\eta}(k), \quad \forall k \in \mathbb{Z}_M^2\,. \end{aligned} \tag{3.44}$$

We are not interested in solving this for every $k \in \mathbb{Z}_M^2$ as to calculate bulk diffusivity we only need to know $\tilde{\mathcal{N}}_k^{1,M}[u_{t_i}]$ and due to the Fourier cut-off at 1 we only need to calculate $\tilde{\mathcal{N}}_k^{1,M}[u_{t_i}]$ for $k \in \mathbb{Z}_M^2, |k| \leq 1$. Hence it suffices to calculate inductively for every t_i over all $k \in \mathbb{Z}_M^2$ such that $|k| \leq 1$. This still poses difficulty from computational stand point because the nonlinear term is effectively a convolution (as given in (3.2)) and hence its computationally costly to calculate it for every t_i and for every $k \in \mathbb{Z}_M^2, |k| \leq 1$. One approach to this is to use a pseudo-spectral method that utilises the fact that Fourier transform of a convolution of two terms is equal to Fourier transform of one term times the Fourier transform of another term. This may not seem so useful however calculating convolution comes at a computational cost of $O(N^2)$ where as Fourier transform comes at a computational cost of $O(N\log(N))$, here N describes the size of the array we are working with. In our case we can utilise the fact that the Fourier transform of the non-linearity $\tilde{\mathcal{N}}_k^{1,M}[u_{t_i}]$ is a convolution of $\hat{u_{t_i}}$. We can then multiply u_{t_i}'s which we already have, one can then Fourier transform the product to obtain to obtain $\tilde{\mathcal{N}}_k^{1,M}[u_{t_i}]$ with some error which one can then partially eliminate by applying Orszag's Two-Thirds rule which results in setting $\tilde{\mathcal{N}}_k^{1,M}[u_{t_i}]$ to 0 for $|k| > 2/3$ or calculating $\hat{u}$ up to $|k| = \frac{3}{2}$ and then applying cut-off at 1. This method is superior from computational efficiency standpoint. Sadly upon review it turns out that the logarithmic nature of the bulk diffusivity is too fragile for such approximations. Although one obtains a clear logarithmic curve for bulk diffusivity, the error causes the power to be uncertain. Thankfully the process of solving (3.44) via calculating the convolution directly is highly pararellizable that is we can perform a lot of calculations simultaneously. In our case for each time increment every space point can be calculated at the same time because each space point only requires the value of the solution from previous time increment and not the current. Recent development in technological sector allows one to use the graphic computing unit (GPU) rather than the central processing unit (CPU) which was the common way to compute such matters in the past. This provides vastly improved performance for such highly pararellizable tasks, in this case the computations are approximately 1000 times faster than via classical CPU computations, it is important to stress that such programming is significantly more difficult as one has to manipulate the connection between system memory and the GPU memory and decide which parts of the code to run on CPU and which on GPU. From now on we will use alternative expression for the bulk diffusivity given by

$$tD_M(t) = t + \mathbf{E}\left[\left(\int \lambda \tilde{\mathcal{N}}_0^{1,M}[u_s]\,\mathrm{d}s\right)^2\right]$$

Approximating $\mathbf{E}$ via classical Monte Carlo and discretising time and the integral we have

$$\begin{aligned} D(t) &= 1 + \frac{1}{t}\mathbf{E}\left[\left(\int_0^t \sum_{\ell=(\ell_1,\ell_2)\,:\,|\ell|\leq 1} \frac{(\ell_2)^2-(\ell_1)^2}{|\ell|^2}\hat{u}_s(\ell)\hat{u}_s(-\ell)\,\mathrm{d}s\right)^2\right] \\ &\approx 1 + \frac{1}{t}\frac{1}{M}\sum_{i=1}^{M}\left[\left(\int_0^t \sum_{\ell=(\ell_1,\ell_2)\,:\,|\ell|\leq 1} \frac{(\ell_2)^2-(\ell_1)^2}{|\ell|^2}\hat{u}_s(l)\hat{u}_s(-\ell)\,\mathrm{d}s\right)^2\right] \\ &\approx 1 + \frac{1}{t}\frac{1}{M}\sum_{i=1}^{M}\left[\left(\sum_{t_i}\Delta_t \sum_{\ell=(\ell_1,\ell_2)\,:\,|\ell|\leq 1} \frac{(\ell_2)^2-(\ell_1)^2}{|\ell|^2}\hat{u}(t_i,\ell)\hat{u}(t_i,-\ell)\,\mathrm{d}s\right)^2\right]. \end{aligned}$$

The greater N is the closer we are to the real expectation, due to small changes after 1000 iterations and diminishing error a total of 10000 instances were calculated up to time $T = 20000$. Before moving the code specifics first, the code is written in python 3.8 although it is compatible with 3.9 and the newest 3.10 and likely backward compatible with other python 3 versions. Python often offers a way to write code quickly and clearly, sometimes at cost of performance when compared to other languages such as C++. In this case this performance issue was addressed by using python packages that actually execute C++ code in the background making performance loss insignificant while preserving the simplicity, one might further try to simplify the matters by using Matlab, the performance penalty is to grand sadly. Google Colab made it possible to have access to professional grade GPUs for free, downside being that one can only run on 3 machines 12 hours at a time after which a 6 hour lock is administered, as such the code was written with the intent to calculate instances over 12 hour windows and save it to a file, after having enough instances we gather them all together and plot the graph, this has no impact on the final outcome, only on the way the code was written. As such the code is broken in two parts, first part computes instances and second part uses the precomputed instances to plot. Starting with the code for computing instances

```python

# pip3 install --upgrade numba
# pip3 install --upgrade tbb
# pip3 install --upgrade cupy-cuda110
# pip3 install --upgrade gupload

#@title Load packages.
import math
import csv
import multiprocessing as mp
import matplotlib.pyplot as plt
import numpy as np
#import cupy as cp
import scipy
from numba import njit, prange, cuda, jit
from timeit import default_timer as timer
import random
from tqdm import tqdm
import os
from numba.cuda.random import create_xoroshiro128p_states,
    xoroshiro128p_uniform_float32, xoroshiro128p_normal_float64,
    xoroshiro128p_normal_float32
from scipy.optimize import curve_fit



#Asking user for parameters.
batchsize = 10
number_of_instances = int(input("How many instances would you like to run ?
     :" ))
R = float(input("Pick radius of the tours R :" ))
Time_Sample_Rate  = float(input("Pick time sample size :"))
T = int(input("Time will be over [0,T], T (T should be at least R^2 ) : "))
```

```
la = float(input("Lambda : "))

#Calculating useful sets based on the parameters.
TTerms = int(T*Time_Sample_Rate)
RLen = int(2*R)
FSpace=int(RLen/2) #The above happens due to NyquistâĂŞShannon sampling
    theorem. (we only get half the points of original space)
Ts = np.linspace (0.0 , T, TTerms)
TsError = np.linspace (0.0 , T, T)


filenamestart = int(input("What filenumber would you like to start with? ")
    )

seednumber = 1007 # Pick a different number on each rerun

@cuda.jit()
def GPUSolver(gpuU,gpuN,rng_states ,gpuInt, t, a ):
  #Assinging thread position to variables k and inst where k is the fourier
     term we are currently working on and inst is the instance
  k,inst = cuda.grid(2)
  k1, k2 = int(k % RLen- int(R)), int((k - k % RLen)/RLen)

  #Initializing if t = 0
  if t == 0:
    gpuU[0+2*inst][k1][k2] = 0
    gpuU[0+2*inst][-k1][-k2] = 0

  #Working out noise terms
  ReNoise = xoroshiro128p_normal_float64(rng_states, k+inst*int(R*R*2)   )
  ImNoise = 1j*xoroshiro128p_normal_float64(rng_states, k+inst*int(R*R*2)
    )

  #Working out non-linearity
  gpuN[inst][k1][k2] = 0
  gpuN[inst][-k1][-k2] = 0
  if k1**2 + k2**2 < R**2:
    for l1 in  range(-R,R):
      for l2 in range(-R,R):
        if (l1**2 + l2**2 > 0) and (l1**2 + l2**2 < R**2) :
          if ((k1-l1)**2 + (k2-l2)**2 < R**2 ) and ((k1-l1)**2 + (k2-l2)**2
```

```
 > 0):
          gpuN[inst][k1][k2] +=  pow(R,-2) * pow(l1**2+l2**2,-0.5) * pow((k1-l1)**2 + (k2-l2)**2,-0.5) *(a[0][0]*(-l1*k1 + l1*l1) + a[1][1]*(l2*l2-l2*k2) + (a[0][1]+a[1][0])*(l1*l2-l1*k2) )*gpuU[t%2+ 2*inst][l1][l2] * gpuU[t%2+ 2*inst][k1-l1][k2-l2] #1/R^2 for scaling, 1/R for 1/|k| then scaling of 1/|k-l| and 1/|l| cancels with the l2k2... stuff
        if ((k1+l1)**2 + (k2+l2)**2 < R**2 ) and ((k1+l1)**2 + (k2+l2)**2 > 0) and (k1 != 0 or k2!= 0):
          gpuN[inst][-k1][-k2] += pow(R,-2) * pow(l1**2+l2**2,-0.5) * pow((k1+l1)**2 + (k2+l2)**2,-0.5) *( a[0][0]*(l1*k1 + l1*l1) + a[1][1]*(l2*l2+l2*k2) + (a[0][1]+a[1][0])*(l1*l2+l1*k2) )*gpuU[t%2+ 2*inst][l1][l2] * gpuU[t%2+ 2*inst][-k1-l1][-k2-l2] #1/R^2 for scaling, 1/R for 1/|k| then scaling of 1/|k-l| and 1/|l| cancels with the l2k2... stuff

  #Adding 0-th fourier term to the Int
  if k1 == 0 and k2 == 0:
    gpuInt[inst][t] = gpuInt[inst][t-1] + 0.5*(gpuN[inst][0][0])*pow(Time_Sample_Rate,-1) #its double counted so dividing by two.

  #Bringing it all together to calculate the next time-increment at k
  gpuU[(t+1)%2 + 2*inst][k1][k2] = (1-0.5*pow(Time_Sample_Rate,-1)*(k1**2 + k2**2)*pow(R,-2))*gpuU[t%2+ 2*inst][k1][k2] + pow(Time_Sample_Rate,-1)*la*gpuN[inst][k1][k2]*pow(k1**2 + k2**2, 0.5)*(1/R) + pow(Time_Sample_Rate,-0.5)*(1/R)*pow(k1**2+k2**2,0.5)*(ReNoise + ImNoise )
  gpuU[(t+1)%2 + 2*inst][-k1][-k2] = (1-0.5*pow(Time_Sample_Rate,-1)*(k1**2 +k2**2)*pow(R,-2))*gpuU[t%2+ 2*inst][-k1][-k2] + pow(Time_Sample_Rate,-1)*la*gpuN[inst][-k1][-k2]*pow(k1**2 + k2**2, 0.5)*(1/R) + pow(Time_Sample_Rate,-0.5)*(1/R)*pow(k1**2+k2**2,0.5)*(ReNoise - ImNoise )


def Dcounter(Dsum, Ds, Int,size,Time_Sample):
  for inst in range(size):
    for t in prange(1,TTerms):
      nonlinsq = (Time_Sample/float(t))*pow(Int.real[inst][t],2)
      if t % int(Time_Sample) == 0:
        Ds[inst][int(t/Time_Sample)] =1+ nonlinsq
      Dsum[t] += nonlinsq
  return Dsum, Ds


rng_states = create_xoroshiro128p_states(int(2*R*R)*int(batchsize), seed=
```

```
    seednumber)
b = np.zeros((2,2), dtype = np.float64)

#Choosing non-linearity
b[0][0],b[0][1] = 1,0
b[1][0],b[1][1] = 0,-1



for batch in range(int(number_of_instances/batchsize)):
  #Fourier of U created and sent to the GPU.
  gpuU = cuda.to_device(np.zeros((2*batchsize,RLen,RLen), dtype=np.complex128))
  gpuInt = cuda.to_device(np.zeros((batchsize,TTerms), dtype = np.complex128))
  gpuN = cuda.to_device(np.zeros((batchsize,RLen,RLen), dtype=np.complex128))

  Dsum = np.ones(TTerms, np.float64)
  Dsum = Dsum*batchsize
  Ds = np.ones((batchsize,T), np.float64)


  start = timer()

  if hasattr(tqdm, '_instances'): tqdm._instances.clear() # clear if it exists
  for t in  tqdm(range(TTerms-1)):
    GPUSolver[(int(R*R/5),int(batchsize/10) ), (10,10) ](gpuU,gpuN,rng_states, gpuInt, t,b) #we send the works to the GPU for it to crunch this time step.
  print("\n Took", timer() - start, " seconds.")

  Int = gpuInt.copy_to_host()
  Dsum, Ds = Dcounter(Dsum,Ds,Int,batchsize,Time_Sample_Rate)
  with open(str(filenamestart + batch) + ".npy", 'wb') as f:
    np.save(f, batchsize)
    np.save(f, Dsum)
    np.save(f, Ds)
    print("Saved: " + str(filenamestart + batch) + ".npy")
```

Listing 3.1: Computing instances

The code above first asks user for certain inputs, how many instances to compute overall how many at once, time increment and so on. Two functions are then defined, first one computes the instances at a specific time increment using GPU, second sums up final results, this functions are not executed just yet. We then create variables pass them to the GPU memory and initiate computation of instances, one time increment at a time, after completing batch size number of instances we then compute sum of bulk diffusivities for each instance and their average and save it into a file. Moving onto plotting code we have

```
import matplotlib.pyplot as plt
from scipy.optimize import curve_fit
import math
import numpy as np
import scipy
from timeit import default_timer as timer
import random
from tqdm import tqdm
import os

#Asking user for parameters.
batchsize = 10
Time_Sample_Rate  = float(input("Pick time sample size :"))
T = int(input("Time will be over [0,T], T (T should be at least R^2 ) : "))
TTerms = int(T*Time_Sample_Rate)
TsError = np.linspace (0.0 , T, T)

location= str(input("File location?"))
filelist = range(0,99) # what file numbres do we want to go over?
def Dfilereader(n):
  with open(location + str(n) + ".npy", 'rb') as f:
    batchsize = np.load(f)
    Dsum = np.load(f)
    Ds = np.load(f)
  return batchsize, Dsum, Ds

def Dfit(t,delta,c):
    return c*pow(np.log(t),delta)


Dsum = np.zeros(TTerms, dtype = np.float64)
number_of_instances = 0.0
Ds = []
```

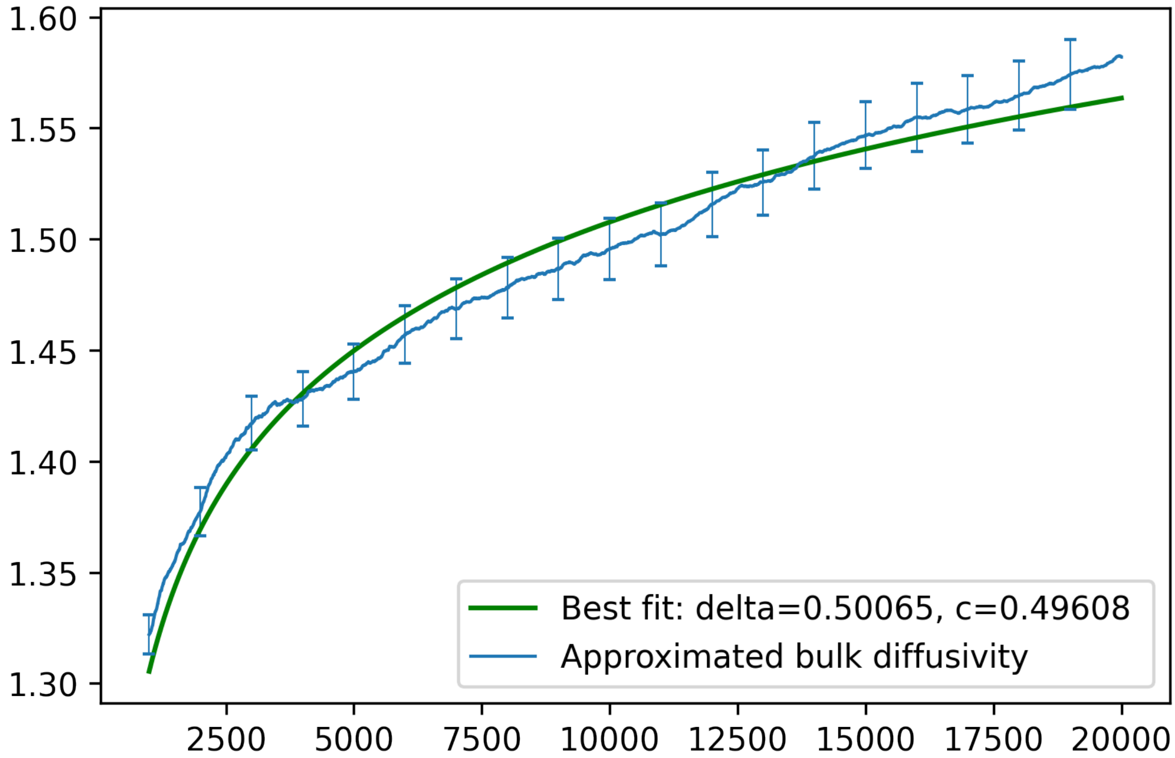
```
for i in filelist:
  batchsize, Dsumtemp,Dstemp = Dfilereader(i)
  Dsum += Dsumtemp
  if len(Ds) == 0:
    Ds = Dstemp
  else:
    Ds = np.concatenate((Ds, Dstemp), axis=0)
  number_of_instances += batchsize
  print(number_of_instances)
Davg = Dsum/number_of_instances
print(Davg[100])

@njit(parallel = True)
def ErrorCalc(number_of_instances,Davg,Ds):
  StdError = np.zeros(T, dtype = np.float64)
  SampleVar = np.zeros(T, dtype = np.float64)
  for t in prange(2,T):
    SampleVar[t] = -pow(number_of_instances,1)*pow(Davg[int(t*
   Time_Sample_Rate)],2)
    for inst in range(int(number_of_instances)):
      SampleVar[t] += pow(Ds[inst][t],2)
    SampleVar[t] = pow(number_of_instances-1,-1)*SampleVar[t]
    StdError[t] = pow(number_of_instances,-0.5)*pow(SampleVar[t],0.5)
  StdError = StdError*1.96 #This is the 95 % confidence interval
  return StdError,SampleVar
StdError,SampleVar = ErrorCalc(number_of_instances,Davg,Ds)
print(number_of_instances)




approxstart =250
start = 250

plt.figure(dpi=300)

plt.plot(TsError[approxstart:], Dfit(TsError[approxstart:], 0.5,0.5), '-',
    color='purple' , label='Conjecture delta=0.5, c=0.5 ')

Dhighavg =Davg[::20]+StdError
popt, pcov = curve_fit(Dfit, TsError[approxstart:], Dhighavg[approxstart:],
```

```
    bounds = ([0,-2.5], [1,2.5])  )
plt.plot(TsError[approxstart:], Dfit(TsError[approxstart:], *popt), 'r-',
    label='HIGH delta=%5.5f, c=%5.5f ' % tuple(popt))

Dlowavg =Davg[::20]-StdError
popt, pcov = curve_fit(Dfit, TsError[approxstart:], Dlowavg[approxstart:],
    bounds = ([0,-2.5], [1,2.5])  )
plt.plot(TsError[approxstart:], Dfit(TsError[approxstart:], *popt), 'b-',
    label='LOW delta=%5.5f, c=%5.5f ' % tuple(popt))

popt, pcov = curve_fit(Dfit, Ts[int(1+approxstart*Time_Sample_Rate):-1],
    Davg[int(1+approxstart*Time_Sample_Rate):-1], bounds = ([0,-25], [2,25])
     )
plt.plot(Ts[int(1+start*Time_Sample_Rate):-1], Dfit(Ts[int(1+start*
    Time_Sample_Rate):-1], *popt), 'g-', label='delta=%5.5f, c=%5.5f' %
    tuple(popt))


plt.plot(Ts[int(start*Time_Sample_Rate):-1], Davg[int(start*
    Time_Sample_Rate):-1], linewidth=1 , label="Bulk diffusivity at lambda="
     + str(la))

plt.errorbar(TsError[int(start)::1000], Davg[int(start*Time_Sample_Rate)::
    int(Time_Sample_Rate)*1000], fmt=' ' ,markersize=1, elinewidth=0.5,
    color='tab:blue' ,capsize=2 , yerr=StdError[int(start)::1000])
plt.legend()
#plt.xscale("log")
```

Listing 3.2: Graph plotting

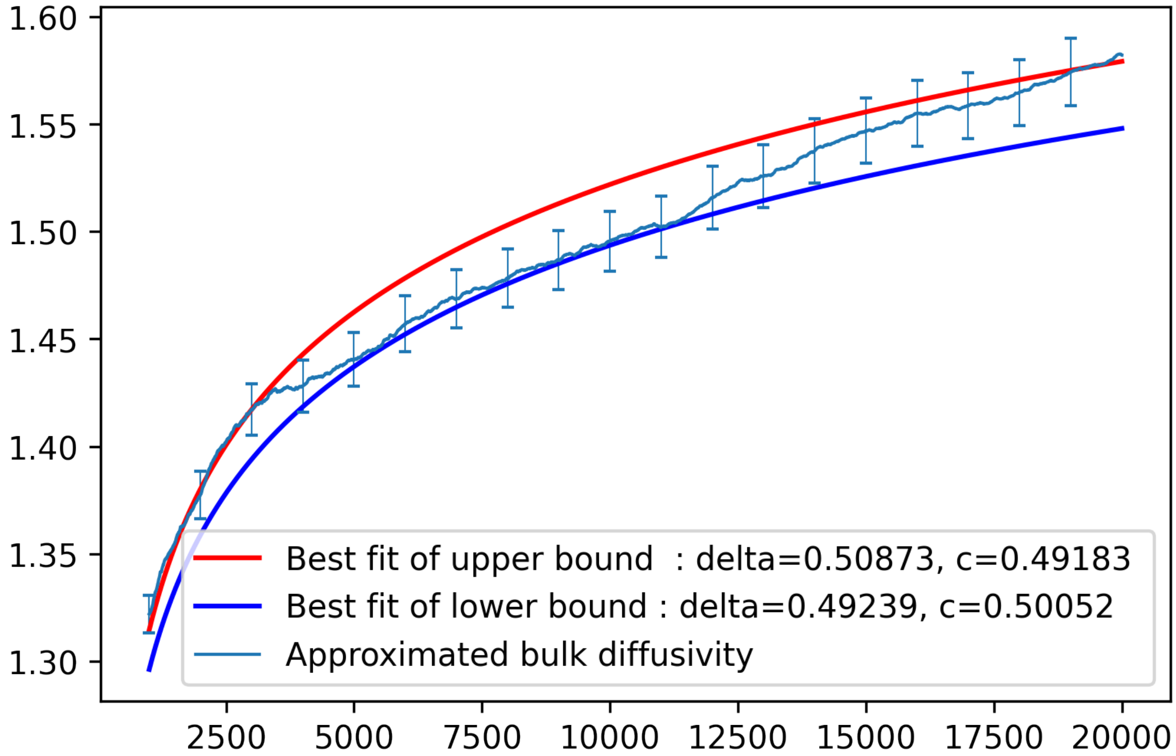
We first ask user where the files are then we load the files. Compute error bars using data we have and then plot variety of graphs. Below we plot best fit of calculated bulk diffusivity, calculated bulk diffusivity and error bars

Figure 3.1: Blue line is the numerically computed bulk diffusivity while the green line is a line of best fit of the form $c\log(t)^{\delta}$.



To put the error in perspective, below is a graph best fitting the numerically computed bulk diffusivity plus the error and numerically computed bulk diffusivity minus the error

Figure 3.2: Blue line is the numerically computed bulk diffusivity. The red line is a line best fit of the form $c\log(t)^\delta$ of the numerically computed bulk diffusivity plus error while the green line is a line of best fit of the form $c\log(t)^\delta$ of the numerically computed bulk diffusivity minus error.



Here one can look at the blue line as what is the slowest growth scenario while red line is the fastest growth scenario with respect to the error. These deviations when compared to the computed bulk diffusivity in fact vary c and δ by very little, a margin of error one expect to occur with the approximations one had to make to compute such quantity placing δ in the $(0.49, 0.51)$ range, this suggests that $\delta = \frac{1}{2}$ conjecture is likely to be true.

Appendix

Proposition 3.3.3 *For $N, M \in \mathbb{N}$ let $\mathcal{N}^{N,M}$ be the non-linearity defined in (2.2) and $\mathcal{N}_k^{N,M}$ be its corresponding fourier transform. Then*

$$\mathcal{N}_k^{N,M}[\omega^{N,M}] = \frac{1}{M^2} \sum_{\ell+m=k} \mathcal{K}_{\ell,m}^N \omega_\ell^{N,M} \omega_m^{N,M} \,,$$

for

$$\mathcal{K}_{\ell,m}^N \stackrel{def}{=} \frac{1}{2\pi} \hat{\varrho}_{\ell,m}^N \frac{(\ell^\perp \cdot (\ell+m))(m \cdot (\ell+m))}{|\ell|^2 |m|^2} \,, \qquad \textit{with} \quad \hat{\varrho}_{\ell,m}^N \stackrel{def}{=} \hat{\varrho}_\ell^N \hat{\varrho}_m^N \hat{\varrho}_{\ell+m}^N \,.$$

Proof. Starting with definition of non-linearity given in (1.9)

$$\begin{aligned}
\mathcal{N}_k^{N,M}(\omega^{N,M}) &= \mathcal{F}\Big(\operatorname{div} \varrho^N * \Big((K * (\varrho^N * \omega^{N,M}))(\varrho^N * \omega^{N,M}) \Big) \Big)(k) \\
&= \mathcal{F}\Big(\partial_1 \varrho^N * \Big((K_1 * (\varrho^N * \omega^{N,M}))(\varrho^N * \omega^{N,M}) \Big) \Big)(k) \\
&\quad + \mathcal{F}\Big(\partial_2 \varrho^N * \Big((K_2 * (\varrho^N * \omega^{N,M}))(\varrho^N * \omega^{N,M}) \Big) \Big)(k) \,,
\end{aligned}$$

here $K = (K_1, K_2)$ is the *Biot-Savart* kernel defined in (1.5). We now consider the two terms in the last equation separately. Starting with the first term

$$\begin{aligned}
&\mathcal{F}\Big(\partial_1 \varrho^N * \Big((K_1 * (\varrho^N * \omega^{N,M}))(\varrho^N * \omega^{N,M}) \Big) \Big)(k) \\
&= \mathcal{F}(\partial_1 \varrho^N)(k) \mathcal{F}\Big((K_1 * (\varrho^N * \omega^{N,M}))(\varrho^N * \omega^{N,M}) \Big)(k) \\
&= 2\pi \iota k_1 \hat{\varrho}_k^N \sum_{\ell+m=k} \mathcal{F}(K_1 * (\varrho^N * \omega^{N,M}))(\ell) \mathcal{F}(\varrho^N * \omega^{N,M})(m) \\
&= 2\pi \iota k_1 \hat{\varrho}_k^N \sum_{\ell+m=k} \mathcal{F}(K_1)(\ell) \mathcal{F}(\varrho^N * \omega^{N,M})(\ell) \mathcal{F}(\varrho^N * \omega^{N,M})(m) \\
&= 2\pi \iota k_1 \hat{\varrho}_k^N \sum_{\ell+m=k} \mathcal{F}(K_1)(\ell) \hat{\varrho}_\ell^N \hat{\varrho}_m^N \omega_\ell^{N,M} \omega_m^{N,M} \\
&= \sum_{\ell+m=k} \Big[2\pi \iota k_1 \mathcal{F}(K_1)(\ell) \Big] \hat{\varrho}_{\ell,m}^N \omega_\ell^{N,M} \omega_m^{N,M} \,.
\end{aligned}$$

Similarly

$$\mathcal{F}\Big(\partial_2 \varrho^N * \Big((K_2 * (\varrho^N * \omega^{N,M}))(\varrho^N * \omega^{N,M}) \Big) \Big)(k) = \sum_{\ell+m=k} \Big[2\pi \iota k_2 \mathcal{F}(K_2)(\ell) \hat{\varrho}_{\ell,m}^N \Big] \omega_\ell^{N,M} \omega_m^{N,M} \,.$$

Given (1.5) we can calculate $\mathscr{F}(K_1)(\ell)$ as follows

$$\mathscr{F}(K_1)(\ell) = \mathscr{F}_z\left(\frac{1}{2\pi\iota}\int_{\mathbb{R}^2}\frac{y_2}{|y|^2}e^{-\iota y\cdot z}\,\mathrm{d}y\right)(\ell) = \frac{1}{2\pi\iota}\frac{\ell_2}{|\ell|^2}\,,$$

similarly

$$\mathscr{F}(K_2)(\ell) = \mathscr{F}_z\left(\frac{1}{2\pi\iota}\int_{\mathbb{R}^2}\frac{-y_1}{|y|^2}e^{-\iota y\cdot z}\,\mathrm{d}y\right)(\ell) = \frac{1}{2\pi\iota}\frac{-\ell_1}{|\ell|^2}\,.$$

Bringing the results of the calculations above together we have

$$\begin{aligned}
\mathcal{N}_k^{N,M}(\omega^{N,M}) &= 2\pi\iota\sum_{\ell+m=k}[k_1\mathscr{F}(K_1)(\ell)+k_2\mathscr{F}(K_2)(\ell)]\hat{\varrho}^N_{\ell,m}\omega_\ell^{N,M}\omega_m^{N,M}\\
&= 2\pi\sum_{\ell+m=k}[k_1\mathscr{F}(K_1)(\ell)+k_2\mathscr{F}(K_2)(\ell)]\hat{\varrho}^N_{\ell,m}\omega_\ell^{N,M}\omega_m^{N,M}\\
&= \frac{1}{2\pi}\sum_{\ell+m=k}\frac{\ell_2k_1-\ell_1k_2}{|\ell|^2}\hat{\varrho}^N_{\ell,m}\omega_\ell^{N,M}\omega_m^{N,M}\\
&= \frac{1}{2\pi}\sum_{\ell+m=k}\frac{\ell^\perp\cdot k}{|\ell|^2}\hat{\varrho}^N_{\ell,m}\omega_\ell^{N,M}\omega_m^{N,M}\\
&= \frac{1}{2\pi}\sum_{\ell+m=k}\frac{\ell^\perp\cdot m}{|\ell|^2}\hat{\varrho}^N_{\ell,m}\omega_\ell^{N,M}\omega_m^{N,M}
\end{aligned}$$

By the Lemma 3.3.4 the proof is concluded. □

Lemma 3.3.4 *For $k\in\mathbb{Z}^2_M$ and $v\in\mathbf{L}^2$ we have*

$$\sum_{\substack{\ell+m=k\\ \ell,m\in\mathbb{Z}^2_M}}\frac{\ell^\perp\cdot m}{|\ell|^2}v_\ell v_m = \sum_{\substack{\ell+m=k\\ \ell,m\in\mathbb{Z}^2_M}}\frac{(\ell^\perp\cdot(\ell+m))(m\cdot(\ell+m))}{|\ell|^2|m|^2}v_\ell v_m\,.$$

Proof. By splitting into two same terms and exchanging m with ℓ in the second term we get

$$\begin{aligned}
\sum_{\ell+m=k}\frac{\ell^\perp\cdot m}{|\ell|^2}v_\ell v_m &= \frac{1}{2}\sum_{\ell+m=k}\frac{\ell^\perp\cdot m}{|\ell|^2}v_\ell v_m + \frac{1}{2}\sum_{\ell+m=k}\frac{m^\perp\cdot \ell}{|m|^2}v_\ell v_m\\
&= \sum_{\ell+m=k}\frac{1}{2}\left(\frac{(\ell^\perp\cdot m)(m\cdot m)+(m^\perp+\ell)(\ell\cdot\ell)}{|\ell|^2|m|^2}\right)v_\ell v_m\,.
\end{aligned}$$

Focusing on numerator part of the fraction and using $m^\perp \cdot \ell = -\ell^\perp \cdot m$ we get

$$
\begin{aligned}
(\ell^\perp \cdot m)(m \cdot m) + (m^\perp + \ell)(\ell \cdot \ell) &= (\ell^\perp \cdot m)(m \cdot m - \ell \cdot \ell) \\
&= (\ell^\perp \cdot m)(m \cdot (\ell + m)) - (\ell^\perp \cdot m)(\ell \cdot (\ell + m)) \\
&= (\ell^\perp \cdot m)(m \cdot (\ell + m)) + (m^\perp \cdot \ell)(\ell \cdot (\ell + m))\,,
\end{aligned}
$$

note the second term is in fact the first term with ℓ and m swapped, a swapping we can do within the sum as everything else is symmetric,

$$
\begin{aligned}
&\sum_{\ell+m=k} \frac{1}{2}\left(\frac{(\ell^\perp \cdot m)(m \cdot (\ell + m)) + (m^\perp \cdot \ell)(\ell \cdot (\ell + m))}{|\ell|^2|m|^2}\right) v_\ell v_m \\
&\quad = \frac{1}{2}\sum_{\ell+m=k} \frac{(\ell^\perp \cdot m)(m \cdot (\ell + m))}{|\ell|^2|m|^2} v_\ell v_m + \frac{1}{2}\sum_{\ell+m=k} \frac{(m^\perp \cdot \ell)(\ell \cdot (\ell + m))}{|\ell|^2|m|^2} v_\ell v_m \\
&\quad = \sum_{\ell+m=k} \frac{(\ell^\perp \cdot m)(m \cdot (\ell + m))}{|\ell|^2|m|^2} v_\ell v_m \\
&\quad = \sum_{\ell+m=k} \frac{(\ell^\perp \cdot (\ell + m))(m \cdot (\ell + m))}{|\ell|^2|m|^2} v_\ell v_m\,,
\end{aligned}
$$

last equality holds as $\ell^\perp \cdot \ell = 0$. □

Proposition 3.3.5 *Suppose ω^1 solves the equation from* (1.6). *For $N \in \mathbb{N}$ the rescaled function given by*

$$\omega^N(t,x) \stackrel{def}{=} N^2 \omega^1(tN^{2\theta}, xN)\,,$$

solves

$$\partial_t \omega^N = -\tfrac{1}{2}(-\Delta)^\theta \omega^N - \hat{\lambda} N^{2\theta-2} \mathcal{N}^N[\omega^N] + (-\Delta)^{\frac{1+\theta}{2}} \xi\,,$$

here the non-linearity $\mathcal{N}^N$ is defined according to

$$\mathcal{N}^N[\omega] \stackrel{def}{=} \operatorname{div} \varrho^N * \Big((K * (\varrho^N * \omega))(\varrho^N * \omega)\Big)\,.$$

where $\varrho^N(\cdot) \stackrel{def}{=} N^2 \varrho(N \cdot)$.

Proof. We now focus on proving (1.8). By (1.7) we know that $\Big(\partial_t \omega^N\Big)(t,x)$ is equal to

$$
\begin{aligned}
&N^2\partial_t\Big(\omega(N^{2\theta}t,Nx)\Big) = N^{2+2\theta}(\partial_t\omega)(N^{2\theta}t,Nx)\\
&= N^{2+2\theta}\Big(-\frac{1}{2}(-\Delta)^{\theta}\omega - \hat{\lambda}\varrho^1 * \Big((K*(\varrho^1*\omega))\cdot\nabla(\varrho^1*\omega)\Big) + (-\Delta)^{\frac{1+\theta}{2}}\Big)(N^{2\theta}t,Nx)\\
&= \Big(\tilde{\xi} - \frac{1}{2}(-\Delta)\omega^N\Big)(t,x) - N^{2+2\theta}\hat{\lambda}\Big(\varrho^1 * \Big((K*(\varrho^1*\omega))\cdot\nabla(\varrho^1*\omega)\Big)\Big)(N^{2\theta}t,Nx)\,,
\end{aligned}
\tag{3.45}
$$

here $\tilde{\xi}$ is different white-noise of the same distribution as ξ. In particular the only term that is left to be considered is the last one. Firstly notice that for a smooth function $F \in \mathbb{R}_{\geq 0}\times\mathbb{R}^2$ we have

$$
\begin{aligned}
(\varrho^1 * F)(N^{2\theta},Nx) &= \int_{\mathbb{R}^2}\varrho^1(Nx-\tau)F(N^{2\theta}t,\tau)\,\mathrm{d}\tau\\
&= \int_{\mathbb{R}^2} N^2\varrho^1(N(x-\bar{\tau}))F(N^{2\theta}t,N\bar{\tau})\,\mathrm{d}\bar{\tau}\\
&= \int_{\mathbb{R}^2}\varrho^N(x-\bar{\tau})F(N^{2\theta}t,N\bar{\tau})\,\mathrm{d}\bar{\tau}\,.
\end{aligned}
\tag{3.46}
$$

Here in the second passage the $\tau = N\bar{\tau}$ substitution was used. Focusing on the last term from (3.45) and using (3.46) we have

$$
\begin{aligned}
&\Big(\varrho^1 * \Big((K*(\varrho^1*\omega))\cdot\nabla(\varrho^1*\omega)\Big)\Big)(N^{2\theta}t,Nx)\\
&= \int_{\mathbb{R}^2}\varrho^N(x-\tau)\Big((K*(\varrho^1*\omega))\cdot\nabla(\varrho^1*\omega)\Big)(N^{2\theta}t,N\tau)\,\mathrm{d}\tau\\
&= \int_{\mathbb{R}^2}\varrho^N(x-\tau)(K*(\varrho^1*\omega))(N^{2\theta}t,N\tau)\cdot(\nabla(\varrho^1*\omega))(N^{2\theta}t,N\tau)\,\mathrm{d}\tau\,.
\end{aligned}
$$

We will firstly focus on $\nabla(\varrho^1*\omega)(N^{2\theta}t,N\tau)$ and then $(K*(\varrho^1*\omega))(N^{2\theta}t,N\tau)$. Once again using (3.46) we have

$$
\begin{aligned}
\nabla(\varrho^1*\omega)(N^{2\theta}t,N\tau) &= \frac{1}{N}\nabla((\varrho^1*\omega)(N^{2\theta}t,N\tau))\\
&= \frac{1}{N}\nabla\Big(\int_{\mathbb{R}^2}\varrho^N(\tau-\bar{\tau})\omega(N^{2\theta}t,N\bar{\tau})\,\mathrm{d}\bar{\tau}\Big)\\
&= \frac{1}{N^3}\nabla\Big(\int_{\mathbb{R}^2}\varrho^N(\tau-\bar{\tau})\omega^N(t,\bar{\tau})\Big)\\
&= \frac{1}{N^3}\nabla(\varrho^N*\omega^N)(t,\tau)\,.
\end{aligned}
\tag{3.47}
$$

Focusing $(K*(\varrho^1*\omega))(N^{2\theta}t,N\tau)$ we once again apply (3.46)

$$
(K*(\varrho^1*\omega))(N^{2\theta}t,N\tau)
$$

$$
\begin{aligned}
&= \int_{\mathbb{R}^2} K(\bar\tau)(\varrho^1 * \omega)(N^{2\theta}t, N\tau - \bar\tau)\,\mathrm{d}\bar\tau \\
&= \int_{\mathbb{R}^2} K(Nz)N^2(\varrho^1 * \omega)(N^{2\theta}t, N(\tau - z))\,\mathrm{d}z
\end{aligned}
$$

in similar fashion as in (3.47) the above is equal to

$$
\begin{aligned}
&\int_{\mathbb{R}^2} K(Nz)(\varrho^N * \omega^N)(t, \tau - z)\,\mathrm{d}z \\
&= \int_{\mathbb{R}^2} \frac{1}{N} K(z)(\varrho^N * \omega^N)(t, \tau - z)\,\mathrm{d}z \\
&= \frac{1}{N}\Big(K * (\varrho^N * \omega^N)\Big)(t, \tau)\,,
\end{aligned}
$$

the first passage holds because

$$
K(Nz) = \frac{1}{\iota}\int_{\mathbb{R}^2} \frac{y^\perp}{|y|^2} e^{-y\cdot Nz}\,\mathrm{d}y = \frac{1}{\iota}\int_{\mathbb{R}^2} \frac{1}{N^2}\frac{(x/N)^\perp}{|(x/N)|^2} e^{-x\cdot z}\,\mathrm{d}x = \frac{1}{N}K(z)\,.
$$

In the calculation above $x = Ny$ substitution was used. □

Proposition 3.3.6 *Suppose u^1 solves the equation from* (1.15)*. For $N \in \mathbb{N}$ the rescaled function given by*

$$
u^N(t,x) \stackrel{\text{def}}{=} N^2 u^1(tN^{2\theta}, xN)\,, \tag{3.48}
$$

solves

$$
\partial_t u^N = -\tfrac{1}{2}(-\Delta)^\theta u^N - \hat\lambda N^{2\theta-2}\tilde{\mathcal{N}}^N[u^N] + (-\Delta)^{\frac{\theta}{2}}\xi\,,
$$

here the non-linearity $\tilde{\mathcal{N}}^N$ is defined according to

$$
\tilde{\mathcal{N}}^N[u] \stackrel{\text{def}}{=} \varrho^N * (-\Delta)^{\frac{1}{2}}((\partial_1(-\Delta)^{-\frac{1}{2}}\varrho^N * u)^2 - (\partial_2(-\Delta)^{-\frac{1}{2}}\varrho^N * u)^2)\,.
$$

where $\varrho^N(\cdot) \stackrel{\text{def}}{=} N^2\varrho(N\cdot)$.

Proof. We start by directly imposing the scaling onto the original equation (1.15)

$$
\begin{aligned}
&\partial_t u^N(t,x) = N^{2+2\theta}(\partial_t u^1)(tN^{2\theta}, xN) \\
&= N^{2+2\theta}\bigg(- \tfrac{1}{2}(-\Delta)^\theta u^1 + (-\Delta)^{\frac{\theta}{2}}\xi \\
&- \hat\lambda\varrho^1 * (-\Delta)^{\frac{1}{2}}\Big((\partial_1(-\Delta)^{-\frac{1}{2}}\varrho^1 * u^1)^2 - (\partial_2(-\Delta)^{-\frac{1}{2}}\varrho^1 * u^1)^2\Big)\bigg)(tN^{2\theta}, xN)
\end{aligned}
$$

$$
\begin{aligned}
&= -\tfrac{1}{2}(-\Delta)^{\theta} u^N(t,x) + (-\Delta)^{\frac{\theta}{2}}\xi(t,x) \\
&- N^{2+2\theta}\Bigg(\hat{\lambda}\varrho^1 * (-\Delta)^{\frac{1}{2}}\Big((\partial_1(-\Delta)^{-\frac{1}{2}}\varrho^1 * u^1)^2 - (\partial_2(-\Delta)^{-\frac{1}{2}}\varrho^1 * u^1)^2\Big)\Bigg)(tN^{2\theta}, xN)\,.
\end{aligned}
$$

The second equality is strictly speaking not true, in fact its not the same space-time white noise in the previous equation but a different space-time white noise of the same distribution which is sufficient here. The (t,x) on the space-time white noise are of course not formal as its a distribution its only there to signal that the scaling was already accounted for. Focusing on the nonlinearity now

$$
\begin{aligned}
&N^{2+2\theta}\hat{\lambda}(-\Delta)^{\frac{1}{2}}\Bigg(\hat{\lambda}\varrho^1 * \Big((\partial_1(-\Delta)^{-\frac{1}{2}}\varrho^1 * u^1)^2 - (\partial_2(-\Delta)^{-\frac{1}{2}}\varrho^1 * u^1)^2\Big)\Bigg)(tN^{2\theta}, xN) \qquad (3.49)\\
&N^{2+2\theta}\hat{\lambda}(-\Delta)^{\frac{1}{2}}\int \varrho^1(Nx-y)\Big((\partial_1(-\Delta)^{-\frac{1}{2}}\varrho^1 * u^1)^2 - (\partial_2(-\Delta)^{-\frac{1}{2}}\varrho^1 * u^1)^2\Big)(tN^{2\theta}, y)\,\mathrm{d}y \\
&N^{2+2\theta}\hat{\lambda}(-\Delta)^{\frac{1}{2}}\int N^2\varrho^1(Nx-N\bar{y})\Big((\partial_1(-\Delta)^{-\frac{1}{2}}\varrho^1 * u^1)^2 - (\partial_2(-\Delta)^{-\frac{1}{2}}\varrho^1 * u^1)^2\Big)(tN^{2\theta}, N\bar{y})\,\mathrm{d}\bar{y} \\
&N^{2+2\theta}\hat{\lambda}(-\Delta)^{\frac{1}{2}}\int \varrho^N(x-\bar{y})\Big((\partial_1(-\Delta)^{-\frac{1}{2}}\varrho^1 * u^1)^2 - (\partial_2(-\Delta)^{-\frac{1}{2}}\varrho^1 * u^1)^2\Big)(tN^{2\theta}, N\bar{y})\,\mathrm{d}\bar{y}\,,
\end{aligned}
$$

here we used substitution $N\bar{y} = y$ on $\mathbb{R}^2$ (hence the N^2 appearing). Focusing now on the second term within the integral above

$$
\begin{aligned}
&\Big((\partial_1(-\Delta)^{-\frac{1}{2}}\varrho^1 * u^1)^2 - (\partial_2(-\Delta)^{-\frac{1}{2}}\varrho^1 * u^1)^2\Big)(tN^{2\theta}, N\bar{y}) \\
&= ((\partial_1(-\Delta)^{-\frac{1}{2}}\varrho^1 * u^1)(tN^{2\theta}, N\bar{y}))^2 - ((\partial_2(-\Delta)^{-\frac{1}{2}}\varrho^1 * u^1)(tN^{2\theta}, N\bar{y}))^2\,,
\end{aligned}
$$

in particular obtaining formulation in terms of u^N of $(\partial_1(-\Delta)^{-\frac{1}{2}}\varrho^1 * u^1)(tN^{2\theta}, N\bar{y})$ will give us the analogous expression for $(\partial_2(-\Delta)^{-\frac{1}{2}}\varrho^1 * u^1)(tN^{2\theta}, N\bar{y})$ which in turn will be sufficient to express the equation above in therms of u^N.

$$
\begin{aligned}
&(\partial_1(-\Delta)^{-\frac{1}{2}}\varrho^1 * u^1)(tN^{2\theta}, N\bar{y}) \\
&= \partial_1(-\Delta)^{-\frac{1}{2}}\int \varrho^1(N\bar{y}-z)u^1(tN^{2\theta}, z)\,\mathrm{d}z \\
&= \partial_1(-\Delta)^{-\frac{1}{2}}\int N^2\varrho^1(N\bar{y}-N\bar{z})u^1(tN^{2\theta}, N\bar{z})\,\mathrm{d}\bar{z} \\
&= \partial_1(-\Delta)^{-\frac{1}{2}}\int \varrho^N(\bar{y}-\bar{z})u^1(tN^{2\theta}, N\bar{z})\,\mathrm{d}\bar{z} \\
&= \frac{1}{N^2}\partial_1(-\Delta)^{-\frac{1}{2}}\int \varrho^N(\bar{y}-\bar{z})u^N(t, \bar{z})\,\mathrm{d}\bar{z}
\end{aligned}
$$

$$= \frac{1}{N^2}\partial_1(-\Delta)^{-\frac{1}{2}}(\varrho^N * u^N)(t,\bar{y}) \,.$$

Bringing it all together we can now express (3.49) as

$$N^{2+2\theta}\hat{\lambda}(-\Delta)^{\frac{1}{2}}\left(\hat{\lambda}\varrho^1 * \left((\partial_1(-\Delta)^{-\frac{1}{2}}\varrho^1 * u^1)^2 - (\partial_2(-\Delta)^{-\frac{1}{2}}\varrho^1 * u^1)^2\right)\right)(tN^{2\theta}, xN)$$
$$= N^{2+2\theta}\hat{\lambda}(-\Delta)^{\frac{1}{2}}\left(\hat{\lambda}\varrho^N * \left(\left(\frac{1}{N^2}\partial_1(-\Delta)^{-\frac{1}{2}}(\varrho^N * u^N)\right)^2 - \left(\frac{1}{N^2}\partial_2(-\Delta)^{-\frac{1}{2}}(\varrho^N * u^N)\right)^2\right)\right)(t,x)$$
$$= N^{2\theta-2}\hat{\lambda}(-\Delta)^{\frac{1}{2}}\left(\hat{\lambda}\varrho^N * \left((\partial_1(-\Delta)^{-\frac{1}{2}}(\varrho^N * u^N))^2 - (\partial_2(-\Delta)^{-\frac{1}{2}}(\varrho^N * u^N))^2\right)\right)(t,x) \,.$$

□

Proposition 3.3.7 *For $N, M \in \mathbb{N}$ let $\mathcal{N}^{N,M}$ be the non-linearity defined in (3.1) and $\tilde{\mathcal{N}}^{N,M}$ be its corresponding fourier transform. Then*

$$\tilde{\mathcal{N}}_k^{N,M}[u^{N,M}](x) = \frac{1}{M^2}\sum_{\substack{\ell,m\in\mathbb{Z}_M^2 \\ \ell+m=k}} \tilde{\mathcal{K}}_{\ell,m}^N u_{-\ell}^{N,M} u_{-m}^{N,M} e_{\ell+m}(x) \,,$$

for

$$\tilde{\mathcal{K}}_{\ell,m}^N \stackrel{def}{=} \frac{1}{2\pi}\hat{\varrho}_{\ell,m}^N|\ell+m|\frac{c(\ell,m)}{|\ell||m|} \,, \quad c(\ell,m) \stackrel{def}{=} m_2\ell_2 - m_1\ell_1 \,,$$

and the variables ℓ and m appearing in the previous equations range over $\mathbb{Z}_M^2$.

Proof. We will start by directly considering the Fourier transform of the expression of the nonlinearity from (1.18). For the sake of clarity in this I will ommit the M, N subscripts in this proof.

$$\begin{aligned}\tilde{\mathcal{N}}_k[u] &= \mathcal{F}\left(\varrho * (-\Delta)^{\frac{1}{2}}\left((\partial_1(-\Delta)^{\frac{1}{2}}(\varrho * u))^2 - (\partial_2(-\Delta)^{\frac{1}{2}}(\varrho * u))^2\right)\right)(k) \\ &= \varrho_k|k|\left(\mathcal{F}((\partial_1(-\Delta)^{\frac{1}{2}}(\varrho * u))^2)(k) - \mathcal{F}((\partial_2(-\Delta)^{\frac{1}{2}}(\varrho * u))^2)(k)\right) \,.\end{aligned} \tag{3.50}$$

We will focus on calculating $\mathcal{F}((\partial_1(-\Delta)^{\frac{1}{2}}(\varrho*u))^2)(k)$ realising the term $\mathcal{F}((\partial_2(-\Delta)^{\frac{1}{2}}(\varrho*u)))(k)$

can be obtained analogously.

$$\begin{aligned}&\mathscr{F}((\partial_1(-\Delta)^{\frac{1}{2}}(\varrho * u))^2)(k)\\&=\frac{1}{2\pi M^2}\sum_{\ell+m=k}\frac{|\ell_1|}{|\ell|}\varrho_\ell u_\ell\frac{|m_1|}{|m|}\varrho_m u_m\\&=\frac{1}{2\pi M^2}\sum_{\ell+m=k}\frac{|\ell_1||m_1|}{|\ell||m|}\varrho_\ell\varrho_m u_\ell u_m\,.\end{aligned} \tag{3.51}$$

Bringing (3.50) and (3.51) together gives us the result. □

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