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An algebraic model for rational $\mathbb{T}^2$-equivariant elliptic cohomology

by

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To mum and dad

Contents

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Declarations

The main results of this thesis are contained in Chapters 4 and 5.

Chapter 4 covers the results in [Bar22a], while chapter 5 covers [Bar22b]. Both these articles have been submitted for publication.

I declare that the material in this thesis is, to the best of my knowledge, my own, except where otherwise indicated or cited in the text. This material has not been submitted for any other degree of qualification.

Abstract

We construct a rational $\mathbb{T}^2$-equivariant elliptic cohomology theory for the 2-torus $\mathbb{T}^2$, starting from an elliptic curve $\mathcal{C}$ over $\mathbb{C}$ and a coordinate data around the identity. The theory is defined by constructing an object $E\mathcal{C}_{\mathbb{T}^2}$ in the algebraic model category $d\mathcal{A}(\mathbb{T}^2)$, which by Greenlees and Shipley [GS18] is Quillen-equivalent to rational $\mathbb{T}^2$-spectra. This result is a generalisation to the 2-torus of the construction [Gre05] for the circle $\mathbb{T}$. The object $E\mathcal{C}_{\mathbb{T}^2}$ is directly built using geometric inputs coming from the Cousin complex of the structure sheaf of the complex abelian surface $\mathcal{X} = \mathcal{C} \times \mathcal{C}$.

We use this construction to compute rational $\mathbb{T}$-equivariant elliptic cohomology of $\mathbb{C}\mathrm{P}(V)$: the complex projective space of a finite dimensional complex representation V of $\mathbb{T}$. More precisely we prove that $E\mathcal{C}_{\mathbb{T}}$ built in [Gre05] and $E\mathcal{C}_{\mathbb{T}^2}$ satisfy a split condition implying $E\mathcal{C}_{\mathbb{T}}(\mathbb{C}\mathrm{P}(V)_+) \cong E\mathcal{C}_{\mathbb{T}^2}(S(V \otimes w)_+)$ where $S(\_)$ is the sphere of vectors with unit norm and w is the natural representation of $\mathbb{T}$. The rational $\mathbb{T}^2$-elliptic cohomology of this space can be deduced from the one on spheres of complex representations S^V of $\mathbb{T}^2$ that we compute in the construction of $E\mathcal{C}_{\mathbb{T}^2}$.

Chapter 1

Introduction

1.1 Motivation

1.1.1 Elliptic cohomology

Cohomology theories are among the best tools to study topological spaces up to homotopy equivalence. In the most classical sense a cohomology theory is a contravariant functor from topological spaces to graded abelian groups satisfying some axioms (Eilenberg-Steenrod axioms), that encode the invariance under homotopy equivalence and makes the theory more computable. Via stabilization, cohomology theories are represented by certain objects called spectra. Indeed given a cohomology theory $E^*(\_)$ we have an associated spectrum E, such that the cohomology of the space X is

$$E^*(X) = [\Sigma^\infty X, E]_*$$

where $[\_,\_]_*$ denotes the graded set of homotopy classes of maps, and $\Sigma^\infty X$ is the suspension spectrum of X (the corresponding stabilized object). Conversely every spectrum E defines a cohomology theory in the same way.

Ordinary cohomology and complex K-theory are the most prominent examples of cohomology theories, but since their definition mathematicians have asked how to build more of them and how can they be classified. For sufficiently nice theories (namely complex orientable) one can associate a power series in two variables called formal group law [Qui69]. Let us specify that the formal group law is not canonically associated to the cohomology theory, but it depends on the choice of a power series generator for the cohomology ring of $\mathbb{C}\mathrm{P}^\infty$. Under this association ordinary cohomology is associated with the additive formal group law and complex K-theory is associated with the multiplicative formal group law. A great source of formal group laws is the formal completion of a one dimensional algebraic group at the identity. If we are working over an algebraically closed field for example, we are soon in short supply of one dimensional algebraic groups, except for the additive and multiplicative ones all the others are elliptic curves. These more exotic theories whose associated formal group law arises as the formal completion of an elliptic curve are called elliptic cohomology theories. In contrast with ordinary cohomology and complex K-theory that enjoyed wide geometric applications, for long time the known constructions of elliptic cohomology [Lan88] [LRS95], and more recently [AHS01], have been purely algebraic, lacking geometric interpretation.

In many concrete cases we have a group G (compact Lie group) acting on the space X, and it would be desirable to have cohomology theories that take this group action into account, namely we would like to have a G-equivariant cohomology theory. Often given a (non-equivariant) cohomology theory we can readapt the definition to obtain an equivariant theory with similar properties but that takes the group action into consideration: for example Borel cohomology, equivariant K-theory and equivariant cobordism, are equivariant counterparts of the respective non-equivariant theories. Exactly as in the non-equivariant world for G a compact Lie group, via stabilization one can construct a category of G-spectra where every such cohomology theory $E^*_G(\_)$ is represented by a G-spectrum E, in the sense that for any based G-space X we have

$$E^*_G(X) = [\Sigma^\infty X, E]^G_*. \tag{1.1.1}$$

The gain with the category of G-spectra is in the structure, in particular one can do homotopy theory in it.

For many years elliptic cohomology has begged for an equivariant counterpart, but in contrast with ordinary cohomology and complex K-theory where a geometric definition could be readapted, it wasn't even clear what a "good theory" of equivariant elliptic cohomology should satisfy. If one takes complex K-theory as a model, than we can expect a good theory of equivariant elliptic cohomology to encode the full algebraic group $\mathcal{C}$, in contrast with the non-equivariant theory simply encoding the formal completion of $\mathcal{C}$ around the identity. This is in analogy to how equivariant K-theory works. By the Atiyah-Segal completion theorem [AS69] if we complete the equivariant K-theory of the point at its augmentation ideal, we obtain the K-theory of BG, the classifying space of G:

$$KU_G(*)^\wedge_I \cong KU(BG).$$

In 1994 Growjnowski [Gro07] proposed the first definition of equivariant elliptic cohomology $E\mathcal{C}^*_G(\_)$ for any compact lie group G with complex coefficients. He defined $E\mathcal{C}^*_G(\_)$ as a coherent holomorphic sheaf over a certain variety $\mathcal{X}_G$ constructed from the given elliptic curve $\mathcal{C}$. Growjnowski was interested in implications for the representation theory of certain elliptic algebras where a sheaf valued theory defined for finite complexes was enough. Around the same years Ginzburg-Kapranov-Vasserot [GKV95] gave an axiomatic description of equivariant elliptic cohomology. The theory they had in mind was a collection of functors going from pairs of G-complexes to the abelian category $\mathrm{Coh}(\mathcal{X}_G)$ of coherent sheaves over the same variety $\mathcal{X}_G$ built from the elliptic curve $\mathcal{C}$. When G is a compact abelian lie group [GKV95, example 1.4.4] then $\mathcal{X}_G$ is the variety:

$$\mathcal{X}_G := \mathrm{Hom}_{\mathrm{Ab}}(G^*, \mathcal{C}) \tag{1.1.2}$$

where G^* is the character group of G: the continuous group homomorphisms from G to the circle group $\mathbb{T}$. To use the full apparatus of stable equivariant homotopy theory it is essential to have a conventional group valued G-equivariant cohomology theory defined on G-spaces, and represented by a G-spectrum $E\mathcal{C}_G$. This is precisely the point of view taken by Greenlees in [Gre05], starting from an elliptic curve $\mathcal{C}$ over a $\mathbb{Q}$-algebra and a coordinate around the identity, he builds a rational $\mathbb{T}$-equivariant elliptic cohomology theory $E\mathcal{C}_{\mathbb{T}}$ using algebraic models [Gre99]. The connection with the elliptic curve $\mathcal{C}$ resides in the cohomology of the one point compactification S^V for a complex representation V of $\mathbb{T}$:

$$\begin{aligned} E\mathcal{C}^{\mathrm{even}}_{\mathbb{T}}(S^V) &\cong H^{\mathrm{even}}(\mathcal{X}_{\mathbb{T}}, \mathcal{O}(-D_V)) \\ E\mathcal{C}^{\mathrm{odd}}_{\mathbb{T}}(S^V) &\cong H^{\mathrm{odd}}(\mathcal{X}_{\mathbb{T}}, \mathcal{O}(-D_V)) \end{aligned} \tag{1.1.3}$$

where $\mathcal{O}(-D_V)$ is a coherent sheaf over $\mathcal{X}_{\mathbb{T}}$, associated to the representation V.

This PhD thesis is a contribution to this last approach. We will build a rational $\mathbb{T}^2$-equivariant elliptic cohomology theory $E\mathcal{C}_{\mathbb{T}^2}$ using algebraic models [GS18], and we forge the connection with the elliptic curve $\mathcal{C}$ we started with, computing the values on spheres of complex representations S^V. Exactly as in the circle-equivariant theory (1.1.3) we obtain the cohomology of certain coherent sheaves $\mathcal{O}(-D_V)$ over the prescribed variety $\mathcal{X}_{\mathbb{T}^2}$. Moreover we will prove that the circle-equivariant theory $E\mathcal{C}_{\mathbb{T}}$ of Greenlees and our $\mathbb{T}^2$-equivariant construction $E\mathcal{C}_{\mathbb{T}^2}$ satisfy the useful split condition (Theorem 1.2.5), allowing us to compute $E\mathcal{C}^*_{\mathbb{T}}(\mathbb{C}\mathrm{P}(V))$ for the $\mathbb{T}$-space of complex lines of a $\mathbb{T}$-representation V. We believe that this is the first time where $\mathbb{T}$-equivariant elliptic cohomology of the complex projective spaces $\mathbb{C}\mathrm{P}(V)$ is computed (rationally).

We only discussed the starting point of the influential field of equivariant elliptic cohomology. Right from the start a different school emerged with the work of Devoto [Dev96], focusing on the case of G finite, particularly for organizing moonshine phenomena. The equivariant theory shares also an interesting connection with physics (see for example the Stolz-Teichner program [ST11], or work of Berwick-Evans [Ber21]). Moreover it contributed to give a more comprehensible picture towards a geometric interpretation of elliptic cohomology [BT18]. It was only recently with a big program started by Lurie [Lur18a], [Lur18b], [Lur19], [GM20] that they were able to define an integral theory of equivariant elliptic cohomology for any compact lie group. Lurie moved the construction entirely in the land of derived algebraic geometry, which seems necessary to enjoy an integral theory [Lur09].

1.1.2 Algebraic models

Studying the full collection of G-equivariant cohomology theories and their invariants has always been a major driving force in Algebraic Topology. Being able to package this information in the more structured category of G-spectra was a big achievement in the field, and allowed to define a closed symmetric monoidal structure under the smash product, as well as allowing to do homotopy theory in it. Still the category of G-spectra is a difficult category to work with. Even when G is the trivial group, the endomorphism ring of the unit object (the sphere spectrum) is the ring of stable homotopy groups of spheres, which is a notoriously complicated ring. Therefore trying to find an algebraic model for the full category of G-spectra seems an impossible task, and we can try to reassess the problem discarding the torsion part in the category. More precisely we can consider only G-equivariant cohomology theories that take values in graded rational vector spaces, and that are classified by rational G-spectra, and we can try to model them with a nicer abelian category where we gain understanding in the structure and computational power.

To make the problem more precise we need to introduce the framework of model categories. A model category $\mathcal{C}$ is a category with a distinguished class of morphisms called weak equivalences, and additional structure and axioms that consent the construction of another category $\mathrm{Ho}(\mathcal{C})$ called the homotopy category of $\mathcal{C}$. A map in the homotopy category $\mathrm{Ho}(\mathcal{C})$ is an isomorphism if and only if can be represented in the model category $\mathcal{C}$ by a weak equivalence. This structure tries to mimic what happens for the classical homotopy theory of topological spaces or chain complexes of R-modules for a commutative ring R. For example the category of chain complexes of R-modules with the weak equivalences being the homology-isomorphisms can carry the structure of a model category, and the homotopy category turns out to be the derived category of R exactly as we can expect. A Quillen-equivalence between two model categories $\mathcal{C}$ and $\mathcal{D}$ is a pair of adjoint functors between the two categories such that the model structure is copied from one category to the other, and it is the right notion to express when $\mathcal{C}$ and $\mathcal{D}$ have the same homotopy theory. In particular a Quillen equivalence $\mathcal{C} \simeq_Q \mathcal{D}$ implies an equivalence of the respective homotopy categories. We will present an introduction to Model categories in 2.1.

A great way to condense scope and power of algebraic models for rational G-spectra is the following conjecture of Greenlees [Gre99]:

Conjecture 1.1.4 (Greenlees)**.** *For every compact Lie group G there is a graded abelian category $\mathcal{A}(G)$ whose injective dimension equals the rank of G, and an homology functor*

$$\pi_*^{\mathcal{A}} : G\text{-Spectra}_{\mathbb{Q}} \to \mathcal{A}(G) \tag{1.1.5}$$

from rational G-spectra, equipped with an Adams spectral sequence converging for every pair of rational G-spectra X and Y:

$$\mathrm{Ext}_{\mathcal{A}}^{*,*}(\pi_*^{\mathcal{A}}(X), \pi_*^{\mathcal{A}}(Y)) \Longrightarrow [X, Y]_*^G. \tag{1.1.6}$$

Furthermore there is a Quillen-equivalence

$$G\text{-Spectra}_{\mathbb{Q}} \simeq_Q d\mathcal{A}(G) \tag{1.1.7}$$

between rational G-spectra and the category $d\mathcal{A}(G)$ of differential graded objects in $\mathcal{A}(G)$.

For example when G is the trivial group, Serre's computations of the stable homotopy groups of spheres [Ser51] combined with Morita theory [SS03] imply that the algebraic model $\mathcal{A}(G)$ may be taken to be the category of graded $\mathbb{Q}$-vector spaces, and the functor $\pi_*^{\mathcal{A}}$ the homotopy groups functor. Furthermore rational spectra are Quillen-equivalent to differential graded $\mathbb{Q}$-vector spaces.

Conjecture 1.1.4 has been proved for various groups: G finite [GM95b], the circle group $\mathbb{T}$ [Gre99] [Shi02], $SO(3)$ [Ked16], $O(2)$ [Bar17], tori of any rank $\mathbb{T}^r$ [GS18], and various classes of G-spectra. The main computational gain in $\mathcal{A}(G)$ is the Adams spectral sequence (1.1.6) that can be used to compute the values of a theory since it converges to (1.1.1). The Quillen equivalence (1.1.7) can be used to build entirely new G-equivariant cohomology theories simply constructing objects in the category $d\mathcal{A}(G)$. This is precisely the method used in [Gre05]: building an object $EC_{\mathbb{T}}$ in $d\mathcal{A}(\mathbb{T})$ and using the Quillen-equivalence (1.1.7) for the circle $G = \mathbb{T}$

to define a $\mathbb{T}$-equivariant elliptic cohomology theory. Moreover once the object $E\mathcal{C}_{\mathbb{T}}$ is built, Greenlees uses the Adams spectral sequence for the circle group, to compute (1.1.3).

1.2 Statement of results

1.2.1 Building $\mathbb{T}^2$-equivariant elliptic cohomology

In 2018 Greenlees and Shipley proved the Quillen-equivalence (1.1.7) for $G = \mathbb{T}^r$ a torus of any rank [GS18, Theorem 1.1]. It is therefore natural to try to generalize the construction of [Gre05] to higher dimensional tori. The first step of this project is to build an object $E\mathcal{C}_{\mathbb{T}^2} \in d\mathcal{A}(\mathbb{T}^2)$ representing $\mathbb{T}^2$-equivariant elliptic cohomology, which is precisely the main goal of this PhD thesis.

To be more precise exactly as in [Gre05] we start from the data of an elliptic curve $\mathcal{C}$ over the complex numbers and a coordinate $t_e \in \mathcal{O}_{\mathcal{C},e}$ in the local ring at the identity of $\mathcal{C}$, vanishing to first order at e. From this we build our object $E\mathcal{C}_{\mathbb{T}^2}$: this is the main theorem of this thesis. The construction of the object can be found in Section 4.4 while the computation on spheres is Theorem 4.5.1.

Theorem 1.2.1. *For every elliptic curve $\mathcal{C}$ over $\mathbb{C}$ and coordinate $t_e \in \mathcal{O}_{\mathcal{C},e}$, there exists an object $E\mathcal{C}_{\mathbb{T}^2} \in \mathcal{A}(\mathbb{T}^2)$ whose associated rational $\mathbb{T}^2$-equivariant cohomology theory $E\mathcal{C}^*_{\mathbb{T}^2}(\_)$ is 2-periodic. The value on the one point compactification S^V for a complex $\mathbb{T}^2$-representation V with no fixed points is given in terms of the sheaf cohomology of a line bundle $\mathcal{O}(-D_V)$ over $\mathcal{X} = \mathcal{X}_{\mathbb{T}^2} = \mathcal{C} \times \mathcal{C}$:*

$$E\mathcal{C}^n_{\mathbb{T}^2}(S^V) \cong \begin{cases} H^0(\mathcal{X}, \mathcal{O}(-D_V)) \oplus H^2(\mathcal{X}, \mathcal{O}(-D_V)) & n \text{ even} \\ H^1(\mathcal{X}, \mathcal{O}(-D_V)) & n \text{ odd.} \end{cases} \tag{1.2.2}$$

This theorem suggests the following conjecture.

Conjecture 1.2.3. *There exists an exact functor of triangulated categories $\mathbf{Sp}^{\mathbb{T}^2}_{\mathbb{Q}} \to D(\mathrm{QCoh}(\mathcal{X}))$ that sends S^V to $\mathcal{O}(-D_V)$. From this one could recover Theorem 1.2.1 by applying the cohomology functor $D(\mathrm{QCoh}(\mathcal{X})) \to \mathrm{QCoh}(\mathcal{X})_*$.*

For every complex $\mathbb{T}^2$-representation V the associated divisor D_V is defined as follows. The definition (1.1.2) for the associated variety defines for us a functor $\mathfrak{X}$ from closed subgroups of $\mathbb{T}^2$ to subvarieties of $\mathcal{X}$

$$\mathfrak{X}(H) := \mathrm{Hom}_{\mathrm{Ab}}(H^*, \mathcal{C})$$

where as before $H^* := \mathrm{Hom}(H, \mathbb{T})$ is the character group of H. Note this is an exact functor inducing an embedding $\mathfrak{X}(K) \hookrightarrow \mathfrak{X}(H)$ for every containment $K \hookrightarrow H$, and that $\mathfrak{X}(H)$ has the same dimension as H. Denote $z^{\underline{n}}$ the one dimensional complex representation of $\mathbb{T}^2$ with weight vector $\underline{n} = (n_1, n_2) \in \mathrm{Hom}(\mathbb{T}^2, \mathbb{T})$. If $V = \bigoplus_{\underline{n}} \alpha_{\underline{n}} z^{\underline{n}}$, then the associated divisor of V is defined as:

$$D_V := \sum_{\underline{n}} \alpha_{\underline{n}} \mathfrak{X}(\mathrm{Ker}(z^{\underline{n}})).$$

The $\mathbb{T}^2$-case is somehow separated from the general $\mathbb{T}^r$ case. Namely the Adams spectral sequence (1.1.6) collapses at the second page for $\mathbb{T}^2$ resulting in a neat and explicit description of the values of the theory on spheres of complex representations (1.2.2). Even if a similar description is expected to be true for higher dimensional tori, the Adams spectral sequence is not expected to collapse at the second page, and a substantial study of it may be necessary. Moreover the construction for $\mathbb{T}^2$ is complicated enough to shed some light over compatibility constraints among connected subgroups of the same codimension, not visible in the circle case (like for example the use of completed coordinates needed for Lemma 4.4.36). At the same time the situation is still simple enough to allow explicit visualization of the objects and to avoid use of combinatorics and inductive arguments necessary for higher tori, that would complicate the comprehension of the main ideas of the construction.

The point of this method is that the construction of an object in $d\mathcal{A}(\mathbb{T}^2)$ so closely corresponds to the algebra of functions over the algebraic variety $\mathcal{X} = \mathcal{X}_{\mathbb{T}^2} = \mathcal{C} \times \mathcal{C}$. The contact point is the Cousin complex (as introduced by Grothendieck [Har66, Proposition 2.3]) of the structure sheaf $\mathcal{O}_{\mathcal{X}}$. From the algebraic geometry side this Cousin complex computes the cohomology of coherent sheaves over $\mathcal{X}$, since it is a flabby resolution of $\mathcal{O}_{\mathcal{X}}$. While in $\mathcal{A}(\mathbb{T}^2)$ this Cousin complex clearly matches the terms of an injective resolution of our object $E\mathcal{C}_{\mathbb{T}^2}$, and therefore computes the values of the cohomology theory $E\mathcal{C}^*_{\mathbb{T}^2}(\_)$ via the Adams spectral sequence. As a consequence calculations of the cohomology theory $E\mathcal{C}_{\mathbb{T}^2}$ are directly reduced to the cohomology of sheaves of the algebraic variety $\mathcal{X}$.

Two main subjects can benefit from our construction of $E\mathcal{C}_{\mathbb{T}^2}$. From the algebraic models perspective, this is the first non-trivial explicit construction of an object in $d\mathcal{A}(G)$ for higher dimensional tori, that is directly built in the algebraic model and does not come from a spectrum through the homology functor $\pi^{\mathcal{A}}_*$. This is interesting since it is a first example of use of these algebraic models for higher tori as a building tool for new theories. Many steps of the construction can be replicated with different geometric inputs to potentially define new and interesting rational equivariant cohomology theories. From the elliptic cohomology perspective, our construction of $E\mathcal{C}_{\mathbb{T}^2}$ is a conventional group valued theory represented by a rational $\mathbb{T}^2$-spectrum which maintains a really close connection to the actual geometry of the curve $\mathcal{C}$. Moreover this construction can be used to enhance computations in the $\mathbb{T}$-equivariant case as we present in the final part.

1.2.2 Computing elliptic cohomology of complex projective spaces

Our construction of $E\mathcal{C}_{\mathbb{T}^2}$ opens the door to new computations also in the $\mathbb{T}$-equivariant case, since $E\mathcal{C}_{\mathbb{T}}$ built in [Gre05] and $E\mathcal{C}_{\mathbb{T}^2}$ satisfy a useful "split condition" (Theorem 1.2.5). The following is an example of a new computation that we can achieve in $\mathbb{T}$-equivariant elliptic cohomology, it is the main computational result of this thesis and to the author's knowledge the first time it appears in literature. It can be found in the Thesis as Theorem 5.0.1.

Theorem 1.2.4. *For every elliptic curve $\mathcal{C}$ over $\mathbb{C}$, if $E\mathcal{C}_{\mathbb{T}}$ is the rational $\mathbb{T}$-equivariant elliptic cohomology theory built in [Gre05], and V is a finite dimensional complex representation of $\mathbb{T}$, then:*

1. *If V has one isotypic component, $V = \alpha z^n$ with $\alpha \geq 0$:*

$$E\mathcal{C}^k_{\mathbb{T}}(\mathbb{C}P(V)) \cong \mathbb{C}^{\alpha-1}$$

for every $k \in \mathbb{Z}$.

2. *If V has more than one isotypic component, $V = \bigoplus_n \alpha_n z^n$:*

$$E\mathcal{C}^k_{\mathbb{T}}(\mathbb{C}P(V)) \cong \begin{cases} 0 & k \text{ even} \\ \mathbb{C}^d & k \text{ odd.} \end{cases}$$

where $d = \sum_{i<j} \alpha_i \alpha_j (i-j)^2$.

We denote z^n the one dimensional complex representation of $\mathbb{T}$ of weight $n \in \mathrm{Hom}(\mathbb{T}, \mathbb{T})$, while $\mathbb{C}\mathrm{P}(V)$ is the $\mathbb{T}$-space of complex lines in V.

To achieve this new computation, we will reduce it to the cohomology of spheres of complex representations in the $\mathbb{T}^2$-case. More precisely let $G = \mathbb{T}^2$, $H_1 = \{1\} \times \mathbb{T}$ and $H_2 = \mathbb{T} \times \{1\}$ the two privileged subgroups, and denote the quotient $\bar{G} := G/H_1 \cong H_2 \cong \mathbb{T}$. The cohomology theories $E\mathcal{C}_{\bar{G}}$ and $E\mathcal{C}_G$ are H_1-split:

Theorem 1.2.5. *For every elliptic curve $\mathcal{C}$ over $\mathbb{C}$ and coordinate $t_e \in \mathcal{O}_{\mathcal{C},e}$, there is a natural transformation of G-cohomology theories*

$$\varepsilon : \mathrm{Inf}_{\bar{G}}^{G} E\mathcal{C}_{\bar{G}} \longrightarrow E\mathcal{C}_G$$

which induces an isomorphism

$$[{}^{G}/{}_{H+}, \mathrm{Inf}_{\bar{G}}^{G} E\mathcal{C}_{\mathbb{T}}]_*^G \cong [{}^{G}/{}_{H+}, E\mathcal{C}_G]_*^G$$

for every subgroup H of G such that $H \cap H_1 = \{1\}$. The functor $\mathrm{Inf}_{\bar{G}}^{G} : \bar{G}\text{-Spectra} \to G\text{-Spectra}$ is the inflation functor on spectra (2.4.10).

This is the majority of the work needed to prove Theorem 1.2.4 and we will prove it in Sections 5.3 and 5.4. As an immediate consequence we obtain the following useful Corollary:

Corollary 1.2.6. *For any H_1-free G-space X:*

$$E\mathcal{C}_G^*(X) \cong E\mathcal{C}_{\bar{G}}^*(X/H_1) \tag{1.2.7}$$

We can apply this corollary to the $\bar{G}$-space $\mathbb{C}\mathrm{P}(V)$, noticing the isomorphism of $\bar{G}$-spaces:

$$\mathbb{C}\mathrm{P}(V) \cong S(V \otimes_{\mathbb{C}} w)/H_1$$

where w is the natural one dimensional complex representation of H_1 and $S(V \otimes_{\mathbb{C}} w)$ is the G-space of vectors of unit norm in the complex vector space $V \otimes_{\mathbb{C}} w$. Notice that $V \otimes_{\mathbb{C}} w$ is now a complex representation of G of the same dimension of V, and that H_1 acts freely on it. Therefore we only need to compute $E\mathcal{C}_G^*(S(V \otimes w)_+)$. The computation of this last cohomology is direct consequence of the one on spheres of complex representations (1.2.2) since we have the cofibre sequence of G-spaces:

$$S(V \otimes w)_+ \longrightarrow S^0 \longrightarrow S^{V \otimes w}, \tag{1.2.8}$$

inducing a long exact sequence in elliptic cohomology.

1.3 Structure of the Thesis

We start in Chapter 2 with a solid background in equivariant stable homotopy theory. We introduce the framework of model categories that will be our context when doing homotopy theory. We then move in defining G-equivariant orthogonal spectra that represent G-equivariant cohomology theories, and discuss change of group functors and localization. We conclude the chapter with a self-contained account of algebraic models.

Chapter 3 contains prerequisites from the algebraic geometry side. We start with the theory of sheaf Cousin complexes as introduced by Grothendieck that will be our main algebraic geometry tool in the construction of $E\mathcal{C}_{\mathbb{T}^2}$. We then define algebraic groups, abelian varieties and formal group laws. Using formal group laws we can define non-equivariant elliptic cohomology. We conclude the chapter with some facts on complex algebraic surfaces that will come at handy when computing $E\mathcal{C}_{\mathbb{T}}(\mathbb{C}\mathrm{P}(V))$.

Chapter 4 is the first chapter of original work and where we construct $E\mathcal{C}_{\mathbb{T}^2}$ starting from an elliptic curve $\mathcal{C}$ over $\mathbb{C}$ and a coordinate around the identity. After changing the topology on the abelian surface $\mathcal{X} = \mathcal{C} \times \mathcal{C}$ we consider the sheaf Cousin complex of its structure sheaf. The local cohomology modules appearing in this complex will constitute all the geometric inputs needed to build $E\mathcal{C}_{\mathbb{T}^2}$. We conclude the chapter computing the values of $E\mathcal{C}_{\mathbb{T}^2}$ on spheres of complex representations using the Adams spectral sequence of the algebraic model $\mathcal{A}(\mathbb{T}^2)$.

In Chapter 5 we present an application of our theory $E\mathcal{C}_{\mathbb{T}^2}$ by computing rational $\mathbb{T}$-equivariant elliptic cohomology of the complex projective space $\mathbb{C}\mathrm{P}(V)$ for a finite dimensional complex representation of $\mathbb{T}$. More precisely we prove that $E\mathcal{C}_{\mathbb{T}}$ built in [Gre05] and $E\mathcal{C}_{\mathbb{T}^2}$ satisfy a split condition allowing us to perform the computation with the $\mathbb{T}^2$-equivariant theory. To achieve this result we define a natural transformation of cohomology theories and prove the transformation to be an H-equivalence for certain subgroups H of $\mathbb{T}^2$.

We conclude the Thesis presenting in Chapter 6 three possible future directions. In particular we discuss some ideas on how to generalize the construction to higher dimensional tori, by building $E\mathcal{C}_{\mathbb{T}^k} \in \mathcal{A}(\mathbb{T}^k)$. We discuss the possibility to generalize the construction to more general complex abelian surfaces and not only $\mathcal{X}_{\mathbb{T}^2} = \mathcal{C} \times \mathcal{C}$. We also hint how to extend the computation of circle-equivariant elliptic cohomology of $\mathbb{C}\mathrm{P}(V)$ to Grassmanians $\mathrm{Gr}_n(V)$.

1.4 Notation and Conventions

In general G denotes the group of equivariance that we are working on, that will always be a compact Lie group. Depending on the section we will sometimes restrict our attention to $G = \mathbb{T}^r$ a torus of rank r: compact connected Lie group of rank r. In chapters 4 and 5 we fix $G = \mathbb{T}^2$ the 2-torus. By $\mathbb{T}$ we denote the circle group:

the torus of rank 1. By subgroup of a compact Lie group we always mean closed subgroup, and we generically denote them with H and K, while F denotes a finite subgroup.

The collection of connected closed codimension 1 subgroups of the 2-torus $\mathbb{T}^2$ is $\{H_i\}_{i\geq 1}$ indexed with $i \geq 1$, and with $H_1 = 1 \times \mathbb{T}$ and $H_2 = \mathbb{T} \times 1$ being the two privileged subgroups. We denote H_i^j the subgroup with j connected components and identity component H_i: we will refer to the subgroups with identity component H_i as being along the i-th direction. In general $z_i : \mathbb{T}^2 \to \mathbb{T}$ is a character of $\mathbb{T}^2$ with kernel H_i.

By representation of a compact Lie group G we always mean a finite dimensional real orthogonal representation of G, and sometimes we will restrict attention to complex representations. We denote S^V the one point compactification of a G-representation V with the added point being the basepoint, these are called representation spheres and when $V = \mathbb{R}^n$ with fixed G-action we obtain the n-th sphere S^n. We denote X_+ the G-space X with a fixed basepoint added and BG denotes the classifying space of G. For a G-space X we implement the convention to denote X also the associated suspension spectrum.

Given a module M we will denote $\overline{M}$ the 2-periodic version of M: it is a graded module with M in each even degree and zero in odd degrees. We denote elements in direct sums and products in the following way: $x = \{x_i\}_i \in \bigoplus_{i\geq 1} M_i$: this identifies the element x in the direct sum that has i-th component $x_i \in M_i$.

We will freely use the standard notation of schemes as well as the notation for sheaves from [Har66] that we recall in Section 3.1. We denote $\mathcal{K}(\mathcal{X})$ the ring of meromorphic function for the algebraic variety $\mathcal{X}$, and $\eta(C)$ the generic point of a closed set C. We denote $\mathcal{C}$ our fixed elliptic curve over $\mathbb{C}$, e is the identity of the elliptic curve and for a positive integer n: $\mathcal{C}[n]$ is the subgroup of elements of n-torsion, while $\mathcal{C}\langle n\rangle$ is the subset of elements of exact order n. We will use P to denote a point of $\mathcal{C}$ of finite order.

Whenever algebraic models are involved (chapters 4 and 5 and section 2.5), everything is rationalized without comment (Example 2.4.30): this means that all spectra are meant localized at the rational sphere spectrum and all the homology and cohomology is meant with $\mathbb{Q}$ coefficients. Tensor products $\otimes$ are meant over $\mathbb{Q}$, or over the graded ring with only $\mathbb{Q}$ in degree zero and zero elsewhere. We will freely use the standard notation for algebraic models , and we recall it in Section 2.5. In particular $\mathcal{A}(G)$ is an abelian category with graded objects and no differentials, while $d\mathcal{A}(G)$ is the category of objects of $\mathcal{A}(G)$ with differentials. Cohomology is unreduced unless indicated to the contrary with a tilde, so that $H^*(BG/H) = \tilde{H}^*(BG/H_+)$ is the unreduced cohomology ring. To ease the notation sometimes we will omit the base ring we are taking the tensor product over and denote it with an index: $\otimes_i$. This in turn means that we are considering the tensor product over the ring $\mathcal{O}_{\mathcal{F}/H_i}$ or its F-th component $H^*(BG/H_i^{n_i})$. We will make extensive use of the coordinates $H^*(BG/H_i^j) \cong \mathbb{Q}[c_{ij}]$ (2.5.17) and $H^*(BG/F) \cong \mathbb{Q}[x_A, x_B]$ (2.5.21).

Chapter 2

Prerequisites: Equivariant stable homotopy theory

2.1 Model categories

We provide a brief introduction about model categories following the appendix of [BR20].

2.1.1 The homotopy category

Definition 2.1.1. A model category $\mathcal{C}$ is a category with 3 distinguished classes of morphisms, closed under composition and all contain the identity: weak equivalences, fibrations and cofibrations. These data should satisfy 5 axioms (MC(1) - MC(5) [BR20, pag. 372]).

A model category has all small products and coproducts as well as an initial object $\emptyset$ and a final object $*$, when they are isomorphic the category is pointed. A morphism that is both a weak equivalence and a cofibration is called an acyclic cofibration, a morphism that is both a weak equivalence and a fibration is called acyclic fibration.

Definition 2.1.2. An object X in a model category is cofibrant if the only morphism $\emptyset \to X$ is a cofibration. An object X in a model category is fibrant if the only morphism $X \to *$ is a fibration.

Definition 2.1.3. For every object X in a model category there is an object CX called cofibrant replacement of X, such that CX is cofibrant and there is a weak equivalence $CX \to X$. symmetrically there is an object RX called fibrant replacement of X such that RX is fibrant and there is a weak equivalence $X \to RX$. In all our cases fibrant and cofibrant replacements are functorial.

In a model category we can formulate the notion of homotopy without using a unit interval. Namely we can define when two morphisms $f, g : X \to Y$ are homotopic [BR20, definition A.2.5], and we denote it $f \simeq g$. If X is cofibrant and Y is fibrant than being homotopic is an equivalence relation $\sim$ in $\mathcal{C}(X, Y)$. Moreover when the objects are both fibrant and cofibrant being homotopic is compatible with the composition of morphisms.

Definition 2.1.4. Let $\mathcal{C}$ be a model category, the homotopy category of $\mathcal{C}$ denoted $\mathrm{Ho}(\mathcal{C})$ is defined as follows. It has for objects the same objects as $\mathcal{C}$, and as morphisms:

$$\mathrm{Ho}(\mathcal{C})(X, Y) := \mathcal{C}(RCX, RCY)/\sim .$$

Namely the homotopy classes between the respective fibrant-cofibrant replacements. We denote $\mathrm{Ho}(\mathcal{C})(X, Y)$ by $[X, Y]$.

The most important feature of the homotopy category (and what characterize it) is the following:

Lemma 2.1.5. *A morphism $[f] \in \mathrm{Ho}(\mathcal{C})$ is an isomorphism if and only if f is a weak equivalence.*

In conclusion the homotopy category is the localization $\mathcal{C}[\mathcal{W}^{-1}]$ at the class of weak equivalences $\mathcal{W}$.

2.1.2 Quillen equivalences

Definition 2.1.6. A functor between model categories is said to be:

- A left Quillen functor if it preserves cofibrations and acyclic cofibrations.
- A right Quillen functor if it preserves fibrations and acyclic fibrations.

Moreover a pair of functors

$$F : \mathcal{C} \rightleftarrows \mathcal{D} : G$$

between model categories is a Quillen adjunction if F is a left Quillen functor and G is a right Quillen functor.

A Quillen adjunction is the right notion of morphism between model categories:

Lemma 2.1.7. *Let $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ be an adjunction, TFAE:*

- *F preserves cofibrations and G preserves fibrations.*
- *F is left Quillen.*
- *G is right Quillen.*
- *(F, G) is a Quillen adjunction.*
- *F preserves acyclic cofibrations and cofibrations between cofibrant objects.*
- *G preserves acyclic fibrations and fibrations between fibrant objects.*

A left Quillen functor takes weak equivalences between cofibrant objects to weak equivalences. A right Quillen functor takes weak equivalences between fibrant objects to weak equivalences.

Definition 2.1.8. If $F : \mathcal{C} \to \mathcal{D}$ is a left Quillen functor, define the total left derived functor

$$LF : \mathrm{Ho}(\mathcal{C}) \to \mathrm{Ho}(\mathcal{D})$$

to be $LF(X) := F(CX)$. Dually if $G : \mathcal{D} \to \mathcal{C}$ is a right Quillen functor, define the total right derived functor

$$RG : \mathrm{Ho}(\mathcal{D}) \to \mathrm{Ho}(\mathcal{C})$$

to be $RG(X) := G(RX)$.

A Quillen adjunction induces an adjunction on the respective model categories [BR20, Theorem A.4.6]:

Theorem 2.1.9. *If (F, G) is a Quillen adjunction, then the derived functors (LF, RG) form an adjunction for the respective homotopy categories.*

Definition 2.1.10. A Quillen adjunction is called a Quillen equivelence if the derived adjunction is an adjoint equivalence of the respective homotopy categories.

2.2 Closed symmetric monoidal categories

We briefly fix the notation for closed symmetric monoidal categories that will provide the natural setting for duality statements. We follow [Blu17, pag. 30].

Let $(\mathcal{C}, \wedge, S^0)$ be a symmetric monoidal category (the flip map $\tau : X \wedge Y \cong Y \wedge X$ is an isomorphism).

Definition 2.2.1. A symmetric monoidal category is closed if for every $X \in \mathcal{C}$ the functor $\_ \wedge X$ has a right adjoint $F(X, \_)$.

When this happens there is a unique functor (called internal Hom functor) on the product category

$$F(\_,\_) : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{C}$$

such that there is a natural isomorphism in all three variables:

$$\mathcal{C}(X \wedge Y, Z) \cong \mathcal{C}(X, F(Y, Z)).$$

Lemma 2.2.2. *The internal Hom functor preserves limits in the second variable and sends colimits in the first variable to limits.*

$$F(X, \varprojlim_j Y_j) \cong \varprojlim_j F(X, Y_j)$$

$$F(\underrightarrow{\operatorname{colim}}_j X_j, Y) \cong \varprojlim_j F(X_j, Y)$$

Definition 2.2.3. The evaluation map is the unit $\varepsilon : F(X,Y) \wedge X \to Y$, and the coevaluation map is the counit $\eta : X \to F(Y, X \wedge Y)$. The dual of X is $DX := F(X, S^0)$.

Remark 2.2.4. The map $\eta : X \to F(S^0, X \wedge S^0) = F(S^0, X)$ is always an isomorphism, with inverse ε.

There are various natural transformations implicit in the structure of a closed symmetric monoidal category:

1. The natural map $\wedge : F(X,Y) \wedge F(X', Y') \to F(X \wedge X', Y \wedge Y')$, whose adjoint is the composite:

$$F(X,Y) \wedge F(X',Y') \wedge X \wedge X' \to F(X,Y) \wedge X \wedge F(X',Y') \wedge X' \to Y \wedge Y'$$

2. The natural map $\nu : F(X,Y) \wedge Z \xrightarrow{\mathrm{Id} \wedge \eta} F(X,Y) \wedge F(S^0, Z) \to F(X, Y \wedge Z)$. Adjoint of the evaluation map $F(X,Y) \wedge X \wedge Z \to Y \wedge Z$.

3. The natural map $\rho : X \to DDX$. Obtained taking the adjoint of the composition $X \wedge DX \xrightarrow{\tau} DX \wedge X \xrightarrow{\varepsilon} S^0$.

4. The natural isomorphism $\mu : F(X \wedge Y, Z) \xrightarrow{\cong} F(X, F(Y,Z))$ obtained applying adjunction twice to the evaluation map $F(X \wedge Y, Z) \wedge X \wedge Y \to Z$.

Definition 2.2.5. An object X is said to be finite or strongly dualizable if there exists a "coevaluation map" $\eta' : S^0 \to X \wedge DX$ such that the following diagram commutes

$$\begin{array}{ccc} S^0 & \xrightarrow{\eta'} & X \wedge DX \\ \downarrow{\scriptstyle\eta} & & \downarrow{\scriptstyle\tau} \\ F(X,X) & \xleftarrow[\nu]{} & DX \wedge X. \end{array}$$

This implies that the map ν is an isomorphism so that we have an explicit description of $\eta' = \tau \circ \nu^{-1} \circ \eta$.

Remark 2.2.6. For a finite object X the functor $\_ \wedge DX$ is right adjoint to $\_ \wedge X$, so by uniqueness of adjoints there is a natural isomorphism $\_ \wedge DX \cong F(X, \_)$.

Proposition 2.2.7. *The following are true:*

1. *If X and X' are both finite or if X is finite and $Y = S^0$ or if X' is finite and $Y' = S^0$ then*

$$\wedge : F(X, Y) \wedge F(X', Y') \to F(X \wedge X', Y \wedge Y')$$

   is a natural isomorphism.

2. *If either X or Z is finite then*

$$\nu : F(X, Y) \wedge Z \to F(X, Y \wedge Z)$$

   is a natural isomorphism.

3. *If X is finite then $\rho : X \to DDX$ is a natural isomorphism.*

2.3 Equivariant orthogonal spectra

In this section G is a compact Lie group. We want to give an introduction to orthogonal G-spectra. There are various sources for the topic, with different definitions, different notation and different model structures. Since we will need different bits from different sources we will point out the differences in definitions and notations for our sources. For this first part we will mainly follow [HHR16] Section 2 and Appendix A. We will point out the differences in definition and notation with [MM02].

2.3.1 G-spaces

Definition 2.3.1. Let $(\mathcal{T}, \wedge, S^0)$ be the symmetric monoidal category of pointed, compactly generated weak hausdorff spaces, with the smash product of pointed spaces and unit the 0-sphere S^0. A topological category is a category enriched over $(\mathcal{T}, \wedge, S^0)$.

Definition 2.3.2. Let $(\mathcal{T}^G, \wedge, S^0)$ be the topological symmetric monoidal category of pointed spaces with a left G-action (the action should fix the basepoint) and spaces of equivariant maps as morphisms. We have an internal mapping space $\mathcal{T}_G(X, Y)$ which is simply the space of continuous maps from X to Y which is a G-space with the conjugation action (In [HHR16] is denoted $\underline{\mathcal{T}}_G$). The internal mapping space makes $\mathcal{T}^G$ into a closed symmetric monoidal category. A topological G-category, is a category enriched over $(\mathcal{T}^G, \wedge, S^0)$. We use $\mathcal{T}_G$ to denote the G-category with G-spaces of nonequivariant maps and $\mathcal{T}^G$ the topological category with equivariant maps, so that:

$$\mathcal{T}^G(X, Y) \cong (\mathcal{T}_G(X, Y))^G.$$

Definition 2.3.3. The homotopy set (group for $n > 0$) $\pi_n^H(X)$ of a pointed G-space X is defined to be the set of H-equivariant homotopy classes of pointed maps $S^n \to X$.

This is the same as the ordinary homotopy groups for the topological space $\pi_n(X^H)$ of H-fixed points.

Definition 2.3.4. A map $X \to Y$ in $\mathcal{T}^G$ is a weak equivalence if for all subgroups $H \subseteq G$ the induced map on the H-fixed points $X^H \to Y^H$ is an ordinary weak equivalence of topological spaces.

With this class of weak equivalences $\mathcal{T}^G$ carries the structure of a topological model category where a fibration is a map $X \to Y$ which is a Serre fibration on fixed points $X^H \to Y^H$ for every subgroup H. We denote $\mathrm{Ho}(\mathcal{T}^G)(X, Y)$ by $[X, Y]^G$.

We will make extensive use of representation spheres S^V, which are the one-point compactification of a representation V of G. When $V = \mathbb{R}^n$ with the trivial action we have the n-sphere S^n. Associated to S^V we have the equivariant homotopy set:

$$\pi_V^G(X) := [S^V, X]^G$$

which is a group if $\dim(V) > 0$ and an abelian group if $\dim(V^G) > 1$. We have also the equivariant suspension and the equivariant loop space:

$$\Sigma^V X := S^V \wedge X$$
$$\Omega^V X := \mathcal{T}_G(S^V, X).$$

Definition 2.3.5. A continuous G-functor $X : \mathcal{C} \to \mathcal{D}$ between topological G-categories is a functor X sucht that

$$X : \mathcal{C}(A, B) \to \mathcal{D}(X(A), X(B))$$

is a G-map of G-spaces for all pairs of objects A, B (maps in $\mathcal{T}^G$).

Definition 2.3.6. A G-natural transformation $\alpha : X \to Y$ between topological G-functors is a natural transformation of functors which consists of G-maps $\alpha : X(A) \to Y(A)$.

Let $\mathfrak{Cat}_G$ be the collection of topological G-categories (G-categories for short), which are categories enriched over $\mathcal{T}^G$, that is to say that $\mathfrak{Cat}_G(\mathcal{C}, \mathcal{D})$ is the G-category of G-functors $\mathcal{C} \to \mathcal{D}$ and left G-spaces of G-natural transformations. Denote $\mathfrak{Cat}_G(\mathcal{C}, \mathcal{D})^G$ the topological category of functors and spaces of equivariant natural transformations.

2.3.2 Equivariant orthogonal spectra

By representation of G we mean a real orthogonal representation V of G. Let $O(V)$ be the orthogonal group of non-equivariant linear isometric maps of V into itself. Given representations V and W, $O(V, W)$ is the Stiefel manifold of linear isometric embeddings of V into W with conjugation action.

Definition 2.3.7. The basic indexing category $\mathcal{J}_G$ is the topological G-category whose objects are finite dimensional real orthogonal representations of G and with G-space of morphisms the Thom complex:

$$\mathcal{J}_G(V, W) := \mathrm{Thom}(O(V, W); W - V).$$

Where $W - V$ is the orthogonal complement of V in W

As usual denote the underlying topological category with the symbol $\mathcal{J}^G$. When G is the trivial group the topological category and the G-category are the same, denote them with $\mathcal{J}$.

Remark 2.3.8. In [HHR16] and [MM02] a more calligraphic and fancy $\mathscr{J}$ is used instead of $\mathcal{J}$. It is possible to decorate this category specifying the G-universe $\mathcal{U}$ you are taking the representations from: $\mathcal{J}_G^{\mathcal{U}}$. A G-universe is a real G-inner product space which contains the trivial representation ($\mathbb{R}$ with the trivial action), and such that if it contains a finite dimensional representation, than it contains an infinite sum of copies of that representation. It is complete when it contains all irreducible representations, it is trivial if it contains only trivial representations. In Definition 2.3.7 we are implicitly using a complete G-universe $\mathcal{U}$ where all the finite dimensional representation embeds.

Definition 2.3.9. An orthogonal G-spectrum is a functor of topological G-categories:

$$X : \mathcal{J}_G \to \mathcal{T}_G.$$

The topological G-category of orthogonal G-spectra is

$$\mathbf{Sp}_G := \mathfrak{Cat}_G(\mathcal{J}_G, \mathcal{T}_G).$$

The underlying topological category of Orthogonal G-spectra is:

$$\mathbf{Sp}^G := \mathfrak{Cat}_G(\mathcal{J}_G, \mathcal{T}_G)^G$$

When G is the trivial group the category of Orthogonal spectra is:

$$\mathbf{Sp} := \mathfrak{Cat}_G(\mathcal{J}, \mathcal{T}).$$

In [HHR16] the letter $\mathcal{S}$ is used instead of $\mathbf{Sp}$.

Notation 2.3.10. We will sometimes use X_V for the value $X(V)$ of an orthogonal G-spectrum at a representation V.

There is a lot of freedom in deciding which G-universe consider for the indexing category, and we can use $\mathbf{Sp}_G^{\mathcal{U}}$ to specify $\mathcal{J}_G^{\mathcal{U}}$ as indexing category. We will always assume a complete G-universe without indication for most of this thesis, but to define certain functors (for example inflation) changing universe is essential. Luckily for us our objects are determined at the level of trivial representations, more precisely.

Remark 2.3.11. If V and W are two representations of the same dimension, then

$$O(V, W)_+ \wedge_{O(V)} X_V \xrightarrow{\cong} X_W$$

is a G-equivariant homeomorphism. In particular this gives a way to extend the values of a G-spectrum indexed on the trivial G-universe $\mathcal{J}$ to all G-representations.

Definition 2.3.12. Let $\mathcal{U}$ and $\mathcal{U}'$ be 2 G-universes, define the change of universe functor

$$I_{\mathcal{U}}^{\mathcal{U}'} \mathbf{Sp}_G^{\mathcal{U}} \to \mathbf{Sp}_G^{\mathcal{U}'}$$

extending the values for every representation $V \subset \mathcal{U}'$ of dimension n:

$$X_V := O(\mathbb{R}^n, V)_+ \wedge_{O(\mathbb{R}^n)} X_{\mathbb{R}^n}$$

The following Lemma is due to Mandell-May [MM02, Lemma V.1.5]:

Lemma 2.3.13. *The functors $I_{\mathcal{U}}^{\mathcal{U}'}$ and $I_{\mathcal{U}'}^{\mathcal{U}}$ are inverse equivalences of categories.*

Definition 2.3.14. Given a G-spectrum X and a G-space K the suspension spectrum and the 0-space functors are defined by:

$$(\Sigma^\infty K)_V := S^V \wedge K$$
$$\Omega^\infty X := X_{\{0\}}$$

Where $\{0\}$ is the zero vector space.

The suspension spectrum functor is left adjoint to the 0-space functor. The functors Σ^∞ and Ω^∞ are topological functors between $\mathcal{T}^G$ and $\mathbf{Sp}^G$ or $\mathcal{T}^G$-enriched functors between $\mathcal{T}_G$ and $\mathbf{Sp}_G$.

Notation 2.3.15. We will denote $\Sigma^\infty K$ simply with K.

Definition 2.3.16. For every G-representation V there is an orthogonal G-spectrum S^{-V} characterized by the functorial isomorphism of G-spaces:

$$\mathbf{Sp}_G(S^{-V}, X) \cong X_V$$

We now compare the different approaches to the definition of orthogonal G-spectra from the various sources we are going to use.

Remark 2.3.17 (Comparison with $\mathcal{I}_G$-spaces)**.** In [MM02] a slightly different approach is taken in the definition of orthogonal spectra. They define the topological G-category $\mathcal{I}_G$ on the same objects of $\mathcal{J}_G$: finite dimensional real orthogonal representations of G, but with morphisms simply the G-space $O(V, W)$. They then define an $\mathcal{I}_G$-space to be a continuous G-functor

$$X : \mathcal{I}_G \to \mathcal{T}_G$$

and an orthogonal G-spectrum to be an $\mathcal{I}_G$-space with additional structure maps

$$S^V \wedge X(W) \to X(V \oplus W).$$

The category $\mathcal{J}_G$ encodes already the data of the structure maps. They call these $\mathcal{I}_G$-spectra. The Topological category of $\mathcal{I}_G$-spectra and non-equivariant natural transformations is denoted in [MM02] with $\mathcal{I}_G\mathcal{S}$. The topological category of $\mathcal{I}_G$-spectra and equivariant natural transformations is denoted $G\mathcal{I}\mathcal{S}$.

Remark 2.3.18 (Comparison with G-prespectra)**.** A G-prespectrum X as defined in [MM02] and [Blu17] is simply the data of a G-space $X(V)$ for every V in our universe, and associative structure G-maps:

$$\sigma : S^V \wedge X(W) \to X(V \oplus W)$$

i.e. you don't require functoriality in V. Arrows of G-prespectra are simply based maps $f(V) : X(V) \to Y(V)$ that commute with the structure maps, this gives us the G-category of G-prespectra, and the topological category of G-prespectra if we consider equivariant maps. When the adjoints of the structure maps

$$\tilde{\sigma} : X(V) \to \Omega^W X(V \oplus W)$$

are homeomoprphisms of G-spaces those are called Ω-prespectra in [Blu17] and G-spectra in [MM02] (as opposite to othogonal). All our orthogonal G-spectra are G-prespectra by forgetting the functoriality, so we have a forgetful functor. The definition of homotopy groups for G-prespectra is straightforward [Blu17, pag. 42]. Moreover G-prespectra admits a model structure with weak equivalences the π_*-isomorphisms, its homotopy category is the Equivariant stable homotopy category.

2.3.3 The smash product

In this subsection we mainly follow [MM02, Chapter II].

The symmetric monoidal structures of $\mathcal{J}_G$ and $\mathcal{T}_G$ combine to give $\mathbf{Sp}^G$ a symmetric monoidal structure (the day convolution) denoted $\wedge$, with unit the sphere spectrum S^0. This construction works in general for diagram spaces [Man+01].

Definition 2.3.19. The smash product of two orthogonal G-spectra X and Y is defined to be the left Kan extension of the external smash product

$$\begin{aligned} \bar{\wedge} : \mathcal{J}_G \times \mathcal{J}_G &\to \mathcal{T}_G \\ (V, W) &\mapsto X_V \wedge Y_W \end{aligned}$$

Along the direct sum map $\oplus : \mathcal{J}_G \times \mathcal{J}_G \to \mathcal{J}_G$.

Remark 2.3.20. It is characterized (universal property) by the isomorphism of G-spaces

$$\mathbf{Sp}_G(X \wedge Y, Z) \cong (X \bar{\wedge} Y, Z \circ \oplus)$$

where on the right we have the G-space of natural transformations of functors $\mathcal{J}_G \times \mathcal{J}_G \to \mathcal{T}_G$.

We can explicitly describe the smash product on a G-representation V by

$$(X \wedge Y)_V = \underset{W \oplus W' \to V}{\underrightarrow{\operatorname{colim}}} X_W \wedge Y_{W'}$$

where the colimit is indexed over the over-category whose objects are maps $W \oplus W' \to V$ and whose morphisms are pairs of maps making the appropriate diagram commutes.

Definition 2.3.21. The internal function spectrum of two orthogonal G-spectra X and Y is defined on a G-representation V to be

$$\begin{aligned} F(X,Y) : \mathcal{J}_G &\to \mathcal{T}_G \\ V &\mapsto \mathbf{Sp}_G(X, Y(V \oplus \_)). \end{aligned}$$

Namely the G-space of continous natural transformations between X and $Y(V \oplus \_)$.

Remark 2.3.22. If Z is a functor $\mathcal{J}_G \times \mathcal{J}_G \to \mathcal{T}_G$, while X and Y are functors $\mathcal{J}_G \to \mathcal{T}_G$ then we have an isomorphism of G-spaces

$$(X \bar{\wedge} Y, Z) \cong (X, \bar{F}(Y, Z))$$

where on the left we are considering the G-space of natural transformations of functors from the product $\mathcal{J}_G \times \mathcal{J}_G$, and on the right from $\mathcal{J}_G$. This gives immediately the desired adjunction with the internal smash and internal function spectra. The external function spectrum is defined

$$\begin{aligned} \bar{F}(Y,Z) : \mathcal{J}_G &\to \mathcal{T}_G \\ V &\mapsto \mathbf{Sp}_G(Y, Z(V, \_)). \end{aligned}$$

These 2 definitions endow the category of orthogonal G-spectra with the structure of a closed symmetric monoidal category [MM02, Theroem 3.1]:

Lemma 2.3.23. *The categories $\mathbf{Sp}^G$ and $\mathbf{Sp}_G$ are both closed symmetric monoidal categories under the smash product $\wedge$ of orthogonal G-spectra and unit object the sphere spectrum S^0. The internal Hom functor is the internal function spectrum F.*

Remark 2.3.24. Both $\mathbf{Sp}_G$ and $\mathbf{Sp}^G$ are tensored and cotensored over G-spaces. For a G-space K and a G-spectrum X we have

$$(X \wedge K)_V = X_V \wedge K$$
$$F(K, X)_V = F(K, X_V).$$

So that smash product and function spectrum with suspension spectra can be computed level-wise with smash and function space of G-spaces. The suspension spectrum Σ^∞ is symmetric monoidal.

2.3.4 Model Structures

We can put a model structure on the category of G-spectra For a representation V and $k \in \mathbb{Z}$ we write $V > k$ when $\dim(V^G) > k + 1$. The following is [HHR16, Definition 2.14]:

Definition 2.3.25. For a G-spectrum X and $k \in \mathbb{Z}$, define for every subgroup $H \subseteq G$ the H-equivariant k-th stable homotopy group of X as:

$$\pi_k^H(X) := \underset{V>-k}{\underrightarrow{\operatorname{colim}}}\, \pi_{V+k}^H(X_V)$$

where the colimit is taken over the partially ordered set of orthogonal G-representations V satisfying $V > -k$.

Remark 2.3.26. An increasing sequence $\dots \subset V_n \subset V_{n+1} \subset \dots$ of finite dimensional representations of G is exhausting if any finite dimensional representation V of G admits an equivariant embedding in some V_n. We can use any exhausting sequence to compute the stable homotopy groups:

$$\pi_k^H(X) = \underset{n}{\underrightarrow{\operatorname{colim}}}\, \pi_{V_n+k}^H(X_{V_n})$$

Definition 2.3.27. A stable weak equivalence is a map $X \to Y$ in $\mathbf{Sp}^G$ inducing an isomorphism of stable homotopy groups π_k^H for all $k \in \mathbb{Z}$ and subgroups $H \subseteq G$ (they are called $\underline{\pi}_*$-isomorphisms in [MM02]).

We can define a model structure on $\mathbf{Sp}^G$ where the weak equivalences are precisely the stable weak equivalences [HHR16, Proposition B.63]:

Proposition 2.3.28. *The category $\mathbf{Sp}^G$ equipped with the stable weak equivalences, the positive complete cofibrations and positive complete fibrations forms a cofibrantly generated model category.*

We call this the positive complete model structure on orthogonal G-spectra. This defines for us (Definition 2.1.4) the homotopy category of G-spectra, inverting precisely the stable weak equivalences. As for G-spaces we use $[X,Y]^G$ for $\mathrm{Ho}(\mathbf{Sp}^G)(X,Y)$.

There are various model structures on G-spectra depending on someone's scope and objectives. We point out other 2 model structures in [MM02]. Recall that in [MM02] the category $\mathbf{Sp}^G$ is denoted $G\mathcal{IS}$ and indexed on a complete G-universe $\mathcal{U}$. We have the Level model structure [MM02, Theorem 2.4] with class of weak equivalences the level equivalences of G-spectra. There is also the stable model structure [MM02, Theorem 4.2 and 7.5] with weak equivalences the stable weak equivalences.

2.3.5 Homology and Cohomology

A G-spectrum $E \in \mathbf{Sp}^G$ defines a $\mathbb{Z}$-graded G-equivariant cohomology theory:

$$E_G^k(X) := [S^{-k} \wedge X, E]^G = [X, E]_k^G = [S^{-k}, F(X,E)]^G = \pi_{-k}^G(F(X,E)).$$

As well as a $\mathbb{Z}$-graded homology theory:

$$E_k^G(X) := [S^k, X \wedge E]^G = \pi_k^G(X \wedge E).$$

The $\mathrm{RO}(G)$-graded versions are defined by:

$$\begin{aligned} E_G^V(X) &:= [S^{-V} \wedge X, E]^G = [S^{-V}, F(X,E)]^G = \pi_{-V}^G(F(X,E)), \\ E_V^G(X) &:= [S^V, X \wedge E]^G = \pi_V^G(E \wedge X), \end{aligned}$$

for $V = V_0 - V_1 \in \mathrm{RO}(G)$ a virtual representation of G, that is a formal difference of isomorphism classes of representations.

2.4 Change of group functors

For this section we follow [MM02, Chapter V].

2.4.1 Restriction, induction, coinduction

Let $i_H : H \hookrightarrow G$ be a subgroup of G, the restriction functor $i_H^* : \mathcal{T}^G \to \mathcal{T}^H$ has

1. A continuous left adjoint called induction. Sending an H-space Y:

$$Y \mapsto G_+ \wedge_H Y$$

where in the wedge we quotient the H-action to obtain an induced G-action.

2. A continuous right adjoint called coinduction. Sending an H-space Y:

$$Y \mapsto \mathcal{T}^H(G_+, Y)$$

where the H-equivariant maps have the coinduced G-action.

We can extend these functors to orthogonal G and H spectra, maintaining the adjunctions.

Definition 2.4.1. For X a G-spectrum, Y an H-spectrum, V a G-representation define respectively restriction, induction and coinduction as:

$$\begin{aligned} (i_H^* X)(i_H^* V) &:= i_H^*(X_V) \\ (G_+ \wedge_H Y)_V &:= G_+ \wedge_H (Y_V) \\ F_H(G_+, Y)_V &:= \mathcal{T}^H(G_+, Y_V) \end{aligned} \tag{2.4.2}$$

Remark 2.4.3. They all preserve stable weak equivalences so they all induce functors between the respective homotopy categories. They all commute with suspension spectra. Restriction is strong symmetric monoidal.

Lemma 2.4.4. *There are Quillen-adjunctions:*

$$\begin{aligned} \mathbf{Sp}^G(G_+ \wedge Y, X) &\cong \mathbf{Sp}^H(Y, i_H^* X) \\ \mathbf{Sp}^G(X, F_H(G_+, Y)) &\cong \mathbf{Sp}^H(i_H^* X, Y) \end{aligned} \tag{2.4.5}$$

relating level model structures and stable model structures.

We have two important natural isomorphisms:

$$\begin{aligned} {}^{G}/_{H+} \wedge X &\cong G_+ \wedge i_H^* X \\ F({}^{G}/_{H+}, X) &\cong F_H(G_+, i_H^* X). \end{aligned} \tag{2.4.6}$$

2.4.2 Inflation, Categorical fixed points, orbits

Let $i_N : N \hookrightarrow G$ be a normal subgroup of G with quotient $\varepsilon : G \twoheadrightarrow G/N = Q$. The inflation functor on spaces $\mathrm{Inf}_Q^G = \varepsilon^* : \mathcal{T}^Q \to \mathcal{T}^G$ has the N-fixed points as right adjoint and the N-orbits as a left adjoint. Let us extend these definitions and adjunctions to spectra.

Let $\mathcal{U}$ be our complete G-universe and $\mathcal{U}^N$ the N-fixed sub-universe that is also called the G-universe of N-trivial G-representations. Notice that we change the universe to $\mathcal{U}^N$ to allow the use of different model structures, and that $\mathcal{U}^N$ is also a complete Q-universe. The following results [MM02, pp. V.1.7, V.1.8] are fundamental at this point.

Lemma 2.4.7. *For G-universes $\mathcal{U} \subset \mathcal{U}'$ there is a $\mathcal{U}$-stable model structure on $\mathbf{Sp}^G_{\mathcal{U}'}$ in which the functor creates the $\mathcal{U}$-stable weak equivalences and the $\mathcal{U}$-fibrations. Moreover the pair $(I^{\mathcal{U}'}_{\mathcal{U}}, I^{\mathcal{U}}_{\mathcal{U}'})$ is a Quillen equivalence between $\mathbf{Sp}^G_{\mathcal{U}}$ with the stable model structure and $\mathbf{Sp}^G_{\mathcal{U}'}$ with the $\mathcal{U}$-stable model structure. The $\mathcal{U}$-stable weak equivalences are those maps that induces isomorphisms on the stable homotopy groups defined using only representations $V \subset \mathcal{U}$.*

Corollary 2.4.8. *For $\mathcal{U} \subset \mathcal{U}'$ the identity functor of $\mathbf{Sp}^G_{\mathcal{U}'}$ is the right Quillen functor of a Quillen adjunction relating the stable model structure and the $\mathcal{U}$-stable model structure.*

We can apply these results to $\mathcal{U}^N \subset \mathcal{U}$. For a Q-representation V let $\varepsilon^* V$ be V regarded as an N-trivial G-representation.

Definition 2.4.9. For a Q-spectrum Y, define $\varepsilon^* Y \in \mathbf{Sp}^G_{\mathcal{U}^N}$ to be the G-spectrum indexed on $V \subset \mathcal{U}^N$:

$$(\varepsilon^* Y)_V = \varepsilon^*(Y_V).$$

Define the inflation functor $\mathrm{Inf}^G_Q : \mathbf{Sp}^Q_{\mathcal{U}^N} \to \mathbf{Sp}^G_{\mathcal{U}}$ simply post-composing with the change of universe functor:

$$\mathrm{Inf}^G_Q(Y) := I^{\mathcal{U}}_{\mathcal{U}^N} \circ \varepsilon^*(Y) \tag{2.4.10}$$

Remark 2.4.11. Inflation is strong symmetric monoidal and commutes with suspension spectra.

Definition 2.4.12. For a G-spectrum $X \in \mathbf{Sp}^G_{\mathcal{U}^N}$ indexed on N-trivial representations, define the Q-spectrum X^N by passage to N-fixed points level-wise:

$$(X^N)_V = (X_V)^N.$$

Define the categorical fixed point functor $\Psi^N : \mathbf{Sp}^G_{\mathcal{U}} \to \mathbf{Sp}^Q_{\mathcal{U}^N}$ simply pre-composing with the change of universe functor. For a a generic G-spectrum X indexed on all G-representations:

$$\Psi^N(X) := (I^{\mathcal{U}^N}_{\mathcal{U}} X)^N.$$

We denote $\Psi^N(X)$ by X^N also for a generic spectrum X.

Remark 2.4.13. The categorical fixed points functor is a right Quillen functor but does not preserve stable weak equivalences in general, and therefore needs to be right derived in the homotopy category. The reason for this is that it does not commute with fibrant replacement.

Remark 2.4.14. Some sources define categorical fixed points also when N is not normal in G as $X^N := (i_N^* X)^N$ which is the underlying non-equivariant spectrum of our definition of a Q-spectrum.

Remark 2.4.15. Categorical fixed points are not strong symmetric monoidal, for example:

$$(S^0)^G \wedge (S^0)^G \not\simeq (S^0 \wedge S^0)^G$$

but they are lax symmetric monoidal: there is a natural map:

$$X^N \wedge Y^N \to (X \wedge Y)^N$$

Inflation and categorical fixed points are still adjoint [MM02, Proposition 3.10]:

Proposition 2.4.16. *There is a Quillen adjunction*

$$\mathbf{Sp}^G(\mathrm{Inf}_Q^G Y, X) \cong \mathbf{Sp}^Q(Y, X^N)$$

relating the respective level and stable model structures. The spectrum X is indexed on a complete G-universe.

Corollary 2.4.17. *For any orthogonal G-spectrum X:*

$$\pi_*^H(X) \cong [{}^G/_{H+} \wedge S^0, X]^G \cong [S^0, i_H^* X]^H \cong [S^0, X^H] = \pi_*(X^H)$$

Definition 2.4.18. For a G-spectrum $X \in \mathbf{Sp}^G_{\mathcal{U}^N}$ indexed on N-trivial representations V, define the orbit Q-spectrum X/N quotienting by N level-wise:

$$({}^X/_N)_V = {}^{X_V}/_N$$

The inflation-orbit adjunction is maintained only for N-trivial G-representations [MM02, Proposition 3.12]

Proposition 2.4.19. *There is a Quillen adjunction*

$$\mathbf{Sp}^G_{\mathcal{U}^N}(X, \varepsilon^* Y) \cong \mathbf{Sp}^Q(X/N, Y) \tag{2.4.20}$$

relating the respective level and stable model structures.

Remark 2.4.21. The orbit functor is a left adjoint while change of universe to N-trivial representations is a right adjoint, so the composition is of no practical use and the adjunction is relevant only on G-spectra indexed on N-trivial G-representations.

2.4.3 Localization of spectra

We follow [MM02, p. IV.6]

Definition 2.4.22. A map $f : X \to Y$ of orthogonal G-spectra is called an H-equivalence if the induced map on the restricted spectra

$$\iota_H^*(f) : \iota_H^* X \to \iota_H^* Y$$

is a stable weak equivalence of orthogonal H-spectra.

Notation 2.4.23. When H is the trivial subgroup we use the terms 1-equivalence or non-equivariant equivalence, and denote it $X \simeq_1 Y$.

Let $\mathcal{F}$ be a family of subgroups of G, and $E\mathcal{F}$ be the universal $\mathcal{F}$-space. It is a G-CW complex characterized up to weak equivalence of G-spaces by the weak equivalences of topological spaces:

$$(E\mathcal{F}_+)^H \simeq \begin{cases} S^0 & \text{if } H \in \mathcal{F} \\ * & \text{if } H \notin \mathcal{F} \end{cases}$$

such a pointed G-CW complex can be build using cells of the form ${}^{G}/_{H+} \wedge D^n_+$.

Let $\tilde{E}\mathcal{F}$ be the mapping cone of the map $E\mathcal{F}_+ \to S^0$ quotienting $E\mathcal{F}$ to a point. The pointed G-CW complex $\tilde{E}\mathcal{F}$ is characterized up to weak equivalence of G-spaces by the weak equivalences of topological spaces:

$$(\tilde{E}\mathcal{F})^H \simeq \begin{cases} * & \text{if } H \in \mathcal{F} \\ S^0 & \text{if } H \notin \mathcal{F} \end{cases}$$

Definition 2.4.24. The isotropy separation cofibre sequence for $\mathcal{F}$ is the cofiber sequence of G-spaces:

$$E\mathcal{F}_+ \to S^0 \to \tilde{E}\mathcal{F}$$

that induces an isotropy separation cofibre sequence for any orthogonal G-spectrum X:

$$X \wedge E\mathcal{F}_+ \to X \to X \wedge \tilde{E}\mathcal{F}. \tag{2.4.25}$$

Definition 2.4.26. A map $f : X \to Y$ is an $\mathcal{F}$-equivalence if it is an H-equivalence for every $H \in \mathcal{F}$. An orthogonal G-spectrum X is said to be an $\mathcal{F}$-object if the first map of its isotropy separation cofibre sequence $X \wedge E\mathcal{F}_+ \to X$ is a stable equivalence.

Definition 2.4.27. Let E be a cofibrant spectrum or a cofibrant based G-space.

- A map $X \to Y$ is an E-equivalence if $E \wedge X \to E \wedge Y$ is a weak equivalence.
- An object Z is E-local if for all E-equivalences $f : X \to Y$ the map $f^* : [Y, Z]^G \to [X, Z]^G$ is an isomorphism.
- An E-localisation of X is an E-equivalence to an E-local object.
- An object X is E-acyclic if the map from the zero object $* \to X$ is an E-equivalence.

Remark 2.4.28. An E-equivalence between E-local objects is a stable weak equivalence.

The following is [MM02, Theorem 6.3]:

Theorem 2.4.29. *Let E be a cofibrant spectrum or a cofibrant based G-space. The category of orthogonal G-spectra $\mathbf{Sp}^G$ admits an E-model structure with:*

- *weak equivalences the E-equivalences,*
- *Same cofibrations,*
- *E-fibrant objects the fibrant objects of $\mathbf{Sp}^G$ that are E-local.*

Moreover the E-fibrant approximation $X \mapsto L_E X$ constructs a Bousfield localization of X at E. The notation for this E-model structure is $L_E\mathbf{Sp}^G$.

The map $X \to F(E\mathcal{F}_+, X)$ induced by $E\mathcal{F}_+ \to S^0$ is an $E\mathcal{F}_+$-localization of X.

Example 2.4.30. The most important example to us, is when $E = S^0\mathbb{Q}$ is the rational sphere spectrum [Bar08, p. 1.5.2]. We call the E-model structure rational G-spectra and denote it by $\mathbf{Sp}^G_{\mathbb{Q}}$, we call the E-equivalences rational equivalences.

Notation 2.4.31. In chapters 4 and 5 and section 2.5 we will always assume that everything is rationalized without comment. This means that all spectra are meant localized at $S^0\mathbb{Q}$, the weak equivalences are the rational equivalences, and so on.

2.4.4 Geometric fixed points

Let N be a normal subgroup of G with quotient Q, and let $\mathcal{F} = [N \not\subseteq]$ be the family of subgroups that do not contain N (For $N = G$ this is the family of proper subgroups).

Definition 2.4.32. Define the geometric fixed point functor $\Phi^N : \mathbf{Sp}^G \to \mathbf{Sp}^Q$ on an orthogonal G-spectrum X:

$$\Phi^N X := (R(\tilde{E}\mathcal{F} \wedge X))^N$$

where R is the functorial fibrant replacement.

In [MM02, Definition V.4.3] a more intuitive definition in terms of N-fixed points spaces for certain extended category of $\mathcal{J}_Q$ is used. The definition we just gave is equivalent to that one [MM02, Proposition V.4.17].

Proposition 2.4.33. *The functor Φ^N sends stable weak equivalences to stable weak equivalences and commutes with filtered homotopy colimits. Moreover for a based G-space K and G-spectra X and Y:*

- $\Phi^N(\Sigma^\infty K) \cong \Sigma^\infty(K^N)$,
- $\Phi^N(X \wedge Y) \simeq \Phi^N X \wedge \Phi^N Y$.

2.5 Algebraic models

In this section we present a self-contained account of algebraic models for tori of any rank, therefore in all this section $G = \mathbb{T}^r$ is an r-dimensional torus, with $r \geq 0$. We also specify that all the modules over graded rings are graded and all the maps between graded modules are graded maps. Here as in Chapters 4 and 5, everything is rationalized without comment.

Algebraic models are a useful tool to study rational equivariant cohomology theories. The main idea is to define an abelian category $\mathcal{A}(G)$ and homology functor from the category of rational G-equivariant orthogonal spectra (Example 2.4.30):

$$\pi_*^{\mathcal{A}} : \mathbf{Sp}_{\mathbb{Q}}^G \to \mathcal{A}(G) \tag{2.5.1}$$

equipped with an Adams spectral sequence to compute maps in the homotopy category of rational G-spectra. More precisely the values of the theory may be calculated by a spectral sequence:

$$\mathrm{Ext}^{*,*}_{\mathcal{A}}(\pi^{\mathcal{A}}_*(X), \pi^{\mathcal{A}}_*(Y)) \Longrightarrow [X, Y]^G_*. \tag{2.5.2}$$

In the case of tori we have a zig-zag of Quillen-equivalences [GS18, Theorem 1.1]):

$$\mathbf{Sp}^G_{\mathbb{Q}} \simeq_Q d\mathcal{A}(G), \tag{2.5.3}$$

where $d\mathcal{A}(G)$ is the model category of differential graded objects in $\mathcal{A}(G)$. Therefore we can build rational G-equivariant cohomology theories simply constructing objects in $d\mathcal{A}(G)$.

2.5.1 Definition of the rings

We start by defining the rings needed for the construction of $\mathcal{A}(G)$ [Gre08, Section 3.A.]. We write $\mathcal{F}$ for the family of finite subgroups of G.

Definition 2.5.4. For every connected subgroup H of G define the collection:

$$\mathcal{F}/H := \{\tilde{H} \leq G \mid H \text{ finite index in } \tilde{H}\}$$

and the ring:

$$\mathcal{O}_{\mathcal{F}/H} := \prod_{\tilde{H} \in \mathcal{F}/H} H^*\left(B\left(G/\tilde{H}\right)\right) \tag{2.5.5}$$

Remark 2.5.6. Note $\mathcal{O}_{\mathcal{F}/G} = \mathbb{Q}$ and $\mathcal{O}_{\mathcal{F}/1} = \mathcal{O}_{\mathcal{F}}$.

Any containment of connected subgroups $K \subseteq H$ induces an inflation map $\mathcal{O}_{\mathcal{F}/H} \to \mathcal{O}_{\mathcal{F}/K}$, defined in the following way.

Definition 2.5.7. The inclusion $K \subseteq H$ of connected subgroups defines a quotient map $q : G/K \to G/H$, and hence

$$\begin{aligned} q_* : \mathcal{F}/K &\to \mathcal{F}/H \\ \tilde{K} &\mapsto \langle H, \tilde{K} \rangle \end{aligned} \tag{2.5.8}$$

For any $\tilde{K} \in \mathcal{F}/K$ define the $\tilde{K}$-th component of the inflation map $\mathcal{O}_{\mathcal{F}/H} \to \mathcal{O}_{\mathcal{F}/K}$ to be the composition:

$$\mathcal{O}_{\mathcal{F}/H} = \prod_{\tilde{H} \in \mathcal{F}/H} H^*(BG/\tilde{H}) \to H^*(BG/q_*\tilde{K}) \to H^*(BG/\tilde{K}) \tag{2.5.9}$$

given by projection onto the term $H^*(BG/q_*\tilde{K})$ followed by the inflation map induced by the quotient ${G}/_{\tilde{K}} \to {G}/_{q_*\tilde{K}}$.

Remark 2.5.10. In particular for any connected subgroup H we have an inflation map induced by the inclusion of the trivial subgroup:

$$i_H : \mathcal{O}_{\mathcal{F}/H} \to \mathcal{O}_{\mathcal{F}} \tag{2.5.11}$$

which is a split monomorphism of $\mathcal{O}_{\mathcal{F}/H}$-modules [Gre12, Proposition 3.1]. As a consequence $\mathcal{O}_{\mathcal{F}}$ is an $\mathcal{O}_{\mathcal{F}/H}$-module for every connected subgroup H.

2.5.2 Euler classes

Fundamental elements of these rings are Euler classes of representations of G, used in the localization process. For any complex representation V of G we want to define its Euler class $e(V) \in \mathcal{O}_{\mathcal{F}}$ [Gre08, Section 3.B.]. We require them to be multiplicative: $e(V \oplus W) = e(V)e(W)$, therefore it's enough to define Euler classes for one dimensional complex representations V.

Definition 2.5.12. For a one dimensional complex representation V of G, define its Euler class $e(V) \in \mathcal{O}_{\mathcal{F}}$ as follows. For every finite subgroup F the F-th component $e(V)_F \in H^*(BG/F)$ is:

$$e(V)_F = \begin{cases} 1 & \text{if } V^F = 0 \\ \bar{e}(V^F) & \text{if } V^F \neq 0, \end{cases} \tag{2.5.13}$$

where $\bar{e}(V^F) \in H^2(BG/F)$ is the classical equivariant Euler class for the G/F representation V^F.

Definition 2.5.14. For any connected subgroup H of G define the multiplicatively closed subset of $\mathcal{O}_{\mathcal{F}}$:

$$\mathcal{E}_H := \{e(V) \mid V^H = 0\}. \tag{2.5.15}$$

We would like now to localize $\mathcal{O}_{\mathcal{F}}$ at the multiplicatively closed subset $\mathcal{E}_H$, but the problem is that we would invert also non-homogeneous elements. To sanitize this let $\mathcal{K}_{G/F}$ be the ring obtained from $H^*(BG/F)$ by inverting all nonzero elements in degree 2.

Definition 2.5.16. Define $\mathcal{E}_H^{-1}\mathcal{O}_{\mathcal{F}}$ to be the following subring of homogeneous elements of $\prod_F \mathcal{K}_{G/F}$:

$$\mathcal{E}_H^{-1}\mathcal{O}_{\mathcal{F}} := \{u \in \prod_F \mathcal{K}_{G/F} \mid \exists V, V^H = 0 \text{ and } e(V^F)u_F \in H^*(BG/F) \forall F \text{ finite } \}$$

2.5.3 The 2-torus

We are mainly interested in the case of the 2-torus, therefore let us compute explicitly rings and Euler classes in this case. Recall that $\{H_i\}_{i\geq 1}$ is the collection of connected closed codimension 1 subgroups of $\mathbb{T}^2$ and H_i^j is the subgroup with j-components and identity component H_i. Exactly as in Definition 4.1.9 and (4.1.7) z_i is a character of $\mathbb{T}^2$ with kernel H_i and z_i^j is a character of $\mathbb{T}^2$ with kernel H_i^j. In this case:

- $\mathcal{O}_{\mathcal{F}/\mathbb{T}^2} = \mathbb{Q}$
- $\mathcal{O}_{\mathcal{F}/H_i} = \prod_{j\geq 1} H^*(B\mathbb{T}^2/H_i^j)$
- $\mathcal{O}_{\mathcal{F}} = \prod_F H^*(B\mathbb{T}^2/F)$, where F runs through all the finite subgroups of $\mathbb{T}^2$.

For every $i, j \geq 1$:

$$\begin{aligned} H^*(B\mathbb{T}^2/H_i) &\cong \mathbb{Q}[c_i] \\ H^*(B\mathbb{T}^2/H_i^j) &\cong \mathbb{Q}[c_{ij}] \end{aligned} \tag{2.5.17}$$

where $c_i = e(z_i)$ and $c_{ij} = e(z_i^j)$ both of degree -2 are the Euler classes of the characters z_i and z_i^j (more precisely of the one dimensional complex representations defined by those characters).

Definition 2.5.18. For every finite subgroup F of $\mathbb{T}^2$ and every index $i \geq 1$ define $n_i = n_i(F)$ to be the only positive integer such that $H_i^{n_i}$ is generated by H_i and F: $\langle F, H_i\rangle = H_i^{n_i}$.

Every finite subgroup F can be written as the intersection of two codimension one subgroups of $\mathbb{T}^2$. Therefore for every F there exists two different integers $A = A(F) \geq 1$ and $B = B(F) \geq 1$ such that

$$F = H_A^{n_A} \cap H_B^{n_B}. \tag{2.5.19}$$

Choice 2.5.20. For any finite subgroup F we choose a pair of positive integers (A, B) that give the decomposition (2.5.19).

By (2.5.19) we obtain the decomposition:

$$H^*(B\mathbb{T}^2/F) \cong H^*(B\mathbb{T}^2/H_A^{n_A}) \otimes H^*(B\mathbb{T}^2/H_B^{n_B}) \cong \mathbb{Q}[x_A, x_B]. \tag{2.5.21}$$

Where $x_A := e(z_A^{n_A})$, $x_B := e(z_B^{n_B})$ have both degree -2 are the Euler classes respectively of $z_A^{n_A}$ and $z_B^{n_B}$.

Definition 2.5.22. For every $i \geq 1$ define

$$x_i := e(z_i^{n_i}) \in H^2(B\mathbb{T}^2/F). \tag{2.5.23}$$

Notice it is an integral linear combination of x_A and x_B.

Remark 2.5.24. With these choices of coordinates (2.5.17) and (2.5.21) the inflation map (2.5.11) on the F-th component of the target $\mathcal{O}_\mathcal{F}$ can be easily described:

$$\mathcal{O}_{\mathcal{F}/H_i} = \prod_{j\geq 1} H^*(B\mathbb{T}^2/H_i^j) \to H^*(B\mathbb{T}^2/H_i^{n_i}) \rightarrowtail H^*(B\mathbb{T}^2/F). \tag{2.5.25}$$

The first map of (2.5.25) is the projection onto the n_i-th component since $q_*(F) = \langle H_i, F\rangle = H_i^{n_i}$ by definition of the index n_i. The second map of (2.5.25) is the natural inclusion of $\mathbb{Q}$-algebras sending the generator c_{i,n_i} to x_i, since by (2.5.23) they are the same Euler class $e(z_i^{n_i})$ for the two different rings.

2.5.4 Description of $\mathcal{A}(G)$

We briefly recap the description of $\mathcal{A}(G)$ [Gre08, Definition 3.9]. The objects of $\mathcal{A}(G)$ are sheaves of modules over the poset of connected subgroups of G with inclusions.

Definition 2.5.26. An object $X \in \mathcal{A}(G)$ is specified by the following pieces of data:

1. For every connected subgroup H an $\mathcal{O}_{\mathcal{F}/H}$-module $\varphi^H X$.

2. For every containment of connected subgroups $K \subseteq H$ an $\mathcal{O}_{\mathcal{F}/K}$-modules map:

$$\varphi^K X \to \mathcal{E}_{H/K}^{-1}\mathcal{O}_{\mathcal{F}/K} \underset{\mathcal{O}_{\mathcal{F}/H}}{\otimes} \varphi^H X. \tag{2.5.27}$$

Then X is a sheaf over the space of connected subgroups of G. This specifically means that for every connected subgroup H, the sheaf X has value the $\mathcal{O}_\mathcal{F}$-module:

$$X(H) := \mathcal{E}_H^{-1}\mathcal{O}_\mathcal{F} \underset{\mathcal{O}_{\mathcal{F}/H}}{\otimes} \varphi^H X, \tag{2.5.28}$$

and that for every containment $K \subseteq H$ of connected subgroups, X has a structure map of $\mathcal{O}_\mathcal{F}$-modules:

$$\beta_K^H : X(K) \to X(H). \tag{2.5.29}$$

The map (2.5.29) is obtained tensoring the $\mathcal{O}_{\mathcal{F}/K}$-modules map (2.5.27) with the $\mathcal{O}_{\mathcal{F}/K}$-module $\mathcal{E}_K^{-1}\mathcal{O}_\mathcal{F}$. Moreover X satisfies the condition that for every connected subgroup H the $\mathcal{O}_{\mathcal{F}}$-modules structure map $\beta_1^H : X(1) \to X(H)$ is the map inverting the multiplicatively closed subset of Euler classes $\mathcal{E}_H$ (2.5.15).

Remark 2.5.30. Notice that by Remark 2.5.10 the inflation map i_K makes $\mathcal{O}_{\mathcal{F}}$ an $\mathcal{O}_{\mathcal{F}/K}$-module. Moreover the structure map (2.5.27) is well defined from (2.5.27), since:

$$\mathcal{E}_K^{-1}\mathcal{O}_{\mathcal{F}} \underset{\mathcal{O}_{\mathcal{F}/K}}{\otimes} \mathcal{E}_{H/K}^{-1}\mathcal{O}_{\mathcal{F}/K} \cong \mathcal{E}_H^{-1}\mathcal{O}_{\mathcal{F}}. \tag{2.5.31}$$

Example 2.5.32. For the 2-torus an object $X \in \mathcal{A}(\mathbb{T}^2)$ has the shape:

$$\begin{tikzcd} & X(\mathbb{T}^2) & \\ X(H_1) \arrow[ur, "\beta^{\mathbb{T}^2}_{H_1}"] & X(H_2) \arrow[u] & \dots \arrow[ul] \\ & X(1) \arrow[ul, "\beta^{H_1}_1"] \arrow[u] \arrow[ur] & \end{tikzcd} = \left[\begin{tikzcd} X(\mathbb{T}^2) \\ X(H_i) \arrow[u] \\ X(1) \arrow[u] \end{tikzcd} \right]$$

with infinitely many values $X(H_i)$ in the middle row, one vertex $X(\mathbb{T}^2)$ and one value $X(1)$ at the bottom level. By (2.5.28):

$$\begin{aligned} X(\mathbb{T}^2) &= \mathcal{E}_{\mathbb{T}^2}^{-1} \underset{\mathbb{Q}}{\otimes} \varphi^{\mathbb{T}^2} X \\ X(H_i) &= \mathcal{E}_{\mathbb{T}^2/H_i}^{-1}\mathcal{O}_{\mathcal{F}} \underset{\mathcal{O}_{\mathcal{F}/H_i}}{\otimes} \varphi^{H_i} X \\ X(1) &= \mathcal{O}_{\mathcal{F}} \underset{\mathcal{O}_{\mathcal{F}}}{\otimes} \varphi^1 X = \varphi^1 X \end{aligned} \tag{2.5.33}$$

Notation 2.5.34. a tensor product with no ring specified will always mean over $\mathbb{Q}$. For any $i \geq 1$ we denote $\otimes_i = \otimes_{\mathcal{O}_{\mathcal{F}/H_i}}$ the tensor product over the ring $\mathcal{O}_{\mathcal{F}/H_i}$ or when we are considering the F-th component: $\otimes_i = \otimes_{H^*(B\mathbb{T}^2/H_i^{n_i})}$.

Example 2.5.35. For the 2-torus using the coordinates we have defined ((2.5.17), (2.5.21), and (2.5.23)), we can easily describe the localizations at the Euler classes:

- In $\mathcal{E}_{G/H_i}^{-1} H^*(B\mathbb{T}^2/H_i^j)$ we are inverting the Euler class c_{ij}.
- In $\mathcal{E}_{H_i}^{-1} H^*(B\mathbb{T}^2/F)$ we are inverting all the Euler classes x_j with $j \geq 1$ and $j \neq i$.
- In $\mathcal{E}_{\mathbb{T}^2}^{-1} H^*(B\mathbb{T}^2/F)$ we are inverting all the Euler classes x_j with $j \geq 1$.

Example 2.5.36. There is a structure sheaf $\mathcal{O} \in \mathcal{A}(G)$ [Gre08, Definition 3.3] obtained using as modules the base rings: $\varphi^H \mathcal{O} = \mathcal{O}_{\mathcal{F}/H}$, and as structure maps the natural inclusions:

$$\mathcal{O}_{\mathcal{F}/K} \to \mathcal{E}_{H/K}^{-1}\mathcal{O}_{\mathcal{F}/K} \underset{\mathcal{O}_{\mathcal{F}/H}}{\otimes} \mathcal{O}_{\mathcal{F}/H}.$$

Definition 2.5.37. A morphism $f : X \to Y$ in the category $\mathcal{A}(G)$ is the data of a (graded) $\mathcal{O}_{\mathcal{F}/H}$-module map $\varphi^H f : \varphi^H X \to \varphi^H Y$ for every connected subgroup H, compatible with the structure maps of X and Y (it makes the evident commutative diagrams between different levels commute [Gre08, Definition 3.6]).

Remark 2.5.38. A morphism $f : X \to Y$ in $\mathcal{A}(G)$ is almost determined by what it does at the trivial subgroup level $f(1) : X(1) \to Y(1)$ (that we will call bottom level). This is because for any connected subgroup H the map f at the H-th level $f(H) : X(H) \to Y(H)$ is then $\mathcal{E}_H^{-1} f(1)$. Therefore properties like injectivity, surjectivity or exactness for a sequence of morphisms can be checked at the bottom level.

2.5.5 Injectives in $\mathcal{A}(G)$

The injective objects in $\mathcal{A}(G)$ that we will use are constant below a certain connected subgroup H, and zero elsewhere [Gre08, Section 4.A.].

Definition 2.5.39. An object $X \in \mathcal{A}(G)$ is concentrated below a connected subgroup H if $X(K) = 0$ for every connected subgroup $K \nsubseteq H$. We denote $\mathcal{A}(G)_H$ the full subcategory of $\mathcal{A}(G)$ of objects concentrated below H.

Definition 2.5.40. If H is a connected subgroup of G, and T is a graded torsion $\mathcal{O}_{\mathcal{F}/H}$-module, define $f_H(T) \in \mathcal{A}(G)$ to be the constant sheaf below H with the following values:

$$f_H(T)(K) := \begin{cases} \mathcal{E}_H^{-1}\mathcal{O}_{\mathcal{F}} \otimes_{\mathcal{O}_{\mathcal{F}/H}} T & \text{if } K \subseteq H \\ 0 & \text{if } K \nsubseteq H \end{cases} \tag{2.5.41}$$

and structure maps either identities or zero.

Remark 2.5.42. We require T to be torsion so that when we invert $\mathcal{E}_K^{-1}$ for $K \nsubseteq H$ we obtain zero. Therefore this requirement can be dropped when $H = G$.

Lemma 2.5.43 (Lemma 4.1 of [Gre08])**.** *For any connected subgroup H of G there is an adjunction:*

$$\begin{tikzcd} \mathcal{A}(G)_H \arrow[r, bend left, "\varphi^H"] \arrow[r, phantom, "\perp"] & \textit{Tors-}\mathcal{O}_{\mathcal{F}/H}\textit{-Mod} \arrow[l, bend left, "f_H"] \end{tikzcd}$$

where the left adjoint is the evaluation φ^H. For any torsion $\mathcal{O}_{\mathcal{F}/H}$-module T, and object $X \in \mathcal{A}(G)$ we have:

$$\mathrm{Hom}_{\mathcal{O}_{\mathcal{F}/H}}(\varphi^H X, T) \cong \mathrm{Hom}_{\mathcal{A}(G)}(X, f_H(T)). \tag{2.5.44}$$

This in order allows us to transfer torsion injectives $\mathcal{O}_{\mathcal{F}/H}$-modules into injective objects in $\mathcal{A}(G)$. First notice that if we are given for every $\tilde{H} \in \mathcal{F}/H$ an $H^*(BG/\tilde{H})$-torsion module $T(\tilde{H})$, then $\bigoplus_{\mathcal{F}/H} T(\tilde{H})$ is naturally a torsion $\mathcal{O}_{\mathcal{F}/H}$-module, with the action given component by component.

Corollary 2.5.45 (Lemma 5.1 of [Gre08])**.** *Suppose $\{H_i\}_{i\geq 1}$ is the collection of all connected subgroups of G of a fixed dimension. If $f_{H_i}(T_i)$ is injective for every i, then so is $\bigoplus_{i\geq 1} f_{H_i}(T_i)$.*

Corollary 2.5.46 (Corollary 5.2 of [Gre08])**.** *If for every $\tilde{H} \in \mathcal{F}/H$, $T(\tilde{H})$ is a graded torsion injective $H^*(BG/\tilde{H})$-module. Then $f_H(\bigoplus_{\mathcal{F}/H} T(\tilde{H}))$ is injective in $\mathcal{A}(G)$.*

2.5.6 Spheres of complex representations

We can now define the fundamental homology functor $\pi_*^{\mathcal{A}}$ [Gre08, Definition 1.4]. Given a rational G-spectrum X we can define the sheaf $\pi_*^{\mathcal{A}}(X) \in \mathcal{A}(G)$ that on a connected subgroup H takes the value

$$\begin{aligned} \pi_*^{\mathcal{A}}(X)(H) &:= \pi_*^G(DE\mathcal{F}_+ \wedge S^{\infty V(H)} \wedge X) \\ &\cong \mathcal{E}_H^{-1}\mathcal{O}_{\mathcal{F}} \underset{\mathcal{O}_{\mathcal{F}/H}}{\otimes} \pi_*^{G/H}(DE\mathcal{F}/H_+ \wedge \Phi^H X). \end{aligned} \tag{2.5.47}$$

We denote Φ^H the geometric fixed point functor, $E\mathcal{F}_+$ is the universal space for the family $\mathcal{F}$ of finite subgroups with a disjoint basepoint added, and $DE\mathcal{F}_+ = F(E\mathcal{F}_+, S^0)$ is its functional dual (The function spectrum of maps from $E\mathcal{F}_+$ to S^0). The space $S^{\infty V(H)}$ is a convenient construction for $\tilde{E}[\not\supseteq H]$ [Gre08, Section 1.C]:

$$S^{\infty V(H)} := \varinjlim_{V^H=0} S^V$$

when $K \subseteq H$ there is a map $S^{\infty V(K)} \to S^{\infty V(H)}$ inducing the structure map $\pi_*^{\mathcal{A}}(X)(K) \to \pi_*^{\mathcal{A}}(X)(H)$. The isomorphism (2.5.47) is proven in [Gre08, Lemma 9.2], and it is one of the steps in the proof that the functor $\pi_*^{\mathcal{A}}$ takes value in the abelian category $\mathcal{A}(G)$. Another key ingredient is to understand $\pi_*^{\mathcal{A}}(S^0)$:

Lemma 2.5.48 (Theorem 1.5 of [Gre08])**.** *The image of S^0 in $\mathcal{A}(G)$ is the structure sheaf $\mathcal{O}$:*

$$\pi_*^{\mathcal{A}}(S^0)(H) = \mathcal{E}_H^{-1}\mathcal{O}_{\mathcal{F}} \underset{\mathcal{O}_{\mathcal{F}/H}}{\otimes} \mathcal{O}_{\mathcal{F}/H}.$$

Corollary 2.5.49 (Corollary 1.6 of [Gre08])**.** *The functor $\pi_*^{\mathcal{A}}$ takes values in the abelian category $\mathcal{A}(G)$.*

Given a complex representation V of G we want to make explicit the object $\pi_*^{\mathcal{A}}(S^V)$ [Gre12, Section 2.B.]. To do so we need first to introduce suspensions:

Definition 2.5.50. If V is an n-dimensional complex representation of G, divide the family $\mathcal{F}$ of finite subgroups of G into $n+1$ disjoint sets $\mathcal{F}_i$, where

$$\mathcal{F}_i := \{F \in \mathcal{F} \mid \dim_{\mathbb{C}}(V^F) = i\}.$$

If M is an $\mathcal{O}_{\mathcal{F}}$-module, define the V-th suspension of M to be the $\mathcal{O}_{\mathcal{F}}$-module:

$$\Sigma^V M := \bigoplus_{i=0}^{n} \Sigma^{2i} e_{\mathcal{F}_i} M,$$

where $e_{\mathcal{F}_i} \in \mathcal{O}_{\mathcal{F}}$ is the idempotent associated to $\mathcal{F}_i$ (it has a one in the F-th component if $F \in \mathcal{F}_i$ and zero everywhere else).

The value of the sheaf $\pi_*^{\mathcal{A}}(S^V)$ at a connected subgroup H is:

$$\pi_*^{\mathcal{A}}(S^V)(H) = \mathcal{E}_H^{-1}\mathcal{O}_{\mathcal{F}} \underset{\mathcal{O}_{\mathcal{F}/H}}{\otimes} \Sigma^{V^H} \mathcal{O}_{\mathcal{F}/H}. \tag{2.5.51}$$

To describe the structure maps it is convenient to use the suspension of the units:

$$\iota_{V^H} := \Sigma^{V^H}(1) \in \Sigma^{V^H}\mathcal{O}_{\mathcal{F}/H}$$

so that for every inclusion of connected subgroups $K \subseteq H$ the structure map β_K^H is determined by the suspended unit:

$$\beta_K^H(\iota_{V^K}) = e(V^K - V^H)^{-1} \otimes \iota_{V^H} \tag{2.5.52}$$

where the difference of the two representations simply means the orthogonal complement:

$$V^K = V^H \oplus (V^K - V^H).$$

Remark 2.5.53. The content of this section applies also in the case of a virtual complex representation $V = V_0 - V_1$. The only thing to specify is the Euler class

$e(V) = e(V_0)/e(V_1)$. As a result (2.5.51) becomes:

$$\pi_*^{\mathcal{A}}(S^V)(H) = \mathcal{E}_H^{-1}\mathcal{O}_{\mathcal{F}} \underset{\mathcal{O}_{\mathcal{F}/H}}{\otimes} \Sigma^{V_0^H - V_1^H}\mathcal{O}_{\mathcal{F}/H}.$$

Chapter 3

Prerequisites: Algebraic Geometry

3.1 Cousin Complex

We give an introduction to the Cousin complex following [Har66, Chapter IV]. Let X be a topological space and Z a generic subset. We need to work with the notion of support in Z in a more general context than when Z is a closed subset. Therefore let us revise the definitions in this more general setting.

Definition 3.1.1. The codimension of a point $x \in X$ is the largest integer n such that there exists a sequence of points $x_0, x_1, x_2, \ldots, x_n = x$ of X where each x_{i+1} is a proper specialization of x_i, i.e. $x_{i+1} \in \overline{\{x_i\}}$ and $x_{i+1} \neq x_i$.

Definition 3.1.2. The support of a sheaf of abelian groups $\mathcal{F}$ on X is defined to be the subset:

$$\operatorname{supp}(\mathcal{F}) := \{x \in X \mid \mathcal{F}_x \neq 0\}.$$

Notice it is not necessarily closed. Given a section $s \in \Gamma(X, \mathcal{F})$ the support of s is:

$$\operatorname{supp}(s) := \{x \in X \mid s_x \neq 0 \in \mathcal{F}_x\}.$$

Definition 3.1.3. Define the global sections with support in $Z \subseteq X$ to be

$$\Gamma_Z(X, \mathcal{F}) := \{s \in \mathcal{F}(X) \mid \operatorname{supp}(s) \subseteq Z\}. \tag{3.1.4}$$

And the corresponding sheaf $\underline{\Gamma}_Z(\mathcal{F})$ to be the sheaf with value on each open subset U of X:

$$U \mapsto \Gamma_{Z \cap U}(U, \mathcal{F}|_U).$$

More explicitly $\underline{\Gamma}_Z(\mathcal{F})(U)$ is the set of sections $s \in \mathcal{F}(U)$ such that $s|_V = 0$ for some open V with $U \setminus Z \subseteq V \subseteq U$. Denote $H^n_Z(\mathcal{F})$ and $\underline{H}^n_Z(\mathcal{F})$ the respective n-th right derived functors.

Definition 3.1.5. For every subset $Z' \subseteq Z$ define the sheaf

$$\underline{\Gamma}_{Z/Z'}(\mathcal{F}) := \underline{\Gamma}_Z(\mathcal{F})/\underline{\Gamma}_{Z'}(\mathcal{F})$$

and denote $\underline{H}^n_{Z/Z'}(\mathcal{F})$ its n-th right derived functor.

Definition 3.1.6. For every point $x \in X$, with closure $Z = \overline{\{x\}}$, define the functor $\mathcal{F} \mapsto \Gamma_x(\mathcal{F})$, that associates to a sheaf $\mathcal{F}$ the subgroup of $\mathcal{F}_x$:

$$\Gamma_x(\mathcal{F}) := \{\alpha \in \mathcal{F}_x \mid \alpha \text{ has a representative } (s, U), \text{ with } \mathrm{supp}(s) \subseteq Z \cap U\}. \quad (3.1.7)$$

Denote $\mathcal{H}^n_x(\mathcal{F})$ its n-th right derived functor.

Remark 3.1.8. If $Z = \overline{\{x\}}$ then by [Har66, Variation 8]:

$$\mathcal{H}^n_x(\mathcal{F}) \cong (\underline{H}^n_Z(\mathcal{F}))_x. \quad (3.1.9)$$

Notation 3.1.10. Denote $\mathcal{H}^n_Z(\mathcal{F}) := \mathcal{H}^n_{\eta(Z)}(\mathcal{F})$.

To apply in full the machinery of Cousin complexes we need a topological space X and a filtration satisfying the following hypothesis.

Hypothesis 3.1.11. Let X be a sober (i.e. every closed irreducible subset has a unique generic point), locally Noetherian topological space, endowed with a filtration by subsets $X = Z^0 \supseteq Z^1 \supseteq \dots$ which is separated ($\cap_{n \geq 0} Z^n = \emptyset$) and strictly exhausting ($Z^0 = X$). Moreover suppose the filtration is stable under specialization (if $x \in Z^n$ then all its specializations are in Z^n), and that for every $n \geq 0$ every element in $Z^n \setminus Z^{n+1}$ is maximal in Z^n under specialization (i.e. if $x \in Z^n \setminus Z^{n+1}$ and y is a nontrivial specialization of x, then $y \in Z^{n+1}$).

Example 3.1.12. The prototypical example we have in mind is the codimension filtration of a topological space

$$Z^n := \{x \in X \mid \mathrm{codim}(x) \geq n\}. \quad (3.1.13)$$

Proposition 3.1.14 (Proposition 2.3 and 2.5 of [Har66])**.** *Let X be a topological space with a filtration by subsets satisfying Hypothesis 3.1.11. Then for every sheaf* *of abelian groups $\mathcal{F}$ there is a unique augmented complex of sheaves:*

$$\mathcal{F} \longrightarrow \underline{H}^0_{Z^0/Z^1}(\mathcal{F}) \xrightarrow{d_0} \underline{H}^1_{Z^1/Z^2}(\mathcal{F}) \xrightarrow{d_1} \underline{H}^2_{Z^2/Z^3}(\mathcal{F}) \longrightarrow \ldots . \quad (3.1.15)$$

called the Cousin Complex of $\mathcal{F}$. Moreover by [Har66, Variation 8 Motif F pg. 225] there is a canonical functorial isomorphism

$$\underline{H}^n_{Z^n/Z^{n+1}}(\mathcal{F}) \cong \coprod_{x \in Z^n \setminus Z^{n+1}} \iota_x(\mathcal{H}^n_x(\mathcal{F})). \quad (3.1.16)$$

where for a group M, the sheaf $\iota_x(M)$ denotes the constant sheaf with value M on the closure of the point x (or the constant sheaf on the closed subset D in case of $\iota_D(M)$).

We now ask when the Cousin complex is a resolution of $\mathcal{F}$.

Proposition 3.1.17 (Proposition 2.6 of [Har66])**.** *Under the Hypothesis 3.1.11 for a sheaf of abelian groups $\mathcal{F}$ the following are equivalent:*

1. *$\underline{H}^i_{Z^n}(\mathcal{F}) = 0$ for all $i \neq n$.*
2. *$\underline{H}^i_{Z^n/Z^{n+1}}(\mathcal{F}) = 0$ for all $i \neq n$.*
3. *The Cousin complex of $\mathcal{F}$ is a flabby resolution of $\mathcal{F}$.*

The sheaf $\mathcal{F}$ is said to be Cohen-Macaulay when it satisfies any of these equivalent conditions.

Definition 3.1.18. Given an ideal I of a commutative ring R, define for any R-module M:

$$\Gamma_I(M) := \{s \in M \mid \exists n \geq 0, I^n s = 0\}. \quad (3.1.19)$$

Denote $H^n_I(M)$ its n-th right derived functors.

This definition is analogous to the sheaf version of cohomology with support since these two cohomologies coincide on affine schemes. More precisely [Har67, Theorem 2.3]:

Theorem 3.1.20. *Let R be a Noetherian ring, $U = \mathrm{Spec}(R)$, I a finitely generated ideal of R with corresponding closed subset $V(I)$, and M an R-module. Then*

$$H^*_{V(I)}(U, \widetilde{M}) \cong H^*_I(M).$$

This is because the two support functors identify the same submodule of M: $s \in \Gamma(\mathrm{Spec}(R), \widetilde{M})$ is a section with support in $V(I)$ if and only if $\exists n \geq 0$ such that $I^n s = 0$.

3.2 Algebraic groups and Abelian varieties

We give an introduction to algebraic groups and Abelian varieties following [Lom18] and [Mil08].

A variety over a field is a geometrically integral (i.e. reduced and irreducible), separated scheme of finite type over that field.

Definition 3.2.1. Let S be a scheme. A group scheme over S is an S-scheme X together with three morphisms:

$$\begin{aligned} m : X \times_S X \to& X \quad \text{(multiplication)} \\ i : X \to& X \quad \text{(inverse)} \\ e : S \to& X \quad \text{(unit)} \end{aligned} \tag{3.2.2}$$

such that they induce a group structure on the set of Y-valued points $X(Y)$, for any S-scheme Y.

Notice that when Y is an S-scheme, Y-valued points $X(Y)$ are defined to be maps $Y \to X$ over S. If Y is a ring then we mean $\operatorname{Spec} Y$-valued points.

Example 3.2.3. The following affine group schemes are the main examples.

- The multiplicative group $\mathbb{G}_m$. Consider as a base scheme the integers $S = \operatorname{Spec}\mathbb{Z}$. The underlying scheme is $\mathbb{G}_m = \operatorname{Spec}\mathbb{Z}[t, t^{-1}]$. The identity section e is the map of affine schemes induced by the map of rings

  $$\begin{aligned} \mathbb{Z}[t, t^{-1}] \to& \mathbb{Z} \\ t \mapsto& 1 \end{aligned}$$

  the inverse i is induced by the map of rings

  $$\begin{aligned} \mathbb{Z}[t, t^{-1}] \to& \mathbb{Z}[t, t^{-1}] \\ t \mapsto& t^{-1} \end{aligned}$$

  and multiplication m is induced by

  $$\begin{aligned} \mathbb{Z}[t, t^{-1}] \to& \mathbb{Z}[t_1, t_1^{-1}] \bigotimes_{\mathbb{Z}} \mathbb{Z}[t_2, t_2^{-1}] \\ t \mapsto& t_1 t_2. \end{aligned}$$

  One can check that with these morphisms for any scheme Y we obtain $\mathbb{G}_m(Y) = H^0(Y, \mathcal{O}_Y)^\times$, justifying the name multiplicative group. Indeed if R is a ring we obtain the multiplicative group of the ring: $\mathbb{G}_m(\operatorname{Spec} R) = R^\times$. For a general base scheme S the multiplicative group over S is simply $\mathbb{G}_{m,S} = \mathbb{G}_m \times_{\operatorname{Spec}\mathbb{Z}} S$.

- The additive group $\mathbb{G}_a$. As before it is enough to define it over the integers $\operatorname{Spec}\mathbb{Z}$. The underlying scheme is $\mathbb{G}_a = \operatorname{Spec}\mathbb{Z}[t]$, and the morphisms are the ones induce by the ring maps:

$$\begin{aligned} e : \mathbb{Z}[t] &\to \mathbb{Z} \\ t &\mapsto 0 \\ i : \mathbb{Z}[t] &\to \mathbb{Z}[t] \\ t &\mapsto -t \\ m : \mathbb{Z}[t] &\to \mathbb{Z}[t_1, t_2] \\ t &\mapsto t_1 + t_2 \end{aligned}$$

  One can check that with these morphisms for any scheme Y we obtain $\mathbb{G}_a(Y) = H^0(Y, \mathcal{O}_Y)$ with its additive structure, justifying the name additive group.

Remark 3.2.4. Over the complex numbers $S = \operatorname{Spec}\mathbb{C}$, a group scheme over $\mathbb{C}$ is a complex variety X together with morphisms

$$\begin{aligned} m : X \times_{\mathbb{C}} X &\to X \\ i : X &\to X \end{aligned} \tag{3.2.5}$$

and a $\mathbb{C}$-valued point $e \in X(\mathbb{C})$ such that the structure induced on $X(\mathbb{C})$ by m and i is a group with identity e. Group schemes over $\mathbb{C}$ are often called complex algebraic groups.

Definition 3.2.6. A complex abelian variety is a connected proper group scheme A over $\mathbb{C}$.

Remark 3.2.7. A complex abelian variety is automatically reduced, projective, nonsingular, irreducible and commutative. The set of $\mathbb{C}$-valued points $A(\mathbb{C})$ inherits a complex structure as a submanifold of $\mathbb{P}^n(\mathbb{C})$. It is a compact connected complex manifold with a commutative group structure. If A has dimension d then $A(\mathbb{C})$ is a complex torus $\mathbb{C}^d/L$ for some full lattice L of $\mathbb{C}^d$.

A morphism of abelian varieties is a morphism of the underlying algebraic varieties that preserves the identity element for the group structure. A morphism of abelian varieties is called an isogeny if it is surjective, and has finite kernel.

Example 3.2.8.
- An elliptic curve over $\mathbb{C}$ is a smooth projective variety of dimension 1 and genus 1 over the complex numbers, with a marked $\mathbb{C}$-valued point e. They are abelian varieties: the point e uniquely determines the group law, and serves as a neutral element for it.
- If $E_1, \ldots, E_g$ are elliptic curves, then $E_1 \times \cdots \times E_g$ is a group scheme which is connected, smooth and projective, hence an abelian variety of dimension g.

3.3 Formal Group Laws and Elliptic cohomology

3.3.1 Formal group laws

We introduce Formal group laws following [Str19].

Definition 3.3.1. A (one dimensional, commutative) formal group law (FGL) over a commutative ring with unit R is a formal power series $F(x, y) \in R[[x, y]]$ such that:

1. $F(x, 0) = x \in R[[x]]$
2. $F(x, y) = F(y, x) \in R[[x, y]]$
3. $F(x, F(y, z)) = F(F(x, y), z) \in R[[x, y, z]]$
4. There is a power series $i(x) \in R[[x]]$ such that $i(0) = 0$ and $F(x, i(x)) = 0$.

Remark 3.3.2. Condition (4) can be deduced from the other properties. Notice as well that $F(x, y) \equiv x + y \mod (x, y)^2$.

Example 3.3.3.
- The additive FGL is defined to be $F_a(x, y) = x + y$ and can be defined over any ring R. It can be obtained from the additive group $\mathbb{G}_a$ that we have previously defined as follows. Pick a coordinate at the identity element $0 \in \mathbb{G}_a(R) = R$, and write down the formal power series expansion of the product map:

$$\begin{aligned} \mathbb{G}_a(R) \times \mathbb{G}_a(R) \xrightarrow{m} & \mathbb{G}_a(R) \\ (x, y) \mapsto & x + y. \end{aligned} \tag{3.3.4}$$

- The multiplicative FGL is defined to be $F_m(x, y) = x + y + xy$ and can be defined over any ring R. It can be obtained from the multiplicative group $\mathbb{G}_m$ picking as coordinate $1 + x$ for $\mathbb{G}_m(R)$ so that for $x = 0$ we obtain the identity element 1 for the multiplicative group of R. Under this coordinate choice:

$$\begin{aligned} \mathbb{G}_m(R) \times \mathbb{G}_m(R) &\to \mathbb{G}_m(R) \\ (1+x, 1+y) &\mapsto (1+x)(1+y) = 1 + F_m(x,y). \end{aligned} \tag{3.3.5}$$

- More generally we can construct a one dimensional commutative FGL from any one dimensional algebraic group, again simply picking coordinates at the identity element and considering the power series expansion of the product map. In this way we obtain a formal group law associated to any elliptic curve.

Definition 3.3.6. A morphism $f : F \to G$ between two formal group laws is $f(x) \in R[[x]]$ with no constant term, such that

$$f(F(x,y)) = G(f(x), f(y)).$$

It's an isomorphism if the coefficient of degree 1 is invertible in R, and a strict isomorphism if it's precisely the unit 1.

In characteristic zero we can completely classify isomorphism classes of formal group laws (see for example [Str19, Proposition 3.1]).

Proposition 3.3.7. *Let R be a $\mathbb{Q}$-algebra, then for every formal group law $F(x,y) \in R[[x,y]]$, there exists a unique $f(x) = x + O(x^2) \in R[[x]]$, giving a strict isomorphism with the additive formal group law. That is such that*

$$f(F(x,y)) = f(x) + f(y).$$

The series $f(x)$ is called a logarithm for F.

Example 3.3.8. Let $F(x,y) = x + y + xy$ be the multiplicative formal group law. If R is a $\mathbb{Q}$-algebra, then F is isomorphic to the additive formal group law via the isomorphism

$$g^{-1}(t) = e^t - 1 = t + t/2 + t/6 + \dots$$

3.3.2 Non-equivariant elliptic cohomology

We give a brief introduction to non-equivariant elliptic cohomology following [Lur09], and we explain the connection with formal group laws.

Let A be a "nice" (non-equivariant) cohomology theory. More precisely we want A to be multiplicative: $A^*(X)$ is a graded commutative ring for every topological space X, even: $A^*(*)$ is concentrated in even degrees, and periodic:

there exists an invertible element in $A^{-2}(*)$. Under these assumptions the Atiyah-Hirzebruch spectral sequence for the space $\mathbb{C}\mathrm{P}^\infty$ degenerates at the second page and its cohomology is (noncanonically) isomorphic to a formal power series ring

$$A^*(\mathbb{C}\mathrm{P}^\infty) \cong R[[t]]$$

over the commutative ring $R := A^0(*)$. The parameter t is a complex orientation for A, and can be seen as the first Chern class of the universal line bundle $\mathcal{O}(1)$ over $\mathbb{C}\mathrm{P}^\infty$. Moreover once we make a choice for t we can define first Chern classes for any complex line bundle over any space X. The space $\mathbb{C}\mathrm{P}^\infty$ then is of fundamental importance since it classifies complex line bundles. In particular we have an associative and commutative multiplication

$$m : \mathbb{C}\mathrm{P}^\infty \times \mathbb{C}\mathrm{P}^\infty \to \mathbb{C}\mathrm{P}^\infty \tag{3.3.9}$$

which classifies the operation of forming tensor product of two line bundles. The space $\mathbb{C}\mathrm{P}^\infty \times \mathbb{C}\mathrm{P}^\infty$ is the classifying space for pairs of complex line bundles, and again by the Atiyah-Hirzebruch spectral sequence:

$$A^*(\mathbb{C}\mathrm{P}^\infty \times \mathbb{C}\mathrm{P}^\infty) \cong R[[t_1, t_2]]$$

where t_1 and t_2 are the pullbacks of t along the projections $\mathbb{C}\mathrm{P}^\infty \times \mathbb{C}\mathrm{P}^\infty \to \mathbb{C}\mathrm{P}^\infty$ into the first and second factor. Under this last isomorphism the map (3.3.9) induces:

$$\begin{aligned} R[[t]] \cong A^*(\mathbb{C}\mathrm{P}^\infty) &\xrightarrow{m^*} A^*(\mathbb{C}\mathrm{P}^\infty \times \mathbb{C}\mathrm{P}^\infty) \cong R[[t_1, t_2]] \\ t &\xmapsto{m^*} F(t_1, t_2) \end{aligned}$$

sending the parameter t to a certain power series $F(t_1, t_2)$. Commutativity and associativity of (3.3.9) imply that $F(t_1, t_2)$ is a one dimensional commutative formal group law over R.

Example 3.3.10.
- When A is periodic ordinary cohomology, the associated formal group law is the additive formal group law $F_a(t_1, t_2)$.
- When A is complex K-theory, the associated formal group law is the multiplicative formal group law $F_m(t_1, t_2)$.

Definition 3.3.11. An elliptic cohomology theory is an even periodic multiplicative cohomology theory together with a choice of elliptic curve and a choice of isomorphism between the formal group associated to the cohomology theory and the formal group associated to the elliptic curve.

3.4 Complex abelian surfaces

In this section we want to recollect some results from [Bea96] about complex abelian surfaces. Therefore in all this section $S = E_1 \times E_2$ is the product of two elliptic curves, namely a complex abelian variety of dimension 2. By "sheaf" we mean coherent algebraic sheaf, and by Serre's "GAGA theorem" [Ser56] there is a bijection between algebraic and analytic coherent sheaves which preserves exactness and cohomology. We denote $\mathcal{O}_S$ the structure sheaf of S, and $\mathcal{K}(S)$ the function field of S: the stalk of the structure sheaf $\mathcal{O}_S$ at the generic point of S.

Since we are working on a smooth variety Cartier and Weil divisors are the same and we can use the generic term divisor D on S, i.e. D is a finite sum with integer coefficients of irreducible closed subvarieties of S of codimension 1 (curves). The divisor D is said to be effective when all the coefficients are ≥ 0 and $D \geq D'$ if $D - D'$ is effective. The divisor D is said to be principal if there is a rational function $f \in \mathcal{K}(S)$ such that $\mathrm{Div}(f) = D$. Two divisors are said to be linearly equivalent when they differ by a principal one.

The Picard group of S, $\mathrm{Pic}(S)$, is the group of isomorphism classes of invertible sheaves (or of line bundles) on S. To every divisor D on S there corresponds an invertible sheaf $\mathcal{O}_S(D)$ that associates to any open U of S:

$$\Gamma(U, \mathcal{O}_S(D)) = \{f \in \mathcal{K}(S) \mid \mathrm{Div}(f) + D \geq 0 \text{ on } U\} \cup \{0\}$$

The map $D \mapsto \mathcal{O}_S(D)$ identifies $\mathrm{Pic}(S)$ with the group of linear equivalence classes of divisors on S.

The Picard group of a surface carries a symmetric bilinear form.

Definition 3.4.1. Let C and C' be two distinct irreducible curves on a surface S, and $x \in C \cap C'$. If f and g are respectively an equation for C and C' in $\mathcal{O}_{S,x}$ then the intersection multiplicity of C and C' at x is defined to be:

$$m_x(C \cap C') = \dim_{\mathbb{C}} \mathcal{O}_{S,x}/(f, g).$$

Remark 3.4.2. We notice immediately that $m_x(C \cap C') = 1$ if and only if f and g generate the maximal ideal, i.e. they form a system of local coordinates in a neighbourhood of x. In this case C and C' are said to be transverse in x.

Definition 3.4.3. If C and C' are two distinct irreducible curves on S, the intersection number $(C.C')$ is defined by:

$$(C.C') = \sum_{x \in C \cap C'} m_x(C \cap C'). \tag{3.4.4}$$

For any sheaf L on S let $\chi(L) = \sum_i (-1)^i h^i(S, L)$ be the Euler-Poincaré characteristic of L, where $h^i(L) = \dim_{\mathbb{C}} H^i(S, L)$.

Definition 3.4.5. For L and L' in $\mathrm{Pic}(S)$, define:

$$(L.L') := \chi(\mathcal{O}_S) - \chi(L^{-1}) - \chi(L'^{-1}) + \chi(L^{-1} \otimes L'^{-1}). \tag{3.4.6}$$

Theorem 3.4.7 (Theorem I.4 of [Bea96])**.** *The equation (3.4.6) defines a symmetric bilinear form on* $\mathrm{Pic}(S)$ *such that if C and C' are two distinct irreducible curves on S with associated line bundles $\mathcal{O}_S(C)$ and $\mathcal{O}_S(C')$, then:*

$$(\mathcal{O}_S(C).\mathcal{O}_S(C')) = (C.C').$$

With the right hand side defined by (3.4.4).

Definition 3.4.8. If D and D' are two divisors on S, define:

$$(D.D') := (\mathcal{O}_S(D).\mathcal{O}_S(D')) \tag{3.4.9}$$

Remark 3.4.10. By Theorem 3.4.7 to compute (3.4.9) we can replace any of the divisors with a linearly equivalent one.

Let ω_S be the line bundle of differential 2-forms on S. It is common to denote K_S any divisor such that $\mathcal{O}_S(K_S) = \omega_S$, and call K_S a canonical divisor. Serre duality is one of the most used tools in cohomology [Bea96, Theorem I.1]. For any line bundle L on S, the cup-product pairing defines a duality

$$H^i(S, L) \otimes H^{2-i}(S, \omega_S \otimes L^{-1}) \to H^2(S, \omega_S) \cong \mathbb{C}. \tag{3.4.11}$$

In terms of divisors we have for $0 \leq i \leq 2$:

$$h^i(D) = h^{2-i}(-D) \tag{3.4.12}$$

where $h^i(D) = h^i(\mathcal{O}_S(D))$. This is because since S is an abelian surface By [Bea96, Corollary VIII.7] the canonical divisor K_S is linearly equivalent to zero and $\omega_S \cong \mathcal{O}_S$.

By [Bea96, Theorem VIII.2] we have the cohomology groups for the structure sheaf:

$$h^0(\mathcal{O}_S) = 1 \qquad h^1(\mathcal{O}_S) = 2 \qquad h^2(\mathcal{O}_S) = 1 \tag{3.4.13}$$

implying $\chi(\mathcal{O}_S) = 0$. Since both the canonical divisor and the Euler-Poincaré characteristic of the structure sheaf are zero, Riemann-Roch for surfaces [Bea96, Theorem I.12] simplifies in:

$$\chi(\mathcal{O}_S(D)) = \frac{1}{2}(D.D). \tag{3.4.14}$$

The genus formula [Bea96, p. I.15] provides us with another tool. If C is an irreducible curve on the surface S, then the genus of the curve, defined as $g(C) = h^1(C, \mathcal{O}_C)$, is given by:

$$g(C) = 1 + \frac{1}{2}(C.C). \tag{3.4.15}$$

Chapter 4

Building $\mathbb{T}^2$-equivariant elliptic cohomology

Let $G = \mathbb{T}^2$ be our fixed group of equivariance. This chapter is devoted in building our theory for rational G-equivariant elliptic cohomology $E\mathcal{C}_G \in \mathcal{A}(G)$, and it consists of the entirety of [Bar22a]. More precisely we want to prove the following, which is the main Theorem of the chapter.

Theorem 4.0.1. *For every elliptic curve $\mathcal{C}$ over $\mathbb{C}$ and coordinate $t_e \in \mathcal{O}_{\mathcal{C},e}$, there exists an object $E\mathcal{C}_G \in \mathcal{A}(G)$ whose associated rational G-equivariant cohomology theory $E\mathcal{C}^*_G(\_)$ is 2-periodic. The value on the one point compactification S^V for a complex G-representation V with $V^G = 0$ is given in terms of the sheaf cohomology of a line bundle $\mathcal{O}(-D_V)$ over the complex abelian surface $\mathcal{X} = \mathcal{X}_G = \mathcal{C} \times \mathcal{C}$:*

$$E\mathcal{C}^n_G(S^V) \cong \begin{cases} H^0(\mathcal{X}, \mathcal{O}(-D_V)) \oplus H^2(\mathcal{X}, \mathcal{O}(-D_V)) & n \text{ even} \\ H^1(\mathcal{X}, \mathcal{O}(-D_V)) & n \text{ odd.} \end{cases} \tag{4.0.2}$$

The construction of the object can be found in Section 4.4 while the computation on spheres is Theorem 4.5.1. This theorem suggests the following conjecture.

Conjecture 4.0.3. *There exists an exact functor of triangulated categories $\mathbf{Sp}^G_{\mathbb{Q}} \to D(\mathrm{QCoh}(\mathcal{X}))$ that sends S^V to $\mathcal{O}(-D_V)$. From this one could recover Theorem 4.0.1 by applying the cohomology functor $D(\mathrm{QCoh}(\mathcal{X})) \to \mathrm{QCoh}(\mathcal{X})_*$.*

To construct $E\mathcal{C}_G$ we proceed as follows. We build an exact sequence of three injective objects in $\mathcal{A}(G)$:

$$\mathbb{I}_0 \xrightarrow{\varphi_0} \mathbb{I}_1 \xrightarrow{\varphi_1} \mathbb{I}_2 \to 0. \tag{4.0.4}$$

and we define $E\mathcal{C}_G$ as the kernel of φ_0, so that this sequence is an injective resolution of our theory in $\mathcal{A}(G)$. In this way in computing $E\mathcal{C}_G(S^V)$ via the Adams Spectral sequence we use (4.0.4), so we only need to build it with the right geometric inputs.

The main input is the Cousin complex of the structure sheaf of the variety $\mathcal{X}$. To every subgroup H of G we associate a subvariety $\bar{\mathfrak{X}}(H)$ of $\mathcal{X}$ of the same dimension. We need to change the topology on $\mathcal{X}$ from the Zarisky one to a new one we call TP-topology (torsion point topology) that focuses only on the subvarieties $\bar{\mathfrak{X}}(H)$. The TP-topology $\mathcal{X}^{\mathrm{TP}}$ basically copies the poset structure of the subgroups of G with a poset of irreducible closed subsets. An essential aspect is that Zarisky coherent sheaves have the same cohomology in the TP-topology.

Consider the sheaf Cousin complex of the pushforward of the Zariski structure sheaf of $\mathcal{X}$ that we denote $\mathcal{O}_{\mathcal{X}}^{\mathrm{TP}}$. We show this is a flabby resolution of $\mathcal{O}_{\mathcal{X}}^{\mathrm{TP}}$, whose n-th term decomposes as a direct sum over the irreducible closed subsets of $\mathcal{X}^{\mathrm{TP}}$ of codimension n. In complete analogy the n-th term of the sequence (4.0.4) we want to build will encode information of the cohomology theory at the subgroups of codimension n. Therefore for every subgroup H we can use the term in the Cousin complex over $\bar{\mathfrak{X}}(H)$ to build (4.0.4).

To compute the value of the theory on a sphere of complex representation S^V, with $V^G = 0$ we use the Adams Spectral sequence, and the injective resolution (4.0.4). Computations are directly reduced to the Cousin complex of $\mathcal{O}_{\mathcal{X}}^{\mathrm{TP}}$ twisted by the coherent sheaf $\mathcal{O}(-D_V)$, giving its cohomology as a result.

4.0.1 Structure of the chapter

For all the chapter $G = \mathbb{T}^2$ is our fixed group of equivariance, and we have also fixed an elliptic curve $\mathcal{C}$ over $\mathbb{C}$. In Section 4.1 we associate to every subgroup H of G a subvariety $\bar{\mathfrak{X}}(H)$ of $\mathcal{X}$ (Definition 4.1.3) and we prove its properties: the most important property is Lemma 4.1.17. In Section 4.2 we define the TP-topology on $\mathcal{X}$ (Definition 4.2.1), and the main result of the section is Corollary 4.2.8. In Section 4.3 we introduce the sheaf Cousin complex (4.3.2), and we prove it is a flabby resolution (Corollary 4.3.7). Section 4.4 is the core of the construction. We start by defining $E\mathcal{C}_G$ (Definition 4.4.2) and discussing its formality, while the rest of the section deals with the hard work of building (4.0.4). We conclude by proving exactness of the injective resolution: Lemma 4.4.48. In the final section we compute $E\mathcal{C}_G(S^V)$: in Theorem 4.5.11 we have that the second page of the Adams spectral sequence is the cohomology of the sheaf $\mathcal{O}(-D_V)$, forging the direct link with the geometry of $\mathcal{C}$. We conclude the section extending the computations for virtual negative complex representations (4.5.34).

4.1 The correspondence subgroups-subvarieties

The goal of this section is to specify a correspondence between subgroups H of G and certain subvarieties $\bar{\mathfrak{X}}(H)$ of $\mathcal{X}$. We have fixed an elliptic curve $\mathcal{C}$ over the complex numbers which defines a functor $\mathfrak{X}$ from compact abelian Lie groups to complex manifolds.

Definition 4.1.1. If H is a compact abelian Lie group and $\mathcal{C}$ our fixed elliptic curve, define

$$\mathfrak{X}(H) := \mathrm{Hom}_{\mathrm{Ab}}(H^*, \mathcal{C}). \tag{4.1.2}$$

Where we are considering group homomorphisms, and $H^* := \mathrm{Hom}(H, \mathbb{T})$ is the character group of H: continuous group homomorphisms into the circle group $\mathbb{T}$.

Let $\mathcal{X} := \mathfrak{X}(G)$ be the complex abelian surface defined by the 2-torus. The functor $\mathfrak{X}$ is exact and induces an embedding $\mathfrak{X}(H) \hookrightarrow \mathcal{X}$ for every subgroup H of G. Moreover $\mathfrak{X}(H)$ has the same dimension as H and is a subgroup of $\mathcal{X}$. We will only be interested in the functor (4.1.2) on subgroups of G, and therefore all the varieties $\mathfrak{X}(H)$ will be subvarieties of $\mathcal{X}$.

Definition 4.1.3. For every subgroup H of G define

$$\bar{\mathfrak{X}}(H) := \mathfrak{X}(H) \setminus \bigcup_K \mathfrak{X}(K) \tag{4.1.4}$$

where the union is over all the proper subgroups K of H of finite index in H.

Since the union in (4.1.4) is finite, $\bar{\mathfrak{X}}(H)$ is a subvariety of $\mathcal{X}$, that for G itself coincide with the all surface $\mathcal{X}$.

Lemma 4.1.5. *For H and K subgroups of G the following properties are satisfied:*

- $\mathfrak{X}(H \times K) = \mathfrak{X}(H) \times \mathfrak{X}(K)$.
- $\mathfrak{X}(H \cap K) = \mathfrak{X}(H) \cap \mathfrak{X}(K)$

Proof. The first property follows immediately applying the functor $\mathfrak{X}$ to the exact sequence

$$H \rightarrowtail H \times K \twoheadrightarrow K.$$

For the second one we only need to prove the containment $\mathfrak{X}(H) \cap \mathfrak{X}(K) \subseteq \mathfrak{X}(H \cap K)$, since the other containment is immediate from $\mathfrak{X}$ being a functor. Apply the exact functor $\mathfrak{X}$ to the commutative diagram:

$$\begin{tikzcd} H \cap K \arrow[r, tail] & G \arrow[r, two heads] \arrow[dr, "p_0 \times p_1"'] & G/_{H \cap K} \arrow[d, tail] \\ & & G/_H \times G/_K. \end{tikzcd}$$

In doing so the right vertical maps remains injective, and therefore the kernel of $\mathfrak{X}(p_0 \times p_1)$ is $\mathfrak{X}(H \cap K)$. Now it is enough to notice that every element in $\mathfrak{X}(H) \cap \mathfrak{X}(K)$ is sent to zero by $\mathfrak{X}(p_0 \times p_1)$. $\square$

Remark 4.1.6. Applying this Lemma to $H = 1 \times \mathbb{T}$ and $K = \mathbb{T} \times 1$, we obtain $\mathcal{X} = \mathfrak{X}(G) = \mathfrak{X}(H) \times \mathfrak{X}(K) = \mathcal{C} \times \mathcal{C}$

4.1.1 The codimension 1 case

Let $\{H_i\}_{i \geq 1}$ be the collection of connected codimension 1 subgroups of G, with $H_1 = 1 \times \mathbb{T}$ and $H_2 = \mathbb{T} \times 1$. Each one of the H_i can be written as the kernel of a nonzero character $z_i : G \to \mathbb{T}$ of G:

$$H_i \rightarrowtail G \xrightarrow{z_i} \mathbb{T}. \tag{4.1.7}$$

Moreover we may choose $z_i = z_1^{\lambda_i} z_2^{\mu_i}$ for a pair of coprime integers (λ_i, μ_i) not both zero and with $\mu_i \geq 0$. Applying the functor $\mathfrak{X}$ to (4.1.7), the subvariety $\mathfrak{X}(H_i)$ can be described in the same way as the kernel of the projection $\pi_i := \mathfrak{X}(z_i)$:

$$\mathfrak{X}(H_i) \rightarrowtail \mathcal{X} \xrightarrow{\pi_i} \mathcal{C}. \tag{4.1.8}$$

Where the relation $\pi_i = \lambda_i \pi_1 + \mu_i \pi_2$ holds now by the group law of the elliptic curve.

Definition 4.1.9. For every $i \geq 1$ and $j \in \mathbb{Z} \setminus \{0\}$ we define the character z_i^j of G post-composing z_i with the j-th power map of $\mathbb{T}$. We also define $\pi_i^j := \mathfrak{X}(z_i^j)$, note that this map is obtained post-composing π_i with the j-th power map in $\mathcal{C}$.

Definition 4.1.10. For every direction $i \geq 1$ and every $j \geq 1$ define the (i, j)-divisor:

$$D_{ij} := \bar{\mathfrak{X}}(H_i^j). \tag{4.1.11}$$

Where H_i^j is the subgroup of G with j connected components and identity component H_i.

Definition 4.1.12. For $P \in \mathcal{C}$ a point of finite order define:

$$D_{i,P} := \pi_i^{-1}(P)$$

Remark 4.1.13. Notice that $\mathfrak{X}(H_i^j) = \pi_i^{-1}(\mathcal{C}[j])$, while $D_{ij} = \pi_i^{-1}(\mathcal{C}\langle j\rangle)$. Therefore we have the decompositions:

$$\begin{aligned} \bar{\mathfrak{X}}(H_i^j) = D_{ij} &= \coprod_{P \in \mathcal{C}\langle j\rangle} D_{i,P} \\ \mathfrak{X}(H_i^n) &= \coprod_{j|n} D_{ij}. \end{aligned} \tag{4.1.14}$$

Remark 4.1.15. From now on we will always refer to π_i as the projection along the i-direction. All the varieties $\mathfrak{X}(H_i^j)$, D_{ij} and $D_{i,P}$ will all be referred as "along the i-direction". We denote $D_i = D_{i,1} = D_{i,e} = \mathfrak{X}(H_i)$. Note from (4.1.14) that the subvarieties along the i-direction D_{ij} are all parallel, disjoint and made up of disjoint pieces $D_{i,P}$ isomorphic to a single copy of $\mathcal{C}$.

4.1.2 The codimension 2 case

If F is a finite subgroup of G, then $\mathfrak{X}(F)$ is a finite collection of closed points of $\mathcal{X}$. The subset $\bar{\mathfrak{X}}(F) \subset \mathfrak{X}(F)$ satisfies some desirable properties.

Lemma 4.1.16. *If $F \neq F'$ are finite subgroups of G, then $\bar{\mathfrak{X}}(F) \cap \bar{\mathfrak{X}}(F') = \emptyset$*

Proof. Suppose $Q \in \bar{\mathfrak{X}}(F) \cap \bar{\mathfrak{X}}(F')$, then $Q \in \mathfrak{X}(F) \cap \mathfrak{X}(F') = \mathfrak{X}(F \cap F')$. Without loss of generality $F \cap F'$ is a proper subgroup of F, and therefore $Q \notin \bar{\mathfrak{X}}(F)$. □

Lemma 4.1.17. *Given $F < G$ finite, then for every direction $i \geq 1$ there exists one and only one index $n_i = n_i(F) \geq 1$ such that*

$$D_{i,n_i} \cap \bar{\mathfrak{X}}(F) \neq \emptyset,$$

precisely the only index n_i such that $H_i^{n_i}$ is the subgroup generated by H_i and F (Definition 2.5.18). Moreover

$$\bar{\mathfrak{X}}(F) = \bigcap_{i \geq 1} D_{i,n_i} \tag{4.1.18}$$

Proof. Given a finite subgroup F, for every direction $i \geq 1$ let n_i be the integer such that $H_i^{n_i} = \langle H_i, F\rangle$. Since $F \subseteq H_i^{n_i}$ then $\mathfrak{X}(F) \subseteq \mathfrak{X}(H_i^{n_i})$. Recall from (4.1.14) the decomposition:

$$\mathfrak{X}(H_i^{n_i}) = \coprod_{j|n_i} D_{ij}. \tag{4.1.19}$$

We start by proving $D_{i,n} \cap \bar{\mathfrak{X}}(F) = \emptyset$ for $n \neq n_i$:

- If $n \nmid n_i$ then $D_{i,n}$ is disjoint from $\bar{\mathfrak{X}}(F)$ since from (4.1.19) it is disjoint from $\mathfrak{X}(H_i^{n_i})$.
- If $n \mid n_i$ but $n \neq n_i$ then $H_i^n \subsetneq H_i^{n_i}$ and $F \nsubseteq H_i^n$ since n_i is the minimum integer for which the containment is true. Therefore $F' := H_i^n \cap F$ is a proper subgroup of F, and as such:

$$D_{i,n} \cap \mathfrak{X}(F) \subseteq \mathfrak{X}(F')$$

  which implies $D_{i,n} \cap \bar{\mathfrak{X}}(F) = \emptyset$.

To prove $D_{i,n_i} \cap \bar{\mathfrak{X}}(F) \neq \emptyset$ and the second part of the statement we use again the decomposition (4.1.19). Since $\bar{\mathfrak{X}}(F)$ can intersect only D_{i,n_i} and it is contained in $\mathfrak{X}(H_i^{n_i})$, it must be contained in D_{i,n_i}. It also follows $\bar{\mathfrak{X}}(F)$ is contained in the intersection (4.1.18).

We are left to prove that if $Q \in \bigcap_{i\geq 1} D_{i,n_i}$, then $Q \in \bar{\mathfrak{X}}(F)$. First, for every direction $i \geq 1$, $Q \in \mathfrak{X}(H_i^{n_i})$, therefore

$$Q \in \bigcap_{i\geq 1} \mathfrak{X}(H_i^{n_i}) = \mathfrak{X}(\bigcap_{i\geq 1} H_i^{n_i}) = \mathfrak{X}(F).$$

Now suppose $Q \in \mathfrak{X}(F')$ for a proper subgroup F' of F. For every $i \geq 1$ define n_i' such that $H_i^{n_i'} = \langle H_i, F' \rangle$. Then

$$\bigcap_{i\geq 1} H_i^{n_i} = F \neq F' = \bigcap_{i\geq 1} H_i^{n_i'}.$$

Therefore it exists an index s for which $n_s' \neq n_s$, but then $Q \in D_{s,n_s}$ and $Q \in D_{s,n_s'}$ which is absurd since they are disjoint. In conclusion $Q \in \bar{\mathfrak{X}}(F)$. □

Lemma 4.1.20. *For every finite subgroup $F \leq G$, the subset $\bar{\mathfrak{X}}(F)$ is non empty.*

Proof. If $F = \{1\} \times \{1\}$ is the trivial subgroup we have $\bar{\mathfrak{X}}(\{1\} \times \{1\}) = \mathfrak{X}(\{1\} \times \{1\}) = \{e\} \times \{e\} \neq \emptyset$.

If $F = \mathbb{Z}_{p^n} \times \mathbb{Z}_{p^m}$ is a p-group, with p prime and $n \leq m$, then applying Lemma 4.1.5:

$$\mathfrak{X}(F) \cong \mathfrak{X}(\mathbb{Z}_{p^n}) \times \mathfrak{X}(\mathbb{Z}_{p^m}) \cong \mathcal{C}[p^n] \times \mathcal{C}[p^m].$$

Every proper finite subgroup of F is contained in a maximal one (i.e. proper subgroup not contained into any other proper subgroup), therefore we can simply consider the maximal subgroups of F. If F is cyclic ($n = 0$ and $m > 0$) the only maximal subgroup of $\mathbb{Z}_{p^m}$ is $\mathbb{Z}_{p^{m-1}}$ and since $\mathfrak{X}(\mathbb{Z}_{p^m}) \cong \mathcal{C}[p^m]$ has p^{2m} points while $\mathfrak{X}(\mathbb{Z}_{p^{m-1}})$ has only p^{2m-2} we have

$$\bar{\mathfrak{X}}(\mathbb{Z}_{p^m}) = \mathfrak{X}(\mathbb{Z}_{p^m}) - \mathfrak{X}(\mathbb{Z}_{p^{m-1}}) \neq \emptyset.$$

If F is not cyclic, we need to use the following computations and facts. Every maximal subgroup F' of F has index p, therefore $\mathfrak{X}(F')$ has $p^{2n+2m-2}$ points. There are exactly $p+1$ maximal subgroups in F. Therefore there are at most $(p+1)p^{2m+2n-2}$ points contained in $\bigcup_{F'} \mathfrak{X}(F')$ where F' ranges on all maximal subgroups of F. Since $\mathfrak{X}(F)$ has p^{2m+2n} points and $p+1 < p^2$ it follows $\bar{\mathfrak{X}}(F) \neq \emptyset$.

For the general case of a finite subgroup F, decompose it into a product of p-groups:

$$F \cong F_{p_1} \times \cdots \times F_{p_k}.$$

Applying the previous case we can pick for each prime p_i a point $Q_i \in \bar{\mathfrak{X}}(F_{p_i})$. The point

$$(Q_1, \ldots, Q_k) \in \mathfrak{X}(F) \cong \mathfrak{X}(F_{p_1}) \times \cdots \times \mathfrak{X}(F_{p_k})$$

is a point in $\bar{\mathfrak{X}}(F)$. Indeed any maximal subgroup F' of F is of the following form: pick one of the factors $1 \leq i \leq k$, and a maximal subgroup $F'_{p_i} < F_{p_i}$, replace F_{p_i} with F'_{p_i} in the product

$$F' = F_{p_1} \times \cdots \times F'_{p_i} \times \cdots \times F_{p_k}.$$

The point $(Q_1, \ldots, Q_k)$ cannot be in any of these $\mathfrak{X}(F')$ since $Q_i \notin \mathfrak{X}(F'_{p_i})$ for each i. □

The bottom line is that we have associated to subgroups of G, certain subvarieties of $\mathcal{X}$ of the same dimension:

Codimension	Subgroups of G	subvarieties of $\mathcal{X}$
0	G	$\mathcal{X}$
1	H_i^j	D_{ij}
2	F	$\bar{\mathfrak{X}}(F)$

4.2 Change of topology

In this section we change the topology on the algebraic variety $\mathcal{X}^{\mathrm{Zar}} = \mathcal{C} \times \mathcal{C}$. We use $\mathcal{X}^{\mathrm{Zar}}$ to denote the algebraic variety with the usual Zariski topology.

Definition 4.2.1. Over the set $\mathcal{X}^{\mathrm{Zar}}$ define the torsion point topology $\mathcal{X}^{\mathrm{TP}}$ with generating closed subsets $\{D_{ij}\}_{ij}$, where $i \geq 1$ and $j \geq 1$ (recall the definition of D_{ij} in (4.1.11)).

Remark 4.2.2. The irreducible closed subsets of $\mathcal{X}^{\mathrm{TP}}$ are precisely the sets $\bar{\mathfrak{X}}(H)$ for every subgroup H of G. In codimension zero we have only $\bar{\mathfrak{X}}(G) = \mathcal{X}$. In codimension one this is due to the fact that the generating closed sets D_{ij} are disjoint when they have the same index i and transverse for different values of i. In codimension two it is precisely the content of Lemma 4.1.17: the various $\bar{\mathfrak{X}}(F)$ are disjoint, and they represent all the possible intersections of the codimension one closed subsets.

Remark 4.2.3. A delicate remark is imperative here. The topological space $\mathcal{X}^{\mathrm{TP}}$ is not sober: for example every point in $\bar{\mathfrak{X}}(F)$ is a generic point for $\bar{\mathfrak{X}}(F)$. To apply in full the theory of Cousin complexes we will need a sober topological space: i.e. every closed irreducible subset has a unique generic point. This can be fixed considering the Kolmogorov quotient of $\mathcal{X}^{\mathrm{TP}}$: $\mathrm{KQ}(\mathcal{X}^{\mathrm{TP}})$, the space obtained from $\mathcal{X}^{\mathrm{TP}}$ by quotienting together the points that belong to exactly the same open subsets. In this way we obtain a sober topological space: the closed irreducible subsets of $\mathrm{KQ}(\mathcal{X}^{\mathrm{TP}})$ are still the sets $\bar{\mathfrak{X}}(H)$ and each of these has exactly one generic point. For the sake of clarity we will work on $\mathcal{X}^{\mathrm{TP}}$ in this section, since it has the same underlying set as $\mathcal{X}^{\mathrm{Zar}}$. We ask the reader to keep in mind that all the results we prove for $\mathcal{X}^{\mathrm{TP}}$ in this section apply word by word to its Kolmogorov quotient $\mathrm{KQ}(\mathcal{X}^{\mathrm{TP}})$. This is because topologically undistinguishable points have exactly the same stalks and nothing changes from the point of view of sheaves. In conclusion we prove all the sheaves result using $\mathcal{X}^{\mathrm{TP}}$, they apply as well to $\mathrm{KQ}(\mathcal{X}^{\mathrm{TP}})$, and we use $\mathrm{KQ}(\mathcal{X}^{\mathrm{TP}})$ when we need a sober topological space for the Cousin complex.

Notice that every subset in the generating collection $\{D_{ij}\}_{ij}$ is a closed subset also in the Zariski topology. Therefore every TP-open is also a Zariski-open, and we have a well defined continuous map $\varphi : \mathcal{X}^{\mathrm{Zar}} \to \mathcal{X}^{\mathrm{TP}}$.

4.2.1 The pushforward is exact on quasi-coherent sheaves

Since φ is continuous it induces a pair of adjoint functors between the respective categories of abelian sheaves called pushforward and pullback (or direct and inverse image) of sheaves:

$$\begin{tikzcd} \mathrm{Ab}(\mathcal{X}^{\mathrm{TP}}) \arrow[r, bend left, "\varphi^{-1}"] \arrow[r, phantom, "\perp"] & \mathrm{Ab}(\mathcal{X}^{\mathrm{Zar}}) \arrow[l, bend left, "\varphi_*"] \end{tikzcd}$$

For $\mathcal{F} \in \mathrm{Ab}(\mathcal{X}^{\mathrm{Zar}})$ its pushforward sheaf is defined on a TP-open U by:

$$(\varphi_*\mathcal{F})(U) := \mathcal{F}(\varphi^{-1}(U))$$

This defines indeed an abelian sheaf on $\mathcal{X}^{\mathrm{TP}}$. Denote by

$$\mathcal{O}_{\mathcal{X}}^{\mathrm{TP}} := \varphi_*(\mathcal{O}_{\mathcal{X}}^{\mathrm{Zar}})$$

the pushforward of the Zariski structure sheaf. The space $(\mathcal{X}^{\mathrm{TP}}, \mathcal{O}_{\mathcal{X}}^{\mathrm{TP}})$ is a ringed topological space, but not a locally ringed space in contrast with the usual expectation from algebraic geometry. The map φ is also a morphism of ringed spaces, and therefore the functor φ_* takes $\mathcal{O}_{\mathcal{X}}^{\mathrm{Zar}}$-modules to $\mathcal{O}_{\mathcal{X}}^{\mathrm{TP}}$-modules. This gives us another pair of adjoint functors between the respective categories of modules:

$$\begin{tikzcd} \mathrm{Mod}(\mathcal{X}^{\mathrm{TP}}) \arrow[r, bend left, "\varphi^*"] \arrow[r, phantom, "\perp"] & \mathrm{Mod}(\mathcal{X}^{\mathrm{Zar}}) \arrow[l, bend left, "\varphi_*"] \end{tikzcd}$$

Remark 4.2.4. The map $\varphi : \mathcal{X}^{\mathrm{Zar}} \to \mathcal{X}^{\mathrm{TP}}$ is a flat map of ringed spaces: i.e. for every $x \in \mathcal{X}^{\mathrm{Zar}}$ the map of rings $\mathcal{O}_{\mathcal{X},x}^{\mathrm{TP}} \to \mathcal{O}_{\mathcal{X},x}^{\mathrm{Zar}}$ is flat. This is because by Remark 4.2.16 to obtain the Zarisky-stalk we simply need to invert more elements in the TP-stalk, and localizations are flat maps.

Corollary 4.2.5. *The pushforward map φ_* sends injective objects in $\mathrm{Mod}(\mathcal{O}_{\mathcal{X}}^{Zar})$ to injectives objects in $\mathrm{Mod}(\mathcal{O}_{\mathcal{X}}^{TP})$*

Proof. By [Stacks, Tag 02N4] the pullback map $\varphi^* : \mathrm{Mod}(\mathcal{O}_{\mathcal{X}}^{\mathrm{TP}}) \to \mathrm{Mod}(\mathcal{O}_{\mathcal{X}}^{\mathrm{Zar}})$ is exact since φ is a flat morphism of ringed spaces. Therefore φ_* preserves injectives since its left adjoint φ^* is exact. $\square$

The functor φ_* is exact if restricted to the subcategory $\mathrm{QCoh}(\mathcal{O}_{\mathcal{X}}^{\mathrm{Zar}})$ of quasi-coherent $\mathcal{O}_{\mathcal{X}}^{\mathrm{Zar}}$-modules since in the TP-topology we still have an open cover of Zariski affines:

Lemma 4.2.6. *Every point in $\mathcal{X}^{TP}$ is contained in a TP-open which is an open affine in the Zariski topology.*

Proof. In $\mathcal{X}^{\mathrm{Zar}}$ the complement of the union of two TP-closed subsets D_{ij} and D_{rs} along different directions ($i \neq r$) is a TP-open which is affine in the Zariski topology. $\square$

Corollary 4.2.7. *The functor φ_* restricted to* $\mathrm{QCoh}(\mathcal{O}_{\mathcal{X}}^{Zar})$ *is exact.*

Proof. Consider a point $x \in \mathcal{X}^{\mathrm{TP}}$, applying Lemma 4.2.6 we can compute the TP-stalk at x as a colimit over TP-opens that are open affines in the Zariski topology. By [Har77, Theorem 3.5] taking sections over an open affine $\Gamma(\mathrm{Spec}(R), \mathcal{F})$ for a quasi-coherent Zariski sheaf $\mathcal{F}$ is an exact functor. Therefore the functor $\mathcal{F} \to (\varphi_*(\mathcal{F}))_x$ is exact since it is a colimit of exact functors. $\square$

Corollary 4.2.8. *If $\mathcal{F} \in$* $\mathrm{QCoh}(\mathcal{O}_{\mathcal{X}}^{Zar})$ *then:*

$$H^*(\mathcal{X}^{Zar}, \mathcal{F}) \cong H^*(\mathcal{X}^{TP}, \varphi_*(\mathcal{F})) \tag{4.2.9}$$

Proof. By Gabber's result [Stacks, Tag 077K] the category $\mathrm{QCoh}(\mathcal{X}^{\mathrm{Zar}})$ has enough injectives. Therefore consider an injective resolution of $\mathcal{F}$ in the category $\mathrm{QCoh}(\mathcal{X}^{\mathrm{Zar}})$:

$$0 \to \mathcal{F} \to I_0 \to I_1 \to \dots$$

Applying the functor φ_* we obtain an injective resolution of $\varphi_*\mathcal{F}$ in $\mathrm{Mod}(\mathcal{O}_{\mathcal{X}}^{\mathrm{TP}})$:

$$0 \to \varphi_*\mathcal{F} \to \varphi_* I_0 \to \varphi_* I_1 \to \dots \tag{4.2.10}$$

This is because $\mathcal{X}^{\mathrm{Zar}}$ is a noetherian scheme, and therefore the injective objects in $\mathrm{QCoh}(\mathcal{X}^{\mathrm{Zar}})$ are precisely the injective objects in $\mathrm{Mod}(\mathcal{O}_{\mathcal{X}}^{\mathrm{Zar}})$ that are quasi-coherent. By Corollary 4.2.5 φ_* preserves injectives in $\mathrm{Mod}(\mathcal{O}_{\mathcal{X}}^{\mathrm{Zar}})$ and by Corollary 4.2.7 is exact on $\mathrm{QCoh}(\mathcal{O}_{\mathcal{X}}^{\mathrm{Zar}})$. It is enough now to notice that

$$\Gamma(\mathcal{X}^{\mathrm{Zar}}, I_n) = \Gamma(\mathcal{X}^{\mathrm{TP}}, \varphi_* I_n).$$

$\square$

Notation 4.2.11. We will denote by $H^*(\mathcal{X}, \mathcal{F})$ the common value of these two cohomologies.

We are interested in explicitly computing the TP-topology stalks of the pushforward of a Zariski-sheaf $\mathcal{F}$.

- If $x = \eta(\mathcal{X}^{\mathrm{TP}})$ is a generic point of the whole space, we can take a colimit of complements of increasingly bigger unions of the D_{ij}:

$$(\varphi_*\mathcal{F})_x = \varinjlim_{n\to\infty} \mathcal{F}(\mathcal{X}^{\mathrm{Zar}} \setminus \bigcup_{i,j\leq n} D_{ij}). \tag{4.2.12}$$

  For $\mathcal{F} = \mathcal{O}_{\mathcal{X}}^{\mathrm{Zar}}$ the colimit above picks the regular functions on the complement of increasingly bigger unions of the D_{ij}. This yields those meromorphic functions on $\mathcal{X}^{\mathrm{Zar}}$ that are allowed poles only in the collection $\{D_{ij}\}$.

$$\mathcal{K} := \mathcal{O}_{\mathcal{X},x}^{\mathrm{TP}} = \{f \in \mathcal{K}(\mathcal{X}^{\mathrm{Zar}}) \mid f \text{ is allowed poles only at } \{D_{ij}\}\}. \tag{4.2.13}$$

- If $x = \eta(D_{ij})$ is a generic point of a generating closed subset, simply skip D_{ij} itself in (4.2.12). For $\mathcal{F} = \mathcal{O}_{\mathcal{X}}^{\mathrm{Zar}}$ this yields:

$$\mathcal{O}_{D_{ij}} := \mathcal{O}_{\mathcal{X},x}^{\mathrm{TP}} = \{f \in \mathcal{K} \mid f \text{ is regular at } D_{ij}\}. \tag{4.2.14}$$

  We also denote $m_{ij} < \mathcal{O}_{D_{ij}}$ the ideal of those functions vanishing at D_{ij}.

- If $x \in \bar{\mathfrak{X}}(F)$ (which automatically makes it also a generic point for $\bar{\mathfrak{X}}(F)$), then in (4.2.12) simply skip all the D_{ij} containing $\bar{\mathfrak{X}}(F)$. By Lemma 4.1.17 for every direction $i \geq 1$, only D_{i,n_i} contains $\bar{\mathfrak{X}}(F)$. Therefore:

$$\mathcal{O}_F := \mathcal{O}_{\mathcal{X},x}^{\mathrm{TP}} = \{f \in \mathcal{K} \mid \forall i \geq 1,\ f \text{ is regular at } D_{i,n_i}\}. \tag{4.2.15}$$

  We also denote $m_F < \mathcal{O}_F$ the ideal of those functions vanishing at $\bar{\mathfrak{X}}(F)$.

Remark 4.2.16. When $\mathcal{F}$ is a quasi-coherent $\mathcal{O}_{\mathcal{X}}^{\mathrm{Zar}}$-module we can use commutative algebra to compute the stalk at a point $x \in \mathcal{X}^{\mathrm{TP}}$. Pick a TP-open containing the point which is an open affine in the Zariski topology (Lemma 4.2.6): $U = \mathrm{Spec}(R)$. Then $\mathcal{F}$ restricted to that open is isomorphic to the sheaf $\widetilde{M}$ for an R-module M. Modulo restricting the affine open U, deleting the closed subset D_{ij} from U corresponds to inverting those elements in R that vanish at D_{ij}. Therefore the stalk at x in the TP-topology is $S^{-1}M$ for the multiplicatively closed subset S generated by those elements vanishing at TP-closed subsets. Notice that it is exactly as in the Zariski topology, with the only difference that instead of inverting everything outside that prime, we invert just those elements outside that prime corresponding to the generating closed subsets for the TP-topology.

4.2.2 Choice of coordinates

The aim of this subsection is to build a set of uniformizers for the subvarieties D_{ij} with respect to the TP-topology. We construct them over the algebraic variety $\mathcal{X}$ with its normal Zariski topology, but notice that they are defined for the TP-topology as well (they belong in $\mathcal{K}$). All the uniformizers we build here only depend upon a choice of a coordinate $t_e \in \mathcal{O}_{\mathcal{C},e}$ vanishing to the first order at e, and with poles only at points of finite order of $\mathcal{C}$.

We begin by recalling the definition of the TP-topology for the single elliptic curve [Gre05, Definition 7.1]:

Definition 4.2.17. Over the set $\mathcal{C}$ define the torsion point topology $\mathcal{C}^{\mathrm{TP}}$ with generating closed subsets $\{\mathcal{C}\langle n\rangle\}_{n\geq 1}$, where $\mathcal{C}\langle n\rangle$ are the elements of exact order n in $\mathcal{C}$.

This is a ringed topological space with the pushforward of the structure sheaf: $\mathcal{O}_{\mathcal{C}}^{\mathrm{TP}}$.

Definition 4.2.18. The fundamental ring is the stalk at the generic point:

$$\mathcal{K}_{\mathbb{T}} := \mathcal{O}^{\mathrm{TP}}_{\mathcal{C},\eta(\mathcal{C})} = \{f \in \mathcal{K}(\mathcal{C}^{\mathrm{Zar}}) \mid f \text{ has poles only at points of finite order of } \mathcal{C}\}. \tag{4.2.19}$$

Likewise [Gre05, Definition 8.2] choose a coordinate for $\mathcal{C}$ at e:

Definition 4.2.20. Define the function $t_e \in \mathcal{O}^{\mathrm{TP}}_{\mathcal{C},e} \subset \mathcal{O}_{\mathcal{C},e}$ with divisor $e - 3\mathcal{C}\langle 2\rangle + \mathcal{C}\langle 3\rangle$. The existence is guaranteed by Abel-Jacobi (Lemma 4.4.51), and moreover it is unique up to scalar multiple. Notice it vanishes to the first order at e.

Remark 4.2.21. We denote $m_e < \mathcal{O}_{\mathcal{C},e}$ the maximal ideal of those functions vanishing at e. Then m_e is principal with generator t_e, and the same is true if we restrict m_e to $\mathcal{O}^{\mathrm{TP}}_{\mathcal{C},e}$. Note that by [Hoc11, Proposition 8.1] we obtain the same result if we complete those two rings with respect to those two maximal ideals:

$$(\mathcal{O}^{\mathrm{TP}}_{\mathcal{C},e})^{\wedge}_{m_e} \cong (\mathcal{O}_{\mathcal{C},e})^{\wedge}_{m_e} \cong \mathbb{C}[[t_e]]. \tag{4.2.22}$$

Elements in (4.2.22) are functions defined in a formal neighbourhood of the identity of $\mathcal{C}$, and can be written as formal power series with complex coefficients in the variable t_e.

The isomorphism $\mathcal{X} = \mathcal{C} \times \mathcal{C}$ is given through the two projections $\pi_1 : \mathcal{X} \to \mathcal{C}$ and $\pi_2 : \mathcal{X} \to \mathcal{C}$. The pullbacks of the coordinate: $t_1 := \pi_1^*(t_e)$ and $t_2 := \pi_2^*(t_e)$

respectively define uniformizers for $D_1 = \{e\} \times \mathcal{C}$ and $D_2 = \mathcal{C} \times \{e\}$ and together they generate the maximal ideal m in the stalk $\mathcal{O}_{\mathcal{X},O}$ of those functions that vanishes at $O = (e, e)$. Exactly as before when we complete with respect to m we get $(\mathcal{O}_{\mathcal{X},O})^{\wedge}_{m} \cong \mathbb{C}[[t_1, t_2]]$.

This is a way to manifest the formal group law of the elliptic curve $\mathcal{C}$. If $g : \mathcal{X} = \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is the group law of the elliptic curve, we have an induced map on the completed rings

$$\begin{aligned} g^* : \mathbb{C}[[t_e]] \cong (\mathcal{O}_{\mathcal{C},e})^{\wedge}_{m_e} &\to (\mathcal{O}_{\mathcal{X},O})^{\wedge}_{m} \cong \mathbb{C}[[t_1, t_2]] \\ t_e &\mapsto F(t_1, t_2) \end{aligned}$$

The element $F(t_1, t_2)$ is the formal group law of the elliptic curve $\mathcal{C}$ with respect to the uniformizer t_e. Since we are over a field of characteristic zero by Proposition 3.3.7 there exists a unique logarithm for F, namely a strict isomorphism with the additive formal group law:

Lemma 4.2.23. *There exists a unique element $\hat{t}_e \in (\mathcal{O}_{\mathcal{C},e})^{\wedge}_{m_e}$ that can be written as a formal power series with complex coefficients:*

$$\hat{t}_e := f(t_e) = \sum_{k=1}^{\infty} \alpha_k t_e^k \in \mathbb{C}[[t_e]] \tag{4.2.24}$$

with $\alpha_1 = 1$ and such that

$$f(F(t_1, t_2)) = f(t_1) + f(t_2).$$

As an immediate corollary:

Corollary 4.2.25. *Given two integers $r, s \in \mathbb{Z}$ the linear map*

$$\begin{aligned} \mathcal{X} \cong \mathcal{C} \times \mathcal{C} &\xrightarrow{(r,s)} \mathcal{C} \\ (x, y) &\mapsto rx + sy \end{aligned} \tag{4.2.26}$$

induces on the completed local rings a map:

$$(r, s)^* : (\mathcal{O}_{\mathcal{C},e})^{\wedge}_{m_e} \to (\mathcal{O}_{\mathcal{X},O})^{\wedge}_{m}$$

such that

$$(r, s)^*(f(t_e)) = rf(t_1) + sf(t_2).$$

We can now simply pullback t_e and $\hat{t}_e$ along the various projections $\pi_i^j : \mathcal{X} \to \mathcal{C}$ (Definition 4.1.9). Note that $\pi_i^j = (j\lambda_i, j\mu_i)$ is a linear map of the kind of (4.2.26), and therefore it induces maps on the completed and uncompleted local rings at the identities.

Definition 4.2.27. For every $i \geq 1$ and $j \geq 1$ define the coordinate

$$t_{ij} := (\pi_i^j)^*(t_e) \in \mathcal{O}_{\mathcal{X},O} \tag{4.2.28}$$

Remark 4.2.29. Since $t_e \in \mathcal{O}_{\mathcal{C},e}^{\mathrm{TP}}$ we have that $t_{ij} \in \mathcal{K}$, namely it has poles only in the collection of generating closed for the TP-topology. Moreover t_{ij} vanishes at first order at $\mathfrak{X}(H_i^j)$ and therefore at D_{ij}. This yields $t_{ij} \in \mathcal{O}_{D_{ij}}$ (defined in (4.2.14)), and that it generates the principal ideal m_{ij} of those functions vanishing at D_{ij}.

Definition 4.2.30. For every $i \geq 1$ and $j \geq 1$ define the completed coordinate

$$\hat{t}_{ij} := (\pi_i^j)^*(\hat{t}_e) \in (\mathcal{O}_{\mathcal{X},O})_m^\wedge = (\mathcal{O}_{\mathcal{X},O}^{\mathrm{TP}})_m^\wedge \tag{4.2.31}$$

Remark 4.2.32. Note that $(\pi_i^j)^* : (\mathcal{O}_{\mathcal{C},e})_{m_e}^\wedge \to (\mathcal{O}_{\mathcal{X},O})_m^\wedge$ is a continuous map of completed rings over $\mathbb{C}$, therefore $\hat{t}_{ij}$ can be expressed using the power series (4.2.24) in the variable t_{ij}:

$$\hat{t}_{ij} = (\pi_i^j)^*(\sum_{k=1}^{\infty} \alpha_k t_e^k) = \sum_{k=1}^{\infty} \alpha_k t_{ij}^k. \tag{4.2.33}$$

From this expansion it is transparent that $\hat{t}_{ij} \in (\mathcal{O}_{D_{ij}})_{m_{ij}}^\wedge$.

4.3 Cousin complex

The aim of this section is to prove that the Cousin complex of the structure sheaf $\mathcal{O} = \mathcal{O}_{\mathcal{X}}^{\mathrm{TP}}$ for the TP-topology is a flabby resolution of $\mathcal{O}$. We conveniently already introduced all the results we need from [Har66, Chapter 4] in Section 3.1. To apply in full that section we need a sober topological space, and $\mathcal{X}^{\mathrm{TP}}$ is not sober as discussed in Remark 4.2.3. To fix this issue we substitute $\mathcal{X}^{\mathrm{TP}}$ with its Kolmogorov quotient to obtain a sober space. We note as explained in Remark 4.2.3 that all the other results including the ones in the previous section apply indifferently to $\mathcal{X}^{\mathrm{TP}}$ and its Kolmogorov quotient. We do not change notation for it, and we highlight that being sober is necessary in particular for the filtration (3.1.13), and for the splitting (3.1.16), where we need to index on the generic points of the irreducible closed subsets. Everything we say about stalks, sheaves, sections and supports apply unchanged to $\mathcal{X}^{\mathrm{TP}}$ and its Kolmogorov quotient.

On the Kolmogorov quotient $\mathcal{X}^{\mathrm{TP}}$ we consider the codimension filtration

$\mathcal{X}^{\mathrm{TP}} = Z^0 \supset Z^1 \supset Z^2 \supset Z^3 = \emptyset$ where

$$Z^n := \{x \in \mathcal{X}^{\mathrm{TP}} \mid \operatorname{codim}(x) \geq n\}.$$

they satisfy Hypothesis 3.1.11, so we obtain the following Corollary to Proposition 3.1.14:

Corollary 4.3.1. *Since the Kolmogorov quotient $\mathcal{X}^{TP}$ with the codimension filtration satisfy Hypothesis 3.1.11, by Proposition 3.1.14 we can consider the Cousin complex of $\mathcal{O} = \mathcal{O}_{\mathcal{X}}^{TP}$:*

$$\mathcal{O} \longrightarrow \iota_{\mathcal{X}}(\mathcal{H}^0_{\mathcal{X}}(\mathcal{O})) \xrightarrow{d_0} \bigoplus_{i,j\geq 1} \iota_{D_{ij}}(\mathcal{H}^1_{D_{ij}}(\mathcal{O})) \xrightarrow{d_1} \bigoplus_{F} \iota_F(\mathcal{H}^2_F(\mathcal{O})) \longrightarrow 0. \tag{4.3.2}$$

where the sheaf $\iota_Z(M)$ denotes the constant sheaf with value M on the closed subset Z, and $\mathcal{H}^n_x(\mathcal{F})$ is defined in 3.1.6.

Notation 4.3.3. When the point x is the generic point of a closed subset Z we will abbreviate $\mathcal{H}^n_{\eta(Z)}(\mathcal{F})$ with $\mathcal{H}^n_Z(\mathcal{F})$. Moreover we use $\iota_F(\mathcal{H}^2_F(\mathcal{O}))$ for $\iota_{\bar{\mathfrak{X}}(F)}(\mathcal{H}^2_{\bar{\mathfrak{X}}(F)}(\mathcal{O}))$.

Proof. The Kolmogorov quotient $\mathcal{X}^{\mathrm{TP}}$ is sober and locally Noetherian (we have explicitly forced the first condition), and the remaining conditions are satisfied by the codimension filtration. Moreover notice that $Z^3 = \emptyset$ and

- $Z^0 \setminus Z^1 = \{\eta(\mathcal{X}^{\mathrm{TP}})\}$.
- $Z^1 \setminus Z^2 = \{\eta(D_{ij})\}_{ij\geq 1}$.
- $Z^2 \setminus Z^3 = \{\eta(\bar{\mathfrak{X}}(F))\}_F$, where F ranges over all finite subgroups of G.

so that the decomposition (3.1.16) gives us the terms of (4.3.2). □

4.3.1 The Cousin complex is a flabby resolution

We want to show that the Cousin complex (4.3.2) of $\mathcal{O}$ is a flabby resolution of $\mathcal{O}$, namely that $\mathcal{O}$ is Cohen-Macaulay. We use Proposition 3.1.17 proving that $\mathcal{O}$ satisfies the second condition. First we need the following Lemma.

Lemma 4.3.4. *Let Z be an irreducible TP-closed subset. Then in a TP-open subset $U = \operatorname{Spec}(R)$ which is affine in the Zariski topology, for any quasi-coherent $\mathcal{O}_{\mathcal{X}}^{Zar}$-module $\mathcal{F}$ the sections in $\mathcal{F}(U)$ with support in Z are the same in both topologies:*

$$\Gamma_Z^{Zar}(U, \mathcal{F}) = \Gamma_Z^{TP}(U, \varphi_* \mathcal{F}).$$

Moreover their right derived functors are isomorphic:

$$H^*_Z(U,\mathcal{F}) \cong H^*_Z(U,\varphi_*\mathcal{F})$$

Proof. First let us prove the containment $\Gamma^{\mathrm{TP}}_Z(U,\varphi_*\mathcal{F}) \subseteq \Gamma^{\mathrm{Zar}}_Z(U,\mathcal{F})$. If $s \in \mathcal{F}(U)$ is a section whose TP-support is contained in Z then its Zariski-support is also contained in Z. If $x \in U \setminus Z$, by Remark 4.2.16, modulo restricting U, the section s is zero in the Zariski-stalk at x since it is already zero in the TP-stalk at x where less elements are inverted.

Let us prove the other containment $\Gamma^{\mathrm{Zar}}_Z(U,\mathcal{F}) \subseteq \Gamma^{\mathrm{TP}}_Z(U,\varphi_*\mathcal{F})$. If $s \in \mathcal{F}(U)$ is a section whose Zariski-support is contained in Z, then by Lemma 4.2.6 there exists $n \geq 0$ such that $\mathcal{I}(U)^n s = 0$, where $\mathcal{I}$ is the ideal sheaf of Z. If $x \in U \setminus Z$, by Remark 4.2.16, modulo restricting U, to obtain the TP-stalk at x from $\mathcal{F}(U)$ we are inverting at least one element in $\mathcal{I}(U)$. If none of the elements in $\mathcal{I}(U)$ were inverted, then every D_{ij} containing Z will also contain x, implying $x \in Z$ since Z is irreducible in the TP-topology, giving us a contradiction. In conclusion the section s is zero in the TP-stalk at x since $\mathcal{I}(U)^n s = 0$ and at least one element of $\mathcal{I}(U)$ is inverted in the TP-stalk.

To show that their right derived functors are isomorphic use the same argument of Corollary 4.2.8. Pick an injective resolution of $\mathcal{F}$ in quasi-coherent $\mathcal{O}^{\mathrm{Zar}}_{\mathcal{X}}$-modules. Applying the pushforward φ_* to this injective resolution we obtain an injective resolution of $\varphi_*\mathcal{F}$ in $\mathcal{O}^{\mathrm{TP}}_{\mathcal{X}}$-modules. By [Har66, IV, theme motif C] flabby sheaves are acyclic with respect to the functor $\Gamma_Z(U,\_)$ and therefore we can use these two resolutions to compute local cohomology. □

Corollary 4.3.5. *If $\mathcal{F}$ is a quasi-coherent $\mathcal{O}^{Zar}_{\mathcal{X}}$-module Cohen-Macaulay with respect to the codimension filtration in $\mathcal{X}^{Zar}$, then $\varphi_*\mathcal{F}$ is Cohen-Macaulay with respect to the codimension filtration in $\mathcal{X}^{TP}$.*

Proof. We prove that for the sheaf $\varphi_*(\mathcal{F})$ condition (2) of Proposition 3.1.17 is satisfied. Since the Hypothesis 3.1.11 are satisfied then by [Har66, Lemma 2.4] we only need to prove condition (2) when $i < n$ since for $i > n$ is automatically satisfied. Therefore we need to show that for every $x \in \mathcal{X}^{\mathrm{TP}}$ with closure Z and $i < \mathrm{codim}(x)$ we have:

$$\mathcal{H}^i_x(\varphi_*\mathcal{F}) \cong (\underline{H}^i_Z(\varphi_*\mathcal{F}))_x = 0 \tag{4.3.6}$$

where the first isomorphism is (3.1.9). By Lemma 4.2.6 the TP-stalk (4.3.6) can be computed using TP-open subsets which are open affines in the Zariski topology, and given such an open U, using Lemma 4.3.4:

$$\underline{H}^i_Z(\varphi_*\mathcal{F})(U) = H^i_Z(U, \varphi_*(\mathcal{F})) = H^i_Z(U, \mathcal{F}) = 0.$$

This equals zero since $\mathcal{F}$ is Cohen-Macaulay with respect to the codimension filtration in the scheme $\mathcal{O}_{\mathcal{X}}^{\mathrm{Zar}}$, therefore condition (1) of Proposition 3.1.17 is satisfied and $i < \mathrm{codim}(Z)$. □

Corollary 4.3.7. *The Cousin complex (4.3.2) of $\mathcal{O}$ is a Flabby resolution of $\mathcal{O}$.*

Proof. The structure sheaf $\mathcal{O}_{\mathcal{X}}^{\mathrm{Zar}}$ is Cohen-Macaulay with respect to the codimension filtration in $\mathcal{X}^{\mathrm{Zar}}$ [Har66, Example pg. 239], therefore by Corollary 4.3.5 its pushforward $\varphi_*(\mathcal{O}_{\mathcal{X}}^{\mathrm{Zar}}) = \mathcal{O}$ is Cohen-Macaulay with respect to the codimension filtration in $\mathcal{X}^{\mathrm{TP}}$. This means that $\mathcal{O}$ satisfies condition (3) of Proposition 3.1.17 and its Cousin complex is a Flabby resolution of $\mathcal{O}$. □

4.3.2 Explicit description of the Cousin complex

We want to give an explicit description of the local cohomology terms appearing in the Cousin complex of $\mathcal{O}$. For this task let us extend Theorem 3.1.20 to the TP-topology.

Lemma 4.3.8. *Let x be a point in $\mathcal{X}^{TP}$ with TP-closure Z, and $\mathcal{I}$ be the $\mathcal{O}_{\mathcal{X}}^{Zar}$-ideal sheaf associated to Z. The ideal $m := (\varphi_*\mathcal{I})_x$ is a well defined ideal of the ring $\mathcal{O}_{\mathcal{X},x}^{TP}$. Then the two local cohomology functors*

$$\Gamma_x(\varphi_*\mathcal{F}) = \Gamma_m((\varphi_*\mathcal{F})_x) \tag{4.3.9}$$

agree on pushforward of quasi-coherent $\mathcal{O}_{\mathcal{X}}^{Zar}$-modules $\mathcal{F}$. As a consequence also their right derived functors agree on the same class:

$$\mathcal{H}^*_x(\varphi_*\mathcal{F}) \cong H^*_m((\varphi_*\mathcal{F})_x) \tag{4.3.10}$$

Proof. Let us first prove the containment $\Gamma_x(\varphi_*\mathcal{F}) \subseteq \Gamma_m((\varphi_*\mathcal{F})_x)$. If $\alpha \in \Gamma_x(\varphi_*\mathcal{F})$, then by definition of Γ_x (3.1.7) there exists a TP-open U (that by Lemma 4.2.6 we can take to be affine for the Zariski-topology) and a section $s \in \mathcal{F}(U)$ representing the germ α such that $\mathrm{supp}^{\mathrm{TP}}(s) \subseteq Z$. Therefore

$$s \in \Gamma_Z^{\mathrm{TP}}(U, \varphi_*\mathcal{F}) = \Gamma_Z^{\mathrm{Zar}}(U, \mathcal{F}) = \Gamma_{\mathcal{I}(U)}(\mathcal{F}(U)) \tag{4.3.11}$$

where the first equality is Lemma 4.3.4 and the second one is Theorem 3.1.20. By definition of $\Gamma_{\mathcal{I}(U)}$ (3.1.19) there is $n \geq 0$ such that $\mathcal{I}(U)^n s = 0$. By Remark 4.2.16 simply invert the appropriate elements to obtain the equality $m^n \alpha = 0$ in the TP-stalk at x.

Let us prove the other containment $\Gamma_m((\varphi_* \mathcal{F})_x) \subseteq \Gamma_x(\varphi_* \mathcal{F})$. If $\alpha \in \Gamma_m((\varphi_* \mathcal{F})_x)$, there exists $n \geq 0$ such that $m^n \alpha = 0$. The ideal m is finitely generated, therefore also m^n is finitely generated as well. This implies that we can find a TP-open U affine for the Zariski topology containing x and a section s representing the germ α such that $\mathcal{I}(U)^n s = 0$. Using again the chain of equalities (4.3.11) the TP-support of s is contained in Z and since (s, U) represents the germ α we obtain $\alpha \in \Gamma_x(\varphi_* \mathcal{F})$.

To prove (4.3.10) use the same argument of Corollary 4.2.8. Pick an injective resolution of $\mathcal{F}$ in quasi-coherent $\mathcal{O}_{\mathcal{X}}^{\text{Zar}}$-modules. Applying the pushforward φ_* to this injective resolution we obtain an injective resolution of $\varphi_* \mathcal{F}$ in $\mathcal{O}_{\mathcal{X}}^{\text{TP}}$-modules. The TP-stalk at x of this last resolution is an injective resolution of $(\varphi_* \mathcal{F})_x$ in $\mathcal{O}_{\mathcal{X},x}^{\text{TP}}$-modules, since taking the stalk preserves injectives. By the equality (4.3.9) just proven we obtain the isomorphism between the respective right-derived functors (4.3.10). □

Corollary 4.3.12. *For every $ij \geq 1$ we have the isomorphism*

$$\mathcal{H}^1_{D_{ij}}(\mathcal{O}) \cong \mathcal{K}/\mathcal{O}_{D_{ij}}. \tag{4.3.13}$$

With $\mathcal{K}$ and $\mathcal{O}_{D_{ij}}$ defined in (4.2.13), and (4.1.11).

Proof. Notice the chain of isomorphisms:

$$\mathcal{H}^1_{D_{ij}}(\mathcal{O}) \cong H^1_{m_{ij}}(\mathcal{O}_{D_{ij}}) \cong \frac{\mathcal{O}_{D_{ij}}[{t_{ij}}^{-1}]}{\mathcal{O}_{D_{ij}}} = \frac{\mathcal{K}}{\mathcal{O}_{D_{ij}}}$$

where the first isomorphism is Lemma 4.3.8, and the second one is the computation of local cohomology by means of the stable Koszul complex (see for example [Hun07, pag. 7]), since t_{ij} defined in (4.2.28) generates the principal ideal m_{ij} of those vanishing at D_{ij}. □

Proposition 4.3.14. *Let F be a finite subgroup of G and $x = \eta(\bar{\mathfrak{X}}(F))$ be the generic (and only) point of $\bar{\mathfrak{X}}(F)$ in the Kolmogorov quotient $\mathcal{X}^{TP}$. Then the TP-stalk at x of the Cousin complex (4.3.2) is*

$$\mathcal{O}_F \rightarrowtail \mathcal{K} \xrightarrow{d_0} \bigoplus_{i \geq 1} \mathcal{K}/\mathcal{O}_{D_{i,n_i}} \xrightarrow{d_1} \mathcal{H}^2_F(\mathcal{O}) \to 0. \tag{4.3.15}$$

Moreover the sequence (4.3.15) *is an exact sequence of* $\mathcal{O}_F = \mathcal{O}_x$*-modules.*

Proof. First of all $(CC^0(\mathcal{O}))_x = \mathcal{K}$ simply computing

$$\mathcal{H}^0_{\mathcal{X}}(\mathcal{O}) = \mathcal{O}_{\eta(\mathcal{X}^{\mathrm{TP}})} = \mathcal{K}. \tag{4.3.16}$$

The next term is $(CC^1(\mathcal{O}))_x = \bigoplus_{i\geq 1} \mathcal{K}/\mathcal{O}_{D_{i,n_i}}$, since by Lemma 4.1.17 for every $i \geq 1$ the only D_{ij} containing $\bar{\mathfrak{X}}(F)$ is D_{i,n_i}, and the local cohomology is described in (4.3.13).

The last term is $(CC^2(\mathcal{O}))_x = \mathcal{H}^2_F(\mathcal{O})$ since by Lemma 4.1.16 if $F' \neq F$ is another finite subgroup of G, then $\bar{\mathfrak{X}}(F')$ and $\bar{\mathfrak{X}}(F)$ are disjoint.

The sequence of $\mathcal{O}_F$ modules (4.3.15) is exact since by Corollary 4.3.7 the Cousin complex of $\mathcal{O}$ is a flabby resolution of $\mathcal{O}$. $\square$

We can use exactness of (4.3.15) to explicitly describe also the last local cohomology term:

$$\mathcal{H}^2_F(\mathcal{O}) \cong \Big(\bigoplus_{i\geq 1} \mathcal{K}/\mathcal{O}_{D_{i,n_i}}\Big)/\mathcal{K}. \tag{4.3.17}$$

We conclude the section considering the global sections of the Cousin complex (4.3.2), which will provide all the geometric inputs needed later in the construction of $E\mathcal{C}_G$:

$$\Gamma(\mathcal{O}) \longrightarrow \mathcal{K} \xrightarrow{d_0} \bigoplus_{i\geq 1}\Big(\bigoplus_{j\geq 1} \mathcal{K}/\mathcal{O}_{D_{ij}}\Big) \xrightarrow{d_1} \bigoplus_{F} \mathcal{H}^2_F(\mathcal{O}) \to 0 \tag{4.3.18}$$

4.4 The main construction

We are now ready to construct $E\mathcal{C}_G$. Recall that $G = \mathbb{T}^2$ is the 2-torus, and that we have fixed an elliptic curve $\mathcal{C}$ over $\mathbb{C}$ together with a coordinate $t_e \in \mathcal{O}^{\mathrm{TP}}_{\mathcal{C},e} \subset \mathcal{O}_{\mathcal{C},e}$ (Definition 4.2.20). We define $E\mathcal{C}_G \in \mathcal{A}(G)$ from an exact sequence of injective objects in $\mathcal{A}(G)$:

$$\mathbb{I}_0 \xrightarrow{\varphi_0} \mathbb{I}_1 \xrightarrow{\varphi_1} \mathbb{I}_2 \xrightarrow{0} 0. \tag{4.4.1}$$

Definition 4.4.2. Define $E\mathcal{C}_G := \mathrm{Ker}(\varphi_0) \in \mathcal{A}(G)$.

Remark 4.4.3. The sequence

$$0 \longrightarrow E\mathcal{C}_G \longrightarrow \mathbb{I}_0 \xrightarrow{\varphi_0} \mathbb{I}_1 \xrightarrow{\varphi_1} \mathbb{I}_2 \xrightarrow{0} 0 \tag{4.4.4}$$

is exact and it is an injective resolution of $E\mathcal{C}_G$ in $\mathcal{A}(G)$.

Remark 4.4.5. A priori since $E\mathcal{C}_G$ needs to be defined in the model category $d\mathcal{A}(G)$ if we define it as the kernel of φ_0 and we see it as an object in $d\mathcal{A}(G)$ without differential, still our construction will depend upon choices of lifts of φ_0 and φ_1 to the model category $d\mathcal{A}(G)$ since they are maps in the homotopy category $\mathcal{A}(G)$. A more generally applicable method is to define our theory as the following object $E\mathcal{C}'_G$, which is equivalent to $E\mathcal{C}_G$. It is a special feature of the object $E\mathcal{C}_G \in \mathcal{A}(G)$ that it is intrinsically formal, so that we are able to give the simpler definition 4.4.2.

Definition 4.4.6. Consider (4.4.1) as a sequence in $d\mathcal{A}(G)$. The map φ_0 factors through a map $\tilde{\varphi_0} : \mathbb{I}_0 \to \mathrm{Fib}(\varphi_1)$ (where $\mathrm{Fib}(\varphi_1)$ is the fibre of φ_1 in $d\mathcal{A}(G)$). Define $E\mathcal{C}'_G := \mathrm{Fib}(\tilde{\varphi_0})$. Moreover by Lemma 4.4.15 $E\mathcal{C}'_G \simeq E\mathcal{C}_G$, so they represent the same cohomology theory.

In turn the injectives are constructed by 2.5.41:

$$\mathbb{I}_0 := f_G(V(G)) \qquad \mathbb{I}_1 := \bigoplus_{i\geq 1} f_{H_i}(\bigoplus_{j\geq 1} V(H_i^j)) \qquad \mathbb{I}_2 := f_1(\bigoplus_F V(F)) \tag{4.4.7}$$

for a graded injective $H^*(BG/G)$-module $V(G)$, a graded torsion injective $H^*(BG/H_i^j)$-module $V(H_i^j)$ for every $ij \geq 1$ and a graded torsion injective $H^*(BG/F)$-module $V(F)$ for every finite subgroup F of G.

Remark 4.4.8. Notice that the objects are indeed injective by 2.5.45 and 2.5.46.

We start in 4.4.1 by proving formality of $E\mathcal{C}'_G$. The bulk of the section is the explicit construction of (4.4.1): the objects are built in 4.4.2, 4.4.3 and 4.4.4, while the maps are built in 4.4.5 and 4.4.6. We conclude proving exactness of (4.4.1) in 4.4.7. All the inputs needed to build (4.4.1) come from the global sections of the Cousin complex (4.3.18).

Notation 4.4.9. In all this section the local cohomology modules are the ones in the Cousin complex (4.3.2), therefore we will omit the sheaf $\mathcal{O}$ from the notation and denote $\mathcal{H}^2_F(\mathcal{O})$ by $\mathcal{H}^2_F$.

4.4.1 Formality

Starting from the exact sequence of injectives (4.4.1) we detail the construction of $E\mathcal{C}'_G$ and prove it is formal, so that $E\mathcal{C}'_G \simeq E\mathcal{C}_G$.

Note there is a natural inclusion $\iota : \mathcal{A}(G) \to d\mathcal{A}(G)$ obtained simply regarding an object of $\mathcal{A}(G)$ as an object of $d\mathcal{A}(G)$ with zero differential. There is also another functor $H_* : d\mathcal{A}(G) \to \mathcal{A}(G)$ obtained taking the homology of the object with differential.

Definition 4.4.10. An object $X \in d\mathcal{A}(G)$ is said to be formal when it is quasi-isomorphic to its homology $H_*(X)$.

Definition 4.4.11. For a map $\varphi : X \to Y$ in $d\mathcal{A}(G)$ the fibre $\mathrm{Fib}(\varphi) \in d\mathcal{A}(G)$ is defined at the level n by: $\mathrm{Fib}(\varphi)_n = X_n \oplus Y_{n+1}$, with differential $d_{\mathrm{Fib}(\varphi)}$ that on the Y component is simply the differential of Y, and on the X component is the direct sum of the differential of X with the map φ itself:

$$\begin{tikzcd} X_n \arrow[d, "d_X"] \arrow[drr, "\varphi_n"] & \oplus & Y_{n+1} \arrow[d, "d_Y"] \\ X_{n-1} & \oplus & Y_n \end{tikzcd} \tag{4.4.12}$$

Remark 4.4.13. For a map $\varphi : X \to Y$ between objects in $\mathcal{A}(G)$ we have two ways to consider the kernel. We have a well defined kernel $\mathrm{Ker}(\varphi) \in \mathcal{A}(G)$ with no differential in the abelian category $\mathcal{A}(G)$, or we can consider φ as a map in $d\mathcal{A}(G)$ through the inclusion ι and consider the fibre of that map $\mathrm{Fib}(\varphi) \in d\mathcal{A}(G)$ which is an object with differential.

When the map is surjective these two objects are equivalent:

Lemma 4.4.14. *If $\varphi : X \to Y$ is a surjective map in $\mathcal{A}(G)$, then there is an homology isomorphism $\iota(\mathrm{Ker}(\varphi)) \xrightarrow{\simeq} \mathrm{Fib}(\varphi)$.*

Proof. Since X and Y have no differential, in (4.4.12) we only have the map φ as differential. There is an obvious inclusion map $i : \mathrm{Ker}(\varphi) \to \mathrm{Fib}(\varphi)$ that on each level includes the kernel in the X-component of the fibre. It is easy to check it commutes with the differentials since the kernel has zero differential and the composition $d_{\mathrm{Fib}(\varphi)} \circ i$ is zero as well. Since φ is surjective, the image of $d_{\mathrm{Fib}(\varphi)}$ is the full Y_n component at each level, while the kernel is $\mathrm{Ker}(\varphi)_n \subseteq X_n$ at each level. Therefore when we take the homology of $d_{\mathrm{Fib}(\varphi)}$, the inclusion i induces an isomorphism. $\square$

Consider the following diagram:

$$\begin{tikzcd} \mathrm{Fib}(\tilde{\varphi}_0) \arrow[dr] & & & \\ \mathrm{Ker}(\varphi_0) \arrow[r] & \mathbb{I}_0 \arrow[r, "\varphi_0"] \arrow[d, "\bar{\varphi}_0"'] \arrow[dr, "\tilde{\varphi}_0"] & \mathbb{I}_1 \arrow[r, "\varphi_1"] & \mathbb{I}_2 \\ & \mathrm{Ker}(\varphi_1) \arrow[r, "i"] & \mathrm{Fib}(\varphi_1) \arrow[u] & \end{tikzcd}$$

the map φ_0 factors trough a map $\tilde{\varphi}_0 := i \circ \bar{\varphi}_0$ where $\bar{\varphi}_0$ is the map that φ_0 induces to the kernel. Recall $E\mathcal{C}'_G = \mathrm{Fib}(\tilde{\varphi}_0)$ and $E\mathcal{C}_G = \mathrm{Ker}(\varphi_0)$

Lemma 4.4.15. *The object $E\mathcal{C}'_G$ is formal, therefore $E\mathcal{C}'_G \simeq H^*(E\mathcal{C}'_G) = E\mathcal{C}_G$.*

Proof. Consider the fibre of $\bar{\varphi_0}$, which fits in a commutative diagram in $d\mathcal{A}(G)$:

$$\begin{tikzcd}
\mathrm{Fib}(\tilde{\varphi_0}) \arrow[r] & \mathbb{I}_0 \arrow[r, "\tilde{\varphi_0}"] & \mathrm{Fib}(\varphi_1) \\
\mathrm{Fib}(\bar{\varphi_0}) \arrow[r] \arrow[u] & \mathbb{I}_0 \arrow[r, "\bar{\varphi_0}"] \arrow[u, "="] & \mathrm{Ker}(\varphi_1) \arrow[u, "i", "\simeq"']
\end{tikzcd}$$

Both rows induce a long exact sequence in homology, and applying Lemma 4.4.14 the right vertical inclusion is an homology isomorphism, since φ_1 is surjective. As a consequence also the left vertical map is an homology isomorphism. Now it is enough to notice that also $\bar{\varphi_0}$ is a surjective map in $\mathcal{A}(G)$ because the sequence is exact at $\mathbb{I}_1$, therefore applying again the lemma we have an homology isomorphism $\mathrm{Ker}(\bar{\varphi_0}) \xrightarrow{\simeq} \mathrm{Fib}(\bar{\varphi_0})$. Summing up we have a chain of homology isomorphisms:

$$E\mathcal{C}_G = \mathrm{Ker}(\varphi_0) = \mathrm{Ker}(\bar{\varphi_0}) \xrightarrow{\simeq} \mathrm{Fib}(\bar{\varphi_0}) \xrightarrow{\simeq} \mathrm{Fib}(\tilde{\varphi_0}) = E\mathcal{C}'_G.$$

Since $E\mathcal{C}_G$ is without differential we obtain $H_*(E\mathcal{C}'_G) = E\mathcal{C}_G$, and $E\mathcal{C}'_G$ is formal. □

4.4.2 Building $\mathbb{I}_0$.

Associated to the whole group G in the Cousin complex we have the codimension 0 piece $\mathcal{H}^0_{\mathcal{X}} = \mathcal{K}$ (defined in (4.2.13)). This is a $\mathbb{Q}$-vector space, that we can make graded 2-periodic. Simply consider the 2-periodic version $\overline{\mathcal{K}}$ that has a copy of $\mathcal{K}$ in each even dimension and zero in odd dimensions (in general we will always use this notation for this 2-periodic operation). This is our graded injective $H^*(BG/G) \cong \mathbb{Q}$-module $V(G) := \overline{\mathcal{K}}$:

$$\mathbb{I}_0 := f_G(\overline{\mathcal{K}}) \tag{4.4.16}$$

4.4.3 Building $\mathbb{I}_1$.

Associated to every codimension one subgroup H^j_i we have the codimension one subvariety D_{ij}, and the associated codimension one piece in the Cousin complex $\mathcal{H}^1_{D_{ij}}$. Recall from (2.5.17) the isomorphism $H^*(BG/H^j_i) \cong \mathbb{Q}[c_{ij}]$ and the definition of the ring $\mathcal{O}_{D_{ij}}$ and ideal m_{ij} in (4.2.14).

Lemma 4.4.17. *The module $\mathcal{H}^1_{D_{ij}}$ is a torsion injective $H^*(BG/H^j_i) \cong \mathbb{Q}[c_{ij}]$-module, where the action is defined by restriction along the ring map*

$$\mathbb{Q}[c_{ij}] \to (\mathcal{O}_{D_{ij}})^{\wedge}_{m_{ij}}$$

that sends $c_{ij} = e(z_i^j)$ to $\hat{t}_{ij}$ (defined in (4.2.31)).

To define this action we will need to switch to the completed rings. For the task we will use the following well-known fact:

Lemma 4.4.18. *If I is a finitely generated ideal of the Noetherian ring R, then for every finitely generated R-module M:*

$$H_I^i(M) \cong H_{\hat{I}}^i(M_I^\wedge).$$

The second local cohomology is computed with respect to the completed ring $R_I^\wedge$.

Proof. By [Gre07, Lemma 2.4] since I and M are finitely generated $H_I^i(M) \cong H_I^i(M_I^\wedge)$. Now simply change the base ring along the map $R \to R_I^\wedge$ which preserves local cohomology [Hun07, Proposition 2.14]: $H_I^i(M_I^\wedge) \cong H_{\hat{I}}^i(M_I^\wedge)$. □

Proof of Lemma 4.4.17. Note the chain of isomorphisms:

$$\mathcal{H}^1_{D_{ij}} \cong H^1_{m_{ij}}(\mathcal{O}_{D_{ij}}) \cong H^1_{\hat{m}_{ij}}((\mathcal{O}_{D_{ij}})^\wedge_{m_{ij}}) \cong \frac{(\mathcal{O}_{D_{ij}})^\wedge_{m_{ij}}[\hat{t}_{ij}^{-1}]}{(\mathcal{O}_{D_{ij}})^\wedge_{m_{ij}}}. \tag{4.4.19}$$

The first isomorphism is Lemma 4.3.8. The second one is Lemma 4.4.18. The third isomorphism is simply the computation of local cohomology by means of the stable Koszul complex ([Hun07]), since $\hat{t}_{ij}$ generates $\hat{m}_{ij}$. From this chain of isomorphisms it is clear the module $\mathcal{H}^1_{D_{ij}}$ admits an action of $\hat{t}_{ij}$, and also that is $\hat{t}_{ij}$-divisible. Since $\mathbb{Q}[c_{ij}]$ is a PID, divisible implies injective and $\mathcal{H}^1_{D_{ij}}$ is injective.

The more transparent form of the module (4.4.19) and the one we will use is (4.3.13). From (4.3.13) it is immediate to see that the module is torsion, since $\hat{t}_{ij}$ adds a zero at D_{ij} to every class, making it regular after a finite number of iterations. □

By the previous lemma the module $\mathcal{H}^1_{D_{ij}}$ is torsion injective. Consider the 2-periodic version $\overline{\mathcal{H}^1_{D_{ij}}}$ as before, where c_{ij} acts as an element of degree -2. This is the graded torsion injective $H^*(BG/H_i^j)$-module that we use: $V(H_i^j) := \overline{\mathcal{H}^1_{D_{ij}}} = \overline{\mathcal{K}/\mathcal{O}_{D_{ij}}}$.

To build $\mathbb{I}_1$, define the direct sum

$$T_i := \bigoplus_{j \geq 1} \overline{\mathcal{K}/\mathcal{O}_{D_{ij}}}. \tag{4.4.20}$$

by Corollary 2.5.45 and Corollary 2.5.46 the object

$$\mathbb{I}_1 := \bigoplus_{i \geq 1} f_{H_i}(T_i) \tag{4.4.21}$$

is injective and well defined in $\mathcal{A}(G)$.

4.4.4 Building $\mathbb{I}_2$

Associated to every finite subgroup F of G we have the codimension two subvariety $\bar{\mathfrak{X}}(F)$, and the codimension two piece in the Cousin complex $\mathcal{H}^2_F$. Recall from (2.5.21) the isomorphism $H^*(BG/F) \cong \mathbb{Q}[x_A, x_B]$, and the definition of the ring $\mathcal{O}_F$ and ideal m_F in (4.2.15).

Lemma 4.4.22. *The module $\mathcal{H}^2_F$ is a torsion injective $H^*(BG/F) \cong \mathbb{Q}[x_A, x_B]$-module, where the action is defined by restriction along the ring map*

$$\mathbb{Q}[x_A, x_B] \to (\mathcal{O}_F)^{\wedge}_{m_F}$$

that sends $x_A = e(z_A^{n_A})$ to $\hat{t}_{A,n_A}$, and $x_B = e(z_B^{n_B})$ to $\hat{t}_{B,n_B}$ (defined in (4.2.31)).

Proof. We have the chain of isomorphisms:

$$\mathcal{H}^2_F \cong H^2_{m_F}(\mathcal{O}_F) \cong H^2_{\hat{m}_F}((\mathcal{O}_F)^{\wedge}_{m_F}). \tag{4.4.23}$$

The first isomorphism is Lemma 4.3.8, while the second one is Lemma 4.4.18. From this chain of isomorphisms it is immediate that $\mathcal{H}^2_F$ is an $(\mathcal{O}_F)^{\wedge}_{m_F}$-module, and therefore we can define the action of x_A and x_B as the action of $\hat{t}_{A,n_A}$ and $\hat{t}_{B,n_B}$ since $m_F = \langle t_{A,n_A}, t_{B,n_B}\rangle$. Moreover the module is torsion because $\hat{m}_F = \langle \hat{t}_{A,n_A}, \hat{t}_{B,n_B}\rangle$.

To prove that $\mathcal{H}^2_F$ is an injective module we will split it as a direct sum of local cohomology modules for the Zariski topology and then use regularity. Consider a TP-open $\mathrm{Spec}(R)$ (that we can take to be affine in the Zariski topology) containing $\bar{\mathfrak{X}}(F) = \{Q_1, \dots Q_k\}$ (which is a finite collection of closed points in $\mathcal{X}^{\mathrm{Zar}}$). Modulo restricting the open we can consider m_i to be the maximal ideal in R associated to the closed point Q_i. By Remark 4.2.16: $\mathcal{O}_F = S^{-1}R$ for a multiplicatively closed subset S, therefore in $\mathcal{O}_F$ the ideal $S^{-1}m_i$ remains maximal, and we can write $m_F = S^{-1}m_1 \cdot S^{-1}m_2 \dots S^{-1}m_k$. We can now apply Mayer-Vietoris for local cohomology [Hun07, Theorem 2.3] since $\{Q_1, \dots Q_k\}$ are disjoints, obtaining the desired splitting:

$$H^2_{m_F}(\mathcal{O}_F) \cong H^2_{S^{-1}m_1}(\mathcal{O}_F) \oplus \cdots \oplus H^2_{S^{-1}m_k}(\mathcal{O}_F).$$

We prove now that each factor $H^2_{S^{-1}m_i}(\mathcal{O}_F)$ is injective, so that the direct sum is injective as well. First of all notice that by Remark 4.2.16 if we localize at

$S^{-1}m_i$ we obtain the local ring for the Zariski structure sheaf at the point Q_i:

$$(\mathcal{O}_F)_{S^{-1}m_i} \cong R_{m_i} = \mathcal{O}^{\mathrm{Zar}}_{\mathcal{X},Q_i}.$$

Since the ideal $S^{-1}m_i$ is maximal we obtain:

$$H^2_{S^{-1}m_i}(\mathcal{O}_F) \cong H^2_{m_i}(R_{m_i}) \cong H^2_{\hat{m}_i}((R_{m_i})^{\wedge}_{m_i}) \tag{4.4.24}$$

where first we localize at $S^{-1}m_i$ obtaining local cohomology of a local ring with respect to its maximal ideal [Hoc11, Proposition 8.1], and then we complete with respect to that maximal ideal. The final module of (4.4.24) is injective since $\mathcal{X}^{\mathrm{Zar}}$ is regular. To prove this notice that $(R_{m_i})^{\wedge}_{m_i}$ is a completed Noetherian local ring whose residue field is $\mathbb{C}$, that is therefore isomorphic to a ring of power series $\mathbb{C}[[x_A, x_B]]$ (here we can use x_A and x_B since through the action just defined they are sent to two generators of $\hat{m}_i$). The module 4.4.24 is injective over $\mathbb{C}[[x_A, x_B]]$: the base ring being regular local of dimension 2 is in particular Gorenstein local of dimension 2 and therefore its top local cohomology $H^2_{(x_A,x_B)}(\mathbb{C}[[x_A, x_B]])$ is an injective hull of its residue field [Hoc11, Proposition 11.8]. The module (4.4.24) remains injective over $\mathbb{Q}[x_A, x_B]$ since the scalar extension to $\mathbb{C}[[x_A, x_B]]$ is faithfully flat. $\square$

By the previous lemma the module $\mathcal{H}^2_F$ is torsion injective. Consider the 2-periodic version $\overline{\mathcal{H}^2_F}$ as before, where x_A and x_B act as elements of degree -2. This is the graded torsion injective $H^*(BG/F)$-module that we use: $V(F) := \overline{\mathcal{H}^2_F}$.

To build $\mathbb{I}_2$, simply define the direct sum over all the finite subgroups

$$N := \bigoplus_F \overline{\mathcal{H}^2_F} \tag{4.4.25}$$

by Corollary 2.5.46 the object

$$\mathbb{I}_2 := f_1(N) \tag{4.4.26}$$

is injective and well defined in $\mathcal{A}(G)$.

4.4.5 Building φ_0.

Now we have constructed the relevant injective objects, we turn to constructing maps

$$f_G(\overline{\mathcal{K}}) \xrightarrow{\varphi_0} \bigoplus_{i\geq 1} f_{H_i}(T_i) \xrightarrow{\varphi_1} f_1(N) \to 0 \tag{4.4.27}$$

between them. Recall the definition of $\mathcal{K}$ (4.2.13), T_i (4.4.20), and N (4.4.25).

Each component φ_0^i is determined by an $\mathcal{O}_{\mathcal{F}/H_i}$-map

$$\varphi_0^i : \mathcal{E}_{G/H_i}^{-1}\mathcal{O}_{\mathcal{F}/H_i} \otimes \overline{\mathcal{K}} \to T_i \tag{4.4.28}$$

tensored with $\mathcal{E}_{H_i}^{-1}\mathcal{O}_{\mathcal{F}}$ (Where the rings are defined in (2.5.5)). Moreover we want this map to extend the one in the Cousin complex (4.3.18):

Lemma 4.4.29. *For every $i \geq 1$ there exists an $\mathcal{O}_{\mathcal{F}/H_i}$-map φ_0^i, making the following diagram commute:*

$$\begin{tikzcd} \overline{\mathcal{K}} \arrow[r, "d_0^i"] \arrow[d, "1\otimes\_"'] & T_i \\ \mathcal{E}_{G/H_i}^{-1}\mathcal{O}_{\mathcal{F}/H_i} \otimes \overline{\mathcal{K}} \arrow[ur, dotted, "\varphi_0^i"'] & \end{tikzcd} \tag{4.4.30}$$

where d_0^i is the i-th component of the map in the Cousin complex (4.3.18).

Proof. The first step is to extend d_0^i to an $\mathcal{O}_{\mathcal{F}/H_i}$-map: $\mathcal{O}_{\mathcal{F}/H_i} \otimes \overline{\mathcal{K}} \to T_i$. This is completely determined by the action of $\mathcal{O}_{\mathcal{F}/H_i}$ on the target and the fact that d_0^i is a $\mathbb{Q}$-map.

We need to extend it further to the localization $\mathcal{E}_{G/H_i}^{-1}$, and we can define it for every j-th component $H^*(BG/H_i^j) \cong \mathbb{Q}[c_{ij}]$ (2.5.17). Notice that inverting $\mathcal{E}_{G/H_i}$ in $\mathbb{Q}[c_{ij}]$ means inverting the Euler class $c_{ij} = e(z_i^j)$ (Example 2.5.35). Therefore extend the map for negative powers of c_{ij} in the following way:

$$\begin{aligned} (\varphi_0^i)_j : \mathbb{Q}[{c_{ij}}^{\pm 1}] \otimes \overline{\mathcal{K}} &\to \overline{\mathcal{K}/\mathcal{O}_{D_{ij}}} \\ {c_{ij}}^{-k} \otimes f &\mapsto [\hat{t}_{ij}^{-k} \cdot f] \end{aligned} \tag{4.4.31}$$

where $\hat{t}_{ij}^{-1}$ is the power series inverse of $\hat{t}_{ij}$ (see (4.2.33), (4.2.31), and (4.2.28)) with complex coefficients:

$$\hat{t}_{ij}^{-1} = t_{ij}^{-1} + a_0 + a_1 t_{ij} + a_2 t_{ij}^2 + \dots \tag{4.4.32}$$

and the ring $\mathcal{O}_{D_{ij}}$ and ideal m_{ij} are defined in (4.2.14). The map in (4.4.31) is well defined since for every element in the target there is a power of the ideal m_{ij} that annihilates that element and $m_{ij} = \langle t_{ij} \rangle$. Moreover the map in the Cousin complex

$$d_0^i : \mathcal{K} \to \mathcal{K}/\mathcal{O}_{D_{ij}}$$

is an $\mathcal{O}_{D_{ij}}$-module map, so that after a certain power of t_{ij} the terms in (4.4.32) do not contribute and the sum is finite. $\square$

4.4.6 Building φ_1.

The map φ_1 is also determined by its components

$$\varphi_1^i : \mathcal{E}_{H_i}^{-1}\mathcal{O}_{\mathcal{F}} \underset{\mathcal{O}_{\mathcal{F}/H_i}}{\otimes} T_i \to N.$$

Recall the definition of T_i (4.4.20), N (4.4.25), and the notation 2.5.34 $\otimes_i = \otimes_{\mathcal{O}_{\mathcal{F}/H_i}}$. As before we want this map to extend the one in the Cousin complex (4.3.18):

Lemma 4.4.33. *For every $i \geq 1$ there exists an $\mathcal{O}_{\mathcal{F}}$-map φ_1^i, making the following diagram commute:*

$$\begin{tikzcd} T_i \arrow[r, "d_1^i"] \arrow[d, "1\otimes\_"] & N \\ \mathcal{E}_{H_i}^{-1}\mathcal{O}_{\mathcal{F}} \otimes_i T_i \arrow[ur, dotted, "\varphi_1^i"'] & \end{tikzcd} \tag{4.4.34}$$

where d_1^i is the i-th component of the map in the Cousin complex (4.3.18)*.*

Remark 4.4.35. First of all d_1^i should be an $\mathcal{O}_{\mathcal{F}/H_i}$-map. This is more delicate than it looks, we have defined an $\mathcal{O}_{\mathcal{F}/H_i}$-action on T_i in Lemma 4.4.17 and an $\mathcal{O}_{\mathcal{F}}$-action on N in Lemma 4.4.22 using specific coordinates for every subgroup F. A priori it is not guaranteed that d_1^i coming from the Cousin complex commutes with the action of $\mathcal{O}_{\mathcal{F}/H_i}$ on the source and on the target. Here we are considering $\mathcal{O}_{\mathcal{F}/H_i}$ acting on N by restriction along the ring map $\mathcal{O}_{\mathcal{F}/H_i} \to \mathcal{O}_{\mathcal{F}}$ of Remark 2.5.10. Nonetheless with our choice of coordinates this is indeed the case. In the argument it is crucial that $\hat{t}_e$ is an isomorphism of the formal group law of $\mathcal{C}$ with the additive formal group law.

Lemma 4.4.36. *The i-th component of d_1 in* (4.3.18)*:*

$$d_1^i : T_i \to N \tag{4.4.37}$$

is an $\mathcal{O}_{\mathcal{F}/H_i}$-map with respect to the actions defined in Lemma 4.4.17 and Lemma 4.4.22.

Proof. To simplify the notation in this proof we omit the bar to indicate the 2-periodic version of the various modules. First, the j-th component in the source is $\mathcal{H}^1_{D_{ij}}$ and it is sent into the j-th component in the target: the direct sum on all F finite such that F and H_i generate H_i^j:

$$\mathcal{H}^1_{D_{ij}} \to \bigoplus_{\langle F,H_i\rangle = H_i^j} \mathcal{H}^2_F.$$

This is because the induced map (4.4.37) $\mathcal{H}^1_{D_{ij}} \to \mathcal{H}^2_F$ is non zero if and only if $\bar{\mathfrak{X}}(F) \subseteq D_{ij}$ if and only if $\langle F, H_i \rangle = H_i^j$ by Lemma 4.1.17. Fixed F, by the same lemma the map is non-zero only at the index $j = n_i$.

Therefore it is enough to show that for any finite subgroup F the induced map (4.4.37)

$$d_1^i : \mathcal{H}^1_{D_{i,n_i}} \to \mathcal{H}^2_F \tag{4.4.38}$$

is an $H^*(BG/H_i^{n_i}) \cong \mathbb{Q}[c_{i,n_i}]$-map, namely it commutes with the action of the Euler class $c_{i,n_i} = e(z_i^{n_i})$ on the source and on the target.

By Lemma 4.4.17: c_{i,n_i} acts as $\hat{t}_{i,n_i}$ on $\mathcal{H}^1_{D_{i,n_i}}$. On $\mathcal{H}^2_F$ the action is defined in Lemma 4.4.22: $\mathcal{H}^2_F$ is an $H^*(BG/F) \cong \mathbb{Q}[x_A, x_B]$-module, and $H^*(BG/H_i^{n_i})$ acts by restriction along the F-th component of the inflation map (2.5.25). Therefore c_{i,n_i} acts as

$$x_i = r \cdot x_A + s \cdot x_B \tag{4.4.39}$$

defined in (2.5.23), where r and s are the two integers such that

$$z_i^{n_i} = (z_A^{n_A})^r \cdot (z_B^{n_B})^s \tag{4.4.40}$$

in the character group of G/F. The equality (4.4.39) can be obtained by taking Euler classes on both members of (4.4.40) and noticing that the group operation in the character group translates into sum in $\mathbb{Q}[x_A, x_B]$. Therefore by Lemma 4.4.22 the Euler class c_{i,n_i} acts on $\mathcal{H}^2_F$ as $r \cdot \hat{t}_{A,n_A} + s \cdot \hat{t}_{B,n_B}$.

We are only left to show that these two actions commute with (4.4.38). This translates in proving that the equality

$$\hat{t}_{i,n_i} = r \cdot \hat{t}_{A,n_A} + s \cdot \hat{t}_{B,n_B} \tag{4.4.41}$$

holds in $(\mathcal{O}_F)^{\wedge}_{m_F}$, since $\hat{t}_{i,n_i}$ commutes with the $\mathcal{O}_F$-module map (4.4.38) (recall the definition of the ring $\mathcal{O}_F$ and the ideal m_F in (4.2.15)).

To prove (4.4.41) apply the functor $\mathfrak{X}$ to the characters equality (4.4.40), to obtain the same linear relation with the projections (Definition 4.1.9):

$$\pi_i^{n_i} = r \cdot \pi_A^{n_A} + s \cdot \pi_B^{n_B}.$$

Note that in this last equality the group operation is the one of $\mathcal{C}$. Recalling the definition of the completed coordinates (4.2.31), combined with Lemma 4.2.25 we obtain precisely:

$$\hat{t}_{i,n_i} = (\pi_i^{n_i})^*(\hat{t}_e) = (\pi_A^{n_A}, \pi_B^{n_B})^* \circ (r,s)^*(\hat{t}_e) = r \cdot \hat{t}_{A,n_A} + s \cdot \hat{t}_{B,n_B}$$

□

Remark 4.4.42. This proof clarifies the need of using completed coordinates $\hat{t}_{i,n_i}$ instead of the uncompleted ones t_{i,n_i}. For the uncompleted functions the relation (4.4.41) is not true (compute zeroes and poles on both sides). As a consequence the algebra of the Euler classes simply does not match the algebra of the uncompleted functions.

Proof of Lemma 4.4.33. Proceed as we did in Lemma 4.4.29 and first extend d_1^i to an $\mathcal{O}_{\mathcal{F}}$-map

$$\mathcal{O}_{\mathcal{F}} \underset{\mathcal{O}_{\mathcal{F}/H_i}}{\otimes} T_i \to N.$$

This is completely determined by the action of $\mathcal{O}_{\mathcal{F}}$ on N and Lemma 4.4.36.

We need to extend it further to the localization $\mathcal{E}_{H_i}^{-1}$, and we can define it for every F-th component of $\mathcal{O}_{\mathcal{F}}$: $H^*(BG/F) \cong \mathbb{Q}[x_A, x_B]$. Recall from Example 2.5.35 that inverting $\mathcal{E}_{H_i}^{-1}$ in $\mathbb{Q}[x_A, x_B]$ means inverting all the Euler classes x_j with $j \geq 1$ and $j \neq i$. Therefore define the map for negative powers of the Euler classes in the following way:

$$\begin{aligned} (\varphi_1^i)_F : \mathcal{E}_{H_i}^{-1}\mathbb{Q}[x_A, x_B] \underset{\mathbb{Q}[x_i]}{\otimes} \overline{\mathcal{K}/\mathcal{O}_{D_{i,n_i}}} &\to \overline{\mathcal{H}_F^2} \\ \frac{1}{x_1^{k_1} x_2^{k_2} \dots x_r^{k_r}} \underset{\mathbb{Q}[x_i]}{\otimes} [f] &\mapsto d_1^i([\hat{t}_{1,n_1}^{-k_1} \cdot \hat{t}_{2,n_2}^{-k_2} \dots \hat{t}_{r,n_r}^{-k_r} \cdot f]) \end{aligned} \tag{4.4.43}$$

and extend it to be a $\mathbb{Q}[x_A, x_B]$-module map (see (4.4.32) for the power series inverses).

Remark 4.4.44. The main issue with this definition is that the element

$$\hat{t}_{1,n_1}^{-k_1} \cdot \hat{t}_{2,n_2}^{-k_2} \dots \hat{t}_{r,n_r}^{-k_r} \tag{4.4.45}$$

is a priori an infinite sum of monomials $t_{1,n_1}^{j_1} \cdot t_{2,n_2}^{j_2} \dots t_{r,n_r}^{j_r}$ and not a well defined element in $\mathcal{K}$. To address this issue note that every element in $\mathcal{H}_F^2$ is annihilated by a power of the ideal m_F, that every coordinate t_{j,n_j} is in m_F, and that d_1^i is an $\mathcal{O}_F$-module map. Therefore there exists a positive integer d such that in the infinite sum (4.4.45), elements of total degree higher than d do not contribute in any way in (4.4.43). In this way we can cap the infinite sum (4.4.45) accordingly so that the sum is finite and therefore is a well defined element of $\mathcal{K}$.

It is immediate now to check (4.4.43) is compatible with the already defined action of the positive powers of x_j for $j \neq i$. Namely t_{j,n_j} commutes with d_1^i, and they simplify with the capped $\hat{t}_{j,n_j}^{-1}$ anyway since we are left with terms of higher enough degree to not contribute to the result.

It is immediate also to check that the definition (4.4.43) does not depend on the representative f picked for the class, since if $[f+g]$ is another representative with g regular on D_{i,n_i}, then $g \cdot \hat{t}_{j,n_j}^{-1}$ is regular on D_{i,n_i} for any $j \neq i$ and does not contribute in any way. $\square$

4.4.7 Exactness

We turn now to prove that the sequence (4.4.1) is exact. From Remark 2.5.38 we only need to check exactness at the bottom level (the level at the trivial subgroup). Therefore it is enough to prove that the bottom level of (4.4.1):

$$\mathcal{E}_G^{-1}\mathcal{O}_{\mathcal{F}} \otimes \overline{\mathcal{K}} \xrightarrow{\varphi_0} \bigoplus_{i \geq 1} \mathcal{E}_{H_i}^{-1}\mathcal{O}_{\mathcal{F}} \underset{i}{\otimes} T_i \xrightarrow{\varphi_1} N \to 0 \tag{4.4.46}$$

is an exact sequence of $\mathcal{O}_{\mathcal{F}}$-modules. We can do it for each F-th component of $\mathcal{O}_{\mathcal{F}}$ at a time. Namely prove that for every finite subgroup F of G, the F-th component of (4.4.46):

$$\mathcal{E}_G^{-1}\mathbb{Q}[x_A, x_B] \otimes \overline{\mathcal{K}} \xrightarrow{\varphi_0} \bigoplus_{i \geq 1} \mathcal{E}_{H_i}^{-1}\mathbb{Q}[x_A, x_B] \underset{i}{\otimes} \overline{\mathcal{K}/\mathcal{O}_{D_{i,n_i}}} \xrightarrow{\varphi_1} \overline{\mathcal{H}_F^2} \to 0. \tag{4.4.47}$$

is an exact sequence of $H^*(BG/F) \cong \mathbb{Q}[x_A, x_B]$-modules.

The strategy is to use exactness of (4.3.15). For this task we will need the "Moving" Lemma 4.4.50. In the interest of the exposition we start in proving exactness of (4.4.47) and will devote the rest of the section in proving Lemma 4.4.50.

Lemma 4.4.48. *The sequence (4.4.47) is an exact sequence of $\mathbb{Q}[x_A, x_B]$-modules.*

Proof. We use exactness of the sequence (4.3.15) which is the first row of the following commutative diagram:

$$\begin{tikzcd}
\overline{\mathcal{K}} \arrow[r, "d_0"] \arrow[d, "1\otimes_"] & \bigoplus_{i\geq 1} \overline{\mathcal{K}/\mathcal{O}_{D_{i,n_i}}} \arrow[r, "d_1"] \arrow[d, "\iota"] & \overline{\mathcal{H}_F^2} \arrow[r] \arrow[d, "\mathrm{Id}"] & 0 \\
\mathcal{E}_G^{-1}\mathbb{Q}[x_A, x_B] \otimes \overline{\mathcal{K}} \arrow[r, "\varphi_0"] & \bigoplus_{i\geq 1} \mathcal{E}_{H_i}^{-1}\mathbb{Q}[x_A, x_B] \otimes_i \overline{\mathcal{K}/\mathcal{O}_{D_{i,n_i}}} \arrow[r, "\varphi_1"] & \overline{\mathcal{H}_F^2} \arrow[r] & 0
\end{tikzcd} \tag{4.4.49}$$

The vertical map ι is the natural inclusion $\alpha \mapsto 1 \otimes_i \alpha$ on every i-th component. Note that the commutativity of (4.4.49) is due to the commutativity of (4.4.30) and (4.4.34).

First of all the composition $\varphi_1 \circ \varphi_0$ is zero. This is because on pure tensor elements in the source:

$$\varphi_1 \circ \varphi_0(\frac{1}{x_1^{k_1} x_2^{k_2} \dots x_r^{k_r}} \otimes f) = d_1 \circ d_0(\hat{t}_{1,n_1}^{-k_1} \cdot \hat{t}_{2,n_2}^{-k_2} \dots \hat{t}_{r,n_r}^{-k_r} \cdot f) = 0.$$

The first equality comes from the two definitions (4.4.31), and (4.4.43). This all equals zero since the composition $d_1 \circ d_0$ is zero in the first row.

The map φ_1 is surjective being an extension of d_1, which is surjective.

We are left to prove that $\mathrm{Ker}(\varphi_1) \subseteq \mathrm{Im}(\varphi_0)$. Pick $X \in \mathrm{Ker}(\varphi_1)$. By Lemma 4.4.50, modulo $\mathrm{Im}(\varphi_0)$ we can suppose $X \in \mathrm{Im}(\iota)$. We can now conclude with diagram chasing: it exists Y such that $\iota(Y) = X$, $d_1(Y) = 0$ and since the first row of (4.4.49) is exact it also exists $F \in \overline{\mathcal{K}}$ such that $d_0(F) = Y$. Therefore $\varphi_0(1 \otimes F) = X$. □

We devote the rest of this section in proving the "Moving" lemma:

Lemma 4.4.50. *In the diagram* (4.4.49)*, every element*

$$X \in \bigoplus_{i \geq 1} \mathcal{E}_{H_i}^{-1} \mathbb{Q}[x_A, x_B] \underset{i}{\otimes} \overline{\mathcal{K}/\mathcal{O}_{D_{i,n_i}}}$$

is equivalent, modulo the image of φ_0 to an element in the image of ι.

To prove this technical lemma we will need to work with the irreducible components $D_{i,P}$ of the D_{ij} in the Zariski topology, and build different uniformizers for them. We will also require somme lemmas about the specific geometry of $\mathcal{X}$.

We start recalling Abel's Theorem [Sil09, Corollary 3.5]:

Lemma 4.4.51 (Abel's Theorem for elliptic curves)**.** *Let $\mathcal{C}$ be an elliptic curve with identity element e, and let $D = \sum n_P(P)$ be a divisor of $\mathcal{C}$. Then there exists a meromorphic function f such that* $\mathrm{Div}(f) = D$ *if and only if*

$$\sum n_P = 0 \quad \textit{and} \quad \sum n_P P = e.$$

Where the first sum is in $\mathbb{Z}$ and the second one is addition in $\mathcal{C}$.

Lemma 4.4.52. *Given $P \in \mathcal{C}\langle n \rangle$ and a direction $i \geq 1$ there exists $h_{i,P} \in \mathcal{K}$ such that:*

1. *for $n \neq 1$ the function $h_{i,P}$ has zeroes only at $D_{i,P}$ and D_i both at first order.*

2. *for $n = 1$ the function $h_{i,e}$ has zeroes only at D_i and $D_{i,Q}$ for Q a point of exact order 2, both at first order.*

Proof. If $P = e$ and $\{P_1, P_2, P_3\}$ are the three points of $\mathcal{C}\langle 2 \rangle$ then the divisor

$$D_e := (e) + (P_1) - (P_2) - (P_3)$$

satisfies Lemma 4.4.51 so there is a meromorphic function h_e with divisor D_e. If we now pullback h_e along the projection π_i the function

$$h_{i,e} := \pi_i^*(h_e) \in \mathcal{K}$$

satisfies (2).

In the same way if $P \neq e$ pick a point P' such that $2P' = P$, then the divisor

$$D_P := (e) + (P) - 2(P')$$

satisfies Lemma 4.4.51 so there's a meromorphic function h_P with divisor D_P. As before the pullback along the projection π_i the function

$$h_{i,P} := \pi_i^*(h_P) \in \mathcal{K}$$

satisfies (1). $\square$

Lemma 4.4.53. *Consider two different arbitrary directions $a, b \geq 1$ such that D_a and D_b intersect only in the origin $O = \{e\} \times \{e\}$ of $\mathcal{X}$. Given a class $[f] \in \mathcal{K}/\mathcal{O}_{D_{a,n}}$ for $n \geq 1$, there exists a representative $f \in \mathcal{K}$ of that class such that f has poles only along the a and b directions (i.e. it's regular on all D_{ij} such that $i \neq a, b$).*

Proof. Since D_a and D_b intersect only in O, we can change coordinates (autoisogeny) on $\mathcal{X}$ in such a way that $D_a = D_1 = \{e\} \times \mathcal{C}$ and $D_b = D_2 = \mathcal{C} \times \{e\}$.

For any point P of finite order of $\mathcal{C}$ and direction $i \geq 1$, denote $\mathcal{O}_{D_{i,P}}$ the subring of $\mathcal{K}$ of those functions that are regular at $D_{i,P}$, and $m_{i,P}$ the ideal of those functions that vanishes at $D_{i,P}$. The pullback along the projection $\pi_2 : \mathcal{X} \to \mathcal{C}$ induces an isomorphism

$$\pi_2^* : \mathcal{K}_{\mathbb{T}} \xrightarrow{\cong} \mathcal{O}_{D_{1,P}}/m_{1,P} \tag{4.4.54}$$

where $\mathcal{K}_{\mathbb{T}}$ is defined in (4.2.19): the ring of those meromorphic functions on $\mathcal{C}$ that have poles only at points of finite order of $\mathcal{C}$. First notice that with the TP-topology on $\mathcal{C}$ we pullback functions in $\mathcal{K}$ that are regular on $D_{1,P}$. To prove injectivity of (4.4.54) consider the inclusion $\iota_{1,P} : D_{1,P} \cong \mathcal{C} \hookrightarrow \mathcal{X}$ and the pullback

$$\mathcal{O}_{D_{1,P}}/m_{1,P} \xrightarrow{\iota^*_{1,P}} \mathcal{K}_{\mathbb{T}}.$$

Composing the two maps $\iota^*_{1,P} \circ \pi^*_2 = (\pi_2 \circ \iota_{1,P})^* = \mathrm{Id}^*_{\mathcal{C}}$, so π^*_2 is injective. To prove surjectivity of (4.4.54) pick $f \in \mathcal{O}_{D_{1,P}}$, then

$$f - \pi^*_2 \circ \iota^*_{1,P}(f) \in m_{1,P}$$

because $\iota^*_{1,P}(f - \pi^*_2 \circ \iota^*_{1,P}(f)) = \iota^*_{1,P}(f) - \iota^*_{1,P}(f) = 0$.

Consider now a class $[f] \in \mathcal{K}/\mathcal{O}_{D_{1,n}}$ and an arbitrary representative $f \in \mathcal{K}$ of that class. If the class is the trivial one, pick any pullback $\pi^*_2(g)$ as representative. Otherwise f is not regular on $D_{1,n}$, which means it has a pole on some of its components $D_{1,P}$ with $P \in \mathcal{C}\langle n\rangle$. Enumerate those components $\{D_{1,P_1}, D_{1,P_2}, \ldots, D_{1,P_r}\}$, and suppose f has a pole of order $k_j \geq 1$ on D_{1,P_j}. Expand f into its principal parts at every pole D_{1,P_j} using coefficients in $\mathcal{O}_{D_{1,P_j}}/m_{1,P_j} \cong \mathcal{K}_{\mathbb{T}}$ and the uniformizer h_{1,P_j} of D_{1,P_j} (By Lemma 4.4.52: $m_{1,P_j} = \langle h_{1,P_j}\rangle$). Recursively:

$$f \cdot h^{k_1}_{1,P_1} \in \mathcal{O}_{D_{1,P_1}} \setminus m_{1,P_1} \tag{4.4.55}$$

therefore using the isomorphism (4.4.54) there exists $g \in \mathcal{K}_{\mathbb{T}}$ whose pullback represents the same class modulo the ideal m_{1,P_1}:

$$f \cdot h^{k_1}_{1,P_1} \equiv \pi^*_2 g \quad (m_{1,P_1})$$

so that dividing by h_{1,P_1}:

$$f \cdot h^{k_1-1}_{1,P_1} - \frac{\pi^*_2 g}{h_{1,P_1}} \in \mathcal{O}_{D_{1,P_1}} \setminus m_{1,P_1}$$

we can reapply (4.4.55) until we obtain an expansion at the pole D_{1,P_1} with coefficients that are pullbacks of functions $g_{i,1} \in \mathcal{K}_{\mathbb{T}}$:

$$f - \sum_{i=-k_1}^{-1} \frac{\pi^*_2(g_{i,1})}{h^i_{1,P_1}} \in \mathcal{O}_{D_{1,P_1}} \tag{4.4.56}$$

Now move on to the next pole D_{1,P_2} and do the same using (4.4.56) instead of f. Notice that by Lemma 4.4.52 h_{1,P_j} does not vanish at D_{1,P_i} for $P_j \neq P_i$ both in $\mathcal{C}\langle n\rangle$, in this way we can deal with the pole at each D_{1,P_j} separately without changing the order of pole on the other components.

Continue in this way expanding all the poles $\{D_{1,P_1}, D_{2,P_2}, \dots, D_{r,P_r}\}$ until we have something that is regular on the whole $D_{1,n}$:

$$f - \sum_{i=-k_1}^{-1} \frac{\pi_2^*(g_{i,1})}{h_{1,P_1}^i} - \dots - \sum_{i=-k_r}^{-1} \frac{\pi_2^*(g_{i,r})}{h_{1,P_r}^i} \in \mathcal{O}_{D_{1,n}}. \tag{4.4.57}$$

This gives us an explicit function in the same class of f modulo $\mathcal{O}_{D_{1,n}}$, but with poles only along the directions 1 and 2. $\square$

Proof of Lemma 4.4.50. Doing it for one i-component at a time we can suppose X has only one component different from zero. Therefore X has all components equal to zero except the s-th component:

$$X_s = \frac{1}{x_{c_1}^{k_1} x_{c_2}^{k_2} \dots x_{c_r}^{k_r}} \underset{s}{\otimes} [f] \in \mathcal{E}_{H_s}^{-1}\mathbb{Q}[x_A, x_B] \underset{s}{\otimes} \overline{\mathcal{K}/\mathcal{O}_{D_{s,n_s}}} \tag{4.4.58}$$

where at the denominator we have inverted $r \geq 0$ Euler classes $x_{c_1}, \dots x_{c_r} \in \mathcal{E}_{H_s}$ (Example 2.5.35) with $c_j \geq 1$ for $1 \leq j \leq r$. Let us prove the lemma by induction on r. If there are no variables at the denominator ($r = 0$), then X is in the image of ι. Therefore we only need to prove we can reduce r by one.

Let us first do the case when one of the directions $\{D_{c_1}, \dots, D_{c_r}\}$ intersects D_s only in the origin $O = \{e\} \times \{e\}$, and without lost of generality suppose it is the direction D_{c_1}. Applying Lemma 4.4.53 we can find a representative $f \in \mathcal{K}$ of the class $[f] \in \mathcal{K}/\mathcal{O}_{D_{s,n_s}}$ such that f has poles only along the directions s and c_1. Consider the element

$$X' := \frac{1}{x_{c_1}^{k_1} x_{c_2}^{k_2} \dots x_{c_r}^{k_r}} \otimes f \in \mathcal{E}_G^{-1}\mathbb{Q}[x_A, x_B] \otimes \overline{\mathcal{K}}.$$

Then the element $X - \varphi_0(X')$:

- It has a zero in the component s.
- In the component c_j, with $1 \leq j \leq r$ it has less than r Euler classes inverted at the denominator, since x_{c_j} itself is not inverted any more.
- In all the other components it has a zero, since f has poles only along the directions s and c_j.

In case none of the directions $\{D_{c_1}, \dots, D_{c_r}\}$ intersects D_s trivially, simply pick an extra direction D_d that intersects D_s and D_{c_1} only in O. Now apply the same argument twice. The first time use D_d instead of D_{c_1} and notice that we have decreased the Euler classes inverted in all components except in the component d. Then apply the same argument with the component d instead of s, noticing that now D_d and D_{c_1} intersect in only one point.

We can apply the same argument in case of a numerator in the component s of X (4.4.58) different from 1. Note that $\mathbb{Q}[x_A, x_B] = \mathbb{Q}[x_s, x_{c_1}]$, and distributing sums and products we are left with the case

$$X_s = x_{c_1}^k \underset{s}{\otimes} [f]$$

with D_{c_1} intersecting D_s only in O. Apply again Lemma 4.4.53 with the directions s and c_1 and proceed exactly as before. □

4.5 Values on spheres of complex representations

The goal of this section is to determine the values of $E\mathcal{C}_G$ (constructed in Section 4.4 for the 2-torus $G = \mathbb{T}^2$) on spheres of complex representations. Here resides the connection with the geometry of the curve:

Theorem 4.5.1. *If V is a finite dimensional complex representation of G with $V^G = 0$, then:*

$$E\mathcal{C}_G^n(S^V) \cong \begin{cases} H^0(\mathcal{X}, \mathcal{O}(-D_V)) \oplus H^2(\mathcal{X}, \mathcal{O}(-D_V)) & n \text{ even} \\ H^1(\mathcal{X}, \mathcal{O}(-D_V)) & n \text{ odd.} \end{cases} \tag{4.5.2}$$

Where $\mathcal{X} = \mathcal{C} \times \mathcal{C}$, $\mathcal{C}$ is our fixed elliptic curve, and D_V is a divisor of $\mathcal{X}$ defined as follows:

Definition 4.5.3. If V is a finite dimensional complex representation of G with $V^G = 0$ and dimension function:

$$v_{ij} := \dim_{\mathbb{C}}(V^{H_i^j}) \qquad ij \geq 1 \tag{4.5.4}$$

where H_i^j is the subgroup of G with j connected components and identity component H_i, then

$$D_V = \sum_{i,j\geq 1} v_{ij} D_{ij} \tag{4.5.5}$$

Remark 4.5.6. Notice that if we write V using characters z_i^n (see Definition 4.1.9):

$$V = \sum_{i\geq 1} \sum_{n\neq 0} a_{i,n} z_i^n$$

then the associated divisor is obtained simply applying the functor $\mathfrak{X}$ (see (4.1.7) and (4.1.8)):

$$D_V := \sum_{i\geq 1}\sum_{n\neq 0} a_{in}\mathfrak{X}(H_i^{|n|}).$$

Definition 4.5.7. Given a divisor D_V of the kind (4.5.5), define the sheaf $\mathcal{O}(-D_V)$ on $\mathcal{X}^{\mathrm{TP}}$ twisted by the line bundle D_V to be the subsheaf of the structure sheaf $\mathcal{O} = \mathcal{O}_{\mathcal{X}}^{\mathrm{TP}}$ with values on TP-opens U:

$$\mathcal{O}(-D_V)(U) = \{f \in \mathcal{K} \mid \mathrm{Div}(f) - D_V \geq 0 \text{ on } U\}.$$

(Recall the definition of $\mathcal{K}$ (4.2.13)).

The strategy to prove (4.5.2) is to use the Adams spectral sequence (2.5.2) for the homology functor $\pi_*^{\mathcal{A}}$ [Gre08, Theorem 1.1]. In our case we obtain a strongly convergent Adams spectral sequence:

$$\mathrm{Ext}_{\mathcal{A}}^{s,t}(S^V, E\mathcal{C}_G) \Longrightarrow [S^V, E\mathcal{C}_G]^G_{t-s} = E\mathcal{C}_G^{t-s}(S^V). \tag{4.5.8}$$

Notation 4.5.9. In (4.5.8) and for the rest of the section we denote S^V both the spectrum and the corresponding object $\pi_*^{\mathcal{A}}(S^V)$ in $\mathcal{A}(G)$.

Remark 4.5.10. Recall that the index s refers to the s-th Ext-group in the graded abelian category $\mathcal{A}(G)$ while t index the grading of these groups, that are graded since the objects $\pi_*^{\mathcal{A}}(\_)$ are.

The bulk of the section will be the computation of the Ext groups of (4.5.8):

Theorem 4.5.11. *For V as in Theorem 4.5.1 we have the isomorphism of graded groups:*

$$\mathrm{Ext}_{\mathcal{A}}^{s}(S^V, E\mathcal{C}_G) \cong \overline{H^s(\mathcal{X}, \mathcal{O}(-D_V))}. \tag{4.5.12}$$

Where on the right in (4.5.12) we have the 2-periodic version of those cohomology groups.

As an immediate consequence we obtain (4.5.2):

proof of Theorem 4.5.1. The isomorphism (4.5.12) computes the second page of the Adams Spectral sequence (4.5.8). This second page has only the first three rows different from zero (namely for $s = 0, 1, 2$ in (4.5.12)), because the algebraic surface $\mathcal{X}$ has dimension 2 and therefore its cohomology groups vanishes in higher degrees. Moreover in these three rows we have a chess pattern of zeroes and cohomology groups since the groups (4.5.12) are 2-periodic. This yields the Adams spectral sequence (4.5.8) to collapse at the second page, since all the differentials in this page are trivial. In conclusion in this second page we find in the even columns:

$$H^0(\mathcal{X}, \mathcal{O}(-D_V)) \oplus H^2(\mathcal{X}, \mathcal{O}(-D_V))$$

while in the odd ones:

$$H^1(\mathcal{X}, \mathcal{O}(-D_V)).$$

□

We devote the rest of this section in proving Theorem 4.5.11. For this task in 4.5.1 we discuss the Cousin complex of the sheaf $\mathcal{O}(-D_V)$ needed in 4.5.2 to compute the second page of the Adams spectral sequence. We conclude in 4.5.3 briefly extending the computation on virtual negative complex representations (4.5.34).

4.5.1 Cousin complex of $\mathcal{O}(-D_V)$

Here we briefly construct the Cousin complex for $\mathcal{O}(-D_V)$ exactly as in Sections 4.2 and 4.3 for the structure sheaf $\mathcal{O}$. Therefore we recall from those sections how the results change for the sheaf twisted by the line bundle $\mathcal{O}(-D_V)$.

Let us start in computing the stalks at the various points $x \in \mathcal{X}^{\mathrm{TP}}$:

- If $x = \eta(\mathcal{X}^{\mathrm{TP}})$ is a generic point of the whole space, in (4.2.12) nothing changes and we still obtain (4.2.13):

$$\mathcal{O}(-D_V)_x = \mathcal{K} \tag{4.5.13}$$

- If $x = \eta(D_{ij})$ is a generic point of a generating closed subset, by (4.5.5) D_{ij} appears with coefficient v_{ij} in D_V, so (4.2.14) becomes:

$$\mathcal{O}(-D_V)_x = \{\mathrm{Div}(f) \geq v_{ij} \text{ on } D_{ij}\} = t_{ij}^{v_{ij}} \mathcal{O}_{D_{ij}} \tag{4.5.14}$$

  since t_{ij} defined in (4.2.28) generates m_{ij}.

- If $x \in \bar{\mathfrak{X}}(F)$ (which automatically makes it also a generic point for $\bar{\mathfrak{X}}(F)$), by Lemma 4.1.17 for every direction $i \geq 1$, only D_{i,n_i} contains $\bar{\mathfrak{X}}(F)$ which by (4.5.5) appears with coefficient v_{i,n_i} in D_V. Therefore (4.2.15) becomes:

$$\mathcal{O}(-D_V)_x = \{f \in \mathcal{K} \mid \mathrm{Div}(f) \geq v_{i,n_i} \text{ on } D_{i,n_i} \text{ for all } i \geq 1\} = (\prod_{i \geq 1} t_{i,n_i}^{v_{i,n_i}})\mathcal{O}_F \tag{4.5.15}$$

Proposition 4.3.1 apply as well to the sheaf $\mathcal{O}(-D_V)$ and therefore we can consider its Cousin complex, that can be written as the one for $\mathcal{O}$ (4.3.2):

$$\mathcal{O}(-D_V) \to \iota_{\mathcal{X}}(\mathcal{H}^0_{\mathcal{X}}(\mathcal{O}(-D_V))) \xrightarrow{d_0^V} \bigoplus_{ij\geq 1} \iota_{D_{ij}}(\mathcal{H}^1_{D_{ij}}(\mathcal{O}(-D_V))) \xrightarrow{d_1^V} \bigoplus_{F} \iota_F(\mathcal{H}^2_F(\mathcal{O}(-D_V))) \tag{4.5.16}$$

For each local cohomology term appearing in (4.5.16) we explicit an isomorphism with the corresponding term in (4.3.2):

- In codimension 0 by (4.5.13):

$$\mathcal{H}^0_{\mathcal{X}}(\mathcal{O}(-D_V)) = \mathcal{O}(-D_V)_{\eta(\mathcal{X}^{\mathrm{TP}})} = \mathcal{K} \tag{4.5.17}$$

- In codimension 1 we have the chain of isomorphisms:

$$\mathcal{H}^1_{D_{ij}}(\mathcal{O}(-D_V)) \cong H^1_{m_{ij}}((\mathcal{O}(-D_V))_{\eta(D_{ij})}) \cong \frac{t_{ij}^{v_{ij}}\mathcal{O}_{D_{ij}}[t_{ij}^{-1}]}{t_{ij}^{v_{ij}}\mathcal{O}_{D_{ij}}} = \frac{\mathcal{K}}{t_{ij}^{v_{ij}}\mathcal{O}_{D_{ij}}} \cong \frac{\mathcal{K}}{\mathcal{O}_{D_{ij}}}. \tag{4.5.18}$$

  The first isomorphism is due by (4.3.10) since $\mathcal{O}(-D_V)$ is the pushforward along φ of the Zariski twisted sheaf $\mathcal{O}^{\mathrm{Zar}}_{\mathcal{X}}(-D_V)$. Te second is the computation of local cohomology by means of the stable Koszul complex ([Hun07, pag. 7]), since t_{ij} generates m_{ij}, and we have computed the stalk in (4.5.14). The final isomorphism we define it in the following way using the completed coordinates (4.2.31):

$$\begin{aligned} \mathcal{K}/t_{ij}^{v_{ij}}\mathcal{O}_{D_{ij}} &\xrightarrow{\cong} \mathcal{K}/\mathcal{O}_{D_{ij}} \\ [f] &\mapsto [\hat{t}_{ij}^{-v_{ij}} \cdot f]. \end{aligned} \tag{4.5.19}$$

  Notice it is well defined exactly as in (4.4.31).

- In codimension 2 in the same way we have the chain of isomorphisms:

$$\mathcal{H}^2_F(\mathcal{O}(-D_V)) \cong H^2_{m_F}((\mathcal{O}(-D_V))_{\eta(\bar{\mathfrak{x}}(F))}) \cong H^2_{m_F}((\prod_{i\geq 1} t_{i,n_i}^{v_{i,n_i}})\mathcal{O}_F) \cong H^2_{m_F}(\mathcal{O}_F). \tag{4.5.20}$$

  Like (4.5.18) the first isomorphism is due by (4.3.10) and the second one is the computation of the stalk (4.5.15). We can define the final isomorphism in the following way:

$$\begin{aligned} H^2_{m_F}((\prod_{i\geq 1} t_{i,n_i}^{v_{i,n_i}})\mathcal{O}_F) &\xrightarrow{\cong} H^2_{m_F}(\mathcal{O}_F) \\ \alpha &\mapsto (\prod_{i\geq 1} \hat{t}_{i,n_i}^{-v_{i,n_i}})\alpha. \end{aligned} \tag{4.5.21}$$

Notice it is well defined as in Remark 4.4.44.

Now Corollary 4.3.7 holds as well for $\mathcal{O}(-D_V)$ since $\mathcal{O}_{\mathcal{X}}^{\mathrm{Zar}}(-D_V)$ is Cohen-Macaulay with respect to the codimension filtration:

Corollary 4.5.22. *The Cousin complex (4.5.16) of $\mathcal{O}(-D_V)$ is a flabby resolution of $\mathcal{O}(-D_V)$.*

As a consequence we have also Proposition 4.3.14 for $\mathcal{O}(-D_V)$ where we use the isomorphisms (4.5.17), (4.5.18) and (4.5.20) to describe the local cohomology terms:

Proposition 4.5.23. *Let F be a finite subgroup of G and $x = \eta(\bar{\mathfrak{X}}(F))$ be the generic (and only) point of $\bar{\mathfrak{X}}(F)$ in the Kolmogorov quotient $\mathcal{X}^{TP}$. Then the TP-stalk at x of the Cousin complex (4.5.16) is the exact sequence:*

$$\mathcal{O}(-D_V)_x \rightarrowtail \mathcal{K} \xrightarrow{d_0^V} \bigoplus_{i\geq 1} \mathcal{K}/\mathcal{O}_{D_{i,n_i}} \xrightarrow{d_1^V} \mathcal{H}^2_F(\mathcal{O}) \to 0. \tag{4.5.24}$$

Remark 4.5.25. Notice (4.5.24) has the same terms as (4.3.15) but different maps:

$$d_0^V(f) = [\hat{t}_{ij}^{-v_{ij}} \cdot f] \in \mathcal{K}/\mathcal{O}_{D_{ij}}. \tag{4.5.26}$$

$$d_1^V(\{[f_i]\}_i) = [\{[f_i \cdot \prod_{s\neq i} \hat{t}_{s,n_s}^{-v_{s,n_s}}]\}_i] \in \mathcal{H}^2_F(\mathcal{O}). \tag{4.5.27}$$

Where we have used (4.3.17) to describe $\mathcal{H}^2_F(\mathcal{O})$.

We conclude with the global sections of the Cousin complex (4.5.16):

$$\Gamma(\mathcal{O}(-D_V)) \longrightarrow \mathcal{K} \xrightarrow{d_0^V} \bigoplus_{i\geq 1}(\bigoplus_{j\geq 1} \mathcal{K}/\mathcal{O}_{D_{ij}}) \xrightarrow{d_1^V} \bigoplus_F \mathcal{H}^2_F(\mathcal{O}) \to 0 \tag{4.5.28}$$

Notice that the terms are the same as the global sections (4.3.18) for $\mathcal{O}$ but with different maps.

4.5.2 Computing the Adams spectral sequence

We turn now in computing the second page of the Adams spectral sequence (4.5.8), namely proving Theorem 4.5.11.

Proof of Theorem 4.5.11. The aim is to explicitly compute the following sequence:

$$\mathrm{Hom}_{\mathcal{A}}(S^V, \mathbb{I}_0) \xrightarrow{\varphi_0'} \mathrm{Hom}_{\mathcal{A}}(S^V, \mathbb{I}_1) \xrightarrow{\varphi_1'} \mathrm{Hom}_{\mathcal{A}}(S^V, \mathbb{I}_2) \to 0 \tag{4.5.29}$$

obtained applying the functor $\mathrm{Hom}_{\mathcal{A}}(S^V, \_)$ to the injective resolution (4.4.4) of $E\mathcal{C}_G$ (the maps φ_0' and φ_1' are the ones induced by φ_0 and φ_1). For this task it is essential the algebraic model $S^V = \pi_*(S^V) \in \mathcal{A}(G)$ that we discussed in detail in 2.5.6.

By using the adjunction (2.5.44) and the explicit form (4.4.27) of the injective resolution of $E\mathcal{C}_G$ we can compute each term of (4.5.29):

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{A}}(S^V, f_G(\overline{\mathcal{K}})) &\cong \mathrm{Hom}_{\mathbb{Q}}(\mathbb{Q}, \overline{\mathcal{K}}) \cong \overline{\mathcal{K}} \\
\mathrm{Hom}_{\mathcal{A}}(S^V, \bigoplus_{i\geq 1} f_{H_i}(T_i)) &\cong \bigoplus_{i\geq 1} \mathrm{Hom}_{\mathcal{O}_{\mathcal{F}/H_i}}(\Sigma^{V^{H_i}} \mathcal{O}_{\mathcal{F}/H_i} T_i) \cong \bigoplus_{i\geq 1} \Sigma^{-V^{H_i}} T_i, \\
\mathrm{Hom}_{\mathcal{A}}(S^V, f_1(N)) &\cong \mathrm{Hom}_{\mathcal{O}_{\mathcal{F}}}(\Sigma^V \mathcal{O}_{\mathcal{F}}, N) \cong \Sigma^{-V} N
\end{aligned} \tag{4.5.30}$$

where for the second equation we obtain the direct sum for $i \geq 1$ instead of the product since S^V is a small object.

The sequence (4.5.29) than takes the form

$$\overline{\mathcal{K}} \xrightarrow{\varphi_0'} \bigoplus_{i\geq 1} \Sigma^{-V^{H_i}} T_i \xrightarrow{\varphi_1'} \Sigma^{-V} N \to 0 \tag{4.5.31}$$

with maps:

$$\begin{aligned}
\varphi_0'(f) &= \{\varphi_0^i(e(V^{H_i})^{-1} \otimes f)\}_{i\geq 1} \\
\varphi_1'(\{\alpha_i\}_{i\geq 1}) &= \varphi_1(\{e(V - V^{H_i})^{-1} \underset{i}{\otimes} \alpha_i\}_i)
\end{aligned} \tag{4.5.32}$$

where φ_0 and φ_1 satisfy the commutative diagrams (4.4.30) and (4.4.34) and the Euler classes $e(V^{H_i})^{-1} \in \mathcal{E}_{G/H_i}^{-1} \mathcal{O}_{\mathcal{F}/H_i}$ and $e(V - V^{H_i})^{-1} \in \mathcal{E}_{H_i}^{-1} \mathcal{O}_{\mathcal{F}}$ pop out from the structure maps of the object S^V (2.5.52).

The sequence (4.5.31) is 2-periodic, therefore we can work at the 0-th level. Note as well that the desuspensions in the second and third term shift 2-periodic objects by an even degree, therefore the term at each level does not change. The

0-th level of (4.5.31) is:

$$\mathcal{K} \xrightarrow{\varphi_0'} \bigoplus_{i\geq 1}(\bigoplus_{j\geq 1}\mathcal{K}/\mathcal{O}_{D_{ij}}) \xrightarrow{\varphi_1'} \bigoplus_{F}\mathcal{H}^2_F(\mathcal{O}) \to 0. \tag{4.5.33}$$

To conclude, we only need to prove that (4.5.33) is the sequence of global sections (4.5.28) of the Cousin complex of $\mathcal{O}(-D_V)$, since by Corollary (4.5.22) the Cousin complex is a flabby resolution and therefore the homology of its global sections (4.5.33) gives us the desired cohomology groups $H^*(\mathcal{X}, \mathcal{O}(-D_V))$ (Recall from Corollary 4.2.8 that the cohomology of $\mathcal{O}(-D_V)$ is the same for both topologies).

We only need to check that the maps in (4.5.33) and (4.5.28) are the same. By (4.5.32) the map φ_0' on every component (i,j) is the map:

$$\varphi_0'(f) = \varphi_0(c_{ij}^{-v_{ij}} \otimes f) = [\hat{t}_{ij}^{-v_{ij}} \cdot f] = d_0^V(f),$$

where the second equality follows by definition of φ_0 (4.4.31), and the last one is (4.5.26).

By (4.5.32) the F-th component of φ_1' for every finite subgroup F of G is the map:

$$\varphi_1'(\{[f_i]\}_i) = \varphi_1(\{\prod_{s\neq i} x_s^{-v_{s,n_s}} \otimes_i [f_i]\}_i) = [\{[f_i \cdot \prod_{s\neq i}\hat{t}_{s,n_s}^{-v_{s,n_s}}]\}_i] = d_1^V(\{[f_i]\}_i),$$

where the second equality follows by definition of φ_1 (4.4.43), the last one is (4.5.27), and we have used (4.3.17) to describe elements in $\mathcal{H}^2_F(\mathcal{O})$. □

4.5.3 Virtual negative complex representations

All the computations presented in this section work exactly in the same way for virtual negative complex representations. If V is a genuine complex representation with $V^G = 0$, Theorem 4.5.1 changes sign:

$$E\mathcal{C}^n_G(S^{-V}) \cong \begin{cases} H^0(\mathcal{X}, \mathcal{O}(D_V)) \oplus H^2(\mathcal{X}, \mathcal{O}(D_V)) & n \text{ even} \\ H^1(\mathcal{X}, \mathcal{O}(D_V)) & n \text{ odd.} \end{cases} . \tag{4.5.34}$$

Everything starts with the changes in the object $S^{-V} \in \mathcal{A}(G)$ explained in Remark 2.5.53 where the desuspensions change sign into suspensions, and therefore (4.5.31) changes into:

$$\overline{\mathcal{K}} \xrightarrow{\varphi_0'} \bigoplus_{i\geq 1}\Sigma^{V^{H_i}}T_i \xrightarrow{\varphi_1'} \Sigma^V N \to 0$$

where in the maps (4.5.32) positive powers of the Euler classes appear instead of the negative ones. As a consequence Theorem 4.5.11 changes sign:

$$\mathrm{Ext}^s_{\mathcal{A}}(S^{-V}, E\mathcal{C}_G) \cong \overline{H^s(\mathcal{X}, \mathcal{O}(D_V))}.$$

and we conclude exactly as before.

Chapter 5

Circle-equivariant elliptic cohomology of $\mathbb{C}\mathrm{P}(V)$

The aim of this chapter is to compute rational $\mathbb{T}$-equivariant elliptic cohomology of $\mathbb{C}\mathrm{P}(V)$: the $\mathbb{T}$-space of complex lines in V, and it consists of the entirety of [Bar22b]. More precisely we want to prove the following which is the main theorem of the chapter.

Theorem 5.0.1. *For every elliptic curve $\mathcal{C}$ over $\mathbb{C}$, if $E\mathcal{C}_{\mathbb{T}}$ (5.2.12) is the rational $\mathbb{T}$-equivariant elliptic cohomology theory built in [Gre05], and V is a finite dimensional complex representation of $\mathbb{T}$, then:*

1. *If V has one isotypic component, $V = \alpha z^n$ with $\alpha \geq 1$,*

$$E\mathcal{C}^k_{\mathbb{T}}(\mathbb{C}P(V)) \cong \mathbb{C}^{\alpha-1}$$

for every $k \in \mathbb{Z}$.

2. *If V has more than one isotypic component, $V = \bigoplus_n \alpha_n z^n$,*

$$E\mathcal{C}^k_{\mathbb{T}}(\mathbb{C}P(V)) \cong \begin{cases} 0 & k \text{ even} \\ \mathbb{C}^d & k \text{ odd.} \end{cases}$$

where $d = \sum_{i<j} \alpha_i \alpha_j (i-j)^2$, and z is the natural representation of $\mathbb{T}$.

Part (1) of this Theorem simply checks our methods on previously done computations since $\mathbb{C}\mathrm{P}(\alpha z^n) \cong \mathbb{C}\mathrm{P}^{\alpha-1}$ with the trivial $\mathbb{T}$-action. In contrast part (2) is the main computational result, and to the knowledge of the author it is the first time it appears in the literature. We refer to Remark 5.4.36 for a discussion of the geometry of this Theorem. We revise the construction of $E\mathcal{C}_{\mathbb{T}}$ in Section 5.2.

In all this chapter $G = \mathbb{T}^2$, and $H_1 = 1 \times \mathbb{T}$ and $H_2 = \mathbb{T} \times 1$ are the two privileged subgroups. We have also the quotient $\bar{G} := G/H_1 \cong H_2 \cong \mathbb{T}$ as group of equivariance. We have fixed an elliptic curve $\mathcal{C}$ over $\mathbb{C}$ together with a coordinate $t_e \in \mathcal{O}_{\mathcal{C},e}^{\mathrm{TP}} \subset \mathcal{O}_{\mathcal{C},e}$ (Definition 4.2.20). This gives us a rational $\bar{G}$-equivariant elliptic cohomology theory $E\mathcal{C}_{\bar{G}} \in \mathcal{A}(\bar{G})$ (5.2.12), as well as a rational G-equivariant elliptic cohomology theory $E\mathcal{C}_G \in \mathcal{A}(G)$ (Theorem 4.0.1). We have also the complex abelian surface $\mathcal{X} = \mathcal{X}_G = \mathcal{C} \times \mathcal{C}$ associated to G and $\mathcal{C}$.

5.1 Elliptic cohomology of $\mathbb{C}\mathrm{P}(V)$

Given a $\bar{G}$ complex representation V we want to compute the reduced cohomology of the pointed space:

$$E\mathcal{C}^*_{\bar{G}}(\mathbb{C}\mathrm{P}(V))$$

where $\mathbb{C}\mathrm{P}(V)$ is the $\bar{G}$-space of complex lines in V. We start pointing out the isomorphism of $\bar{G}$-spaces:

$$\mathbb{C}\mathrm{P}(V) \cong S(V \otimes_{\mathbb{C}} w)/H_1 \tag{5.1.1}$$

where w is the natural complex representation of H_1 and $S(V \otimes_{\mathbb{C}} w)$ is the $\bar{G}$-space of vectors of unit norm in the complex vector space $V \otimes_{\mathbb{C}} w$. Notice that $V \otimes_{\mathbb{C}} w$ is a complex representation of G of the same dimension of V, where H_1 acts on the second factor of the tensor product, while $\bar{G}$ acts on the first one. The computation is made possible since $E\mathcal{C}_{\bar{G}}$ and $E\mathcal{C}_G$ are H_1-split:

Theorem 5.1.2. *Let $E\mathcal{C}_G$ be G-elliptic cohomology (Theorem 4.0.1), and $E\mathcal{C}_{\bar{G}}$ be $\bar{G}$-elliptic cohomology* (5.2.12)*. Then there is a natural transformation of G-cohomology theories*

$$\varepsilon : \mathrm{Inf}_{\bar{G}}^{G} E\mathcal{C}_{\bar{G}} \longrightarrow E\mathcal{C}_G$$

which induces an isomorphism

$$[G/H_+, \mathrm{Inf}_{\bar{G}}^{G} E\mathcal{C}_{\bar{G}}]^G_* \cong [G/H_+, E\mathcal{C}_G]^G_* \tag{5.1.3}$$

for every subgroup H of G such that $H \cap H_1 = \{1\}$.

We prove this Theorem in Sections 5.3 and 5.4. More precisely in Section 5.3 we build the map ε while in Section 5.4 we prove the H-equivalence (5.1.3). We have an immediate Corollary:

Corollary 5.1.4. *For any H_1-free G-space X:*

$$\mathcal{EC}^*_G(X) \cong \mathcal{EC}^*_{\bar{G}}(X/H_1) \tag{5.1.5}$$

Proof. Since X is H_1-free, it is built using cells G/H_+ with $H \cap H_1 = \{1\}$, for which we have the equivalence (5.1.3). Therefore:

$$\mathcal{EC}^*_G(X) = [X, \mathcal{EC}_G]^G_* \cong [X, \mathrm{Inf}^G_{\bar{G}}\, \mathcal{EC}_{\bar{G}}]^G_* \cong [X/H_1, \mathcal{EC}_{\bar{G}}]^{\bar{G}}_* = \mathcal{EC}^*_{\bar{G}}(X/H_1)$$

where the second isomorphism is the orbits-inflation adjunction (2.4.20). □

Applying this corollary to (5.1.1) allows us to reduce the computation of $\bar{G}$-equivariant elliptic cohomology to a computation of G-equivariant elliptic cohomology:

$$\mathcal{EC}^*_{\bar{G}}(\mathbb{C}\mathrm{P}(V)_+) \cong \mathcal{EC}^*_{\bar{G}}(S(V\otimes w)_+/H_1) \cong \mathcal{EC}^*_G(S(V\otimes w)_+). \tag{5.1.6}$$

Remark 5.1.7. Notice we have to add a disjoint basepoint to $S(V\otimes w)$, leading us to compute $\mathcal{EC}^*_{\bar{G}}(\mathbb{C}\mathrm{P}(V)_+)$. To obtain $\mathbb{C}\mathrm{P}(V)$ without the added basepoint notice that stably:

$$\mathbb{C}\mathrm{P}(V)_+ \cong \mathbb{C}\mathrm{P}(V) \vee S^0$$

and therefore

$$\mathcal{EC}^*_{\bar{G}}(\mathbb{C}\mathrm{P}(V)_+) \cong \mathcal{EC}^*_{\bar{G}}(\mathbb{C}\mathrm{P}(V)) \oplus \mathcal{EC}^*_{\bar{G}}(S^0) \cong \mathcal{EC}^*_{\bar{G}}(\mathbb{C}\mathrm{P}(V)) \oplus \mathbb{C}, \tag{5.1.8}$$

since by [Gre05, Theorem 1.1] we have

$$\mathcal{EC}^k_{\bar{G}}(S^0) \cong \begin{cases} H^0(\mathcal{C}, \mathcal{O}_\mathcal{C}) \cong \mathbb{C} & k \text{ even} \\ H^1(\mathcal{C}, \mathcal{O}_\mathcal{C}) \cong \mathbb{C} & k \text{ odd.} \end{cases}$$

G-equivariant elliptic cohomology of $S(V\otimes w)_+$ is easier to compute in view of the cofibre sequence of G-spaces:

$$S(V\otimes w)_+ \longrightarrow S^0 \longrightarrow S^{V\otimes w} \tag{5.1.9}$$

inducing a long exact sequence:

$$\mathcal{EC}^*_G(S^{V\otimes w}) \longrightarrow \mathcal{EC}^*_G(S^0) \longrightarrow \mathcal{EC}^*_G(S(V\otimes w)_+). \tag{5.1.10}$$

The first two terms of (5.1.10) are computed in Theorem 4.5.1 from the previous chapter, therefore we can deduce the third term from kernel and cokernel of the first map.

Remark 5.1.11. One might argue that to compute rational $\mathbb{T}$-equivariant elliptic cohomology of $\mathbb{C}\mathrm{P}(V)$ a more straightforward approach can be taken. Namely we have $E\mathcal{C}_{\mathbb{T}} \in \mathcal{A}(\mathbb{T})$ and if we explicitly compute the algebraic model $\pi_*^{\mathcal{A}(\mathbb{T})}(\mathbb{C}\mathrm{P}(V)_+)$, then we can use the Adams spectral sequence for the circle to compute directly $E\mathcal{C}^*_{\mathbb{T}}(\mathbb{C}\mathrm{P}(V)_+)$. Originally this was the method tried on this project, with a double motivation: first as a warm-up in the circle case before generalizing to higher tori, and second, once $E\mathcal{C}_{\mathbb{T}^2}$ has been constructed, compute $E\mathcal{C}^*_{\mathbb{T}^2}(S(V\otimes w)_+)$, and see if the two values match. If indeed $E\mathcal{C}^*_{\mathbb{T}}(\mathbb{C}\mathrm{P}(V)_+) \cong E\mathcal{C}^*_{\mathbb{T}^2}(S(V\otimes w)_+)$, then one might expect that $E\mathcal{C}_{\mathbb{T}}$ and $E\mathcal{C}_{\mathbb{T}^2}$ are H_1-split (Theorem 5.1.2). The problem encountered with this method is that after computing the algebraic model $\pi_*^{\mathcal{A}(\mathbb{T})}(\mathbb{C}\mathrm{P}(V)_+)$ then computing maps in $\mathcal{A}(\mathbb{T}^2)$ from this object into an injective resolution of $E\mathcal{C}_{\mathbb{T}}$ still retains a lot of complexity. More precisely it is still difficult to compute $\mathcal{O}_{\mathcal{F}}$-module maps between arbitrary $\mathcal{O}_{\mathcal{F}}$-module when none of them is a suspension of the base ring $\mathcal{O}_{\mathcal{F}}$. Therefore this more conceptual approach of proving H_1-splitness and computing $E\mathcal{C}^*_{\mathbb{T}^2}(S(V\otimes w)_+)$ has been taken. Moreover we hope to replicate and generalise this to other spaces such as Grassmannians of n-planes $\mathrm{Gr}_n(V)$ of a complex $\mathbb{T}$-representation.

To find the associated divisor $D_{V\otimes w}$ decompose V as a sum of one dimensional complex representations $V = \bigoplus_n \alpha_n z^n$ where z is the natural representation of $\bar{G}$ and $\alpha_n \geq 0$. Notice $z^n \otimes w$ is a one dimensional representation of G with kernel the connected codimension 1 subgroup:

$$H_{d_n} := \mathrm{Ker}(z^n \otimes w) = \{(x,y) \in G = \mathbb{T}\times\mathbb{T} \mid x^n y = 1\}$$

and corresponding divisor:

$$D_{d_n} = \mathfrak{X}(H_{d_n}) = \{(P,Q) \in \mathcal{X} = \mathcal{C}\times\mathcal{C} \mid nP + Q = e\}. \tag{5.1.12}$$

Therefore $V\otimes w = \bigoplus_n \alpha_n(z^n\otimes w)$ is a complex representation of G of the same dimension as V, whose associated divisor is:

$$D_{V\otimes w} = \sum_n \alpha_n D_{d_n}. \tag{5.1.13}$$

We have two distinct proves depending if the divisor (5.1.13) has only one coefficient different from zero, or more than one coefficient different from zero.

When V has only one isotypic component, then $\mathbb{C}\mathrm{P}(V) \cong \mathbb{C}\mathrm{P}^{\alpha-1}$ with trivial $\mathbb{T}$-action. We can prove part (1) of Theorem 5.0.1:

Lemma 5.1.14. *If V has only one isotypic component: $V = \alpha z^n$ with $\alpha \geq 1$ then*

$$E\mathcal{C}^k_G(\mathbb{C}P(V)) \cong \mathbb{C}^{\alpha-1} \tag{5.1.15}$$

for every $k \in \mathbb{Z}$.

Proof. Combining (5.1.6) and (5.1.10) we only need to understand kernel and cokernel of the map

$$E\mathcal{C}^*_G(S^{V\otimes w}) \longrightarrow E\mathcal{C}^*_G(S^0). \tag{5.1.16}$$

By (5.1.13) the divisor associated to $V \otimes w$ is $D_{V\otimes w} = \alpha D_{d_n}$, denote $D := D_{d_n}$ the smooth irreducible curve. By Theorem 4.5.1 it is enough to understand kernel and cokernel for the map

$$H^*(\mathcal{X}, \mathcal{O}(-\alpha D)) \longrightarrow H^*(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \tag{5.1.17}$$

induced by the inclusion of sheaves $\mathcal{O}(-\alpha D) \hookrightarrow \mathcal{O}_{\mathcal{X}}$. We compute this map for the various degrees.

In degree zero $H^0(\mathcal{X}, \mathcal{O}(-\alpha D)) = 0$, since $\mathcal{X}$ is a compact complex abelian surface, therefore a function regular on all the surface is constant and since it has a zero at D it is the constant zero. Therefore (5.1.17) in degree zero has zero kernel and cokernel $\mathbb{C}$.

In degree 2 by Serre duality

$$H^2(\mathcal{X}, \mathcal{O}(-\alpha D)) \cong H^0(\mathcal{X}, \mathcal{O}(\alpha D))^\vee \cong \mathbb{C}^\alpha \tag{5.1.18}$$

where the second isomorphism is obtained as follows. Consider the divisor $D' = \alpha(e)$ on the single elliptic curve $\mathcal{C}$, defining the line bundle $\mathcal{L} := \mathcal{O}_{\mathcal{C}}(\alpha(e))$. The projective morphism $f := \pi_{d_n} : \mathcal{X} \to \mathcal{C}$ is such that $f^*\mathcal{L} \cong \mathcal{O}(\alpha D)$, and $f_*\mathcal{O}_{\mathcal{X}} \cong \mathcal{O}_{\mathcal{C}}$ by [Liu02, Exercise 3.12 Chapter 5]. As a consequence

$$f_*f^*\mathcal{L} = f_*(\mathcal{O}_{\mathcal{X}} \otimes_{\mathcal{O}_{\mathcal{X}}} f^*\mathcal{L}) \cong f_*(\mathcal{O}_{\mathcal{X}}) \otimes_{\mathcal{O}_{\mathcal{C}}} \mathcal{L} \cong \mathcal{O}_{\mathcal{C}} \otimes_{\mathcal{O}_{\mathcal{C}}} \mathcal{L} \cong \mathcal{L} \tag{5.1.19}$$

by the projection formula [Har77, exercise 5.1]. Therefore

$$H^0(\mathcal{X}, \mathcal{O}(\alpha D)) \cong H^0(\mathcal{X}, f^*\mathcal{L}) \cong H^0(\mathcal{C}, f_*f^*\mathcal{L}) \cong H^0(\mathcal{C}, \mathcal{L}) \cong \mathbb{C}^\alpha, \tag{5.1.20}$$

where the second isomorphism is due to the equality of functors $\Gamma(\mathcal{X}, \_\_) =$

$\Gamma(\mathcal{C}, f_*(\_))$, while the last one is Riemann-Roch for the elliptic curve $\mathcal{C}$. Moreover the map

$$\mathbb{C} \cong H^0(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \rightarrowtail H^0(\mathcal{X}, \mathcal{O}(\alpha D)) \cong \mathbb{C}^{\alpha}$$

is injective being the inclusion of the constant functions. By Serre duality the dual map, (5.1.17) in degree 2, is surjective with kernel $\mathbb{C}^{\alpha-1}$.

In degree 1 by Serre duality

$$H^1(\mathcal{X}, \mathcal{O}(-\alpha D)) \cong H^1(\mathcal{X}, \mathcal{O}(\alpha D))^{\vee} \cong \mathbb{C}^{\alpha}$$

since by Riemann-Roch (3.4.14)

$$h^0(\mathcal{X}, \alpha D) - h^1(\mathcal{X}, \alpha D) = \frac{1}{2}\alpha^2(D.D) = 0. \tag{5.1.21}$$

We can see $(D.D) = 0$ since D is linearly equivalent to the translated $D + \lambda$, which is an irreducible curve disjoint from D, or alternatively simply using the genus formula (3.4.15) since D is irreducible of genus 1 being isomorphic to the elliptic curve $\mathcal{C}$. By Lemma 5.1.22 the image of the map (5.1.17) in degree 1 is precisely $H^1(\mathcal{X}, \mathcal{O}(-D)) \cong \mathbb{C}$, combining this with the computations (5.1.17) we obtain that the kernel in degree 1 of (5.1.17) is $\mathbb{C}^{\alpha-1}$ while the cokernel is $\mathbb{C}$.

In conclusion we obtain that $E\mathcal{C}^k_{\tilde{G}}(\mathbb{C}\mathrm{P}(V)_+)$ has in even degrees the degree zero cokernel and the degree one kernel of (5.1.17), therefore is isomorphic to $\mathbb{C}^{\alpha}$. In odd degrees we have the degree 1 cokernel and the degree 2 kernel of (5.1.17), so $\mathbb{C}^{\alpha}$ as well. By (5.1.8) $E\mathcal{C}^k_{\tilde{G}}(\mathbb{C}\mathrm{P}(V)) \cong \mathbb{C}^{\alpha-1}$ for every $k \in \mathbb{Z}$. □

Lemma 5.1.22. *Let $D = D_{d_n}$ be any of the smooth divisors (5.1.12). For every integer $\alpha \geq 1$ the image of the map*

$$H^1(\mathcal{X}, \mathcal{O}(-\alpha D)) \to H^1(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \tag{5.1.23}$$

is precisely $H^1(\mathcal{X}, \mathcal{O}(-D)) \subset H^1(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$.

Proof. When $\alpha = 1$ we have that D is a subvariety of $\mathcal{X}$ whose ideal sheaf is precisely $\mathcal{O}_{\mathcal{X}}(-D)$, and structure sheaf $\mathcal{O}_D$ that we can see as a sheaf on $\mathcal{X}$ via the inclusion map $\iota : D \to \mathcal{X}$. We have the closed subscheme short exact sequence [Vak17, 14.3.B]:

$$0 \to \mathcal{O}(-D) \to \mathcal{O}_{\mathcal{X}} \to \iota_*\mathcal{O}_D \to 0 \tag{5.1.24}$$

inducing a long exact sequence in cohomology. Since $H^0(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \xrightarrow{\cong} H^0(\mathcal{X}, \iota_*\mathcal{O}_D) \cong \mathbb{C}$ then $H^1(\mathcal{X}, \mathcal{O}(-D)) \to H^1(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ is injective.

When $\alpha \geq 2$ we can tensor the exact sequence (5.1.24) with the invertible sheaf $\mathcal{O}(-(\alpha-1)D)$ obtaining the exact sequence of sheaves:

$$0 \to \mathcal{O}(-\alpha D) \to \mathcal{O}(-(\alpha-1)D) \to \iota_*\mathcal{O}_D(-(\alpha-1)D) \to 0 \tag{5.1.25}$$

inducing a long exact sequence in cohomology. By the adjunction formula [Liu02, Theorem 1.37]:

$$(\omega_{\mathcal{X}} \otimes \mathcal{O}(D)) \restriction_D = \omega_D$$

where in our case $\omega_{\mathcal{X}} \cong \mathcal{O}_{\mathcal{X}}$ and $\omega_D \cong \mathcal{O}_D$, resulting in $\iota_*\mathcal{O}_D(D) \cong \iota_*\mathcal{O}_D$ and by induction $\iota_*\mathcal{O}_D(\beta D) \cong \iota_*\mathcal{O}_D$ for every integer β.

Therefore we have that the last term of (5.1.25) is $\iota_*\mathcal{O}_D$, and we can analyse the degree 2 piece of the long exact sequence induced:

$$H^1(\iota_*\mathcal{O}_D) \to H^2(\mathcal{O}(-\alpha D)) \to H^2(\mathcal{O}(-(\alpha-1)D)) \to H^2(\iota_*\mathcal{O}_D) = 0. \tag{5.1.26}$$

The first map in (5.1.26) is injective simply by dimension computation: $H^1(\iota_*\mathcal{O}_D) \cong \mathbb{C}$, and the other two terms are computed in (5.1.18). Consequently the map $H^1(\mathcal{O}(-\alpha D)) \twoheadrightarrow H^1(\mathcal{O}(-(\alpha-1)D))$ is surjective. In conclusion we have the chain of maps:

$$H^1(\mathcal{O}(-\alpha D)) \twoheadrightarrow H^1(\mathcal{O}(-(\alpha-1)D)) \twoheadrightarrow \cdots \twoheadrightarrow H^1(\mathcal{O}(-D)) \rightarrowtail H^1(\mathcal{O}_{\mathcal{X}})$$

where all the maps are surjective except the last one that is injective, giving us the desired result. $\square$

When V has more than one isotypic component, then the proof of part (2) of Theorem 5.0.1 has a different approach:

Theorem 5.1.27. *If V has more than one isotypic component: $V = \bigoplus_n \alpha_n z^n$ with $\alpha_n \geq 0$ then*

$$E\mathcal{C}^k_{\tilde{G}}(\mathbb{C}P(V)) \cong \begin{cases} 0 & k \text{ even} \\ \mathbb{C}^d & k \text{ odd.} \end{cases} \tag{5.1.28}$$

where $d = \sum_{i<j} \alpha_i\alpha_j(i-j)^2$.

Remark 5.1.29. Notice that there are different representations, with different complex projective spaces that nonetheless have the same value for d and therefore rational $\mathbb{T}$-equivariant elliptic cohomology does not distinguish them. For example if ε is $\mathbb{C}$ with the trivial action, $V = \varepsilon \oplus 4z$ and $V' = \varepsilon \oplus z^2$ have $d = 4$, or $W = \varepsilon \oplus 16z$,

$W' = \varepsilon \oplus z^4$ and $W'' = \varepsilon \oplus z \oplus 3z^2$ have $d = 16$. Therefore they have the same elliptic cohomology even if their complex projective spaces are quite different.

Proof. Exactly as in the proof of Theorem 5.1.14 we only need to understand kernel and cokernel of the map

$$H^*(\mathcal{X}, \mathcal{O}(-D)) \longrightarrow H^*(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \tag{5.1.30}$$

where $D := D_{V \otimes w} = \sum_n \alpha_n D_{d_n}$ is the divisor associated to $V \otimes w$ (5.1.13).

When V has more than one isotypic components, the associated divisor D is an ample divisor. To show this we can use the Nakai-Moishezon criterion [Har77, Theorem V.1.10]. This criterion states that D is an ample divisor on $\mathcal{X}$ if and only if $D.D > 0$ and $D.\mathcal{C}' > 0$ for every irreducible curve $\mathcal{C}'$ on $\mathcal{X}$. First let us prove that $D_{d_r}.D_{d_s} = (r-s)^2$ for any two integers r and s. If $r = s$ then $D_{d_r}.D_{d_r} = 0$ since D_{d_r} is linearly equivalent to the translated $D_{d_r} + \lambda$, which is an irreducible curve disjoint from D_{d_r}, or alternatively simply using the genus formula (3.4.15) since D_{d_r} is irreducible of genus 1 being isomorphic to the elliptic curve $\mathcal{C}$. If $r \neq s$ than the two curves D_{d_r} and D_{d_s} are transverse in each point of intersection therefore we simply need to count the intersection points [Bea96, Definition I.3]. There is an autoisogeny of $\mathcal{X}$ bringing D_{d_r} to $D_{d_0} = D_2 = \{(P,Q) \in \mathcal{X} \mid Q = e\}$. Under this isogeny D_{d_s} is brought to $D_{d_{s-r}} = \{(P,Q) \in \mathcal{X} \mid (s-r)P + Q = e\}$. The intersection of these last two curves is easily computed to be $\mathcal{C}[s-r]$ which has cardinality $(s-r)^2$. Therefore:

$$D.D = \sum_{i,j \in \mathbb{Z}} \alpha_i \alpha_j (D_{d_i}.D_{d_j}) = 2 \sum_{i<j} \alpha_i \alpha_j (i-j)^2 > 0 \tag{5.1.31}$$

since there are at least two different integers $r \neq s$ such that $\alpha_r, \alpha_s > 0$ (V has more than one isotypic component). It remains to show that for every irreducible curve $\mathcal{C}'$ in $\mathcal{X}$, the curve $\mathcal{C}'$ intersects either D_{d_r} or D_{d_s}, so that $D.\mathcal{C}' > 0$. If $\mathcal{C}' \cap D_{d_r} = \emptyset$ than $\mathcal{C}'$ is necessarily parallel to D_{d_r}, meaning $\mathcal{C}' = D_{d_r} + \lambda$ is a translated of D_{d_r}, but all these curves intersect D_{d_s} since $s \neq r$.

The sheaf $\mathcal{O}(D)$ is invertible and ample, so we can use Kodaira vanishing theorem [Har77, Remark III.7.15] obtaining $H^i(\mathcal{X}, \mathcal{O}(D)) = 0$ for $i \geq 1$. Therefore only the zeroth cohomology is nonzero and we can compute it with Riemann-Roch (3.4.14):

$$h^0(\mathcal{X}, D) = \frac{1}{2}(D.D) = \sum_{i<j} \alpha_i \alpha_j (i-j)^2 = d$$

where $(D.D)$ is computed in (5.1.31). Moreover the map

$$\mathbb{C} \cong H^0(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \rightarrowtail H^0(\mathcal{X}, \mathcal{O}(D)) \cong \mathbb{C}^d$$

is injective being the inclusion of the constant functions. By Serre duality the dual map, (5.1.30) in degree 2, is surjective with kernel $\mathbb{C}^{d-1}$.

In conclusion we obtain that $E\mathcal{C}^k_{\bar{G}}(\mathbb{C}\mathrm{P}(V)_+)$ has in even degrees the degree zero cokernel of (5.1.30): $H^0(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \cong \mathbb{C}$. In odd degrees we have the degree 1 cokernel and the degree 2 kernel of (5.1.30), so $\mathbb{C}^{d+1}$. By (5.1.8) $E\mathcal{C}^k_{\bar{G}}(\mathbb{C}\mathrm{P}(V))$ has zero in even degrees and $\mathbb{C}^d$ in odd degrees. □

5.2 The circle case revisited

We revise the construction of circle equivariant elliptic cohomology from [Gre05]. Recall from Definition 4.2.17 the torsion point topology $\mathcal{C}^{\mathrm{TP}}$ on the elliptic curve $\mathcal{C}$ and the choice of a coordinate function $t_e \in \mathcal{O}^{\mathrm{TP}}_{\mathcal{C},e}$ (Definition 4.2.20). By Theorem 4.2.23 we have a unique logarithm for the formal group law of $\mathcal{C}$: $\hat{t}_e = f(t_e) \in (\mathcal{O}^{\mathrm{TP}}_{\mathcal{C},e})^{\wedge}_{m_e} \cong \mathbb{C}[[t_e]]$.

Denoting $[n] : \mathcal{C} \to \mathcal{C}$ the multiplication by n map in the elliptic curve, we can pullback coordinate for the various $\mathcal{C}\langle n \rangle$:

Definition 5.2.1. For every integer $n \geq 1$ define the coordinate and completed coordinate:

$$\begin{aligned} t_n &:= [n]^*(t_e) \in \mathcal{O}^{\mathrm{TP}}_{\mathcal{C},e} \\ \hat{t}_n &:= [n]^*(\hat{t}_e) = f(t_n) \in (\mathcal{O}^{\mathrm{TP}}_{\mathcal{C},e})^{\wedge}_{m_e} \end{aligned} \tag{5.2.2}$$

Remark 5.2.3. Notice $t_n \in \mathcal{O}_{\mathcal{C}\langle n \rangle}$ since it has a zero of degree one on all points of $\mathcal{C}[n]$, so we can use it as a coordinate for the irreducible closed subset $\mathcal{C}\langle n \rangle$. In the same way $\hat{t}_n$ is an element in the completed ring $(\mathcal{O}_{\mathcal{C}\langle n \rangle})^{\wedge}$.

Definition 5.2.4. The fundamental rings are the stalks in the TP-topology:

$$\begin{aligned} \mathcal{K}_{\mathbb{T}} &:= \mathcal{O}^{\mathrm{TP}}_{\mathcal{C},\eta(\mathcal{C})} = \{ f \in \mathcal{K}(\mathcal{C}^{\mathrm{Zar}}) \mid f \text{ has poles only at points of finite order of } \mathcal{C} \} \\ \mathcal{O}_{\mathcal{C}\langle n \rangle} &:= \mathcal{O}^{\mathrm{TP}}_{\mathcal{C},\eta(\mathcal{C}\langle n \rangle)} = \{ f \in \mathcal{K}_{\mathbb{T}} \mid f \text{ is regular at } \mathcal{C}\langle n \rangle \} \end{aligned} \tag{5.2.5}$$

Denote $\bar{\mathcal{F}}$ the family of finite subgroups of $\bar{G}$: for every $n \geq 1$ we have the cyclic subgroup C_n of order n. The fundamental ring for the algebraic models $\mathcal{A}(\bar{G})$

is

$$\mathcal{O}_{\bar{\mathcal{F}}} = \prod_{n\geq 1} H^*(B\bar{G}/C_n) = \prod_{n\geq 1} \mathbb{Q}[c_n]$$

where $c_n \in H^*(B\bar{G}/C_n)$ of degree -2 is the Euler class (2.5.13) of a character having kernel C_n.

Definition 5.2.6. Define the torsion injective $\mathcal{O}_{\bar{\mathcal{F}}}$-module

$$T_{\mathbb{T}} := \bigoplus_{n\geq 1} \overline{\mathcal{K}_{\mathbb{T}}/\mathcal{O}_{\mathcal{C}\langle n\rangle}} \tag{5.2.7}$$

where the action is defined on the n-th component as follows. The Euler class c_n acts as $\hat{t}_n$:

$$c_n \cdot [f] = [\hat{t}_n \cdot f] \in \mathcal{K}_{\mathbb{T}}/\mathcal{O}_{\mathcal{C}\langle n\rangle}.$$

Notice the action is well defined since t_n vanishes at first order at $\mathcal{C}\langle n\rangle$, so that powers of $\hat{t}_n$ do not contribute after a certain integer and the sum is finite.

Definition 5.2.8. Define the graded surjective $\mathcal{O}_{\bar{\mathcal{F}}}$-module map

$$q : \mathcal{E}_{\bar{G}}^{-1}\mathcal{O}_{\bar{\mathcal{F}}} \otimes \overline{\mathcal{K}_{\mathbb{T}}} \twoheadrightarrow T_{\mathbb{T}} \tag{5.2.9}$$

as follows. On the n-th component on pure tensor elements:

$$c_n^k \otimes f \overset{q}{\mapsto} [\hat{t}_n^k \cdot f].$$

Notice this is well defined also for negative powers of the euler class simply considering the inverse power series $\hat{t}_n^{-1}$.

Definition 5.2.10. Let $N_{\mathbb{T}} := \mathrm{Ker}(q)$ be the kernel of the map q (5.2.9). We have the exact sequence of $\mathcal{O}_{\bar{\mathcal{F}}}$-module:

$$N_{\mathbb{T}} \rightarrowtail \mathcal{E}_{\bar{G}}^{-1}\mathcal{O}_{\bar{\mathcal{F}}} \otimes \overline{\mathcal{K}_{\mathbb{T}}} \xrightarrow{\ q\ } \!\!\!\!\twoheadrightarrow T_{\mathbb{T}}. \tag{5.2.11}$$

Define the algebraic model for circle-equivariant elliptic cohomology $E\mathcal{C}_{\bar{G}} \in \mathcal{A}(\bar{G})$ to be the object:

$$E\mathcal{C}_{\bar{G}} := \begin{bmatrix} \mathcal{E}_{\bar{G}}^{-1}\mathcal{O}_{\bar{\mathcal{F}}} \otimes \overline{\mathcal{K}_{\mathbb{T}}} \\ \uparrow \\ N_{\mathbb{T}} \end{bmatrix} \in \mathcal{A}(\bar{G}) \tag{5.2.12}$$

with structure map the natural inclusion of (5.2.11).

Remark 5.2.13. Since $T_{\mathbb{T}}$ is torsion injective we have immediately the injective resolution of $E\mathcal{C}_{\bar{G}}$ in $\mathcal{A}(\bar{G})$:

$$\begin{bmatrix} \mathcal{E}_{\bar{G}}^{-1}\mathcal{O}_{\bar{\mathcal{F}}} \otimes \overline{\mathcal{K}_{\mathbb{T}}} \\ \uparrow \\ N_{\mathbb{T}} \end{bmatrix} \rightarrowtail \begin{bmatrix} \mathcal{E}_{\bar{G}}^{-1}\mathcal{O}_{\bar{\mathcal{F}}} \otimes \overline{\mathcal{K}_{\mathbb{T}}} \\ \uparrow \\ \mathcal{E}_{\bar{G}}^{-1}\mathcal{O}_{\bar{\mathcal{F}}} \otimes \overline{\mathcal{K}_{\mathbb{T}}} \end{bmatrix} \twoheadrightarrow \begin{bmatrix} 0 \\ \uparrow \\ T_{\mathbb{T}} \end{bmatrix}.$$

In this way the object (5.2.12) we have defined here has the same values on spheres of complex representations as the object constructed in [Gre05]. The analogy with our construction of $\mathbb{T}^2$-equivariant presented in the previous chapter goes further. Namely the Cousin complex for $\mathcal{O}_{\mathcal{C}}^{\mathrm{TP}}$ is the following short exact sequence of sheaves that we can find in [Gre05, Corollary 9.3]:

$$\mathcal{O}_{\mathcal{C}}^{\mathrm{TP}} \rightarrowtail \iota_{\mathcal{C}}(\mathcal{K}_{\mathbb{T}}) \twoheadrightarrow \bigoplus_{n\geq 1} \iota_{\mathcal{C}\langle n\rangle}(\mathcal{K}_{\mathbb{T}}/\mathcal{O}_{\mathcal{C}\langle n\rangle}) \tag{5.2.14}$$

where we denote $\iota_Z(M)$ the sheaf of constant value M on the closed subset Z. Exactly as in the previous chapter, (5.2.14) is a flabby resolution of the structure sheaf that we can use to link the values of (5.2.12) on spheres of complex representations with the appropriate cohomology of the associated line bundle on $\mathcal{C}$ [Gre05, Theorem 1.1].

5.3 Building the map

The aim of this section is to build the map $\varepsilon : \mathrm{Inf}_{\bar{G}}^{G} E\mathcal{C}_{\bar{G}} \to E\mathcal{C}_G$ of Theorem 5.1.2. The first step is to identify the functor $\mathrm{Inf}_{\bar{G}}^{G} : \mathcal{A}(\bar{G}) \to \mathcal{A}(G)$ and we do this in general for tori of any rank.

5.3.1 The inflation functor

Only for this subsection $G = \mathbb{T}^r$ is a generic torus of some rank r, we fix a connected subgroup K and we define the quotient group $\bar{G} := G/K$, with quotient map $q : G \to \bar{G}$. For a generic connected subgroup H of G denote $L := \langle H, K\rangle$ the subgroup generated by H and K, and $\bar{H} := q(L)$ the image subgroup of H in $\bar{G}$.

Proposition 5.3.1. *For a $\bar{G}$-spectrum $\bar{X}$, the value of $\pi_*^{\mathcal{A}(G)}(\mathrm{Inf}_{\bar{G}}^{G} \bar{X})$ (2.5.47) at a connected subgroup H is given by:*

$$\varphi^H(\mathrm{Inf}_{\bar{G}}^{G} \bar{X}) \cong \mathcal{O}_{\mathcal{F}/H} \underset{\mathcal{O}_{\mathcal{F}/L}}{\otimes} \varphi^{\bar{H}}(\bar{X}), \tag{5.3.2}$$

with structure maps induced by the structure maps of $\pi_*^{\mathcal{A}(\bar{G})}(\bar{X})$.

To prove this result we will need some lemmas about G-spectra that holds more generally for compact Lie groups (with the appropriate assumptions of subgroups being normal). Moreover these results do not rely on the fact that we have rationalized everything and therefore holds also if we do not localize at $S^0\mathbb{Q}$.

Given a G-spectrum X we can pick a strictly increasing indexing sequence $V_1 \subsetneq V_2 \subsetneq \cdots \subsetneq \mathcal{U}$ for the complete G-universe $\mathcal{U}$, and use the canonical presentation of X [HHR16, pg. 12], namely there is a weak equivalence of G-spectra

$$X \simeq \underset{\overrightarrow{V_n}}{\operatorname{hocolim}}\, \Sigma^{-V_n}\Sigma^\infty X(V_n) \tag{5.3.3}$$

where the homotopy colimit is taken over the indexing sequence $\{V_n\}$ and $\Sigma^\infty X(V_n)$ is the suspension spectrum of the G-space $X(V_n)$. In general we will prove that two spectra X and Y are weakly equivalent proving that for every V_n in an indexing sequence the two G-spaces $X(V_n)$ and $Y(V_n)$ are isomorphic.

Definition 5.3.4. If $H \subseteq G$ is a subgroup of G, we say that the indexing sequence $\{V_n\}$ is an H-separated indexing sequence, if the sequence of H-fixed points $V_1^H \subsetneq V_2^H \subsetneq \cdots \subseteq \mathcal{U}^H$ is a strictly increasing indexing sequence for the G/H-universe $\mathcal{U}^H$.

When this happens we have a nice description for the geometric fixed point:

$$\Phi^H X \simeq \underset{\overrightarrow{V_n^H}}{\operatorname{hocolim}}\, \Sigma^{-V_n^H}\Sigma^\infty((X(V_n))^H). \tag{5.3.5}$$

In this sense $(\Phi^H X)(V_n^H) = (X(V_n))^H$ as G/H-spaces.

Lemma 5.3.6. *Given a* $\bar{G}$*-spectrum* $\bar{X}$ *and a* K*-separated indexing sequence* $\{V_n\}$ *the values of* (5.3.3) *for the inflated spectrum are:*

$$(\operatorname{Inf}_{\bar{G}}^G \bar{X})(V_n) = \Sigma^{V_n - V_n^K} \operatorname{Inf}_{\bar{G}}^G(\bar{X}(V_n^K)) \tag{5.3.7}$$

where $V_n - V_n^K$ *is the orthogonal complement of* V_n^K.

Proof. We prove the statement for suspension spectra, and using (5.3.3) it holds in general for all spectra. If $\bar{S}$ is a $\bar{G}$-space:

$$\begin{aligned}(\Sigma^\infty \operatorname{Inf}_{\bar{G}}^G \bar{S})(V_n) = \Sigma^{V_n} \operatorname{Inf}_{\bar{G}}^G \bar{S} = \Sigma^{V_n - V_n^K}\Sigma^{V_n^K}(\operatorname{Inf}_{\bar{G}}^G \bar{S}) = \Sigma^{V_n - V_n^K} \operatorname{Inf}_{\bar{G}}^G(\Sigma^{V_n^K} \bar{S}) = \\ = \Sigma^{V_n - V_n^K} \operatorname{Inf}_{\bar{G}}^G((\Sigma^\infty \bar{S})(V_n^K)).\end{aligned} \tag{5.3.8}$$

The first equality is the definition of suspension spectrum 2.3.14, the third equality is because V_n^K is a G/K-representation and commutes with inflation, while the last one is again the definition of suspension spectrum. □

Lemma 5.3.9. *for every $\bar{G}$-spectrum $\bar{X}$ we have a weak equivalence of G/H-spectra:*

$$\Phi^H(\mathrm{Inf}_{\bar{G}}^G \bar{X}) \simeq \mathrm{Inf}_{\bar{G}/\bar{H}}^{G/H}(\Phi^{\bar{H}}\bar{X}) \tag{5.3.10}$$

Proof. Denote $L := \langle H, K\rangle$ the subgroup generated by H and K in G, and notice that the inflation on the left hand side of (5.3.10) makes sense since $\bar{G}/\bar{H} \cong (G/K)/(L/K) \cong G/L \cong (G/H)/(L/H)$. Choose an indexing sequence $\{V_n\}$ for the complete G-universe $\mathcal{U}$ that is H-separated and K-separated, and such that the indexing sequence $\{V_n^H\}$ is L/H-separated (It is an indexing sequence for the group G/H). We will show that the two spectra (5.3.10) have isomorphic values on the indexing sequence $\{V_n^H\}$.

$$\begin{aligned}(\Phi^H(\mathrm{Inf}_{\bar{G}}^G \bar{X}))(V_n^H) = ((\mathrm{Inf}_{\bar{G}}^G \bar{X})(V_n))^H = (\Sigma^{V_n - V_n^K}(\mathrm{Inf}_{\bar{G}}^G(\bar{X}(V_n^K))))^H = \\ = \Sigma^{V_n^H - V_n^L}(\mathrm{Inf}_{\bar{G}}^G(\bar{X}(V_n^K)))^H.\end{aligned} \tag{5.3.11}$$

The first isomorphism is (5.3.5), while the second one is (5.3.10). For the right hand side:

$$(\mathrm{Inf}_{\bar{G}/\bar{H}}^{G/H}(\Phi^{\bar{H}}\bar{X}))(V_n^H) = \Sigma^{V_n^H - V_n^L}\,\mathrm{Inf}_{\bar{G}/\bar{H}}^{G/H}((\Phi^{\bar{H}}\bar{X})(V_n^L)) = \Sigma^{V_n^H - V_n^L}\,\mathrm{Inf}_{\bar{G}/\bar{H}}^{G/H}((\bar{X}(V_n^K))^{\bar{H}}). \tag{5.3.12}$$

The G/H-spaces (5.3.11) and (5.3.12) are isomorphic since (5.3.10) is an isomorphism for the $\bar{G}$-space $\bar{X}(V_n^K)$:

$$(\mathrm{Inf}_{\bar{G}}^G(\bar{X}(V_n^K)))^H = \mathrm{Inf}_{\bar{G}/\bar{H}}^{G/H}((\bar{X}(V_n^K))^{\bar{H}})$$

□

Proof of Proposition 5.3.1. Let us first prove the case when $H = \{1\}$ is the trivial subgroup, by (2.5.47) we need to prove the isomorphism:

$$\pi_*^G(\mathrm{Inf}_{\bar{G}}^G(\bar{X}) \wedge DE\mathcal{F}_+) \cong \mathcal{O}_{\mathcal{F}} \underset{\mathcal{O}_{\mathcal{F}/K}}{\otimes} \pi_*^{\bar{G}}(\bar{X} \wedge DE\mathcal{F}/K_+). \tag{5.3.13}$$

The proof works exactly as the proof of [Gre08, Lemma 9.2]. Both sides of (5.3.13) are homology theories of $\bar{X}$ (for the right hand side of (5.3.13): by [Gre08, Corollary 5.7] the ring $\mathcal{O}_{\mathcal{F}}$ is flat over $\mathcal{O}_{\mathcal{F}/K}$ so that tensoring with it is an exact functor) and we have a natural transformation of homology theories induced by:

$$[S^0, \bar{X} \wedge DE\mathcal{F}/K_+]^{\bar{G}}_* \to [\mathrm{Inf}^G_{\bar{G}} S^0, \mathrm{Inf}^G_{\bar{G}}(\bar{X} \wedge DE\mathcal{F}/K_+)]^G_* \to [S^0, (\mathrm{Inf}^G_{\bar{G}} \bar{X}) \wedge DE\mathcal{F}_+]^G_*$$

The first map is induced by the inflation map, and the second one is induced by a map of G-spectra $\mathrm{Inf}^G_{\bar{G}} DE\mathcal{F}/K_+ \to DE\mathcal{F}_+$ described in [Gre08, pag.22]. We only need to prove that this natural transformation of homology theories is an isomorphism for the various cells of $\bar{G}$. When $\bar{X} = S^0$ is the sphere spectrum (5.3.13) holds since by [Gre08, Theorem 7.4]: $\pi^G_*(DE\mathcal{F}_+) = \mathcal{O}_\mathcal{F}$. For more general cells of $\bar{G}$ the isomorphism (5.3.13) follows from the case of the sphere spectrum by the "Rep(G)-iso argument" [Gre08, Theorem 11.2] since both homology theories satisfy Thom isomorphism (smashing with $DE\mathcal{F}_+$ gives Thom isomorphism by [Gre08, Corollary 8.5]).

For the case when H is an arbitrary connected subgroup of G we need to prove the more general isomorphism:

$$\pi^{G/H}_*(\Phi^H(\mathrm{Inf}^G_{\bar{G}} \bar{X}) \wedge DE\mathcal{F}/H_+) \cong \mathcal{O}_{\mathcal{F}/H} \bigotimes_{\mathcal{O}_{\mathcal{F}/L}} \pi^{\bar{G}/\bar{H}}_*(\Phi^{\bar{H}} \bar{X} \wedge DE\mathcal{F}/\bar{H}_+). \quad (5.3.14)$$

We can reduce it to the case of the trivial subgroup since by (5.3.10) the left hand side of (5.3.14) becomes

$$\pi^{G/H}_*(\Phi^H(\mathrm{Inf}^G_{\bar{G}} \bar{X}) \wedge DE\mathcal{F}/H_+) \cong \pi^{G/H}_*(\mathrm{Inf}^{G/H}_{\bar{G}/\bar{H}}(\Phi^{\bar{H}} \bar{X}) \wedge DE\mathcal{F}/H_+).$$

With this substitution (5.3.14) is precisely (5.3.13) for the new ambient group G/H, with inflation along the quotient map $G/H \to G/L$ and the G/L-spectrum $\Phi^{\bar{H}} \bar{X}$. $\square$

5.3.2 Building the map

We can now return to our case $G = \mathbb{T}^2$ and $\bar{G} = G/H_1$, and apply proposition 5.3.1 to $E\mathcal{C}_{\bar{G}} \in \mathcal{A}(\bar{G})$ (5.2.12) to explicitly obtain

$$\mathrm{Inf}^G_{\bar{G}}(E\mathcal{C}_{\bar{G}}) = \left[\begin{tikzcd} \mathcal{E}^{-1}_G \mathcal{O}_\mathcal{F} \otimes \overline{\mathcal{K}_\mathbb{T}} & \\ \mathcal{E}^{-1}_{H_1} \mathcal{O}_\mathcal{F} \otimes_1 N_\mathbb{T} \arrow[u] & \mathcal{E}^{-1}_{H_i} \mathcal{O}_\mathcal{F} \otimes \overline{\mathcal{K}_\mathbb{T}} \arrow[ul] \\ \mathcal{O}_\mathcal{F} \otimes_1 N_\mathbb{T} \arrow[u] \arrow[ur] & \end{tikzcd} \right] \quad (5.3.15)$$

where in codimension 1 (middle row) the value at H_1 (middle row left) is the only one different from the values at all the other connected codimension one subgroups H_i with $i \neq 1$ (middle row right).

We can also explicitly compute $E\mathcal{C}_G \in \mathcal{A}(G)$ (Definition 4.4.2) as the kernel of the map φ_0 (4.4.4).

Definition 5.3.16. For every connected codimension 1 subgroup H_i of G define N_i to be the kernel of the surjective $\mathcal{O}_{\mathcal{F}/H_i}$-map φ_0^i (4.4.28):

$$N_i \rightarrowtail \mathcal{E}_{G/H_i}^{-1}\mathcal{O}_{\mathcal{F}/H_i} \otimes \overline{\mathcal{K}} \xrightarrow{\varphi_0^i} T_i. \tag{5.3.17}$$

Then we can explicitly compute $E\mathcal{C}_G$:

$$E\mathcal{C}_G = \begin{bmatrix} \mathcal{E}_G^{-1}\mathcal{O}_{\mathcal{F}} \otimes \overline{\mathcal{K}} \\ \uparrow \\ \mathcal{E}_{H_i}^{-1}\mathcal{O}_{\mathcal{F}} \otimes_i N_i \\ \uparrow \\ \mathrm{Ker}(\varphi_0(1)) \end{bmatrix} \tag{5.3.18}$$

Definition 5.3.19. Define the map $\varepsilon : \mathrm{Inf}_{\bar{G}}^{G} E\mathcal{C}_{\bar{G}} \longrightarrow E\mathcal{C}_G$ in $\mathcal{A}(G)$ to be the map that at the vertex:

$$\varphi^G(\varepsilon) := \pi_1^* : \overline{\mathcal{K}_{\mathbb{T}}} \to \overline{\mathcal{K}} \tag{5.3.20}$$

is the graded map that has the pullback π_1^* in each even degree, where $\pi_1 : \mathcal{X} \to \mathcal{C}$ is the projection (4.1.8) defining H_1.

Lemma 5.3.21. *The map* (5.3.20) *extends to a well defined map* $\varepsilon : \mathrm{Inf}_{\bar{G}}^{G} E\mathcal{C}_{\bar{G}} \longrightarrow E\mathcal{C}_G$ *in* $\mathcal{A}(G)$.

Proof. Notice that the structure maps of $E\mathcal{C}_G$ (5.3.18) are all injective, therefore (5.3.20) determines the map ε at each level. We only need to verify that for each connected subgroup the target is the correct one.

For the subgroup H_1, the induced map $\varphi^{H_1}(\varepsilon) : N_{\mathbb{T}} \to N_1$ is the only map that completes the diagram:

$$\begin{tikzcd} N_{\mathbb{T}} \arrow[r, tail] \arrow[d, dashed, "\varphi^{H_1}(\varepsilon)"] & \mathcal{E}_{G/H_1}^{-1}\mathcal{O}_{\mathcal{F}/H_1} \otimes \overline{\mathcal{K}_{\mathbb{T}}} \arrow[r, two heads, "q"] \arrow[d, "\mathrm{Id} \otimes \pi_1^*"] & T_{\mathbb{T}} \arrow[d, "\pi_1^*"] \\ N_1 \arrow[r, tail] & \mathcal{E}_{G/H_1}^{-1}\mathcal{O}_{\mathcal{F}/H_1} \otimes \overline{\mathcal{K}} \arrow[r, two heads, "\varphi_0^1"] & T_1 \end{tikzcd} \tag{5.3.22}$$

where the top row is the exact sequence (5.2.11) whose pullback π_1^* gives the bottom row which is the exact sequence (5.3.17) for the subgroup H_1. The right vertical map is induced by π_1^* since for every integer $j \geq 1$ we have $\pi_1^{-1}(\mathcal{C}\langle j\rangle) = D_{1,j}$ so that in the j-th component we have an induced map $\pi_1^* : \mathcal{K}_{\mathbb{T}}/\mathcal{O}_{\mathcal{C}\langle j\rangle} \to \mathcal{K}/\mathcal{O}_{D_{1,j}}$. The right square of (5.3.22) commutes due to our particular choice of coordinates $\hat{t}_j$ for $\mathcal{C}\langle j\rangle$ and $\hat{t}_{1,j}$ for $D_{1,j}$ (4.2.28) (4.2.31)

$$\begin{aligned} t_{1,j} &= (\pi_1^j)^*(t_e) = \pi_1^* \circ [j]^*(t_e) = \pi_1^*(t_j) \\ \hat{t}_{1,j} &= (\pi_1^j)^*(\hat{t}_e) = \pi_1^* \circ [j]^*(\hat{t}_e) = \pi_1^*(\hat{t}_j). \end{aligned} \tag{5.3.23}$$

For all the other connected codimension one subgroups H_i with $i \neq 1$ to show the existence of the induced dotted arrow $\varphi^{H_i}(\varepsilon)$:

$$\begin{tikzcd} \mathcal{O}_{\mathcal{F}/H_i} \otimes \overline{\mathcal{K}_{\mathbb{T}}} \arrow[r, tail] \arrow[d, dashed, "\varphi^{H_i}(\varepsilon)"] & \mathcal{E}_{G/H_i}^{-1}\mathcal{O}_{\mathcal{F}/H_i} \otimes \overline{\mathcal{K}_{\mathbb{T}}} \arrow[d, "\mathrm{Id}\otimes\pi_1^*"] & \\ N_i \arrow[r, tail] & \mathcal{E}_{G/H_i}^{-1}\mathcal{O}_{\mathcal{F}/H_i} \otimes \overline{\mathcal{K}} \arrow[r, two heads, "\varphi_0^i"] & T_i \end{tikzcd} \tag{5.3.24}$$

we only need to verify that the zig-zag from the top left to the bottom right is the zero map, since the bottom row is exact. The map φ_0^i (4.4.28) is zero on every element in $\mathcal{O}_{\mathcal{F}/H_i} \otimes \mathrm{Im}(\pi_1^*)$ since in the image of π_1^* there are only meromorphic functions with poles at D_{1j}, so they are regular on all D_{ij} with $i \neq 1$.

For the trivial subgroup to show the existence of the induced dotted arrow $\varphi^1(\varepsilon)$:

$$\begin{tikzcd} \mathcal{O}_{\mathcal{F}} \otimes_1 N_{\mathbb{T}} \arrow[r, tail] \arrow[d, dashed, "\varphi^1(\varepsilon)"] & \mathcal{E}_G^{-1}\mathcal{O}_{\mathcal{F}} \otimes \overline{\mathcal{K}_{\mathbb{T}}} \arrow[d, "\mathrm{Id}\otimes\pi_1^*"] & \\ \mathrm{Ker}(\varphi_0(1)) \arrow[r, tail] & \mathcal{E}_G^{-1}\mathcal{O}_{\mathcal{F}} \otimes \overline{\mathcal{K}} \arrow[r, "\varphi_0(1)"] & \bigoplus_{i\geq 1} \mathcal{E}_{H_i}^{-1}\mathcal{O}_{\mathcal{F}} \otimes_i T_i \end{tikzcd} \tag{5.3.25}$$

we need to verify that the zig-zag from the top left corner to the bottom right corner is zero. The bottom row of (5.3.25) is the beginning of the trivial subgroup level of the injective resolution (4.4.4) of $E\mathcal{C}_G$. This zig-zag is indeed zero because of the previous diagrams. For the component $i = 1$ the map is zero because the same zig-zag in (5.3.22) is zero. In the same way for all the other components $i \neq 1$ the map is zero because the same zig-zag in (5.3.24) is zero. □

5.4 Proving the H-equivalence

To finish the proof of Theorem 5.1.2 we are left to show that for every subgroup H of G such that $H \cap H_1 = 1$ the map ε (Definition 5.3.19) induces an isomorphism

$$[{}^{G}/{}_{H_+}, \operatorname{Inf}_{\bar{G}}^{G} E\mathcal{C}_{\bar{G}}]_*^G \cong [{}^{G}/{}_{H_+}, E\mathcal{C}_G]_*^G. \tag{5.4.1}$$

To do so we will use the Adams spectral sequence (2.5.2) for the homology functor $\pi_*^{\mathcal{A}}$ (2.5.1), and show that ε induces an isomorphism of the second page of the two Adams spectral sequences:

$$\operatorname{Ext}_{\mathcal{A}}^{*,*}(\pi_*^{\mathcal{A}}({}^{G}/{}_{H_+}), \operatorname{Inf}_{\bar{G}}^{G} E\mathcal{C}_{\bar{G}}) \cong \operatorname{Ext}_{\mathcal{A}}^{*,*}(\pi_*^{\mathcal{A}}({}^{G}/{}_{H_+}), E\mathcal{C}_G). \tag{5.4.2}$$

The first step is to identify the algebraic model for the natural cells G/H_+ which is done in 5.4.1. We turn then in building an injective resolution for $\operatorname{Inf}_{\bar{G}}^{G} E\mathcal{C}_{\bar{G}}$ in 5.4.2 to compute the Ext groups in Theorem 5.4.26. Recall that we already have an injective resolution of $E\mathcal{C}_G$ (4.4.4) that we recall in 5.4.3.

5.4.1 Algebraic model for natural cells

We explicitly compute the algebraic model for the natural cells: $\pi_*^{\mathcal{A}}(G/H_{i_+})$ for the connected codimension one subgroups H_i of G, and $\pi_*^{\mathcal{A}}(G/F_+)$ for the finite subgroups F.

Lemma 5.4.3. *For every $i \geq 1$ the algebraic model $\pi_*^{\mathcal{A}}(G/H_{i_+})$ has values:*

$$\varphi^{H_i}(\pi_*^{\mathcal{A}}(G/H_{i_+})) = \Sigma H^*(BG/H_i)/e(z_i) = \Sigma\mathbb{Q} \tag{5.4.4}$$

$$\varphi^1(\pi_*^{\mathcal{A}}(G/H_{i_+})) = \bigoplus_{F \subseteq H_i} \Sigma H^*(BG/F)/e(z_i) \tag{5.4.5}$$

while $\varphi^G(\pi_*^{\mathcal{A}}(G/H_{i_+})) = \varphi^{H_j}(\pi_*^{\mathcal{A}}(G/H_{i_+})) = 0$ for $j \neq i$. The structure maps are the ones induced by the suspended sphere $\pi_*^{\mathcal{A}}(S^{z_i})$ (2.5.52). Here $e(z_i)$ is the Euler class (2.5.13) of the character z_i (4.1.7) having H_i as kernel.

Proof. We start from the cofibre sequence of G-spaces:

$$G/H_{i_+} \longrightarrow S^0 \xrightarrow{e(z_i)} S^{z_i}. \tag{5.4.6}$$

Applying the suspension functor we obtain a cofibre sequence in G-spectra, which induces a long exact sequence for the homology functor $\pi_*^{\mathcal{A}}$:

$$\pi_*^{\mathcal{A}}(G/H_{i+}) \longrightarrow \pi_*^{\mathcal{A}}(S^0) \xrightarrow{e(z_i)} \pi_*^{\mathcal{A}}(S^{z_i}). \tag{5.4.7}$$

By (2.5.51) the induced map $e(z_i)$ in (5.4.7) at the levels of the subgroups G and H_j with $j \neq i$ is the identity since $z_i^G = z_i^{H_j} = 0$, therefore we obtain zero as kernel and cokernel at those levels. At the level of the subgroup H_i since $z_i^{H_i} = z_i$ the map $e(z_i)$ is induced by the map

$$\varphi^{H_i}(e(z_i)) : \mathcal{O}_{\mathcal{F}/H_i} \to \Sigma^{z_i}\mathcal{O}_{\mathcal{F}/H_i} \tag{5.4.8}$$

that sends the unit $\iota \in \mathcal{O}_{\mathcal{F}/H_i}$ to precisely the Euler class $e(z_i^{H_i}) \in \mathcal{O}_{\mathcal{F}/H_i}$ as defined in (2.5.13). Using the coordinate c_i (2.5.17), the Euler class $e(z_i^{H_i})$ has c_i in the H_i-th component and 1 in all the other components. Therefore the map (5.4.8) is injective so it has no kernel, while the cokernel is $\Sigma^2\mathbb{Q}[c_i]/(c_i)$, which is the only contribution to (5.4.4).

At the bottom level we can apply the same argument, the map:

$$\varphi^1(e(z_i)) : \mathcal{O}_{\mathcal{F}} \to \Sigma^{z_i}\mathcal{O}_{\mathcal{F}} \tag{5.4.9}$$

sends the unit $\iota \in \mathcal{O}_{\mathcal{F}}$ to precisely the Euler class $e(z_i) \in \mathcal{O}_{\mathcal{F}}$, which by (2.5.13) and (2.5.23) has components:

$$e(z_i)_F = \begin{cases} 1 & \text{if } F \nsubseteq H_i \\ x_i & \text{if } F \subseteq H_i \end{cases} \tag{5.4.10}$$

Therefore the map (5.4.9) is injective and the cokernel is precisely the suspension of (5.4.5). $\square$

Lemma 5.4.11. *For every finite subgroup F of G the algebraic model $\pi_*^{\mathcal{A}}(G/F_+)$ has a zero at each subgroup level except at the bottom level* (2.5.33) *where it has the value:*

$$\pi_*^{\mathcal{A}}(G/F_+)(1) = \bigoplus_{F' \subseteq F} \Sigma^2 H^*(BG/F')/(x_A, x_B) = \bigoplus_{F' \subseteq F} \Sigma^2\mathbb{Q} \tag{5.4.12}$$

the suspended rationalized Burnside ring of F. We are using the coordinates $H^*(BG/F') \cong \mathbb{Q}[x_A, x_B]$ (2.5.21)*.*

Proof. In this case we use directly the definition of the homology functor $\pi_*^{\mathcal{A}}$ (2.5.47). If H is a connected subgroup of G different from the trivial one, then

$\varphi^H(\pi_*^{\mathcal{A}}(G/F_+)) = 0$, since $\Phi^H(G/F_+) = 0$ because the H-fixed points of the G-space G/F_+ is only the basepoint.

At the bottom level we have the rationalized Burnside ring of F [Bar08, Definition 1.3.5]:

$$\pi_*^G(DE\mathcal{F}_+ \wedge G/F_+) \cong \pi_*^G(G/F_+) \cong \Sigma^2 A(F) \cong \bigoplus_{F' \subseteq F} \Sigma^2 \mathbb{Q}. \tag{5.4.13}$$

For the first isomorphism of (5.4.13) we use that natural cells are strongly dualizable 2.2.5, more precisely by [GM95a, (4.16)]:

$$D(G/F_+) = F(G/F_+, S^0) \simeq S^{-L(F)} \wedge G/F_+ \tag{5.4.14}$$

where $L(F)$ is the tangent F-representation at the identity coset G/F. Therefore:

$$\begin{aligned} \pi_*^G(DE\mathcal{F}_+ \wedge G/F_+) &\cong \pi_*^G(DE\mathcal{F}_+ \wedge S^{L(F)} \wedge D(G/F_+)) \\ &\cong \pi_*^G(S^{L(F)} \wedge D(E\mathcal{F}_+ \wedge G/F_+)) \\ &\cong \pi_*^G(S^{L(F)} \wedge D(G/F_+)) \cong \pi_*^G(G/F_+). \end{aligned} \tag{5.4.15}$$

The weak equivalence $E\mathcal{F}_+ \wedge G/F_+ \simeq G/F_+$ can be obtained from the isotropy separation cofibre sequence of G/F_+ (2.4.25):

$$E\mathcal{F}_+ \wedge G/F_+ \to G/F_+ \to \tilde{E}\mathcal{F} \wedge G/F_+ \tag{5.4.16}$$

noticing the last term is nullhomotopic since G/F_+ has stable isotropy in $\mathcal{F}$. The second isomorphism of (5.4.13) can be obtained with (2.4.6) and the restriction-coinduction adjunction (2.4.5):

$$\begin{aligned} [S^0, G/F_+]^G &\cong [S^0, S^{L(F)} \wedge F(G/F_+, S^0)]^G \\ &\cong [S^{-L(F)}, F_F(G_+, S^0)]^G \\ &\cong [i_F^* S^{-L(F)}, S^0]^F \cong \Sigma^2[S^0, S^0]^F = \Sigma^2 A(F) \end{aligned}$$

□

5.4.2 Injective resolution of $\mathrm{Inf}_{\bar{G}}^G E\mathcal{C}_{\bar{G}}$

We build an injective resolution of $\mathrm{Inf}_{\bar{G}}^G E\mathcal{C}_{\bar{G}}$ (5.3.15) in the abelian category $\mathcal{A}(G)$:

$$0 \longrightarrow \mathrm{Inf}_{\bar{G}}^G E\mathcal{C}_{\bar{G}} \longrightarrow \mathbb{I}'_0 \xrightarrow{\varphi'_0} \mathbb{I}'_1 \xrightarrow{\varphi'_1} \mathbb{I}'_2 \longrightarrow 0. \tag{5.4.17}$$

We first rewrite $\mathrm{Inf}_{\bar{G}}^{G} E\mathcal{C}_{\bar{G}}$ in a more convenient form:

$$\mathrm{Inf}_{\bar{G}}^{G} E\mathcal{C}_{\bar{G}} = \begin{bmatrix} \mathcal{E}_G^{-1}\mathcal{O}_{\mathcal{F}} \otimes_1 N_{\mathbb{T}} \\ \uparrow \\ \mathcal{E}_{H_i}^{-1}\mathcal{O}_{\mathcal{F}} \otimes_1 N_{\mathbb{T}} \\ \uparrow \\ \mathcal{O}_{\mathcal{F}} \otimes_1 N_{\mathbb{T}} \end{bmatrix} \tag{5.4.18}$$

where at each level we are tensoring the $\mathcal{O}_{\mathcal{F}}$-module of $\pi_*^{\mathcal{A}}(S^0)$ (Lemma 2.5.48) with the nub $N_{\mathbb{T}}$ (5.2.12) of $E\mathcal{C}_{\bar{G}}$:

Remark 5.4.19. To obtain (5.4.18) from (5.3.15) simply notice that when $i \neq 1$ we have $\mathcal{E}_{G/H_1} \subseteq \mathcal{E}_{H_i} \subseteq \mathcal{E}_G$ and therefore:

$$\mathcal{E}_{H_i}^{-1}\mathcal{O}_{\mathcal{F}} \underset{1}{\otimes} N_{\mathbb{T}} \cong \mathcal{E}_{H_i}^{-1}\mathcal{O}_{\mathcal{F}} \underset{1}{\otimes} \mathcal{E}_{G/H_1}^{-1} N_{\mathbb{T}} \cong \mathcal{E}_{H_i}^{-1}\mathcal{O}_{\mathcal{F}} \underset{1}{\otimes} (\mathcal{E}_{G/H_1}^{-1}\mathcal{O}_{\mathcal{F}/H_1} \otimes \mathcal{K}_{\mathbb{T}}) \cong \mathcal{E}_{H_i}^{-1}\mathcal{O}_{\mathcal{F}} \otimes \mathcal{K}_{\mathbb{T}}.$$

The same is true for $\mathcal{E}_G^{-1}\mathcal{O}_{\mathcal{F}} \otimes_1 N_{\mathbb{T}} \cong \mathcal{E}_G^{-1}\mathcal{O}_{\mathcal{F}} \otimes \mathcal{K}_{\mathbb{T}}$.

Notice that for the family of all subgroups of G the universal space is precisely $E[All]_+ \simeq S^0$. Therefore combining [Gre08, p. 12.3] and [Gre08, p. 10.2] we obtain the injective resolution of S^0 in $\mathcal{A}(G)$:

$$S^0 \longrightarrow f_G(\mathbb{Q}) \longrightarrow \bigoplus_{i\geq 1} f_{H_i}(\bigoplus_{j\geq 1} \Sigma^2 H_*(BG/H_i^j)) \longrightarrow f_1(\bigoplus_{F\in\mathcal{F}} \Sigma^4 H_*(BG/F)) \longrightarrow 0.$$

We can rewrite this last sequence in a more convenient form for us:

$$S^0 \longrightarrow f_G(\mathbb{Q}) \longrightarrow \bigoplus_{i\geq 1} f_{H_i}(\frac{\mathcal{E}_{G/H_i}^{-1}\mathcal{O}_{\mathcal{F}/H_i}}{\mathcal{O}_{\mathcal{F}/H_i}}) \longrightarrow f_1(M) \longrightarrow 0, \tag{5.4.20}$$

where M is the $\mathcal{O}_{\mathcal{F}}$-module:

$$M := (\bigoplus_{i\geq 1} \frac{\mathcal{E}_G^{-1}\mathcal{O}_{\mathcal{F}}}{\mathcal{E}_{H_i}^{-1}\mathcal{O}_{\mathcal{F}}})/\Delta \tag{5.4.21}$$

and Δ is the diagonal submodule: the image of the map

$$\mathcal{E}_G^{-1}\mathcal{O}_{\mathcal{F}} \longrightarrow \bigoplus_{i\geq 1} \frac{\mathcal{E}_G^{-1}\mathcal{O}_{\mathcal{F}}}{\mathcal{E}_{H_i}^{-1}\mathcal{O}_{\mathcal{F}}}.$$

Since $N_{\mathbb{T}}$ is flat over $\mathcal{O}_{\mathcal{F}/H_1}$ [Gre05, Lemma 5.3], we can simply tensor over this ring every $\mathcal{O}_{\mathcal{F}}$-module of (5.4.20) to obtain an injective resolution of $\operatorname{Inf}_{\bar{G}}^{G} E\mathcal{C}_{\bar{G}}$ in $\mathcal{A}(G)$. More precisely the terms of the injective resolution (5.4.17) are:

$$\begin{aligned}
\mathbb{I}'_0 &= f_G(\overline{\mathcal{K}_{\mathbb{T}}}) \\
\mathbb{I}'_1 &= f_{H_1}(T_{\mathbb{T}}) \bigoplus_{i>1} f_{H_i}\big(\frac{\mathcal{E}^{-1}_{G/H_i}\mathcal{O}_{\mathcal{F}/H_i}}{\mathcal{O}_{\mathcal{F}/H_i}} \otimes \overline{\mathcal{K}_{\mathbb{T}}}\big) \\
\mathbb{I}'_2 &= f_1(M \underset{1}{\otimes} N_{\mathbb{T}}),
\end{aligned} \tag{5.4.22}$$

where M is defined in (5.4.21), $N_{\mathbb{T}}$ in (5.2.11), and we have used Remark 5.4.19. Notice we obtain $f_{H_1}(T_{\mathbb{T}})$ in $\mathbb{I}'_1$ since by [Gre12, Proposition 3.1] the module $\mathcal{E}^{-1}_{H_1}\mathcal{O}_{\mathcal{F}}$ is flat over $\mathcal{O}_{\mathcal{F}/H_1}$. Moreover $\mathbb{I}'_1$ is indeed injective by 2.5.45 and 2.5.46 since $T_{\mathbb{T}}$ is a torsion injective $\mathcal{O}_{\mathcal{F}/H_1}$-module and

$$\Sigma^2 H_*(BG/H_i^j) \otimes \overline{\mathcal{K}_{\mathbb{T}}}$$

is a torsion injective $H^*(B{}^{G}/_{H_i^j})$-module.

5.4.3 Injective resolution of $E\mathcal{C}_G$

For convenience we recall the injective resolution built in 4.4 of $E\mathcal{C}_G$. By (4.4.4) we have an injective resolution of $E\mathcal{C}_G$ in $\mathcal{A}(G)$:

$$0 \longrightarrow E\mathcal{C}_G \longrightarrow \mathbb{I}_0 \xrightarrow{\varphi_0} \mathbb{I}_1 \xrightarrow{\varphi_1} \mathbb{I}_2 \xrightarrow{0} 0 \tag{5.4.23}$$

where the terms are:

$$\begin{aligned}
\mathbb{I}_0 &:= f_G(\overline{\mathcal{K}}) \\
\mathbb{I}_1 &:= \bigoplus_{i\geq 1} f_{H_i}(T_i) \\
\mathbb{I}_2 &:= f_1(N),
\end{aligned} \tag{5.4.24}$$

with

$$\begin{aligned}
N &:= \bigoplus_{F} \overline{\mathcal{H}^2_F} \\
T_i &:= \bigoplus_{j\geq 1} \overline{\mathcal{K}/\mathcal{O}_{D_{ij}}}
\end{aligned} \tag{5.4.25}$$

as defined in (4.4.25) and (4.4.20), with all local cohomology modules coming from the Cousin complex of $\mathcal{O}^{\mathrm{TP}}_{\mathcal{X}}$ (4.3.18).

5.4.4 Computing the Ext groups

We are finally ready to finish the proof of Theorem 5.1.2, namely to show the isomorphism of the Ext groups in the Adams spectral sequences (5.4.1). To do so we will need some technical results that we prove after the main Theorem, and that we use in the proof.

Theorem 5.4.26. *For every subgroup H of G such that $H \cap H_1 = \{1\}$ the map ε (5.3.20) induces an isomorphism between the terms in the Adams spectral sequence:*

$$\mathrm{Ext}^{*,*}_{\mathcal{A}}(G/H_+, \mathrm{Inf}^G_{\bar{G}}(E\mathcal{C}_{\bar{G}})) \cong \mathrm{Ext}^{*,*}_{\mathcal{A}}(G/H_+, E\mathcal{C}_G) \tag{5.4.27}$$

Proof. First recall the injective resolutions of $\mathrm{Inf}^G_{\bar{G}} E\mathcal{C}_{\bar{G}}$ (5.4.17), and $E\mathcal{C}_G$ (5.4.23), as well as the algebraic model for the natural cells (Lemma 5.4.3, and Lemma 5.4.11).

It is enough to show that for every such H, in the following commutative diagram (obtained taking morphisms into the injective resolutions):

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{A}}(G/H_+, \mathbb{I}'_0) \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathcal{A}}(G/H_+, \mathbb{I}'_1) \arrow[r, "\delta'_1"] \arrow[d, "\varepsilon_1"] & \mathrm{Hom}_{\mathcal{A}}(G/H_+, \mathbb{I}'_2) \arrow[r] \arrow[d, "\varepsilon_2"] & 0 \\
\mathrm{Hom}_{\mathcal{A}}(G/H_+, \mathbb{I}_0) \arrow[r] & \mathrm{Hom}_{\mathcal{A}}(G/H_+, \mathbb{I}_1) \arrow[r, "\delta_1"] & \mathrm{Hom}_{\mathcal{A}}(G/H_+, \mathbb{I}_2) \arrow[r] & 0
\end{tikzcd} \tag{5.4.28}$$

the vertical maps (induced by ε) are all isomorphisms. We will first show this when H has codimension 1, and then when H is finite. Recall the definition of the injective objects in the first row (5.4.22), and in the second one (5.4.24).

If H has codimension 1, then necessarily H is connected and without loss of generality we can suppose $H = H_2$. The first column of (5.4.28) is zero since by the adjunction (2.5.44):

$$\mathrm{Hom}_{\mathcal{A}}(G/H_{2+}, f_G(\overline{\mathcal{K}_{\mathbb{T}}})) \cong \mathrm{Hom}_{\mathbb{Q}}(0, \overline{\mathcal{K}_{\mathbb{T}}}) = 0$$

and with the same argument: $\mathrm{Hom}_{\mathcal{A}}(G/H_{2+}, f_G(\overline{\mathcal{K}})) = 0$.

Moving to the second column, by Lemma 5.4.3 the object G/H_{2+} has a zero at each subgroup H_s with $s \neq 2$. Therefore the only summands of $\mathbb{I}'_1$ and $\mathbb{I}_1$ that do contribute are the ones constant below H_2, using the coordinate $H^*(BG/H_2) \cong \mathbb{Q}[c_2]$ (2.5.17) by the adjunction (2.5.44) we obtain:

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{A}}(G/H_{2+}, \mathbb{I}'_1) &\cong \mathrm{Hom}_{\mathcal{A}}(G/H_{2+}, f_{H_2}(\frac{\mathcal{E}^{-1}_{G/H_2}\mathcal{O}_{\mathcal{F}/H_2}}{\mathcal{O}_{\mathcal{F}/H_2}} \otimes \overline{\mathcal{K}_{\mathbb{T}}})) \\
&\cong \mathrm{Hom}_{\mathbb{Q}[c_2]}(\Sigma\mathbb{Q}, \mathbb{Q}[c_2^{\pm 1}]/\mathbb{Q}[c_2] \otimes \overline{\mathcal{K}_{\mathbb{T}}}) \cong \Sigma\overline{\mathcal{K}_{\mathbb{T}}}.
\end{aligned}$$

Explicitly the morphism $f \in \overline{\mathcal{K}_{\mathbb{T}}}$ is the one that sends the unit to the element $[c_2^{-1}] \otimes f$. In the same way:

$$\begin{aligned}\mathrm{Hom}_{\mathcal{A}}(G/H_{2+}, \mathbb{I}_1) &\cong \mathrm{Hom}_{\mathcal{A}}(G/H_{2+}, f_{H_2}(T_2)) \\ &\cong \mathrm{Hom}_{\mathbb{Q}[c_2]}(\Sigma\mathbb{Q}, \overline{\mathcal{K}/\mathcal{O}_{D_2}}) \cong \Sigma\overline{\mathcal{O}_{D_2}/m_2}\end{aligned}$$

where m_2 is the ideal of $\mathcal{O}_{D_2}$ of those functions vanishing at D_2 and explicitly the morphism $[f] \in \overline{\mathcal{O}_{D_2}/m_2}$ is the one that sends the unit to the element $[\hat{t}_2^{-1} f] \in \overline{\mathcal{K}/\mathcal{O}_{D_2}}$.

By Lemma 5.4.38 the map ε_1 (5.4.28) is an isomorphism, since in each even degree it is the map:

$$\pi_1^* : \mathcal{K}_{\mathbb{T}} \xrightarrow{\cong} \mathcal{O}_{D_2}/m_2.$$

Moving to the third column of (5.4.28), using the adjunction (2.5.44):

$$\begin{aligned}\mathrm{Hom}_{\mathcal{A}}(G/H_{2+}, \mathbb{I}'_2) &\cong \mathrm{Hom}_{\mathcal{A}}(G/H_{2+}, f_1(M \underset{1}{\otimes} N_{\mathbb{T}})) \\ &\cong \mathrm{Hom}_{\mathcal{O}_{\mathcal{F}}}(\bigoplus_{F\subseteq H_2} \Sigma H^*(BG/F)/(x_2), M \underset{1}{\otimes} N_{\mathbb{T}}) \\ &\cong \bigoplus_{F\subseteq H_2} \mathrm{Hom}_{H^*(BG/F)}(\Sigma H^*(BG/F)/(x_2), (M \underset{1}{\otimes} N_{\mathbb{T}})_F) \\ &\cong \bigoplus_{n_1\geq 1} \Sigma\overline{\mathcal{K}_{\mathbb{T}}/\mathcal{O}_{\mathcal{C}\langle n_1\rangle}}.\end{aligned} \tag{5.4.29}$$

where M is defined in (5.4.21), $N_{\mathbb{T}}$ in (5.2.11) and x_2 in (2.5.23). The last isomorphism of (5.4.29) is obtained as follows. First for every $n_1 \geq 1$ we have the finite cyclic subgroup $F = C_{n_1} = H_1^{n_1} \cap H_2 \subseteq H_2$. By Lemma 5.4.41 an element in the F-th component of the third row of (5.4.29) is a map sending the unit of $\Sigma H^*(BG/F)/(x_2)$ to a sum of elements of the form (5.4.42) with $h = 1$. Therefore the F-th component of the map δ'_1 (5.4.28):

$$(\delta'_1)_F : \Sigma\overline{\mathcal{K}_{\mathbb{T}}} \to \mathrm{Hom}_{H^*(BG/F)}(\Sigma H^*(BG/F)/(x_2), (M \underset{1}{\otimes} N_{\mathbb{T}})_F) \tag{5.4.30}$$

is surjective with kernel $\Sigma\overline{\mathcal{O}_{\mathcal{C}\langle n_1\rangle}}$, since (5.4.30) sends f to the map which sends the unit to the element

$$[(0, [x_1^{-k} x_2^{-1}], 0, \dots)] \underset{\mathbb{Q}[x_1]}{\otimes} (x_1^k \otimes f) \in (M \underset{1}{\otimes} N_{\mathbb{T}})_F$$

with $f t_{n_1}^k$ regular on $\mathcal{C}\langle n_1\rangle$. Therefore $(\delta'_1)_F$ induces the last isomorphism of (5.4.29).

In the same way:

$$\begin{aligned}\mathrm{Hom}_{\mathcal{A}}(G/H_{2+}, \mathbb{I}_2) &\cong \bigoplus_{F\subseteq H_2} \mathrm{Hom}_{H^*(BG/F)}(\Sigma H^*(BG/F)/(x_2), N_F) \\ &\cong \bigoplus_{F\subseteq H_2} \Sigma\overline{(\mathcal{O}_{D_2}/m_2)/(\mathcal{O}_F/m_2)}.\end{aligned} \tag{5.4.31}$$

where by (4.3.17) the F-th component of N (5.4.25) can be described as:

$$N_F = \overline{\mathcal{H}_F^2} \cong \overline{\bigoplus_{i\geq 1}(\mathcal{K}/\mathcal{O}_{D_{i,n_i}})/\mathcal{K}}, \tag{5.4.32}$$

and $\mathcal{O}_F$ is defined in (4.2.15). The last isomorphism of (5.4.31) is obtained as before. The F-th component of the map δ_1 (5.4.31):

$$(\delta_1)_F : \Sigma\overline{\mathcal{O}_{D_2}/m_2} \to \mathrm{Hom}_{H^*(BG/F)}(\Sigma H^*(BG/F)/(x_2), N_F) \tag{5.4.33}$$

is surjective, with kernel $\Sigma\overline{\mathcal{O}_F/m_2}$. This is because By Lemma 4.4.53 every element in (5.4.32) admits a representative $[(0, [g], 0, \dots)]$ with only the second component different from zero. Therefore an element in the F-th component of (5.4.31) is a map sending the unit to an element of the form: $[(0, [\hat{t}_2^{-1} f], 0, \dots)]$ with $[f] \in \overline{\mathcal{O}_{D_2}/m_2}$, which is precisely the F-th component of $\delta_1([f])$. This shows that $(\delta_1)_F$ induces the last isomorphism of (5.4.31).

By Lemma 5.4.38 the map ε_2 (5.4.28) is an isomorphism, since in every F-th component and in each even degree it is the map:

$$\pi_1^* : \mathcal{K}_{\mathbb{T}}/\mathcal{O}_{\mathcal{C}\langle n_1\rangle} \xrightarrow{\cong} (\mathcal{O}_{D_2}/m_2)/(\mathcal{O}_F/m_2).$$

where $F = H_1^{n_1} \cap H_2$.

Let us prove the case when H is a finite subgroup of G such that $H \cap H_1 = \{1\}$. By Lemma 5.4.37 we have $H \cong C_n$ is cyclic and without loss of generality $H \subseteq H_2$. Notice that $H = H_1^n \cap H_2$. Using the adjunction (2.5.44), the first two columns of (5.4.28) are zero, since by Lemma 5.4.11 the algebraic model for G/H_+ is zero everywhere except at the trivial subgroup. Therefore we are only left to prove that the map ε_2 (5.4.28) is an isomorphism. By the adjunction (2.5.44):

$$\mathrm{Hom}_{\mathcal{A}}(G/H_+, \mathbb{I}_2') \cong \bigoplus_{F\subseteq H} \mathrm{Hom}_{H^*(BG/F)}(\Sigma^2\mathbb{Q}, (M \underset{1}{\otimes} N_{\mathbb{T}})_F) \cong \bigoplus_{n_1|n} \overline{\mathcal{O}_{\mathcal{C}\langle n_1\rangle}/m}, \tag{5.4.34}$$

where $m < \mathcal{O}_{\mathcal{C}\langle n_1\rangle}$ is the ideal of those functions vanishing at $\mathcal{C}\langle n_1\rangle$. The last isomorphism of (5.4.34) can be proven similarly to (5.4.29) term by term for every

$F = H_1^{n_1} \cap H_2$. An element in the F-th component of (5.4.34) is an $H^*(BG/F) \cong \mathbb{Q}[x_1, x_2]$-module map sending the unit of $\mathbb{Q}$ to an element of $(M \otimes_1 N_{\mathbb{T}})_F$ which is zero if multiplied by x_1 or x_2 (2.5.23). By Lemma 5.4.41 such an element is sum of elements of the form:

$$[([x_1^{-1}x_2^{-1}], 0, 0, \dots)] \underset{\mathbb{Q}[x_1]}{\otimes} (x_1 \otimes f/t_{n_1})$$

with $[f] \in \overline{\mathcal{O}_{\mathcal{C}\langle n_1 \rangle}/m}$, hence we obtain (5.4.34). Similarly:

$$\mathrm{Hom}_{\mathcal{A}}(G/H_+, \mathbb{I}_2) \cong \bigoplus_{F \subseteq H} \mathrm{Hom}_{H^*(BG/F)}(\Sigma^2\mathbb{Q}, N_F) \cong \bigoplus_{F \subseteq H} \overline{\mathcal{O}_F/\langle m_1, m_2 \rangle}, \tag{5.4.35}$$

where $m_i < \mathcal{O}_F$ is the ideal of those functions vanishing at D_{i,n_i}. The second isomorphism of (5.4.35) is proven for every F similarly to (5.4.31). The element $[f] \in \overline{\mathcal{O}_F/\langle m_1, m_2 \rangle}$ defines the $H^*(BG/F) \cong \mathbb{Q}[x_1, x_2]$-module map that sends the unit of $\mathbb{Q}$ to the following element with only the first component different from zero:

$$[([\frac{f}{\hat{t}_{1,n_1}\hat{t}_{2,1}}], 0, 0, \dots)] \in N_F,$$

where we have used (5.4.32).

By Lemma 5.4.38 the map ε_2 (5.4.28) is an isomorphism, since in every F-th component and in each even degree it is the map:

$$\pi_1^* : \mathcal{O}_{\mathcal{C}\langle n_1 \rangle}/m \xrightarrow{\cong} \mathcal{O}_F/\langle m_1, m_2 \rangle$$

where $\pi_1^{-1}(\mathcal{C}\langle n_1 \rangle) = D_{1,n_1}$, so that a function in m has pullback in m_1. $\square$

Remark 5.4.36. The form of this proof suggests a geometric counterpart of the statement directly in Algebraic Geometry. The Lie group homomorphism $z_1 : G \twoheadrightarrow \bar{G}$ induces the projection $\pi_1 = \mathfrak{X}(z_1) : \mathcal{X} \twoheadrightarrow \mathcal{C}$, while for every subgroup H such that $H \cap H_1 = 1$ we have the inclusion $i_H : H \hookrightarrow G$ inducing the immersion $\iota = \mathfrak{X}(i_H) : \mathfrak{X}(H) \hookrightarrow \mathcal{X}$. This induces the following dictionary between Topology and Algebraic geometry:

Topology	Algebraic Geometry
$E\mathcal{C}_G$	$\mathcal{O}_{\mathcal{X}}$
$E\mathcal{C}_{\bar{G}}$	$\mathcal{O}_{\mathcal{C}}$
$\mathrm{Inf}_{\bar{G}}^{G} E\mathcal{C}_{\bar{G}}$	$\pi_1^* \mathcal{O}_{\mathcal{C}}$
S^W	$\mathcal{O}_{\mathcal{X}}(D_W)$
$S(V \otimes w)_+$	$\Sigma^{-1}\mathcal{O}_{\mathcal{X}}(D_{V\otimes w})/\mathcal{O}_{\mathcal{X}}$
$\mathbb{C}\mathrm{P}(V)_+$	$(\pi_1)_*(\Sigma^{-1}\mathcal{O}_{\mathcal{X}}(D_{V\otimes w})/\mathcal{O}_{\mathcal{X}})$
$i_H^* \mathrm{Inf}_{\bar{G}}^{G} E\mathcal{C}_{\bar{G}} \simeq i_H^* E\mathcal{C}_G$	$\iota^*\pi_1^*\mathcal{O}_{\mathcal{C}} \cong \iota^*\mathcal{O}_{\mathcal{X}}$

where V is a $\mathbb{T}$-representation and W is a G-representation. Denoting $\mathcal{F} = \Sigma^{-1}\mathcal{O}_{\mathcal{X}}(D_{V\otimes w})/\mathcal{O}_{\mathcal{X}}$, we can also translate the statement of Corollary 5.1.4 (using homology instead of cohomology):

$$\pi_*^G(S(V \otimes w)_+ \wedge E\mathcal{C}_G) \cong \pi_*^{\bar{G}}(\mathbb{C}\mathrm{P}(V)_+ \wedge E\mathcal{C}_{\bar{G}})$$

into its Algebraic Geometry counterpart:

$$H^*(\mathcal{X}, \mathcal{F} \otimes \mathcal{O}_{\mathcal{X}}) \cong H^*(\mathcal{C}, (\pi_1)_*(\mathcal{F} \otimes \mathcal{O}_{\mathcal{X}})) \cong H^*(\mathcal{C}, (\pi_1)_*\mathcal{F} \otimes \mathcal{O}_{\mathcal{C}}),$$

that possibly can be proven directly using geometric arguments like (5.1.19), (5.1.20), and working on the higher images functors.

We conclude with the results needed for the proof of Theorem 5.4.26.

Lemma 5.4.37. *If $F \subseteq G$ is a finite subgroup of G such that $F \cap H_1 = \{1\}$, then F is cyclic and there is a connected codimension 1 subgroup H_i such that $F \subseteq H_i$ and $H_i \cap H_1 = \{1\}$.*

Proof. Suppose F is not cyclic, then it contains at least a p-group of the form $F' \cong \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$ for a prime p. But inside G there is only one copy of such a p-group, namely the subgroup of elements of order p in G: $G[p] = \mathbb{T}[p] \times \mathbb{T}[p]$. This is immediate to see since every element in F' has order p, therefore $F' \subseteq G[p]$ and they have the same cardinality. This gives us a contradiction since $|F' \cap H_1| = p > 1$.

To prove the existence of such an H_i let us think about G as the quotient $G = \mathbb{T} \times \mathbb{T} \cong (\mathbb{R} \times \mathbb{R})/(\mathbb{Z} \times \mathbb{Z})$. Every connected codimension 1 subgroup H_i is determined by a point $P_i = (\lambda_i, \mu_i) \in \mathbb{Z} \times \mathbb{Z}$ for a pair of coprime integers λ_i and μ_i, namely H_i is the image in the quotient $(\mathbb{R} \times \mathbb{R})/(\mathbb{Z} \times \mathbb{Z})$ of the line in $\mathbb{R} \times \mathbb{R}$ connecting the origin to the point P_i. For example H_1 is determined by the point $P_1 = (0, 1)$. By the first part of the Lemma F is cyclic and generated by a point $Q = [(a/n, b/n)] \in (\mathbb{R} \times \mathbb{R})/(\mathbb{Z} \times \mathbb{Z})$ with $a, b \in \mathbb{Z}$ and where n is the order of F.

Since $F \cap H_1 = \{1\}$ we have $a \neq 0$ and that a and n are coprimes, otherwise we can find a non-trivial multiple of Q not in $\mathbb{Z} \times \mathbb{Z}$ but that lies in H_1. Therefore we can find $r, s \in \mathbb{Z}$ such that $rn + sa = 1$. For obvious reason s and n are coprimes, so the s-th multiple of Q:

$$Q' = sQ = [(\frac{sa}{n}, \frac{sb}{n})] = [(r + \frac{sa}{n}, \frac{sb}{n})] = [(\frac{1}{n}, \frac{sb}{n})]$$

is a generator for the subgroup F. The subgroup H_i defined by the point $P_i = (1, sb)$ satisfies all the requirements. We have $H_i \cap H_1 = \{1\}$ since the line connecting the origin to P_i does not intersect any integer vertical line. Moreover the line connecting the origin to P_i passes through Q', therefore $F \subseteq H_i$ since Q' generates F. □

Lemma 5.4.38. *The pullback along the projection $\pi_1 : \mathcal{X} \to \mathcal{C}$ induces an isomorphism:*

$$\pi_1^* : \mathcal{K}_{\mathbb{T}} \xrightarrow{\cong} \mathcal{O}_{D_2}/m_2, \tag{5.4.39}$$

that restricted to $\mathcal{O}_{\mathcal{C}\langle n_1 \rangle}$ for any integer $n_1 \geq 1$ gives an isomorphism:

$$\pi_1^* : \mathcal{O}_{\mathcal{C}\langle n_1 \rangle} \xrightarrow{\cong} \mathcal{O}_F/m_2 \tag{5.4.40}$$

where $F = H_1^{n_1} \cap H_2$.

Proof. The isomorphism (5.4.39) is (4.4.54) for the point $P = \{e\}$ and for the projection π_1 instead of π_2. To prove (5.4.40) it is enough to notice that $\pi_1^{-1}(\mathcal{C}\langle n_1 \rangle) = D_{1,n_1}$ and therefore (5.4.39) restricted to $\mathcal{O}_{\mathcal{C}\langle n_1 \rangle}$ gives functions that are regular also at D_{1,n_1}, by Lemma 4.4.53 we obtain that the image is precisely $\mathcal{O}_F$ (4.2.15). □

Lemma 5.4.41. *For every finite subgroup $F = H_1^{n_1} \cap H_2 \subseteq H_2$, every element in the F-th component $(M \otimes_1 N_{\mathbb{T}})_F$ is sum of elements of the form*

$$[([x_1^{-k} x_2^{-h}], 0, 0, \dots)] \underset{\mathbb{Q}[x_1]}{\otimes} (x_1^k \otimes f) \in (M \underset{1}{\otimes} N_{\mathbb{T}})_F \tag{5.4.42}$$

such that $h, k \geq 1$ and $ft_{n_1}^k \in \mathcal{O}_{\mathcal{C}\langle n_1 \rangle}$. Here M is defined in (5.4.21), $N_{\mathbb{T}}$ in (5.2.11), and we have used the coordinates $H^*(BG/F) \cong \mathbb{Q}[x_1, x_2]$ (2.5.23).

Proof. Every element in the F-th component of M (5.4.21) can be written as:

$$[([\frac{p(x_1, x_2)}{x_1^k x_2^h}], 0, 0, \dots)] \in (\bigoplus_{i \geq 1} \frac{\mathcal{E}_G^{-1}\mathbb{Q}[x_1, x_2]}{\mathcal{E}_{H_i}^{-1}\mathbb{Q}[x_1, x_2]})/\Delta$$

with $k, h \geq 1$ and only the first component of the representative of the class different from zero.

Every element in $(N_{\mathbb{T}})_{n_1}$ can be written as $x_1^s x_1^r \otimes f$ for $s \geq 0$, with $t_{n_1}^r f$ regular at $\mathcal{C}\langle n_1 \rangle$ and that does not vanish on $\mathcal{C}\langle n_1 \rangle$.

Therefore every element in $(M \otimes_1 N_{\mathbb{T}})_F$ is sum of elements of the form:

$$[([x_1^{-k} x_2^{-h}], 0, 0, \dots)] \underset{\mathbb{Q}[x_1]}{\otimes} (x_1^r \otimes f) = [([x_1^{-k} x_2^{-h}], 0, 0, \dots)] \underset{\mathbb{Q}[x_1]}{\otimes} (x_1^k \otimes (f t_{n_1}^{r-k})). \quad (5.4.43)$$

The equality (5.4.43) is true since under the isomorphism:

$$\frac{\mathcal{E}_G^{-1}\mathbb{Q}[x_1, x_2]}{\mathcal{E}_{H_1}^{-1}\mathbb{Q}[x_1, x_2]} \underset{\mathbb{Q}[x_1]}{\otimes} (N_{\mathbb{T}})_{n_1} \cong \mathcal{E}_{H_1}^{-1}\mathbb{Q}[x_1, x_2] \underset{\mathbb{Q}[x_1]}{\otimes} \mathcal{K}_{\mathbb{T}} / \mathcal{O}_{\mathcal{C}\langle n_1 \rangle} \quad (5.4.44)$$

the two elements in the left hand side: $[x_1^{-k} x_2^{-h}] \otimes_{\mathbb{Q}[x_1]} (x_1^r \otimes f)$ and $[x_1^{-k} x_2^{-h}] \otimes_{\mathbb{Q}[x_1]} (x_1^k \otimes (f t_{n_1}^{r-k}))$ are sent to the same element in the right hand side. $\square$

Chapter 6

Future directions

6.1 Higher Tori

In Chapter 4 we built rational $\mathbb{T}^2$-equivariant elliptic cohomology $E\mathcal{C}_{\mathbb{T}^2} \in \mathcal{A}(\mathbb{T}^2)$. A really natural question is how to generalize this construction and the construction for the circle $E\mathcal{C}_{\mathbb{T}} \in \mathcal{A}(\mathbb{T})$ to tori of any rank $\mathbb{T}^k$, namely building rational $\mathbb{T}^k$-equivariant elliptic cohomology $E\mathcal{C}_{\mathbb{T}^k} \in \mathcal{A}(\mathbb{T}^k)$ starting from an elliptic curve $\mathcal{C}$ over $\mathbb{C}$ and a coordinate $t_e \in \mathcal{O}_{\mathcal{C},e}$. First we need to ask which kind of properties we want $E\mathcal{C}_{\mathbb{T}^k}$ to satisfy. An obvious thing to ask is for $E\mathcal{C}_{\mathbb{T}^k}$ to be 2-periodic, and the value on spheres of complex representations S^V with $V^{\mathbb{T}^k} = 0$ to be given in terms of the sheaf cohomology of a line bundle $\mathcal{O}(-D_V)$ over $\mathcal{X}_{\mathbb{T}^k} \cong \mathcal{C}^k$:

$$\begin{aligned} E\mathcal{C}^{\text{even}}_{\mathbb{T}^k}(S^V) &\cong H^{\text{even}}(\mathcal{X}_{\mathbb{T}^k}, \mathcal{O}(-D_V)) \\ E\mathcal{C}^{\text{odd}}_{\mathbb{T}^k}(S^V) &\cong H^{\text{odd}}(\mathcal{X}_{\mathbb{T}^k}, \mathcal{O}(-D_V)). \end{aligned} \tag{6.1.1}$$

The isomorphism in (6.1.1) might be too much to ask for our theory, namely in the $\mathbb{T}^2$-case we obtained an isomorphism since by (4.5.12) the second page of the Adams spectral sequence:

$$\mathrm{Ext}^s_{\mathcal{A}}(S^V, E\mathcal{C}_{\mathbb{T}^2}) \cong \overline{H^s(\mathcal{X}_{\mathbb{T}^2}, \mathcal{O}(-D_V))}. \tag{6.1.2}$$

has only three rows different from zero (namely for $s = 0, 1, 2$), since $\mathcal{X}_{\mathbb{T}^2}$ has dimension 2. As a consequence all the differentials in the second page are zero and the Adams spectral sequence degenerates at the second page. For $k \geq 3$ the complex abelian variety $\mathcal{X}_{\mathbb{T}^k}$ has dimension k, and therefore in the Adams spectral sequence (6.1.2) we obtain more than 3 rows different from zero and the differential might be non-trivial. Therefore we might get simply a convergent spectral sequence in (6.1.1) instead of an isomorphism. The associated divisor D_V might be defined in the same way, for every subgroup H of $\mathbb{T}^k$ we have the associated subvariety of $\mathcal{X}_{\mathbb{T}^k}$:

$$\mathfrak{X}(H) := \mathrm{Hom}_{\mathrm{Ab}}(H^*, \mathcal{C}).$$

where $H^* := \mathrm{Hom}(H, \mathbb{T})$ is the character group of H. Therefore if $V = \bigoplus_{\underline{n}} \alpha_{\underline{n}} z^{\underline{n}}$, where $z^{\underline{n}}$ is the one dimensional complex representation of $\mathbb{T}^k$ with weight vector $\underline{n} = (n_1, \ldots, n_k) \in \mathrm{Hom}(\mathbb{T}^k, \mathbb{T})$, the associated divisor of V can still be defined as:

$$D_V := \sum_{\underline{n}} \alpha_{\underline{n}} \mathfrak{X}(\mathrm{Ker}(z^{\underline{n}})).$$

The main difficulty in building $E\mathcal{C}_{\mathbb{T}^k}$ resides in the combinatorics of the intersections of the subvarieties $\mathfrak{X}(H)$. For $E\mathcal{C}_{\mathbb{T}^2}$ we started by defining the codimension 1 subvarieties D_{ij} (4.1.11) associated to the codimension one subgroups H_i^j of $\mathbb{T}^2$. In codimension 2 the way the H_i^j intersect in the finite subgroups F, is mirrored in geometry in how the D_{ij} intersect in the $\bar{\mathfrak{X}}(F)$ (Lemma 4.1.17). For $\mathbb{T}^2$ we don't need to go further, but for $\mathbb{T}^k$ we might need an inductive argument to replace Lemma 4.1.17, and starting from the subvarieties in codimension 1, define the lower dimensional subvarieties as the appropriate intersections of the one above so to mirror the poset of subgroups in $\mathbb{T}^k$. Moreover all the Lemmas to fit the geometric inputs from the Cousin complex as modules in the algebraic model (Lemmas 4.4.17, 4.4.22, 4.4.29, 4.4.33), are somehow a doc constructions for the $\mathbb{T}^2$-case. For general tori we need a more rigorous framework to obtain those results at each dimension. This goal can probably be achieved by a more generic use of Local cohomology modules and their properties (for $\mathbb{T}^2$ we have often used a specific form of those modules (4.3.17), (4.3.13)).

6.2 General complex abelian surfaces

We can also consider a totally different direction, and try to generalize the construction of $E\mathcal{C}_{\mathbb{T}^2}$ towards more general geometric inputs instead of generalizing the group of equivariance. Namely one can consider a complex abelian surface S that might not be $\mathcal{X} = \mathcal{C} \times \mathcal{C}$ for a complex elliptic curve $\mathcal{C}$, for example a good first generalization could be a surface S which is isogeneous to the product of two elliptic curves. One can then try to define a poset of subvarieties to mirror the intersection pattern of subgroups of $\mathbb{T}^2$ 4.1, consider the Cousin complex (which will still have the same length) and try to define the action (Lemmas 4.4.17, 4.4.22) on the pieces of the cousin complex to build an exact sequence of injective objects (4.4.1) in $\mathcal{A}(\mathbb{T}^2)$. From this exact sequence we can define a theory $ES_{\mathbb{T}^2} \in \mathcal{A}(\mathbb{T}^2)$ whose values on spheres of complex representations S^V are given in terms of the sheaf cohomology of a line bundle $\mathcal{O}(-D_V)$ on the surface S. It would be really interesting to see if such theories can be defined, which properties do they have and if they are bonded in some sense to elliptic cohomology or they differ from it in a significant way. It seems unlikely that every complex abelian surface S gives rise to a $\mathbb{T}^2$-equivariant cohomology theory, probably it could be done for surfaces satisfying certain conditions. Investigating the right hypothesis for a surface to generate a rational $\mathbb{T}^2$-equivariant elliptic cohomology theory could lead to interesting results.

6.3 Grassmanians

In Chapter 5 we computed rational $\mathbb{T}$-equivariant elliptic cohomology of the space of complex lines $\mathbb{C}\mathrm{P}(V)$ for a $\mathbb{T}$-representation V. The next natural step would be to compute elliptic cohomology of grassmanians of n-planes $\mathrm{Gr}_n(V)$ for $n > 1$ (notice $\mathrm{Gr}_1(V) = \mathbb{C}\mathrm{P}(V)$). To generalize the same method used in Chapter 5 notice that $\mathrm{Gr}_n(V) \cong \mathrm{Fr}_n(V)/_{U(n)}$ is the quotient of the space of n-frames of V by the free $U(n)$-action. Therefore we would need first to build unitary versions of rational equivariant elliptic cohomology $E\mathcal{C}_G$ for $G = \mathbb{T} \times U(n)$, and then prove a $U(n)$-splitness result like 5.1.2.

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