Fluctuations of polymer partition functions and collision local times

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Declarations

I declare that the work contained in this PhD thesis is entirely my own and have been completed under the supervision of Nikos Zygouras, for the degree of Doctor of Philosophy in Mathematics and Statistics. This thesis has not been submitted for a degree at another university. The material presented in Chapters 2-4 is based on the following works published in collaboration with N. Zygouras:

[LZ22a] D. Lygkonis, N. Zygouras. Edwards-Wilkinson fluctuations for the directed polymer in the full L^2-regime for dimensions $d \geqslant 3$, *Annales de l'Institut Henri Poincaré, Probabilités et Statistiques*, 58 (1). pp. 65-104, (2022).

[LZ21+] D. Lygkonis, N. Zygouras. Moments of the $2d$ directed polymer and a generalisation of the Erdös-Taylor theorem, arXiv:2109.06115, (2021).

[LZ22b+] D. Lygkonis, N. Zygouras. A multivariate extension of the Erdös-Taylor theorem, arXiv:2202.08145, (2022).

Abstract

In this work we study the directed polymer in random environment and some associated problems. In Chapter 2, we focus in spatial dimensions $d \geqslant 3$ and study the spatial fluctuations of the field of partition functions and log-partition functions in the subregion of the weak disorder regime called L^2 regime. We prove convergence of the two fields, under centering and suitable scaling, to the solution of the Edwards-Wilkinson model, thus establishing Gaussian fluctuations, in the full L^2 regime.

In Chapter 3 we study the directed polymer in random environment in the case of spatial dimension $d = 2$ and in the so-called subcritical regime. We establish that all moments of the partition function are bounded in the full subcritical regime and compute their limit. As a byproduct, we obtain that the logarithmically scaled total collision local time between h, $(h \in \mathbb{N}, h \geqslant 3)$, independent simple symmetric random walks on $\mathbb{Z}^2$ converges in distribution to a Gamma random variable. Based on this result, we formulate the conjecture that the joint distribution of the $h(h-1)/2$ logarithmically scaled collision local times between h simple symmetric random walks on $\mathbb{Z}^2$ converges to that of a vector of $h(h-1)/2$ independent exponential random variables.

Last, in Chapter 4, we prove the aforementioned conjecture on the logarithmically scaled collision local times by exactly computing their limiting joint Laplace transform. In order to prove this result, we build on tools developed in Chapter 3 and further analyse the microscopic structure of the collision local times.

Notation

We note that throughout Chapters 2, 3 and 4 we will use the letters $c, c', C, C', \dots$ to denote constants that may change from line to line. Furthermore, given any two positive sequences $(a_N)_{N\in\mathbb{N}}, (b_N)_{N\in\mathbb{N}}$, we will write $a_N \overset{N\to\infty}{\approx} b_N$ or simply $a_N \approx b_N$ when $\lim_{N\to\infty} \frac{a_N}{b_N} = 1$.

Below, we summarise the main notation we will use in the following chapters.

Notation	Definition
S	symmetric simple random walk in $\mathbb{Z}^d$
P_x, E_x	probability and expectation w.r.t. the law of S starting from x
$q_n(x)$	$\mathrm{P}(S_n = x)$
ω	random environment
$\mathbb{P}$, $\mathbb{E}$	probability and expectation w.r.t. the law of ω
β	inverse temperature / strength of disorder parameter
DPRE_d	d-dimensional directed polymer in random environment
$\lambda(\beta)$	$\log \mathbb{E}[e^{\beta\omega}]$
$\lambda_2(\beta)$	$\lambda(2\beta) - 2\lambda(\beta)$
$\sigma(\beta)$	$\sqrt{e^{\lambda_2(\beta)} - 1}$, (Chapter 2)
$\sigma_{N,\hat{\beta}}$	$\frac{\hat{\beta}}{\sqrt{R_N}}$, (Chapter 3)
$\sigma_N^{i,j}(\beta)$	$e^{\frac{\pi\beta_{i,j}}{\log N}} - 1$, (Chapter 4)
$\xi_{n,z}$	$\frac{e^{\beta\omega_{n,z}-\lambda(\beta)}-1}{\sigma(\beta)}$, ($e^{\beta_N\omega_{n,z}-\lambda(\beta_N)} - 1$ in Chapter 3)
$Z_{N,\beta}(x)$	$\mathrm{E}_x\Big[e^{\sum_{n=1}^N \{\beta\omega_{n,S_n} - \lambda(\beta)\}}\Big]$
$Z_{N,\beta}^{\Lambda}(x)$	$\mathrm{E}_x\Big[e^{\sum_{(n,z)\in\Lambda}\{\beta\omega_{n,z}-\lambda(\beta)\}\mathbb{1}_{\{S_n=z\}}}\Big]$ for $\Lambda \subset \mathbb{N}\times\mathbb{Z}^d$
$\bar{Z}_{N,\beta}(\varphi)$	$\displaystyle\sum_{x\in\mathbb{Z}^d} \frac{\varphi\big(\frac{x}{\sqrt{N}}\big)}{N^{\frac{d}{2}}}\big(Z_{N,\beta}(x) - 1\big)$ for $\varphi \in C_c(\mathbb{R}^d)$
$\varphi_N(x)$	$\varphi\big(\frac{x}{\sqrt{N}}\big)$
π_d	return probability of d-dimensional simple random walk
L_N	$\sum_{n=1}^N \mathbb{1}_{\{S_{2n}=0\}}$
R_N	$\mathrm{E}[\mathsf{L}_N]$
$\mathsf{L}_N^{(i,j)}$	$\sum_{n=1}^N \mathbb{1}_{\{S_n^{(i)}=S_n^{(j)}\}}$ - collision local time between random walks $S^{(i)}, S^{(j)}$

We note that in Chapter 3, the sums used to define $Z_{N,\beta}(x)$, L_N and $\mathsf{L}_N^{(i,j)}$ will run from $n = 1$ to $n = N - 1$ instead of N.

Bibliography. Citations of works that, to the best of our knowledge, have not yet been published to a scientific journal will be denoted ending with a + symbol (e.g. [L21+]) to avoid any chronological confusion.

CHAPTER 1

Introduction

1.1. The directed polymer in random environment

The main focus of this work is the study of the directed polymer in random environment in dimensions $d \geqslant 2$, DPRE_d for short, which is a model consisting of a random walk interacting with a space-time random environment placed on the vertices of the d-dimensional lattice $\mathbb{Z}^d$. In particular, consider $S = (S_n)_{n \geqslant 0}$ to be a d-dimensional simple symmetric random walk, whose law and expectation we will denote by P_x, E_x, respectively, when starting from $x \in \mathbb{Z}^d$ and let also $\omega = (\omega_{n,x})_{(n,x) \in \mathbb{N} \times \mathbb{Z}^d}$ be a family of independent and identically distributed random variables with law $\mathbb{P}$ and expectation $\mathbb{E}$ such that

$$\mathbb{E}[\omega] = 0, \qquad \mathbb{E}[\omega^2] = 1, \qquad \lambda(\beta) := \log \mathbb{E}[e^{\beta\omega}] < \infty\,, \quad \forall \beta \in (0,\infty).$$

The law of DPRE_d of length N, starting from $x \in \mathbb{Z}^d$ and at inverse temperature $\beta \in (0,\infty)$ is defined by

$$\frac{\mathrm{d}\mathrm{P}_{N,\beta,x}}{\mathrm{d}\mathrm{P}_x}(S) := \frac{1}{Z_{N,\beta}(x)}\, e^{\sum_{n=1}^N \{\beta\omega_{n,S_n} - \lambda(\beta)\}}\,, \tag{1.1.1}$$

where

$$Z_{N,\beta}(x) = \mathrm{E}_x\Big[e^{\sum_{n=1}^N \{\beta\omega_{n,S_n} - \lambda(\beta)\}}\Big]. \tag{1.1.2}$$

$Z_{N,\beta}(x)$ is a (random) normalising constant which makes the polymer measure a probability measure. It is called the *partition function* of the model and its importance stems form the fact that it contains crucial information about all the thermodynamic quantities of interest, see [Bov06]. The study of its statistical properties in the infinite volume limit, i.e. as $N \to \infty$, will be the main interest of this work. When the starting point of the random walk is $0 \in \mathbb{Z}^d$ we will simply write $Z_{N,\beta}$ instead of $Z_{N,\beta}(x)$. Note that, due to the translation invariant nature of the random environment ω, $Z_{N,\beta}(x)$ has the same law with $Z_{N,\beta}$ for every $x \in \mathbb{Z}^d$. $Z_{N,\beta}$. We note that including the factor $\lambda(\beta)$ in the exponential in (1.1.1) and (1.1.2) turns $Z_{N,\beta}$ into a martingale with respect to the natural filtration $(\mathcal{F}_n)_{n \geqslant 0}$ with $\mathcal{F}_n = \sigma\big(\omega_{\ell,x} : 1 \leqslant \ell \leqslant n, x \in \mathbb{Z}^d\big)$, such that $\mathbb{E}[Z_{N,\beta}(x)] = 1$ for all $N \in \mathbb{N}$. The significance of this modification will become apparent in Section 1.2.

The DPRE_d models a competition between *entropy*, stemming from the underlying random walk and *energy*,which takes the form of rewards provided by the environment ω that are collected by the random walk as it traverses the lattice $\mathbb{Z}^d$. One can almost immediately distinguish two extreme cases of the directed polymer, one that is dominated by entropy and one that is dominated by energy. In particular, when $\beta = 0$ in (1.1.1) one recovers the law of the simple random walk under which every path of length N has the same probability $(2d)^{-N}$. On the other hand, by (1.1.1) we see that as β increases (temperature decreases) the polymer measure tends to assign larger probability to directed paths along which the environment ω is more favorable, i.e. it attains higher values. In particular, in the limit $\beta \to \infty$ the polymer measure is concentrated on the directed random walk paths π along which the energy $\sum_{\alpha \in \pi} \omega_\alpha$ is maximized. Therefore, the DPRE_d at inverse temperature $\beta \in (0,\infty)$ can be seen as an interpolation between those two extreme cases, and as such it is interesting to study how the transition between the entropy dominated phase to the energy dominated phase happens as one varies the inverse temperature β as well as the spatial dimension d. Before we delve into a more detailed exposition of the main established results about the DPRE_d we present some of the main questions around it as well as links with other models that further motivate its study.

1.1.1. DPRE_d **as a disordered system.** One of the main questions the directed polymer in random environment poses is whether the introduction of the disorder, i.e. the random environment ω, is sufficient to alter the large scale statistical properties of the underlying random walk, a question which falls under the more general scope of disorder relevance/irrelevance in the field of disordered systems. In particular, if even a small amount of disorder is enough to change the large scale properties of a system we say that disorder is *relevant*, otherwise we say that disorder is *irrelevant*.

In the context of the polymer, one can imagine two possible scenarios for the DPRE_d depending on the dimension d and inverse temperature β. If the spatial dimension d is large and β is small then the environment should not have much effect on the polymer, because there is enough space for the polymer to avoid large values of the environment and also the environment is weak due to high temperature. On the other hand, if the dimension is small or the strength of the disorder β is high, there is not much room for the polymer to avoid the influence of the environment and it will have an advantage to travel to atypically far distances to collect disorder that is more favourable.

According to a powerful but heuristic criterion due to Harris [Har74] which was first formulated in the context of the ferromagnetic Ising model with random impurities, the question of whether disorder is relevant or irrelevant for a statistical physics model can be determined by looking at a suitably defined correlation length exponent ν and the effective dimension d_{eff} of the pure model. In particular, if $\nu < \frac{2}{d_{\text{eff}}}$, disorder is deemed irrelevant and a small amount of random impurities is not sufficient to alter the large scale properties of the model, if $\nu > \frac{2}{d_{\text{eff}}}$ disorder is relevant and even a small amount of external randomness is sufficient to change the macroscopic behaviour of the system, while for the case $\nu = \frac{2}{d_{\text{eff}}}$, the Harris criterion is inconclusive and one has to look at the fine details of each specific model to rule whether disorder is relevant or not. For the simple random walk on $\mathbb{Z}^d$, diffusivity suggests that we have $d_{\text{eff}} = d + 2$ and $\nu = \frac{1}{2}$. Therefore, for DPRE_d, according to the Harris criterion, disorder is relevant when $\frac{1}{2} < \frac{2}{d+2}$ that is, $d < 2$ and disorder is irrelevant when $\frac{1}{2} > \frac{2}{d+2}$ or equivalently, $d > 2$. The case of dimension $d = 2$ is dubbed marginal and the Harris criterion is inconclusive.

In the present work we will be concerned with the disorder irrelevant case of $d \geqslant 3$ in Chapter 2 and the marginal case of $d = 2$ in Chapters 3 and 4.

1.1.2. DPRE_d **and singular SPDEs.** Besides the question of disorder relevance/irrelevance, one of the main reasons to study the DPRE_d is its close connection with certain singular stochastic partial differential equations. In particular, let ξ denote space-time white noise, that is the *generalised* centred Gaussian process with covariance structure

$$\mathbb{E}\big[\xi(t,x)\xi(s,y)\big] = \delta(t-s)\delta(x-y) \qquad t,s > 0,\ x,y \in \mathbb{R}^d \tag{1.1.3}$$

and consider the stochastic heat equation with multiplicative white noise ξ, (mSHE), that is

$$\text{(mSHE)} \qquad \begin{cases} \partial_t u(t,x) = \frac{1}{2}\Delta u(t,x) + \beta(u \cdot \xi)(t,x) & t > 0,\ x \in \mathbb{R}^d \\ u(0,x) \equiv 1 \end{cases}. \tag{1.1.4}$$

The physical interpretation of the solution to mSHE (1.1.4) is that it represents the density, at a given time t and point in space $x \in \mathbb{R}^d$, of independent particles performing diffusions in an environment where particles can be generated or killed independently in space and time with a rate that depends on β and the sign of ξ.

Notice that, while in dimension $d = 1$, one can make sense of equation (1.1.4) by using classical Itô theory, this is no longer possible in dimensions $d \geqslant 2$, due to the very singular nature of space-time white noise which makes the product $u \cdot \xi$ ill-defined. More specifically, the d-dimensional space-time white noise ξ is a random distribution that belongs to the (parabolically scaled) Hölder space $\mathcal{C}_{\mathfrak{s}}^{-\frac{d}{2}-1-\kappa}$ for every $\kappa > 0$, see [CW17], Section 2. Taking into account the smoothing effect of the Laplacian operator which improves spatial regularity by 2 degrees, see for example [CW17], the solution is expected to have the regularity of $\mathcal{C}_{\mathfrak{s}}^{-\frac{d}{2}+1-\kappa}$ for all $\kappa > 0$, or lower. This suggests that in dimensions $d \geqslant 2$, the solution to mSHE should be a random distribution leading to the aforementioned ambiguities, see [CW17], Theorem 2.13.

Nevertheless, a first investigation of what properties a solution to mSHE should satisfy can be carried out by a scaling argument which probes the large scale behaviour of the solution to (1.1.4). More specifically, let ε denote a small positive parameter and consider the parabolically rescaled version of u, that is

$$\widetilde{u}_\varepsilon(t,x) := u\left(\tfrac{t}{\varepsilon^2}, \tfrac{x}{\varepsilon}\right). \tag{1.1.5}$$

Note that this tranformation leaves the standard heat equation ($\partial_t u = \frac{1}{2}\Delta u$) invariant. An easy calculation shows that $\widetilde{u}_\varepsilon$ satisfies the equation

$$\begin{cases} \partial_t \widetilde{u}_\varepsilon(t,x) = \frac{1}{2}\Delta \widetilde{u}_\varepsilon(t,x) + \beta\, \varepsilon^{\frac{d-2}{2}} (\widetilde{u}_\varepsilon \cdot \widetilde{\xi})(t,x) \qquad t > 0,\ x \in \mathbb{R}^d \\ \widetilde{u}_\varepsilon(0,x) \equiv 1 \end{cases}, \tag{1.1.6}$$

where $\widetilde{\xi}$ is a space-time white noise which has the same distribution with ξ and appears due to the fundamental scaling property of space-time white noise

$$\xi(t,x) \overset{\text{dist}}{=} \varepsilon^{-\frac{d}{2}-1}\, \xi\left(\tfrac{t}{\varepsilon^2}, \tfrac{x}{\varepsilon}\right), \tag{1.1.7}$$

which is understood as a distributional equality when testing against L^2 functions.

Observe that when $\varepsilon \downarrow 0$, the coefficient in front of the noise $\widetilde{\xi}$ in (1.1.6), vanishes when $d \geqslant 3$, blows up when $d = 1$ and it is constant and equal to 1 when $d = 2$. This suggests that the noise should have a non-trivial effect in dimension $d = 1$ when one moves to larger and larger scales, while the opposite should be true in dimensions $d \geqslant 3$. This heuristic argument fails whatsoever to make any prediction in dimension $d = 2$. A similar argument in small scales, that is considering $\widetilde{u}_\varepsilon(t,x) := u(\varepsilon^2 t, \varepsilon x)$ produces a coefficient $\varepsilon^{-\frac{d-2}{2}}$ in front of the noise thus yielding analoguous predictions as in the case of the large scales, but reversed. Notice that this picture matches exactly the disorder relevance/irrelevance picture based on the Harris criterion that we discussed in the previous subsection. In the language of SPDEs, dimension $d = 1$ corresponds to the subcritical dimension, dimensions $d \geqslant 3$ correspond to the supercritical dimensions while $d = 2$ is the critical dimension.

A similar scaling argument can also be derived after first centering and then scaling u. More specifically, let

$$\widetilde{v}_\varepsilon(t,x) := \varepsilon^{-(\frac{d}{2}-1)}\Big(u\left(\tfrac{t}{\varepsilon^2}, \tfrac{x}{\varepsilon}\right) - 1\Big). \tag{1.1.8}$$

Note that $\widetilde{v}_\varepsilon(0,x) \equiv 0$ and the scaling (1.1.8) is chosen because it is the scaling which leaves the additive stochastic heat equation (also referred as Edwards-Wilkinson model [EW82])

$$\text{(aSHE)} \qquad \begin{cases} \partial_t v(t,x) = \frac{1}{2}\Delta v(t,x) + \beta\, \xi(t,x) \qquad t > 0,\ x \in \mathbb{R}^d \\ v(0,x) \equiv 0 \end{cases} \tag{1.1.9}$$

invariant. A simple (formal) calculation shows then that $\widetilde{v}_\varepsilon$ satisfies the equation

$$\begin{cases} \partial_t \widetilde{v}_\varepsilon(t,x) = \frac{1}{2}\Delta \widetilde{v}_\varepsilon(t,x) + \beta\,\varepsilon^{\frac{d-2}{2}}(\widetilde{v}_\varepsilon \cdot \widetilde{\xi}\,)(t,x) + \beta\widetilde{\xi}(t,x) & t>0,\, x\in\mathbb{R}^d \\ \widetilde{v}_\varepsilon(0,x) \equiv 0 \end{cases},$$

Then, similar conclusions can be drawn regarding the classification of the equation depending on the spatial dimension d. In particular, in dimensions $d \geqslant 3$, the vanishing coefficient $\varepsilon^{\frac{d-2}{2}}$ in front of $\widetilde{v}_\varepsilon \cdot \widetilde{\xi}$ suggests that $\widetilde{v}_\varepsilon$ should converge to the solution of the Edwards-Wilkinson model (1.1.9). As we will see in the next section, this is only *partly* true. In dimension $d = 2$, the coefficient $\varepsilon^{\frac{d-2}{2}} = \varepsilon^0$ is constant and equal to 1. We will see in the next section though, that the correct interpretation is not a constant but a *logarithmically vanishing* coefficient.

To bypass the analytical obstacles and be able to define some notion of solution to (1.1.4) in dimensions $d \geqslant 2$, one resorts to a regularisation procedure which is carried out by replacing the original noise ξ with a spatially mollified version ξ_ε and considering the corresponding regularised equation. More specifically, given a probability density $j \in C_c(\mathbb{R}^d)$, with $j(x) = j(-x)$ for $x \in \mathbb{R}^d$ and $\varepsilon > 0$, we define $j_\varepsilon(x) := \varepsilon^{-d} j\big(\frac{x}{\varepsilon}\big)$ and

$$\xi_\varepsilon(t,x) := (\xi * j_\varepsilon)(t,x) = \varepsilon^{-d}\int_{\mathbb{R}^d} \mathrm{d}z\, \xi(t,z)\, j\big(\tfrac{x-z}{\varepsilon}\big)\,. \tag{1.1.10}$$

Then, for every $\varepsilon > 0$ and fixed $t > 0$, $x \mapsto \int_0^t \xi_\varepsilon(s,x)\mathrm{d}s$ is a smooth function while for fixed $x \in \mathbb{R}^d$, the process $t \mapsto \int_0^t \xi_\varepsilon(s,x)\,\mathrm{d}s$ is a Brownian motion with variance $\|j\|_2^2$. In that case, replacing ξ by ξ_ε in (1.1.4) leads to a well-posed equation by Itô theory.

In order to reveal the link with directed polymers, let us consider mSHE with mollified noise and at large scales, that is the equation

$$(\mathrm{mSHE}_\varepsilon) \quad \begin{cases} \partial_t u_\varepsilon(t,x) = \frac{1}{2}\Delta u_\varepsilon(t,x) + \beta\,\varepsilon^{\frac{d-2}{2}}(u_\varepsilon \cdot \xi_\varepsilon)(t,x)\,, & t>0, x\in\mathbb{R}^d \\ u_\varepsilon(0,x) \equiv 1 \end{cases}. \tag{1.1.11}$$

The solution u_ε satisfies, by [BC95], the following Feynman-Kac formula

$$\begin{aligned} u_\varepsilon(t,x) &= \mathrm{E}_x\Big[e^{\beta\,\varepsilon^{\frac{d-2}{2}}\int_0^t \xi_\varepsilon(t-s,B_s)\,\mathrm{d}s - \frac{1}{2}\beta^2\varepsilon^{d-2}\,\mathbb{E}\big[\big(\int_0^t \xi_\varepsilon(t-s,B_s)\,\mathrm{d}s\big)^2\big]}\Big] \\ &\overset{\text{dist}}{=} \mathrm{E}_{\varepsilon^{-1}x}\Big[e^{\beta\int_0^{\varepsilon^{-2}t}\int_{\mathbb{R}^d}\xi(s,u)j(u-B_s)\,\mathrm{d}s\,\mathrm{d}u - \frac{1}{2}\beta^2\varepsilon^{-2}\,t\,\|j\|_2^2}\Big] \end{aligned} \tag{1.1.12}$$

where $B = (B_s)_{s>0}$ is a d-dimensional Brownian motion starting from $B_0 := x \in \mathbb{R}^d$ and to derive (1.1.12), we used that the distribution of ξ is invariant under time-reversal and satisfies the scaling relation (1.1.7). Therefore, under the natural identification $N = \varepsilon^{-2}t$, $\xi \leftrightarrow \omega$ and $B \leftrightarrow S$, we see that $Z_{N,\beta}(x)$ can be regarded as the discrete analogue to the solution of equation (1.1.11) and in that sense discretisation is equivalent to mollification of the noise.

The study of (1.1.4) is also motivated by the fact that $h := \log u$ (Cole-Hopf transformation) formally solves the Kardar-Parisi-Zhang equation

$$(\mathrm{KPZ}) \quad \begin{cases} \partial_t h(t,x) = \frac{1}{2}\Delta h(t,x) + \frac{1}{2}|\nabla h|^2(t,x) + \beta\xi(t,x) & t>0,\, x\in\mathbb{R}^d \\ h(0,x) \equiv 0\,, \end{cases} \tag{1.1.13}$$

Equation (1.1.13), which was introduced in [KPZ86] by the physicists Kardar, Parisi and Zhang is by now considered to be the universal model for random growth phenomena. It has attracted a lot of interest recently after the celebrated work of M. Hairer [H13] and his subsequent theory of *regularity structures* [H14], as well as the theories of *paracontrolled distributions* [GP17] and *energy solutions* [GJ14] for the case of dimension $d = 1$. Contrary to the case of (1.1.4), the KPZ equation is ill-posed in any dimension $d \geqslant 1$ due the irregularity of the noise which causes the $|\nabla h|^2$ term in (1.1.13) to be apriori ill-defined.

Although the Cole-Hopf transformation providing the link between mSHE and KPZ consists in a formal calculation that does not apriori make sense, there are important reasons why one should consider it to be the correct notion of solution to the KPZ equation, originating from the case of one spatial dimension $d = 1$. First, as we stressed out previously, in dimension $d = 1$, equation (1.1.4), is well posed in its mild form and the solution u is positive [Mü91]. Moreover, if we consider the solution u_ε to the counterpart of (1.1.4) where the noise has been mollified, then $u_\varepsilon \to u$ uniformly on compact sets and and since $u(t,x) > 0$ for $t > 0$ we may define $h_\varepsilon := \log u_\varepsilon(t,x)$. Then, by Itô's formula h_ε satisfies the equation

$$\partial_t h_\varepsilon(t,x) = \tfrac{1}{2}\Delta h_\varepsilon(t,x) + \tfrac{1}{2}|\nabla h_\varepsilon|^2(t,x) + \beta\,\xi_\varepsilon(t,x) - C_\varepsilon$$

where $C_\varepsilon := \beta^2\varepsilon^{-2}\,\|j\|_2^2$ is the Itô correction. Second, it was proven some years ago in a seminal work by Bertini and Giacomin [BG97], that the fluctuations of a discrete particle system, the stationary weakly asymmetric simple exclusion (WASEP), under a suitable rescaling, are governed by the Cole-Hopf solution. An additional argument in favour of the Cole-Hopf solution as the canonical solution to the KPZ equation is that it has the conjectured in [KPZ86] scaling exponents as it was established in [BQS11].

1.2. Overview of the existing literature and our results

We will now present in more details some established results for DPRE_d in the case of the supercritical/disorder irrelevant dimensions $d \geqslant 3$ for the directed polymer, mSHE and KPZ equations, as well as in the case of the critical/marginal dimension $d = 2$. We will also present our results and explain how they fit in the existing literature.

1.2.1. The case of dimensions $d \geqslant 3$. The first contributions in this direction came from the works of Imbrie-Spencer [IS88] and Bolthausen [B89] who showed the existence of a *weak disorder* regime for DPRE_d in dimensions $d \geqslant 3$ when the strength of disorder β is small enough. In particular, it was shown that almost surely, paths weighted by the polymer measure (1.1.1) are diffusive in the large scale limit. The regime of β that was considered in these works was what we call here the L^2 regime, which is characterised by the $L^2(\mathbb{P})$ boundedness of the partition function $Z_{N,\beta}$ as $N \to \infty$. This regime can be explicitly characterised as follows. Let $\lambda_2(\beta) := \lambda(2\beta) - 2\lambda(\beta)$ and denote by π_d the probability that a simple symmetric random walk on $\mathbb{Z}^d$ starting from the origin, will return to the origin. Then, the L^2 regime corresponds to the interval $\big(0, \beta_{L^2}(d)\big)$, where

$$\beta_{L^2} := \beta_{L^2}(d) := \sup\left\{\beta : \lambda_2(\beta) < \log\left(\tfrac{1}{\pi_d}\right)\right\}.$$

This characterisation is achieved by the simple and standard computation

$$\mathbb{E}\big[(Z_{N,\beta})^2\big] = \mathrm{E}^{\otimes 2}\big[e^{\lambda_2(\beta)\mathsf{L}_N^{(1,2)}}\big] = \mathrm{E}\big[e^{\lambda_2(\beta)\mathsf{L}_N}\big], \tag{1.2.1}$$

where $\mathsf{L}_N^{(1,2)} := \sum_{n=1}^N \mathbb{1}_{\{S_n^{(1)}=S_n^{(2)}\}} \overset{\text{law}}{=} \mathsf{L}_N := \sum_{n=1}^N \mathbb{1}_{\{S_{2n}=0\}}$. Since, the simple random walk is transient in dimensions $d \geqslant 3$, L_N converges almost surely to a random variable L_∞ as $N \to \infty$ and L_∞ follows a geometric distribution with success probability $\pi_d < 1$. Specifically, it is not hard to see that $\lim_{N\to\infty} \mathbb{E}\big[(Z_{N,\beta})^2\big] = \mathrm{E}\big[e^{\lambda_2(\beta)\mathsf{L}_\infty}\big]$ and

$$\mathrm{E}\big[e^{\lambda_2(\beta)\mathsf{L}_\infty}\big] = \begin{cases} \frac{1-\pi_d}{1-\pi_d e^{\lambda_2(\beta)}}, & \text{if } \lambda_2(\beta) < \log(\frac{1}{\pi_d}) \\ \infty\ , & \text{otherwise.} \end{cases} \tag{1.2.2}$$

In the L^2-regime it was also proven by Sinai [S95] and later by Vargas [V06], also in the continuum, that a local limit theorem holds for the polymer.

The weak disorder regime was subsequently characterised by the works of Comets, Shiga, Yoshida [CSY03, CSY04, CY06] as the regime of $\beta < \beta_c(d)$, such that $Z_{N,\beta}$ is a uniformly integrable martingale sequence and as such converges almost surely to a strictly positive random variable $Z_{\infty,\beta}$. It was proven in [CY06] that the polymer is diffusive in this regime, extending previous results that were limited to the L^2 regime. For $\beta > \beta_c(d)$, $Z_{N,\beta}$ converges to 0 as $N \to \infty$. The latter is called the *strong disorder* regime. Clearly, one has $\beta_c(d) \geqslant \beta_{L^2}(d)$ and in fact it took some time to resolve the nontriviality of the interval $\big(\beta_{L^2}(d), \beta_c(d)\big)$, see [BS10, BS11, BT10, BGH11]. The parameter β_c marks the transition to a stronger disorder phase where the polymer localises in a few regions where the environment is more favorable, see [Ch19], [BC20a], [BC20b], [Ba21]. In the strong disorder phase it is expected that the polymer exhibits super-diffusive behaviour but this has yet to be proven. Additional limitations for studying the directed polymer above or at the weak/strong disorder transition poses the fact that that a concrete characterisation of β_c is still missing. Some indirect descriptions have been given in [CY06] in terms of the overlap between two independent paths under the polymer measure and more recently by Junk, in [J22], in the case of bounded environment ω, in terms of the integrability of the running supremum $\sup_{N\in\mathbb{N}} Z_{N,\beta}$. Let us note that although little is known for the limit of the partition function $Z_{\infty,\beta}$ in dimensions $d \geqslant 3$, limiting theorems have been established for the difference $Z_{N,\beta} - Z_{\infty,\beta}$ for small $\beta > 0$, by Comets and Liu in [CL17] and later extended in the full L^2 regime by Cosco and Nakajima [CN21]. See also [CCM22] for results of similar flavour in the continuum.

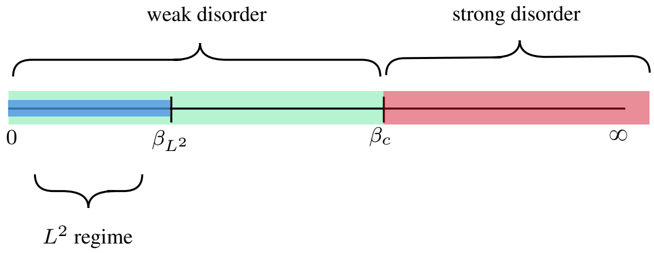


FIGURE 1.2.1. The phase diagram of DPRE_d in dimensions $d \geqslant 3$.

A weak/strong disorder transition similar to that of the directed polymer in dimensions $d \geqslant 3$ has been established for the solution of the regularised mSHE (1.1.6) by Muhkerjee, Shamov and Zeitouni [MSZ16]. Particular focus has been devoted to studying spatial correlations of the solution to the regularised mSHE as $\varepsilon \downarrow 0$, when viewed as a field

$$\Big\{u_\varepsilon(t,x) : x \in \mathbb{R}^d\Big\}$$

or equivalently for the diffusively rescaled polymer partition function field

$$\Big\{Z_{N,\beta}\big(\lfloor\sqrt{N}x\rfloor\big) : x \in \mathbb{R}^d\Big\},$$

as well as the corresponding questions for the solution to the KPZ equation and log-partition function. The first contribution in this direction in the supercritical dimensions $d \geqslant 3$ was the work of by Magnen and Unterberger [MU18] for the KPZ equation. In particular, the authors of [MU18] considered a regularised KPZ equation (with noise regularised both in space and time) and proved, using the Cole-Hopf mapping to the solution of the mSHE, that as the regularisation is removed the solution to the KPZ equation converges as a field to the Edwards-Wilkinson model, that is, the fluctuations of the limiting field are described by the solution to the additive stochastic heat equation (1.1.9), but with an effective noise strength. Their work was based on rigorous adaptation of ideas originating in Quantum Field Theory, in particular, perturbation expansions and multi-scale analysis via the renormalisation group. It was later shown by Gu, Ryzhik and Zeitouni in [GRZ18] that when centred and scaled appropriately, the solution of the regularised mSHE also converges as a field to the Edwards-Wilkinson model, again with an effective noise strength, strictly larger than the noise strength parameter used to define the original equation. Moreover, the Edwards-Wilkinson fluctuations for the KPZ equation obtained by Magnen and Unterberger [MU18] was also proved by Dunlap et al. in [DGRZ18] using Malliavin calculus techniques. Both works were restriced in a small β regime.

Our first contribution, contained in Chapter 2 is the proof of the limiting Edwards-Wilkinson fluctuations for the diffusively rescaled, centred and scaled random field

$$\left\{ N^{\frac{d-2}{4}} \Big(Z_{N,\beta}\big(\lfloor \sqrt{N} x \rfloor\big) - 1 \Big) \,:\, x \in \mathbb{R}^d \right\}, \tag{1.2.3}$$

(corresponding to the solution of the regularised mSHE at fixed time $t = 1^{\dagger}$) and for the diffusively rescaled, centred and scaled random field of log-partition functions

$$\left\{ N^{\frac{d-2}{4}} \Big(\log Z_{N,\beta}\big(\lfloor \sqrt{N} x \rfloor\big) - \mathbb{E}\big[\log Z_{N,\beta}\big(\lfloor \sqrt{N} x \rfloor\big)\big] \Big) \,:\, x \in \mathbb{R}^d \right\}, \tag{1.2.4}$$

(corresponding to the solution of the regularised KPZ equation at time $t = 1$). In particular, if $\varphi \in C_c(\mathbb{R}^d)$ is a test function, we prove that the sequences

$$N^{\frac{d-2}{4}} \sum_{x \in \mathbb{Z}^d} \Big(Z_{N,\beta}(x) - 1 \Big) \frac{\varphi(\frac{x}{\sqrt{N}})}{N^{\frac{d}{2}}}$$

and

$$N^{\frac{d-2}{4}} \sum_{x \in \mathbb{Z}^d} \Big(\log Z_{N,\beta}(x) - \mathbb{E}\big[\log Z_{N,\beta}(x) \big] \Big) \frac{\varphi(\frac{x}{\sqrt{N}})}{N^{\frac{d}{2}}}$$

both converge as $N \to \infty$ in distribution to the same limiting Gaussian random variable $\mathscr{Z}_\beta(\varphi)$. Our result unlike the previous works [MU18], [GRZ18] and [DGRZ18] covers the full L^2 regime $\beta \in (0, \beta_{L^2})$ and is in some sense optimal since the variance of the limit, $\mathscr{Z}_\beta(\varphi)$, blows up at L^2 critical point $\beta = \beta_{L^2}$. Our methods, as we will explain in more detail Chapter 2, are based on analysis of chaos expansions inspired by works on scaling limits of disordered systems [CSZ17a, CSZ16] and two dimensional polymers, SHE and KPZ [CSZ17b, CSZ20] (alternative methods to the two dimensional case, which however do not cover the whole L^2 - in this case also subcritical - regime, are those of [CD20, G20]).

Let us also mention that analogous to our results, for regularisations of SHE and KPZ as in (1.1.11), (1.1.13) were simultaneously and independently established by Cosco, Nakajima and Nakashima [CNN22] via quite different methods than ours, based on stochastic calculus and local limit theorems for polymers inspired by earlier works of Comets, Neveu [CNe95] and of Sinai [S95] (see also [V06, CN21, CCM22]).

†we stick to time $t = 1$ for simplicity, while the case of general time t is recovered by replacing $Z_{N,\beta}$ with $Z_{Nt,\beta}$ in (1.2.3)

A very interesting, open problem is to go beyond the L^2 regime in dimension $d \geqslant 3$. Currently, the only work in this direction is a recent paper by Junk [J22+], in the case of bounded environment ω, where it is shown that for $\beta \in (\beta_{L^2}, \beta_c)$ the centred, diffusively rescaled field averages with respect to test functions $\varphi \in C_c(\mathbb{R}^d)$,

$$\bar{Z}_{N,\beta}(\varphi) := \sum_{x \in \mathbb{Z}^d} \Big(Z_{N,\beta}(x) - 1 \Big) \frac{\varphi(\frac{x}{\sqrt{N}})}{N^{\frac{d}{2}}} \tag{1.2.5}$$

converge to zero, as $N \to \infty$, at a rate $N^{-\mathsf{h}(\beta,d)+o(1)}$, with $\mathsf{h}(\beta,d) > \frac{d-2}{4}$. This however leaves open the question of what the limiting fluctuations of the sequence $\big(N^{\mathsf{h}(\beta,d)}\, \bar{Z}_{N,\beta}(\varphi)\big)_{N \geqslant 1}$ are as $N \to \infty$.

1.2.2. The case of dimension $d = 2$. As we mentioned earlier, the case of dimension $d = 2$ is called marginal since the Harris criterion can not rule whether in this case the polymer exhibits disorder relevance or irrelevance and similarly $d = 2$ is called the critical dimension in the language of SPDE. It was proved in [CY06] that DPRE_2 exhibits strong disorder for every $\beta \in (0, \infty)$, that is the partition function $Z_{N,\beta}$ converges almost surely to 0 as $N \to \infty$.

An underlying transition was later unveiled by the work of Caravenna, Sun and Zygouras in [CSZ17b], when one focuses on a regime where the strength of the disorder is tuned down to 0 as $N \to \infty$. More specifically, the authors of [CSZ17b] showed that if one chooses $\beta_N := \frac{\hat{\beta}}{\sqrt{R_N}} \overset{N\to\infty}{\approx} \hat{\beta}\sqrt{\frac{\pi}{\log N}}$, where $R_N = \mathrm{E}\big[\mathsf{L}_N^{(1,2)}\big] = \sum_{n=1}^{N} q_{2n}(0)$ is the expected collision local time between two independent random walks $S^{(1)}, S^{(2)}$ and the asymptotic $R_N \overset{N\to\infty}{\approx} \frac{1}{\pi} \log N$ follows by [ET60], it is true that as $N \to \infty$,

$$Z_{N,\beta_N} \xrightarrow{(d)} \begin{cases} \exp\big(\varrho_{\hat{\beta}} \mathsf{X} - \frac{1}{2}\varrho_{\hat{\beta}}^2\big)\,, & \text{if } \hat{\beta} \in (0,1) \\ 0\,, & \text{if } \hat{\beta} \geqslant 1\,, \end{cases} \tag{1.2.6}$$

where X follows a standard normal distribution $\mathcal{N}(0,1)$ and $\varrho_{\hat{\beta}} := \log\Big(\frac{1}{1-\hat{\beta}^2}\Big)$. Their result encompasses also a large class of so called marginally relevant disordered systems which display the same universal behaviour, along with the regularised mSHE with logarithmically attenuating disorder strength

$$\partial_t u_\varepsilon(t,x) = \tfrac{1}{2}\Delta u_\varepsilon(t,x) + \hat{\beta}\sqrt{\tfrac{2\pi}{\log \varepsilon^{-1}}}\,(u_\varepsilon \cdot \xi_\varepsilon)(t,x)\,. \tag{1.2.7}$$

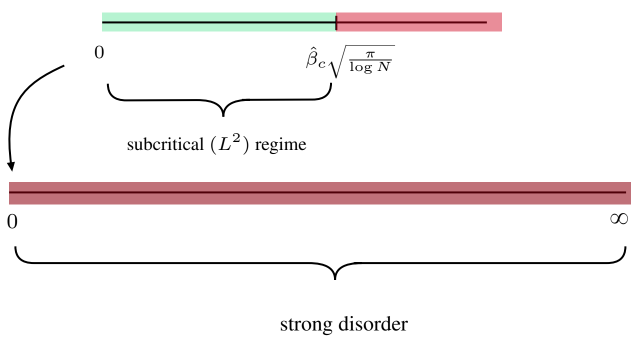


FIGURE 1.2.2. The phase diagram of DPRE_2, highlighting the presence of the intermediate disorder regime when $\beta_N = \hat{\beta}\sqrt{\frac{\pi}{\log N}}$ with $\hat{\beta}_c = 1$ marking the transition from the subcritical to the critical ($\hat{\beta}_c = 1$) and supercritical ($\hat{\beta}_c \geqslant 1$) regimes.

Note that (1.2.6) indicates a weak/strong disorder transition in the intermediate disorder regime reminiscent of the weak/strong disorder transition observed in higher dimensions. Such a transition can be guessed by a second moment computation. In particular, we have that in spatial dimension $d = 2$, as in (1.2.1),

$$\mathbb{E}\big[(Z_{N,\beta})^2\big] = \mathrm{E}\big[e^{\lambda_2(\beta)\mathsf{L}_N^{(1,2)}}\big] = \mathrm{E}\big[e^{\lambda_2(\beta)\mathsf{L}_N}\big]\,,$$

where we recall that if $S^{(1)}, S^{(2)}, S$ are independent simple symmetric random walks on $\mathbb{Z}^2$, then $\mathsf{L}_N^{(1,2)} := \sum_{n=1}^N \mathbb{1}_{\{S_n^{(1)}=S_n^{(2)}\}}$, $\mathsf{L}_N := \sum_{n=1}^N \mathbb{1}_{\{S_{2n}=0\}}$ and $\mathsf{L}_N^{(1,2)} \overset{\text{law}}{=} \mathsf{L}_N$, by the symmetry of S. Contrary to the case of dimensions $d \geqslant 3$, in the 2-dimensional case, due to the recurrence of the simple random walk, L_N does not converge as $N \to \infty$. Instead, due to a classical result of Erdös and Taylor [ET60], we have that

$$\tfrac{\pi}{\log N}\,\mathsf{L}_N \xrightarrow{(d)} Y\,, \tag{1.2.8}$$

where Y is a random variable having exponential distribution with parameter 1, namely the density of Y is given by $f_Y(y) = e^{-y}\,\mathbb{1}_{y>0}$. It is not hard to prove then, that for the second moment of the partition function Z_{N,β_N} with $\beta_N \approx \hat{\beta}\sqrt{\frac{\pi}{\log N}}$ we have

$$\mathbb{E}\big[(Z_{N,\beta_N})^2\big] \xrightarrow{N\to\infty} \frac{1}{1-\hat{\beta}^2}\,,$$

which evidently blows up at $\hat{\beta} = 1$. The regime $\hat{\beta} \in (0,1)$ is called the subcritical regime while the regime $\hat{\beta} \geqslant 1$ is called the supercritical regime and $\hat{\beta} = 1$, the critical point of the transition. Let us note that contrary to the case of dimensions $d \geqslant 3$, in dimension $d = 2$ and in this intermediate disorder regime, the L^2 regime coincides with the subcritical regime.

In the same work [CSZ17b], Caravenna, Sun and Zygouras showed that in the subcritical regime the limiting fluctuations of the centred, diffusively scaled and logarithmically rescaled field

$$\Big\{\sqrt{\log N}\Big(Z_{N,\beta_N}(\lfloor\sqrt{N}x\rfloor) - 1\Big) : x \in \mathbb{R}^2\Big\} \tag{1.2.9}$$

are Gaussian (together with the analoguous result on mSHE). Both the pointwise and averaged results (1.2.6) and (1.2.9) relied on polynomial chaos expansions of the partition function $Z_{N,\beta_N}(x)$ (see next Chapters for more details), multi-scale analysis and the celebrated Fourth moment theorem to show that certain multilinear polynomials of disorder variables are asymptotically Gaussian, see also [CC22] for more recent results in this direction. What crucially underlies the analysis carried out in [CSZ17b, CC22] is the exponential time scale induced by the logarithmic scaling (1.2.8).

We also note a more recent generalisation due to Dunlap and Gu [DG22], who studied the semilinear regularised mSHE

$$\begin{cases} \partial_t u_{\varepsilon,a}(t,x) = \frac{1}{2}\Delta u_{\varepsilon,a}(t,x) + \frac{1}{\sqrt{\log \varepsilon^{-1}}}\,\sigma\big(u_{\varepsilon,a}(t,x)\big)\,\xi_\varepsilon(t,x)\,, & t > 0, x \in \mathbb{R}^2 \\ u_{\varepsilon,a}(0,x) \equiv a & \end{cases} \tag{1.2.10}$$

with $\sigma : [0,\infty) \to [0,\infty)$ Lipschitz with $\sigma(0) = 0$ and Lipschitz constant $\sigma_{\text{Lip}} < \sqrt{2\pi}$ and flat initial condition $a > 0$. They showed that the limiting one-point distribution of the solution $u_{\varepsilon,a}$ is described by a forward-backward SDE (FBSDE), recovering the log-normal fluctuations (1.2.6) when $\sigma(x) = x$.

In the context of the KPZ equation, Chatterjee and Dunlap in [CD20] showed that the solution of the regularised KPZ equation

$$\begin{cases} \partial_t \mathfrak{h}_\varepsilon(t,x) = \frac{1}{2}\Delta \mathfrak{h}_\varepsilon(t,x) + \frac{1}{2}\beta_\varepsilon |\nabla \mathfrak{h}_\varepsilon|^2(t,x) + \xi_\varepsilon(t,x) & t>0,\ x\in\mathbb{R}^2 \\ \mathfrak{h}_\varepsilon(0,x) \equiv 0\,, \end{cases} \tag{1.2.11}$$

with $\beta_\varepsilon = \hat{\beta}\sqrt{\frac{2\pi}{\log \varepsilon^{-1}}}$ and $\hat{\beta}$ small enough, when viewed as a random field, centred and rescaled, is tight as $\varepsilon \downarrow 0$. Moreover, any subsequential limit is not the solution of aSHE one obtains by naively dropping the nonlinearity. Shortly after, Caravenna, Sun and Zygouras showed in [CSZ20] considered the regularised KPZ equation

$$\begin{cases} \partial_t h_\varepsilon(t,x) = \frac{1}{2}\Delta h_\varepsilon(t,x) + |\nabla h_\varepsilon|^2(t,x) + \beta_\varepsilon\, \xi_\varepsilon(t,x) - C_\varepsilon & t>0,\ x\in\mathbb{R}^2 \\ h_\varepsilon(0,x) \equiv 0\,, \end{cases} \tag{1.2.12}$$

with $\beta_\varepsilon = \hat{\beta}\sqrt{\frac{2\pi}{\log \varepsilon^{-1}}}$ and $C_\varepsilon := \beta_\varepsilon^2 \varepsilon^{-2}\, \|j\|_2^2$, and showed that the centred and scaled solution

$$\tfrac{1}{\beta_\varepsilon}\Big(h_\varepsilon(t,x) - \mathbb{E}\big[h_\varepsilon(t,x)\big]\Big)$$

converges when $\varepsilon \downarrow 0$ (as random fields) for all $\hat{\beta} \in (0,1)$ to the solution of the Edwards-Wilkinson equation

$$\begin{cases} \partial_t v(t,x) = \frac{1}{2}\Delta v(t,x) + c_{\hat{\beta}}\, \xi(t,x) & t>0,\ x\in\mathbb{R}^2 \\ v(0,x) \equiv 0 \end{cases} \tag{1.2.13}$$

with $c_{\hat{\beta}} := \sqrt{\frac{1}{1-\hat{\beta}^2}}$, (see also [G20] where the same result was proven but for $\hat{\beta}$ sufficiently small). The two equations (1.2.11) and (1.2.12) are equivalent, which can be seen through the relation

$$\tfrac{1}{\beta_\varepsilon}\Big(h_\varepsilon(t,x) - \mathbb{E}\big[h_\varepsilon(t,x)\big]\Big) \overset{\text{dist}}{=} \mathfrak{h}_\varepsilon(t,x) - \mathbb{E}\big[\mathfrak{h}_\varepsilon(t,x)\big]\,,$$

see [CSZ20], Appendix A.

As we discussed earlier, outside the subcritical regime, namely, for $\beta_N = \hat{\beta}\sqrt{\pi(\log N)^{-1}}$ with $\hat{\beta} \geqslant 1$ the partition function Z_{N,β_N} converges in distribution to 0, while for all $h \in \mathbb{N}$ with $h \geqslant 2$ the h^{th} moment $\mathbb{E}\big[(Z_{N,\beta_N})^h\big]$ blows up as $N \to \infty$, see [CSZ19a]. This suggests that, at criticality, the field

$$\Big\{ Z_{N,\beta_N}\big(\lfloor \sqrt{N} x\rfloor\big) \,:\, x \in \mathbb{R}^2 \Big\} \tag{1.2.14}$$

becomes rough as $N \to \infty$ and the correct point of view is therefore to look at it as a random distribution, that is when tested against test functions. The first work in this direction was carried out by Bertini and Cancrini [BC98], who showed that in the context of the critical mSHE, there exists a *critical window* of disorder strength around $\hat{\beta}_c = 1$ for which the field (1.2.14) is tight, and explicitly computed the limiting covariance structure. Their analysis was based on the spectral theory of Schrödinger operators with point interactions.

The result of Bertini and Cancrini for the critical 2-dimensional mSHE was later rediscovered by Caravenna, Sun and Zygouras via probabilistic methods in the context of the directer polymer [CSZ19a, CSZ19b], where they also showed that the third moment of the field tested against test functions is bounded as $N \to \infty$ and as a consequence all subsequential limits are non-trivial and have the covariance structure computed by Bertini and Cancrini. The work of Gu, Quastel and Tsai [GQT21] further showed that all centred positive integer moments of the field (1.2.14) are bounded as $N \to \infty$, inspired by the works of Dell'Antonio, Figari, Teta [DFT94] and Dimock, Rajeev [DR04] on the 2-dimensional delta Bose gas. However, these moment estimates are not sufficient to determine the distribution because the moments grow too fast. More recently, the question of uniqueness of the limiting field was settled by Caravenna, Sun and Zygouras. Utilising chaos expansions, a space-time renewal structure, moment estimates and a Lindeberg principle for multilinear polynomials of dependent variables, the authors showed that, indeed, there exists a unique limiting field, named thereafter, the *Critical* $2d$ *Stochastic Heat Flow*, which is the natural candidate for the long sought solution to the critical 2-dimensional mSHE.

Let us also mention that there has been significant progress in understanding the so called Anisotropic KPZ (aKPZ) equation in dimension $d = 2$, which is formally given by

$$\partial_t h = \tfrac{1}{2}\Delta h + \lambda\big((\partial_1 h)^2 - (\partial_2 h)^2\big) + \xi\,. \tag{1.2.15}$$

The first work in this direction is due to Cannizzaro, Erhard and Schönbauer who showed in [CES21] that the regularised aKPZ

$$\partial_t h^N = \tfrac{1}{2}\Delta h^N + \lambda\mathcal{N}^N[h^N] + \xi \tag{1.2.16}$$

where $\mathcal{N}^N[h^N] := \Pi_N\big((\Pi_N\partial_1 h^N)^2 - (\Pi_N\partial_2 h^N)^2\big)$ and Π_N cuts the Fourier modes larger than N, has non-trivial subsequential limits when λ is going to 0 as $\lambda = \frac{\hat\lambda}{\sqrt{\log N}}$. Note that, instead of discretisation or mollification of the noise, the regularisation of (1.2.15) is done by replacing the nonlinearity with the regularised version $\mathcal{N}^N[h^N]$, where the regularisation is taking place in Fourier space. Furthermore, Cannizzaro, Erhard and Toninelli showed in [CET21+] that the solution to the regularised aKPZ (1.2.16) with $\lambda = \frac{\hat\lambda}{\sqrt{\log N}}$ viewed as a random field converges to the Edwards-Wilkinson model with non-trivial coefficients. Focusing on a different scaling regime, Cannizzaro, Erhard and Toninelli showed in [CET20a+], [CET20b+] that when λ is being kept fixed and not varying with N, the solution to aKPZ is logarithmically superdiffusive. We stress that contrary to the isotropic KPZ (1.1.13), there is no Cole-Hopf transform for the anisotropic KPZ and therefore it cannot be reduced to a problem involving directed polymers. One important ingredient crucially utilised in the above works however, is that aKPZ in the form (1.2.15) (the anisotropic problem can be formulated more generally, see [CES21]) admits the Gaussian Free Field as an invariant measure.

We will devote the rest of this introduction to a specific problem concerning the 2-dimensional directed polymer partition function in the intermediate disorder regime which was the motivation for the material presented in Chapters 3 and 4, and draw the connection with other models that are of interest.

1.2.3. Moments of the polymer partition function and collisions of independent random walks in $d = 2$. It is an interesting and non-trivial question whether all moments of the partition function Z_{N,β_N} remain uniformly bounded as $N \to \infty$ in the same regime of $\hat\beta$ where the second moment remains uniformly bounded. Information on moments higher than two in the subcritical regime has already appeared necessary in a number of situations, in particular in proving tightness and regularity properties of the approximations to the solutions of the $2d$-KPZ [CD20] or Edwards-Wilkinson universality for the $2d$-KPZ [CSZ20, G20]. The lack of control on higher moments was resulting into restrictions to strict subsets of the subcritical regime in [CD20, G20], while this was circumvented in [CSZ20] by employing hypercontractivity to show, for any $\hat\beta < 1$, the uniform boundedness of moments *up to* certain order $h(\hat\beta) > 2$ with $\lim_{\hat\beta\uparrow 1} h(\hat\beta) = 2$.

In Chapter 3 we resolve this question, showing that all moments of Z_{N,β_N} are uniformly bounded as $N \to \infty$ in the subcritical regime $\hat{\beta} \in (0,1)$. Combining this result with the distributional convergence (1.2.6) we can actually compute the limit of all moments. More specifically, we show that for $\beta_N = \hat{\beta}\sqrt{\frac{\pi}{\log N}}$ with $\hat{\beta} \in (0,1)$ and for all $h \geqslant 0$†,

$$\lim_{N\to\infty} \mathbb{E}\Big[\big(Z_{N,\beta_N}\big)^h\Big] = \left(\frac{1}{1-\hat{\beta}^2}\right)^{\frac{h(h-1)}{2}} . \tag{1.2.17}$$

We also apply our techniques to show that the moments of the averaged field are uniformly bounded and therefore converge to those of a Gaussian free field due to [CSZ17b]. Our approach for proving moment boundedness generalises moment bounds that were established in [CSZ21+] and used therein to prove uniqueness of the scaling limit of the polymer field at the critical temperature scaling. The work of Caravenna, Sun and Zygouras was inspired by the previous work of Gu, Quastel and Tsai [GQT21] in the context of the critical $2d$ mSHE, which was based on the works of Dell'Antonio, Figari, Teta [DFT94] and Dimock, Rajeev [DR04] on the delta Bose gas. The main idea used in [GQT21], [CSZ21+] and also in our setting is to expand the centred h^{th} moment of the partition function into a chaos series, and then rewrite this expansion into the form of a composition of certain transition operators applied to an initial condition and a terminal condition. The required moment bounds are then a result of norm operator estimates. In [GQT21] these norm operator estimates are carried out in an L^2 setting, while in [CSZ21+] they are extended, in a discrete setting, to ℓ^q for all $q \in (0,\infty)$. In order to be able to prove moment boundedness and consequently convergence (1.2.17), it was necessary to compute sharp asymptotics of these operator norms as $q \to \infty$, see Chapter 3 for a detailed outline of our proof.

As we further explain in Chapter 3, moment convergence (1.2.17) has more implications beyond the directed polymer and in particular in the context of collisions between independent random walks on the 2-dimensional lattice $\mathbb{Z}^2$. More specifically, let $h \in \mathbb{N}$ and $S^{(1)}, S^{(2)}, \ldots, S^{(h)}$ denote independent simple symmetric random walks on $\mathbb{Z}^2$, all starting from the origin. If we choose the law of ω to be standard Gaussian $\mathcal{N}(0,1)$ then a standard computation shows that

$$\mathbb{E}\Big[\big(Z_{N,\beta_N}\big)^h\Big] = \mathrm{E}^{\otimes h}\left[e^{\hat{\beta}^2 \sum_{1\leqslant i<j\leqslant h} \frac{\pi}{\log N} \mathsf{L}_N^{(i,j)}}\right] \tag{1.2.18}$$

where $\mathsf{L}_N^{(i,j)} := \sum_{n=1}^N \mathbb{1}_{S_n^{(i)}=S_n^{(j)}}$ denotes the *collision local time* of walks $S^{(i)}$ and $S^{(j)}$. Therefore, as we show in Chapter 3, (1.2.18) in conjunction with (1.2.17) implies that the logarithmically scaled total pairwise collision local time between $S^{(1)}, S^{(2)}, \ldots, S^{(h)}$, namely,

$$\frac{\pi}{\log N} \sum_{1\leqslant i<j\leqslant h} \mathsf{L}_N^{(i,j)} ,$$

converges in distribution as $N \to \infty$ to a $\Gamma\big(\frac{h(h-1)}{2}, 1\big)$ distributed random variable, where $\Gamma(a,1)$ is the law with density function $\frac{1}{\Gamma(a)} x^{a-1} e^{-x} \mathbb{1}_{\{x>0\}}$ and in the last expression $\Gamma(a)$ denotes the gamma function.

Given that a gamma distribution $\Gamma(k,1)$, with parameter $k \geqslant 1$, arises as the distribution of the sum of k independent random variables each one distributed according to an exponential random variable with parameter one (denoted as $\mathrm{Exp}(1)$), the convergence of the total collision local time, $\frac{\pi}{\log N} \sum_{1\leqslant i<j\leqslant h} \mathsf{L}_N^{(i,j)}$ to a $\Gamma\big(\frac{h(h-1)}{2}, 1\big)$ distributed random variable raises the question

† The result extends to all $h < 0$, provided that the law of ω satisfies a concentration condition, see the statement of Theorem 3.0.1 for more details.

as to whether the joint distribution of the individual rescaled collision times $\left\{\frac{\pi}{\log N}\,\mathsf{L}_N^{(i,j)}\right\}_{1\leqslant i<j\leqslant h}$ converges to that of a family of independent $\mathrm{Exp}(1)$ random variables. Chapter 4 is devoted to the proof of this fact.

An intuitive way to understand the convergence of the individual collision times, or equivalently of the local time of a planar walk, to an exponential variable is the following. By (1.2.8), the number of visits to zero of a planar walk, which starts at zero, is $O(\log N)$ and, thus, much smaller than the time horizon $2N$. Typically, also, these visits happen within a short time, much smaller than $2N$, so that every time the random walk is back at zero, the probability that it will return there again before time $2N$ is not essentially altered. This results in the local time L_N being close to a geometric random variable with parameter of order $(\log N)^{-1}$ (as also manifested by (1.2.8)), which when rescaled suitably converges to an exponential random variable.

The fact that the joint distribution of $\left\{\frac{\pi}{\log N}\,\mathsf{L}_N^{(i,j)}\right\}_{1\leqslant i<j\leqslant h}$ converges to that of *independent* exponentials is much less apparent as the collision times have obvious correlations. A way to understand this is, again, through the fact that collisions happen at time scales much shorter than the time horizon N and, thus, every time two walks start colliding they have essentially 'forgotten' their previous collisions with other walks. More crucially, the logarithmic scaling, as indicated via (1.2.8), introduces a separation of scales between collisions of different pairs of walks, which is what, essentially, leads to the asymptotic factorisation of the joint Laplace transform of $\left\{\frac{\pi}{\log N}\,\mathsf{L}_N^{(i,j)}\right\}_{1\leqslant i<j\leqslant h}$. This intuition is reflected in the two main steps in our proof, which are carried out in Sections 4.2.3 and 4.2.4.

Even though the Erdös-Taylor theorem (1.2.8) appeared a long time ago, the multivariate extension we establish in Chapter 4 appears to be new. In [GS09] it was shown that the law of $\frac{\pi}{\log N}\,\mathsf{L}_N^{(1,2)}$, conditioned on $S^{(1)}$, converges a.s. to that of an $\mathrm{Exp}(1)$ random variable. This implies that $\left\{\frac{\pi}{\log N}\,\mathsf{L}_N^{(1,i)}\right\}_{1<i\leqslant h}$ converge to independent exponentials. However, it does not address the full independence of the family of *all* pairwise collisions $\left\{\frac{\pi}{\log N}\,\mathsf{L}_N^{(i,j)}\right\}_{1\leqslant i<j\leqslant h}$.

In the continuum, phenomena of independence in functionals of planar Brownian motions have appeared in works around *log-scaling laws* see [PY86] (where the term *log-scaling laws* was introduced) as well as [Y91] and [Kn93]. These works are mostly concerned with the problem of identifying the limiting distribution of *windings* of a planar Brownian motion around a number of points $z_1, ..., z_k$, different than the starting point of the Brownian motion, or the winding around the origin of the differences $B^{(i)} - B^{(j)}$ between k independent Brownian motions $B^{(1)}, ..., B^{(k)}$, starting all from different points, which are also different than zero. Without getting into details, we mention that the results of [PY86, Y91, Kn93] establish that the windings (as well as some other functionals that fall within the class of *log-scaling laws*) converge, when logarithmically scaled, to independent Cauchy variables. [Kn93] outlines a proof that the local times of the differences $B^{(i)} - B^{(j)}, 1 \leqslant i < j \leqslant k$, on the unit circle $\{z \in \mathbb{R}^2 \colon |z| = 1\}$ converge, jointly, to independent exponentials $\mathrm{Exp}(1)$, when logarithmically scaled, in a fashion similar to the scaling of Theorem 4.0.1. The methods employed in the above works rely heavily on continuous techniques (Itô calculus, time changes etc.), which do not have discrete counterparts. In fact, the passage from continuous to discrete is not straightforward either at a technical level (see e.g. the discussion on page 41 of [Kn93] and [Kn94]) or at a phenomenological level (see e.g. discussion on page 736 of [PY86]).

Exponential moments of collision times arise naturally when one looks at moments of partition functions of the model of directed polymer in a random environment, see (1.2.18). We note that the asymptotic independence of the logarithmically scaled collision local times $\left\{\frac{\pi}{\log N}\,\mathsf{L}_N^{(i,j)}\right\}_{1\leqslant i<j\leqslant h}$ that we establish provides an explanation for the exponent $\frac{h(h-1)}{2}$ in (1.2.18) since asymptotic independence implies the asymptotic factorisation

$$\mathrm{E}^{\otimes h}\left[e^{\hat{\beta}^2\sum_{1\leqslant i<j\leqslant h}\frac{\pi}{\log N}\mathsf{L}_N^{(i,j)}}\right]\overset{N\to\infty}{\approx}\prod_{1\leqslant i<j\leqslant h}\mathrm{E}\left[e^{\hat{\beta}^2\frac{\pi}{\log N}\mathsf{L}_N^{(i,j)}}\right]$$

which produces the exponent $\frac{h(h-1)}{2}$ in (1.2.18).

Let us close this introduction with some possible further prospects of the work we develop in Chapters 3 and 4. A first application of the overall methodology that we develop here could be used to investigate the growth of the moments of the point-to-plane partition (or equivalently the solution of the SHE with delta initial conditions) at the critical temperature scaling $\hat{\beta}=1$. It is known [CSZ19a] that the second moment of this quantity grows as $\log N$. Moreover, boundedness of the moments of the averaged field (3.0.8) at this critical temperature scaling has been established in [GQT21, Che21+] for all moments and in [CSZ19b] for the third moment. See also [CSZ22+] for explicit moment lower bounds for the averaged field at the critical temperature scaling. However, the rate of growth of the h^{th} moment of the point-to-plane partition function, in this case, is not known. It is expected to be of the form $(\log N)^{\mathfrak{m}(h)}$ but the exponent $\mathfrak{m}(h)$ has not been determined, yet. We believe that the approach we develop here can shed some light to this question.

Moment estimates are also important in establishing fine properties, such as structure of maxima, of the field of log-partition functions

$$\left\{\sqrt{\log N}\Big(\log Z_{N,\beta}\big(\lfloor\sqrt{N}x\rfloor\big)-\mathbb{E}\big[\log Z_{N,\beta}\big(\lfloor\sqrt{N}x\rfloor\big)\big]\Big): x\in\mathbb{R}^2\right\},$$

which is known to converge to a log-correlated Gaussian field [CSZ20]. We refer to [CZ21+] for more details. We expect that the independence structure of the collision local times, that we establish here, to be useful towards these investigations. An interesting problem, in relation to this (but also of broader interest), is how large can the number h of random walks be (depending on N), before we start seeing correlations in the limit of the rescaled collisions. The work of Cosco-Zeitouni [CZ21+] has shown that there exists $\beta_0\in(0,1)$ such that for all $\beta\in(0,\beta_0)$ and $h=h_N\in\mathbb{N}$ such that

$$\limsup_{N\to\infty}\frac{3\beta}{1-\beta}\frac{1}{\log N}\binom{h}{2}<1\,,$$

one has that

$$\mathrm{E}^{\otimes h}\left[e^{\frac{\pi\beta}{\log N}\sum_{1\leqslant i<j\leqslant h}\mathsf{L}_N^{(i,j)}}\right]\leqslant c(\beta)\left(\frac{1}{1-\beta}\right)^{\binom{h}{2}(1+\varepsilon_N)},$$

with $c(\beta)\in(0,\infty)$ and $0\leqslant\varepsilon_N=\varepsilon(\beta,N)\downarrow 0$ as $N\to\infty$. This suggests that the threshold might be $h=h_N=O(\sqrt{\log N})$.

CHAPTER 2

Edwards-Wilkinson fluctuations for the directed polymer in the full L^2-regime for dimensions $d \geqslant 3$

In this chapter we study the directed polymer in random environment (DPRE_d) in dimensions $d \geqslant 3$. We recall that the random environment $(\omega_{n,x})_{(n,x)\in\mathbb{N}\times\mathbb{Z}^d}$ is a collection of i.i.d. random variables with law $\mathbb{P}$ such that

$$\mathbb{E}[\omega] = 0, \qquad \mathbb{E}[\omega^2] = 1, \qquad \lambda(\beta) := \log \mathbb{E}[e^{\beta\omega}] < \infty\,, \quad \forall \beta \in (0,\infty).$$

and S is a simple symmetric random walk on $\mathbb{Z}^d$, whose distribution we denote by P_x when starting from $x \in \mathbb{Z}^d$. When starting from 0 we will refrain from using the subscript and just write P. We will use the notation $q_n(x) := \mathrm{P}(S_n = x)$ for the transition kernel of the random walk. The partition function is defined as

$$Z_{N,\beta}(x) := \mathrm{E}_x\Big[e^{\sum_{n=1}^N \{\beta\omega_{n,S_n} - \lambda(\beta)\}}\Big]. \tag{2.0.1}$$

We work in the so called L^2 regime $\beta \in (0, \beta_{L^2})$, where

$$\beta_{L^2} := \beta_{L^2}(d) := \sup\big\{\beta\colon \lambda_2(\beta) < \log\big(\tfrac{1}{\pi_d}\big)\big\}\,,$$

with $\lambda_2(\beta) := \lambda(2\beta) - 2\lambda(\beta)$. In this regime, the $L^2(\mathbb{P})$ norm of $Z_{N,\beta}$ is uniformly bounded. Recall that

$$\mathbb{E}\big[(Z_{N,\beta}(x))^2\big] = \mathrm{E}^{\otimes 2}\Big[e^{\lambda_2(\beta)\sum_{n=1}^N \mathbb{1}_{S_n^{(1)} = S_n^{(2)}}}\Big] = \mathrm{E}\big[e^{\lambda_2(\beta)\mathsf{L}_N}\big]\,, \tag{2.0.2}$$

where $S^{(1)}, S^{(2)}$ are two independent copies of the simple random walk, starting from the origin, with joint law denoted by $\mathrm{P}^{\otimes 2}$. Moreover, $\mathsf{L}_N := \sum_{n=1}^N \mathbb{1}_{S_{2n}=0}$ denotes the number of times that a d-dimensional simple random walk returns to zero and for the second equality we made use of the equality in law $\sum_{n=1}^N \mathbb{1}_{S_n^1 = S_n^2} \overset{\text{law}}{=} \sum_{n=1}^N \mathbb{1}_{S_{2n}=0}$. In particular, we have that $\mathbb{E}\big[(Z_{N,\beta}(x))^2\big] \xrightarrow{N\to\infty} \mathrm{E}\big[e^{\lambda_2(\beta)\mathsf{L}_\infty}\big]$ and

$$\mathrm{E}\big[e^{\lambda_2(\beta)\mathsf{L}_\infty}\big] = \begin{cases} \frac{1-\pi_d}{1-\pi_d e^{\lambda_2(\beta)}}, & \text{if } \lambda_2(\beta) < \log(\frac{1}{\pi_d}) \\ \infty\,, & \text{otherwise.} \end{cases} \tag{2.0.3}$$

Our first result is the Edwards-Wilkinson fluctuations for the field of the partition functions, namely,

Theorem 2.0.1. *Let $d \geqslant 3$, $\beta \in (0, \beta_{L^2}(d))$ and consider the field of partition functions of the d-dimensional directed polymer $\big(Z_{N,\beta}(x)\big)_{x\in\mathbb{Z}^d}$. If $\varphi \in C_c(\mathbb{R}^d)$ is a test function, denote by*

$$\begin{aligned} \bar{Z}_{N,\beta}(\varphi) &:= \sum_{x\in\mathbb{Z}^d} \Big(Z_{N,\beta}(x) - \mathbb{E}\big[Z_{N,\beta}(x)\big]\Big) \frac{\varphi\big(\frac{x}{\sqrt{N}}\big)}{N^{\frac{d}{2}}} \\ &= \sum_{x\in\mathbb{Z}^d} \big(Z_{N,\beta}(x) - 1\big) \frac{\varphi\big(\frac{x}{\sqrt{N}}\big)}{N^{\frac{d}{2}}}, \end{aligned} \tag{2.0.4}$$

the centred and averaged partition function over φ. The rescaled sequence $\big(N^{\frac{d-2}{4}}\bar{Z}_{N,\beta}(\varphi)\big)_{N\geqslant 1}$ converges in distribution to a centred Gaussian random variable $\mathscr{Z}_\beta(\varphi)$ with variance given by

$$\mathbb{V}\mathrm{ar}[\mathscr{Z}_\beta(\varphi)] = \mathcal{C}_\beta \int_0^1 \mathrm{d}t \int_{\mathbb{R}^d\times\mathbb{R}^d} \mathrm{d}x\mathrm{d}y\, \varphi(x) g_{\frac{2t}{d}}(x-y)\varphi(y)\,, \tag{2.0.5}$$

where $g(\cdot)$ is the d-dimensional heat kernel, $\mathcal{C}_\beta = \sigma^2(\beta)\,\mathrm{E}[e^{\lambda_2(\beta)\mathsf{L}_\infty}]$ and $\sigma^2(\beta) = e^{\lambda_2(\beta)} - 1$.

We also establish a similar result for the field of log-partition functions. In this case we will additionally require that the disorder satisfies a (mild) concentration property (2.2.1). More precisely,

Theorem 2.0.2. *Let $d \geqslant 3$, $\beta \in (0, \beta_{L^2}(d))$ and consider the fields of log-partition functions of the d-dimensional directed polymer $\big(\log Z_{N,\beta}(x)\big)_{x\in\mathbb{Z}^d}$, with disorder that satisfies concentration property (2.2.1). If $\varphi \in C_c(\mathbb{R}^d)$ is a test function, we have that*

$$N^{\frac{d-2}{4}} \sum_{x\in\mathbb{Z}^d} \Big(\log Z_{N,\beta}(x) - \mathbb{E}\big[\log Z_{N,\beta}(x)\big]\Big) \frac{\varphi\big(\frac{x}{\sqrt{N}}\big)}{N^{\frac{d}{2}}}\,, \tag{2.0.6}$$

converges in distribution to the centred Gaussian random variable $\mathcal{Z}_\beta(\varphi)$ defined in Theorem 2.0.1.

Remark 2.0.3. We remark that, in fact, the sequences defined in (2.0.4) and (2.0.6) converge jointly to the random vector $(\mathcal{Z}_\beta(\varphi), \mathcal{Z}_\beta(\varphi))$. This follows from the proof of Theorem 2.0.2 which shows, after a series of approximations, that the difference of the two sequences converges to 0 in $L^1(\mathbb{P})$ as $N \to \infty$.

We will now describe the method we follow as well as the new ideas required. The basis of our analysis is the chaos expansion of the polymer partition function as

$$Z_{N,\beta}(x) = 1 + \sum_{k=1}^{N} \sigma^k \sum_{\substack{1\leqslant n_1<\ldots<n_k\leqslant N,\\ z_1,\ldots,z_k\in\mathbb{Z}^d}} q_{n_1}(z_1 - x) \prod_{i=2}^{k} q_{n_i-n_{i-1}}(z_i - z_{i-1}) \prod_{i=1}^{k} \xi_{n_i,z_i}, \tag{2.0.7}$$

where $q_n(x) = \mathrm{P}(S_n = x)$, $\sigma = \sigma(\beta) := \sqrt{e^{\lambda_2(\beta)} - 1}$ and $\xi_{n,z} := \sigma^{-1}\big(e^{\beta\omega_{n,z} - \lambda(\beta)} - 1\big)$, see (2.1.1) for the details of this derivation.

To prove the central limit theorem for $(N^{\frac{d-2}{4}}\,\overline{Z}_{N,\beta}(\varphi))_{N\geqslant 1}$ we make use of the so-called Fourth Moment Theorem [dJ87, NP05, NPR10, CSZ17b], which states that a sequence of random variables in a fixed Wiener chaos, normalised to have mean zero and variance one, converges to a standard normal random variable if its fourth moment converges to 3. Of course, in order to be able to reduce ourselves to a fixed chaos, we need to perform truncation and for this, the assumption of bounded second moments (L^2 regime) plays an important role. This approach of analysing chaos expansions of partition functions was first used in [CSZ17b] in a framework that also included the analysis of the two dimensional directed polymer and SHE. The work, which is needed to carry out this approach in $d \geqslant 3$, is actually easier than the $d = 2$ case in [CSZ17b]. The reason for this is that the variance of $Z_{N,\beta}$ is a functional of the local time L_N, see (2.0.2), which stays bounded in $d \geqslant 3$ but grows logarithmically in $d = 2$, introducing, in the latter case, a certain multiscale structure. Still, a careful combinatorial accounting and analytical estimates, which actually deviate from those in [CSZ17b], are needed to handle the $d \geqslant 3$ case. The detailed analysis of such expansion is what allows to go all the way to the L^2 critical temperature, as compared to the previous works [GRZ18], [MU18].

For the Edwards-Wilkinson fluctuations of the log-partition function, namely Theorem 2.0.2, we also adapt the approach of "linearisation" via chaos expansion proposed in [CSZ20]. However, the analysis in $d \geqslant 3$, required to achieve the goal of going all the way to $\beta_{L^2}(d)$, is rather more subtle. The reason is that the power law prefactor $N^{\frac{d-2}{4}}$ in (2.0.6) (as opposed to the corresponding $\log N$ prefactor in [CSZ20]) does not allow for any "soft" (or even more intricate) bounds à la

Cauchy-Schwarz or triangle inequalities in the approximations. Instead, we have to look carefully at the correlation structure that will cancel the $N^{\frac{d-2}{4}}$. This correlation structure is rather obvious in the case of the partition function and can be already understood by looking at the first term of the chaos expansion of $N^{\frac{d-2}{4}}\bar{Z}_{N,\beta}(\varphi)$ as derived from (2.0.7), which is

$$N^{\frac{d-2}{4}}\sum_{x\in\mathbb{Z}^d}\frac{\varphi(\frac{x}{\sqrt{N}})}{N^{\frac{d}{2}}}\sum_{z\in\mathbb{Z}^d,1\leqslant n\leqslant N}q_n(z-x)\xi_{n,z}\,,$$

and whose variance is easily computed as

$$\begin{aligned}&N^{\frac{d-2}{2}}\sum_{x,y\in\mathbb{Z}^d}\frac{\varphi(\frac{x}{\sqrt{N}})\varphi(\frac{y}{\sqrt{N}})}{N^d}\sum_{z\in\mathbb{Z}^d,1\leqslant n\leqslant N}q_n(z-x)q_n(z-y)\\=&N^{\frac{d-2}{2}}\sum_{x,y\in\mathbb{Z}^d}\frac{\varphi(\frac{x}{\sqrt{N}})\varphi(\frac{y}{\sqrt{N}})}{N^d}\sum_{1\leqslant n\leqslant N}q_{2n}(x-y)\,.\end{aligned}$$

The factor $N^{\frac{d-2}{2}}$ is then absorbed by the sum $\sum_n q_{2n}(x-y)$ in a Riemann sum approximation. What underlies the above computation is that correlations are captured by two independent copies of the random walk, one starting at x and another at y, meeting at some point by time N. The probability of such a coincidence event compensates for the $N^{\frac{d-2}{2}}$.

When considering the log-partition functions, the above described mechanism is not obvious, as $\log Z_{N,\beta}$ does not admit an equally nice and tractable chaos expansion. Nevertheless, it is necessary (which was not the case in [CSZ20]) to tease out the aforementioned correlation structure, in order to absorb $N^{\frac{d-2}{4}}$ and carry out the approximation. The way we do this is by writing $\log Z_{N,\beta}$ (or more accurately a certain approximation, which we call $\log Z^A_{N,\beta}$, see (2.2.9)) as a martingale difference:

$$\log Z_{N,\beta}-\mathbb{E}\big[\log Z_{N,\beta}\big]=\sum_{j\geqslant 1}\Big(\mathbb{E}\big[\log Z_{N,\beta}\,|\,\mathcal{F}_j\big]-\mathbb{E}\big[\log Z_{N,\beta}\,|\,\mathcal{F}_{j-1}\big]\Big)\,,$$

where $\{\mathcal{F}_j\colon j\geqslant 1\},\mathcal{F}_0=\{\varnothing,\Omega\}$ is a filtration generated as $\mathcal{F}_j=\sigma(\omega_{a_i}\colon i=1,...,j)$ with $\{a_1,a_2,...\}$ an enumeration of $\mathbb{N}\times\mathbb{Z}^d$. By adding the information from the disorder at a single additional site at each time, we keep track of how the polymer explores the disorder and this allows (after a certain "resampling" procedure) to keep track of the correlations. The martingale difference approach we introduce has in some sense some similarity to the Clark-Ocone formula, which was used in the work of [GRZ18, DGRZ18]. However, our approach of exploring a single new site disorder at a time seems to be necessary for the precise estimates that we need, in order to reach the whole L^2 regime. Along the way, a fine use of concentration and negative tail estimates of the log-partition function (e.g. Proposition 2.2.1) is made.

Once all the necessary approximations to the log-partition function are completed, the task is then reduced to a central limit theorem for a partition function of certain sorts, thus bringing us back to the context of Theorem 2.0.1. The previous work of [DGRZ18] seems to be necessarily restricted to a small subregion of $(0,\beta_{L^2})$, as a consequence of both the linearisation approach employed but also more importantly (as far as we can tell) due to the use of the so-called "second order Poincaré inequality" for the central limit theorem, which requires higher moment estimates that lead outside the L^2 regime, if β is not restricted to be small enough.

2.1. The Central Limit theorem for $\bar{Z}_{N,\beta}(\varphi)$

This section is devoted to the proof of Theorem 2.0.1. Throughout this chapter we rely on polynomial chaos expansions of the partition function. Specifically, consider the partition function of a polymer chain of length N starting from x at time zero. We can write

$$\begin{aligned} Z_{N,\beta}(x) &= \mathrm{E}_x\Big[\prod_{1\leqslant n\leqslant N,\, z\in\mathbb{Z}^d} e^{\{\beta\omega_{n,z}-\lambda(\beta)\}\mathbb{1}_{S_n=z}}\Big] \\ &= \mathrm{E}_x\Big[\prod_{1\leqslant n\leqslant N,\, z\in\mathbb{Z}^d} \big(1+(e^{\beta\omega_{n,z}-\lambda(\beta)}-1)\mathbb{1}_{S_n=z}\big)\Big] \\ &= 1+\sum_{k=1}^{N}\sigma^k \sum_{\substack{1\leqslant n_1<\ldots<n_k\leqslant N,\\ z_1,\ldots,z_k\in\mathbb{Z}^d}} q_{n_1}(z_1-x)\prod_{i=2}^{k} q_{n_i-n_{i-1}}(z_i-z_{i-1})\prod_{i=1}^{k}\xi_{n_i,z_i}\,. \end{aligned} \tag{2.1.1}$$

For $(n,z)\in\mathbb{N}\times\mathbb{Z}^d$ we have denoted by $\xi_{n,z}$ the centred random variables

$$\xi_{n,z} := \frac{e^{\beta\omega_{n,z}-\lambda(\beta)}-1}{\sigma}\,. \tag{2.1.2}$$

The number $\sigma=\sigma(\beta)$ is chosen so that for $(n,z)\in\mathbb{N}\times\mathbb{Z}^d$ the centred random variables $\xi_{n,z}$ have unit variance. A simple calculation shows that $\sigma=\sqrt{e^{\lambda(2\beta)-2\lambda(\beta)}-1}$. Also, the last equality in (2.1.1) comes from expanding the product in the second line of (2.1.1) and interchanging the expectation with the summation. By using the expansion (2.1.1) we can derive an expression for the averaged partition function. Let us fix a test function $\varphi\in C_c(\mathbb{R}^d)$. In the following we shall use the notation

$$\varphi_N(x_1,\ldots,x_k) := \prod_{i=1}^{k}\varphi\big(\tfrac{x_i}{\sqrt{N}}\big)\,, \qquad k\geqslant 1\,. \tag{2.1.3}$$

We have

$$\begin{aligned} \bar{Z}_{N,\beta}(\varphi) &:= \sum_{x\in\mathbb{Z}^d}(Z_{N,\beta}(x)-1)\,\frac{\varphi_N(x)}{N^{\frac{d}{2}}} \\ &= \sum_{k=1}^{N}\sigma^k \sum_{\substack{1\leqslant n_1<\ldots<n_k\leqslant N,\\ z_1,\ldots,z_k\in\mathbb{Z}^d}} \Big(\sum_{x\in\mathbb{Z}^d}\frac{\varphi_N(x)}{N^{\frac{d}{2}}}\,q_{n_1}(x,z_1)\Big)\prod_{i=2}^{k}q_{n_i-n_{i-1}}(z_{i-1},z_i)\prod_{i=1}^{k}\xi_{n_i,z_i} \\ &= \sum_{k=1}^{N}\bar{Z}^{(k)}_{N,\beta}(\varphi)\,, \end{aligned} \tag{2.1.4}$$

where

$$\bar{Z}^{(k)}_{N,\beta}(\varphi) := \sigma^k \sum_{\substack{1\leqslant n_1<\ldots<n_k\leqslant N,\\ z_1,\ldots,z_k\in\mathbb{Z}^d}} \Big(\sum_{x\in\mathbb{Z}^d}\frac{\varphi_N(x)}{N^{\frac{d}{2}}}q_{n_1}(x,z_1)\Big)\prod_{i=2}^{k}q_{n_i-n_{i-1}}(z_{i-1},z_i)\prod_{i=1}^{k}\xi_{n_i,z_i}\,. \tag{2.1.5}$$

2.1.1. Computation of the limiting variance. The first step towards the proof of Theorem 2.0.1 is the following proposition which identifies the limiting variance of the scaled sequence of centred and averaged over φ partition functions, $(N^{\frac{d-2}{4}}\bar{Z}_{N,\beta}(\varphi))_{N\geqslant 1}$.

Proposition 2.1.1. *Let $d \geqslant 3$, $\beta \in (0, \beta_{L^2})$ and fix $\varphi \in C_c(\mathbb{R}^d)$ to be a test function. Consider the sequence $(N^{\frac{d-2}{4}} \bar{Z}_{N,\beta}(\varphi))_{N\geqslant 1}$, where $\bar{Z}_{N,\beta}(\varphi)$ is defined in (2.0.4). Then, one has that*

$$\mathbb{V}\mathrm{ar}\left[N^{\frac{d-2}{4}} \bar{Z}_{N,\beta}(\varphi)\right] \xrightarrow{N\to\infty} \mathcal{C}_\beta \int_0^1 \mathrm{d}t \int_{\mathbb{R}^d\times\mathbb{R}^d} \mathrm{d}x\mathrm{d}y\, \varphi(x) g_{\frac{2t}{d}}(x-y)\varphi(y)\,,$$

where $\mathcal{C}_\beta = \sigma^2(\beta)\,\mathrm{E}[e^{\lambda_2(\beta)\mathsf{L}_\infty}]$, $\sigma^2(\beta) = e^{\lambda_2(\beta)} - 1$ and g denotes the d-dimensional heat kernel.

For the proof of Proposition 2.1.1, we will need the following standard consequence of the local limit theorem, which we prove for completeness.

Lemma 2.1.2. *For any test function $\varphi \in C_c(\mathbb{R}^d)$ we have that*

$$\lim_{N\to\infty} N^{\frac{d}{2}-1} \sum_{n=1}^{N} \sum_{x,y\in\mathbb{Z}^d} \frac{\varphi_N(x,y)}{N^d}\, q_{2n}(x-y)$$
$$= \int_0^1 \mathrm{d}t \int_{\mathbb{R}^d\times\mathbb{R}^d} \mathrm{d}x\mathrm{d}y\, \varphi(x) g_{\frac{2t}{d}}(x-y)\varphi(y)\,.$$

Proof. Recall that by the local limit theorem for the d-dimensional simple random walk, see [LL10], one has that $q_{2n}(x) = 2\big(g_{\frac{2n}{d}}(x) + o(n^{-\frac{d}{2}})\big)\mathbb{1}_{x\in\mathbb{Z}^d_{\text{even}}}$, uniformly in $x \in \mathbb{Z}^d$, as $n \to \infty$, where $\mathbb{Z}^d_{\text{even}} := \{x = (x_1, \dots, x_d) \in \mathbb{Z}^d : x_1 + \dots + x_d \in 2\mathbb{Z}\}$. The factor 2 comes from the periodicity of the random walk. The kernel $g_{\frac{2n}{d}}(x)$ appears instead of $g_{2n}(x)$, because after n steps the d-dimensional simple random walk S_n has covariance matrix $\frac{n}{d}I$. Let us fix $\vartheta \in (0,1)$. Let us also use the notation

$$A_{\vartheta,N} := N^{\frac{d}{2}-1} \sum_{n=1}^{\vartheta N} \sum_{x,y\in\mathbb{Z}^d} \frac{\varphi_N(x,y)}{N^d}\, q_{2n}(x-y)\,,$$
$$B_{\vartheta,N} := N^{\frac{d}{2}-1} \sum_{n>\vartheta N}^{N} \sum_{x,y\in\mathbb{Z}^d} \frac{\varphi_N(x,y)}{N^d}\, q_{2n}(x-y)\,.$$

Observe that if we bound $\varphi(\frac{y}{\sqrt{N}})$ in $\varphi_N(x,y)$ by its supremum norm and use that $\sum_{z\in\mathbb{Z}^d} q_{2n}(z) = 1$ we obtain that

$$A_{\vartheta,N} \leqslant \frac{\|\varphi\|_\infty}{N} \sum_{n=1}^{\vartheta N} \sum_{x\in\mathbb{Z}^d} \frac{\varphi_N(x)}{N^{\frac{d}{2}}} \sum_{y\in\mathbb{Z}^d} q_{2n}(x-y) \leqslant \frac{\|\varphi\|_\infty}{N} \sum_{n=1}^{\vartheta N} \sum_{x\in\mathbb{Z}^d} \frac{\varphi_N(x)}{N^{\frac{d}{2}}} \leqslant \|\varphi\|_\infty \|\varphi\|_1\, \vartheta\,.$$

On the other hand, by using the local limit theorem and Riemann approximation one obtains that

$$B_{\vartheta,N} \xrightarrow{N\to\infty} \int_\vartheta^1 \mathrm{d}t \int_{\mathbb{R}^d\times\mathbb{R}^d} \mathrm{d}x\mathrm{d}y\, \varphi(x) g_{\frac{2t}{d}}(x-y)\varphi(y)\,.$$

By combining those two facts and letting $\vartheta \to 0$, one obtains the desired result. $\square$

We are now ready to present the proof of Proposition 2.1.1.

Proof of Proposition 2.1.1. Recalling (2.1.4) and using also the fact that terms of different degree in the chaos expansion are orthogonal in $L^2(\mathbb{P})$, one arrives into the following expression for the variance of $Z_{N,\beta}(\varphi)$

$$\mathbb{V}\mathrm{ar}\left[\bar{Z}_{N,\beta}(\varphi)\right] = \sum_{k=1}^{N} \sigma^{2k} \sum_{1\leqslant n_1<\dots<n_k\leqslant N} \sum_{x,y\in\mathbb{Z}^d} \frac{\varphi_N(x,y)}{N^d}\, q_{2n_1}(x-y) \prod_{i=2}^{k} q_{2(n_i-n_{i-1})}(0)\,.$$

We can factor out the $k = 1$ term and change variables to obtain the expression:

$$\sum_{n=1}^{N} \sigma^2 \sum_{x,y\in\mathbb{Z}^d} \frac{\varphi_N(x,y)}{N^d} q_{2n}(x-y) \left(1 + \sum_{k=1}^{N-n} \sigma^{2k} \sum_{1\leqslant \ell_1 < \ldots < \ell_k \leqslant N-n} \prod_{i=1}^{k} q_{2(\ell_i - \ell_{i-1})}(0) \right), \quad (2.1.6)$$

where, by convention if $n = N$ the sum on the rightmost parenthesis is equal to 1. Furthermore, one can observe that the right parenthesis is exactly equal to $\mathrm{E}\big[e^{\lambda_2(\beta)\mathsf{L}_{N-n}}\big]$, where we recall that $\mathsf{L}_N := \sum_{k=1}^{N} \mathbb{1}_{S_{2k}=0}$ denotes the number of times a random walk returns to 0 up to time N. Thus,

$$\mathbb{V}\mathrm{ar}\left[N^{\frac{d-2}{4}} \bar{Z}_{N,\beta}(\varphi)\right] = N^{\frac{d}{2}-1} \sum_{n=1}^{N} \sigma^2 \sum_{x,y\in\mathbb{Z}^d} \frac{\varphi_N(x,y)}{N^d} q_{2n}(x-y) \mathrm{E}[e^{\lambda_2(\beta)\mathsf{L}_{N-n}}]. \quad (2.1.7)$$

The heuristic idea here is that, if in the expression (2.1.7) we ignore n in the expectation, then the sum would factorise. Then, by noticing that $\mathrm{E}[e^{\lambda_2(\beta)\mathsf{L}_N}]$ converges and by using also Lemma 2.1.2, we obtain the conclusion of Proposition 2.1.1. Let us justify this heuristic idea rigorously. We have that

$$\mathrm{E}\big[e^{\lambda_2(\beta)\mathsf{L}_{N-n}}\big] = \mathrm{E}\big[e^{\lambda_2(\beta)\mathsf{L}_N}\big] + \mathrm{E}\big[(e^{\lambda_2(\beta)\mathsf{L}_{N-n}} - e^{\lambda_2(\beta)\mathsf{L}_N})\mathbb{1}_{\mathsf{L}_N > \mathsf{L}_{N-n}}\big]. \quad (2.1.8)$$

Also,

$$\Big|\mathrm{E}\big[(e^{\lambda_2(\beta)\mathsf{L}_{N-n}} - e^{\lambda_2(\beta)\mathsf{L}_N})\mathbb{1}_{\mathsf{L}_N > \mathsf{L}_{N-n}}\big]\Big| \leqslant 2\mathrm{E}\big[e^{\lambda_2(\beta)\mathsf{L}_N}\mathbb{1}_{\mathsf{L}_N > \mathsf{L}_{N-n}}\big], \quad (2.1.9)$$

by triangle inequality and because L_N is non-decreasing. Using Hölder inequality we can further bound the error in (2.1.8) as follows: We choose $p > 1$ very close to 1, such that $p\lambda_2(\beta) < \log(\frac{1}{\pi_d})$, thus $\mathrm{E}[e^{p\lambda_2(\beta)\mathsf{L}_N}] < \infty$, for every $N \in \mathbb{N}$. This is only possible when β is in the L^2-regime. Then, by Hölder:

$$\mathrm{E}\big[e^{\lambda_2(\beta)\mathsf{L}_N}\mathbb{1}_{\mathsf{L}_N > \mathsf{L}_{N-n}}\big] \leqslant \mathrm{E}\big[e^{p\lambda_2(\beta)\mathsf{L}_N}\big]^{\frac{1}{p}} \mathrm{P}\big(\mathsf{L}_N > \mathsf{L}_{N-n}\big)^{\frac{1}{q}}.$$

Hence,

$$\Big|\mathrm{E}\big[(e^{\lambda_2(\beta)\mathsf{L}_{N-n}} - e^{\lambda_2(\beta)\mathsf{L}_N})\mathbb{1}_{\mathsf{L}_N > \mathsf{L}_{N-n}}\big]\Big| \leqslant C_{p,\beta}\, \mathrm{P}\big(\mathsf{L}_N > \mathsf{L}_{N-n}\big)^{\frac{1}{q}},$$

where $C_{p,\beta} := 2\mathrm{E}\big[e^{p\lambda_2(\beta)\mathsf{L}_\infty}\big]^{\frac{1}{p}} < \infty$.

Now, we split the sum in (2.1.7) into two parts. Let $\vartheta \in (0,1)$. We distinguish two cases:
(Case 1) If $n \leqslant \vartheta N$, then $N - n \geqslant (1-\vartheta)N$. Thus,

$$\Big|\mathrm{E}\big[(e^{\lambda_2(\beta)\mathsf{L}_{N-n}} - e^{\lambda_2(\beta)\mathsf{L}_N})\mathbb{1}_{\mathsf{L}_N > \mathsf{L}_{N-n}}\big]\Big| \leqslant C_{p,\beta}\, \mathrm{P}(\mathsf{L}_N > \mathsf{L}_{(1-\vartheta)N})^{\frac{1}{q}},$$

since L_N is non-decreasing in N. We also have that

$$\mathrm{P}(\mathsf{L}_N > \mathsf{L}_{(1-\vartheta)N}) \leqslant \mathrm{P}\big(\exists\, n > (1-\vartheta)N :\ S_{2n} = 0\big) \leqslant \sum_{n > (1-\vartheta)N}^{\infty} q_{2n}(0) \xrightarrow[N\to\infty]{} 0,$$

since $\sum_{n=1}^{\infty} q_{2n}(0) < \infty$, because $d \geqslant 3$. Therefore, in this case we obtain that,

$$\begin{aligned} &N^{\frac{d}{2}-1} \sum_{n=1}^{\vartheta N} \sigma^2 \sum_{x,y\in\mathbb{Z}^d} \frac{\varphi_N(x,y)}{N^d} q_{2n}(x-y)\mathrm{E}[e^{\lambda_2(\beta)\mathsf{L}_{N-n}}] \\ &= N^{\frac{d}{2}-1} \sum_{n=1}^{\vartheta N} \sigma^2 \sum_{x,y\in\mathbb{Z}^d} \frac{\varphi_N(x,y)}{N^d} q_{2n}(x-y)\Big(\mathrm{E}[e^{\lambda_2(\beta)\mathsf{L}_N}] + o(1)\Big). \end{aligned}$$

(Case 2) If $n > \vartheta N$, we have that:

$$
\begin{aligned}
& N^{\frac{d}{2}-1} \sum_{n>\vartheta N} \sigma^2 \sum_{x,y\in\mathbb{Z}^d} \frac{\varphi_N(x,y)}{N^d}\, q_{2n}(x-y)\mathrm{E}[e^{\lambda_2(\beta)\mathsf{L}_{N-n}}] \\
& \leqslant N^{\frac{d}{2}-1} \sum_{n>\vartheta N} \sigma^2 \sum_{x,y\in\mathbb{Z}^d} \frac{\varphi_N(x,y)}{N^d}\, q_{2n}(x-y)\mathrm{E}[e^{\lambda_2(\beta)\mathsf{L}_{\infty}}]\,.
\end{aligned}
$$

By combining the two cases above we get that, for every $\vartheta \in (0,1)$

$$
\limsup_{N\to\infty} \mathbb{V}\mathrm{ar}[N^{\frac{d-2}{4}} \bar{Z}_{N,\beta}(\varphi)] \leqslant \sigma^2 \int_0^{\vartheta} \mathrm{d}t \int_{\mathbb{R}^d\times\mathbb{R}^d} \mathrm{d}x\mathrm{d}y\, \varphi(x) g_{\frac{2t}{d}}(x-y)\varphi(y)\mathrm{E}[e^{\lambda_2(\beta)\mathsf{L}_\infty}] + k(\vartheta)\,,
$$

where

$$
k(\vartheta) \leqslant \mathrm{E}[e^{\lambda_2(\beta)\mathsf{L}_\infty}]\, \sigma^2 \int_{\vartheta}^{1} \mathrm{d}t \int_{\mathbb{R}^d\times\mathbb{R}^d} \mathrm{d}x\mathrm{d}y\, \varphi(x) g_{\frac{2t}{d}}(x-y)\varphi(y)\,,
$$

and

$$
\liminf_{N\to\infty} \mathbb{V}\mathrm{ar}[N^{\frac{d-2}{4}} \bar{Z}_{N,\beta}(\varphi)] \geqslant \sigma^2 \int_0^{\vartheta} \mathrm{d}t \int_{\mathbb{R}^d\times\mathbb{R}^d} \mathrm{d}x\mathrm{d}y\, \varphi(x) g_{\frac{2t}{d}}(x-y)\varphi(y)\mathrm{E}[e^{\lambda_2(\beta)\mathsf{L}_\infty}]\,.
$$

It is clear that $k(\vartheta) \longrightarrow 0$ as $\vartheta \longrightarrow 1$, hence we obtain the desired result. □

2.1.2. Reduction to finite chaoses. We proceed towards the proof of the Central Limit Theorem for the sequence $\big(\bar{Z}_{N,\beta}(\varphi)\big)_{N\geqslant 1}$ of the averaged partition functions. In order to determine the limiting distribution of the sequence $\big(N^{\frac{d-2}{4}} \bar{Z}_{N,\beta}(\varphi)\big)_{N\geqslant 1}$, we use the Fourth Moment Theorem, see [dJ87, NP05, NPR10, CSZ17b]. The strategy we deploy is the following: First, we show that it suffices to consider a large $M \in \mathbb{N}$ and work with a truncated version of the partition function, namely

$$
\bar{Z}^{\leqslant M}_{N,\beta}(\varphi) := \sum_{k=1}^{M} \sigma^k \sum_{\substack{1\leqslant n_1<\ldots<n_k\leqslant N,\\ z_1,\ldots,z_k\in\mathbb{Z}^d}} \left(\sum_{x\in\mathbb{Z}^d} \frac{\varphi_N(x)}{N^{\frac{d}{2}}}\, q_{n_1}(z_1-x) \right) \prod_{i=2}^{k} q_{n_i-n_{i-1}}(z_i-z_{i-1}) \prod_{i=1}^{k} \xi_{n_i,z_i}\,. \tag{2.1.10}
$$

To do this it is enough to show that for any $\varepsilon > 0$ we can choose a large $M = M(\varepsilon)$ such that $N^{\frac{d-2}{4}} \bar{Z}^{\leqslant M}_{N,\beta}(\varphi)$ and $N^{\frac{d-2}{4}} \bar{Z}_{N,\beta}(\varphi)$ are ε-close in $L^2(\mathbb{P})$, uniformly for $N \in \mathbb{N}$ large. Then, by using the Fourth Moment Theorem and the Crámer-Wold device, we show that the random vector $N^{\frac{d-2}{4}} \big(\bar{Z}^{(1)}_{N,\beta}(\varphi), \ldots, \bar{Z}^{(M)}_{N,\beta}(\varphi)\big)$ converges in distribution to a centred Gaussian random vector. This allows us to conclude that the limiting distribution of $N^{\frac{d-2}{4}} \bar{Z}^{\leqslant M}_{N,\beta}(\varphi)$ is a centred Gaussian. After removing the truncation in M, we obtain the desired result for $N^{\frac{d-2}{4}} \bar{Z}_{N,\beta}(\varphi)$, namely Theorem 2.0.1.

We begin by proving that we can approximate $\bar{Z}_{N,\beta}(\varphi)$ in $L^2(\mathbb{P})$, uniformly for large enough N, by $\bar{Z}^{\leqslant M}_{N,\beta}(\varphi)$ for some large $M \in \mathbb{N}$.

Lemma 2.1.3. *For every $\varepsilon > 0$, there exists $M_0 \in \mathbb{N}$, such that for all $M > M_0$*

$$
\limsup_{N\to\infty} \left\| N^{\frac{d-2}{4}} \bar{Z}_{N,\beta}(\varphi) - N^{\frac{d-2}{4}} \bar{Z}^{\leqslant M}_{N,\beta}(\varphi) \right\|_{L^2(\mathbb{P})} \leqslant \varepsilon\,.
$$

Proof. Consider $\varepsilon > 0$. One has that

$$\bar{Z}_{N,\beta}(\varphi) - \bar{Z}^{\leqslant M}_{N,\beta}(\varphi)$$
$$= \sum_{k>M}^{N} \sigma^k \sum_{\substack{1\leqslant n_1<\ldots<n_k\leqslant N,\\ z_1,\ldots,z_k\in\mathbb{Z}^d}} \left(\sum_{x\in\mathbb{Z}^d} \frac{\varphi_N(x)}{N^{\frac{d}{2}}} q_{n_1}(z_1-x)\right) \prod_{i=2}^{k} q_{n_i-n_{i-1}}(z_i-z_{i-1}) \prod_{i=1}^{k} \xi_{n_i,z_i}\,.$$

By an analogous computation as in Proposition 2.1.1 we have that

$$\left\| N^{\frac{d-2}{4}} \bar{Z}_{N,\beta}(\varphi) - N^{\frac{d-2}{4}} \bar{Z}^{\leqslant M}_{N,\beta}(\varphi) \right\|^2_{L^2(\mathbb{P})}$$
$$\leqslant N^{\frac{d}{2}-1} \sum_{n=1}^{N} \sigma^2 \sum_{x,y\in\mathbb{Z}^d} \frac{\varphi_N(x,y)}{N^d} q_{2n}(x-y) \Big(\sum_{k\geqslant M}^{N-n} \sigma^{2k} \sum_{1\leqslant \ell_1<\ldots<\ell_k\leqslant N-n} \prod_{i=1}^{k} q_{2(\ell_i-\ell_{i-1})}(0)\Big)$$
$$\leqslant N^{\frac{d}{2}-1} \sum_{n=1}^{N} \sigma^2 \sum_{x,y\in\mathbb{Z}^d} \frac{\varphi_N(x,y)}{N^d} q_{2n}(x-y) \Big(\sum_{k\geqslant M}^{N} \sigma^{2k} \sum_{1\leqslant \ell_1<\ldots<\ell_k\leqslant N} \prod_{i=1}^{k} q_{2(\ell_i-\ell_{i-1})}(0)\Big)\,.$$

By Lemma 2.1.2 we have that

$$N^{\frac{d}{2}-1} \sum_{n=1}^{N} \sigma^2 \sum_{x,y\in\mathbb{Z}^d} \frac{\varphi_N(x,y)}{N^d} q_{2n}(x-y) \xrightarrow[N\to\infty]{} \int_0^1 \mathrm{d}t \int_{\mathbb{R}^d\times\mathbb{R}^d} \mathrm{d}x\mathrm{d}y\, \varphi(x) g_{\frac{2t}{d}}(x-y)\varphi(y)\,.$$

On the other hand, the sum in the rightmost parenthesis can be bounded by

$$\Big(\sum_{k\geqslant M}^{N} \sigma^{2k} \sum_{1\leqslant \ell_1<\ldots<\ell_k\leqslant N} \prod_{i=1}^{k} q_{2(\ell_i-\ell_{i-1})}(0)\Big) \leqslant \sum_{k\geqslant M}^{N} \sigma^{2k} R_N^k \leqslant \sum_{k\geqslant M}^{N} \sigma^{2k} R_\infty^k \leqslant \sum_{k\geqslant M}^{\infty} \sigma^{2k} R_\infty^k\,,$$

where $R_N = \sum_{k=1}^{N} q_{2n}(0)$ is the expected number of visits to zero before time N of the simple random walk and $R_\infty = \lim_{N\to\infty} R_N = \sum_{n=1}^{\infty} q_{2n}(0)$. Since β is in the L^2-regime, the series $\sum_{k\geqslant 1}^{\infty} \sigma(\beta)^{2k} R_\infty^k$ is convergent. Therefore, we have that

$$\sum_{k\geqslant M}^{\infty} \sigma^{2k} R_\infty^k \xrightarrow[M\to\infty]{} 0\,.$$

Therefore, we conclude that if we take M to be sufficiently large we have that

$$\left\| N^{\frac{d-2}{4}} \bar{Z}_{N,\beta}(\varphi) - N^{\frac{d-2}{4}} \bar{Z}^{\leqslant M}_{N,\beta}(\varphi) \right\|_{L^2(\mathbb{P})} \leqslant \varepsilon\,,$$

uniformly for all large enough $N \in \mathbb{N}$, hence there exists $M_0 \in \mathbb{N}$, so that for $M > M_0$:

$$\limsup_{N\to\infty} \left\| N^{\frac{d-2}{4}} \bar{Z}_{N,\beta}(\varphi) - N^{\frac{d-2}{4}} \bar{Z}^{\leqslant M}_{N,\beta}(\varphi) \right\|_{L^2(\mathbb{P})} \leqslant \varepsilon\,.$$

□

2.1.3. Joint convergence of chaoses of bounded degree. We proceed by showing that for any $M \in \mathbb{N}$, the random vector $N^{\frac{d-2}{4}} \big(\bar{Z}^{(1)}_{N,\beta}(\varphi), \ldots, \bar{Z}^{(M)}_{N,\beta}(\varphi)\big)$ converges in distribution to a Gaussian vector. To do this we employ the Cramér-Wold device. Namely, we prove that for any M-tuple of real numbers $(t_1, \ldots, t_M)$ the linear combination $N^{\frac{d-2}{4}} \sum_{k=1}^{M} t_k \bar{Z}^{(k)}_{N,\beta}(\varphi)$ converges in distribution to a Gaussian random variable.

Proposition 2.1.4. *For all $M \in \mathbb{N}$ and $(t_1, \ldots, t_M) \in \mathbb{R}^M$, $N^{\frac{d-2}{4}} \sum_{k=1}^{M} t_k \bar{Z}^{(k)}_{N,\beta}(\varphi)$ converges in distribution to a Gaussian random variable with mean zero and variance equal to*

$$\sum_{k=1}^{M} t_k^2\, \mathcal{C}^{(k)}_{\beta} \int_0^1 \mathrm{d}t \int_{\mathbb{R}^{2d}} \mathrm{d}x\, \mathrm{d}y\, \varphi(x) g_{\frac{2t}{d}}(x-y)\varphi(y)\,,$$

where $\mathcal{C}_\beta^{(k)} = \sigma(\beta)^{2k} \sum_{0:=\ell_0<\ell_1<\ldots<\ell_{k-1}} \prod_{i=1}^{k-1} q_{2(\ell_i-\ell_{i-1})}(0)$ *for* $k>1$ *and* $\mathcal{C}_\beta^{(1)} = \sigma(\beta)^2$.

Proof. We start by introducing some shorthand notation that is going to be useful for a concise presentation of the rest of the proof. For any $u \in \mathbb{Z}^d$, $\tau_u^{(k)}$ will denote a time-increasing sequence of $(k+1)$ space-time points $(n_i, z_i)_{0\leqslant i\leqslant k} \subset \mathbb{N}\times\mathbb{Z}^d$ with a starting point $(n_0, z_0) := (0, u)$. We will use the convention that for two sequences $\tau_x^{(k)} = (n_i, z_i)_{0\leqslant i\leqslant k}$ and $\tau_y^{(\ell)} = (m_i, w_i)_{0\leqslant i\leqslant \ell}$, the equality $\tau_x^{(k)} = \tau_y^{(\ell)}$ means that $k=\ell$ and $(n_i, z_i) = (m_i, w_i)$ for $i = 1, ..., k$, that is for all points in the sequences $\tau_x^{(k)}$ and $\tau_y^{(\ell)}$ except the starting ones.

Given a sequence $\tau_u^{(k)} = (n_i, z_i)_{1\leqslant i\leqslant k}$, we will use the following notation

$$q(\tau_u^{(k)}) := q_{n_1}(z_1-u)\prod_{i=2}^{k} q_{n_i-n_{i-1}}(z_i-z_{i-1}) \qquad \text{and} \qquad \xi(\tau_u^{(k)}) := \prod_{i=1}^{k}\xi_{n_i,z_i}\,.$$

Furthermore, recall from (2.1.3), that for a finite set $\{x_1, ..., x_k\} \subset \mathbb{Z}^d$ we use the notation

$$\varphi_N(x_1, ..., x_k) := \prod_{u\in\{x_1,\ldots,x_k\}} \varphi\big(\tfrac{u}{\sqrt{N}}\big)\,. \tag{2.1.11}$$

We start by deriving the limiting variance of $N^{\frac{d-2}{4}}\sum_{k=1}^M t_k\, \bar{Z}_{N,\beta}^{(k)}(\varphi)$. We have that

$$\mathbb{V}\mathrm{ar}\bigg(N^{\frac{d-2}{4}}\sum_{k=1}^M t_k \bar{Z}_{N,\beta}^{(k)}(\varphi)\bigg) = \sum_{k=1}^M t_k^2\, N^{\frac{d}{2}-1}\,\mathbb{E}\Big[\big(\bar{Z}_{N,\beta}^{(k)}(\varphi)\big)^2\Big]\,,$$

because for every $k\geqslant 1$, $\mathbb{E}\Big[\bar{Z}_{N,\beta}^{(k)}(\varphi)\Big] = 0$ and if $1\leqslant k<\ell$, we have that

$$\mathbb{E}\Big[\bar{Z}_{N,\beta}^{(k)}(\varphi)\,\bar{Z}_{N,\beta}^{(\ell)}(\varphi)\Big] = 0\,,$$

see (2.1.5). One can follow the steps of the proof of Proposition 2.1.1 to conclude that

$$\lim_{N\to\infty} N^{\frac{d}{2}-1}\,\mathbb{E}\Big[\big(\bar{Z}_{N,\beta}^{(k)}(\varphi)\big)^2\Big] = \mathcal{C}_\beta^{(k)}\int_0^1 \mathrm{d}t \int_{\mathbb{R}^{2d}} \mathrm{d}x\,\mathrm{d}y\,\varphi(x) g_{\frac{2t}{d}}(x-y)\varphi(y)\,,$$

where $\mathcal{C}_\beta^{(k)} := \sigma(\beta)^{2k} \sum_{0:=\ell_0<\ell_1<\ldots<\ell_{k-1}} \prod_{i=1}^{k-1} q_{2(\ell_i-\ell_{i-1})}(0)$ for $k>1$ and $\mathcal{C}_\beta^{(1)} := \sigma(\beta)^2$.

In order to show that $N^{\frac{d-2}{4}}\sum_{k=1}^M t_k\, \bar{Z}_{N,\beta}^{(k)}(\varphi)$ converges in distribution to a Gaussian limit we will employ the Fourth Moment Theorem, which states that a sequence of random variables in a fixed Wiener chaos or multilinear polynomials of finite degree converge to a Gaussian random variable if the 4th moment converges to three times the square of the variance, see [dJ87, NP05, NPR10, CSZ17b] for more details. Namely, we will show that as $N\to\infty$,

$$\mathbb{E}\Bigg[\bigg(N^{\frac{d-2}{4}}\sum_{k=1}^M t_k\bar{Z}_{N,\beta}^{(k)}(\varphi)\bigg)^4\Bigg] = 3\,\mathbb{V}\mathrm{ar}\Bigg[N^{\frac{d-2}{4}}\sum_{k=1}^M t_k\bar{Z}_{N,\beta}^{(k)}(\varphi)\Bigg]^2 + o(1)\,.$$

that is, the fourth moment of $N^{\frac{d-2}{4}}\sum_{k=1}^{M} t_k \bar{Z}^{(k)}_{N,\beta}(\varphi)$ converges to 3 times its variance, squared. In view of the chaos expansion (2.1.5) we have that

$$\begin{aligned}
&\mathbb{E}\Bigg[\bigg(N^{\frac{d-2}{4}}\sum_{k=1}^{M} t_k \bar{Z}^{(k)}_{N,\beta}(\varphi)\bigg)^4\Bigg] \\
&\qquad = N^{d-2}\sum_{1\leqslant \mathsf{a},\mathsf{b},\mathsf{c},\mathsf{d}\leqslant M} t_{\mathsf{a}}t_{\mathsf{b}}t_{\mathsf{c}}t_{\mathsf{d}}\, \mathbb{E}\Big[\bar{Z}^{(\mathsf{a})}_{N,\beta}(\varphi)\bar{Z}^{(\mathsf{b})}_{N,\beta}(\varphi)\bar{Z}^{(\mathsf{c})}_{N,\beta}(\varphi)\bar{Z}^{(\mathsf{d})}_{N,\beta}(\varphi)\Big] \\
&\qquad = N^{d-2}\sum_{1\leqslant \mathsf{a},\mathsf{b},\mathsf{c},\mathsf{d}\leqslant M} t_{\mathsf{a}}t_{\mathsf{b}}t_{\mathsf{c}}t_{\mathsf{d}}\, \sigma^{\mathsf{a}+\mathsf{b}+\mathsf{c}+\mathsf{d}} \sum_{x,y,z,w\in\mathbb{Z}^d} \frac{\varphi_N(x,y,z,w)}{N^{2d}} \\
&\qquad\qquad \times \sum_{\tau^{(\mathsf{a})}_x,\tau^{(\mathsf{b})}_y,\tau^{(\mathsf{c})}_z,\tau^{(\mathsf{d})}_w} \prod_{\substack{(u,\mathsf{s})\in\{(x,\mathsf{a}),(y,\mathsf{b}),\\ (z,\mathsf{c}),(w,\mathsf{d})\}}} q(\tau^{(\mathsf{s})}_u)\,\mathbb{E}\Bigg[\prod_{\substack{(u,\mathsf{s})\in\{(x,\mathsf{a}),(y,\mathsf{b}),\\ (z,\mathsf{c}),(w,\mathsf{d})\}}} \xi(\tau^{(\mathsf{s})}_u)\Bigg].
\end{aligned} \tag{2.1.12}$$

Since M is finite, we can fix a quadruple $(\mathsf{a},\mathsf{b},\mathsf{c},\mathsf{d})$ and deal with the rest of the sum which varies as $N\to\infty$. Thus, we will focus on the sum

$$\begin{aligned}
&N^{d-2}\sum_{x,y,z,w\in\mathbb{Z}^d} \frac{\varphi_N(x,y,z,w)}{N^{2d}}\,\sigma^{\mathsf{a}+\mathsf{b}+\mathsf{c}+\mathsf{d}} \\
&\qquad \times \sum_{\tau^{(\mathsf{a})}_x,\tau^{(\mathsf{b})}_y,\tau^{(\mathsf{c})}_z,\tau^{(\mathsf{d})}_w} \prod_{\substack{(u,\mathsf{s})\in\{(x,\mathsf{a}),(y,\mathsf{b}),\\ (z,\mathsf{c}),(w,\mathsf{d})\}}} q(\tau^{(\mathsf{s})}_u)\,\mathbb{E}\Bigg[\prod_{\substack{(u,\mathsf{s})\in\{(x,\mathsf{a}),(y,\mathsf{b}),\\ (z,\mathsf{c}),(w,\mathsf{d})\}}} \xi(\tau^{(\mathsf{s})}_u)\Bigg],
\end{aligned} \tag{2.1.13}$$

instead of (2.1.12). We note that the expectation

$$\mathbb{E}\Bigg[\prod_{\substack{(u,\mathsf{s})\in\{(x,\mathsf{a}),(y,\mathsf{b}),\\ (z,\mathsf{c}),(w,\mathsf{d})\}}} \xi(\tau^{(\mathsf{s})}_u)\Bigg], \tag{2.1.14}$$

is non-zero only if the random variables ξ appearing in the product, are matched to each other. This is because, if a random variable ξ stands alone in the expectation (2.1.14), then due to independence and the fact that every ξ has mean zero, the expectation is trivially zero. The possible matchings among the ξ variables can be double, triple or quadruple. We cannot have more than quadruple matchings, because points in a sequence $\tau^{(\mathsf{s})}_u$ are strictly increasing in time, thus they cannot match with each other.

We will show that when $N\to\infty$, only one type of matchings contributes to (2.1.13) and hence also to (2.1.12). Specifically, the only configuration that contributes, asymptotically, is the one where four random walk paths meet in pairs without switching their pair. In terms of the sequences $\tau^{(\mathsf{a})}_x,\tau^{(\mathsf{b})}_y,\tau^{(\mathsf{c})}_z,\tau^{(\mathsf{d})}_w$, this condition translates to that $\tau^{(\mathsf{a})}_x,\tau^{(\mathsf{b})}_y,\tau^{(\mathsf{c})}_z,\tau^{(\mathsf{d})}_w$ must be pairwise equal to two sequences which do not share any common points. For the rest of the proof, when we say pairwise equal we will always mean pairwise equal to two distinct sequences which do not share any common points. We will first focus on sequences $\tau^{(\mathsf{a})}_x,\tau^{(\mathsf{b})}_y,\tau^{(\mathsf{c})}_z,\tau^{(\mathsf{d})}_w$, which do not satisfy this condition and show that their contribution is negligible.

Consider sequences $\tau^{(\mathsf{a})}_x,\tau^{(\mathsf{b})}_y,\tau^{(\mathsf{c})}_z,\tau^{(\mathsf{d})}_w$ and let $\tau := \tau^{(\mathsf{a})}_x\cup\tau^{(\mathsf{b})}_y\cup\tau^{(\mathsf{c})}_z\cup\tau^{(\mathsf{d})}_w = (f_i,h_i)_{1\leqslant i\leqslant|\tau|}$ with $f_1\leqslant f_2\leqslant\cdots\leqslant f_{|\tau|}$. Let $1\leqslant i_\star\leqslant|\tau|$ be the first index, so that for all $(u,\mathsf{s})\in\{(x,\mathsf{a}),(y,\mathsf{b}),(z,\mathsf{c}),(w,\mathsf{d})\}$, the sequences $\tau^{(\mathsf{s})}_u\cap\big([1,f_{i_\star})\times\mathbb{Z}^d\big)$ are pairwise equal, but this fails to hold for $\tau^{(\mathsf{s})}_u\cap\big([1,f_{i_\star}]\times\mathbb{Z}^d\big)$, see figures 2.1.1, 2.1.2.

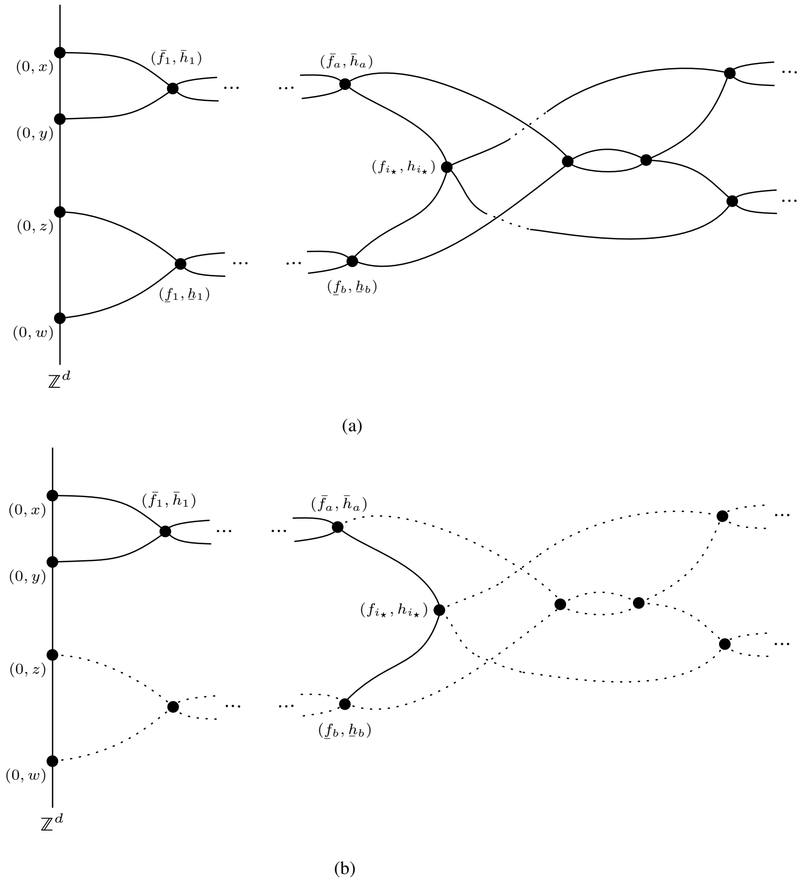


FIGURE 2.1.1. (a) A sample T_1 configuration. The walks start matching in pairs ($x \leftrightarrow y, z \leftrightarrow w$), but then switch pair at $(f_{i_\star}, h_{i_\star})$. (b) The same configuration after summation of all the possible values of the points $(f_i, h_i)_{i > i_\star}$, of the initial positions $(0, z), (0, w)$ and of all the points $(\underline{f}_i, \underline{h}_i)_{1 \leqslant i < b}$.

If there does not exist such index $1 \leqslant i_\star \leqslant |\tau|$, then the four sequences $\tau_x^{(\mathsf{a})}, \tau_y^{(\mathsf{b})}, \tau_z^{(\mathsf{c})}, \tau_w^{(\mathsf{d})}$ have to be pairwise equal. Their contribution to (2.1.12) is

$$3N^{d-2} \sum_{1 \leqslant \mathsf{a},\mathsf{b} \leqslant M} t_{\mathsf{a}}^2 t_{\mathsf{b}}^2 \, \sigma^{2(\mathsf{a}+\mathsf{b})} \times \sum_{x,y,z,w \in \mathbb{Z}^d} \frac{\varphi_N(x,y,z,w)}{N^{2d}} \sum_{\substack{\tau_x^{(\mathsf{a})} = \tau_y^{(\mathsf{a})},\, \tau_w^{(\mathsf{b})} = \tau_z^{(\mathsf{b})}, \\ \tau_x^{(\mathsf{a})} \cap \tau_z^{(\mathsf{b})} = \varnothing}} q(\tau_x^{(\mathsf{a})})\, q(\tau_y^{(\mathsf{a})})\, q(\tau_w^{(\mathsf{b})})\, q(\tau_z^{(\mathsf{b})}) \,. \tag{2.1.15}$$

The factor 3 accounts for the number of ways we can pair the sequences $\tau_x^{(\mathsf{a})}, \tau_y^{(\mathsf{b})}, \tau_z^{(\mathsf{c})}, \tau_w^{(\mathsf{d})}$. The sum in (2.1.15) equals $3N^{d-2}\, \mathbb{E}\Big[\big(\sum_{k=1}^M t_k \bar{Z}_{N,\beta}^{(k)}(\varphi) \big)^2 \Big]^2 + o(1)$ as $N \to \infty$. The $o(1)$ factor is a consequence of the restriction $\tau_x^{(\mathsf{a})} \cap \tau_z^{(\mathsf{b})} \neq \varnothing$ in (2.1.15), which excludes configurations of the four random walk paths such that four walks meet simultaneously at a single point. It is part of the proof below to show that the contribution of these configurations is negligible in the large N limit.

Hence, for now we can focus on the cases for which such a point $(f_{i_\star}, h_{i_\star})$ exists and show that their contribution is negligible for (2.1.12).

We distinguish the following cases for such sequences $\tau_x^{(\mathsf{a})}, \tau_y^{(\mathsf{b})}, \tau_z^{(\mathsf{c})}, \tau_w^{(\mathsf{d})}$:

- **Type 1** (T_1). For all $(u,\mathsf{s}) \in \{(x,\mathsf{a}),(y,\mathsf{b}),(z,\mathsf{c}),(w,\mathsf{d})\}$, we have $\tau_u^{(\mathsf{s})} \cap \big([1,f_{i_\star}) \times \mathbb{Z}^d\big) \neq \varnothing$.

- **Type 2** (T_2). For exactly two of the points $(u,\mathsf{s}) \in \{(x,\mathsf{a}),(y,\mathsf{b}),(z,\mathsf{c}),(w,\mathsf{d})\}$, we have that $\tau_u^{(\mathsf{s})} \cap \big([1,f_{i_\star}) \times \mathbb{Z}^d\big) \neq \varnothing$.

- **Type 3** (T_3). For all $(u,\mathsf{s}) \in \{(x,\mathsf{a}),(y,\mathsf{b}),(z,\mathsf{c}),(w,\mathsf{d})\}$ we have that $\tau_u^{(\mathsf{s})} \cap \big([1,f_{i_\star}) \times \mathbb{Z}^d\big) = \varnothing$.

Note that we have not included the case that three of the sets $\tau_u^{(\mathsf{s})} \cap \big([1,f_{i_\star}) \times \mathbb{Z}^d\big)$ are non-empty. This is because, in this case, by the definition of $i_\star$, we have that $\tau_u^{(\mathsf{s})} \cap \big([1,f_{i_\star}) \times \mathbb{Z}^d\big)$ have to be pairwise equal, therefore all four of them are non-empty. Thus, this is the case of T_1 sequences.

(T_1 **sequences**). We begin with the case of T_1 sequences $\tau_x^{(\mathsf{a})}, \tau_y^{(\mathsf{b})}, \tau_z^{(\mathsf{c})}, \tau_w^{(\mathsf{d})}$. In this case, the four random walks meet pairwise without switching their pair before time $f_{i_\star}$. Let us suppose at first that the walk starting from $(0,x)$ is paired to the walk starting from $(0,y)$ and the walk starting from $(0,z)$ is paired to the walk starting from $(0,w)$, that is

$$\tau_x^{(\mathsf{a})} \cap \big([1,f_{i_\star}) \times \mathbb{Z}^d\big) = \tau_y^{(\mathsf{b})} \cap \big([1,f_{i_\star}) \times \mathbb{Z}^d\big)$$

and

$$\tau_z^{(\mathsf{c})} \cap \big([1,f_{i_\star}) \times \mathbb{Z}^d\big) = \tau_w^{(\mathsf{d})} \cap \big([1,f_{i_\star}) \times \mathbb{Z}^d\big)\,.$$

We shall refer to this type of sequences as $\mathsf{T}_1^{x\leftrightarrow y}$. Analogously, we define $\mathsf{T}_1^{x\leftrightarrow z}$ and $\mathsf{T}_1^{x\leftrightarrow w}$. By symmetry it only suffices to consider $\mathsf{T}_1^{x\leftrightarrow y}$. We will first show how we can perform the summation

$$N^{d-2} \sum_{x,y,z,w\in\mathbb{Z}^d} \frac{\varphi_N(x,y,z,w)}{N^{2d}}\, \sigma^{\mathsf{a}+\mathsf{b}+\mathsf{c}+\mathsf{d}} \\ \times \sum_{\tau_x^{(\mathsf{a})},\tau_y^{(\mathsf{b})},\tau_z^{(\mathsf{c})},\tau_w^{(\mathsf{d})} \in \mathsf{T}_1^{x\leftrightarrow y}} \prod_{\substack{(u,\mathsf{s})\in\{(x,\mathsf{a}),(y,\mathsf{b}),\\(z,\mathsf{c}),(w,\mathsf{d})\}}} q(\tau_u^{(\mathsf{s})})\, \mathbb{E}\Bigg[\prod_{\substack{(u,\mathsf{s})\in\{(x,\mathsf{a}),(y,\mathsf{b}),\\(z,\mathsf{c}),(w,\mathsf{d})\}}} \xi(\tau_u^{(\mathsf{s})})\Bigg]. \tag{2.1.16}$$

Since the ξ variables have to be paired to each other, we can bound the expectation in (2.1.16) as

$$\mathbb{E}\Bigg[\prod_{\substack{(u,\mathsf{s})\in\{(x,\mathsf{a}),(y,\mathsf{b}),\\(z,\mathsf{c}),(w,\mathsf{d})\}}} \xi(\tau_u^{(\mathsf{s})})\Bigg] \leqslant C^{2M}\,, \qquad\qquad C = \max\Big\{1, \mathbb{E}[\xi^3], \mathbb{E}[\xi^4]\Big\}\,. \tag{2.1.17}$$

Moreover, since M is fixed and $1 \leqslant \mathsf{a},\mathsf{b},\mathsf{c},\mathsf{d} \leqslant M$ we have that $\sigma^{\mathsf{a}+\mathsf{b}+\mathsf{c}+\mathsf{d}} \leqslant (\sigma \vee 1)^{4M}$. Therefore,

$$
\begin{aligned}
&N^{d-2} \sum_{x,y,z,w\in\mathbb{Z}^d} \frac{\varphi_N(x,y,z,w)}{N^{2d}}\, \sigma^{\mathsf{a}+\mathsf{b}+\mathsf{c}+\mathsf{d}} \\
&\qquad\qquad \times \sum_{\tau_x^{(\mathsf{a})},\tau_y^{(\mathsf{b})},\tau_z^{(\mathsf{c})},\tau_w^{(\mathsf{d})}\in \mathsf{T}_1^{x\leftrightarrow y}} \prod_{\substack{(u,\mathsf{s})\in\{(x,\mathsf{a}),(y,\mathsf{b}),\\ (z,\mathsf{c}),(w,\mathsf{d})\}}} q(\tau_u^{(\mathsf{s})})\, \mathbb{E}\Bigg[\prod_{\substack{(u,\mathsf{s})\in\{(x,\mathsf{a}),(y,\mathsf{b}),\\ (z,\mathsf{c}),(w,\mathsf{d})\}}} \xi(\tau_u^{(\mathsf{s})}) \Bigg] \\
&\leqslant C^{2M}(\sigma\vee 1)^{4M} N^{d-2} \sum_{x,y,z,w\in\mathbb{Z}^d} \frac{\varphi_N(x,y,z,w)}{N^{2d}} \sum_{\tau_x^{(\mathsf{a})},\tau_y^{(\mathsf{b})},\tau_z^{(\mathsf{c})},\tau_w^{(\mathsf{d})}\in \mathsf{T}_1^{x\leftrightarrow y}} \prod_{\substack{(u,\mathsf{s})\in\{(x,\mathsf{a}),(y,\mathsf{b}),\\ (z,\mathsf{c}),(w,\mathsf{d})\}}} q(\tau_u^{(\mathsf{s})})\,.
\end{aligned}
\tag{2.1.18}
$$

By the definition of T_1 sequences, we have that for a given $\mathsf{T}_1^{x\leftrightarrow y}$ sequence $\tau_x^{(\mathsf{a})}, \tau_y^{(\mathsf{b})}, \tau_z^{(\mathsf{c})}, \tau_w^{(\mathsf{d})}$, with $\tau = \tau_x^{(\mathsf{a})} \cup \tau_y^{(\mathsf{b})} \cup \tau_z^{(\mathsf{c})} \cup \tau_w^{(\mathsf{d})} = (f_i,h_i)_{1\leqslant i\leqslant p}$ and $p = |\tau|$, we can decompose the sequence $(f_i,h_i)_{1\leqslant i< i_\star}$ into two disjoint subsequences $(\bar f_1,\bar h_1),\dots,(\bar f_a,\bar h_a)$ and $(\underline{f}_1,\underline{h}_1),\dots,(\underline{f}_b,\underline{h}_b)$, see Figure 2.1.1, so that

$$
\begin{aligned}
\prod_{\substack{(u,\mathsf{s})\in\{(x,\mathsf{a}),(y,\mathsf{b}),\\ (z,\mathsf{c}),(w,\mathsf{d})\}}} q(\tau_u^{(\mathsf{s})}) &= q_{\bar f_1}(\bar h_1 - x) q_{\bar f_1}(\bar h_1 - y) \prod_{i=2}^{a} q^2_{(\bar f_i - \bar f_{i-1})}(\bar h_i - \bar h_{i-1}) \\
&\quad \times q_{\underline{f}_1}(\underline{h}_1 - z) q_{\underline{f}_1}(\underline{h}_1 - w) \prod_{i=2}^{b} q^2_{(\underline{f}_i - \underline{f}_{i-1})}(\underline{h}_i - \underline{h}_{i-1}) \\
&\quad \times q^{\nu_a}_{(f_{i_\star} - \bar f_a)}(h_{i_\star} - \bar h_a)\, q^{\nu_b}_{(f_{i_\star} - \underline{f}_b)}(h_{i_\star} - \underline{h}_b) \\
&\quad \times \prod_{m=1}^{\mathsf{m}_{i_\star+1}} q_{f_{i_\star+1} - f^{(i_\star+1)}_{r_m}}(h_{i_\star+1} - h^{(i_\star+1)}_{r_m}) \dots \prod_{m=1}^{\mathsf{m}_p} q_{f_p - f^{(p)}_{r_m}}(h_p - h^{(p)}_{r_m})\,.
\end{aligned}
\tag{2.1.19}
$$

For every $i_\star + 1 \leqslant j \leqslant p$, the number m_j ranges from 2 to 4 and indicates whether (f_j,h_j) is a double, triple or quadruple matching. Furthermore, for every $i_\star + 1 \leqslant j \leqslant p$ and $1 \leqslant m \leqslant \mathsf{m}_j$, $(f^{(j)}_{r_m}, h^{(j)}_{r_m})$ is some space-time point which belongs to the sequence $(f_i,h_i)_{i_\star\leqslant i\leqslant p} \cup \{(\bar f_a,\bar h_a),(\underline{f}_b,\underline{h}_b)\}$, such that $f^{(j)}_{r_m} < f_j$. Also, the exponents ν_a,ν_b in (2.1.19) can take values in $\{1,2\}$ and indicate whether the matching in $(f_{i_\star},h_{i_\star})$ was double, triple or quadruple. In any case the product above is bounded by the corresponding expression for $\nu_a,\nu_b = 1$, since we have $q_n(x) \leqslant 1$.

In order to perform the summation in (2.1.16) for $\mathsf{T}_1^{x\leftrightarrow y}$ sequences we make the following observation. We can start by summing the last point (f_p,h_p) as follows: We use the fact that

$q_n(x) \leqslant 1$ and Cauchy-Schwarz to obtain that

$$
\begin{aligned}
&\sum_{(f_p,h_p)} \prod_{m=1}^{\mathsf{m}_p} q_{f_p - f^{(p)}_{r_m}}(h_p - h^{(p)}_{r_m}) \leqslant \sum_{(f_p,h_p)} q_{f_p - f^{(p)}_{r_1}}(h_p - h^{(p)}_{r_1}) q_{f_p - f^{(p)}_{r_2}}(h_p - h^{(p)}_{r_2}) \\
\leqslant& \Big(\sum_{(f_p,h_p)} q^2_{f_p - f^{(p)}_{r_1}}(h_p - h^{(p)}_{r_1}) \Big)^{\frac{1}{2}} \Big(\sum_{(f_p,h_p)} q^2_{f_p - f^{(p)}_{r_2}}(h_p - h^{(p)}_{r_2}) \Big)^{\frac{1}{2}} \\
=& \Big(\sum_{f_p} q_{2(f_p - f^{(p)}_{r_1})}(0) \Big)^{\frac{1}{2}} \Big(\sum_{f_p} q_{2(f_p - f^{(p)}_{r_2})}(0) \Big)^{\frac{1}{2}} \\
\leqslant& (\sqrt{R_N})^2 = R_N \leqslant R_\infty = \frac{\pi_d}{1-\pi_d} < 1 \,.
\end{aligned}
\tag{2.1.20}
$$

For the last inequality, we used that the range of $f_p - f^{(p)}_{r_i}$ is contained in $\{1, 2, ..., N\}$ and the fact that, $\pi_d < \frac{1}{2}$ for $d \geqslant 3$, since $\pi_3 \approx 0.34$, see [Sp76], and $\pi_{d+1} < \pi_d$ for $d \geqslant 3$, see [OS96]. We can successively iterate this estimate for all values of (f_i, h_i) as long as $i > i_\star$. Therefore, by recalling (2.1.16), (2.1.18) and (2.1.19) we deduce that

$$
\begin{aligned}
&(\sigma \vee 1)^{4M} C^{2M} N^{d-2} \sum_{x,y,z,w \in \mathbb{Z}^d} \frac{\varphi_N(x,y,z,w)}{N^{2d}} \sum_{\tau_x^{(\mathsf{a})}, \tau_y^{(\mathsf{b})}, \tau_z^{(\mathsf{c})}, \tau_w^{(\mathsf{d})} \in \mathsf{T}_1^{x \leftrightarrow y}} \prod_{\substack{(u,\mathsf{s}) \in \{(x,\mathsf{a}),(y,\mathsf{b}), \\ (z,\mathsf{c}),(w,\mathsf{d})\}}} q(\tau_u^{(\mathsf{s})}) \\
\leqslant& \, c_M \, (\sigma \vee 1)^{4M} \, C^{2M} N^{d-2} \sum_{x,y,z,w \in \mathbb{Z}^d} \frac{\varphi_N(x,y,z,w)}{N^{2d}} \\
\times& \sum_{a,b=1}^{2M} \Big(\sum_{(\bar{f}_i, \bar{h}_i)_{1 \leqslant i \leqslant a}} q_{\bar{f}_1}(\bar{h}_1 - x) q_{\bar{f}_1}(\bar{h}_1 - y) \prod_{i=2}^{a} q^2_{(\bar{f}_i - \bar{f}_{i-1})}(\bar{h}_i - \bar{h}_{i-1}) \Big) \\
\times& \Big(\sum_{(\underline{f}_i, \underline{h}_i)_{1 \leqslant i \leqslant b}} q_{\underline{f}_1}(\underline{h}_1 - z) q_{\underline{f}_1}(\underline{h}_1 - w) \prod_{i=2}^{b} q^2_{(\underline{f}_i - \underline{f}_{i-1})}(\underline{h}_i - \underline{h}_{i-1}) \Big) \\
\times& \Big(\sum_{(f_{i_\star}, h_{i_\star})} q_{(f_{i_\star} - \bar{f}_a)}(h_{i_\star} - \bar{h}_a) q_{(f_{i_\star} - \underline{f}_b)}(h_{i_\star} - \underline{h}_b) \Big) \,,
\end{aligned}
\tag{2.1.21}
$$

where c_M is a constant combinatorial factor which bounds the number of different ways that the points of $\mathsf{T}_1^{x \leftrightarrow y}$ can be mapped to a fixed sequence $(f_i, h_i)_{1 \leqslant i \leqslant p}$, for all $p \leqslant \frac{\mathsf{a}+\mathsf{b}+\mathsf{c}+\mathsf{d}}{2} \leqslant 2M$. Therefore, the last step for showing that the sum (2.1.16) has negligible contribution in (2.1.12) is to show that for all fixed a, b the following sum vanishes when N goes to infinity:

$$
\begin{aligned}
\widetilde{C}_M N^{d-2} \sum_{x,y,z,w \in \mathbb{Z}^d} \frac{\varphi_N(x,y,z,w)}{N^{2d}} & \Big(\sum_{(\bar{f}_i, \bar{h}_i)_{1 \leqslant i \leqslant a}} q_{\bar{f}_1}(\bar{h}_1 - x) q_{\bar{f}_1}(\bar{h}_1 - y) \prod_{i=2}^{a} q^2_{(\bar{f}_i - \bar{f}_{i-1})}(\bar{h}_i - \bar{h}_{i-1}) \Big) \\
\times & \Big(\sum_{(\underline{f}_i, \underline{h}_i)_{1 \leqslant i \leqslant b}} q_{\underline{f}_1}(\underline{h}_1 - z) q_{\underline{f}_1}(\underline{h}_1 - w) \prod_{i=2}^{b} q^2_{(\underline{f}_i - \underline{f}_{i-1})}(\underline{h}_i - \underline{h}_{i-1}) \Big) \\
\times & \Big(\sum_{(f_{i_\star}, h_{i_\star})} q_{(f_{i_\star} - \bar{f}_a)}(h_{i_\star} - \bar{h}_a) q_{(f_{i_\star} - \underline{f}_b)}(h_{i_\star} - \underline{h}_b) \Big) \,,
\end{aligned}
\tag{2.1.22}
$$

where $\widetilde{C}_M = c_M \, (\sigma \vee 1)^{4M} \, C^{2M}$. Let us describe how this can be done. In (2.1.22), we can bound $\varphi(\frac{z}{\sqrt{N}})\varphi(\frac{w}{\sqrt{N}})$ by $\|\varphi\|_\infty^2$ and sum out z, w using that $\sum_{u \in \mathbb{Z}^d} q_n(u) = 1$ so that we bound

(2.1.22) by

$$\frac{\widetilde{C}_M \|\varphi\|_\infty^2}{N^2} \sum_{x,y\in\mathbb{Z}^d} \frac{\varphi_N(x,y)}{N^d}\Big(\sum_{(\bar f_i,\bar h_i)_{1\leqslant i\leqslant a}} q_{\bar f_1}(\bar h_1 - x) q_{\bar f_1}(\bar h_1 - y) \prod_{i=2}^{a} q^2_{(\bar f_i - \bar f_{i-1})}(\bar h_i - \bar h_{i-1})\Big)$$
$$\times \Big(\sum_{(\underline{f}_i,\underline{h}_i)_{1\leqslant i\leqslant b}} \prod_{i=2}^{b} q^2_{(\underline{f}_i - \underline{f}_{i-1})}(\underline{h}_i - \underline{h}_{i-1})\Big)$$
$$\times \Big(\sum_{(f_{i_\star},h_{i_\star})} q_{(f_{i_\star}-\bar f_a)}(h_{i_\star} - \bar h_a) q_{(f_{i_\star}-\underline{f}_b)}(h_{i_\star} - \underline{h}_b)\Big) . \tag{2.1.23}$$

We sum out all points $(\underline{f}_{i-1}, \underline{h}_{i-1})_{2\leqslant i<b}$ successively, starting from $(\underline{f}_1, \underline{h}_1)$ and moving forward. The contribution of each of these summations is bounded by $R_N < 1$, since for each $2 \leqslant i < b$,

$$\sum_{(\underline{f}_{i-1},\underline{h}_{i-1})} q^2_{(\underline{f}_i - \underline{f}_{i-1})}(\underline{h}_i - \underline{h}_{i-1}) = \sum_{\underline{f}_{i-1}} q_{2(\underline{f}_i - \underline{f}_{i-1})}(0) \leqslant R_N < 1 . \tag{2.1.24}$$

because the range of $\underline{f}_i - \underline{f}_{i-1}$ is contained in $\{1, ..., N\}$. Therefore, we are left with estimating

$$\frac{\widetilde{C}_M \|\varphi\|_\infty^2}{N^2} \sum_{x,y\in\mathbb{Z}^d} \frac{\varphi_N(x,y)}{N^d}\Big(\sum_{(\bar f_i,\bar h_i)_{1\leqslant i\leqslant a}} q_{\bar f_1}(\bar h_1 - x) q_{\bar f_1}(\bar h_1 - y) \prod_{i=2}^{a} q^2_{(\bar f_i - \bar f_{i-1})}(\bar h_i - \bar h_{i-1})\Big)$$
$$\times \Big(\sum_{(f_{i_\star},h_{i_\star})} \sum_{(\underline{f}_b,\underline{h}_b)} q_{(f_{i_\star}-\bar f_a)}(h_{i_\star} - \bar h_a) q_{(f_{i_\star}-\underline{f}_b)}(h_{i_\star} - \underline{h}_b)\Big) .$$

The contribution of the sums over $(\underline{f}_b, \underline{h}_b)$ and $(f_{i_\star}, h_{i_\star})$ is

$$\sum_{(f_{i_\star},h_{i_\star})} q_{(f_{i_\star}-\bar f_a)}(h_{i_\star} - \bar h_a) \sum_{(\underline{f}_b,\underline{h}_b)} q_{(f_{i_\star}-\underline{f}_b)}(h_{i_\star} - \underline{h}_b) \leqslant N^2 . \tag{2.1.25}$$

by summing first over space, using that $\sum_{u\in\mathbb{Z}^d} q_n(u) = 1$ and then summing over time using that the range of $f_{i_\star} - \bar f_a$ and $f_{i_\star} - \underline{f}_b$ is contained in $\{1, ..., N\}$. Therefore, it remains to show that the following sum vanishes as $N \to \infty$:

$$\widetilde{C}_M \|\varphi\|_\infty^2 \sum_{x,y\in\mathbb{Z}^d} \frac{\varphi_N(x,y)}{N^d}\Big(\sum_{(\bar f_i,\bar h_i)_{1\leqslant i\leqslant a}} q_{\bar f_1}(\bar h_1 - x) q_{\bar f_1}(\bar h_1 - y) \prod_{i=2}^{a} q^2_{(\bar f_i - \bar f_{i-1})}(\bar h_i - \bar h_{i-1})\Big) .$$

We perform the summation over $(\bar f_i, \bar h_i)$ for $2 \leqslant i \leqslant a$ starting from $(\bar f_a, \bar h_a)$ and moving backward. The contribution of each of these summations is bounded by $R_N < 1$. Consequently, we need to show that

$$\widetilde{C}_M \|\varphi\|_\infty^2 \sum_{x,y\in\mathbb{Z}^d} \frac{\varphi_N(x,y)}{N^d} \sum_{(\bar f_1,\bar h_1)} q_{\bar f_1}(\bar h_1 - x) q_{\bar f_1}(\bar h_1 - y) \xrightarrow[N\to\infty]{} 0 .$$

By summing out the points $\bar h_1 \in \mathbb{Z}^d$ it suffices to show that

$$\widetilde{C}_M \|\varphi\|_\infty^2 \sum_{x,y\in\mathbb{Z}^d} \frac{\varphi_N(x,y)}{N^d} \sum_{\bar f_1} q_{2\bar f_1}(x - y) \xrightarrow[N\to\infty]{} 0 .$$

But it follows from Lemma 2.1.2 that the last sum is $O(N^{1-\frac{d}{2}})$ hence vanishes as $N \to \infty$, since $d \geqslant 3$. Therefore, we have proved that the sum (2.1.16) vanishes as $N \to \infty$. It is exactly the same to prove the analogous sums for $\mathsf{T}_1^{x\leftrightarrow z}$ and $\mathsf{T}_1^{x\leftrightarrow w}$ sequences vanish as $N \to \infty$.

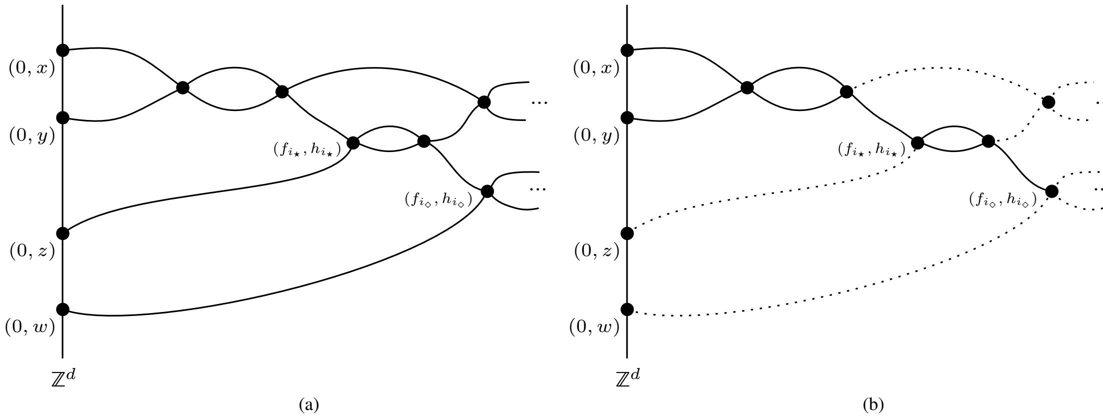


FIGURE 2.1.2. (a) A sample T_2 configuration. (b) The same configuration after summation of all possible values of the points $(f_i, h_i)_{i>i_\diamond}$ and of the initial positions $(0,z),(0,w)$.

(T_2 **sequences)** Recall that by the definition of T_2 sequences we have that for exactly two of the points $(u,\mathsf{s}) \in \{(x,\mathsf{a}),(y,\mathsf{b}),(z,\mathsf{c}),(w,\mathsf{d})\}$, it holds for the corresponding sets $\tau_u^{(\mathsf{s})} \cap \big([1,f_{i_\star}) \times \mathbb{Z}^d\big) \neq \varnothing$ that

$$\tau_x^{(\mathsf{a})} \cap \big([1,f_{i_\star}) \times \mathbb{Z}^d\big) = \tau_y^{(\mathsf{b})} \cap \big([1,f_{i_\star}) \times \mathbb{Z}^d\big) \neq \varnothing$$

and

$$\tau_z^{(\mathsf{c})} \cap \big([1,f_{i_\star}) \times \mathbb{Z}^d\big) = \tau_w^{(\mathsf{d})} \cap \big([1,f_{i_\star}) \times \mathbb{Z}^d\big) = \varnothing\,.$$

We will refer to this type of T_2 sequences as $\mathsf{T}_2^{x\leftrightarrow y}$. Analogously, we can define $\mathsf{T}_2^{x\leftrightarrow z}$ and $\mathsf{T}_2^{x\leftrightarrow w}$. We will show that the sum

$$N^{d-2} \sum_{x,y,z,w\in\mathbb{Z}^d} \frac{\varphi_N(x,y,z,w)}{N^{2d}}\, \sigma^{\mathsf{a}+\mathsf{b}+\mathsf{c}+\mathsf{d}} \sum_{\tau_x^{(\mathsf{a})},\tau_y^{(\mathsf{b})},\tau_z^{(\mathsf{c})},\tau_w^{(\mathsf{d})}\in\mathsf{T}_2^{x\leftrightarrow y}} \prod_{\substack{(u,\mathsf{s})\in\{(x,\mathsf{a}),(y,\mathsf{b}),\\(z,\mathsf{c}),(w,\mathsf{d})\}}} q(\tau_u^{(\mathsf{s})})\, \mathbb{E}\Bigg[\prod_{\substack{(u,\mathsf{s})\in\{(x,\mathsf{a}),(y,\mathsf{b}),\\(z,\mathsf{c}),(w,\mathsf{d})\}}} \xi(\tau_u^{(\mathsf{s})})\Bigg] \tag{2.1.26}$$

vanishes as $N\to\infty$. By using (2.1.17) and the bound $\sigma^{\mathsf{a}+\mathsf{b}+\mathsf{c}+\mathsf{d}} \leqslant (\sigma\vee 1)^{4M}$ we obtain that

$$\begin{aligned} &N^{d-2} \sum_{x,y,z,w\in\mathbb{Z}^d} \frac{\varphi_N(x,y,z,w)}{N^{2d}}\, \sigma^{\mathsf{a}+\mathsf{b}+\mathsf{c}+\mathsf{d}} \\ &\qquad\times \sum_{\tau_x^{(\mathsf{a})},\tau_y^{(\mathsf{b})},\tau_z^{(\mathsf{c})},\tau_w^{(\mathsf{d})}\in\mathsf{T}_2^{x\leftrightarrow y}} \prod_{\substack{(u,\mathsf{s})\in\{(x,\mathsf{a}),(y,\mathsf{b}),\\(z,\mathsf{c}),(w,\mathsf{d})\}}} q(\tau_u^{(\mathsf{s})})\, \mathbb{E}\Bigg[\prod_{\substack{(u,\mathsf{s})\in\{(x,\mathsf{a}),(y,\mathsf{b}),\\(z,\mathsf{c}),(w,\mathsf{d})\}}} \xi(\tau_u^{(\mathsf{s})})\Bigg] \\ &\leqslant C^{2M}(\sigma\vee 1)^{4M} N^{d-2} \sum_{x,y,z,w\in\mathbb{Z}^d} \frac{\varphi_N(x,y,z,w)}{N^{2d}} \sum_{\tau_x^{(\mathsf{a})},\tau_y^{(\mathsf{b})},\tau_z^{(\mathsf{c})},\tau_w^{(\mathsf{d})}\in\mathsf{T}_2^{x\leftrightarrow y}} \prod_{\substack{(u,\mathsf{s})\in\{(x,\mathsf{a}),(y,\mathsf{b}),\\(z,\mathsf{c}),(w,\mathsf{d})\}}} q(\tau_u^{(\mathsf{s})})\,. \end{aligned} \tag{2.1.27}$$

By the definition of $(f_{i_\star},h_{i_\star})$ we have that $(f_{i_\star},h_{i_\star})$ is the first point of at least one of the sequences $\tau_z^{(\mathsf{c})},\tau_w^{(\mathsf{d})}$. Let us assume that it is the first point of exactly one of them. We will refer to this type of sequences, $\tau_x^{(\mathsf{a})},\tau_y^{(\mathsf{b})},\tau_z^{(\mathsf{c})},\tau_w^{(\mathsf{d})}$, as $\mathsf{T}_{2,\diamond}^{x\leftrightarrow y}$ sequences, see figure 2.1.2. Without loss of generality, we may assume that $(f_{i_\star},h_{i_\star})$ is the first point of $\tau_z^{(\mathsf{c})}$. In that case, $(f_{i_\star},h_{i_\star})$ can be a double or triple matching. Let $(f_{i_\diamond}, h_{i_\diamond})$ be the first point of $\tau_w^{(\mathsf{d})}$. We have that $f_{i_\star} \leqslant f_{i_\diamond}$. Therefore, we first show that

$$\begin{aligned} &(\sigma \vee 1)^{4M} C^{2M} N^{d-2} \sum_{x,y,z,w \in \mathbb{Z}^d} \frac{\varphi_N(x,y,z,w)}{N^{2d}} \\ &\qquad \times \sum_{\tau_x^{(\mathsf{a})}, \tau_y^{(\mathsf{b})}, \tau_z^{(\mathsf{c})}, \tau_w^{(\mathsf{d})} \in \mathsf{T}_{2,\diamond}^{x \leftrightarrow y}} \prod_{\substack{(u,\mathsf{s}) \in \{(x,\mathsf{a}),(y,\mathsf{b}), \\ (z,\mathsf{c}),(w,\mathsf{d})\}}} q(\tau_u^{(\mathsf{s})}) \xrightarrow[N\to\infty]{} 0\,. \end{aligned} \tag{2.1.28}$$

Similarly to the case of T_1 sequences, for given $\mathsf{T}_{2,\diamond}^{x\leftrightarrow y}$ sequences $\tau_x^{(\mathsf{a})}, \tau_y^{(\mathsf{b})}, \tau_z^{(\mathsf{c})}, \tau_w^{(\mathsf{d})}$ with $\tau = \tau_x^{(\mathsf{a})} \cup \tau_y^{(\mathsf{b})} \cup \tau_z^{(\mathsf{c})} \cup \tau_w^{(\mathsf{d})} = (f_i, h_i)_{1 \leqslant i \leqslant p}$ and $p = |\tau|$, the cardinality of τ, we have that (see Figure 2.1.2)

$$\begin{aligned} \prod_{\substack{(u,\mathsf{s}) \in \{(x,\mathsf{a}),(y,\mathsf{b}), \\ (z,\mathsf{c}),(w,\mathsf{d})\}}} q(\tau_u^{(\mathsf{s})}) &= q_{\bar f_1}(\bar h_1 - x) q_{\bar f_1}(\bar h_1 - y) \prod_{i=2}^{a} q^2_{(\bar f_i - \bar f_{i-1})}(\bar h_i - \bar h_{i-1}) \\ &\quad \times q^{\nu_a}_{(f_{i_\star} - \bar f_a)}(h_{i_\star} - \bar h_a)\, q_{f_{i_\star}}(h_{i_\star} - z)\, q_{f_{i_\diamond}}(h_{i_\diamond} - w) \\ &\quad \times \prod_{m=1}^{\mathsf{m}_{i_\star+1}} q_{f_{i_\star+1} - f^{(i_\star+1)}_{r_m}}(h_{i_\star+1} - h^{(i_\star+1)}_{r_m}) \dots \prod_{m=1}^{\mathsf{m}_{i_\diamond}-1} q_{f_{i_\diamond} - f^{(i_\diamond)}_{r_m}}(h_{i_\diamond} - h^{(i_\diamond)}_{r_m}) \\ &\quad \times \prod_{m=1}^{\mathsf{m}_{i_\diamond+1}} q_{f_{i_\diamond+1} - f^{(i_\diamond+1)}_{r_m}}(h_{i_\diamond+1} - h^{(i_\diamond+1)}_{r_m}) \dots \prod_{m=1}^{\mathsf{m}_p} q_{f_p - f^{(p)}_{r_m}}(h_p - h^{(p)}_{r_m})\,, \end{aligned} \tag{2.1.29}$$

where, for every $i_\star + 1 \leqslant j \leqslant p$, the number m_j ranges from 2 to 4 and indicates whether (f_j, h_j) was a double, triple or quadruple matching. Also, for every $i_\star + 1 \leqslant j \leqslant p$ and $1 \leqslant m \leqslant \mathsf{m}_j$, $(f^{(j)}_{r_m}, h^{(j)}_{r_m})$ is some space-time point which belongs to the sequence $(f_i, h_i)_{i_\star \leqslant i \leqslant p} \cup \{(\bar f_a, \bar h_a)\}$, such that $f^{(j)}_{r_m} < f_j$. However, note that in the third line of (2.1.29), the product for $(f_{i_\diamond}, h_{i_\diamond})$ runs from $m = 1$ to $\mathsf{m}_{i_\diamond} - 1$, since $q_{f_{i_\diamond}}(h_{i_\diamond} - w)$ appears in the second line. The exponent ν_a in the second line of (2.1.29) can take values 1 or 2 and indicates whether $(f_{i_\star}, h_{i_\star})$ is a double or triple matching; it cannot be a quadruple matching since we assumed that it is contained only in $\tau_z^{(c)}$ and not in $\tau_w^{(d)}$. In any case, we can bound $q^{\nu_a}_{(f_{i_\star} - \bar f_a)}(h_{i_\star} - \bar h_a)$ by $q_{(f_{i_\star} - \bar f_a)}(h_{i_\star} - \bar h_a)$.

We first make some observations so that the presentation is more concise. By iterating (2.1.20) we obtain that

$$\sum_{(f_{i_\diamond+1}, h_{i_\diamond+1})} \prod_{m=1}^{\mathsf{m}_{i_\diamond+1}} q_{f_{i_\diamond+1} - f^{(i_\diamond+1)}_{r_m}}(h_{i_\diamond+1} - h^{(i_\diamond+1)}_{r_m}) \dots \sum_{(f_p, h_p)} \prod_{m=1}^{\mathsf{m}_p} q_{f_p - f^{(p)}_{r_m}}(h_p - h^{(p)}_{r_m}) \leqslant 1\,. \tag{2.1.30}$$

We also have that

$$\begin{aligned} \sum_{w \in \mathbb{Z}^d} \frac{\varphi_N(w)}{N^{\frac{d}{2}}} q_{f_{i_\diamond}}(h_{i_\diamond} - w) &= \frac{1}{N^{\frac{d}{2}}} \sum_{w \in \mathbb{Z}^d} \varphi\big(\tfrac{w}{\sqrt{N}}\big)\, q_{f_{i_\diamond}}(h_{i_\diamond} - w) \\ &\leqslant \frac{\|\varphi\|_\infty}{N^{\frac{d}{2}}} \sum_{w \in \mathbb{Z}^d} q_{f_{i_\diamond}}(h_{i_\diamond} - w) = \frac{\|\varphi\|_\infty}{N^{\frac{d}{2}}}\,, \end{aligned} \tag{2.1.31}$$

and then we can sum

$$\sum_{(f_{i_\diamond}, h_{i_\diamond})} \prod_{m=1}^{\mathsf{m}_{i_\diamond}-1} q_{f_{i_\diamond} - f^{(i_\diamond)}_{r_m}}(h_{i_\diamond} - h^{(i_\diamond)}_{r_m}) \leqslant \sum_{(f_{i_\diamond}, h_{i_\diamond})} q_{f_{i_\diamond} - f^{(i_\diamond)}_{r_1}}(h_{i_\diamond} - h^{(i_\diamond)}_{r_1}) \leqslant N\,, \tag{2.1.32}$$

Having summed out the points $(f_i,h_i)_{i\geqslant i_\diamond}$, we can iterate estimate (2.1.20) again to obtain that

$$\begin{aligned}\sum_{(f_{i_\star+1},h_{i_\star+1})}\prod_{m=1}^{\mathsf{m}_{i_\star+1}} q_{f_{i_\star+1}-f^{(i_\star+1)}_{r_m}}(h_{i_\star+1}-h^{(i_\star+1)}_{r_m})\,\dots\\ \times\sum_{(f_{i_\diamond-1},h_{i_\diamond-1})}\prod_{m=1}^{\mathsf{m}_{i_\diamond-1}} q_{f_{i_\diamond-1}-f^{(i_\diamond-1)}_{r_m}}(h_{i_\diamond-1}-h^{(i_\diamond-1)}_{r_m})\leqslant 1\,.\end{aligned}\tag{2.1.33}$$

Therefore, in view of (2.1.29), (2.1.28) and by using (2.1.30), (2.1.31), (2.1.32) and (2.1.33) in their respective order, we get that

$$\begin{aligned}&(\sigma\vee 1)^{4M}C^{2M}N^{d-2}\sum_{x,y,z,w\in\mathbb{Z}^d}\frac{\varphi_N(x,y,z,w)}{N^{2d}}\sum_{\tau_x^{(\mathsf{a})},\tau_y^{(\mathsf{b})},\tau_z^{(\mathsf{c})},\tau_w^{(\mathsf{d})}\in\mathsf{T}_{2,\diamond}^{x\leftrightarrow y}}\prod_{\substack{(u,\mathsf{s})\in\{(x,\mathsf{a}),(y,\mathsf{b}),\\(z,\mathsf{c}),(w,\mathsf{d})\}}}q(\tau_u^{(\mathsf{s})})\\
&\leqslant\|\varphi\|_\infty\,c_{M,\diamond}\,(\sigma\vee 1)^{4M}\,C^{2M}N^{\frac d2-1}\sum_{x,y,z\in\mathbb{Z}^d}\frac{\varphi_N(x,y,z)}{N^{\frac{3d}{2}}}\\
&\times\sum_{a=1}^{2M}\Big(\sum_{(\bar f_i,\bar h_i)_{1\leqslant i\leqslant a}}q_{\bar f_1}(\bar h_1-x)q_{\bar f_1}(\bar h_1-y)\prod_{i=2}^{a}q^2_{(\bar f_i-\bar f_{i-1})}(\bar h_i-\bar h_{i-1})\Big)\\
&\times\Big(\sum_{(f_{i_\star},h_{i_\star})}q_{(f_{i_\star}-\bar f_a)}(h_{i_\star}-\bar h_a)\,q_{f_{i_\star}}(h_{i_\star}-z)\Big)\,,\end{aligned}$$

where $c_{M,\diamond}$ is a constant combinatorial factor which bounds the number of possible assignments of $\mathsf{T}_{2,\diamond}^{x\leftrightarrow y}$ sequences, $\tau_x^{(\mathsf{a})},\tau_y^{(\mathsf{b})},\tau_z^{(\mathsf{c})},\tau_w^{(\mathsf{d})}$ to $(f_i,h_i)_{1\leqslant i\leqslant p}$. We set $\widetilde{C}_{M,\diamond}:=c_{M,\diamond}\,(\sigma\vee 1)^{4M}\,C^{2M}$. In order to establish (2.1.28), we need to show that for all fixed $a\leqslant 2M$

$$\begin{aligned}\|\varphi\|_\infty\,\widetilde{C}_{M,\diamond}\,N^{\frac d2-1}\times\Big(\sum_{(\bar f_i,\bar h_i)_{1\leqslant i\leqslant a}}q_{\bar f_1}(\bar h_1-x)q_{\bar f_1}(\bar h_1-y)\prod_{i=2}^{a}q^2_{(\bar f_i-\bar f_{i-1})}(\bar h_i-\bar h_{i-1})\Big)\\ \times\Big(\sum_{(f_{i_\star},h_{i_\star})}q_{(f_{i_\star}-\bar f_a)}(h_{i_\star}-\bar h_a)\,q_{f_{i_\star}}(h_{i_\star}-z)\Big)\xrightarrow[N\to\infty]{}0\,.\end{aligned}$$

In analogy to (2.1.31), we have that

$$\sum_{z\in\mathbb{Z}^d}\frac{\varphi_N(z)}{N^{\frac d2}}\,q_{f_{i_\star}}(h_{i_\star}-z)\leqslant\frac{\|\varphi\|_\infty}{N^{\frac d2}}\,.$$

Furthermore, by summing over $(f_{i_\star},h_{i_\star})$ we deduce that

$$\sum_{(f_{i_\star},h_{i_\star})}q_{(f_{i_\star}-\bar f_a)}(h_{i_\star}-\bar h_a)\leqslant N\,,$$

since the spatial sum is equal to 1 and $f_{i_\star}-\bar f_a\in\{1,\dots,N\}$. Therefore, the last step in order to establish (2.1.28) is to show that

$$\widetilde{C}_{M,\diamond}\,\|\varphi\|^2_\infty\sum_{x,y\in\mathbb{Z}^d}\frac{\varphi_N(x,y)}{N^d}\sum_{(\bar f_i,\bar h_i)_{1\leqslant i\leqslant a}}q_{\bar f_1}(\bar h_1-x)q_{\bar f_1}(\bar h_1-y)\prod_{i=2}^{a}q^2_{(\bar f_i-\bar f_{i-1})}(\bar h_i-\bar h_{i-1})\xrightarrow[N\to\infty]{}0\,.$$

By summing over the points $(\bar f_i,\bar h_i)_{2\leqslant i\leqslant a}$, this amounts to proving that

$$\widetilde{C}_{M,\diamond}\,\|\varphi\|^2_\infty\sum_{x,y\in\mathbb{Z}^d}\frac{\varphi_N(x,y)}{N^d}\sum_{\bar f_1}q_{2\bar f_1}(x-y)\xrightarrow[N\to\infty]{}0\,,$$

which is true by Lemma 2.1.2. The same procedure can be followed for sequences of type $\mathsf{T}_{2,\diamond}^{x\leftrightarrow z}$ and $\mathsf{T}_{2,\diamond}^{x\leftrightarrow w}$. So, this concludes the estimate for $\mathsf{T}_2^{x\leftrightarrow y}$ sequences in the case that $(f_{i_\star},h_{i_\star})$ is the first point of only one of the sequences $\tau_z^{(\mathsf{c})}, \tau_w^{(\mathsf{d})}$ and by symmetry also for the analogous cases for $\mathsf{T}_2^{x\leftrightarrow z}$ and $\mathsf{T}_2^{x\leftrightarrow w}$.

Let us treat the case where $(f_{i_\star}, h_{i_\star})$ is the first point of both sequences $\tau_z^{(\mathsf{c})}, \tau_w^{(\mathsf{d})}$. Then, $(f_{i_\star}, h_{i_\star})$ is a triple or quadruple matching, i.e. either $(f_{i_\star}, h_{i_\star}) \in \tau_x^{(\mathsf{a})}, \tau_z^{(\mathsf{c})}, \tau_w^{(\mathsf{d})}$, or $(f_{i_\star}, h_{i_\star}) \in \tau_y^{(\mathsf{b})}, \tau_z^{(\mathsf{c})}, \tau_w^{(\mathsf{d})}$, or $(f_{i_\star}, h_{i_\star}) \in \tau_x^{(\mathsf{a})}, \tau_y^{(\mathsf{b})}, \tau_z^{(\mathsf{c})}, \tau_w^{(\mathsf{d})}$. Both cases can be treated as we did for T_1 sequences. Namely, we can first restrict ourselves to the sequence $(f_i, h_i)_{1 \leqslant i \leqslant i_\star}$ by using the bound we used in (2.1.20). After following the procedure we described for T_1 sequences we get that the sum in this case is either $O(N^{-\frac{d}{2}})$ if $(f_{i_\star}, h_{i_\star})$ is a triple matching and $O(N^{-1-\frac{d}{2}})$ when $(f_{i_\star}, h_{i_\star})$ is a quadruple matching. Thus, in total the contribution of T_2 sequences to (2.1.12), is $O(N^{1-\frac{d}{2}})$.

(T_3 **sequences**). For all $(u, \mathsf{s}) \in \{(x, \mathsf{a}), (y, \mathsf{b}), (z, \mathsf{c}), (w, \mathsf{d})\}$ we have that $\tau_u^{(\mathsf{s})} \cap \big([1, f_{i_\star}) \times \mathbb{Z}^d\big) = \varnothing$. This implies that $i_\star = 1$ and $(f_{i_\star}, h_{i_\star})$ is a triple or quadruple matching. It is easy to see, using the technique for T_1 and T_2 sequences, that the contribution of T_3 sequences to (2.1.12) is $O(N^{-\frac{d}{2}})$.

Therefore, we have showed that the part of the sum (2.1.12) which is over sequences of Type 1 (T_1), Type 2 (T_2) or Type 3 (T_3) is negligible in the $N \to \infty$ limit. Thus, the proof is complete.

□

2.1.4. Proof of the CLT.

Proof of Theorem 2.0.1. By Proposition 2.1.4 we obtain that $N^{\frac{d-2}{4}} \bar{Z}_{N,\beta}^{\leqslant M}(\varphi)$ converges in distribution to a centred Gaussian random variable $\mathcal{G}_M$ as $N \to \infty$, with variance equal to

$$\mathbb{V}\mathrm{ar}\left[\mathcal{G}_M\right] = \sum_{k=1}^{M} \mathcal{C}_\beta^{(k)} \int_0^1 \mathrm{d}t \int_{\mathbb{R}^{2d}} \mathrm{d}x\, \mathrm{d}y\, \varphi(x) g_{\frac{2t}{d}}(x-y) \varphi(y).$$

We also have that

$$\lim_{M \to \infty} \mathbb{V}\mathrm{ar}\left[\mathcal{G}_M\right] = \sum_{k=1}^{\infty} \mathcal{C}_\beta^{(k)} \int_0^1 \mathrm{d}t \int_{\mathbb{R}^{2d}} \mathrm{d}x\, \mathrm{d}y\, \varphi(x) g_{\frac{2t}{d}}(x-y) \varphi(y) = \mathbb{V}\mathrm{ar}\, \mathcal{Z}_\beta(\varphi)\,,$$

where $\mathcal{Z}_\beta(\varphi)$ is the random variable defined by Theorem 2.0.1, since

$$\sum_{k=1}^{\infty} \mathcal{C}_\beta^{(k)} = \sigma^2(\beta) \sum_{k=1}^{\infty} \sigma(\beta)^{2(k-1)} \sum_{0 := \ell_0 < \ell_1 < \ldots < \ell_{k-1}} \prod_{i=1}^{k-1} q_{2(\ell_i - \ell_{i-1})}(0) = \sigma^2(\beta) \mathrm{E}[e^{\lambda_2(\beta) \mathsf{L}_\infty}]\,.$$

Combining this with Lemma 2.1.3, we obtain the conclusion of Theorem 2.0.1, that is

$$N^{\frac{d-2}{4}} \bar{Z}_{N,\beta}(\varphi) \xrightarrow[N \to \infty]{(d)} \mathcal{Z}_\beta(\varphi)\,.$$

□

2.2. Edwards-Wilkinson fluctuations for the log-partition function

In this section we prove Theorem 2.0.2, namely, the Edwards-Wilkinson fluctuations for the log-partition function.

We will need to impose one more condition to the random environment for technical reasons. Specifically, we require that the law of the random environment satisfies a concentration inequality. In particular, we assume that there exists an exponent $\gamma > 1$ and constants $C_1, C_2 > 0$, such that for every $n \in \mathbb{N}$, 1-Lipschitz function $f : \mathbb{R}^n \to \mathbb{R}$ and i.i.d. random variables $\omega_1, \ldots, \omega_n$ having law $\mathbb{P}$, we have that

$$\mathbb{P}\Big(\big|f(\omega_1,...,\omega_n) - M_f\big| \geqslant t\Big) \leqslant C_1 \exp\Big(-\frac{t^\gamma}{C_2}\Big), \tag{2.2.1}$$

where M_f denotes a median of $f(\omega_1,...,\omega_n)$. One can replace the median by $\mathbb{E}[f(\omega_1,...,\omega_n)]$, by changing the constants C_1, C_2 appropriately. Condition (2.2.1) is satisfied if ω is bounded or has a density of the form $\exp(-V(\cdot)+U(\cdot))$, where V is uniformly strictly convex and U is bounded, see [Led01].

Condition (2.2.1) allows us to formulate the following estimate. For $\Lambda \subset \mathbb{N}\times\mathbb{Z}^d$, let $Z^\Lambda_{N,\beta}(x)$ denote the partition function which contains disorder only from Λ, that is

$$Z^\Lambda_{N,\beta}(x) = \mathrm{E}_x\left[e^{\sum_{(n,z)\in\Lambda}\{\beta\omega_{n,z}-\lambda(\beta)\}\mathbb{1}_{S_n=z}}\right].$$

Then, we have the following Proposition:

Proposition 2.2.1 (Left-tail estimate)**.** *For every $\beta \in (0,\beta_{L^2})$ there exists a constant $c_\beta > 0$, such that: for every $N \in \mathbb{N}$, $\Lambda \subset \mathbb{N}\times\mathbb{Z}^d$, one has that $\forall t \geqslant 0$*

$$\mathbb{P}\Big(\log Z^\Lambda_{N,\beta}(x) \leqslant -t\Big) \leqslant c_\beta \exp\Big(-\frac{t^\gamma}{c_\beta}\Big),$$

where γ, is the exponent in (2.2.1)*.*

As a consequence of Proposition 2.2.1 we also get the following boundedness of moments.

Proposition 2.2.2. *For every $\beta \in (0,\beta_{L^2})$, $\Lambda \subset \mathbb{N}\times\mathbb{Z}^d$ and $p \geqslant 0$,*

$$\sup_{N\in\mathbb{N}} \mathbb{E}\Big[\big(Z^\Lambda_{N,\beta}(x)\big)^{-p}\Big] < \infty,$$

$$\sup_{N\in\mathbb{N}} \mathbb{E}\Big[\big|\log Z^\Lambda_{N,\beta}(x)\big|^p\Big] < \infty.$$

We refer to [CSZ20] for the proofs of Propositions 2.2.1, 2.2.2, as the method presented there can be followed exactly to give those results in our case. For Proposition 2.2.1 see also [CTT17], where this method appeared in the context of pinning models.

We will also need the existence of higher than 2 moments for the partition function. This can be established with the use of hypercontractivity, for which we refer to Section 3 of [CSZ20] for a detailed exposition. In particular, we have the following proposition:

Proposition 2.2.3. *For every $\beta \in (0,\beta_{L^2})$ and $\Lambda \subset \mathbb{N}\times\mathbb{Z}^d$ there exists $p = p(\beta) \in (2,\infty)$, such that*

$$\sup_{N\in\mathbb{N}} \mathbb{E}\Big[\big(Z^\Lambda_{N,\beta}(x)\big)^p\Big] < \infty.$$

Let us proceed to the sketch of the proof for the Edwards-Wilkinson fluctuations for the log-partition function. For every $x \in \mathbb{Z}^d$ we define a microscopic space-time window around x as follows

$$A^x_N = \Big\{(n,z):\ 1 \leqslant n \leqslant N^\varepsilon, |x-z| < N^{\frac{\varepsilon}{2}+\alpha_\varepsilon}\Big\}, \tag{2.2.2}$$

for $\varepsilon \in \big(\frac{7}{8},1\big)$ and $\alpha_\varepsilon = \varepsilon\cdot\delta_\varepsilon$ with $\delta_\varepsilon \in \big(0,\frac{1-\varepsilon}{8}\big)$. In particular, $\alpha_\varepsilon \in \big(0,\frac{\varepsilon}{64}\big)$. We decompose the partition function as:

$$Z_{N,\beta}(x) = Z^A_{N,\beta}(x) + \hat{Z}^A_{N,\beta}(x),$$

where

$$Z^A_{N,\beta}(x) = \mathrm{E}_x\left[e^{\sum_{(n,z)\in A^x_N}\{\beta\omega_{n,z}-\lambda(\beta)\}\mathbb{1}_{S_n=z}}\right],$$

is the partition function which contains disorder indexed only from the set A_N^x, while the remainder, $\hat{Z}^A_{N,\beta}(x) = Z_{N,\beta}(x) - Z^A_{N,\beta}(x)$, necessarily contains disorder from points outside of A_N^x in its chaos decomposition, see also [CSZ20], Section 2, for analoguous definitions. The chaos expansions of $Z^A_{N,\beta}(x), \hat{Z}^A_{N,\beta}(x)$ are

$$Z^A_{N,\beta}(x) = 1 + \sum_{k \geqslant 1} \sigma^k \sum_{(n_i,z_i)_{1\leqslant i\leqslant k} \subset A_N^x} q_{n_1}(z_1 - x) \prod_{i=2}^k q_{n_i - n_{i-1}}(z_i - z_{i-1}) \prod_{i=1}^k \xi_{n_i,z_i}\,, \tag{2.2.3}$$

and

$$\hat{Z}^A_{N,\beta}(x) = \sum_{k \geqslant 1} \sigma^k \sum_{(n_i,z_i)_{1\leqslant i\leqslant k} \cap (A_N^x)^c \neq \varnothing} q_{n_1}(z_1 - x) \prod_{i=2}^k q_{n_i - n_{i-1}}(z_i - z_{i-1}) \prod_{i=1}^k \xi_{n_i,z_i}\,. \tag{2.2.4}$$

We can then write, for every $x \in \mathbb{Z}^d$,

$$\log Z_{N,\beta}(x) = \log Z^A_{N,\beta}(x) + \log\left(1 + \frac{\hat{Z}^A_{N,\beta}(x)}{Z^A_{N,\beta}(x)}\right). \tag{2.2.5}$$

The first step we take is to show that the contribution of the term $\log Z^A_{N,\beta}(x)$ to the fluctuations of $\log Z_{N,\beta}(x)$ is negligible, when averaged over x, in the following sense

Proposition 2.2.4. *Let $\varphi \in C_c(\mathbb{R}^d)$ be a test function. Then,*

$$N^{\frac{d-2}{4}} \sum_{x \in \mathbb{Z}^d} \frac{\varphi_N(x)}{N^{\frac{d}{2}}} \Big(\log Z^A_{N,\beta}(x) - \mathbb{E}\big[\log Z^A_{N,\beta}(x)\big]\Big) \xrightarrow[N\to\infty]{L^2(\mathbb{P})} 0\,. \tag{2.2.6}$$

The second step is to prove that we can replace $\log\left(1 + \frac{\hat{Z}^A_{N,\beta}(x)}{Z^A_{N,\beta}(x)}\right)$ by $\frac{\hat{Z}^A_{N,\beta}(x)}{Z^A_{N,\beta}(x)}$. In particular, if we define

$$O_N(x) := \log\left(1 + \frac{\hat{Z}^A_{N,\beta}(x)}{Z^A_{N,\beta}(x)}\right) - \frac{\hat{Z}^A_{N,\beta}(x)}{Z^A_{N,\beta}(x)}\,,$$

then we will show that

Proposition 2.2.5. *Let $\varphi \in C_c(\mathbb{R}^d)$ be a test function. Then,*

$$N^{\frac{d-2}{4}} \sum_{x \in \mathbb{Z}^d} \frac{\varphi_N(x)}{N^{\frac{d}{2}}} \Big(O_N(x) - \mathbb{E}[O_N(x)]\Big) \xrightarrow[N\to\infty]{L^1(\mathbb{P})} 0\,.$$

Therefore, we need to identify the fluctuations of the quotient $\hat{Z}^A_{N,\beta}(x)/Z^A_{N,\beta}(x)$. Note that this quantity has mean zero since each term in the chaos expansion of $\hat{Z}^A_{N,\beta}(x)$ contains disorder outside A_N^x, see (2.2.4). To study the fluctuations of $\hat{Z}^A_{N,\beta}(x)/Z^A_{N,\beta}(x)$ we define, for a suitable $\varrho \in (\varepsilon, 1)$, the set

$$B_N^{\geqslant} = \big([N^{\varrho}, N] \cap \mathbb{N}\big) \times \mathbb{Z}^d\,, \tag{2.2.7}$$

and show, employing the local limit theorem for random walks, that the asymptotic factorisation $\hat{Z}^A_{N,\beta}(x) \approx Z^A_N(x)\big(Z^{B^{\geqslant}}_{N,\beta}(x) - 1\big)$ takes place when we average over x, namely

Proposition 2.2.6. *Let $\varphi \in C_c(\mathbb{R}^d)$ be a test function. Then,*

$$N^{\frac{d-2}{4}} \sum_{x \in \mathbb{Z}^d} \frac{\varphi_N(x)}{N^{\frac{d}{2}}} \left(\frac{\hat{Z}^A_{N,\beta}(x)}{Z^A_{N,\beta}(x)} - \big(Z^{B^{\geqslant}}_{N,\beta}(x) - 1\big)\right) \xrightarrow[N\to\infty]{L^1(\mathbb{P})} 0\,.$$

The last step is to show that the fluctuations of $Z^{B^{\geqslant}}_{N,\beta}(x) - 1$ when averaged over x, are Gaussian with variance equal to that of Theorem 2.0.1, namely

Proposition 2.2.7. *Let $\varphi \in C_c(\mathbb{R}^d)$ be a test function. Then, we have the following convergence in distribution,*

$$N^{\frac{d-2}{4}} \sum_{x \in \mathbb{Z}^d} \frac{\varphi_N(x)}{N^{\frac{d}{2}}} \left(Z^{B^{\geqslant}}_{N,\beta}(x) - 1\right) \xrightarrow[N\to\infty]{(d)} \mathcal{Z}_\beta(\varphi)\,,$$

where $\mathcal{Z}_\beta(\varphi)$ is the centred normal random variable appearing in Theorem 2.0.1.

2.2.1. The contribution of $\log Z^A_{N,\beta}$ through martingale difference decomposition. We begin with the proof of Proposition 2.2.4.

Proof of Proposition 2.2.4. It suffices to restrict the summation and show that

$$N^{\frac{d}{2}-1} \sum_{|x-y| \leqslant 2N^{\frac{\varepsilon}{2}+\alpha_\varepsilon}} \frac{\varphi_N(x,y)}{N^d} \operatorname{Cov}\left(\log Z^A_{N,\beta}(x), \log Z^A_{N,\beta}(y)\right) \xrightarrow[N\to\infty]{} 0\,, \tag{2.2.8}$$

because, by the definition of the sets A^x_N, if $|x-y| > 2N^{\frac{\varepsilon}{2}+\alpha_\varepsilon}$, then $\log Z^A_{N,\beta}(x)$ and $\log Z^A_{N,\beta}(y)$ are independent, so their covariance is zero. The proof will be divided in four steps.

(Step 1) - Martingale decomposition. We will expand the covariance appearing in (2.2.8) by using a martingale difference decomposition. Let $\{\omega_{a_1}, \omega_{a_2}, ...\}$ be an arbitrary enumeration of the disorder indexed by $\mathbb{N} \times \mathbb{Z}^d$. We can then define a filtration $(\mathcal{F}_j)_{j\geqslant 1}$, such that $\mathcal{F}_j = \sigma(\omega_{a_1}, ..., \omega_{a_j})$. We define also $\mathcal{F}_0 = \{\varnothing, \Omega\}$, where Ω is the underlying sample space where the random variables $(\omega_{n,z})_{(n,z)\in\mathbb{N}\times\mathbb{Z}^d}$, are defined. Using this filtration we can write the difference $\log Z^A_{N,\beta}(x) - \mathbb{E}[\log Z^A_{N,\beta}(x)]$ as a telescoping sum, namely

$$\log Z^A_{N,\beta}(x) - \mathbb{E}\big[\log Z^A_{N,\beta}(x)\big] = \sum_{j\geqslant 1}\Big(\mathbb{E}[\log Z^A_{N,\beta}(x)|\mathcal{F}_j] - \mathbb{E}[\log Z^A_{N,\beta}(x)|\mathcal{F}_{j-1}]\Big)\,. \tag{2.2.9}$$

Then, using the shorthand notation $D_{j,N}(x) = \mathbb{E}[\log Z^A_{N,\beta}(x)|\mathcal{F}_j] - \mathbb{E}[\log Z^A_{N,\beta}(x)|\mathcal{F}_{j-1}]$ we have that:

$$\operatorname{Cov}\left(\log Z^A_{N,\beta}(x), \log Z^A_{N,\beta}(y)\right) = \sum_{k,j\geqslant 1} \mathbb{E}[D_{k,N}(x) D_{j,N}(y)]\,.$$

In fact, all the non-diagonal terms in the above sum are zero, since, if $j < k$,

$$\mathbb{E}\big[D_{j,N}(x)D_{k,N}(y)\big] = \mathbb{E}\Big[\mathbb{E}\big[D_{j,N}(x)D_{k,N}(y)|\mathcal{F}_j\big]\Big] = \mathbb{E}\Big[D_{j,N}(x)\,\mathbb{E}\big[D_{k,N}(y)|\mathcal{F}_j\big]\Big] = 0\,,$$

because $D_{j,N}(x)$ is $\mathcal{F}_j$-measurable and also

$$\begin{aligned}
\mathbb{E}\big[D_{k,N}(y)|\mathcal{F}_j\big] &= \mathbb{E}\Big[\mathbb{E}\big[\log Z^A_{N,\beta}(y)|\mathcal{F}_k\big]\Big|\mathcal{F}_j\Big] - \mathbb{E}\Big[\mathbb{E}\big[\log Z^A_{N,\beta}(y)|\mathcal{F}_{k-1}\big]\Big|\mathcal{F}_j\Big] \\
&= \mathbb{E}\big[\log Z^A_{N,\beta}(y)|\mathcal{F}_j\big] - \mathbb{E}\big[\log Z^A_{N,\beta}(y)|\mathcal{F}_j\big] \\
&= 0\,,
\end{aligned}$$

since $\mathcal{F}_j \subset \mathcal{F}_{k-1}, \mathcal{F}_k$. Therefore, we can rewrite the sum in (2.2.8) as

$$N^{\frac{d}{2}-1} \sum_{|x-y|\leqslant 2N^{\frac{\varepsilon}{2}+\alpha_\varepsilon}} \frac{\varphi_N(x,y)}{N^d} \sum_{j\geqslant 1} \mathbb{E}[D_{j,N}(x)D_{j,N}(y)]\,. \tag{2.2.10}$$

One has to make an important observation at this point. If a_j is not contained in A^x_N, then $D_{j,N}(x) = 0$. Hence, the rightmost expectation in (2.2.10) is non-zero only for $j \geqslant 1$, such that $a_j \in A^x_N \cap A^y_N$.

(Step 2) - Resampling. In this step we derive a closed form of the martingale differences $D_{j,N}(x)$. In particular, let $j \geqslant 1$ such that $a_j \in A_N^x \cap A_N^y$. Then, we claim that

$$D_{j,N}(x) = \int \prod_{k>j} \mathbb{P}(\mathrm{d}\omega_{a_k}) \int \mathbb{P}(\mathrm{d}\widetilde{\omega}_{a_j}) \big[\log Z_{N,\beta}^{A}(x) - \log Z_{N,\beta}^{A,\mathsf{T}_{a_j}}(x) \big] , \tag{2.2.11}$$

where $Z_{N,\beta}^{A,\mathsf{T}_{a_j}}(x)$ denotes the partition function in the environment $\{\omega_{a_k}\}_{k\neq j} \cup \widetilde{\omega}_{a_j}$, where $\widetilde{\omega}_{a_j}$ is an independent copy of ω_{a_j}.

Note that if $f(\omega)$ is a function of the i.i.d. family of random variables $\omega = \{\omega_{a_k}\}_{k=1}^{\infty}$ and $\mathcal{F}_j = \sigma\big(\{\omega_{a_k}\}_{1\leqslant k\leqslant j}\big)$, then

$$\mathbb{E}\big[f \,|\mathcal{F}_j \big] = \int \prod_{k>j} \mathbb{P}(\mathrm{d}\omega_{a_k})\, f(\omega) \,. \tag{2.2.12}$$

Applying this observation to $\log Z_{N,\beta}^{A}(x)$ we obtain that

$$\mathbb{E}\big[\log Z_{N,\beta}^{A}(x) | \mathcal{F}_j \big] = \int \prod_{k>j} \mathbb{P}(\mathrm{d}\omega_{a_k}) \, \log Z_{N,\beta}^{A}(x) \,.$$

By resampling ω_{a_j} with an independent copy $\widetilde{\omega}_{a_j}$ we can also write

$$\mathbb{E}\big[\log Z_{N,\beta}^{A}(x) | \mathcal{F}_{j-1} \big] = \int \prod_{k>j} \mathbb{P}(\mathrm{d}\omega_{a_k}) \int \mathbb{P}(\mathrm{d}\widetilde{\omega}_{a_j}) \, \log Z_{N,\beta}^{A,\mathsf{T}_{a_j}}(x) \,.$$

Therefore,

$$\begin{aligned} D_{j,N}(x) &= \int \prod_{k>j} \mathbb{P}(\mathrm{d}\omega_{a_k}) \Big[\log Z_{N,\beta}^{A}(x) - \int \mathbb{P}(\mathrm{d}\widetilde{\omega}_{a_j}) \log Z_{N,\beta}^{A,\mathsf{T}_{a_j}}(x) \Big] \\ &= \int \prod_{k>j} \mathbb{P}(\mathrm{d}\omega_{a_k}) \int \mathbb{P}(\mathrm{d}\widetilde{\omega}_{a_j}) \big[\log Z_{N,\beta}^{A}(x) - \log Z_{N,\beta}^{A,\mathsf{T}_{a_j}}(x) \big] , \end{aligned} \tag{2.2.13}$$

since $Z_{N,\beta}^{A}(x)$ does not depend on $\widetilde{\omega}_{a_j}$. This concludes the proof of equation (2.2.11). The next step shows how we can remove the logarithms.

(Step 3) - Removing the logarithms. We fix a positive number $\mathsf{h} \in (0, \frac{1-\varepsilon}{2})$ and for $x \in \mathbb{Z}^d$, we define

$$E_j(x) := \Big\{ Z_{N,\beta}^{A}(x), Z_{N,\beta}^{A,\mathsf{T}_{a_j}}(x) \geqslant N^{-\mathsf{h}} \Big\} . \tag{2.2.14}$$

We then decompose $D_{j,N}(x)$ as follows

$$\begin{aligned} D_{j,N}(x) = &\int \prod_{k>j} \mathbb{P}(\mathrm{d}\omega_{a_k}) \int \mathbb{P}(\mathrm{d}\widetilde{\omega}_{a_j}) \big[\log Z_{N,\beta}^{A}(x) - \log Z_{N,\beta}^{A,\mathsf{T}_{a_j}}(x) \big] \mathbb{1}_{E_j(x)} \\ &+ \int \prod_{k>j} \mathbb{P}(\mathrm{d}\omega_{a_k}) \int \mathbb{P}(\mathrm{d}\widetilde{\omega}_{a_j}) \big[\log Z_{N,\beta}^{A}(x) - \log Z_{N,\beta}^{A,\mathsf{T}_{a_j}}(x) \big] \mathbb{1}_{E_j^c(x)} \,. \end{aligned} \tag{2.2.15}$$

We hereafter use the notation $D_{j,N}^{(\mathsf{big})}(x), D_{j,N}^{(\mathsf{small})}(x)$ for the two summands on the right hand side of (2.2.15), respectively. The corresponding superscripts refer to the events $E_j(x)$ (2.2.14). We then have that

$$\sum_{j\geqslant 1\,:\, a_j\in A_N^x\cap A_N^y} \mathbb{E}\Big[D_{j,N}(x) D_{j,N}(y) \Big] = \sum_{\mathsf{l},\mathsf{l}'\in\{\mathsf{big},\mathsf{small}\}} \sum_{j\geqslant 1:\, a_j\in A_N^x\cap A_N^y} \mathbb{E}\big[D_{j,N}^{(\mathsf{l})}(x) D_{j,N}^{(\mathsf{l}')}(y) \big] \tag{2.2.16}$$

and in view of (2.2.8) the rest of the proof will be devoted to showing that every sum in (2.2.16) converges to zero after testing against φ and scaling by $N^{\frac{d}{2}-1}$. We will first prove that

$$\lim_{N\to\infty} N^{\frac{d}{2}-1} \sum_{|x-y|\leqslant 2N^{\frac{\varepsilon}{2}+\alpha\varepsilon}} \frac{\varphi_N(x,y)}{N^d} \sum_{j\geqslant 1\,:\,a_j\in A_N^x\cap A_N^y} \mathbb{E}\Big[D^{(\mathsf{big})}_{j,N}(x)D^{(\mathsf{big})}_{j,N}(y)\Big] = 0\,. \tag{2.2.17}$$

Using that if $x,y\in[t,\infty)$ for some positive $t>0$, then $|\log x-\log y|\leqslant \frac{1}{t}|x-y|$, implies that

$$\begin{aligned} \big|D^{(\mathsf{big})}_{j,N}(x)\big| &\leqslant \int\prod_{k>j}\mathbb{P}(\mathrm{d}\omega_{a_k})\int\mathbb{P}(\mathrm{d}\widetilde{\omega}_{a_j})\big|\log Z^A_{N,\beta}(x)-\log Z^{A,\mathsf{T}_{a_j}}_{N,\beta}(x)\big|\mathbb{1}_{E_j(x)} \\ &\leqslant N^{\mathsf{h}}\int\prod_{k>j}\mathbb{P}(\mathrm{d}\omega_{a_k})\int\mathbb{P}(\mathrm{d}\widetilde{\omega}_{a_j})\big|Z^A_{N,\beta}(x)-Z^{A,\mathsf{T}_{a_j}}_{N,\beta}(x)\big|\mathbb{1}_{E_j(x)} \\ &\leqslant N^{\mathsf{h}}\int\prod_{k>j}\mathbb{P}(\mathrm{d}\omega_{a_k})\int\mathbb{P}(\mathrm{d}\widetilde{\omega}_{a_j})\big|Z^A_{N,\beta}(x)-Z^{A,\mathsf{T}_{a_j}}_{N,\beta}(x)\big|\,, \end{aligned} \tag{2.2.18}$$

where we dropped the indicator function $\mathbb{1}_{E_j(x)}$ to obtain the third inequality. For the sake of the presentation, we adopt the notation

$$w_{j,N}(x) := \int\prod_{k>j}\mathbb{P}(\mathrm{d}\omega_{a_k})\int\mathbb{P}(\mathrm{d}\widetilde{\omega}_{a_j})\big|Z^A_{N,\beta}(x)-Z^{A,\mathsf{T}_{a_j}}_{N,\beta}(x)\big|\,, \tag{2.2.19}$$

omitting the dependence in N. Using the estimate (2.2.18) and summing over $j\geqslant 1$, such that $a_j\in A_N^x\cap A_N^y$ we deduce that

$$\sum_{j\geqslant 1:\,a_j\in A_N^x\cap A_N^y} \mathbb{E}\Big[\big|D^{(\mathsf{big})}_{j,N}(x)D^{(\mathsf{big})}_{j,N}(y)\big|\Big] \leqslant N^{2\mathsf{h}} \sum_{j\geqslant 1\,:\,a_j\in A_N^x\cap A_N^y} \mathbb{E}\Big[w_{j,N}(x)w_{j,N}(y)\Big]. \tag{2.2.20}$$

If we denote by S^x the path of a random walk starting at x we have

$$Z^A_{N,\beta}(x)-Z^{A,\mathsf{T}_{a_j}}_{N,\beta}(x) = \sigma(\beta)(\xi_{a_j}-\widetilde{\xi}_{a_j})\mathrm{E}_x\big[e^{\mathsf{H}^x_{A\setminus a_j}}\mathbb{1}_{a_j\in S^x}\big]\,, \tag{2.2.21}$$

where

$$\mathsf{H}^x_{A\setminus a_j}(\omega) := \sum_{a\in A_N^x\setminus\{a_j\}} \big[\beta\omega_a-\lambda(\beta)\big]\mathbb{1}_{a\in S^x}\,, \tag{2.2.22}$$

and recall from (2.1.2) that

$$\xi_{a_j} = \frac{e^{\beta\omega_{a_j}-\lambda(\beta)}-1}{\sigma(\beta)} \quad\text{ and }\quad \widetilde{\xi}_{a_j} = \frac{e^{\beta\widetilde{\omega}_{a_j}-\lambda(\beta)}-1}{\sigma(\beta)}\,.$$

At this point, we will bound $w_{j,N}(x)$. By (2.2.19) and (2.2.21) we have that

$$\begin{aligned} w_{j,N}(x) &= \int\prod_{k>j}\mathbb{P}(\mathrm{d}\omega_{a_k})\int\mathbb{P}(\mathrm{d}\widetilde{\omega}_{a_j})\,\big|Z^A_{N,\beta}(x)-Z^{A,\mathsf{T}_{a_j}}_{N,\beta}(x)\big| \\ &= \int\prod_{k>j}\mathbb{P}(\mathrm{d}\omega_{a_k})\int\mathbb{P}(\mathrm{d}\widetilde{\omega}_{a_j})\,\sigma(\beta)\,|\xi_{a_j}-\widetilde{\xi}_{a_j}|\,\mathrm{E}_x\big[e^{\mathsf{H}^x_{A\setminus a_j}(\omega)}\mathbb{1}_{a_j\in S^x}\big]\,. \end{aligned}$$

We will perform this integration in steps. The expectation, $\mathrm{E}_x\big[e^{\mathsf{H}^x_{A\setminus a_j}(\omega)}\mathbb{1}_{a_j\in S^x}\big]$, does not depend on $\widetilde{\omega}_{a_j}$ by (2.2.22), and we have by triangle inequality

$$\int\mathbb{P}(\mathrm{d}\widetilde{\omega}_{a_j})\,\sigma(\beta)\,|\xi_{a_j}-\widetilde{\xi}_{a_j}| \leqslant \sigma(\beta)\Big(|\xi_{a_j}|+1\Big)\,. \tag{2.2.23}$$

Furthermore, by exchanging the integral and the expectation we deduce that

$$\int \prod_{k>j} \mathbb{P}(\mathrm{d}\omega_{a_k}) \mathrm{E}_x\big[e^{\mathsf{H}^x_{A\setminus a_j}(\omega)} \mathbb{1}_{a_j \in S^x}\big] = \mathrm{E}_x\big[e^{\mathsf{H}^x_{A\cap\{a_1,\dots,a_{j-1}\}}(\omega)} \mathbb{1}_{a_j\in S^x}\big], \tag{2.2.24}$$

where

$$\mathsf{H}^x_{A\cap\{a_1,\dots,a_{j-1}\}}(\omega) := \sum_{\substack{1\leqslant k\leqslant j-1,\\ a_k\in A^x_N}} \big[\beta\omega_{a_k} - \lambda(\beta)\big]\mathbb{1}_{a_k\in S^x}\,.$$

If $j = 1$, we set the corresponding energy to be equal to 0. Hence, combining (2.2.23) and (2.2.24) we obtain that

$$w_{j,N}(x) \leqslant \sigma(\beta)\Big(|\xi_{a_j}| + 1\Big)\mathrm{E}_x\big[e^{\mathsf{H}^x_{A\cap\{a_1,\dots,a_{j-1}\}}(\omega)}\mathbb{1}_{a_j\in S^x}\big].$$

Therefore, by Fubini we get that

$$w_{j,N}(x)\, w_{j,N}(y) \leqslant \sigma^2(\beta)\Big(|\xi_{a_j}|+1\Big)^2\, \mathrm{E}_{x,y}\big[e^{\mathsf{H}^x_{A\cap\{a_1,\dots,a_{j-1}\}}(\omega)+\mathsf{H}^y_{A\cap\{a_1,\dots,a_{j-1}\}}(\omega)}\mathbb{1}_{a_j\in S^x\cap S^y}\big],$$

which after taking the expectation $\mathbb{E}[\,\cdot\,]$ leads to

$$\mathbb{E}\Big[w_{j,N}(x)\, w_{j,N}(y)\Big] \leqslant 4\sigma^2(\beta)\mathrm{E}_{x,y}\big[e^{\lambda_2(\beta)\mathsf{L}_N(x,y)}\mathbb{1}_{a_j\in S^x\cap S^y}\big]. \tag{2.2.25}$$

Therefore, by summing over $j \geqslant 1$ such that $a_j \in A^x_N \cap A^y_N$ we deduce that

$$\sum_{j\geqslant 1\,:\, a_j\in A^x_N\cap A^y_N} \mathbb{E}\Big[w_{j,N}(x)\, w_{j,N}(y)\Big] \leqslant 4\sigma^2(\beta)\mathrm{E}_{x,y}\big[e^{\lambda_2(\beta)\mathsf{L}_N(x,y)}\mathsf{L}_{N^\varepsilon}(x,y)\big]. \tag{2.2.26}$$

Note that the rightmost overlap, $\mathsf{L}_{N^\varepsilon}(x,y)$, goes up to time N^ε, since by (2.2.2), for every $j\geqslant 1$, such that $a_j \in A^x_N \cap A^y_N$, a_j has time index $t \leqslant N^\varepsilon$, therefore

$$\sum_{j\geqslant 1\,:\, a_j\in A^x_N\cap A^y_N} \mathbb{1}_{a_j\in S^x\cap S^y} \leqslant \sum_{n=1}^{N^\varepsilon} \mathbb{1}_{S^x_n = S^y_n} := \mathsf{L}_{N^\varepsilon}(x,y)\,.$$

Recalling (2.2.20) we get that

$$\sum_{j\geqslant 1\,:\, a_j\in A^x_N\cap A^y_N} \mathbb{E}\Big[\big|D^{(\mathsf{big})}_{j,N}(x) D^{(\mathsf{big})}_{j,N}(y)\big|\Big] \leqslant N^{2\mathsf{h}}\, 4\sigma^2(\beta)\, \mathrm{E}_{x,y}\big[e^{\lambda_2(\beta)\mathsf{L}_N(x,y)}\, \mathsf{L}_{N^\varepsilon}(x,y)\big].$$

So far, we have shown that

$$\begin{aligned} &N^{\frac{d}{2}-1} \sum_{|x-y|\leqslant 2N^{\frac{\varepsilon}{2}+\alpha_\varepsilon}} \frac{\varphi_N(x,y)}{N^d} \sum_{j\geqslant 1\,:\, a_j\in A^x_N\cap A^y_N} \mathbb{E}\Big[D^{(\mathsf{big})}_{j,N}(x) D^{(\mathsf{big})}_{j,N}(y)\Big] \\ &\leqslant 4\sigma^2(\beta)\, N^{\frac{d}{2}-1+2\mathsf{h}} \sum_{|x-y|\leqslant 2N^{\frac{\varepsilon}{2}+\alpha_\varepsilon}} \frac{\varphi_N(x,y)}{N^d}\, \mathrm{E}_{x,y}\big[e^{\lambda_2(\beta)\mathsf{L}_N(x,y)}\mathsf{L}_{N^\varepsilon}(x,y)\big]. \end{aligned} \tag{2.2.27}$$

Therefore, to establish (2.2.17), we derive an upper bound for $\mathrm{E}_{x,y}\big[e^{\lambda_2(\beta)\mathsf{L}_N(x,y)}\mathsf{L}_{N^\varepsilon}(x,y)\big]$. Let us denote by $\tau_{x,y}$ the first meeting time of two independent random walks starting from $x, y \in \mathbb{Z}^d$, respectively. By conditioning on $\tau_{x,y}$ we obtain

$$\mathrm{E}_{x,y}\Big[e^{\lambda_2(\beta)\mathsf{L}_N(x,y)}\mathsf{L}_{N^\varepsilon}(x,y)\Big] = \sum_{n=1}^{N^\varepsilon} \mathrm{E}_{x,y}\Big[e^{\lambda_2(\beta)\mathsf{L}_N(x,y)}\mathsf{L}_{N^\varepsilon}(x,y)|\tau_{x,y}=n\Big]\mathrm{P}(\tau_{x,y}=n)\,.$$

Using the Markov property we obtain

$$\sum_{n=1}^{N^{\varepsilon}} \mathrm{E}_{x,y}\Big[e^{\lambda_2(\beta)\mathsf{L}_N(x,y)}\mathsf{L}_{N^{\varepsilon}}(x,y)|\tau_{x,y}=n\Big]\mathrm{P}(\tau_{x,y}=n)$$
$$= \sum_{n=1}^{N^{\varepsilon}} \mathrm{E}\Big[e^{\lambda_2(\beta)(1+\mathsf{L}_{N-n})}\big(1+\mathsf{L}_{N^{\varepsilon}-n}\big)\Big]\mathrm{P}(\tau_{x,y}=n)\,.$$

For every $1 \leqslant n \leqslant N^{\varepsilon}$, we can bound the expectation

$$\mathrm{E}\Big[e^{\lambda_2(\beta)(1+\mathsf{L}_{N-n})}\big(1+\mathsf{L}_{N^{\varepsilon}-n}\big)\Big] \leqslant e^{\lambda_2(\beta)}\Big(\mathrm{E}\big[e^{\lambda_2(\beta)\mathsf{L}_\infty}\big]+\mathrm{E}\big[e^{\lambda_2(\beta)\mathsf{L}_\infty}\mathsf{L}_\infty\big]\Big) := c(\beta) < \infty\,,$$

because $\beta \in (0, \beta_{L^2})$, see (2.0.3). Moreover, we have that

$$\mathrm{P}(\tau_{x,y}=n) \leqslant \sum_{z\in\mathbb{Z}^d} q_n(z-x)q_n(z-y) = q_{2n}(x-y)\,.$$

Therefore,

$$\mathrm{E}_{x,y}\Big[e^{\lambda_2(\beta)\mathsf{L}_N(x,y)}\mathsf{L}_{N^{\varepsilon}}(x,y)\Big] \leqslant c(\beta)\sum_{n=1}^{N^{\varepsilon}} q_{2n}(x-y)\,. \tag{2.2.28}$$

Recalling (2.2.17), (2.2.27) and (2.2.28), in order to conclude Step 3, we need to show that

$$N^{\frac{d}{2}-1+2\mathsf{h}} \sum_{|x-y|\leqslant 2N^{\frac{\varepsilon}{2}+\alpha_\varepsilon}} \frac{\varphi_N(x,y)}{N^d}\sum_{n=1}^{N^{\varepsilon}} q_{2n}(x-y) \xrightarrow[N\to\infty]{} 0\,.$$

We bound $\varphi(\frac{y}{\sqrt{N}})$ by its supremum norm and use the fact that $\sum_{z\in\mathbb{Z}^d} q_{2n}(z)=1$, to obtain that

$$\begin{aligned} N^{\frac{d}{2}-1+2\mathsf{h}} \sum_{|x-y|\leqslant 2N^{\frac{\varepsilon}{2}+\alpha_\varepsilon}} \frac{\varphi_N(x,y)}{N^d}\sum_{n=1}^{N^{\varepsilon}} q_{2n}(x-y) &\leqslant \|\varphi\|_\infty N^{2\mathsf{h}+\varepsilon-1}\sum_{x\in\mathbb{Z}^d}\frac{\varphi_N(x)}{N^{\frac{d}{2}}} \\ &\leqslant \|\varphi\|_\infty \|\varphi\|_1 N^{2\mathsf{h}+\varepsilon-1}\,. \end{aligned} \tag{2.2.29}$$

Since $\mathsf{h} \in (0, \frac{1-\varepsilon}{2})$, we have that $2\mathsf{h}+\varepsilon<1$, hence the last bound vanishes as $N\to\infty$, which concludes the proof of (2.2.17).

(Step 4) - Events of small partition functions. Let us see how one can treat the rest of the terms in the expansion (2.2.16), which involve the complementary events $E_j^c(x)$, defined in (2.2.14). We need to show that

$$\begin{aligned} N^{\frac{d}{2}-1} \sum_{|x-y|\leqslant 2N^{\frac{\varepsilon}{2}+\alpha_\varepsilon}} \frac{\varphi_N(x,y)}{N^d} \sum_{j\geqslant 1\,:\, a_j\in A_N^x\cap A_N^y} \mathbb{E}\Big[D_{j,N}^{(\mathsf{small})}(x)D_{j,N}^{(\mathsf{big})}(y)\Big] &\xrightarrow[N\to\infty]{} 0\,, \\ N^{\frac{d}{2}-1} \sum_{|x-y|\leqslant 2N^{\frac{\varepsilon}{2}+\alpha_\varepsilon}} \frac{\varphi_N(x,y)}{N^d} \sum_{j\geqslant 1\,:\, a_j\in A_N^x\cap A_N^y} \mathbb{E}\Big[D_{j,N}^{(\mathsf{small})}(x)D_{j,N}^{(\mathsf{small})}(y)\Big] &\xrightarrow[N\to\infty]{} 0\,. \end{aligned} \tag{2.2.30}$$

It suffices to show one of the these results, since all of them can be treated with similar arguments. Let us present for example the proof that

$$\lim_{N\to\infty} N^{\frac{d}{2}-1} \sum_{|x-y|\leqslant 2N^{\frac{\varepsilon}{2}+\alpha_\varepsilon}} \frac{\varphi_N(x,y)}{N^d} \sum_{j\geqslant 1\,:\, a_j\in A_N^x\cap A_N^y} \mathbb{E}\Big[D_{j,N}^{(\mathsf{big})}(x)D_{j,N}^{(\mathsf{small})}(y)\Big] = 0\,.$$

Recall from (2.2.15) that

$$D_{j,N}^{(\mathsf{big})}(x) = \int\prod_{k>j}\mathbb{P}(\mathrm{d}\omega_{a_k})\int\mathbb{P}(\mathrm{d}\widetilde{\omega}_{a_j})\big[\log Z_{N,\beta}^{A}(x) - \log Z_{N,\beta}^{A,\mathsf{T}_{a_j}}(x)\big]\mathbb{1}_{E_j(x)}\,,$$

and
$$D_{j,N}^{(\mathsf{small})}(y) = \int \prod_{k>j} \mathbb{P}(\mathrm{d}\omega_{a_k}) \int \mathbb{P}(\mathrm{d}\widetilde{\omega}_{a_j})\big[\log Z_{N,\beta}^{A}(x) - \log Z_{N,\beta}^{A,\mathsf{T}_{a_j}}(x)\big]\mathbb{1}_{E_j^c(x)}\,.$$
By Cauchy-Schwarz one has that
$$\mathbb{E}\Big[D_{j,N}^{(\mathsf{big})}(x)D_{j,N}^{(\mathsf{small})}(y)\Big] \leqslant \mathbb{E}\Big[\big(D_{j,N}^{(\mathsf{big})}(x)\big)^2\Big]^{\frac{1}{2}}\,\mathbb{E}\Big[\big(D_{j,N}^{(\mathsf{small})}(y)\big)^2\Big]^{\frac{1}{2}}\,.$$
Note that then,
$$\begin{aligned}
&\mathbb{E}\Big[\big(D_{j,N}^{(\mathsf{big})}(x)\big)^2\Big]\\
&\quad= \int \prod_{k\geqslant 1} \mathbb{P}(\mathrm{d}\omega_{a_k})\bigg(\int \prod_{k>j} \mathbb{P}(\mathrm{d}\omega_{a_k}) \int \mathbb{P}(\mathrm{d}\widetilde{\omega}_{a_j})\big[\log Z_{N,\beta}^{A}(x) - \log Z_{N,\beta}^{A,\mathsf{T}_{a_j}}(x)\big]\cdot \mathbb{1}_{E_j(x)}\bigg)^2\\
&\quad\leqslant \int \prod_{k\geqslant 1} \mathbb{P}(\mathrm{d}\omega_{a_k}) \int \mathbb{P}(\mathrm{d}\widetilde{\omega}_{a_j})\big[\log Z_{N,\beta}^{A}(x) - \log Z_{N,\beta}^{A,\mathsf{T}_{a_j}}(x)\big]^2\cdot \mathbb{1}_{E_j(x)}\\
&\quad\leqslant \int \prod_{k\geqslant 1} \mathbb{P}(\mathrm{d}\omega_{a_k}) \int \mathbb{P}(\mathrm{d}\widetilde{\omega}_{a_j})\big[\log Z_{N,\beta}^{A}(x) - \log Z_{N,\beta}^{A,\mathsf{T}_{a_j}}(x)\big]^2
\end{aligned} \tag{2.2.31}$$
by Jensen's inequality and because $\mathbb{1}_{E_j(x)} \leqslant 1$. Therefore, using the elementary inequality $(a-b)^2 \leqslant 2(a^2+b^2)$ and Proposition (2.2.2) we deduce that there exists a constant $C = C(\beta) \in (0,\infty)$ such that
$$\mathbb{E}\Big[\big(D_{j,N}^{(\mathsf{big})}(x)\big)^2\Big]^{\frac{1}{2}} \leqslant \big(2\mathbb{E}\big[(\log Z_{N,\beta}^{A})^2\big]\big)^{\frac{1}{2}} \leqslant C\,, \tag{2.2.32}$$
Similarly, we obtain that
$$\mathbb{E}\Big[\big(D_{j,N}^{(\mathsf{small})}(y)\big)^2\Big] \leqslant \int \prod_{k\geqslant 1} \mathbb{P}(\mathrm{d}\omega_{a_k}) \int \mathbb{P}(\mathrm{d}\widetilde{\omega}_{a_j})\big[\log Z_{N,\beta}^{A}(x) - \log Z_{N,\beta}^{A,\mathsf{T}_{a_j}}(x)\big]^2\cdot \mathbb{1}_{E_j^c(x)}\,.$$
and via Cauchy-Schwarz we deduce that
$$\begin{aligned}
&\mathbb{E}\Big[\big(D_{j,N}^{(\mathsf{small})}(y)\big)^2\Big]\\
&\leqslant \bigg(\int \prod_{k\geqslant 1} \mathbb{P}(\mathrm{d}\omega_{a_k}) \int \mathbb{P}(\mathrm{d}\widetilde{\omega}_{a_j})\big[\log Z_{N,\beta}^{A}(x) - \log Z_{N,\beta}^{A,\mathsf{T}_{a_j}}(x)\big]^4\bigg)^{\frac{1}{2}}\cdot \mathbb{P}_{\omega,\widetilde{\omega}}\big(E_j^c(x)\big)^{\frac{1}{2}}\\
&\leqslant 4\,\mathbb{E}\big[(\log Z_{N,\beta}^{A})^4\big]\,\mathbb{P}_{\omega,\widetilde{\omega}}\big(E_j^c(x)\big)^{\frac{1}{2}}\\
&\leqslant C\,\mathbb{P}_{\omega,\widetilde{\omega}}\big(E_j^c(x)\big)^{\frac{1}{2}}\,,
\end{aligned} \tag{2.2.33}$$
where we used the shorthand notation $\mathbb{P}_{\omega,\widetilde{\omega}}\big(E_j^c(x)\big)$ for
$$\int \prod_{k\geqslant 1} \mathbb{P}(\mathrm{d}\omega_{a_k}) \int \mathbb{P}(\mathrm{d}\widetilde{\omega}_{a_j})\mathbb{1}_{E_j^c(x)}\,.$$
By a union bound we have that
$$\mathbb{P}_{\omega,\widetilde{\omega}}\big(E_j^c(y)\big) \leqslant 2\mathbb{P}\big(Z_{N,\beta}^{A}(y) < N^{-\mathsf{h}}\big) = 2\mathbb{P}\big(Z_{N,\beta}^{A} < N^{-\mathsf{h}}\big)\,.$$
Therefore, by (2.2.32) and (2.2.33) there exists a constant $C = C(\beta) \in (0,\infty)$, such that for all $j \geqslant 1$,
$$\mathbb{E}\Big[D_{j,N}^{(\mathsf{big})}(x)D_{j,N}^{(\mathsf{small})}(y)\Big] \leqslant C\,\mathbb{P}\big(Z_{N,\beta}^{A} < N^{-\mathsf{h}}\big)^{\frac{1}{4}}\,.$$

Substituting the above upper bound to (2.2.30) we obtain that

$$
\begin{aligned}
&N^{\frac{d}{2}-1} \sum_{|x-y|\leqslant 2N^{\frac{\varepsilon}{2}+\alpha_\varepsilon}} \frac{\varphi_N(x,y)}{N^d} \sum_{j\geqslant 1\,:\,a_j\in A_N^x\cap A_N^y} \mathbb{E}\Big[D_{j,N}^{(\mathsf{big})}(x)D_{j,N}^{(\mathsf{small})}(y)\Big] \\
&\leqslant C\,N^{\frac{d}{2}-1} \sum_{|x-y|\leqslant 2N^{\frac{\varepsilon}{2}+\alpha_\varepsilon}} \frac{\varphi_N(x,y)}{N^d} \sum_{j\geqslant 1\,:\,a_j\in A_N^x\cap A_N^y} \mathbb{P}\big(Z_{N,\beta}^A < N^{-\mathsf{h}}\big)^{\frac{1}{4}}\,. \\
&\leqslant C\,\mathbb{P}\big(Z_{N,\beta}^A < N^{-\mathsf{h}}\big)^{\frac{1}{4}}\,N^{\frac{d}{2}-1} \sum_{|x-y|\leqslant 2N^{\frac{\varepsilon}{2}+\alpha_\varepsilon}} \frac{\varphi_N(x,y)}{N^d}\,\big|A_N^x\cap A_N^y\big|
\end{aligned}
$$

From definition (2.2.2), we have $|A_N^x \cap A_N^y| = O\big(N^{\varepsilon+d(\frac{\varepsilon}{2}+\alpha_\varepsilon)}\big) = O\big(N^{1+d}\big)$. We also have that the probability $\mathbb{P}(Z_{N,\beta}^A < N^{-\mathsf{h}})$ decays super-polynomially by Proposition 2.2.1 and so does $\mathbb{P}\big(Z_{N,\beta}^A < N^{-\mathsf{h}}\big)^{\frac{1}{4}}$. Indeed, by Proposition 2.2.1,

$$
\mathbb{P}\Big(Z_{N,\beta}^A < N^{-\mathsf{h}}\Big)^{\frac{1}{4}} \leqslant c_\beta^{\frac{1}{4}} \exp\left(-\frac{(\mathsf{h}\log N)^\gamma}{4c_\beta}\right), \quad \gamma > 1\,,
$$

Thus, we have that

$$
\begin{aligned}
&N^{\frac{d}{2}-1} \sum_{|x-y|\leqslant 2N^{\frac{\varepsilon}{2}+\alpha_\varepsilon}} \frac{\varphi_N(x,y)}{N^d} \sum_{j\geqslant 1\,:\,a_j\in A_N^x\cap A_N^y} \mathbb{E}\Big[D_{j,N}^{(\mathsf{big})}(x)D_{j,N}^{(\mathsf{small})}(y)\Big] \\
&\leqslant C\,\mathbb{P}\big(Z_{N,\beta}^A < N^{-\mathsf{h}}\big)^{\frac{1}{4}}\,|A_N^x\cap A_N^y|\,N^{\frac{d}{2}-1} \sum_{|x-y|\leqslant 2N^{\frac{\varepsilon}{2}+\alpha_\varepsilon}} \frac{\varphi_N(x,y)}{N^d} \\
&\leqslant C\,\|\varphi\|_1^2\,\mathbb{P}\big(Z_{N,\beta}^A < N^{-\mathsf{h}}\big)^{\frac{1}{4}}\,N^{\frac{d}{2}+d} \\
&= O(N^{\frac{3d}{2}})e^{-O\left((\log N)^\gamma\right)}\,.
\end{aligned}
$$

Since $\gamma > 1$, the last bound vanishes and therefore we conclude that

$$
N^{\frac{d}{2}-1} \sum_{|x-y|\leqslant 2N^{\frac{\varepsilon}{2}+\alpha_\varepsilon}} \frac{\varphi_N(x,y)}{N^d} \sum_{j\geqslant 1\,:\,a_j\in A_N^x\cap A_N^y} \mathbb{E}\Big[D_{j,N}^{(\mathsf{big})}(x)D_{j,N}^{(\mathsf{small})}(y)\Big] \xrightarrow[N\to\infty]{} 0\,.
$$

$\square$

2.2.2. Taylor approximation. We now proceed to the proof of Proposition 2.2.5. We will need the following lemma which provides a bound on the rate of decay of $\mathbb{E}\Big[\big(\hat{Z}_{N,\beta}^A(x)\big)^2\Big]$.

Lemma 2.2.8. *For every $\beta \in (0, \beta_{L^2})$ there exists a constant $C = C(\beta, d, \varepsilon) \in (0, \infty)$, such that $\mathbb{E}\Big[\big(\hat{Z}_{N,\beta}^A(x)\big)^2\Big] \leqslant C\,N^{-\varepsilon(\frac{d}{2}-1)}$.*

Proof. By (2.2.4), the chaos expansion of $\mathbb{E}\Big[\big(\hat{Z}_{N,\beta}^A(x)\big)^2\Big]$ is as follows.

$$
\mathbb{E}\Big[\big(\hat{Z}_{N,\beta}^A(x)\big)^2\Big] = \sum_{k=1}^N \sigma^{2k} \sum_{\substack{1\leqslant n_1<\ldots<n_k\leqslant N,\\ x:=z_0,z_1,\ldots,z_k\in\mathbb{Z}^d,\\ \exists\, i\in\{1,\ldots,k\}:\,(n_i,z_i)\notin A_N^x}} \prod_{i=1}^k q_{n_i-n_{i-1}}^2(z_i - z_{i-1})\,.
$$

Since the rightmost summation is over sequences of k space-time points $(n_i, z_i)_{1\leqslant i\leqslant k}$, such that at least one of the points $(n_i, z_i)_{1\leqslant i\leqslant k}$ is not in A_N^x, for every such sequence, there exists at least one index $i \in \{1, \ldots, k\}$, such that $|n_i - n_{i-1}| > \frac{1}{k}N^\varepsilon$ or $|z_i - z_{i-1}| > \frac{1}{k}N^{\frac{\varepsilon}{2}+\alpha_\varepsilon}$; recall the definition of A_N^x from (2.2.2). Thus, by changing variables $w_i := z_i - z_{i-1}$, $\ell_i := n_i - n_{i-1}$ and extending the range of summation from $1 \leqslant \ell_1 + \ldots + \ell_k \leqslant N$ to $\ell_1, \ldots, \ell_k \in \{1, \ldots, N\}$, we obtain that

$$
\begin{aligned}
&\mathbb{E}\Big[\big(\hat{Z}^{A}_{N,\beta}(x)\big)^{2}\Big] \\
&\leqslant \sum_{k=1}^{N}\sigma^{2k}\sum_{\substack{\ell_1,\dots,\ell_k\in\{1,\dots,N\},\\ w_1,\dots,w_k\in\mathbb{Z}^d}}\sum_{j=1}^{k}\Big(\mathbb{1}_{\{\ell_j>\frac{1}{k}N^{\varepsilon}\}}+\mathbb{1}_{\big\{\ell_j\leqslant\frac{1}{k}N^{\varepsilon},|w_j|>\frac{1}{k}N^{\frac{\varepsilon}{2}+\alpha_\varepsilon}\big\}}\Big)\prod_{i=1}^{k}q^2_{\ell_i}(w_i).
\end{aligned}
$$

By changing the order of summation, for each $i \neq j$ we have that

$$
\sum_{\ell_i=1}^{N}\sum_{w_i\in\mathbb{Z}^d}q^2_{\ell_i}(w_i)=\sum_{n=1}^{N}q_{2n}(0)=R_N\,.
$$

Thus,

$$
\mathbb{E}\Big[\big(\hat{Z}^{A}_{N,\beta}(x)\big)^{2}\Big]\leqslant\sum_{k=1}^{N}k\,\sigma^{2k}R_N^{k-1}\sum_{\substack{1\leqslant n\leqslant N,\\ w\in\mathbb{Z}^d}}\Big(\mathbb{1}_{\{n>\frac{1}{k}N^{\varepsilon}\}}+\mathbb{1}_{\big\{n\leqslant\frac{1}{k}N^{\varepsilon},|w|>\frac{1}{k}N^{\frac{\varepsilon}{2}+\alpha_\varepsilon}\big\}}\Big)q^2_n(w)\,. \tag{2.2.34}
$$

Let us consider the contribution of the two indicator functions separately. For the first one, by summing $w\in\mathbb{Z}^d$, we obtain that

$$
\sum_{k=1}^{N}k\,\sigma^{2k}R_N^{k-1}\sum_{\substack{1\leqslant n\leqslant N,\\ w\in\mathbb{Z}^d}}\mathbb{1}_{\{n>\frac{1}{k}N^{\varepsilon}\}}q^2_n(w)=\sum_{k=1}^{N}k\,\sigma^{2k}R_N^{k-1}\sum_{\frac{N^{\varepsilon}}{k}<n\leqslant N}q_{2n}(0) \tag{2.2.35}
$$

By the local limit theorem we have that $q_{2n}(0)\leqslant\frac{C}{n^{\frac{d}{2}}}$ for a constant $C=C(d)\in(0,\infty)$ and moreover using the standard estimate

$$
\sum_{n\geqslant A}\frac{1}{n^{\frac{d}{2}}}\leqslant A^{-\frac{d}{2}}+\int_A^{\infty}x^{-\frac{d}{2}}\,\mathrm{d}x=A^{-\frac{d}{2}}+\frac{A^{-(\frac{d}{2}-1)}}{\frac{d}{2}-1}\leqslant CA^{-(\frac{d}{2}-1)}\,.
$$

we obtain that there exists a constant $C(d)\in(0,\infty)$ so that

$$
\sum_{\frac{N^{\varepsilon}}{k}<n\leqslant N}q_{2n}(0)\leqslant C\,k^{\frac{d}{2}-1}N^{-\varepsilon(\frac{d}{2}-1)}\,.
$$

Consequently, the contribution of the first indicator function (2.2.35) is bounded by

$$
C\,N^{-\varepsilon(\frac{d}{2}-1)}\cdot\Big(\sum_{k\geqslant 1}k^{\frac{d}{2}}\big(\sigma^2(\beta)R_\infty\big)^k\Big)\,,
$$

where $R_\infty:=\sum_{n\geqslant 1}q_{2n}(0)<1$. Since β lies in the L^2-region, we have that $\sigma^2(\beta)R_\infty<1$, and therefore

$$
\sum_{k=1}^{\infty}k^{\frac{d}{2}}\big(\sigma^2(\beta)R_\infty\big)^k<\infty\,.
$$

Therefore, the contribution of the first indicator function to $\mathbb{E}\big[(\hat{Z}_{N,\beta}(x))^2\big]$ that is, (2.2.35), is bounded by $C\,N^{-\varepsilon(\frac{d}{2}-1)}$ for some constant $C=C(\beta,d)\in(0,\infty)$.

For the contribution of the second indicator function in (2.2.34), namely the sum

$$
\sum_{k=1}^{N}\sigma^{2k}R_N^{k-1}k\sum_{\substack{n\in\{1,\dots,N\},\\ w\in\mathbb{Z}^d}}\mathbb{1}_{\big\{n\leqslant\frac{1}{k}N^{\varepsilon},|w|>\frac{1}{k}N^{\frac{\varepsilon}{2}+\alpha_\varepsilon}\big\}}q^2_n(w)\,, \tag{2.2.36}
$$

we have that

$$\sum_{\substack{1\leqslant n\leqslant N,\\ w\in\mathbb{Z}^d}} \mathbb{1}_{\left\{n\leqslant\frac{1}{k}N^{\varepsilon},|w|>\frac{1}{k}N^{\frac{\varepsilon}{2}+\alpha\varepsilon}\right\}} q_n^2(w) \leqslant C \sum_{1\leqslant n\leqslant\frac{1}{k}N^{\varepsilon}} n^{-\frac{d}{2}}\, \mathrm{P}(|S_n|>\vartheta_{k,N}\sqrt{n})\,,$$

with $\vartheta_{k,N} := \frac{1}{\sqrt{k}}N^{\alpha_\varepsilon}$, which by the following moderate deviation estimate

$$\mathrm{P}\Big(\max_{0\leqslant k\leqslant n}|S_k|>\vartheta\sqrt{n}\Big)\leqslant C\, e^{-c\vartheta^2}\,,$$

with $\vartheta=\vartheta_{k,N}$ implies that

$$\sum_{\substack{1\leqslant n\leqslant N,\\ w\in\mathbb{Z}^d}} \mathbb{1}_{\left\{n\leqslant\frac{1}{k}N^{\varepsilon},|w|>\frac{1}{k}N^{\frac{\varepsilon}{2}+\alpha\varepsilon}\right\}} q_n^2(w) \leqslant Ce^{-\frac{c}{k}N^{2\alpha_\varepsilon}} \sum_{1\leqslant n\leqslant\frac{1}{k}N^{\varepsilon}} n^{-\frac{d}{2}}\,.$$

for a constant $C(d)\in(0,\infty)$. Notice that since $d\geqslant 3$, the sum $\sum_{n\geqslant 1} n^{-\frac{d}{2}}$ is finite therefore, there exists a constant $C(d)\in(0,\infty)$ such that (2.2.36) can be bounded as

$$\sum_{k=1}^{N}\sigma^{2k}R_N^{k-1}k\sum_{\substack{n\in\{1,\ldots,N\},\\ w\in\mathbb{Z}^d}} \mathbb{1}_{\left\{n\leqslant\frac{1}{k}N^{\varepsilon},|w|>\frac{1}{k}N^{\frac{\varepsilon}{2}+\alpha\varepsilon}\right\}} q_n^2(w)\leqslant C\sum_{k\geqslant 1}k\,\sigma^{2k}R_N^{k-1}e^{-\frac{c}{k}N^{2\alpha_\varepsilon}}\,.$$

Moreover,

$$\begin{aligned}\sum_{k\geqslant 1}k\,\sigma^{2k}R_N^{k-1}e^{-\frac{c}{k}N^{2\alpha_\varepsilon}} &\leqslant \sum_{k\leqslant N^{\alpha_\varepsilon}}k\,\sigma^{2k}R_N^{k-1}e^{-\frac{c}{k}N^{2\alpha_\varepsilon}}+\sum_{k>N^{\alpha_\varepsilon}}k\,\sigma^{2k}R_N^{k-1}e^{-\frac{c}{k}N^{2\alpha_\varepsilon}}\\ &\leqslant e^{-cN^{\alpha_\varepsilon}}\sum_{1\leqslant k\leqslant N^{\alpha_\varepsilon}}k\,\sigma^{2k}\,R_N^{k-1}+\sum_{k>N^{\alpha_\varepsilon}}k\,\sigma^{2k}\,R_N^{k-1}\\ &\leqslant C\,\eta^{N^{\alpha_\varepsilon}}\,,\end{aligned}$$

for constants $\eta=\eta(\beta)\in(0,1)$ and $C=C(\beta,\varepsilon)\in(0,\infty)$. Therefore, we deduce that there exists a constant $C=C(\beta,d,\varepsilon)\in(0,\infty)$ such that $\mathbb{E}\Big[\big(\hat{Z}_{N,\beta}^A(x)\big)^2\Big]\leqslant C\,N^{-\varepsilon(\frac{d}{2}-1)}$. □

Proof of Proposition 2.2.5. It suffices to prove that:

$$\lim_{N\to\infty}N^{\frac{d-2}{4}}\,\mathbb{E}\Big[\big|O_N(x)\big|\Big]=0\,.$$

As in [CSZ20] this is a careful Taylor estimate. We define

$$D_N^{\pm}:=\left\{\pm\frac{\hat{Z}_{N,\beta}^A(x)}{Z_{N,\beta}^A(x)}>N^{-p}\right\}\quad\text{and}\quad D_N:=D_N^+\cup D_N^-=\left\{\left|\frac{\hat{Z}_{N,\beta}^A(x)}{Z_{N,\beta}^A(x)}\right|>N^{-p}\right\},$$

for $p=\frac{d-2}{4}p^*$, with $0<p^*<1$ to be defined later. For $q=\frac{d-2}{4}q^*$ with $0<q^*<1$, also to be specified later, we have that

$$\begin{aligned}\mathbb{P}(D_N)&\leqslant\mathbb{P}\Big(D_N\cap\Big\{Z_{N,\beta}^A(x)\geqslant N^{-q}\Big\}\Big)+\mathbb{P}\Big(D_N\cap\Big\{Z_{N,\beta}^A(x)<N^{-q}\Big\}\Big)\\ &\leqslant\mathbb{P}\Big(\big|\hat{Z}_{N,\beta}^A(x)\big|>N^{-(p+q)}\Big)+\mathbb{P}\Big(Z_{N,\beta}^A(x)<N^{-q}\Big)\\ &\leqslant N^{2(p+q)}\mathbb{E}\Big[(\hat{Z}_{N,\beta}^A(x))^2\Big]+\mathbb{P}\Big(Z_{N,\beta}^A(x)<N^{-q}\Big)\,.\end{aligned}\tag{2.2.37}$$

For the last inequality we used Chebyshev's inequality. By Lemma 2.2.8 we have that

$$\mathbb{E}\Big[(\hat{Z}_{N,\beta}^A(x))^2\Big]\leqslant C\,N^{-\varepsilon(\frac{d}{2}-1)}$$

for some constant $C = C(\beta) \in (0, \infty)$. By Proposition 2.2.1 we have that $\mathbb{P}\Big(Z^A_{N,\beta}(x) < N^{-q}\Big)$ vanishes super-polynomially i.e.

$$\mathbb{P}\Big(Z^A_{N,\beta}(x) < N^{-q}\Big) \leqslant c_\beta \exp\left(\frac{-q^\gamma(\log N)^\gamma}{c_\beta}\right), \qquad \gamma > 1\,.$$

Therefore, by plugging those estimates into (2.2.37) we get that for a constant $C = C(\beta) \in (0, \infty)$,

$$\mathbb{P}(D_N) \leqslant C\, N^{2(p+q)-\varepsilon(\frac{d}{2}-1)}\,. \tag{2.2.38}$$

Furthermore, for a constant $C \in (0, \infty)$, it is true that,

$$|\log(1+y) - y| \leqslant C \cdot \begin{cases} \sqrt{\frac{|y|}{1+y}} & \text{if } -1 < y < 0 \\ y^2 & \text{if } -\frac{1}{2} \leqslant y \leqslant \frac{1}{2} \\ |y| & \text{if } 0 < y < \infty \end{cases}.$$

Hence,

$$\mathbb{E}\Big[\big|O_N(x)\big|\Big] \leqslant \mathbb{E}\bigg[\bigg(\frac{\hat{Z}^A_{N,\beta}(x)}{Z^A_{N,\beta}(x)}\bigg)^2 \mathbb{1}_{D_N^c}\bigg] + \mathbb{E}\bigg[\bigg|\frac{\hat{Z}^A_{N,\beta}(x)}{Z^A_{N,\beta}(x)}\bigg| \mathbb{1}_{D_N^+}\bigg] + \mathbb{E}\bigg[\sqrt{\bigg|\frac{\hat{Z}^A_{N,\beta}(x)}{Z_{N,\beta}(x)}\bigg|}\mathbb{1}_{D_N^-}\bigg]. \tag{2.2.39}$$

Let us deal with each term separately. We have that

$$\mathbb{E}\bigg[\bigg(\frac{\hat{Z}^A_{N,\beta}(x)}{Z^A_{N,\beta}(x)}\bigg)^2 \mathbb{1}_{D_N^c}\bigg] \leqslant N^{-2p}\,, \tag{2.2.40}$$

by the definition of D_N. We split the second term as follows:

$$\mathbb{E}\bigg[\bigg|\frac{\hat{Z}^A_{N,\beta}(x)}{Z^A_{N,\beta}(x)}\bigg| \mathbb{1}_{D_N^+}\bigg] = \mathbb{E}\bigg[\bigg|\frac{\hat{Z}^A_{N,\beta}(x)}{Z^A_{N,\beta}(x)}\bigg| \mathbb{1}_{D_N^+ \cap \{Z^A_{N,\beta}(x) \geqslant N^{-q}\}}\bigg] + \mathbb{E}\bigg[\bigg|\frac{\hat{Z}^A_{N,\beta}(x)}{Z^A_{N,\beta}(x)}\bigg| \mathbb{1}_{D_N^+ \cap \{Z^A_{N,\beta}(x) < N^{-q}\}}\bigg]. \tag{2.2.41}$$

For the first summand of (2.2.40) we have that

$$\begin{aligned}
\mathbb{E}\bigg[\bigg|\frac{\hat{Z}^A_{N,\beta}(x)}{Z^A_{N,\beta}(x)}\bigg| \mathbb{1}_{D_N^+ \cap \{Z^A_{N,\beta}(x) \geqslant N^{-q}\}}\bigg] &\leqslant N^q\, \mathbb{E}\Big[\big|\hat{Z}^A_{N,\beta}(x)\big| \mathbb{1}_{D_N^+ \cap \{Z^A_{N,\beta}(x) \geqslant N^{-q}\}}\Big] \\
&\leqslant N^q\, \mathbb{E}\Big[\big|\hat{Z}^A_{N,\beta}(x)\big| \mathbb{1}_{D_N^+}\Big] \\
&\leqslant N^q\, \mathbb{E}\Big[\big(\hat{Z}^A_{N,\beta}(x)\big)^2\Big]^{\frac{1}{2}} \mathbb{P}(D_N)^{\frac{1}{2}}\,,
\end{aligned}$$

by Cauchy-Schwarz. By Lemma 2.2.8, we get that $\mathbb{E}\Big[(\hat{Z}^A_{N,\beta}(x))^2\Big] \leqslant C\, N^{-\varepsilon(\frac{d}{2}-1)}$ and $\mathbb{P}(D_N) \leqslant C\, N^{2(p+q)-\varepsilon(\frac{d}{2}-1)}$ by (2.2.38). Hence,

$$\begin{aligned}
\mathbb{E}\bigg[\bigg|\frac{\hat{Z}^A_{N,\beta}(x)}{Z^A_{N,\beta}(x)}\bigg| \mathbb{1}_{D_N^+ \cap \{Z^A_{N,\beta}(x) \geqslant N^{-q}\}}\bigg] &\leqslant C\, N^q N^{-\varepsilon(\frac{d-2}{4})} N^{p+q-\varepsilon(\frac{d-2}{4})} \\
&= C\, N^{p+2q-2\varepsilon(\frac{d-2}{4})}\,.
\end{aligned}$$

for some constant $C = C(\beta) \in (0, \infty)$. For the second summand of (2.2.40) we use Hölder inequality with exponents $a = \frac{1}{2}, b = c = \frac{1}{4}$ to obtain that

$$\mathbb{E}\bigg[\bigg|\frac{\hat{Z}^A_{N,\beta}(x)}{Z^A_{N,\beta}(x)}\bigg| \mathbb{1}_{D_N^+ \cap \{Z^A_{N,\beta}(x) < N^{-q}\}}\bigg] \leqslant \mathbb{E}\Big[(\hat{Z}^A_{N,\beta}(x))^2\Big]^{\frac{1}{2}} \mathbb{E}\Big[\frac{1}{(Z^A_{N,\beta}(x))^4}\Big]^{\frac{1}{4}} \mathbb{P}(Z^A_{N,\beta}(x) < N^{-q})^{\frac{1}{4}}\,.$$

The term $\mathbb{P}(Z^A_{N,\beta}(x) < N^{-q})^{\frac{1}{4}}$ vanishes super-polynomially therefore, recalling (2.2.41) we conclude that

$$\mathbb{E}\bigg[\bigg|\frac{\hat{Z}^A_{N,\beta}(x)}{Z^A_{N,\beta}(x)}\bigg|\mathbb{1}_{D_N^+}\bigg] \leqslant C\, N^{p+2q-2\varepsilon(\frac{d-2}{4})}\,. \tag{2.2.42}$$

for some constant $C = C(\beta) \in (0,\infty)$. The second summand of (2.2.39) can be treated similarly. In particular, we split it as follows

$$\begin{aligned}\mathbb{E}\bigg[\bigg|\frac{\hat{Z}^A_{N,\beta}(x)}{Z_{N,\beta}(x)}\bigg|^{\frac{1}{2}}\mathbb{1}_{D_N^-}\bigg] = \mathbb{E}\bigg[\bigg|\frac{\hat{Z}^A_{N,\beta}(x)}{Z_{N,\beta}(x)}\bigg|^{\frac{1}{2}}\mathbb{1}_{D_N^-\cap\{Z_{N,\beta}(x)\geqslant N^{-q}\}}\bigg]\\ + \mathbb{E}\bigg[\bigg|\frac{\hat{Z}^A_{N,\beta}(x)}{Z_{N,\beta}(x)}\bigg|^{\frac{1}{2}}\mathbb{1}_{D_N^-\cap\{Z_{N,\beta}(x)< N^{-q}\}}\bigg]\,.\end{aligned} \tag{2.2.43}$$

For the first term we have that

$$\begin{aligned}\mathbb{E}\bigg[\bigg|\frac{\hat{Z}^A_{N,\beta}(x)}{Z_{N,\beta}(x)}\bigg|^{\frac{1}{2}}\mathbb{1}_{D_N^-\cap\{Z_{N,\beta}(x)\geqslant N^{-q}\}}\bigg] &\leqslant N^{\frac{q}{2}}\mathbb{E}\Big[|\hat{Z}^A_{N,\beta}(x)|^{\frac{1}{2}}\mathbb{1}_{D_N^-}\Big]\\ &\leqslant N^{\frac{q}{2}}\mathbb{E}\Big[|\hat{Z}^A_{N,\beta}(x)|^{\frac{1}{2}}\mathbb{1}_{D_N}\Big]\\ &\leqslant N^{\frac{q}{2}}\mathbb{E}\Big[(\hat{Z}^A_{N,\beta}(x))^2\Big]^{\frac{1}{4}}\mathbb{P}(D_N)^{\frac{3}{4}}\,.\end{aligned} \tag{2.2.44}$$

by Hölder inequality. Using aforementioned upper bounds on $\mathbb{E}\big[(\hat{Z}_{N,\beta}(x))^2\big]$ and $\mathbb{P}(D_N)$ we get that for a constant $C = C(\beta) \in (0,\infty)$,

$$\begin{aligned}N^{\frac{q}{2}}\mathbb{E}\Big[(\hat{Z}^A_{N,\beta}(x))^2\Big]^{\frac{1}{4}}\mathbb{P}(D_N)^{\frac{3}{4}} &\leqslant C\,N^{\frac{q}{2}}N^{-\frac{\lambda}{2}(\frac{d-2}{4})}N^{\frac{3}{2}(p+q-\lambda(\frac{d-2}{4}))}\\ &= C\,N^{\frac{3}{2}p+2q-2\lambda(\frac{d-2}{4})}\,,\end{aligned} \tag{2.2.45}$$

where we used Hölder inequality for the last inequality as well as bound (2.2.38) and Lemma 2.2.8. For the second term in (2.2.43) we can proceed as before, namely

$$\begin{aligned}&\mathbb{E}\bigg[\bigg|\frac{\hat{Z}^A_{N,\beta}(x)}{Z_{N,\beta}(x)}\bigg|^{\frac{1}{2}}\mathbb{1}_{D_N^-\cap\{Z_{N,\beta}(x)< N^{-q}\}}\bigg]\\ &\qquad\leqslant \mathbb{E}\Big[(\hat{Z}^A_{N,\beta}(x))^2\Big]^{\frac{1}{4}}\mathbb{E}\bigg[\frac{1}{(Z^A_{N,\beta}(x))^4}\bigg]^{\frac{1}{4}}\mathbb{P}(Z_{N,\beta}(x) < N^{-q})^{\frac{1}{2}}\,,\end{aligned} \tag{2.2.46}$$

by Hölder inequality. The super-polynomial decay of $\mathbb{P}(Z_{N,\beta}(x) < N^{-q})$ together with the bounds (2.2.38), (2.2.44), (2.2.45), (2.2.46) and Proposition 2.2.1, allows us to conclude that

$$\mathbb{E}\bigg[\bigg|\frac{\hat{Z}^A_{N,\beta}(x)}{Z_{N,\beta}(x)}\bigg|^{\frac{1}{2}}\mathbb{1}_{D_N^-}\bigg] \leqslant C\,N^{\frac{3}{2}p+2q-2\varepsilon(\frac{d-2}{4})}\,, \tag{2.2.47}$$

for some constant $C = C(\beta) \in (0,\infty)$. Recall now that we wanted to prove that

$$\lim_{N\to\infty} N^{\frac{d-2}{4}}\mathbb{E}\Big[|O_N(x)|\Big] = 0\,.$$

By the estimates (2.2.40), (2.2.42) and (2.2.47) respectively, we see that it suffices to find exponents p^*, q^* and ε, so that

$$1-2p^* < 0, \qquad 1-2\varepsilon+p^*+2q^* < 0, \qquad 1-2\varepsilon+\frac{3}{2}p^*+2q^* < 0\,.$$

The second inequality is implied by the third therefore, it suffices to find exponents p^*, q^* and ε, so that

$$1-2p^*<0, \qquad 1-2\varepsilon+\frac{3}{2}p^*+2q^*<0\,.$$

This would lead to $\varepsilon>\frac{1}{2}(1+\frac{3}{2}p^*+2q^*)$ and since we can take $p^*>\frac{1}{2}$ arbitrarily close to $\frac{1}{2}$ and $q^*>0$ arbitrarily small, it suffices to choose $\varepsilon>\frac{7}{8}$ in the definition of the sets A_N^x, recall (2.2.2). □

2.2.3. Main contribution and identification of the fluctuations. We proceed now to the proof of Proposition 2.2.6.

Proof of Proposition 2.2.6. We need to prove that

$$N^{\frac{d-2}{4}}\sum_{x\in\mathbb{Z}^d}\frac{\varphi_N(x)}{N^{\frac{d}{2}}}\left(\frac{\hat{Z}^A_{N,\beta}(x)}{Z^A_{N,\beta}(x)}-\left(Z^{B^{\geqslant}_N}_{N,\beta}(x)-1\right)\right)\xrightarrow[N\to\infty]{L^1(\mathbb{P})}0\,. \tag{2.2.48}$$

We remind the reader that $B_N^{\geqslant}:=\big([N^\varrho,N]\cap\mathbb{N}\big)\times\mathbb{Z}^d$ for some $\varrho\in(\varepsilon,1)$, the choice of which is specified by (2.2.76). We also define the sets

$$\begin{aligned}
B_N &:= \big((N^\varepsilon,N]\cap\mathbb{N}\big)\times\mathbb{Z}^d\,,\\
C_N^x &:= \left\{(n,z)\in\mathbb{N}\times\mathbb{Z}^d:\ 1\leqslant n\leqslant N^\varepsilon,\ |z-x|\geqslant N^{\frac{\varepsilon}{2}+\alpha_\varepsilon}\right\}.
\end{aligned}$$

We decompose $\hat{Z}^A_{N,\beta}(x)$ into two parts

$$\hat{Z}^A_{N,\beta}(x)=\hat{Z}^{A,B}_{N,\beta}(x)+\hat{Z}^{A,C}_{N,\beta}(x)\,,$$

where

$$\begin{aligned}
\hat{Z}^{A,B}_{N,\beta}(x) &:= \sum_{\tau\subset A_N^x\cup B_N:\,\tau\cap B_N\neq\varnothing}\sigma^{|\tau|}q^{(0,x)}(\tau)\xi(\tau)\,,\\
\hat{Z}^{A,C}_{N,\beta}(x) &:= \sum_{\tau\subset\{1,\dots,N\}\times\mathbb{Z}^d:\,\tau\cap C_N^x\neq\varnothing}\sigma^{|\tau|}q^{(0,x)}(\tau)\xi(\tau)\,.
\end{aligned} \tag{2.2.49}$$

and if $\tau=(n_i,z_i)_{1\leqslant i\leqslant k}$,

$$q^{(0,x)}(\tau):=q_{n_1}(z_1-x)\prod_{i=2}^k q_{n_i-n_{i-1}}(z_i-z_{i-1})\,.$$

The proof will consist of three steps.

(Step 1) The first task is to show that $\hat{Z}^{A,C}_{N,\beta}(x)$ has a negligible contribution to (2.2.48). The proof of this is based on the fact that $\hat{Z}^{A,C}_{N,\beta}(x)$ consists of random walk paths which are super-diffusive: the walk will have to travel at distance greater than $N^{\frac{\varepsilon}{2}+\alpha_\varepsilon}$ from x within time N^ε. Therefore, by standard moderate deviation estimates one can show that

$$N^{\frac{d-2}{4}}\sum_{x\in\mathbb{Z}^d}\frac{\varphi_N(x)}{N^{\frac{d}{2}}}\cdot\frac{\hat{Z}^{A,C}_{N,\beta}(x)}{Z^A_{N,\beta}(x)}\xrightarrow[N\to\infty]{L^2(\mathbb{P})}0\,,$$

super-polynomially.The proof follows the same lines of the proof of Proposition 2.3. in [CSZ20] and for this reason we omit the details.

(Step 2) The second step will be to show that in the chaos expansion of $\hat{Z}^{A,B}_{N,\beta}(x)$, the contribution from sampling disorder $\xi_{r,z}$, with $r<N^\varrho$ is negligible, for every $\varrho\in(\varepsilon,1)$. In particular,

let us denote by B_N^{strip} the set

$$B_N^{\mathsf{strip}} := \Big\{(n,z) \in (N^{\varepsilon}, N^{\varrho}) \times \mathbb{Z}^d\Big\}. \tag{2.2.50}$$

We can decompose $\hat{Z}^{A,B}_{N,\beta}(x)$ into two parts $\hat{Z}^{A,B}_{N,\beta}(x) = \hat{Z}^{A,B^<}_{N,\beta}(x) + \hat{Z}^{A,B^\geqslant}_{N,\beta}(x)$ such that

$$\hat{Z}^{A,B^<}_{N,\beta}(x) := \sum_{k=1}^{N} \sigma^k \sum_{\substack{0:=n_0<n_1<\ldots<n_k\leqslant N,\\ x:=z_0,z_1,\ldots,z_k\in\mathbb{Z}^d,\\ (n_i,z_i)_{i=1}^k\subset A_N^x\cup B_N,\, (n_i,z_i)_{i=1}^k\cap B_N^{\mathsf{strip}}\neq\varnothing}} \prod_{i=1}^{k} q_{n_i-n_{i-1}}(z_i-z_{i-1})\xi_{n_i,z_i}\,. \tag{2.2.51}$$

and

$$\hat{Z}^{A,B^\geqslant}_{N,\beta}(x) := \sum_{k=1}^{N} \sigma^k \sum_{\substack{0:=n_0<n_1<\ldots<n_k\leqslant N\\ x:=z_0,z_1,\ldots,z_k\in\mathbb{Z}^d\\ (n_i,z_i)_{i=1}^k\subset A_N^x\cup B_N,\, (n_i,z_i)_{1\leqslant i\leqslant k}\cap B_N^{\mathsf{strip}}=\varnothing}} \prod_{i=1}^{k} q_{n_i-n_{i-1}}(z_i-z_{i-1})\xi_{n_i,z_i}\,. \tag{2.2.52}$$

In this step we will show that

$$N^{\frac{d-2}{4}} \sum_{x\in\mathbb{Z}^d} \frac{\varphi_N(x)}{N^{\frac{d}{2}}} \cdot \frac{\hat{Z}^{A,B^<}_{N,\beta}(x)}{Z^{A}_{N,\beta}(x)} \xrightarrow[N\to\infty]{L^2(\mathbb{P})} 0\,,$$

or equivalently

$$N^{\frac{d}{2}-1} \sum_{x,y\in\mathbb{Z}^d} \frac{\varphi_N(x,y)}{N^d}\, \mathbb{E}\left[\frac{\hat{Z}^{A,B^<}_{N,\beta}(x)}{Z^{A}_{N,\beta}(x)} \cdot \frac{\hat{Z}^{A,B^<}_{N,\beta}(y)}{Z^{A}_{N,\beta}(y)}\right] \xrightarrow[N\to\infty]{} 0\,. \tag{2.2.53}$$

Let us denote by S^x, S^y the paths of two independent random walks starting from x, y respectively. We will use the notation

$$\mathsf{C}^{A,B}_{N,\beta}(x,y) := \mathrm{E}_{x,y}\Big[(e^{\mathsf{H}^x_{A,B}(\omega)}-1)(e^{\mathsf{H}^y_{A,B}(\omega)}-1)\mathbb{1}_{S^x\cap S^y\cap B_N^{\mathsf{strip}}\neq\varnothing}\Big], \tag{2.2.54}$$

where

$$\mathsf{H}^x_{A,B}(\omega) := \sum_{(n,z)\in A_N^x\cup B_N} \{\beta\omega_{n,z}-\lambda(\beta)\}\mathbb{1}_{S_n^x=z}\,,$$

and

$$\mathsf{C}^{A,B^\geqslant}_{N,\beta}(x,y) := \mathrm{E}_{x,y}\Big[(e^{\mathsf{H}^x_{A,B^\geqslant}(\omega)}-1)(e^{\mathsf{H}^y_{A,B^\geqslant}(\omega)}-1)\mathbb{1}_{S^x\cap S^y\cap B_N^{\mathsf{strip}}\neq\varnothing}\Big], \tag{2.2.55}$$

where

$$\mathsf{H}^x_{A,B^\geqslant}(\omega) := \sum_{(n,z)\in A_N^x\cup(B_N\setminus B_N^{\mathsf{strip}})} \{\beta\omega_{n,z}-\lambda(\beta)\}\mathbb{1}_{S_n^x=z}$$

is the energy which does not contain disorder indexed by space-time points in the region B_N^{strip}. Note that, even though in the definition (2.2.54) of $\mathsf{C}^{A,B^\geqslant}_{N,\beta}(x,y)$, $\mathsf{H}^x_{A,B^\geqslant}(\omega)$ and $\mathsf{H}^y_{A,B^\geqslant}(\omega)$ do not contain disorder indexed by B_N^{strip}, there is still the constraint that the two random walks S^x, S^y meet at some point in B_N^{strip}.

We will control (2.2.53), by showing that

$$\mathbb{E}\left[\frac{\hat{Z}^{A,B^<}_{N,\beta}(x)}{Z^{A}_{N,\beta}(x)} \cdot \frac{\hat{Z}^{A,B^<}_{N,\beta}(y)}{Z^{A}_{N,\beta}(y)}\right] = \mathbb{E}\left[\frac{\mathsf{C}^{A,B}_{N,\beta}(x,y)-\mathsf{C}^{A,B^\geqslant}_{N,\beta}(x,y)}{Z^{A}_{N,\beta}(x)Z^{A}_{N,\beta}(y)}\right], \tag{2.2.56}$$

and then showing that when the right-hand side is inserted into (2.2.53), it leads to vanishing contribution. Let us first check equality (2.2.56). The chaos expansion of $\mathsf{C}^{A,B}_{N,\beta}$ is

$$
\begin{aligned}
\mathsf{C}^{A,B}_{N,\beta}(x,y) =&\mathrm{E}_{x,y}\big[(e^{\mathsf{H}^x_{A,B}(\omega)}-1)(e^{\mathsf{H}^y_{A,B}(\omega)}-1)\mathbb{1}_{S^x\cap S^y\cap B^{\mathsf{strip}}_N\neq\varnothing}\big] \\
&= \sum_{1\leqslant k,\ell\leqslant N}\sigma^{k+\ell}\sum_{\substack{(n_i,z_i)_{i=1}^k\subset A^x_N\cup B_N,\\(m_j,w_j)_{j=1}^\ell\subset A^y_N\cup B_N}}\mathrm{E}_{x,y}\Bigg[\prod_{\substack{1\leqslant i\leqslant k,\\1\leqslant j\leqslant\ell}}\mathbb{1}_{S^x_{n_i}=z_i}\mathbb{1}_{S^y_{m_j}=w_j}\mathbb{1}_{S^x\cap S^y\cap B^{\mathsf{strip}}_N\neq\varnothing}\Bigg] \\
&\times\prod_{\substack{1\leqslant i\leqslant k\\1\leqslant j\leqslant\ell}}\xi_{n_i,z_i}\,\xi_{m_j,w_j}\,.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&\mathsf{C}^{A,B^{\geqslant}}_{N,\beta}(x,y) \\
=&\mathrm{E}_{x,y}\big[(e^{\mathsf{H}^x_{A,B^{\geqslant}}(\omega)}-1)(e^{\mathsf{H}^y_{A,B^{\geqslant}}(\omega)}-1)\mathbb{1}_{S^x\cap S^y\cap B^{\mathsf{strip}}_N\neq\varnothing}\big] \\
=&\sum_{1\leqslant k,\ell\leqslant N}\sigma^{k+\ell}\sum_{\substack{(n_i,z_i)_{i=1}^k\subset A^x_N\cup(B_N\backslash B^{\mathsf{strip}}_N),\\(m_j,w_j)_{j=1}^\ell\subset A^y_N\cup(B_N\backslash B^{\mathsf{strip}}_N)}}\mathrm{E}_{x,y}\Bigg[\prod_{\substack{1\leqslant i\leqslant k,\\1\leqslant j\leqslant\ell}}\mathbb{1}_{S^x_{n_i}=z_i}\mathbb{1}_{S^y_{m_j}=w_j}\mathbb{1}_{S^x\cap S^y\cap B^{\mathsf{strip}}_N\neq\varnothing}\Bigg] \\
&\times\prod_{\substack{1\leqslant i\leqslant k,\\1\leqslant j\leqslant\ell}}\xi_{n_i,z_i}\,\xi_{m_j,w_j}\,.
\end{aligned}
$$

The constraints $(n_i,z_i)_{i=1}^k\subset A^x_N\cup(B_N\backslash B^{\mathsf{strip}}_N)$ and $(m_j,w_j)_{j=1}^\ell\subset A^y_N\cup(B_N\backslash B^{\mathsf{strip}}_N)$ come from the fact that $\mathsf{H}^x_{A,B^{\geqslant}}(\omega),\mathsf{H}^y_{A,B^{\geqslant}}(\omega)$ do not sample ξ indexed by points in B^{strip}_N. The chaos expansion of the difference, $\mathsf{C}^{A,B}_{N,\beta}(x,y)-\mathsf{C}^{A,B^{\geqslant}}_{N,\beta}(x,y)$, is then

$$
\begin{aligned}
&\mathsf{C}^{A,B}_{N,\beta}(x,y)-\mathsf{C}^{A,B^{\geqslant}}_{N,\beta}(x,y) \\
=&\sum_{1\leqslant k,\ell\leqslant N}\sigma^{k+\ell}\sum_{\substack{(n_i,z_i)_{i=1}^k\subset A^x_N\cup B_N,\,(n_i,z_i)_{i=1}^k\cap B^{\mathsf{strip}}_N\neq\varnothing\\ \textbf{or}\\(m_j,w_j)_{j=1}^\ell\subset A^y_N\cup B_N,\,(m_j,w_j)_{j=1}^\ell\cap B^{\mathsf{strip}}_N\neq\varnothing}} \\
&\qquad\times\mathrm{E}_{x,y}\Bigg[\prod_{\substack{1\leqslant i\leqslant k,\\1\leqslant j\leqslant\ell}}\mathbb{1}_{S^x_{n_i}=z_i}\mathbb{1}_{S^y_{m_j}=w_j}\mathbb{1}_{S^x\cap S^y\cap B^{\mathsf{strip}}_N\neq\varnothing}\Bigg]\prod_{\substack{1\leqslant i\leqslant k,\\1\leqslant j\leqslant\ell}}\xi_{n_i,z_i}\,\xi_{m_j,w_j}\,.
\end{aligned}
$$

Therefore, the expansion of $\mathbb{E}\left[\dfrac{\mathsf{C}^{A,B}_{N,\beta}(x,y)-\mathsf{C}^{A,B^{\geqslant}}_{N,\beta}(x,y)}{Z^A_{N,\beta}(x)Z^A_{N,\beta}(y)}\right]$ is

$$
\begin{aligned}
&\mathbb{E}\left[\frac{\mathsf{C}^{A,B}_{N,\beta}(x,y)-\mathsf{C}^{A,B^{\geqslant}}_{N,\beta}(x,y)}{Z^A_{N,\beta}(x)Z^A_{N,\beta}(y)}\right] \\
=&\mathbb{E}\Bigg[\frac{1}{Z^A_{N,\beta}(x)Z^A_{N,\beta}(y)}\sum_{1\leqslant k,\ell\leqslant N}\sigma^{k+\ell}\sum_{\substack{(n_i,z_i)_{i=1}^k\subset A^x_N\cup B_N,\,(n_i,z_i)_{i=1}^k\cap B^{\mathsf{strip}}_N\neq\varnothing\\ \textbf{or}\\(m_j,w_j)_{j=1}^\ell\subset A^y_N\cup B_N,\,(m_j,w_j)_{j=1}^\ell\cap B^{\mathsf{strip}}_N\neq\varnothing}} \\
&\qquad\times\mathrm{E}_{x,y}\Bigg[\prod_{\substack{1\leqslant i\leqslant k\\1\leqslant j\leqslant\ell}}\mathbb{1}_{S^x_{n_i}=z_i}\mathbb{1}_{S^y_{m_j}=w_j}\mathbb{1}_{S^x\cap S^y\cap B^{\mathsf{strip}}_N\neq\varnothing}\Bigg]\prod_{\substack{1\leqslant i\leqslant k\\1\leqslant j\leqslant\ell}}\xi_{n_i,z_i}\,\xi_{m_j,w_j}\Bigg].
\end{aligned}
\tag{2.2.57}
$$

Note that if for example $(n_i, z_i)_{i=1}^k \cap B_N^{\mathsf{strip}} \neq \varnothing$, the expectation $\mathbb{E}[\cdot]$ will impose that also $(m_j, z_j)_{j=1}^\ell \cap B_N^{\mathsf{strip}} \neq \varnothing$ and in particular, $(n_i, z_i)_{i=1}^k \cap B_N^{\mathsf{strip}} = (m_j, w_j)_{j=1}^\ell \cap B_N^{\mathsf{strip}}$, due to the fact that the ξ variables indexed by space-time points with time index $t > N^\varepsilon$ appearing in the expansion of $\mathsf{C}_{N,\beta}^{A,B}(x,y) - \mathsf{C}_{N,\beta}^{A,B^\geqslant}(x,y)$ have to match pairwise, because they are independent of $Z_{N,\beta}^A(x), Z_{N,\beta}^A(y)$, and so if a disorder variable ξ_{n_i,z_i} or ξ_{m_j,w_j} is unmatched, their mean zero property will lead to vanishing of the whole expectation $\mathbb{E}[\cdot]$. Thus, the indicator $\mathbb{1}_{S^x \cap S^y \cap B_N^{\mathsf{strip}} \neq \varnothing}$ will always be equal to 1 for every summand of the last expansion, since we are summing space-time sequences, such that $(n_i, z_i)_{i=1}^k \cap (m_j, z_j)_{j=1}^\ell \cap B_N^{\mathsf{strip}} \neq \varnothing$. Therefore, the expansion of $\mathbb{E}\Bigg[\dfrac{\mathsf{C}_{N,\beta}^{A,B}(x,y) - \mathsf{C}_{N,\beta}^{A,B^\geqslant}(x,y)}{Z_{N,\beta}^A(x) Z_{N,\beta}^A(y)}\Bigg]$ is actually equal to

$$
\begin{aligned}
&\mathbb{E}\Bigg[\frac{\mathsf{C}_{N,\beta}^{A,B}(x,y) - \mathsf{C}_{N,\beta}^{A,B^\geqslant}(x,y)}{Z_{N,\beta}^A(x) Z_{N,\beta}^A(y)}\Bigg] \\
=&\mathbb{E}\Bigg[\frac{1}{Z_{N,\beta}^A(x) Z_{N,\beta}^A(y)} \sum_{1 \leqslant k,\ell \leqslant N} \sigma^{k+\ell} \sum_{\substack{(n_i,z_i)_{i=1}^k \subset A_N^x \cup B_N,\, (n_i,z_i)_{i=1}^k \cap B_N^{\mathsf{strip}} \neq \varnothing, \\ (m_j,w_j)_{j=1}^\ell \subset A_N^y \cup B_N,\, (m_j,w_j)_{j=1}^\ell \cap B_N^{\mathsf{strip}} \neq \varnothing}} \\
&\qquad \times \mathrm{E}_{x,y}\Bigg[\prod_{\substack{1 \leqslant i \leqslant k \\ 1 \leqslant j \leqslant \ell}} \mathbb{1}_{S_{n_i}^x = z_i} \mathbb{1}_{S_{m_j}^y = w_j}\Bigg] \prod_{\substack{1 \leqslant i \leqslant k \\ 1 \leqslant j \leqslant \ell}} \xi_{n_i,z_i} \xi_{m_j,w_j}\Bigg],
\end{aligned}
$$

which matches exactly the expansion of $\mathbb{E}\Bigg[\dfrac{\hat{Z}_{N,\beta}^{A,B^<}(x)}{Z_{N,\beta}^A(x)} \cdot \dfrac{\hat{Z}_{N,\beta}^{A,B^<}(y)}{Z_{N,\beta}^A(y)}\Bigg]$, by (2.2.51), thus allowing us to conclude that

$$
\mathbb{E}\Bigg[\frac{\hat{Z}_{N,\beta}^{A,B^<}(x)}{Z_{N,\beta}^A(x)} \cdot \frac{\hat{Z}_{N,\beta}^{A,B^<}(y)}{Z_{N,\beta}^A(y)}\Bigg] = \mathbb{E}\Bigg[\frac{\mathsf{C}_{N,\beta}^{A,B}(x,y) - \mathsf{C}_{N,\beta}^{A,B^\geqslant}(x,y)}{Z_{N,\beta}^A(x) Z_{N,\beta}^A(y)}\Bigg].
$$

Having established this equality, to finish the proof of (2.2.53), we will prove that

$$
N^{\frac{d}{2}-1} \sum_{x,y \in \mathbb{Z}^d} \frac{\varphi_N(x,y)}{N^d} \, \mathbb{E}\Bigg[\frac{\mathsf{C}_{N,\beta}^{A,B}(x,y)}{Z_{N,\beta}^A(x) Z_{N,\beta}^A(y)}\Bigg] \xrightarrow[N \to \infty]{} 0\,, \tag{2.2.58}
$$

and

$$
N^{\frac{d}{2}-1} \sum_{x,y \in \mathbb{Z}^d} \frac{\varphi_N(x,y)}{N^d} \, \mathbb{E}\Bigg[\frac{\mathsf{C}_{N,\beta}^{A,B^\geqslant}(x,y)}{Z_{N,\beta}^A(x) Z_{N,\beta}^A(y)}\Bigg] \xrightarrow[N \to \infty]{} 0\,. \tag{2.2.59}
$$

We start by showing the validity of (2.2.58), since (2.2.59) can be treated with the same arguments. In view of (2.2.54) we have that

$$
\begin{aligned}
\mathsf{C}_{N,\beta}^{A,B}(x,y) =& \mathrm{E}_{x,y}\big[(e^{\mathsf{H}_{A,B}^x(\omega)} - 1)(e^{\mathsf{H}_{A,B}^y(\omega)} - 1)\mathbb{1}_{S^x \cap S^y \cap B_N^{\mathsf{strip}} \neq \varnothing}\big] \\
=& \mathrm{E}_{x,y}\big[e^{\mathsf{H}_{A,B}^x(\omega) + \mathsf{H}_{A,B}^y(\omega)} \mathbb{1}_{S^x \cap S^y \cap B_N^{\mathsf{strip}} \neq \varnothing}\big] - \mathrm{E}_{x,y}\big[e^{\mathsf{H}_{A,B}^x(\omega)} \mathbb{1}_{S^x \cap S^y \cap B_N^{\mathsf{strip}} \neq \varnothing}\big] \\
&- \mathrm{E}_{x,y}\big[e^{\mathsf{H}_{A,B}^y(\omega)} \mathbb{1}_{S^x \cap S^y \cap B_N^{\mathsf{strip}} \neq \varnothing}\big] + \mathrm{P}_{x,y}\big(S^x \cap S^y \cap B_N^{\mathsf{strip}} \neq \varnothing\big)\,.
\end{aligned} \tag{2.2.60}
$$

We begin by showing that

$$
N^{\frac{d}{2}-1} \sum_{x,y \in \mathbb{Z}^d} \frac{\varphi_N(x,y)}{N^d} \, \mathbb{E}\Bigg[\frac{\mathrm{E}_{x,y}\big[e^{\mathsf{H}_{A,B}^x(\omega) + \mathsf{H}_{A,B}^y(\omega)} \mathbb{1}_{S^x \cap S^y \cap B_N^{\mathsf{strip}} \neq \varnothing}\big]}{Z_{N,\beta}^A(x) Z_{N,\beta}^A(y)}\Bigg] \xrightarrow[N \to \infty]{} 0\,.
$$

The main point here will be to remove the denominators. Consider the set

$$E_N := \left\{Z^A_{N,\beta}(x), Z^A_{N,\beta}(y) \geqslant N^{-\mathsf{h}}\right\}$$

for some $\mathsf{h} \in (0, \frac{1-\varrho}{2})$. We have that

$$\begin{aligned}
&\mathbb{E}\Bigg[\frac{\mathrm{E}_{x,y}\big[e^{\mathsf{H}^x_{A,B}(\omega)+\mathsf{H}^y_{A,B}(\omega)}\mathbb{1}_{S^x\cap S^y\cap B^{\mathsf{strip}}_N\neq\varnothing}\big]}{Z^A_{N,\beta}(x)Z^A_{N,\beta}(y)}\Bigg] \\
&=\mathbb{E}\Bigg[\frac{\mathrm{E}_{x,y}\big[e^{\mathsf{H}^x_{A,B}(\omega)+\mathsf{H}^y_{A,B}(\omega)}\mathbb{1}_{S^x\cap S^y\cap B^{\mathsf{strip}}_N\neq\varnothing}\big]}{Z^A_{N,\beta}(x)Z^A_{N,\beta}(y)}\mathbb{1}_{E_N}\Bigg] \\
&\qquad + \mathbb{E}\Bigg[\frac{\mathrm{E}_{x,y}\big[e^{\mathsf{H}^x_{A,B}(\omega)+\mathsf{H}^y_{A,B}(\omega)}\mathbb{1}_{S^x\cap S^y\cap B^{\mathsf{strip}}_N\neq\varnothing}\big]}{Z^A_{N,\beta}(x)Z^A_{N,\beta}(y)}\mathbb{1}_{E^c_N}\Bigg].
\end{aligned} \tag{2.2.61}$$

We can bound the first summand using the definition of the sets E_N, as follows

$$\begin{aligned}
&\mathbb{E}\Bigg[\frac{\mathrm{E}_{x,y}\big[e^{\mathsf{H}^x_{A,B}(\omega)+\mathsf{H}^y_{A,B}(\omega)}\mathbb{1}_{S^x\cap S^y\cap B^{\mathsf{strip}}_N\neq\varnothing}\big]}{Z^A_{N,\beta}(x)Z^A_{N,\beta}(y)}\mathbb{1}_{E_N}\Bigg] \\
&\qquad\leqslant N^{2\mathsf{h}}\,\mathbb{E}\Big[\mathrm{E}_{x,y}\big[e^{\mathsf{H}^x_{A,B}(\omega)+\mathsf{H}^y_{A,B}(\omega)}\mathbb{1}_{S^x\cap S^y\cap B^{\mathsf{strip}}_N\neq\varnothing}\big]\Big] \\
&\qquad\leqslant N^{2\mathsf{h}}\,\mathrm{E}_{x,y}\big[e^{\lambda_2(\beta)\mathsf{L}_N(x,y)}\mathbb{1}_{S^x\cap S^y\cap B^{\mathsf{strip}}_N\neq\varnothing}\big].
\end{aligned} \tag{2.2.62}$$

We condition on the first time, $\tau_{x,y}$, that the two random walk paths meet, to obtain that

$$\begin{aligned}
&\mathrm{E}_{x,y}\big[e^{\lambda_2(\beta)\mathsf{L}_N(x,y)}\mathbb{1}_{S^x\cap S^y\cap B^{\mathsf{strip}}_N\neq\varnothing}\big] \\
&\qquad=\sum_{n=1}^{N^\varrho}\mathrm{E}_{x,y}\big[e^{\lambda_2(\beta)\mathsf{L}_N(x,y)}\mathbb{1}_{S^x\cap S^y\cap B^{\mathsf{strip}}_N\neq\varnothing}\big|\tau_{x,y}=n\big]\mathrm{P}(\tau_{x,y}=n) \\
&\qquad\leqslant\sum_{n=1}^{N^\varrho}\mathrm{E}_{x,y}\big[e^{\lambda_2(\beta)\mathsf{L}_N(x,y)}\big|\tau_{x,y}=n\big]\mathrm{P}(\tau_{x,y}=n).
\end{aligned}$$

By the Markov property

$$\begin{aligned}
\sum_{n=1}^{N^\varrho}\mathrm{E}_{x,y}\big[e^{\lambda_2(\beta)\mathsf{L}_N(x,y)}\big|\tau_{x,y}=n\big]\mathrm{P}_{x,y}(\tau_{x,y}=n) &= \sum_{n=1}^{N^\varrho}\mathrm{E}\big[e^{\lambda_2(\beta)(\mathsf{L}_{N-n}+1)}\big]\mathrm{P}_{x,y}(\tau_{x,y}=n) \\
&= \sum_{n=1}^{N^\varrho}e^{\lambda_2(\beta)}\,\mathrm{E}\big[e^{\lambda_2(\beta)\mathsf{L}_{N-n}}\big]\mathrm{P}_{x,y}(\tau_{x,y}=n) \\
&\leqslant e^{\lambda_2(\beta)}\,\mathrm{E}\big[e^{\lambda_2(\beta)\mathsf{L}_\infty}\big]\sum_{n=1}^{N^\varrho}q_{2n}(x-y).
\end{aligned} \tag{2.2.63}$$

We remind the reader that $\mathrm{E}\big[e^{\lambda_2(\beta)\mathsf{L}_\infty}\big] < \infty$ because $\beta \in (0, \beta_{L^2})$. Therefore, if we combine (2.2.62), (2.2.63), we deduce the estimate

$$\begin{aligned}
&N^{\frac{d}{2}-1} \sum_{x,y\in\mathbb{Z}^d} \frac{\varphi_N(x,y)}{N^d}\, \mathbb{E}\left[\frac{\mathrm{E}_{x,y}\big[e^{\mathsf{H}^x_{A,B}(\omega)+\mathsf{H}^y_{A,B}(\omega)}\mathbb{1}_{S^x\cap S^y\cap B_N^{\mathsf{strip}}\neq\varnothing}\big]}{Z^A_{N,\beta}(x)Z^A_{N,\beta}(y)}\mathbb{1}_{E_N}\right] \\
&\leqslant C\, N^{\frac{d}{2}-1+2\mathsf{h}} \sum_{x,y\in\mathbb{Z}^d} \frac{\varphi_N(x,y)}{N^d} \sum_{n=1}^{N^\varrho} q_{2n}(y-x)\,.
\end{aligned}$$

The last bound vanishes because $\mathsf{h} \in (0, \frac{1-\varrho}{2})$, see (2.2.29) for the derivation of this fact.

We now deal with the complementary event E_N^c in (2.2.61). Recall that

$$E_N^c = \big\{Z^A_{N,\beta}(x) < N^{-\mathsf{h}}\big\} \cup \big\{Z^A_{N,\beta}(y) < N^{-\mathsf{h}}\big\}\,.$$

By Proposition 2.2.1 and a union bound we obtain that

$$\mathbb{P}(E_N^c) \leqslant 2\mathbb{P}\big(Z^A_{N,\beta}(x) < N^{-\mathsf{h}}\big) \leqslant 2c_\beta \exp\left(\frac{-\mathsf{h}^\gamma(\log N)^\gamma}{c_\beta}\right). \tag{2.2.64}$$

Recall that we need to show that

$$N^{\frac{d}{2}-1} \sum_{x,y\in\mathbb{Z}^d} \frac{\varphi_N(x,y)}{N^d}\, \mathbb{E}\left[\frac{\mathrm{E}_{x,y}\big[e^{\mathsf{H}^x_{A,B}(\omega)+\mathsf{H}^y_{A,B}(\omega)}\mathbb{1}_{S^x\cap S^y\cap B_N^{\mathsf{strip}}\neq\varnothing}\big]}{Z^A_{N,\beta}(x)Z^A_{N,\beta}(y)}\mathbb{1}_{E_N^c}\right] \xrightarrow[N\to\infty]{} 0\,.$$

We have that

$$\begin{aligned}
\mathbb{E}\left[\frac{\mathrm{E}_{x,y}\big[e^{\mathsf{H}^x_{A,B}(\omega)+\mathsf{H}^y_{A,B}(\omega)}\mathbb{1}_{S^x\cap S^y\cap B_N^{\mathsf{strip}}\neq\varnothing}\big]}{Z^A_{N,\beta}(x)Z^A_{N,\beta}(y)}\mathbb{1}_{E_N^c}\right] &\leqslant \mathbb{E}\left[\frac{\mathrm{E}_{x,y}\big[e^{\mathsf{H}^x_{A,B}(\omega)+\mathsf{H}^y_{A,B}(\omega)}\big]}{Z^A_{N,\beta}(x)Z^A_{N,\beta}(y)}\mathbb{1}_{E_N^c}\right] \\
&= \mathbb{E}\left[\frac{Z^{A,B}_{N,\beta}(x)}{Z^A_{N,\beta}(x)}\cdot\frac{Z^{A,B}_{N,\beta}(y)}{Z^A_{N,\beta}(y)}\mathbb{1}_{E_N^c}\right],
\end{aligned}$$

where

$$Z^{A,B}_{N,\beta}(x) = \mathrm{E}_x\Big[e^{\sum_{(n,z)\in A_N^x\cup B_N}\{\beta\omega_{n,z}-\lambda(\beta)\}\mathbb{1}_{\{S_n^x=z\}}}\Big]\,.$$

In order to bound the last expectation, we use Hölder inequality with exponents $p,p,q > 1$, so that $\frac{2}{p}+\frac{1}{q} = 1$, with $p \in (2,\infty)$ sufficiently close to 2 so that $\sup_{N\in\mathbb{N}} \mathbb{E}\big[(Z^{A,B}_{N,\beta}(x))^p\big] < \infty$, thanks to Proposition 2.2.3. In particular, we obtain that

$$\mathbb{E}\left[\frac{Z^{A,B}_{N,\beta}(x)}{Z^A_{N,\beta}(x)}\cdot\frac{Z^{A,B}_{N,\beta}(y)}{Z^A_{N,\beta}(y)}\cdot\mathbb{1}_{E_N^c}\right] \leqslant \mathbb{E}\left[\left(\frac{Z^{A,B}_{N,\beta}}{Z^A_{N,\beta}}\right)^p\right]^{\frac{2}{p}}\mathbb{P}(E_N^c)^{\frac{1}{q}}\,.$$

We apply Hölder inequality again on the first term, with exponents $r,s > 1$, so that $\frac{1}{r}+\frac{1}{s} = 1$ and $r > 1$ is sufficiently close to 1 so that we have $\sup_{N\in\mathbb{N}} \mathbb{E}\Big[\big(Z^{A,B}_{N,\beta}\big)^{pr}\Big] < \infty$, by Proposition 2.2.3. This way, we obtain that

$$\mathbb{E}\left[\left(\frac{Z^{A,B}_{N,\beta}}{Z^A_{N,\beta}}\right)^p\right]^{\frac{2}{p}} \leqslant \mathbb{E}\Big[\big(Z_{N,\beta}\big)^{pr}\Big]^{\frac{2}{pr}}\mathbb{E}\Big[\big(Z^A_{N,\beta}\big)^{-ps}\Big]^{\frac{2}{ps}}\,.$$

By Proposition 2.2.2, we also have that $\sup_{N\in\mathbb{N}} \mathbb{E}\Big[\big(Z^A_{N,\beta}\big)^{-ps}\Big] < \infty$. Therefore, we have showed that there exists a constant $C = C(\beta) \in (0,\infty)$, such that

$$\mathbb{E}\left[\frac{\mathrm{E}_{x,y}\big[e^{\mathsf{H}^x_{A,B}(\omega)+\mathsf{H}^y_{A,B}(\omega)}\mathbb{1}_{S^x\cap S^y\cap B_N^{\mathsf{strip}}\neq\varnothing}\big]}{Z^A_{N,\beta}(x)Z^A_{N,\beta}(y)}\mathbb{1}_{E_N^c}\right] \leqslant C\,\mathbb{P}(E_N^c)^{\frac{1}{q}}\,.$$

for some $q > 1$. Thus,

$$N^{\frac{d}{2}-1} \sum_{x,y \in \mathbb{Z}^d} \frac{\varphi_N(x,y)}{N^d} \mathbb{E}\left[\frac{\mathrm{E}_{x,y}\left[e^{\mathsf{H}^x_{A,B}(\omega)+\mathsf{H}^y_{A,B}(\omega)} \mathbb{1}_{S^x \cap S^y \cap B_N^{\mathsf{strip}} \neq \varnothing}\right]}{Z^A_{N,\beta}(x) Z^A_{N,\beta}(y)} \mathbb{1}_{E_N^c} \right]$$
$$\leqslant C\, N^{\frac{d}{2}-1} \exp\left(\frac{-\mathsf{h}^\gamma (\log N)^\gamma}{q\, c_\beta} \right) \sum_{x,y \in \mathbb{Z}^d} \frac{\varphi_N(x,y)}{N^d} \xrightarrow[N\to\infty]{} 0\,,$$

because $\gamma > 1$ and $\sum_{x,y\in\mathbb{Z}^d} \frac{\varphi_N(x,y)}{N^d} \leqslant C\, \|\varphi\|_1^2$. Recall now decomposition (2.2.60). We have shown that

$$N^{\frac{d}{2}-1} \sum_{x,y \in \mathbb{Z}^d} \frac{\varphi_N(x,y)}{N^d} \mathbb{E}\left[\frac{\mathrm{E}_{x,y}\left[e^{\mathsf{H}^x_{A,B}(\omega)+\mathsf{H}^y_{A,B}(\omega)} \mathbb{1}_{S^x \cap S^y \cap B_N^{\mathsf{strip}} \neq \varnothing}\right]}{Z^A_{N,\beta}(x) Z^A_{N,\beta}(y)} \right] \xrightarrow[N\to\infty]{} 0\,. \tag{2.2.65}$$

Similarly, we can show that

$$\begin{aligned} &N^{\frac{d}{2}-1} \sum_{x,y \in \mathbb{Z}^d} \frac{\varphi_N(x,y)}{N^d} \mathbb{E}\left[\frac{\mathrm{E}_{x,y}\left[e^{\mathsf{H}^x_{A,B}(\omega)} \mathbb{1}_{S^x \cap S^y \cap B_N^{\mathsf{strip}} \neq \varnothing}\right]}{Z^A_{N,\beta}(x) Z^A_{N,\beta}(y)} \right] \xrightarrow[N\to\infty]{} 0\,, \\ &N^{\frac{d}{2}-1} \sum_{x,y \in \mathbb{Z}^d} \frac{\varphi_N(x,y)}{N^d} \mathbb{E}\left[\frac{\mathrm{E}_{x,y}\left[e^{\mathsf{H}^y_{A,B}(\omega)} \mathbb{1}_{S^x \cap S^y \cap B_N^{\mathsf{strip}} \neq \varnothing}\right]}{Z^A_{N,\beta}(x) Z^A_{N,\beta}(y)} \right] \xrightarrow[N\to\infty]{} 0\,, \\ &N^{\frac{d}{2}-1} \sum_{x,y \in \mathbb{Z}^d} \frac{\varphi_N(x,y)}{N^d} \mathbb{E}\left[\frac{1}{Z^A_{N,\beta}(x) Z^A_{N,\beta}(y)} \right] \mathrm{P}_{x,y}\left(S^x \cap S^y \cap B_N^{\mathsf{strip}} \neq \varnothing\right) \xrightarrow[N\to\infty]{} 0\,. \end{aligned} \tag{2.2.66}$$

The steps to do that are quite similar to the steps we followed to prove (2.2.65). Therefore, the proof of (2.2.58) has been completed. Then, the proof of (2.2.59) follows exactly the same lines, since $\mathsf{C}^{A,B^{\geqslant}}_{N,\beta}(x,y)$ admits a similar decomposition to (2.2.60).

(Step 3) Recall from (2.2.48) that we have to show that

$$N^{\frac{d-2}{4}} \sum_{x \in \mathbb{Z}^d} \frac{\varphi_N(x)}{N^{\frac{d}{2}}} \left(\frac{\hat{Z}^A_{N,\beta}(x)}{Z^A_{N,\beta}(x)} - \left(Z^{B^{\geqslant}}_{N,\beta}(x) - 1 \right) \right) \xrightarrow[N\to\infty]{L^1(\mathbb{P})} 0\,.$$

In Steps 1 and 2 we showed that if one decomposes $\hat{Z}^A_{N,\beta}(x)$ as

$$\hat{Z}^A_{N,\beta}(x) = \hat{Z}^{A,C}_{N,\beta}(x) + \hat{Z}^{A,B^<}_{N,\beta}(x) + \hat{Z}^{A,B^{\geqslant}}_{N,\beta}(x)$$

(recall their definitions from (2.2.49), (2.2.51), (2.2.52)) then one has that

$$N^{\frac{d-2}{4}} \sum_{x \in \mathbb{Z}^d} \frac{\varphi_N(x)}{N^{\frac{d}{2}}} \frac{\hat{Z}^{A,C}_{N,\beta}(x)}{Z^A_{N,\beta}(x)} \xrightarrow[N\to\infty]{L^2(\mathbb{P})} 0$$

and

$$N^{\frac{d-2}{4}} \sum_{x \in \mathbb{Z}^d} \frac{\varphi_N(x)}{N^{\frac{d}{2}}} \frac{\hat{Z}^{A,B^<}_{N,\beta}(x)}{Z^A_{N,\beta}(x)} \xrightarrow[N\to\infty]{L^2(\mathbb{P})} 0\,.$$

Therefore, this last step will be devoted to showing that

$$N^{\frac{d-2}{4}} \sum_{x \in \mathbb{Z}^d} \frac{\varphi_N(x)}{N^{\frac{d}{2}}} \left(\frac{\hat{Z}^{A,B^{\geqslant}}_{N,\beta}(x)}{Z^A_{N,\beta}(x)} - \left(Z^{B^{\geqslant}}_{N,\beta}(x) - 1 \right) \right) \xrightarrow[N\to\infty]{L^1(\mathbb{P})} 0\,.$$

We can rewrite the expansion of $Z^{A,B^{\geqslant}}_{N,\beta}(x)$, according to the last point that the polymer samples inside A^x_N and the first point that it samples in $B^{\geqslant}_N$, where we recall the definition of $B^{\geqslant}_N$, from

(2.2.7). In particular,

$$\hat{Z}^{A,B^{\geqslant}}_{N,\beta}(x) = \sum_{(t,w)\in A^x_N,\,(r,z)\in B^{\geqslant}_N} Z^A_{0,t,\beta}(x,w)\cdot\, q_{r-t}(z-w)\,\cdot\sigma\,\xi_{r,z}\cdot\; Z_{r,N,\beta}(z)\,. \tag{2.2.67}$$

where $Z^A_{0,t,\beta}(x,w)$ is the point-to-point partition function from $(0,x)$ to (t,w), defined by

$$Z^A_{0,t,\beta}(x,w) := \sum_{\tau\subset A^x_N\cap([0,t]\times\mathbb{Z}^d):\tau\ni(t,w)} \sigma^{|\tau|} q^{(0,x)}(\tau)\xi(\tau)\,. \tag{2.2.68}$$

and by $Z^A_{0,t,\beta}(x,w) := 1$ if $(t,w) = (0,x)$. We will show that if we replace $q_{r-t}(z-w)$ by $q_r(z-x)$ in the expansion of

$$N^{\frac{d-2}{4}}\sum_{x\in\mathbb{Z}^d}\frac{\varphi_N(x)}{N^{\frac{d}{2}}}\cdot\frac{\hat{Z}^{A,B^{\geqslant}}_{N,\beta}(x)}{Z^A_{N,\beta}(x)}\,,$$

via (2.2.67), then the corresponding error vanishes in $L^1(\mathbb{P})$, as $N\to\infty$. Note that if we perform this replacement, then the right hand side of (2.2.67) becomes exactly equal to

$$Z^A_{N,\beta}(x)(Z^{B^{\geqslant}}_{N,\beta}(x)-1)$$

and this will lead to the cancellation of the corresponding denominator. We define the set

$$B^{\geqslant}_N(x) := \big\{(r,z)\in B^{\geqslant}_N:\ |z-x| < r^{\frac{1}{2}+\delta_\varepsilon}\big\}\,.$$

where α_ε is defined in (2.2.2). Then by first restricting to $(r,z)\in B^{\geqslant}_N(x)$, we want to show that the $L^1(\mathbb{P})$ norm of

$$N^{\frac{d-2}{4}}\sum_{x\in\mathbb{Z}^d}\frac{\varphi_N(x)}{N^{\frac{d}{2}}}\sum_{\substack{(t,w)\in A^x_N,\\(r,z)\in B^{\geqslant}_N(x)}}\frac{Z^A_{0,t,\beta}(x,w)}{Z^A_{N,\beta}(x)}\Big(q_{r-t}(z-w)-q_r(z-x)\Big)\cdot\sigma\,\xi_{r,z}\cdot Z_{r,N,\beta}(z)\,, \tag{2.2.69}$$

vanishes as $N\to\infty$. We note that the rightmost sum in (2.2.69) is essentially over points $(t,w)\in A^x_N$, so that $q_t(x-w)\neq 0$, because otherwise the point to point partition function $Z^A_{0,t,\beta}(x,w)$ is zero. In that case, we observe that if due to the periodicity of the random walk, $q_{r-t}(z-w)=0$ then we also have that $q_r(z-x)=0$, since $q_t(x-w)\neq 0$. Therefore, we shall assume that $q_{r-t}(z-w), q_r(z-x)\neq 0$ from now on. By Theorem 2.3.11 in [LL10], we have that for $(r,z)\in B^{\geqslant}_N(x)$,

$$\begin{aligned} q_r(z-x) =& 2g_{\frac{r}{d}}(z-x)\exp\Big(O\big(\tfrac{1}{r}+\tfrac{|z-x|^4}{r^3}\big)\Big)\cdot\mathbb{1}_{q_r(z-x)\neq 0}\\ =& 2g_{\frac{r}{d}}(z-x)\exp\big(O(r^{-1+4\delta_\varepsilon})\big)\cdot\mathbb{1}_{q_r(z-x)\neq 0}\,. \end{aligned} \tag{2.2.70}$$

Furthermore, for $(t,w)\in A^x_N$ we have that

$$\begin{aligned} q_{r-t}(z-w) =& 2g_{\frac{r-t}{d}}(z-w)\exp\Big(O\big(\tfrac{1}{r-t}+\tfrac{|z-w|^4}{(r-t)^3}\big)\Big)\cdot\mathbb{1}_{q_{r-t}(z-w)\neq 0}\\ =& 2g_{\frac{r-t}{d}}(z-w)\exp\big(O(r^{-1+4\delta_\varepsilon})\big)\cdot\mathbb{1}_{q_{r-t}(z-w)\neq 0}\,, \end{aligned} \tag{2.2.71}$$

because we have that

$$|z-w|\leqslant|z-x|+|x-w|\leqslant r^{\frac{1}{2}+\delta_\varepsilon}+N^{\frac{\varepsilon}{2}+\alpha_\varepsilon} = r^{\frac{1}{2}+\delta_\varepsilon}+(N^\varepsilon)^{\frac{1}{2}+\delta_\varepsilon}\leqslant 2r^{\frac{1}{2}+\delta_\varepsilon}\,,$$

for large N since $r\in[N^\varrho,N]$. Also, we have that for large N, $|r-t|\geqslant\frac{1}{2}r$, since $t\leqslant N^\varepsilon$.

Let us derive some bounds for

$$\sup\left\{\left|\frac{q_r(z-x)}{q_{r-t}(z-w)}-1\right| : r > N^{\varrho}\,,\, t \leqslant N^{\varepsilon}\,, |w-x| < N^{\frac{\varepsilon}{2}+\alpha_\varepsilon}, |z-x| < r^{\frac{1}{2}+\delta_\varepsilon}\right\}. \tag{2.2.72}$$

We have that

$$\frac{q_r(z-x)}{q_{r-t}(z-w)} = \frac{g_{\frac{r}{d}}(z-x)}{g_{\frac{r-t}{d}}(z-w)} \cdot e^{O(r^{-1+4\delta_\varepsilon})} = \left(\tfrac{r-t}{r}\right)^{\frac{d}{2}} \cdot e^{\frac{2}{d}\left(\frac{|z-w|^2}{r-t}-\frac{|z-x|^2}{r}\right)} \cdot e^{O(r^{-1+4\delta_\varepsilon})}\,, \tag{2.2.73}$$

by (2.2.70) and (2.2.71). First, we have that $e^{O(r^{-1+4\delta_\varepsilon})} \leqslant e^{O(N^{\varrho(4\delta_\varepsilon-1)})}$ and for large N

$$1 \geqslant \left(\tfrac{r-t}{r}\right)^{\frac{d}{2}} = \left(1-\tfrac{t}{r}\right)^{\frac{d}{2}} \geqslant \left(1-N^{\varepsilon-\varrho}\right)^{\frac{d}{2}} \geqslant 1-\tfrac{d}{2}N^{\varepsilon-\varrho}\,,$$

using the inequality $(1+x)^\gamma \geqslant 1+\gamma x$ for $x \geqslant -1$ and $\gamma > 0$. Moreover, looking at the exponent in (2.2.73) we have

$$\tfrac{|z-w|^2}{r-t} - \tfrac{|z-x|^2}{r} = \left(\tfrac{|z-w|}{\sqrt{r-t}} - \tfrac{|z-x|}{\sqrt{r}}\right) \cdot \left(\tfrac{|z-w|}{\sqrt{r-t}} + \tfrac{|z-x|}{\sqrt{r}}\right). \tag{2.2.74}$$

Then, for the first factor in the right hand side of (2.2.74) we have by triangle inequality that

$$\tfrac{|z-w|}{\sqrt{r-t}} - \tfrac{|z-x|}{\sqrt{r}} \leqslant \tfrac{|z-x|}{\sqrt{r-t}} - \tfrac{|z-x|}{\sqrt{r}} + \tfrac{|x-w|}{\sqrt{r-t}}\,. \tag{2.2.75}$$

For the first summand on the right hand side of (2.2.75),

$$\begin{aligned}\tfrac{|z-x|}{\sqrt{r-t}} - \tfrac{|z-x|}{\sqrt{r}} = |z-x| \cdot \left(\tfrac{1}{\sqrt{r-t}} - \tfrac{1}{\sqrt{r}}\right) &= |z-x| \cdot \tfrac{t}{\sqrt{r}\sqrt{r-t}\,(\sqrt{r}+\sqrt{r-t})} \\ &\leqslant \tfrac{2}{1+\sqrt{2}}\,\tfrac{t\cdot|z-x|}{r^{3/2}}\,,\end{aligned}$$

where we used that $r-t \geqslant \frac{1}{2}r$ for large N. Since, $|z-x| \leqslant r^{\frac{1}{2}+\delta_\varepsilon}$ and $r > N^\varrho$ we have that

$$\tfrac{|z-x|}{\sqrt{r-t}} - \tfrac{|z-x|}{\sqrt{r}} \leqslant \tfrac{2}{1+\sqrt{2}}\,\tfrac{t\cdot|z-x|}{r^{3/2}} \leqslant \tfrac{2}{1+\sqrt{2}} \cdot t\, r^{\delta_\varepsilon-1} \leqslant \tfrac{2}{1+\sqrt{2}} \cdot N^{\varepsilon+\varrho\,(\delta_\varepsilon-1)}\,.$$

For the second summand on the right hand side of (2.2.75) we have, using $r-t \geqslant \frac{1}{2}r$

$$\tfrac{|x-w|}{\sqrt{r-t}} \leqslant \sqrt{2} \cdot \tfrac{|x-w|}{\sqrt{r}} \leqslant \sqrt{2} \cdot N^{\frac{\varepsilon-\varrho}{2}+\alpha_\varepsilon}\,,$$

since $r > N^\varrho$ and $|x-w| \leqslant N^{\frac{\varepsilon}{2}+\alpha_\varepsilon}$. Moreover, for the second factor in the right hand side of (2.2.74) we have that

$$\tfrac{|z-w|}{\sqrt{r-t}} + \tfrac{|z-x|}{\sqrt{r}} \leqslant 2\sqrt{2}r^{\delta_\varepsilon} + r^{\delta_\varepsilon} \leqslant (1+2\sqrt{2})\,N^{\delta_\varepsilon}\,.$$

Therefore, we have that for the left hand side of (2.2.74) that there exists a constant $C \in (0,\infty)$ such that

$$\tfrac{|z-w|^2}{r-t} - \tfrac{|z-x|^2}{r} \leqslant C\,N^{\delta_\varepsilon+\max\left\{\varepsilon+\varrho(\delta_\varepsilon-1),\frac{\varepsilon-\varrho}{2}+\alpha_\varepsilon\right\}} \leqslant C\,N^{\varepsilon-\varrho+2\delta_\varepsilon}\,.$$

Therefore,

$$\begin{aligned}\frac{q_r(z-x)}{q_{r-t}(z-w)} - 1 &\leqslant \frac{g_{\frac{r}{d}}(z-x)}{g_{\frac{r-t}{d}}(z-w)} \cdot e^{O(N^{\varrho(4\delta_\varepsilon-1)})} - 1 \\ &\leqslant e^{O(N^{\varepsilon-\varrho+2\delta_\varepsilon})+O(N^{\varrho(4\delta_\varepsilon-1)})} - 1 = O(N^{\varepsilon-\varrho+4\delta_\varepsilon})\,.\end{aligned}$$

Through similar reasoning we obtain that

$$\begin{aligned}\frac{q_r(z-x)}{q_{r-t}(z-w)}-1 &\geqslant \frac{g_{\frac{r}{d}}(z-x)}{g_{\frac{r-t}{d}}(z-w)}\cdot e^{-O(N^{\varrho(4\delta_\varepsilon-1)})}-1\\ &\geqslant \left(1-\tfrac{d}{2}N^{\varepsilon-\varrho}\right)\cdot e^{-O(N^{\frac{\varepsilon-\varrho}{2}+3\delta_\varepsilon})-O(N^{\varrho(4\delta_\varepsilon-1)})}-1\\ &\geqslant \left(1-\tfrac{d}{2}N^{\varepsilon-\varrho}\right)\cdot\left(1-O(N^{\frac{\varepsilon-\varrho}{2}+4\delta_\varepsilon})\right)-1\\ &\geqslant -O(N^{\frac{\varepsilon-\varrho}{2}+4\delta_\varepsilon})\,.\end{aligned}$$

Combining both upper and lower bounds we conclude that

$$\begin{aligned}&\sup\left\{\left|\frac{q_r(z-x)}{q_{r-t}(z-w)}-1\right|: r>N^\varrho\,,\, t\leqslant N^\varepsilon\,, |w-x|<N^{\frac{\varepsilon}{2}+\alpha_\varepsilon}, |z-x|<r^{\frac{1}{2}+c}\right\}\\ =&O(N^{\frac{\varepsilon-\varrho}{2}+4\delta_\varepsilon})\,.\end{aligned}$$

By Cauchy-Schwarz we obtain the following estimate for the L^1-norm of (2.2.69),

$$\begin{aligned}&N^{\frac{d-2}{4}}\sum_{x\in\mathbb{Z}^d}\left|\frac{\varphi_N(x)}{N^{\frac{d}{2}}}\right|\mathbb{E}\Bigg[\frac{1}{Z^A_{N,\beta}(x)}\\ &\qquad\times\Bigg|\sum_{\substack{(t,w)\in A^x_N,\\(r,z)\in B^{\geqslant}_N(x)}}Z^A_{0,t,\beta}(x,w)\Big(q_r(z-x)-q_{r-t}(z-w)\Big)\cdot\sigma\,\xi_{r,z}\cdot Z_{r,N,\beta}(z)\Bigg|\Bigg]\\ \leqslant&N^{\frac{d-2}{4}}\sum_{x\in\mathbb{Z}^d}\left|\frac{\varphi_N(x)}{N^{\frac{d}{2}}}\right|\mathbb{E}\Big[\frac{1}{Z^A_{N,\beta}(x)^2}\Big]^{1/2}\\ &\qquad\times\mathbb{E}\Bigg[\Bigg(\sum_{\substack{(t,w)\in A^x_N,\\(r,z)\in B^{\geqslant}_N(x)}}Z^A_{0,t,\beta}(x,w)\Big(q_r(z-x)-q_{r-t}(z-w)\Big)\cdot\sigma\,\xi_{r,z}\cdot Z_{r,N,\beta}(z)\Bigg)^2\Bigg]^{1/2}.\end{aligned}$$

By the negative moment estimate, i.e. Proposition 2.2.2 we have that

$$\sup_{N\in\mathbb{N}}\mathbb{E}\Big[Z^A_{N,\beta}(x)^{-2}\Big]<\infty\,.$$

Also, by expanding the square in the second expectation we have that it is equal to

$$\begin{aligned}&\sum_{\substack{(t,w)\in A^x_N,\\(r,z)\in B^{\geqslant}_N(x)}}\mathbb{E}\Big[Z^A_{0,t,\beta}(x,w)^2\Big]\,\Big(q_r(z-x)-q_{r-t}(z-w)\Big)^2\sigma^2\,\mathbb{E}\Big[Z_{r,N,\beta}(z)^2\Big]\\ =&\sum_{\substack{(t,w)\in A^x_N,\\(r,z)\in B^{\geqslant}_N(x)}}\mathbb{E}\Big[Z^A_{0,t,\beta}(x,w)^2\Big]\left\{1-\frac{q_r(z-x)}{q_{r-t}(z-w)}\right\}^2q^2_{r-t}(z-w)\sigma^2\,\mathbb{E}\Big[Z_{r,N,\beta}(z)^2\Big]\\ \leqslant&O(N^{\frac{\varepsilon-\varrho}{2}+4\delta_\varepsilon})\sum_{\substack{(t,w)\in A^x_N,\\(r,z)\in B^{\geqslant}_N(x)}}\mathbb{E}\Big[Z^A_{0,t,\beta}(x,w)^2\Big]\,q^2_{r-t}(z-w)\sigma^2\,\mathbb{E}\Big[Z_{r,N,\beta}(z)^2\Big]\,,\end{aligned}$$

by using estimate (2.2.72) and (2.2.70), (2.2.71). The last sum is bounded by $\mathbb{E}\Big[\big(\hat{Z}^{A,B^{\geqslant}}_{N,\beta}\big)^2\Big]$. By adapting the proof of Lemma 2.2.8, one can show that $\mathbb{E}\Big[\big(\hat{Z}^{A,B^{\geqslant}}_{N,\beta}\big)^2\Big] = O\big(N^{-\varrho(\frac{d}{2}-1)}\big)$. Therefore,

$$\begin{aligned}
&N^{\frac{d-2}{4}} \sum_{x\in\mathbb{Z}^d} \left|\frac{\varphi_N(x)}{N^{\frac{d}{2}}}\right| \mathbb{E}\Bigg[\frac{1}{Z^A_{N,\beta}(x)} \\
&\qquad \times \Bigg| \sum_{\substack{(t,w)\in A^x_N,\\ (r,z)\in B^{\geqslant}_N(x)}} Z^A_{0,t,\beta}(x,w) \left\{1 - \frac{q_r(z-x)}{q_{r-t}(z-w)}\right\} q_{r-t}(z-w)\cdot\sigma\,\xi_{r,z}\cdot Z_{r,N,\beta}(z)\Bigg|\Bigg] \\
&\leqslant C\,\|\varphi\|_1\, \mathbb{E}\Big[\big(Z^A_{N,\beta}(x)\big)^{-2}\Big]^{\frac{1}{2}} N^{\frac{d-2}{4}}\cdot N^{\frac{\varepsilon-\varrho}{2}+4\delta_\varepsilon}\cdot N^{-\varrho(\frac{d-2}{4})}\,.
\end{aligned}$$

In order for the last bound to vanish we need that

$$\tfrac{d-2}{4} + \tfrac{\varepsilon-\varrho}{2} + 4\delta_\varepsilon - \varrho(\tfrac{d-2}{4}) < 0\,.$$

Rearranging this inequality, we need that

$$\frac{\frac{d-2}{4}+\frac{\varepsilon}{2}+4\delta_\varepsilon}{\frac{d-2}{2}+\frac{1}{2}} < \varrho\,. \tag{2.2.76}$$

This is possible since, first, $\dfrac{\frac{d-2}{4}+\frac{\varepsilon}{2}+4\delta_\varepsilon}{\frac{d-2}{2}+\frac{1}{2}} \in (0,1)$ because $\delta_\varepsilon \in (0,\frac{1-\varepsilon}{8})$ and second, because given a choice of $\varepsilon\in(0,1)$, we proved in Step 2 that (2.2.53) is valid for any $\varrho\in(\varepsilon,1)$, therefore we can choose ϱ, large enough, so that (2.2.76) is satisfied. To complete Step 3, one needs to show that we can lift the restriction $(r,z)\in B^{\geqslant}_N(x)$, that is, allow $(r,z)\in B^{\geqslant}_N$, such that $|z-x|\geqslant r^{\frac{1}{2}+\delta_\varepsilon}$ but this follows by standard moderate deviation estimates and is quite to similar to the proof of [CSZ20], thus we omit the details. $\square$

In order to complete the steps needed to prove Theorem 2.0.2, one has to show that also Proposition 2.2.7 is valid. But, this is a corollary of Theorem 2.0.1. Since we are using the diffusive scaling, the fact that $Z^{B^{\geqslant}}_{N,\beta}(x)$ is the partition function of a polymer which starts sampling noise after time N^ϱ for some $\varrho\in(0,1)$, does not change the asymptotic distribution.

Proof of Proposition 2.2.7. This Proposition is a corollary of Theorem 2.0.1, since one can see that the difference of

$$N^{\frac{d-2}{4}} \sum_{x\in\mathbb{Z}^d} \frac{\varphi_N(x)}{N^{\frac{d}{2}}}\big(Z_{N,\beta}(x)-1\big) \quad \text{and} \quad N^{\frac{d-2}{4}} \sum_{x\in\mathbb{Z}^d} \frac{\varphi_N(x)}{N^{\frac{d}{2}}}\big(Z^{B^{\geqslant}}_{N,\beta}(x)-1\big)\,.$$

vanishes in $L^2(\mathbb{P})$. More specifically, we have that

$$\begin{aligned}
&\left\| N^{\frac{d-2}{4}} \sum_{x\in\mathbb{Z}^d} \frac{\varphi_N(x)}{N^{\frac{d}{2}}}\big(Z_{N,\beta}(x)-1\big) - N^{\frac{d-2}{4}} \sum_{x\in\mathbb{Z}^d} \frac{\varphi_N(x)}{N^{\frac{d}{2}}}\big(Z^{B^{\geqslant}}_{N,\beta}(x)-1\big)\right\|^2_{L^2(\mathbb{P})} \\
&\leqslant N^{\frac{d}{2}-1}\sum_{n=1}^{N^\varrho}\sigma^2 \sum_{x,y\in\mathbb{Z}^d} \frac{\varphi_N(x,y)}{N^d}\, q_{2n}(x-y)\mathrm{E}[e^{\lambda_2(\beta)\mathsf{L}_{N-n}}]\,.
\end{aligned}$$

by recalling expression (2.1.7). We can bound the last quantity as follows

$$N^{\frac{d}{2}-1}\sum_{n=1}^{N^{\varrho}}\sigma^2\sum_{x,y\in\mathbb{Z}^d}\frac{\varphi_N(x,y)}{N^d}\,q_{2n}(x-y)\mathrm{E}[e^{\lambda_2(\beta)\mathsf{L}_{N-n}}]$$

$$\leqslant \mathrm{E}[e^{\lambda_2(\beta)\mathsf{L}_{\infty}}]\,N^{\frac{d}{2}-1}\sum_{n=1}^{N^{\varrho}}\sigma^2\sum_{x,y\in\mathbb{Z}^d}\frac{\varphi_N(x,y)}{N^d}\,q_{2n}(x-y)\,.$$

By Lemma 2.1.2 the main contribution to the sum

$$N^{\frac{d}{2}-1}\sum_{n=1}^{N^{\varrho}}\sigma^2\sum_{x,y\in\mathbb{Z}^d}\frac{\varphi_N(x,y)}{N^d}\,q_{2n}(x-y)\,.$$

comes from $n\in[\vartheta N, N]$ for ϑ small, therefore it converges to 0 as $N\to\infty$. $\square$

CHAPTER 3

Moments of the $2d$ directed polymer in the subcritical regime and a generalisation of the Erdös-Taylor theorem

In this chapter we study DPRE_d in the critical dimension $d = 2$. Since the framework we are going to work with presents slight variations compared to the previous chapter we describe it in detail in the following. Let $S = (S_n)_{n\geqslant 0}$ be a two-dimensional simple symmetric random walk and $\big(\omega_{n,z}\big)_{(n,z)\in\mathbb{N}\times\mathbb{Z}^2}$ a space-time field of i.i.d. random variables with $\mathbb{E}[\omega] = 0$, $\mathbb{E}[\omega^2] = 1$ and $\lambda(\beta) := \log \mathbb{E}[e^{\beta\omega}] < \infty$ for all $\beta > 0$. We use the notation $\mathrm{P}_{a,x}$ and $\mathrm{E}_{a,x}$ to denote the probability and the expectation with respect to the distribution of the random walk when the walk starts from $x \in \mathbb{Z}^2$ at time $a \in \mathbb{N}$. If either a or x are zero, we will omit them from the subscripts. We consider the (point-to-point) partition function

$$Z_{N,\beta}(x,y) = \mathrm{E}_x\Big[e^{\sum_{n=1}^{N-1}\{\beta\omega_{n,S_n}-\lambda(\beta)\}}\,\mathbb{1}_{\{S_N=y\}}\Big] \tag{3.0.1}$$

of the directed polymer, i.e. random walk, in the random environment ω, at inverse temperature $\beta > 0$. We also denote the point-to-plane partition function

$$Z_{N,\beta}(x) := \sum_{y\in\mathbb{Z}^2} Z_{N,\beta}(x,y), \tag{3.0.2}$$

and simply write $Z_{N,\beta}$ if $x = 0$.

We are going to focus on the intermediate disorder regime where inverse temperature vanishes as

$$\beta_N \approx \hat\beta\sqrt{\frac{\pi}{\log N}} \qquad \text{with } \hat\beta > 0\,. \tag{3.0.3}$$

We remind the reader that it was shown in [CSZ17b], that for $\beta_N \approx \hat\beta\sqrt{\frac{\pi}{\log N}}$ with $\hat\beta \in (0,1)$,

$$Z_{N,\beta_N} \xrightarrow[N\to\infty]{(d)} \exp(\varrho_{\hat\beta}\mathsf{X} - \tfrac{1}{2}\varrho_{\hat\beta}^2)\,,$$

where $\mathsf{X} \sim \mathcal{N}(0,1)$ and $\varrho_{\hat\beta}^2 = \log\big(\frac{1}{1-\hat\beta^2}\big)$, while for $\hat\beta \geqslant 1$, Z_{N,β_N} converges in distribution to 0. In particular, the *subcritical* regime $\hat\beta \in (0,1)$ coincides with the range of $\hat\beta$ for which the second moment of the partition function is uniformly bounded, that is $\sup_{N\geqslant 1}\mathbb{E}\big[(Z_{N,\beta_N})^2\big] < \infty$.

We recall that the emergence of such intermediate scaling can be guessed as follows. Using Gaussian environment for simplicity one has that

$$\mathbb{E}\Big[\big(Z_{N,\beta}\big)^2\Big] = \mathrm{E}^{\otimes 2}\Big[e^{\beta^2\,\mathsf{L}_N^{(1,2)}}\Big] = \mathrm{E}\Big[e^{\beta^2\,\mathsf{L}_N}\Big]\,, \tag{3.0.4}$$

where $\mathrm{E}^{\otimes 2}$ denotes the law of two independent, $2d$ simple random walks starting both at the origin, $\mathsf{L}_N^{(1,2)} := \sum_{n=1}^{N-1}\mathbb{1}_{\{S_n^1=S_n^2\}}$ denotes their *collision local time* up to time $N-1$ and $\mathsf{L}_N := \sum_{n=1}^{N-1}\mathbb{1}_{\{S_{2n}=0\}}$ denotes the number of returns to zero, up to time $N-1$, of a single random walk starting at 0. The second equality in (3.0.4) follows since $\mathsf{L}_N^{1,2} \overset{\text{law}}{=} \mathsf{L}_N$. A classical result of Erdös-Taylor [ET60] states that

$$\frac{\pi}{\log N}\,\mathsf{L}_N \xrightarrow[N\to\infty]{(d)} \mathrm{Exp}(1)\,, \tag{3.0.5}$$

where $\mathrm{Exp}(1)$ denotes an exponential random variable of parameter 1. Thus, it is not hard to see that under (3.0.3), one has $\sup_{N\geqslant 1} \mathbb{E}\big[(Z_{N,\beta_N})^2\big] < \infty$ if and only if $\hat\beta < 1$.

The first result of this chapter is to show that *all* positive moments of the *point-to-plane* partition function Z_{N,β_N} are uniformly bounded in the whole subcritical regime $\hat\beta < 1$ while, obviously, no moment higher than one exists in the limit at $\hat\beta \geqslant 1$. Combining this with the distributional convergence (3.0.3) allows us compute the limit of all moments. In particular, our first theorem is stated as:

Theorem 3.0.1. *Consider the point-to-plane partition function Z_{N,β_N} defined in (3.0.2) with an intermediate disorder scaling β_N as in (3.1.1), which is asymptotically equivalent to (3.0.3). Then, for every $\hat\beta \in (0,1)$ and $h \geqslant 0$, it holds that*

$$\lim_{N\to\infty} \mathbb{E}\Big[\big(Z_{N,\beta_N}\big)^h\Big] = \left(\frac{1}{1-\hat\beta^2}\right)^{\frac{h(h-1)}{2}} = \left(\lim_{N\to\infty} \mathbb{E}\Big[\big(Z_{N,\beta_N}\big)^2\Big]\right)^{\frac{h(h-1)}{2}}. \tag{3.0.6}$$

Furthermore, (3.0.6) is valid also for all $h < 0$ if we assume that the law of ω satisfies the following concentration property:

There exists $\gamma > 1$ and constants $c_1, c_2 \in (0,\infty)$ such that for all $n \in \mathbb{N}$, $(\omega_1,\dots,\omega_n)$ i.i.d. and all convex, 1-Lipschitz functions $f : \mathbb{R}^n \to \mathbb{R}$,

$$\mathbb{P}\Big(\big|f(\omega_1,\dots,\omega_n) - M_f\big| \geqslant t\Big) \leqslant c_1 \exp\Big(-\frac{t^\gamma}{c_2}\Big), \tag{3.0.7}$$

where M_f is a median of f.

Remark 3.0.2. We note that (3.0.7) is satisfied if ω is bounded or if it has a density of the form $\exp(-V+U)$ for $V, U : \mathbb{R} \to \mathbb{R}$, where V is strictly convex and U is bounded, see [Led01].

The above theorem in combination with an analogous to (3.0.4) computation for the h moment will, almost immediately, lead us to a generalisation of the Erdös-Taylor theorem (see [ET60] and [GS09] for a quenched path generalisation), to the case of the rescaled, total pairwise collision times of h (instead of just two as in [ET60, GS09]) independent, two-dimensional simple random walks. More specifically, let $\Gamma(a,1)$ denote the Gamma distribution, which is the law with density function $\frac{1}{\Gamma(a)} x^{a-1} e^{-x}\, \mathbb{1}_{\{x>0\}}$ and in the last expression $\Gamma(a)$ is the gamma function. Then,

Theorem 3.0.3. *Consider $h \in \mathbb{N}$ such that $h \geqslant 2$ and for $i = 1,\dots,h$ let $S^{(i)} = \big(S^{(i)}_n\big)_{n\geqslant 0}$ be independent simple symmetric random walks in $\mathbb{Z}^2$ starting all from the origin at time zero. Moreover, for $1 \leqslant i < j \leqslant h$ let*

$$\mathsf{L}_N^{(i,j)} := \sum_{n=1}^{N} \mathbb{1}_{\{S_n^{(i)} = S_n^{(j)}\}},$$

denote the collision local time of $S^{(i)}$ and $S^{(j)}$ until time N. Then

$$\frac{\pi}{\log N} \sum_{1\leqslant i<j\leqslant h} \mathsf{L}_N^{(i,j)} \xrightarrow[N\to\infty]{(d)} \Gamma\Big(\frac{h(h-1)}{2}, 1\Big),$$

More precisely, if $Y_N := \frac{\pi}{\log N} \sum_{1\leqslant i<j\leqslant h} \mathsf{L}_N^{(i,j)}$, Y is a random variable with law $\Gamma\big(\frac{h(h-1)}{2},1\big)$ and $M_{Y_N}(t)$, $M_Y(t)$ denote the associated moment generating functions, respectively, we have that

$$M_{Y_N}(t) \xrightarrow[N\to\infty]{} M_Y(t),$$

for all $t \in (0,1) := I$, which is the maximum interval $I \subset (0,\infty)$ where $M_Y(t) < \infty$, $t \in I$.

The main step towards the above two theorems is to establish that, in the subcritical regime, the moments of the two-dimensional *point-to-plane* partition function Z_{N,β_N} are uniformly bounded. To state the corresponding theorem, let us briefly introduce the *averaged partition functions*. For test functions $\varphi, \psi : \mathbb{R}^2 \to \mathbb{R}$ such that φ has compact support and ψ is bounded, we define the averaged partition function to be

$$Z_{N,\beta_N}(\varphi,\psi) := \frac{1}{N}\sum_{x,y} \varphi(\tfrac{x}{\sqrt{N}})\, Z_{N,\beta_N}(x,y)\, \psi(\tfrac{y}{\sqrt{N}}) \tag{3.0.8}$$

and introduce its centred version as $\bar{Z}_{N,\beta_N}(\varphi,\psi) := Z_{N,\beta_N}(\varphi,\psi) - \mathbb{E}\big[Z_{N,\beta_N}(\varphi,\psi)\big]$. Similarly, we introduce the centred version of the point-to-plane partition function as

$$\bar{Z}_{N,\beta_N} := Z_{N,\beta_N} - \mathbb{E}[Z_{N,\beta_N}] = Z_{N,\beta_N} - 1\,.$$

Theorem 3.0.4. *Let $\varphi, \psi : \mathbb{R}^2 \to \mathbb{R}$ be such that φ has compact support and ψ is bounded and consider the centred, averaged field $\bar{Z}_{N,\beta_N}(\varphi,\psi)$ with respect to φ, ψ, as in (3.0.8). Let also $w : \mathbb{R}^2 \to \mathbb{R}$ be a weight function such that $\log w$ is Lipschitz continuous. Then, for every $h \in \mathbb{N}$ with $h \geqslant 3$, $\hat{\beta} \in (0,1)$, there exist $\mathsf{a}_* = \mathsf{a}_*(h,\hat{\beta},w) \in (0,1)$ and $C = C(h,\hat{\beta},w) \in (0,\infty)$ such that for any $p,q \in (1,\infty)$ that satisfy $\frac{1}{p}+\frac{1}{q} = 1$ and $p\,q \leqslant \mathsf{a}_* \log N$, the following inequality holds:*

$$\left|\mathbb{E}\Big[\bar{Z}_{N,\beta_N}(\varphi,\psi)^h\Big]\right| \leqslant \Big(\frac{C\,p\,q}{\log N}\Big)^{\frac{h}{2}} \cdot \frac{1}{N^h} \cdot \left\|\frac{\varphi_N}{w_N}\right\|_{\ell^p}^h \|\psi_N\|_\infty^h \, \|w_N\|_{\ell^q}^h\,, \tag{3.0.9}$$

where for $x \in \mathbb{Z}^2$ we have $\varphi_N(x) := \varphi(x/\sqrt{N})$, $\psi_N(x) := \psi(x/\sqrt{N})$ and $w_N(x) := w(x/\sqrt{N})$. Moreover, for $\bar{Z}_{N,\beta_N}$ being the centred, point-to-plane partition function, it holds that

$$\sup_{N\in\mathbb{N}} \left|\mathbb{E}\Big[\big(\bar{Z}_{N,\beta_N}\big)^h\Big]\right| < \infty. \tag{3.0.10}$$

A version of inequality (3.0.9) at the *critical temperature* was established in [CSZ21+], where inequalities of this type were used as an input to prove uniqueness of the scaling limit of the polymer field at the critical temperature scaling. Here, we had, first, to extend this methodology to cover the subcritical regime and, most importantly, we had to pull out the explicit dependence of the constant on the right-hand-side of (3.0.9) on the parameters p and q. The subcriticality assumption, $\hat{\beta} \in (0,1)$, is reflected on the fact that the constant C is finite, compared to the critical case where it grows logarithmically with N and gets cancelled out by the logarithmically attenuating factor seen in (3.0.9). The precise knowledge of this dependence is crucial in order to derive the moment estimate (3.0.10) of the point-to-plane partition function. This is because in order to obtain the point-to-plane moment estimate, we would need to insert in (3.0.9) a delta-like function $\varphi_N(x) := N\mathbb{1}_{\{x=0\}}$ (as well as $\psi_N(x) \equiv 1$, but this is innocuous), which, however, leads to a blowing in N constant in the right hand side of (3.0.9). The idea to overcome this difficulty is to optimise the choice of p, q of the corresponding ℓ^p and ℓ^q norms and for this one needs to have the dependence of the right-hand constant on p, q. The latter turns out to be of the form pq leading to an optimal choice depending on N as $q := \mathsf{a}\log N$, which washes out the dependence on N.

As already mentioned, the general framework towards (3.0.9) is inspired by estimates in [CSZ21+], The latter was subsequently inspired by and generalised the work of Gu, Quastel and Tsai [GQT21], who introduced methods from spectral theory of Schrödinger operators with point interactions of Dell'Antonio, Figari and Teta [DFT94] and Dimock, Rajeev [DR04] to prove existence of all moments for the solution of the $2d$ stochastic heat equation (mSHE) at the critical temperature with L^2 initial data ψ, when averaged against a smooth test function φ. A novelty here (as well as in [CSZ21+], with the latter having a different focus and scope) is the extension from an L^2 setting to an ℓ^q setting* with $q \in (1, \infty)$, which, in combination with the optimisation idea introduced here, allows to also reach the case $q = \infty$. The desired extension comes from a combination of a renewal framework (see Section 3.1.2 and Proposition 3.2.4) as well as an extension of an inequality of Dell'Antonio, Figari and Teta (Proposition 3.1 in [DFT94]) (using a different and more robust methodology than [DFT94]) from an L^2 to an ℓ^q setting with $q \in (1, \infty)$.

The method we develop also allows to compute the asymptotics of the moments of the logarithmically scaled and averaged field

$$\frac{\sqrt{\log N}}{N} \sum_{x \in \mathbb{Z}^2} \varphi(\tfrac{x}{\sqrt{N}}) \big(Z_{N,\beta_N}(x) - 1 \big),$$

for $\varphi \in C_c(\mathbb{R}^2)$, thus allowing for the computation of higher moment correlations, answering, in the discrete setting, a question of Gu, Quastel and Tsai (see Remark 1.10 in [GQT21]). In particular, we establish that

Theorem 3.0.5. *Let $\varphi \in C_c(\mathbb{R}^2)$ and consider the centred and averaged field with respect to φ, that is*

$$\bar{Z}_{N,\beta_N}(\varphi, 1) := \frac{1}{N} \sum_{x \in \mathbb{Z}^2} \varphi(\tfrac{x}{\sqrt{N}}) \, \big(Z_{N,\beta_N}(x) - 1 \big) \; .$$

Then, for every $h \in \mathbb{N}$ with $h \geqslant 2$ and $\hat{\beta} \in (0,1)$,

$$\lim_{N \to \infty} (\log N)^{\frac{h}{2}} \, \mathbb{E}\Big[\bar{Z}_{N,\beta_N}(\varphi)^h \Big] = \begin{cases} \varrho_\varphi(\hat{\beta})^h \cdot (h-1)!! & , \text{ if } h \text{ is even} \\ 0 & , \text{ if } h \text{ is odd}, \end{cases}$$

where $\varrho_\varphi(\hat{\beta})$ is defined by

$$\varrho_\varphi^2(\hat{\beta}) := \frac{\pi \, \hat{\beta}^2}{1 - \hat{\beta}^2} \int_0^1 \mathrm{d}t \int_{(\mathbb{R}^2)^2} \mathrm{d}x \, \mathrm{d}y \, \varphi(x) g_t(x-y) \varphi(y),$$

with $g_t(x) := \frac{1}{2\pi t} e^{-|x|^2/2t}$ the two-dimensional heat kernel.

3.1. Auxiliary tools

In this section we develop all the necessary machinery for the proof of the main results.

3.1.1. Partition functions and chaos expansion. Let us start by denoting the transition probability kernel of the underlying, two-dimensional, simple random walk S by $q_n(x)$ for $n \in \mathbb{N}$ and $x \in \mathbb{Z}^2$, that is $q_n(x) := \mathrm{P}(S_n = x)$. Recall from (3.0.1) the definition of the *point-to-plane* partition function

$$Z_{N,\beta_N}(x) := \mathrm{E}_x \Big[e^{\sum_{n=1}^{N-1} \{ \beta_N \, \omega_{n,S_n} - \lambda(\beta_N) \}} \Big],$$

where β_N is chosen so that

$$\sigma^2_{N,\hat{\beta}} := e^{\lambda(2\beta_N) - 2\lambda(\beta_N)} - 1 = \frac{\hat{\beta}^2}{R_N}, \tag{3.1.1}$$

where

$$R_N := \mathrm{E}^{\otimes 2} \Big[\sum_{n=1}^{N} \mathbb{1}_{\{S_n^{(1)} = S_n^{(2)}\}} \Big] = \sum_{n=1}^{N} \sum_{z \in \mathbb{Z}^2} q_n(z)^2 = \sum_{n=1}^{N} q_{2n}(0) \, , \tag{3.1.2}$$

*the adaptation to the continuous L^q spaces is also possible denotes the expected collisions until time N of two independent, two-dimensional, simple random walks, starting from the origin. Note that [ET60]

$$R_N = \frac{\log N}{\pi} + \frac{\alpha}{\pi} + o(1)\,,$$

where $\alpha := \gamma + \log 16 - \pi \simeq 0.208$ and $\gamma \simeq 0.577$ is the Euler constant. By Taylor expansion in (3.1.1), this implies the asymptotic scaling of β_N as $\beta_N \sim \hat{\beta}\sqrt{\frac{\pi}{\log N}}$ for $N \to \infty$.

We shall also need the definition of the *point-to-point* partition functions. In particular, for $a, b \in \mathbb{N}$ with $a < b$ and $x, y \in \mathbb{Z}^2$, we define the *point-to-point* partition function from the space-time point (a, x) to (b, y) by

$$Z_{a,b,\beta_N}(x,y) := \mathrm{E}_{a,x}\Big[e^{\sum_{n=a+1}^{b-1} \{\beta_N\, \omega_{n,S_n} - \lambda(\beta_N)\}}\, \mathbb{1}_{\{S_b = y\}} \Big]\,, \tag{3.1.3}$$

Note that with these definitions,

$$Z_{N,\beta_N}(x) = \sum_{y\in\mathbb{Z}^2} Z_{0,N,\beta_N}(x,y)\,.$$

Given $\varphi, \psi : \mathbb{R}^2 \to \mathbb{R}$ such that φ has compact support and ψ is bounded, we can further define the *averaged partition functions* by,

$$\begin{aligned} Z_{a,b,\beta_N}(\varphi, y) &:= \sum_{x\in\mathbb{Z}^2} \varphi(\tfrac{x}{\sqrt{N}})\, Z_{a,b,\beta_N}(x,y)\,, \\ Z_{a,b,\beta_N}(x, \psi) &:= \sum_{y\in\mathbb{Z}^2} Z_{a,b,\beta_N}(x,y)\, \psi(\tfrac{y}{\sqrt{N}}) \end{aligned}$$

and

$$Z_{a,b,\beta_N}(\varphi, \psi) := \frac{1}{N}\sum_{x,y} \varphi(\tfrac{x}{\sqrt{N}})\, Z_{a,b,\beta_N}(x,y)\, \psi(\tfrac{y}{\sqrt{N}})\,. \tag{3.1.4}$$

For $(a, x), (b, y) \in \mathbb{N} \times \mathbb{Z}^2$ with $a < b$, the mean of each of the quantities above is computed as

$$\begin{aligned} \mathbb{E}\big[Z_{a,b,\beta_N}(\varphi, y)\big] &= q^N_{a,b}(\varphi, y) := \sum_{x\in\mathbb{Z}^2} \varphi(\tfrac{x}{\sqrt{N}})\, q_{a,b}(x,y)\,, \\ \mathbb{E}\big[Z_{a,b,\beta_N}(x, \psi)\big] &= q^N_{a,b}(x, \psi) := \sum_{y\in\mathbb{Z}^2} q_{a,b}(x,y)\, \psi(\tfrac{y}{\sqrt{N}}) \end{aligned} \tag{3.1.5}$$

and

$$\mathbb{E}\big[Z_{a,b,\beta_N}(\varphi, \psi)\big] = q^N_{a,b}(\varphi, \psi) := \frac{1}{N}\sum_{x,y\in\mathbb{Z}^2} \varphi(\tfrac{x}{\sqrt{N}})\, q_{a,b}(x,y)\psi(\tfrac{y}{\sqrt{N}})\,.$$

Next, we derive an expansion for the point-to-point partition function $Z_{a,b,\beta_N}(x, y)$ as a multilinear polynomial, which goes by the name of *chaos expansion*. This is the starting point of our analysis. Recalling (3.1.3) we have

$$Z_{a,b,\beta_N}(x,y) = \mathrm{E}_{a,x}\Big[\prod_{a<n<b}\prod_{z\in\mathbb{Z}^2} e^{\{\beta_N\, \omega_{n,z} - \lambda(\beta_N)\}\, \mathbb{1}_{\{S_n = z\}}}\, \mathbb{1}_{\{S_b = y\}} \Big]$$

and by using the fact that for $\lambda \in \mathbb{R}$, $e^{\lambda\, \mathbb{1}_{\{S_n=z\}}} = 1 + (e^{\lambda} - 1)\mathbb{1}_{\{S_n=z\}}$ we obtain

$$Z_{a,b,\beta_N}(x,y) = \mathrm{E}_{a,x}\Big[\prod_{a<n<b}\prod_{z\in\mathbb{Z}^2} \big(1 + \xi_{n,z}\mathbb{1}_{\{S_n=z\}}\big)\, \mathbb{1}_{\{S_b = y\}} \Big] \tag{3.1.6}$$

where $\xi_{n,z} := e^{\beta_N \omega_{n,z} - \lambda(\beta_N)} - 1$ are i.i.d. random variables with

$$\mathbb{E}[\xi] = 0\,, \quad \mathbb{E}[\xi^2] = e^{\lambda(2\beta_N) - 2\lambda(\beta_N)} - 1 =: \sigma_{N,\hat\beta}^2 \overset{N\to\infty}{\sim} \beta_N^2\,, \quad \mathbb{E}\big[|\xi|^k\big] \leqslant C_k\, \sigma_{N,\hat\beta}^k \quad \text{for } k \geqslant 3\,, \tag{3.1.7}$$

for some constants $C_k \in (0,\infty)$, $k \geqslant 3$. The asymptotic and the bound in (3.1.7) follow by Taylor expansion. Expanding the product in (3.1.6) yields the following expansion of $Z_{a,b,\beta_N}(x,y)$ as a multilinear polynomial of the variables $\xi_{n,z}$,

$$\begin{aligned} Z_{a,b,\beta_N}(x,y) &= q_{a,b}(x,y) \\ &+ \sum_{k\geqslant 1} \sum_{\substack{a<n_1<\cdots<n_k<b \\ z_1,\ldots,z_k\in\mathbb{Z}^2}} q_{a,n_1}(x,z_1)\,\xi_{n_1,z_1} \left\{ \prod_{j=2}^k q_{n_{j-1},n_j}(z_{j-1},z_j)\,\xi_{n_j,z_j} \right\} q_{n_k,b}(z_k,y)\,, \end{aligned} \tag{3.1.8}$$

which also leads to

$$\begin{aligned} Z_{a,b,\beta_N}(\varphi,\psi) &:= q_{a,b}^N(\varphi,\psi) \\ &+ \frac{1}{N} \sum_{k\geqslant 1} \sum_{\substack{a<n_1<\cdots<n_k<b \\ z_1,\ldots,z_k\in\mathbb{Z}^2}} q_{a,n_1}^N(\varphi,z_1)\xi_{n_1,z_1} \left\{ \prod_{j=2}^k q_{n_{j-1},n_j}(z_{j-1},z_j)\xi_{n_j,z_j} \right\} q_{n_k,b}^N(z_k,\psi) \end{aligned}$$

for the averaged point-to-point partition function. Using the notation

$$\bar Z_{N,\beta_N}(\varphi,\psi) := Z_{N,\beta_N}(\varphi,\psi) - \mathbb{E}\big[Z_{N,\beta_N}(\varphi,\psi)\big]$$

for the centred averaged partition function we have that

$$\begin{aligned} &\bar Z_{N,\beta_N}(\varphi,\psi) \\ &= \frac{1}{N} \sum_{k\geqslant 1} \sum_{\substack{z_1,z_2,\ldots,z_k \\ 0<n_1<\cdots<n_k<N}} q_{0,n_1}^N(\varphi,z_1)\xi_{n_1,z_1} \left\{ \prod_{j=2}^k q_{n_{j-1},n_j}(z_{j-1},z_j)\xi_{n_j,z_j} \right\} q_{n_k,N}^N(z_k,\psi). \end{aligned} \tag{3.1.9}$$

For simplicity, we will use the notation $Z_{N,\beta_N}(\varphi) := Z_{N,\beta_N}(\varphi,1)$ and $\bar Z_{N,\beta_N}(\varphi) := \bar Z_{N,\beta_N}(\varphi,1)$.

3.1.2. Renewal representation. We will also need certain renewal representations for the second moment of the point-to-point partition functions. These were introduced in [CSZ19b] but only mainly studied in the context of the critical directed polymer therein. Let $(a,x),(b,y) \in \mathbb{N}\times\mathbb{Z}^2$ with $a<b$. We define

$$U_N^{\beta_N}\big((a,x),(b,y)\big) := \sigma_{N,\hat\beta}^2\, \mathbb{E}\Big[Z_{a,b,\beta_N}(x,y)^2\Big]\,. \tag{3.1.10}$$

By translation invariance

$$U_N^{\beta_N}\big((a,x),(b,y)\big) = U_N^{\beta_N}(b-a,y-x) := \sigma_{N,\hat\beta}^2\, \mathbb{E}\Big[Z_{0,b-a,\beta_N}(y-x)^2\Big],$$

therefore it suffices to work with $U_N^{\beta_N}(n,x)$. We furthermore define $U_N^{\beta_N}(n,x) := \mathbb{1}_{\{x=0\}}$ if $n=0$. Using (3.1.8) and (3.1.7) we derive the expansion

$$\begin{aligned} U_N^{\beta_N}(n,x) &= \sigma_{N,\hat\beta}^2\, q_n^2(x) \\ &+ \sum_{k\geqslant 1} \sigma_{N,\hat\beta}^{2(k+1)} \sum_{\substack{0<n_1<\cdots<n_k<n \\ z_1,z_2,\ldots,z_k\in\mathbb{Z}^2}} q_{0,n_1}^2(0,z_1) \Big\{ \prod_{j=2}^k q_{n_{j-1},n_j}^2(z_{j-1},z_j) \Big\}\, q_{n_k,n}^2(z_k,x)\,. \end{aligned} \tag{3.1.11}$$

Moreover, for $0 \leqslant n \leqslant N$ we define

$$U_N^{\beta_N}(n) := \sum_{x \in \mathbb{Z}^2} U_N^{\beta_N}(n,x)\,. \tag{3.1.12}$$

We will, now, recast $U_N^{\beta_N}(n,x)$ and $U_N^{\beta_N}(n)$ in a renewal theory framework. We define a family of i.i.d. random vectors $(\mathsf{t}_i^{(N)}, \mathsf{x}_i^{(N)})_{i \geqslant 1}$, such that

$$\mathrm{P}\Big(\big(\mathsf{t}_1^{(N)}, \mathsf{x}_1^{(N)}\big) = (n,x) \Big) = \frac{q_n^2(x)}{R_N}\, \mathbb{1}_{\{n \leqslant N\}}$$

and moreover we let $\tau_k^{(N)} := \mathsf{t}_1^{(N)} + \cdots + \mathsf{t}_k^{(N)}$ and $S_k^{(N)} := \mathsf{x}_1^{(N)} + \cdots + \mathsf{x}_k^{(N)}$ if $k \geqslant 1$. For $k = 0$ we set $(\tau_0, S_0) := (0,0)$. Using this framework we see by (3.1.11) and (3.1.12) that

$$U_N^{\beta_N}(n,x) = \sum_{k \geqslant 0} \hat\beta^{2k}\, \mathrm{P}\big(\tau_k^{(N)} = n, S_k^{(N)} = x\big)$$

and

$$U_N^{\beta_N}(n) = \sum_{k \geqslant 0} \hat\beta^{2k}\, \mathrm{P}\big(\tau_k^{(N)} = n\big)\,.$$

Finally, we remark that

$$\sum_{n=0}^{N} U_N^{\beta_N}(n) = \mathbb{E}\Big[(Z_{N+1,\beta_N})^2\Big]\,. \tag{3.1.13}$$

3.1.3. Some useful results. We will make use of the following results on the limiting distribution of Z_{N,β_N} and the fluctuations of $\bar{Z}_{N,\beta_N}(\varphi)$, which were established in [CSZ17b].

Theorem A. [CSZ17b] Fix $\hat\beta \in (0,1)$ and let $\varrho_{\hat\beta}^2 := \log\big(\frac{1}{1-\hat\beta^2}\big)$. Then,

$$Z_{N,\beta_N} \xrightarrow[N\to\infty]{(d)} \exp\big(\varrho_{\hat\beta}\, \mathsf{X} - \tfrac{1}{2}\,\varrho_{\hat\beta}^2\big)\,,$$

where X has a standard normal distribution $\mathcal{N}(0,1)$.

Theorem B. [CSZ17b] Fix $\hat\beta \in (0,1)$ and $\varphi \in C_c(\mathbb{R}^2)$. Then,

$$\sqrt{\log N}\, \bar{Z}_{N,\beta_N}(\varphi) \xrightarrow[N\to\infty]{(d)} \mathcal{N}\big(0, \varrho_\varphi^2(\hat\beta)\big)\,,$$

where $\bar{Z}_{N,\beta_N}(\varphi) := \bar{Z}_{N,\beta_N}(\varphi, 1)$ is defined in (3.1.4),

$$\varrho_\varphi^2(\hat\beta) := \frac{\pi\,\hat\beta^2}{1-\hat\beta^2} \int_0^1 \mathrm{d}t \int_{(\mathbb{R}^2)^2} \mathrm{d}x\,\mathrm{d}y\, \varphi(x) g_t(x-y) \varphi(y)$$

and $g_t(x) := \frac{1}{2\pi t}\, e^{-|x|^2/2t}$ denotes the two-dimensional heat kernel.

3.2. Expansion of moments and integral inequalities

We shall hereafter use the notation

$$M_{N,h}^{\varphi,\psi} := \mathbb{E}\Big[\bar{Z}_{N,\beta_N}(\varphi,\psi)^h\Big]\,,$$

for the h^{th} centred moments of the averaged field (3.1.4).

3.2.1. Chaos expansion of moments. By (3.1.9) we have

$$M_{N,h}^{\varphi,\psi} = \frac{1}{N^h} \times \mathbb{E}\Bigg[\Bigg(\sum_{k\geqslant 1} \sum_{\substack{z_1,z_2,\ldots,z_k\in\mathbb{Z}^2\\ 0<n_1<\cdots<n_k<N}} q_{0,n_1}^N(\varphi,z_1)\xi_{n_1,z_1} \times \Big\{\prod_{j=2}^{k} q_{n_{j-1},n_j}(z_{j-1},z_j)\xi_{n_j,z_j}\Big\} q_{n_k,N}^N(z_k,\psi)\Bigg)^h\Bigg]. \tag{3.2.1}$$

When $h \in \mathbb{N}$, the power h on the right hand side of (3.2.1) can be expanded as

$$\sum_{k_1,\ldots,k_h\geqslant 1} \sum_{\substack{(n_i^{(r)},z_i^{(r)})\in\mathbb{N}\times\mathbb{Z}^2,\\ 1\leqslant i\leqslant k_r,\,1\leqslant r\leqslant h,\\ 0<n_1^{(r)}<\cdots<n_{k_r}^{(r)}<N}} \prod_{r=1}^{h} q_{0,n_1^{(r)}}^N(\varphi,z_1^{(r)})\xi_{n_1^{(r)},z_1^{(r)}} \times\Big\{\prod_{j=2}^{k_r} q_{n_{j-1}^{(r)},n_j^{(r)}}(z_{j-1}^{(r)},z_j^{(r)})\xi_{n_j^{(r)},z_j^{(r)}}\Big\} q_{n_k^{(r)},N}^N(z_{k_r}^{(r)},\psi)\,. \tag{3.2.2}$$

Note that every term in that expansion contains a product of disorder variables of the form

$$\prod_{r=1}^{h}\prod_{j=1}^{k_r} \xi_{n_j^{(r)},x_j^{(r)}}\,.$$

Therefore, after taking the expectation with respect to the environment and taking into account that the ξ variables have mean zero and are independent if they are indexed by different space time points, see (3.1.7), we see that the non-zero terms of the expansion of (3.2.1) will be those such that for every point $(n_j^{(r)},x_j^{(r)})$, $1\leqslant j\leqslant k_r$, $1\leqslant r\leqslant h$ there exists (at least one) $1\leqslant r'\leqslant h$, $1\leqslant j\leqslant k_{r'}$ such that $r\neq r'$ and $(n_j^{(r)},x_j^{(r)}) = (n_{j'}^{(r')},x_{j'}^{(r')})$, that is, every disorder variable $\xi_{n_j^{(r)},x_j^{(r)}}$ should appear at least twice in a product of disorder variables. Hence, a natural way to parametrise the sum (3.2.1) is to sum over the space-time locations of these coincidence points along with all the possible coincidence configurations. We will also use iteratively the Chapman-Kolmogorov equation $q_{t_1,t_2}(x,y) = \sum_{z\in\mathbb{Z}^2} q_{t_1,s}(x,z)\,q_{s,t_2}(z,y)$, $t_1<s<t_2$, for the simple random walk, to break down 'long range jumps', appearing in (3.2.2) via their transition probabilities, into smaller jumps, so that we can track the location of each random walk at each time t, see Figure 3.2.1. Let us introduce the framework which will allow to formalise the above.

For $h\geqslant 3$, let $I\vdash\{1,\ldots,h\}$ denote a partition $I = I_1\sqcup I_2\sqcup\cdots\sqcup I_m$ of $\{1,\ldots,h\}$ into disjoint subsets $I_1,\ldots,I_m$ with cardinality $|I|=m$. Given $I\vdash\{1,\ldots,h\}$, we define the equivalence relation $\overset{I}{\sim}$ such that for $k,\ell\in\{1,\ldots,h\}$, we have $k\overset{I}{\sim}\ell$ if k and ℓ belong to the same component of the partition I. For $\boldsymbol{x}=(x_1,\ldots,x_h)\in(\mathbb{Z}^2)^h$ and a partition I we will denote $\boldsymbol{x}\sim I$ if $x_k = x_\ell$ for all $k\overset{I}{\sim}\ell$. We shall also use the notation $(\mathbb{Z}^2)^h_I := \{\boldsymbol{x}\in(\mathbb{Z}^2)^h : \boldsymbol{x}\sim I\}$.

For $p\in(1,\infty)$ we define the I-restricted ℓ^p spaces $\ell^p\big((\mathbb{Z}^2)^h_I\big)$ via the norm

$$\|f\|_{\ell^p((\mathbb{Z}^2)^h_I)} := \Big(\sum_{x\in(\mathbb{Z}^2)^h_I} |f(x)|^p\Big)^{1/p}$$

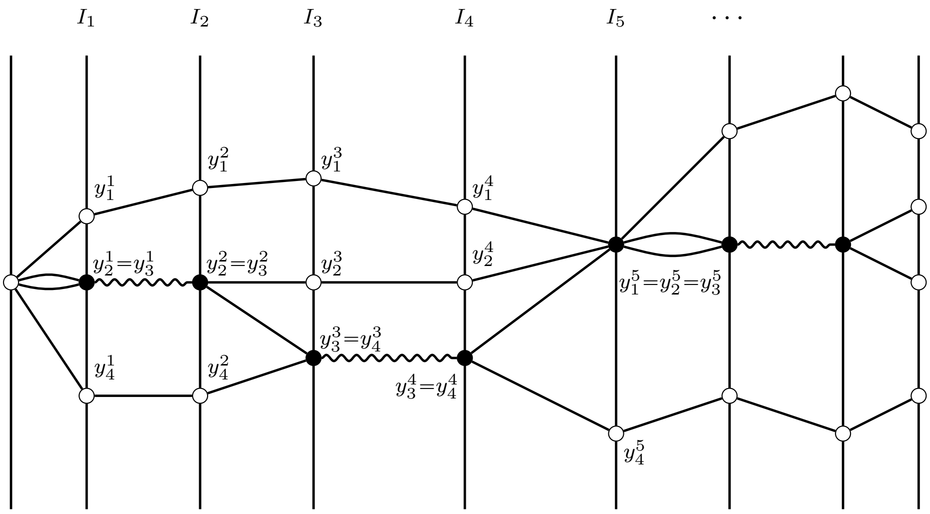


FIGURE 3.2.1. A diagrammatic representation of the expansion (3.2.10) for $\mathbb{E}\big[(\bar{Z}_{N,\beta_N})^4\big]$. The horizontal direction is the time direction, while the vertical lines correspond to different time slices, $\{n\} \times \mathbb{Z}^2$, $n \in \mathbb{N}$. We use straight lines to represent free evolution (3.2.5) and wiggly lines to represent replica evolution, see (3.2.8). We use filled dots to represent space-time points where disorder ξ is sampled.

for functions $f\colon (\mathbb{Z}^2)^h_I \to \mathbb{R}$. In shorthand, we will often write ℓ^p_I or just ℓ^p if there is no risk of confusion. For an integral operator $\mathsf{T} : \ell^q\big((\mathbb{Z}^2)^h_J\big) \to \ell^q\big((\mathbb{Z}^2)^h_I\big)$, we define the pairing

$$\langle f, \mathsf{T} g\rangle := \sum_{\boldsymbol{x}\in(\mathbb{Z}^2)^h_I, \boldsymbol{y}\in(\mathbb{Z}^2)^h_J} f(\boldsymbol{x})\mathsf{T}(\boldsymbol{x},\boldsymbol{y})g(\boldsymbol{y})\,. \tag{3.2.3}$$

The operator norm will be given by

$$\|\mathsf{T}\|_{\ell^q\to\ell^q} := \sup_{\|g\|_{\ell^q_J}\leqslant 1} \|\mathsf{T}g\|_{\ell^q_I} = \sup_{\|f\|_{\ell^p_I}\leqslant 1,\, \|g\|_{\ell^q_J}\leqslant 1} \langle f, \mathsf{T}g\rangle \tag{3.2.4}$$

for $p, q \in (1, \infty)$ conjugate exponents, i.e. $\frac{1}{p} + \frac{1}{q} = 1$.

For two partitions $I, J \vdash \{1, \ldots, h\}$ and $\boldsymbol{x}, \boldsymbol{y} \in (\mathbb{Z}^2)^h$ with $\boldsymbol{x} \sim I$ and $\boldsymbol{y} \sim J$ we define the *free evolution subject to constraints* I, J as

$$Q^{I,J}_n(\boldsymbol{x},\boldsymbol{y}) := \mathbb{1}_{\{\boldsymbol{x}\sim I\}} \prod_{i=1}^h q_n(y_i - x_i)\, \mathbb{1}_{\{\boldsymbol{y}\sim J\}}\,, \qquad \text{for } n \in \mathbb{N}\,. \tag{3.2.5}$$

$Q^{I,*}_n$ and $Q^{*,J}_n$ will denote the particular cases where I and J, respectively, are the partitions consisting only of singletons, i.e. $I = \{1\} \sqcup \cdots \sqcup \{h\}$. Moreover, if $I, J \vdash \{1, \ldots, h\}$, $\varphi, \psi : \mathbb{R}^2 \to \mathbb{R}$ and $n \in \mathbb{N}$ we define

$$\begin{aligned} Q^{*,J}_n(\varphi^{\otimes h}, \boldsymbol{y}) &:= \prod_{i=1}^h q^N_n(\varphi, y_i)\cdot \mathbb{1}_{\{\boldsymbol{y}\sim J\}} \\ Q^{I,*}_n(\boldsymbol{x}, \psi^{\otimes h}) &:= \mathbb{1}_{\{\boldsymbol{x}\sim I\}} \cdot \prod_{i=1}^h q^N_n(x_i, \psi)\,, \end{aligned} \tag{3.2.6}$$

see also (3.1.5). The mixed moment subject to a partition $I = I_1 \sqcup \cdots \sqcup I_m$ will be denoted by

$$\mathbb{E}[\xi^I] := \prod_{1\leqslant j\leqslant |I|,\, |I_j|\geqslant 2} \mathbb{E}[\xi^{|I_j|}]\,. \tag{3.2.7}$$

Using this formalism, we can then write

$$M_{N,h}^{\varphi,\psi} = \frac{1}{N^h} \sum_{k\geqslant 1} \sum_{\substack{0:=n_0<n_1<\cdots<n_k\leqslant N,\\ (I_1,\ldots,I_k)\in\mathcal{I},\\ m_i:=|I_i|<h,\ \boldsymbol{y}_i\in(\mathbb{Z}^2)^{m_i}}} Q_{n_1}^{*,I_1}(\varphi^{\otimes h}, \boldsymbol{y}_1)\mathbb{E}\Big[\xi^{I_1}\Big]$$
$$\times \prod_{i=2}^{k} Q_{n_i-n_{i-1}}^{I_{i-1},I_i}(\boldsymbol{y}_{i-1}, \boldsymbol{y}_i)\mathbb{E}\Big[\xi^{I_i}\Big] \cdot Q_{N-n_k}^{I_k,*}(\boldsymbol{y}_k, \psi^{\otimes h}),$$

where $\mathcal{I}$ is the set of all finite sequences of partitions of $\{1,\ldots,h\}$, $(I_1,\ldots,I_k)$, which satisfy the following condition: For every $r \in \{1,\ldots,h\}$ there exists $1 \leqslant i \leqslant k$ such that the block of I_i that contains r is non-trivial, i.e. it has cardinality equal or larger than 2. This restriction comes from the fact that $M_{N,h}^{\varphi,\psi}$ are centred moments and the fact that all terms in the expansion of (3.2.2) that contain a standalone ξ variable, vanish after taking the expectation $\mathbb{E}$, see also the discussion below (3.2.2).

Let $B = B(0,r) \subset \mathbb{R}^2$ be a ball containing the support of ψ (allowing the possibility of $r = \infty$, in case $\operatorname{supp}\psi = \mathbb{R}^2$). We then have that

$$Q_{N-n_k}^{I_k,*}\big(\boldsymbol{y}_k, \psi^{\otimes h}\big) \leqslant Q_{N-n_k}^{I_k,*}\big(\boldsymbol{y}_k, \|\psi\|_\infty^h\, \mathbb{1}_B^{\otimes h}\big) \leqslant \frac{c}{N} \sum_{n_{k+1}\in\{N+1,\ldots,2N\}} Q_{n_{k+1}-n_k}^{I_k,*}\big(\boldsymbol{y}_k, \|\psi\|_\infty^h\, \mathbb{1}_B^{\otimes h}\big),$$

with the latter inequality following because the probability that a random walk starts inside the ball $B(0,\sqrt{N}r) \subset \mathbb{R}^2$ at time $N - n_k$ and is still inside $B(0,\sqrt{N}r)$ at time $n_{k+1} - n_k$ with $n_{k+1} \in \{N+1,\ldots,2N\}$ is uniformly bounded away from zero.

Thus,

$$\big|\, M_{N,h}^{\varphi,\psi} \,\big| \leqslant \frac{c\,\|\psi\|_\infty^h}{N^{h+1}} \sum_{k\geqslant 1} \sum_{\substack{0:=n_0<n_1<\cdots<n_{k+1}\leqslant 2N,\\ (I_1,\ldots,I_k)\in\mathcal{I},\\ m_i:=|I_i|<h,\ \boldsymbol{y}_i\in(\mathbb{Z}^2)^{m_i}}} Q_{n_1}^{*,I_1}(\varphi^{\otimes h}, \boldsymbol{y}_1)\mathbb{E}\Big[|\xi|^{I_1}\Big]$$
$$\times \prod_{i=2}^{k} Q_{n_i-n_{i-1}}^{I_{i-1},I_i}(\boldsymbol{y}_{i-1}, \boldsymbol{y}_i)\mathbb{E}\Big[|\xi|^{I_i}\Big] \cdot Q_{n_{k+1}-n_k}^{I_k,*}(\boldsymbol{y}_k, \mathbb{1}_B^{\otimes h}).$$

We also need to define the *replica evolution*. For $I \vdash \{1,\ldots,h\}$ of the form $I = \{k,\ell\} \sqcup \bigsqcup_{j\neq k,\ell}\{j\}$

$$\mathsf{U}_n^I(\boldsymbol{x},\boldsymbol{y}) := \mathbb{1}_{\{\boldsymbol{x},\boldsymbol{y}\sim I\}} \cdot U_N^{\beta_N}(n, y_k - x_k) \cdot \prod_{i\neq k,\ell} q_n(y_i - x_i), \tag{3.2.8}$$

where $U_N^{\beta_N}(n, y_k - x_k)$ is defined in (3.1.10). The replica evolution operator will be used to contract consecutive appearances of the same partition I, with $|I| = h-1$ in the right-hand side of (3.2.8). In particular, note that if $I \vdash \{1,\ldots,h\}$, such that $|I| = h-1$, then

$$\mathsf{U}_n^I(\boldsymbol{x},\boldsymbol{y}) = \sum_{k\geqslant 0} \mathbb{E}[\xi^2]^k \sum_{0:=n_0<n_1<\cdots<n_k:=n} \sum_{\substack{\boldsymbol{y}_i\in(\mathbb{Z}^2)_I^h,\,1\leqslant i\leqslant k-1\\ \boldsymbol{y}_0:=\boldsymbol{x},\,\boldsymbol{y}_k:=\boldsymbol{y}}} \prod_{i=1}^{k} Q_{n_i-n_{i-1}}^{I;I}(\boldsymbol{y}_{i-1}, \boldsymbol{y}_i).$$

To be able to estimate the right-hand side of (3.2.8) we will upper bound it by enlarging the domain of the temporal sum in the right-hand side of (3.2.8) from $1 \leqslant n_1 < \cdots < n_{k+1} \leqslant 2N$ to $n_i - n_{i-1} \in \{1,\ldots,2N\}$ for all $1 \leqslant i \leqslant k+1$. This enlargement of the domain of summation deconvolves the temporal sum in the right-hand side of (3.2.8).

On this account, we introduce the discrete Laplace transforms of the operators Q and U,

$$\mathsf{Q}^{I,J}_{N,\lambda}(\boldsymbol{y},\boldsymbol{z}) := \sum_{n=1}^{2N} e^{-\lambda\frac{n}{N}}\, Q^{I,J}_{n}(\boldsymbol{y},\boldsymbol{z}), \qquad \boldsymbol{y},\boldsymbol{z}\in(\mathbb{Z}^2)^h,$$

$$\mathsf{U}^{I}_{N,\lambda}(\boldsymbol{y},\boldsymbol{z}) := \sum_{n=0}^{2N} e^{-\lambda\frac{n}{N}}\, \mathsf{U}^{I}_{n}(\boldsymbol{y},\boldsymbol{z}), \qquad \boldsymbol{y},\boldsymbol{z}\in(\mathbb{Z}^2)^h\,,$$

for $\lambda \geqslant 0$. In our case, it will be sufficient to work with $\lambda = 0$.

Let us define

$$\mathsf{P}^{I;J}_{N,\hat\beta} = \begin{cases} \mathsf{Q}^{I;J}_{N,0} & , \text{ if } |J| < h-1 \\ \mathsf{Q}^{I;J}_{N,0}\,\mathsf{U}^{J}_{N,0} & , \text{ if } |J| = h-1\,. \end{cases}$$

Note that the appearance of the operator $\mathsf{U}^{J}_{N,0}$ is necessarily preceded by a free evolution operator $\mathsf{Q}^{I;J}_{N,0}$, with $|J| = h-1$, see also Figure 3.2.1. In view of (3.2.8) and the discussion above we can now write

$$\big|M^{\varphi,\psi}_{N,h}\big| \leqslant \frac{c\,\|\psi\|^h_\infty}{N^{h+1}} \sum_{k\geqslant 1} \sum_{(I_1,\dots,I_k)\in\mathcal{I}} \Big\langle \varphi_N^{\otimes h}, \mathsf{P}^{*,I_1}_{N,\hat\beta}\,\mathsf{P}^{I_1,I_2}_{N,\hat\beta}\cdots \mathsf{P}^{I_k,*}_{N,\hat\beta}\mathbb{1}^{\otimes h}_{\sqrt{N}\,B}\Big\rangle \prod_{i=1}^{k}\mathbb{E}\Big[|\xi|^{I_i}\Big]\,,$$

where we recall the definition of the pairing $\langle\cdot\,,\cdot\rangle$ form (3.2.3) and note that the sum runs over partitions $I_1,\dots,I_k$ such that $I_j \neq I_{j+1}$ if $|I_j| = |I_{j+1}| = h-1$ for $1\leqslant j\leqslant k-1$.

Because of the assumption of Theorem 3.0.4 on ψ being merely a bounded function we will need to introduce weighted versions of the operators $\mathsf{U}^{I}_{N,\lambda}$, $\mathsf{Q}^{I;J}_{N,\lambda}$ and $\mathsf{P}^{I;J}_{N,\hat\beta}$. In particular, if $w:\mathbb{R}^2\to\mathbb{R}$ is such that $\log w$ is Lipschitz continuous with Lipschitz constant denoted by $C_w>0$, $w_N(x) = w\big(\frac{x}{\sqrt{N}}\big)$, we define for $\lambda\geqslant 0$,

$$\widehat{\mathsf{Q}}^{I;J}_{N,\lambda}(\boldsymbol{x},\boldsymbol{y}) := \frac{w_N^{\otimes h}(\boldsymbol{x})}{w_N^{\otimes h}(\boldsymbol{y})}\,\mathsf{Q}^{I;J}_{N,\lambda}(\boldsymbol{x},\boldsymbol{y})\,,$$

$$\widehat{\mathsf{U}}^{I}_{N,\lambda}(\boldsymbol{x},\boldsymbol{y}) := \frac{w_N^{\otimes h}(\boldsymbol{x})}{w_N^{\otimes h}(\boldsymbol{y})}\,\mathsf{U}^{I}_{N,\lambda}(\boldsymbol{x},\boldsymbol{y})\,,$$

where we recall that $w_N^{\otimes h}(\boldsymbol{x}) = w_N(x_1)\cdots w_N(x_h)$, if $\boldsymbol{x} = (x_1,\dots,x_h)$. We modify accordingly the operator $\mathsf{P}^{I;J}_{N,\hat\beta}$ into a new operator $\widehat{\mathsf{P}}^{I;J}_{N,\hat\beta}$,

$$\widehat{\mathsf{P}}^{I;J}_{N,\hat\beta} = \begin{cases} \widehat{\mathsf{Q}}^{I;J}_{N,0} & , \text{ if } |J| < h-1 \\ \widehat{\mathsf{Q}}^{I;J}_{N,0}\,\widehat{\mathsf{U}}^{J}_{N,0} & , \text{ if } |J| = h-1\,. \end{cases} \tag{3.2.9}$$

Therefore, we can now write

$$\begin{aligned} \big|M^{\varphi,\psi}_{N,h}\big| &\leqslant \frac{c\,\|\psi\|^h_\infty}{N^{h+1}} \sum_{k\geqslant 1} \sum_{(I_1,\dots,I_k)\in\mathcal{I}} \Big\langle \frac{\varphi_N^{\otimes h}}{w_N^{\otimes h}}, \widehat{\mathsf{P}}^{*,I_1}_{N,\hat\beta}\,\widehat{\mathsf{P}}^{I_1,I_2}_{N,\hat\beta}\cdots \widehat{\mathsf{P}}^{I_k,*}_{N,\hat\beta}\mathbb{1}^{\otimes h}_{\sqrt{N}\,B}w_N^{\otimes h}\Big\rangle \prod_{i=1}^{k}\mathbb{E}\Big[|\xi|^{I_i}\Big] \\ &\leqslant \frac{c\,\|\psi\|^h_\infty}{N^{h+1}} \sum_{k\geqslant 1} \sum_{(I_1,\dots,I_k)\in\mathcal{I}} \Big\langle \frac{\varphi_N^{\otimes h}}{w_N^{\otimes h}}, \widehat{\mathsf{P}}^{*,I_1}_{N,\hat\beta}\,\widehat{\mathsf{P}}^{I_1,I_2}_{N,\hat\beta}\cdots \widehat{\mathsf{P}}^{I_k,*}_{N,\hat\beta}w_N^{\otimes h}\Big\rangle \prod_{i=1}^{k}\mathbb{E}\Big[|\xi|^{I_i}\Big] \end{aligned} \tag{3.2.10}$$

where we bounded the indicator function $\mathbb{1}^{\otimes h}_{\sqrt{N}B}$ by 1 to obtain the second inequality. Passing to the operator norms (see (3.2.4)) we estimate

$$\left|M^{\varphi,\psi}_{N,h}\right| \leqslant \frac{c\,\|\psi\|^h_\infty}{N^{h+1}} \sum_{k\geqslant 1} \sum_{(I_1,\dots,I_k)\in\mathcal{I}} \left\| \widehat{\mathsf{P}}^{*,I_1}_{N,\hat\beta} \frac{\varphi^{\otimes h}_N}{w^{\otimes h}_N} \right\|_{\ell^p} \prod_{i=2}^k \left\| \widehat{\mathsf{P}}^{I_{i-1},I_i}_{N,\hat\beta} \right\|_{\ell^q\to\ell^q} \left\| \widehat{\mathsf{P}}^{I_k,*}_{N,\hat\beta} w^{\otimes h}_N \right\|_{\ell^q} \prod_{i=1}^k \mathbb{E}\Big[|\xi|^{I_i}\Big]. \tag{3.2.11}$$

This is the key expansion we will use for the Proof of Theorem 3.0.4.

3.2.2. Integral inequalities for the operators $\widehat{\mathsf{Q}}^{I;J}_{N,0}$ and $\widehat{\mathsf{U}}^{I}_{N,0}$. At this point, we will prove some intermediate results about the operators $\widehat{\mathsf{Q}}^{I;J}_{N,0}$, $\widehat{\mathsf{U}}^{I}_{N,0}$ that we will need along the way. In what follows we shall use the letter C to denote constants that may depend only on $h, \hat\beta$ and w but not on p and q. We will also use the letter c to denote absolute constants, i.e. constants that do not depend on $h, \hat\beta, w$ or p, q. Their value may change from line to line.

We start this subsection by stating a lemma from [CSZ21+] on the operator

$$\mathsf{Q}_{N,\lambda}(\boldsymbol{x},\boldsymbol{y}) := \sum_{n=1}^{2N} e^{-\frac{\lambda n}{N}} Q_n(\boldsymbol{x},\boldsymbol{y}).$$

Lemma 3.2.1 ([CSZ21+])**.** *Let $N \geqslant 1$, $h \geqslant 2$ and $\boldsymbol{x},\boldsymbol{y} \in (\mathbb{Z}^2)^h$. Then, there exists a constant $C \in (0,\infty)$ such that uniformly in N, $\boldsymbol{x},\boldsymbol{y}$ and $\lambda \geqslant 0$,*

$$\mathsf{Q}_{N,\lambda}(\boldsymbol{x},\boldsymbol{y}) \leqslant \begin{cases} \dfrac{C}{\left(1+|\boldsymbol{x}-\boldsymbol{y}|^2\right)^{h-1}} & \text{for all } \boldsymbol{x},\boldsymbol{y} \in (\mathbb{Z}^2)^h, \\[2ex] \dfrac{C}{N^{h-1}} \exp\left(\dfrac{-|\boldsymbol{x}-\boldsymbol{y}|^2}{C\,N}\right) & \text{if } |\boldsymbol{x}-\boldsymbol{y}| > \sqrt{N}. \end{cases}$$

We will use Lemma 3.2.1 to prove the following operator norm estimate.

The next proposition contains the central estimate. It is on the operator norm of operator $\widehat{\mathsf{Q}}^{I;J}_{N,0}$, as an operator from an $\ell^q \to \ell^q$, containing the explicit dependence on the parameters p, q.

Proposition 3.2.2. *Let $p, q \in (1,\infty)$ such that $\frac{1}{p}+\frac{1}{q}=1$. There exists a constant $C = C(h,w) \in (0,\infty)$, independent of p and q, such that for all $I, J \vdash \{1,\dots,h\}$ with $1 \leqslant |I|,|J| \leqslant h-1$ and $I \neq J$ when $|I| = |J| = h-1$,*

$$\left\| \widehat{\mathsf{Q}}^{I;J}_{N,0} \right\|_{\ell^q\to\ell^q} \leqslant C\,p\,q\,. \tag{3.2.12}$$

Proof. Let $I, J \vdash \{1,\dots,h\}$ with $1 \leqslant |I|,|J| \leqslant h-1$ and $I \neq J$ when $|I| = |J| = h-1$ and consider $f \in \ell^p\big((\mathbb{Z}^2)^h_I\big)$, $g \in \ell^q\big((\mathbb{Z}^2)^h_J\big)$. In view of (3.2.4), in order to prove (3.2.12), we need to prove that there exists a constant $C \in (0,\infty)$ such that

$$\sum_{\boldsymbol{x}\in(\mathbb{Z}^2)^h_I,\boldsymbol{y}\in(\mathbb{Z}^2)^h_J} f(\boldsymbol{x}) \mathsf{Q}^{I;J}_{N,0}(\boldsymbol{x},\boldsymbol{y}) \frac{w^{\otimes h}_N(\boldsymbol{x})}{w^{\otimes h}_N(\boldsymbol{y})} g(\boldsymbol{y}) \leqslant C\,p\,q\, \|f\|_{\ell^p} \|g\|_{\ell^q}\,. \tag{3.2.13}$$

Let

$$E_N := \Big\{ (\boldsymbol{x},\boldsymbol{y}) \in (\mathbb{Z}^2)^h_I \times (\mathbb{Z}^2)^h_J : |\boldsymbol{x}-\boldsymbol{y}| \leqslant C_0\sqrt{N} \Big\}. \tag{3.2.14}$$

for some $C_0 > 0$ to be determined. By the second inequality in Lemma 3.2.1 and the Lipschitz condition on $\log w$, we can choose C_0 large enough so that for all $(\boldsymbol{x},\boldsymbol{y}) \in E^c_N$ we have

$$\mathsf{Q}^{I;J}_{N,0}(\boldsymbol{x},\boldsymbol{y}) \frac{w^{\otimes h}_N(\boldsymbol{x})}{w^{\otimes h}_N(\boldsymbol{y})} \leqslant \frac{C}{N^{h-1}} \exp\big(-\tfrac{|\boldsymbol{x}-\boldsymbol{y}|}{\sqrt{N}}\big).$$

Therefore,

$$\sum_{(\boldsymbol{x},\boldsymbol{y})\in E_N^c} f(\boldsymbol{x})\mathsf{Q}^{I;J}_{N,0}(\boldsymbol{x},\boldsymbol{y})\frac{w_N^{\otimes h}(\boldsymbol{x})}{w_N^{\otimes h}(\boldsymbol{y})}g(\boldsymbol{y}) \leqslant \frac{C}{N^{h-1}} \sum_{(\boldsymbol{x},\boldsymbol{y})\in E_N^c} f(\boldsymbol{x}) \exp\big(-\tfrac{|\boldsymbol{x}-\boldsymbol{y}|}{\sqrt{N}}\big) g(\boldsymbol{y})$$

and by Hölder's inequality,

$$\begin{aligned}
&\frac{1}{N^{h-1}} \sum_{(\boldsymbol{x},\boldsymbol{y})\in E_N^c} f(\boldsymbol{x}) \exp\big(-\tfrac{|\boldsymbol{x}-\boldsymbol{y}|}{\sqrt{N}}\big) g(\boldsymbol{y}) \\
\leqslant & \frac{1}{N^{h-1}} \Bigg(\sum_{\boldsymbol{x}\in(\mathbb{Z}^2)^h_I, \boldsymbol{y}\in(\mathbb{Z}^2)^h_J} |f(\boldsymbol{x})|^p \exp\Big(-\tfrac{|\boldsymbol{x}-\boldsymbol{y}|}{\sqrt{N}}\Big)\Bigg)^{\frac{1}{p}} \Bigg(\sum_{\boldsymbol{x}\in(\mathbb{Z}^2)^h_I, \boldsymbol{y}\in(\mathbb{Z}^2)^h_J} |g(\boldsymbol{y})|^q \exp\Big(-\tfrac{|\boldsymbol{x}-\boldsymbol{y}|}{\sqrt{N}}\Big)\Bigg)^{\frac{1}{q}} \\
\leqslant & \, C\, N^{\frac{|J|}{p} + \frac{|I|}{q} - (h-1)} \, \|f\|_{\ell^p} \, \|g\|_{\ell^q} \\
\leqslant & \, C\, \|f\|_{\ell^p} \, \|g\|_{\ell^q} \,,
\end{aligned} \tag{3.2.15}$$

where the inequality in the last line of (3.2.15) follows by the assumption $|I|, |J| \leqslant h-1$. Thus,

$$\sum_{(\boldsymbol{x},\boldsymbol{y})\in E_N^c} f(\boldsymbol{x})\mathsf{Q}^{I;J}_{N,0}(\boldsymbol{x},\boldsymbol{y})\frac{w_N^{\otimes h}(\boldsymbol{x})}{w_N^{\otimes h}(\boldsymbol{y})}g(\boldsymbol{y}) \leqslant C\, \|f\|_{\ell^p} \, \|g\|_{\ell^q} \,,$$

for a constant $C \in (0,\infty)$. On the other hand, recalling that $\log w$ is Lipschitz with Lipschitz constant C_w and (3.2.14), we get that

$$\sum_{(\boldsymbol{x},\boldsymbol{y})\in E_N} f(\boldsymbol{x})\mathsf{Q}^{I;J}_{N,0}(\boldsymbol{x},\boldsymbol{y})\frac{w_N^{\otimes h}(\boldsymbol{x})}{w_N^{\otimes h}(\boldsymbol{y})}g(\boldsymbol{y}) \leqslant e^{C_w\, C_0} \sum_{(\boldsymbol{x},\boldsymbol{y})\in E_N} f(\boldsymbol{x})\mathsf{Q}^{I;J}_{N,0}(\boldsymbol{x},\boldsymbol{y}) g(\boldsymbol{y}) \,.$$

Therefore, using the first inequality of Lemma 3.2.1, the key step is to show that there exists a constant $C \in (0,\infty)$ that may depend on h and w but not on p and q, such that

$$\sum_{\boldsymbol{x}\in(\mathbb{Z}^2)^h_I,\, \boldsymbol{y}\in(\mathbb{Z}^2)^h_J} \frac{f(\boldsymbol{x})g(\boldsymbol{y})}{\Big(1+\sum_{i=1}^h |x_i-y_i|^2\Big)^{h-1}} \leqslant C\, p\, q\, \|f\|_{\ell^p} \, \|g\|_{\ell^q} \,. \tag{3.2.16}$$

By assumption there exist $1 \leqslant k, \ell \leqslant h$ such that $k \overset{I}{\sim} \ell$ and $1 \leqslant m, n \leqslant h$ such that $m \overset{J}{\sim} n$. Since we have assumed that $I \neq J$ when $|I| = |J| = h-1$, we may assume without loss of generality that $m \neq k, \ell$. Let $a \in \big(0, \min\{p^{-1}, q^{-1}\}\big)$ to be determined later. By multiplying and dividing by $\frac{1+|x_m-x_n|^{2a}}{1+|y_k-y_\ell|^{2a}}$ and using Hölder's inequality, the left-hand side of (3.2.16) is upper bounded by

$$\begin{aligned}
&\Bigg(\sum_{\boldsymbol{x}\in(\mathbb{Z}^2)^h_I,\, \boldsymbol{y}\in(\mathbb{Z}^2)^h_J} \frac{|f(\boldsymbol{x})|^p}{\Big(1+\sum_{i=1}^h |x_i-y_i|^2\Big)^{h-1}} \cdot \frac{\big(1+|x_m-x_n|^{2a}\big)^p}{\big(1+|y_k-y_\ell|^{2a}\big)^p} \Bigg)^{\frac{1}{p}} \\
&\qquad \times \Bigg(\sum_{\boldsymbol{x}\in(\mathbb{Z}^2)^h_I,\, \boldsymbol{y}\in(\mathbb{Z}^2)^h_J} \frac{|g(\boldsymbol{y})|^q}{\Big(1+\sum_{i=1}^h |x_i-y_i|^2\Big)^{h-1}} \cdot \frac{\big(1+|y_k-y_\ell|^{2a}\big)^q}{\big(1+|x_m-x_n|^{2a}\big)^q} \Bigg)^{\frac{1}{q}} .
\end{aligned} \tag{3.2.17}$$

By symmetry, it is enough to bound one of the two factors in (3.2.17). By triangle inequality and the fact that $m \overset{J}{\sim} n$, which means that $y_m = y_n$, we have

$$|x_m - y_m|^2 + |x_n - y_n|^2 \geqslant \frac{|x_m - x_n|^2 + |x_n - y_n|^2}{4} \,.$$

Therefore,

$$\left(\sum_{\boldsymbol{x}\in(\mathbb{Z}^2)^h_I,\,\boldsymbol{y}\in(\mathbb{Z}^2)^h_J}\frac{|f(\boldsymbol{x})|^p}{\left(1+\sum_{i=1}^h|x_i-y_i|^2\right)^{h-1}}\cdot\frac{(1+|x_m-x_n|^{2a})^p}{(1+|y_k-y_\ell|^{2a})^p}\right)^{\frac{1}{p}}$$
$$\leqslant 4^{\frac{h-1}{p}}\Bigg(\sum_{\boldsymbol{x}\in(\mathbb{Z}^2)^h_I}|f(\boldsymbol{x})|^p\,(1+|x_m-x_n|^{2a})^p$$
$$\times\sum_{\boldsymbol{y}\in(\mathbb{Z}^2)^h_J}\frac{1}{\left(1+|x_m-x_n|^2+\sum_{i\neq m}|x_i-y_i|^2\right)^{h-1}(1+|y_k-y_\ell|^{2a})^p}\Bigg)^{\frac{1}{p}}. \tag{3.2.18}$$

By using (3.2.39) of Lemma 3.2.5 and summing successively the y_i variables for $i\neq k,\ell$ we obtain that

$$\sum_{\boldsymbol{y}\in(\mathbb{Z}^2)^h_J}\frac{1}{\left(1+|x_m-x_n|^2+\sum_{i\neq m}|x_i-y_i|^2\right)^{h-1}(1+|y_k-y_\ell|^{2a})^p}$$
$$\leqslant c^{|J|-2}\sum_{y_k,\,y_\ell\in\mathbb{Z}^2}\frac{1}{\left(1+|x_m-x_n|^2+|y_k-x_k|^2+|y_\ell-x_\ell|^2\right)^{h+1-|J|}(1+|y_k-y_\ell|^{2a})^p}.$$

We make a change of variables $w_1=y_k-y_\ell$ and $w_2=y_k+y_\ell-2x_k$ and observe that $\frac{w_1^2+w_2^2}{2}=|y_k-x_k|^2+|y_\ell-x_\ell|^2$, where we used that $k\overset{I}{\sim}\ell$ thus $x_k=x_\ell$. Therefore, we have

$$c^{|J|-2}\sum_{y_k,\,y_\ell\in\mathbb{Z}^2}\frac{1}{\left(1+|x_m-x_n|^2+|y_k-x_k|^2+|y_\ell-x_\ell|^2\right)^{h+1-|J|}(1+|y_k-y_\ell|^{2a})^p}$$
$$\leqslant 2^{h+1-|J|}\,c^{|J|-2}\sum_{w_1,\,w_2\in\mathbb{Z}^2}\frac{1}{\left(1+|x_m-x_n|^2+|w_1|^2+|w_2|^2\right)^{h+1-|J|}(1+|w_1|^{2a})^p}.$$

By summing w_2 and using (3.2.39) of Lemma 3.2.5 we have,

$$2^{h+1-|J|}\,c^{|J|-2}\sum_{w_1,\,w_2\in\mathbb{Z}^2}\frac{1}{\left(1+|x_m-x_n|^2+|w_1|^2+|w_2|^2\right)^{h+1-|J|}(1+|w_1|^{2a})^p}$$
$$\leqslant 2^{h+1-|J|}\,c^{|J|-1}\sum_{w_1\in\mathbb{Z}^2}\frac{1}{\left(1+|x_m-x_n|^2+|w_1|^2\right)^{h-|J|}(1+|w_1|^{2a})^p}$$

By (3.2.40) of Lemma 3.2.5 we have that

$$2^{h+1-|J|}\,c^{|J|-1}\sum_{w_1\in\mathbb{Z}^2}\frac{1}{\left(1+|x_m-x_n|^2+|w_1|^2\right)^{h-|J|}(1+|w_1|^{2a})^p}$$
$$\leqslant 2^{h+1-|J|}\,c^{|J|}\frac{1}{ap(1-ap)}\frac{1}{\left(1+|x_m-x_n|^2\right)^{ap+h-1-|J|}}$$
$$\leqslant 2^{h+1-|J|}\,c^{|J|}\frac{1}{ap(1-ap)}\frac{1}{\left(1+|x_m-x_n|^2\right)^{ap}},$$

where in the last inequality we used that $|J| \leqslant h-1$ by assumption. Therefore, the right-hand side of (3.2.18) is bounded by

$$\begin{aligned}
&\left(4^{h-1}\,2^{h+1}\left(\frac{c}{2}\right)^{|J|}\frac{1}{ap(1-ap)}\right)^{\frac{1}{p}}\cdot\left(\sum_{\boldsymbol{x}\in(\mathbb{Z}^2)^h_I}|f(\boldsymbol{x})|^p\,\frac{(1+|x_m-x_n|^{2a})^p}{\left(1+|x_m-x_n|^2\right)^{ap}}\right)^{\frac{1}{p}}\\
=&\left(2^{3h-1}\left(\frac{c}{2}\right)^{|J|}\frac{1}{ap(1-ap)}\right)^{\frac{1}{p}}\cdot\left(\sum_{\boldsymbol{x}\in(\mathbb{Z}^2)^h_I}|f(\boldsymbol{x})|^p\,\frac{(1+|x_m-x_n|^{2a})^p}{\left(1+|x_m-x_n|^2\right)^{ap}}\right)^{\frac{1}{p}}.
\end{aligned}\tag{3.2.19}$$

Note furthermore, that

$$\frac{\left(1+|x_m-x_n|^{2a}\right)^p}{\left(1+|x_m-x_n|^2\right)^{ap}} \leqslant \frac{2^p\max\left\{1,|x_m-x_n|\right\}^{2ap}}{\left(1+|x_m-x_n|^2\right)^{ap}} \leqslant 2^p\,,$$

therefore,

$$\left(\sum_{\boldsymbol{x}\in(\mathbb{Z}^2)^h_I}|f(\boldsymbol{x})|^p\,\frac{(1+|x_m-x_n|^{2a})^p}{\left(1+|x_m-x_n|^2\right)^{ap}}\right)^{\frac{1}{p}} \leqslant 2\left(\sum_{\boldsymbol{x}\in(\mathbb{Z}^2)^h_I}|f(\boldsymbol{x})|^p\right)^{\frac{1}{p}} = 2\,\|f\|_{\ell^p}\,.$$

Hence, setting

$$C^J_{p,h} := 2\cdot\left(2^{3h-1}\left(\frac{c}{2}\right)^{|J|}\frac{1}{ap(1-ap)}\right)^{\frac{1}{p}}$$

and recalling (3.2.18), (3.2.19) we get that

$$\left(\sum_{\boldsymbol{x}\in(\mathbb{Z}^2)^h_I,\,\boldsymbol{y}\in(\mathbb{Z}^2)^h_J}\frac{|f(\boldsymbol{x})|^p}{\left(1+\sum_{i=1}^h|x_i-y_i|^2\right)^{h-1}}\cdot\frac{\left(1+|x_m-x_n|^{2a}\right)^p}{\left(1+|y_k-y_\ell|^{2a}\right)^p}\right)^{\frac{1}{p}} \leqslant C^J_{p,h}\,\|f\|_{\ell^p}\,. \tag{3.2.20}$$

By symmetry we also obtain that

$$\left(\sum_{\boldsymbol{x}\in(\mathbb{Z}^2)^h_I,\,\boldsymbol{y}\in(\mathbb{Z}^2)^h_J}\frac{|g(\boldsymbol{y})|^q}{\left(1+\sum_{i=1}^h|x_i-y_i|^2\right)^{h-1}}\cdot\frac{\left(1+|y_k-y_\ell|^{2a}\right)^q}{\left(1+|x_m-x_n|^{2a}\right)^q}\right)^{\frac{1}{q}} \leqslant C^I_{q,h}\,\|g\|_{\ell^q}\,, \tag{3.2.21}$$

with

$$C^I_{q,h} := 2\cdot\left(2^{3h-1}\left(\frac{c}{2}\right)^{|I|}\frac{1}{aq(1-aq)}\right)^{\frac{1}{q}}.$$

Consequently, recalling (3.2.16) and using (3.2.20), (3.2.21) we deduce that

$$\sum_{\boldsymbol{x}\in(\mathbb{Z}^2)^h_I,\,\boldsymbol{y}\in(\mathbb{Z}^2)^h_J}\frac{f(\boldsymbol{x})g(\boldsymbol{y})}{\left(1+\sum_{i=1}^h|x_i-y_i|^2\right)^{h-1}} \leqslant C^J_{p,h}\,C^I_{q,h}\,\|f\|_{\ell^p}\,\|g\|_{\ell^q}\,.$$

We optimise by choosing $a=(p\,q)^{-1}$ so as to obtain

$$C^J_{p,h}=2\cdot\left(2^{3h-1}\left(\frac{c}{2}\right)^{|J|}p\,q\right)^{\frac{1}{p}} \text{ and } C^I_{q,h}=2\cdot\left(2^{3h-1}\left(\frac{c}{2}\right)^{|I|}p\,q\right)^{\frac{1}{q}},$$

which implies that

$$C^J_{p,h}\,C^I_{q,h}=2^{3h+1}\left(\frac{c}{2}\right)^{\frac{|J|}{p}+\frac{|I|}{q}}p\,q\,.$$

Noting that $\left(\frac{c}{2}\right)^{\frac{|J|}{p}+\frac{|I|}{q}} \leqslant \max\left\{1, \left(\frac{c}{2}\right)^{h-1}\right\}$, we deduce that there exists $C = C(h,w) \in (0,\infty)$ such that

$$\sum_{\boldsymbol{x}\in(\mathbb{Z}^2)^h_I,\, \boldsymbol{y}\in(\mathbb{Z}^2)^h_J} \frac{f(\boldsymbol{x})g(\boldsymbol{y})}{\left(1+\sum_{i=1}^h |x_i-y_i|^2\right)^{h-1}} \leqslant C\,p\,q\, \|f\|_{\ell^p} \|g\|_{\ell^q}\ ,$$

which together with (3.2.15) imply (3.2.13). □

The next proposition is the analogue of Proposition 3.2.2 for the boundary operators.

Proposition 3.2.3. *Let $p,q \in (1,\infty)$ such that $\frac{1}{p}+\frac{1}{q}=1$. There exists a constant $C = C(h,w) \in (0,\infty)$, independent of p and q, such that for all $I \vdash \{1,\dots,h\}$ with $|I| \leqslant h-1$ and $g \in \ell^q(\mathbb{Z}^2)$,*

$$\left\| \widehat{\mathsf{Q}}^{I;*}_{N,0}\, g^{\otimes h} \right\|_{\ell^q} \leqslant C\,p\,N^{\frac{1}{p}} \|g\|^h_{\ell^q}\ .$$

Proof. Let $I \vdash \{1,\dots,h\}$ with $|I| \leqslant h-1$. In order to prove Proposition 3.2.3, we need to show that

$$\sum_{\boldsymbol{x}\in(\mathbb{Z}^2)^h_I,\, \boldsymbol{y}\in(\mathbb{Z}^2)^h} f(\boldsymbol{x}) \mathsf{Q}^{I;*}_{N,0}(\boldsymbol{x},\boldsymbol{y}) \frac{w_N^{\otimes h}(\boldsymbol{x})}{w_N^{\otimes h}(\boldsymbol{y})} g^{\otimes h}(\boldsymbol{y}) \leqslant C\,p\,N^{\frac{1}{p}} \|f\|_{\ell^p} \|g\|^h_{\ell^q}\ .$$

for any $f \in \ell^p\big((\mathbb{Z}^2)^{|I|}\big)$. The proof of this Proposition is a modification of the proof of Proposition 3.2.2. Let

$$E_N := \left\{(\boldsymbol{x},\boldsymbol{y}) \in (\mathbb{Z}^2)^h_I \times (\mathbb{Z}^2)^h : |\boldsymbol{x}-\boldsymbol{y}| \leqslant C_0\sqrt{N}\right\}.$$

For $(\boldsymbol{x},\boldsymbol{y}) \in E^c_N$, following (3.2.15) we have

$$\begin{aligned}\sum_{(\boldsymbol{x},\boldsymbol{y})\in E^c_N} f(\boldsymbol{x}) \frac{w_N^{\otimes h}(\boldsymbol{x})}{w_N^{\otimes h}(\boldsymbol{y})} \mathsf{Q}^{I;*}_{N,0}(\boldsymbol{x},\boldsymbol{y}) g^{\otimes h}(\boldsymbol{y}) &\leqslant C\,N^{\frac{h}{p}+\frac{|I|}{q}-(h-1)} \|f\|_{\ell^p} \|g\|^h_{\ell^q} \\ &\leqslant C\,N^{\frac{1}{p}} \|f\|_{\ell^p} \|g\|^h_{\ell^q}\ ,\end{aligned}$$

since $|I| \leqslant h-1$. Therefore, in light of the first inequality of Lemma 3.2.1, it remains to show that

$$\sum_{(\boldsymbol{x},\boldsymbol{y})\in E_N} \frac{f(\boldsymbol{x})g^{\otimes h}(\boldsymbol{y})}{\left(1+\sum_{i=1}^h |x_i-y_j|^2\right)^{h-1}} \leqslant C\,p\,N^{\frac{1}{p}}\ \|f\|_{\ell^p} \|g\|^h_{\ell^q}\ . \tag{3.2.22}$$

We can assume without loss of generality that $1 \overset{I}{\sim} 2$, that is $x_1 = x_2$. We multiply and divide by the factor $\left(\log\left(1+\frac{C_0^2 N}{1+|y_1-y_2|^2}\right)\right)^{\frac{1}{q}}$ in (3.2.22) and apply Hölder's inequality, namely

$$\begin{aligned}&\sum_{(\boldsymbol{x},\boldsymbol{y})\in E_N} \frac{f(\boldsymbol{x})g^{\otimes h}(\boldsymbol{y})}{\left(1+\sum_{i=1}^h |x_i-y_i|^2\right)^{h-1}} \\ &\leqslant \left(\sum_{(\boldsymbol{x},\boldsymbol{y})\in E_N} \frac{|f(\boldsymbol{x})|^p \left(\log\left(1+\frac{C_0^2 N}{1+|y_1-y_2|^2}\right)\right)^{\frac{p}{q}}}{\left(1+\sum_{i=1}^h |x_i-y_i|^2\right)^{h-1}}\right)^{\frac{1}{p}} \\ &\quad\times\left(\sum_{(\boldsymbol{x},\boldsymbol{y})\in E_N} \frac{|g^{\otimes h}(\boldsymbol{y})|^q}{\left(1+\sum_{i=1}^h |x_i-y_i|^2\right)^{h-1} \log\left(1+\frac{C_0^2 N}{1+|y_1-y_2|^2}\right)}\right)^{\frac{1}{q}}.\end{aligned} \tag{3.2.23}$$

By triangle inequality and using that $x_1 = x_2$ we have that

$$|x_1-y_1|^2 + |x_2-y_2|^2 \geqslant \frac{|y_1-y_2|^2 + |x_2-y_2|^2}{4}\ ,$$

therefore

$$\left(\sum_{(\boldsymbol{x},\boldsymbol{y})\in E_N}\frac{|g^{\otimes h}(\boldsymbol{y})|^q}{\left(1+\sum_{i=1}^h|x_i-y_i|^2\right)^{h-1}\log\left(1+\frac{C_0^2N}{1+|y_1-y_2|^2}\right)}\right)^{\frac{1}{q}}$$
$$\leqslant 4^{\frac{h-1}{q}}\left(\sum_{(\boldsymbol{x},\boldsymbol{y})\in E_N}\frac{|g^{\otimes h}(\boldsymbol{y})|^q}{\left(1+|y_1-y_2|^2+\sum_{i=2}^h|x_i-y_i|^2\right)^{h-1}\log\left(1+\frac{C_0^2N}{1+|y_1-y_2|^2}\right)}\right)^{\frac{1}{q}}. \tag{3.2.24}$$

We sum the x_i variables for $i>2$ successively, so that by inequality (3.2.39) of Lemma 3.2.5,

$$\sum_{\boldsymbol{x}\in(\mathbb{Z}^2)^h_I:\,(\boldsymbol{x},\boldsymbol{y})\in E_N}\frac{1}{\left(1+|y_1-y_2|^2+\sum_{i=2}^h|x_i-y_i|^2\right)^{h-1}\log\left(1+\frac{C_0^2N}{1+|y_1-y_2|^2}\right)}$$
$$\leqslant c^{|I|-1}\frac{1}{\log\left(1+\frac{C_0^2N}{1+|y_1-y_2|^2}\right)}\sum_{\substack{x_2\in\mathbb{Z}^2\\|x_2-y_2|\leqslant C_0\sqrt{N}}}\frac{1}{\left(1+|y_1-y_2|^2+|x_2-y_2|^2\right)^{h-|I|}}. \tag{3.2.25}$$

We also note that since $|I|\leqslant h-1$,

$$\sum_{\substack{x_2\in\mathbb{Z}^2\\|x_2-y_2|\leqslant C_0\sqrt{N}}}\frac{1}{\left(1+|y_1-y_2|^2+|x_2-y_2|^2\right)^{h-|I|}}\leqslant\sum_{\substack{x_2\in\mathbb{Z}^2\\|x_2-y_2|\leqslant C_0\sqrt{N}}}\frac{1}{1+|y_1-y_2|^2+|x_2-y_2|^2}$$
$$\leqslant c\log\left(1+\frac{C_0^2N}{1+|y_1-y_2|^2}\right), \tag{3.2.26}$$

where the last inequality in (3.2.26) follows from inequality (3.2.48) of Lemma 3.2.6. Thus, taking into account (3.2.25) and (3.2.26) we deduce that

$$\sum_{\boldsymbol{x}\in(\mathbb{Z}^2)^h_I:\,(\boldsymbol{x},\boldsymbol{y})\in E_N}\frac{1}{\left(1+|y_1-y_2|^2+\sum_{i=2}^h|x_i-y_i|^2\right)^{h-1}\log\left(1+\frac{C_0^2N}{1+|y_1-y_2|^2}\right)}\leqslant c^{|I|}\leqslant c^{h-1},$$

since $|I|\leqslant h-1$. By (3.2.24) we obtain that

$$\left(\sum_{(\boldsymbol{x},\boldsymbol{y})\in E_N}\frac{|g^{\otimes h}(\boldsymbol{y})|^q}{\left(1+\sum_{i=1}^h|x_i-y_i|^2\right)^{h-1}\log\left(1+\frac{C_0^2N}{1+|y_1-y_2|^2}\right)}\right)^{\frac{1}{q}}$$
$$\leqslant(4c)^{\frac{h-1}{q}}\left(\sum_{\boldsymbol{y}\in(\mathbb{Z}^2)^h}|g^{\otimes h}(\boldsymbol{y})|^q\right)^{\frac{1}{q}}=(4c)^{\frac{h-1}{q}}\,\|g\|_{\ell^q}^h\,. \tag{3.2.27}$$

On the other hand, for the first term in (3.2.23), using that $x_1=x_2$, by (3.2.39) of Lemma 3.2.5, we have that

$$\sum_{\boldsymbol{y}\in(\mathbb{Z}^2)^h}\frac{\left(\log\left(1+\frac{C_0^2N}{1+|y_1-y_2|^2}\right)\right)^{\frac{p}{q}}}{\left(1+\sum_{i=1}^h|x_i-y_i|^2\right)^{h-1}}\leqslant c^{h-2}\sum_{\substack{y_1,y_2\in\mathbb{Z}^2\\|y_1-x_1|,|y_2-x_1|\leqslant C_0\sqrt{N}}}\frac{\left(\log\left(1+\frac{C_0^2N}{1+|y_1-y_2|^2}\right)\right)^{\frac{p}{q}}}{\left(1+|x_1-y_1|^2+|x_1-y_2|^2\right)}. \tag{3.2.28}$$

We make the change of variables $w_1 := y_1 - y_2$ and $w_2 := y_1 + y_2 - 2x_1$, so that $|w_1|, |w_2| \leqslant 2C_0\sqrt{N}$ and $|w_1|^2 + |w_2|^2 = 2|y_1 - x_1|^2 + 2|y_2 - x_1|^2$. Note that then,

$$c^{h-2} \sum_{\substack{y_1, y_2 \in \mathbb{Z}^2 \\ |y_1 - x_1|, |y_2 - x_1| \leqslant C_0\sqrt{N}}} \frac{\left(\log\left(1 + \frac{C_0^2 N}{1+|y_1-y_2|^2}\right)\right)^{\frac{p}{q}}}{\left(1 + |x_1 - y_1|^2 + |x_1 - y_2|^2\right)}$$

$$\leqslant 2c^{h-2} \sum_{\substack{w_1, w_2 \in \mathbb{Z}^2 \\ |w_1|, |w_2| \leqslant 2\, C_0\sqrt{N}}} \frac{\left(\log\left(1 + \frac{C_0^2 N}{1+|w_1|^2}\right)\right)^{\frac{p}{q}}}{1 + |w_1|^2 + |w_2|^2} .$$

Next, we sum over w_2 and use inequality (3.2.48) of Lemma 3.2.6 to obtain

$$2c^{h-2} \sum_{\substack{w_1, w_2 \in \mathbb{Z}^2 \\ |w_1|, |w_2| \leqslant 2\, C_0\sqrt{N}}} \frac{\left(\log\left(1 + \frac{C_0^2 N}{1+|w_1|^2}\right)\right)^{\frac{p}{q}}}{1 + |w_1|^2 + |w_2|^2} \leqslant 2c^{h-1} \sum_{\substack{w_1 \in \mathbb{Z}^2 \\ |w_1| \leqslant 2\, C_0\sqrt{N}}} \left(\log\left(1 + \frac{C_0^2 N}{1+|w_1|^2}\right)\right)^{\frac{p}{q}+1} . \tag{3.2.29}$$

By (3.2.50) of Lemma 3.2.6 and noting that $\frac{p}{q} + 1 = p$ we have

$$\sum_{\substack{w_1 \in \mathbb{Z}^2 \\ |w_1| \leqslant 2C_0\sqrt{N}}} \left(\log\left(1 + \frac{C_0^2 N}{1+|w_1|^2}\right)\right)^{p} \leqslant c\, C_0^2\, N\, p^p . \tag{3.2.30}$$

Therefore, by (3.2.28), (3.2.29) and (3.2.30) we have that

$$\left(\sum_{(\boldsymbol{x},\boldsymbol{y}) \in E_N} \frac{|f(\boldsymbol{x})|^p \left(\log\left(1 + \frac{C_0^2 N}{1+|y_1-y_2|^2}\right)\right)^{\frac{p}{q}}}{\left(1 + \sum_{i=1}^h |x_i - y_i|^2\right)^{h-1}} \right)^{\frac{1}{p}} \leqslant \left(2\, c^h\, C_0^2\right)^{\frac{1}{p}} N^{\frac{1}{p}}\, p \left(\sum_{\boldsymbol{x} \in (\mathbb{Z}^2)^h_I} |f(\boldsymbol{x})|^p \right)^{\frac{1}{p}}$$

$$\leqslant \left(2\, c^h\, C_0^2\right)^{\frac{1}{p}} N^{\frac{1}{p}}\, p\, \|f\|_{\ell^p} . \tag{3.2.31}$$

Taking into account (3.2.27), (3.2.31) and (3.2.23) we obtain that there exists $C = C(h, w) \in (0, \infty)$ such that

$$\sum_{(\boldsymbol{x},\boldsymbol{y}) \in E_N} \frac{f(\boldsymbol{x}) g^{\otimes h}(\boldsymbol{y})}{\left(1 + \sum_{i=1}^h |x_i - y_j|^2\right)^{h-1}} \leqslant C\, p\, N^{\frac{1}{p}}\, \|f\|_{\ell^p}\, \|g\|_{\ell^q}^h ,$$

which concludes the proof of (3.2.22) and thus, the proof of Proposition 3.2.3. □

Proposition 3.2.4. *Let $p, q \in (1, \infty)$ such that $\frac{1}{p} + \frac{1}{q} = 1$. There exists a constant $C = C(h, \hat{\beta}, w) \in (0, \infty)$, independent of p and q, such that for all $I \vdash \{1, \ldots, h\}$ with $|I| = h - 1$,*

$$\left\| \hat{\mathsf{U}}^I_{N,0} \right\|_{\ell^q \to \ell^q} \leqslant C .$$

Proof. Using (3.2.4) it suffices to prove that if $f \in \ell^p\big((\mathbb{Z}^2)^h_I\big)$, $g \in \ell^q\big((\mathbb{Z}^2)^h_I\big)$, then we have

$$\sum_{\boldsymbol{x}, \boldsymbol{y} \in (\mathbb{Z}^2)^h_I} f(\boldsymbol{x}) \mathsf{U}^I_{N,0}(\boldsymbol{x}, \boldsymbol{y}) \frac{w_N^{\otimes h}(\boldsymbol{x})}{w_N^{\otimes h}(\boldsymbol{y})} g(\boldsymbol{y}) \leqslant C\, \|f\|_{\ell^p}\, \|g\|_{\ell^q} .$$

By the Lipschitz condition on $\log w$ we first have

$$\sum_{\boldsymbol{x},\boldsymbol{y}\in(\mathbb{Z}^2)^h_I} f(\boldsymbol{x})\mathsf{U}^I_{N,0}(\boldsymbol{x},\boldsymbol{y})\frac{w_N^{\otimes h}(\boldsymbol{x})}{w_N^{\otimes h}(\boldsymbol{y})}g(\boldsymbol{y}) \leqslant \sum_{\boldsymbol{x},\boldsymbol{y}\in(\mathbb{Z}^2)^h_I} f(\boldsymbol{x})\mathsf{U}^I_{N,0}(\boldsymbol{x},\boldsymbol{y})e^{C_w\frac{|\boldsymbol{x}-\boldsymbol{y}|}{\sqrt{N}}}g(\boldsymbol{y})\,,$$

which by Hölder's inequality is bounded by

$$\Bigg(\sum_{\boldsymbol{x},\boldsymbol{y}\in(\mathbb{Z}^2)^h_I} |f(\boldsymbol{x})|^p\,\mathsf{U}^I_{N,0}(\boldsymbol{x},\boldsymbol{y})e^{C_w\frac{|\boldsymbol{x}-\boldsymbol{y}|}{\sqrt{N}}}\Bigg)^{\frac{1}{p}}\cdot\Bigg(\sum_{\boldsymbol{x},\boldsymbol{y}\in(\mathbb{Z}^2)^h_I} |g(\boldsymbol{y})|^q\,\mathsf{U}^I_{N,0}(\boldsymbol{x},\boldsymbol{y})e^{C_w\frac{|\boldsymbol{x}-\boldsymbol{y}|}{\sqrt{N}}}\Bigg)^{\frac{1}{q}}.$$

Therefore, in order to conclude the proof of (3.2.4) it suffices to prove that there exists a constant C such that uniformly in $\boldsymbol{x}\in(\mathbb{Z}^2)^h_I$,

$$\sum_{\boldsymbol{y}\in(\mathbb{Z}^2)^h_I}\mathsf{U}^I_{N,0}(\boldsymbol{x},\boldsymbol{y})e^{C_w\frac{|\boldsymbol{x}-\boldsymbol{y}|}{\sqrt{N}}}\leqslant C\,. \tag{3.2.32}$$

Recall from (3.2.8) that if I is of the form $I=\{k,\ell\}\sqcup\bigsqcup_{j\neq k,\ell}\{j\}$ then for $\boldsymbol{x},\boldsymbol{y}\in(\mathbb{Z}^2)^h_I$ the operator $\mathsf{U}^I_{N,0}(\boldsymbol{x},\boldsymbol{y})$ is defined as

$$\mathsf{U}^I_{N,0}(\boldsymbol{x},\boldsymbol{y})=\sum_{n=0}^{2N}\mathsf{U}^I_n(\boldsymbol{x},\boldsymbol{y})=\mathbb{1}_{\{\boldsymbol{x},\boldsymbol{y}\sim I\}}\cdot\sum_{n=0}^{2N}U_N^{\beta_N}(n,y_k-x_k)\cdot\prod_{i\neq k,\ell}q_n(y_i-x_i)\,.$$

Therefore, in view of (3.2.32), we shall prove that uniformly in $0\leqslant n\leqslant 2N$,

$$\sum_{z\in\mathbb{Z}^2}U_N^{\beta_N}(n,z)e^{C_w\frac{|z|}{\sqrt{N}}}\leqslant C\,U_N^{\beta_N}(n) \tag{3.2.33}$$

and

$$\sum_{z\in\mathbb{Z}^2}q_n(z)e^{C_w\frac{|z|}{\sqrt{N}}}\leqslant C\,q_n(z)\,. \tag{3.2.34}$$

Inequality (3.2.34) follows easily by the local CLT, see [LL10] and Gaussian concentration. For the sake of the presentation, we will prove (3.2.33) for $0\leqslant n\leqslant N$, that is,

$$\sum_{z\in\mathbb{Z}^2}U_N^{\beta_N}(n,z)e^{C_w\frac{|z|}{\sqrt{N}}}\leqslant C\,U_N^{\beta_N}(n)\,,\qquad\forall\ 0\leqslant n\leqslant N\,. \tag{3.2.35}$$

Note that, by (3.1.13) we have,

$$\sum_{n=0}^{N}U_N^{\beta_N}(n)\leqslant\mathbb{E}\Big[(Z^{\beta_N}_{N+1})^2\Big]\leqslant\frac{C}{1-\hat\beta^2}\,. \tag{3.2.36}$$

Moreover, following the renewal framework we introduced in Section 3.1, we have

$$\begin{aligned}
&\sum_{z\in\mathbb{Z}^2}U_N^{\beta_N}(n,z)e^{C_w\frac{|z|}{\sqrt{N}}}\\
&=\sum_{k\geqslant0}\hat\beta^{2k}\,\mathrm{E}\Big[e^{C_w\frac{|S_k^{(N)}|}{\sqrt{N}}};\,\tau_k^{(N)}=n\Big]\\
&=\sum_{k\geqslant0}\hat\beta^{2k}\sum_{n_1+\cdots+n_k=n}\mathrm{E}\Big[e^{C_w\frac{|S_k^{(N)}|}{\sqrt{N}}}\,\Big|\,\mathsf{t}_i^{(N)}=n_i\,,1\leqslant i\leqslant k\Big]\prod_{i=1}^k\mathrm{P}\big(\mathsf{t}_i^{(N)}=n_i\big)\,.
\end{aligned} \tag{3.2.37}$$

Therefore, in order to establish (3.2.35) it suffices to prove that there exists $C\in(0,\infty)$, such that for all $k\geqslant1$,

$$\mathrm{E}\Big[e^{C_w\frac{|S_k^{(N)}|}{\sqrt{N}}}\,\Big|\,\mathsf{t}_i^{(N)}=n_i\,,1\leqslant i\leqslant k\Big]\leqslant C\,. \tag{3.2.38}$$

We note that when we condition on the times $\big(\mathsf{t}_i^{(N)}\big)_{1\leqslant i\leqslant k}$, the space increments $\big(\mathsf{x}_i^{(N)}\big)_{1\leqslant i\leqslant k}$ are independent with distribution

$$\mathrm{P}\big(\mathsf{x}_1^{(N)} = x \,\big|\, \mathsf{t}_1^{(N)} = n_1\big) = \frac{q_{n_1}^2(x)}{q_{2n_1}(0)} \mathbb{1}_{\{n_1 \leqslant N\}} .$$

Let $\lambda \geqslant 0$ and $(\xi_i)_{1\leqslant i\leqslant k}$ independent random variables such that $\xi_i \stackrel{\text{law}}{=} \mathsf{x}_i^{(N)} \big| \mathsf{t}_i^{(N)} = n_i$. We will show that

$$\mathrm{E}\Big[e^{\lambda |\sum_{i=1}^k \xi_i|}\Big] \leqslant 2e^{4c\lambda^2 n} ,$$

for some $c > 0$. Therefore, taking $\lambda = \frac{C_w}{\sqrt{N}}$ will lead to (3.2.38). To this end, for each $1 \leqslant i \leqslant k$, let $\xi_{i,1}, \xi_{i,2} \in \mathbb{Z}$ be the two components of $\xi_i \in \mathbb{Z}^2$ .Then we can find $c > 0$ such that

$$\mathrm{E}\big[e^{\pm\lambda \xi_{i,j}}\big] \leqslant e^{c\lambda^2 n_i}$$

for $j = 1, 2$, since by the local CLT we have

$$\mathrm{P}(\xi_i = x) = \frac{q_{n_i}^2(x)}{q_{2n_i}(0)} \leqslant \Big(\frac{\sup_{x\in\mathbb{Z}^2} q_{n_i}(x)}{q_{2n_i}(0)}\Big) q_{n_i}(x) \leqslant C' q_{n_i}(x)$$

and $q_{n_i}(x) = 2(g_{n_i/2}(x) + o(1))$, thus q_{n_i} has Gaussian tail decay. By Cauchy-Schwarz we

$$\mathrm{E}\Big[e^{\lambda |\sum_{i=1}^k \xi_i|}\Big] \leqslant \mathrm{E}\Big[e^{2\lambda |\sum_{i=1}^k \xi_{i,1}|}\Big]^{\frac{1}{2}} \Big[e^{2\lambda |\sum_{i=1}^k \xi_{i,2}|}\Big]^{\frac{1}{2}} .$$

Also, by the inequality $e^{|x|} \leqslant e^x + e^{-x}$ and independence, we obtain for $j = 1, 2$

$$\mathrm{E}\Big[e^{2\lambda |\sum_{i=1}^k \xi_{i,j}|}\Big]^{\frac{1}{2}} \leqslant \left(\prod_{i=1}^k \mathrm{E}[e^{2\lambda\xi_{i,j}}] + \prod_{i=1}^k \mathrm{E}[e^{-2\lambda\xi_{i,j}}]\right)^{\frac{1}{2}} \leqslant \Big(2e^{4c\lambda^2 n}\Big)^{\frac{1}{2}} ,$$

therefore,

$$\mathrm{E}\Big[e^{\lambda |\sum_{i=1}^k \xi_i|}\Big] \leqslant 2e^{4c\lambda^2 n} .$$

Given the inequality above and choosing $\lambda = \frac{C_w}{\sqrt{N}}$ we get that

$$\mathrm{E}\left[e^{C_w \frac{|S_k^{(N)}|}{\sqrt{N}}} \,\Big|\, \mathsf{t}_i^{(N)} = n_i \,, 1 \leqslant i \leqslant k\right] \leqslant 2e^{4cC_w^2} ,$$

since $1 \leqslant n \leqslant N$. Therefore, recalling (3.2.36) and (3.2.37),we have

$$\sum_{\substack{z\in\mathbb{Z}^2,\\ 0\leqslant n\leqslant N}} U_N^{\beta_N}(n,z) e^{C_w \frac{|z|}{\sqrt{N}}} \leqslant 2e^{4cC_w^2} \sum_{n=0}^N U_N^{\beta_N}(n) \leqslant 2e^{4cC_w^2}\, \mathbb{E}\Big[(Z_{N+1,\beta_N})^2\Big] \leqslant C ,$$

for a constant $C = C(h, \hat{\beta}, w) \in (0, \infty)$. $\square$

3.2.3. Some technical estimates. We state here the integral estimates we used for proving Propositions 3.2.2 and 3.2.3.

Lemma 3.2.5. *Let $\lambda \geqslant 1$, $p > 1$, $a < \frac{1}{p}$. Then,*

$$\sum_{y\in\mathbb{Z}^2} \frac{1}{(\lambda + |y|^2)^r} \leqslant \frac{c}{\lambda^{r-1}} \quad \text{if } r \geqslant 2 , \tag{3.2.39}$$

$$\sum_{y\in\mathbb{Z}^2} \frac{1}{(\lambda + |y|^2)^r (1 + |y|^{2a})^p} \leqslant \frac{c}{ap\,(1-ap)\lambda^{r-1+ap}} \quad \text{if } r \geqslant 1 , \tag{3.2.40}$$

for a constant $c \in (0, \infty)$, that does not depend on λ, p, a or r.

Proof. We note that since $y \mapsto \dfrac{1}{\left(\lambda+|y|^2\right)^r}$ and $y \mapsto \dfrac{1}{\left(\lambda+|y|^2\right)^r\left(1+|y|^{2a}\right)^p}$ are decreasing in the radial direction we have that

$$\sum_{y\in\mathbb{Z}^2}\frac{1}{\left(\lambda+|y|^2\right)^r} \leqslant \frac{1}{\lambda^r}+\int_{\mathbb{R}^2}\frac{1}{\left(\lambda+|y|^2\right)^r}\,\mathrm{d}y \tag{3.2.41}$$

and

$$\sum_{y\in\mathbb{Z}^2}\frac{1}{\left(\lambda+|y|^2\right)^r\left(1+|y|^{2a}\right)^p} \leqslant \frac{1}{\lambda^r}+\int_{\mathbb{R}^2}\frac{1}{\left(\lambda+|y|^2\right)^r|y|^{2ap}}\,\mathrm{d}y\,. \tag{3.2.42}$$

In order to prove (3.2.39), we switch to polar coordinates in (3.2.41), so that

$$\int_{\mathbb{R}^2}\frac{1}{\left(\lambda+|y|^2\right)^r}\,\mathrm{d}y = 2\pi\int_0^\infty\frac{\varrho}{(\lambda+\varrho^2)^r}\,\mathrm{d}\varrho = \pi\cdot\frac{(\lambda+\varrho^2)^{1-r}}{1-r}\bigg|_{\varrho=0}^{\varrho=\infty} = \frac{\pi}{r-1}\frac{1}{\lambda^{r-1}}\,. \tag{3.2.43}$$

Therefore, by (3.2.41) and (3.2.43) we get that

$$\sum_{y\in\mathbb{Z}^2}\frac{1}{\left(\lambda+|y|^2\right)^r} \leqslant \frac{1}{\lambda^r}+\frac{\pi}{r-1}\frac{1}{\lambda^{r-1}} = \frac{1}{\lambda^{r-1}}\left(\frac{1}{\lambda}+\frac{\pi}{r-1}\right).$$

Thus, since $r\geqslant 2$ and $\lambda\geqslant 1$ we conclude (3.2.39) with $c=\pi+1$.

For (3.2.40) we split the integral in (3.2.42) into two regions,

$$\int_{\mathbb{R}^2}\frac{1}{\left(\lambda+|y|^2\right)^r|y|^{2ap}}\,\mathrm{d}y = \underbrace{\int_{|y|\leqslant\sqrt{\lambda}}\frac{1}{\left(\lambda+|y|^2\right)^r|y|^{2ap}}\,\mathrm{d}y}_{:=I_1} + \underbrace{\int_{|y|>\sqrt{\lambda}}\frac{1}{\left(\lambda+|y|^2\right)^r|y|^{2ap}}\,\mathrm{d}y}_{:=I_2}\,.$$

First,

$$I_1 \leqslant \frac{1}{\lambda^r}\int_{|y|\leqslant\sqrt{\lambda}}\frac{1}{|y|^{2ap}}\,\mathrm{d}y = \frac{2\pi}{\lambda^r}\int_0^{\sqrt{\lambda}}\frac{1}{\varrho^{2ap-1}}\,\mathrm{d}\varrho = \frac{\pi}{\lambda^r}\frac{\lambda^{1-ap}}{1-ap} = \frac{\pi}{1-ap}\frac{1}{\lambda^{r-1+ap}}\,.$$

Similarly,

$$\begin{aligned} I_2 \leqslant \int_{|y|>\sqrt{\lambda}}\frac{1}{|y|^{2r+2ap}}\mathrm{d}y = 2\pi\int_{\sqrt{\lambda}}^\infty\frac{1}{\varrho^{2r+2ap-1}}\,\mathrm{d}\varrho &= \frac{\pi}{r-1+ap}\frac{-1}{\varrho^{2r+2ap-2}}\bigg|_{\varrho=\sqrt{\lambda}}^{\varrho=\infty} \\ &= \frac{\pi}{r-1+ap}\frac{1}{\lambda^{r-1+ap}}\,. \end{aligned}$$

Therefore,

$$\begin{aligned} \int_{\mathbb{R}^2}\frac{1}{\left(\lambda+|y|^2\right)^r|y|^{2ap}}\,\mathrm{d}y = I_1+I_2 &\leqslant \frac{\pi}{1-ap}\frac{1}{\lambda^{r-1+ap}}+\frac{\pi}{r-1+ap}\frac{1}{\lambda^{r-1+ap}} \\ &= \frac{\pi\,r}{(1-ap)(r-1+ap)}\frac{1}{\lambda^{r-1+ap}}\,. \end{aligned} \tag{3.2.44}$$

By (3.2.42) and (3.2.44) we thus obtain

$$\begin{aligned} \sum_{y\in\mathbb{Z}^2}\frac{1}{\left(\lambda+|y|^2\right)^r\left(1+|y|^{2a}\right)^p} &\leqslant \frac{1}{\lambda^r}+\int_{\mathbb{R}^2}\frac{1}{\left(\lambda+|y|^2\right)^r|y|^{2ap}}\,\mathrm{d}y \\ &\leqslant \frac{1}{\lambda^r}+\frac{\pi\,r}{(1-ap)(r-1+ap)}\frac{1}{\lambda^{r-1+ap}}\,. \end{aligned} \tag{3.2.45}$$

Note that

$$\frac{\pi\,r}{(1-ap)(r-1+ap)} \leqslant \frac{\pi}{(1-ap)ap}\,,$$

since that inequality is equivalent to $(r-1)(ap-1) \leqslant 0$, which is valid since we have assumed that $a\,p < 1$ and $r \geqslant 2$. Therefore,

$$\begin{aligned}\frac{1}{\lambda^r} + \frac{\pi\, r}{(1-ap)(r-1+ap)}\frac{1}{\lambda^{r-1+ap}} &\leqslant \frac{1}{\lambda^r} + \frac{\pi}{(1-ap)ap}\frac{1}{\lambda^{r-1+ap}} \\ &= \frac{1}{\lambda^{r-1+ap}}\left(\frac{1}{\lambda^{1-ap}} + \frac{\pi}{(1-ap)ap}\right) \\ &\leqslant \frac{1}{\lambda^{r-1+ap}}\left(1 + \frac{\pi}{(1-ap)ap}\right),\end{aligned} \tag{3.2.46}$$

since $\lambda \geqslant 1$ and $1 - ap > 0$, by assumption. Last, we have that

$$1 + \frac{\pi}{(1-ap)ap} = \frac{\pi + ap(1-ap)}{ap(1-ap)} \leqslant \frac{1+\pi}{ap(1-ap)}\,. \tag{3.2.47}$$

Hence, by (3.2.45), (3.2.46) and (3.2.47),

$$\sum_{y\in\mathbb{Z}^2} \frac{1}{\left(\lambda + |y|^2\right)^r \left(1 + |y|^{2a}\right)^p} \leqslant \frac{c}{(1-ap)\, ap\, \lambda^{r-1+ap}}\,,$$

with $c = 1 + \pi$, thus concluding the proof of (3.2.40). □

Lemma 3.2.6. *There exists a constant $c \in (0,\infty)$ such that uniformly in $A, \lambda, p \geqslant 1$,*

$$\sum_{\substack{y\in\mathbb{Z}^2\\ |y|\leqslant\sqrt{A}}} \frac{1}{\lambda + |y|^2} \leqslant c \log\left(1 + \frac{A}{\lambda}\right), \tag{3.2.48}$$

$$\left(\int_1^A \left(\log\left(\tfrac{A}{x}\right)\right)^p \mathrm{d}x\right)^{\frac{1}{p}} \leqslant p\, A^{\frac{1}{p}}\,, \tag{3.2.49}$$

and

$$\sum_{\substack{y\in\mathbb{Z}^2\\ |y|\leqslant 2\sqrt{A}}} \left(\log\left(1 + \frac{A}{1+|y|^2}\right)\right)^p \leqslant c\, A\, p^p\,. \tag{3.2.50}$$

Proof. For (3.2.48), using the same reasoning as in the proof of Lemma 3.2.5 we have

$$\sum_{\substack{y\in\mathbb{Z}^2\\ |y|\leqslant\sqrt{A}}} \frac{1}{\lambda + |y|^2} \leqslant \frac{1}{\lambda} + \int_{|y|\leqslant\sqrt{A}} \frac{1}{\lambda + |y|^2}\,\mathrm{d}y \tag{3.2.51}$$

Switching to polar coordinates in (3.2.51) we have

$$\int_{|y|\leqslant\sqrt{A}} \frac{1}{\lambda + |y|^2}\,\mathrm{d}y = 2\pi\int_0^{\sqrt{A}} \frac{\varrho}{\lambda + \varrho^2}\,\mathrm{d}\varrho = \pi\,\log(\lambda + \varrho^2)\Big|_{\varrho=0}^{\varrho=\sqrt{A}} = \pi\log\left(1 + \frac{A}{\lambda}\right). \tag{3.2.52}$$

A simple computation shows that when $\lambda \geqslant 1$, one has that $\frac{1}{\lambda} \leqslant 2\log\left(1 + \frac{1}{\lambda}\right) \leqslant 2\log\left(1 + \frac{A}{\lambda}\right)$, the latter following since $A \geqslant 1$, by assumption. Therefore, by (3.2.51) and (3.2.52) we have that

$$\sum_{\substack{y\in\mathbb{Z}^2\\ |y|\leqslant\sqrt{A}}} \frac{1}{\lambda + |y|^2} \leqslant \frac{1}{\lambda} + \int_{|y|\leqslant\sqrt{A}} \frac{1}{\lambda + |y|^2}\,\mathrm{d}y \leqslant (2+\pi)\log\left(1 + \frac{A}{\lambda}\right),$$

which implies (3.2.48) with $c = 2 + \pi$.

Let us now prove (3.2.49) and (3.2.50). First, we prove (3.2.49). We have

$$\frac{1}{A}\int_1^A \left(\log\left(\tfrac{A}{x}\right)\right)^p \mathrm{d}x \overset{y=A\,e^{-u}}{=\!=} \int_0^{\log A} e^{-u}\, u^p\,\mathrm{d}u \leqslant \Gamma(p+1) \leqslant p^p\,, \tag{3.2.53}$$

since $\Gamma(p+1) \leqslant p^p$ for $p \geqslant 1$. After raising both sides of (3.2.53) to the $\frac{1}{p}$ we get (3.2.49).

To prove (3.2.50) we first note that

$$\sum_{\substack{y\in\mathbb{Z}^2\\|y|\leqslant 2\sqrt{A}}} \Big(\log\Big(1+\frac{A}{1+|y|^2}\Big)\Big)^p \leqslant (\log(1+A))^p + \int_{|y|\leqslant 2\sqrt{A}} \Big(\log\Big(1+\frac{A}{1+|y|^2}\Big)\Big)^p \,\mathrm{d}y\,. \tag{3.2.54}$$

Using polar coordinates in (3.2.54) we compute

$$\begin{aligned}\int_{|y|\leqslant 2\sqrt{A}} \Big(\log\Big(1+\frac{A}{1+|y|^2}\Big)\Big)^p \,\mathrm{d}y &= 2\pi\int_0^{2\sqrt{A}} \varrho\Big(\log\Big(1+\frac{A}{1+\varrho^2}\Big)\Big)^p \,\mathrm{d}\varrho\\ &\overset{u=1+\varrho^2}{=\!=} \pi\int_1^{1+4A} \Big(\log\Big(1+\frac{A}{u}\Big)\Big)^p \,\mathrm{d}u\,.\end{aligned}$$

Furthermore,

$$\begin{aligned}\pi\int_1^{1+4A} \Big(\log\Big(1+\frac{A}{u}\Big)\Big)^p \,\mathrm{d}u &\leqslant \pi\int_1^{1+4A} \Big(\log\Big(\frac{1+5A}{u}\Big)\Big)^p \,\mathrm{d}u\\ &\leqslant \pi\int_1^{1+5A} \Big(\log\Big(\frac{1+5A}{u}\Big)\Big)^p \mathrm{d}u\,.\end{aligned}$$

Note that by (3.2.49), we further have that

$$\pi\int_1^{1+5A} \Big(\log\Big(\frac{1+5A}{u}\Big)\Big)^p \mathrm{d}u \leqslant (1+5A)\,\pi\,p^p \;\leqslant 6\,A\,\pi\,p^p\,,$$

since $A \geqslant 1$. Combining this inequality with (3.2.54) we get that

$$\sum_{\substack{y\in\mathbb{Z}^2\\|y|\leqslant 2\sqrt{A}}} \Big(\log\Big(1+\frac{A}{1+|y|^2}\Big)\Big)^p \leqslant \log(1+A)^p + 6\,A\,\pi\,p^p\,. \tag{3.2.55}$$

We are going to prove that for all $A \geqslant 1$,

$$(\log(1+A))^p \leqslant \frac{1}{\sqrt{e}-1}\,A\,p^p\,,$$

thus deducing inequality (3.2.50), via (3.2.55), with $c = \frac{1}{\sqrt{e}-1} + 6\pi$. To this end, consider $k_p(x) := \frac{(\log(1+x))^p}{x}$ for $x \geqslant 0$ and $p \geqslant 1$. We have that

$$k_p'(x) := \frac{(\log(1+x))^{p-1}}{x}\Big(\frac{p}{1+x} - \frac{\log(1+x)}{x}\Big)\,,$$

therefore, k_p is increasing in $[0, x_p]$ and decreasing in $[x_p, \infty)$, where $x_p \geqslant 0$ is the solution to the equation $k_p'(x_p) = 0$, or equivalently

$$p = \frac{(1+x_p)\,\log(1+x_p)}{x_p}\,. \tag{3.2.56}$$

By working with $g(x) := \frac{(1+x)\log(1+x)}{x}$, one can see that equation (3.2.56) has a unique solution $x_p \geqslant 0$ for every $p \geqslant 1$, since $g'(x) > 0$ for all $x > 0$, $\lim_{x\downarrow 0} g(x) = 1$ and $\lim_{x\to\infty} g(x) = \infty$. We distinguish two cases:

Suppose first that $x_p \geqslant 1$. Then

$$\log(1+x_p) \leqslant p \leqslant 2\,\log(1+x_p)\,, \tag{3.2.57}$$

by (3.2.56) and since $x_p \geqslant 1$. Therefore, in this case, for all $x \geqslant 1$,

$$\frac{(\log(1+x))^p}{x} = k_p(x) \leqslant k_p(x_p) = \frac{(\log(1+x_p))^p}{x_p} \leqslant \frac{p^p}{x_p} \leqslant \frac{p^p}{e^{\frac{p}{2}}-1},$$

where the last two inequalities follow by the first and second inequality in (3.2.57), respectively. Since, $p \geqslant 1$ we have that $e^{\frac{p}{2}} - 1 \geqslant \sqrt{e} - 1$, thus we conclude that in the case where $x_p \geqslant 1$ we have for all $x \geqslant 1$,

$$k_p(x) = \frac{(\log(1+x))^p}{x} \leqslant \frac{1}{\sqrt{e}-1}\, p^p\,. \tag{3.2.58}$$

Moving to the second case, i.e. $0 \leqslant x_p < 1$, we have that since k_p is decreasing in $[1,\infty) \subset [x_p,\infty)$, we have that for all $x \geqslant 1$,

$$k_p(x) = \frac{(\log(1+x))^p}{x} \leqslant k_p(1) = (\log(2))^p < 1\,, \tag{3.2.59}$$

since $p \geqslant 1$ and $\log 2 < 1$. Therefore, by (3.2.58), (3.2.59) and since $p \geqslant 1$, for all $x \geqslant 1$ and $p \geqslant 1$ we have that

$$k_p(x) \leqslant \max\left\{1, \frac{1}{\sqrt{e}-1}\right\} p^p = \frac{1}{\sqrt{e}-1}\, p^p\,. \tag{3.2.60}$$

Recalling that $k_p(x) = \frac{(\log(1+x))^p}{x}$ and applying (3.2.60) to (3.2.55) for $x = A \geqslant 1$ we get that

$$\sum_{\substack{y\in\mathbb{Z}^2\\ |y|\leqslant 2\sqrt{A}}} \left(\log\left(1+\frac{A}{1+|y|^2}\right)\right)^p \leqslant (\log(1+A))^p + 6\,A\,\pi\,p^p \leqslant \frac{1}{\sqrt{e}-1}\,A\,p^p + 6\,A\,\pi\,p^p = c\,A\,p^p\,,$$

with $c = \frac{1}{\sqrt{e}-1} + 6\pi$, thus concluding the proof of (3.2.50). □

3.3. Proofs of Theorems 3.0.1, 3.0.3, 3.0.4 and 3.0.5.

Now we have all the ingredients to prove the main results. We begin with Theorem 3.0.4.

Proof of Theorem 3.0.4. We first prove (3.0.9). Recall from (3.2.11) that

$$\left|M_{N,h}^{\varphi,\psi}\right| \leqslant \frac{c\,\|\psi\|_\infty^h}{N^{h+1}} \sum_{k\geqslant 1} \sum_{(I_1,\dots,I_k)\in\mathcal{I}} \left\| \widehat{\mathsf{Q}}_{N,0}^{*;I_1} \frac{\varphi_N^{\otimes h}}{w_N^{\otimes h}} \right\|_{\ell^p} \prod_{i=2}^k \left\| \widehat{\mathsf{P}}_{N,\hat\beta}^{I_{i-1};I_i} \right\|_{\ell^q\to\ell^q} \left\| \widehat{\mathsf{Q}}_{N,0}^{I_k;*} w_N^{\otimes h} \right\|_{\ell^q} \prod_{i=1}^k \mathbb{E}\Big[|\xi|^{I_i}\Big]\,. \tag{3.3.1}$$

By Proposition 3.2.3, we have the following bounds on the boundary operator norms

$$\left\| \widehat{\mathsf{Q}}_{N,0}^{*;I_1} \frac{\varphi_N^{\otimes h}}{w_N^{\otimes h}} \right\|_{\ell^p} \leqslant C\,q\,N^{\frac{1}{q}} \left\| \frac{\varphi_N}{w_N} \right\|_{\ell^p}^h \quad \text{and} \quad \left\| \widehat{\mathsf{Q}}_{N,0}^{I_k;*} w_N^{\otimes h} \right\|_{\ell^q} \leqslant C\,p\,N^{\frac{1}{p}}\, \|w_N\|_{\ell^q}^h\,, \tag{3.3.2}$$

for a constant $C = C(h,w) \in (0,\infty)$. By Propositions 3.2.2 and 3.2.4 we also have that for all $2 \leqslant i \leqslant k$, there exists a constant $C = C(h,\hat\beta,w) \in (0,\infty)$, such that

$$\left\| \widehat{\mathsf{P}}_{N,\hat\beta}^{I_{i-1};I_i} \right\|_{\ell^q\to\ell^q} \leqslant C\,p\,q\,. \tag{3.3.3}$$

By inserting the bounds (3.3.2) and (3.3.3) in (3.3.1) we obtain that

$$\left|M_{N,h}^{\varphi,\psi}\right| \leqslant \frac{\|\psi\|_\infty^h}{N^h} \left\| \frac{\varphi_N}{w_N} \right\|_{\ell^p}^h \|w_N\|_{\ell^q}^h \sum_{k\geqslant 1} (C\,p\,q)^k \sum_{(I_1,\dots,I_k)\in\mathcal{I}} \prod_{i=1}^k \mathbb{E}\Big[|\xi|^{I_i}\Big]\,. \tag{3.3.4}$$

We now distinguish two cases depending on the range of k.

(Case 1). If $k > \lfloor \frac{h}{2} \rfloor$ we use the bound

$$\prod_{i=1}^{k} \mathbb{E}\big[|\xi|^{I_i}\big] \leqslant \big(\tfrac{C}{\log N}\big)^k ,$$

which is a consequence of the fact that $\mathbb{E}\big[|\xi|^{I_i}\big] \leqslant C\,\sigma_{N,\hat{\beta}}^2 = O(1/\log N)$, see (3.2.7) and (3.1.7). Therefore, in this case

$$\sum_{k > \lfloor \frac{h}{2} \rfloor} (C\,p\,q)^k \sum_{(I_1,\dots,I_k)\in\mathcal{I}} \prod_{i=1}^{k} \mathbb{E}\Big[|\xi|^{I_i}\Big] \leqslant \sum_{k>\lfloor \frac{h}{2}\rfloor} \Big(\frac{\widetilde{C}\,p\,q}{\log N}\Big)^k , \tag{3.3.5}$$

for a constant $\widetilde{C} = \widetilde{C}(h,\hat{\beta},w) \in (0,\infty)$, which also incorporates the fact that the number of possible choices for a sequence of partitions $(I_1,\dots,I_k)$ is bounded by C^k where $C = C(h)$ is some positive constant.

(Case 2). The second case is when $1 \leqslant k \leqslant \lfloor \frac{h}{2} \rfloor$, for which we claim that there exists a constant $C = C(h,\hat{\beta}) \in (0,\infty)$ such that

$$\prod_{i=1}^{k} \mathbb{E}\big[|\xi|^{I_i}\big] \leqslant C^k \,(\log N)^{-\frac{h}{2}} .$$

To see this fix $1 \leqslant k \leqslant \lfloor \frac{h}{2} \rfloor$ and $(I_1,\dots,I_k) \in \mathcal{I}$, and let $I_i = \bigsqcup_{1\leqslant j\leqslant |I_i|} I_{i,j}$. By (3.2.7) and (3.1.7), we have that

$$\prod_{i=1}^{k} \mathbb{E}\big[|\xi|^{I_i}\big] \leqslant C^k\,(\sigma_{N,\hat{\beta}})^{\sum_{1\leqslant i\leqslant k}\sum_{1\leqslant j\leqslant |I_i|;|I_{i,j}|\geqslant 2} |I_{i,j}|} .$$

From the definition of $\mathcal{I}$ (see below (3.1.7)), we have that

$$\sum_{1\leqslant i\leqslant k}\ \sum_{1\leqslant j\leqslant |I_i|;|I_{i,j}|\geqslant 2} |I_{i,j}| \geqslant h ,$$

since every $r \in \{1,\dots,h\}$ necessarily belongs to a non-trivial block of some partition I_i, $1 \leqslant i \leqslant k$. Therefore, as before we have that there exists a constant $\widetilde{C} = \widetilde{C}(h,\hat{\beta},w) \in (0,\infty)$ such that

$$\sum_{1\leqslant k\leqslant \lfloor \frac{h}{2}\rfloor} (C\,p\,q)^k \sum_{(I_1,\dots,I_k)\in\mathcal{I}} \prod_{i=1}^{k} \mathbb{E}\Big[|\xi|^{I_i}\Big] \leqslant (\log N)^{-\frac{h}{2}} \sum_{1\leqslant k\leqslant \lfloor \frac{h}{2}\rfloor} (\widetilde{C}\,p\,q)^k . \tag{3.3.6}$$

Combining estimates (3.3.5) and (3.3.5) we deduce from (3.3.4) that

$$\big|M_{N,h}^{\varphi,\psi}\big| \leqslant C\,\frac{\|\psi\|_\infty^h}{N^h} \left\|\frac{\varphi_N}{w_N}\right\|_{\ell^p}^h \|w_N\|_{\ell^q}^h \left(\sum_{k>\lfloor \frac{h}{2}\rfloor} \Big(\frac{\widetilde{C}\,p\,q}{\log N}\Big)^k + (\log N)^{-\frac{h}{2}} \sum_{1\leqslant k\leqslant\lfloor \frac{h}{2}\rfloor} (\widetilde{C}\,p\,q)^k \right). \tag{3.3.7}$$

Let $p, q > 1$, conjugate exponents, that satisfy the growth condition

$$\frac{\widetilde{C}\,p\,q}{\log N} < \frac{1}{2} . \tag{3.3.8}$$

In particular, $p\,q \leqslant \mathsf{a}_* \log N$ with $\mathsf{a}_* = \mathsf{a}_*(h,\hat{\beta},w) \in (0,1)$ defined as $\mathsf{a}_* := (2\widetilde{C})^{-1}$. We then have that

$$\sum_{k>\lfloor \frac{h}{2}\rfloor} \Big(\frac{\widetilde{C}\,p\,q}{\log N}\Big)^k \leqslant 2\,\Big(\frac{\widetilde{C}\,p\,q}{\log N}\Big)^{\lfloor \frac{h}{2}\rfloor+1} \tag{3.3.9}$$

by summing the tail of the geometric series, which is possible due to the growth condition (3.3.8) imposed on p, q. On the other hand, we have that

$$
\begin{aligned}
(\log N)^{-\frac{h}{2}} \sum_{1 \leqslant k \leqslant \lfloor \frac{h}{2} \rfloor} (\widetilde{C}\, p\, q)^k &\leqslant (\log N)^{-\frac{h}{2}} \cdot \frac{(\widetilde{C}\, p\, q)^{\lfloor \frac{h}{2} \rfloor + 1} - \widetilde{C}\, p\, q}{\widetilde{C}\, p\, q - 1} \\
&\leqslant (\log N)^{-\frac{h}{2}} \cdot \frac{(\widetilde{C}\, p\, q)^{\lfloor \frac{h}{2} \rfloor + 1}}{\widetilde{C}\, p\, q - 1} \\
&\leqslant 2 (\log N)^{-\frac{h}{2}} \, (\widetilde{C}\, p\, q)^{\lfloor \frac{h}{2} \rfloor}, \qquad (3.3.10)
\end{aligned}
$$

since $\widetilde{C}\, p\, q > 2$ ($pq \geqslant 4$ because $\frac{1}{p} + \frac{1}{q} = 1$ and we can choose $\widetilde{C} > 1$). Combining estimates (3.3.9) and (3.3.10) we obtain that

$$
\begin{aligned}
\left(\sum_{k > \lfloor \frac{h}{2} \rfloor} \left(\frac{\widetilde{C}\, p\, q}{\log N} \right)^k + (\log N)^{-\frac{h}{2}} \sum_{1 \leqslant k \leqslant \lfloor \frac{h}{2} \rfloor} (\widetilde{C}\, p\, q)^k \right) &\leqslant 2 \left(\frac{\widetilde{C}\, p\, q}{\log N} \right)^{\lfloor \frac{h}{2} \rfloor + 1} + 2 (\log N)^{-\frac{h}{2}} \, (\widetilde{C}\, p\, q)^{\lfloor \frac{h}{2} \rfloor} \\
&\leqslant 4 \left(\frac{\widetilde{C}\, p\, q}{\log N} \right)^{\frac{h}{2}},
\end{aligned}
$$

by using that $\frac{\widetilde{C}\, p\, q}{\log N} \leqslant \frac{1}{2}$ and $\lfloor \frac{h}{2} \rfloor \leqslant \frac{h}{2} < \lfloor \frac{h}{2} \rfloor + 1$. Inserting this bound to (3.3.7) we finally obtain that

$$
\left| M_{N,h}^{\varphi,\psi} \right| \leqslant \left(\frac{C\, p\, q}{\log N} \right)^{\frac{h}{2}} \frac{1}{N^h} \left\| \frac{\varphi_N}{w_N} \right\|_{\ell^p}^h \|\psi\|_\infty^h \, \|w_N\|_{\ell^q}^h \,, \qquad (3.3.11)
$$

for a constant $C = C(h, \hat{\beta}, w) > \widetilde{C}$, which establishes (3.0.9).

Let us now prove (3.0.10). By choosing $\varphi := \delta_0^{(N)} := N\, \mathbb{1}_{\{x=0\}}$, $\psi \equiv 1$ and $w(x) = e^{-|x|}$, we deduce from (3.0.9) that

$$
\left| \mathbb{E}\big[(\bar{Z}_{N,\beta_N})^h \big] \right| \leqslant \left(\frac{C\, p\, q}{\log N} \right)^{\frac{h}{2}} \|w_N\|_{\ell^q}^h = \left(\frac{C\, p\, q}{\log N} \right)^{\frac{h}{2}} \cdot N^{\frac{h}{q}} \cdot \frac{1}{N^{\frac{h}{q}}} \|w_N\|_{\ell^q}^h \,. \qquad (3.3.12)
$$

Since $w(x) = e^{-|x|}$ is decreasing in the radial direction we have

$$
\begin{aligned}
\frac{1}{N^{\frac{h}{q}}} \|w_N\|_{\ell^q}^h \leqslant \left(\frac{1}{N} + \frac{1}{N} \int_{\mathbb{R}^2} e^{-q \frac{|x|}{\sqrt{N}}} \, \mathrm{d}x \right)^{\frac{h}{q}} &= \left(\frac{1}{N} + \int_{\mathbb{R}^2} e^{-q|x|} \, \mathrm{d}x \right)^{\frac{h}{q}} \\
&= \left(\frac{1}{N} + \frac{2\pi}{q^2} \right)^{\frac{h}{q}} \qquad (3.3.13) \\
&\leqslant e^{(2\pi h)/q^3} \,.
\end{aligned}
$$

We choose $q = q_N := \mathsf{a} \log N$ with $\mathsf{a} = \mathsf{a}(h, \hat{\beta}, w) \in (0,1)$ small enough such that $\frac{C\, p\, q}{\log N} < \frac{1}{2}$ (and therefore (3.3.8) is satisfied). For this choice of q we have by (3.3.13) that

$$
\frac{1}{N^{\frac{h}{q}}} \|w_N\|_{\ell^q}^h \leqslant e^{O\left((\log N)^{-3} \right)} \leqslant C \,. \qquad (3.3.14)
$$

Furthermore, again with $q = q_N = \mathsf{a} \log N$ and thus $p = p_N = 1 + o(1)$, since $\frac{1}{p} + \frac{1}{q} = 1$, we get

$$
\left(\frac{C\, p\, q}{\log N} \right)^{\frac{h}{2}} \cdot N^{\frac{h}{q}} \leqslant 2^{-\frac{h}{2}} \exp\left(\tfrac{h}{\mathsf{a}} \right) < \infty \,, \qquad (3.3.15)
$$

since $\frac{C\, p\, q}{\log N} < \frac{1}{2}$. We note that the parameter $\mathsf{a} = \mathsf{a}(h, \hat{\beta}, w)$ on the right-hand side of (3.3.15) depends non-trivially on h, and therefore the true order of the bound in (3.3.15) is not exponential in h. Finally, by (3.3.12), (3.3.14) and (3.3.15), we obtain that

$$\sup_{N\in\mathbb{N}} \mathbb{E}\big[(\bar{Z}_{N,\beta_N})^h\big] < \infty\,.$$

$\square$

Proof of Theorem 3.0.1. By binomial expansion, for $h \in \mathbb{N}$ we have that

$$\mathbb{E}\big[(Z_N^{\beta_N})^h\big] = \sum_{k=0}^{h} \binom{h}{k} \mathbb{E}\big[(\bar{Z}_N^{\beta_N})^k\big] \leqslant \sum_{k=0}^{h} \binom{h}{k} \Big|\mathbb{E}\big[(\bar{Z}_N^{\beta_N})^k\big]\Big|.$$

Therefore, by estimate (3.0.10) of Theorem 3.0.4, for every $h \geqslant 3$ we obtain that

$$\sup_{N\in\mathbb{N}} \mathbb{E}\big[(Z_N^{\beta_N})^h\big] < \infty.$$

Hence, for every $h \geqslant 0$ the sequence $\Big\{(Z_N^{\beta_N})^h\Big\}_{N\geqslant 1}$ is uniformly integrable and therefore, by Theorem A for every $h \geqslant 0$,

$$\lim_{N\to\infty} \mathbb{E}\big[(Z_N^{\beta_N})^h\big] = \mathbb{E}\big[\exp\big(\varrho_{\hat{\beta}}\, h\, \mathsf{X} - \tfrac{1}{2}\varrho_{\hat{\beta}}^2\, h\big)\big] = \exp\Big(\tfrac{h(h-1)}{2}\,\varrho_{\hat{\beta}}^2\Big) = \left(\frac{1}{1-\hat{\beta}^2}\right)^{\frac{h(h-1)}{2}}.$$

As can be seen in [CSZ20], section 3, (3.0.7) implies that for all $h > 0$,

$$\sup_{N\in\mathbb{N}} \mathbb{E}\big[(Z_N^{\beta_N})^{-h}\big] < \infty,$$

which in combination with Theorem A implies the convergence of negative moments. $\square$

Proof of Theorem 3.0.3. We note that if we choose the law of the environment ω to be Gaussian, i.e. $\omega \sim \mathcal{N}(0,1)$, then for $h \in \mathbb{N}$

$$\begin{aligned}\mathbb{E}\big[(Z_{N,\beta_N})^h\big] &= \mathrm{E}^{\otimes h}\Bigg[\exp\Big(\beta_N^2 \sum_{1\leqslant i<j\leqslant h} \mathsf{L}_N^{(i,j)}\Big)\Bigg]\\ &= \mathrm{E}^{\otimes h}\Bigg[\exp\Big(\frac{\hat{\beta}^2\,\pi}{\log N}\big(1+o(1)\big) \sum_{1\leqslant i<j\leqslant h} \mathsf{L}_N^{(i,j)}\Big)\Bigg].\end{aligned}$$

Therefore, by Theorem 3.0.1 we have that

$$\mathrm{E}^{\otimes h}\Bigg[\exp\Big(\frac{\hat{\beta}^2\,\pi}{\log N} \sum_{1\leqslant i<j\leqslant h} \mathsf{L}_N^{(i,j)}\Big)\Bigg] \xrightarrow{N\to\infty} \left(\frac{1}{1-\hat{\beta}^2}\right)^{\frac{h(h-1)}{2}}, \tag{3.3.16}$$

for all $\hat{\beta} \in [0,1)$. The right-hand side of (3.3.16) is equal to $M_Y(\hat{\beta}^2)$, where $M_Y(t) := \mathrm{E}[e^{tY}]$ denotes the moment generating function of a random variable Y with law $\Gamma\big(\frac{h(h-1)}{2},1\big)$. By exercise 9, chapter 4 in [K97], (3.3.16) implies the convergence of $\frac{\pi}{\log N}\sum_{1\leqslant i<j\leqslant h} \mathsf{L}_N^{(i,j)}$ in law, to a $\Gamma\big(\frac{h(h-1)}{2},1\big)$ distribution. $\square$

Proof of Theorem 3.0.5. We are going to show that for all $h \in \mathbb{N}$ with $h \geqslant 3$ we have that

$$\sup_{N\in\mathbb{N}} (\log N)^{\frac{h}{2}}\, |M_{N,h}^{\varphi,\psi}| < \infty\,. \tag{3.3.17}$$

In that case we obtain uniform integrability of $(\log N)^{\frac{h}{2}}\,\big(\bar{Z}_{N,\beta_N}(\varphi,\psi)\big)^h$ for all $h \in \mathbb{N}$ and the convergence of moments in Theorem 3.0.5 follows by Theorem B. But, (3.3.17) is an immediate consequence of (3.0.9) of Theorem 3.0.4. Indeed, let us fix $p,q \in (1,\infty)$ such that $\frac{1}{p}+\frac{1}{q}=1$. By

(3.0.9) of Theorem 3.0.4 we have that

$$(\log N)^{\frac{h}{2}} \big| M^{\varphi,\psi}_{N,h} \big| \leqslant \, (C\,p\,q)^{\frac{h}{2}} \frac{1}{N^h} \left\| \frac{\varphi_N}{w_N} \right\|^h_{\ell^p} \|w_N\|^h_{\ell^q} \, \|\psi_N\|^h_\infty \,. \tag{3.3.18}$$

Furthermore, by Riemann approximation we have that

$$\frac{1}{N^h} \left\| \frac{\varphi_N}{w_N} \right\|^h_{\ell^p} \|w_N\|^h_{\ell^q} \, \|\psi_N\|^h_\infty = \frac{1}{N^{\frac{h}{p}}} \left\| \frac{\varphi_N}{w_N} \right\|^h_{\ell^p} \frac{1}{N^{\frac{h}{q}}} \|w_N\|^h_{\ell^q} \, \|\psi_N\|^h_\infty \leqslant \, C \, \left\| \frac{\varphi}{w} \right\|^h_{\ell^p} \|w\|^h_{\ell^q} \, \|\psi\|^h_\infty \,. \tag{3.3.19}$$

Therefore, by (3.3.18) and (3.3.19) we obtain that

$$\sup_{N \in \mathbb{N}} (\log N)^{\frac{h}{2}} \, |M^{\varphi,\psi}_{N,h}| < \infty \,,$$

which concludes the proof. $\square$

CHAPTER 4

A multivariate extension of the Erdös-Taylor theorem

Let $S^{(1)}, \dots, S^{(h)}$ be independent, simple, symmetric random walks on $\mathbb{Z}^2$ starting at the origin. As in previous chapters, we will use P_x and E_x to denote the probability and expectation with respect to the law of the simple random walk when starting from $x \in \mathbb{Z}^2$ and we will omit the subscripts when the walk starts from 0. For $1 \leqslant i < j \leqslant h$ we define the *collision local time* between $S^{(i)}$ and $S^{(j)}$ up to time N by

$$\mathsf{L}_N^{(i,j)} := \sum_{n=1}^{N} \mathbb{1}_{\{S_n^{(i)} = S_n^{(j)}\}} .$$

Notice that given $1 \leqslant i < j \leqslant h$, $\mathsf{L}_N^{(i,j)}$ has the same law as the number of returns to zero, before time $2N$, for a single simple, symmetric random walk S on $\mathbb{Z}^2$, that is $\mathsf{L}_N^{(i,j)} \overset{\text{law}}{=} \mathsf{L}_N := \sum_{n=1}^{N} \mathbb{1}_{\{S_{2n}=0\}}$. This equality is a consequence of the independence of $S^{(i)}$, $S^{(j)}$ and the symmetry of the simple random walk. A first moment calculation shows that

$$R_N := \mathrm{E}\big[\mathsf{L}_N\big] = \sum_{n=1}^{N} \mathrm{P}(S_{2n} = 0) \overset{N\to\infty}{\approx} \frac{\log N}{\pi} , \tag{4.0.1}$$

see Section 4.1 for more details. We recall from chapter 3, the classical result of Erdös and Taylor, [ET60], which establishes that under normalisation (4.0.1), L_N satisfies the following limit theorem.

Theorem A ([ET60])**.** Let $\mathsf{L}_N := \sum_{n=1}^{N} \mathbb{1}_{\{S_{2n}=0\}}$ be the local time at zero, up to time $2N$, of a two-dimensional, simple, symmetric random walk $(S_n)_{n\geqslant 1}$ starting at 0. Then, as $N \to \infty$,

$$\frac{\pi}{\log N} \mathsf{L}_N \xrightarrow{(d)} Y ,$$

where Y has an exponential distribution with parameter 1.

Theorem A was recently generalised in [LZ21+], see Chapter 3. In particular,

Theorem B ([LZ21+])**.** Let $h \in \mathbb{N}$ with $h \geqslant 2$ and $S^{(1)}, \dots, S^{(h)}$ be h independent two-dimensional, simple random walks starting all at zero. Then, for all $\beta \in (0,1)$, it holds that the total collision time $\sum_{1\leqslant i<j\leqslant h} \mathsf{L}_N^{(i,j)}$ satisfies

$$\mathrm{E}^{\otimes h}\Big[e^{\frac{\pi\beta}{\log N} \sum_{1\leqslant i<j\leqslant h} \mathsf{L}_N^{(i,j)}} \Big] \xrightarrow[N\to\infty]{} \Big(\frac{1}{1-\beta}\Big)^{\frac{h(h-1)}{2}} ,$$

and, consequently,

$$\frac{\pi}{\log N} \sum_{1\leqslant i<j\leqslant h} \mathsf{L}_N^{(i,j)} \xrightarrow[N\to\infty]{(d)} \Gamma\big(\tfrac{h(h-1)}{2}, 1\big) ,$$

where $\Gamma\big(\frac{h(h-1)}{2}, 1\big)$ denotes a Gamma variable, with density $\Gamma(h(h-1)/2)^{-1} x^{\frac{h(h-1)}{2}-1} e^{-x}$; $\Gamma(\cdot)$, in the expression of the density, denotes the Gamma function.

Given the fact that a gamma distribution $\Gamma(k,1)$, with parameter $k \geqslant 1$, arises as the distribution of the sum of k independent random variables each one distributed according to an exponential random variable with parameter one (denoted as $\mathrm{Exp}(1)$), Theorem B raises the question as to whether the joint distribution of the individual rescaled collision times

$$\Big\{\frac{\pi}{\log N}\,\mathsf{L}_N^{(i,j)}\Big\}_{1\leqslant i<j\leqslant h}$$

converges to that of a family of independent $\mathrm{Exp}(1)$ random variables. This is what we prove in this chapter. In particular,

Theorem 4.0.1. *Let $h \in \mathbb{N}$ with $h \geqslant 2$ and $\boldsymbol{\beta} := \{\beta_{i,j}\}_{1\leqslant i<j\leqslant h} \in \mathbb{R}^{\frac{h(h-1)}{2}}$ with $\beta_{i,j} < 1$ for all $1 \leqslant i < j \leqslant h$. Then we have that*

$$\mathrm{E}^{\otimes h}\Big[e^{\frac{\pi}{\log N}\sum_{1\leqslant i<j\leqslant h}\beta_{i,j}\mathsf{L}_N^{(i,j)}}\Big] \xrightarrow[N\to\infty]{} \prod_{1\leqslant i<j\leqslant h}\frac{1}{1-\beta_{i,j}} \tag{4.0.2}$$

and, consequently,

$$\Big\{\tfrac{\pi}{\log N}\,\mathsf{L}_N^{(i,j)}\Big\}_{1\leqslant i<j\leqslant h} \xrightarrow[N\to\infty]{(d)} \big\{Y^{(i,j)}\big\}_{1\leqslant i<j\leqslant h},$$

where $\big\{Y^{(i,j)}\big\}_{1\leqslant i<j\leqslant h}$ are independent and identically distributed random variables following an $\mathrm{Exp}(1)$ distribution.

We remark that the additional difficulty in proving theorem 4.0.1 stems form the fact that we need an exact computation of the joint Laplace transform of the collision local times as opposed to Theorem B which was derived through moment bounds and a distributional convergence result on the directed polymer partition function, see Chapter 3 for more details.

The approach we follow towards proving asymptotic independence of the family

$$\Big\{\tfrac{\pi}{\log N}\,\mathsf{L}_N^{(i,j)}\Big\}_{1\leqslant i<j\leqslant h}$$

starts with expanding the joint Laplace transform in the form of *chaos series*, which take the form of Feynman-type diagrams. To control (and simplify) these diagrams, we start by inputing a renewal representation as well as a functional analytic framework. The renewal theoretic framework was originally introduced in [CSZ19a] in the context of scaling limits of random polymers (we will come back to the connection with polymers later on) and it captures the stream of collisions within a single pair of walks. The functional analytic framework can be traced back to works on spectral theory of *delta-Bose gases* [DFT94, DR04] and was also recently used in works on random polymers [GQT21, CSZ21+, LZ21+]. The core of this framework is to establish operator norm bounds for the total Green's functions of a set of planar random walks conditioned on a subset of them starting at the same location and on another subset of them ending up at the same location. Roughly speaking, the significance of these operator estimates is to control the redistribution of collisions when walks switch pairs. The operator framework (together with the renewal one) allows to reduce the number of Feynman-type diagrams that need to be considered. For the reduced Feynman diagrams one, then, needs to look into the logarithmic structure, which induces the separation of scales and leads to the fact that, asymptotically, the structure of the Feynman diagrams becomes that of the product of Feynman diagrams corresponding to Laplace transforms of single pairs of random walks.

The structure of this chapter is as follows: In Section 4.1 we set the framework of the chaos expansion, its graphical representations in terms of Feynman-type diagrams, as well as the renewal and functional analytic frameworks. In Section 4.2 we carry out the approximation steps, which lead to our theorem. At the beginning of Section 4.2 we also provide an outline of the scheme.

4.1. Chaos expansions and auxiliary results

In this section we will introduce the framework, within which we work, and which consists of setting chaos expansions for the joint Laplace transform

$$M_{N,h}^{\boldsymbol{\beta}} := \mathrm{E}^{\otimes h}\Big[e^{\sum_{1\leqslant i<j\leqslant h} \frac{\pi\beta_{i,j}}{\log N} \mathsf{L}_N^{(i,j)}}\Big], \tag{4.1.1}$$

for a fixed collection of numbers $\boldsymbol{\beta} := \{\beta_{i,j}\}_{1\leqslant i<j\leqslant h} \in \mathbb{R}^{\frac{h(h-1)}{2}}$ with $\beta_{i,j} \in (0,1)$ for all $1 \leqslant i < j \leqslant h$. We denote by

$$\bar{\beta} := \max_{1\leqslant i<j\leqslant h} \beta_{i,j} < 1\,, \tag{4.1.2}$$

and define

$$\sigma_N^{i,j} := \sigma_N^{i,j}(\beta_{i,j}) := e^{\beta_N^{i,j}} - 1 \qquad \text{with} \qquad \beta_N^{i,j} := \frac{\pi\,\beta_{i,j}}{\log N}. \tag{4.1.3}$$

We will use the notation $q_n(x) := \mathrm{P}(S_n = x)$ for the transition probability of the simple, symmetric random walk. The expected collision local time between two independent simple, symmetric random walks will be

$$R_N := \mathrm{E}^{\otimes 2}\Big[\sum_{n=1}^N \mathbb{1}_{S_n^{(1)}=S_n^{(2)}}\Big] = \sum_{n=1}^N q_{2n}(0) \tag{4.1.4}$$

and by Proposition 3.2 in [CSZ19a] we have that in the two-dimensional setting

$$R_N = \frac{\log N}{\pi} + \frac{\alpha}{\pi} + o(1)\,, \tag{4.1.5}$$

as $N \to \infty$, with $\alpha = \gamma + \log 16 - \pi \simeq 0.208$ and $\gamma \simeq 0.577$ is the Euler constant.

4.1.1. Chaos expansion for two-body collisions and renewal framework. We start with the Laplace transform of the simple case of two-body collisions $\mathrm{E}\Big[e^{\beta_N^{i,j}\,\mathsf{L}_N^{(i,j)}}\Big]$ and deduce its chaos expansion as follows:

$$\begin{aligned}
\mathrm{E}\Big[e^{\beta_N^{i,j}\,\mathsf{L}_N^{(i,j)}}\Big] &= \mathrm{E}\Big[e^{\beta_N^{i,j}\,\sum_{n=1}^N \sum_{x\in\mathbb{Z}^2} \mathbb{1}_{\{S_n^{(i)}=x\}}\mathbb{1}_{\{S_n^{(j)}=x\}}}\Big] \\
&= \mathrm{E}\Big[\prod_{\substack{1\leqslant n\leqslant N\\ x\in\mathbb{Z}^2}} \Big(1 + \Big(e^{\beta_N^{i,j}} - 1\Big)\,\mathbb{1}_{\{S_n^{(i)}=x\}}\mathbb{1}_{\{S_n^{(j)}=x\}}\Big)\Big] \\
&= 1 + \sum_{k\geqslant 1} (\sigma_N^{i,j})^k \sum_{\substack{1\leqslant n_1<\dots<n_k\leqslant N\\ x_1,\dots,x_k\in\mathbb{Z}^2}} \mathrm{E}\Big[\prod_{a=1}^k \mathbb{1}_{\{S_{n_a}^{(i)}=x_a\}}\mathbb{1}_{\{S_{n_a}^{(j)}=x_a\}}\Big] \\
&= 1 + \sum_{k\geqslant 1} (\sigma_N^{i,j})^k \sum_{\substack{1\leqslant n_1<\dots<n_k\leqslant N,\\ x_1,\dots,x_k\in\mathbb{Z}^2}} \prod_{a=1}^k q_{n_a-n_{a-1}}^2(x_a - x_{a-1})
\end{aligned} \tag{4.1.6}$$

where in the last equality we used the Markov property, in the third we expanded the product and in the second we used the simple fact that

$$\begin{aligned}
e^{\beta_N^{i,j}\,\mathbb{1}_{\{S_n^{(i)}=S_n^{(j)}=x\}}} &= 1 + \Big(e^{\beta_N^{i,j}\,\mathbb{1}_{\{S_n^{(i)}=S_n^{(j)}=x\}}} - 1\Big) = 1 + \Big(e^{\beta_N^{i,j}} - 1\Big)\mathbb{1}_{\{S_n^{(i)}=S_n^{(j)}=x\}} \\
&= 1 + \sigma_N^{i,j}\,\mathbb{1}_{\{S_n^{(i)}=S_n^{(j)}=x\}},
\end{aligned}$$

with $\sigma_N^{i,j}$ defined in (4.1.3). We will express (4.1.6) in terms of the following quantity $U_N^{\beta}(n,x)$, which plays an important role in our formulation. For $\beta > 0$, $\sigma_N := \sigma_N(\beta) := e^{\frac{\pi\beta}{\log N}} - 1$ and

$(n,x) \in \mathbb{N} \times \mathbb{Z}^2$, we define

$$U_N^\beta(n,x) := \sigma_N\, q_n^2(x) + \sum_{k\geqslant 1} \sigma_N^{k+1} \sum_{\substack{0<n_1<\dots<n_k<n \\ z_1,z_2,\dots,z_k\in\mathbb{Z}^2}} q_{n_1}^2(z_1) \Big\{ \prod_{j=2}^{k} q_{n_j-n_{j-1}}^2(z_j - z_{j-1}) \Big\}\, q_{n-n_k}^2(x - z_k). \tag{4.1.7}$$

and $U_N^\beta(n,x) := \mathbb{1}_{\{x=0\}}$, if $n = 0$. Moreover, for $n \in \mathbb{N}$ we define

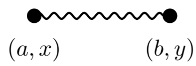
$$U_N^\beta(n) := \sum_{x\in\mathbb{Z}^2} U_N^\beta(n,x)\,.$$

$U_N^\beta(n,x)$ represents the Laplace transform of the two-body collisions, scaled by β, between a pair of random walks that are constrained to end at the spacetime point $(n,x) \in \{1,\dots,N\} \times \mathbb{Z}^2$, starting from $(0,0)$. In particular, for any $1 \leqslant i < j \leqslant h$, we can write (4.1.6) as

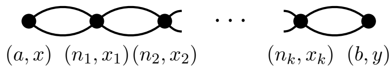
$$\mathrm{E}\Big[e^{\beta_N^{i,j}\, \mathsf{L}_N^{(i,j)}}\Big] = \sum_{n=0}^{N} \sum_{x\in\mathbb{Z}^2} U_N^{\beta_{i,j}}(n,x) = \sum_{n=0}^{N} U_N^{\beta_{i,j}}(n).$$

We will call $U_N^\beta(n,x)$ a **replica** and for $\sigma_N(\beta) = e^{\frac{\pi\beta}{\log N}} - 1$ we will graphically represent $\sigma_N(\beta)\, U_N^\beta(n,x)$ as

$$\sigma_N(\beta)\, U_N^\beta(b-a, y-x) \equiv$$



$$:= \sum_{k\geqslant 1} \sum_{\substack{n_1<\dots<n_k \\ x_1,\dots,x_k}}$$



In the second line we have assigned weights $q_{n'-n}(x'-x)$ to the solid lines going from (n,x) to (n',x') and we have assigned the weight $\sigma_N(\beta) = e^{\frac{\pi\beta}{\log N}} - 1$ to every solid dot.

$U_N^\beta(n)$ and $U_N^\beta(n,x)$ admit a very useful probabilistic interpretation in terms of certain renewal processes. More specifically, consider the family of i.i.d. random variables $(T_i^{(N)}, X_i^{(N)})_{i\geqslant 1}$ with law

$$\mathrm{P}\Big(\big(T_1^{(N)}, X_1^{(N)}\big) = (n,x) \Big) = \frac{q_n^2(x)}{R_N}\, \mathbb{1}_{\{n\leqslant N\}}\,.$$

and R_N defined in (4.1.4). Define the random variables $\tau_k^{(N)} := T_1^{(N)} + \dots + T_k^{(N)}$, $S_k^{(N)} := X_1^{(N)} + \dots + X_k^{(N)}$, if $k \geqslant 1$, and $(\tau_0, S_0) := (0,0)$, if $k = 0$. It is not difficult to see that $U_N^\beta(n,x)$ and $U_N^\beta(n)$ can, now, be written as

$$U_N^\beta(n,x) = \sum_{k\geqslant 0} (\sigma_N R_N)^k\, \mathrm{P}\big(\tau_k^{(N)} = n, S_k^{(N)} = x\big) \quad \text{and} \quad U_N^\beta(n) = \sum_{k\geqslant 0} (\sigma_N R_N)^k\, \mathrm{P}\big(\tau_k^{(N)} = n\big) \tag{4.1.8}$$

This formalism was developed in [CSZ19a] and is very useful in obtaining sharp asymptotic estimates. In particular, it was shown in [CSZ19a] that the rescaled process $\Big(\frac{\tau_{\lfloor s\log N\rfloor}^{(N)}}{N}, \frac{S_{\lfloor s\log N\rfloor}^{(N)}}{\sqrt{N}}\Big)$ converges in distribution for $N \to \infty$ with the law of the marginal limiting process for $\frac{\tau_{\lfloor s\log N\rfloor}^{(N)}}{N}$ being the *Dickman subordinator*, which was defined in [CSZ19a] as a truncated, zero-stable Lévy process.

An estimate that follows easily from this framework, which is useful for our purposes here, is the following: for $\beta < 1$, it holds

$$\begin{aligned}\limsup_{N\to\infty} \sum_{n=0}^{N} U_N^{\beta}(n) &= \limsup_{N\to\infty} \sum_{k\geqslant 0} (\sigma_N R_N)^k \, \mathrm{P}\big(\tau_k^{(N)} \leqslant N\big) \\ &\leqslant \limsup_{N\to\infty} \sum_{k\geqslant 0} (\sigma_N R_N)^k \\ &= \limsup_{N\to\infty} \frac{1}{1-\sigma_N R_N} = \frac{1}{1-\beta},\end{aligned} \tag{4.1.9}$$

where we used the fact that

$$\sigma_N R_N = \big(e^{\frac{\pi\beta}{\log N}} - 1\big) \cdot \Big(\frac{\log N}{\pi} + \frac{\alpha}{\pi} + o(1)\Big) \xrightarrow[N\to\infty]{} \beta < 1. \tag{4.1.10}$$

4.1.2. Chaos expansion for many-body collisions. We now move to the expansion of the Laplace transform of the many-body collisions $M_{N,h}^{\boldsymbol{\beta}}$. The goal is to obtain an expansion in the form of products of certain Markovian operators. The desired expression will be presented in (4.1.17). This expansion will be instrumental in obtaining some important estimates in Section 4.1.3.

The first steps are similar as in the expansion for the two-body collisions, above. In particular, we have

$$\begin{aligned}&\mathrm{E}^{\otimes h}\Big[e^{\sum_{1\leqslant i<j\leqslant h} \beta_N^{i,j}\, \mathsf{L}_N^{(i,j)}}\Big] \\ =&\mathrm{E}\Big[\prod_{1\leqslant i<j\leqslant h} \prod_{\substack{1\leqslant n\leqslant N \\ x\in\mathbb{Z}^2}} \Big(1 + \sigma_N^{i,j}\, \mathbb{1}_{\{S_n^{(i)}=x\}} \mathbb{1}_{\{S_n^{(j)}=x\}}\Big)\Big] \\ =&1 + \sum_{k\geqslant 1} \sum_{\substack{(i_a,j_a,n_a,x_a)\in\mathcal{A}_h,\text{ for } a=1,\dots,k \\ \text{distinct}}} \mathrm{E}\Big[\prod_{a=1}^{k} \sigma_N^{i_a,j_a}\, \mathbb{1}_{\{S_{n_a}^{(i_a)}=x_a\}} \mathbb{1}_{\{S_{n_a}^{(j_a)}=x_a\}}\Big]\end{aligned} \tag{4.1.11}$$

where the last sum is over k distinct elements of the set

$$\mathcal{A}_h := \big\{(i,j,n,x) \in \mathbb{N}^3 \times \mathbb{Z}^2 \colon 1 \leqslant i < j \leqslant h\big\}.$$

The graphical representation of expansion (4.1.11) is depicted in Figure 4.1.1. There, we have marked with black dots the space-time points (n,x) where some of the walks collide and have assigned to such each one the weight $\prod_{1\leqslant i<j\leqslant h} \sigma_N^{i,j}\, \mathbb{1}_{\{S_n^{(i)}=S_n^{(j)}=x\}}$.

We now want to write the above expansion as a convolution of Markovian operators, following the Markov property of the simple random walks. We can partition the time interval $\{0,1,...,N\}$ according to the times when collisions take place; these are depicted in Figure 4.1.1 by vertical lines. In between two successive times m,n, the walks will move from their locations $(x^{(i)})_{i=1,...,h}$ at time m to their new locations $(y^{(i)})_{i=1,...,h}$ at time n (some of which might coincide) according to their transition probabilities, giving a total weight to this transition of $\prod_{i=1}^h q_{n-m}(y^{(i)} - x^{(i)})$. We, now, want to encode in this product the coincidences that may take place within the sets $(x^{(i)})_{i=1,...,h}$ and $(y^{(i)})_{i=1,...,h}$. To this end, we consider partitions I of the set of indices $\{1,...,h\}$, which we denote by $I \vdash \{1,...,h\}$. We also denote by $|I|$ the number of parts of I. Given a partition $I \vdash \{1,\dots,h\}$, we define an equivalence relation $\overset{I}{\sim}$ in $\{1,\dots,h\}$ such that $k \overset{I}{\sim} \ell$ if and only if k and ℓ belong to the same part of partition I. Given a vector $\boldsymbol{y} = (y_1,\dots,y_h) \in (\mathbb{Z}^2)^h$ and $I \vdash \{1,\dots,h\}$, we shall use the notation $\boldsymbol{y} \sim I$ to mean that $y_k = y_\ell$ for all pairs $k \overset{I}{\sim} \ell$.

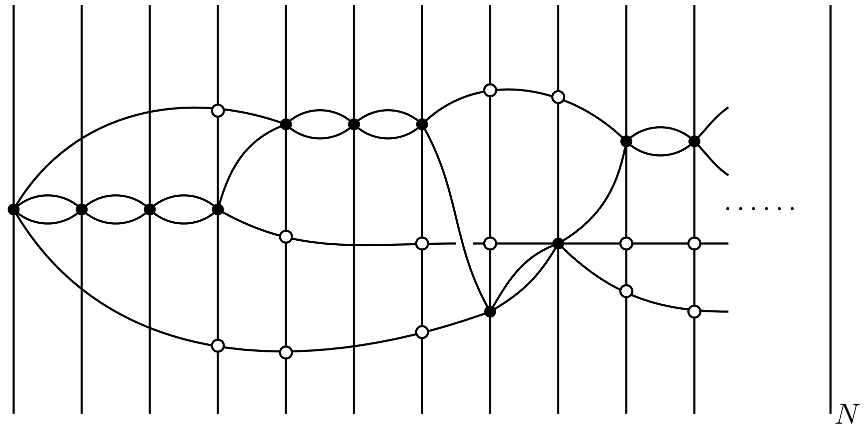


FIGURE 4.1.1. This is a graphical representation of expansion (4.1.11) corresponding to the collisions of four random walks, each starting from the origin. Each solid line will be marked with the label of the walk that it corresponds to throughout the diagram. The solid dots, which mark a collision among a subset A of the random walks, is given a weight $\prod_{i,j\in A}\sigma_N^{i,j}$. Any solid line between points $(m,x),(n,y)$ is assigned the weight of the simple random walk transition kernel $q_{m-n}(y-x)$. The hollow dots are assigned weight 1 and they mark the places where we simply apply the Chapman-Kolmogorov formula.

We use the symbol $\circ$ to denote the one-part partition*, that is, $\circ := \{1,\dots,h\}$, and $*$ to denote the partition consisting only of singletons, that is $* := \bigsqcup_{i=1}^{h}\{i\}$. Moreover, given $I \vdash \{1,\dots,h\}$ such that $|I| = h-1$ and $I = \{i,j\} \sqcup \bigsqcup_{k\neq i,j}\{k\}$, by slightly abusing notation, we may identify and denote I by its non-trivial part $\{i,j\}$.

Given this formalism, we denote the total transition weight of the h walks, from points $\boldsymbol{x} = (x^{(1)},...,x^{(h)}) \in (\mathbb{Z}^2)^h$, subject to constraints $\boldsymbol{x} \sim I$ at time m, to points $\boldsymbol{y} = (y^{(1)},...,y^{(h)}) \in (\mathbb{Z}^2)^h$, subject to constraints $\boldsymbol{y} \sim J$ at time n, by

$$Q_{n-m}^{I,J}(\boldsymbol{x},\boldsymbol{y}) := \mathbb{1}_{\{\boldsymbol{x}\sim I\}} \prod_{i=1}^{h} q_{n-m}(y^{(i)}-x^{(i)})\, \mathbb{1}_{\{\boldsymbol{y}\sim J\}}\,. \tag{4.1.12}$$

We will call this operator the **constrained evolution**. Furthermore, for a partition $I \vdash \{1,\dots,h\}$ and $\boldsymbol{\beta} = \{\beta_{i,j}\}_{1\leqslant i<j\leqslant h}$ we define the **mixed collision weight** subject to I as

$$\sigma_N(I) := \sigma_N(I,\{\beta_{i,j}\}_{1\leqslant i<j\leqslant h}) = \prod_{\substack{1\leqslant i<j\leqslant h,\\ i\overset{I}{\sim}j}} \sigma_N^{i,j}, \tag{4.1.13}$$

with $\sigma_N^{i,j}$ as defined in (4.1.3). We can then rewrite (4.1.11) in the form

$$1 + \sum_{r=1}^{\infty} \sum_{\circ:=I_0,I_1,\dots,I_r} \prod_{i=1}^{r} \sigma_N(I_i) \sum_{\substack{1\leqslant n_1<\cdots<n_r\leqslant N\\ 0:=\boldsymbol{x}_0,\boldsymbol{x}_1,\dots,\boldsymbol{x}_r\in(\mathbb{Z}^2)^h}} \prod_{i=1}^{r} Q_{n_i-n_{i-1}}^{I_{i-1};I_i}(\boldsymbol{x}_{i-1},\boldsymbol{x}_i)\,. \tag{4.1.14}$$

We want to make one more simplification in this representation, which, however, contains an important structural feature. This is to group together consecutive constrained evolution operators $\sigma_N(I_i)Q_{n_i-n_{i-1}}^{I_{i-1};I_i}(\boldsymbol{x}_{i-1},\boldsymbol{x}_i)$ for which $I_{i-1} = I_i$. An example in Figure 4.1.1 is the sequence of evolutions in the first three strips and another one is the group of evolutions in strips five and six.

*the notation $\circ$, with which we denote the one-part partition, here, should not be confused with the $\circ$ that appears in the figures, where it just marks places where we apply the Chapman-Kolmogorov.

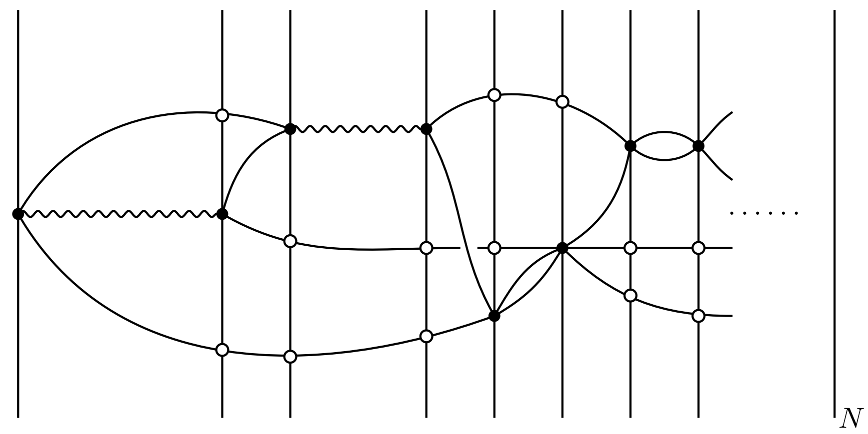


FIGURE 4.1.2. This is the simplified version of Figure's 4.1.1 graphical representation of the expansion (4.1.14), where we have grouped together the blocks of consecutive collisions between the same pair of random walks. These are now represented by the wiggle lines (**replicas**) and we call the evolution in strips that contain only one replica as **replica evolution** (although strip seven is the beginning of another wiggle line, we have not represented it as such since we have not completed the picture beyond that point). The wiggle lines (replicas) between points $(n,x),(m,y)$, corresponding to collisions of a single pair of walks $S^{(k)}, S^{(\ell)}$, are assigned weight $U_N^{\beta_{k,\ell}}(m-n, y-x)$. A solid line between points $(m,x),(n,y)$ is assigned the weight of the simple random walk transition kernel $q_{m-n}(y-x)$.

Such groupings can be captured by the following definition: For a partition $I \vdash \{1,\dots,h\}$ of the form $I = \{k,\ell\} \sqcup \bigsqcup_{j\neq k,\ell}\{j\}$ and $\boldsymbol{x} = (x^{(1)},\dots,x^{(h)})$, $\boldsymbol{y} = (y^{(1)},\dots,y^{(h)}) \in (\mathbb{Z}^2)^h$, we define the **replica evolution** as

$$\mathsf{U}_n^I(\boldsymbol{x},\boldsymbol{y}) := \mathbb{1}_{\{\boldsymbol{x},\boldsymbol{y}\sim I\}} \cdot U_N^{\beta_{k,\ell}}(n, y^{(k)} - x^{(k)}) \cdot \prod_{i\neq k,\ell} q_n(y^{(i)} - x^{(i)})\,, \tag{4.1.15}$$

with $U_N^{\beta}(n, y^{(k)} - x^{(k)})$ defined in (4.1.7). We name this *replica evolution* since in the time interval $[0,n]$ we see a stream of collisions between only two of the random walks. The simplified version of expansion (4.1.14) (and Figure 4.1.1) is presented in Figure 4.1.2.

In order to re-express (4.1.14) with the reduction of the replica evolution (4.1.15), we need to introduce one more formalism, which is

$$P_n^{I;J}(\boldsymbol{x},\boldsymbol{y}) := \begin{cases} \displaystyle\sum_{\substack{m_1\geqslant 1,\, m_2\geqslant 0:\ m_1+m_2=n\,,\\ \boldsymbol{z}\in(\mathbb{Z}^2)^h}} Q_{m_1}^{I;J}(\boldsymbol{x},\boldsymbol{z}) \cdot \mathsf{U}_{m_2}^J(\boldsymbol{z},\boldsymbol{y})\,, & \text{if } |J| = h-1\,, \\ Q_n^{I;J}(\boldsymbol{x},\boldsymbol{y})\,, & \text{if } |J| < h-1\,, \end{cases} \tag{4.1.16}$$

where we recall that $|J|$ is the number of parts of J and so $|J| = h-1$ means that J has the form $\{k,\ell\} \bigsqcup \sqcup_{i\neq k,\ell}\{i\}$, corresponding to a pairwise collision, while $|J| < h-1$ means that there are multiple collisions (the latter would correspond to the end of the eighth strip in Figure 4.1.1). In other words, the operator $P_n^{I;J}$ groups together the replica evolutions with its preceding constrained evolution.

We, finally, arrive to the desired expression for the Laplace transform of the many-body collisions:

$$M^{\boldsymbol{\beta}}_{N,h} = 1 + \sum_{r=1}^{\infty} \sum_{\circ := I_0, I_1, \dots, I_r} \prod_{i=1}^{r} \sigma_N(I_i) \sum_{\substack{1 \leqslant n_1 < \dots < n_r \leqslant N, \\ 0 := \boldsymbol{x}_0, \boldsymbol{x}_1, \dots, \boldsymbol{x}_r \in (\mathbb{Z}^2)^h}} \prod_{i=1}^{r} P^{I_{i-1}; I_i}_{n_i - n_{i-1}}(\boldsymbol{x}_{i-1}, \boldsymbol{x}_i) \,. \tag{4.1.17}$$

4.1.3. Functional analytic framework and some auxiliary estimates. Let us start with some, fairly easy, bounds on operators Q and U (with the estimate on the latter being an upgrade of estimate (4.1.9)).

Lemma 4.1.1. *Let the operators $Q^{I;J}_n$, U^J_n be defined in (4.1.12) and (4.1.15), respectively. For all partitions $I \neq J$ with $|J| = h-1$, $\bar\beta < 1$ defined in (4.1.2) and $\sigma_N(I)$ defined in (4.1.13), we have the bounds*

$$\sum_{0 \leqslant n \leqslant N,\, \boldsymbol{y} \in (\mathbb{Z}^2)^h} \mathsf{U}^J_n(\boldsymbol{x}, \boldsymbol{y}) \leqslant \frac{1}{1 - \bar\beta\,'} \quad \textit{and} \quad \sigma_N(J) \cdot \Bigg(\sum_{1 \leqslant n \leqslant N, \boldsymbol{y} \in (\mathbb{Z}^2)^h} Q^{I;J}_n(\boldsymbol{x}, \boldsymbol{y}) \Bigg) \leqslant \bar\beta\,' \,, \tag{4.1.18}$$

for all large enough N and a $\bar\beta\,' \in (\bar\beta, 1)$.

Proof. We start by proving the first bound in (4.1.18). By definition (4.1.15) we have that

$$\begin{aligned} \sum_{n \geqslant 0, \boldsymbol{y} \in (\mathbb{Z}^2)^h} \mathsf{U}^J_n(\boldsymbol{x}, \boldsymbol{y}) &:= \sum_{n \geqslant 0,\, \boldsymbol{y} \in (\mathbb{Z}^2)^h} \mathbb{1}_{\{\boldsymbol{x}, \boldsymbol{y} \sim J\}} \cdot U^{\beta_{k,\ell}}_N(n, y^{(k)} - x^{(k)}) \cdot \prod_{j \neq k,\ell} q_n(y^{(j)} - x^{(j)}) \\ &= \sum_{n \geqslant 0} U^{\beta_{k,\ell}}_N(n) \,, \end{aligned}$$

by using that $\sum_{z \in \mathbb{Z}^2} q_n(z) = 1$ to sum all the kernels $q_n(y^{(j)} - x^{(j)})$ for $j \neq k, \ell$ and

$$\sum_{z \in \mathbb{Z}^2} U^{\beta}_N(n, z) = U^{\beta}_N(n) \,.$$

Moreover, by definition (4.1.7) and (4.1.3), since $\beta_{k,\ell} \leqslant \bar\beta$, we have

$$\sum_{n \geqslant 0} U^{\beta_{k,\ell}}_N(n) \leqslant \sum_{n \geqslant 0} U^{\bar\beta}_N(n) = \sum_{k \geqslant 0} (\sigma_N(\bar\beta) R_N)^k \, \mathrm{P}(\tau^{(N)}_k = n) \,,$$

and by (4.1.10) we have that for any $\bar\beta\,' \in (\bar\beta, 1)$ and all N large enough

$$\sum_{n \geqslant 0} U^{\beta_{k,\ell}}_N(n) \leqslant \sum_{k \geqslant 0} (\bar\beta\,')^k \, \mathrm{P}(\tau^{(N)}_k = n) \leqslant \sum_{k \geqslant 0} (\bar\beta\,')^k = \frac{1}{1 - \bar\beta\,'} \,.$$

Therefore,

$$\sum_{n \geqslant 0, \boldsymbol{y} \in (\mathbb{Z}^2)^h} \mathsf{U}^J_n(\boldsymbol{x}, \boldsymbol{y}) \leqslant (1 - \bar\beta\,')^{-1} \,.$$

For the second bound in (4.1.18) we recall from (4.1.12) that when $J = \{k, \ell\} \sqcup \bigsqcup_{j \neq k,\ell} \{j\}$, then

$$Q^{I;J}_n(\boldsymbol{x}, \boldsymbol{y}) := \Bigg(\mathbb{1}_{\{\boldsymbol{x} \sim I\}} \prod_{j \neq k, \ell} q_n(y^{(j)} - x^{(j)}) \Bigg) \cdot q_n(y^{(k)} - x^{(k)}) \cdot q_n(y^{(k)} - x^{(\ell)}) \,,$$

since $\boldsymbol{y} \sim J$ means that $y_k = y_\ell$. Therefore, $\sigma_N(J) = \sigma_N(\beta_{i,j}) \leqslant \sigma_N(\bar\beta)$. We, now, use that $\sum_{z \in \mathbb{Z}^2} q_n(z) = 1$ in order to sum the kernels $q_n(y^{(j)} - x^{(j)})$, $j \neq k, \ell$, while we also have by Cauchy-Schwarz that

$$\sigma_N(J) \cdot \Bigg(\sum_{1 \leqslant n \leqslant N,\, y_k \in \mathbb{Z}^2} q_n(y^{(k)} - x^{(k)}) \cdot q_n(y^{(k)} - x^{(\ell)}) \Bigg) \leqslant \sigma_N(\bar\beta) \cdot \Big(\sum_{n=1}^{N} q_{2n}(0) \Big) \leqslant \bar\beta\,' \,,$$

by (4.1.10), for all N large enough, thus establishing the second bound in (4.1.18). $\square$

Next, in Proposition 4.1.2, we are going to recall some norm estimates from [LZ21+], presented also in detail in Chapter 3, on the Laplace transform of operators $P_n^{I;J}$, defined (4.1.16). For this, we need to set up the functional analytic framework. We start by defining $(\mathbb{Z}^2)^h_I := \{\boldsymbol{y} \in (\mathbb{Z}^2)^h_I : \boldsymbol{y} \sim I\}$ and, for $q \in (1,\infty)$, the $\ell^q((\mathbb{Z}^2)^h_I)$ space of functions $f : (\mathbb{Z}^2)^h_I \to \mathbb{R}$ which have finite norm

$$\|f\|_{\ell^q((\mathbb{Z}^2)^h_I)} := \Bigg(\sum_{\boldsymbol{y} \in ((\mathbb{Z}^2)^h_I)} |f(\boldsymbol{y})|^q\Bigg)^{\frac{1}{q}}.$$

For $q \in (1,\infty)$ and for an integral operator $\mathsf{T} : \ell^q\big((\mathbb{Z}^2)^h_J\big) \to \ell^q\big((\mathbb{Z}^2)^h_I\big)$, one can define the pairing

$$\langle f, \mathsf{T} g\rangle := \sum_{\boldsymbol{x} \in (\mathbb{Z}^2)^h_I, \boldsymbol{y} \in (\mathbb{Z}^2)^h_J} f(\boldsymbol{x}) \mathsf{T}(\boldsymbol{x},\boldsymbol{y}) g(\boldsymbol{y})\,. \tag{4.1.19}$$

The operator norm will be given by

$$\|\mathsf{T}\|_{\ell^q \to \ell^q} := \sup_{\|g\|_{\ell^q_J} \leqslant 1} \|\mathsf{T} g\|_{\ell^q_I} = \sup_{\|f\|_{\ell^p_I} \leqslant 1,\, \|g\|_{\ell^q_J} \leqslant 1} \langle f, \mathsf{T} g\rangle\,, \tag{4.1.20}$$

for $p,q \in (1,\infty)$ conjugate exponents, i.e. $\frac{1}{p} + \frac{1}{q} = 1$.

We introduce the weighted Laplace transforms of operators $Q^{I,J}$ and U^J. In particular, let $w(x)$ be any continuous function in $L^\infty(\mathbb{R}^2) \cap L^1(\mathbb{R}^2)$ such that $\log w(x)$ is Lipschitz (one can think of $w(x) = e^{-|x|}$) and define $w_N(x) := w(x/\sqrt{N})$. Also, for a function $g \colon \mathbb{R}^2 \to \mathbb{R}$ we define the tensor product $g^{\otimes h}(x_1,...,x_h) = g(x_1)\cdots g(x_h)$, The weighted Laplace transforms are now defined as

$$\begin{aligned}
\widehat{\mathsf{Q}}^{I;J}_{N,\lambda}(\boldsymbol{x},\boldsymbol{y}) &:= \Bigg(\sum_{n \geqslant 1}^{N} e^{-\lambda \frac{n}{N}} Q^{I;J}_n(\boldsymbol{x},\boldsymbol{y})\Bigg) \cdot \frac{w_N^{\otimes h}(\boldsymbol{x})}{w_N^{\otimes h}(\boldsymbol{y})}\,,\\
\widehat{\mathsf{U}}^{J}_{N,\lambda}(\boldsymbol{x},\boldsymbol{y}) &:= \Bigg(\sum_{n \geqslant 0}^{N} e^{-\lambda \frac{n}{N}} \mathsf{U}^{J}_n(\boldsymbol{x},\boldsymbol{y})\Bigg) \cdot \frac{w_N^{\otimes h}(\boldsymbol{x})}{w_N^{\otimes h}(\boldsymbol{y})}\,.
\end{aligned} \tag{4.1.21}$$

The passage to a Laplace transform will help to estimate convolutions involving $Q^{I;J}_n(\boldsymbol{x},\boldsymbol{y})$ and $\mathsf{U}^J_n(\boldsymbol{x},\boldsymbol{y})$ and the introduction of the weight comes handy in improving integrability when these operators are applied to functions which are not in $\ell^1((\mathbb{Z}^2)^h)$. We will see this in Lemma 4.1.3 below. We also define the Laplace transform operator of the combined evolution (4.1.16):

$$\widehat{\mathsf{P}}^{I;J}_{N,\lambda} = \begin{cases} \widehat{\mathsf{Q}}^{I;J}_{N,\lambda}\,, & \text{if } |J| < h-1 \\ \widehat{\mathsf{Q}}^{I;J}_{N,\lambda}\,\widehat{\mathsf{U}}^{J}_{N,\lambda}\,, & \text{if } |J| = h-1\,. \end{cases} \tag{4.1.22}$$

For our purposes, it will be sufficient to take $\lambda = 0$ and consider operators $\widehat{\mathsf{Q}}^{I;J}_{N,0}$, $\widehat{\mathsf{U}}^{J}_{N,0}$ and $\widehat{\mathsf{P}}^{I;J}_{N,0}$.

Using the above formalism we summarise in the next proposition some key estimates of [LZ21+], which are refinements of estimates in [CSZ21+] (Section 6) and [DFT94] (Section 3). These are also presented in detail in Chapter 3, Section 3.2.

Proposition 4.1.2. *Consider the operators* $\widehat{\mathsf{Q}}^{I;J}_{N,0}$ *and* $\widehat{\mathsf{P}}^{I;J}_{N,0}$ *defined in* (4.1.21) *and* (4.1.22) *with* $\lambda = 0$ *and a weight function* $w \in L^\infty(\mathbb{R}^2) \cap L^1(\mathbb{R}^2)$ *such that* $\log w(x)$ *is Lipschitz. Then there exists a constant* $C = C(h, \bar{\beta}, w) \in (0,\infty)$ *(recall* $\bar{\beta}$ *from* (4.1.2)*) such that for all* $p, q \in (1,\infty)$ *with* $\frac{1}{p} + \frac{1}{q} = 1$ *and all partitions* $I, J \vdash \{1,\dots,h\}$*, such that* $I \neq J$ *and* $|I|, |J| \leqslant h-1$*, we* *have that*

$$\left\| \widehat{\mathsf{P}}^{I;J}_{N,0} \right\|_{\ell^q \to \ell^q} \leqslant C\, p\, q\,. \tag{4.1.23}$$

Moreover, if $g \in \ell^q(\mathbb{Z}^2)$,

$$\left\| \widehat{\mathsf{Q}}^{*;I}_{N,0}\, g^{\otimes h} \right\|_{\ell^p} \leqslant C\, q\, N^{\frac{1}{q}}\, \|g\|^h_{\ell^p}\ , \tag{4.1.24}$$

for $g^{\otimes h}(x_1, ..., x_h) := g(x_1) \cdots g(x_h)$.

Let us now present the following lemma, which demonstrates how the above functional analytic framework will be used. This lemma will be useful in the first approximation, that we will perform in the next Section, in showing that contributions from multiple, i.e. three or more, collisions are negligible.

Lemma 4.1.3. *Let $H_{r,N}$ be the r^{th} term in the expansion (4.1.17), that is,*

$$H_{r,N} := \sum_{\circ := I_0, I_1, \ldots, I_r} \prod_{i=1}^{r} \sigma_N(I_i) \sum_{\substack{1 \leqslant n_1 < \cdots < n_r \leqslant N, \\ 0 := \boldsymbol{x}_0, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_r \in (\mathbb{Z}^2)^h}} \prod_{i=1}^{r} P^{I_{i-1};I_i}_{n_i - n_{i-1}}(\boldsymbol{x}_{i-1}, \boldsymbol{x}_i)\,, \tag{4.1.25}$$

and $H^{(\mathsf{multi})}_{r,N}$ be the corresponding term with the additional constraint that there is at least one multiple collision (i.e. at some point, three or more walks meet), that is,

$$H^{(\mathsf{multi})}_{r,N} := \sum_{\circ := I_0, I_1, \ldots, I_r} \left(\prod_{i=1}^{r} \sigma_N(I_i) \right) \mathbb{1}_{\{\exists\, 1 \leqslant j \leqslant r \,:\, |I_j| < h-1\}}$$

$$\times \sum_{\substack{1 \leqslant n_1 < \cdots < n_r \leqslant N \\ 0 := \boldsymbol{x}_0, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_r \in (\mathbb{Z}^2)^h}} \prod_{i=1}^{r} P^{I_{i-1};I_i}_{n_i - n_{i-1}}(\boldsymbol{x}_{i-1}, \boldsymbol{x}_i)\,.$$

Then the following bounds hold:

$$H_{r,N} \leqslant \Big(\frac{C\, p\, q}{\log N} \Big)^r N^{\frac{h+1}{q}} \qquad \text{and} \qquad H^{(\mathsf{multi})}_{r,N} \leqslant \frac{r}{\log N} \Big(\frac{C\, p\, q}{\log N} \Big)^r N^{\frac{h+1}{q}}\,. \tag{4.1.26}$$

for any $p, q \in (1, \infty)$ with $\frac{1}{p} + \frac{1}{q} = 1$ and a constant C that depends on h and $\bar{\beta}$ but is independent of N, r, p, q.

Proof. We start by considering $w(x) = e^{-|x|}$, $w_N(x) := w(\frac{x}{\sqrt{N}})$ and

$$w_N^{\otimes h}(x_1, ..., x_h) = \prod_{i=1}^{h} w_N(x_i)$$

and by including in the expression (4.1.25) the term

$$\frac{1}{w_N^{\otimes h}(\boldsymbol{x}_0)} \Big(\prod_{i=1}^{r} \frac{w_N^{\otimes h}(\boldsymbol{x}_{i-1})}{w_N^{\otimes h}(\boldsymbol{x}_i)} \Big) w_N^{\otimes h}(\boldsymbol{x}_r) = 1,$$

thus rewriting $H_{r,N}$ as

$$H_{r,N} = \sum_{\circ := I_0, I_1, \ldots, I_r} \prod_{i=1}^{r} \sigma_N(I_i)$$

$$\times \sum_{\substack{1 \leqslant n_1 < \cdots < n_r \leqslant N \\ 0 := \boldsymbol{x}_0, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_r \in (\mathbb{Z}^2)^h}} \frac{1}{w_N^{\otimes h}(\boldsymbol{x}_0)} \prod_{i=1}^{r} P^{I_{i-1};I_i}_{n_i - n_{i-1}}(\boldsymbol{x}_{i-1}, \boldsymbol{x}_i) \frac{w_N^{\otimes h}(\boldsymbol{x}_{i-1})}{w_N^{\otimes h}(\boldsymbol{x}_i)} \cdot w_N^{\otimes h}(\boldsymbol{x}_r)\,.$$

We can extend the summation on $\boldsymbol{x}_0$ from $\boldsymbol{x}_0 = 0$ to $\boldsymbol{x}_0 \in \mathbb{Z}^2$ by introducing a delta function $\delta_0^{\otimes h}$ at zero. Then

$$
\begin{aligned}
H_{r,N} = &\sum_{*:=I_0,I_1,\ldots,I_r} \prod_{i=1}^{r} \sigma_N(I_i) \\
&\times \sum_{\substack{1\leqslant n_1<\cdots<n_r\leqslant N \\ \boldsymbol{x}_0,\boldsymbol{x}_1,\ldots,\boldsymbol{x}_r\in(\mathbb{Z}^2)^h}} \frac{\delta_0^{\otimes h}(\boldsymbol{x}_0)}{w_N^{\otimes h}(\boldsymbol{x}_0)} \prod_{i=1}^{r} P_{n_i-n_{i-1}}^{I_{i-1};I_i}(\boldsymbol{x}_{i-1},\boldsymbol{x}_i) \frac{w_N^{\otimes h}(\boldsymbol{x}_{i-1})}{w_N^{\otimes h}(\boldsymbol{x}_i)} \cdot w_N^{\otimes h}(\boldsymbol{x}_r)\,.
\end{aligned}
$$

We can, now, bound the last expression by extending the temporal range of summations from $1 \leqslant n_1 < \cdots < n_r \leqslant N$ to $n_i - n_{i-1} \in \{1,\ldots,N\}$ for all $i = 1,\ldots,r$. Recalling the definition of the Laplace transforms of the operators (4.1.21), (4.1.22), we, thus, obtain the upper bound

$$
H_{r,N} \leqslant \sum_{*:=I_0,I_1,\ldots,I_r} \prod_{i=1}^{r} \sigma_N(I_i) \sum_{\boldsymbol{x}_0,\boldsymbol{x}_1,\ldots,\boldsymbol{x}_r\in(\mathbb{Z}^2)^h} \frac{\delta_0^{\otimes h}(\boldsymbol{x}_0)}{w_N^{\otimes h}(\boldsymbol{x}_0)} \prod_{i=1}^{r} \widehat{\mathsf{P}}_{N,0}^{I_{i-1};I_i}(\boldsymbol{x}_{i-1},\boldsymbol{x}_i) \cdot w_N^{\otimes h}(\boldsymbol{x}_r)\,,
$$

which we can write in the more compact and useful notation, using the brackets (4.1.19), as

$$
H_{r,N} \leqslant \sum_{I_1,\ldots,I_r} \Big\langle \frac{\delta_0^{\otimes h}}{w_N^{\otimes h}}, \widehat{\mathsf{Q}}_{N,0}^{*;I_1} \widehat{\mathsf{P}}_{N,0}^{I_1;I_2} \cdots \widehat{\mathsf{P}}_{N,0}^{I_{r-1};I_r} w_N^{\otimes h} \Big\rangle \prod_{i=1}^{r} \sigma_N(I_i).
$$

We note, here, that in the right-hand side we set the I_0 partition to be equal to $I_0 = \{1\} \sqcup \cdots \sqcup \{h\}$. The delta function $\delta_0^{\otimes h}(\boldsymbol{x}_0)$ will force all points of $\boldsymbol{x}_0$ to coincide at zero, thus, forcing I_0 to be equal to the partition $\circ = \{1,\ldots,h\}$ but, at the stage of operators, we do not yet need to enforce this constraint. At this stage we can proceed with the estimate using the operator norms (4.1.20) as

$$
H_{r,N} \leqslant \sum_{I_1,\ldots,I_r} \left\| \widehat{\mathsf{Q}}_{N,0}^{*,I_1} \frac{\delta_0^{\otimes h}}{w_N^{\otimes h}} \right\|_{\ell^p} \prod_{i=2}^{r} \left\| \widehat{\mathsf{P}}_{N,0}^{I_{i-1};I_i} \right\|_{\ell^q\to\ell^q} \left\| w_N^{\otimes h} \right\|_{\ell^q} \cdot \prod_{i=1}^{r} \sigma_N(I_i), \tag{4.1.27}
$$

By (4.1.24) of Proposition 4.1.2 we have that

$$
\left\| \widehat{\mathsf{Q}}_{N,0}^{*,I_1} \frac{\delta_0^{\otimes h}}{w_N^{\otimes h}} \right\|_{\ell^p} \leqslant C\, q\, N^{\frac{1}{q}} \left\| \frac{\delta_0}{w_N} \right\|_{\ell^p}^{h} = C\, q\, N^{\frac{1}{q}},
$$

and by (4.1.23) we have that for all $1 \leqslant i \leqslant r-1$,

$$
\left\| \widehat{\mathsf{P}}_{N,0}^{I_{i-1};I_i} \right\|_{\ell^q\to\ell^q} \leqslant C\, p\, q\,.
$$

Inserting these estimates in (4.1.27) we deduce that

$$
\begin{aligned}
H_{r,N} &\leqslant (C\,p\,q)^r N^{\frac{1}{q}} \left\| \frac{\delta_0}{w_N} \right\|_{\ell^p}^{h} \|w_N\|_{\ell^q}^{h} \sum_{I_1,\ldots,I_r} \prod_{i=1}^{r} \sigma_N(I_i) \\
&= (C\,p\,q)^r N^{\frac{1}{q}} \|w_N\|_{\ell^q}^{h} \sum_{I_1,\ldots,I_r} \prod_{i=1}^{r} \sigma_N(I_i)\,,
\end{aligned} \tag{4.1.28}
$$

for a constant $C = C(h,\bar{\beta}) \in (0,\infty)$, not depending on p,q,r,N. We now notice that for any partition $I \vdash \{1,\ldots,h\}$, it holds that $\sigma_N(I) \leqslant C/\log N$ (recall definitions (4.1.13) and (4.1.3)) and that, by Riemann summation, $N^{-h/q}\,\|w_N\|_{\ell^q}^{h}$ is bounded uniformly in N. Therefore, applying these on (4.1.28) we arrive at the bound

$$
H_{r,N} \leqslant \Big(\frac{C\,p\,q}{\log N} \Big)^{r} N^{\frac{h+1}{q}}
$$

which is the first claimed estimate in (4.1.26). For the second estimate in (4.1.26) we follow the same steps until we arrive at the bound

$$H_{r,N}^{(\mathsf{multi})} \leqslant (C\,p\,q)^r N^{\frac{1}{q}}\, \|w_N\|_{\ell^q}^h \sum_{I_1,\dots,I_r} \prod_{i=1}^r \sigma_N(I_i) \mathbb{1}_{\{\exists\, 1 \leqslant j \leqslant r : |I_j| < h-1\}}.$$

Then we notice that for a partition $I \vdash \{1,\dots,h\}$ with $|I| < h-1$ it will hold that $\sigma_N(I) \leqslant C(\log N)^{-2}$ (recall definitions (4.1.13) and (4.1.3)). This fact, together with the fact that there are r possible choices among the partitions $I_1,\dots,I_r$ that can be chosen so that $|I_j| < h-1$, leads to the second bound in (4.1.26). □

4.2. Approximation steps and proof of the main theorem

In this section we prove Theorem 4.0.1 through a series of approximations on the chaos expansion (4.1.11), (4.1.17). The first step, in Section 4.2.1, is to establish that the series in the chaos expansion (4.1.17) can be truncated up to a finite order and that the main contribution comes from diagrams where, at any fixed time, we only have at most two walks colliding. The second step, Section 4.2.2, is to show that the main contribution to the expansion and to diagrams like in Figure 4.2.1, comes when all jumps between marked dots (see Figure 4.2.1) happen within diffusive scale. The third step, in Section 4.2.3, captures the important feature of *scale separation*. This is intrinsic to the two-dimensionality and can be seen as the main feature that leads to the asymptotic independence of the collision times. With reference to Figure 4.2.1, this says that the time between two consecutive replicas, say $a_4 - b_3$ in Figure 4.2.1 must be much larger than the time between the previous replicas, say $b_3 - b_2$. This would then lead to the next step in Section 4.2.4, see also Figure 4.2.2, which is that we can *rewire* the links so that the solid lines connect only replicas between the *same* pairs of walks. The final step, which is performed in Section 4.2.5 is to reverse all the above approximations within the rewired diagrams, to which we arrived in the previous step. The summation, then, of all rewired diagrams leads, in the limit, to the right hand of (4.0.2), thus completing the proof of the theorem.

4.2.1. Reduction to 2-body collisions and finite order chaoses. In this step, we use the functional analytic framework and estimates of the previous section to show that for each $r \geqslant 1$, $H_{r,N}$ decays exponentially in r, uniformly in $N \in \mathbb{N}$ and that it is concentrated on configurations which contain only two-body collisions between the h random walks.

Proposition 4.2.1. *There exist constants* $\mathsf{a} \in (0,1)$ *and* $\bar{C} = C(h, \bar{\beta}, \mathsf{a}) \in (0,\infty)$ *and such that for all* $r \geqslant 1$*,*

$$\sup_{N\in\mathbb{N}} H_{r,N} \leqslant \bar{C}\,\mathsf{a}^r\,, \qquad \textit{and} \qquad H_{r,N}^{(\mathsf{multi})} \leqslant \frac{\bar{C}}{\log N}\, r\,\mathsf{a}^r\,. \tag{4.2.1}$$

Proof. We use the estimates in (4.1.26) and make the choice $q = q_N := \frac{\mathsf{a}}{C_1}\log N$ with $\mathsf{a} \in (0,1)$ and a constant C_1 such that $\frac{Cpq}{\log N} < \mathsf{a}$ (recall that $\frac{1}{p} + \frac{1}{q} = 1$). Moreover, this choice of q implies that

$$N^{\frac{h+1}{q}} = e^{\frac{h+1}{q}\log N} = e^{\frac{C_1\,(h+1)}{\mathsf{a}}}.$$

Therefore, choosing $\bar{C} = e^{\frac{C_1\,(h+1)}{\mathsf{a}}}$ implies the first estimate in (4.2.1).

The second estimate follows from the same procedure and the same choice of $q = q_N := \frac{\mathsf{a}}{C_1}\log N$ in the second bound of (4.1.26). □

Proposition 4.2.2. *If $M^{\boldsymbol{\beta}}_{N,h}$ is the joint Laplace transform of the collision local times*

$$\left\{\tfrac{\pi}{\log N}\,\mathsf{L}^{(i,j)}_N\right\}_{1\leqslant i<j\leqslant h},$$

as defined in (4.1.1) *and $H_{r,N}$ is the r^{th} term in its chaos expansion* (4.1.25)*, then for any $\varepsilon>0$ there exists $K=K_\varepsilon$ such that*

$$\left|M^{\boldsymbol{\beta}}_{N,h}-\sum_{r=0}^{K}H_{r,N}\right|\leqslant\varepsilon\,,$$

uniformly for all $N\in\mathbb{N}$.

Proof. By Proposition, 4.2.1, $H_{r,N}$ decay exponentially in r, uniformly in $N\in\mathbb{N}$ and therefore

$$\limsup_{K\to\infty}\left(\sup_{N\geqslant 1}\sum_{r>K}H_{r,N}\right)=0\,,$$

which means that we can truncate the expansion of $M^{\boldsymbol{\beta}}_{N,h}$ to a finite number of terms K depending only on ε. $\square$

By Proposition 4.2.1 we can focus on only two-body collisions, since higher order collisions bear a negligible contribution as $N\to\infty$. Let us introduce some notation to conveniently describe the expansion of $H_{r,N}$, after the reduction to only two-body collisions, which we will use in the sequel. Given $r\geqslant 1$ we will denote by $a_i,b_i\in\mathbb{N}\cup\{0\}$, $a_i\leqslant b_i$, $i=1,\dots,r$ the times where replicas start and end respectively, see (4.1.15) and Figure 4.1.2, where replicas are represented by wiggle lines. Thus, a_i will be the time marking the beginning of the i^{th} wiggle line and b_i the time marking its end. Note that, $a_1=0$. Moreover, we use the notation $\vec{\boldsymbol{x}}=(\boldsymbol{x}_1,\boldsymbol{x}_2,\dots,\boldsymbol{x}_r)\in(\mathbb{Z}^2)^{hr}$ to denote the starting points of the r replicas and $\vec{\boldsymbol{y}}=(\boldsymbol{y}_1,\dots,\boldsymbol{y}_r)\in(\mathbb{Z}^2)^{hr}$ the corresponding ending points. Again, notice that $\boldsymbol{x}_1=\boldsymbol{0}$. We then define the set

$$\mathsf{C}_{r,N}:=\left\{(\vec{a},\vec{b},\vec{\boldsymbol{x}},\vec{\boldsymbol{y}})\,\middle|\,0:=a_1\leqslant b_1<a_2\leqslant\cdots<a_r\leqslant b_r\leqslant N,\ \vec{\boldsymbol{x}},\vec{\boldsymbol{y}}\in(\mathbb{Z}^2)^{hr}\,,\boldsymbol{x}_1=\boldsymbol{0}\right\}. \tag{4.2.2}$$

We also define a set of finite sequences of partitions

$$\mathcal{I}^{(2)}=\bigcup_{r=0}^{\infty}\left\{(I_1,\dots,I_r):I_j\neq I_{j+1}\text{ and }|I_j|=h-1\,,\forall j\in\{1,\dots,r\}\right\}.$$

Using the notational conventions outlined above we can write $H_{r,N}=H^{(2)}_{r,N}+H^{(\mathsf{multi})}_{r,N}$ with

$$H^{(2)}_{r,N}:=\sum_{(I_1,\dots,I_r)\in\mathcal{I}^{(2)}}\ \sum_{(\vec{a},\vec{b},\vec{\boldsymbol{x}},\vec{\boldsymbol{y}})\in\mathsf{C}_{r,N}}\mathsf{U}^{I_1}_{b_1}(0,\boldsymbol{y}_1)\prod_{i=2}^{r}Q^{I_{i-1};I_i}_{a_i-b_{i-1}}(\boldsymbol{y}_{i-1},\boldsymbol{x}_i)\mathsf{U}^{I_i}_{b_i-a_i}(\boldsymbol{x}_i,\boldsymbol{y}_i)\sigma_N(I_i). \tag{4.2.3}$$

In the next sections will focus on $H^{(2)}_{r,N}$, which by Proposition 4.2.1 contains the main contributions.

4.2.2. Diffusive spatial truncation. In this step we show that we can introduce diffusive spatial truncations in all the kernels appearing in (4.2.3) which originate from the diffusive behaviour of the simple random walk in $\mathbb{Z}^2$. For a vector $\boldsymbol{x}=(x^{(1)},\dots,x^{(h)})\in(\mathbb{Z}^2)^h$, we shall use the notation

$$\|\boldsymbol{x}\|_\infty=\max_{1\leqslant j\leqslant h}|x^{(j)}|\,,$$

where $|\cdot|$ denotes the usual Euclidean norm on $\mathbb{R}^2$. For each $r \in \mathbb{N}$, define $H^{(\mathsf{diff})}_{r,N}$ to be the sum in (4.2.3) where $\mathsf{C}_{r,N}$ is replaced by

$$\mathsf{C}^{(\mathsf{diff})}_{r,N,R} := \mathsf{C}_{r,N} \cap \Big\{(\vec{a},\vec{b},\vec{\boldsymbol{x}},\vec{\boldsymbol{y}}) : \|\boldsymbol{y}_i - \boldsymbol{x}_i\|_\infty \leqslant R\sqrt{b_i - a_i} \\ \text{and } \|\boldsymbol{x}_i - \boldsymbol{y}_{i-1}\|_\infty \leqslant R\sqrt{a_i - b_{i-1}} \text{ for all } 1 \leqslant i \leqslant r\Big\} \tag{4.2.4}$$

and similarly we define

$$H^{(\mathsf{superdiff})}_{r,N,R} = \sum_{(I_1,\dots,I_r)\in\mathcal{I}^{(2)}} \sum_{(\vec{a},\vec{b},\vec{\boldsymbol{x}},\vec{\boldsymbol{y}})\in\mathsf{C}^{(\mathsf{superdiff})}_{r,N,R}} \mathsf{U}^{I_1}_{b_1}(0,\boldsymbol{y}_1)\prod_{i=2}^r Q^{I_{i-1};I_i}_{a_i-b_{i-1}}(\boldsymbol{y}_{i-1},\boldsymbol{x}_i)\,\mathsf{U}^{I_i}_{b_i-a_i}(\boldsymbol{x}_i,\boldsymbol{y}_i)\,\sigma_N(I_i), \tag{4.2.5}$$

where

$$\mathsf{C}^{(\mathsf{superdiff})}_{r,N,R} := \mathsf{C}_{r,N} \cap \Big\{(\vec{a},\vec{b},\vec{\boldsymbol{x}},\vec{\boldsymbol{y}}) : \exists\, 1 \leqslant i \leqslant r : \|\boldsymbol{y}_i - \boldsymbol{x}_i\|_\infty > R\sqrt{b_i - a_i} \\ \text{or } \|\boldsymbol{x}_i - \boldsymbol{y}_{i-1}\|_\infty > R\sqrt{a_i - b_{i-1}}\Big\}.$$

Note that then we have that

$$H^{(2)}_{r,N} = H^{(\mathsf{diff})}_{r,N,R} + H^{(\mathsf{superdiff})}_{r,N,R}.$$

We have the following Proposition.

Proposition 4.2.3. *For all $r \geqslant 1$,*

$$\lim_{R\to\infty}\sup_{N\in\mathbb{N}} H^{(\mathsf{superdiff})}_{r,N,R} = 0. \tag{4.2.6}$$

Proof. We use the bounds established in Lemma 4.2.4, below, and (4.1.18) to show (4.2.6). We can use a union bound for (4.2.5) to obtain that

$$\begin{aligned} &H^{(\mathsf{superdiff})}_{r,N,R} \\ &= \sum_{(I_1,\dots,I_r)\in\mathcal{I}^{(2)}} \sum_{(\vec{a},\vec{b},\vec{\boldsymbol{x}},\vec{\boldsymbol{y}})\in\mathsf{C}^{(\mathsf{superdiff})}_{r,N,R}} \mathsf{U}^{I_1}_{b_1}(0,\boldsymbol{y}_1)\prod_{i=2}^r Q^{I_{i-1};I_i}_{a_i-b_{i-1}}(\boldsymbol{y}_{i-1},\boldsymbol{x}_i)\,\mathsf{U}^{I_i}_{b_i-a_i}(\boldsymbol{x}_i,\boldsymbol{y}_i)\,\sigma_N(I_i) \\ &\leqslant \sum_{j=1}^r \sum_{(I_1,\dots,I_r)\in\mathcal{I}^{(2)}} \sum_{\substack{0:=a_1\leqslant b_1<a_2\leqslant\dots<a_r\leqslant b_r\leqslant N,\\ 0:=\boldsymbol{x}_1,\boldsymbol{y}_1,\dots,\boldsymbol{x}_r,\boldsymbol{y}_r\in(\mathbb{Z}^2)^h}} \\ &\qquad \Big(\mathbb{1}_{\{\|\boldsymbol{y}_j-\boldsymbol{x}_j\|_\infty>R\sqrt{b_j-a_j}\}} + \mathbb{1}_{\{\|\boldsymbol{x}_j-\boldsymbol{y}_{j-1}\|_\infty>R\sqrt{a_j-b_{j-1}}\}}\Big) \\ &\qquad\qquad \times \mathsf{U}^{I_1}_{b_1}(0,\boldsymbol{y}_1)\prod_{i=2}^r Q^{I_{i-1};I_i}_{a_i-b_{i-1}}(\boldsymbol{y}_{i-1},\boldsymbol{x}_i)\,\mathsf{U}^{I_i}_{b_i-a_i}(\boldsymbol{x}_i,\boldsymbol{y}_i)\,\sigma_N(I_i). \end{aligned} \tag{4.2.7}$$

We split the sum on the last three lines of (4.2.7) according to the two indicator functions that appear therein. By repeated successive application of the bounds from (4.1.18) for $j < i \leqslant r$ and then by using (4.2.10), which reads as

$$\sum_{\boldsymbol{y}_j\in(\mathbb{Z}^2)^h,\, a_j\leqslant b_j\leqslant N} \mathsf{U}^{I_j}_{b_j-a_j}(\boldsymbol{x}_j,\boldsymbol{y}_j)\,\mathbb{1}_{\{\|\boldsymbol{y}_j-\boldsymbol{x}_j\|_\infty>R\sqrt{b_j-a_j}\}} \leqslant e^{-\kappa R},$$

we deduce that

$$\sum_{\substack{0:=a_1\leqslant b_1<a_2\leqslant\cdots<a_r\leqslant b_r\leqslant N,\\ 0:=\boldsymbol{x}_1,\boldsymbol{y}_1,\ldots,\boldsymbol{x}_r,\boldsymbol{y}_r\in(\mathbb{Z}^2)^h}} \mathbb{1}_{\left\{\|\boldsymbol{y}_j-\boldsymbol{x}_j\|_\infty>R\sqrt{b_j-a_j}\right\}} \times \mathsf{U}^{I_1}_{b_1}(0,\boldsymbol{y}_1)\prod_{i=2}^{r} Q^{I_{i-1};I_i}_{a_i-b_{i-1}}(\boldsymbol{y}_{i-1},\boldsymbol{x}_i)\,\mathsf{U}^{I_i}_{b_i-a_i}(\boldsymbol{x}_i,\boldsymbol{y}_i)\,\sigma_N(I_i)$$

$$\leqslant e^{-\kappa R}\left(\frac{\bar\beta'}{1-\bar\beta'}\right)^{r-j} \times \sum_{\substack{0:=a_1\leqslant b_1<a_2\leqslant\cdots<b_{j-1}<a_j\leqslant N,\\ 0:=\boldsymbol{x}_1,\boldsymbol{y}_1,\ldots,\boldsymbol{x}_j\in(\mathbb{Z}^2)^h}} \mathsf{U}^{I_1}_{b_1}(0,\boldsymbol{y}_1)\prod_{i=2}^{j-1} Q^{I_{i-1};I_i}_{a_i-b_{i-1}}(\boldsymbol{y}_{i-1},\boldsymbol{x}_i)\,\mathsf{U}^{I_i}_{b_i-a_i}(\boldsymbol{x}_i,\boldsymbol{y}_i)\,\sigma_N(I_i) \times Q^{I_{j-1};I_j}_{a_j-b_{j-1}}(\boldsymbol{y}_{j-1},\boldsymbol{x}_j)\,\sigma_N(I_j)\,. \tag{4.2.8}$$

We then continue the summation using the bounds from (4.1.18), to obtain that the right-hand side of the inequality in (4.2.8) is bounded by

$$e^{-\kappa R}\left(\frac{\bar\beta'}{1-\bar\beta'}\right)^{r-j}\left(\frac{\bar\beta'}{1-\bar\beta'}\right)^{j-1} = e^{-\kappa R}\left(\frac{\bar\beta'}{1-\bar\beta'}\right)^{r-1}.$$

Similarly, for the sum involving the second indicator function in (4.2.7) we obtain by using (4.1.18) and (4.2.9) of Lemma 4.2.4 that

$$\sum_{\substack{0:=a_1\leqslant b_1<a_2\leqslant\cdots<a_r\leqslant b_r\leqslant N\,,\\ 0:=\boldsymbol{x}_1,\boldsymbol{y}_1,\ldots,\boldsymbol{x}_r,\boldsymbol{y}_r\in(\mathbb{Z}^2)^h}} \mathbb{1}_{\left\{\|\boldsymbol{x}_j-\boldsymbol{y}_{j-1}\|_\infty>R\sqrt{a_j-b_{j-1}}\right\}}\mathsf{U}^{I_1}_{b_1}(0,\boldsymbol{y}_1) \times\prod_{i=2}^{r} Q^{I_{i-1};I_i}_{a_i-b_{i-1}}(\boldsymbol{y}_{i-1},\boldsymbol{x}_i)\,\mathsf{U}^{I_i}_{b_i-a_i}(\boldsymbol{x}_i,\boldsymbol{y}_i)\,\sigma_N(I_i)$$

$$\leqslant e^{-\kappa R^2}\left(\frac{1}{1-\bar\beta'}\right)^{r}(\bar\beta')^{r-1}\,.$$

Therefore, the right-hand side of the inequality in (4.2.7) is bounded by

$$\sum_{j=1}^{r}\sum_{(I_1,\ldots,I_r)\in\mathcal{I}^{(2)}} e^{-\kappa R}\left(\left(\frac{\bar\beta'}{1-\bar\beta'}\right)^{r-1}+\left(\frac{1}{1-\bar\beta'}\right)^{r}(\bar\beta')^{r-1}\right)\leqslant e^{-\kappa R}\left(2r\cdot\frac{(\bar\beta')^{r-1}}{(1-\bar\beta')^r}\cdot\binom{h}{2}^{r}\right),$$

where the $\binom{h}{2}^r$ factor comes from the fact that there are at most $\binom{h}{2}^r$ choices for the sequence $(I_1,\ldots,I_r)\in\mathcal{I}^{(2)}$. Thus, recalling (4.2.7) we get that

$$\sup_{N\in\mathbb{N}} H^{(\mathsf{superdiff})}_{r,N,R}\leqslant e^{-\kappa R}\left(2r\cdot\frac{(\bar\beta')^{r-1}}{(1-\bar\beta')^r}\cdot\binom{h}{2}^{r}\right)\xrightarrow{R\to\infty}0.$$

□

Lemma 4.2.4. *Let $I,J\vdash\{1,\ldots,h\}$ such that $|I|=|J|=h-1$ and $I\neq J$. For large enough $R\in(0,\infty)$ and uniformly in $\boldsymbol{x}\in(\mathbb{Z}^2)^h_I$ we have that for a constant $\kappa=\kappa(h,\bar\beta)\in(0,\infty)$,*

$$\sigma_N(J)\cdot\left(\sum_{1\leqslant n\leqslant N,\,\boldsymbol{y}\in(\mathbb{Z}^2)^h} Q^{I;J}_n(\boldsymbol{x},\boldsymbol{y})\cdot\mathbb{1}_{\left\{\|\boldsymbol{x}-\boldsymbol{y}\|_\infty>R\sqrt{n}\right\}}\right)\leqslant e^{-\kappa R^2} \tag{4.2.9}$$

and

$$\sum_{1 \leqslant n \leqslant N,\, \boldsymbol{y} \in (\mathbb{Z}^2)^h} \mathsf{U}_n^J(\boldsymbol{x}, \boldsymbol{y}) \cdot \mathbb{1}_{\left\{\|\boldsymbol{x}-\boldsymbol{y}\|_\infty > R\sqrt{n}\right\}} \leqslant e^{-\kappa R}. \tag{4.2.10}$$

Proof. We start with the proof of (4.2.9). Since $|J| = h-1$, let us assume without loss of generality that $J = \{k, \ell\} \sqcup \bigsqcup_{j \neq k,\ell} \{j\}$. In this case, $Q_n^{I;J}(\boldsymbol{x}, \boldsymbol{y})$ contains $h-2$ random walk jumps with free endpoints $y^{(j)}$, $j \neq k, \ell$, that is

$$\prod_{j \neq k,\ell} q_n(y^{(j)} - x^{(j)}).$$

Moreover, J imposes the constraint that $y^{(k)} = y^{(\ell)}$, which appears in $Q_n^{I;J}(\boldsymbol{x}, \boldsymbol{y})$ through the product of transition kernels

$$q_n(y^{(k)} - x^{(k)}) \cdot q_n(y^{(k)} - x^{(\ell)}),$$

recall (4.1.12). The constraint $\|\boldsymbol{x} - \boldsymbol{y}\|_\infty > R\sqrt{n}$ implies that there exists $1 \leqslant j \leqslant h$ such that $|x^{(j)} - y^{(j)}| > R\sqrt{n}$. We distinguish two cases:

(1) There exists $j \neq k, \ell$ such that $|x^{(j)} - y^{(j)}| > R\sqrt{n}$, or
(2) $|x^{(j)} - y^{(j)}| > R\sqrt{n}$ for $j = k$ or $j = \ell$.

In both cases, we can use $\sum_{z \in \mathbb{Z}^2} q_n(z) = 1$ to sum the kernels $q_n(y^{(j)} - x^{(j)})$, $j \neq k, \ell$ to which we do not impose any super-diffusive constraints. By symmetry and translation invariance we can upper bound the left-hand side of (4.2.9) by

$$\sigma_N(J) \cdot \Bigg((h-2) \sum_{1 \leqslant n \leqslant N,\, z \in \mathbb{Z}^2} q_n(z) \cdot \mathbb{1}_{\{|z| > R\sqrt{n}\}} \cdot \Big\{ \sup_{u \in \mathbb{Z}^2} \sum_{z \in \mathbb{Z}^2} q_n(z) q_n(z+u) \Big\} \\ + 2 \sup_{u \in \mathbb{Z}^2} \sum_{1 \leqslant n \leqslant N,\, z \in \mathbb{Z}^2} q_n(z) q_n(z+u) \cdot \mathbb{1}_{\{|z| > R\sqrt{n}\}} \Bigg). \tag{4.2.11}$$

Looking at the first summand in (4.2.11) we have by Cauchy-Schwarz that

$$\sup_{u \in \mathbb{Z}^2} \sum_{z \in \mathbb{Z}^2} q_n(z)\, q_n(z+u) \leqslant \Big(\sum_{z \in \mathbb{Z}^2} q_n^2(z) \Big)^{1/2} \cdot \sup_{u \in \mathbb{Z}^2} \Big(\sum_{z \in \mathbb{Z}^2} q_n^2(z+u) \Big)^{1/2} \leqslant q_{2n}(0), \tag{4.2.12}$$

since $\sum_{z \in \mathbb{Z}^2} q_n^2(z) = q_{2n}(0)$. Let us recall the deviation estimate for the simple random walk, which can be found in [LL10], that is

$$\mathrm{P}\Big(\max_{0 \leqslant k \leqslant n} |S_k| > R\sqrt{n} \Big) \leqslant e^{-cR^2}, \tag{4.2.13}$$

for a constant $c \in (0, \infty)$ and all $R \in (0, \infty)$. By using bound (4.2.12) and subsequently (4.2.13) on the first summand of (4.2.11) we get that

$$\begin{aligned} &\sum_{1 \leqslant n \leqslant N,\, z \in \mathbb{Z}^2} q_n(z) \cdot \mathbb{1}_{\{|z| > R\sqrt{n}\}} \cdot \Big\{ \sup_{u \in \mathbb{Z}^2} \sum_{z \in \mathbb{Z}^2} q_n(z) q_n(z+u) \Big\} \\ \leqslant &\sum_{1 \leqslant n \leqslant N,\, z \in \mathbb{Z}^2} q_{2n}(0) \cdot q_n(z) \mathbb{1}_{\{|z| > R\sqrt{n}\}} \\ \leqslant &\, e^{-cR^2} R_N. \end{aligned}$$

We recall from (4.1.4) and (4.1.5) that $R_N = \sum_{n=1}^{N} q_{2n}(0) \overset{N\to\infty}{\approx} \frac{\log N}{\pi}$, therefore,

$$\begin{aligned} &\sigma_N(J) \cdot \left((h-2) \sum_{1 \leqslant n \leqslant N,\, z \in \mathbb{Z}^2} q_n(z) \cdot \mathbb{1}_{\{|z|>R\sqrt{n}\}} \cdot \left\{ \sup_{u \in \mathbb{Z}^2} \sum_{z \in \mathbb{Z}^2} q_n(z) q_n(z+u) \right\} \right) \\ &\leqslant (h-2)\, \sigma_N(J)\, R_N\, e^{-cR^2} \\ &\leqslant (h-2)\, \bar{\beta}'\, e^{-cR^2}, \end{aligned} \tag{4.2.14}$$

for some $\bar{\beta}' \in (\bar{\beta}, 1)$. The second summand in the parenthesis in (4.2.11) can be bounded via Cauchy-Schwarz by

$$\left(\sum_{1 \leqslant n \leqslant N,\, z \in \mathbb{Z}^2} q_n^2(z) \cdot \mathbb{1}_{\{|z|>R\sqrt{n}\}} \right)^{\frac{1}{2}} \cdot \left(\sum_{1 \leqslant n \leqslant N,\, z \in \mathbb{Z}^2} q_n^2(z+u) \right)^{\frac{1}{2}}. \tag{4.2.15}$$

For the first term in (4.2.15), using that $\sup_{z \in \mathbb{Z}^2} q_n(z) \leqslant \frac{C}{n}$ we get

$$\sum_{1 \leqslant n \leqslant N,\, z \in \mathbb{Z}^2} q_n^2(z) \cdot \mathbb{1}_{\{|z|>R\sqrt{n}\}} \leqslant C \sum_{1 \leqslant n \leqslant N,\, z \in \mathbb{Z}^2} \frac{q_n(z)}{n} \cdot \mathbb{1}_{\{|z|>R\sqrt{n}\}} \leqslant C\, e^{-cR^2} \log N \tag{4.2.16}$$

For the second term in (4.2.11), we have that for all $u \in \mathbb{Z}^2$

$$\sum_{1 \leqslant n \leqslant N,\, z \in \mathbb{Z}^2} q_n^2(z+u) = \sum_{n=1}^{N} q_{2n}(0) \overset{N\to\infty}{\approx} \frac{\log N}{\pi}.$$

Thus, by (4.2.15) together with (4.2.16) we conclude that for the second summand in (4.2.11) we have

$$\sigma_N(J) \cdot \left(2 \sup_{u \in \mathbb{Z}^2} \sum_{1 \leqslant n \leqslant N,\, z \in \mathbb{Z}^2} q_n(z) q_n(z+u) \cdot \mathbb{1}_{\{|z|>R\sqrt{n}\}} \right) \leqslant C\, e^{-\frac{cR^2}{2}}.$$

Therefore, recalling (4.2.14) we deduce that there exists a constant $\kappa(h, \bar{\beta}) \in (0, \infty)$ such that

$$\sigma_N(J) \cdot \left(\sum_{1 \leqslant n \leqslant N,\, \boldsymbol{y} \in (\mathbb{Z}^2)^h} Q_n^{I;J}(\boldsymbol{x}, \boldsymbol{y}) \cdot \mathbb{1}_{\left\{\|\boldsymbol{x}-\boldsymbol{y}\|_\infty > R\sqrt{n}\right\}} \right) \leqslant e^{-\kappa R^2}.$$

We move to the proof of (4.2.10). Similar to the proof of (4.2.9), we can bound the left-hand side of (4.2.10) by

$$(h-1) \sum_{1 \leqslant n \leqslant N,\, w,\, z \in \mathbb{Z}^2} U_N^{\bar{\beta}}(n, w) \cdot q_n(z) \mathbb{1}_{\{|z|>R\sqrt{n}\}} + \sum_{1 \leqslant n \leqslant N, z \in \mathbb{Z}^2} U_N^{\bar{\beta}}(n, z) \cdot \mathbb{1}_{\{|z|>R\sqrt{n}\}}. \tag{4.2.17}$$

For the first summand in (4.2.17), by (4.2.13) we have that

$$\sum_{z \in \mathbb{Z}^2} q_n(z) \mathbb{1}_{\{|z|>R\sqrt{n}\}} \leqslant e^{-cR^2},$$

and $\sum_{1 \leqslant n \leqslant N,\, w \in \mathbb{Z}^2} U_N^{\bar{\beta}}(n, w) \leqslant \frac{1}{1-\bar{\beta}'}$, therefore

$$(h-1) \sum_{1 \leqslant n \leqslant N,\, w,\, z \in \mathbb{Z}^2} U_N^{\bar{\beta}}(n, w) \cdot q_n(z) \mathbb{1}_{\{|z|>R\sqrt{n}\}} \leqslant \frac{h-1}{1-\bar{\beta}'}\, e^{-cR^2}. \tag{4.2.18}$$

For the second summand, we use the renewal representation of $U_N^{\bar{\beta}}(\cdot,\cdot)$ introduced in (4.1.8). In particular, we have that

$$\sum_{1\leqslant n\leqslant N, z\in\mathbb{Z}^2} U_N^{\bar{\beta}}(n,z)\cdot \mathbb{1}_{\{|z|>R\sqrt{n}\}} = \sum_{k\geqslant 0}(\sigma_N(\bar{\beta})R_N)^k \sum_{n=0}^{N} \mathrm{P}\Big(\big|S_k^{(N)}\big| > R\sqrt{n}, \tau_k^{(N)} = n\Big)\,. \tag{4.2.19}$$

Then, by conditioning on the times $(T_i^{(N)})_{1\leqslant i\leqslant k}$ for which $\tau_k^{(N)} = T_1^{(N)}+\cdots+T_k^{(N)}$ we have that

$$\begin{aligned}\mathrm{P}\Big(\big|S_k^{(N)}\big| > &R\sqrt{n},\, \tau_k^{(N)} = n\Big)\\ &= \sum_{n_1+\cdots+n_k=n} \mathrm{P}\Big(\big|S_k^{(N)}\big| > R\sqrt{n}\,\Big|\, \cap_{i=1}^k \{T_i^{(N)} = n_i\}\Big) \prod_{i=1}^k \mathrm{P}\big(T_i^{(N)} = n_i\big)\,.\end{aligned} \tag{4.2.20}$$

Note that when we condition on $\cap_{i=1}^k\{T_i^{(N)} = n_i\}$, $S_k^{(N)}$ is a sum of k independent random variables $(\xi_i)_{1\leqslant i\leqslant k}$ taking values in $\mathbb{Z}^2$, with law

$$\mathrm{P}(\xi_i = x) = \frac{q_{n_i}^2(x)}{q_{2n_i}(0)}\,.$$

The proof of Proposition 3.2.4 in Chapter 3 showed that there exists a constant $C\in(0,\infty)$ such that for all $\lambda\geqslant 0$

$$\mathrm{E}\Big[e^{\lambda|\sum_{i=1}^k \xi_i|}\Big] \leqslant 2e^{4C\lambda^2 n}\,. \tag{4.2.21}$$

Therefore, by (4.2.21) with $\lambda = \frac{1}{\sqrt{n}}$ and Markov's inequality we obtain that

$$\mathrm{P}\Big(\big|S_k^{(N)}\big| > R\sqrt{n}\,\Big|\, \cap_{i=1}^k \{T_i^{(N)} = n_i\}\Big) \leqslant 2e^{4C-R}\,.$$

Thus, looking back at (4.2.20) we have that for all $k\geqslant 0$,

$$\mathrm{P}\Big(\big|S_k^{(N)}\big| > R\sqrt{n},\, \tau_k^{(N)} = n\Big) \leqslant 2e^{4C-R}\,\mathrm{P}(\tau_k^{(N)} = n)\,,$$

therefore, plugging the last inequality into (4.2.19), we get that

$$\sum_{1\leqslant n\leqslant N, z\in\mathbb{Z}^2} U_N^{\bar{\beta}}(n,z)\cdot \mathbb{1}_{\{|z|>R\sqrt{n}\}} \leqslant 2e^{4C-R}\sum_{n=1}^N U_N^{\bar{\beta}}(n) \leqslant \frac{2e^{4C-R}}{1-\bar{\beta}'}\,, \tag{4.2.22}$$

therefore by (4.2.18) and (4.2.22) we have that there exists a constant $\kappa(h,\bar{\beta})\in(0,\infty)$ such that

$$\sum_{1\leqslant n\leqslant N,\, \boldsymbol{y}\in(\mathbb{Z}^2)^h} \mathsf{U}_n^J(\boldsymbol{x},\boldsymbol{y})\cdot \mathbb{1}_{\left\{\|\boldsymbol{x}-\boldsymbol{y}\|_\infty > R\sqrt{n}\right\}} \leqslant e^{-\kappa R}\,,$$

for large enough $R\in(0,\infty)$, thus concluding the proof of (4.2.10). □

4.2.3. Scale separation. In this step we show that given $r\in\mathbb{N}$, $r\geqslant 2$, the main contribution to $H_{r,N}^{(\mathsf{diff})}$ comes from configurations where $a_{i+1}-b_i > M(b_i-b_{i-1})$ for all $1\leqslant i\leqslant r$ and large M, as $N\to\infty$. Recall from (4.2.4) that

$$H_{r,N,R}^{(\mathsf{diff})} = \sum_{(I_1,\ldots,I_r)\in\mathcal{I}^{(2)}} \sum_{(\vec{a},\vec{b},\vec{\boldsymbol{x}},\vec{\boldsymbol{y}})\in\mathsf{C}_{r,N,R}^{(\mathsf{diff})}} \mathsf{U}_{b_1}^{I_1}(0,\boldsymbol{y}_1)\prod_{i=2}^r Q_{a_i-b_{i-1}}^{I_{i-1};I_i}(\boldsymbol{y}_{i-1},\boldsymbol{x}_i)\,\mathsf{U}_{b_i-a_i}^{I_i}(\boldsymbol{x}_i,\boldsymbol{y}_i)\sigma_N(I_i)\,.$$

Define the set

$$\mathsf{C}_{r,N,R,M}^{(\mathsf{main})} := \mathsf{C}_{r,N,R}^{(\mathsf{diff})}\cap\Big\{(\vec{a},\vec{b},\vec{\boldsymbol{x}},\vec{\boldsymbol{y}}) :\, a_{i+1}-b_i > M(b_i-b_{i-1}) \text{ for all } 1\leqslant i\leqslant r-1\Big\}\,, \tag{4.2.23}$$

with the convention $b_0 := 0$ and accordingly define

$$H^{(\mathsf{main})}_{r,N,R,M} := \sum_{(I_1,\dots,I_r)\in\mathcal{I}^{(2)}} \sum_{(\vec{a},\vec{b},\vec{\boldsymbol{x}},\vec{\boldsymbol{y}})\in \mathsf{C}^{(\mathsf{main})}_{r,N,R,M}} \mathsf{U}^{I_1}_{b_1}(0,\boldsymbol{y}_1)\prod_{i=2}^{r} Q^{I_{i-1};I_i}_{a_i-b_{i-1}}(\boldsymbol{y}_{i-1},\boldsymbol{x}_i)\mathsf{U}^{I_i}_{b_i-a_i}(\boldsymbol{x}_i,\boldsymbol{y}_i)\,\sigma_N(I_i)\,. \tag{4.2.24}$$

We then have the following approximation proposition:

Proposition 4.2.5. *For all fixed $r\in\mathbb{N}$, $r\geqslant 2$ and $M\in(0,\infty)$,*

$$\lim_{N\to\infty}\sup_{R\in(0,\infty)}\left|H^{(\mathsf{diff})}_{r,N,R}-H^{(\mathsf{main})}_{r,N,R,M}\right| = 0\,. \tag{4.2.25}$$

Proof. Fix $M>0$. Let us begin by showing (4.2.25) for the simplest case which is $r=2$. We have

$$H^{(\mathsf{diff})}_{2,N,R}-H^{(\mathsf{main})}_{2,N,R,M} \leqslant \sum_{(I_1,I_2)\in\mathcal{I}^{(2)}} \sum_{\substack{0\leqslant b_1<a_2\leqslant N,\, a_2-b_1\leqslant Mb_1,\\ \boldsymbol{y}_1,\boldsymbol{x}_2,\boldsymbol{y}_2\in(\mathbb{Z}^2)^h}} \mathsf{U}^{I_1}_{b_1}(0,\boldsymbol{y}_1)Q^{I_1;I_2}_{a_2-b_1}(\boldsymbol{y}_1,\boldsymbol{x}_2)\,\sigma_N(I_2)\mathsf{U}^{I_2}_{b_2-a_2}(\boldsymbol{x}_2,\boldsymbol{y}_2).$$

We can bound $\sigma_N(I_2)$ by $\frac{\pi\bar{\beta}'}{\log N}$, for some $\bar{\beta}'\in(\bar{\beta},1)$ and use (4.1.18) to bound the last replica, i.e. the sum over $(b_2,\boldsymbol{y}_2)$, thus getting

$$H^{(\mathsf{diff})}_{2,N,R}-H^{(\mathsf{main})}_{2,N,R,M} \leqslant \frac{\pi\bar{\beta}'(1-\bar{\beta}')^{-1}}{\log N} \sum_{(I_1,I_2)\in\mathcal{I}^{(2)}} \sum_{\substack{0\leqslant b_1<a_2\leqslant N,\, a_2-b_1\leqslant Mb_1\,,\\ \boldsymbol{y}_1,\boldsymbol{x}_2,\boldsymbol{y}_2\in(\mathbb{Z}^2)^h}} \mathsf{U}^{I_1}_{b_1}(0,\boldsymbol{y}_1)Q^{I_1;I_2}_{a_2-b_1}(\boldsymbol{y}_1,\boldsymbol{x}_2)\,. \tag{4.2.26}$$

Notice that at this stage we can sum out the spatial endpoints of the free kernels in (4.2.26) and bound the coupling strength $\beta_{k,\ell}$ of any replica $\mathsf{U}^{I_1}_{b_1}(0,\boldsymbol{y}_1)$ with $I_1=\{k,\ell\}\sqcup\bigsqcup_{j\neq k,\ell}\{j\}$ by $\bar{\beta}$ to obtain

$$\begin{aligned} H^{(\mathsf{diff})}_{2,N,R}-H^{(\mathsf{main})}_{2,N,R,M} &\leqslant \frac{\pi\bar{\beta}'(1-\bar{\beta}')^{-1}}{\log N} \sum_{(I_1,I_2)\in\mathcal{I}^{(2)}} \sum_{\substack{0\leqslant b_1<a_2\leqslant N,\, a_2-b_1\leqslant Mb_1\,,\\ y_1,x_2\in\mathbb{Z}^2}} U^{\bar{\beta}}_N(b_1,y_1)q_{a_2-b_1}(x_2-y_1)\,q_{a_2}(x_2) \\ &\leqslant \frac{\pi\bar{\beta}'(1-\bar{\beta}')^{-1}}{\log N}\binom{h}{2}^2 \sum_{\substack{0\leqslant b_1<a_2\leqslant N,\, a_2-b_1\leqslant Mb_1\,,\\ y_1,x_2\in\mathbb{Z}^2}} U^{\bar{\beta}}_N(b_1,y_1)q_{a_2-b_1}(x_2-y_1)\,q_{a_2}(x_2)\,. \end{aligned} \tag{4.2.27}$$

For the last inequality we also have used that the number of possible partitions $(I_1,I_2)\in\mathcal{I}^{(2)}$ is bounded by $\binom{h}{2}^2$. For every fixed value of b_1 in (4.2.27), we use Cauchy-Schwarz for the sum over

$(a_2, x_2) \in (0, (1+M)b_1] \times \mathbb{Z}^2$ in (4.2.27) to obtain that

$$
\begin{aligned}
&\sum_{b_1 < a_2 \leqslant (1+M)b_1,\, x_2 \in \mathbb{Z}^2} q_{a_2 - b_1}(x_2 - y_1)\, q_{a_2}(x_2) \\
&\qquad \leqslant \Bigg(\sum_{0 < a_2 \leqslant (1+M)b_1,\, x_2 \in \mathbb{Z}^2} q^2_{a_2 - b_1}(x_2 - y_1) \Bigg)^{\frac{1}{2}} \Bigg(\sum_{b_1 < a_2 \leqslant (1+M)b_1,\, x_2 \in \mathbb{Z}^2} q^2_{a_2}(x_2) \Bigg)^{\frac{1}{2}} \\
&\qquad = \Bigg(\sum_{0 < a_2 \leqslant (1+M)b_1} q_{2(a_2 - b_1)}(0) \Bigg)^{\frac{1}{2}} \Bigg(\sum_{b_1 < a_2 \leqslant (1+M)b_1} q_{2a_2}(0) \Bigg)^{\frac{1}{2}} .
\end{aligned}
\tag{4.2.28}
$$

We can bound the leftmost parenthesis in the last line of (4.2.28) by $R_N^{1/2} = \Big(\sum_{n=1}^N q_{2n}(0) \Big)^{1/2} = O(\sqrt{\log N})$. For the other term we have

$$
\sum_{b_1 < a_2 \leqslant (1+M)b_1} q_{2a_2}(0) \leqslant c \sum_{b_1 < a_2 \leqslant (1+M)b_1} \tfrac{1}{a_2} \leqslant c \log(1+M) . \tag{4.2.29}
$$

Therefore, using (4.2.28) and (4.2.29) along with $\sum_{0 \leqslant b_1 \leqslant N,\, y_1 \in \mathbb{Z}^2} U_N^{\bar{\beta}}(b_1, y_1) \leqslant (1 - \bar{\beta}\,')^{-1}$ in (4.2.27) we obtain that

$$
H^{(\mathsf{diff})}_{2,N,R} - H^{(\mathsf{main})}_{2,N,R,M} \leqslant C\pi \bar{\beta}\,'(1 - \bar{\beta}\,')^{-2} \sqrt{\frac{\log(1+M)}{\log N}} \xrightarrow{N \to \infty} 0 .
$$

Let us show how this argument can be extended to work for general $r \in \mathbb{N}$. The key observation is that for every fresh collision between two random walks, that is $I_{i+1} = \{k, \ell\} \sqcup \bigsqcup_{j \neq k, \ell} \{j\}$, happening at time $0 < a_{i+1} \leqslant N$, we have $I_i \neq I_{i+1}$, therefore one of the two colliding walks with labels k, ℓ has to have travelled freely, for time at least $a_{i+1} - b_{i-1}$ from its previous collision. More precisely, every term in the expansion of $H^{(\mathsf{diff})}_{r,N,R} - H^{(\mathsf{main})}_{r,N,R,M}$ contains for every $1 \leqslant i \leqslant r-1$ a product of the form

$$
q_{a_{i+1} - b_i}(x_{i+1} - y_i) \cdot q_{a_{i+1} - b_{i-1}}(x_{i+1} - y_{i-1}) ,
$$

see Figure 4.2.1. Recall from (4.2.2) and (4.2.24) that we have the expansion

$$
\begin{aligned}
&H^{(\mathsf{diff})}_{r,N,R} - H^{(\mathsf{main})}_{r,N,R,M} \\
&= \sum_{(I_1, \dots, I_r) \in \mathcal{I}^{(2)}} \ \sum_{(\vec{a}, \vec{b}, \vec{\boldsymbol{x}}, \vec{\boldsymbol{y}}) \in \mathsf{C}^{(\mathsf{diff})}_{r,N,R} \smallsetminus \mathsf{C}^{(\mathsf{main})}_{r,N,R,M}} \mathsf{U}^{I_1}_{b_1}(0, \boldsymbol{y}_1) \prod_{i=2}^r Q^{I_{i-1}; I_i}_{a_i - b_{i-1}}(\boldsymbol{y}_{i-1}, \boldsymbol{x}_i)\, \mathsf{U}^{I_i}_{b_i - a_i}(\boldsymbol{x}_i, \boldsymbol{y}_i)\, \sigma_N(I_i),
\end{aligned}
\tag{4.2.30}
$$

where by definition (4.2.23) we have that

$$
\mathsf{C}^{(\mathsf{diff})}_{r,N,R} \smallsetminus \mathsf{C}^{(\mathsf{main})}_{r,N,R,M} = \mathsf{C}^{(\mathsf{diff})}_{r,N,R} \cap \bigcup_{i=1}^{r-1} \Big\{ (\vec{a}, \vec{b}, \vec{\boldsymbol{x}}, \vec{\boldsymbol{y}}) :\ a_{i+1} - b_i \leqslant M(b_i - b_{i-1}) \Big\} . \tag{4.2.31}
$$

The strategy we are going to follow is to start the summation of (4.2.30) from the end until we find the index $1 \leqslant i \leqslant r-1$ for which the sum over a_{i+1} is restricted to $\big(b_i, b_i + M(b_i - b_{i-1})\big]$, in agreement with (4.2.31), using (4.1.18) to bound the contribution of the sums over b_j, a_{j+1} and the corresponding spatial points for $i < j \leqslant r-1$. Next, notice that we can bound the contribution of the sum over $a_{i+1} \in \big(b_i, b_i + M(b_i - b_{i-1})\big]$ and $x_{i+1} \in \mathbb{Z}^2$, using a change of variables, by a

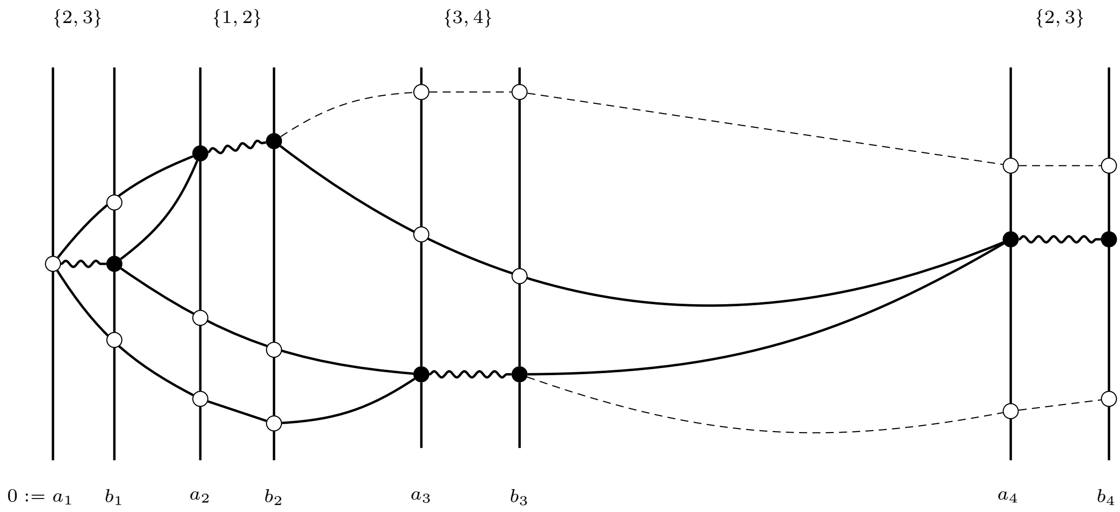


FIGURE 4.2.1. A diagramatic representation of a configuration of collisions between 4 random walks in $H^{(2)}_{4,N}$ with $I_1 = \{2,3\}$, $I_2 = \{1,2\}$, $I_3 = \{3,4\}$ and $I_4 = \{2,3\}$. Wiggly lines represent replica evolution, see (4.1.15).

factor of

$$\frac{C}{\log N}\left(\sup_{1\leqslant t\leqslant N,\, u\in\mathbb{Z}^2}\sum_{1\leqslant n\leqslant Mt,\, z\in\mathbb{Z}^2} q_n(z)q_{n+t}(z+u)\right) \leqslant C\sqrt{\frac{\log(1+M)}{\log N}}\,,$$

using Cauchy-Schwarz as in (4.2.28) and (4.2.29). The remaining sums over b_j, a_{j-1}, $1 \leqslant j \leqslant i$ can be bounded again via (4.1.18). Therefore, taking into account that by (4.2.31) there are $r-1$ choices for the index i such that the sum over a_{i+1} is restricted to $\big(b_i, b_i + M(b_i - b_{i-1})\big]$, we can give an upper bound to $H^{(\mathsf{diff})}_{r,N,R} - H^{(\mathsf{main})}_{r,N,R,M}$ as follows:

$$H^{(\mathsf{diff})}_{r,N,R} - H^{(\mathsf{main})}_{r,N,R,M} \leqslant C(r-1)\binom{h}{2}^r\left(\frac{\bar{\beta}'}{1-\bar{\beta}'}\right)^r\sqrt{\frac{\log(1+M)}{\log N}} \xrightarrow{N\to\infty} 0\,,$$

where we also used that the number of distinct sequences $(I_1,\ldots,I_r) \in \mathcal{I}^{(2)}$ is bounded by $\binom{h}{2}^r$. $\square$

4.2.4. Rewiring. Recall the expansion of $H^{(\mathsf{main})}_{r,N,R,M}$,

$$H^{(\mathsf{main})}_{r,N,R,M} := \sum_{(I_1,\ldots,I_r)\in\mathcal{I}^{(2)}}\ \sum_{(\vec{a},\vec{b},\vec{\boldsymbol{x}},\vec{\boldsymbol{y}})\in\mathsf{C}^{(\mathsf{main})}_{r,N,R,M}} \mathsf{U}^{I_1}_{b_1}(0,\boldsymbol{y}_1)\prod_{i=2}^{r} Q^{I_{i-1};I_i}_{a_i-b_{i-1}}(\boldsymbol{y}_{i-1},\boldsymbol{x}_i)\,\mathsf{U}^{I_i}_{b_i-a_i}(\boldsymbol{x}_i,\boldsymbol{y}_i)\,\sigma_N(I_i)\,. \tag{4.2.32}$$

We also remind the reader that we may identify a partition $I = \{k,\ell\} \sqcup \bigsqcup_{j\neq k,\ell}\{j\}$ with its non-trivial part $\{k,\ell\}$. Moreover, if $\big(\{i_1,j_1\},\cdots,\{i_r,j_r\}\big)\in\mathcal{I}^{(2)}$ we will use the notation

$$\mathsf{p}(m) := \max\big\{k < m : \{i_m,j_m\} = \{i_k,j_k\}\big\}\,,$$

with the convention that $\mathsf{p}(m) = 0$ if $\{i_k,j_k\} \neq \{i_m,j_m\}$ for all $1 \leqslant k < m$. Given this definition, the time $b_{\mathsf{p}(m)}$ represents the last time walks i_m, j_m collided before their new collision at time a_m. Note that since we always have $\{i_k,j_k\} \neq \{i_{k+1},j_{k+1}\}$ by construction, $\mathsf{p}(m) < m-1$.

Consider a sequence of partitions $\big(\{i_1, j_1\}, \ldots, \{i_m, j_m\}\big) \in \mathcal{I}^{(2)}$ and let $m \in \{2, \ldots, r\}$. The goal of this step will be to show that we can make the replacement of kernels

$$q_{a_m - b_{m-1}}\Big(x_m^{(i_m)} - y_{m-1}^{(i_m)}\Big) \cdot q_{a_m - b_{m-1}}\Big(x_m^{(j_m)} - y_{m-1}^{(j_m)}\Big) \quad \longleftrightarrow \quad q^2_{a_m - b_{\mathsf{p}(m)}}\Big(x_m^{(i_m)} - y_{\mathsf{p}(m)}^{(i_m)}\Big) . \tag{4.2.33}$$

by inducing an error which is negligible when $M \to \infty$. We iterate this procedure for all partitions $I_1, \ldots, I_r$. We call the procedure described above **rewiring**, see Figures 4.2.1 and 4.2.2. The first step towards the full rewiring is to show the following lemma which quantifies the error of a single replacement (4.2.33).

Lemma 4.2.6. *Let $r \geqslant 2$ fixed and $m \in \{2, \ldots, r\}$ with $I_m = \{i_m, j_m\}$. Then, for every fixed $R \in (0, \infty)$ and uniformly in $(\vec{a}, \vec{b}, \vec{\boldsymbol{x}}, \vec{\boldsymbol{y}}) \in \mathsf{C}^{(\mathsf{main})}_{r,N,R,M}$ and all sequences of partitions $(I_1, \ldots, I_r) \in \mathcal{I}^{(2)}$,*

$$q_{a_m - b_{m-1}}\Big(x_m^{(i_m)} - y_{m-1}^{(i_m)}\Big) \cdot q_{a_m - b_{m-1}}\Big(x_m^{(j_m)} - y_{m-1}^{(j_m)}\Big) = q^2_{a_m - b_{\mathsf{p}(m)}}\Big(x_m^{(i_m)} - y_{\mathsf{p}(m)}^{(i_m)}\Big) \cdot e^{o_M(1)} , \tag{4.2.34}$$

where $o_M(1)$ denotes a quantity such that $\lim_{M \to \infty} o_M(1) = 0$.

Proof. We will show that

$$q_{a_m - b_{m-1}}\Big(x_m^{(i_m)} - y_{m-1}^{(i_m)}\Big) = q_{a_m - b_{\mathsf{p}(m)}}\Big(x_m^{(i_m)} - y_{\mathsf{p}(m)}^{(i_m)}\Big) \cdot e^{o_M(1)} \tag{4.2.35}$$

and by symmetry we will get (4.2.34). To this end, we invoke the local limit theorem for simple random walks, which we recall from [LL10]. In particular, by Theorem 2.3.11 [LL10], we have that there exists $\varrho > 0$ such that for all $n \geqslant 0$ and $x \in \mathbb{Z}^2$ with $|x| < \varrho n$,

$$q_n(x) = g_{\frac{n}{2}}(x) \cdot e^{O\left(\frac{1}{n} + \frac{|x|^4}{n^3}\right)} \cdot 2 \cdot \mathbb{1}_{\left\{(n,x) \in \mathbb{Z}^3_{\mathsf{even}}\right\}} , \tag{4.2.36}$$

where $g_t(x) = \frac{e^{-\frac{|x|^2}{2t}}}{2\pi t}$ denotes the 2-dimensional heat kernel and

$$\mathbb{Z}^3_{\mathsf{even}} := \{(n, x) \in \mathbb{Z} \times \mathbb{Z}^2 : n + x_1 + x_2 = 0 \ (\mathrm{mod}\ 2)\} .$$

The last constraint in (4.2.36) is a consequence of the periodicity of the simple random walk. Let us proceed with the proof of Lemma 4.2.6.

First, we derive some inequalities which are going to be useful for the approximations using the local limit theorem. We claim that

$$a_m - b_{m-1} > (M - 1)\left(b_{m-1} - b_{\mathsf{p}(m)}\right) . \tag{4.2.37}$$

The proof of the latter is done by a finite recursion. In particular, notice that since the inequality $a_{k+1} - b_k > M(b_k - b_{k-1})$ holds for all $1 \leqslant k \leqslant r - 1$, we have

$$a_m - b_{m-1} > M(b_{m-1} - b_{m-2}) = M(b_{m-1} - a_{m-1}) + M(a_{m-1} - b_{m-2}) . \tag{4.2.38}$$

We can then write

$$\begin{aligned} M(a_{m-1} - b_{m-2}) &= (M - 1)(a_{m-1} - b_{m-2}) + (a_{m-1} - b_{m-2}) \\ &> (M - 1)(a_{m-1} - b_{m-2}) + M(b_{m-2} - b_{m-3}) \end{aligned}$$

and repeat the same steps for $M(b_{m-2}-b_{m-3})$ as we did in (4.2.38) for $M(b_{m-1}-b_{m-2})$. After telescoping we get the lower bound in (4.2.37). Moreover, by triangle inequality we have that

$$\Big| |x_m^{(i_m)}-y_{\mathsf{p}(m)}^{(i_m)}|-|x_m^{(i_m)}-y_{m-1}^{(i_m)}| \Big| \leqslant |y_{m-1}^{(i_m)}-x_{m-1}^{(i_m)}|+|x_{m-1}^{(i_m)}-y_{m-2}^{(i_m)}|+\cdots+|x_{\mathsf{p}(m)+1}^{(i_m)}-y_{\mathsf{p}(m)}^{(i_m)}| . \tag{4.2.39}$$

Note that by the diffusivity constraints of $\mathsf{C}_{r,N,R,M}^{(\mathsf{main})}$ we have that

$$\begin{aligned}|y_{m-1}^{(i_m)}-x_{m-1}^{(i_m)}|+|x_{m-1}^{(i_m)}-y_{m-2}^{(i_m)}|+\cdots+&|x_{\mathsf{p}(m)+1}^{(i_m)}-y_{\mathsf{p}(m)}^{(i_m)}|\\ &\leqslant R\cdot\Bigg(\sum_{k=\mathsf{p}(m)}^{m-2}\sqrt{a_{k+1}-b_k}+\sum_{k=\mathsf{p}(m)+1}^{m-1}\sqrt{b_k-a_k}\Bigg).\end{aligned} \tag{4.2.40}$$

By Cauchy-Schwarz on the right hand side of (4.2.40), (4.2.37) and the fact that $m\leqslant r$, we furthermore have that

$$\begin{aligned}&\Bigg(\sum_{k=\mathsf{p}(m)}^{m-2}\sqrt{a_{k+1}-b_k}+\sum_{k=\mathsf{p}(m)+1}^{m-1}\sqrt{b_k-a_k}\Bigg)\\ \leqslant&\sqrt{2r-1}\Bigg(\sum_{k=\mathsf{p}(m)}^{m-2}(a_{k+1}-b_k)+\sum_{k=\mathsf{p}(m)+1}^{m-1}(b_k-a_k)\Bigg)^{1/2}\\ =&\sqrt{2r-1}\cdot\sqrt{b_{m-1}-b_{\mathsf{p}(m)}}\\ \leqslant&\sqrt{2r-1}\cdot\sqrt{\frac{a_m-b_{m-1}}{M-1}}\,.\end{aligned}$$

Thus, taking into account (4.2.39) we conclude that

$$\Big| |x_m^{(i_m)}-y_{\mathsf{p}(m)}^{(i_m)}|-|x_m^{(i_m)}-y_{m-1}^{(i_m)}| \Big| \leqslant R\cdot\sqrt{2r-1}\cdot\sqrt{\frac{a_m-b_{m-1}}{M-1}}\,. \tag{4.2.41}$$

Now, we are ready to show approximation (4.2.35). By (4.2.36) we have

$$\begin{aligned}&\frac{q_{a_m-b_{m-1}}\big(x_m^{(i_m)}-y_{m-1}^{(i_m)}\big)}{q_{a_m-b_{\mathsf{p}(m)}}\big(x_m^{(i_m)}-y_{\mathsf{p}(m)}^{(i_m)}\big)}\\ &=e^{-\frac{|x_m^{(i_m)}-y_{m-1}^{(i_m)}|^2}{a_m-b_{m-1}}+\frac{|x_m^{(i_m)}-y_{\mathsf{p}(m)}^{(i_m)}|^2}{a_m-b_{\mathsf{p}(m)}}}\cdot\left(\frac{a_m-b_{\mathsf{p}(m)}}{a_m-b_{m-1}}\right)\cdot e^{O\left(\frac{1}{a_m-b_{m-1}}+\frac{|x_m^{(i_m)}-y_{m-1}^{(i_m)}|^4+|x_m^{(i_m)}-y_{\mathsf{p}(m)}^{(i_m)}|^4}{(a_m-b_{m-1})^3}\right)}.\end{aligned} \tag{4.2.42}$$

Let us look at each term on the right hand side of (4.2.42), separately. First, by (4.2.41) we have

$$\begin{aligned}&e^{-\frac{\left|x_m^{(i_m)}-y_{m-1}^{(i_m)}\right|^2}{a_m-b_{m-1}}+\frac{\left|x_m^{(i_m)}-y_{\mathsf{p}(m)}^{(i_m)}\right|^2}{a_m-b_{\mathsf{p}(m)}}}\\ \leqslant\,&e^{\frac{1}{a_m-b_{m-1}}\left(\left|x_m^{(i_m)}-y_{\mathsf{p}(m)}^{(i_m)}\right|^2-\left|x_m^{(i_m)}-y_{m-1}^{(i_m)}\right|^2\right)}\\ =\,&e^{\frac{1}{a_m-b_{m-1}}\left(\left|x_m^{(i_m)}-y_{\mathsf{p}(m)}^{(i_m)}\right|+\left|x_m^{(i_m)}-y_{m-1}^{(i_m)}\right|\right)\cdot\left(\left|x_m^{(i_m)}-y_{\mathsf{p}(m)}^{(i_m)}\right|-\left|x_m^{(i_m)}-y_{m-1}^{(i_m)}\right|\right)}\,.\end{aligned}$$

Using (4.2.41) we have

$$\begin{aligned}\big|x_m^{(i_m)} - y_{\mathsf{p}(m)}^{(i_m)}\big| + \big|x_m^{(i_m)} - y_{m-1}^{(i_m)}\big| &\leqslant 2\big|x_m^{(i_m)} - y_{m-1}^{(i_m)}\big| + R\sqrt{2r-1}\sqrt{\frac{a_m - b_{m-1}}{M-1}} \\ &\leqslant 2R\sqrt{a_m - b_{m-1}} + R\sqrt{2r-1}\sqrt{\frac{a_m - b_{m-1}}{M-1}} \\ &= R\sqrt{a_m - b_{m-1}}\Big(2 + \sqrt{\frac{2r-1}{M-1}}\Big)\,.\end{aligned}$$

Therefore, by (4.2.41) we get

$$e^{\frac{1}{a_m - b_{m-1}}\left(\left|x_m^{(i_m)} - y_{\mathsf{p}(m)}^{(i_m)}\right| + \left|x_m^{(i_m)} - y_{m-1}^{(i_m)}\right|\right)\cdot\left(\left|x_m^{(i_m)} - y_{\mathsf{p}(m)}^{(i_m)}\right| - \left|x_m^{(i_m)} - y_{m-1}^{(i_m)}\right|\right)} \leqslant e^{R^2\left(2 + \sqrt{\frac{2r-1}{M-1}}\right)\sqrt{\frac{2r-1}{M-1}}}\,.$$

Similarly, we can get a lower bound of

$$e^{-\frac{\left|x_m^{(i_m)} - y_{m-1}^{(i_m)}\right|^2}{a_m - b_{m-1}} + \frac{\left|x_m^{(i_m)} - y_{\mathsf{p}(m)}^{(i_m)}\right|^2}{a_m - b_{\mathsf{p}(m)}}} \geqslant e^{-R^2\left(1 - \frac{1}{M}\right)\left(2 + \sqrt{\frac{2r-1}{M-1}}\right)\sqrt{\frac{2r-1}{M-1}}}\,,$$

since $a_m - b_{\mathsf{p}(m)} < \big(1 + \frac{1}{M-1}\big)(a_m - b_{m-1})$ by (4.2.37). The second term in (4.2.42) can be handled by (4.2.37) as

$$1 \leqslant \Big(\frac{a_m - b_{\mathsf{p}(m)}}{a_m - b_{m-1}}\Big) = \Big(1 + \frac{b_{m-1} - b_{\mathsf{p}(m)}}{a_m - b_{m-1}}\Big) < 1 + \frac{1}{M-1} \xrightarrow{M\to\infty} 1\,.$$

For the last term in (4.2.42) we have that

$$\begin{aligned}&\frac{\big|x_m^{(i_m)} - y_{m-1}^{(i_m)}\big|^4 + \big|x_m^{(i_m)} - y_{\mathsf{p}(m)}^{(i_m)}\big|^4}{(a_m - b_{m-1})^3} \\ &\qquad\leqslant \frac{\big|x_m^{(i_m)} - y_{m-1}^{(i_m)}\big|^4}{(a_m - b_{m-1})^3} + \frac{\Big(\big|x_m^{(i_m)} - y_{m-1}^{(i_m)}\big| + R\cdot\sqrt{2r-1}\cdot\sqrt{\frac{(a_m - b_{m-1})}{M-1}}\Big)^4}{(a_m - b_{m-1})^3} \\ &\qquad\leqslant \frac{9\,R^4}{(a_m - b_{m-1})} + \frac{8R^4(2r-1)^2}{(a_m - b_{m-1})\cdot(M-1)}\,,\end{aligned}$$

where we used (4.2.41) along with the inequality $(x+y)^4 \leqslant 8(x^4 + y^4)$ for $x, y \in \mathbb{R}$. Therefore,

$$e^{\frac{\left|x_m^{(i_m)} - y_{m-1}^{(i_m)}\right|^4 + \left|x_m^{(i_m)} - y_{\mathsf{p}(m)}^{(i_m)}\right|^4}{(a_m - b_{m-1})^3}} \leqslant e^{\frac{9\,R^4}{(a_m - b_{m-1})} + \frac{8R^4(2r-1)^2}{(a_m - b_{m-1})\cdot(M-1)}} \leqslant e^{\frac{9\,R^4}{M} + \frac{8R^4(2r-1)^2}{M\cdot(M-1)}} \xrightarrow{M\to\infty} 1\,,$$

where we used in the last inequality that $a_m - b_{m-1} > M(b_{m-1} - b_{m-2}) \geqslant M$ by (4.2.23). $\square$

4.2.5. Final step. Now that we have Lemma 4.2.6 at our disposal, we can prove the main approximation result of this step. Recall from (4.2.32) that

$$H_{r,N,R,M}^{(\mathsf{main})} = \sum_{(I_1,\ldots,I_r)\in\mathcal{I}^{(2)}}\ \sum_{(\vec{a},\vec{b},\vec{\boldsymbol{x}},\vec{\boldsymbol{y}})\in\mathsf{C}_{r,N,R,M}^{(\mathsf{main})}} \mathsf{U}_{b_1}^{I_1}(0,\boldsymbol{y}_1)\prod_{i=2}^{r} Q_{a_i - b_{i-1}}^{I_{i-1};I_i}(\boldsymbol{y}_{i-1},\boldsymbol{x}_i)\,\mathsf{U}_{b_i - a_i}^{I_i}(\boldsymbol{x}_i,\boldsymbol{y}_i)\,\sigma_N(I_i)\,.$$

Define $H_{r,N,R,M}^{(\mathsf{rew})}$ to be the resulting sum after rewiring has been applied to every term of $H_{r,N,R,M}^{(\mathsf{main})}$, that is, given a sequence of partitions $(I_1,\ldots,I_r)\in\mathcal{I}^{(2)}$ and $(\vec{a},\vec{b},\vec{\boldsymbol{x}},\vec{\boldsymbol{y}}) \in \mathsf{C}_{r,N,R,M}^{(\mathsf{main})}$, we apply the kernel replacement (4.2.33) to all partitions $I_1,\ldots,I_r$ starting from I_r and moving backward. We remind the reader that that we may denote a partition $I = \{i,j\} \sqcup \bigsqcup_{k\neq i,j}\{k\} \in \mathcal{I}^{(2)}$ by its non-trivial part $\{i,j\}$, see subsection 4.1.2.

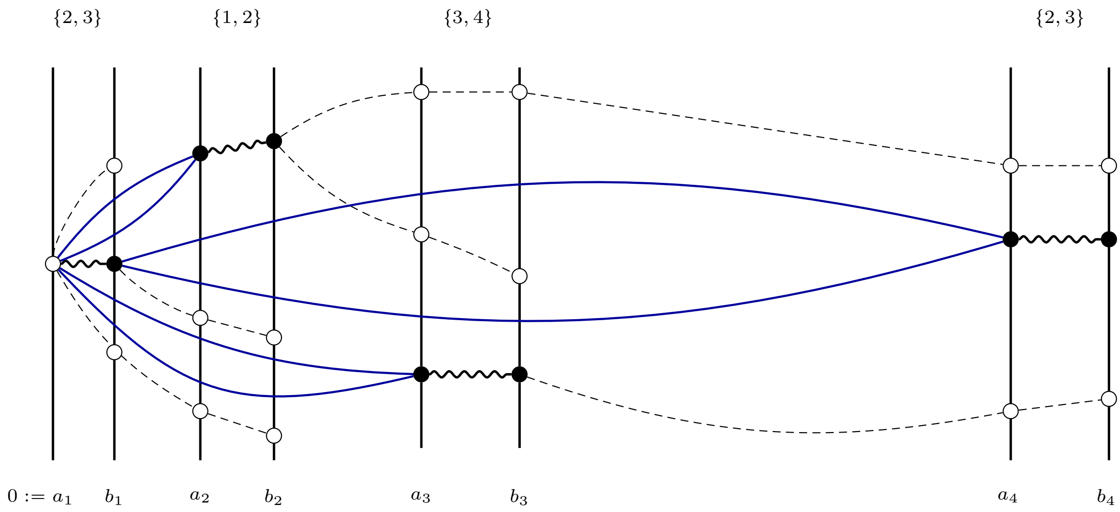


FIGURE 4.2.2. Figure 4.2.1 after *rewiring*. We use blue lines to represent the new kernels produced by rewiring. The dashed lines represent remaining free kernels from the rewiring procedure as well as kernels coming from using the Chapman-Kolmogorov formula for the simple random walk.

Proposition 4.2.7. *Fix* $0 \leqslant r \leqslant K$. *We have that*

$$H^{(\mathsf{rew})}_{r,N,R,M} = e^{K \cdot o_M(1)}\, H^{(\mathsf{main})}_{r,N,R,M} \tag{4.2.43}$$

and

$$\begin{aligned} &H^{(\mathsf{rew})}_{r,N,R,M} \\ &= \sum_{(\{i_1,j_1\},\ldots,\{i_r,j_r\}) \in \mathcal{I}^{(2)}} \; \sum_{(\vec{a},\vec{b},\vec{\boldsymbol{x}},\vec{\boldsymbol{y}}) \in \mathsf{C}^{(\mathsf{main})}_{r,N,R,M}} \prod_{k=1}^{r} U_N^{\beta_{i_k,j_k}}\big(b_k - a_k, y_k^{(i_k)} - x_k^{(i_k)}\big) \\ &\qquad \times \prod_{\substack{1 \leqslant \ell \leqslant h, \\ \ell \neq i_k, j_k}} q_{b_k - a_k}\big(y_k^{(\ell)} - x_k^{(\ell)}\big) \\ &\qquad \times \prod_{m=2}^{r} \sigma_N^{i_m,j_m} \cdot q^2_{a_m - b_{\mathsf{p}(m)}}\big(x_m^{(i_m)} - y_{\mathsf{p}(m)}^{(i_m)}\big) \cdot \prod_{\substack{1 \leqslant \ell \leqslant h, \\ \ell \neq i_m, j_m}} q_{a_m - b_{m-1}}\big(x_m^{(\ell)} - y_{m-1}^{(\ell)}\big)\,. \end{aligned} \tag{4.2.44}$$

Proof. Equation (4.2.43) is a consequence of Lemma 4.2.6 and the fact that $r \leqslant K$, while expansion (4.2.44) is a direct consequence of the rewiring procedure we described in the previous step, see also Figures 4.2.1 and 4.2.2. $\square$

Next, we derive upper and lower bounds for $H^{(\mathsf{rew})}_{r,N,R,M}$. We begin with the upper bound.

Proposition 4.2.8. *We have that*

$$\begin{aligned} H^{(\mathsf{rew})}_{r,N,R,M} \leqslant \sum_{(\{i_1,j_1\},\ldots,\{i_r,j_r\}) \in \mathcal{I}^{(2)}} \; \sum_{\substack{0 := a_1 \leqslant b_1 < a_2 \leqslant \cdots < a_r \leqslant b_r \leqslant N, \\ 0 := x_1, y_1, \ldots, x_r, y_r \in \mathbb{Z}^2}} \prod_{k=1}^{r} U_N^{\beta_{i_k,j_k}}\big(b_k - a_k, y_k - x_k\big) \\ \times \prod_{m=2}^{r} \sigma_N^{i_m,j_m} \cdot q^2_{a_m - b_{\mathsf{p}(m)}}\big(x_m - y_{\mathsf{p}(m)}\big)\,. \end{aligned}$$

Proof. Fix $r \geqslant 1$ and from (4.2.44) recall that

$$
\begin{aligned}
&H^{(\mathsf{rew})}_{r,N,R,M} \\
&\quad = \sum_{\left(\{i_1,j_1\},\ldots,\{i_r,j_r\}\right)\in\mathcal{I}^{(2)}} \ \sum_{(\vec{a},\vec{b},\vec{\boldsymbol{x}},\vec{\boldsymbol{y}})\in \mathsf{C}^{(\mathsf{main})}_{r,N,R,M}} \ \prod_{k=1}^{r} U_N^{\beta_{i_k,j_k}}\big(b_k-a_k, y_k^{(i_k)}-x_k^{(i_k)}\big) \\
&\qquad\qquad \times \prod_{\substack{1\leqslant \ell\leqslant h,\\ \ell\neq i_k,j_k}} q_{b_k-a_k}(y_k^{(\ell)}-x_k^{(\ell)}) \\
&\qquad\qquad \times \prod_{m=2}^{r} \sigma_N^{i_m,j_m}\cdot q^2_{a_m-b_{\mathsf{p}(m)}}\big(x_m^{(i_m)}-y_{\mathsf{p}(m)}^{(i_m)}\big)\cdot \prod_{\substack{1\leqslant \ell\leqslant h,\\ \ell\neq i_m,j_m}} q_{a_m-b_{m-1}}\big(x_m^{(\ell)}-y_{m-1}^{(\ell)}\big)\,.
\end{aligned}
\tag{4.2.45}
$$

For the sake of obtaining an upper bound on $H^{(\mathsf{rew})}_{r,N,R,M}$ we can sum $(\vec{a},\vec{b},\vec{\boldsymbol{x}},\vec{\boldsymbol{y}})$ in (4.2.45) over $\mathsf{C}_{r,N}$, see definition in (4.2.2), instead of $\mathsf{C}^{(\mathsf{main})}_{r,N,R,M}$. We start the summation of the right hand side of (4.2.45) from the end. Using that for $n\in\mathbb{N}$, $\sum_{z\in\mathbb{Z}^2} q_n(z)=1$ we deduce that

$$
\sum_{\substack{y_r^{(\ell)}\in\mathbb{Z}^2:\\ 1\leqslant \ell\leqslant h,\, \ell\neq i_r,j_r}} \ \prod_{\ell\neq i_r,j_r} q_{b_r-a_r}(y_r^{(\ell)}-x_r^{(\ell)}) = 1\,.
$$

We leave the sum $\sum_{b_r\in[a_r,N],\, y_r^{(i_r)}\in\mathbb{Z}^2} U_N\big(b_r-a_r, y_r^{(i_r)}-x_r^{(i_r)}\big)$ intact and move on to the time interval $[b_{r-1},a_r]$. We use again that for $n\in\mathbb{N}$, $\sum_{z\in\mathbb{Z}^2} q_n(z)=1$, to deduce that

$$
\sum_{\substack{x_r^{(\ell)}\in\mathbb{Z}^2:\\ 1\leqslant \ell\leqslant h,\, \ell\neq i_r,j_r}} \ \prod_{\ell\neq i_r,j_r} q_{a_r-b_{r-1}}(x_r^{(\ell)}-y_{r-1}^{(\ell)}) = 1\,.
$$

Again, we leave the sum $\sum_{a_r\in(b_{r-1},b_r],\, x_r^{(i_r)}\in\mathbb{Z}^2} q^2_{a_r-b_{\mathsf{p}(r)}}\big(x_r^{(i_r)}-y_{\mathsf{p}(r)}^{(i_r)}\big)$ intact. We can iterate this procedure inductively since due to rewiring all the spatial variables $y_{r-1}^{(\ell)}$, $\ell\neq i_{r-1},j_{r-1}$ are free, that is, there are no outgoing laces starting off $y_{r-1}^{(\ell)}$, $\ell\neq i_{r-1},j_{r-1}$ at time b_{r-1}. The summations we have performed correspond to getting rid of the dashed lines in Figure 4.2.2. Iterating this procedure inductively then implies the following upper bound for $H^{(\mathsf{rew})}_{r,N,R,M}$.

$$
\begin{aligned}
H^{(\mathsf{rew})}_{r,N,R,M} \leqslant \sum_{\left(\{i_1,j_1\},\ldots,\{i_r,j_r\}\right)\in\mathcal{I}^{(2)}} \ \sum_{\substack{0:=a_1\leqslant b_1<a_2\leqslant\cdots<a_r\leqslant b_r\leqslant N\,,\\ 0:=x_1,y_1,\ldots,x_r,y_r\in\mathbb{Z}^2}} \ \prod_{k=1}^{r} U_N^{\beta_{i_k,j_k}}\big(b_k-a_k, y_k-x_k\big) \\
\times \prod_{m=2}^{r} \sigma_N^{i_m,j_m}\cdot q^2_{a_m-b_{\mathsf{p}(m)}}\big(x_m-y_{\mathsf{p}(m)}\big)\,.
\end{aligned}
$$

$\square$

In the next proposition we derive complementary lower bounds for $H^{(\mathsf{rew})}_{r,N,R,M}$. Given $0\leqslant r\leqslant K$ and a sequence of partitions $\vec{I}=(\{i_1,j_1\},\ldots,\{i_r,j_r\})\in\mathcal{I}^{(2)}$ we define the set $\mathsf{C}^{(\mathsf{rew})}_{r,N,R,M}(\vec{I})$ to be $\mathsf{C}^{(\mathsf{main})}_{r,N,R,M}$ where for every $2\leqslant m\leqslant r$ we replace the diffusivity constraint $\big\|\boldsymbol{x}_m-\boldsymbol{y}_{m-1}\big\|_\infty\leqslant R\sqrt{a_m-b_{m-1}}$ by the constraints

$$
|x_m^{(\ell)}-y_{m-1}^{(\ell)}|\leqslant R\sqrt{a_m-b_{m-1}},\ \ell\in\{1,\ldots,h\}\smallsetminus\{i_m,j_m\} \quad \text{and}
$$

$$
|x_m^{(\ell')}-y_{\mathsf{p}(m)}^{(\ell')}|\leqslant R\sqrt{1-\frac{1}{M}}\Bigg(1-\sqrt{\frac{2K-1}{M-1}}\Bigg)\sqrt{a_m-b_{\mathsf{p}(m)}},\ \ \ell'\in\{i_m,j_m\}\,.
$$

This replacement transforms the diffusivity constraints imposed on the jumps of two walks $\{i_m, j_m\}$ from their respective positions at time b_{m-1} to time a_m, which is the time they (re)start colliding, to a diffusivity constraint connecting their common position at time $b_{\mathsf{p}(m)}$, which is the last time they collided before time a_m, to their common position at time a_m when they start colliding again.

We have the following Lemma.

Lemma 4.2.9. *Let $0 \leqslant r \leqslant K$ and $M > 2K$. For all $\vec{I} = \big(\{i_1, j_1\}, \ldots, \{i_r, j_r\}\big) \in \mathcal{I}^{(2)}$ we have that*

$$\mathsf{C}^{(\mathsf{rew})}_{r,N,R,M}(\vec{I}) \subset \mathsf{C}^{(\mathsf{main})}_{r,N,R,M}\,.$$

Proof. Fix $0 \leqslant r \leqslant K$, a sequence $\vec{I} = \big(\{i_1, j_1\}, \ldots, \{i_r, j_r\}\big) \in \mathcal{I}^{(2)}$ and $(\vec{a}, \vec{b}, \vec{\boldsymbol{x}}, \vec{\boldsymbol{y}}) \in \mathsf{C}^{(\mathsf{rew})}_{r,N,M,R}(\vec{I})$. Moreover, let $2 \leqslant m \leqslant r$. By symmetry it suffices to prove that

$$\mathbb{1}_{\Big\{|x^{(i_m)}_m - y^{(i_m)}_{\mathsf{p}(m)}| \leqslant R\sqrt{1-\frac{1}{M}}\Big(1-\sqrt{\frac{2r-1}{M-1}}\Big)\sqrt{a_m - b_{\mathsf{p}(m)}}\Big\}} \leqslant \mathbb{1}_{\Big\{|x^{(i_m)}_m - y^{(i_m)}_{m-1}| \leqslant R\sqrt{a_m - b_{m-1}}\Big\}}\,.$$

Indeed, by the definition of $\mathsf{C}^{(\mathsf{rew})}_{r,N,M,R}(\vec{I})$ and (4.2.41) we have that for $(\vec{a}, \vec{b}, \vec{\boldsymbol{x}}, \vec{\boldsymbol{y}}) \in \mathsf{C}^{(\mathsf{rew})}_{r,N,M,R}(\vec{I})$,

$$\Big|\,|x^{(i_m)}_m - y^{(i_m)}_{\mathsf{p}(m)}| - |x^{(i_m)}_m - y^{(i_m)}_{m-1}|\,\Big| \leqslant R \cdot \sqrt{2r-1} \cdot \sqrt{\frac{a_m - b_{m-1}}{M-1}}\,. \tag{4.2.46}$$

Moreover by (4.2.37) we have that

$$a_m - b_{m-1} > (M-1)(b_{m-1} - b_{\mathsf{p}(m)}) \Rightarrow a_m - b_{\mathsf{p}(m)} > M(b_{m-1} - b_{\mathsf{p}(m)})\,.$$

Therefore,

$$\begin{aligned} a_m - b_{m-1} = a_m - b_{\mathsf{p}(m)} - (b_{m-1} - b_{\mathsf{p}(m)}) &> a_m - b_{\mathsf{p}(m)} - \frac{1}{M}(a_m - b_{\mathsf{p}(m)}) \\ &= \Big(1 - \frac{1}{M}\Big)(a_m - b_{\mathsf{p}(m)})\,. \end{aligned} \tag{4.2.47}$$

Combining inequalities (4.2.46) and (4.2.47) we get the result.

$\square$

Proposition 4.2.10. *Let $0 \leqslant r \leqslant K$. For $M > 2K$ we have that*

$$\begin{aligned} &H^{(\mathsf{rew})}_{r,N,R,M} \\ &\quad \geqslant (1 - e^{-cR^2})^{2Kh} \sum_{\vec{I} = \big(\{i_1,j_1\},\ldots,\{i_r,j_r\}\big) \in \mathcal{I}^{(2)}} \ \sum_{\substack{0 := a_1 \leqslant b_1 < a_2 \leqslant \cdots < a_r \leqslant b_r \leqslant N, \\ a_{i+1} - b_i > M(b_i - b_{i-1}),\, 1 \leqslant i \leqslant r-1, \\ 0 := x_1, y_1, \ldots, x_r, y_r \in \mathbb{Z}^2}} \\ &\quad \times \prod_{k=1}^r U_N^{\beta_{i_k,j_k}}\big(b_k - a_k, y_k - x_k\big) \\ &\quad \times \prod_{m=2}^r \sigma_N^{i_m,j_m} \cdot q^2_{a_m - b_{\mathsf{p}(m)}}\big(x_m - y_{\mathsf{p}(m)}\big) \cdot \mathbb{1}_{\Big\{|y_k - x_k| \leqslant R\sqrt{b_k - a_k},\, |x_m - y_{\mathsf{p}(m)}| \leqslant R\, C_{K,M}\sqrt{a_m - b_{\mathsf{p}(m)}}\Big\}}, \end{aligned}$$

with $C_{K,M} := \sqrt{1 - \frac{1}{M}}\Big(1 - \sqrt{\frac{2K-1}{M-1}}\Big)$.

Proof. Recall from (4.2.44) that

$$
\begin{aligned}
H^{(\mathsf{rew})}_{r,N,R,M} & \\
&= \sum_{\vec{I}=\left(\{i_1,j_1\},\ldots,\{i_r,j_r\}\right)\in\mathcal{I}^{(2)}} \ \sum_{(\vec{a},\vec{b},\vec{\boldsymbol{x}},\vec{\boldsymbol{y}})\in \mathsf{C}^{(\mathsf{main})}_{r,N,R,M}} \\
&\times \prod_{k=1}^{r} U_N^{\beta_{i_k,j_k}}\big(b_k-a_k, y_k^{(i_k)}-x_k^{(i_k)}\big)\cdot \prod_{\ell\neq i_k,j_k} q_{b_k-a_k}(y_k^{(\ell)}-x_k^{(\ell)}) \\
&\quad \times \prod_{m=2}^{r} \sigma_N^{i_m,j_m}\cdot q^2_{a_m-b_{\mathsf{p}(m)}}\big(x_m^{(i_m)}-y_{\mathsf{p}(m)}^{(i_m)}\big)\cdot \prod_{\ell\neq i_m,j_m} q_{a_m-b_{m-1}}\big(x_m^{(\ell)}-y_{m-1}^{(\ell)}\big)\,.
\end{aligned}
$$

By Lemma 4.2.9 we have that

$$
\begin{aligned}
H^{(\mathsf{rew})}_{r,N,R,M} & \\
&\geqslant \sum_{\vec{I}=\left(\{i_1,j_1\},\ldots,\{i_r,j_r\}\right)\in\mathcal{I}^{(2)}} \ \sum_{(\vec{a},\vec{b},\vec{\boldsymbol{x}},\vec{\boldsymbol{y}})\in \mathsf{C}^{(\mathsf{rew})}_{r,N,R,M}(\vec{I})} \\
&\times \prod_{k=1}^{r} U_N^{\beta_{i_k,j_k}}\big(b_k-a_k, y_k^{(i_k)}-x_k^{(i_k)}\big)\cdot \prod_{\ell\neq i_k,j_k} q_{b_k-a_k}(y_k^{(\ell)}-x_k^{(\ell)}) \\
&\quad \times \prod_{m=2}^{r} \sigma_N^{i_m,j_m}\cdot q^2_{a_m-b_{\mathsf{p}(m)}}\big(x_m^{(i_m)}-y_{\mathsf{p}(m)}^{(i_m)}\big)\cdot \prod_{\ell\neq i_m,j_m} q_{a_m-b_{m-1}}\big(x_m^{(\ell)}-y_{m-1}^{(\ell)}\big)\,.
\end{aligned}
\tag{4.2.48}
$$

The first step in getting a lower bound for $H^{(\mathsf{rew})}_{r,N,R,M}$ is to get rid of the dashed lines, see Figure 4.2.2. We follow the steps we took in the proof of Proposition 4.2.8 for the upper bound. In particular, we start the summation of (4.2.48) beginning from the end. Using that for $n\in\mathbb{N}$ and $R\in(0,\infty)$

$$
\sum_{z\in\mathbb{Z}^2:\,|z|\leqslant R\sqrt{n}} q_n(z) = 1-\sum_{z\in\mathbb{Z}^2:\,|z|>R\sqrt{n}} q_n(z)\geqslant 1-e^{-cR^2}\,,
\tag{4.2.49}
$$

by (4.2.13), we get that

$$
\sum_{\substack{y_r^{(\ell)}\in\mathbb{Z}^2:|y_r^{(\ell)}-x_r^{(\ell)}|\leqslant R\sqrt{b_r-a_r},\\ 1\leqslant \ell\leqslant h,\,\ell\neq i_r,j_r}} \ \prod_{\ell\neq i_r,j_r} q_{b_r-a_r}(y_r^{(\ell)}-x_r^{(\ell)})\geqslant (1-e^{-cR^2})^h\,.
$$

We leave the sum

$$
\sum_{\substack{b_r\in[a_r,N],\\ y_r^{(i_r)}\in\mathbb{Z}^2:\,|y_r^{(i_r)}-x_r^{(i_r)}|\leqslant R\sqrt{b_r-a_r}}} U_N\big(b_r-a_r, y_r^{(i_r)}-x_r^{(i_r)}\big)
$$

as is and move on to the time interval $[b_{r-1},a_r]$. We use (4.2.49) to deduce that

$$
\sum_{\substack{x_r^{(\ell)}\in\mathbb{Z}^2:|x_r^{(\ell)}-y_{r-1}^{(\ell)}|\leqslant R\sqrt{a_r-b_{r-1}},\\ 1\leqslant \ell\leqslant h,\,\ell\neq i_r,j_r}} \ \prod_{\ell\neq i_r,j_r} q_{a_r-b_{r-1}}(x_r^{(\ell)}-y_{r-1}^{(\ell)})\geqslant (1-e^{-cR^2})^h\,.
$$

Again, we leave the sum $\sum_{a_r\in(b_{r-1},b_r],\,x_r^{(i_r)}\in\mathbb{Z}^2} q^2_{a_r-b_{\mathsf{p}(r)}}\big(x_r^{(i_r)}-y_{\mathsf{p}(r)}^{(i_r)}\big)$ intact. We can continue this procedure since due to rewiring all the spatial variables $y_{r-1}^{(\ell)}$, $\ell\neq i_{r-1},j_{r-1}$ are free, i.e. there are no outgoing laces starting off $y_{r-1}^{(\ell)}$, $\ell\neq i_{r-1},j_{r-1}$ at time b_{r-1}, and there are no diffusivity constraints linking $x_r^{(i_r)}=x_r^{(j_r)}$ with $y_{r-1}^{(i_r)},y_{r-1}^{(j_r)}$ by definition of $\mathsf{C}^{(\mathsf{rew})}_{r,N,R,M}(\vec{I})$. Iterating this procedure we obtain that

$$
\begin{aligned}
&H^{(\mathsf{rew})}_{r,N,R,M} \\
&\geqslant (1-e^{-cR^2})^{2Kh} \sum_{\vec{I}=\left(\{i_1,j_1\},\dots,\{i_r,j_r\}\right)\in\mathcal{I}^{(2)}} \quad \sum_{\substack{0:=a_1\leqslant b_1<a_2\leqslant\cdots<a_r\leqslant b_r\leqslant N,\\ a_{i+1}-b_i>M(b_i-b_{i-1}),\,1\leqslant i\leqslant r-1,\\ 0:=x_1,y_1,\dots,x_r,y_r\in\mathbb{Z}^2}} \\
&\quad \times \prod_{k=1}^{r} U_N^{\beta_{i_k,j_k}}\left(b_k-a_k, y_k-x_k\right) \\
&\quad \times \prod_{m=2}^{r} \sigma_N^{i_m,j_m}\cdot q^2_{a_m-b_{\mathsf{p}(m)}}\left(x_m-y_{\mathsf{p}(m)}\right)\cdot \mathbb{1}_{\left\{|y_k-x_k|\leqslant R\sqrt{b_k-a_k},\,|x_m-y_{\mathsf{p}(m)}|\leqslant R\,C_{K,M}\sqrt{a_m-b_{\mathsf{p}(m)}}\right\}},
\end{aligned}
$$

with $C_{K,M}=\sqrt{1-\frac{1}{M}}\left(1-\sqrt{\frac{2K-1}{M-1}}\right)$. □

Proposition 4.2.11. *We have that*

$$
\lim_{K\to\infty}\lim_{M\to\infty}\lim_{R\to\infty}\lim_{N\to\infty}\sum_{r=0}^{K} H^{(\mathsf{rew})}_{r,N,R,M} = \prod_{1\leqslant i<j\leqslant h}\frac{1}{1-\beta_{i,j}}\,.
$$

Proof. We are going to prove this Proposition via means of the lower and upper bounds established in Propositions 4.2.8 and 4.2.10. By Proposition 4.2.8 we have that

$$
\begin{aligned}
H^{(\mathsf{rew})}_{r,N,R,M} \leqslant \sum_{\left(\{i_1,j_1\},\dots,\{i_r,j_r\}\right)\in\mathcal{I}^{(2)}} \quad \sum_{\substack{0:=a_1\leqslant b_1<a_2\leqslant\cdots<a_r\leqslant b_r\leqslant N\,,\\ 0:=x_1,y_1,\dots,x_r,y_r\in\mathbb{Z}^2}} \prod_{k=1}^{r} U_N^{\beta_{i_k,j_k}}\left(b_k-a_k, y_k-x_k\right) \\
\times \prod_{m=2}^{r}\sigma_N^{i_m,j_m}\cdot q^2_{a_m-b_{\mathsf{p}(m)}}\left(x_m-y_{\mathsf{p}(m)}\right).
\end{aligned}
\tag{4.2.50}
$$

Summing the spatial points on the right hand side of (4.2.50) we obtain that

$$
\begin{aligned}
H^{(\mathsf{rew})}_{r,N,R,M} \leqslant \sum_{\left(\{i_1,j_1\},\dots,\{i_r,j_r\}\right)\in\mathcal{I}^{(2)}} \quad \sum_{0:=a_1\leqslant b_1<a_2\leqslant\cdots<a_r\leqslant b_r\leqslant N} \\
\times \prod_{k=1}^{r} U_N^{\beta_{i_k,j_k}}\left(b_k-a_k\right)\prod_{m=2}^{r}\sigma_N^{i_m,j_m} q_{2(a_m-b_{\mathsf{p}(m)})}(0)\,.
\end{aligned}
\tag{4.2.51}
$$

Note that the right hand sides of (4.2.50) and (4.2.51) describe a system of $\binom{h}{2}$ pairs of random walks which collide only between themselves. The times $a_i\leqslant b_i$ mark when a pair of random walks starts and terminates colliding (temporarily) before the next pair starts colliding. The order of these collision events is encoded in $\vec{I}=\left(\{i_1,j_1\},\dots,\{i_r,j_r\}\right)\in\mathcal{I}^{(2)}$. Using (4.1.6) and (4.1.7) one can deduce that

$$
\sum_{r\geqslant 0} H^{(\mathsf{rew})}_{r,N,R,M} \leqslant \prod_{1\leqslant i<j\leqslant h}\mathrm{E}\Big[e^{\frac{\pi\beta_{i,j}}{\log N}\mathsf{L}_N^{(i,j)}}\Big] = \big(1+o_N(1)\big)\prod_{1\leqslant i<j\leqslant h}\frac{1}{1-\beta_{i,j}}\,. \tag{4.2.52}
$$

Next, by Proposition 4.2.10 we have that

$$
\begin{aligned}
& H^{(\mathsf{rew})}_{r,N,R,M} \\
& \geqslant (1-e^{-cR^2})^{2Kh} \sum_{\vec{I}=\left(\{i_1,j_1\},\dots,\{i_r,j_r\}\right)\in\mathcal{I}^{(2)}} \ \sum_{\substack{0:=a_1\leqslant b_1<a_2\leqslant\cdots<a_r\leqslant b_r\leqslant N,\\ a_{i+1}-b_i>M(b_i-b_{i-1}),\,1\leqslant i\leqslant r-1,\\ 0:=x_1,y_1,\dots,x_r,y_r\in\mathbb{Z}^2}} \\
& \quad \times \prod_{k=1}^{r} U_N^{\beta_{i_k,j_k}}\left(b_k-a_k, y_k-x_k\right) \\
& \quad \times \prod_{m=2}^{r} \sigma_N^{i_m,j_m}\cdot q^2_{a_m-b_{\mathsf{p}(m)}}\left(x_m-y_{\mathsf{p}(m)}\right)\cdot \mathbb{1}_{\left\{|y_k-x_k|\leqslant R\sqrt{b_k-a_k},\ |x_m-y_{\mathsf{p}(m)}|\leqslant R\,C_{K,M}\sqrt{a_m-b_{\mathsf{p}(m)}}\right\}},
\end{aligned}
\tag{4.2.53}
$$

Lifting the diffusivity conditions imposed on the right-hand side of (4.2.53) can be done using arguments already present in Lemma 4.2.4. More specifically, we use that for $0\leqslant m\leqslant N$, $w\in\mathbb{Z}^2$ and $1\leqslant i<j\leqslant h$,

$$
\begin{aligned}
& \sum_{\substack{n\in[m,N],\\ z\in\mathbb{Z}^2:\,|z-w|\leqslant R\sqrt{n-m}}} U_N^{\beta_{i,j}}\left(n-m, z-w\right) \\
& = \sum_{n\in[m,N]} U_N^{\beta_{i,j}}\left(n-m\right) - \sum_{\substack{n\in[m,N],\\ z\in\mathbb{Z}^2:\,|z-w|>R\sqrt{n-m}}} U_N^{\beta_{i,j}}\left(n-m,z-w\right) \\
& \geqslant \sum_{n\in[m,N]} U_N^{\beta_{i,j}}\left(n-m\right) - e^{-\kappa R}\sum_{n\in[m,N]} U_N^{\beta_{i,j}}\left(n-m\right) \\
& \geqslant \left(1-e^{-\kappa R}\right)\sum_{n\in[m,N]} U_N^{\beta_{i,j}}\left(n-m\right),
\end{aligned}
$$

where in the first inequality we used (4.2.22) from Lemma 4.2.4 with a suitable constant $\kappa(\bar{\beta})\in(0,\infty)$. Similarly we have that

$$
\begin{aligned}
& \sum_{\substack{n\in[m,N],\\ z\in\mathbb{Z}^2:|z-w|\leqslant R\,C_{K,M}\sqrt{n-m}}} q^2_{n-m}(z-w) \\
& = \sum_{n\in[m,N]} q_{2(n-m)}(0) - \sum_{\substack{n\in[m,N],\\ z\in\mathbb{Z}^2:|z-w|>R\,C_{K,M}\sqrt{n-m}}} q^2_{n-m}(z-w) \\
& \geqslant \left(1-e^{-\kappa\,R^2\,C^2_{K,M}}\right)\sum_{n\in[m,N]} q_{2(n-m)}(0)
\end{aligned}
$$

by tuning the constant κ if needed. Therefore, we finally obtain that

$$
\begin{aligned}
H^{(\mathsf{rew})}_{r,N,R,M} & \geqslant (1-e^{-cR^2})^{2Kh}\,(1-e^{-\kappa\,R})^K\,(1-e^{-\kappa\,R^2\,C^2_{K,M}})^K \\
& \quad\times \sum_{\left(\{i_1,j_1\},\dots,\{i_r,j_r\}\right)\in\mathcal{I}^{(2)}} \ \sum_{\substack{0:=a_1\leqslant b_1<a_2\leqslant\cdots<a_r\leqslant b_r\leqslant N,\\ a_{i+1}-b_i>M(b_i-b_{i-1}),\,1\leqslant i\leqslant r-1}} . \\
& \quad\quad \times \prod_{k=1}^{r} U_N^{\beta_{i_k,j_k}}\left(b_k-a_k\right)\prod_{m=2}^{r}\sigma_N^{i_m,j_m}\cdot q_{2(a_m-b_{\mathsf{p}(m)})}(0)\,.
\end{aligned}
\tag{4.2.54}
$$

The last restriction we need to lift is the restriction $a_{i+1}-b_i>M(b_i-b_{i-1})$, $1\leqslant i\leqslant r-1$. This can be done via the arguments used in Proposition 4.2.5, so we do not repeat it here, but only note that there exists a constant $\widetilde{C}_K=\widetilde{C}_K(\bar{\beta},h)\in(0,\infty)$ such that for all $0\leqslant r\leqslant K$, the corresponding sum to the right-hand side of (4.2.54), but with its temporal range of summation be such that there exists $1 \leqslant i \leqslant r-1:\ a_{i+1} - b_i \leqslant M(b_i - b_{i-1})$, satisfies the bound

$$
\begin{aligned}
&\sum_{\left(\{i_1,j_1\},\dots,\{i_r,j_r\}\right)\in\mathcal{I}^{(2)}} \ \sum_{\substack{0:=a_1\leqslant b_1<a_2\leqslant\cdots<a_r\leqslant b_r\leqslant N,\\ \exists\, 1\leqslant i\leqslant r-1:\, a_{i+1}-b_i\leqslant M(b_i-b_{i-1})}} \\
&\qquad\qquad \times \prod_{k=1}^{r} U_N^{\beta_{i_k,j_k}}\big(b_k-a_k\big) \prod_{m=2}^{r} \sigma_N^{i_m,j_m}\cdot q_{2(a_m-b_{\mathsf{p}(m)})}(0) \\
&\leqslant \widetilde{C}_K\cdot\varepsilon_{N,M}\,,
\end{aligned}
$$

where $\varepsilon_{N,M}$ is such that $\lim_{N\to\infty}\varepsilon_{N,M} = 0$ for any fixed $M \in (0,\infty)$. Therefore, the resulting resulting lower bound on $H^{(\mathsf{rew})}_{r,N,R,M}$ will be then

$$
\begin{aligned}
H^{(\mathsf{rew})}_{r,N,R,M} &\geqslant (1-e^{-cR^2})^{2Kh}\,(1-e^{-\kappa\,R})^K\,(1-e^{-\kappa\,R^2\,C^2_{K,M}})^K \\
&\quad\times\Bigg(\sum_{\left(\{i_1,j_1\},\dots,\{i_r,j_r\}\right)\in\mathcal{I}^{(2)}} \ \sum_{0:=a_1\leqslant b_1<a_2\leqslant\cdots<a_r\leqslant b_r\leqslant N} \ \prod_{k=1}^{r} U_N^{\beta_{i_k,j_k}}\big(b_k-a_k\big) \\
&\qquad\qquad \times \prod_{m=2}^{r}\sigma_N^{i_m,j_m}\cdot q_{2(a_m-b_{\mathsf{p}(m)})}(0) - \widetilde{C}_K\cdot\varepsilon_{N,M}\Bigg)\,.
\end{aligned}
\tag{4.2.55}
$$

Note that

$$
\begin{aligned}
&\prod_{1\leqslant i<j\leqslant h}\mathrm{E}\Big[e^{\frac{\pi\beta_{i,j}}{\log N}\,\mathsf{L}_N^{(i,j)}}\Big] \\
&=\sum_{r=0}^{K}\ \sum_{\left(\{i_1,j_1\},\dots,\{i_r,j_r\}\right)\in\mathcal{I}^{(2)}} \ \sum_{0:=a_1\leqslant b_1<a_2\leqslant\cdots<a_r\leqslant b_r\leqslant N} \ \prod_{k=1}^{r} U_N^{\beta_{i_k,j_k}}\big(b_k-a_k\big) \\
&\qquad \times\prod_{m=2}^{r}\sigma_N^{i_m,j_m}\cdot q_{2(a_m-b_{\mathsf{p}(m)})}(0) + A_N^{(1)} + A_{N,K}^{(2)}\,,
\end{aligned}
\tag{4.2.56}
$$

where $A_N^{(1)}$ denotes the part of the chaos expansion of $\prod_{1\leqslant i<j\leqslant h}\mathrm{E}\Big[e^{\frac{\pi\beta_{i,j}}{\log N}\,\mathsf{L}_N^{(i,j)}}\Big]$ which contains multiple collisions for at least some time $1 \leqslant n \leqslant N$ and $A^{(2)}_{N,K}$ denotes the corresponding sum on the right-hand side of (4.2.56) but from $r = K+1$ to ∞, that is

$$
\begin{aligned}
A^{(2)}_{N,K} = \sum_{r>K}\ \sum_{\left(\{i_1,j_1\},\dots,\{i_r,j_r\}\right)\in\mathcal{I}^{(2)}} \ \sum_{0:=a_1\leqslant b_1<a_2\leqslant\cdots<a_r\leqslant b_r\leqslant N} \ \prod_{k=1}^{r} U_N^{\beta_{i_k,j_k}}\big(b_k-a_k\big) \\
\times\prod_{m=2}^{r}\sigma_N^{i_m,j_m} q_{2(a_m-b_{\mathsf{p}(m)})}(0)\,.
\end{aligned}
$$

Next, we will give bounds for $A_N^{(1)}$ and $A^{(2)}_{N,K}$. Beginning with $A^{(2)}_{N,K}$, let $\varrho_K := \Big\lfloor \frac{K}{2\binom{h}{2}}\Big\rfloor$. Since we are summing over $r > K$, there has to be a pair $1 \leqslant i < j \leqslant h$ which has recorded more than ϱ_K collisions. We recall from (4.1.8) that $U_N^{\beta}(\cdot)$ admits the renewal representation

$$
U_N^{\beta}(n) = \sum_{k\geqslant 0}(\sigma_N(\beta)\,R_N)^k\,\mathrm{P}\big(\tau_k^{(N)} = n\big)\,.
$$

There are $\binom{h}{2}$ choices for the pair with more than ϱ_K collisions. We can also use the bound (4.1.9) to bound the contribution of the rest $\binom{h}{2}-1$ pairs in $A^{(2)}_{N,K}$. Therefore, we can then write

$$A^{(2)}_{N,K} \leqslant \frac{\binom{h}{2}}{(1-\bar\beta')^{\binom{h}{2}}} \sum_{k>\varrho_K} (\sigma_N(\bar\beta)R_N)^k \, \mathrm{P}(\tau^{(N)}_k \leqslant N) \leqslant \frac{\binom{h}{2}}{(1-\bar\beta')^{\binom{h}{2}}} \sum_{k>\varrho_K} (\bar\beta')^k \xrightarrow{K\to\infty} 0\,,$$

uniformly in N, where $\bar\beta' \in (\bar\beta, 1)$.

Similarly, for $A^{(1)}_N$, we can choose two pairs which collide at the same time in $\binom{h}{2}\cdot\Big(\binom{h}{2}-1\Big) \leqslant \binom{h}{2}^2$ ways and we can use bound (4.1.9) to bound the contribution in $A^{(1)}_N$ of the rest $\binom{h}{2}-2$ pairs. Therefore, we obtain that

$$A^{(1)}_N \leqslant \frac{\binom{h}{2}^2}{(1-\bar\beta')^{\binom{h}{2}}} \sum_{n\geqslant 0,\, x,y\in\mathbb{Z}^2} U^{\bar\beta}_N(n,x)\, U^{\bar\beta}_N(n,y) \leqslant \frac{\binom{h}{2}^2}{(1-\bar\beta')^{\binom{h}{2}}} \sum_{n\geqslant 0} \big(U^{\bar\beta}_N\big)^2(n)\,.$$

By Proposition 1.5 of [CSZ19a] we get the estimate

$$\mathrm{P}(\tau^{(N)}_k = n) \leqslant \frac{C\,k\,q_{2n}(0)}{R_N} \leqslant \frac{C'\,k}{n(\log N)}\,,$$

where the second inequality follows by the local limit theorem. Therefore, by (4.1.8) and the aforementioned estimate we get that

$$\big(U^{\bar\beta}_N\big)^2(n) = \sum_{k,\ell\geqslant 0} (\sigma_N(\bar\beta)R_N)^{k+\ell}\, \mathrm{P}(\tau^{(N)}_k = n)\, \mathrm{P}(\tau^{(N)}_\ell = n) \leqslant \frac{(C')^2}{n^2(\log N)^2}\Big(\sum_{k\geqslant 0} k\cdot(\bar\beta')^k\Big)^2.$$

for some $\bar\beta' \in (\bar\beta,1)$. Since $\bar\beta' < 1$ we have that $\sum_{k\geqslant 0} k\cdot(\bar\beta')^k < \infty$, therefore we deduce that there exists a constant $C = C(\bar\beta')$ such that $\big(U^{\bar\beta}_N\big)^2(n) \leqslant \frac{C}{n^2\,(\log N)^2}$. Since $\sum_{n\geqslant 1}\frac{1}{n^2} < \infty$, there exists a constant $C = C(\bar\beta') \in (0,\infty)$ such that

$$\sum_{n\geqslant 0} \big(U^{\bar\beta}_N\big)^2(n) \leqslant \frac{C}{(\log N)^2} \xrightarrow{N\to\infty} 0\,,$$

The two bounds above, in combination with (4.2.55) and (4.2.56), allow us to write:

$$\sum_{r=0}^{K} H^{(\mathsf{rew})}_{r,N,R,M} \geqslant (1-e^{-cR^2})^{2Kh}\,(1-e^{-\kappa\,R})^K\,(1-e^{-\kappa\,R^2\,C^2_{K,M}})^K$$
$$\times\left(\prod_{1\leqslant i<j\leqslant h} \mathrm{E}\Big[e^{\frac{\pi\beta_{i,j}}{\log N}\mathsf{L}^{(i,j)}_N}\Big] - K\cdot\widetilde{C}_K\cdot\varepsilon_{N,M} - o_N(1) - o_K(1)\right),$$

which together with upper bound (4.2.52) entail that

$$\lim_{K\to\infty}\lim_{M\to\infty}\lim_{R\to\infty}\lim_{N\to\infty}\sum_{r=0}^{K} H^{(\mathsf{rew})}_{r,N,R,M} = \prod_{1\leqslant i<j\leqslant h}\frac{1}{1-\beta_{i,j}}\,.$$

□

We are now ready to put all pieces together and prove the main result of the paper, Theorem 4.0.1.

Proof of Theorem 4.0.1. Let $\varepsilon > 0$. There exists large $K = K_\varepsilon \in \mathbb{N}$ such that uniformly in $N \in \mathbb{N}$

$$\Big|M^{\boldsymbol\beta}_{N,h} - \sum_{r=0}^{K} H_{r,N}\Big| \leqslant \varepsilon\,, \tag{4.2.57}$$

by Proposition 4.2.2. We have

$$\Big|\sum_{r=0}^{K} H_{r,N} - \sum_{r=0}^{K} H^{(\mathsf{rew})}_{r,N,R,M}\Big| \leqslant \Big(\sum_{r=0}^{K}(H^{(\mathsf{superdiff})}_{r,N,R} + H^{(\mathsf{multi})}_{r,N})\Big) + \Big|\sum_{r=0}^{K}(H^{(\mathsf{diff})}_{r,N,R} - H^{(\mathsf{main})}_{r,N,R,M})\Big| + \Big|\sum_{r=0}^{K}(H^{(\mathsf{main})}_{r,N,R,M} - H^{(\mathsf{rew})}_{r,N,R,M})\Big|.$$

By Propositions 4.2.1, 4.2.3 we have that

$$\lim_{R\to\infty}\lim_{N\to\infty}\Big(\sum_{r=0}^{K}(H^{(\mathsf{superdiff})}_{r,N,R} + H^{(\mathsf{multi})}_{r,N})\Big) \leqslant \lim_{R\to\infty}\sup_{N\in\mathbb{N}}\sum_{r=0}^{K} H^{(\mathsf{superdiff})}_{r,N,R} + \lim_{N\to\infty}\sum_{r=0}^{K} H^{(\mathsf{multi})}_{r,N} = 0\,.$$

Moreover, by Proposition 4.2.5 we have that

$$\lim_{R\to\infty}\lim_{M\to\infty}\lim_{N\to\infty}\Big|\sum_{r=0}^{K} H^{(\mathsf{diff})}_{r,N,R} - \sum_{r=0}^{K} H^{(\mathsf{main})}_{r,N,R,M}\Big| = 0\,.$$

Last, by Proposition 4.2.7 we have that

$$\lim_{R\to\infty}\lim_{M\to\infty}\lim_{N\to\infty}\Big|\sum_{r=0}^{K} H^{(\mathsf{main})}_{r,N,R,M} - \sum_{r=0}^{K} H^{(\mathsf{rew})}_{r,N,R,M}\Big| = 0\,,$$

therefore

$$\lim_{R\to\infty}\lim_{M\to\infty}\lim_{N\to\infty}\Big|\sum_{r=0}^{K} H_{r,N} - \sum_{r=0}^{K} H^{(\mathsf{rew})}_{r,N,R,M}\Big| = 0\,. \tag{4.2.58}$$

By Proposition 4.2.11 we have that

$$\lim_{K\to\infty}\lim_{R\to\infty}\lim_{M\to\infty}\lim_{N\to\infty}\sum_{r=0}^{K} H^{(\mathsf{rew})}_{r,N,R,M} = \prod_{1\leqslant i<j\leqslant h}\frac{1}{1-\beta_{i,j}}\,, \tag{4.2.59}$$

Therefore, by (4.2.57), (4.2.58) and (4.2.59) we obtain that

$$\lim_{N\to\infty} M^{\boldsymbol{\beta}}_{N,h} = \prod_{1\leqslant i<j\leqslant h}\frac{1}{1-\beta_{i,j}}\,.$$

$\square$

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