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The Bethe ansatz for sticky Brownian motions

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Abstract

We consider a multi-dimensional diffusion whose coordinates behave as one-dimensional Brownian motions, evolving independently when apart, but with a sticky interaction when they coincide. We derive the Kolmogorov backwards equation and show that for a specific choice of interaction it can be solved exactly with the Bethe ansatz. The diffusion in $\mathbb{R}^n$ can be viewed as the n-point motions of a stochastic flow of kernels. We use our formulae to study the flow of kernels and show that atoms in the flow are asymptotically exponentially distributed in size at large times.

Keywords: Bethe ansatz; Sticky Brownian motions; Stochastic flows

1. Introduction

In this paper we study an $\mathbb{R}^n$-valued diffusion whose coordinates evolve as independent one-dimensional Brownian motions when they are distinct and have an attractive, so called sticky interaction when they are equal. The diffusion can be interpreted as the evolving positions of n particles on the real line, which interact when they meet. In particular, the difference between two coordinates is described by a one-dimensional sticky Brownian motion, which has been studied as the weak solution to an SDE in [4,6]. Sticky Brownian motion with parameter $\theta > 0$ is a one-dimensional diffusion in natural scale and with speed measure $m(dx) = 2dx + \frac{2}{\theta}\delta_0(dx)$, see [14] for a review of scale functions and speed measures. The $\mathbb{R}^n$-valued diffusion can visit the diagonal $\{x \in \mathbb{R}^n | \ x_1 = \cdots = x_n\}$ for a set of times with positive Lebesgue measure,

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quite unlike a standard Brownian motion in $\mathbb{R}^n$. The interaction between coordinates at such times is not determined solely by specifying the parameter θ describing the stickiness between pairs of particles. It was shown in [9] that the possible interactions can be specified by a finite measure on $[0, 1]$ called the characteristic, or splitting, measure. The characteristic measure determines, via its moments, the rate and direction at which the diffusion leaves the diagonal, with the directions corresponding to the sizes of two clusters which are formed as the cluster of n particles breaks up. As n varies, these multidimensional diffusions are consistent, in that for any $k < n$, any k coordinates of the sticky Brownian motions in $\mathbb{R}^n$ with characteristic measure ν, are sticky Brownian motions in $\mathbb{R}^k$ with the same characteristic measure, ν. An example of such diffusions was originally investigated by Le Jan and Raimond [12] using Dirichlet forms (on the torus rather than Euclidean space), and then the more general case was studied by Howitt and Warren [9] via a martingale problem which we describe later.

The consistency property means that we can also consider such systems of sticky Brownian motions to be the n-point motions of a stochastic flow of kernels. A flow of kernels $(K_{s,t}(x, dy))_{s\leq t}$ is essentially a random family of transition probability measures for a Markov process. Le Jan and Raimond introduced flows of kernels in [11] as a generalisation of flows of maps to study stationary evolutions of turbulent fluids. The n-point motions can then be thought of as describing the behaviour of n particles thrown into the fluid. Stochastic flows of kernels whose n-point motions are described by sticky Brownian motions are called Howitt–Warren flows in [16], where their properties are studied in detail. Gawedzki and Horvai, [7], discovered that for two particles, sticky behaviour arises in certain limits of the Kraichnan model for turbulent advection. For the same model, Warren then proved the convergence of n particles towards sticky Brownian motions with an explicit characteristic measure [19]. Sun, Swart and Schertzer studied general Howitt–Warren flows in [16], where they constructed the flows directly as flows of mass in the Brownian web by marking special separation points and attaching extra random variables to them that tells the mass following a path in the web how to split. The law of these additional random variables is described by the characteristic measure, as we alluded to earlier. Amongst other results, they showed that the Howitt–Warren flows are almost surely purely atomic at deterministic times.

The Howitt–Warren flow can be thought of as the continuum analogue to the random transition probabilities of the random walk in a random environment (RWRE) with a space–time i.i.d. environment. Consistent with this, sticky Brownian motions arise as scaling limits of the n-point motions of random walks in space–time i.i.d. random environments. This was first proved by Le Jan and Lemaire, [10], when the RWRE takes values on the circle and the environment is Beta distributed. Howitt and Warren proved the result for RWREs on $\mathbb{Z}$ for general environments, [9], before a simplified proof was given by Sun, Swart and Schertzer, [16]. A special case of the RWRE, where the environment is Beta distributed, was shown by Barraquand and Corwin, [2], to be exactly solvable; in particular, they found exact solutions for the point to half line probabilities. This was shown using the Bethe ansatz and a non-commutative binomial formula from [13]. These exact solutions were then used to establish that there are GUE Tracy–Widom fluctuations in the large deviations of the random walk in a beta random environment. In another work, Balázs, Rassoul-Agha and Seppeläläinen, [1], showed that when the Beta RWRE is conditioned to escape at an atypical velocity it obeys the wandering exponent $\frac{2}{3}$ that is characteristic of models in the KPZ universality class.

In this paper, we will derive the Kolmogorov backwards equation for the sticky Brownian motions with ordered coordinates from the martingale problem characterisation. In the case that the characteristic measure is uniform, we apply the Bethe ansatz to find an exact formula for the transition density of this process. The choice of uniform characteristic measure seems to be essential, only in this case is the diffusion exactly solvable by the Bethe ansatz. Further, this seems to be the only case the diffusion is reversible, at least with respect to a measure we can write down explicitly; we comment further on this in Remark 3.11. Note that we are finding the transition density for the process with ordered coordinates. It is clearly possible to retrieve the transition density of the two particle version of the original process; however, it is much more difficult for an arbitrary number of particles; indeed, it is not clear that an explicit formula will exist for the unordered process, and we do not pursue one here. Our method is similar to that used by Tracy and Widom for the delta Bose gas [18]; however, the importance of interactions between more than two particles adds significant complexity.

Another approach is to take limits of the exactly solvable model for the RWRE. It is a straightforward application of the scaling limit result from [16], see Section 5.1, to show the scaling limit of random walks in a Beta random environment corresponds to the sticky Brownian motions with a uniform characteristic measure. Barraquand and Rychnovsky [3], working independently of us, derived exact solutions for the point to half-line probabilities of sticky Brownian motions with uniform characteristic measure by taking limits of the exact formulae for the Beta RWRE. An asymptotic analysis then led to the discovery of GUE Tracy–Widom fluctuations in the large deviations of sticky Brownian motions as well.

Before we introduce our main result, we must define some terms. We use the notations $\mathbb{W}^n := \{x \in \mathbb{R}^n | \ x_1 > x_2 > \cdots > x_n\}$ and $\overline{\mathbb{W}^n} := \{x \in \mathbb{R}^n | \ x_1 \geq \cdots \geq x_n\}$ for the principal Weyl chamber, the images of this set under a permutation are called simply Weyl chambers; however, we may sometimes refer to the principal Weyl chamber as just the Weyl chamber. By $C_0^2(\overline{\mathbb{W}^n})$ we mean the set of functions $f : \overline{\mathbb{W}^n} \to \mathbb{R}$ that have a C^2 extension to some open set containing $\overline{\mathbb{W}^n}$ such that f and all of its first and second partial derivatives vanish at infinity. Let Π_n denote the collection of ordered partitions, $(\pi_1, \ldots, \pi_k)$, of $\{1, \ldots, n\}$ such that if $a \in \pi_j$, $b \in \pi_k$ and $j < k$ then $a < b$. That is the elements of the partition each consist of intervals intersected with $\mathbb{Z}$ and are indexed according to the size of their elements.

We want to define what will be the invariant measure for the ordered sticky Brownian motions. Because the coordinates of the process spend a positive amount of time being equal, due to the sticky interactions, this measure takes the form of a linear combination of the Lebesgue measure and lower dimensional copies of the Lebesgue measure on subspaces where some combination of the coordinates are equal. Below we define these measures precisely, before we define the invariant measure itself.

To each partition $\pi \in \Pi_n$ we associate a subset of $\overline{\mathbb{W}^n}$ defined by

$$\mathbb{W}^n_\pi := \{x \in \overline{\mathbb{W}^n} | \ x_\alpha = x_\beta \text{ if and only if there is a } \pi_i \in \pi \text{ such that } \alpha, \beta \in \pi_i\}.$$

In other words, the set of all points in $\overline{\mathbb{W}^n}$ whose coordinates are equal if and only if their indices are in the same element of π. Notice, for $\pi = \{\{1\}, \ldots, \{n\}\}$ we have $\mathbb{W}^n_\pi = \mathbb{W}^n$; in addition, $\overline{\mathbb{W}^n} = \cup_{\pi \in \Pi_n} \mathbb{W}^n_\pi$, and the sets $\mathbb{W}^n_\pi$ are disjoint. It is clear that there is a natural continuous bijection $I^\pi : \mathbb{W}^n_\pi \to \mathbb{W}^{|\pi|}$, given by $I^\pi(x) = (x_{p_1}, \ldots, x_{p_{|\pi|}})$ for some choice of $p_i \in \pi_i$. We can now define a Borel measure on $\mathbb{W}^n_\pi$ as the pushforward of the Lebesgue measure λ on $\mathbb{W}^{|\pi|}$, $\lambda^\pi := I^\pi_* \lambda$. This extends to a Borel measure on $\overline{\mathbb{W}^n}$ via the formula $\lambda^\pi(A) := \lambda^\pi(A \cap \mathbb{W}^n_\pi)$.

Definition 1.1. For $\theta > 0$ the Borel measure $m^{(n)}_\theta$ on $\overline{\mathbb{W}^n}$ is defined as

$$m^{(n)}_\theta := \sum_{\pi \in \Pi_n} \theta^{|\pi| - n} \left(\prod_{\pi_\iota \in \pi} \frac{1}{|\pi_\iota|} \right) \lambda^\pi.$$

Suppose $\theta > 0$ and that, under $\mathbb{P}_x$, $X = (X(t))_{t\geq 0}$ is a solution to the Howitt–Warren martingale problem (we will define this in Section 2) in $\mathbb{R}^n$ with characteristic measure $\frac{\theta}{2}\mathbb{1}_{[0,1]}dx$, zero drift and initial condition $x \in \overline{\mathbb{W}^n}$. We choose to consider only the case of zero drift for convenience, and the main result can be generalised to non-zero drifts without difficulty. Define $Y = (Y(t))_{t\geq 0}$ as the process obtained by ordering the coordinates of X, i.e. for each $t \geq 0$ $Y(t) = (Y^1(t), \dots, Y^n(t)) = (X^{\sigma(1)}(t), \dots, X^{\sigma(n)}(t))$ for some permutation $\sigma \in S_n$ such that $Y^1(t) \geq \cdots \geq Y^n(t)$, where for each $n \in \mathbb{N}$, S_n denotes the set of permutations on $\{1, \dots, n\}$. Note that Y is a diffusion taking values in $\overline{\mathbb{W}^n}$. We now state our main result: an explicit formula for the transition density of the process Y in terms of the Bethe ansatz.

Theorem 1.2. *For every bounded Lipschitz continuous function $f : \overline{\mathbb{W}^n} \to \mathbb{R}$, $x \in \overline{\mathbb{W}^n}$ and $t > 0$*

$$\mathbb{E}_x[f(Y_t)] = \int u_t(x, y) f(y) m_\theta^{(n)}(dy),$$

where $u_t : \overline{\mathbb{W}^n} \times \overline{\mathbb{W}^n} \to \mathbb{R}$ is defined for each $t > 0$ by

$$u_t(x, y) := \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{-\frac{1}{2}t|k|^2} \sum_{\sigma \in S_n} e^{\mathrm{i}k_\sigma \cdot (x - y_\sigma)} \prod_{\substack{\alpha<\beta:\\ \sigma(\beta)<\sigma(\alpha)}} \frac{\mathrm{i}\theta(k_{\sigma(\alpha)} - k_{\sigma(\beta)}) + k_{\sigma(\beta)}k_{\sigma(\alpha)}}{\mathrm{i}\theta(k_{\sigma(\alpha)} - k_{\sigma(\beta)}) - k_{\sigma(\beta)}k_{\sigma(\alpha)}} dk,$$

where, as before, S_n denotes the group of permutations on $\{1, \dots, n\}$, $k_\sigma = (k_{\sigma(1)}, \dots, k_{\sigma(n)})$ and $\mathrm{i} = \sqrt{-1}$.

Furthermore, we prove that $m_\theta^{(n)}$, (Definition 1.1), is in fact a stationary measure of the ordered sticky Brownian motions, and that the process is reversible with respect to $m_\theta^{(n)}$.

Remark 1.3. Note that the function u_t is well defined (the integral always converges), because for every $t > 0$, $x, y, k \in \mathbb{R}^n$, and every permutation $\sigma \in S_n$

$$\left| e^{\mathrm{i}k_\sigma \cdot (x - y_\sigma)} \prod_{\substack{\alpha<\beta:\\ \sigma(\beta)<\sigma(\alpha)}} \frac{\mathrm{i}\theta(k_{\sigma(\alpha)} - k_{\sigma(\beta)}) + k_{\sigma(\beta)}k_{\sigma(\alpha)}}{\mathrm{i}\theta(k_{\sigma(\alpha)} - k_{\sigma(\beta)}) - k_{\sigma(\beta)}k_{\sigma(\alpha)}} \right| = 1.$$

Also note that substituting $-k$ for k in the integral defining $u_t(x, y)$ proves that $\overline{u_t(x, y)} = u_t(x, y)$, and thus, $u_t(x, y)$ is always real valued. The integrand appearing in the formula for $u_t(x, y)$ is not defined where there are distinct α, β such that $k_\alpha = k_\beta = 0$ (where the denominator vanishes), but such points have measure zero. It is easily seen that we can pass derivatives under the integral, and thus we have $u_t(\cdot, y) \in C_0^2(\mathbb{R}^n)$ for all $t > 0$ and $y \in \mathbb{R}^n$. In particular, $u_t(\cdot, y) \in C_0^2(\overline{\mathbb{W}^n})$ for all $t > 0$ and $y \in \overline{\mathbb{W}^n}$, when restricted to $x \in \overline{\mathbb{W}^n}$.

Remark 1.4. Another representation for $u_t(x, y)$ is in terms of eigenfunctions of the generator of the ordered sticky Brownian motions. For each $k \in \mathbb{R}^n$ we have an eigenfunction given by

$$E_k(x) := \sum_{\sigma \in S_n} e^{\mathrm{i}k_\sigma \cdot x} \prod_{\substack{\alpha<\beta:\\ \sigma(\beta)<\sigma(\alpha)}} \frac{\mathrm{i}\theta(k_{\sigma(\alpha)} - k_{\sigma(\beta)}) + k_{\sigma(\beta)}k_{\sigma(\alpha)}}{\mathrm{i}\theta(k_{\sigma(\alpha)} - k_{\sigma(\beta)}) - k_{\sigma(\beta)}k_{\sigma(\alpha)}}.$$

The eigenfunctions give the following representation for $u_t(x, y)$.

$$u_t(x, y) = \frac{1}{(2\pi)^n} \int_{\mathbb{W}^n} e^{-\frac{1}{2}t|k|^2} E_k(x) \overline{E_k(y)} dk.$$

The proof that the above expression for $u_t(x, y)$ agrees with the one previously given is straightforward; we will provide a short sketch here, but leave the details to the reader. To begin the proof, we note that the factors that appear in the product defining $E_k(x)$ have modulus one, and therefore, their conjugate is also their inverse. This fact leads to cancellation between products, when we multiply $E_k(x)$ and $\overline{E_k(y)}$, which gives the following equalities.

$$E_k(x)\overline{E_k(y)} = \sum_{\sigma,\tilde{\sigma}\in S_n} e^{i(k_\sigma\cdot x - k_{\tilde{\sigma}}\cdot y)} \prod_{\substack{\alpha<\beta:\\ \tilde{\sigma}^{-1}\circ\sigma(\beta)<\tilde{\sigma}^{-1}\circ\sigma(\alpha)}} \frac{\mathrm{i}\theta(k_{\sigma(\alpha)}-k_{\sigma(\beta)})+k_{\sigma(\beta)}k_{\sigma(\alpha)}}{\mathrm{i}\theta(k_{\sigma(\alpha)}-k_{\sigma(\beta)})-k_{\sigma(\beta)}k_{\sigma(\alpha)}}$$
$$= \sum_{\sigma,\tilde{\sigma}\in S_n} e^{i(k_{\tilde{\sigma}\circ\sigma}\cdot x - k_{\tilde{\sigma}}\cdot y)} \prod_{\substack{\alpha<\beta:\\ \sigma(\beta)<\sigma(\alpha)}} \frac{\mathrm{i}\theta(k_{\tilde{\sigma}\circ\sigma(\alpha)}-k_{\tilde{\sigma}\circ\sigma(\beta)})+k_{\tilde{\sigma}\circ\sigma(\beta)}k_{\tilde{\sigma}\circ\sigma(\alpha)}}{\mathrm{i}\theta(k_{\tilde{\sigma}\circ\sigma(\alpha)}-k_{\tilde{\sigma}\circ\sigma(\beta)})-k_{\tilde{\sigma}\circ\sigma(\beta)}k_{\tilde{\sigma}\circ\sigma(\alpha)}}.$$

If we apply the above formula to our above expression for $u_t(x, y)$, then we can recover the original expression for $u_t(x, y)$ from Theorem 1.2 by noticing that $\tilde{\sigma}$ is simply permuting the coordinates of the integration variable, k. Thus, when we sum over every $\tilde{\sigma} \in S_n$, the integral over $\mathbb{W}^n$ simply becomes an integral over $\mathbb{R}^n$, resulting in the expression from Theorem 1.2. Since we know the expression from Theorem 1.2 is real valued, we also know that the above expression for $u_t(x, y)$ is real. Notice that in the above expression, it is clear that $\overline{u_t(x, y)} = u_t(y, x)$. But, since $u_t(x, y)$ is real, this tells us that $u_t(x, y) = u_t(y, x)$. We will provide a full proof of this fact in Lemma 4.10.

The Howitt–Warren flows are almost surely purely atomic; we will show in Proposition 5.4 that the size of an atom, when conditioned to be at a fixed location x, is a random variable whose moments can be written in terms of the transition densities of the ordered sticky Brownian motions, the process Y above. Using this identity, we show that the rescaled sizes of the atoms are asymptotically exponentially distributed, as $t \to \infty$, with parameter determined by θ. This result is similar to that found for the point to point probabilities of the Beta random walk in a random environment studied by Thiery and Le Doussal [17] where the asymptotic distribution is a Gamma distribution. In the same paper, the authors found that in the large deviation regime, these point to point probabilities have Tracy–Widom GUE fluctuations, just as for the point to half-line probabilities. Thus, it seems reasonable to conjecture the same fluctuations appear in the size of atoms of the Howitt–Warren flows, but we do not pursue the necessary asymptotic analysis here.

The outline of the paper is as follows: In Section 2 we define the diffusion via a martingale problem. In Section 3, we derive the Kolmogorov backwards equation for the ordered n-point motions, and show that the generator of the process is symmetric with respect to the measure $m_\theta^{(n)}$ when restricted to a certain class of C^2 functions. In Section 4, we show that the backwards equation is solvable by the Bethe ansatz, and as a consequence, we show that the ordered n point motions are reversible with respect to $m_\theta^{(n)}$. Finally, in Section 5, we introduce stochastic flows of kernels and apply the exact formula to study the fluctuations of the sizes of atoms in the Howitt–Warren flow.

2. A consistent family of sticky Brownian motions

We introduce the Howitt–Warren martingale problem in $\mathbb{R}^n$ with drift $\beta \in \mathbb{R}$ and characteristic measure ν (a finite measure on [0, 1]), as formulated in [9]. Solutions are processes in $\mathbb{R}^n$ representing the positions of n particles each moving as one dimensional Brownian motions with drift β. When two or more particles meet, they undergo sticky interactions determined by ν. The solutions are consistent, in the sense that if X is the solution to martingale problem in $\mathbb{R}^n$

with characteristic measure ν and drift β, then for any choice of distinct $i_1, \dots, i_k \in \{1, \dots, n\}$ with $k < n$, $(X^{i_j})_{j=1}^k$ is a solution to the martingale problem in $\mathbb{R}^k$ with characteristic measure ν and drift β.

To each point $x \in \mathbb{R}^n$ we associate a partition of the set $\{1, \dots, n\}$, $\pi(x)$, where $i, j \in \{1, \dots, n\}$ are in the same component of $\pi(x)$ if and only if $x_i = x_j$. Next, for each pair of disjoint subsets $I, J \subset \{1, \dots, n\}$, we define the vectors $v_{I,J} \in \mathbb{R}^n$ as

$$(v_{I,J})_i = \begin{cases} 1, & \text{if } i \in I; \\ -1, & \text{if } i \in J; \\ 0, & \text{otherwise.} \end{cases}$$

Note that I and J are allowed to be empty. Then, we define the set of vectors $\mathcal{V}(x)$ as

$$\mathcal{V}(x) := \{v_{I,J} : \quad I \cup J \in \pi(x),\ I \cap J = \emptyset.\}.$$

$\mathcal{V}(x)$ keeps track of the directions in which the process can infinitesimally move from the point $x \in \mathbb{R}^n$, and will be used to describe the interactions between particles. Now we define the parameters $(\theta(k,l))_{k,l \in \mathbb{N}_0}$, which can be thought of as representing the rate, in a certain excursion theoretic sense, that a cluster of $k + l$ particles break into two clusters of k and l particles. For $k, l \geq 1$, set

$$\theta(k,l) := \int_0^1 x^{k-1}(1-x)^{l-1}\nu(dx). \tag{1}$$

For $k, l \geq 0$, first set $\theta(1,0) - \theta(0,1) = \beta$ and $\theta(0,0) = 0$, imposing the consistency property, $\theta(k,l) = \theta(k+1,l) + \theta(k,l+1)$ for all $k, l \geq 0$, gives definition to $\theta(k,l)$ for all $k, l \geq 0$.

Definition 2.1. Let D_n be the collection of functions $f : \mathbb{R}^n \to \mathbb{R}$ which are continuous and are such that for all Weyl chambers $A \subset \mathbb{R}^n$ the restriction of f to A is linear, so that if $A \subset \mathbb{R}$ is a Weyl chamber, $x, y \in A$ and $a, b \in \mathbb{R}$ are such that $ax + by \in A$, then $f(ax + by) = af(x) + bf(y)$.

For functions $f \in D_n$, we define the operator $\mathcal{A}_n^\theta$ by

$$\mathcal{A}_n^\theta f(x) := \sum_{v_{I,J} \in \mathcal{V}(x)} \theta(|I|, |J|) \nabla_{v_{I,J}} f(x),$$

where $\nabla_{v_{I,J}}$ denotes the one sided derivative in direction $v_{I,J}$. Since we assumed f to be linear when restricted to Weyl chambers, the directional derivatives are constant on the interior of each Weyl chamber. Going further, the linearity condition ensures, together with continuity, ensures the value of $\mathcal{A}_n^\theta f(x)$ is constant on any connected set on which $\pi(x)$ is constant.

Definition 2.2. Let $(X(t))_{t\geq 0} = \big((X^1(t), \dots, X^n(t))\big)_{t \geq 0} \subset \mathbb{R}^n$ be a continuous square-integrable semi-martingale with initial condition $X(0) = x \in \mathbb{R}^n$, defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$. Then $(X(t))_{t\geq 0}$ is a solution to the Howitt–Warren martingale problem with drift β and characteristic measure ν if for any $i, j \in \{1, \dots, n\}$:

$$\langle X^i, X^j \rangle(t) = \int_0^t \mathbb{1}_{\{X^i(s) = X^j(s)\}} ds,$$

and the following process is a martingale with respect to the filtration generated by X, for every function $F \in D_n$,

$$F(X(t)) - \int_0^t \mathcal{A}_n^\theta F(X(s)) ds.$$

Note that the first condition implies that $\langle X^i, X^i\rangle(t) = t$, and it follows from the second condition and the definition of $\mathcal{A}_n^\theta$ that $X^i(t) - \beta t$ is a martingale for each i. Hence each coordinate must be a Brownian motion with drift β. The well posedness of this martingale problem and that the solutions do indeed form a consistent family of Feller processes is shown in [9].

3. The backwards equation

3.1. The generator of ordered sticky Brownian motions

Define the functions $F^{(i)} : \mathbb{R}^n \to \mathbb{R}$ by $F^{(i)}(x) = x_j$ where x_j is the ith largest coordinate of x, and $F : \mathbb{R}^n \to \overline{\mathbb{W}^n}$ by $F(x) := (F^{(1)}(x), \dots, F^{(n)}(x))$. Note that these functions are in D_n. Further, suppose $X = (X(t))_{t\geq 0}$ is a solution to the Howitt–Warren martingale problem in $\mathbb{R}^n$ with characteristic measure ν, drift $\beta = 0$ and initial condition $x \in \overline{\mathbb{W}^n}$. Define the process $Y = (Y(t))_{t\geq 0}$ by $Y(t) := F(X(t))$. Note that we defined Y from x started inside the Weyl chamber, so that $Y(0) = x$. The process, Y, lies entirely in the Weyl chamber $\overline{\mathbb{W}^n}$, which will allow us to apply the Bethe ansatz in the same way as Tracy and Widom in [18]. This section aims to identify the Kolmogorov backwards equation for Y and from it the invariant measure for Y.

Remark 3.1. Before talking about its Kolmogorov backward equation, we need to know Y is a Markov process. For this, we refer to Dynkin's criterion [15]. In particular, we only need to show that $\mathbb{E}_x[f \circ F(X(t))] = \mathbb{E}_{F(x)}[f(Y(t))]$ for every $x \in \mathbb{R}^n$ and every bounded measurable function $f : \overline{\mathbb{W}^n} \to \mathbb{R}$. The equality holds by definition for $x \in \overline{\mathbb{W}^n}$; for $x \in \mathbb{R}^n \setminus \overline{\mathbb{W}^n}$, we need to show that for any permutation $\sigma \in S_n$ $\sigma(X(t)) := (X^{\sigma(1)}(t), \dots, X^{\sigma(n)}(t))$ remains a solution to the same Howitt–Warren martingale problem, but with initial condition $\sigma(x)$. It is clear $\sigma(X)$ remains a continuous square-integrable semi-martingale and has initial condition $\sigma(x)$. Further, it is immediate that $\sigma(X)$ has the correct quadratic variations. Finally, because the function σ is continuous, linear, and maps Weyl chambers to Weyl chambers, i.e. $\{F \circ \sigma : F \in D_n\} = D_n$, the martingale problem is still satisfied by $\sigma(X)$. For each $x \in \mathbb{R}^n$, there exists a permutation $\sigma \in S_n$ such that $\sigma(x) \in \overline{\mathbb{W}^n}$, and by definition, $\sigma(x) = F(x)$. By uniqueness of solutions to the martingale problem, we have $\mathbb{E}_x[f \circ F(X(t))] = \mathbb{E}_x[f \circ F \circ \sigma^{-1} \circ \sigma(X(t))] = \mathbb{E}_{\sigma(x)}[f \circ F \circ \sigma^{-1}(X(t))]$ but clearly $F \circ \sigma^{-1} = F$. Hence, $\mathbb{E}_x[f \circ F(X(t))] = \mathbb{E}_{\sigma(x)}[f \circ F(X(t))] = \mathbb{E}_{F(x)}[f(Y(t))]$ as required; thus, $Y = F(X)$ is a Markov process.

We proceed by defining a subset of C^2 functions that is in the domain of the generator of Y; then, we will show that the action of the generator on this set is given by the Laplacian. In Proposition 4.2, we will show that the Bethe ansatz given in Theorem 1.2 is in this set of functions; this will be a major step in showing that it does give us the transition probabilities of Y.

Definition 3.2. Let $\mathcal{D}_\theta$ denote the set of functions $f \in C_0^2(\overline{\mathbb{W}^n})$ such that for any $a, b \in \{1, \dots, n\}$ with $a < b$, $x_a = x_b$ implies

$$\frac{1}{2} \sum_{\substack{a\leq i,j\leq b: \\ i\neq j}} \frac{\partial^2 f}{\partial x_i \partial x_j}(x) = \sum_{i=a}^{b} \frac{\partial f}{\partial x_i}(x) \sum_{k=0}^{b-a+1} \binom{b-a+1}{k} \theta(k, b-a+1-k)\, \mathrm{sign}(k-i+a-1), \tag{2}$$

where sign(0) is taken to be 1 here.

In order to show that the action of the generator of Y on the set $\mathcal{D}_\theta$ we first need to compute the quadratic covariation processes for Y. It turns out that the quadratic covariations for Y take the same form as those of X, which are prescribed by the martingale problem in Definition 2.2.

Lemma 3.3. *For each $t > 0$ the following equality holds almost surely.*

$$\langle Y^i, Y^j\rangle(t) = \int_0^t \mathbb{1}_{\{Y^i(s)=Y^j(s)\}} ds.$$

Proof. We will calculate the quadratic covariations for Y by referring to [8, Proposition 8], which provides a stochastic integral representation for the local martingale part of the semi-martingale given by applying a convex function to a local martingale. To apply this proposition, we will first decompose the function F into a composition of convex and concave functions. Denoting $P_i = \{A \subset \{1,\dots,n\} \mid |A| = n-i+1\}$, we can define $f_A : \mathbb{R}^n \to \mathbb{R}$ as $f_A(x) = \max_{a\in A} x_a$ and $g_i : \mathbb{R}^{P_i} \to \mathbb{R}$ as $g((y_A)_{A\in P_i}) = \min_{A\in P_i} y_A$. Then $F^{(i)}(x) = g_i\left((f_A(x))_{A\in P_i}\right)$, where f_A is a convex function and g_i is a concave function. Now we can apply [8, Proposition 8] to write the local martingale part of $F^{(i)}(X)$ in terms of a linear combination of stochastic integrals with respect to the X^i. In particular, we can write

$$f_A(X(t)) = f_A(x) + \sum_{a\in A}\int_0^t \mathbb{1}_{B_a^A}(X(s))dX^a(s) + C_t,$$

where C_t has finite variation, and $B_a^A = \{x : \min_{k\in A}\{k : \max_{j\in A} x_j = x_k\} = a\}$. Notice that for a fixed x and A there is only one a such that $\mathbb{1}_{B_a^A}(x)$ is non zero.

Now we put an ordering on the set P_i. The specific ordering does not matter; we just need to be able to minimise over the indices of elements in $\mathbb{R}^{P_i}$. Suppose $A, B \in P_i$ are distinct, define $(a_j)_{j=1}^{n-i+1}$ and $(b_j)_{j=1}^{n-i+1}$ as the elements of A and B respectively in increasing order. We say $A < B$ if for $l := \min\{k \in \mathbb{N} : b_k \neq a_k,\ 1 \le k \le n-i+1\}$ we have $a_l < b_l$; if instead $b_l < a_l$, then $B < A$. This ordering is a total ordering for P_i. Suppose Z is a semi-martingale taking values in $\mathbb{R}^{P_i}$ with decomposition $Z_t = Z_0 + M_t + K_t$, where M is a local martingale and K a process with finite variation. Then, using that for $y \in \mathbb{R}^{P_i}$ $-g_i(-y) = -\max_{A\in P_i}(-y_A)$ we have

$$-g_i(-Z_t) = -g_i(-Z_0) + \sum_{A\in P_i}\int_0^t \mathbb{1}_{B_A}(Z_s)dZ_s^A + D_t,$$

where D has finite variation and $B_A := \{z \in \mathbb{R}^{P_i} : \min\{B \in P_i : \inf_{C\in P_i} z_C = z_B\} = A\}$ with the minimum understood in terms of the ordering we just defined on P_i. That is, B_A is the subset of $z \in \mathbb{R}^{P_i}$ such that $z_A \le z_B$ for any $B \in P_i$, and for any $B < A$ (according to the ordering defined in the previous paragraph) $z_B > z_A$. Notice that for a fixed z there is only one A such that $\mathbb{1}_{B_A}(z)$ is non zero. The local martingale part of $Y^i = g_i((f_A(X))_{A\in P_i})$ is given by

$$\sum_{A\in P_i}\sum_{a\in A}\int_0^t \mathbb{1}_{B_a^A}(X(s))\mathbb{1}_{B_A}((f_C(X(s)))_{C\in P_i})dX^a(s).$$

Therefore, we can find the quadratic covariation processes.

$$\begin{aligned}&\langle Y^i, Y^j\rangle(t)\\ &= \sum_{\substack{A\in P_i,\\ B\in P_j}}\sum_{\substack{a\in A,\\ b\in B}}\int_0^t \mathbb{1}_{B_a^A}(X(s))\mathbb{1}_{B_A}((f_C(X(s)))_{C\in P_i})\mathbb{1}_{B_b^B}(X(s))\mathbb{1}_{B_B}((f_C(X(s)))_{C\in P_i})\mathbb{1}_{\{X^a(s)=X^b(s)\}}ds.\end{aligned}$$

Recall $f_C(x) = \max_{c\in C} x_c$, so that $\mathbb{1}_{B_A}((f_C(x))_{C\in P_i})$ is non-zero precisely when A is the subset of $\{1,\dots,n\}$ with indices corresponding to the first $i-1$ largest coordinates of $X(s)$ removed, call this set $A_i(X(s))$. Then, $\mathbb{1}_{B_a^{A_i(X(s))}}(X(s))$ is non zero if and only if a is the smallest element of $\{1,\dots,n\}$ such that $X^a(s)$ is equal to the ith largest coordinate of $X(s)$, i.e. $Y^i(s)$. Hence, we have the desired equality

$$\langle Y^i, Y^j\rangle(t) = \int_0^t \mathbb{1}_{\{Y^i(s)=Y^j(s)\}} ds. \quad \square$$

With the above lemma, we can determine the action of the generator of Y on the set $\mathcal{D}_\theta$.

Proposition 3.4. *Suppose $f \in \mathcal{D}_\theta$ then, denoting the generator of the process Y by $\mathcal{G}_\theta$ (in the sense of [14]), we have*

$$\mathcal{G}_\theta f = \frac{1}{2}\Delta f.$$

As a consequence of this proposition, we can derive the backwards equation for the process.

Proposition 3.5. *Suppose $g \in C^2(\mathbb{R}_{>0} \times \overline{\mathbb{W}^n})$, and $g(t,\cdot) \in \mathcal{D}_\theta$ for all $t > 0$. Further, suppose that g satisfies the PDE*

$$\frac{\partial g}{\partial t} = \frac{1}{2}\Delta g, \text{ for all } t>0,\ x \in \overline{\mathbb{W}^n}, \tag{3}$$

with the initial condition $g(0,x) = f(x)$ for some function $f \in C_b(\overline{\mathbb{W}^n})$. To be precise, we require that $g(t,\cdot) \to f$ uniformly as $t \to 0$. Then for each $t > 0$ $(g(t-s, Y(s)))_{s\in[0,t]}$ is a continuous local martingale.

Proof of Proposition 3.4. Since X solves the martingale problem (Definition 2.2), and $F^{(i)} \in D_n$, Y is a semi-martingale. For $f \in C_0^2(\overline{\mathbb{W}^n})$, Itô's formula gives

$$\mathbb{E}_x[f(Y(t))] = f(x) + \sum_{i=1}^n \mathbb{E}_x\left[\int_0^t \frac{\partial f}{\partial x_i}(Y(s))dY^i(s)\right] + \frac{1}{2}\sum_{i,j=1}^n \mathbb{E}_x\left[\int_0^t \frac{\partial^2 f}{\partial x_i \partial x_j}(Y(s))d\langle Y^i, Y^j\rangle(s)\right].$$

From Lemma 3.3, we know that

$$d\langle Y^i, Y^j\rangle(s) = \mathbb{1}_{\{Y^i(s)=Y^j(s)\}} ds.$$

Another consequence of the martingale problem is that for each i,

$$Y^i(t) - \int_0^t \mathcal{A}_n^\theta F^{(i)}(X(s))ds$$

is a martingale. Recall $f \in C_0^2(\overline{\mathbb{W}^n})$, thus $\frac{\partial f}{\partial x_i}$ is bounded on $\overline{\mathbb{W}^n}$ so that the stochastic integral with respect to the martingale part of Y is a true martingale. Thus, we can rewrite the expectation as

$$\begin{aligned}\mathbb{E}_x[f(Y(t))] = f(x) &+ \sum_{i=1}^n \mathbb{E}_x\left[\int_0^t \frac{\partial f}{\partial x_i}(Y(s))\mathcal{A}_n^\theta F^{(i)}(X(s))ds\right] \\ &+ \frac{1}{2}\sum_{i,j=1}^n \mathbb{E}_x\left[\int_0^t \frac{\partial^2 f}{\partial x_i \partial x_j}(Y(s))\mathbb{1}_{\{Y^i(s)=Y^j(s)\}} ds\right].\end{aligned} \tag{4}$$

By evaluating $\mathcal{A}_n^\theta F^{(i)}$, and then differentiating Eq. (4) in time, we can determine the generator of Y.

Let $x \in \mathbb{R}^n$ and denote $y = F(x) \in \overline{\mathbb{W}^n}$. We have

$$\mathcal{A}_n^\theta F^{(i)}(x) = \sum_{v \in \mathcal{V}(x)} \theta(v) \nabla_v F^{(i)}(x), \tag{5}$$

where ∇_v is the directional derivative in direction v. Recall $v \in \mathcal{V}(x)$ is defined by the disjoint subsets $I, J \subset \{1, \dots, n\}$ such that $I \cup J \in \pi(x)$, with $v_i = 1$ if $i \in I$, -1 if $i \in J$, and 0 otherwise. For each element, B, of the partition $\pi(x)$ there is a corresponding element, C, of the partition $\pi(y)$ such that for each $i \in B$ there is a $j_i \in C$ with $x_i = y_{j_i}$, and the j_i can be chosen so that the mapping $i \mapsto i_j$ is injective. Letting C denote the element of $\pi(y)$ corresponding to $I \cup J \in \pi(x)$, it is clear that if $i \notin C$ then $\nabla_v F^{(i)}(x) = 0$, and for $i \in C$ the derivative is either 1 or -1 depending only on the sizes of I and J. Since $y \in \overline{\mathbb{W}^n}$ there is an $a \in \{1, \dots, n\}$ and $m > 0$ such that $C = \{a, \dots, a + m - 1\}$. Hence, line (5) is equal to

$$\sum_{k=0}^{m} \binom{m}{k} \theta(k, m-k) \operatorname{sign}(k - i + a - 1),$$

where sign(0) is taken to be 1 here. In particular, this means that when y_i is distinct from all other coordinates, the above equals $\theta(1, 0) - \theta(0, 1) = \beta = 0$.

$$\begin{aligned} &\sum_{i=1}^{n} \frac{\partial f}{\partial y_i}(y) \mathcal{A}_n^\theta F^{(i)}(x) \\ &\qquad = \sum_{C \in \pi(y)} \sum_{i \in C} \frac{\partial f}{\partial y_i}(y) \sum_{k=0}^{|C|} \binom{|C|}{k} \theta(k, |C| - k) \operatorname{sign}(k - i + \inf C - 1), \end{aligned} \tag{6}$$

where each of the partial derivatives are evaluated at y. Putting (6) into (4) we can compute the limit.

$$\begin{aligned} &\lim_{t \to 0} \frac{1}{t} \left(\mathbb{E}_x \left[f(Y(t)) \right] - f(x) \right) \\ &= \lim_{t \to 0} \frac{1}{2t} \int_0^t \mathbb{E}_x [\Delta f(Y(s))] ds \\ &\quad + \frac{1}{t} \Bigg(\int_0^t \mathbb{E}_x \Bigg[\sum_{C \in \pi(y)} \sum_{i \in C} \frac{\partial f}{\partial y_i}(y) \sum_{k=0}^{|C|} \binom{|C|}{k} \theta(k, |C| - k) \operatorname{sign}(k - i + \inf C - 1) \Bigg] \\ &\qquad + \frac{1}{2} \sum_{i \neq j} \mathbb{E}_x \left[\frac{\partial^2 f}{\partial y_i \partial y_j} (Y(s)) \mathbb{1}_{\{Y^i(s) = Y^j(s)\}} \right] ds \Bigg). \end{aligned}$$

In particular, if we have $f \in \mathcal{D}_\theta$, then the term in the bracket cancels to 0 leaving only first term after the equality, whose limit we now calculate. Recall that $F : \mathbb{R}^n \to \overline{\mathbb{W}^n}$ is continuous and $Y(t) = F(X(t))$; since $\Delta f \in C_0(\overline{\mathbb{W}^n})$, we also have $\Delta f \circ F \in C_0(\mathbb{R}^n)$ (since $F(x) \to \infty$ as $|x| \to \infty$). Thus, the Feller property of X implies that $\frac{1}{2t} \int_0^t \mathbb{E}_x [\Delta f(Y(s))] ds$ converges uniformly to $\frac{1}{2} \Delta f(y)$ as $t \to 0$. Hence, for $f \in \mathcal{D}_\theta$,

$$\lim_{t \to 0} \frac{1}{t} \left(\mathbb{E}_x \left[f(Y(t)) \right] - f(x) \right) = \frac{1}{2} \Delta f(y), \text{ with respect to the uniform norm.}$$

Therefore, we have proved that if $f \in \mathcal{D}_\theta$, then it is in the domain of the generator of Y and $\mathcal{G}_\theta f = \frac{1}{2} \Delta f$. $\square$

We now apply Proposition 3.4 to prove Proposition 3.5.

Proof of Proposition 3.5. Applying Proposition 3.4, we see that for any function g satisfying the assumptions of the proposition, there is an adapted process $(M(u))_{u\in[0,t]}$ that is a continuous local martingale on $[0, s]$ for each $s < t$ such that

$$\begin{aligned} g(t-s, Y(s)) &= -\int_0^s \frac{\partial g}{\partial t}(t-u, Y(u))du + \int_0^s \Delta g(t-u, Y(u))du + M(s), \\ &= M(s). \end{aligned}$$

Now we just need to show that $M(s)$ is a local martingale on $[0, t]$. Since $g(t, \cdot) \to f$ uniformly as $t \to 0$, we have

$$|M(s)| = |g(t-s, Y(s))| \le \underbrace{\|g(t-s, \cdot) - f\|_\infty}_{\to 0 \text{ as } s\to t} + \|f\|_\infty.$$

Thus, there is an $\varepsilon > 0$ such that $M(s)$ is bounded on $[t-\varepsilon, t]$. Therefore, $M(s) - M(t-\varepsilon)$ is a martingale on $[t-\varepsilon, t]$, and it follows that $M(s)$ is a local martingale on $[0, t]$. Clearly, $M(0) = g(t, x)$ and $M(t) = f(Y(t))$ since

$$\begin{aligned} |M(s) - f(Y(t))| &= |g(t-s, Y(s)) - f(Y(t))| \\ &\le \|g(t-s, \cdot) - f\|_\infty + |f(Y(s)) - f(Y(t))|. \end{aligned}$$

The first term vanishes as $s \to t$ due to the uniform convergence of g to f, and the second almost surely due to the continuity of f and Y. □

Hence, we can find the transition probabilities of Y by looking for the Green's function for (3), providing solutions are sufficiently regular to make $g(t-s, Y(s))$ a true martingale. In general, it is not clear that (3) should have solutions in $\mathcal{D}_\theta$; it is not even clear whether $\mathcal{D}_\theta$ is non-trivial. In the rest of the paper, we focus on the case of a uniform characteristic measure: $\nu = \frac{1}{2}\theta \mathbb{1}_{[0,1]}dx$. Since we know ν, we can calculate the constants $\theta(k, l)$. By definition we have

$$\begin{aligned} \theta(k, l) &= \frac{\theta}{2}\int_0^1 x^{k-1}(1-x)^{l-1}dx, \\ &= \frac{\theta}{2}\frac{(l-1)!(k-1)!}{(k+l-1)!}. \end{aligned} \tag{7}$$

In this case, we also have $\theta(k, 0) = \theta(0, k)$ for all $k \in \mathbb{N}$. Hence, for the characteristic measure $\nu = \frac{1}{2}\theta \mathbb{1}_{[0,1]}dx$, (2) can be rewritten as

$$\frac{1}{2}\sum_{\substack{a\le i,j\le b:\\ i\ne j}} \frac{\partial^2 f}{\partial x_i \partial x_j}(x) = -\frac{\theta}{2}\sum_{i=a}^{b} \frac{\partial f}{\partial x_i}(x)c(a, b, i), \quad \text{whenever } x_a = x_b, \tag{8}$$

where the coefficients are defined

$$c(a, b, i) := \sum_{k=1}^{b-a} \frac{b-a+1}{k(b-a+1-k)} \operatorname{sign}(k-i+a-1). \tag{9}$$

In the following section, this particular form of the constants $\theta(k, l)$ will allow us to replace the conditions in line (8) with a simplified set of conditions, where each condition will only involve a single second derivative, rather than a sum. This replacement will simplify the combinatorics we need to do to show that the Bethe ansatz, given in Theorem 1.2, satisfies the conditions.

Remark 3.6. If we try to derive the Kolmogorov backwards equation for the original process X, we run into problems: the action of the generator of X within the set of C_0^2 functions does not determine the process. We can see this by considering a pair of sticky Brownian motions with parameter $\theta > 0$ X^1, X^2. We have by Itô's formula for all $f \in C_0^2(\mathbb{R}^2)$

$$\mathbb{E}_x[f(X^1(t), X^2(t))] = f(x_1, x_2) + \frac{1}{2}\int_0^t \mathbb{E}_x[\Delta f(X^1(s), X^2(s))]ds + \int_0^t \mathbb{E}_x[\mathbb{1}_{\{X^1(s)=X^2(s)\}}\frac{\partial^2 f}{\partial x_1 \partial x_2}(X^1(s), X^2(s))]ds,$$

so that f is in the domain of the generator if $\frac{\partial^2 f}{\partial x_1 \partial x_2}(x_1, x_2) = 0$ whenever $x_1 = x_2$. But, there is no dependence on the parameter θ in either the condition for f to be in the domain, or in the action of the generator on this subset of C_0^2 functions that are in the domain; thus, the generator restricted to this set cannot determine the law of the sticky Brownian motions.

3.2. Rearranging the boundary conditions

Henceforth, we consider the case where the characteristic measure is uniform, i.e. $\nu(dx) = \frac{\theta}{2}\mathbb{1}_{[0,1]}dx$. Let us first note that if we set $|C| = 2$ in (8), we see $f \in \mathcal{D}_\theta$ satisfies

$$\frac{\partial^2 f}{\partial x_a \partial x_{a+1}} = \theta\left(\frac{\partial f}{\partial x_{a+1}} - \frac{\partial f}{\partial x_a}\right), \quad \text{whenever } x_a = x_{a+1}.$$

In the next lemma, will show that we can replace the full boundary conditions with equivalent ones of the above form.

Lemma 3.7.

$$\mathcal{D}_\theta = \left\{ f \in C_0^2(\overline{\mathbb{W}^n}) | \textit{ for } \ 1 \le a < b \le n, \right. \\ \left. \textit{if } x_a = x_b \textit{ then } \frac{\theta}{b-a}\left(\frac{\partial f}{\partial x_b} - \frac{\partial f}{\partial x_a}\right) = \frac{\partial^2 f}{\partial x_a \partial x_b}\right\}.$$

Remark 3.8. Essentially we are solving for the second derivatives of functions in $\mathcal{D}_\theta$, given their first derivatives. Whilst this should be possible for any characteristic measure, our method relies on the special form of the parameters $\theta(k, l)$ in the case of the uniform characteristic measure.

Proof. Note that because we are in the Weyl chamber, $x_a = x_b$ implies $x_a = x_{a+1} = \cdots = x_b$. Thus, if $x_a = x_b$ then we also have $x_a = \cdots = x_b$. Using an inductive argument, we prove that the original conditions, (8), are equivalent to the new conditions. To begin, we prove that the new condition for $x_a = x_b$ is implied by the old conditions, when we also assume the new conditions for $x_c = x_d$ are satisfied for all $a \le c < d \le b$ such that $d - c < b - a$.

Hence, we assume that the boundary conditions (8) for $x_c = x_d$ are satisfied for all $a \le c < d \le b$ and that for all $a \le c < d \le b$ with $d - c < b - a$

$$\frac{\partial^2 f}{\partial x_c \partial x_d}(x) = \frac{\theta}{d-c}\left(\frac{\partial f}{\partial x_d}(x) - \frac{\partial f}{\partial x_c}(x)\right), \quad \text{if } x_c = \cdots = x_d. \tag{10}$$

Without loss of generality, we can relabel $(x_a, \ldots, x_b)$ as $(x_1, \ldots, x_m)$ for $m = b-a+1$. Then for $u \in \mathcal{D}_\theta$, we can rewrite the sum over mixed derivatives.

$$\frac{1}{2}\sum_{i\neq j}\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{1}{2}\sum_{\substack{i\neq j\\ i,j\neq m}}\frac{\partial^2 f}{\partial x_i \partial x_j} + \sum_{k=2}^{m-1}\frac{\partial^2 f}{\partial x_k \partial x_m} + \frac{\partial^2 f}{\partial x_1 \partial x_m}.$$

Using Eqs. (8) and (10), when $x_1 = \cdots = x_m$ we have the equality

$$\frac{\partial^2 f}{\partial x_1 \partial x_m} = -\frac{\theta}{2}\sum_{j=1}^{m}\frac{\partial f}{\partial y_j}(y)\sum_{k=1}^{m-1}\frac{m}{k(m-k)}\operatorname{sign}(k-j) - \sum_{i<j}\frac{\theta}{j-i}\left(\frac{\partial f}{\partial x_j} - \frac{\partial f}{\partial x_i}\right) + \frac{\theta}{m-1}\left(\frac{\partial f}{\partial x_m} - \frac{\partial f}{\partial x_1}\right). \tag{11}$$

We have the following equalities

$$\begin{aligned}\sum_{i<j}\frac{\theta}{j-i}\left(\frac{\partial f}{\partial x_j} - \frac{\partial f}{\partial x_i}\right) &= \sum_{j=2}^{m}\sum_{i=1}^{j-1}\frac{\theta}{j-i}\frac{\partial f}{\partial x_j} - \sum_{j=2}^{m-1}\sum_{i=1}^{j-1}\frac{\theta}{j-i}\frac{\partial f}{\partial x_i}\\ &= \sum_{j=2}^{m}\sum_{i=1}^{j-1}\frac{\theta}{j-i}\frac{\partial f}{\partial x_j} + \sum_{j=1}^{m-1}\sum_{i=j+1}^{m}\frac{\theta}{j-i}\frac{\partial f}{\partial x_j}\\ &= \theta\sum_{j=1}^{m}\frac{\partial f}{\partial x_j}\sum_{i\neq j}\frac{1}{j-i}.\end{aligned}$$

Therefore, the proof is finished if for each $j \in \{1, \ldots, n\}$,

$$\frac{1}{2}\sum_{k=1}^{m-1}\frac{m}{k(m-k)}\operatorname{sign}(k-j) + \sum_{i\neq j}\frac{1}{j-i} = 0.$$

Noting that we have $\frac{m}{k(m-k)} = \frac{1}{k} + \frac{1}{m-k}$, we get

$$\begin{aligned}\frac{1}{2}\sum_{k=1}^{m-1}\frac{m}{k(m-k)}\operatorname{sign}(k-j) &= \frac{1}{2}\sum_{k=j}^{m-j}\frac{m}{k(m-k)}\\ &= \frac{1}{2}\sum_{k=j}^{m-j}\left(\frac{1}{k} + \frac{1}{m-k}\right)\\ &= \sum_{k=j}^{m-j}\frac{1}{k}.\end{aligned} \tag{12}$$

In addition,

$$\begin{aligned}\sum_{i\neq j}\frac{1}{j-i} &= \sum_{i=1}^{j-1}\frac{1}{j-i} - \sum_{i=j+1}^{m}\frac{1}{i-j}\\ &= \sum_{k=1}^{j-1}\frac{1}{k} - \sum_{k=1}^{m-j}\frac{1}{k} = -\sum_{k=j}^{m-j}\frac{1}{k},\end{aligned}$$

with the convention that when $a < b$ the sum changes sign $\sum_{k=b}^{a} c_k = -\sum_{k=a}^{b} c_k$. Putting this into line (11), we see

$$\frac{\partial^2 f}{\partial x_1 \partial x_m} = \frac{\theta}{m-1}\left(\frac{\partial f}{\partial x_m} - \frac{\partial f}{\partial x_1}\right).$$

As noted previously, for $m = 2$ both conditions are equivalent; thus, by induction the old conditions imply the new conditions. Finally it is easy to see that, assuming the new conditions hold on $x_c = x_d$ for all $a \le c < d \le b$ and the old conditions hold on $x_c = x_d$ for all $a \le c < d \le b$ such that $d - c < b - a$, we can follow the above argument in reverse to prove the new conditions imply the old ones. Hence, the equivalence of the two sets of conditions is proved. □

As a consequence we can reframe Proposition 3.5 in terms of the new conditions.

Proposition 3.9. *Suppose that $g \in C_0^2(\mathbb{R}_{>0} \times \overline{\mathbb{W}^n})$ satisfying the PDE*

$$\begin{cases} \frac{\partial g}{\partial t} = \frac{1}{2}\Delta g, & \text{for } x \in \overline{\mathbb{W}^n}; \\ \frac{\partial^2 u}{\partial x_a \partial x_b} = \frac{\theta}{b-a}\left(\frac{\partial g}{\partial x_b} - \frac{\partial g}{\partial x_a}\right), & \text{if } b > a \text{ and } x_a = x_b, \end{cases}$$

with initial condition $g(t,\cdot) \to f$ uniformly as $t \to 0$, where $f \in C_b(\overline{\mathbb{W}^n})$. Then, we have $g(t,x) = \mathbb{E}_x[f(Y(t))]$.

This rearrangement will simplify the combinatorics required to show that we can solve the PDE with the Bethe ansatz.

3.3. Invariant measure

In this section, we prove an integration by parts formula for the generator of the ordered n-point motion of the Howitt–Warren flow with uniform characteristic measure. First, we introduce some useful notation.

Recall that for $\pi \in \Pi_n$, $\mathbb{W}_\pi^n$ consists of all $x \in \overline{\mathbb{W}^n}$ such that if i and j are in the same element of π, then $x_i = x_j$, and otherwise, $x_i \neq x_j$. Thus, by replacing the multiple indices in each block of π with a single index, as the corresponding x_i are all equal, we can map $\mathbb{W}_\pi^n$ into $\overline{\mathbb{W}^{|\pi|}}$, providing a natural bijection between $\mathbb{W}^{|\pi|}$ and $\mathbb{W}_\pi^n$ which we will denote $I^\pi : \mathbb{W}_\pi^n \to \mathbb{W}^{|\pi|}$. To be precise, let $\underline{\pi_i} = \min\{a \in \pi_i\}$ and set $I^\pi(x)_i = x_{\underline{\pi_i}}$. For a function $u : \overline{\mathbb{W}^n} \to \mathbb{R}$, denote by $u_\pi : \mathbb{W}^{|\pi|} \to \mathbb{R}$ the function defined by $u_\pi(x) := u \circ (I^\pi)^{-1}(x)$ for all $x \in \mathbb{W}^{|\pi|}$. For $u, v \in C^1(\overline{\mathbb{W}^n})$ such that the below integrals converge, we define

$$(u,v)_\theta := \sum_{\pi \in \Pi_n} \theta^{|\pi|-n} \left(\prod_{\pi_\iota \in \pi} \frac{1}{|\pi_\iota|}\right) \int_{\mathbb{W}^{|\pi|}} \nabla u_\pi \cdot \nabla v_\pi dx. \tag{13}$$

Now we can state the integration by parts formula for the measure $m_\theta^{(n)}$ from Definition 1.1.

Proposition 3.10. *Suppose $u \in \mathcal{D}_\theta$ and $v \in C_b^1(\overline{\mathbb{W}^n})$, such that there exists $a, c > 0$ such that $|\nabla u(x)| \le a e^{-c|x|}$. We have*

$$\int_{\overline{\mathbb{W}^n}} \Delta u(x) v(x) m_\theta^{(n)}(dx) = -(u,v)_\theta, \tag{14}$$

whenever the above integrals are finite.

Proof. Since $u \in \mathcal{D}_\theta$ we can relate Δu_π and $(\Delta u)_\pi$. Clearly, we have

$$\Delta u_\pi = \sum_{\pi_\iota \in \pi} \sum_{j,k \in \pi_\iota} \left(\frac{\partial^2 u}{\partial x_j \partial x_k} \right)_\pi .$$

Hence,

$$\Delta u_\pi - (\Delta u)_\pi = \sum_{\pi_\iota \in \pi} \sum_{\substack{j,k \in \pi_\iota \\ j \neq k}} \left(\frac{\partial^2 u}{\partial x_j \partial x_k} \right)_\pi .$$

The second sum on the right-hand side of the above expression is empty whenever $|\pi_\iota| = 1$, so we can exclude those terms from the first sum. Using Eqs. (8), (9) and the notations $\underline{\pi_\iota} := \inf \pi_\iota$, $\overline{\pi_\iota} := \sup \pi_\iota = |\pi_\iota| + \underline{\pi_\iota} - 1$, the previous expression is equal to

$$-\theta \sum_{\substack{\pi_\iota \in \pi: \\ |\pi_\iota| > 1}} \sum_{j \in \pi_\iota} \left(\frac{\partial u}{\partial x_j} \right)_\pi c(\underline{\pi_\iota}, \overline{\pi_\iota}, j).$$

Definition 1.1 allows us to rewrite the left hand side of Eq. (14) as

$$\sum_{\pi \in \Pi_n} \theta^{|\pi|-n} \left(\prod_{\pi_\iota \in \pi} \frac{1}{|\pi_\iota|} \right) \int_{\mathbb{W}^n_\pi} \Delta u(x) v(x) \lambda^\pi (dx). \tag{15}$$

By the definition of λ^π, given above Definition 1.1, we can rewrite the integral in the summand above in terms of a Lebesgue integral over a lower dimensional space; the result is the equality:

$$\begin{aligned} &\sum_{\pi \in \Pi_n} \theta^{|\pi|-n} \left(\prod_{\pi_\iota \in \pi} \frac{1}{|\pi_\iota|} \right) \int_{\mathbb{W}^{|\pi|}} (\Delta u)_\pi (x) v_\pi (x) dx \\ &= \sum_{\pi \in \Pi_n} \theta^{|\pi|-n} \left(\prod_{\pi_\iota \in \pi} \frac{1}{|\pi_\iota|} \right) \int_{\mathbb{W}^{|\pi|}} \left(\Delta u_\pi (x) + \theta \sum_{\substack{\pi_\iota \in \pi: \\ |\pi_\iota| > 1}} \sum_{j \in \pi_\iota} \left(\frac{\partial u}{\partial x_j} \right)_\pi c(\underline{\pi_\iota}, \overline{\pi_\iota}, j) \right) v_\pi (x) dx. \end{aligned} \tag{16}$$

Since the Weyl chamber has a piecewise smooth boundary, we can apply Green's identity to the first term in each integral. Applying the identity on $\overline{\mathbb{W}^{|\pi|}} \cap \{x \in \overline{\mathbb{W}^n} : \ |x| < R\}$ and then taking $R \to \infty$, the exponential bound on $|\nabla u|$ together with the boundedness of v ensures the only boundary term to survive in the limit will be the integral over $\partial \overline{\mathbb{W}^{|\pi|}}$.

The smooth part of the boundary of the Weyl chamber $\mathbb{W}^{|\pi|}$ can be written in terms of the disjoint union of $\mathbb{W}^{|\pi|}_{\tilde{\pi}}$ over the set $M_\pi := \{\tilde{\pi} \in \Pi_{|\pi|} : \ |\tilde{\pi}| = |\pi| - 1\}$. Note that if $|\pi| = 1$ this union is empty, and the boundary integral vanishes. Each $\tilde{\pi}$ in M_π consists of $|\pi| - 2$ singletons and one set $\{l, l+1\}$ for some $l \in \{1, \ldots, |\pi|\}$. Further, the outward unit normal on $\mathbb{W}^{|\pi|}_{\tilde{\pi}}$ is given by

$$\underline{n}(x)_r = \begin{cases} -\frac{1}{\sqrt{2}}, & \text{if } r = l; \\ \frac{1}{\sqrt{2}}, & \text{if } r = l+1; \\ 0, & \text{otherwise.} \end{cases}$$

Finally, the boundary measure is given by $\sum_{\tilde{\pi}\in M_\pi} \sqrt{2}\lambda^{\tilde{\pi}}$, so that the integral in (16) equals

$$\Bigg(\sum_{\tilde{\pi}\in M_\pi} \int_{\mathbb{W}^{|\pi|}_{\tilde{\pi}}} \left(\frac{\partial u_\pi}{\partial y_{l+1}} - \frac{\partial u_\pi}{\partial y_l}\right) v_\pi d\lambda^{\tilde{\pi}} - \int_{\mathbb{W}^{|\pi|}} \nabla u_\pi(x)\cdot\nabla v_\pi(x)dx$$
$$+\ \theta \sum_{\substack{\pi_\iota\in\pi:\\ |\pi_\iota|>1}} \sum_{j\in\pi_\iota} \int_{\mathbb{W}^{|\pi|}} \left(\frac{\partial u}{\partial x_j}\right)_\pi c(\underline{\pi_\iota}, \overline{\pi_\iota}, j) v_\pi(x)dx\Bigg),$$

where l depends on $\tilde{\pi}$ and is defined as above. We have written the partial derivatives of u_π with respect to $y \in \overline{\mathbb{W}^{|\pi|}}$ to emphasise the fact that u_π is a function on $\overline{\mathbb{W}^{|\pi|}}$ rather than $\overline{\mathbb{W}^n}$. Hence, to complete the proof it is enough to show that the first and third terms cancel when we put this expression back into (16). Rewriting the integrals with respect to the Lebesgue measure, the first term is equal to

$$\sum_{\tilde{\pi}\in M_\pi} \int_{\mathbb{W}^{|\tilde{\pi}|}} \left(\frac{\partial u_\pi}{\partial y_{l+1}} - \frac{\partial u_\pi}{\partial y_l}\right)_{\tilde{\pi}} (v_\pi)_{\tilde{\pi}} dx,$$

which is equal to

$$\sum_{\tilde{\pi}\in M_\pi} \int_{\mathbb{W}^{|\tilde{\pi}|}} \left(\left(\sum_{j\in\pi_{l+1}} \frac{\partial u}{\partial x_j}\right)_\pi - \left(\sum_{j\in\pi_l} \frac{\partial u}{\partial x_j}\right)_\pi\right)_{\tilde{\pi}} (x)\ (v_\pi)_{\tilde{\pi}}(x)dx.$$

Summing this over $\pi \in \Pi_n$ with the appropriate coefficients, we see that (15) is equal to

$$\sum_{\pi\in\Pi_n}\sum_{\tilde{\pi}\in M_\pi} \theta^{|\pi|-n} \left(\prod_{\pi_\iota\in\pi} \frac{1}{|\pi_\iota|}\right) \int_{\mathbb{W}^{|\pi|-1}} \sum_{j\in\pi_{l+1}\cup\pi_l} \left(\left(\frac{\partial u}{\partial x_j}\right)_\pi\right)_{\tilde{\pi}} (x)\,\mathrm{sign}(j-\underline{\pi_{l+1}})\ (v_\pi)_{\tilde{\pi}}(x)dx.$$

Notice that for each $\pi \in \Pi_n$ and $\tilde{\pi} \in M_\pi$ we can rewrite the summand in terms of a new partition, $\hat{\pi}$, formed from π by merging two adjacent blocks to form the $\pi_{l+1} \cup \pi_l$ block. Further, because the partitions are in Π_n, there are exactly $|\pi_{l+1} \cup \pi_l| - 1$ partitions that yield $\hat{\pi}$ by merging two blocks to form $\pi_{l+1} \cup \pi_l$. Rewriting the sum in terms of $\hat{\pi}$ we get

$$\sum_{\hat{\pi}\in\Pi_n} \theta^{|\hat{\pi}|+1-n} \left(\prod_{\hat{\pi}_\iota\in\hat{\pi}} \tfrac{1}{|\hat{\pi}_\iota|}\right) \sum_{\substack{\hat{\pi}_\iota\in\hat{\pi}:\\ |\hat{\pi}_\iota|>1}} \int_{\mathbb{W}^{|\hat{\pi}|}} \sum_{k=1}^{|\hat{\pi}_\iota|-1} \tfrac{|\hat{\pi}_\iota|}{k(|\hat{\pi}_\iota|-k)} \sum_{j\in\hat{\pi}_\iota} \left(\frac{\partial u}{\partial x_j}\right)_{\hat{\pi}} (x)\,\mathrm{sign}(j-\underline{\hat{\pi}_\iota}-k)\ v_{\hat{\pi}}(x)dx.$$

Here, the sum over j is over the partitions whose blocks have been merged to get $\hat{\pi}$, with k corresponding to the size of the lower block. The extra factor $\frac{|\hat{\pi}_\iota|}{k(|\hat{\pi}_\iota|-k)}$ is simply a correction to the product to write it in terms of $\hat{\pi}$ rather than the π partition whose blocks we merged.

Recalling that $\mathrm{sign}(0) = 1$ here, Eq. (9) yields that the above is precisely equal to

$$-\sum_{\pi\in\Pi_n} \theta^{|\pi|+1-n} \left(\prod_{\pi_\iota\in\pi} \frac{1}{|\pi_\iota|}\right) \sum_{\substack{\pi_\iota\in\pi:\\ |\pi_\iota|>1}} \sum_{j\in\pi_\iota} \int_{\mathbb{W}^{|\pi|}} \left(\frac{\partial u}{\partial x_j}\right)_\pi (x) c(\underline{\pi_\iota}, \overline{\pi_\iota}, j) v_\pi(x)dx.$$

Hence, (16) is equal to

$$-\sum_{\pi\in\Pi_n} \theta^{|\pi|-n} \prod_{\pi_\iota\in\pi} \frac{1}{|\pi_\iota|} \int_{\mathbb{W}^{|\pi|}} \nabla u_\pi(x)\cdot\nabla v_\pi(x)dx$$
$$= -(u, v)_\theta. \quad \square$$

Thus, if we denote by $L^2(m_\theta^{(n)})$ the L^2 space on $\overline{\mathbb{W}^n}$ with respect to the measure $m_\theta^{(n)}$ and the standard L^2 inner product, then the generator is symmetric on $\mathcal{D}_\theta \cap L^2(m_\theta^{(n)})$. This symmetry suggests the process is reversible with respect to this measure, but because our calculations are only done for $u \in \mathcal{D}_\theta$, and we do not know how rich the set $\mathcal{D}_\theta$ is, this is not enough for a proof.

Remark 3.11. We can also ask whether the same calculations can be done for choices of the characteristic measure other than uniform. A different choice of characteristic measure leads to a different invariant measure, and it turns out that if we suppose that this invariant measure takes the form $\sum_{\pi \in \Pi_n} c_\pi \lambda^\pi$ (with the coefficients c_π determined by the characteristic measure), as we would expect, then the only characteristic measure for which the above integration by parts argument works is the uniform measure. This suggests that the process is only reversible for a uniform characteristic measure.

If we take $v = 1$, the right-hand side of (14) vanishes, giving us the following helpful corollary.

Corollary 3.12. *For $u \in \mathcal{D}_\theta$ such that there are $a, c > 0$ with $|\nabla u(x)| \le ae^{-c|x|}$ we have*

$$\frac{1}{2} \int \Delta u(x) m_\theta^{(n)}(dx) = 0.$$

4. Bethe ansatz for sticky Brownian motions

In this section, we will introduce the Bethe ansatz and show that it solves the backwards equation with a delta initial condition, and thus, is the transition density for the process (with respect to the measure $m_\theta^{(n)}$). Using this we can prove that $m_\theta^{(n)}$ is the stationary measure, and that Y is reversible with respect to $m_\theta^{(n)}$. We are trying to find a solution to the PDE from Proposition 3.9, which we recall now. For each fixed $y \in \mathbb{W}^n$, θ some positive constant and with the initial condition $u_0(x, y) = \delta(x - y)$, where δ is the Dirac delta distribution, we wish to solve

$$\begin{cases} \frac{\partial u_t}{\partial t} = \frac{1}{2}\Delta u_t, & \text{for all } x \in \overline{\mathbb{W}^n}; \\ \theta\left(\frac{\partial u}{\partial x_b} - \frac{\partial u}{\partial x_a}\right) = (b-a)\frac{\partial^2 u}{\partial x_a \partial x_b}, & \text{when } x_a = x_b, \text{ for some } a < b. \end{cases} \tag{17}$$

The Bethe ansatz suggests that we can construct a solution for general $n \in \mathbb{N}$ by first considering the $n = 2$ problem. The main idea is to try to combine solutions with permuted coordinates in such a way that the boundary conditions are satisfied.

$$u_t(x, y) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^2} e^{-\frac{1}{2}t|k|^2} \left(A(k)e^{\mathrm{i}k\cdot(x-y)} + B(k)e^{\mathrm{i}k\cdot((x_2,x_1)-y)}\right) dk. \tag{18}$$

Notice that when $x_1 = x_2$, the exponential terms become equal. Thus, the boundary conditions will be satisfied if we have

$$(\mathrm{i}\theta(k_2 - k_1) + k_1k_2)\, A(k) + (\mathrm{i}\theta(k_1 - k_2) + k_1k_2)\, B(k) = 0.$$

It turns out that setting $A(k) = 1$ and $B(k) = \frac{\mathrm{i}\theta(k_2-k_1)+k_1k_2}{\mathrm{i}\theta(k_2-k_1)-k_1k_2}$ ensures the correct initial condition is satisfied. The Bethe ansatz then suggests that if we define

$$\mathrm{R}_{\alpha,\beta}(k) := \frac{\mathrm{i}\theta\left(k_\beta - k_\alpha\right) + k_\alpha k_\beta}{\mathrm{i}\theta\left(k_\beta - k_\alpha\right) - k_\alpha k_\beta}, \tag{19}$$

then the solution for general n is given by the following function.

$$u_t(x, y) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{-\frac{1}{2}t|k|^2} \sum_{\sigma \in S_n} e^{ik_\sigma \cdot (x - y_\sigma)} \prod_{\substack{\alpha<\beta: \\ \sigma(\beta)<\sigma(\alpha)}} \mathrm{R}_{\sigma(\beta),\sigma(\alpha)}(k) dk, \tag{20}$$

where S_n denotes the group of permutations on $\{1, \ldots, n\}$ and $k_\sigma = (k_{\sigma(1)}, \ldots, k_{\sigma(n)})$.

This construction ensures that boundary conditions where $b - a = 1$ are satisfied; however, the additional conditions in (17), corresponding to $b - a > 1$, have played no role. Therefore, we will need to do additional work to verify that the extra conditions, where $b - a > 1$, are indeed satisfied by (20).

Barraquand and Rychnovsky conjectured in [3] that the backwards equation for the system of sticky Brownian motions was the heat equation with the boundary conditions corresponding to $b - a = 1$ in (17), based on the Bethe ansatz answer for the system. It is important to note that for any other choice of characteristic measure ν with $\nu([0, 1]) = \frac{\theta}{2}$, the boundary conditions corresponding to $b-a = 1$ would be the same, so we do not expect these boundary conditions alone to give uniqueness of the PDE. However, we should note that it is possible that, under the additional regularity assumption that the solution is C^2 in space, the $b-a = 1$ boundary conditions do determine the solution, and the transition densities for all of the other systems of sticky Brownian motions are not C^2 in space. We do not know if this is the case or not.

It is clear that (20) satisfies the first condition in (17) and our choice of (19) guarantees the second condition holds when $b - a = 1$. However, when $b - a > 1$, it is not clear that it is still satisfied. Fortunately, and surprisingly, the second condition turns out to be satisfied in its entirety. Moreover, we can show the initial condition holds; hence, we obtain our main result, Theorem 1.2.

In the rest of the section, we shall prove Theorem 1.2. First, we show the boundary conditions are satisfied, and then the initial condition. To ensure we can perform the necessary exchanges of integral and derivative, we start by collecting some bounds on the Bethe ansatz.

4.1. Bounds for dominated convergence

Lemma 4.1. *For every $x \in \overline{\mathbb{W}^n}$ and $t > 0$ we have $u_t(x, \cdot) \in L^1(m_\theta^{(n)})$, where $u_t(x, \cdot)$ is defined as in (20). Further, for each $x \in \overline{\mathbb{W}^n}$ and $t > 0$, there exist $a, c > 0$ such that $|\nabla_y u_t(x, y)| \le ae^{-c|y|}$ for all $y \in \overline{\mathbb{W}^n}$. The same statement holds if we instead consider the x derivative and vary x with y being fixed. Similarly, for each $x \in \overline{\mathbb{W}^n}$ and $s > 0$ we can find $a, c > 0$ such that $|u_t(x, y)|,\ |\partial_t u_t(x, y)| \le ae^{-c|y|/\sqrt{t}}$ for all $t > s$ and $y \in \overline{\mathbb{W}^n}$.*

We leave the proof of this lemma to the end of Section 4.3, as it is a simplification of the methods used in that section.

The second part of the above lemma provides the necessary bounds to justify passing derivatives through the integral in $\int u_t(x, y) f(y) m_\theta^{(n)}(dy)$. Further, it is easy to see we can apply dominated convergence to find

$$\frac{\partial u_t}{\partial x_a} = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{-\frac{1}{2}t|k|^2} \sum_{\sigma \in S_n} ik_{\sigma(a)} e^{ik_\sigma \cdot (x - y_\sigma)} \prod_{\substack{\alpha<\beta: \\ \sigma(\beta)<\sigma(\alpha)}} \mathrm{R}_{\sigma(\beta),\sigma(\alpha)}(k) dk,$$

$$\frac{\partial^2 u_t}{\partial x_a \partial x_b} = -\frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{-\frac{1}{2}t|k|^2} \sum_{\sigma \in S_n} k_{\sigma(a)} k_{\sigma(b)} e^{ik_\sigma \cdot (x - y_\sigma)} \prod_{\substack{\alpha<\beta: \\ \sigma(\beta)<\sigma(\alpha)}} \mathrm{R}_{\sigma(\beta),\sigma(\alpha)}(k) dk,$$

$$\frac{\partial u_t}{\partial t} = -\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\frac{1}{2}|k|^2 e^{-\frac{1}{2}t|k|^2}\sum_{\sigma\in S_n} e^{\mathrm{i}k_\sigma\cdot(x-y_\sigma)}\prod_{\substack{\alpha<\beta:\\ \sigma(\beta)<\sigma(\alpha)}}\mathrm{R}_{\sigma(\beta),\sigma(\alpha)}(k)dk.$$

This allows us to not only confirm that $\int u_t(x,y)f(y)m_\theta^{(n)}(dy)$ solves the heat equation, but also to reduce the boundary conditions to a combinatorial problem.

4.2. Boundary conditions

Proposition 4.2.

$$\int u_t(x,y)f(y)m_\theta^{(n)}(dy)\in\mathcal{D}_\theta.$$

Due to Lemma 4.1, we know that $\int u_t(\cdot,y)f(y)m_\theta^{(n)}(dy)$ is in $C_0^2(\overline{\mathbb{W}^n})$. Hence, we just need to show it satisfies the correct boundary conditions for the PDE (17). The proof will follow from several lemmas. To begin, we derive the combinatorial identity that implies the above proposition.

Fix $a,b\in\{1,\dots,n\}$ with $a<b$, then for $t>0$ we can differentiate under the integral, as noted in the previous subsection, to see that the corresponding boundary condition is satisfied if for all $a<b$, $x_a=x_b$ implies

$$\sum_{\sigma\in S_n} e^{\mathrm{i}k_\sigma\cdot(x-y_\sigma)}\left(\mathrm{i}\theta(k_{\sigma(b)}-k_{\sigma(a)})+(b-a)k_{\sigma(b)}k_{\sigma(a)}\right)\prod_{\substack{\alpha<\beta:\\ \sigma(\beta)<\sigma(\alpha)}}\mathrm{R}_{\sigma(\beta),\sigma(\alpha)}(k)=0. \tag{21}$$

This can be simplified by splitting the summand into parts dependent on $\sigma(a),\dots,\sigma(b)$ and on the remaining values σ takes. Noting that we have $x_a=\cdots=x_b$

$$\prod_{c=a}^{b} e^{\mathrm{i}k_{\sigma(c)}(x_c-y_{\sigma(c)})}=\prod_{c=a}^{b} e^{\mathrm{i}k_{\sigma(c)}(x_a-y_{\sigma(c)})}=\prod_{\tilde{c}\in\{\sigma(a),\dots,\sigma(b)\}} e^{\mathrm{i}k_{\tilde{c}}(x_a-y_{\tilde{c}})}.$$

Notice that the last expression above depends only on the set $\{\sigma(a),\dots,\sigma(b)\}=\sigma(\{a,\dots,b\})$, and not the order of the values σ takes on $\{a,\dots,b\}$. Thus, the exponential factor of the summand in (21) only depends on $\sigma(\{a,\dots,b\})$ and not $\sigma(a),\dots,\sigma(b)$ themselves. Now we split the product

$$\prod_{\substack{\alpha<\beta:\\ \sigma(\beta)<\sigma(\alpha)}}\mathrm{R}_{\sigma(\beta),\sigma(\alpha)}(k)=\prod_{\substack{\alpha<a\le\beta\le b:\\ \sigma(\beta)<\sigma(\alpha)}}\mathrm{R}_{\sigma(\beta),\sigma(\alpha)}(k)\prod_{\substack{a\le\alpha\le b<\beta:\\ \sigma(\beta)<\sigma(\alpha)}}\mathrm{R}_{\sigma(\beta),\sigma(\alpha)}(k)$$
$$\prod_{\substack{\alpha,\beta\in\{a,\dots,b\}^c:\\ \alpha<\beta,\\ \sigma(\beta)<\sigma(\alpha)}}\mathrm{R}_{\sigma(\beta),\sigma(\alpha)}(k)\prod_{\substack{a\le\alpha<\beta\le b:\\ \sigma(\beta)<\sigma(\alpha)}}\mathrm{R}_{\sigma(\beta),\sigma(\alpha)}(k).$$

Note that $\mathrm{R}_{\sigma(\beta),\sigma(\alpha)}$ does not depend on α and β directly, but on $\sigma(\alpha)$ and $\sigma(\beta)$. Suppose, for a given permutation σ, $\mathrm{R}_{\sigma(\beta),\sigma(\alpha)}$ appears in the first product, then for any permutation τ with $\sigma(c)=\tau(c)$ for every $c\in\{a,\dots,b\}^c$, we have $\sigma(\beta)\in\{\sigma(a),\dots,\sigma(b)\}=\{\tau(a),\dots,\tau(b)\}$. Thus, there exists $\gamma\in\{a,\dots,b\}$ such that $\tau(\gamma)=\sigma(\beta)$, and so we have $\tau(\alpha)=\sigma(\alpha)>\sigma(\beta)=\tau(\gamma)$ and $\alpha<a\le\gamma$. Hence, $\mathrm{R}_{\tau(\gamma),\tau(\alpha)}=\mathrm{R}_{\sigma(\beta),\sigma(\alpha)}$ appears in the product for τ. This shows the first product does not depend on $\{\sigma(a),\dots,\sigma(b)\}$, and similarly the second does not either. The third product clearly does not depend on $\{\sigma(a),\dots,\sigma(b)\}$, leaving only the fourth product. Finally, we note that the fourth product does not depend on the values σ takes outside

$\{a, \dots, b\}$. Hence, we can split the sum into a sum over possibilities for the permutation outside $\{a, \dots, b\}$ and a sum over possibilities inside $\{a, \dots, b\}$. Pulling the parts depending only on the values of σ outside $\{a, \dots, b\}$ out of the second sum we see that it is sufficient for the second sum to vanish; thus, our condition will hold if

$$\sum_{\sigma \in S_m} \left(\mathrm{i}\theta(k_{\sigma(m)} - k_{\sigma(1)}) + (m-1) k_{\sigma(m)} k_{\sigma(1)} \right) \prod_{\substack{1 \le \alpha < \beta \le m: \\ \sigma(\beta) < \sigma(\alpha)}} \mathrm{R}_{\sigma(\beta),\sigma(\alpha)}(k) = 0,$$

where we have relabelled so that we sum over permutations in S_m, rather than over bijections between $\{a, \dots, b\}$ and $\{\sigma(a), \dots, \sigma(b)\}$. Hence, it is enough to prove the following

Proposition 4.3. *For every $n \in \mathbb{N}$ we have the identity*

$$\sum_{\sigma \in S_n} \left(\mathrm{i}\theta \left(k_{\sigma(n)} - k_{\sigma(1)}\right) + (n-1) k_{\sigma(n)} k_{\sigma(1)} \right) \prod_{\substack{\alpha < \beta: \\ \sigma(\beta) < \sigma(\alpha)}} R_{\sigma(\beta),\sigma(\alpha)} = 0. \tag{22}$$

We will prove this statement in several steps, first we will simplify Equality (22), by showing that it is equivalent to a polynomial equation. Then, we will show the resultant polynomial is alternating and is the product of the Vandermonde determinant and a symmetric polynomial. To finish, we will prove the symmetric polynomial to be 0, proving the proposition.

As discussed above, we begin simplifying the left hand side by pulling out the common denominator. Recalling (19)

$$\begin{aligned}
&\prod_{\sigma(\beta)<\sigma(\alpha)} \left(\mathrm{i}\theta(k_{\sigma(\alpha)} - k_{\sigma(\beta)}) - k_{\sigma(\beta)} k_{\sigma(\alpha)} \right) \prod_{\substack{\alpha < \beta: \\ \sigma(\beta) < \sigma(\alpha)}} \mathrm{R}_{\sigma(\beta),\sigma(\alpha)} \\
&= \prod_{\sigma(\beta)<\sigma(\alpha)} \left(\mathrm{i}\theta(k_{\sigma(\alpha)} - k_{\sigma(\beta)}) - k_{\sigma(\beta)} k_{\sigma(\alpha)} \right) \prod_{\substack{\alpha < \beta: \\ \sigma(\beta) < \sigma(\alpha)}} \frac{\mathrm{i}\theta(k_{\sigma(\alpha)} - k_{\sigma(\beta)}) + k_{\sigma(\alpha)} k_{\sigma(\beta)}}{\mathrm{i}\theta(k_{\sigma(\alpha)} - k_{\sigma(\beta)}) - k_{\sigma(\alpha)} k_{\sigma(\beta)}} \\
&= \prod_{\substack{\beta < \alpha: \\ \sigma(\beta) < \sigma(\alpha)}} \left(\mathrm{i}\theta(k_{\sigma(\alpha)} - k_{\sigma(\beta)}) - k_{\sigma(\alpha)} k_{\sigma(\beta)} \right) \prod_{\substack{\alpha < \beta: \\ \sigma(\beta) < \sigma(\alpha)}} \left(\mathrm{i}\theta(k_{\sigma(\alpha)} - k_{\sigma(\beta)}) + k_{\sigma(\alpha)} k_{\sigma(\beta)} \right).
\end{aligned}$$

Thus, multiplying both sides of (22) by $\prod_{\sigma(\beta)<\sigma(\alpha)} \left(\mathrm{i}\theta(k_{\sigma(\alpha)} - k_{\sigma(\beta)}) - k_{\sigma(\beta)} k_{\sigma(\alpha)} \right)$ (since permutations are bijections, this does not depend on σ) gives the equivalent equation

$$\begin{aligned}
&\sum_{\sigma \in S_n} \left(\mathrm{i}\theta \left(k_{\sigma(n)} - k_{\sigma(1)}\right) + (n-1) k_{\sigma(n)} k_{\sigma(1)} \right) \\
&\qquad \prod_{\substack{\beta < \alpha: \\ \sigma(\beta) < \sigma(\alpha)}} \left(\mathrm{i}\theta(k_{\sigma(\alpha)} - k_{\sigma(\beta)}) - k_{\sigma(\alpha)} k_{\sigma(\beta)} \right) \prod_{\substack{\alpha < \beta: \\ \sigma(\beta) < \sigma(\alpha)}} \left(\mathrm{i}\theta(k_{\sigma(\alpha)} - k_{\sigma(\beta)}) + k_{\sigma(\alpha)} k_{\sigma(\beta)} \right) = 0.
\end{aligned}$$

We can get rid of the $\mathrm{i}\theta$ factors by replacing each k_j with $\mathrm{i}\theta k_j$, since $\theta > 0$ this change of variables is invertible. This results in a factor of $(\mathrm{i}\theta)^{2(\binom{n}{2}+1)}$ appearing before the sum, which we can cancel off. We are left with the following equivalent equation, which we will prove for $k \in \mathbb{C}^n$.

$$\begin{aligned}
&\sum_{\sigma \in S_n} \left(\left(k_{\sigma(n)} - k_{\sigma(1)}\right) + (n-1) k_{\sigma(n)} k_{\sigma(1)} \right) \\
&\qquad \prod_{\substack{\alpha < \beta: \\ \sigma(\alpha) < \sigma(\beta)}} \left((k_{\sigma(\beta)} - k_{\sigma(\alpha)}) - k_{\sigma(\alpha)} k_{\sigma(\beta)} \right) \prod_{\substack{\alpha < \beta: \\ \sigma(\beta) < \sigma(\alpha)}} \left((k_{\sigma(\alpha)} - k_{\sigma(\beta)}) + k_{\sigma(\alpha)} k_{\sigma(\beta)} \right) = 0.
\end{aligned}$$

Now, we split the equation into two parts and simplify before showing they cancel. Making the following rearrangements, and defining the polynomial B

$$\prod_{\substack{\beta<\alpha:\\ \sigma(\beta)<\sigma(\alpha)}} \left((k_{\sigma(\alpha)} - k_{\sigma(\beta)}) - k_{\sigma(\alpha)}k_{\sigma(\beta)}\right) \prod_{\substack{\alpha<\beta:\\ \sigma(\beta)<\sigma(\alpha)}} \left((k_{\sigma(\alpha)} - k_{\sigma(\beta)}) + k_{\sigma(\alpha)}k_{\sigma(\beta)}\right). \tag{23}$$

$$= \prod_{\alpha<\beta} \operatorname{sign}(\sigma(\beta)-\sigma(\alpha)) \left(k_{\sigma(\beta)} - k_{\sigma(\alpha)} - k_{\sigma(\alpha)}k_{\sigma(\beta)}\right)$$

$$= \operatorname{sign}(\sigma) \prod_{\alpha<\beta} \left(k_{\sigma(\beta)} - k_{\sigma(\alpha)} - k_{\sigma(\alpha)}k_{\sigma(\beta)}\right) =: \operatorname{sign}(\sigma)B(k_\sigma).$$

We proceed by considering the expressions

$$\sum_{\sigma\in S_n} \operatorname{sign}(\sigma)(n-1)k_{\sigma(n)}k_{\sigma(1)}B(k_\sigma); \tag{24}$$

$$\sum_{\sigma\in S_n} \operatorname{sign}(\sigma)\left(k_{\sigma(n)} - k_{\sigma(1)}\right)B(k_\sigma). \tag{25}$$

It is clear that both (24) and (25) are polynomials in the k_j; we will now make some more general statements about polynomials of the form above; that is, given by an alternating sum of $f(k_\sigma)B(k_\sigma)$, for a polynomial function f.

It is clear that if $f : \mathbb{R}^n \to \mathbb{R}$ is a polynomial, then

$$\sum_{\sigma\in S_n} \operatorname{sign}(\sigma) f(k_\sigma)B(k_\sigma) \tag{26}$$

is an alternating polynomial. To see this suppose $a < b$ and we exchange k_a and k_b in the above expression. Then k_σ becomes $k_{(a,b)\circ\sigma}$ giving

$$\sum_{\sigma\in S_n} \operatorname{sign}(\sigma) f(k_{(a,b)\circ\sigma})B(k_{(a,b)\circ\sigma}) = -\sum_{\sigma\in S_n} \operatorname{sign}((a,b)\circ\sigma) f(k_{(a,b)\circ\sigma})B(k_{(a,b)\circ\sigma})$$

$$= -\sum_{\sigma\in S_n} \operatorname{sign}(\sigma) f(k_\sigma)B(k_\sigma).$$

In particular, whenever we have $k_\alpha = k_\beta$, for $\alpha \neq \beta$, any such polynomial must vanish. Hence we must be able to take the Vandermonde determinant, $\prod_{\alpha<\beta}(k_\beta - k_\alpha)$, out as a factor; since this is itself alternating, whatever remains must be symmetric. Thus for any polynomial $f : \mathbb{R}^n \to \mathbb{R}$, there exists a symmetric polynomial $g : \mathbb{R}^n \to \mathbb{R}$ such that

$$\sum_{\sigma\in S_n} \operatorname{sign}(\sigma) f(k_\sigma)B(k_\sigma) = g(k) \prod_{\alpha<\beta}(k_\beta - k_\alpha). \tag{27}$$

In the case of (24) and (25), the polynomial f is also multilinear (no variable appears with exponent higher than one), and depends only on two variables. In the following lemma, we will use these additional assumptions on f to show that the polynomial operators $H_{i,j}$, which map polynomials on $\mathbb{R}^n$ to polynomials on $\mathbb{R}^{n-2}$, defined by

$$H_{i,j} f(k) = f(k_1, \ldots, k_{i-1}, -1, k_{i+1}, \ldots, k_{j-1}, 1, k_{j+1}, \ldots, k_n)$$

reduce the degree of B by 2 when $i, j \in \{2, \ldots, n-1\}$.

Lemma 4.4. *Suppose $i, j \in \{2, \ldots, n-1\}$ with $i \neq j$ then $H_{i,j}B$ has degree at most $n-2$ in k_1 or in k_n.*

Proof. Recalling the formula for $B(k)$, (23), we have

$$H_{i,j}B(k) = \prod_{\substack{\alpha<\beta:\\ \alpha,\beta\neq i,j}} \left(k_\beta - k_\alpha - k_\alpha k_\beta\right) \prod_{\alpha\neq i,j} (\operatorname{sign}(j-\alpha)(1-k_\alpha) - k_\alpha)$$
$$\times \prod_{\alpha\neq i,j} (\operatorname{sign}(i-\alpha)(-1-k_\alpha) + k_\alpha)\ \ (2\operatorname{sign}(j-i)+1)\,.$$

The first product contains $(n-3)$ factors with k_1 and k_n each. The second and third contribute a factor of the form:

$$(1-2k_1)(-1)$$

for k_1, and a factor of the form

$$(-1)(2k_n+1) \tag{28}$$

for k_n. Leaving a total of $n-2$ factors involving k_1 and k_n each, which proves the statement. □

Now we can apply the above lemma to the expressions we are interested in.

Lemma 4.5. *If $f : \mathbb{R}^2 \to \mathbb{R}$ is a multilinear polynomial, then there exists constants C_0, C_1 and C_2 such that*

$$\sum_{\sigma\in S_n} \operatorname{sign}(\sigma) f(k_{\sigma(1)}, k_{\sigma(n)}) B(k_\sigma) = \prod_{\alpha<\beta} (k_\beta - k_\alpha)$$
$$\times \left(C_0 + \sum_{m=1}^{\lfloor n/2\rfloor} \left(C_1 \sum_{\alpha_1<\cdots<\alpha_{2m}} k_{\alpha_1}\ldots k_{\alpha_{2m}} + C_2 \sum_{\alpha_1<\cdots<\alpha_{2m+1}} k_{\alpha_1}\ldots k_{\alpha_{2m+1}} \right)\right).$$

Proof. The discussion preceding Lemma 4.4 shows that we at least have Eq. (27), and that g must be symmetric. To get the form given in the statement, we will show that g is also multilinear. This tells us we can write it as a linear combination of elementary symmetric polynomials; thus, in the last step, we only need to show that the coefficients in this combination are of the form given above. Both of these arguments proceed by considering the exponents of the variables k_j.

To show multilinearity, we note that for each k_j, $\prod_{\alpha<\beta}(k_\beta - k_\alpha)$ contains $n-1$ linear factors of k_j. Furthermore, each $B(k_\sigma)$ also contains exactly $n-1$ linear factors of k_j. But f is multilinear, so in the summand $\operatorname{sign}(\sigma) f(k_{\sigma(1)}, k_{\sigma(n)}) B(k_\sigma)$ the largest possible power of k_j is n. Hence, the largest possible power of k_j in $g(k)$ is 1. This holds for each j; thus, $g(k)$ is multilinear. Since $g(k)$ is multilinear and symmetric it must be of the form

$$g(k) = C_0 + \sum_{m=1}^{n} C_m \sum_{\alpha_1<\cdots<\alpha_m} k_{\alpha_1}\ldots k_{\alpha_m}.$$

Now we show that the constants C_m satisfy $C_1 = C_{2m+1}$ and $C_2 = C_{2m}$ for all $m \leq n/2$. We will use the operator $H_{n-1,n}$, as defined prior to Lemma 4.4, on the symmetric polynomial g. Since g is a symmetric polynomial, if one of its terms contains k_{n-1} but not k_n, there is a term otherwise equal where k_{n-1} is replaced with k_n, and vice versa. Thus, in $H_{n-1,n}g$ these terms will cancel leaving only the terms that contain both or neither; but, we can also see that

$k_{n-1}k_n$ will be evaluated as -1 when we apply $H_{n-1,n}$, so that we have the following.

$$H_{n-1,n}g(k) = C_0 + \sum_{m=1}^{n-2}(C_m - C_{m+2}) \sum_{\alpha_1<\cdots<\alpha_m<n-1} k_{\alpha_1}\ldots k_{\alpha_m}.$$

The next step is to consider the exponents on the left hand side of the expression appearing in Eq. (27) when we apply $H_{n-1,n}$ to it. We will use throughout that the operators $H_{i,j}$ are ring homomorphisms. We aim to show that $H_{n-1,n}g$ must be constant, which we will achieve by comparing degrees. First $B(k_\sigma)$ contains $(n-1)$ linear factors of each k_j, so the only way a k_j with exponent n can appear is if it also occurs in $f(k_{\sigma(1)}, k_{\sigma(n)})|_{k_n=1,k_{n-1}=-1}$; hence, only if $j = \sigma(n)$ or $\sigma(1)$. But the previous lemma tells us that $H_{n-1,n}B(k_\sigma)$ has degree at most $n-2$ in $k_{\sigma(1)}$ or in $k_{\sigma(n)}$. Thus, the highest possible power of any of the k_j that can appear when we apply $H_{n-1,n}$ to the left hand side of (27) is $n-1$. However, when we apply $H_{n-1,n}$ to the right hand side of (27), the product alone contains $n-1$ linear factors of each k_j, so $H_{n-1,n}g$ must be constant. Hence, $C_m = C_{m+2}$ for every $m > 0$, proving the result. □

Remark 4.6. Using the general formula for the sum of elementary symmetric polynomials on n variables, $\prod_{j=1}^n(1+x_j)$, together with the above lemma, gives us that for a multilinear polynomial $f : \mathbb{R}^2 \to \mathbb{R}$, there are constants C_m and D_m such that

$$\begin{aligned}
&\sum_{\sigma\in S_n} \operatorname{sign}(\sigma) f(k_{\sigma(1)}, k_{\sigma(n)}) B(k_\sigma)\\
&= \prod_{\alpha<\beta}(k_\beta - k_\alpha)\Bigg(C_0 + \frac{1}{2}C_1\left(\prod_{j=1}^n(1+k_j) + \prod_{j=1}^n(1-k_j) - 2\right)\\
&\qquad + \frac{1}{2}C_2\left(\prod_{j=1}^n(1+k_j) - \prod_{j=1}^n(1-k_j)\right)\Bigg)\\
&= \prod_{\alpha<\beta}(k_\beta - k_\alpha)\left(D_0 + D_1\prod_{j=1}^n(1+k_j) + D_2\prod_{j=1}^n(1-k_j)\right)\\
&= \det\left(k_i^{j-1}\right)\left(D_0 + D_1 \det\left((1+k_j)\delta_{ij}\right) + D_2 \det\left((1-k_j)\delta_{ij}\right)\right).
\end{aligned}$$

Now we can return to our original expressions (24) and (25). These two lemmas imply that we have constants $C_0^{(n)}, \tilde{C}_0^{(n)}, C_1, \tilde{C}_1^{(n)}, C_2^{(n)}$ and $\tilde{C}_2^{(n)}$ such that

$$\sum_{\sigma\in S_n} \operatorname{sign}(\sigma)(k_{\sigma(n)} - k_{\sigma(1)})B(k_\sigma) = \prod_{\alpha<\beta}(k_\beta - k_\alpha) \left(C_0^{(n)} + \sum_{m=1}^{\lfloor n/2\rfloor}\left(C_1^{(n)} \sum_{\alpha_1<\cdots<\alpha_{2m}} k_{\alpha_1}\ldots k_{\alpha_{2m}} + C_2^{(n)} \sum_{\alpha_1<\cdots<\alpha_{2m+1}} k_{\alpha_1}\ldots k_{\alpha_{2m+1}}\right)\right), \tag{29}$$

and

$$\sum_{\sigma\in S_n} \operatorname{sign}(\sigma)(n-1)k_{\sigma(n)}k_{\sigma(1)}B(k_\sigma) = \prod_{\alpha<\beta}(k_\beta - k_\alpha) \left(\tilde{C}_0^{(n)} + \sum_{m=1}^{\lfloor n/2\rfloor}\left(\tilde{C}_1^{(n)} \sum_{\alpha_1<\cdots<\alpha_{2m}} k_{\alpha_1}\ldots k_{\alpha_{2m}} + \tilde{C}_2^{(n)} \sum_{\alpha_1<\cdots<\alpha_{2m+1}} k_{\alpha_1}\ldots k_{\alpha_{2m+1}}\right)\right). \tag{30}$$

The next lemma provides a link between these constants for different values of n that will allow us to find their value inductively.

Lemma 4.7. *For $m = 0, 1, 2$ and $n \geq 3$, we have that $C_m^{(n)} = (n-1)C_m^{(n-1)}$ and $\tilde{C}_m^{(n)} = (n-1)\tilde{C}_m^{(n-1)}$.*

Proof. If we take $k_n = 0$ in (29), then we get the equality

$$\sum_{\sigma \in S_n} \operatorname{sign}(\sigma)(k_{\sigma(n)} - k_{\sigma(1)})B(k_\sigma)|_{k_n=0} = \prod_{\alpha=1}^{n-1}(-k_\alpha) \prod_{\alpha<\beta<n}(k_\beta - k_\alpha)$$
$$\left(C_0^{(n)} + \sum_{m=1}^{\lfloor n/2 \rfloor}\left(C_1^{(n)} \sum_{\alpha_1 < \cdots < \alpha_{2m} < n} k_{\alpha_1} \ldots k_{\alpha_{2m}} + C_2^{(n)} \sum_{\alpha_1 < \cdots < \alpha_{2m+1} < n} k_{\alpha_1} \ldots k_{\alpha_{2m+1}} \right)\right).$$

Recalling how we defined the polynomial B in line (23), we see that the left-hand side of the above equality is equal to

$$\sum_{\substack{\sigma \in S_n: \\ \sigma(1), \sigma(n) \neq n}} (k_{\sigma(n)} - k_{\sigma(1)}) \left(\prod_{\alpha=1}^{n-1}(-k_\alpha) \right) D_\sigma(k) \tag{31}$$
$$+ \sum_{\substack{\sigma \in S_n: \\ \sigma(1) = n}} k_{\sigma(n)} \left(\prod_{\alpha=1}^{n-1}(-k_\alpha) \right) D_\sigma(k) - \sum_{\substack{\sigma \in S_n: \\ \sigma(n) = n}} k_{\sigma(1)} \left(\prod_{\alpha=1}^{n-1}(-k_\alpha) \right) D_\sigma(k),$$

where we have used the shorthand

$$D_\sigma(k) = \prod_{\alpha<\beta<n} \left(k_\beta - k_\alpha - \operatorname{sign}\left(\sigma^{-1}(\beta) - \sigma^{-1}(\alpha)\right) k_\beta k_\alpha\right)$$
$$= \prod_{\substack{\alpha<\beta: \\ \alpha,\beta \neq \sigma^{-1}(n)}} \operatorname{sign}(\sigma(\beta) - \sigma(\alpha))(k_{\sigma(\beta)} - k_{\sigma(\alpha)} - k_{\sigma(\alpha)}k_{\sigma(\beta)}).$$

Note that $\sigma^{-1}(n)$ plays no role in the terms of the first sum on line (31). Thus, we can relabel each permutation, σ in that sum to a new one, $\tilde{\sigma}$, in S_{n-1} defined as follows

$$\tilde{\sigma}(\alpha) = \begin{cases} \sigma(\alpha), & \text{if } \alpha < \sigma^{-1}(n), \\ \sigma(\alpha+1), & \text{if } \alpha \geq \sigma^{-1}(n). \end{cases}$$

As an example of this relabelling, when $n = 4$, we would replace the permutations $\left(\begin{smallmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 3 & 2 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 4 & 3 \end{smallmatrix}\right)$ with $\left(\begin{smallmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{smallmatrix}\right)$ respectively. Note that each permutation in S_{n-1} occurs $n-2$ times (it is $n-2$, rather than n, because the sum excludes the cases where $\sigma^{-1}(n)$ is 1 or n). Importantly, this replacement does not change the value of sign $\left(\sigma^{-1}(\beta) - \sigma^{-1}(\alpha)\right)$, and thus does not change the summand. We can do the same with the two sums on the next line, these have no repeats as $\sigma^{-1}(n)$ must be 1 or n depending on the sum. Under this relabelling, $D_\sigma(k)$ becomes $\operatorname{sign}(\sigma)B(k_\sigma)$. Thus, we get

$$(n-2)\prod_{\alpha=1}^{n-1}(-k_\alpha) \sum_{\sigma \in S_{n-1}} \operatorname{sign}(\sigma)(k_{\sigma(n-1)} - k_{\sigma(1)})B(k_\sigma)$$

$$+\prod_{\alpha=1}^{n-1}(-k_\alpha)\sum_{\sigma\in S_{n-1}}\operatorname{sign}(\sigma)k_{\sigma(n-1)}B(k_\sigma)-\prod_{\alpha=1}^{n-1}(-k_\alpha)\sum_{\sigma\in S_{n-1}}\operatorname{sign}(\sigma)k_{\sigma(1)}B(k_\sigma),$$

which is equal to

$$(n-1)\prod_{\alpha=1}^{n-1}(-k_\alpha)\sum_{\sigma\in S_{n-1}}\operatorname{sign}(\sigma)(k_{\sigma(n-1)}-k_{\sigma(1)})B(k_\sigma).$$

Applying Eq. (29) in the $n-1$ case, we get that the above is equal to

$$(n-1)\prod_{\alpha=1}^{n-1}(-k_\alpha)\prod_{\alpha<\beta<n}(k_\beta-k_\alpha)\Bigg(C_0^{(n-1)}$$
$$+\sum_{m=1}^{\lfloor (n-1)/2\rfloor}\Bigg(C_1^{(n-1)}\sum_{\alpha_1<\cdots<\alpha_{2m}}k_{\alpha_1}...k_{\alpha_{2m}}+C_2^{(n-1)}\sum_{\alpha_1<\cdots<\alpha_{2m+1}}k_{\alpha_1}...k_{\alpha_{2m+1}}\Bigg)\Bigg).$$

Comparing coefficients with what we started with, it is clear that $C_m^{(n)}=(n-1)C_m^{(n-1)}$ for $m=0,1,2$ as required.

The proof for the $\tilde{C}_m^{(n)}$ follows the same lines as above. □

Finally, we just need to establish the values $C_0^{(2)}$, $C_1^{(2)}$, $C_2^{(2)}$, $\tilde{C}_0^{(2)}$, $\tilde{C}_1^{(2)}$ and $\tilde{C}_2^{(2)}$ to find all the remaining values by induction. Eq. (24) in the $n=2$ case is

$$k_1k_2(k_2-k_1-k_1k_2)+k_1k_2(k_2-k_1+k_1k_2)=2(k_2-k_1)k_1k_2.$$

Thus $C_0^{(2)}=0$, $C_1^{(2)}=0$ and $C_2^{(2)}=2$. Combining the two lemmas above this implies for $m=0,1$ $C_0^{(n)}=0$ for every n, and $C_2^{(n)}=2(n-1)!$ for every n. (25) in the $n=2$ case is

$$(k_2-k_1)(k_2-k_1-k_1k_2)+(k_1-k_2)(k_2-k_1+k_1k_2)=-2(k_2-k_1)k_1k_2.$$

Thus $\tilde{C}_0^{(2)}=0$, $\tilde{C}_1^{(2)}=0$ and $\tilde{C}_2^{(2)}=-2$. Combining the two lemmas above this implies for $m=0,1$ $\tilde{C}_m^{(n)}=0$ for every n, and $\tilde{C}_2^{(n)}=-2(n-1)!$ for every n. In particular, this shows that the sum of (24) and (25) is 0, proving Proposition 4.3. As a consequence, we have proved Proposition 4.2, concluding this subsection.

4.3. Initial condition

Proposition 4.8. *For any bounded Lipschitz continuous function $f:\overline{\mathbb{W}^n}\to\mathbb{R}$, we have*

$$\int u_t(\cdot,y)f(y)m_\theta^{(n)}(dy)\to f \text{ uniformly, as } t\to 0,$$

where the definitions of $m_\theta^{(n)}$ and u_t are given in Definition 1.1 and Eq. (20) respectively.

The proof of this proposition will be the focus of the rest of the section. We begin by proving two useful properties of u_t, namely, that it integrates to 1 under $m_\theta^{(n)}$, and that it is symmetric: $u_t(x,y)=u_t(y,x)$.

Lemma 4.9.

$$\int u_t(x,y)m_\theta^{(n)}(dy)=1 \quad \text{for all } x\in\overline{\mathbb{W}^n},\ t>0.$$

Proof. Lemma 4.1 allows us to calculate the time derivative by passing it through the integral:

$$\frac{\partial}{\partial t}\int u_t(x,y)m_\theta^{(n)}(dy) = \int \frac{1}{2}\Delta u_t(x,y)m_\theta^{(n)}(dy)$$
$$= 0.$$

The first equality is clear from the definition of u. The second equality follows from Corollary 3.12 and Lemma 4.1. This shows the integral is constant, to finish we shall show convergence to 1 as $t \to \infty$. Scaling y by $t^{\frac{1}{2}}$, and then k by $t^{-\frac{1}{2}}$, we see the following equalities.

$$\int u_t(x,y)m_\theta^{(n)}(dy) = \sum_{\pi\in\Pi_n}\theta^{|\pi|-n}\left(\prod_{\pi_\iota\in\pi}\frac{1}{|\pi_\iota|}\right)\int t^{\frac{|\pi|}{2}}u_t(x,\sqrt{t}y)\lambda^\pi(dy)$$
$$= \sum_{\pi\in\Pi_n}\theta^{|\pi|-n}\left(\prod_{\pi_\iota\in\pi}\frac{1}{|\pi_\iota|}\right)\frac{1}{(2\pi)^n t^{\frac{1}{2}(n-|\pi|)}}\int\int_{\mathbb{R}^n} e^{-\frac{1}{2}|k|^2}\sum_{\sigma\in S_n}e^{\mathrm{i}k_\sigma\cdot(x/\sqrt{t}-y_\sigma)}$$
$$\prod_{\substack{\alpha<\beta:\\ \sigma(\beta)<\sigma(\alpha)}}\mathrm{R}_{\sigma(\beta),\sigma(\alpha)}(k/\sqrt{t})dk\ \lambda^\pi(dy).$$

We can calculate the limit as $t \to \infty$ above by first looking at the right-hand side of the first line, and applying dominated convergence to pass the limit through the first integral; this is justified by Lemma 4.1. We can then do the rescaling of the integral in the k variable to get to the second line; we can calculate the limit of the k integral by applying dominated convergence again; to find the limit of the integrand, note that $\mathrm{R}_{\sigma(\beta),\sigma(\alpha)}(\frac{k}{\sqrt{t}}) \to 1$ as $t \to \infty$ for almost every k. Now we can calculate the limits of the summand above, all terms with $|\pi| < n$ in the sum over partitions vanish in the limit, because of the $t^{\frac{1}{2}(n-|\pi|)}$ that appears in the denominator, leaving only the partition consisting exclusively of singletons; for this partition, λ^π is just the Lebesgue measure on the Weyl chamber. Thus, we have

$$\int u_t(x,y)m_\theta^{(n)}(dy) = \frac{n!}{(2\pi)^n}\int_{\mathbb{W}^n}\int_{\mathbb{R}^n} e^{-\frac{1}{2}|k|^2-\mathrm{i}k\cdot y}dkdy$$
$$= 1.$$

The $n!$ comes from the sum over permutations, the resulting integral in k is just the Fourier transform of a Gaussian; hence, the integral over the Weyl chamber is easily calculated. □

Now we can write

$$\int u_t(x,y)f(y)m_\theta^{(n)}(dy) - f(x) = \int u_t(x,y)\left(f(y)-f(x)\right)m_\theta^{(n)}(dy).$$

It follows directly from the definition of $m_\theta^{(n)}$ that

$$\left|\int u_t(x,y)\left(f(y)-f(x)\right)m_\theta^{(n)}(dy)\right|$$
$$\leq \sum_{\pi\in\Pi_n}\theta^{|\pi|-n}\prod_{\pi_i\in\pi}\frac{1}{|\pi_i|}\left|\int u_t(x,y)\left(f(y)-f(x)\right)\lambda^\pi(dy)\right|. \tag{32}$$

Thus, we can restrict our attentions to the integral with respect to λ^π for a fixed $\pi \in \Pi_n$.

Let us briefly outline the proof. We wish to estimate $u_t(x,y)$, which we recall from 1.2, is given as a sum of Fourier integrals, indexed by the permutation group on n elements. For each permutation, σ, we can try to estimate the Fourier integral by following the same idea used to calculate the Fourier transform of the Gaussian density; that is, we rely on completing the square: $-\frac{1}{2}tk^2+\mathrm{i}kx = -\frac{1}{2t}(tk-\mathrm{i}x)^2-\frac{1}{2t}x^2$ and then making the relevant contour shift to allow us to rewrite the Fourier integral as a Gaussian integral multiplied by a Gaussian density. This step is complicated by the presence of poles in the integrand of the integral defining $u_t(x, y)$. However, because we are in the Weyl chamber, we do not need to shift the contour of every integration variable, and indeed, we will instead determine a subset of the indices for which the poles are restricted to one complex half-plane, allowing us, for certain valued of x and y, to make the contour shifts without encountering any poles. Together, these contour shifts will prove to be sufficient to get the desired bound, Proposition 4.14.

Another complication is caused by the dimension of space we are integrating over, that is, the size of the partition π. Coarser partitions will require finer control over $u_t(x, y)$, and to achieve this we will need to show that, on the parts of the boundary corresponding to these partitions, there is cancellation occurring between terms in the sum over permutations defining $u_t(x, y)$. We will achieve this with a combinatorial formula we state in Lemma 4.11. The estimate we describe above, will actually be on the combined terms between which there is cancellation.

In the final step, we combine the estimate on the Fourier integral defining $u_t(x, y)$ with the Lipschitz property for f to derive the desired uniform convergence. This requires bounding of the contribution from $\overline{\mathbb{W}^\pi}$ to $\int |u_t(x, y)|m_\theta^{(n)}(dy)$ and some care in considering what happens when x is near, but not in, $\overline{\mathbb{W}^\pi}$ to ensure we get uniform convergence.

We start with a fact that will allow us to make a useful rearrangement: $u_t(x, y)$ is symmetric under swaps of x and y.

Lemma 4.10. *For every $x, y \in \overline{\mathbb{W}^n}$ and $t > 0$*

$$u_t(x, y) = u_t(y, x).$$

Proof. Recall that u is defined in (20) as

$$u_t(x, y) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{-\frac{1}{2}t|k|^2}\sum_{\sigma\in S_n} e^{\mathrm{i}k_\sigma\cdot(x-y_\sigma)}\prod_{\substack{\alpha<\beta:\\ \sigma(\alpha)>\sigma(\beta)}}\frac{\mathrm{i}\theta(k_{\sigma(\alpha)}-k_{\sigma(\beta)})+k_{\sigma(\alpha)}k_{\sigma(\beta)}}{\mathrm{i}\theta(k_{\sigma(\alpha)}-k_{\sigma(\beta)})-k_{\sigma(\alpha)}k_{\sigma(\beta)}}dk.$$

If we first take the sum outside the integral, then perform the change of variables in the k integral, $k \to -k_{\sigma^{-1}}$, this becomes

$$\frac{1}{(2\pi)^n}\sum_{\sigma\in S_n}\int_{\mathbb{R}^n} e^{-\frac{1}{2}t|k|^2+\mathrm{i}k_{\sigma^{-1}}\cdot(x_{\sigma^{-1}}-y)}\prod_{\substack{\alpha<\beta:\\ \sigma(\alpha)>\sigma(\beta)}}\frac{\mathrm{i}\theta(k_\beta-k_\alpha)+k_\alpha k_\beta}{\mathrm{i}\theta(k_\beta-k_\alpha)-k_\alpha k_\beta}dk.$$

Notice that we can relabel the product as follows

$$\prod_{\substack{\alpha<\beta:\\ \sigma(\alpha)>\sigma(\beta)}}\frac{\mathrm{i}\theta(k_\beta-k_\alpha)+k_\alpha k_\beta}{\mathrm{i}\theta(k_\beta-k_\alpha)-k_\alpha k_\beta} = \prod_{\substack{\alpha<\beta:\\ \sigma^{-1}(\alpha)>\sigma^{-1}(\beta)}}\frac{\mathrm{i}\theta(k_{\sigma^{-1}(\alpha)}-k_{\sigma^{-1}(\beta)})+k_{\sigma^{-1}(\alpha)}k_{\sigma^{-1}(\beta)}}{\mathrm{i}\theta(k_{\sigma^{-1}(\alpha)}-k_{\sigma^{-1}(\beta)})-k_{\sigma^{-1}(\alpha)}k_{\sigma^{-1}(\beta)}}.$$

Hence, by relabelling the sum to be over $\sigma^{-1} \in S_n$, we see that we get $u_t(y, x)$ as desired. □

Now, we proceed with the proof of Proposition 4.8. We start by writing $u_t(x, y)$ in (32) in terms of a sum over permutations (as in Theorem 1.2), and then combining those terms in this sum which cancel as $t \to 0$. That is, we can rewrite the summand of (32) (ignoring the constants) as

$$\left| \int u_t(y,x)\,(f(y)-f(x))\,\lambda^\pi(dy) \right|$$
$$= \left| \int \int_{\mathbb{R}^n} e^{-\frac{1}{2}t|k|^2} \sum_{\sigma \in S_n} e^{\mathrm{i}k_\sigma \cdot (y-x_\sigma)} \prod_{\substack{\alpha<\beta:\\ \sigma(\alpha)>\sigma(\beta)}} \mathrm{R}_{\sigma(\beta),\sigma(\alpha)}(k)\, dk\, (f(y)-f(x))\, \frac{\lambda^\pi(dy)}{(2\pi)^n} \right|. \tag{33}$$

For a partition $\pi \in \Pi_n$ and permutation $\sigma \in S_n$, define the set of ordered pairs

$$\sigma(\pi) := \{(\pi_1, \sigma(\pi_1)), \ldots, (\pi_{|\pi|}, \sigma(\pi_{|\pi|}))\},$$

where $\sigma(A)$ denotes the image of A under σ. We can rewrite the sum appearing on line (33) as follows

$$\sum_{\substack{\tau \in S_n:\\ \tau|_{\pi_\iota} \text{ is increasing } \forall \iota}} \sum_{\substack{\sigma \in S_n:\\ \sigma(\pi)=\tau(\pi)}} e^{\mathrm{i}k_\sigma \cdot (y-x_\sigma)} \prod_{\substack{\alpha<\beta:\\ \sigma(\alpha)>\sigma(\beta)}} \mathrm{R}_{\sigma(\beta),\sigma(\alpha)}(k).$$

Let $\overline{\pi_\iota} := \sup \pi_\iota$, and $\underline{\pi_\iota} := \sup \pi_\iota$. Let us consider $e^{\mathrm{i}k_\sigma \cdot (y-x_\sigma)} = e^{-\mathrm{i}k\cdot x} \prod_{j=1}^n e^{\mathrm{i}k_{\tau(j)} y_j}$. We know that for each $\pi_\iota \in \pi$, $\alpha, \beta \in \pi_\iota$ implies $y_\alpha = y_\beta$ λ^π-a.e. Hence, $\prod_{j=1}^n e^{\mathrm{i}k_{\sigma(j)} y_j} = \prod_{\pi_\iota \in \pi} \prod_{\alpha \in \pi_\iota} e^{\mathrm{i}k_{\sigma(\alpha)} y_{\overline{\pi_\iota}}}$ λ^π-a.e. But since $\sigma(\pi) = \tau(\pi)$, this is just equal to $\prod_{\pi_\iota \in \pi} \prod_{\alpha \in \pi_\iota} e^{\mathrm{i}k_{\tau(\alpha)} y_{\overline{\pi_\iota}}}$, which equals $e^{\mathrm{i}k_\tau \cdot y}$. Hence, we can pull the exponential out of the second sum to make the previous expression equal λ^π-a.e. to

$$\sum_{\substack{\tau \in S_n:\\ \tau|_{\pi_\iota} \text{ is increasing } \forall \iota}} e^{\mathrm{i}k_\tau \cdot (y-x_\tau)} \sum_{\substack{\sigma \in S_n:\\ \sigma(\pi)=\tau(\pi)}} \prod_{\substack{\alpha<\beta:\\ \sigma(\alpha)>\sigma(\beta)}} \mathrm{R}_{\sigma(\beta),\sigma(\alpha)}(k).$$

Now, consider the product in the expression above; in particular, we can show that if α and β are in different elements of π, then the appearance of $\mathrm{R}_{\sigma(\beta),\sigma(\alpha)}(k)$ in the product depends only on τ, and not on σ. Suppose $\alpha < \beta$ are in different elements of π and that $\sigma(\beta) < \sigma(\alpha)$. Since π is an ordered partition, there exists $\iota < j$ such that $\alpha \in \pi_\iota$ and $\beta \in \pi_j$. But $\sigma(\pi) = \tau(\pi)$, so there must exist $\gamma \in \pi_\iota$ and $\delta \in \pi_j$ (thus $\gamma < \delta$) such that $\tau(\gamma) = \sigma(\alpha) > \sigma(\beta) = \tau(\delta)$. Hence, the appearance of $\mathrm{R}_{\sigma(\beta),\sigma(\alpha)}(k)$ in the product depends only on τ, as desired. Similarly, we can go in the other direction, so that if α and β are in different elements of π, then $(\sigma(\beta), \sigma(\alpha))$ is an inversion for σ if and only if it is an inversion for τ (that is, if α and β are in different elements of π, then $\alpha < \beta$ with $\sigma(\beta) < \sigma(\alpha)$ occurs if and only if $\tau^{-1}(\sigma(\alpha)) < \tau^{-1}(\sigma(\beta))$ with $\sigma(\beta) < \sigma(\alpha)$). Hence, we can split off the part of the product where α and β are in different elements of π and rewrite it entirely in terms of τ. Thus, the previous expression is equal to

$$\sum_{\substack{\tau \in S_n:\\ \tau|_{\pi_\iota} \text{ is increasing } \forall \iota}} e^{\mathrm{i}k_\tau \cdot (y-x_\tau)} \left(\prod_{\iota<j} \prod_{\substack{\alpha \in \pi_\iota,\ \beta \in \pi_j:\\ \tau(\beta)<\tau(\alpha)}} \mathrm{R}_{\tau(\beta),\tau(\alpha)}(k) \right) \sum_{\substack{\sigma \in S_n:\\ \sigma(\pi)=\tau(\pi)}} A^o_{\sigma,\pi}(k), \tag{34}$$

where $A^o_{\sigma,\pi}$ is shorthand for the summand of the second sum and is defined as follows.

$$A^o_{\sigma,\pi}(k) := \prod_{\pi_\iota \in \pi} \prod_{\substack{\alpha<\beta:\\ \sigma(\alpha)>\sigma(\beta),\\ \alpha,\beta \in \pi_\iota}} \mathrm{R}_{\sigma(\beta),\sigma(\alpha)}(k).$$

We can now combine all the terms in the second sum in (34), using the formula in the following lemma.

Lemma 4.11. *Suppose $m \in \mathbb{N}$ and $\theta > 0$, then for all $k \in \mathbb{R}^m$ such that $k_\alpha \neq 0$ for all $\alpha \in \{1, \dots, m\}$,*

$$\sum_{\sigma \in S_m} \prod_{\substack{\alpha<\beta: \\ \sigma(\alpha)>\sigma(\beta)}} R_{\sigma(\beta),\sigma(\alpha)}(k) = m! \prod_{\alpha<\beta} \frac{\mathrm{i}\theta(k_\beta - k_\alpha)}{\mathrm{i}\theta(k_\beta - k_\alpha) - k_\alpha k_\beta}.$$

Proof. First, we prove the following equality holds for all $\xi \in \mathbb{C}^m$:

$$\sum_{\sigma \in S_m} \left(\prod_{\substack{\alpha<\beta: \\ \sigma(\beta)<\sigma(\alpha)}} (-1) \right) \prod_{\alpha<\beta} \left(\xi_{\sigma(\alpha)} - \xi_{\sigma(\beta)} - 1 \right) = m! \prod_{\alpha<\beta} (\xi_\alpha - \xi_\beta). \tag{35}$$

It is clear that the left hand side is a degree $\binom{m}{2}$ polynomial, which we shall denote $P(\xi)$. Thus, if we can prove that $P(\xi)$ is also alternating, it must be a constant multiple of the right hand side. We then just need to check the constant to finish the proof.

To prove the left hand side is alternating it is enough to consider swaps of consecutive variables, e.g. ξ_j and ξ_{j+1} for some $j \in \{1, \dots, m-1\}$. Let $s_j = (j, j+1) \in S_n$, i.e. the permutation that swaps j and $j+1$ leaving everything else fixed. Clearly, for all $\sigma \in S_n$

$$\prod_{\substack{\alpha<\beta: \\ \sigma(\beta)<\sigma(\alpha)}} (-1) = - \prod_{\substack{\alpha<\beta: \\ \sigma\circ s_j(\beta)<\sigma\circ s_j(\alpha)}} (-1). \tag{36}$$

It follows by relabelling the sum in its definition on the left-hand side of (35) that $P(\xi_{s_j}) = -P(\xi)$. Hence, P is an alternating polynomial and there is a $c \in \mathbb{R}$ such that

$$\sum_{\sigma \in S_m} \left(\prod_{\substack{\alpha<\beta: \\ \sigma(\beta)<\sigma(\alpha)}} (-1) \right) \prod_{\alpha<\beta} \left(\xi_{\sigma(\alpha)} - \xi_{\sigma(\beta)} - 1 \right) = c \prod_{\alpha<\beta} (\xi_\alpha - \xi_\beta).$$

To finish, we just have to note that if we expand the bracket on the left hand side we get $m! \prod_{\alpha<\beta} (\xi_\alpha - \xi_\beta)$ plus additional terms of lower degree. But we know that the left hand side, P, is a constant multiple of $\prod_{\alpha<\beta} (\xi_\beta - \xi_\alpha)$; thus, the lower degree terms must cancel. This proves (35).

To prove the lemma, we just need to divide both sides of (35) by $\prod_{\alpha<\beta} (\xi_\alpha - \xi_\beta - 1)$, and then, set $\xi_j = \mathrm{i}\theta/k_j$ for each j. An application of the following equality tp the left hand side and some simple rearrangements give the desired identity.

$$\left(\prod_{\alpha<\beta} \xi_\alpha - \xi_\beta - 1 \right) = \left(\prod_{\substack{\alpha<\beta: \\ \sigma(\beta)<\sigma(\alpha)}} \xi_{\sigma(\beta)} - \xi_{\sigma(\alpha)} - 1 \right) \left(\prod_{\substack{\alpha<\beta: \\ \sigma(\alpha)<\sigma(\beta)}} \xi_{\sigma(\alpha)} - \xi_{\sigma(\beta)} - 1 \right). \quad \square$$

Hence, we get that (34) is equal to

$$\left(\prod_{\iota=1}^{|\pi|} |\pi_\iota|! \right) \sum_{\substack{\tau \in S_n: \\ \tau|_{\pi_\iota} \text{ is increasing } \forall \iota}} e^{\mathrm{i}k_\tau \cdot (y - x_\tau)} T^{\tau,\pi}(k),$$

where $T^{\tau,\pi} : \mathbb{C}^n \to \mathbb{C}^n$ is defined (for a.e. $k \in \mathbb{C}^n$) below.

$$T^{\tau,\pi}(k) := \left(\prod_{\iota<j} \prod_{\substack{\alpha\in\pi_\iota,\ \beta\in\pi_j: \\ \tau(\beta)<\tau(\alpha)}} \mathrm{R}_{\tau(\beta),\tau(\alpha)}(k) \right) \left(\prod_{\pi_\iota\in\pi} \prod_{\substack{\alpha<\beta: \\ \alpha,\beta\in\pi_\iota}} \frac{\mathrm{i}\theta(k_{\tau(\beta)}-k_{\tau(\alpha)})}{\mathrm{i}\theta(k_{\tau(\beta)}-k_{\tau(\alpha)})-k_{\tau(\alpha)}k_{\tau(\beta)}} \right). \tag{37}$$

This rearrangement, together with the triangle inequality, gives us that (33) is bounded above by

$$\frac{\prod_{\iota=1}^{|\pi|}|\pi_\iota|!}{(2\pi)^n} \sum_{\substack{\tau\in S_n: \\ \tau|_{\pi_\iota} \text{ is increasing } \forall\iota}} \int \left| \int_{\mathbb{R}^n} e^{-\frac{1}{2}t|k|^2+\mathrm{i}k_\tau\cdot(y-x_\tau)} T^{\tau,\pi}(k)dk \right| |f(y)-f(x)|\lambda^\pi(dy). \tag{38}$$

Now we can move on to the next step, which we now discuss in a bit more detail. We want to get control on the k integral in (38), and we need the bound to be integrable in y with respect to λ^π and to be vanishing as $t \to 0$ whenever $y \neq x$. As discussed earlier in this section, we can complete the square in the exponent:

$$-\frac{1}{2}t|k|^2 + \mathrm{i}k_\tau\cdot(y-x_\tau) = -\frac{1}{2}t\sum_{\alpha=1}^n (k_{\tau(\alpha)} - \frac{i}{t}(y_\alpha - x_{\tau(\alpha)}))^2 - \frac{(y_\alpha - x_{\tau(\alpha)})^2}{2t}.$$

The above calculation suggests that we should use Cauchy's residue theorem to shift the $k_{\tau(\alpha)}$ contour from $\mathbb{R}$ to $C_\alpha := \{z \in \mathbb{C} :\ z - \frac{i}{t}(y_\alpha - x_{\tau(\alpha)}) \in \mathbb{R}\}$ for each $\alpha \in \{1,\ldots,n\}$, and then parameterise the resulting contour integral as an integral over $\mathbb{R}$. Supposing we can do this without encountering any poles, the exponent becomes

$$-\frac{1}{2}t\sum_{\alpha=1}^n \tilde{k}_{\tau(\alpha)}^2 - \frac{(y_\alpha - x_{\tau(\alpha)})^2}{2t},$$

where $\tilde{k}_{\tau(\alpha)} \in \mathbb{R}$ is our new integration variable. The second term of the summand gives us the necessary control in the y variable, and the first term should allow us to control the resulting k integral. However, this approach is complicated by $T^{\tau,\pi}$, which contribute poles that hinder our contour shifting. We end up not being able to shift the integration contours for all of the k variables without encountering poles; nevertheless, we are still able to make some of the desired contour shifts. To see which shifts can be made, we need check where these poles occur. The following lemma provides us with the desired information.

Lemma 4.12. *Let $\mathbb{H} = \{x + \mathrm{i}y \in \mathbb{C}|\ x \in \mathbb{R},\ y \in \mathbb{R}_{>0}\}$ be the upper half complex plane. The function $(z, w) \mapsto \mathrm{i}\theta(z - w) - zw$ has no zeros in the set $\mathbb{H} \times -\mathbb{H}$.*

Proof. For $w \in -\mathbb{H}$ there are $a \in \mathbb{R}$ and $b \in \mathbb{R}_{>0}$ such that $w = a - bi$. It is easily checked that $\mathrm{i}\theta(z - w) - zw = 0$ if and only if we have

$$z = \frac{\theta^2 a - \mathrm{i}\theta((\theta+b)b + a^2)}{(\theta+b)^2 + a^2} \in -\mathbb{H}.$$

Thus, there are no zeros inside $\mathbb{H} \times -\mathbb{H}$ as claimed. □

Observing the structure of the products in (37), we define the set $E^{\tau,\pi} \subset \mathbb{C}^n$ to avoid any poles. $E^{\tau,\pi}$ is defined as $\times_{k=1}^n E_k^{\tau,\pi}$, where $E_k^{\tau,\pi}$ is given in terms of the following two conditions. For (τ, π, k), say condition (A) is satisfied if there is $\pi_\iota \in \pi$ such that $k = \sup \pi_\iota$

and $\tau(\alpha) < \tau(k)$ for all $\alpha < k$, and say condition (B) is satisfied if there is $\pi_\iota \in \pi$ such that $k = \inf \pi_\iota$ and $\tau(\beta) > \tau(k)$ for all $\beta > k$. Then,

$$E_k^{\tau,\pi} := \begin{cases} \overline{\mathbb{H}}, & \text{if condition (A) is satisfied, but not (B);} \\ -\overline{\mathbb{H}}, & \text{if condition (B) is satisfied, but not (A);} \\ \mathbb{C}, & \text{if both (A) and (B) are satisfied;} \\ \mathbb{R}, & \text{if both (A) and (B) are not satisfied.} \end{cases}$$

Lemma 4.12 shows the denominator of $T^{\tau,\pi}$, as in (37), has no zeros in the set $E^{\tau,\pi}$(37), and thus, we can perform the desired contour shifts as long as the contours remain within this set. To simplify our notation, we will once again write $\overline{\pi_\iota} := \sup \pi_\iota$ and $\underline{\pi_\iota} := \inf \pi_\iota$ for each $\pi_\iota \in \pi$.

We will state the estimate that results from shifting contours in Proposition 4.14. But first, we need to find a family of indices for which contour shifts can be made, that is, a collection of α such that $E_\alpha^{\tau,\pi}$ contains at least one complex half plane.

Lemma 4.13. *Suppose $\pi \in \Pi_n$ and $\tau \in S_n$ such that $\tau|_{\pi_\iota}$ is increasing for every $\pi_\iota \in \pi$. For each $\pi_\iota \in \pi$ there are $a_\iota \le \iota \le b_\iota$ such that $\tau(\underline{\pi_{b_\iota}}) \le \tau(\overline{\pi_{a_\iota}})$, and the following properties hold*

- *$\tau(\underline{\pi_{b_\iota}}) < \tau(\beta)$ for every $\beta > \underline{\pi_{b_\iota}}$;*
- *and $\tau(\overline{\pi_{a_\iota}}) > \tau(\alpha)$ for all $\alpha < \overline{\pi_\iota}$.*

Further, given such a (a_ι, b_ι), if $\overline{\pi_{a_\iota}} < \underline{\pi_{b_\iota}}$, then we define $m_\iota := \sup\{\tau(\alpha) | \overline{\pi_{a_\iota}} \le \alpha \le \underline{\pi_{b_\iota}}\}$ and $l_\iota := \inf\{\tau(\beta) | \overline{\pi_{a_\iota}} \le \beta \le \underline{\pi_{b_\iota}}\}$; if instead $\overline{\pi_{a_\iota}} \ge \underline{\pi_{b_\iota}}$, then we define $m_\iota := \tau(\overline{\pi_{a_\iota}})$ and $l_\iota := \tau(\underline{\pi_{b_\iota}})$. The following properties hold for m_ι and l_ι:

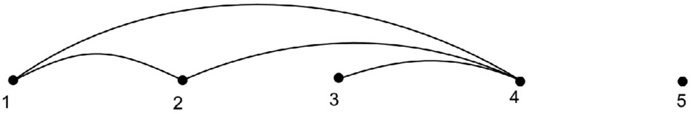
- *there are $\pi_c, \pi_d \in \pi$ such that $\tau^{-1}(m_\iota) = \overline{\pi_d}$ and $\tau^{-1}(l_\iota) = \underline{\pi_c}$;*
- *for all $\alpha < \tau^{-1}(m_\iota)$ we have $\tau(\alpha) < m_\iota$;*
- *and for all $\beta > \tau^{-1}(l_\iota)$ we have $\tau(\beta) > l_\iota$.*

Proof. First we define $\mu_\iota := \overline{\pi_{a_\iota}}$, where $a_\iota := \inf\{a \le \iota : \ \tau(\overline{\pi_a}) \ge \tau(\underline{\pi_\iota})\}$, and then from it, we define $\nu_\iota := \underline{\pi_{b_\iota}}$, where $b_\iota := \sup\{b \ge \iota : \ \tau(\mu_\iota) \ge \tau(\underline{\pi_b})\}$. μ_ι and ν_ι are introduced for convenience and will be used throughout this section. In Fig. 1 we provide an example of a permutation and partition and the resulting values of μ_ι and ν_ι.

It is easy to see that the a_ι and b_ι satisfy the first two properties we claimed for them, namely that $a_\iota \le \iota \le b_\iota$ and $\tau(\nu_\iota) = \tau(\underline{\pi_{b_\iota}}) \le \tau(\overline{\pi_{a_\iota}}) = \tau(\mu_\iota)$.

We will show $\tau(\nu_\iota) < \tau(\beta)$ for all $\beta > \nu_\iota$, and $\tau(\mu_\iota) > \tau(\alpha)$ for all $\alpha < \mu_\iota$. Starting with μ_ι, if there is an $\alpha < \mu_\iota$ such that $\tau(\mu_\iota) < \tau(\alpha)$, then by definition of μ_ι, α must be in a different element of π to μ_ι, say π_c, with $c < a_\iota$. Since τ is increasing on every element of π, this means we must have $\tau(\overline{\pi_c}) > \tau(\alpha) > \tau(\mu_\iota) = \tau(\overline{\pi_{a_\iota}})$, which contradicts the definition of a_ι, so no such α exists. By a similar argument, there is no $\beta > \nu_\iota$ such that $\tau(\nu_\iota) > \tau(\beta)$.

It remains to prove the second set of statements, those about m_ι and l_ι. Suppose we are given (a_ι, b_ι) as in the first part of the lemma, and once more define $\mu_\iota := \overline{\pi_{a_\iota}}$ and $\nu_\iota := \underline{\pi_{b_\iota}}$. The first property for m_ι and l_ι follows immediately from the fact that $\tau|_{\pi_j}$ is increasing for all $\pi_j \in \pi$, the definitions of m_ι and l_ι, and from $\pi \in \Pi_n$. For the second and third statements, we consider two cases separately: $\mu_\iota < \nu_\iota$ and $\nu_\iota \le \mu_\iota$. For the latter case, we have $m_\iota = \tau(\nu_\iota)$ and $l_\iota = \tau(\mu_\iota)$, so the statements are the same as those we just proved. If instead we have $\mu_\iota < \nu_\iota$, we can argue the second statement as follows. Clearly, for all α such that $\mu_\iota \le \alpha \le \nu_\iota$



π_1	π_2	π_3	π_4
(1,4)	(1,4)	(1,4)	(5,5)

Fig. 1. The bottom row displays the values of (μ_ι, ν_ι) for the permutation $\tau = (1, 3, 4)$ in S_5 and partition $\pi = (\{1\}, \{2, 3\}, \{4\}, \{5\})$. Lines connect pairs of indices which τ inverts, so $i < j$ are connected if $\tau(j) < \tau(i)$. μ_ι is the largest element of the leftmost block of π connected to π_ι by a line. Similarly, ν_ι is the least element of the rightmost block of π connected to π_ι by a line.

we have $\tau(\alpha) < m_\iota$; thus, we only need to check that $\alpha < \mu_\iota$ implies $\tau(\alpha) < m_\iota$. Suppose this is false, i.e. there is an $\alpha < \mu_\iota$ such that $\tau(\alpha) > m_\iota$. Since $m_\iota > \tau(\mu_\iota)$, this implies $\tau(\alpha) > \tau(\mu_\iota)$; since we also have $\alpha < \mu_\iota$, this is a contradiction, as we know from previously that $\tau(\mu_\iota) > \tau(\alpha)$ whenever $\mu_\iota > \alpha$. A similar argument proves the third statement, thereby proving the lemma. □

In the following proposition, we will assume we have a $\pi \in \Pi_n$ with a family $(a_\iota, b_\iota)_{\pi_\iota \in \pi}$ given by the above lemma, and adopt the notation of the above proof, namely $\mu_\iota := \overline{\pi_{a_\iota}}$ and $\nu_\iota := \underline{\pi_{b_\iota}}$. The above lemma ensures that whenever $\alpha = \mu_\iota$ or $\tau^{-1}(m_\iota)$, the set $E_\alpha^{\tau,\pi}$ contains the upper half complex plane, and if $\beta = \nu_\iota$ or $\tau^{-1}(l_\iota)$, then $E_\beta^{\tau,\pi}$ contains the lower half complex plane. It turns out that it is sufficient to consider the contour shifts corresponding only to these indices, because of the fact that $x, y \in \overline{\mathbb{W}^n}$, and because we are only interested in estimating integrals with respect to the measure λ^π, which is supported on sets where certain coordinates are always equal. As a result, we will get the desired estimate, which we state now.

Proposition 4.14. *Suppose $\pi \in \Pi_n$ and $\tau \in S_n$ such that $\tau|_{\pi\iota}$ is increasing for every $\pi_\iota \in \pi$, and for each $\pi_\iota \in \pi$ we have $a_\iota \le \iota \le b_\iota$ as in the above lemma. There is a constant $C > 0$, depending only on π and n, such that the following bound holds for all $x, y \in \overline{\mathbb{W}^n}$.*

$$\left|\int_{\mathbb{R}^n} e^{-\frac{1}{2}t|k|^2 + ik_\tau\cdot(y - x_\tau)} T^{\tau,\pi}(k)dk\right|$$
$$\le C t^{-\frac{1}{2}|\pi|} |\log(t)|^{|\pi|} e^{-\frac{|y-\chi|^2}{12nt}} \prod_{\pi_\iota \in \pi} e^{-\frac{1}{24nt}\left((x_{m_\iota} - \chi^\iota)^2 + (x_{l_\iota} - \chi^\iota)^2\right)}, \tag{39}$$

where $\chi = \chi(x) \in \mathbb{R}^n$ is defined by $\chi_\alpha := \chi^\iota := \frac{1}{2}(x_{\tau(\mu_\iota)} + x_{\tau(\nu_\iota)})$ for all $\alpha \in \pi_\iota$.

We begin the proof with an intermediate bound, which is achieved by making the contour shifts we have been discussing. Let $\Gamma_{\alpha,x,y} = C_\alpha$ if $x, y \in \overline{\mathbb{W}^n}$ are such that the C_α contour lies in $E_\alpha^{\tau,\pi}$, and $\mathbb{R}$ otherwise.

Lemma 4.15.

$$\left|\int_{\mathbb{R}^n} e^{-\frac{1}{2}t|k|^2+\mathrm{i}k_\tau\cdot(y-x_\tau)} T^{\tau,\pi}(k)dk\right| \le e^{-\frac{|y-\chi|^2}{12nt}} \left(\prod_{\pi_\iota\in\pi} e^{-\frac{1}{24nt}\left((x_{m_\iota}-\chi^\iota)^2+(x_{l_\iota}-\chi^\iota)^2\right)}\right) \int_{\times_{\alpha=1}^n \Gamma_{\alpha,x,y}} e^{-\frac{1}{2}t\sum_{\alpha=1}^n Re(k_{\tau(\alpha)})^2} |T^{\tau,\pi}(k)|\,dk. \quad (40)$$

Proof. On the left-hand side of (40), we can apply Cauchy's residue theorem to shift the contours of the integral into the complex plane, onto the contours $\Gamma_{\alpha,x,y}$. This is possible, because we have defined the contours $\Gamma_{\alpha,x,y}$ in such a way that they are either $\mathbb{R}$, and thus no deformation is required, or the integrand is analytic in whichever half plane they occupy. The result is the following equality.

$$\int_{\mathbb{R}^n} e^{-\frac{1}{2}t|k|^2+\mathrm{i}k_\tau\cdot(y-x_\tau)} T^{\tau,\pi}(k)dk = e^{-\frac{1}{2t}\sum_{\alpha:\Gamma_{\alpha,x,y}\neq\mathbb{R}}(y_\alpha-x_{\tau(\alpha)})^2} \times \int_{\times_{\alpha=1}^n \Gamma_{\alpha,x,y}} e^{-\frac{1}{2}t\sum_{\alpha=1}^n \mathrm{Re}(k_{\tau(\alpha)})^2+i\sum_{\alpha:\Gamma_{\alpha,x,y}=\mathbb{R}} k_{\tau(\alpha)}(y_\alpha-x_{\tau(\alpha)})} T^{\tau,\pi}(k)dk. \quad (41)$$

Note that on the right hand side, when the contour for k_j is not the real line, we have rewritten the exponential by completing the square: $k_j^2 - \frac{2i}{t}k_j(y_{\tau^{-1}(j)}-x_j) = (k_i - \frac{i}{t}(y_{\tau^{-1}(i)}-x_i))^2 + \frac{1}{t^2}(y_{\tau^{-1}(i)}-x_i)^2$.

From (41), we see that to prove Lemma 4.15 we must bound the exponential appearing in front of the integral, which means we need to consider which contour shifts have been made. In particular, we want to check when the condition for $\Gamma_{\alpha,x,y} = C_\alpha$ is true, for $\alpha = \mu_\iota, \nu_\iota$. Thus, we want to check when C_α lies inside $E_\alpha^{\tau,\pi}$. We know from Lemma 4.13 that $E_{\mu_\iota}^{\tau,\pi}$ contains the upper half complex plane, and $E_{\nu_\iota}^{\tau,\pi}$ contains the lower half complex plane. Thus, $C_{\mu_\iota} \subset E_{\mu_\iota}^\tau$ when $y_{\mu_\iota} \ge x_{\tau(\mu_\iota)}$, and $C_{\nu_\iota} \subset E_{\nu_\iota}^\tau$ when $y_{\nu_\iota} \le x_{\tau(\nu_\iota)}$. Hence, we have the following inequalities:

$$-\frac{1}{2t}\mathbb{1}_{\{\Gamma_{\mu_\iota,x,y}\neq\mathbb{R}\}}(y_{\mu_\iota}-x_{\tau(\mu_\iota)})^2 \le -\frac{1}{2t}\mathbb{1}_{\{(y_{\mu_\iota}\ge x_{\tau(\mu_\iota)})\}}(y_{\mu_\iota}-x_{\tau(\mu_\iota)})^2;$$
$$-\frac{1}{2t}\mathbb{1}_{\{\Gamma_{\nu_\iota,x,y}\neq\mathbb{R}\}}(y_{\nu_\iota}-x_{\tau(\nu_\iota)})^2 \le -\frac{1}{2t}\mathbb{1}_{\{(y_{\nu_\iota}\le x_{\tau(\nu_\iota)})\}}(y_{\nu_\iota}-x_{\tau(\nu_\iota)})^2.$$

There are two cases of interest, the first is when $\mu_\iota < \nu_\iota$. In this case, the two indices are in different elements of π. The second case is when $\mu_\iota \ge \nu_\iota$, for which the two indices are in the same element of π. Let us deal now with the first case.

By definition, we have $\tau(\mu_\iota) \ge \tau(\nu_\iota)$; and since $x, y \in \overline{\mathbb{W}^n}$, $\mu_\iota < \nu_\iota$ implies that $y_{\nu_\iota} \le y_{\mu_\iota}$. Hence, if we have both $y_{\nu_\iota} > x_{\tau(\nu_\iota)}$ and $y_{\mu_\iota} < x_{\tau(\mu_\iota)}$, it follows that $x_{\tau(\nu_\iota)} < x_{\tau(\mu_\iota)}$, but since $x \in \overline{\mathbb{W}^n}$ this is a contradiction. Hence, for all $x, y \in \overline{\mathbb{W}^n}$ at least one of $y_{\mu_\iota} \ge x_{\tau(\mu_\iota)}$ and $y_{\nu_\iota} \le x_{\tau(\nu_\iota)}$ must be true. This means we have the equality,

$$-\frac{1}{2t}\mathbb{1}_{\{(y_{\mu_\iota}\ge x_{\tau(\mu_\iota)})\}}(y_{\mu_\iota}-x_{\tau(\mu_\iota)})^2 - \frac{1}{2t}\mathbb{1}_{\{(y_{\nu_\iota}\le x_{\tau(\nu_\iota)})\}}(y_{\nu_\iota}-x_{\tau(\nu_\iota)})^2$$
$$= \begin{cases} -\frac{1}{2t}(y_{\mu_\iota}-x_{\tau(\mu_\iota)})^2, & \text{if } y_{\mu_\iota} \ge x_{\tau(\mu_\iota)} \text{ and } y_{\nu_\iota} > x_{\tau(\nu_\iota)}; \\ -\frac{1}{2t}(y_{\nu_\iota}-x_{\tau(\nu_\iota)})^2, & \text{if } y_{\mu_\iota} < x_{\tau(\mu_\iota)} \text{ and } y_{\nu_\iota} \le x_{\tau(\nu_\iota)}; \\ -\frac{1}{2t}(y_{\mu_\iota}-x_{\tau(\mu_\iota)})^2 - \frac{1}{2t}(y_{\nu_\iota}-x_{\tau(\nu_\iota)})^2, & \text{if } y_{\nu_\iota} \le x_{\tau(\nu_\iota)} \text{ and } y_{\mu_\iota} \ge x_{\tau(\mu_\iota)}. \end{cases} \quad (42)$$

Let $\chi^\iota := \frac{1}{2}(x_{\tau(\mu_\iota)} + x_{\tau(\nu_\iota)})$, we can rewrite the first line as

$$-\frac{1}{2t}\left((y_{\mu_\iota}-\chi^\iota)^2 + \frac{1}{4}(x_{\tau(\mu_\iota)}-x_{\tau(\nu_\iota)})^2\right) + \frac{1}{2t}(y_{\mu_\iota}-\chi^\iota)(x_{\tau(\mu_\iota)}-x_{\tau(\nu_\iota)}).$$

We have $\tau(\mu_\iota) > \tau(\nu_\iota)$ and $x \in \overline{\mathbb{W}^n}$, so that $(x_{\tau(\mu_\iota)} - x_{\tau(\nu_\iota)}) \leq 0$. From $y \in \overline{\mathbb{W}^n}$ and $\mu_\iota < \nu_\iota$, it follows that $y_{\mu_\iota} \geq \frac{1}{2}(y_{\mu_\iota} + y_{\nu_\iota})$, which, under the conditions of the first line in (42), is bounded below by $\chi^\iota = \frac{1}{2}(x_{\tau(\mu_\iota)} + x_{\tau(\nu_\iota)})$. Thus $y_{\mu_\iota} - \chi^\iota > 0$, and the last term in the above expression is negative. We also have $y_{\nu_\iota} - \chi_\iota \geq y_{\nu_\iota} - x_{\tau(\nu_\iota)} > 0$, under the condition of the first line; thus, using $y_{\nu_\iota} \leq y_{\mu_\iota}$, we get $-(y_{\mu_\iota} - \chi^\iota)^2 \leq -(y_{\nu_\iota} - \chi^\iota)^2$. It follows that the above expression is bounded above by

$$-\frac{1}{4t}\left((y_{\mu_\iota} - \chi^\iota)^2 + (y_{\nu_\iota} - \chi^\iota)^2 + \frac{1}{2}(x_{\tau(\mu_\iota)} - x_{\tau(\nu_\iota)})^2\right).$$

The same ideas yield the same bound on the cases of the second and third lines of (42), so that the above expression is an upper bound for (42).

In order to get the precise control of the decay in the x variable, which we need to cover the case where x is near the boundary of $\overline{\mathbb{W}^n}$, we need to look at the contour shifts for m_ι and l_ι. Recall $m_\iota = \sup\{\tau(\alpha)|\ \mu_\iota \leq \alpha \leq \nu_\iota\}$ and $l_\iota = \inf\{\tau(\beta)|\ \mu_\iota \leq \beta \leq \nu_\iota\}$. Note that it is quite possible for $m_\iota = \tau(\mu_\iota)$ or for $l_\iota = \tau(\nu_\iota)$, so that we will need to account for repetitions. We need to check when $C_{\tau^{-1}(m_\iota)} \subset E^{\tau,\pi}_{\tau^{-1}(m_\iota)}$. From Lemma 4.13 we know $E^{\tau,\pi}_{\tau^{-1}(m_\iota)}$ contains the upper half complex plane. Therefore, $\Gamma_{\tau^{-1}(m_\iota),x,y} = C_{\tau^{-1}(m_\iota)}$ if $y_{\tau^{-1}(m_\iota)} \geq x_{m_\iota}$, so that

$$-\frac{1}{2t}\mathbb{1}_{\{\Gamma_{\tau^{-1}(m_\iota),x,y} \neq \mathbb{R}\}}(y_{\tau^{-1}(m_\iota)} - x_{m_\iota})^2 \leq -\frac{1}{2t}\mathbb{1}_{\{(y_{\tau^{-1}(m_\iota)} \geq x_{m_\iota})\}}(y_{\tau^{-1}(m_\iota)} - x_{m_\iota})^2.$$

We can combine this bound with our previous bound to get the following inequality (where we have accounted for the case $m_\iota = \tau(\mu_\iota)$ in the indicators).

$$\begin{aligned}
&-\frac{1}{2t}\mathbb{1}_{\{\Gamma_{\mu_\iota,x,y} \neq \mathbb{R}\}}(y_{\mu_\iota} - x_{\tau(\mu_\iota)})^2 - \frac{1}{2t}\mathbb{1}_{\{\Gamma_{\nu_\iota,x,y} \neq \mathbb{R}\}}(y_{\nu_\iota} - x_{\tau(\nu_\iota)})^2 \\
&\qquad - \frac{1}{2t}\mathbb{1}_{\{\Gamma_{\tau^{-1}(m_\iota),x,y} \neq \mathbb{R},\ m_\iota \neq \tau(\mu_\iota)\}}(y_{\tau^{-1}(m_\iota)} - x_{m_\iota})^2 \\
&\leq -\frac{1}{4t}\left((y_{\mu_\iota} - \chi^\iota)^2 + (y_{\nu_\iota} - \chi^\iota)^2 + (x_{\tau(\mu_\iota)} - x_{\tau(\nu_\iota)})^2\right) \\
&\qquad - \frac{1}{2t}\mathbb{1}_{\{y_{\tau^{-1}(m_\iota)} \geq x_{m_\iota},\ m_\iota \neq \tau(\mu_\iota)\}}(y_{\tau^{-1}(m_\iota)} - x_{m_\iota})^2.
\end{aligned} \tag{43}$$

We aim to show this is bounded above, for some positive constants C_1, C_2, by

$$-\frac{C_1}{t}\left((y_{\mu_\iota} - \chi^\iota)^2 + (y_{\nu_\iota} - \chi^\iota)^2 + (x_{\tau(\mu_\iota)} - x_{\tau(\nu_\iota)})^2\right) - \frac{C_2}{t}(x_{m_\iota} - \chi^\iota)^2.$$

To prove this, we consider the various cases for the indicator in (43).

If $m_\iota = \tau(\mu_\iota)$, then it follows from $x_{m_\iota} \leq \chi_\iota \leq x_{\tau(\nu_\iota)}$ that $(x_{\tau(\mu_\iota)} - x_{\tau(\nu_\iota)})^2 \leq (x_{m_\iota} - \chi^\iota)^2$, so that our desired bound is easily seen.

In the case that $m_\iota \neq \tau(\mu_\iota)$ and $y_{\tau^{-1}(m_\iota)} \geq x_{m_\iota}$, if we further assume $y_{\tau^{-1}(m_\iota)} \geq \chi^\iota$, then it follows that

$$\begin{aligned}
&-(y_{\mu_\iota} - \chi^\iota)^2 - (y_{\tau^{-1}(m_\iota)} - x_{m_\iota})^2 \\
&= -(y_{\mu_\iota} - y_{\tau^{-1}(m_\iota)})^2 - (\chi^\iota - x_{m_\iota})^2 + 2(y_{\mu_\iota} - x_{m_\iota})(\chi^\iota - y_{\tau^{-1}(m_\iota)}) \\
&\leq -(y_{\mu_\iota} - y_{\tau^{-1}(m_\iota)})^2 - (\chi^\iota - x_{m_\iota})^2 \leq -(\chi^\iota - x_{m_\iota})^2,
\end{aligned}$$

where the last line is true because $y \in \overline{\mathbb{W}^n}$, so that $y_{\mu_\iota} \geq y_{\tau^{-1}(m_\iota)}$. Thus, our assumptions imply the last term on the third line above is negative. If instead, $x_{m_\iota} \leq y_{\tau^{-1}(m_\iota)} < \chi^\iota$, then $y \in \overline{\mathbb{W}^n}$

implies that $y_{\nu_\iota} \leq y_{\tau^{-1}(m_\iota)}$; thus, $0 > y_{\tau^{-1}(m_\iota)} - \chi^\iota \geq y_{\nu_\iota} - \chi^\iota$. Hence,

$$\begin{aligned}&-(y_{\nu_\iota} - \chi^\iota)^2 - (y_{\tau^{-1}(m_\iota)} - x_{m_\iota})^2\\ \leq& -(y_{\tau^{-1}(m_\iota)} - \chi^\iota)^2 - (y_{\tau^{-1}(m_\iota)} - x_{m_\iota})^2\\ =& -2(y_{\tau^{-1}(m_\iota)} - \frac{1}{2}(x_{m_\iota} + \chi^\iota))^2 - \frac{1}{2}(x_{m_\iota} - \chi^\iota)^2 \leq -\frac{1}{2}(x_{m_\iota} - \chi^\iota)^2.\end{aligned}$$

Therefore, when $y_{\tau^{-1}(m_\iota)} \geq x_{m_\iota}$ we have the bound on (43)

$$-\frac{1}{8t}(x_{m_\iota} - \chi^\iota)^2. \tag{44}$$

If instead, we have $m_\iota \neq \tau(\mu_\iota)$ and $y_{\tau^{-1}(m_\iota)} < x_{m_\iota}$, then we have $x, y \in \overline{\mathbb{W}^n}$; therefore, $y_{\nu_\iota} \leq y_{\tau^{-1}(m_\iota)} < x_{m_\iota} \leq \chi^\iota$. Thus, $-(y_{\nu_\iota} - \chi^\iota)^2 \leq -(x_{m_\iota} - \chi^\iota)^2$, so that Expression (44) is a bound on (43) for any $x, y \in \overline{\mathbb{W}^n}$, as desired. Following the same steps for l_ι, we get the analogous bound

$$\begin{aligned}&-\frac{1}{2t}\mathbb{1}_{\{\Gamma_{\mu_\iota,x,y}\neq\mathbb{R}\}}(y_{\mu_\iota} - x_{\tau(\mu_\iota)})^2 - \frac{1}{2t}\mathbb{1}_{\{\Gamma_{\nu_\iota,x,y}\neq\mathbb{R}\}}(y_{\nu_\iota} - x_{\tau(\nu_\iota)})^2\\ &\quad -\frac{1}{2t}\mathbb{1}_{\{\Gamma_{\tau^{-1}(l_\iota),x,y}\neq\mathbb{R},\ l_\iota\neq\tau(\nu_\iota)\}}(y_{\tau^{-1}(l_\iota)} - x_{l_\iota})^2\\ \leq& -\frac{1}{4t}\left((y_{\mu_\iota} - \chi^\iota)^2 + (y_{\nu_\iota} - \chi^\iota)^2 + (x_{\tau(\mu_\iota)} - x_{\tau(\nu_\iota)})^2\right)\\ &\quad -\frac{1}{2t}\mathbb{1}_{\{y_{\tau^{-1}(l_\iota)}\geq x_{l_\iota},\ l_\iota\neq\tau(\nu_\iota)\}}(y_{\tau^{-1}(m_\iota)} - x_{m_\iota})^2\\ \leq& -\frac{1}{8t}(x_{l_\iota} - \chi^\iota)^2.\end{aligned} \tag{45}$$

Combining the bounds in (43), (44), and (45), we get the following bound when $\nu_\iota > \mu_\iota$.

$$\begin{aligned}&-\frac{1}{2t}\mathbb{1}_{\{\Gamma_{\mu_\iota,x,y}\neq\mathbb{R}\}}(y_{\mu_\iota} - x_{\tau(\mu_\iota)})^2 - \frac{1}{2t}\mathbb{1}_{\{\Gamma_{\nu_\iota,x,y}\neq\mathbb{R}\}}(y_{\nu_\iota} - x_{\tau(\nu_\iota)})^2\\ &-\frac{1}{2t}\mathbb{1}_{\{\Gamma_{\tau^{-1}(m_\iota),x,y}\neq\mathbb{R},\ m_\iota\neq\tau(\mu_\iota)\}}(y_{\tau^{-1}(m_\iota)} - x_{m_\iota})^2 - \frac{1}{2t}\mathbb{1}_{\{\Gamma_{\tau^{-1}(l_\iota),x,y}\neq\mathbb{R},\ l_\iota\neq\tau(\nu_\iota)\}}(y_{\tau^{-1}(l_\iota)} - x_{l_\iota})^2\\ \leq& -\frac{1}{12t}\left((y_{\mu_\iota} - \chi^\iota)^2 + (y_{\nu_\iota} - \chi^\iota)^2\right) - \frac{1}{24t}\left((x_{m_\iota} - \chi^\iota)^2 + (x_{l_\iota} - \chi^\iota)^2\right)\\ \leq& -\frac{1}{12(\overline{\pi_{b_\iota}} - \underline{\pi_{a_\iota}})t}\sum_{\alpha=\underline{\pi_{a_\iota}}}^{\overline{\pi_{b_\iota}}}(y_\alpha - \chi^\iota)^2 - \frac{1}{24t}\left((x_{m_\iota} - \chi^\iota)^2 + (x_{l_\iota} - \chi^\iota)^2\right),\end{aligned} \tag{46}$$

where for the last line, we have used that by definition $\mu_\iota = \overline{\pi_{a_\iota}}$ and $\nu_\iota = \underline{\pi_{b_\iota}}$, and that under λ^π, we have that for any $\pi_j \in \pi$, if $\alpha, \beta \in \pi_j$, then $y_\alpha = y_\beta$ almost everywhere as well as having that $y \in \overline{\mathbb{W}^n}$, so that $(y_{\mu_\iota} - \chi^\iota) \geq (y_\alpha - \chi_\iota) \geq (y_{\nu_\iota} - \chi_\iota)$ for all $\underline{\pi_{a_\iota}} \leq \alpha \leq \overline{\pi_{b_\iota}}$. Thus, either $-(y_\alpha - \chi^\iota)^2 \leq -(y_{\mu_\iota} - \chi^\iota)^2$ or $-(y_\alpha - \chi^\iota)^2 \leq -(y_{\nu_\iota} - \chi^\iota)^2$.

To finish the argument and get the bound on (38), we just need to deal with the second case: $\nu_\iota \leq \mu_\iota$.

In the second case, μ_ι and ν_ι are both in π_ι, and therefore, under λ^π, we have $y_{\mu_\iota} = y_{\nu_\iota}$ almost everywhere. Further, since τ is increasing on every element of π, it follows that $m_\iota := \tau(\mu_\iota) = \sup\{\tau(\alpha)|\ \nu_\iota \leq \alpha \leq \mu_\iota\}$ and $l_\iota := \tau(\nu_\iota) = \inf\{\tau(\beta)|\ \nu_\iota \leq \beta \leq \mu_\iota\}$. Following the same steps as before, if we assume both $y_{\nu_\iota} > x_{\tau(\nu_\iota)}$ and $y_{\mu_\iota} < x_{\tau(\mu_\iota)}$, then since $y_{\mu_\iota} = y_{\nu_\iota}$, it follows that $x_{\tau(\nu_\iota)} < x_{\tau(\mu_\iota)}$, which is a contradiction because $\tau(\nu_\iota) < \tau(\mu_\iota)$ and $x \in \overline{\mathbb{W}^n}$. Thus, at least one of $y_{\nu_\iota} \leq x_{\tau(\nu_\iota)}$ and $y_{\mu_\iota} \geq x_{\tau(\mu_\iota)}$ must hold for all $x, y \in \overline{\mathbb{W}^n}$. With similar ideas to those used above, we find that

$$
\begin{aligned}
& -\frac{1}{2t}\mathbb{1}_{\{\Gamma_{\mu_\iota,x,y}\neq\mathbb{R}\}}(y_{\mu_\iota} - x_{\tau(\mu_\iota)})^2 - \frac{1}{2t}\mathbb{1}_{\{\Gamma_{\nu_\iota,x,y}\neq\mathbb{R}\}}(y_{\nu_\iota} - x_{\tau(\nu_\iota)})^2 \\
\leq & -\frac{1}{4t}\left((y_{\mu_\iota} - \chi^\iota)^2 + (y_{\nu_\iota} - \chi^\iota)^2 + (x_{m_\iota} - x_{l_\iota})^2\right) \\
\leq & -\frac{1}{12t}\left((y_{\mu_\iota} - \chi^\iota)^2 + (y_{\nu_\iota} - \chi^\iota)^2\right) - \frac{1}{24t}\left((x_{m_\iota} - \chi^\iota)^2 + (x_{l_\iota} - \chi^\iota)^2\right) && (47) \\
\leq & -\frac{1}{12(\mu_\iota - \nu_\iota)t}\sum_{\alpha=\nu_\iota}^{\mu_\iota}(y_\alpha - \chi^\iota)^2 - \frac{1}{24t}\left((x_{m_\iota} - \chi^\iota)^2 + (x_{l_\iota} - \chi^\iota)^2\right). && (48)
\end{aligned}
$$

The idea behind the above bounds is similar to before, but this time we use $y_{\mu_\iota} = y_{\nu_\iota}$, and we used that $x_{l_\iota} \geq \chi^\iota \geq x_{m_\iota}$ for the second inequality. The constants appearing in the denominator have been chosen to be consistent with (46), and so are not optimal.

Applying the bounds (46) and (48) to (41) leads to the following inequality

$$
\begin{aligned}
\left|\int_{\mathbb{R}^n} e^{-\frac{1}{2}t|k|^2 + \mathrm{i}k_\tau\cdot(y-x_\tau)}T^{\tau,\pi}(k)dk\right| \leq & e^{-\frac{1}{12nt}|y-\chi|^2}\left(\prod_{\pi_\iota\in\pi} e^{-\frac{1}{24nt}\left((x_{m_\iota}-\chi^\iota)^2+(x_{l_\iota}-\chi^\iota)^2\right)}\right) \\
& \int_{\times_{\alpha=1}^n \Gamma_{\alpha,x,y}} e^{-\frac{1}{2}t\sum_{\alpha=1}^n \mathrm{Re}(k_{\tau(\alpha)})^2}\left|T^{\tau,\pi}(k)\right| dk, \qquad (49)
\end{aligned}
$$

where we have used $\mu_\iota - \nu_\iota, \overline{\pi_{b_\iota}} - \underline{\pi_{a_\iota}} < n$ for all ι to get the form of the Gaussian bound given above. □

We complete the proof of Proposition 4.14 with the following lemma.

Lemma 4.16. *There is a constant $C > 0$, depending only on π and n, such that*

$$
\int_{\times_{\alpha=1}^n \Gamma_{\alpha,x,y}} e^{-\frac{1}{2}t\sum_{\alpha=1}^n Re(k_{\tau(\alpha)})^2}\left|T^{\tau,\pi}(k)\right| dk \leq Ct^{-\frac{1}{2}|\pi|}|\log(t)|^{|\pi|}. \qquad (50)
$$

Proof. To begin, we need to collect some bounds on the factors appearing in the products (37). We need to make sure the bound covers the new contours; therefore, it is sufficient to bound for $k \in E^{\tau,\pi}$. This can be done for the factors in the first product by bounding for all $h_a, h_b \geq 0$ and $k_a, k_b \in \mathbb{R}$

$$
\begin{aligned}
& \left|\frac{\mathrm{i}\theta((k_a + ih_a) - (k_b - \mathrm{i}h_b)) + ((k_a + ih_a))((k_b - ih_b))}{\mathrm{i}\theta((k_a + ih_a) - (k_b - \mathrm{i}h_b)) - ((k_a + ih_a))((k_b - ih_b))}\right| \\
= & \left|\frac{\mathrm{i}\theta(k_a - k_b) - \theta(h_b + h_a) + i(k_b h_a - k_a h_b) + k_a k_b + h_a h_b}{\mathrm{i}\theta(k_a - k_b) - \theta(h_b + h_a) - i(k_b h_a - k_a h_b) - k_a k_b - h_a h_b}\right| \\
= & \left(\frac{\theta^2(k_a-k_b)^2+(k_bh_a-k_ah_b)^2-2\theta(k_b^2h_a+k_a^2h_b)+\theta^2(h_b+h_a)^2-2\theta h_ah_b(h_b+h_a)+(k_ak_b+h_ah_b)^2}{\theta^2(k_a-k_b)^2+(k_bh_a-k_ah_b)^2+2\theta(k_b^2h_a+k_a^2h_b)+\theta^2(h_b+h_a)^2+2\theta h_ah_b(h_b+h_a)+(k_ak_b+h_ah_b)^2}\right)^{\frac{1}{2}} \\
\leq & 1, \quad \text{because } h_a, h_b \geq 0. \qquad (51)
\end{aligned}
$$

Here, the k variables are the real part of the integration variables, and the h variables are the imaginary part. Hence, we have that for all $k \in E^{\tau,\pi}$

$$\int_{\times_{\alpha=1}^{n}\Gamma_{\alpha,x,y}} e^{-\frac{1}{2}t\sum_{\alpha=1}^{n}\mathrm{Re}(k_{\tau(\alpha)})^2}\,|T^{\tau,\pi}(k)|\,dk$$
$$\le \prod_{\pi_\iota\in\pi}\int_{\times_{\alpha\in\pi_\iota}\Gamma_{\alpha,x,y}} e^{-\frac{1}{2}t\sum_{\alpha\in\pi_\iota}\mathrm{Re}(k_{\tau(\alpha)})^2}\prod_{\substack{\alpha<\beta:\\ \alpha,\beta\in\pi_\iota}}\left|\frac{\mathrm{i}\theta(k_{\tau(\beta)}-k_{\tau(\alpha)})}{\mathrm{i}\theta(k_{\tau(\beta)}-k_{\tau(\alpha)})-k_{\tau(\alpha)}k_{\tau(\beta)}}\right|dk. \tag{52}$$

This expression will be estimated using the following bound on the factors in the product in the integrand. For any $k_a, k_b \in \mathbb{R}$ and $h_a, h_b \ge 0$,

$$\left|\frac{\mathrm{i}\theta((k_a+ih_a)-(k_b-\mathrm{i}h_b))}{\mathrm{i}\theta((k_a+ih_a)-(k_b-\mathrm{i}h_b))-((k_a+ih_a))((k_b-ih_b))}\right|$$
$$=\left(\frac{\theta^2(k_a-k_b)^2+\theta^2(h_b+h_a)^2}{\theta^2(k_a-k_b)^2+(k_bh_a-k_ah_b)^2+2\theta(k_b^2h_a+k_a^2h_b)+\theta^2(h_b+h_a)^2+2\theta h_ah_b(h_b+h_a)+(k_ak_b+h_ah_b)^2}\right)^{\frac{1}{2}}$$
$$\le\begin{cases}1,\\ \theta\left(\frac{|k_a-k_b|}{((k_bh_a-k_ah_b)^2+(k_ak_b+h_ah_b)^2)^{\frac{1}{2}}}\right)+\theta\left(\frac{|h_b+h_a|}{\left((k_bh_a-k_ah_b)^2+(k_ak_b+h_ah_b)^2\right)^{1/2}}\right)\end{cases}$$
$$\le\begin{cases}1,\\ 2\theta\left(\frac{1}{|k_a|}+\frac{1}{|k_b|}\right).\end{cases} \tag{53}$$

The last line follows by expanding the brackets in the denominator, removing some non-negative terms, and then applying the triangle inequality.

Returning to (52), we can divide each contour integral into two parts: one where $|\mathrm{Re}(k_\alpha)| < \varepsilon/\sqrt{t}$ and another where $|\mathrm{Re}(k_\alpha)| \ge \varepsilon/\sqrt{t}$; this gives the following

$$\prod_{\pi_\iota\in\pi}\int_{\times_{\alpha\in\pi_\iota}\Gamma_{\alpha,x,y}} e^{-\frac{1}{2}t\sum_{\alpha\in\pi_\iota}\mathrm{Re}(k_{\tau(\alpha)})^2}$$
$$\prod_{\alpha\in\pi_\iota}\left(\mathbb{1}_{\{|\mathrm{Re}(k_\alpha)|<\varepsilon/\sqrt{t}\}}+\mathbb{1}_{\{|\mathrm{Re}(k_\alpha)|\ge\varepsilon/\sqrt{t}\}}\right)\prod_{\substack{\alpha<\beta:\\ \alpha,\beta\in\pi_\iota}}\left|\frac{\mathrm{i}\theta(k_{\tau(\beta)}-k_{\tau(\alpha)})}{\mathrm{i}\theta(k_{\tau(\beta)}-k_{\tau(\alpha)})-k_{\tau(\alpha)}k_{\tau(\beta)}}\right|dk.$$

We can simplify as follows: expanding the first product, in each term where an indicator for $|k_\alpha| < \varepsilon/\sqrt{t}$ appears, we bound all factors in the second product which depend on k_α by 1, using the first line bound in (53); it is then easy to see that the contribution from the k_α integral to that term is at most $2\varepsilon/\sqrt{t}$. We can then bound any remaining terms in the second product by the second line bound in (53) (remembering that the k in that estimate represents the real part of the complex integration variable), the resulting integral depends only on the number of k_α for which $|\mathrm{Re}(k_\alpha)| \ge \varepsilon/\sqrt{t}$. Relabelling the remaining variables, we see that the previous expression is bounded above by

$$\prod_{\pi_\iota\in\pi}\sum_{j=1}^{|\pi_\iota|}\binom{|\pi_\iota|}{j}2^{|\pi_\iota|-j+\binom{j}{2}}\theta^{\binom{j}{2}}(\varepsilon/\sqrt{t})^{|\pi_\iota|-j}\int_{\substack{\mathbb{R}^j:\\ |k_\alpha|\ge\varepsilon/\sqrt{t},\ \forall\alpha}} e^{-\frac{1}{2}t|k|^2}\prod_{\alpha<\beta}\left(\frac{1}{|k_\alpha|}+\frac{1}{|k_\beta|}\right)dk.$$

Rescaling the k variables by $\frac{1}{\sqrt{t}}$, we see that this equals

$$\prod_{\pi_\iota\in\pi}\sum_{j=1}^{|\pi_\iota|}\binom{|\pi_\iota|}{j}2^{|\pi_\iota|-j+\binom{j}{2}}\theta^{\binom{j}{2}}\varepsilon^{|\pi_\iota|-j}t^{\frac{1}{2}\left(\binom{j}{2}-|\pi_\iota|\right)}\int_{\substack{\mathbb{R}^j:\\ |k_\alpha|\ge\varepsilon,\ \forall\alpha}} e^{-\frac{1}{2}|k|^2}\prod_{\alpha<\beta}\left(\frac{1}{|k_\alpha|}+\frac{1}{|k_\beta|}\right)dk. \tag{54}$$

Since the product runs through all pairs of $\alpha, \beta \in \{1, \ldots, j\}$, upon expanding the brackets, every term will involve a k_γ to a power of at most $1 - j$. Further, in each term, at most one of the k_γ can have exponent -1, with the rest having exponent at most -2. It is clear from repeated integration by parts that for each $y \neq 1$, there is some constant $C > 0$ such that

$$\int_{|x|\geq\varepsilon} \frac{1}{|x|^y} e^{-\frac{1}{2}|x|^2} dx \leq C\varepsilon^{1-y}, \qquad \text{when } \varepsilon \in (0, 1).$$

For $y = 1$, we instead have that there is a constant $C > 0$ such that

$$\int_{|x|\geq\varepsilon} \frac{1}{|x|} e^{-\frac{1}{2}|x|^2} dx \leq C|\log(\varepsilon)|, \qquad \text{when } \varepsilon \in (0, 1).$$

Since the sum of all the powers of all the k_γ in each term of the expanded brackets is $\binom{j}{2}$, and because the product runs through all pairs of indices so that in each term in the expansion there can be at most one k_γ appearing with power -1, there is some constant $C > 0$ depending only on n and π such that for all $\varepsilon \in (0, 1)$, (54) is bounded above by

$$C \prod_{\pi_\iota\in\pi} \sum_{j=1}^{|\pi_\iota|} \varepsilon^{|\pi_\iota|-j+j-1-\binom{j}{2}} |\log(\varepsilon)| t^{\frac{1}{2}\left(\binom{j}{2}-|\pi_\iota|\right)}.$$

If we set $\varepsilon = \sqrt{t}$, then the above expression is bounded above by

$$Ct^{-\frac{1}{2}|\pi|}|\log(t)|^{|\pi|},$$

which is the desired upper bound. □

Proof of Proposition 4.14. Combining the bounds from the above lemma and Lemma 4.15 proves the statement. □

We now have what we need to complete the proof of the main proposition of the subsection.

Proof of Proposition 4.8. Proposition 4.14 implies that (38) is bounded above by

$$Ct^{-\frac{1}{2}|\pi|}|\log(t)|^{|\pi|} \sum_{\substack{\tau\in S_n:\\ \tau|_{\pi_\iota} \text{ is increasing } \forall\iota}} \int e^{\frac{1}{12nt}|y-\chi|^2} \prod_{\pi_\iota\in\pi} e^{-\frac{1}{24nt}\left((x_{m_\iota}-\chi^\iota)^2+(x_{l_\iota}-\chi^\iota)^2\right)} |f(y) - f(x)|\lambda^\pi(dy) \tag{55}$$

We can replace the function $f : \overline{\mathbb{W}^n} \to \mathbb{R}$ with its symmetric extension $\overline{f} : \mathbb{R}^n \to \mathbb{R}$, that is the function $\overline{f} : \mathbb{R}^n \to \mathbb{R}$ such that for any $\sigma \in S_n$, $x \in \mathbb{R}^n$ we have $\overline{f}(x_\sigma) = \overline{f}(x)$ and $\overline{f}|_{\overline{\mathbb{W}^n}} = f$. Then, after rescaling y by $\sqrt{t}$, (55) is bounded above by

$$C|\log(t)|^{|\pi|} \sum_{\substack{\tau\in S_n:\\ \tau|_{\pi_\iota} \text{ is increasing } \forall\iota}} \int_{\overline{\mathbb{W}^{|\pi|}}} e^{-\frac{1}{12n}|y|^2} \left(\prod_{\pi_\iota\in\pi} e^{-\frac{1}{24nt}\left((x_{m_\iota}-\chi^\iota)^2+(x_{l_\iota}-\chi^\iota)^2\right)}\right) |\overline{f}(\sqrt{t}\underline{y} + \chi) - \overline{f}(x_\tau)|dy.$$

In the above, $\chi \in \mathbb{R}^n$ is defined by $\chi_\alpha := \chi^\iota$ when $\alpha \in \pi_\iota$, $\underline{y}$ is defined by $\underline{y}_\alpha = y_\iota$ for all $\alpha \in \pi_\iota$, and we have used that $\overline{f}(x) = \overline{f}(x_\tau)$. We have also rewritten the integral with respect to λ^π as an integral with respect to the Lebesgue measure. Since f is a Lipschitz function, it is straightforward to show that $\overline{f}$ is also Lipschitz; therefore, the above expression is bounded above by

$$C|\log(t)|^{|\pi|} \sum_{\substack{\tau\in S_n:\\ \tau|\pi_\iota \text{ is increasing } \forall\iota}} \int_{\overline{\mathbb{W}^{|\pi|}}} e^{-\frac{1}{12n}|y|^2}\left(\prod_{\pi_\iota\in\pi} e^{-\frac{1}{24nt}\left((x_{m_\iota}-\chi^\iota)^2+(x_{l_\iota}-\chi^\iota)^2\right)}\right)\left(\sqrt{t}|\underline{y}|+|\chi-x_\tau|\right)dy.$$

The integrand is non negative and $|\underline{y}| \le |\pi||y|$; therefore, this is bounded above (for a new constant C) by

$$\begin{aligned} &C|\log(t)|^{|\pi|} \sum_{\substack{\tau\in S_n:\\ \tau|\pi_\iota \text{ is increasing } \forall\iota}} \int_{\mathbb{R}^{|\pi|}} e^{-\frac{1}{12n}|y|^2}\left(\prod_{\pi_\iota\in\pi} e^{-\frac{1}{24nt}\left((x_{m_\iota}-\chi^\iota)^2+(x_{l_\iota}-\chi^\iota)^2\right)}\right)\left(\sqrt{t}|\underline{y}|+|\chi-x_\tau|\right)dy \\ \le &C|\log(t)|^{|\pi|} \sum_{\substack{\tau\in S_n:\\ \tau|\pi_\iota \text{ is increasing } \forall\iota}} \qquad (56) \\ &\int_{\mathbb{R}^{|\pi|}} (|y|+1)e^{-\frac{1}{12n}|y|^2}dy\left(\sqrt{t}+|\chi-x_\tau|e^{-\frac{1}{24nt}\sum_{\pi_\iota\in\pi}\left((x_{m_\iota}-\chi^\iota)^2+(x_{l_\iota}-\chi^\iota)^2\right)}\right). \qquad (57) \end{aligned}$$

Now we note that $|\chi - x_\tau| \le \sum_{\alpha=1}^n |\chi_\alpha - x_{\tau(\alpha)}|$, but for all $\alpha \in [\mu_\iota, \nu_\iota]$ (or $[\nu_\iota, \mu_\iota]$) we have $x_{m_\iota} \le x_{\tau(\alpha)}, \chi_\alpha \le x_{l_\iota}$ (or $x_{l_\iota} \le x_{\tau(\alpha)}, \chi_\alpha \le x_{m_\iota}$). Hence, either $|\chi_\alpha - x_{\tau(\alpha)}| \le |\chi_\alpha - x_{m_\iota}|$ or $|\chi_\alpha - x_{\tau(\alpha)}| \le |\chi_\alpha - x_{l_\iota}|$. Note that for any $c > 0$ and $x \in \mathbb{R}$ we have the inequality $|x|e^{-c|x|^2} \le (2ec)^{-\frac{1}{2}}$. Hence, (57) is bounded by

$$\begin{aligned} &C\sqrt{t}|\log(t)|^{|\pi|} \sum_{\substack{\tau\in S_n:\\ \tau|\pi_\iota \text{ is increasing } \forall\iota}} \int_{\mathbb{R}^{|\pi|}} (|y|+1)e^{-\frac{1}{12n}|y|^2}dy \\ \le &C\sqrt{t}|\log(t)|^{|\pi|}, \end{aligned}$$

where we have bounded the integral independently of $|\pi|$, and the constant C has changed between lines. Summing over $\pi \in \Pi_n$, and using that since Π_n is a finite set the constants C in the above expression have a finite maximum, we get, for a new constant $C > 0$ depending only on n,

$$\begin{aligned} \sup_{x\in\overline{\mathbb{W}^n}} |\int u_t(x,y)f(y)m_\theta^{(n)}(dy) - f(x)| \le &C\sqrt{t}\sum_{\pi\in\Pi_n}|\log(t)|^{|\pi|} \\ \le &C\sqrt{t}\log(t)^n \to 0, \quad \text{as } t\to 0. \end{aligned}$$

Note that the last inequality is valid only for $t < 1/e$. Hence, we have the desired uniform convergence, and Proposition 4.8 is proved. □

Proof of Theorem 1.2. As a consequence of Proposition 4.2 and Proposition 4.8, we can apply Proposition 3.9 to our function $\int u_t(x,y)f(y)m_\theta^{(n)}(dy)$ to prove $\int u_{t-s}(Y_s,y)f(y)m_\theta^{(n)}(dy)$ is a local martingale. Suppose that $f \in C_c^\infty(\overline{\mathbb{W}^n})$, i.e. f has an extension to an open set U containing $\overline{\mathbb{W}^n}$ that is smooth and compactly supported. Then since $\int u_t(x,y)f(y)m_\theta^{(n)}(dy)$ converges uniformly to f as $t \to 0$, and f is bounded, there must be some $\varepsilon > 0$

such that $\int u_t(x,y)f(y)m_\theta^{(n)}(dy)$ is bounded for $t \in [0,\varepsilon]$ and $x \in \overline{\mathbb{W}^n}$. We also have $|\int u_t(x,y)f(y)m_\theta^{(n)}(dy)| \leq \frac{1}{(2\pi t)^{n/2}} \int |f(y)| m_\theta^{(n)}(dy)$, which is bounded for $t \in [\varepsilon, \infty)$. Hence, $\int u_t(x,y)f(y)m_\theta^{(n)}(dy)$ is bounded as a function of $(t,x) \in \mathbb{R}_{>0} \times \overline{\mathbb{W}^n}$. It follows that $\int u_{t-s}(Y_s,y)f(y)m_\theta^{(n)}(dy)$ is a true martingale; thus, $\mathbb{E}_x[f(Y_t)] = \int u_t(x,y)f(y)m_\theta^{(n)}(dy)$ for every $f \in C_c^\infty(\overline{\mathbb{W}^n})$.

To extend the equality to more general functions f, we note that the above argument shows non-negativity of $u_t(x,y)$: if $f(x) \geq 0$ for all $x \in \overline{\mathbb{W}^n}$, then $\int u_t(x,y)f(y)m_\theta^{(n)}(dy) \geq 0$. Since this holds for every $f \in C_c^\infty(\overline{\mathbb{W}^n})$, we have that for each $t > 0$ and $x \in \overline{\mathbb{W}^n}$, $u_t(x,y) \geq 0$ for $m_\theta^{(n)}$ almost every $y \in \overline{\mathbb{W}^n}$.

Returning to the case where f is merely bounded and Lipschitz, we can use the non-negativity of $u_t(x,y)$ and Lemma 4.9 to get the bound $|\int u_t(x,y)f(y)m_\theta^{(n)}(dy)| \leq \|f\|_\infty$. Hence, the local martingale $\int u_{t-s}(Y_t,y)f(y)m_\theta^{(n)}(dy)$ is in fact a true martingale for $s \in [0,t]$, and so $\mathbb{E}_x[f(Y_t)] = \int u_t(x,y)f(y)m_\theta^{(n)}(dy)$. Thus, the proof of Theorem 1.2 is completed. □

As a consequence we can also prove the following.

Theorem 4.17. *$m_\theta^{(n)}$ is a stationary measure for Y, and Y is reversible with respect to $m_\theta^{(n)}$.*

Proof. For f a bounded, integrable, Lipschitz continuous function, we have for all $t > 0$

$$\frac{d}{dt}\int \mathbb{E}_x[f(Y_t)]m_\theta^{(n)}(dx) = \frac{d}{dt}\iint u_t(x,y)f(y)m_\theta^{(n)}(dy)m_\theta^{(n)}(dx) \tag{58}$$

$$= 0. \tag{59}$$

The first equality is a consequence of Theorem 1.2 and the second equality is a consequence of Corollary 3.12 and Fubini's theorem. Lemma 4.1 provides the necessary bounds to pass the derivatives through the integrals and to apply Fubini's theorem. Theorem 1.2 allows us to prove the following limit, with an application of dominated convergence theorem justified by Lemma 4.1.

$$\lim_{t\to 0}\int \mathbb{E}_x[f(Y_t)]m_\theta^{(n)}(dx) = \int f(x)m_\theta^{(n)}(dx). \tag{60}$$

We can extend this to any $L^1(m_\theta^{(n)})$ function by a density argument, proving that $m_\theta^{(n)}$ is the stationary measure for Y.

If f and g are bounded, Lipschitz continuous, and integrable; Fubini's theorem gives

$$\begin{aligned}\int \mathbb{E}_x[f(Y_t)]g(x)m_\theta^{(n)}(dx) &= \iint u_t(x,y)f(y)m_\theta^{(n)}(dy)g(x)m_\theta^{(n)}(dx)\\ &= \iint u_t(x,y)g(x)m_\theta^{(n)}(dx)f(y)m_\theta^{(n)}(dy)\\ &= \int \mathbb{E}_y[g(Y_t)]f(y)m_\theta^{(n)}(dy),\end{aligned}$$

where we have used the symmetry $u_t(x,y) = u_t(y,x)$ in the last line. Hence, Y is reversible with respect to $m_\theta^{(n)}$. □

To finish this section, we return to prove Lemma 4.1.

Proof of Lemma 4.1. The proof of this lemma is a simplified version of the methods we applied in earlier in this section; as such, we omit the main details to avoid repetition and instead sketch the proof. Following the arguments used to prove Proposition 4.14, with $\pi = \{\{1\}, \{2\}, \ldots, \{n\}\}$, we can derive a Gaussian bound on the summand in (20). We can then adapt the arguments in Lemma 4.16 to bound the resulting contour integrals, which will have additional factors of k due to the derivatives. In fact, the proof can be significantly simplified in this case as we do not need to consider the $t \to 0$ limit, and therefore we do not need to ensure we get the optimal exponent for t. The above arguments give us a bound in the form of a finite sum of Gaussian kernels, multiplied by a negative power of t, from which the above bounds follow easily (note that for the bound on the x derivatives, we can apply the bound on the y derivatives, as $u_t(x, y) = u_t(y, x)$ which we proved in Lemma 4.10). □

5. Stochastic flows of kernels

5.1. Random walks in random environments

We will begin by recalling the definitions for the discrete counterparts of Howitt–Warren flows and sticky Brownian motions: Random walks in space–time i.i.d. random environments on $\mathbb{Z}$ and their n-point motions. A random walk in a random environment on $\mathbb{Z}$ is simply a random walk on $\mathbb{Z}$ whose transition probabilities are themselves random variables. We define the random environment as a family of i.i.d. [0, 1] valued random variables $\boldsymbol{\omega} = (\omega_{t,x})_{t,x\in\mathbb{Z}}$ with law and expectation $\mathbb{P}$ and $\mathbb{E}$, respectively. We then define a random walk running through realisation of the environment with transition probabilities:

$$P^{\boldsymbol{\omega}}(X(t+1) = x+1 |\ X(t) = x) = \omega_{x,t};$$
$$P^{\boldsymbol{\omega}}(X(t+1) = x-1 |\ X(t) = x) = 1 - \omega_{x,t}.$$

Here, $P^{\boldsymbol{\omega}}$ denotes the law of the RWRE, and $E^{\boldsymbol{\omega}}$ its expectation, both of which depend on the realisation of the environment. The random transition probabilities, $P^{\omega}(X_t = y |\ X_0 = x)$, can be interpreted as a random flow of mass in a fluid, where the quantities describe how a point mass at x has spread through the fluid by time t.

An important idea for studying such models are the n-point motions, we run n random walks independently through a sampling of the environment, and then average out the environment. The averaging over the law of the environment will break the particles' independence, so that the resultant system has interactions. That is, if $X(t) = (X^1(t), \ldots, X^n(t))$ is the n-point motion, then

$$\mathbb{P}(X(t+1) = y | X(t) = x) = \mathbb{E}\left[\prod_{i=1}^{n} P^{\boldsymbol{\omega}}(X^i(t+1) = y_i | X(t) = x_i)\right].$$

Alternatively, we can view the n-point motions as describing the behaviour of n particles thrown into the fluid. Notice now that since the environment is i.i.d, the coordinate processes of the n-point motion behave independently when they are apart. However, when they meet, they interact. In particular, it is a simple consequence of Jensen's inequality that they are more likely to move in the same direction when together than when apart; if we let ω be a copy of an environment variable, then we see

$$\mathbb{E}[\omega^n] + \mathbb{E}[(1-\omega)^n] \geq \mathbb{E}[\omega]^n + \mathbb{E}[1-\omega]^n.$$

A group of particles situated at the same site, x, at time t can break into at most two groups. The probability of a group of n particles breaking into two groups of size k and l, with the k moving to $x+1$ and the l to $x-1$, is

$$\mathbb{E}[\omega_{x,t}^k (1-\omega_{x,t})^l].$$

Hence, the distribution of ω can be viewed as controlling the rate at which groups of particles break up, and the size of the groups they tend to break into. Of course, when clusters of particles are in different locations the corresponding parts of the environment are independent, and thus, the clusters of particles behave independently of each other, so that the distribution of the environment only affects the behaviour of particles that are already in the same location. At the extreme ends, if the environment variables are chosen to be $\{0, 1\}$ valued Bernoulli random variables, then the n-point motions become coalescing simple random walks. On the other hand, it the environment variables are chosen to be deterministic with value $\frac{1}{2}$, then the n-point motions will simply be independent simple random walks. Thus, the strength of the effect of the environment on the interaction between the n-point motions is related to how probable it is that the environment variables take values near 0 or 1.

If we take the diffusive scaling limit of these n-point motions in an environment having a fixed distribution, then the contribution of the environment is overcome in the limit, and we simply end up with independent Brownian motions (assuming the environment variables are mean $1/2$ so there is no drift).

It was shown by Howitt and Warren [9] that by changing the distribution of the ω as we take the diffusive scaling limit, we can obtain Brownian motions which still interact; specifically, they are sticky when they meet, see also Schertzer, Sun and Swart [16]. To preserve the interaction into the diffusive scaling limit the strength of the interaction has to be increased; this means taking the laws of the environment random variables to be closer to that of a Bernoulli random variable. This requirement is made explicit in the second condition of Howitt and Warren's theorem, stated below.

Theorem 5.1. *Suppose $X(t)$ is the n-point motion of a RWRE, where the environment variables have law $\mu^{(\varepsilon)}$ satisfying the following:*

$$\frac{1}{\varepsilon}\int_0^1 (1-2q)\mu^{(\varepsilon)}(dq) \to \beta, \quad \text{as } \varepsilon \to 0;$$

$$\frac{1}{\varepsilon} q(1-q)\mu^{\varepsilon}(dq) \implies \nu(dq), \quad \text{as } \varepsilon \to 0.$$

Then the laws of the processes $(\varepsilon X(\varepsilon^2 t))_{t\geq 0}$ converge weakly to the law of a solution to the Howitt–Warren martingale problem with drift β and characteristic measure ν.

In the special case of $\nu(dx) = \theta/2dx$, where dx is the Lebesgue measure the above result shows the solution to the Howitt–Warren martingale problem is the scaling limit of the Beta random walk in a random environment. That is, choose $\mu^{(\varepsilon)}(dq) = \frac{\Gamma(2\theta\varepsilon)}{\Gamma(\theta\varepsilon)\Gamma(\theta\varepsilon)} q^{\theta\varepsilon-1}(1-q)^{\theta\varepsilon-1}dq$, then for any function $C_b([0,1])$ the dominated convergence theorem implies

$$\begin{aligned}\frac{1}{\varepsilon}\int_0^1 f(q)q(1-q)\mu^{(\varepsilon)}(dq) &= \frac{\Gamma(2\theta\varepsilon)}{\varepsilon\Gamma(\theta\varepsilon)\Gamma(\theta\varepsilon)}\int_0^1 f(q)q^{\theta\varepsilon}(1-q)^{\theta\varepsilon}dq \\ &\to \frac{\theta}{2}\int_0^1 f(q)dq,\end{aligned}$$

using $\Gamma(x) = \frac{\Gamma(x+1)}{x} \sim \frac{1}{x}$ as $x \to 0$. Hence $\frac{1}{\varepsilon}q(1-q)\mu^{(\varepsilon)} \Rightarrow \frac{\theta}{2}dx$; since we also have $\int_0^1 (1-2q)\mu^{(\varepsilon)}(dq) = 0$ for all $\varepsilon > 0$ the theorem implies the convergence of the n-point motions of the Beta random walk in a random environment to solutions of the Howitt–Warren martingale problem with characteristic measure $\frac{\theta}{2}\mathbb{1}_{[0,1]}dx$ and zero drift. This is the key motivator for looking for exact solutions in the sticky Brownian motion case and was used by Barraquand and Rychnovsky in [3] to find Fredholm determinant expressions in the sticky Brownian motions case by taking limits of those found for the Beta random walk in a random environment in [2].

5.2. The Howitt-Warren process

We now briefly introduce stochastic flows of kernels, these are essentially random transition probabilities $(K_{s,t}(x, dy))_{s\le t}$, with the following additional assumptions: independent increments in the sense that for any $t_0 < \cdots , t_n$ the random kernels $K_{t_0,t_1}, \dots , K_{t_{n-1},t_n}$ are independent; stationarity, that is the law of $K_{s,t}$ depends only on $t - s$. They can be thought of as the continuum version of the random environment that is i.i.d. in space and time we considered in the previous section.

The n-point motions of a stochastic flow of kernels are the family of Markov processes $(X_n)_{n=1}^{\infty}$ with X_n taking values in $\mathbb{R}^n$ with transition probabilities

$$\mathbb{P}(X_n(t) \in E |\ X_n(s) = x) = \mathbb{E}\Big[\int_E \prod_{i=1}^{n} K_{s,t}(x_i, dy_i)\Big], \quad \text{for } x \in \mathbb{R}^n,\ E \in \mathcal{B}(\mathbb{R}^n).$$

Notice that this is very similar to the definition of the n-point motions in the RWRE case, with K taking the place of the random transition probabilities.

Le Jan and Raimond [11] have shown that any consistent family of Feller processes are the n-point motions of some stochastic flow of kernels. A family of Feller processes $(X_n)_{n=1}^{\infty}$, $X_n : \mathbb{R}_{>0} \to \mathbb{R}^n$ is consistent, if for any $k \le n$ and any choice of k coordinates from X_n: $(X_n^{i_1}, \dots , X_n^{i_k})$ is equal in law to X_k. For a more complete introduction to stochastic flows of kernels we refer to [11]. When the family of n-point motions, $(X_n)_{n=1}^{\infty}$, are sticky Brownian motions characterised by a Howitt–Warren martingale problem the resulting flow of kernels is called a Howitt–Warren flow. These flows have been studied extensively by Schertzer, Sun, and Swart [16].

Definition 5.2. The stochastic flow of kernels whose n-point motions solve the Howitt–Warren martingale problem, as stated in Definition 2.2, with characteristic measure ν and drift β is called the Howitt–Warren flow with characteristic measure ν and drift β.

Rather than look at the flow directly, we want to consider the Howitt–Warren process a measure valued process that describes how an initial mass is carried by the flow. In our case, we are interested in the case where all mass starts at the origin; thus, we consider the Howitt–Warren process with initial condition δ_0. That is, for the Howitt–Warren flow $(K_{s,t})_{s\le t}$ with characteristic measure ν and drift β we define the Howitt–Warren process started from δ_0 with characteristic measure ν and drift β to be the measure valued process given by

$$\rho_t(A) := K_{0,t}(0, A), \quad \text{for every Borel set } A \subset \mathbb{R}. \tag{61}$$

If $f : \mathbb{R}^n \to \mathbb{R}$ is a symmetric function, then $\mathbb{E}_x[f(X(t))] = \mathbb{E}_x[f(Y(t))]$ for all $x \in \overline{\mathbb{W}^n}$. Hence, we have the following corollary of our main result, Theorem 1.2, that allows us to study the Howitt–Warren process.

Corollary 5.3. *If $f : \mathbb{R}^n \to \mathbb{R}$ is a symmetric function, and its restriction to $\overline{\mathbb{W}^n}$ is a bounded, Lipschitz continuous function, then for a Howitt–Warren flow $(K_{s,t})_{s\le t}$ with characteristic* *measure $\frac{\theta}{2}dx$ and drift zero we have*

$$\mathbb{E}\left[\int f(y)\prod_{i=1}^{n} K_{s,t}(x_i, dy_i)\right] = \int u_{t-s}(x, y) f(y) m_\theta^{(n)}(dy) \quad \text{for all } x \in \overline{\mathbb{W}^n}.$$

From which it clearly follows that for the Howitt–Warren process started from δ_0 with characteristic measure $\frac{\theta}{2}\mathbb{1}_{[0,1]}$ and drift 0, we have that

$$\mathbb{E}\left[\int f(y)\rho_t^{\otimes n}(dy)\right] = \int u_t(0, y) f(y) m_\theta^{(n)}(dy). \tag{62}$$

This allows us to study the process directly, via u, which we will pursue further in the next subsection.

5.3. Atoms of the Howitt-Warren process

Schertzer, Swart, and Sun proved [16, Theorem 2.8] that any Howitt–Warren process is almost surely purely atomic for fixed times t. Thus, almost surely we can write the Howitt–Warren process at time t as a linear combination of delta measures $\rho_t(dy) = \sum_i w_i \delta_{y_i}(dy)$, where the w_i and y_i are both random. One can think of the Howitt–Warren process as the density of an infinite number of sticky Brownian motions evolving in time. Thus, the fact that the process is atomic shows that when the number of particles is very large, the sticky behaviour leads to the formation of large clusters of particles. This is very different from the behaviour of large numbers of independent Brownian motions.

We can think of the collection of pairs (y_i, w_i) as a point process on $\mathbb{R} \times \mathbb{R}_{>0}$. Note that the Howitt–Warren process conserves mass, so that for any $t > 0$ we will have $\sum_i w_i = 1$. However, due to another result of [16], the total number of points will be infinite almost surely This point process has an associated intensity measure γ_t on $\mathbb{R} \times \mathbb{R}_{>0}$ defined by

$$\gamma_t(A_1 \times A_2) = \mathbb{E}\left[\sum_i \mathbb{1}_{\{y_i \in A_1,\ w_i \in A_2\}}\right].$$

We will use this intensity to study the behaviour of the weight of a single atom at a given point in space. See [5] for an introduction to point processes. For any $n \in \mathbb{N}$ and $f : \mathbb{R} \to \mathbb{R}$ that is bounded and Lipschitz continuous, we have the equalities

$$\begin{aligned}\int_{\mathbb{R}\times\mathbb{R}_{>0}} f(y)w^n\ \gamma_t(dy, dw) =& \mathbb{E}\left[\sum_i f(y_i)w_i^n\right] \\ =& \mathbb{E}\left[\int_{\mathbb{D}^n} f^{\otimes n}(y)\rho_t^{\otimes n}(dy)\right] \\ =& \int_{\mathbb{D}^n} f^{\otimes n}(y)u_t^{(n)}(0, y)m_\theta^{(n)}(dy) \\ =& n^{-1}\theta^{1-n}\int_{\mathbb{R}} f(y)u_t^{(n)}(0, (y, \dots, y))dy.\end{aligned} \tag{63}$$

Above, $\mathbb{D}^n := \{(y, \dots, y) \in \mathbb{R}^n :\ y \in \mathbb{R}\}$ and we have written $u_t^{(n)}$ for the transition density u_t on $\mathbb{R}^n$, which we do for the rest of the section to indicate the dependency on dimension. The first equality can be seen by approximating by simple functions, the second is direct from the definitions, the third is a consequence of Corollary 5.3 and the fourth from Definition 1.1.

Equality (63) also shows that the measure $\gamma_t(dy, dw)$ can be written in the form $\gamma_t(y, dw)dy$, and that we have for each $n \in \mathbb{N}$ and almost every $y \in \mathbb{R}$ the equality

$$\int_{\mathbb{R}_{>0}} w^n \gamma_t(y, dw) = n^{-1}\theta^{1-n} u_t^{(n)}(0, (y, \dots, y)). \tag{64}$$

We will study the asymptotic behaviour of the measure $\gamma_t(y, dw)$ for certain choices of y. We can interpret $\gamma_t(y, dw)$ as describing the distribution of the size of an atom at y. However, $\gamma_t(y, dw)$ is not a probability distribution; the measure of any neighbourhood of $w = 0$ is infinite. Introducing size biasing, and instead considering the measure $w\gamma_t(y, dw)$, we do get a finite measure. If we set $n = 1$ in (64), then we can see that the marginal of $w\gamma_t(y, dw)dy$, when w is integrated out, is just a Gaussian measure. If we sample an atom and its size, $(X, W) \in \mathbb{R} \times \mathbb{R}_{>0}$ from the Howitt–Warren process, ρ_t, with probabilities given by the size of the atoms, then the distribution of (X, W) is given by $w\gamma_t(y, dw)dy$. In the following proposition, we study this distribution in the large time limit, conditioned on $X = \sqrt{t}x$. This is analogous to Thiery and Le Doussal's result in [17], where they found that the fluctuations of the transition probabilities of the Beta RWRE were Gamma distributed in the large t limit.

Proposition 5.4. *For each $x \in \mathbb{R}$, we have as $t \to \infty$*

$$t^{-\frac{1}{2}}\sqrt{2\pi}e^{\frac{x^2}{2}} w\gamma_t\left(\sqrt{t}x, \frac{dw}{\sqrt{t}}\right) \Rightarrow \theta\sqrt{2\pi}e^{\frac{x^2}{2}} e^{-\theta\sqrt{2\pi}e^{\frac{x^2}{2}} w} dw.$$

In particular, the convergence is towards the exponential distribution with rate $\theta\sqrt{2\pi}e^{\frac{x^2}{2}}$.

Proof. Note that the measure on the left hand side in the proposition has been normalised and is a probability measure. Thus, it is enough to show pointwise convergence of the moment generating functions on a neighbourhood of 0. With Theorem 1.2, we can rewrite the expression for the moments derived in line (64) as follows.

$$\int_{\mathbb{R}_{>0}} w^n \sqrt{2\pi} t^{-\frac{1}{2}} e^{\frac{x^2}{2}} w\gamma_t\left(\sqrt{t}x, \tfrac{dw}{\sqrt{t}}\right) = \sqrt{2\pi}e^{\frac{x^2}{2}} t^{\frac{n+1}{2}} \int_{\mathbb{R}_{>0}} w^{n+1}\gamma_t\left(\sqrt{t}x, dw\right)$$

$$= \sqrt{2\pi}e^{\frac{x^2}{2}} t^{\frac{n+1}{2}} \frac{u_t^{(n+1)}((0,\dots,0), \sqrt{t}(x,\dots,x))}{(n+1)\theta^n}$$

$$= \sqrt{2\pi} \frac{e^{\frac{x^2}{2}} t^{\frac{n+1}{2}}}{(n+1)\theta^n(2\pi)^{n+1}} \int_{\mathbb{R}^{n+1}} e^{-\frac{1}{2}t|k|^2 - \mathrm{i}\sqrt{t}k\cdot\underline{x}} \sum_{\sigma\in S_{n+1}} \prod_{\substack{\alpha<\beta: \\ \sigma(\beta)<\sigma(\alpha)}} \frac{\mathrm{i}\theta(k_{\sigma(\alpha)} - k_{\sigma(\beta)}) + k_{\sigma(\beta)}k_{\sigma(\alpha)}}{\mathrm{i}\theta(k_{\sigma(\alpha)} - k_{\sigma(\beta)}) - k_{\sigma(\beta)}k_{\sigma(\alpha)}} dk$$

$$= \sqrt{2\pi} \frac{n! e^{\frac{x^2}{2}} t^{\frac{n+1}{2}}}{\theta^n(2\pi)^{n+1}} \int_{\mathbb{R}^{n+1}} e^{-\frac{1}{2}t|k|^2 - \mathrm{i}\sqrt{t}k\cdot\underline{x}} \prod_{\alpha<\beta} \frac{\mathrm{i}\theta(k_\beta - k_\alpha)}{\mathrm{i}\theta(k_\beta - k_\alpha) - k_\alpha k_\beta} dk$$

$$= \sqrt{2\pi} \frac{n! e^{\frac{x^2}{2}}}{\theta^n(2\pi)^{n+1}} \int_{\mathbb{R}^{n+1}} e^{-\frac{1}{2}|k|^2 - \mathrm{i}k\cdot\underline{x}} \prod_{\alpha<\beta} \frac{\mathrm{i}\theta(k_\beta - k_\alpha)}{\mathrm{i}\theta(k_\beta - k_\alpha) - t^{-\frac{1}{2}} k_\alpha k_\beta} dk.$$

To go from the first to the second line we have used line (64) and to go from the third to the fourth line we have used the summation formula from Lemma 4.11. We can now write the moment generating function in terms of the moments.

$$
\begin{aligned}
&\sqrt{2\pi}t^{-\frac{1}{2}}e^{\frac{x^2}{2}}\int_{\mathbb{R}_{>0}} e^{\lambda w} w\gamma_t\left(\sqrt{t}x, \frac{dw}{\sqrt{t}}\right) \\
&= \sum_{n=0}^{\infty}\sqrt{2\pi}\frac{\lambda^n e^{\frac{x^2}{2}}}{\theta^n(2\pi)^{n+1}}\int_{\mathbb{R}^{n+1}} e^{-\frac{1}{2}|k|^2-\mathrm{i}k\cdot\underline{x}}\prod_{\alpha<\beta}\frac{\mathrm{i}\theta(k_\beta-k_\alpha)}{\mathrm{i}\theta(k_\beta-k_\alpha)-t^{-\frac{1}{2}}k_\alpha k_\beta}dk.
\end{aligned}
$$

To take $t\to\infty$, we want to apply the dominated convergence theorem to pass the limit through both the sum and the integral. Similarly to what we have seen previously, line (53) to be precise, the modulus of the product within the integral is bounded above by 1. With this bound we find that the modulus of the nth term of the series is bounded above for all $t>0$ by $\frac{\lambda^n e^{x^2/2}}{\theta^n}$, which is summable for $|\lambda|<\theta$, and so we can take the limit $t\to\infty$ through the sum. Further the bound on the integral allows us to take the limit through the integral. Hence, for $|\lambda|<\theta$, we have

$$
\begin{aligned}
&\lim_{t\to\infty}\sum_{n=0}^{\infty}\sqrt{2\pi}\frac{\lambda^n e^{\frac{x^2}{2}}}{\theta^n(2\pi)^{n+1}}\int_{\mathbb{R}^{n+1}} e^{-\frac{1}{2}|k|^2-\mathrm{i}k\cdot\underline{x}}\prod_{\alpha<\beta}\frac{\mathrm{i}\theta(k_\beta-k_\alpha)}{\mathrm{i}\theta(k_\beta-k_\alpha)-t^{-\frac{1}{2}}k_\alpha k_\beta}dk \\
&= \sum_{n=0}^{\infty}\sqrt{2\pi}\frac{\lambda^n e^{\frac{x^2}{2}}}{\theta^n(2\pi)^{n+1}}\int_{\mathbb{R}^{n+1}} e^{-\frac{1}{2}|k|^2-\mathrm{i}k\cdot\underline{x}}dk = \sum_{n=0}^{\infty}\left(\frac{\lambda e^{-\frac{x^2}{2}}}{\theta\sqrt{2\pi}}\right)^n.
\end{aligned}
$$

This is exactly the moment generating function of an exponential random variable with parameter $\theta\sqrt{2\pi}e^{x^2/2}$, and thus the statement is proved. □

We can reframe the above proposition as a result concerning the convergence of the intensity measure of the point process. This gives us the following corollary.

Corollary 5.5.

$$
t^{-\frac{1}{2}}w\gamma_t\left(\sqrt{t}x, \frac{dw}{\sqrt{t}}\right)dx \Rightarrow \theta e^{-\theta\sqrt{2\pi}e^{\frac{x^2}{2}}w}dxdw.
$$

Proof. This statement follows from the previous proposition by a simple application of the dominated convergence theorem. □

In a remark, Sun, Swart and Schertzer showed that the stationary distribution of the Howitt–Warren process with a uniform interaction measure is given by a Poisson point process with intensity measure $dx\frac{1}{w}e^{-w}dw$ [16]. This remark was based on a similar result by Le Jan and Raimond for sticky flows on the circle [12]. In the same work, the authors show that when the Howitt–Warren process is started from a distribution with infinite mass, it converges towards the stationary solution. The above corollary concerns the case when the starting mass is instead finite.

Remark 5.6. In the preceding corollary, we do not show that the point process itself is converging, only its intensity. However, given the result of Sun, Swart and Schertzer mentioned above, it is reasonable to expect that the point process should converge a Poisson point process. To prove such a result, one could consider the convergence of the kth correlation measures, for arbitrary $k\in\mathbb{N}$ (rather than just the $k=1$ case considered above). Similar identities to (64) exist for the kth correlation measures, and we believe convergence to a Poisson point process can be shown by generalising the arguments above.

Another aspect of Thiery and Le Doussal's work on the Beta RWRE, [17], was to derive a Fredholm determinant formula, which, after some formal manipulations, was used to analyse the behaviour of the transition probabilities in the large deviation regime. In our case, we can derive the following Fredholm determinant formula, which is analogous to formula (52) in [17].

Proposition 5.7.

$$1+\sum_{n=1}^{\infty}\int_{\mathbb{R}_{>0}}\frac{(\lambda w)^n}{n!(n-1)!}\gamma_t(y,dw)=\theta\det\left(I+\frac{\lambda}{\theta 2\pi}K\right). \tag{65}$$

Above, the determinant is a Fredholm determinant and K is an integral operator on $L^2(\mathbb{R})$ with kernel

$$K(x,y)=\frac{xye^{-\frac{1}{4}t(x^2+y^2)}}{\mathrm{i}\theta(y-x)+xy}. \tag{66}$$

Proof. Eq. (64) and the summation formula in Lemma 4.11 give the equality

$$\int_{\mathbb{R}>0}w^n\gamma_t(y,dw)=\frac{(n-1)!}{\theta^{n-1}(2\pi)^n}\int_{\mathbb{R}^n}e^{-\frac{1}{2}t|k|^2-\mathrm{i}k\cdot\underline{y}}\prod_{\alpha<\beta}\frac{\mathrm{i}\theta(k_\beta-k_\alpha)}{\mathrm{i}\theta(k_\beta-k_\alpha)-k_\alpha k_\beta}dk.$$

The proof is completed by the following identity, which is a consequence of the equalities (A.1) and (D.1) in [17]

$$\sum_{\sigma\in S_n}\prod_{\alpha<\beta}\frac{\mathrm{i}\theta(k_{\sigma(\beta)}-k_{\sigma(\alpha)})}{\mathrm{i}\theta(k_{\sigma(\beta)}-k_{\sigma(\alpha)})-k_{\sigma(\alpha)}k_{\sigma(\beta)}}=n!\det_{1\le\alpha,\beta\le n}\left[\frac{k_\beta k_\alpha}{\mathrm{i}\theta(k_\beta-k_\alpha)+k_\alpha k_\beta}\right].\quad\square \tag{67}$$

It would be interesting to use the above formula to analyse the behaviour of γ_t in the large deviation regime: $\frac{y}{t}$ converges to a non zero number as $t\to\infty$, where we expect the appearance of GUE Tracy–Widom fluctuations. Unfortunately, the above Fredholm determinant is not in an ideal form for asymptotic analysis. We would instead want an analogue of the conjectured formula (92) in [17]. In [3], Barraquand and Rychnovsky considered the tails of the Howitt–Warren process, $\rho_t([tx,\infty])$, and derived a Fredholm determinant formula for the Laplace transform via a scaling limit from the Beta random walk in a random environment, with which they were able to prove the existence of GUE fluctuations. In a non-rigorous work Thiery and Le Doussal [17] show the existence of GUE fluctuations for the transition probabilities of the Beta RWRE evaluated at a point. This suggests the following conjecture for the fluctuations of the individual atoms.

Conjecture 5.8. *If $X_{x,t}$ is a random variable on $\mathbb{R}$ with law $\sqrt{2\pi t}e^{-t\frac{x^2}{2}}w\gamma_t(tx,dw)$, then there are functions $J:\mathbb{R}\to\mathbb{R}$ and $\sigma:\mathbb{R}\to\mathbb{R}$ such that*

$$\lim_{t\to\infty}\mathbb{P}\left(\frac{\log(X_{x,t})+J(x)t}{t^{1/3}\sigma(x)}<z\right)=F_{GUE}(z), \tag{68}$$

where F_{GUE} is the cumulative function for the Tracy–Widom GUE distribution.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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