Using EDT0L systems to solve some equations in the solvable Baumslag-Solitar groups

Andrew Duncan, Alex Evetts, Derek F. Holt, Sarah Rees

March 27, 2023

Abstract

We investigate the solution sets to equations in the solvable Baumslag-Solitar groups $BS(1, k)$, $k \geq 2$, and show that these sets are represented by EDT0L languages in some cases. In particular, we prove that the multiplication table of such a group forms an EDT0L language with respect to a specific natural normal form for group elements.

1 Introduction

This work was motivated in part by Ciobanu, Diekert and Elder's proof [2] that the solution sets of systems of equations over free groups are EDT0L (which was subsequently generalised to virtually free groups by Diekert and Elder, hyperbolic groups by Ciobanu and Elder [3], and right-angled Artin groups by Diekert, Jeż and Kufleitner [6]). Evetts and Levine [9] proved that the same thing is true in virtually abelian groups. Further motivation was provided by the proof of Kharlampovich, López and Myasnikov [13] that it is decidable whether or not an equation over a group G is solvable, for all G in a family of groups that includes the solvable Baumslag-Solitar groups and groups with structure $A \wr \mathbb{Z}$ for A finitely generated abelian (such as the lamplighter group when $|A| = 2$). We wanted to understand for which types of groups solution sets to systems of equations might be EDT0L languages, and why this might be a natural family of languages in which to find such solution sets.

In this article we have attempted to provide an accessible description of this relatively unknown family of languages, and of how we may find solutions for group equatons within it. Using the solvable Baumslag-Solitar groups as our 'testbed', we have examined various rather elementary equations over these groups, proved some to have solution sets that are EDT0L languages, and provided examples of others that seem not to be (although we have not yet proved conclusively that they are not).

The following Section 2 contains the definitions of ET0L and EDT0L systems and related languages, relating them to the better known families of context-free and indexed languages, and listing some of the operations under which the sets of ET0L and EDT0L languages are closed. The Baumslag-Solitar groups are studied from Section 3 onwards, with definitions and the construction of two different normal forms in Section 3, preliminary results in Section 4 and the consideration of particular equations relating to centralisers, conjugacy, multiplication and inversion, in Sections 5–7. In Section 8 we describe a solution set for a centraliser equation that we believe is not EDT0L, and explain why we believe this, while not giving a proof of that fact.

2 ET0L and EDT0L systems

ET0L languages, introduced by Rozenberg [15], generalise both context-free and 0L languages, and are most naturally defined through their grammars, known as ET0L systems. An important component of such a grammar is a set of tables.

Definition 1. *Let $\mathcal{V}$ be a finite alphabet. A* table *for $\mathcal{V}$ is a finite subset of $\mathcal{V} \times \mathcal{V}^*$, considered as a finite collection of rewriting rules of the form $v \to w_1, \ldots, w_r$, for $v \in \mathcal{V}$, $w_i \in \mathcal{V}^*$, whose application replaces each instance of v with any one of $w_1, \cdots, w_r$.*

We use the conventions that (1) when a table is applied to a word it must be applied to every letter within that word and (2) if a table does not specify a rewrite for some letter in $\mathcal{V}$, then applying the table fixes that letter. A table will act on the right of a word, so we write $w\mathrm{T}$ for the result of applying a table T to a word w. Applying one table after another will be denoted by concatenation; note that when applying a string of tables that string should be read from left to right. Note that if the right hand side of each rule in a table contains only one word, then that table defines a free monoid endomorphism of $\mathcal{V}^*$.

Definition 2. *An* ET0L *system is a tuple* $\mathcal{H} = (\mathcal{V}, \mathcal{U}, \mathcal{R}, v_0)$ *where*

1. *$\mathcal{V}$ is a finite alphabet,*
2. *$\mathcal{U} \subset \mathcal{V}$ is the set of terminals,*
3. *$\mathcal{R}$ is a regular subset of $\mathcal{T}^*$ for some finite set of tables $\mathcal{T}$ for $\mathcal{V}$, called the* rational control *of $\mathcal{H}$.*
4. *$v_0 \in \mathcal{V}^*$ is a chosen word called the* start word *or* axiom.

The language $\{v \in \mathcal{U}^* : v = v_0\mathrm{R}$ for some $\mathrm{R} \in \mathcal{R}\}$ is the language of the system $\mathcal{H}$. A language arising from an ET0L *system is called an* ET0L *language.*

It is proved in [4] and [8] that the class of ET0L languages is properly contained in the class of indexed languages, as defined in [1].

Definition 3. *An* EDT0L *system is an* ET0L *system $\mathcal{H} = (\mathcal{V}, \mathcal{U}, \mathcal{R}, v_0)$ where each table in $\mathcal{T}$, the alphabet of $\mathcal{R}$, is a free monoid endomorphism $\mathcal{V}^* \to \mathcal{V}^*$; that is, each rule in a table contains a unique word on the right hand side.*

Let $\mathcal{G} = (\mathcal{V}, \mathcal{U}, \mathcal{P}, s_0)$ be a context-free grammar, with set $\mathcal{V}$ of terminals, $\mathcal{U} \subseteq \mathcal{V}$, $\mathcal{P}$ of productions, and start variable s_0; for such a grammar the left hand side of any production is necessarily in $\mathcal{V} \setminus \mathcal{U}$, then we can form an ET0L system $(\mathcal{V}, \mathcal{U}, \mathcal{R}, s_0)$ with the same language as $\mathcal{G}$ as follows:

For each $v \in \mathcal{V}$, we define $\mathcal{P}_v$ to be the set of all productions in $\mathcal{P}$ with left hand side v. If the productions in $\mathcal{P}_v$ are $v \to u_1, \ldots, v \to u_k$, then we define r_v to be the rule $v \to u_1, u_2, \cdots u_k, v$. Note that the fact that v is on the right-hand side of the rule indicates that we are not obliged to apply a non-trivial production to v.

We define the table T_v to be the singleton set $\{r_v\}$, $\mathcal{T}$ to be $\{\mathrm{T}_v : v \in \mathcal{V}\}$, and then $\mathcal{R}$ to be $\mathcal{T}^*$. The language of the ET0L system $(\mathcal{V}, \mathcal{U}, \mathcal{R}, s_0)$ is the context-free language generated by the grammar $\mathcal{G}$. But note that context-free languages exist that do not arise as the languages of EDT0L systems [7].

The following two lemmas are standard (see [16]).

Lemma 4. *The class of* EDT0L *languages contains the class of regular languages.*

Lemma 5. *The classes of* EDT0L *and* ET0L *languages are both closed under the following operations:*

1. *finite union,*
2. *intersection with regular languages,*
3. *concatenation,*
4. *Kleene star,*
5. *image under free monoid homomorphisms.*

The class of ET0L *languages is additionally closed under taking preimages of free monoid homomorphims.*

3 The solvable Baumslag-Solitar groups: definition and normal forms

We use the notation $\mathbb{N} := \{n : n \in \mathbb{Z}, n > 0\}$, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

The solvable Baumslag-Solitar groups are the groups defined by the presentations

$$\mathsf{BS}(1, k) = \langle a, b \mid b^{-1}ab = a^k \rangle$$

for some fixed $k \in \mathbb{Z} \setminus \{0\}$. We shall assume here that $k > 0$, since the groups with $k < 0$ are similar, but with additional minor complications. Although we have not checked all of the details, we believe that all of the results proved in this section remain true when k is negative, but their proofs require subdivisions into more cases than when $k > 0$. We shall assume further that $k > 1$, since $\mathsf{BS}(1, 1)$ is free abelian. For ease of notation, we shall abbreviate $\mathsf{BS}(1, k)$ as BS.

Note that BS is a split extension of the infinite abelian group $N = \langle a \rangle^{\mathsf{BS}}$ (the normal closure of the subgroup $\langle a \rangle$ in BS) by the infinite cyclic group $\langle b \rangle$.

We denote by $a^{\frac{1}{k}}$ the element bab^{-1} and note that $(a^{\frac{1}{k}})^k = a$. Similarly we represent $b^j a^i b^{-j}$ by $a^{\frac{i}{k^j}}$, where $i, j \in \mathbb{Z}$, $i \neq 0$, $j > 0$. We also write A for a^{-1}, and B for b^{-1}.

With this notation, we observe that any element g of BS can be represented as a product $b^r \alpha^u$ for $\alpha \in \{a, \text{A}\}$, where $r \in \mathbb{Z}$ and

$$u = \left(\frac{i_m}{k^m} + \frac{i_{m-1}}{k^{m-1}} + \cdots + \frac{i_1}{k}\right) + s, \quad m, s \in \mathbb{N}_0,\ i_j \in \{0, \ldots, k-1\},$$

and we interpret α^u as the product

$$(b^m \alpha b^{-m})^{i_m} (b^{m-1} \alpha b^{-m+1})^{i_{m-1}} \cdots (b \alpha b^{-1})^{i_1} \alpha^s.$$

We require that either $m = 0$ (that is, the sequence $i_1, \ldots, i_m$ is empty) or $m > 0$ and $i_m \neq 0$. We shall call $b^r \alpha^u$ the *fractional representation* of the group element, and we shall call s and $i_m/k^m + i_{m-1}/k^{m-1} + \cdots + i_1/k$ the *integral* and *fractional* parts of the exponent u.

The fractional representation provides a mechanism for representing each element of BS by a unique string of symbols from the alphabet $\{b, \text{B}, 0, 1, \ldots, k{-}1, \centerdot, +, -\}$. The group element $b^r \alpha^u$ is represented by the string $\hat{w}$ given by

$$\left.\begin{array}{r} \beta \cdots \beta + i_m i_{m-1} \cdots i_1 \centerdot s_0 s_1 \cdots s_p \text{ when } \alpha = a, \\ \beta \cdots \beta - i_m i_{m-1} \cdots i_1 \centerdot s_0 s_1 \cdots s_p \text{ when } \alpha = \text{A}, \\ \text{where } p \geq -1,\ i_1, \ldots, i_m, s_0, \ldots s_p \in \{0, 1, \ldots, k{-}1\}, \\ \text{and if } p \neq -1, \text{then } s_p \neq 0 \text{ and } s = s_0 + s_1 k + s_2 k^2 + \cdots + s_p k^p, \end{array}\right\} \quad (1)$$

where β is b when $r \geq 0$, B when $r < 0$, and where the prefix $\beta \cdots \beta$ consists of $|r|$ copies of β (and in future will generally be abbreviated as β^r). By convention, we define $p = -1$ if the sequence $s_0, \ldots, s_p$ is empty and $s = 0$.

Then the suffix $\pm i_m i_{m-1} \cdots i_1 \centerdot s_0 \cdots s_p$ of $\hat{w}$ is precisely the representation of u in base k written backwards. The symbol $\centerdot$ (which would be the decimal point when $k = 10$) is known as the *radix point*. We shall call the subwords $i_m i_{m-1} \cdots i_1$ and $s_0 \cdots s_p$ the *fractional* and *integral* parts of $\hat{w}$. Note that either or both of these could be empty, and the representation of the element β^t is $\beta^t + \centerdot$ for $t \geq 0$.

We denote the representation of $g \in \mathsf{BS}$ defined in (1) by $\mathsf{NF}^{\text{frac}}(g)$, and define $\mathsf{NF}^{\text{frac}} := \{\mathsf{NF}^{\text{frac}}(g) : g \in \mathsf{BS}\}$. We can think of $\mathsf{NF}^{\text{frac}}$ as a normal form for

BS, although of course the elements of its alphabet are not all within the group.

Now the product

$$b^r(b^m\alpha b^{-m})^{i_m}(b^{m-1}\alpha b^{-m+1})^{i_{m-1}}\cdots(b\alpha b^{-1})^{i_1}\alpha^s$$

that represents $b^r\alpha^u$ can be freely reduced to give the word

$$w = b^t\alpha^{i_m}\mathrm{B}\alpha^{i_{m-1}}\mathrm{B}\cdots\alpha^{i_1}\mathrm{B}\alpha^s, \tag{2}$$

where $t = r + m$.

When $t \geq 0$, we shall refer to the subwords b^t, $\alpha^{i_m}\mathrm{B}\alpha^{i_{m-1}}\mathrm{B}\cdots\alpha^{i_1}\mathrm{B}$, and α^s of w as its *left, central and right* subwords, respectively, but if $t < 0$ we define the left subword to be empty, the central subword to be $\mathrm{B}^{|t|}\alpha^{i_m}\mathrm{B}\alpha^{i_{m-1}}\mathrm{B}\cdots\alpha^{i_1}\mathrm{B}$ and the right subword to be α^s. We call the subwords $\alpha^{i_j}\mathrm{B}$ together with the letters B in a $\mathrm{B}^{|t|}$ prefix the *components* of the central subword.

Observe that the integral part s of $\hat{w}$ is the base k representation of the exponent of α in the right subword of w, whereas the fractional part of $\hat{w}$, together with the value of t when t is negative, determines the central subword of w.

We denote the representation of $g \in \mathsf{BS}$ defined in (2) by $\mathsf{NF}(g)$, and put $\mathsf{NF} := \{\mathsf{NF}(g) : g \in \mathsf{BS}\}$. Note that both NF and $\mathsf{NF}^{\mathrm{frac}}$ are regular languages.

4 Preliminary results

Proposition 6. *Fix a constant $r \in \mathbb{Z}$. The subset $\mathsf{NF}_r = \{\mathsf{NF}(g) : g = b^r\alpha^u$ for some $u\}$ of words in NF corresponding to this value of r forms an* EDT0L *language.*

Proof. Let $\mathcal{U} = \{a, \mathrm{A}, b, \mathrm{B}\}$ and $\mathcal{V} = \{a, \mathrm{A}, b, \mathrm{B}, S, T\}$, and define a set $\mathcal{T}$ of endomorphisms of $\mathcal{V}^*$ (each of which is the sole entry of a table within $\mathcal{T}$) as follows:

$$\begin{aligned}
\phi_{aj}\colon & S \mapsto Sa^j\textsc{b}, \text{ for each } 0 \le j \le k-1 \\
\phi_{\mathrm{A}j}\colon & S \mapsto S\mathrm{A}^j\textsc{b}, \text{ for each } 0 \le j \le k-1 \\
\psi_{aj}\colon & S \mapsto bSa^j\textsc{b}, \text{ for each } 0 \le j \le k-1 \\
\psi_{\mathrm{A}j}\colon & S \mapsto bS\mathrm{A}^j\textsc{b}, \text{ for each } 0 \le j \le k-1 \\
\theta\colon & S \mapsto bS \\
\mu_a\colon & T \mapsto Ta \\
\mu_{\mathrm{A}}\colon & T \mapsto T\mathrm{A} \\
\nu\colon & S \mapsto \epsilon, T \mapsto \epsilon.
\end{aligned}$$

We define

$$\begin{aligned}
\Phi_a &:= \{\phi_{a0}, \ldots, \phi_{a,k-1}\}, \qquad \Phi_A := \{\phi_{A0}, \ldots, \phi_{A,k-1}\}, \\
\Psi_a &:= \{\psi_{a0}, \ldots, \psi_{a,k-1}\}, \qquad \Psi_A := \{\psi_{A0}, \ldots, \psi_{A,k-1}\}.
\end{aligned}$$

First, suppose that $r \ge 0$. In this case, the language in question is

$$\{b^r b^m \alpha^{i_m}\textsc{b}\alpha^{i_{m-1}}\textsc{b}\cdots\alpha^{i_1}\textsc{b}\alpha^s\colon m \ge 0, s \ge 0\}.$$

This is the language of the EDT0L system $(\mathcal{V}, \mathcal{U}, \mathcal{R}_1, ST)$ with rational control $\mathcal{R}_1$ given by $\mathcal{R}_1 := \mathcal{R}_{1a} \mid \mathcal{R}_{1\mathrm{A}}$ (recall that the symbol '|' denotes union in the standard notation for regular sets) where

$$\mathcal{R}_{1a} = \theta^r\,(\Psi_a^* \setminus \Psi_a^*\psi_{a0})\,\mu_a^*\nu,$$

and $\mathcal{R}_{1\mathrm{A}}$ is defined similarly with A in place of a.

Now suppose that $r < 0$. If we also have $t = m+r < 0$, then the left subword is empty and there are only finitely many possibilities for the central subword, so the language in this case regular. For the case $t \ge 0$, we need to show that the following language is EDT0L:

$$\{b^t\alpha^{i_{t+|r|}}\textsc{b}\alpha^{i_{t+|r|-1}}\textsc{b}\cdots\alpha^{i_{t+1}}\textsc{b}\alpha^{i_t}\textsc{b}\cdots\alpha^{i_1}\textsc{b}\alpha^s\colon t \ge 0, s \ge 0\}.$$

This is the language of the EDT0L system $(\mathcal{V}, \mathcal{U}, \mathcal{R}_2, ST)$ with rational control $\mathcal{R}_2$ given by $\mathcal{R}_2 := \mathcal{R}_{2a} \mid \mathcal{R}_{2\mathrm{A}}$, where

$$\mathcal{R}_{2a} := \Phi_a^{|r|}\,(\Psi_a^* \setminus \Psi_a^*\psi_{a0})\,\mu_a^*\nu,$$

and $\mathcal{R}_{2\mathrm{A}}$ is defined similarly with A in place of a. □

The following lemma will be used several times in the proofs in Section 6 below.

Lemma 7. *Suppose that $r > 0$. Fix constants $n_0 \in \mathbb{N}$, $\lambda, c \in \mathbb{Z}$, let α be equal to either a or to A, and let $w \in (\alpha \mathrm{B}^+ \mid \alpha^2 \mathrm{B}^+ \mid \cdots \mid \alpha^{k-1}\mathrm{B}^+)^*$ be a constant word (with k fixed as above). Let $\{s_n\}_{n\in\mathbb{N}}$ be a fixed sequence of integers such that, for each $n \geq n_0$, we have*

$$s_{n+1} = k^r s_n + \lambda \quad \text{and} \quad s_{n+1} > s_n.$$

Then, for some $n' \in \mathbb{N}$, the following subset of $\{a, \mathrm{A}, b, \mathrm{B}\}^*$ is EDT0L:

$$\{b^{rn+c} w \alpha^{s_n} : n \geq n'\}.$$

Proof. We prove this for $\alpha = a$; the case $\alpha = \mathrm{A}$ is similar. First, consider the case where $\lambda \geq 0$. Define endomorphisms ϕ, ψ of $\{a, \mathrm{A}, b, \mathrm{B}, S\}^*$ as follows:

$$\begin{aligned} &\phi\colon S \mapsto b^r S a^\lambda,\ a \mapsto a^{k^r} \\ &\psi\colon S \mapsto w. \end{aligned}$$

Choose $n' \geq n_0$ large enough so that $rn' + c \geq 0$. The EDT0L system with axiom $b^{rn'+c} S a^{s_{n'}}$ and rational control $\mathcal{R} = \phi^* \psi$ gives the language $\{b^{rn+c} w a^{s_n} : n \geq n'\}$.

Now suppose that $\lambda < 0$. Choose $n' \geq n_0$ large enough so that $rn' + c \geq 0$ and so that $s_{n'} > |\lambda|$. Consider an extended alphabet $\{a, \mathrm{A}, \mathsf{a}, b, \mathrm{B}, S\}$, endomorphisms

$$\begin{aligned} &\phi\colon S \mapsto b^r S, a \mapsto a^{k^r}, \mathsf{a} \mapsto a^{k^r - 2}\mathsf{a} \\ &\psi\colon S \mapsto w, \mathsf{a} \mapsto a, \end{aligned}$$

axiom $b^{rn'+c} S a^{s_{n'} - |\lambda|} \mathsf{a}^{|\lambda|}$, and rational control $\phi^*\psi$. We claim that this EDT0L system produces the language in question. To see this, consider an application of ϕ:

$$b^{rn+c} S a^{s_n - |\lambda|} \mathsf{a}^{|\lambda|} \mapsto b^{r(n+1)+c} S a^{k^r(s_n - |\lambda|)} (a^{k^r - 2}\mathsf{a})^{|\lambda|}.$$

At each such an application, we increase the number of a's from $s_n - |\lambda|$ to $k^r(s_n - |\lambda|) + (k^r - 2)|\lambda| = k^r s_n - 2|\lambda| = s_{n+1} - |\lambda|$, and keep the number of a's fixed at $|\lambda|$. If this were the final application of ϕ, we would then apply ψ to convert the a's to a's, and insert w:

$$b^{r(n+1)+c} S a^{k^r(s_n - |\lambda|)} (a^{k^r-2}\mathsf{a})^{|\lambda|} \mapsto b^{r(n+1)+c} w a^{k^r s_n - |\lambda|} = b^{r(n+1)+c} w a^{s_{n+1}}.$$

$\square$

5 Centralisers and conjugacy

Proposition 8. *Centralisers of elements in* $\mathsf{BS}(1,k)$ *are* EDT0L*; that is, for fixed* $g \in \mathsf{BS}$*, the set* $C_g := \{w \in \mathsf{NF} : wg =_{\mathsf{BS}} gw\}$ *is* EDT0L*.*

Proof. This follows from Lemma 4 when $g = 1$, because then $C_g = \mathsf{NF}$, which is a regular language.

If $1 \neq g \in N$, then $C_g = N$, and C_g consists of those words in NF with $t = m$, which is EDT0L by Proposition 6.

If $g \notin N$, then $C_N(g) = 1$, so $C_{\mathsf{BS}}(g)$ is an infinite cyclic group $\langle g' \rangle$, for which $g' \notin N$ and $C_{\mathsf{BS}}(g) = C_{\mathsf{BS}}(g')$. In this case we replace g by g', for ease of notation.

So we need to prove that $\{\mathsf{NF}(g^n) : n \in \mathbb{Z}\}$ is EDT0L and, since this is a union of the positive and negative powers of g, it is enough (by Lemma 5) to prove that $\{\mathsf{NF}(g^n) : n \geq 0\}$ is EDT0L.

We use the notation introduced above for $w := \mathsf{NF}(g)$, and we have $w = b^{t'} w_c \alpha^s$, where $b^{t'}$, w_c and α^s are its left, central and right subwords, with $\alpha = a$ or A. (So $t' = t$ when $t \geq 0$ and $t' = 0$ when $t < 0$.) Note that the fractional representation of w is of the form $b^r \alpha^u$ with $r := t - m$ and $0 \leq u \in \mathbb{Q}$, and so that of g^n is $b^{rn} \alpha^{u_n}$ with $u_n = u(1 + k^r + \cdots + k^{(n-1)r})$. We consider two cases.

Case 1: $t < m$ (so w has more occurrences of B than of b). So $t - m < 0$. Since the power of B in the fractional representation of g^n is $\mathrm{B}^{n(m-t)}$, the central subword of $\mathsf{NF}(g^n)$ must have at least $n(m - t)$ components and so there exists $n' \geq 0$ such that this number of components is at least t' for all $n \geq n'$.

For $n \geq n'$, let $\mathsf{NF}(g^n) = b^{t'_n} w_{cn} \alpha^{s_n}$, where w_{cn} is its central subword, and let $w_{cn} = \gamma_n \beta_n$, where β_n consists of its final t' components. Note that the sequence u_n is bounded above by $u/(1 - k^r)$ in this case, and so s_n is also bounded above. We claim that $\mathsf{NF}(\beta_n \alpha^{s_n} w)$ has empty left subword and that its central subword has exactly m components. From this it follows that $\mathsf{NF}(g^{n+1}) = b^{t'_n} \gamma_n \mathsf{NF}(\beta_n \alpha^{s_n} w)$, so $w_{c,n+1}$ has $m - t'$ more components than w_{cn}.

To prove the claim, note that, since β_n has exactly t' components, $\beta_n \alpha^{s_n} b^{t'}$ is equal in BS to some power α^{x_n} of α with $x_n \in \mathbb{Z}$ and $x_n \geq 0$. Then, when we put $\beta_n\alpha^{s_n}w =_{\mathsf{BS}} \alpha^{x_n}w_c\alpha^s$ into normal form, we may move some of the m letters $\textsc{b}$ in w_c to the left, but we end up with a central subword consisting of m components followed by a power of α (in fact $\alpha^{s_{n+1}}$).

Now, by considering the fractional representation of g^n described above and noting that $r < 0$ in this case, we see that s_n, which is the integral part of u_n, is a non-decreasing sequence which (as we observed earlier) is bounded above, and so s_n must be constant for sufficiently large n. Also, β_n and hence also the suffix $\beta_n\alpha^{s_n}$ of $\mathsf{NF}(g^n)$ must eventually repeat, and it follows easily that the set $\{\mathsf{NF}(g^n) : n \in \mathbb{Z}\}$ is regular, and hence EDT0L by Lemma 4.

Case 2: $t > m$ (so w has more occurrences of b than of $\textsc{b}$). Then $r = t - m > 0$. Recall that we are denoting the fractional representations of g and g^n by $b^r\alpha^u$ and $b^{rn}\alpha^{u_n}$, respectively, where $u_n = u(1+k^r+\cdots+k^{(n-1)r})$. Since m is the number of components in the central subword of $\mathsf{NF}(g)$, we have $k^m u \in \mathbb{Z}$. So, by choosing $n' > 0$ such that $(n'-1)r \geq m$, we have $k^{(n-1)r}u \in \mathbb{Z}$ for all $n \geq n'$. Then the fractional part of u_n remains constant for all $n \geq n'$, and hence the central subword of $\mathsf{NF}(g^n)$ is the same word w'_{c} for all such n, and also the sequence $s_n = \lfloor u_n \rfloor$ is strictly increasing for $n \geq n'$.

So for $n \geq n'$, if $\mathsf{NF}(g^n) = b^{t_n}w'_{\mathsf{c}}\alpha^{s_n}$, then

$$b^{t_{n+1}}w'_{\mathsf{c}}\alpha^{s_{n+1}} =_{\mathsf{BS}} b^{t_n}w'_{\mathsf{c}}\alpha^{s_n}b^r\alpha^u$$

and so $t_{n+1} = t_n + r$ and

$$\alpha^{s_{n+1}} =_{\mathsf{BS}} w'^{-1}_{\mathsf{c}}b^{-r}w'_{\mathsf{c}}\alpha^{s_n}b^r\alpha^u =_{\mathsf{BS}} [w'_{\mathsf{c}}, b^r]\alpha^{k^r s_n+u}.$$

Now the commutator $[w'_{\mathsf{c}}, b^r]$ is some fixed power α^κ of α for some $\kappa \in \mathbb{Q}$, and $s_{n+1} = k^r s_n + \lambda$ where $\lambda := u + \kappa$ is a constant that must lie in $\mathbb{Z}$.

It follows from $t_{n+1} = t_n + r$ that $t_n = rn + c$ for some constant $c \in \mathbb{Z}$. Since we also have $s_{n+1} = k^r s_n + \lambda$, we can apply Lemma 7 with $w = w'_{\mathsf{c}}$ to deduce that $\{\mathsf{NF}(g^n) : n \geq n'\}$ and hence also $\{\mathsf{NF}(g^n) : n \geq 0\}$ is EDT0L. $\square$

Proposition 9. *The set of conjugators of fixed pairs of elements is* EDT0L*; that is, for fixed $g, h \in \mathsf{BS}$, the set $\mathsf{Con}_{hg} := \{w \in \mathsf{NF} : wh =_{\mathsf{BS}} gw\}$ is* EDT0L.

Proof. The solution set is either empty or a right coset $C_g x$ of the centraliser C_g of g, for some fixed $x \in \mathsf{BS}$. Let $b^{r_x}\alpha^{u_x}$ (where u_x might be negative) be the fractional representation of x.

If $g = 1$ then $\mathsf{Con}_{hg} = \mathsf{NF}$ when $h = 1$ and is empty otherwise and, if $g \in N \setminus \{1\}$, then $C_g = N$, and $\mathsf{Con}_{hg}(hg)$ is either empty, or equal to the set of normal form words for which $t - m = r_x$. The result holds in these cases by Lemma 4 and Proposition 6.

Otherwise, as we saw in the previous proof, $C_g = C_1 \cup C_2$ is the disjoint union of two sets, where C_1 (from Case 1) is regular and C_2 is the union of a finite set with a set C_2' of the form $\{b^{rn+c} w_{\mathsf{c}}' \alpha^{s_n} : n \geq n'\}$ for some fixed $n' \geq 0$ where $r > 0$ and, for $n \geq n'$, we have $s_{n+1} = k^r s_n + \lambda$ and $c, \lambda \in \mathbb{Z}$ are constants.

Suppose that Con_{hg} is nonempty. Then, by a similar argument to that used in the proof of Case 1 of Proposition 8, we see that, for all but finitely many of the words in C_1, multiplication on the right by x affects only a suffix of bounded length, and so $\mathsf{NF}(C_1 x)$ is regular and hence EDT0L by Lemma 4.

It remains to consider the set $\mathsf{NF}(C_2' x)$ with C_2' as above. If $r_x \geq 0$ then, for $n \geq n'$ and $g = b^{rn+c} w_{\mathsf{c}}' \alpha^{s_n} \in C_2$, we have $gx =_{\mathsf{BS}} b^{rn+c} w_{\mathsf{c}}' b^{r_x} \alpha^{u_x} \alpha^{k^{r_x} s_n}$. Now $\mathsf{NF}(b^{rn+c} w_{\mathsf{c}}' b^{r_x} \alpha^{u_x}) = b^{rn+c'} w_{\mathsf{c}}'' \alpha^{\lambda_x}$ for some constant central subword w_{c}'' and $c, \lambda_x \in \mathbb{Z}$ and so, for sufficiently large n, we have $\mathsf{NF}(gx) = b^{rn+c'} w_{\mathsf{c}}'' \alpha^{s_n'}$, where $s_n' = k^{r_x} s_n + \lambda_x$. Then s_n' also satisfies the recurrence relation $s_{n+1}' = k^r s_n' + \lambda'$ for some constant $\lambda' \in \mathbb{Z}$, and so the set $\mathsf{NF}(C_2' x)$ is EDT0L by Lemma 7.

Now suppose that $r_x < 0$. Since, for $t \geq 0$, we have $s_{n+t} = k^{rt} s_n + (k^{r(t-1)} + \cdots + k^r + 1)\lambda$, we see that $s_{n+t} \bmod k^{-r_x}$ is constant for all sufficiently large t (i.e. such that $rt \geq -r_x$). So, for sufficiently large n, we have $\mathsf{NF}(\alpha^{s_n} b^{r_x}) = w_{\mathsf{c}}'' \alpha^{\lfloor k^{r_x} s_n \rfloor}$ for some fixed central subword w_{c}'', and hence, by splitting u_x into its integral and fractional parts, we see that $\mathsf{NF}(gx) = b^{rn+c'} w_{\mathsf{c}}''' \alpha^{s_n'}$ for fixed central subword w_{c}''', where $s_n' = \lfloor k^{r_x} s_n \rfloor + \lambda_x$ for constants $c', \lambda_x \in \mathbb{Z}$. Then since, as we saw above, $s_n \bmod k^{-r_x}$ is constant for sufficiently large n, we have $s_{n+1}' = k^r s_n' + \lambda'$ for some constant $\lambda' \in \mathbb{Z}$, for sufficiently large n, and the result follows from Lemma 7. □

6 Multiplication and inversion

In this section we shall prove the following two results. Note that (informally) the second of these results says that the multiplication table of the group with respect to NF is EDT0L. This result was partly motivated by a related result of Gilman, who proved in [11] that a group is hyperbolic if and only if its multiplication table is context-free with respect to some regular normal form.

Theorem 10. *The language* $\{x\#y\#z : x, y, z \in \mathsf{NF},\ xy =_{\mathsf{BS}} z\}$ *is* EDT0L.

Theorem 11. *The language* $\{x\#y\#z : x, y, z \in \mathsf{NF},\ xy =_{\mathsf{BS}} z^{-1}\}$ *is* EDT0L.

In the proofs, we first prove that the corresponding subsets of $\mathsf{NF}^{\mathrm{frac}}$ are EDT0L, which reduces essentially to addition and subtraction of numbers in base k. In order to derive a corresponding proof for NF, we need to convert a positive integer n written in base k to the string α^n (with $\alpha = a$ or A) and then simulate the above addition and subtraction on the exponents of these strings. It might be helpful to illustrate the conversion process, from $\mathsf{NF}^{\mathrm{frac}}$ to NF, in the special case $x = a$, $y = b^t\alpha^s$, and we shall do that case first.

Lemma 12. *The set*

$$L := \{a\#u\#v : v = \mathsf{NF}(au),\ u = b^t\alpha^s,\ \alpha \in \{a, \mathrm{A}\},\ t, s \in \mathbb{N}_0\}$$

is EDT0L.

Proof. We do this by partitioning L into three subsets, and prove that each of these is EDT0L. The result then follows from Lemma 5.

The first of these subsets is $\{a\#u\#v\colon u = b^ta^s, s, t \geq 0, v = \mathsf{NF}(au)\} = \{a\#b^ta^s\#b^ta^{s+k^t} : s, t \geq 0\}$. To show that this set is EDT0L, we need extra symbols a and T. We start with the start word $T\#\mathsf{a}\#\mathsf{a}a$ and apply $\mathrm{T}_1^*\mathrm{T}_2$, for tables $\mathrm{T}_1 := \{\mathsf{a} \to b\mathsf{a}, a \to a^k\}$ and $\mathrm{T}_2 := \{T \to a\}$. This produces the word $a\#b^t\mathsf{a}\#b^t\mathsf{a}a^{k^t}$, where $t \geq 0$ is the number of applications of T_1. Then we apply $\mathrm{T}_3^*\mathrm{T}_4$, for tables $\mathrm{T}_3 := \mathsf{a} \to a\mathsf{a}$ and $\mathrm{T}_4 := \mathsf{a} \to \epsilon$.

Note that $ab^t\mathrm{A}^s =_{\mathsf{BS}} b^ta^{k^t-s}$, so $\mathsf{NF}(ab^t\mathrm{A}^s) = b^t\mathrm{A}^{s-k^t}$ when $s \geq k^t$, and $\mathsf{NF}(ab^t\mathrm{A}^s) = b^ta^{k^t-s}$ when $k^t > s$.

Our second of the three subsets is $\{a\#b^t\mathrm{A}^s\#b^t\mathrm{A}^{s-k^t} : s, t \geq 0, s \geq k^t\}$. To show that this is EDT0L, observe that we can construct arbitrary words in this language by first constructing words of the form $a\#b^t\mathrm{A}^{k^t}\mathsf{A}\#b^t\mathsf{A}$, using a similar construction as in the preceding case. Then we apply $\mathrm{T}_3^*\mathrm{T}_4$ for tables $\mathrm{T}_3 := \{\mathsf{A} \to \mathrm{A}\mathsf{A}\}$ and $\mathrm{T}_4 := \{\mathsf{A} \to \epsilon\}$.

Proving that our third subset, $\{a\#b^t\mathrm{A}^s\#b^ta^{k^t-s} : s, t > 0, k^t > s\}$ is EDT0L is more difficult, and this is the case that we are using to illustrate how we simulate subtraction of base k numbers. (In fact the previous two cases can also be done using this technique, but they were easier to do directly.)

We shall describe a recipe for constructing all words of this form, which is based on the idea of carrying out the subtraction $k^t - s$ using the representations of k^t and s in base k.

Let $s = s_0 + s_1 k + \cdots + s_{t-1}k^{t-1}$ be the expansion of s in base k. Then, for some j with $0 \le j \le t-1$, we have $s_i = 0$ for $0 \le i < j$, and $s_j \ne 0$ (we are assuming that $s > 0$). Then $k^t - s = s'_0 + s'_1 k + \cdots + s'_{t-1}k^{t-1}$, where:

(i) $s'_i = 0$ for $0 \le i < j$;
(ii) $s'_j = k - s_j$;
(iii) $s'_i = k - s_i - 1$ for $j < i \le t-1$.

The construction of this word involves symbols a and A (variables in the associated EDT0L system), which represent potential occurrences of a and $\textsc{a}$, and will eventually be deleted. The construction consists of t steps, numbered $1, \ldots, t$.

As axiom we use the word $a\#\mathsf{A}\#\mathsf{a}$.

In Step i, for $1 \le i \le t$, we do the following:

(i) apply the rule $\# \to \#b$;
(ii) apply the rule $\mathsf{A} \to \mathsf{A}^k \textsc{a}^{s_{i-1}}$;
(iii) apply the rule $\mathsf{a} \to \mathsf{a}^k a^{s'_{i-1}}$.

Apply $\mathsf{a} \to \epsilon$, and $\mathsf{A} \to \epsilon$.

Note that since $s_i, s'_i \in \{0, 1, \ldots, k-1\}$ the rules we apply come from a finite set.

More formally, define tables as follows.

$$\alpha\colon \# \mapsto \#b,\ \mathsf{A} \mapsto \mathsf{A}^k,\ \mathsf{a} \mapsto \mathsf{a}^k$$
$$\delta\colon \mathsf{a} \mapsto \varepsilon,\ \mathsf{A} \mapsto \varepsilon$$

and for each $0 \le m \le k-1$:

$$\beta_m\colon \# \mapsto \#b,\ \mathsf{A} \mapsto \mathsf{A}^k \textsc{a}^m,\ \mathsf{a} \mapsto \mathsf{a}^k a^{k-m}$$
$$\gamma_m\colon \# \mapsto \#b,\ \mathsf{A} \mapsto \mathsf{A}^k \textsc{a}^m,\ \mathsf{a} \mapsto \mathsf{a}^k a^{k-m-1}.$$

Then the EDT0L system with the following rational control produces the required language in the way described above:

$$\alpha^*(\beta_1 \mid \beta_2 \mid \cdots \mid \beta_{k-1})(\gamma_0, \gamma_1, \ldots, \gamma_{k-1})^*\delta.$$

$\square$

Proof of Theorem 10. For words $x, y, z \in \mathsf{NF}$, we denote the words $\mathsf{NF}^{\mathrm{frac}}(x)$, $\mathsf{NF}^{\mathrm{frac}}(y)$ and $\mathsf{NF}^{\mathrm{frac}}(z)$ by $\hat{x}$, $\hat{y}$ and $\hat{z}$, respectively. We prove first that the language $\mathsf{L}_{\mathrm{frac}} = \{\hat{x}\#\hat{y}\#\hat{z} : x, y, z \in \mathsf{NF}, xy =_{\mathsf{BS}} z\}$ is EDT0L, and then explain how to adapt the arguments to prove the theorem.

Let $x, y, z \in \mathsf{NF}$ with $xy =_{\mathsf{BS}} z$, and suppose that the fractional representations of x and y are $b^{r_x}a^{u_x}$ and $b^{r_y}a^{u_y}$, respectively (where r_x, r_y, u_x and u_y could be positive or negative). Then z has fractional representation $b^{r_x+r_y}a^{u_z}$ with $u_z = k^{r_y}u_x + u_y$. As usual, we aim to construct the language $\mathsf{L}_{\mathrm{frac}}$ using an EDT0L system, starting with the word $\blacksquare\#\blacksquare\#\blacksquare$. Recall that $\hat{x}$ consists of b^{r_x} or B^{-r_x} followed by the base k representation of u_x written backwards, and similarly for $\hat{y}$ and $\hat{z}$.

There are various cases to be considered, depending on the signs of r_x, r_y, u_x and u_y. We need to partition $\mathsf{L}_{\mathrm{frac}}$ into a large number of disjoint subsets depending on these signs, and the EDT0L systems that define these subsets are all slightly different. If u_x and u_y have different signs, then the sign of $u_z = k^{r_y}u_x + u_y$ may be positive or negative, and we need to distinguish between those cases. So there are 24 principal cases. Each of r_x and r_y can be non-negative or negative, and for each of these four possibilities there are six subcases: $u_x, u_y \geq 0$; $u_x, u_y \leq 0$; $u_x > 0, u_y < 0, u_z \geq 0$; $u_x > 0, u_y < 0, u_z < 0$; $u_x < 0, u_y > 0, u_z \geq 0$; and $u_x < 0, u_y > 0, u_z < 0$.

The fractional and integral parts of $\hat{z}$ are computed by addition or subtraction (depending on the signs of u_x and u_y) of those of $\hat{x}$ and $\hat{y}$, where the radix point in that of $\hat{x}$ is shifted r_y places to the left (or $-r_y$ places to the right) before performing this operation. Each of the possible combinations of signs of u_x, u_y and u_z constitutes a separate subcase, and in the first step of the construction we insert the signs of u_x, u_y and u_z for the subcase that we are dealing with.

In the subsequent steps, we carry out the addition or subtraction of the base k numbers, dealing with one base k digit in each step, working from left to right (i.e. from the smallest power of k to the largest). Some of these operations will result in a "carry one" that needs to be handled in the usual way in the following step. So these "carry" steps must be followed by a further step that might itself be a carry step. (In the formal EDT0L system, we need extra variables to indicate that the next step should be a "carry" step. We have left out the details of that process in this proof, but we will present an explicit system for one case of the language L in Subsection 7 below.)

After completing this addition or subtraction, there may be some further steps in which powers of b or of $\textsc{b}$ are inserted at the left of $\hat{x}'$ and $\hat{z}'$ (where $\hat{x}'\#\hat{y}'\#\hat{z}'$ denotes the word that has been constructed so far).

Rather than attempting formal proofs in all cases, we shall content ourselves with providing the constructions of $\hat{x}\#\hat{y}\#\hat{z}$ for three illustrative examples. The value of k is not critical, and we take $k = 3$ in our examples.

The easiest situation is when $r_y = 0$, and we are just calculating $u_x + u_y$ in base k arithmetic. Suppose, for example, that $\hat{x} = \textsc{b}{+}11{\centerdot}21$ and $\hat{y} = -101{\centerdot}2$, so $x = ba\textsc{b}a\textsc{b}a^5 =_{\mathsf{BS}} \textsc{b}a^{49/9}$, $y = b^3\textsc{abbaba}^2 =_{\mathsf{BS}} a^{-64/27}$. Hence $z =_{\mathsf{BS}} \textsc{b}a^{83/27}$, $\hat{z} = \textsc{b}{+}200{\centerdot}01$, and and $z = b^2a^2\textsc{b}^3a^3$. Then the construction is

$$\begin{array}{rcll}
{\centerdot}\#{\centerdot}\#{\centerdot} & \to & +{\centerdot}\#-{\centerdot}\#+{\centerdot} & \to \\
+{\centerdot}\#-1{\centerdot}\#+2{\centerdot} & \to & +1{\centerdot}\#-10{\centerdot}\#+20{\centerdot} & \to \\
+11{\centerdot}\#-101{\centerdot}\#+200{\centerdot} & \to & +11{\centerdot}2\#-101{\centerdot}2\#+200{\centerdot}0 & \to \\
+11{\centerdot}21\#-101{\centerdot}2\#+200{\centerdot}01 & \to & \textsc{b}+11{\centerdot}21\#-101{\centerdot}2\#\textsc{b}+200{\centerdot}01 &
\end{array}$$

In this example, there is just one carry step, namely $+{\centerdot}\#-{\centerdot}\#+{\centerdot} \to +{\centerdot}\#-1{\centerdot}\#+2{\centerdot}$.

Now let us keep the same $\hat{x}$, but replace $\hat{y}$ by $b{-}101{\centerdot}2$, so now $y = b^4\textsc{abbaba}^2 =_{\mathsf{BS}} ba^{-64/27}$, $z = b^3a^2\textsc{b}a^2\textsc{b}a^2\textsc{b}a^{13} =_{\mathsf{BS}} a^{377/27}$ and $\hat{z} = +222{\centerdot}111$. Now the construction is

$$\begin{array}{rcll}
{\centerdot}\#{\centerdot}\#{\centerdot} & \to & +{\centerdot}\#-{\centerdot}\#+{\centerdot} & \to \\
+{\centerdot}\#-1{\centerdot}\#+2{\centerdot} & \to & +{\centerdot}\#-10{\centerdot}\#+22{\centerdot} & \to \\
+1{\centerdot}\#-101{\centerdot}\#+222{\centerdot} & \to & \textsc{b}+11{\centerdot}\#b-101{\centerdot}2\#+222{\centerdot}1 & \to \\
\textsc{b}+11{\centerdot}2\#b-101{\centerdot}2\#+222{\centerdot}11 & \to & \textsc{b}+11{\centerdot}21\#b-101{\centerdot}2\#+222{\centerdot}111 &
\end{array}$$

Note that, when $r_y > 0$, we insert a symbol b at the beginning of $\hat{y}'$ in each of the r_y steps in which we are processing fractional parts of x and integral parts of y. If there are symbols $\textsc{b}$ to be entered at the beginning of $\hat{x}'$, then (as we did in one of the steps in the example above) we insert a $\textsc{b}$ in $\hat{x}'$ in the same step as the b in $\hat{y}'$; otherwise we would insert a symbol b at the beginning of $\hat{z}'$. In this example the first four subtraction steps are all carry steps.

As illustrated in the first and following example, any remaining occurrences of b or $\textsc{b}$ at the beginning of $\hat{x}$ can be inserted into $\hat{x}'$ and $\hat{z}'$ at the end of the construction.

Now we consider an example with $r_y < 0$, and with changed signs for x, y and z, namely $\hat{x} = \text{B}{-}21\text{.}112$ and $\hat{y} = \text{B}{+}221\text{.}01$, so $x = b\text{A}^2\text{BABA}^{22} =_{\mathsf{BS}} \text{B}a^{-203/9}$, $y = b^2a^2\text{B}a^2\text{B}a\text{B}a^3 =_{\mathsf{BS}} \text{B}a^{98/27}$, $z = \text{A}^2\text{BA}^2\text{BA}^3 =_{\mathsf{BS}} \text{B}^2a^{-35/9}$ and $\hat{z} = \text{B}^2{-}22\text{.}01$. The construction is

$$\begin{array}{rcrc} \text{.}\#\text{.}\#\text{.} & \to & -\text{.}\#+\text{.}\#-\text{.} & \to \\ -2\text{.}\#+2\text{.}\#-\text{.} & \to & -21\text{.}\#+22\text{.}\#-2\text{.} & \to \\ -21\text{.}1\#\text{B}+221\text{.}\#\text{B}-22\text{.} & \to & -21\text{.}11\#\text{B}+221\text{.}0\#\text{B}-22\text{.}0 & \to \\ -21\text{.}112\#\text{B}+221\text{.}01\#\text{B}-22\text{.}01 & \to & \text{B}-11\text{.}112\#\text{B}+221\text{.}01\#\text{B}^2-22\text{.}01 & \end{array}$$

Here we inserted B into $\hat{y}'$ in the step dealing with the integral part of $\hat{x}$ and the fractional part of $\hat{y}$. Since there is no b at the beginning of $\hat{x}$, we insert B into $\hat{z}'$ at the same time.

Now we turn to the proof that the language $L = \{x\#y\#z : x, y, z \in \mathsf{NF},\ xy =_{\mathsf{BS}} z\}$ in the theorem statement is EDT0L. Again we denote the subwords of x, y, z that have been inserted into $x\#y\#z$ so far by x', y', z'.

The words x', y', z' are constructed in the same way and roughly in the same order as the corresponding subwords $\hat{x}', \hat{y}', \hat{z}'$ of $\hat{x}, \hat{y}, \hat{z}$; that is, for each step in the process of constructing $\hat{x}', \hat{y}'$ and $\hat{z}'$, there is a corresponding step in the construction of x', y' and z'.

There are two principal issues that arise here. One of these involves the insertion of the right subwords of the words $x, y, z \in \mathsf{NF}$. These subwords are strings in the generators a or A of which lengths are represented by numbers in base k in $\hat{x}, \hat{y}, \hat{z}$. The integral part of $\hat{z}$ is calculated from $\hat{x}$ and $\hat{y}$ by addition or subtraction of numbers in base k. The corresponding process for x, y and z can be carried out using the method illustrated in the proof of Lemma 12 that involves the use of dummy symbols a and A that will be deleted at the end of the process.

The second issue concerns the insertion of the central subwords, together with parts of the left subwords, of x, y, z into x', y', z'. In general, in a step in which we insert digits i with $0 \leq i < k$ into the fractional parts of each of $\hat{x}'$, $\hat{y}'$ and $\hat{z}'$, we insert $a^i\text{B}$ into the central subwords of x, y and z at their right hand ends, immediately before the radix point. However, we want to ensure that the same total powers of b are inserted into $x'y'$ as into z', which we do as follows.

In such a step, if there are letters b in the left subwords of x and/or y that have not yet been inserted into x' and/or y', then we insert b into x' and/or

y' in the same step. If this involves inserting b into both x' and y' then, since we are also inserting terms $a^i\mathrm{B}$ into x' and y', the total power of b inserted in $x'y'$ is zero and, since we are inserting a term $a^i\mathrm{B}$ into z', we insert a b in the left subword of z' to ensure that the total power of b entered into z' is also zero. But if we insert b into just one of x' and y' then the total power of b inserted in $x'y'$ is -1, and we do not insert b into z'.

In some steps, we may be processing digits in the fractional part of one of $\hat{x}$ and $\hat{y}$ and in the integral part of the other, and again we just need to ensure that we insert the same total power of b into $x'y'$ and into z'. We would be in trouble if there was no b to insert either into x' or into y' and we had to insert a term $a^i\mathrm{B}$ into z', but in fact that never happens. That situation could only arise when $r_y < 0$, and in that case we would be combining a fractional digit of $\hat{y}'$ with an integral digit of $\hat{x}'$, so we would not be changing the fractional part of x'. Note also that if the above process should involve inserting $b\mathrm{B}$ at the beginning of the word z' then we would of course not do that (that situation arises in the third example below).

As was the case with $\hat{x}', \hat{z}'$, any remaining occurrences of b or B in the left subwords of x', z' can be inserted at the end of the process.

Let us now illustrate the procedure with the same three examples as before. As in the proof of Lemma 12, we use dummy symbols $\mathsf{a}_x, \mathsf{a}_y, \mathsf{a}_z$ and $\mathsf{A}_x, \mathsf{A}_y, \mathsf{A}_z$, which will eventually be removed, to represent future possible instances of a and of A in x, y, z, respectively. (In fact we have shortened the process by removing these dummy symbols in the final step of the rest of the procedure rather than in a separate final step at the end.) The first of the examples was $x = ba\mathrm{B}a\mathrm{B}a^5$, $y = b^3\mathrm{ABBABA}^2$, $z = b^2a^2\mathrm{B}^3a^3$. The construction is

$$\begin{array}{cccc}
\#\# & \to & \#b\mathrm{AB}\#ba^2\mathrm{B} & \to \\
ba\mathrm{B}\#b^2\mathrm{ABB}\#b^2a^2\mathrm{B}^2 & \to & ba\mathrm{B}a\mathrm{B}\#b^3\mathrm{ABBAB}\#b^2a^2\mathrm{B}^3 & \to \\
ba\mathrm{B}a\mathrm{B}a^2\mathsf{a}_x^3\#b^3\mathrm{ABBABA}^2\mathsf{A}_y^3\#b^2a^2\mathrm{B}^3\mathsf{a}_z^3 & \to & ba\mathrm{B}a\mathrm{B}a^5\#b^3\mathrm{ABBABA}^2\#b^2a^2\mathrm{B}^3a^3 &
\end{array}$$

The second example is $x = ba\mathrm{B}a\mathrm{B}a^5$, $y = b^4\mathrm{ABBABA}^2$, $z = b^3a^2\mathrm{B}a^2\mathrm{B}a^2\mathrm{B}a^{13}$.

$$\begin{array}{cccc}
\#\# & \to & \#b\mathrm{AB}\#ba^2\mathrm{B} & \to \\
\#b^2\mathrm{ABB}\#b^2a^2\mathrm{B}a^2\mathrm{B} & \to & ba\mathrm{B}\#b^3\mathrm{ABBAB}\#b^3a^2\mathrm{B}a^2\mathrm{B}a^2\mathrm{B} & \to \\
ba\mathrm{B}a\mathrm{B}\#b^4\mathrm{ABBABA}^2\mathsf{A}_y^3\#b^3a^2\mathrm{B}a^2\mathrm{B}a^2\mathrm{B}a\mathsf{a}_z^3 & \to & ba\mathrm{B}a\mathrm{B}a^2\mathsf{a}_x^3\#b^4\mathrm{ABBABA}^2\mathsf{A}_y^9\#b^3a^2\mathrm{B}a^2\mathrm{B}a^2\mathrm{B}a^4\mathsf{a}_z^9 & \to \\
ba\mathrm{B}a\mathrm{B}a^5\#b^4\mathrm{ABBABA}^2\#b^3a^2\mathrm{B}a^2\mathrm{B}a^2\mathrm{B}a^{13} & & &
\end{array}$$

The third example is $x = b\mathrm{A}^2\mathrm{BABA}^{22}$, $y = b^2a^2\mathrm{B}a^2\mathrm{B}a\mathrm{B}a^3$, $z = \mathrm{A}^2\mathrm{BA}^2\mathrm{BA}^3$.

$$\begin{array}{cccc}
\#\# & \to & b\textsc{a}^2\textsc{b}\#ba^2\textsc{b}\# & \to \\
b\textsc{a}^2\textsc{bab}\#b^2a^2\textsc{b}a^2\textsc{b}\#\textsc{a}^2\textsc{b} & \to & b\textsc{a}^2\textsc{baba}\mathsf{A}_x^3\#b^2a^2\textsc{b}a^2\textsc{b}a\textsc{b}\#\textsc{a}^2\textsc{ba}^2\textsc{b} & \to \\
b\textsc{a}^2\textsc{baba}^4\mathsf{A}_x^9\#b^2a^2\textsc{b}a^2\textsc{b}a\textsc{b}\mathsf{a}_y^3\#\textsc{a}^2\textsc{ba}^2\textsc{b}\mathsf{A}_z^3 & \to & b\textsc{a}^2\textsc{baba}^{22}\#b^2a^2\textsc{b}a^2\textsc{b}a\textsc{b}a^3\#\textsc{a}^2\textsc{ba}^2\textsc{ba}^3 &
\end{array}$$

An explicit EDT0L system for one case of the proof is presented in Subsection 7 □

Proof of Theorem 11. Let $x, y, z \in \mathsf{NF}$ with $xy =_{\mathsf{BS}} z^{-1}$ and let $\bar{z} = \mathsf{NF}(xy)$. Then, as we saw in the previous theorem, the fractional representations of x and y can be written as $x =_{\mathsf{BS}} b^{r_x}a^{u_x}$, $y =_{\mathsf{BS}} b^{r_y}a^{u_y}$, and then $\bar{z} =_{\mathsf{BS}} b^{r_x+r_y}a^{u_{\bar{z}}}$ with $u_{\bar{z}} = k^{r_y}u_x + u_y$.

Now we have $z =_{\mathsf{BS}} \bar{z}^{-1} =_{\mathsf{BS}} a^{-u_{\bar{z}}}b^{-(r_x+r_y)} = b^{-(r_x+r_y)}a^{u_z}$ with $u_z = -k^{-(r_x+r_y)}u_{\bar{z}}$. So u_z has the opposite sign to that of $u_{\bar{z}}$, and the digits of its (reversed) base k expansion are the same as those of $\bar{z}$, but the radix point is shifted $r_x + r_y$ places to the right or $-(r_x + r_y)$ places to the left.

Let $L := \{x\#y\#z : x, y, z \in \mathsf{NF},\ xy =_{\mathsf{BS}} z^{-1}\}$ and $\mathsf{L}_{\mathrm{frac}} := \{\hat{x}\#\hat{y}\#\hat{z} : \hat{x}, \hat{y}, \hat{z} \in \mathsf{NF}^{\mathrm{frac}},\ xy =_{\mathsf{BS}} z^{-1}\}$ The processes involved in proving first that $\mathsf{L}_{\mathrm{frac}}$ and then that L are EDT0L are similar to those in the previous proof, with the added complication that power b^{r_x} of b in $\hat{x}$ plays a more significant role.

The constructions of $x\#y\#z$ and of $\hat{x}\#\hat{y}\#\hat{z}$ using EDT0L systems split up into four phases, some of which may be empty in some examples. In the first phase we are constructing parts of the fractional parts of $\hat{x}$, $\hat{y}$ and $\hat{z}$, and in the fourth phase we are constructing parts of their integral parts. In the second and third phases, we are constructing parts of the fractional parts of some of them and of the integral parts of the others. As in the previous proof, we denote the words constructed so far by $x'\#y'\#z'$ and $\hat{x}'\#\hat{y}'\#\hat{z}'$.

In the construction of x, y, z, in each step in the first phase we insert a b at the beginning of each of x', y', z', and $\alpha^i\textsc{b}$ for some $0 \le i < k$ at the end of its fractional part, but with the usual proviso that we avoid inserting $b\textsc{b}$ at the beginning of the word.

Unlike in the proof of Theorem 10, in the situation when r_x and r_y have opposite, there are two cases depending on the sign of $r_x + r_y$. So there are six possibilities for r_x and r_y to be considered, each of which splits into six subcases for u_x and u_y as in Theorem 10. So there are 36 cases in total. We shall describe phases 2 and 3 in more detail in three of these possibilities for r_x, r_y, and provide the constructions for an illustrative example with $k = 3$ in each case.

Suppose first that r_x and r_y are both non-negative. In the second phase, we are constructing parts of the fractional parts of $\hat{x}$ and $\hat{z}$, and of the integral part of $\hat{y}$. This phase consists of r_y steps, in each of which we insert a b at the beginning of each of x', y' and $\hat{y}'$, a $\text{\textsc{b}}$ at the beginning of $\hat{z}'$, and one term $a^i\text{\textsc{b}}$ at the end of the fractional parts of each of x' and z'.

In the third phase, we are constructing parts of the integral parts of $\hat{x}$ and $\hat{y}$, and of the fractional part of $\hat{z}$. This phase consists of r_x steps, in each of which we insert a b at the beginning of x' and of $\hat{x}'$, a $\text{\textsc{b}}$ at the beginning of $\hat{z}'$, and a term $a^i\text{\textsc{b}}$ at the end of the fractional part of z'.

As an example, we take $x = b^3\text{\textsc{aba}}^2$, $y = b^2a^2\text{\textsc{b}}a$, so $\hat{x} = b^2{-}1\mathbf{.}2$, $\hat{y} = b{+}2\mathbf{.}1$, $\hat{z} = \text{\textsc{b}}^3{+}1210\mathbf{.}$, $z = ba\text{\textsc{b}}a^2\text{\textsc{b}}a\text{\textsc{b}}^2$.

Here are the derivations of $\hat{x}\#\hat{y}\#\hat{z}$ and of $x\#y\#z$. There is one step in each of the first two phases, two steps in the third phase, and none in the fourth phase. (So the integral part of $\hat{z}$ is 0.) We have saved space by suppressing the first step in which signs are entered into $\hat{x}', \hat{y}'$ and $\hat{z}'$.

$$
\begin{array}{ccc}
-\mathbf{.}\#+\mathbf{.}\#+\mathbf{.} & \to & -\mathbf{.}\#+2\mathbf{.}\#+1\mathbf{.} \quad \to \\
-1\mathbf{.}\#b+2\mathbf{.}1\#\text{\textsc{b}}+12\mathbf{.} & \to & b-1\mathbf{.}2\#b+2\mathbf{.}1\#\text{\textsc{b}}^2+121\mathbf{.} \quad \to \\
b^2-1\mathbf{.}2\#b^2+2\mathbf{.}1\#\text{\textsc{b}}^3+1210\mathbf{.} & &
\end{array}
$$

$$
\begin{array}{ccc}
\#\# & \to & \#ba^2\text{\textsc{b}}\#ba\text{\textsc{b}} \quad \to \\
b\text{\textsc{ab}}\#b^2a^2\text{\textsc{b}}a\#ba\text{\textsc{b}}a^2\text{\textsc{b}} & \to & b^2\text{\textsc{aba}}^2\#b^2a^2\text{\textsc{b}}a\#ba\text{\textsc{b}}a^2\text{\textsc{b}}a\text{\textsc{b}} \quad \to \\
b^3\text{\textsc{aba}}^2\#b^2a^2\text{\textsc{b}}a\#ba\text{\textsc{b}}a^2\text{\textsc{b}}a\text{\textsc{b}}^2 & &
\end{array}
$$

Suppose next that $r_x > 0$ and $r_y < 0$ and that $r_x > |r_y|$. In the second phase, we are constructing parts of the fractional parts of $\hat{y}$ and $\hat{z}$, and the integral part of $\hat{x}$. This phase consists of $|r_y|$ steps, in each of which we insert a b at the beginning of each of x', $\hat{x}'$ and z', a $\text{\textsc{b}}$ at the beginning of $\hat{y}'$, and one term $a^i\text{\textsc{b}}$ into the end of the fractional parts of each of y' and z'.

In the third phase, we are constructing parts of the integral parts of $\hat{x}$ and $\hat{y}$, and the fractional part of $\hat{z}$. This phase consists of $r_x - |r_y|$ steps, in each of which we insert a b at the beginning of $\hat{x}'$ and x', a $\text{\textsc{b}}$ at the beginning of $\hat{z}'$, and a term $a^i\text{\textsc{b}}$ at the end of the fractional part of z'.

As example, we take $x = b^3\text{\textsc{aba}}^5$, $y = b\text{\textsc{aba}}^2\text{\textsc{ba}}^8$, so $\hat{x} = b^2{-}1\mathbf{.}21$, $\hat{y} = \text{\textsc{b}}{-}12\mathbf{.}22$, $\hat{z} = \text{\textsc{b}}{+}211\mathbf{.}01$, $z = b^2a^2\text{\textsc{b}}a\text{\textsc{b}}a\text{\textsc{b}}a^3$.

In the derivations below, there is one step in each of the first three phases, and two in the fourth phase.

$$\begin{array}{ccll}
-\mathbf{.}\#-\mathbf{.}\#+\mathbf{.} & \to & -1\mathbf{.}\#-1\mathbf{.}\#+2\mathbf{.} & \to \\
b-1\mathbf{.}2\#\mathrm{B}-12\mathbf{.}\#+21\mathbf{.} & \to & b^2-1\mathbf{.}21\#\mathrm{B}-12\mathbf{.}2\#\mathrm{B}+211\mathbf{.} & \to \\
b^2-1\mathbf{.}21\#\mathrm{B}-12\mathbf{.}22\#\mathrm{B}+211\mathbf{.}0 & \to & b^2-1\mathbf{.}21\#\mathrm{B}-12\mathbf{.}22\#\mathrm{B}+211\mathbf{.}01 &
\end{array}$$

$$\begin{array}{ccll}
\#\# & \to & b\mathrm{AB}\#b\mathrm{AB}\#ba^2\mathrm{B} & \to \\
b^2\mathrm{ABA}^2\mathsf{A}_x^3\#b\mathrm{ABA}^2\mathrm{B}\#b^2a^2\mathrm{B}a\mathrm{B} & \to & b^3\mathrm{ABA}^5\#b\mathrm{ABA}^2\mathrm{BA}^2\mathsf{A}_y^3\#b^2a^2\mathrm{B}a\mathrm{B}a\mathrm{B} & \to \\
b^3\mathrm{ABA}^5\#b\mathrm{ABA}^2\mathrm{BA}^8\#b^2a^2\mathrm{B}a\mathrm{B}a\mathrm{B}\mathsf{a}_z^3 & \to & b^3\mathrm{ABA}^5\#b\mathrm{ABA}^2\mathrm{BA}^8\#b^2a^2\mathrm{B}a\mathrm{B}a\mathrm{B}a^3 &
\end{array}$$

The final case that we shall consider in detail is $r_x > 0$ and $r_y < 0$ with $r_x < |r_y|$. As in the previous case that we considered, in the second phase, we are constructing parts of the fractional parts of $\hat{y}$ and $\hat{z}$, and the integral part of $\hat{x}$. But now this phase consists of r_x steps, in each of which we insert a b at the beginning of each of x', $\hat{x}'$ and z', a B at the beginning of $\hat{y}'$, and one term $a^i\mathrm{B}$ at the end of the fractional parts of each of y' and z'.

In the third phase, we are constructing parts of the integral parts of $\hat{x}$ and $\hat{z}$, and the fractional part of $\hat{y}$. This phase consists of $|r_y| - r_x$ steps, in each of which we insert a b at the beginning of of z' and $\hat{z}'$, a B at the beginning of $\hat{y}'$, and a term $a^i\mathrm{B}$ at the end of the fractional part of y'.

As example, we take $x = b^3\mathrm{ABA}^5$, $y = \mathrm{B}a^2\mathrm{B}a\mathrm{B}a^5$, so $\hat{x} = b^2{-}1\mathbf{.}21$, $\hat{y} = \mathrm{B}^3{+}21\mathbf{.}21$, $\hat{z} = b{-}200\mathbf{.}121$, $z = b^4\mathrm{A}^2\mathrm{BBBA}^{16}$.

In the derivations below, there is one step in the first and third phases, and two in the second and fourth phases.

$$\begin{array}{ccll}
-\mathbf{.}\#+\mathbf{.}\#-\mathbf{.} & \to & -1\mathbf{.}\#+\mathbf{.}\#-2\mathbf{.} & \to \\
b-1\mathbf{.}2\#\mathrm{B}+\mathbf{.}\#-20\mathbf{.} & \to & b^2-1\mathbf{.}21\#\mathrm{B}^2+2\mathbf{.}\#-200\mathbf{.} & \to \\
b^2-1\mathbf{.}21\#\mathrm{B}^3+21\mathbf{.}\#b-200\mathbf{.}1 & \to & b^2-1\mathbf{.}21\#\mathrm{B}^3+21\mathbf{.}2\#b-200\mathbf{.}12 & \to \\
b^2-1\mathbf{.}21\#\mathrm{B}^3+21\mathbf{.}21\#b-200\mathbf{.}121 & & &
\end{array}$$

$$\begin{array}{ccll}
\#\# & \to & b\mathrm{AB}\#\#b\mathrm{A}^2\mathrm{B} & \to \\
b^2\mathrm{ABA}^2\mathsf{A}_x^3\#\mathrm{B}\#b^2\mathrm{A}^2\mathrm{B}^2 & \to & b^3\mathrm{ABA}^5\#\mathrm{B}a^2\mathrm{B}\#b^3\mathrm{A}^2\mathrm{B}^3 & \to \\
b^3\mathrm{ABA}^5\#\mathrm{B}a^2\mathrm{B}a\mathrm{B}\#b^4\mathrm{A}^2\mathrm{B}^3\mathrm{A}\mathsf{A}_z^3 & \to & b^3\mathrm{ABA}^5\#\mathrm{B}a^2\mathrm{B}a\mathrm{B}a^2\mathsf{a}_y^3\#b^4\mathrm{A}^2\mathrm{B}^3\mathrm{A}^7\mathsf{A}_z^9 & \to \\
b^3\mathrm{ABA}^5\#\mathrm{B}a^2\mathrm{B}a\mathrm{B}a^5\#b^4\mathrm{A}^2\mathrm{BBBA}^{16} & & &
\end{array}$$

$\square$

7 Explicit system

We explicitly construct one of the EDT0L systems described in the proof of Theorem 10, for the case $\mathsf{BS}(1,3)$. The system described in this section generates the language

$$\{x\#y\#z \colon x, y, z \in \mathsf{NF},\ u_x < 0, u_y > 0, u_z < 0\}.$$

Systems that generate the corresponding languages for different signs of u_x, u_y, u_z are analogous, with the only difference occurring in the tables $\alpha_{ij}, \beta_{ij}, \gamma_{ij}, \delta_{ij}$.

- Alphabet: $\tilde{X}, \tilde{Y}, \tilde{Z}, X, Y, Z, Z_1, X_b, Y_b, Z_b, \mathsf{A}_X, \mathsf{a}_Y, \mathsf{A}_Z, \mathsf{A}_{Z1}, a, \text{A}, b, \text{B}, \#$
- Terminals: $a, \text{A}, b, \text{B}, \#$
- Axiom: $\tilde{X}\#\tilde{Y}\#\tilde{Z}$
- Tables: $\alpha_{ij}, \beta_{ij}, \gamma_{ij}, \delta_{ij}$ (for $i, j \in \{0, 1, 2\}$, defined in Appendix A),

$$\begin{aligned}
\sigma &\colon \tilde{X} \mapsto \text{B}X, X \mapsto \text{B}X, \tilde{Z} \mapsto \text{B}Z, Z \mapsto \text{B}Z \\
\rho_X &\colon X_b \mapsto bX_b \\
\rho_Y &\colon Y_b \mapsto bY_b \\
\rho_{XY} &\colon X_b \mapsto bX_b, Y_b \mapsto bY_b, Z_b \mapsto bZ_b \\
\mu &\colon X_b \mapsto bX_b, Z_b \mapsto bZ_b \\
\tau &\colon \tilde{X}, X, X_b, \tilde{Y}, Y, Y_b, \tilde{Z}, Z, Z_b, \mathsf{A}_X, \mathsf{a}_Y, \mathsf{A}_Z \mapsto \epsilon
\end{aligned}$$

- Rational control: as in Figure 1

Figure 1 is a schematic diagram of the finite state automaton defining the rational control. Labelled edges are single transitions in the FSA as usual. Unlabelled edges represent multiple transitions (each starting and ending at the same states as the unlabelled edge) as follows, where x is replaced with $\alpha, \beta, \gamma, \delta$ as indicated in the corresponding dashed box. The arrow with the open triangle represents a transition from a non-carry state to a non-carry state, the open square represents non-carry to carry, the filled triangle carry to carry, and the filled square carry to non-carry.

6 transitions, labelled $x_{00} \mid x_{10} \mid x_{11} \mid x_{20} \mid x_{21} \mid x_{22}$
3 transitions, labelled $x_{01} \mid x_{02} \mid x_{12}$
6 transitions, labelled $x_{00} \mid x_{01} \mid x_{02} \mid x_{11} \mid x_{12} \mid x_{22}$
3 transitions, labelled $x_{10} \mid x_{20} \mid x_{21}$

The edges connecting dashed boxes represent pairs of ε-labelled edges, connecting the shaded states within the boxes. The left hand shaded state in one box is connected to the left hand shaded state in the other box, and similarly for the right hand states. The right hand shaded states (marked with a dot) represent the fact that there is a carry that has yet to be resolved.

The variables X, Y, Z, Z_1 are the 'sources' of the powers of a and A before the radix point, with Z_1 recording a carry. The variables $\mathsf{A}_X, \mathsf{a}_Y, \mathsf{A}_Z, \mathsf{A}_{Z1}$ are the 'sources' of powers of a and A after the radix point, with A_{Z1} again indicating a carry. Note that the rational control ensures that a word can never be completed with an outstanding carry.

The table σ optionally inserts B's at the start of the central subwords of both x' and z'. The table μ optionally inserts b's at the start of the same subwords. Note that the rational control ensures that these tables are never both used for the same word. The table τ occurs at the end of every word in the rational control, with the purpose of deleting all remaining non-terminals.

The tables α_{ij} etc insert powers A^i and a^j at the chosen step, as well as the B that separates them. For example, α_{12} inserts AB to the word representing x, $a^2\mathrm{B}$ to the word representing y, and either $\mathrm{A}^2\mathrm{B}$ with a carry, or B with a carry, to the word representing z, depending on whether or not there was already a carry present. The choice of $\alpha, \beta, \gamma, \delta$ corresponds to both left and right words being before the radix point, only the right hand word being before the radix point, only the left, and neither, respectively. To ensure that the string $b\mathrm{B}$ never gets inserted, the variables $\tilde{X}, \tilde{Y}, \tilde{Z}$ are used initially. Once a non-zero power of a or A has been inserted, the corresponding variable is 'initialised' and the tilde version is replaced.

The tables ρ add powers of b to the left hand side of each word, using the variables X_b, Y_b, Z_b as 'sources'.

Consider the third example in the proof of Theorem 10 above: $x = b\mathrm{A}^2\mathrm{BABA}^{22}$, $y = b^2a^2\mathrm{B}a^2\mathrm{B}a\mathrm{B}a^3$, $z = \mathrm{A}^2\mathrm{BA}^2\mathrm{BA}^3$. This element of the language is produced by the EDT0L system, via the word $\alpha_{22}\rho_{XY}\alpha_{12}\rho_Y\beta_{11}\delta_{10}\delta_{21}\tau$ of the rational

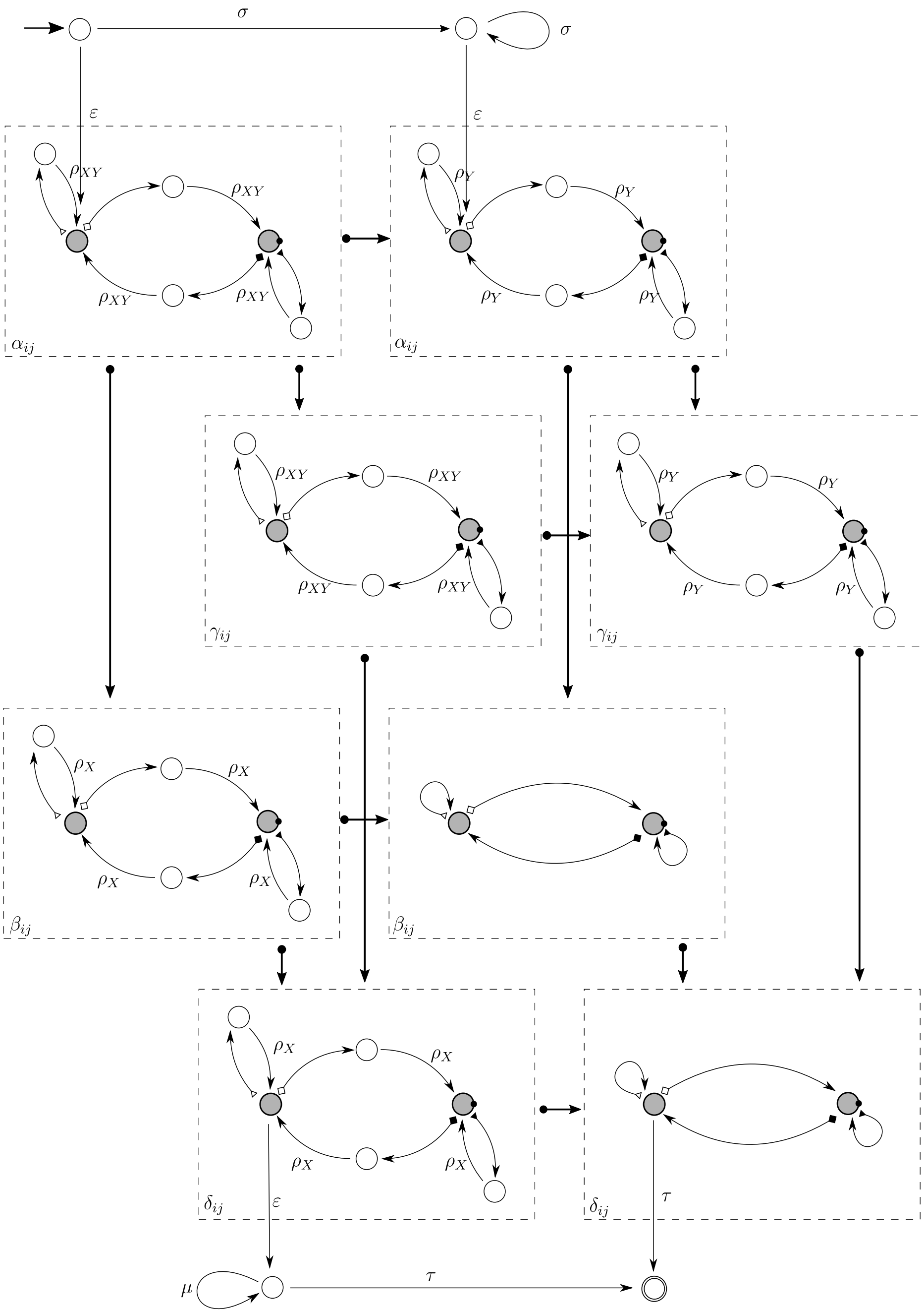

Figure 1: Rational control for the EDT0L system of Theorem 10

control, in the following way:

$$
\begin{array}{lcc}
\tilde{X}\#\tilde{Y}\#\tilde{Z} & \triangleright\!\xrightarrow{\alpha_{22}} & X_b\textsc{a}^2\textsc{b}X\#Y_b a^2\textsc{b}Y\#\tilde{Z} \\
 & \xrightarrow{\rho_{XY}} & bX_b\textsc{a}^2\textsc{b}X\#bY_b a^2\textsc{b}Y\#\tilde{Z} \\
 & \diamond\!\xrightarrow{\alpha_{12}} & bX_b\textsc{a}^2\textsc{bab}X\#bY_b a^2\textsc{b}a^2\textsc{b}Y\#Z_b\textsc{a}^2\textsc{b}Z_1 \\
 & \xrightarrow{\rho_{Y}} & bX_b\textsc{a}^2\textsc{bab}X\#b^2Y_b a^2\textsc{b}a^2\textsc{b}Y\#Z_b\textsc{a}^2\textsc{b}Z_1 \\
 & \blacktriangleright\!\xrightarrow{\beta_{11}} & bX_b\textsc{a}^2\textsc{baba}\mathsf{A}_X^3\#b^2Y_b a^2\textsc{b}a^2\textsc{b}a\textsc{b}Y\#Z_b\textsc{a}^2\textsc{ba}^2\textsc{b}Z_1 \\
 & \blacklozenge\!\xrightarrow{\delta_{10}} & bX_b\textsc{a}^2\textsc{baba}(\textsc{a}\mathsf{A}_X^3)^3\#b^2Y_b a^2\textsc{b}a^2\textsc{b}a\textsc{b}\mathsf{a}_Y^3\#Z_b\textsc{a}^2\textsc{ba}^2\textsc{b}\mathsf{A}_Z^3 \\
 & \diamond\!\xrightarrow{\delta_{21}} & bX_b\textsc{a}^2\textsc{baba}(\textsc{a}(\textsc{a}^2\mathsf{A}_X^3)^3)^3\#b^2Y_b a^2\textsc{b}a^2\textsc{b}a\textsc{b}(a\mathsf{a}_Y^3)^3\#Z_b\textsc{a}^2\textsc{ba}^2\textsc{b}(\textsc{a}\mathsf{A}_Z^3)^3 \\
 & \xrightarrow{\tau} & b\textsc{a}^2\textsc{baba}^{22}\#b^2a^2\textsc{b}a^2\textsc{b}a\textsc{b}a^3\#\textsc{a}^2\textsc{ba}^2\textsc{ba}^3
\end{array}
$$

8 An equation in which the solution set might not be EDT0L.

We conjecture that the set $C = \{x\#y : x, y \in \mathsf{NF},\ xy =_{\mathsf{BS}} yx\}$ is not EDT0L. Elements of $\mathsf{BS}(1,k)$ have the form $g = b^r a^u$, where the exponent u of the fractional part of g lies in the set $\mathbb{Z}[1/k] = \{i/k^m : i, m \in \mathbb{Z}\}$. As we saw earlier, we have $g^n = b^{rn}a^{u_n}$ with $u_n = u(1 + k^r + \cdots + k^{(n-1)r}) = u\left(\frac{k^{rn}-1}{k^r-1}\right)$ for $n \geq 0$. Since $\frac{k^r-1}{k^{rn}-1} \notin \mathbb{Z}[1/k]$ for $n > 1$, it follows that the element $b^r a$ is not a proper power for any $r \geq 0$.

So the set

$$\{b^r a\#\mathsf{NF}((b^r a)^n) : r, n \geq 0\} = \{b^r a\#b^{rn}a^{\frac{k^{rn}-1}{k^r-1}} : r, n \geq 0\}$$

is the intersection of C with the regular set $b^*a\#b^*a^*$, and it would suffice to show that this is not EDT0L.

We conjecture that [10, Theorem A] can be applied to deduce that this set is not indexed; this would of course imply that it is not EDT0L or even ET0L. We have verified by computer that, if the language were indexed, and we applied that result with $m = 2$, then we would deduce that the constant k in [10, Theorem A] satisfies $k \geq 300$.

Acknowledgements

The authors wish to thank Graham Campbell for much useful discussion during the early stages of this project. Thanks are also due to the Heilbronn

Institute for Mathematical Research for facilitating collaboration via their small grants scheme.

A Table definitions

The following are the definitions of the tables $\alpha_{ij}, \beta_{ij}, \gamma_{ij}, \delta_{ij}$ used in Section 7.

$$\alpha_{00}\colon \begin{aligned} X &\mapsto \mathrm{B}X \\ Y &\mapsto \mathrm{B}Y \\ Z &\mapsto \mathrm{B}Z \\ Z_1 &\mapsto \mathrm{A}^2\mathrm{B}Z_1 \end{aligned} \qquad \alpha_{01}\colon \begin{aligned} X &\mapsto \mathrm{B}X \\ Y &\mapsto a\mathrm{B}Y \\ \tilde{Y} &\mapsto Y_b a\mathrm{B}Y \\ Z &\mapsto \mathrm{A}^2\mathrm{B}Z_1 \\ \tilde{Z} &\mapsto Z_b\mathrm{A}^2\mathrm{B}Z_1 \\ Z_1 &\mapsto \mathrm{AB}Z_1 \end{aligned} \qquad \alpha_{02}\colon \begin{aligned} X &\mapsto \mathrm{B}X \\ Y &\mapsto a^2\mathrm{B}Y \\ \tilde{Y} &\mapsto Y_b a^2\mathrm{B}Y \\ Z &\mapsto \mathrm{AB}Z_1 \\ \tilde{Z} &\mapsto Z_b\mathrm{AB}Z_1 \\ Z_1 &\mapsto \mathrm{B}Z_1 \end{aligned}$$

$$\alpha_{10}\colon \begin{aligned} X &\mapsto \mathrm{AB}X \\ \tilde{X} &\mapsto X_b\mathrm{AB}X \\ Y &\mapsto \mathrm{B}Y \\ Z &\mapsto \mathrm{AB}Z \\ \tilde{Z} &\mapsto Z_b\mathrm{AB}Z \\ Z_1 &\mapsto \mathrm{B}Z \end{aligned} \qquad \alpha_{11}\colon \begin{aligned} X &\mapsto \mathrm{AB}X \\ \tilde{X} &\mapsto X_b\mathrm{AB}X \\ Y &\mapsto a\mathrm{B}Y \\ \tilde{Y} &\mapsto Y_b a\mathrm{B}Y \\ Z &\mapsto \mathrm{B}Z \\ Z_1 &\mapsto \mathrm{A}^2\mathrm{B}Z_1 \end{aligned} \qquad \alpha_{12}\colon \begin{aligned} X &\mapsto \mathrm{AB}X \\ \tilde{X} &\mapsto X_b\mathrm{AB}X \\ Y &\mapsto a^2\mathrm{B}Y \\ \tilde{Y} &\mapsto Y_b a^2\mathrm{B}Y \\ Z &\mapsto \mathrm{A}^2\mathrm{B}Z_1 \\ \tilde{Z} &\mapsto Z_b\mathrm{A}^2\mathrm{B}Z_1 \\ Z_1 &\mapsto \mathrm{B}Z_1 \end{aligned}$$

$$\alpha_{20}\colon \begin{aligned} X &\mapsto \mathrm{A}^2\mathrm{B}X \\ \tilde{X} &\mapsto X_b\mathrm{A}^2\mathrm{B}X \\ Y &\mapsto \mathrm{B}Y \\ Z &\mapsto \mathrm{A}^2\mathrm{B}Z \\ \tilde{Z} &\mapsto Z_b\mathrm{A}^2\mathrm{B}Z \\ Z_1 &\mapsto \mathrm{AB}Z \end{aligned} \qquad \alpha_{21}\colon \begin{aligned} X &\mapsto \mathrm{A}^2\mathrm{B}X \\ \tilde{X} &\mapsto X_b\mathrm{A}^2\mathrm{B}X \\ Y &\mapsto a\mathrm{B}Y \\ \tilde{Y} &\mapsto Y_b a\mathrm{B}Y \\ Z &\mapsto \mathrm{AB}Z \\ \tilde{Z} &\mapsto Z_b\mathrm{AB}Z \\ Z_1 &\mapsto \mathrm{B}Z \end{aligned} \qquad \alpha_{22}\colon \begin{aligned} X &\mapsto \mathrm{A}^2\mathrm{B}X \\ \tilde{X} &\mapsto X_b\mathrm{A}^2\mathrm{B}X \\ Y &\mapsto a^2\mathrm{B}Y \\ \tilde{Y} &\mapsto Y_b a^2\mathrm{B}Y \\ Z &\mapsto \mathrm{B}Z \\ Z_1 &\mapsto \mathrm{A}^2\mathrm{B}Z_1 \end{aligned}$$

$$
\begin{aligned}
\beta_{00}\colon \tilde{X}, X, \mathsf{A}_X &\mapsto \mathsf{A}_X^3\\
Y &\mapsto \textsc{b}Y\\
Z &\mapsto \textsc{b}Z\\
Z_1 &\mapsto \textsc{a}^2 Z_1
\end{aligned}
$$

$$
\begin{aligned}
\beta_{01}\colon \tilde{X}, X, \mathsf{A}_X &\mapsto \mathsf{A}_X^3\\
Y &\mapsto a\textsc{b}Y\\
\tilde{Y} &\mapsto Y_b a\textsc{b}Y\\
Z &\mapsto \textsc{a}^2\textsc{b}Z_1\\
\tilde{Z} &\mapsto Z_b\textsc{a}^2\textsc{b}Z_1\\
Z_1 &\mapsto \textsc{ab}Z_1
\end{aligned}
$$

$$
\begin{aligned}
\beta_{02}\colon \tilde{X}, X, \mathsf{A}_X &\mapsto \mathsf{A}_X^3\\
Y &\mapsto a^2\textsc{b}Y\\
\tilde{Y} &\mapsto Y_b a^2\textsc{b}Y\\
Z &\mapsto \textsc{ab}Z_1\\
\tilde{Z} &\mapsto Z_b\textsc{ab}Z_1\\
Z_1 &\mapsto \textsc{b}Z_1
\end{aligned}
$$

$$
\begin{aligned}
\beta_{10}\colon \tilde{X}, X, \mathsf{A}_X &\mapsto \textsc{a}\mathsf{A}_X^3\\
Y &\mapsto \textsc{b}Y\\
Z &\mapsto \textsc{ab}Z\\
\tilde{Z} &\mapsto Z_b\textsc{ab}Z\\
Z_1 &\mapsto \textsc{b}Z
\end{aligned}
$$

$$
\begin{aligned}
\beta_{11}\colon \tilde{X}, X, \mathsf{A}_X &\mapsto \textsc{a}\mathsf{A}_X^3\\
Y &\mapsto a\textsc{b}Y\\
\tilde{Y} &\mapsto Y_b a\textsc{b}Y\\
Z &\mapsto \textsc{b}Z\\
Z_1 &\mapsto \textsc{a}^2\textsc{b}Z_1
\end{aligned}
$$

$$
\begin{aligned}
\beta_{12}\colon \tilde{X}, X, \mathsf{A}_X &\mapsto \textsc{a}\mathsf{A}_X^3\\
Y &\mapsto a^2\textsc{b}Y\\
\tilde{Y} &\mapsto Y_b a^2\textsc{b}Y\\
Z &\mapsto \textsc{a}^2\textsc{b}Z_1\\
Z_1 &\mapsto \textsc{ab}Z_1
\end{aligned}
$$

$$
\begin{aligned}
\beta_{20}\colon \tilde{X}, X, \mathsf{A}_X &\mapsto \textsc{a}^2\mathsf{A}_X^3\\
Y &\mapsto \textsc{b}Y\\
Z &\mapsto \textsc{a}^2\textsc{b}Z\\
\tilde{Z} &\mapsto Z_b\textsc{a}^2\textsc{b}Z\\
Z_1 &\mapsto \textsc{ab}Z
\end{aligned}
$$

$$
\begin{aligned}
\beta_{21}\colon \tilde{X}, X, \mathsf{A}_X &\mapsto \textsc{a}^2\mathsf{A}_X^3\\
Y &\mapsto a\textsc{b}Y\\
\tilde{Y} &\mapsto Y_b a\textsc{b}Y\\
Z &\mapsto \textsc{ab}Z\\
\tilde{Z} &\mapsto Z_b\textsc{ab}Z\\
Z_1 &\mapsto \textsc{b}Z
\end{aligned}
$$

$$
\begin{aligned}
\beta_{22}\colon \tilde{X}, X, \mathsf{A}_X &\mapsto \textsc{a}^2\mathsf{A}_X^3\\
Y &\mapsto a^2\textsc{b}Y\\
\tilde{Y} &\mapsto Y_b a^2\textsc{b}Y\\
Z &\mapsto \textsc{b}Z\\
Z_1 &\mapsto \textsc{a}^2\textsc{b}Z_1
\end{aligned}
$$

$$
\begin{aligned}
\gamma_{00}\colon\ & X \mapsto \textsc{b}X\\
& \tilde{Y}, Y, \mathsf{a}_Y \mapsto \mathsf{a}_Y^3\\
& Z, \tilde{Z}, \mathsf{A}_Z \mapsto \mathsf{A}_Z^3\\
& Z_1, \mathsf{A}_{Z1} \mapsto \textsc{a}^2\mathsf{A}_{Z1}^3
\end{aligned}
\qquad
\begin{aligned}
\gamma_{01}\colon\ & X \mapsto \textsc{b}X\\
& \tilde{Y}, Y, \mathsf{a}_Y \mapsto a\mathsf{a}_Y^3\\
& Z, \tilde{Z}, \mathsf{A}_Z \mapsto \textsc{a}^2\mathsf{A}_{Z1}^3\\
& Z_1, \mathsf{A}_{Z1} \mapsto \textsc{a}\mathsf{A}_{Z1}^3
\end{aligned}
\qquad
\begin{aligned}
\gamma_{02}\colon\ & X \mapsto \textsc{b}X\\
& \tilde{Y}, Y, \mathsf{a}_Y \mapsto a^2\mathsf{a}_Y^3\\
& Z, \tilde{Z}, \mathsf{A}_Z \mapsto \textsc{a}\mathsf{A}_{Z1}^3\\
& Z_1, \mathsf{A}_{Z1} \mapsto \mathsf{A}_{Z1}^3
\end{aligned}
$$

$$
\begin{aligned}
\gamma_{10}\colon\ & X \mapsto \textsc{ab}X\\
& \tilde{X} \mapsto X_b\textsc{ab}X\\
& \tilde{Y}, Y, \mathsf{a}_Y \mapsto \mathsf{a}_Y^3\\
& Z, \tilde{Z}, \mathsf{A}_Z \mapsto \textsc{a}\mathsf{A}_Z^3\\
& Z_1, \mathsf{A}_{Z1} \mapsto \mathsf{A}_Z^3
\end{aligned}
\qquad
\begin{aligned}
\gamma_{11}\colon\ & X \mapsto \textsc{ab}X\\
& \tilde{X} \mapsto X_b\textsc{ab}X\\
& \tilde{Y}, Y, \mathsf{a}_Y \mapsto a\mathsf{a}_Y^3\\
& Z, \tilde{Z}, \mathsf{A}_Z \mapsto \mathsf{A}_Z^3\\
& Z_1, \mathsf{A}_{Z1} \mapsto \textsc{a}^2\mathsf{A}_{Z1}^3
\end{aligned}
\qquad
\begin{aligned}
\gamma_{12}\colon\ & X \mapsto \textsc{ab}X\\
& \tilde{X} \mapsto X_b\textsc{ab}X\\
& \tilde{Y}, Y, \mathsf{a}_Y \mapsto a^2\mathsf{a}_Y^3\\
& Z, \tilde{Z}, \mathsf{A}_Z \mapsto \textsc{a}^2\mathsf{A}_{Z1}^3\\
& Z_1, \mathsf{A}_{Z1} \mapsto \textsc{a}\mathsf{A}_{Z1}^3
\end{aligned}
$$

$$
\begin{aligned}
\gamma_{20}\colon\ & X \mapsto \textsc{a}^2\textsc{b}X\\
& \tilde{X} \mapsto X_b\textsc{a}^2\textsc{b}X\\
& \tilde{Y}, Y, \mathsf{a}_Y \mapsto \mathsf{a}_Y^3\\
& Z, \tilde{Z}, \mathsf{A}_Z \mapsto \textsc{a}^2\mathsf{A}_Z^3\\
& Z_1, \mathsf{A}_{Z1} \mapsto \textsc{a}\mathsf{A}_Z^3
\end{aligned}
\qquad
\begin{aligned}
\gamma_{21}\colon\ & X \mapsto \textsc{a}^2\textsc{b}X\\
& \tilde{X} \mapsto X_b\textsc{a}^2\textsc{b}X\\
& \tilde{Y}, Y, \mathsf{a}_Y \mapsto a\mathsf{a}_Y^3\\
& Z, \tilde{Z}, \mathsf{A}_Z \mapsto \textsc{a}\mathsf{A}_Z^3\\
& Z_1, \mathsf{A}_{Z1} \mapsto \mathsf{A}_Z^3
\end{aligned}
\qquad
\begin{aligned}
\gamma_{22}\colon\ & X \mapsto \textsc{a}^2\textsc{b}X\\
& \tilde{X} \mapsto X_b\textsc{a}^2\textsc{b}X\\
& \tilde{Y}, Y, \mathsf{a}_Y \mapsto a^2\mathsf{a}_Y^3\\
& Z, \tilde{Z}, \mathsf{A}_Z \mapsto \mathsf{A}_Z^3\\
& Z_1, \mathsf{A}_{Z1} \mapsto \textsc{a}^2\mathsf{A}_{Z1}^3
\end{aligned}
$$

$$
\begin{array}{lll}
\delta_{00}\colon \tilde{X}, X, \mathsf{A}_X \mapsto \mathsf{A}_X^3 & \delta_{01}\colon \tilde{X}, X, \mathsf{A}_X \mapsto \mathsf{A}_X^3 & \delta_{02}\colon \tilde{X}, X, \mathsf{A}_X \mapsto \mathsf{A}_X^3 \\
\quad \tilde{Y}, Y, \mathsf{a}_Y \mapsto \mathsf{a}_Y^3 & \quad \tilde{Y}, Y, \mathsf{a}_Y \mapsto a\mathsf{a}_Y^3 & \quad \tilde{Y}, Y, \mathsf{a}_Y \mapsto a^2\mathsf{a}_Y^3 \\
\quad \tilde{Z}, Z, \mathsf{A}_Z \mapsto \mathsf{A}_Z^3 & \quad \tilde{Z}, Z, \mathsf{A}_Z \mapsto \textsc{a}^2\mathsf{A}_{Z1}^3 & \quad \tilde{Z}, Z, \mathsf{A}_Z \mapsto \textsc{a}\mathsf{A}_{Z1}^3 \\
\quad Z_1, \mathsf{A}_{Z1} \mapsto \textsc{a}^2\mathsf{A}_{Z1}^3 & \quad Z_1, \mathsf{A}_{Z1} \mapsto \textsc{a}\mathsf{A}_{Z1}^3 & \quad Z_1, \mathsf{A}_{Z1} \mapsto \mathsf{A}_{Z1}^3
\end{array}
$$

$$
\begin{array}{lll}
\delta_{10}\colon \tilde{X}, X, \mathsf{A}_X \mapsto \textsc{a}\mathsf{A}_X^3 & \delta_{11}\colon \tilde{X}, X, \mathsf{A}_X \mapsto \textsc{a}\mathsf{A}_X^3 & \delta_{12}\colon \tilde{X}, X, \mathsf{A}_X \mapsto \textsc{a}\mathsf{A}_X^3 \\
\quad \tilde{Y}, Y, \mathsf{a}_Y \mapsto \mathsf{a}_Y^3 & \quad \tilde{Y}, Y, \mathsf{a}_Y \mapsto a\mathsf{a}_Y^3 & \quad \tilde{Y}, Y, \mathsf{a}_Y \mapsto a^2\mathsf{a}_Y^3 \\
\quad \tilde{Z}, Z, \mathsf{A}_Z \mapsto \textsc{a}\mathsf{A}_Z^3 & \quad \tilde{Z}, Z, \mathsf{A}_Z \mapsto \mathsf{A}_Z^3 & \quad \tilde{Z}, Z, \mathsf{A}_Z \mapsto \textsc{a}^2\mathsf{A}_{Z1}^3 \\
\quad Z_1, \mathsf{A}_{Z1} \mapsto \mathsf{A}_Z^3 & \quad Z_1, \mathsf{A}_{Z1} \mapsto \textsc{a}^2\mathsf{A}_{Z1}^3 & \quad Z_1, \mathsf{A}_{Z1} \mapsto \textsc{a}\mathsf{A}_{Z1}^3
\end{array}
$$

$$
\begin{array}{lll}
\delta_{20}\colon \tilde{X}, X, \mathsf{A}_X \mapsto \textsc{a}^2\mathsf{A}_X^3 & \delta_{21}\colon \tilde{X}, X, \mathsf{A}_X \mapsto \textsc{a}^2\mathsf{A}_X^3 & \delta_{22}\colon \tilde{X}, X, \mathsf{A}_X \mapsto \textsc{a}^2\mathsf{A}_X^3 \\
\quad \tilde{Y}, Y, \mathsf{a}_Y \mapsto \mathsf{a}_Y^3 & \quad \tilde{Y}, Y, \mathsf{a}_Y \mapsto a\mathsf{a}_Y^3 & \quad \tilde{Y}, Y, \mathsf{a}_Y \mapsto a^2\mathsf{a}_Y^3 \\
\quad \tilde{Z}, Z, \mathsf{A}_Z \mapsto \textsc{a}^2\mathsf{A}_Z^3 & \quad \tilde{Z}, Z, \mathsf{A}_Z \mapsto \textsc{a}\mathsf{A}_Z^3 & \quad \tilde{Z}, Z, \mathsf{A}_Z \mapsto \mathsf{A}_Z^3 \\
\quad Z_1, \mathsf{A}_{Z1} \mapsto \textsc{a}\mathsf{A}_Z^3 & \quad Z_1, \mathsf{A}_{Z1} \mapsto \mathsf{A}_Z^3 & \quad Z_1, \mathsf{A}_{Z1} \mapsto \textsc{a}^2\mathsf{A}_{Z1}^3
\end{array}
$$

References

[1] Alfred V. Aho, Indexed grammars - an extension of context-free grammars, J. Assoc. Comp. Math (15) 1968, 647–671.

[2] L. Ciobanu, V. Diekert and M. Elder. Solution sets for equations over free groups are EDT0L languages. *Internat. J. Algebra Comput.*, **26** no. 5 (2016), 843–866.

[3] L. Ciobanu and M. Elder, The complexity of solution sets to equations in hyperbolic groups, *Israel J. Math.*, **245**, no. 2 (2021):869–920.

[4] K. Culik II, On some families of languages related to developmental systems, *International Journal of Computer Mathematics*, **4** (1974), 31–42.

[5] V. Diekert and M. Elder, Solutions to twisted word equations and equations in virtually free groups, *Internat. J. Algebra Comput.*, **30** no. 4 (2020), 731–819.

[6] V. Diekert, A. Jeż and M. Kufleitner, Solutions of word equations over partially commutative structures, https://arxiv.org/abs/1603.02966

[7] A. Ehrenfeucht and G. Rozenberg, On some context free languages that are not deterministic ETOL languages, RAIRO - Theoretical Informatics and Applications - Informatique Théorique et Applications **11** no. 4 (1977), 273–291.

[8] A. Ehrenfeucht, G. Rozenberg, and S. Skyum, A relationship between ET0L and EDT0L languages, *Theoretical Computer Science*, **1** No 4 (1976), 325–330.

[9] A. Evetts and A. Levine, Equations in virtually abelian groups: languages and growth, *Internat. J. Algebra Comput.*, to appear.

[10] Robert H. Gilman, A shrinking lemma for indexed languages, Theoretical Computer Science **163** (1996) 277–281.

[11] R. H. Gilman, On the definition of word hyperbolic groups. Mathematische Zeitschrift **242**, no. 3 (2002), 529—541.

[12] P. Hall, 'Nilpotent groups', Notes of lectures given at the Canadian Mathematical Congress 1957 Summer Seminar, in "The collected works of Philip Hall" (Clarendon Press, Oxford, 1988) pp. 415—462.

[13] O. Kharlampovich, L. López, and A. Myasnikov, The Diophantine problem in some metabelian groups, *Mathematics of computation*, **89,** no. 325 (2020), 2507—2519.

[14] M.Rabkin, Ogden's lemma for ET0L languages, Language and automata theory and applications, 458–467, Lecture Notes in Comput. Sci., 7183.

[15] G.Rozenberg, Extension of tabled 0L systems and languages, International Journal of Computer and Information Sciences, 2 (1973) 311–334.

[16] G. Rozenberg and A. Salomaa, The mathematical theory of L systems, volume 90 of Pure and Applied Mathematics. Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York-London, 1980.