MEAN CURVATURE FLOW WITH GENERIC LOW-ENTROPY INITIAL DATA

OTIS CHODOSH, KYEONGSU CHOI, CHRISTOS MANTOULIDIS, AND FELIX SCHULZE

Abstract. We prove that sufficiently low-entropy closed hypersurfaces can be perturbed so that their mean curvature flow encounters only spherical and cylindrical singularities. Our theorem applies to all closed surfaces in $\mathbb{R}^3$ with entropy ≤ 2 and to all closed hypersurfaces in $\mathbb{R}^4$ with entropy $\leq \lambda(\mathbb{S}^1 \times \mathbb{R}^2)$. When combined with recent work of Daniels-Holgate, this strengthens Bernstein–Wang's low-entropy Schoenflies-type theorem by relaxing the entropy bound to $\lambda(\mathbb{S}^1 \times \mathbb{R}^2)$.

Our techniques, based on a novel density drop argument, also lead to a new proof of generic regularity result for area-minimizing hypersurfaces in eight dimensions (due to Hardt–Simon and Smale).

1. Introduction

Mean curvature flow is the natural heat equation for submanifolds. A family of hypersurfaces $M(t) \subset \mathbb{R}^{n+1}$ flows by mean curvature flow if

$$\left(\tfrac{\partial}{\partial t}\mathbf{x}\right)^{\perp} = \mathbf{H}_{M(t)}(\mathbf{x}), \tag{1.1}$$

where $\mathbf{H}_{M(t)}(\mathbf{x})$ denotes the mean curvature vector of $M(t)$ at $\mathbf{x}$. When $M(0)$ is compact, mean curvature flow is guaranteed to become singular in finite time. Understanding the potential singularities is thus a fundamental problem. One approach to this issue is to study the flow in the generic case: a well-known conjecture of Huisken suggests that the singularities of a generic mean curvature flow should be as simple as possible, namely, spherical and cylindrical [Ilm03, #8].

The main results of this note completely resolve Huisken's conjecture in three and four dimensions for low-entropy initial data (see (1.2) for the definition of entropy). Informally stated (see Corollaries 1.8 and 1.9 for precise statements) we prove the following results.

Theorem 1.1 (Low-entropy generic flow in $\mathbb{R}^3$, informal)**.** *If $M^2 \subset \mathbb{R}^3$ is a closed embedded surface with entropy $\lambda(M) \leq 2$ then there exist arbitrarily small C^∞ graphs M' over M so that the mean curvature flow starting from M' has only multiplicity-one spherical and cylindrical singularities.*

Theorem 1.2 (Low-entropy generic flow in $\mathbb{R}^4$, informal)**.** *If $M^3 \subset \mathbb{R}^4$ is a closed embedded hypersurface with entropy $\lambda(M) \leq \lambda(\mathbb{S}^1 \times \mathbb{R}^2)$ then there exist arbitrarily small C^∞ graphs M' over M so that the mean curvature* *flow starting from M' has only multiplicity-one spherical and cylindrical singularities.*

In an earlier version of this paper, we conjectured that Theorem 1.2 could be combined with a surgery construction to yield a strengthened version of Bernstein–Wang's low-entropy Schoenflies theorem [BW22a] (cf. Theorem 1.4 below). This surgery construction has been recently carried out by Daniels-Holgate [DH22] who showed that if a mean curvature flow has only spherical and neckpinch singularities, then one can construct a mean curvature flow with surgery. As such, combining these results leads to the following:

Corollary 1.3 (Strengthened low-entropy Schoenflies-type theorem)**.** *If $M^3 \subset \mathbb{R}^4$ is an embedded 3-sphere with entropy $\lambda(M) \leq \lambda(\mathbb{S}^1 \times \mathbb{R}^2)$ then M is smoothly isotopic to the round $\mathbb{S}^3$.*

See Sections 1.2 and 1.4 for an expanded discussion of this result.

1.1. **Previous work on generic mean curvature flow.** Trailblazing work of Colding–Minicozzi demonstrated that spheres and cylinders are the only *linearly stable* singularity models for mean curvature flow [CM12]. In particular, the remaining singularity models are unstable so should not generically occur (as conjectured by Huisken). In a previous paper [CCMS20], the authors introduced new methods to the study of generic mean curvature flow, proving that a large class of singularity models (specifically, singularities with tangent flows modeled on multiplicity one compact or asymptotically conical self-shrinkers) can be indeed avoided by a slight perturbation of the initial conditions.

In particular, our previous work shows that for a generic initial surface in $\mathbb{R}^3$, either the mean curvature flow has only spherical and cylindrical singularities or at the first singular time it has a tangent flow with a cylindrical end or higher multiplicity (both possibilities are conjectured not to happen). We refer the reader to the introduction to our previous article [CCMS20] for further discussion of generic mean curvature flows and related work.

1.1.1. *Relationship between this paper and our previous work.* In [CCMS20], we proved a classification of ancient one-sided flows (analogous to the minimal surface results of Hardt–Simon [HS85]; see Appendix D for further discussion) which led to a complete understanding of flows on either side of a neighborhood of a non-generic (compact or asymptotically conical) singularity. In particular, we showed that nearby flows to either side do not have such singularities nearby.

In $\mathbb{R}^3$, to understand generic mean curvature flow *without* a low-entropy condition (in contrast with this note), one must work at the first non-generic time rather than globally in space-time. However, two serious issues arise when working this way. First, there is no partial regularity known for tangent flows past[1] the first singular time without a low-entropy bound. Second, the possibility that a small perturbation of the initial data increases the first singular time slightly without improving the flow in an effective way. To that end, in [CCMS20] we had to additionally prove that the nearby flows strictly decrease genus as they avoid the non-generic singularity. This genus-loss property is crucial for tackling Huisken's conjecture in $\mathbb{R}^3$ without a low-entropy condition and is a consequence of the classification of ancient one sided flows, as obtained in [CCMS20].

On the other hand, by including a low-entropy condition, here we are able to work *globally* in space-time. This allows for significantly simplified arguments. In fact, the key observation of this paper is that in this setting one can completely avoid the classification of one-sided ancient flows and instead rely on a soft argument based on compactness and a new geometric property of non-generic shrinkers (see Proposition 2.2). We emphasize that a drawback of the methods used in this note as compared to our previous work is that the arguments used here give no indication as to the local dynamics near a non-generic singularity (such information was obtained in [CCMS20] near asymptotically conical and compact shrinkers; see also [CM19, CM22]).

Remark. After the first version of this paper (as well as our previous paper [CCMS20]) were posted, another approach to the generic perturbation of the initial data was pursued by Sun–Xue [SX21b, SX21a]. This approach is in the spirit of local ODE dynamics, as suggested by the Colding–Minicozzi program, cf. [CM19]. The analytic framework in [SX21b, SX21a] has the interesting feature that non-one-sided perturbations are analyzed, but the applications are currently limited to locally perturbing away singularities that arise at the first singular time. Conversely, our geometric approach (first developed in [CCMS20]) is motivated by global results such as the ones stated in Theorems 1.1 and 1.2. Of course, our approach also admits localizations; see Appendix C.

1.2. **Entropy.** To state our main results, we first recall Colding–Minicozzi's definition [CM12] of entropy of $M^n \subset \mathbb{R}^{n+1}$:

$$\lambda(M) := \sup_{\substack{\mathbf{x}_0 \in \mathbb{R}^{n+1} \\ t_0 > 0}} \int_M (4\pi t_0)^{-\frac{n}{2}} e^{-\frac{1}{4t_0}|\mathbf{x}-\mathbf{x}_0|^2}. \tag{1.2}$$

By Huisken's monotonicity of Gaussian area, we see that $t \mapsto \lambda(M(t))$ is non-increasing when $M(t)$ is flowing by mean curvature flow. A computation of Stone [Sto94] shows that the entropies of the self-shrinking cylinders

[1]At the first singular time, work of Ilmanen [Ilm95] and Wang [Wan16] show that the support of any tangent flow is a smooth self-shrinker with only conical/cylindrical ends.

$\mathbb{S}^k(\sqrt{2k}) \times \mathbb{R}^{n-k} \subset \mathbb{R}^{n+1}$ satisfy[2]

$$2 > \lambda(\mathbb{S}^1) = \sqrt{\frac{2\pi}{e}} \approx 1.52 > \frac{3}{2} > \lambda(\mathbb{S}^2) = \frac{4}{e} \approx 1.47 > \cdots > \lambda(\mathbb{S}^n).$$

Several fundamental results have been obtained about hypersurfaces with sufficiently small entropy, starting with work of Colding–Ilmanen–Minicozzi–White [CIMW13] who proved that the round sphere $\mathbb{S}^n(\sqrt{2n})$ has minimal entropy among all closed self-shrinkers. This was extended by Bernstein–Wang [BW16] who showed that the round sphere minimizes entropy among all closed hypersurfaces (see also [Zhu20, HW19]). Moreover, Bernstein–Wang have also proven [BW17] that the cylinder $\mathbb{S}^1(\sqrt{2}) \times \mathbb{R} \subset \mathbb{R}^3$ has second least entropy among all self-shrinkers in $\mathbb{R}^3$ (their result crucially relies on Brendle's classification of genus zero self-shrinkers [Bre16]).

Subsequent work of Bernstein–Wang provides a robust picture of hypersurfaces with sufficiently small entropy [BW18b, BW18a, BW22b] (see also [BW21]). In particular, they obtained the following low-entropy Schoenflies result:

Theorem 1.4 (Bernstein–Wang [BW22a])**.** *If $M^3 \subset \mathbb{R}^4$ has $\lambda(M) \leq \lambda(\mathbb{S}^2 \times \mathbb{R})$ then M is smoothly isotopic to the round $\mathbb{S}^3$.*

In [BW22a], this is proven by flowing M by mean curvature flow and then smoothing out any potential non-generic singularities to construct the desired isotopy. Our previous work [CCMS20] on generic mean curvature flow gave an alternative approach to this result by showing that if one perturbs M slightly, the mean curvature flow directly *provides* the isotopy:

Theorem 1.5 ([CCMS20])**.** *If $M^3 \subset \mathbb{R}^4$ has $\lambda(M) \leq \lambda(\mathbb{S}^2 \times \mathbb{R})$ then after a small C^∞-perturbation to a nearby hypersurface M', the mean curvature flow $M'(t)$ is completely smooth until it disappears in a round point.*

One of the consequences of this paper is a simplified proof of Theorem 1.5 (see also the stronger version stated in Corollary 1.3).

1.3. **Main results.** We now describe our main results in full generality. We construct generic mean curvature flows of sufficiently low-entropy hypersurfaces in all dimension. To quantify the low-entropy condition we make several definitions.[3] Let $\mathcal{S}_n$ denote the set of smooth self-shrinkers in $\mathbb{R}^{n+1}$ with $\lambda(\Sigma) < \infty$, i.e., properly embedded hypersurfaces Σ satisfying $\mathbf{H} + \frac{\mathbf{x}^\perp}{2} = 0$ with finite Gaussian area. Let $\mathcal{S}_n^*$ denote the non-flat elements of $\mathcal{S}_n$. For $\Lambda > 0$, let

$$\mathcal{S}_n(\Lambda) := \{\Sigma \in \mathcal{S}_n : \lambda(\Sigma) < \Lambda\}, \qquad \mathcal{S}_n^*(\Lambda) := \mathcal{S}_n(\Lambda) \cap \mathcal{S}_n^*.$$

[2]Note that $\lambda(\mathbb{S}^k(\sqrt{2k}) \times \mathbb{R}^{n-k}) = \lambda(\mathbb{S}^k)$.

[3]The definitions here are closely related to the hypotheses $(\star_{n,\Lambda}), (\star\star_{n,\Lambda})$ introduced by Bernstein–Wang (cf. [BW18b, BW22a]), but our second hypothesis is less restrictive.

We also define

$$\mathcal{S}_n^{\mathrm{gen}} := \left\{ O(\mathbb{S}^j(\sqrt{2j}) \times \mathbb{R}^{n-j}) \in \mathcal{S}_n : j = 1, \ldots, k,\ O \in O(n+1) \right\}$$

to be the set of (round) self-shrinking spheres and cylinders in $\mathbb{R}^{n+1}$.

Similarly, we let $\mathcal{RMC}_n$ denote the space of regular minimal cones in $\mathbb{R}^{n+1}$, i.e., the set of $\mathcal{C} \subset \mathbb{R}^{n+1}$ with $\mathcal{C} \setminus \{\mathbf{0}\}$ a smooth properly embedded hypersurface invariant under dilations and having vanishing mean curvature. Let $\mathcal{RMC}_n^*$ denote the non-flat elements of $\mathcal{RMC}_n$. Define

$$\mathcal{RMC}_n(\Lambda) := \{\mathcal{C} \in \mathcal{RMC}_n : \lambda(\mathcal{C}) < \Lambda\}, \quad \mathcal{RMC}_n^*(\Lambda) := \mathcal{RMC}_n(\Lambda) \cap \mathcal{RMC}_n^*.$$

For a dimension $n \geq 2$ and entropy bound $\Lambda \in (\lambda(\mathbb{S}^n), 2]$, our first hypothesis is

$$(\dagger_{n,\Lambda}) \qquad \text{For } 3 \leq k \leq n,\ \mathcal{RMC}_k^*(\Lambda) = \emptyset$$

while our second hypothesis is

$$(\dagger\dagger_{n,\Lambda}) \qquad \mathcal{S}_{n-1}^*(\Lambda) \subset \mathcal{S}_{n-1}^{\mathrm{gen}}.$$

Finally, we define certain notation that will be used throughout.

Definition 1.6. For a closed embedded hypersurface $M^n \subset \mathbb{R}^{n+1}$ we denote by $\mathfrak{F}(M)$ the set of cyclic[4] unit-regular integral Brakke flows $\mathcal{M}$ with $\mathcal{M}(0) = \mathcal{H}^n \lfloor M$, and for each $\mathcal{M} \in \mathfrak{F}(M)$, we define $\operatorname{sing}_{\mathrm{gen}} \mathcal{M} \subset \operatorname{sing} \mathcal{M}$ to be the set of singular points $(\mathbf{x}, t)$ so that some[5] tangent flow to $\mathcal{M}$ at $(\mathbf{x}, t)$ is a multiplicity-one flow associated to elements of $\mathcal{S}_n^{\mathrm{gen}}$.

Having given these definitions, we can now state our main technical result. By convention we take $\lambda(\mathbb{S}^0) = 2$. Everywhere below, M is taken to be closed and embedded.

Theorem 1.7. *Assume that $n \geq 2$ and $\Lambda \in (\lambda(\mathbb{S}^n), \lambda(\mathbb{S}^{n-2})]$ satisfy hypothesis $(\dagger_{n,\Lambda})$ and $(\dagger\dagger_{n,\Lambda})$. If $M^n \subset \mathbb{R}^{n+1}$ has $\lambda(M) \leq \Lambda$ then there exist arbitrarily small C^∞ graphs M' over M so that $\lambda(M') < \Lambda$ and all $\mathcal{M}' \in \mathfrak{F}(M')$ have $\operatorname{sing} \mathcal{M}' = \operatorname{sing}_{\mathrm{gen}} \mathcal{M}'$. In particular, the level set flow of M' does not fatten.*

See [CCMS20, Section 1.2] for a discussion of results related to the regularity of flows satisfying $\operatorname{sing} \mathcal{M}' = \operatorname{sing}_{\mathrm{gen}} \mathcal{M}'$.

In low dimensions, the hypothesis $(\dagger_{n,\Lambda})$ and $(\dagger\dagger_{n,\Lambda})$ can be understood more concretely. This leads to the following results.

Corollary 1.8. *If $M^2 \subset \mathbb{R}^3$ has $\lambda(M) \leq 2$ then there exist arbitrarily small C^∞ graphs M' over M so that the level-set flow of M' is non-fattening and the associated Brakke flow $\mathcal{M}' \in \mathfrak{F}(M')$ has $\operatorname{sing} \mathcal{M}' = \operatorname{sing}_{\mathrm{gen}} \mathcal{M}'$.*

---

[4]Recall that a integral varifold V is cyclic if the unique mod 2 flat chain $[V]$ has $\partial[V] = 0$. Work of White [Whi09] shows that this property is preserved under varifold (and Brakke flow) convergence.

[5]Note that if some tangent flow is a multiplicity one element of $\mathcal{S}_n^{\mathrm{gen}}$ then all are by [CIM15, CM15], cf. [BW15].

Proof. Condition ($\dagger_{2,2}$) is vacuous while ($\dagger\dagger_{2,2}$) holds by the classification of self-shrinking curves [AL86]. □

Corollary 1.9. *If $M^3 \subset \mathbb{R}^4$ has $\lambda(M) \leq \lambda(\mathbb{S}^1 \times \mathbb{R}^2)$ then there exist arbitrarily small C^∞ graphs M' over M so that the level-set flow of M' is non-fattening and the associated Brakke flow $\mathcal{M}' \in \mathfrak{F}(M')$ has $\operatorname{sing}\mathcal{M}' = \operatorname{sing}_{\text{gen}}\mathcal{M}'$.*

Proof. By the resolution of the Willmore conjecture [MN14], $\mathcal{RMC}_3^*(\Lambda_{\mathcal{C}}) = \emptyset$ for

$$\Lambda_{\mathcal{C}} = \frac{2\pi^2}{4\pi} \approx 1.57 > \lambda(\mathbb{S}^1) \approx 1.52.$$

Thus ($\dagger_{3,\Lambda}$) holds for all $\Lambda \leq \Lambda_{\mathcal{C}}$. Furthermore, by the classification of low-entropy shrinkers in $\mathbb{R}^3$ from [BW17], it holds that $\mathcal{S}_2^*(\lambda(\mathbb{S}^1)) = \mathcal{S}_2^{\text{gen}}$. Thus ($\dagger\dagger_{3,\lambda(\mathbb{S}^1)}$) holds. □

1.4. **Generic mean curvature flow with surgery.** As already observed in [CCMS20], we can apply Corollary 1.9 to give a direct proof of Theorems 1.4 and 1.5. Moreover, Daniels-Holgate has recently proven that if an initial hypersurface admits a (cyclic, unit-regular, integral) Brakke flow with only[6] spherical and neckpinch type singularities[7] then it is possible to construct a smooth mean curvature flow with surgery starting from this initial condition (see [DH22] for the precise definition of mean curvature flow with surgery).

As such, Corollaries 1.8 and 1.9 combined with [DH22, Theorem 1.2] yields the following generic surgery construction.

Corollary 1.10 (Generic mean curvature flow with surgery)**.** *Assume that $n \geq 2$ and $\Lambda \in (\lambda(\mathbb{S}^n), \lambda(\mathbb{S}^{n-2})]$ satisfy ($\dagger_{n,\Lambda}$) and ($\dagger\dagger_{n,\Lambda}$). If $M^n \subset \mathbb{R}^{n+1}$ has $\lambda(M) \leq \Lambda$, then there is an arbitrarily small C^∞ graph M' over M and a smooth mean curvature flow with surgery starting from M'.*

In particular, when $M^3 \subset \mathbb{R}^4$ is an embedded 3-sphere with $\lambda(M) \leq \lambda(\mathbb{S}^1 \times \mathbb{R}^2)$, the mean curvature flow with surgery can be used (see [DH22, Theorem 6.4]) to construct an isotopy to the round 3-sphere. This yields the strengthened version of the low-entropy Schoenflies theorem stated in Corollary 1.3.

Remark. In the setting of 2-convex mean curvature flow with surgery (see [HS99, HS99, Bre15, BH16, HK17a, HK17b, ADS19, ADS20, BC19, BC21]) the surgery to isotopy construction has been studied in several works [HS09, BHH16, BHH19, Mra18, MW21]. (We also mention related work using Ricci flow with surgery [Mar12, CL19] and singular Ricci flow [BK22, BK23, BK19].)

[6]The spherical and neckpinch singularities are the tangent flows for which a canonical neighborhood theorem is proven, thanks to [CHH22, CHHW22].

[7]Note that if $\mathcal{M}'$ is such a Brakke flow in $\mathbb{R}^{n+1}$ and $\operatorname{sing}\mathcal{M}' = \operatorname{sing}_{\text{gen}}\mathcal{M}'$, then the condition "$\mathcal{M}'$ has only spherical and neckpinch singularities" is a consequence of $\lambda(\mathcal{M}') < \lambda(\mathbb{S}^{n-2})$.

1.5. **Generic regularity of area-minimizing hypersurfaces in eight dimensions.** We remark that the study of generic mean curvature flow in our previous work [CCMS20] can be viewed as the parabolic analogue of the work of Hardt–Simon [HS85] and Smale [Sma93] concerning the generic regularity of area-minimizing hypersurfaces in eight dimensions. In particular, the existence and uniqueness of the ancient one-sided mean curvature flow [CCMS20] is a direct analogue of the existence and uniqueness of the foliation on either side of a regular area minimizing cone, as proven in [HS85] (see also [Wan22]).

In this paper, we develop a new technique based on density drop, that avoids the classification of the ancient one-sided flow. As one might expect, this also yields a new proof of the generic regularity results of Hardt–Simon [HS85] and Smale [Sma93] that avoids the need to classify the foliation. This is discussed further in Appendix D.

1.6. **Organization.** See [CCMS20, Section 2] for the conventions used in this paper. In Section 2 we prove entropy drop near non-generic singularities and we use this to prove Theorem 1.7 in Section 3. Appendices A and B recall some standard stability results. Appendix C contains a localized perturbative result. In Appendix D, we discuss how the arguments here relate to generic regularity of area-minimizing hypersurfaces in eight dimensions.

1.7. **Acknowledgments.** O.C. was partially supported by a Sloan Fellowship, a Terman Fellowship, and NSF grants DMS-1811059 and DMS-2016403. K.C. was supported by KIAS Individual Grant MG078901. C.M. was supported by the NSF grant DMS-2050120 and DMS-2147521. F.S. was supported by a Leverhulme Trust Research Project Grant RPG-2016-174. We would like to thank Richard Bamler for some discussions related to weak flows and surgery constructions. Finally we are grateful to the referees for many helpful suggestions concerning

2. Entropy drop near non-generic singularities

Lemma 2.1. *Assume that* $(\dagger_{n,\Lambda})$ *holds for some* $\Lambda \leq 2$. *Suppose that* V *is a* F*-stationary cyclic integral* n*-varifold in* $\mathbb{R}^{n+1}$ *satisfying* $F(V) < \Lambda$. *Then, there is* $\Sigma \in \mathcal{S}_n(\Lambda)$ *so that* $V = \mathcal{H}^n \lfloor \Sigma$.

Proof. This follows from the proof of [BW18b, Lemma 3.1 and Proposition 3.2] except the cyclic property of V is used to rule out three half-spaces as a potential iterated tangent cone (cf. [Whi09, Corollary 4.5]). □

Recall that Huisken has classified the cylinders $\mathbb{S}^k(\sqrt{2k}) \times \mathbb{R}^{n-k}$ as the unique smooth embedded self-shrinkers with non-negative mean curvature $H \geq 0$ [Hui90, Hui93] (the technical assumption of bounded curvature was later removed by Colding–Minicozzi [CM12]). The following result can be viewed as a geometric consequence of Huisken's result. It will serve as our key mechanism for perturbing away "non-generic" singularities.

Proposition 2.2. *For $\Sigma \in \mathcal{S}_n^*$, fix an open set $\Omega \subset \mathbb{R}^{n+1}$ with $\Sigma = \partial\Omega$. Assume that there is a space-time point $(\mathbf{x}_0, t_0) \in (\mathbb{R}^{n+1} \times \mathbb{R}) \setminus (\mathbf{0}, 0)$ so that*

$$\sqrt{t_0 - t}\,\Sigma + \mathbf{x}_0 \subset \sqrt{-t}\,\bar{\Omega} \tag{2.1}$$

for all $t < \min\{0, t_0\}$. Then, one of the following holds:

1. *$\Sigma = \mathbb{S}^n(\sqrt{2n})$, or*
2. *$\Sigma = O(\hat{\Sigma} \times \mathbb{R})$ for $\hat{\Sigma} \in \mathcal{S}_{n-1}^*$ and $O \in O(n+1)$.*

Note that if we replaced condition (2.1) with

$$\sqrt{t_0 - t}\,\Sigma + \mathbf{x}_0 \subset \sqrt{-t}\,\Omega \tag{2.2}$$

(i.e., if we replaced the closure of Ω with the interior of Ω), we could use an inductive argument to conclude that $\Sigma \in \mathcal{S}_n^{\text{gen}}$.

Let us give the geometric intuition underlying our proof strategy. Let $\mathcal{M}_0$ denote the spacetime track of $t \mapsto \sqrt{-t}\Sigma$ and $\mathcal{M}$ denote the spacetime track of $t \mapsto \sqrt{t_0 - t}\,\Sigma + \mathbf{x}_0$. For $\lambda \in (0, 1]$, let $\mathcal{M}_\lambda$ be the parabolic rescaling of $\mathcal{M}$ by a factor of λ; thus, $\mathcal{M}_1 = \mathcal{M}$ and, as $\lambda \to 0$, $\mathcal{M}_\lambda \to \mathcal{M}_0$ smoothly locally away from $(\mathbf{0}, 0)$. Note that $\mathcal{M}_0$ is invariant under parabolic dilations, so $\mathcal{M}_\lambda$ always lies weakly to one side of $\mathcal{M}_0$.

If $\mathcal{M}_\lambda$ touches $\mathcal{M}_0$ for some $\lambda > 0$ (equivalently, for all $\lambda > 0$ due to $\mathcal{M}_0$'s parabolic dilation invariance), it is then a simple consequence of the strong maximum principle and monotonicity that Σ splits a line.

Otherwise, $\mathcal{M}_\lambda$ was disjoint from $\mathcal{M}_0$ for all $\lambda \in (0, 1]$. It is then standard to use the height of $\mathcal{M}_\lambda$ over $\mathcal{M}_0$ at time $t = -1$, for $\lambda > 0$ small, to produce a kernel element of the linearized operator that is everywhere nonnegative ($\mathcal{M}_\lambda$ always lies weakly to one side of $\mathcal{M}_0$). By studying the geometry of parabolic dilations, the kernel element produced is $\mathbf{x}_0 \cdot \nu_\Sigma$ if $\mathbf{x}_0 \neq \mathbf{0}$ or $\mathbf{x} \cdot \nu_\Sigma$ if $\mathbf{x}_0 = \mathbf{0}$ ($\implies t_0 \neq 0$). It turns out that the former case implies splitting once again, while the latter implies the mean-convexity of Σ.

The proof we give below is a more succinct version of the argument above: it handles both cases in a unified way.

Proof of Proposition 2.2. Observe that the set $\cup_{t<0}\sqrt{-t}\,\bar{\Omega} \times \{t\}$ is invariant under parabolic dilation around the space-time origin. We thus conclude that for all $\lambda \in [0, \infty)$ and $t < \min\{0, \lambda^2 t_0\}$,

$$\sqrt{\lambda^2 t_0 - t}\,\Sigma + \lambda \mathbf{x}_0 \subset \sqrt{-t}\,\bar{\Omega}$$

In particular, taking $t = -1$ and $\lambda \geq 0$ small, we have that

$$\lambda \mapsto \Sigma_\lambda := \sqrt{1 + \lambda^2 t_0}\,\Sigma + \lambda \mathbf{x}_0 \subset \bar{\Omega}$$

is a 1-parameter family of hypersurfaces with $\Sigma_0 = \Sigma = \partial\Omega$. The normal speed at $\lambda = 0$ is $\mathbf{x}_0 \cdot \nu_\Sigma \geq 0$ (where ν_Σ is the unit normal pointing into Ω). Because

$$\Delta_\Sigma(\mathbf{x}_0 \cdot \nu_\Sigma) - \tfrac{1}{2}\mathbf{x} \cdot \nabla_\Sigma(\mathbf{x}_0 \cdot \nu_\Sigma) + |A_\Sigma|^2(\mathbf{x}_0 \cdot \nu_\Sigma) = 0$$

(cf. [CM12, Theorem 5.2]), the maximum principle implies that either $\mathbf{x}_0 \cdot \nu_\Sigma > 0$ along Σ or $\mathbf{x}_0 \cdot \nu_\Sigma = 0$ along Σ. (Note that Σ is connected thanks to the Frankel property of shrinkers, cf. [CCMS20, Corollary C.4].)

In the first case (i.e., $\mathbf{x}_0 \cdot \nu_\Sigma > 0$), each component of Σ is a graph over the $\mathbf{x}_0^\perp$-hyperplane. By [Wan11] (cf. [EH89]), each component of Σ must be a hyperplane, so there is only one component and Σ is a flat hyperplane. This contradicts the assumption that $\Sigma \in \mathcal{S}_n^*$ (the set of non-flat shrinkers).

In the second case (i.e., $\mathbf{x}_0 \cdot \nu_\Sigma = 0$), we see that $\mathbf{x}_0 \in T_p\Sigma$ for all $p \in \Sigma$. In particular, if $\mathbf{x}_0 \neq \mathbf{0}$, then Σ splits a line in the $\mathbf{x}_0$-direction. It thus remains to consider the situation in which $\mathbf{x}_0 = \mathbf{0}$. If this is the case, then it must hold that $t_0 \neq 0$ and we have

$$\tilde{\Sigma}_\mu := (1 + \mu t_0)\Sigma \subset \bar{\Omega}.$$

for $\mu \geq 0$ sufficiently small. The normal speed at $\mu = 0$ is $t_0 \mathbf{x} \cdot \nu_\Sigma \geq 0$. Using the shrinker equation, we thus find that $t_0 H_\Sigma \geq 0$. Since $t_0 \neq 0$, we can assume that $H_\Sigma \geq 0$. Thus, up to a rotation, $\Sigma = \mathbb{S}^k(\sqrt{2k}) \times \mathbb{R}^{n-k}$ for $k = 1, \ldots, n$ by [CM12, Theorem 10.1]. This completes the proof. ☐

Recall the the definition of smoothly crossing Brakke flows in Definition B.1.

Proposition 2.3. *Fix $n \geq 2$, $\varepsilon > 0$ and $\Lambda \in (\lambda(\mathbb{S}^n), 2]$ so that $(\dagger_{n,\Lambda})$ and $(\dagger\dagger_{n,\Lambda})$ hold. There is $\delta = \delta(n, \varepsilon, \Lambda) > 0$ with the following property.*

Consider $\Sigma \in \mathcal{S}_n^*(\Lambda - \varepsilon) \setminus \mathcal{S}_n^{\text{gen}}$ and $\tilde{\mathcal{M}}$ an ancient cyclic unit-regular integral n-dimensional Brakke flow in $\mathbb{R}^{n+1}$ with $\lambda(\tilde{\mathcal{M}}) \leq F(\Sigma)$ so that $\tilde{\mathcal{M}}$ does not smoothly cross the flow $(-\infty, 0) \ni t \mapsto \mathcal{H}^n\lfloor\sqrt{-t}\,\Sigma$. Then, $\Theta_{\tilde{\mathcal{M}}}(\mathbf{x}, t) \leq F(\Sigma) - \delta$ for all $(\mathbf{x}, t) \in (\mathbb{R}^{n+1} \times \mathbb{R}) \setminus (\mathbf{0}, 0)$.

Proof. We argue by contradiction. Consider a sequence of $\Sigma_i \in \mathcal{S}_n^*(\Lambda - \varepsilon) \setminus \mathcal{S}_n^{\text{gen}}$ and $\mathcal{M}_i$ ancient cyclic unit-regular integral Brakke flows in $\mathbb{R}^{n+1}$ with $\lambda(\tilde{\mathcal{M}}_i) \leq F(\Sigma)$ so that $\tilde{\mathcal{M}}_i$ does not smoothly cross the flow $(-\infty, 0) \ni t \mapsto \mathcal{H}^n\lfloor\sqrt{-t}\,\Sigma_i$ and so that there are points $(\mathbf{x}_i, t_i) \in (\mathbb{R}^{n+1} \times \mathbb{R}) \setminus (\mathbf{0}, 0)$ with

$$\Theta_{\tilde{\mathcal{M}}_i}(\mathbf{x}_i, t_i) \geq F(\Sigma_i) - o(1) \tag{2.3}$$

as $i \to \infty$. We can assume that $|(\mathbf{x}_i, t_i)| = 1$.

By Lemma 2.1 and Allard's theorem [All72, Sim83], we can pass to a subsequence so that Σ_i converges in C^∞_{loc} to $\Sigma \in \mathcal{S}_n(\Lambda)$. By Brakke's theorem [Bra75, Whi05], Σ is non-flat. Because cylinders are isolated in C^∞_{loc} by [CIM15], we thus see that $\Sigma \in \mathcal{S}_n^*(\Lambda) \setminus \mathcal{S}_n^{\text{gen}}$. Note that $F(\Sigma_i) \to F(\Sigma)$.

We now pass to a further subsequence so that $(\mathbf{x}_i, t_i) \to (\mathbf{x}_0, t_0) \in \mathbb{R}^{n+1} \times \mathbb{R}$ with $|(\mathbf{x}_0, t_0)| = 1$ and the Brakke flows $\tilde{\mathcal{M}}_i$ converge to an ancient cyclic unit-regular integral Brakke flow $\tilde{\mathcal{M}}$ with $\lambda(\tilde{\mathcal{M}}) \leq F(\Sigma)$. By upper semi-continuity of Gaussian density, (2.3) implies that $\Theta_{\tilde{\mathcal{M}}}(\mathbf{x}_0, t_0) \geq F(\Sigma)$. Because $\lambda(\tilde{\mathcal{M}}) \leq F(\Sigma)$, $\tilde{\mathcal{M}}$ is a self-similar flow around $(\mathbf{x}_0, t_0)$. By stability of smoothly crossing flows, Lemma B.2, $\tilde{\mathcal{M}}$ does not smoothly cross $(-\infty, 0) \ni t \mapsto \mathcal{H}^n\lfloor\sqrt{-t}\,\Sigma$.

Consider any tangent flow to $\tilde{\mathcal{M}}$ at $t = -\infty$. By Huisken's monotonicity formula and Lemma 2.1 there is a smooth shrinker $\tilde{\Sigma}$ so that this tangent flow at $t = -\infty$ corresponds to some $\tilde{\Sigma} \in \mathcal{S}_n(\Lambda)$ with multiplicity-one. By the Frankel property for self-shrinkers (cf. [CCMS20, Corollary C.4]) and the strong maximum principle, if $\tilde{\Sigma} \neq \Sigma$ then the flows $t \mapsto \mathcal{H}^n \lfloor \sqrt{-t}\,\Sigma$ and $t \mapsto \mathcal{H}^n \lfloor \sqrt{-t}\,\tilde{\Sigma}$ smoothly cross each other at some point. This contradicts the stability of smooth crossings.

We conclude that any tangent flow to $\tilde{\mathcal{M}}$ at $t = -\infty$ is the flow associated to Σ. Since $\tilde{\mathcal{M}}$ is self-similar around $(\mathbf{x}_0, t_0)$ we find

$$\tilde{\mathcal{M}}(t) = \mathcal{H}^n \lfloor (\sqrt{t_0 - t}\,\Sigma + \mathbf{x}_0)$$

for $t < t_0$. Since $\tilde{\mathcal{M}}$ does not smoothly cross $t \mapsto \mathcal{H}^n \lfloor \sqrt{-t}\,\Sigma$, we see that there is an open set $\Omega \subset \mathbb{R}^{n+1}$ with $\partial\Omega = \Sigma$ so that

$$\sqrt{t_0 - t}\,\Sigma + \mathbf{x}_0 \subset \sqrt{-t}\,\bar{\Omega}$$

for $t < \min\{0, t_0\}$. We can thus apply Proposition 2.2 to conclude that (up to a rotation) $\Sigma = \hat{\Sigma} \times \mathbb{R}$ for $\hat{\Sigma} \in \mathcal{S}^*_{n-1}(\Lambda)$. By hypothesis $(\dagger\dagger_{n,\Lambda})$, $\hat{\Sigma} \in \mathcal{S}^{\text{gen}}_{n-1}$ so $\Sigma = \hat{\Sigma} \times \mathbb{R} \in \mathcal{S}^{\text{gen}}_n$. This is a contradiction. $\square$

3. Proof of Theorem 1.7

For $M' \subset \mathbb{R}^{n+1}$ a smooth closed hypersurface, recall that $\mathfrak{F}(M')$ is the set of cyclic unit-regular integral Brakke flows $\mathcal{M}'$ with $\mathcal{M}'(0) = \mathcal{H}^n \lfloor M'$. Note that [Ilm93, Whi09] implies that $\mathfrak{F}(M') \neq \emptyset$ (see also [HW20, Appendix B]).

We define

$$\mathcal{D}(M') := \sup\{\Theta_{\mathcal{M}'}(\mathbf{x}, t) : \mathcal{M}' \in \mathfrak{F}(M'), (\mathbf{x}, t) \in \operatorname{sing} \mathcal{M}' \setminus \operatorname{sing}_{\text{gen}} \mathcal{M}'\}.$$

Recall that by convention $\sup \emptyset = -\infty$.

Assume that hypotheses $(\dagger_{n,\Lambda})$ and $(\dagger\dagger_{n,\Lambda})$ hold for $\Lambda \in (\lambda(\mathbb{S}^n), \lambda(\mathbb{S}^{n-2})]$ fixed. Consider a smooth closed hypersurface $M^n \subset \mathbb{R}^{n+1}$ with $\lambda(M) \leq \Lambda$. Flowing M by mean curvature flow for a short time strictly decreases the entropy unless M is homothetic to a self-shrinker. If M is homothetic to a self-shrinker other than $\mathbb{S}^n(\sqrt{2n})$ then by [CM12], a small C^∞-perturbation of M has strictly smaller entropy.

As such, either $M = \mathbb{S}^n(r)$ in which case the Theorem 1.7 trivially holds or we can perform an initial perturbation and assume that $\lambda(M) \leq \Lambda - 2\varepsilon$ for some $\varepsilon > 0$. Choose a foliation $\{M_s\}_{s \in (-1,1)}$ of a tubular neighborhood of M so that $M_0 = M$ and so that $\lambda(M_s) \leq \Lambda - \varepsilon$. Fix $\delta = \delta(n, \varepsilon, \Lambda) > 0$ from Proposition 2.3.

Lemma 3.1. *We have*

$$\limsup_{s \to s_0} \mathcal{D}(M_s) \leq \mathcal{D}(M_{s_0}) - \delta.$$

for all $s_0 \in (-1, 1)$.

Lemma 3.1 implies Theorem 1.7 by a straightforward iteration argument since by Brakke's regularity theorem [Bra75, Whi05], if $\mathcal{D}(M') \leq 1$ then $\mathcal{D}(M') = -\infty$ implying that $\operatorname{sing}\mathcal{M}' = \operatorname{sing}_{\text{gen}}\mathcal{M}'$ for all $\mathcal{M}' \in \mathfrak{F}(M')$. Since $\lambda(M') < \Lambda \leq \lambda(\mathbb{S}^{n-2} \times \mathbb{R}^2)$, any $\mathcal{M}' \in \mathfrak{F}(M')$ has only (multiplicity one) $\mathbb{S}^n$ and $\mathbb{S}^{n-1} \times \mathbb{R}$-type singularities. Thus, the resolution of the mean convex neighborhood conjecture for $\mathbb{S}^{n-1} \times \mathbb{R}$ singularities [CHH22, CHHW22] (cf. [HW20]) implies non-fattening of the flow of M'.

Proof of Lemma 3.1. Assume there is $s_i \to s_0 \in (-1,1)$ with $s_i \neq s_0$ but

$$\lim_{i\to\infty} \mathcal{D}(M_{s_i}) > \mathcal{D}(M_{s_0}) - \delta.$$

Fix $\mathcal{M}_i \in \mathfrak{F}(M_{s_i})$ and $(\mathbf{x}_i, t_i) \in \operatorname{sing}\mathcal{M}_i \setminus \operatorname{sing}_{\text{gen}}\mathcal{M}_i$ with

$$\lim_{i\to\infty} \Theta_{\mathcal{M}_i}(\mathbf{x}_i, t_i) > \mathcal{D}(M_{s_0}) - \delta$$

Pass to a subsequence $\mathcal{M}_i$ converging to $\mathcal{M} \in \mathfrak{F}(M_{s_0})$ and $(\mathbf{x}_i, t_i) \to (\mathbf{x}_0, t_0) \in \operatorname{sing}\mathcal{M}$. Since $s_i \neq s_0$ for all i, we have that M_{s_i} is disjoint from M_{s_0} for all i. In particular, $\operatorname{supp}\mathcal{M}_i \cap \operatorname{supp}\mathcal{M} = \emptyset$ (by the avoidance principle for Brakke flows [Ilm94, 10.6]). Thus, $(\mathbf{x}_i, t_i) \neq (\mathbf{x}_0, t_0)$.

Observe that that if $(\mathbf{x}_0, t_0) \in \operatorname{sing}_{\text{gen}}\mathcal{M}$ then since $\lambda(M) < \Lambda \leq \lambda(\mathbb{S}^{n-2})$, we see that $(\mathbf{x}_0, t_0)$ must be a $\mathbb{S}^n$ or $\mathbb{S}^{n-1} \times \mathbb{R}$-type singularity. Proposition A.1 then implies that $(\mathbf{x}_i, t_i) \in \operatorname{sing}_{\text{gen}}\mathcal{M}_i$, a contradiction. Thus, it must hold that $(\mathbf{x}_0, t_0) \in \operatorname{sing}\mathcal{M} \setminus \operatorname{sing}_{\text{gen}}\mathcal{M}$.

Translate $(\mathbf{x}_0, t_0)$ to the space-time origin and parabolically dilate to yield $\tilde{\mathcal{M}}_i$ and $(\tilde{\mathbf{x}}_i, \tilde{t}_i)$ with $|(\tilde{\mathbf{x}}_i, \tilde{t}_i)| = 1$ and

$$\lim_{i\to\infty} \Theta_{\tilde{\mathcal{M}}_i}(\tilde{\mathbf{x}}_i, \tilde{t}_i) > \mathcal{D}(M_{s_0}) - \delta$$

Pass to a subsequence so that $\tilde{\mathcal{M}}_i \rightharpoonup \tilde{\mathcal{M}}$ and $(\tilde{\mathbf{x}}_i, \tilde{t}_i) \to (\tilde{\mathbf{x}}, \tilde{t}) \in (\mathbb{R}^{n+1} \times \mathbb{R}) \setminus (\mathbf{0}, 0)$. By upper semicontinuity of density

$$\Theta_{\tilde{\mathcal{M}}}(\tilde{\mathbf{x}}, \tilde{t}) > \mathcal{D}(M_{s_0}) - \delta. \tag{3.1}$$

On the other hand, we can perform the same translation and parabolic dilation to $\mathcal{M}$ and by extracting a further subsequence, the resulting flows converge to a tangent flow to $\mathcal{M}$ at $(\mathbf{x}_0, t_0)$. By Lemma 2.1, the tangent flow is the multiplicity-one flow associated to a smooth shrinker Σ. Note that

$$F(\Sigma) \leq \lambda(\mathcal{M}) \leq \limsup_{s\to s_0} \lambda(M_s) \leq \Lambda - \varepsilon.$$

Since $(\mathbf{x}_0, t_0) \in \operatorname{sing}\mathcal{M} \setminus \operatorname{sing}_{\text{gen}}\mathcal{M}$ it must hold that $\Sigma \in \mathcal{S}_n^*(\Lambda - \varepsilon) \setminus \mathcal{S}_n^{\text{gen}}$. Huisken's monotonicity formula implies that $\lambda(\tilde{\mathcal{M}}) \leq F(\Sigma) = \Theta_{\mathcal{M}}(\mathbf{x}_0, t_0)$ (cf. the proof of Proposition 10.6 in [CCMS20]). Finally, since the supports of $\mathcal{M}$ and $\mathcal{M}_i$ are disjoint, $\mathcal{M}_i$ does not smoothly cross $\mathcal{M}$. As such (using Lemma B.2), $\tilde{\mathcal{M}}$ does not smoothly cross $t \mapsto \mathcal{H}^n \lfloor \sqrt{-t}\Sigma$. We can now apply Proposition 2.3 to conclude that

$$\Theta_{\tilde{\mathcal{M}}}(\tilde{\mathbf{x}}, \tilde{t}) \leq F(\Sigma) - \delta = \Theta_{\mathcal{M}}(\mathbf{x}_0, t_0) - \delta \leq \mathcal{D}(M_{s_0}) - \delta.$$

This contradicts (3.1), completing the proof. □

Appendix A. Stability of generic singularities

Based on [CHHW22], the following stability of generic singularities was proven in [SS20, Proposition 2.3] (see [CCMS20, Lemma 10.4] for the simple argument when the singularity is modeled on $\mathbb{S}^n$). When $n = 2$ this also follows via density considerations using [BW17].

Proposition A.1. *Suppose that $\mathcal{M}_i \rightharpoonup \mathcal{M}$ are unit-regular integral Brakke flows in $\mathbb{R}^{n+1}$ and that $(\mathbf{x}_i, t_i) \in \operatorname{sing} \mathcal{M}_i$ converge to $(\mathbf{0}, 0) \in \operatorname{sing}_{\mathrm{gen}} \mathcal{M}$. If the singularity at $(\mathbf{0}, 0)$ is modeled on $\mathbb{S}^n$ or $\mathbb{S}^{n-1} \times \mathbb{R}$, then for i sufficiently large $(\mathbf{x}_i, t_i) \in \operatorname{sing}_{\mathrm{gen}} \mathcal{M}_i$.*

Appendix B. Stability of crossing points

Definition B.1. Given two integral unit Brakke flows $\mathcal{M}^{(1)}$ and $\mathcal{M}^{(2)}$, we say that $\mathcal{M}^{(1)}$ and $\mathcal{M}^{(2)}$ *smoothly cross* at $(\mathbf{x}, t)$ if there is $r > 0$ with

$$\mathcal{M}^{(j)}(s) \lfloor B_r(\mathbf{x}) = \mathcal{H}^n \lfloor \Gamma^{(j)}(s)$$

for $s \in (t - r^2, t + r^2)$ where $\Gamma^{(j)}(s)$ are smooth connected mean curvature flows so that in any small neighborhood of $\mathbf{x}$ there are points of $\Gamma^{(1)}(0)$ on both sides of $\Gamma^{(2)}(0)$.

The following is a straightforward consequence of Brakke's regularity theorem [Bra75, Whi05].

Lemma B.2. *For $j = 1, 2$, suppose that $\mathcal{M}_i^{(j)} \rightharpoonup \mathcal{M}^{(j)}$ are integral unit-regular n-dimensional Brakke flows in $\mathbb{R}^{n+1}$. Assume that $\mathcal{M}^{(1)}$ smoothly crosses $\mathcal{M}^{(2)}$ at $(\mathbf{x}, t)$. Then, for i sufficiently large, there is $(\mathbf{x}_i, t_i) \to (\mathbf{x}, t)$ so that $\mathcal{M}_i^{(1)}$ smoothly crosses $\mathcal{M}_i^{(2)}$ at $(\mathbf{x}_i, t_i)$.*

Appendix C. Local results

In this appendix we prove the following local perturbative result.

Proposition C.1. *Suppose that $M^n \subset \mathbb{R}^{n+1}$ is a closed embedded hypersurface, $\mathcal{M} \in \mathcal{F}(M)$ is a cyclic unit-regular integral Brakke flow starting at M. Assume that for $(\mathbf{x}_0, t_0) \in \operatorname{sing} \mathcal{M}$, the following holds:*

- *$\operatorname{reg} \mathcal{M} \cap \{t < t_0\} \subset \mathbb{R}^{n+1} \times \mathbb{R}$ is connected*
- *any tangent flow $\mathcal{N}$ to $\mathcal{M}$ at $(\mathbf{x}_0, t_0)$ has $\mathcal{N}(-1) = \mathcal{H}^n \lfloor \Sigma$, for $\Sigma \in \mathcal{S}_n^* \setminus \mathcal{S}_n^{\mathrm{gen}}$ that does not split a line.*

Then, there is $r = r(\mathcal{M}, \mathbf{x}_0, t_0) > 0$ so that for $M_j = \operatorname{graph}_M(u_j)$, $u_j > 0$ with $u_j \to 0$ in C^∞ it holds that any

$$(\mathbf{x}, t) \in B_r(\mathbf{x}_0) \times (t_0 - r^2, t_0 + r^2)$$

has $\Theta_{\mathcal{M}_j}(\mathbf{x}, t) \le \Theta_{\mathcal{M}}(\mathbf{x}_0, t_0) - r$ for j sufficiently large.

In particular, no tangent flow to $\mathcal{M}$ at $(\mathbf{x}_0, t_0)$ can arise as the tangent flow to $\mathcal{M}_j$ at some point in $B_r(\mathbf{x}_0) \times (t_0 - r^2, t_0 + r^2)$, for j large.

Proof. If this failed, there is $(\mathbf{x}_j, t_j) \to (\mathbf{x}_0, t_0)$ with

$$\Theta_{\mathcal{M}_j}(\mathbf{x}, t) \geq \Theta_{\mathcal{M}}(\mathbf{x}_0, t_0) - o(1).$$

The assumption on the connectedness of the regular set implies that $\mathcal{M}_j \lfloor \{t < t_0\} \rightharpoonup \mathcal{M} \lfloor \{t < t_0\}$. Thus, by rescaling around $(\mathbf{x}_0, t_0)$ so that $(\mathbf{x}_j, t_j)$ is scaled to a unit distance from $(\mathbf{0}, 0)$, we obtain $\Sigma \in \mathcal{S}_n^* \setminus \mathcal{S}_n^{\mathrm{gen}}$ that does not split a line and an ancient Brakke flow $\tilde{\mathcal{M}}$ that does not smoothly cross $t \mapsto \mathcal{H}^n \lfloor \sqrt{-t}\, \Sigma$, so that $\lambda(\tilde{\mathcal{M}}) \leq F(\Sigma)$, but for some $(\tilde{\mathbf{x}}, \tilde{t}) \in (\mathbb{R}^{n+1} \times \mathbb{R}) \setminus \{(\mathbf{0}, 0)\}$ it holds that $\Theta_{\tilde{\mathcal{M}}}(\mathbf{x}, t) \geq F(\Sigma)$.

The argument in the second half of the proof of Proposition 2.3 carries over without change to show that there is an open set $\Omega \subset \mathbb{R}^{n+1}$ with $\partial\Omega = \Sigma$ and

$$\sqrt{\tilde{t} - t}\, \Sigma + \tilde{\mathbf{x}} \subset \sqrt{-t}\, \overline{\Omega}$$

for $t < \min\{0, t_0\}$. By Proposition 2.2, we have that either $\Sigma = \mathbb{S}^n(\sqrt{2n}) \in \mathcal{S}_n^{\mathrm{gen}}$ or Σ splits a line. Either case contradicts the assumption that $\mathcal{M}$ has no such tangent flow at $(\mathbf{x}_0, t_0)$. This completes the proof. □

Note that Proposition C.1 does not give any indication as to *how* the perturbation avoids the singularity (the trade-off is that the proof is very short). On the other hand, the results in [CCMS20] give a rather *complete* description of how the perturbed flow avoids a compact/asymptotically conical singularity. The works [SX21b, SX21a] also obtain some information along these lines, but only as long as the perturbed flow remains graphical over the original flow.

Appendix D. The setting of area-minimizing hypersurfaces

We recall the following fundamental result:

Theorem D.1 (Hardt–Simon [HS85, Theorem 2.1])**.** *If $\mathcal{C}^n \subset \mathbb{R}^{n+1}$ is a regular area minimizing cone then there exists smooth area-minimizing hypersurfaces $S_\pm$ in each component of $\mathbb{R}^{n+1} \setminus \mathcal{C} = U_+ \cup U_-$ so that if S' is area minimizing and contained in $U_\pm$ then $S' = \lambda S_\pm$.*

The uniqueness statement in Theorem D.1 implies smoothness of solution to the Plateau problem for seven-dimensional currents in $\mathbb{R}^8$ with generic boundary data (see [HS85, Theorem 5.6]). Later, Smale used Theorem D.1 to prove that for (M^8, g) a closed Riemannian manifold and $\alpha \in H_7(M; \mathbb{Z})$, there is a C^k-close metric g' so that the least area representative of α is smooth [Sma93].

Remark. Besides their role in generic regularity of area-minimizing hypersurfaces in eight-dimensional manifolds, the surfaces $S_\pm$ are important objects in their own right, cf. [IW15, CLS22, Wan20, LW20, Sim21, Sim23]. In our previous paper [CCMS20], we proved the parabolic analogue of Theorem

D.1 (for compact/asymptotically-conical self-shrinkers) by constructing and classifying ancient one-sided flows analogous to the surfaces $S_\pm$.

We explain here how the main idea of this note can be used to prove the generic regularity results from [HS85, Sma93] using the following result in lieu of Theorem D.1 (compare with Proposition 2.3):

Proposition D.2. *There is $\delta > 0$ with the following property. Suppose that $\mathcal{C}^7 \subset \mathbb{R}^8$ is a non-flat area-minimizing cone. If S' is area-minimizing with support contained in $\bar{U}_\pm$, where $\mathbb{R}^8 \setminus \mathcal{C} = U_+ \cup U_-$, then*

$$\Theta_{S'}(\mathbf{x}) \leq \Theta_{\mathcal{C}}(0) - \delta.$$

Proof. Using smooth compactness of the links of area minimizing cones in $\mathbb{R}^8$ it suffices to rule out the case where $S' \subset \bar{U}_\pm$ is area-minimizing and there is $|\mathbf{x}_0| = 1$ so that

$$\Theta_{S'}(\mathbf{x}_0) = \Theta_{\mathcal{C}}(\mathbf{0}).$$

Because S' is contained in $\bar{U}_\pm$, its tangent cone at ∞ must be $\mathcal{C}$ (e.g., using the Frankel property of minimal hypersurfaces in $\mathbb{S}^n$). Thus, $S' = \mathcal{C} + \mathbf{x}_0$. This implies that $\mathcal{C} + \lambda\mathbf{x}_0 \subset \bar{U}_\pm$ as $\lambda \to 0$, so $\mathbf{x}_0 \cdot \nu_{\mathcal{C}} \geq 0$. It cannot hold that $\mathbf{x}_0 \cdot \nu_{\mathcal{C}} = 0$ since $\mathcal{C}$ does not split a line, so $\mathbf{x}_0 \cdot \nu_{\mathcal{C}} > 0$. This would imply that $\mathcal{C}$ is a graph, which is impossible since $\mathcal{C}$ is non-flat. □

Using this, we obtain the following density drop result (compare with Lemma 3.1).

Corollary D.3. *There is $\delta > 0$ with the following property. Suppose that $\Sigma = \partial[\Omega] \subset B_2 \subset \mathbb{R}^8$ is an area minimizing boundary with* $\operatorname{sing}\Sigma = \{\mathbf{0}\}$. *Suppose that $\Omega_1, \Omega_2, \dots \supset \Omega$ is a sequence of sets of finite perimeter in B_2 with $\Sigma_i := \partial[\Omega_i]$ area minimizing, $\Sigma_i \cap \Sigma = \emptyset$, and $\Omega_i \to \Omega$. Then, for $\mathbf{x}_i \in \Sigma_i \cap B_1$, we have*

$$\limsup_{i\to\infty} \Theta_{\Sigma_i}(\mathbf{x}_i) \leq \Theta_\Sigma(\mathbf{0}) - \delta$$

Note that this result can be iterated exactly in the proof of Theorem 1.7 to obtain generic regularity of area-minimizing hypersurfaces in eight dimensions:

Corollary D.4 (cf. [HS85, Theorem 5.6])**.** *For $\Gamma^6 \subset \mathbb{R}^8$ a smooth compact oriented submanifold without boundary, there is an arbitrarily small C^∞-perturbation of Γ to Γ' so that any area-minimizing integral current bounded by Γ' is completely smooth.*

Corollary D.5 (cf. [Sma93])**.** *For (M^8, g) a closed oriented Riemannian manifold and $\alpha \in H_7(M; \mathbb{Z})$ a codimension-one integral homology class, there is an arbitrarily small C^k-perturbation of g to g' so that there is a unique g'-area-minimizing representative Σ of α and Σ is completely smooth.*

References

[ADS19] Sigurd Angenent, Panagiota Daskalopoulos, and Natasa Sesum. Unique asymptotics of ancient convex mean curvature flow solutions. *J. Differential Geom.*, 111(3):381–455, 2019.

[ADS20] Sigurd Angenent, Panagiota Daskalopoulos, and Natasa Sesum. Uniqueness of two-convex closed ancient solutions to the mean curvature flow. *Ann. of Math. (2)*, 192(2):353–436, 2020.

[AL86] U. Abresch and J. Langer. The normalized curve shortening flow and homothetic solutions. *J. Differential Geom.*, 23(2):175–196, 1986.

[All72] William K. Allard. On the first variation of a varifold. *Ann. of Math. (2)*, 95:417–491, 1972.

[BC19] Simon Brendle and Kyeongsu Choi. Uniqueness of convex ancient solutions to mean curvature flow in $\mathbb{R}^3$. *Invent. Math.*, 217(1):35–76, 2019.

[BC21] Simon Brendle and Kyeongsu Choi. Uniqueness of convex ancient solutions to mean curvature flow in higher dimensions. *Geom. Topol.*, 25(5):2195–2234, 2021.

[BH16] Simon Brendle and Gerhard Huisken. Mean curvature flow with surgery of mean convex surfaces in $\mathbb{R}^3$. *Invent. Math.*, 203(2):615–654, 2016.

[BHH16] Reto Buzano, Robert Haslhofer, and Or Hershkovits. The moduli space of two-convex embedded spheres. *to appear in J. Differential Geom.*, 2016.

[BHH19] Reto Buzano, Robert Haslhofer, and Or Hershkovits. The moduli space of two-convex embedded tori. *Int. Math. Res. Not. IMRN*, (2):392–406, 2019.

[BK19] Richard H. Bamler and Bruce Kleiner. Ricci flow and contractibility of spaces of metrics. *https://arxiv.org/abs/1909.08710*, 2019.

[BK22] Richard H. Bamler and Bruce Kleiner. Uniqueness and stability of Ricci flow through singularities. *Acta Math.*, 228(1):1–215, 2022.

[BK23] Richard H. Bamler and Bruce Kleiner. Ricci flow and diffeomorphism groups of 3-manifolds. *J. Amer. Math. Soc.*, 36(2):563–589, 2023.

[Bra75] Kenneth Allen Brakke. *The motion of a surface by its mean curvature*. ProQuest LLC, Ann Arbor, MI, 1975. Thesis (Ph.D.)–Princeton University.

[Bre15] Simon Brendle. A sharp bound for the inscribed radius under mean curvature flow. *Invent. Math.*, 202(1):217–237, 2015.

[Bre16] Simon Brendle. Embedded self-similar shrinkers of genus 0. *Ann. of Math. (2)*, 183(2):715–728, 2016.

[BW15] Jacob Bernstein and Lu Wang. A remark on a uniqueness property of high multiplicity tangent flows in dimension 3. *Int. Math. Res. Not. IMRN*, (15):6286–6294, 2015.

[BW16] Jacob Bernstein and Lu Wang. A sharp lower bound for the entropy of closed hypersurfaces up to dimension six. *Invent. Math.*, 206(3):601–627, 2016.

[BW17] Jacob Bernstein and Lu Wang. A topological property of asymptotically conical self-shrinkers of small entropy. *Duke Math. J.*, 166(3):403–435, 2017.

[BW18a] Jacob Bernstein and Lu Wang. Hausdorff stability of the round two-sphere under small perturbations of the entropy. *Math. Res. Lett.*, 25(2):347–365, 2018.

[BW18b] Jacob Bernstein and Lu Wang. Topology of closed hypersurfaces of small entropy. *Geom. Topol.*, 22(2):1109–1141, 2018.

[BW21] Jacob Bernstein and Shengwen Wang. The level set flow of a hypersurface in $\mathbb{R}^4$ of low entropy does not disconnect. *Comm. Anal. Geom.*, 29(7):1523–1543, 2021.

[BW22a] Jacob Bernstein and Lu Wang. Closed hypersurfaces of low entropy in $\mathbb{R}^4$ are isotopically trivial. *Duke Math. J.*, 171(7):1531–1558, 2022.

[BW22b] Jacob Bernstein and Lu Wang. Topological uniqueness for self-expanders of small entropy. *Camb. J. Math.*, 10(4):785–833, 2022.

[CCMS20] Otis Chodosh, Kyeongsu Choi, Christos Mantoulidis, and Felix Schulze. Mean curvature flow with generic initial data. *https://arxiv.org/abs/2003.14344*, 2020.

[CHH22] Kyeongsu Choi, Robert Haslhofer, and Or Hershkovits. Ancient low-entropy flows, mean-convex neighborhoods, and uniqueness. *Acta Math.*, 228(2):217–301, 2022.

[CHHW22] Kyeongsu Choi, Robert Haslhofer, Or Hershkovits, and Brian White. Ancient asymptotically cylindrical flows and applications. *Invent. Math.*, 229(1):139–241, 2022.

[CIM15] Tobias Holck Colding, Tom Ilmanen, and William P. Minicozzi, II. Rigidity of generic singularities of mean curvature flow. *Publ. Math. Inst. Hautes Études Sci.*, 121:363–382, 2015.

[CIMW13] Tobias Holck Colding, Tom Ilmanen, William P. Minicozzi, II, and Brian White. The round sphere minimizes entropy among closed self-shrinkers. *J. Differential Geom.*, 95(1):53–69, 2013.

[CL19] Alessandro Carlotto and Chao Li. Constrained deformations of positive scalar curvature metrics. *to appear in Jour. Diff. Geom., https://arxiv.org/abs/1903.11772*, 2019.

[CLS22] Otis Chodosh, Yevgeny Liokumovich, and Luca Spolaor. Singular behavior and generic regularity of min-max minimal hypersurfaces. *Ars Inven. Anal.*, Paper No. 2, 27 pp., 2022.

[CM12] Tobias H. Colding and William P. Minicozzi, II. Generic mean curvature flow I: generic singularities. *Ann. of Math. (2)*, 175(2):755–833, 2012.

[CM15] Tobias Holck Colding and William P. Minicozzi, II. Uniqueness of blowups and łojasiewicz inequalities. *Ann. of Math. (2)*, 182(1):221–285, 2015.

[CM19] Tobias Holck Colding and William P. Minicozzi, II. Dynamics of closed singularities. *Annales de l'Institut Fourier*, 2019.

[CM22] Kyeongsu Choi and Christos Mantoulidis. Ancient gradient flows of elliptic functionals and Morse index. *Amer. J. Math.*, 144(2):541–573, 2022.

[DH22] J. M. Daniels-Holgate. Approximation of mean curvature flow with generic singularities by smooth flows with surgery. *Adv. Math.*, 410(part A):Paper No. 108715, 42, 2022.

[EH89] Klaus Ecker and Gerhard Huisken. Mean curvature evolution of entire graphs. *Ann. of Math. (2)*, 130(3):453–471, 1989.

[HK17a] Robert Haslhofer and Bruce Kleiner. Mean curvature flow of mean convex hypersurfaces. *Comm. Pure Appl. Math.*, 70(3):511–546, 2017.

[HK17b] Robert Haslhofer and Bruce Kleiner. Mean curvature flow with surgery. *Duke Math. J.*, 166(9):1591–1626, 2017.

[HS85] Robert Hardt and Leon Simon. Area minimizing hypersurfaces with isolated singularities. *J. Reine Angew. Math.*, 362:102–129, 1985.

[HS99] Gerhard Huisken and Carlo Sinestrari. Mean curvature flow singularities for mean convex surfaces. *Calc. Var. Partial Differential Equations*, 8(1):1–14, 1999.

[HS09] Gerhard Huisken and Carlo Sinestrari. Mean curvature flow with surgeries of two-convex hypersurfaces. *Invent. Math.*, 175(1):137–221, 2009.

[Hui90] Gerhard Huisken. Asymptotic behavior for singularities of the mean curvature flow. *J. Differential Geom.*, 31(1):285–299, 1990.

[Hui93] Gerhard Huisken. Local and global behaviour of hypersurfaces moving by mean curvature. In *Differential geometry: partial differential equations on manifolds (Los Angeles, CA, 1990)*, volume 54 of *Proc. Sympos. Pure Math.*, pages 175–191. Amer. Math. Soc., Providence, RI, 1993.

[HW19] Or Hershkovits and Brian White. Sharp entropy bounds for self-shrinkers in mean curvature flow. *Geom. Topol.*, 23(3):1611–1619, 2019.

[HW20] Or Hershkovits and Brian White. Nonfattening of mean curvature flow at singularities of mean convex type. *Comm. Pure Appl. Math.*, 73(3):558–580, 2020.

[Ilm93] Tom Ilmanen. The level-set flow on a manifold. In *Differential geometry: partial differential equations on manifolds (Los Angeles, CA, 1990)*, volume 54 of *Proc. Sympos. Pure Math.*, pages 193–204. Amer. Math. Soc., Providence, RI, 1993.

[Ilm94] Tom Ilmanen. Elliptic regularization and partial regularity for motion by mean curvature. *Mem. Amer. Math. Soc.*, 108(520):x+90, 1994.

[Ilm95] Tom Ilmanen. Singularities of mean curvature flow of surfaces. *https://people.math.ethz.ch/~ilmanen/papers/sing.ps*, 1995.

[Ilm03] Tom Ilmanen. Problems in mean curvature flow. *https://people.math.ethz.ch/~ilmanen/classes/eil03/problems03.ps*, 2003.

[IW15] Tom Ilmanen and Brian White. Sharp lower bounds on density for area-minimizing cones. *Camb. J. Math.*, 3(1-2):1–18, 2015.

[LW20] Yangyang Li and Zhihan Wang. Generic regularity of minimal hypersurfaces in dimension 8. *https://arxiv.org/abs/2012.05401*, 2020.

[Mar12] Fernando Codá Marques. Deforming three-manifolds with positive scalar curvature. *Ann. of Math. (2)*, 176(2):815–863, 2012.

[MN14] Fernando C. Marques and André Neves. Min-max theory and the Willmore conjecture. *Ann. of Math. (2)*, 179(2):683–782, 2014.

[Mra18] Alexander Mramor. A finiteness theorem via the mean curvature flow with surgery. *J. Geom. Anal.*, 28(4):3348–3372, 2018.

[MW21] Alexander Mramor and Shengwen Wang. Low entropy and the mean curvature flow with surgery. *Calc. Var. Partial Differential Equations*, 60(3):Paper No. 96, 28, 2021.

[Sim83] Leon Simon. *Lectures on geometric measure theory*, volume 3 of *Proceedings of the Centre for Mathematical Analysis, Australian National University*. Australian National University, Centre for Mathematical Analysis, Canberra, 1983.

[Sim21] Leon Simon. A Liouville-type theorem for stable minimal hypersurfaces. *Ars Inven. Anal.*, Paper No. 5, 35 pp., 2021.

[Sim23] Leon Simon. Stable minimal hypersurfaces in $\mathbb{R}^{N+1+\ell}$ with singular set an arbitrary closed $K \subset \{0\} \times \mathbb{R}^{\ell}$. *Ann. of Math. (2)*, 197(3):1205–1234, 2023.

[Sma93] Nathan Smale. Generic regularity of homologically area minimizing hypersurfaces in eight-dimensional manifolds. *Comm. Anal. Geom.*, 1(2):217–228, 1993.

[SS20] Felix Schulze and Natasa Sesum. Stability of neckpinch singularities. *https://arxiv.org/abs/2006.06118*, 2020.

[Sto94] Andrew Stone. A density function and the structure of singularities of the mean curvature flow. *Calc. Var. Partial Differential Equations*, 2(4):443–480, 1994.

[SX21a] Ao Sun and Jinxin Xue. Initial perturbation of the mean curvature flow for asymptotical conical limit shrinker. *https://arxiv.org/abs/2107.05066*, 2021.

[SX21b] Ao Sun and Jinxin Xue. Initial perturbation of the mean curvature flow for closed limit shrinker. *https://arxiv.org/abs/2104.03101*, 2021.

[Wan11] Lu Wang. A Bernstein type theorem for self-similar shrinkers. *Geom. Dedicata*, 151:297–303, 2011.

[Wan16] Lu Wang. Asymptotic structure of self-shrinkers. *https://arxiv.org/abs/1610.04904*, 2016.

[Wan20] Zhihan Wang. Deformations of singular minimal hypersurfaces I, isolated singularities. *https://arxiv.org/abs/2011.00548*, 2020.

[Wan22] Zhihan Wang. Mean convex smoothing of mean convex cones. *https://arxiv.org/abs/2202.07851*, 2022.

[Whi05] Brian White. A local regularity theorem for mean curvature flow. *Ann. of Math. (2)*, 161(3):1487–1519, 2005.

[Whi09] Brian White. Currents and flat chains associated to varifolds, with an application to mean curvature flow. *Duke Math. J.*, 148(1):41–62, 2009.

[Zhu20] Jonathan J. Zhu. On the entropy of closed hypersurfaces and singular self-shrinkers. *J. Differential Geom.*, 114(3):551–593, 2020.

OC: Department of Mathematics, Bldg. 380, Stanford University, Stanford, CA 94305, USA
Email address: ochodosh@stanford.edu

KC: School of Mathematics, Korea Institute for Advanced Study, 85 Hoegiro, Dongdaemun-gu, Seoul 02455, Republic of Korea
Email address: choiks@kias.re.kr

CM: Department of Mathematics, Rice University, Houston, TX 77005, USA
Email address: christos.mantoulidis@rice.edu

FS: Department of Mathematics, Zeeman Building, University of Warwick, Gibbet Hill Road, Coventry CV4 7AL, UK
Email address: felix.schulze@warwick.ac.uk