Asymptotic counting problems for periodic orbits and holonomies of rational maps and Kleinian groups

Anastasios Stylianou

Supervisor: Professor Richard Sharp

Mathematics Institute
University of Warwick

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I would like to dedicate this thesis to my parents Andreas and Emily and my siblings Matthaios, Vicky, Andreas and Iro.

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Last but not least, I would like to extend my gratitude to my family and friends for their constant support and encouragement during these last four years.

Declaration

I hereby declare that this thesis is my own work, except where otherwise indicated or cited in the text, or else where the material is widely known. I confirm that this thesis has not been submitted for a degree or qualification at another university. One of the main results of this thesis, Theorem 3.1.1, appeared in [SS22] and was produced in collaboration with Richard Sharp.

Anastasios Stylianou

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Abstract

In this thesis we study orbit counting problems in different setups. We start by considering hyperbolic rational maps on the Riemann sphere. For such a map we study its periodic orbits and obtain an asymptotic counting result as the lengths of the orbits grow. In particular, we associate to each periodic orbit a complex number called the multiplier and we call the normalised complex number in the direction of the multiplier, the holonomy of the periodic orbit. We place restrictions on the magnitudes of the multipliers and the arguments of the holonomies of the orbits. We consider varying and potentially shrinking intervals and arcs and obtain two results which resemble a local central limit theorem for the logarithm of the absolute value of the multipliers and an equidistribution theorem for the holonomies. Using traditional ideas from probability theory and thermodynamic formalism we reduce our proofs to bounding the iterates of a certain family of transfer operators. Then we obtain our results by adapting Dolgopyat–type arguments, obtained by Oh and Winter in the situation we consider.

In the second half of this thesis we prove an analogous result to the one above in the setting of convex-cocompact hyperbolic manifolds. We consider closed geodesics in a hyperbolic manifold of arbitrary dimension and prove an asymptotic equidistribution result. We fix a Markov section for the non-wandering set of the geodesic flow on our manifold and order closed geodesics by their word length with respect to the Poincaré first return map of this section. As before, we place restrictions on the geometric length of the geodesics. Moreover, to each closed geodesic we associate a rotation element, called the holonomy, obtained by parallel transporting a frame around the closed geodesic. Once again, using symbolic dynamics and ideas from thermodynamic formalism we reduce our proofs to obtaining bounds for the iterates of a certain family of transfer operators. Adapting Dolgopyat-type estimates, established by Sarkar and Winter in a similar setup, we prove an asymptotic equidistribution result.

Chapter 1

Introduction

1.1 History and motivation

A major theme in the theory of dynamical systems is the study of the distribution of periodic orbits. This is particularly well-developed for hyperbolic systems, where one finds precise asymptotic and equidistribution results. In this thesis we study equidistribution problems in two setups. Firstly, we consider the periodic orbits of hyperbolic rational maps and prove an equidistribution theorem as well as a local central limit theorem. Our second setup involves the study of closed geodesics on hyperbolic manifolds of dimension at least three where we prove an analogue equidistribution theorem for the holonomies.

1.1.1 How many geodesics are there up to a certain length?

Asymptotic counting problems for closed geodesics in the spirit of the Prime Number Theorem have been studied for many years starting with Huber in [Hub59, Hub61]. He showed that there exists an asymptotic formula, often called a Prime Geodesic Theorem, for the number of closed geodesics in terms of a bound on their length while also providing bounds on the error terms of his asymptotic formula. His results were proved in the setting of a compact surface with constant negative curvature using harmonic analysis tools. See also Hejhal's book [Hej76] for a more detailed approach based on the Selberg trace formula [Sel56]. In Hejhal's second volume [Hej83] the result is extended to non-compact surfaces of finite volume and constant negative curvature. This asymptotic result for the number of closed geodesics on non-compact surfaces of finite volume appeared first in Sarnak's thesis [Sar80].

Following Huber's work, Margulis in his thesis extended the asymptotic counting results of Huber to compact manifolds of arbitrary dimension with variable negative curvature [Mar04, Mar69] but without the bounds on the error terms. Writing $\pi(T)$ for the number of closed geodesics with length less than T, Margulis obtained that

$$\pi(T) \sim \frac{e^{hT}}{hT} \qquad \text{as } T \to \infty,$$

where $h > 0$ is the topological entropy of the geodesic flow. In the special case, of a compact N-dimensional manifold with constant negative curvature $-K^2$ Margulis obtained that

$$\pi(T) \sim \frac{e^{K(N-1)T}}{K(N-1)T} \qquad \text{as } T \to \infty,$$

thus recovering the main term in the asymptotic equivalence from the result of Huber for compact surfaces. In the more general setting of variable negative curvature Margulis resorted to an approach based on the dynamics of the geodesic flow.

Following the work of Margulis, Parry and Pollicott in [PP83] introduced a new method to obtaining asymptotic counting results for $\pi(T)$. Using the work of Bowen they realised Axiom A flows as suspension flows over shifts of finite type. Then, using their understanding of the spectrum of a family of Ruelle transfer operators they obtained information on the the associated Ruelle zeta function and the poles of its logarithmic derivative. After using a classical Tauberian argument they obtained a prime number theorem type of result for the periodic orbits of a topologically weak mixing Axiom A flow restricted to a basic set.

The next advance in the study of the distribution of lengths of closed geodesics on hyperbolic manifolds came from Guillopé. He proved in [Gui86] that there exists an asymptotic formula for the number of closed geodesics up to a certain length for convex-cocompact surfaces of constant negative curvature imposing a condition on the Poincaré exponent of the fundamental group of the surface. Convex-cocompact here refers to the property that there exists a closed subset $\mathcal{C} \subseteq \mathbb{H}^N$ invariant under the action of the fundamental group of the surface such that this action is cocompact. Lalley recovered this result unconditionally in [Lal89] using symbolic dynamics and renewal theory.

We call a discrete subgroup of orientation preserving isometries of $\mathbb{H}^N$ geometrically finite if it has a fundamental domain with finitely many faces. For geometrically finite groups it was shown in [DP96] that a similar asymptotic holds. In this paper, Dal'bo and Peigné adapted Lalley's methods to cover the case where parabolic elements are included in the fundamental group, hence allowing the presence of cusps in the quotient manifold.

Writing δ for the Poincaré exponent of the fundamental group the three results above can be summarised as

$$\pi(T) \sim \frac{e^{\delta T}}{\delta T} \qquad \text{as } T \to \infty.$$

Perry in [Per01] recovered the asymptotic counting result for closed geodesics in hyperbolic convex-cocompact manifolds. He used his joint work with Patterson [PP01] to study the pole with greatest real part of the logarithmic derivative of a dynamical zeta function and concluded his asymptotic result using a Tauberian argument similar to [PP83]. However, his method to study the pole was different to [PP83] as he associated it to the pole of the resolvent of the Laplacian on the quotient manifold, and found that it has residue equal to one.

Following a major advance in the area of dynamics coming from the work of Dolgopyat [Dol98] on the decay of correlations of Anosov flows, Pollicott and Sharp established an improvement of Margulis' result for surfaces of variable negative curvature. They used similar methods to [PP83] involving the dynamics of the geodesic flow and transfer operators together with the improved bounds on the transfer operators obtained by Dolgopyat. In the case of a compact surface with variable negative curvature it is shown in [PS98b] that the asymptotic equivalence enjoys an exponential error term, that is there exists $h > c > 0$ such that

$$\pi(T) = \mathrm{Li}(e^{hT}) + O(e^{cT}) \qquad \text{as } T \to \infty.$$

Here, Li denotes the logarithmic integral $\mathrm{Li}(x) = \int_2^x (\log u)^{-1}\,du \sim x/\log x$, as $x \to \infty$ and we write $f(x) = O(g(x))$ as $x \to \infty$ whenever there exists $C > 0$ and $x_0 \in \mathbb{R}$ such that for all $x \geq x_0$ we have that $|f(x)| \leq Cg(x)$. We also write $f(x) \sim g(x)$ as $x \to \infty$ whenever $\lim_{x\to\infty} f(x)/g(x) = 1$.

Naud in [Nau05] strengthened the Prime Geodesic Theorem on convex-cocompact hyperbolic surfaces. Adapting Dolgopyat's analysis he managed to provide exponential bounds for the error terms for the asymptotic number of closed geodesics up to a certain length on convex-cocompact surfaces of constant negative curvature. He also proved a prime number theorem type of result involving exponential bounds for the error terms for the number of periodic orbits of certain quadratic polynomials with respect to a bound on the magnitudes of their multipliers. Here the magnitude of the multiplier of a periodic orbit gives a notion of a geometric length to the periodic orbits of rational maps.

Subsequently, several generalisations of Naud's results were obtained. Oh and Winter generalised Naud's prime number theorem type of result for quadratic polynomials in [OW17] by proving a prime number theorem with exponential bounds for the error terms for a large class of hyperbolic rational maps on the Riemann sphere. In particular, Naud's result concerned the family of quadratic polynomials $f_c(z) = z^2 + c$ for $c < -2$ but he had already noticed that his result does not hold for $c = 0$ and conjectured that it should hold for a much more 'generic' family of values of $c \in \mathbb{C}$. Oh and Winter proved that the prime number theorem type of result with exponential bounds for the error terms can be generalised to all hyperbolic rational maps not conjugate to a monomial $z \to z^{\pm d}$ by a Möbius transformation. Note that in the case of a rational map that is conjugate to a monomial $z \to z^{\pm d}$ for some integer $d \geq 2$ we already knew of a different asymptotic counting result for periodic orbits with respect to a bound on their multiplier. This result follows from the work of Parry in the more general setup of a weak-mixing suspension of a subshift of finite type in [Par83].

Finally, Oh and Winter proved another asymptotic counting result for periodic orbits of hyperbolic rational maps. This time instead of only counting periodic orbits they also studied the distribution of the holonomies of periodic orbits. Holonomy in this setup refers to the normalised complex number in the direction of the multiplier of the period orbit. If the Julia set of a rational map is included in a circle in the Riemann sphere it is an easy argument to show that all holonomies of periodic orbits are real numbers, that is, equal to ± 1. Oh and Winter showed that this is the only obstruction to the equidistribution of holonomies on the unit circle. Specifically they showed that for a hyperbolic rational map of degree at least two whose Julia set is not included in a circle in the Riemann sphere, the holonomies of periodic orbits equidistribute in the unit circle as the multipliers of the periodic orbits considered tend to infinity while also obtaining exponential bounds for the error terms. In the next subsection we discuss the analogue results for holonomies of closed geodesics.

1.1.2 Equidistribution of holonomies

Another interesting dynamical system generalising the notion of the geodesic flow is the frame flow. A k-frame is an ordered set of k orthonormal tangent vectors of a Riemannian manifold. The frame flow moves a k-frame along the geodesic defined from the first vector and parallel transporting the rest. This system was studied in [BP74, BP73], where Brin and Pesin showed that the frame flow on a closed and connected manifold of constant negative curvature is ergodic. For the ergodicity of the frame flow on closed, connected manifolds of variable negative curvature see [BG80, CLMS21, BP03, BK84]. This more general dynamical system allows us to refine our counting of closed geodesics on hyperbolic manifolds. To every closed geodesic we can relate a conjugacy class of rotation elements called the holonomy, obtained by parallel transporting a frame around the closed geodesic. Parry and Pollicott in [PP86] viewed holonomies of closed geodesics, on oriented Riemannian manifolds of variable negative curvature, as Frobenius classes of periodic orbits of the geodesic flow and proved an equidistribution result. Specifically, they showed that if the frame flow is topologically mixing (or, equivalently, topologically weak mixing) then the holonomies uniformly distribute in conjugacy classes of the special orthonormal group of appropriate dimension as the lengths of the closed geodesics considered tend to infinity.

Sarnak and Wakayama in [SW99] proved the equidistribution of holonomies for any rank-1 locally symmetric space of finite volume. Using harmonic analysis techniques (rather than ergodic theoretic ones) they further managed to provide some bounds for the error terms. Many papers improved on these results in various ways [MMO14, MO15, EO21, Sha18, DM21]. More recently, Sarkar and Winter in [SW21] proved an equidistribution result for the holonomies of closed geodesics on quotients of $\mathbb{H}^N$ by discrete, convex-cocompact, torsion-free and Zariski-dense subgroups of $\mathrm{Isom}^+(\mathbb{H}^N)$ with exponential error bounds. We call a subgroup $\Gamma \leq G$ *Zariski-dense* as a real Lie group if it is not contained in a proper real algebraic subvariety $V \subset G$. In three dimensions this is equivalent to requiring that the limit set of $\Gamma \leq \mathrm{Isom}^+(\mathbb{H}^3)$ is not inside a circle in $\mathbb{S}^2$. (Notice then, the similarity with the assumption on Julia sets discussed above required to prove the equidistribution of holonomies with exponential error terms for hyperbolic rational maps.)

In this thesis, we are studying asymptotic counting problems for periodic orbits of rational maps and for closed geodesics on hyperbolic convex-cocompact manifolds. As a first step we refine our counting to involve the holonomies of periodic orbits or closed geodesics.

Further, following a long tradition of asymptotic counting results [CP20, PS06a, PS13, PS06b, PS98a, PS01] we also refine our counting by associating a discrete length to each periodic orbit or closed geodesic. We use this notion of discrete length to order our periodic orbits and closed geodesics and obtain results that compare these two notions of lengths. In the next section we present our results in more detail.

1.2 Main results

1.2.1 Periodic orbits of hyperbolic rational maps

Let us now be more precise about our setting. Let $f : J \to J$ be a rational map of degree at least two restricted to its Julia set J and let $0 < \delta < 2$ be the Hausdorff dimension of J. (See the next chapter for formal definitions.) A periodic orbit $\tau = \{z, f(z), \ldots, f^{n-1}(z)\}$ (with $f^n(z) = z$) is called *primitive* if $f^m(z) \neq z$ for all $1 \leq m < n$. We denote the set of primitive periodic orbits by $\mathcal{P}$. For each $\tau = \{z, f(z), \ldots, f^{n-1}(z)\}$ in $\mathcal{P}$, we define its *multiplier* to be

$$\lambda(\tau) := (f^n)'(z) \in \mathbb{C},$$

and its *holonomy*

$$\hat{\lambda}(\tau) := \frac{\lambda(\tau)}{|\lambda(\tau)|} \in \mathbb{S}^1,$$

where $\mathbb{S}^1$ denotes the unit circle in $\mathbb{C}$. We call f a hyperbolic rational map if it is eventually expanding on its Julia set, i.e. whenever there exist constants $C > 0$ and $\gamma > 1$ such that

$$|(f^n)'(z)| \geq C\gamma^n,$$

for all $z \in J$ and $n \in \mathbb{N}$. A recent result of Oh and Winter [OW17, Theorem 1.1] states that, for a hyperbolic rational map f of degree at least two which is not Möbius conjugate to a monomial $z \to z^{\pm d}$, there exists $\varepsilon > 0$ such that

$$\#\{\tau \in \mathcal{P} : |\lambda(\tau)| < t\} = \mathrm{Li}(t^\delta) + O(t^{\delta-\varepsilon}), \tag{1.2.1}$$

and moreover if the Julia set of f is not inside a circle in $\widehat{\mathbb{C}}$, then for any $\psi \in C^4(\mathbb{S}^1)$

$$\sum_{\tau\in\mathcal{P}:|\lambda(\tau)|<t} \psi(\hat{\lambda}(\tau)) = \left(\int_0^1 \psi(e^{2\pi i\theta})\, d\theta\right) \mathrm{Li}(t^\delta) + O\left(\|\psi\|_{C^4}\, t^{\delta-\varepsilon}\right).$$

We take a slightly different viewpoint. For a periodic orbit τ we think of the modulus of its multiplier $|\lambda(\tau)|$ as a geometric notion of length for the orbit. Instead of counting periodic orbits $\tau = \{z, f(z), \dots, f^{n-1}(z)\}$ according to their geometric length, that is the modulus of their multiplier, we count by their respective periods $|\tau| = n$. A rational map of degree at least two has $\#\{z \in \mathbb{C} \,:\, f^n(z) = z\} = d^n$ (plus one in the case the highest order term is at the denominator of the rational map f^n) counted with multiplicity. Since as we will see later in Proposition 3.3.1 'most periodic orbits are primitive' it is not hard to show that

$$\#\{\tau \in \mathcal{P} \,:\, |\tau| = n\} \sim \frac{d^n}{n} \qquad \text{as } n \to \infty.$$

We strengthen the asymptotic result above by imposing certain constraints on the geometric lengths $|\lambda(\tau)|$ and the holonomies $\hat{\lambda}(\tau)$ of primitive periodic orbits of length n. For a hyperbolic rational map f it follows that the geometric lengths of primitive periodic orbits grow exponentially fast with respect to their periods. In particular, this motivates us to consider the deviations of the logarithm of the geometric lengths of periodic orbits against their period. More precisely, for $\alpha \in \mathbb{R}$ and an interval $I \subset \mathbb{R}$ we study the distribution of the differences $\log|\lambda(\tau)| - n\alpha$ in the interval I for primitive periodic orbits of period n. Simultaneously we also study the distribution of holonomies $\hat{\lambda}(\tau)$ in an arc $S \subset \mathbb{S}^1$. Writing $\mathcal{P}_n = \{\tau \in \mathcal{P} : |\tau| = n\}$, we aim to study the behaviour of

$$\pi(n, \alpha, I, S) := \#\{\tau \in \mathcal{P}_n : \ \log|\lambda(\tau)| - n\alpha \in I \text{ and } \hat{\lambda}(\tau) \in S\}, \qquad \text{as } n \to \infty.$$

We need to impose a restriction on α and, to do this, define the closed interval

$$\mathcal{I}_f := \left\{ \int \log|f'| \, d\mu \,:\, \mu \in \mathcal{M}_f \right\},$$

where $\mathcal{M}_f$ is the set of f-invariant probability measures on J. We also assume that the Julia set of f is not contained in a circle in $\widehat{\mathbb{C}}$ since otherwise all holonomies are real. We write ℓ for the Lebesgue measure on $\mathbb{R}$ and ν for the normalised Haar measure on $\mathbb{S}^1$.

Theorem 1.2.1 ([SS22]). *Let $f : J \to J$ be a hyperbolic rational map of degree $d \geq 2$ restricted to its Julia set such that J is not contained in a circle in $\widehat{\mathbb{C}}$. Then, for $\alpha \in \mathrm{int}(\mathcal{I}_f)$,* *there exists $\sigma_\alpha > 0$ and $\xi_\alpha \in \mathbb{R}$ such that*

$$\pi(n, \alpha, I, S) \sim \frac{\nu(S)}{\sigma_\alpha \sqrt{2\pi}} \int_I e^{-\xi_\alpha x}\, dx\, \frac{e^{H(\alpha)n}}{n^{3/2}} \qquad \text{as } n \to \infty,$$

where

$$H(\alpha) = \sup \left\{ h_f(\mu) \,:\, \mu \in \mathcal{M}_f \text{ and } \int \log |f'|\, d\mu = \alpha \right\},$$

and $h_f(\mu)$ denotes the entropy of f with respect to μ. In particular, if $\alpha = \int \log |f'|\, d\mu_{\max}$, where $\mu_{\max}$ is the measure of maximal entropy then

$$\pi(n, \alpha, I, S) \sim \frac{\nu(S)\ell(I)}{\sigma_\alpha \sqrt{2\pi}} \frac{d^n}{n^{3/2}} \qquad \text{as } n \to \infty.$$

We can also allow I and S to shrink at suitably slow rates. This generalisation will appear in Chapter 3.

1.2.2 Distribution of ergodic sums for hyperbolic rational maps

We now present a result that involves the statistical properties of the distribution of all orbits of the dynamical system discussed above rather than just the periodic ones. As before let $f : J \to J$ be a hyperbolic rational map restricted to its Julia set. We define the *distortion function* $r : J \to R$ by

$$r(z) = \log |f'(z)|,$$

and the *rotation function* $\theta : J \to \mathbb{R}/2\pi\mathbb{Z}$ by

$$\theta(z) = \arg(f'(z)).$$

For $z \in J$ let $r^n(z) = r(z) + r(f(z)) + \cdots + r(f^{n-1}(z))$ denote the sum of the distortions along the first n points in the orbit of $z \in J$. Similarly, $\theta^n(z) = \theta(z) + \cdots + \theta(f^{n-1}(z))$. Let m be an f-invariant and ergodic probability measure on J. Birkhoff's Ergodic Theorem implies that the sequences $r^n(z)/n$ converge to $\int r\, dm$ for m almost every point $z \in J$. More sophisticated results study the deviations of this sequence from its average. As in the previous subsection consider the closed interval

$$\mathcal{I}_f := \left\{ \int r\, d\mu \,:\, \mu \in \mathcal{M}_f \right\},$$

where $\mathcal{M}_f$ is the set of f-invariant probability measures on J. We show that for each $\alpha \in \mathrm{int}(\mathcal{I}_f)$ we can find a unique measure μ_α such that $\int r\,d\mu_\alpha = \alpha$ and a unique $\xi_\alpha \in \mathbb{R}$ such that μ_α is the equilibrium state of $\xi_\alpha r$. Further, we define the variance of the distortion function with respect to this unique measure μ_a by

$$\sigma_\alpha^2 := \lim_{n\to\infty} \frac{1}{n} \int \left(r^n - n \int r\,d\mu_\alpha \right)^2 d\mu_\alpha.$$

Assuming that $f : J \to J$ is not cohomologous to a monomial $z \to z^{\pm d}$ for any $d \in \mathbb{N}$ we show that the limit above exists and $\sigma_\alpha > 0$. In the current situation it can be shown using arguments from [Lal86, CP90, Rat73a] that the ergodic sums satisfy the stronger property of a central limit theorem. In particular, for a real interval I we have that

$$\lim_{n\to\infty} \mu_\alpha \left\{ z \in J \ : \ \frac{r^n(z) - n\int r\,d\mu_\alpha}{\sqrt{n}} \in I \right\} = \frac{1}{\sigma_\alpha\sqrt{2\pi}} \int_I e^{-x^2/2\sigma_\alpha^2}\,dx.$$

We say a function $g : J \to \mathbb{R}$ satisfies the lattice property whenever there exist constants $a, b \in \mathbb{R}$, a function $\psi : J \to \mathbb{Z}$ and a continuous function $u : J \to \mathbb{R}$ such that

$$g = a + b\psi + u \circ f - u.$$

Oh and Winter proved in [OW17] that if we assume that the Julia set of f is not contained in a circle in the Riemann sphere it follows that the distortion function r does not satisfy the lattice property. This assumption implies through work of Lalley [Lal86] that we can obtain a stronger asymptotic result. We get a local central limit theorem for the distortion function, that is for a real interval I we have that

$$\mu_\alpha \left\{ z \in J \ : \ r^n(z) - n \int r\,d\mu_\alpha \in I \right\} \sim \frac{\ell(I)}{\sigma_\alpha\sqrt{2\pi n}} \qquad \text{as } n \to \infty.$$

Finally, we present our result that strengthens the one above by allowing us to place a further restriction on the ergodic sums of the rotation function θ. Recall that $\int r\,d\mu_\alpha = \alpha$.

Theorem 1.2.2. *Let $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a hyperbolic rational map of degree at least two such that its Julia set is not contained in a circle in $\widehat{\mathbb{C}}$. Let I be an interval in $\mathbb{R}$ and let S be* *an arc in* $\mathbb{S}^1$*. Then, for each* $\alpha \in \mathrm{int}(\mathcal{I}_f)$ *there exists* $\sigma_\alpha > 0$ *such that*

$$\mu_\alpha \{z \in J \ : \ r^n(z) - n\alpha \in I \quad \text{and} \quad \theta^n(z) \in S\} \sim \frac{\ell(I)\nu(S)}{\sigma_\alpha \sqrt{2\pi n}} \qquad \text{as } n \to \infty, \tag{1.2.2}$$

where for a unique $\xi_\alpha \in \mathbb{R}$*,* μ_α *is the equilibrium state of* $\xi_\alpha r$*.*

Again we can also allow I and S to shrink at suitably slow rates. This generalisation will appear in Chapter 3.

1.2.3 Closed geodesics and holonomies on convex-cocompact hyperbolic manifolds

Let $\mathbb{H}^N$ denote the N-dimensional hyperbolic space and write $\mathrm{Isom}^+(\mathbb{H}^N)$ for the group of orientation preserving isometries of $\mathbb{H}^N$. Let Γ be a Kleinian group, that is a discrete subgroup of $\mathrm{Isom}^+(\mathbb{H}^N)$. Assuming that Γ is torsion-free we have that the quotient manifold $X = \Gamma \backslash \mathbb{H}^N$ is a hyperbolic manifold of dimension N. It is a classical problem to study asymptotic counting problems for the closed geodesics on hyperbolic manifolds. Denote by $\mathcal{G}$ the set of primitive closed geodesics in X and for $\gamma \in \mathcal{G}$ write $l(\gamma)$ for its geometric length. For each closed geodesic γ we have an associated holonomy element h_γ which corresponds to a conjugacy class in $\mathrm{SO}(N-1)$. This can be obtained by parallel transporting an N-dimensional oriented frame around γ where we move the first vector by the geodesic flow.

We call a function $\phi : \mathrm{SO}(N-1) \to \mathbb{R}$ a class function if it remains constant in conjugacy classes of $\mathrm{SO}(N-1)$. Recently, Sarkar and Winter obtained the following result in [SW21, Theorem 1.3].

Theorem 1.2.3. *Assume* Γ *is a torsion-free, convex-cocompact and Zariski-dense Kleinian subgroup of* $\mathrm{Isom}^+(\mathbb{H}^N)$*. Then, there exists* $\varepsilon > 0$ *such that for all class functions* $\phi \in C^\infty(\mathrm{SO}(N-1), \mathbb{R})$ *we have that*

$$\sum_{l(\gamma) \le T} \phi(h_g) = \mathrm{Li}(e^{\delta_\Gamma T}) \int_{\mathrm{SO}(N-1)} \phi \, d\mu + O(e^{(\delta_\Gamma - \varepsilon)T}) \qquad \text{as } T \to \infty,$$

where μ *is the probability Haar measure of* $\mathrm{SO}(N-1)$ *and* δ_Γ *denotes the critical exponent of* Γ*.*

Here we take a different approach and order closed geodesics with respect to a discrete length. We restrict to the three dimensional case for the moment to ease notation.

Recall that $\mathcal{G}$ denotes the set of primitive closed geodesics which are in one-to-one correspondence with the primitive periodic orbits of the geodesic flow on the tangent space of the quotient manifold $\Gamma\backslash\mathbb{H}^3$. Considering the geodesic flow on the tangent space of our manifold $X = \Gamma\backslash\mathbb{H}^3$ let $\Omega \subset T^1(X)$ be the non-wandering set. It follows from work of Ratner [Rat73b] that there exist Markov sections of arbitrarily small size for Ω. For each Markov section $\mathcal{R}$ of Ω there exists an associated first return time map $\tau : \Omega \to \mathbb{R}^+$. Since the first return time map is constant along stable leaves, by abusing notation we can think of the first return time map as a function on the union of unstable leaves $U := \bigsqcup_{j=1}^m U_j \to \mathbb{R}^+$ after collapsing the stable leaves. Further, let $P : U \to U$ be the projection of the Poincaré first return map on the union of unstable leaves of the Markov section. (The precise definitions appear in Chapter 4.)

Every primitive periodic orbit $\gamma \in \mathcal{G}$ for the geodesic flow corresponds to a periodic orbit $\{u, P(u), \ldots, P^{n-1}(u)\}$ with $P^n(u) = u$ for the Poincaré first return map $P : U \to U$. In fact, this correspondence is possibly non-unique when the periodic orbit γ is passing through the boundaries of the rectangles of our Markov section $\mathcal{R}$. If $\{u, P(u), \ldots, P^{n-1}(u)\}$ is unique then we define

$$|\gamma|_{\mathcal{R}} = n,$$

that is the period of u. Otherwise, we choose $|\gamma|_{\mathcal{R}}$ to be equal to the smallest period of all the P-orbits corresponding to γ. Crucially, we have the identity

$$l(\gamma) = \tau^n(u) := \sum_{i=0}^{n-1} \tau(P^i(u)),$$

where $\{u, P(u), \ldots, P^{n-1}(u)\}$ is any P-orbit corresponding to γ with period equal to $|\gamma|_{\mathcal{R}}$ and observe that we have the inequality

$$\min_U \tau \le \frac{l(\gamma)}{|\gamma|_{\mathcal{R}}} \le \max_U \tau.$$

Letting T be the transition matrix for the Markov section $\mathcal{R}$ we have that T is topologically mixing. By the Perron-Frobenius Theorem, the matrix T has a positive eigenvalue $\lambda > 1$, with all the other eigenvalues having strictly smaller modulus. Furthermore, λ is related to the topological entropy $h(\sigma)$ of $\sigma : \Sigma^+ \to \Sigma^+$ since $h(\sigma) = \log \lambda$ where (Σ^+, σ) is the one-sided subshift of finite type related to T. The number of periodic points of period n of σ is given by

$$\#\,\mathrm{Fix}_n(\sigma) := \{x \in \Sigma^+ \ : \ \sigma^n(x) = x\} = \mathrm{trace}(T^n) = \lambda^n + O((\theta_0\lambda)^n),$$

where $0 < \theta_0 < 1$. Given any σ-invariant probability measure ν on Σ^+, we may define its entropy $h_\sigma(\nu)$. This always satisfies $h_\sigma(\nu) \leq h(\sigma)$ and there is a unique σ-invariant probability measure μ_0, called the measure of maximal entropy, for which $h(\mu_0) = h(\sigma)$.

In particular, the topological entropy of $P : U \to U$ satisfies $h(P) = h(\sigma) = \log \lambda$. The topological entropy gives the exponential growth rate of periodic points for P. More precisely, if we write $\mathrm{Fix}_n(P) = \{u \in U \ : \ P^n(u) = u\}$ then there exists $0 < \theta_1 < 1$ such that

$$\#\,\mathrm{Fix}_n(P) = \lambda^n + O((\theta_1\lambda)^n). \tag{1.2.3}$$

This next result is due to Bowen [Bow73].

Lemma 1.2.4. *There exists $0 < \theta_2 < 1$ such that*

$$\#\{\gamma \in \mathcal{G} \ : \ |\gamma|_{\mathcal{R}} = n\} = \frac{\#\,\mathrm{Fix}_n(P)}{n} + O((\theta_2\lambda)^n). \tag{1.2.4}$$

The difference between counting closed geodesics of word length n and the number of periodic orbits of length n described above does not cause a problem for our analysis. This follows from [Bow73, Theorem 6.1]. In particular, using (1.2.3) and (1.2.4) we obtain that

$$\#\{\gamma \in \mathcal{G} \ : \ |\gamma|_{\mathcal{R}} = n\} \sim \frac{\lambda^n}{n} \qquad \text{as } n \to \infty.$$

We strengthen this asymptotic result by placing restrictions on the geometric lengths and the holonomies of the closed geodesics. To this end, fix an interval $I \subset \mathbb{R}$, an arc $A \subset \mathbb{S}^1$ and a real number α. (Note that in three dimensions we can think of the holonomy element of a closed geodesic as an element of $\mathbb{S}^1$ through the isomorphism $\mathrm{SO}(2) \cong \mathbb{S}^1$.) Considering closed geodesics of word length n with respect to $|.|_{\mathcal{R}}$ we count those whose deviations of their geometric length $l(\gamma)$ from $n\alpha$ lie in I. Further, we place a restriction on the associated holonomy by requiring that h_γ lies in A. We therefore wish to study the asymptotic growth of the following quantity

$$\pi_{\mathcal{R}}(\alpha, n, I, A) := \#\Big\{|\gamma|_{\mathcal{R}} = n \,:\, l(\gamma) - n\alpha \in I \quad \text{and} \quad h_\gamma \in A\Big\}.$$

To do that, we need to impose some restrictions on α. We define the interval

$$\mathcal{I} := \left\{\int \tau \, d\mu \,:\, \mu \in \mathcal{M}\right\},$$

where $\mathcal{M}$ is the set of P-invariant probability measures on U, that is the union of unstable leaves of our Markov section. Let ν be the normalised Haar measure on $\mathbb{S}^1$ and ℓ be the one dimensional Lebesgue measure. We then get the analogous equidistribution result.

Theorem 1.2.5. *Let Γ be a torsion-free, convex-cocompact discrete subgroup of* $\mathrm{Isom}^+(\mathbb{H}^3)$ *and assume that the limit set of Γ is not included inside a circle in $\mathbb{S}^2$. Let $|.|_{\mathcal{R}}$ denote the word length with respect a fixed Markov section $\mathcal{R}$ for the non-wandering set Ω. For any real value $\alpha \in \operatorname{int} \mathcal{I}$ there exist $\sigma_\alpha > 0$ and $a \in \mathbb{R}$ such that*

$$\pi_{\mathcal{R}}(\alpha, n, I, A) \sim \frac{\int_I e^{-ax} \, dx \; \nu(A)}{\sigma_\alpha \sqrt{2\pi n^3}} e^{H(\alpha) n} \qquad \text{as } n \to \infty,$$

where $H(\alpha) = \sup\{h_\mu(P) \,:\, \mu \in \mathcal{M} \;\text{ and }\; \int \tau \, d\mu = \alpha\}$ and $h_\mu(P)$ demotes the entropy of P with respect to μ.

In this setting Pollicott and Sharp obtained an asymptotic result that did not involve the holonomies in [PS13]. Our result can be stated more generally in any dimension $N \geq 3$ where the condition on the limit set of Γ is placed by the Zariski-density of Γ in $\mathrm{Isom}^+(\mathbb{H}^N)$. Moreover, we can also allow I and A to shrink at suitably slow rates. These generalisations will appear in Chapter 5.

Chapter 2

Preliminaries on hyperbolic rational maps

2.1 Introduction and definitions

In this chapter we present all the main definitions and background knowledge which we will need in order to present our results on hyperbolic rational maps in the following chapter.

2.1.1 Rational maps and periodic orbits

In the study of dynamical systems it is always more convenient to consider orbits of points with respect to a map on a compact space. Hence, we start by considering the space of extended complex numbers $\mathbb{C} \cup \{\infty\}$. Throughout this thesis we denote the extended complex numbers by $\widehat{\mathbb{C}}$ and refer to them as the *Riemann sphere*. Topologically, the resulting space is the one-point compactification of a plane into the sphere. However, the Riemann sphere is not merely a topological sphere. It is a sphere with a well-defined complex structure, so that around every point on the sphere there is a neighbourhood that can be biholomorphically identified with a region in $\mathbb{C}$ through the holomorphic maps $\mathbb{C} \to \mathbb{C} : z \to z$ or $\widehat{\mathbb{C}} \setminus \{0\} \to \mathbb{C} : z \to \frac{1}{z}$.

We call a function $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ a *rational map* if it is given as a quotient of two polynomials with complex coefficients and no common factors, that is

$$f(z) = \frac{p(z)}{q(z)} \qquad \text{where } p,\, q \in \mathbb{C}[z] \text{ and } \operatorname{gcf}(p, q) = 1,$$

where we set $f(\infty) = \lim_{|z|\to\infty} \frac{p(z)}{q(z)}$ and $f(z) = \infty$ for each $z \in \mathbb{C}$ with $q(z) = 0$. We call the number

$$d = \max\{\deg(p), \deg(q)\},$$

the *degree* of f.

The main object of study in our asymptotic counting problem will be the periodic orbits of rational maps. We call a set $\tau = \{z, f(z), \ldots, f^{n-1}(z)\} \subset \widehat{\mathbb{C}}$ a *periodic orbit* of f whenever $f^n(z) = z$ and call n its *period*. If moreover, $f^k(z) \neq z$ for each $k = 1, \ldots, n-1$ we call τ a *primitive* periodic orbit. For each natural number n set $\mathcal{P}_n$ to be the set of all primitive periodic orbits of f of period n. Observe that points in a primitive periodic orbit of period n are roots of the equation $f^n(z) - z = 0$. However, roots of this equation might also include points in non-primitive periodic orbits. We note that for $d \geq 2$

$$\#\{z \in \mathbb{C} \ : \ f^n(z) = z\} = d^n,$$

where these points are counted with multiplicity and that the *topological entropy* of f, denoted by $h(f)$, is in fact equal to $\log d$ [Lyu81].

It is a standard result that not only a rational map is holomorphic but in fact the only holomorphic functions from the Riemann sphere to itself are rational maps. Considering the derivative of f, we call a point $z \in \widehat{\mathbb{C}}$ *critical* whenever $f'(z) = 0$.

Multipliers and holonomies Returning our focus on the periodic orbits of f, in addition to their period we define a different notion of length for each periodic orbit. Considering a periodic orbit $\tau = \{z, \ldots, f^{n-1}(z)\}$ in $\mathcal{P}_n$ we define its *multiplier* to be the complex number

$$\lambda(\tau) := (f^n)'(z) \in \mathbb{C}.$$

Note that using the chain rule it is easy to see that the multiplier of a periodic orbit is well defined since $(f^n)'(z) = \prod_{i=0}^{n-1} f'(f^i(z))$. For each periodic orbit τ we will call the magnitude of its multiplier $|\lambda(\tau)|$ the geometric length of the periodic orbit.

Additionally if τ does not include a critical point (equivalently if $\lambda(\tau) \neq 0$) we define its *holonomy* to be the normalised complex number

$$\hat{\lambda}(\tau) := \frac{\lambda(\tau)}{|\lambda(\tau)|} \in \mathbb{S}^1,$$

where $\mathbb{S}^1$ is the unit circle centred at the origin in $\mathbb{C}$.

We use the multipliers to divide periodic orbits in three classes. We call a primitive periodic orbit τ

$$\begin{aligned} \textit{Attracting} \quad & \text{if} \quad |\lambda(\tau)| < 1, \\ \textit{Repelling} \quad & \text{if} \quad |\lambda(\tau)| > 1, \\ \textit{Indifferent} \quad & \text{if} \quad |\lambda(\tau)| = 1. \end{aligned}$$

Conjugate rational maps We call two rational maps f, g *conjugate* whenever there exists a bijective conformal map from $\widehat{\mathbb{C}}$ to itself, that is a fractional linear transformation $h : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ given by

$$h(z) = \frac{az + b}{cz + d},$$

with $ad - bc \neq 0$, such that

$$f \circ h = h \circ g.$$

Conjugate rational maps enjoy many similarities in the study of their dynamics. Indeed assuming two rational maps f, g are conjugate we can easily show that the periodic orbits of these two maps are in one-to-one correspondence. Further, using the conjugacy relation and the chain rule we can show that not only corresponding periodic orbits have the same periods but also have equal multipliers (and holonomies when these are defined). This fact will allow us to study our asymptotic counting problem for a conjugacy class of rational maps f hence providing us with some flexibility.

2.1.2 Julia sets and hyperbolicity

Julia set Let $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a rational map. We call a point $z_0 \in \widehat{\mathbb{C}}$ *normal* with respect to f if there exists an open neighbourhood U of z_0 such that the family of iterated maps $\{f^n\}_{n \in \mathbb{N}}$ restricted to U forms a normal family. The *Fatou set* of f is the set of normal points of f denoted by $F = F(f)$. We call the complement of F in $\widehat{\mathbb{C}}$ the *Julia set* of f and denote it by $J = J(f)$.

Intuitively, the dynamics of f should be thought of as being tame on the Fatou set and more wild on the Julia set. Indeed later on we will restrict our focus to the study of the dynamics of f on its Julia set. It follows by the definition of these sets that the Fatou set is an open set whereas the Julia set is a closed set and hence in particular a compact subset of $\widehat{\mathbb{C}}$. Indeed the Julia set enjoys many important properties of which some will be useful for our analysis later on. Restricting our attention to a rational map f of degree at least two we have the following

- J is a non-empty, compact subset of $\widehat{\mathbb{C}}$,
- $f(J) = J = f^{-1}(J)$ and $J(f) = J(f^n)$ for $n \in \mathbb{Z} \setminus \{0\}$,
- J is a perfect set,
- for any open set U with $U \cap J \neq \varnothing$ there exists $n \in \mathbb{N}$ so that $J \subset f^n(U)$,
- the periodic orbits of f are dense in J.

Further, whenever the Julia set of a rational map f is not the whole of $\widehat{\mathbb{C}}$ we can, up to considering a conjugate rational map, assume that $\infty \notin J$ and so in particular for simplicity, we can view the Julia set J as a compact subset of $\mathbb{C}$.

Moreover, we have the dichotomy that attracting periodic orbits belong to the Fatou set whereas repelling periodic orbits lie in the Julia set. In fact Fatou showed that only a finite number of periodic orbits are not repelling (Shishikura obtained the best bound for the number of non-repelling periodic orbits given by $2d - 2$, where d is the degree of the rational map). Using this, Fatou also showed that the Julia set of f can be equivalently characterised as the closure of the union of repelling periodic orbits. Proofs and a systematic discussion on all the results mentioned above can be found in [Mil06, Ste93, Bea91, CG93].

Eremenko and van Strien proved in [EvS11] that if the Julia set of a rational map lies inside a smooth curve then in fact it must be included in a circle in $\widehat{\mathbb{C}}$. Up to considering a conjugate map we can assume that this circle is the real line. Assume that we have a map with Julia set included in the real line. This forces all the repelling periodic orbits to have real multipliers. Indeed assuming the Julia set of f is in the real line we can use the basic properties of a Julia set, the fact that it is perfect and f-invariant, to deduce that the derivative of an iterate of f evaluated at a point in the Julia set is real.

Hyperbolicity We call a rational map $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ *hyperbolic* if f is eventually expanding on its Julia set, that is there exist constants $C > 0$ and $\gamma > 1$ such that

$$|(f^n)'(z)| \geq C\gamma^n, \tag{2.1.1}$$

for all $z \in J$ and all $n \geq 1$.

Observe that a hyperbolic rational map cannot have critical points in its Julia set. In fact, a rational map is hyperbolic whenever the orbit of each critical point converges to an attracting periodic orbit in the Fatou set.

The shape and size of Julia sets is a complex problem on its own right. A useful invariant of Julia sets of conjugate rational maps is their Hausdorff dimension. In particular, we know that the Hausdorff dimension of the Julia set of a hyperbolic rational map f lies in $(0, 2)$ [Sul83].

2.2 Thermodynamic formalism for hyperbolic rational maps

The main purpose of this section is to describe how one can study the dynamics of a hyperbolic rational map using symbolic dynamics and ideas from thermodynamic formalism. We begin by recalling the essential features of this approach but for more details the reader is referred to [Rue89]. We then proceed to define the Ruelle transfer operators for which we obtain some decay estimates for their spectral radii in the next chapter. Fix a hyperbolic rational map f of degree $d \geq 2$. Further, assume that the Julia set of f is not contained inside a circle in $\widehat{\mathbb{C}}$.

2.2.1 Markov Partitions

For any small $\varepsilon > 0$, we can find a Markov partition for J: compact subsets $P_1, \dots, P_N$ of J each of diameter at most ε, such that

1. $J = \bigcup_{i=1}^{N} P_i$,
2. $\overline{\mathrm{int}(P_i)} = P_i \quad$ for $i = 1, \dots, N$,

3. $\mathrm{int}(P_i) \cap \mathrm{int}(P_j) = \varnothing \quad$ whenever $i \neq j$,

4. for each $i = 1, \ldots, N$
$$f(P_i) = \bigcup_{j \in \mathcal{N}_i} P_j,$$
where $\mathcal{N}_i = \{j \in \{1, \ldots, N\} : f(P_i) \cap \mathrm{int}(P_j) \neq \varnothing\}$,

(where closure and interior is taken relative to J).

Given a Markov partition $P_1, \ldots, P_N$, we can find open neighbourhoods $U_1, \ldots, U_N$ in $\mathbb{C}$ such that

1. f is injective on the closure of each U_j and on the union $U_i \cup U_j$, whenever $U_i \cap U_j \neq \varnothing$,

2. each $P_i \subseteq U_i$ is *not* contained in $\bigcup_{j \neq i} \overline{U_j}$,

3. for each pair i, j with $P_j \subset f(P_i)$ there is a local inverse $g_{ij} : U_j \to U_i$ for f.

We write
$$U = \bigsqcup_{i=1}^{N} U_i\,,$$
for the *disjoint* union of the neighbourhoods U_i.

The structure of the partition allows us to define an $N \times N$ matrix M with zero-one entries, where
$$M_{ij} = \begin{cases} 1 \text{ if } f(P_i) \supset P_j, \\ 0 \text{ otherwise.} \end{cases}$$

2.2.2 Ruelle Transfer Operators and the Pressure Function

By the hyperbolicity assumption the Julia set of f, and hence U (assuming each U_i is sufficiently small), does not contain any critical points. We can therefore define the following real analytic functions related to f, which will help us in the study of multipliers and holonomies of periodic orbits.

Definition 2.2.1. We define the *distortion function*
$$r(z) = \log |f'(z)|,$$
and the *rotation function*
$$\theta(z) = \arg(f'(z)) \in \mathbb{R}/2\pi\mathbb{Z},$$

which are both defined on U.

For a function $w : U \to \mathbb{R}$ (or $\mathbb{C}$) and $n \geq 1$, we denote the n-th Birkhoff sum by

$$w^n(z) = \sum_{j=0}^{n-1} w(f^j(z)).$$

(The context should make clear that this is not an iterate.) Hence, for a periodic orbit $\tau = \{z, f(z), \ldots, f^{n-1}(z)\}$ in $\mathcal{P}_n$, we have that

$$\lambda(\tau) = (f^n)'(z) = e^{r^n(z)+i\theta^n(z)}.$$

Ruelle transfer operators We proceed to define the Ruelle transfer operators as well as recalling some concepts from thermodynamic formalism. Write $C^1(U)$ for functions in $C^1(U, \mathbb{C})$ with bounded derivatives. Then, for $F \in C^1(U)$, we define the *transfer operator* $\mathcal{L}_F : C^1(U) \to C^1(U)$ by

$$(\mathcal{L}_F\, w)(x) \quad := \quad \sum_{i\,:\,M_{ij}=1} e^{F(g_{ij}x)} w(g_{ij}x) \text{ when } x \in U_j.$$

Let s be a complex parameter and k be an integer. Consider the $C^1(U)$ function given by $s(r - \alpha) + ik\theta : U \to \mathbb{C}$, where $\alpha \in \mathbb{R}$ will be specified later. We will be interested in the spectral properties of a special family of transfer operators parametrised by $s \in \mathbb{C}$ and $k \in \mathbb{Z}$ given below

$$\mathcal{L}_{(s,k)} := \mathcal{L}_{s(r-\alpha)+ik\theta}.$$

We will show how to obtain some bounds on the norms and spectral radii of the transfer operators in this family in order to prove our results. In fact this family of transfer operators is not uniformly bounded using the usual C^1 norm. We thus define a family of modified C^1 norms on $C^1(U)$ by

$$\|w\|_{(t)} := \begin{cases} \|w\|_\infty + \frac{\|w'\|_\infty}{t} & \text{if } t \geq 1, \\ \|w\|_\infty + \|w'\|_\infty & \text{if } 0 < t < 1. \end{cases}$$

These modified norms $\|\cdot\|_{(t)}$ will help us find sufficiently good bounds at least for large values of t.

Pressure function Given a continuous function $g : J \to \mathbb{R}$ we define the *topological pressure* of g by

$$\Pr(g) := \sup \left\{ h_f(\mu) + \int g \, d\mu \, : \, \mu \in \mathcal{M}_f \right\},$$

where $h_f(\mu)$ is the measure theoretic entropy of f with respect to μ and $\mathcal{M}_f$ is the set of f-invariant probability measures on J. We call μ an *equilibrium state* of g if

$$\Pr(g) = h_f(\mu) + \int g \, d\mu.$$

If g is a Hölder continuous function then it has a unique equilibrium state, which is fully supported and ergodic; we denote this by m_g. Given two functions g, h we have the inequality

$$|\Pr(g) - \Pr(h)| \leq \|g - h\|_\infty. \tag{2.2.1}$$

Two continuous functions g and h are called *cohomologous* if there exists a continuous function $u : J \to \mathbb{R}$ such that $g - h = u \circ f - u$. If g and h are Hölder continuous then $m_g = m_h$ if and only if $g - h$ is cohomologous to a constant.

If g and h are Hölder continuous then the function $\mathbb{R} \to \mathbb{R} : t \mapsto \Pr(tg + h)$ is real analytic and

$$\left. \frac{d \Pr(tg + h)}{dt} \right|_{t=0} = \int g \, dm_h, \tag{2.2.2}$$

$$\left. \frac{d^2 \Pr(tg + h)}{dt^2} \right|_{t=0} = \lim_{n \to \infty} \frac{1}{n} \int \left(g^n(x) - n \int g \, dm_h \right)^2 dm_h, \tag{2.2.3}$$

see [PP90, Propositions 4.10 and 4.11] and [Rue04]. Furthermore, as in [PP90, Proposition 4.12], if g is not cohomologous to a constant then $t \mapsto \Pr(tg + h)$ is strictly convex and

$$\left. \frac{d^2 \Pr(tg + h)}{dt^2} \right|_{t=0} > 0. \tag{2.2.4}$$

(The references provided are for the symbolic case but all proofs follow from the spectral gap property which will appear in the following chapter.)

We will now give a more precise version of our results. Recall that $\mathcal{M}_f$ is the set of f-invariant probability measures on J, which is convex and compact with respect to the weak* topology. Hence, the image of $\mathcal{M}_f$ onto the reals under the continuous projection

$$\mu \mapsto \int_J \log|f'|\, d\mu,$$

is an interval, which we denote by $\mathcal{I}_f$. Since we are assuming that f is not Möbius conjugate to a monomial, $\mathcal{I}_f$ has non-empty interior. As we will see later in Lemma 2.2.2 this follows from Zdunik's work [Zdu90].

We define

$$H(\alpha) := \sup\left\{h_f(\mu) \,:\, \mu \in \mathcal{M}_f \text{ with } \int \log|f'|\, d\mu = \alpha\right\},$$

where $h_f(\mu)$ denotes the measure-theoretic entropy. For $\alpha \in \operatorname{int}\mathcal{I}_f$ there is a unique $\mu_\alpha \in \mathcal{M}_f$ that realises this supremum above and a unique $\xi_\alpha \in \mathbb{R}$ such that

$$h_f(\mu_\alpha) + \xi_\alpha \int \log|f'|\, d\mu_\alpha = \sup\left\{h_f(\mu) + \xi_\alpha \int \log|f'|\, d\mu \,:\, \mu \in \mathcal{M}_f\right\}. \tag{2.2.5}$$

We also define the variance of $\log|f'| - \alpha$ by

$$\sigma_\alpha^2 := \lim_{n\to\infty} \frac{1}{n} \int \left(\log|(f^n)'| - n\alpha\right)^2 d\mu_\alpha.$$

Our hypothesis on f implies that the limit exists and $\sigma_\alpha^2 > 0$. These statements will be proved in the next lemma and will allow us to present our results in full detail and generality. Recall that for a Hölder continuous function $g : J \to \mathbb{R}$ we denote by m_g the unique equilibrium state of g. We have the following result.

Lemma 2.2.2. *The interval $\mathcal{I}_f$ is not a singleton. Furthermore, for each $\alpha \in \operatorname{int}(\mathcal{I}_f)$, there is a unique $\xi_\alpha \in \mathbb{R}$ such that $H(\alpha) = h_f(m_{\xi_\alpha r})$ and*

$$\int r\, dm_{\xi_\alpha r} = \alpha.$$

Before proving this lemma we first recall Livšic's (Livshits) Theorem [Liv71] adjusted in our situation.

Theorem 2.2.3. *Let $g, h : J \to \mathbb{R}$ be η-Hölder functions. Then, g and h are cohomologous if and only if their Birkhoff sums agree on each periodic orbit, that is if*

$$g^n(z) = h^n(z),$$

whenever $f^n(z) = z$.

Proof of Lemma 2.2.2. We claim that if $\mathcal{I}_f$ consists of a single point $c \in \mathbb{R}$, then the distortion function r is cohomologous to this constant. Indeed, assume that $\mathcal{I}_f = \{c\}$ and let $\tau = \{z, \ldots, f^{n-1}(z)\} \in \mathcal{P}_n$ be an arbitrary periodic orbit of length n. Consider the f-invariant probability measure supported on this orbit given by

$$\mu_\tau = \frac{1}{n} \sum_{i=0}^{n-1} \delta_{f^i(z)}.$$

Since $\mu_\tau \in \mathcal{M}_f$ we have that $\int_J \log|f'|\, d\mu_\tau = c$ that is

$$\sum_{i=0}^{n-1} \log|f'(f^i(z))| = nc.$$

Since the choice of τ was arbitrary, we have that the Birkhoff sums of the constant function c and the distortion function $r = \log|f'|$ agree for each periodic orbit τ of length n. Livšic's theorem then implies that the distortion function is cohomologous to the constant c. In particular, the function

$$\frac{\log d}{\int r\, d\mu_{\max}} r$$

is cohomologous to $\log d$ where $\mu_{\max}$ is the measure of maximal entropy for f. Zdunik showed in [Zdu90, Corollary in section 7 and Proposition 8] that the only hyperbolic rational maps satisfying this property are monomials and their conjugates. Since we are assuming that the Julia set of f is not contained in a circle in $\widehat{\mathbb{C}}$, f is not conjugate to a monomial and hence $\mathcal{I}_f$ is not a singleton.

Furthermore, we see that r is not cohomologous to any constant since this would imply that $\mathcal{I}_f$ is a singleton. In turn, this implies that the function

$$\mathfrak{p} : \mathbb{R} \to \mathbb{R} \quad \text{defined by} \quad \mathfrak{p}(t) = \Pr(tr),$$

is strictly convex. Now consider the set

$$\mathcal{D} := \{\mathfrak{p}'(\xi) : \xi \in \mathbb{R}\} = \left\{\int r \, dm_{\xi r} : \xi \in \mathbb{R}\right\} \subset \mathcal{I}_f.$$

Since $\mathfrak{p}$ is strictly convex, $\mathcal{D}$ is an open interval. By the definition of pressure, for all $\mu \in \mathcal{M}_f$,

$$\mathfrak{p}(t) \geq h_f(\mu) + t \int r \, d\mu.$$

In particular, the graph of the convex function $\mathfrak{p}$ lies above a line with slope $\int r \, d\mu$ (possibly touching it tangentially) and so $\int r \, d\mu \in \overline{\mathcal{D}}$. Thus, since μ is arbitrary, $\mathrm{int}(\mathcal{I}_f) \subset \overline{\mathcal{D}}$, and so we have $\mathcal{D} = \mathrm{int}(\mathcal{I}_f)$. Thus, for $\alpha \in \mathrm{int}(\mathcal{I}_f)$, since $\mathfrak{p}$ is strictly convex, there is a unique $\xi = \xi_\alpha \in \mathbb{R}$ with

$$\alpha = \mathfrak{p}'(\xi) = \int r \, dm_{\xi r}.$$

Since the map $\mu \mapsto h_f(\mu)$ is upper semi-continuous [Lyu81], the supremum in

$$H(\alpha) = \sup\left\{h_f(\mu) : \int r \, d\mu = \alpha\right\},$$

is attained. Since $m_{\xi r}$ is the equilibrium state for ξr, we have, for any $\mu \in \mathcal{M}_f$ with $\mu \neq m_{\xi r}$,

$$h_f(m_{\xi r}) + \xi \int r \, dm_{\xi r} > h_f(\mu) + \xi \int r \, d\mu.$$

In particular, if $\int r \, d\mu = \alpha$ then $h_f(m_{\xi r}) > h_f(\mu)$. Therefore, $m_{\xi r}$ is the unique measure with the desired properties. □

Setting $\mu_\alpha = m_{\xi_\alpha r}$, we have the measure whose existence is claimed in (2.2.5). Furthermore,

$$\sigma_\alpha^2 = \lim_{n\to\infty} \frac{1}{n} \int (r^n - n\alpha)^2 \, d\mu_\alpha = \mathfrak{p}''(\xi) > 0,$$

where we have used that $m_{\xi r} = m_{\xi(r-\alpha)}$.

For the rest of the paper, we will fix $\alpha \in \mathrm{int}(\mathcal{I}_f)$ and set $\xi = \xi_\alpha$ as in Lemma 2.2.2. We will also write $R := r - \alpha$ and $R^n(x) := r^n(x) - n\alpha$ and note that, by Lemma 2.2.2, we have that

$$H(\alpha) = \Pr(\xi R).$$

Chapter 3

Statistics for periodic orbits and holonomies of hyperbolic rational maps

3.1 Statement of results

Let $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a rational map of degree d at least two. Recalling the definitions in the preliminaries, a periodic orbit can be classified as *repelling, attracting* or *indifferent* depending on whether its multiplier has modulus greater than, less than, or equal to one, respectively. Then, the Julia set of f is given by the closure of the union of repelling periodic orbits and denoted by $J = J(f)$. Recall that it is a compact $f^{\pm 1}$-invariant subset of $\mathbb{C}$. Moreover, we recall that such a map has topological entropy $h(f) = \log d$ and $\#\{z \in \mathbb{C} : f^n(z) = z\} = d^n$.

Further we recall that we say that a rational map $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is *hyperbolic* if f is eventually expanding on J, that is there exist constants $C > 0$ and $\gamma > 1$ such that

$$|(f^n)'(z)| \geq C\gamma^n,$$

for all $z \in J$ and all $n \geq 1$.

For such a map, it is known that at most $2d - 2$ primitive periodic orbits are not repelling. Therefore, to study asymptotic counting problems for periodic orbits of f we can focus, without any loss of generality, to the study of the repelling periodic orbits. We write δ for the Hausdorff dimension of J; this satisfies $0 < \delta < 2$ [Sul83]. We will impose an additional hypothesis on f: we suppose that J is not contained in any circle in $\widehat{\mathbb{C}}$. In particular, this implies that f is not conjugate by a Möbius transformation to a monomial $z \mapsto z^{\pm d}$ for any $d \in \mathbb{N}$. In fact, Eremenko and van Strien in [EvS11] showed that if the Julia set of a hyperbolic rational map is contained in a smooth curve then it is contained in a circle in $\hat{\mathbb{C}}$.

We want to study the asymptotic behaviour of the quantity $\pi(n, \alpha, I, S)$ defined in the introduction. However, we also wish to consider a situation where I and S shrink as $n \to \infty$. To do this, let $K \subset \mathbb{R}$ be a compact set, let $(I_n)_{n=1}^{\infty}$ be a sequence of intervals contained in K and let $(S_n)_{n=1}^{\infty}$ be a sequence of arcs in $\mathbb{S}^1$. We are mainly interested in the two special cases where the sequences $(I_n)_{n=1}^{\infty}$ and $(S_n)_{n=1}^{\infty}$ are constant, corresponding to the case of a fixed interval and a fixed arc as in the introduction, and where the sequences $(\ell(I_n))_{n=1}^{\infty}$ and $(\nu(S_n))_{n=1}^{\infty}$ tend to zero, hence realising shrinking intervals. Similar asymptotic counting problems were considered in [PS06a], [PS06b] and [PS13].

We say that a sequence $(s_n)_{n=1}^{\infty}$ has sub-exponential growth if $\limsup_{n\to\infty} |\log s_n|/n = 0$. Writing

$$\pi(n, \alpha, I_n, S_n) := \# \left\{ \tau \in \mathcal{P}_n \ : \ \lambda(\tau) - n\alpha \in I_n \ \text{ and } \ \hat{\lambda}(\tau) \in S_n \right\},$$

we have the following theorem.

Theorem 3.1.1. *[SS22] Let $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a hyperbolic rational map of degree at least two such that its Julia set is not contained in a circle in $\widehat{\mathbb{C}}$. Let $K \subset \mathbb{R}$ be a compact set, let $(I_n)_{n=1}^{\infty}$ be a sequence of intervals in K and let $(S_n)_{n=1}^{\infty}$ be a sequence of arcs in $\mathbb{S}^1$. Furthermore, suppose that $(\ell(I_n)^{-1})_{n=1}^{\infty}$ and $(\nu(S_n)^{-1})_{n=1}^{\infty}$ have sub-exponential growth. Then, for each $\alpha \in \mathrm{int}(\mathcal{I}_f)$, there exist $\sigma_\alpha > 0$ and $\xi_\alpha \in \mathbb{R}$ so that*

$$\pi(n, \alpha, I_n, S_n) \sim \frac{\nu(S_n)}{\sigma_\alpha \sqrt{2\pi}} \int_{I_n} e^{-\xi_\alpha x}\, dx \, \frac{e^{H(\alpha)n}}{n^{3/2}}, \quad \text{as } n \to \infty. \tag{3.1.1}$$

In particular, if in addition we have that $\lim_{n\to\infty} \ell(I_n) = 0$ and $p_n \in I_n$ is arbitrary then

$$\pi(n, \alpha, I_n, S_n) \sim \frac{\nu(S_n)\ell(I_n)e^{-\xi_\alpha p_n}}{\sigma_\alpha \sqrt{2\pi}} \, \frac{e^{H(\alpha)n}}{n^{3/2}}, \quad \text{as } n \to \infty. \tag{3.1.2}$$

Corollary 3.1.2. *If* $\alpha = \int \log|f'|\, d\mu_{\max}$*, where* $\mu_{\max}$ *is the measure of maximal entropy then*

$$\pi(n, \alpha, I_n, S_n) \sim \frac{\nu(S_n)\ell(I_n)}{\sigma_\alpha\sqrt{2\pi}}\, \frac{d^n}{n^{3/2}}, \qquad \textit{as } n \to \infty. \tag{3.1.3}$$

Finally, we present our generalisation of Theorem 1.2.2.

Theorem 3.1.3. *Let* $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ *be a hyperbolic rational map of degree at least two such that its Julia set is not contained in a circle in* $\widehat{\mathbb{C}}$*. Let* $K \subset \mathbb{R}$ *be a compact set, let* $(I_n)_{n=1}^{\infty}$ *be a sequence of intervals in* K *and let* $(S_n)_{n=1}^{\infty}$ *be a sequence of arcs in* $\mathbb{S}^1$*. Furthermore, suppose that* $(\ell(I_n)^{-1})_{n=1}^{\infty}$ *and* $(\nu(S_n)^{-1})_{n=1}^{\infty}$ *have sub-exponential growth. Then, for each* $\alpha \in \mathrm{int}(\mathcal{I}_f)$ *there exists* $\sigma_\alpha > 0$ *so that*

$$\mu_\alpha \{x \in J \;:\; r^n(x) - n\alpha \in I_n \quad \textit{and} \quad \theta^n(x) \in S_n\} \sim \frac{\nu(S_n)\,\ell(I_n)}{\sigma_\alpha\sqrt{2\pi n}}, \qquad \textit{as } n \to \infty. \tag{3.1.4}$$

3.2 Dolgopyat-type estimates

The approach in this section is motivated by Dolgopyat's work on exponential mixing of Anosov flows in [Dol98]. This work was later used by Pollicott and Sharp to obtain exponential bounds for the error terms on the Prime Geodesic Theorem on compact negatively curved surfaces [PS98b]. Naud adapted Dolgopyat's analysis to prove a similar result for closed geodesics on convex-cocompact hyperbolic surfaces [Nau05] as well as Oh and Winter whose work was in the current setting of hyperbolic rational maps [OW17]. We use a similar approach to obtain bounds on the spectral radii of a family of transfer operators in order to extract our asymptotic result in the final section.

3.2.1 Ruelle–Perron–Frobenius Theorem and Pressure

Recall $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is a hyperbolic rational map of degree at least two and $\alpha \in \mathrm{int}(\mathcal{I}_f)$, $\xi = \xi(\alpha)$ are fixed constants as in Lemma 2.2.2. We now consider the family of Ruelle transfer operators $\mathcal{L}_{(\xi+ib,k)}$, for $b \in \mathbb{R}$ and $k \in \mathbb{Z}$, where

$$\mathcal{L}_{(\xi+ib,k)} := \mathcal{L}_{(\xi+ib)R+ik\theta}.$$

We recall Theorem 3.6 and Corollary 5.2 from [Rue89].

Theorem 3.2.1 (Ruelle–Perron–Frobenius Theorem). *Let $u \in C^1(U)$ be real valued. Then*

- *the operator $\mathcal{L}_u$ has a simple maximal positive eigenvalue $\lambda = e^{\Pr(u)}$ with an associated strictly positive eigenfunction $\psi \in C^1(U)$,*
- *the rest of the spectrum is contained in a disk of radius strictly smaller than $e^{\Pr(u)}$ and*
- *there is a unique probability measure μ on J such that ${\mathcal{L}_u}^*\mu = e^{\Pr(u)}\mu$ and $\int \psi \, d\mu = 1$.*

If $v \in C^1(U)$ is real valued then the spectral radius of $\mathcal{L}_{u+iv}$ is bounded above by $e^{\Pr(u)}$.

Analytic extension of the pressure function We will need to consider the function $s \mapsto e^{\Pr(sR)}$, $s \in \mathbb{R}$. In the previous chapter we defined the pressure of a real valued function using a variational principle. Here, we use the Ruelle–Perron–Frobenius Theorem to extend this definition. We view $e^{\Pr(sR)}$ as the simple maximal positive eigenvalue of the operator $\mathcal{L}_{sR}$ and show that $e^{\Pr(sR)}$ can be analytically extended to a neighbourhood of the real line using the Perturbation Theorem below.

Proposition 3.2.2 ([Kat95]). *Let $B(V)$ denote the Banach algebra of bounded linear operators on a Banach space V. If $\mathcal{L}_0 \in B(V)$ has a simple isolated eigenvalue λ_0 with corresponding eigenvector v_0 then for any $\varepsilon > 0$ we can find $\delta > 0$ such that if $\mathcal{L} \in B(V)$ with $\|\mathcal{L}_0 - \mathcal{L}\| < \delta$ then $\mathcal{L}$ has simple maximal eigenvalue $\lambda(\mathcal{L})$ and corresponding eigenvector $v(\mathcal{L})$ with $\lambda(\mathcal{L}_0) = \lambda_0$, $v(\mathcal{L}_0) = v_0$ and*

- *The functions $\mathcal{L} \to \lambda(\mathcal{L})$ and $\mathcal{L} \to v(\mathcal{L})$ are analytic for $\|\mathcal{L}_0 - \mathcal{L}\| < \delta$,*
- *for $\|\mathcal{L}_0 - \mathcal{L}\| < \delta$ we have $|\lambda(\mathcal{L}) - \lambda_0| < \varepsilon$ and the rest of the spectrum satisfies:*

$$\mathrm{spec}(\mathcal{L}) \setminus \{\lambda(\mathcal{L})\} \subseteq \{z \in \mathbb{C} \ : \ |z - \lambda_0| > \varepsilon\}.$$

Using the theorem above and the spectral gap property for the transfer operator $\mathcal{L}_{\xi R} : C^1(U) \to C^1(U)$ guaranteed from the Ruelle–Perron–Frobenius together with (2.2.2) and (2.2.3) from Chapter 2 we obtain the following result.

Corollary 3.2.3. *[PP90, Proposition 4.7] The function $t \mapsto e^{\Pr((\xi+it)R)}$ is analytic and for some $\varepsilon > 0$ we can write for each $t \in [-\varepsilon, \varepsilon]$*

$$e^{\Pr((\xi+it)R)} = e^{\Pr(\xi R)}\left(1 - \frac{\sigma_\alpha^2 t^2}{2} + O(|t|^3)\right),$$

where the implied constant is uniform on $[-\varepsilon, \varepsilon]$.

3.2.2 Decay estimates

By Theorem 3.2.1, the spectral radius of $\mathcal{L}_{(\xi+ib,k)}$ is bounded above by $e^{\Pr(\xi R)}$. The aim of this subsection is to show that in fact, when the Julia set of f is not contained in a circle in $\widehat{\mathbb{C}}$ we can bound the spectral radii of $\mathcal{L}_{(\xi+ib,k)}$ uniformly away from $e^{\Pr(\xi R)}$ when $(b,k) \neq (0,0)$. To achieve this we fix arbitrary $b \in \mathbb{R}$ and $k \in \mathbb{Z}$ with $(b,k) \neq (0,0)$ and consider the transfer operator $\mathcal{L}_{(\xi+ib,k)}$. As a first step we want to consider a normalised transfer operator. Let ψ be the positive eigenfunction of $\mathcal{L}_{(\xi,0)}$ with corresponding eigenvalue $e^{\Pr(\xi R)}$ guaranteed by Theorem 3.2.1. We define $\widehat{\mathcal{L}}_{(b,k)} : C^1(U) \to C^1(U)$ by

$$\widehat{\mathcal{L}}_{(b,k)}(g)(x) := \frac{\mathcal{L}_{(\xi+ib,k)}(g \cdot \psi)(x)}{e^{\Pr(\xi R)}\psi(x)}. \tag{3.2.1}$$

This is well defined since ψ is strictly positive and in particular it implies that

$$\widehat{\mathcal{L}}_{(0,0)}\mathbb{1} = \mathbb{1}. \tag{3.2.2}$$

It then follows that to show that the spectral radius of $\mathcal{L}_{(\xi+ib,k)}$ is less than $e^{\Pr(\xi R)}$ for $(b,k) \neq (0,0)$ it suffices to show that the spectral radius of $\widehat{\mathcal{L}}_{(b,k)}$ is less than 1. Below we show that the spectral radii $\mathrm{spr}(\widehat{\mathcal{L}}_{(b,k)})$, is strictly less than 1 uniformly for all $(b,k) \neq (0,0)$.

Let μ be the unique probability measure on J satisfying $\widehat{\mathcal{L}}_{(0,0)}{}^{*}\mu = \mu$. Its existence and uniqueness is guaranteed by Theorem 3.2.1 since $\widehat{\mathcal{L}}_{(0,0)}$ is the transfer operator corresponding to the real valued function

$$g = \xi R + \log(\psi) - \log(\psi \circ f) - \Pr(\xi R).$$

It follows from the variational principle that μ is the equilibrium state of the real potential g. Clearly, g is cohomologous to the real potential $\xi R - \Pr(\xi R)$ and hence the measure μ

is in fact the equilibrium state of ξR (potentials that differ only by a constant have the same equilibrium state), that is μ is the measure μ_α.

We regard μ as a measure on U by taking $\mu = \sum \mu_j$ where μ_j is the restriction of μ to the copy of P_j sitting inside U_j. Since the boundary points of μ_j, that is points in $U_j \setminus P_j$, have zero mass, μ is a probability measure on U.

Definition 3.2.4. We say that a probability measure m on J has the doubling property if there exists a positive constant C such that for all $x \in J$ and all $\varepsilon > 0$ we have that

$$m(B(x, 2\varepsilon)) \leq C \cdot m(B(x, \varepsilon)).$$

We know that in fact μ is a doubling measure [PW97, Theorem A2]. Moreover, as in [OW17, Proposition 4.5], it follows that the restrictions μ_j satisfy the doubling property as probability measures on P_j. We therefore have all the properties required to get the following theorem. We discuss the proof of this theorem in the next section.

Theorem 3.2.5 (Theorem 2.7, [OW17])**.** *Suppose that the Julia set of f is not contained in a circle in $\widehat{\mathbb{C}}$. Then there exist $C > 0$ and $\rho \in (0,1)$ such that for any $g \in C^1(U)$ with $\|g\|_{(|b|+|k|)} \leq 1$ and any $n \in \mathbb{N}$*

$$\left\| \widehat{\mathcal{L}}^n_{(b,k)}\, g \right\|_{L^2(\mu)} \leq C\rho^n,$$

whenever $|b| + |k| \geq 1$.

Using a standard argument [Dol98, Nau05] we can convert the bounds on the $\|\cdot\|_{L^2(\mu)}$ norm to bounds for the modified $\|\cdot\|_{(t)}$ norm. Then noting that $\|\cdot\|_{C^1} \leq (|b| + |k|)\, \|\cdot\|_{(|b|+|k|)}$ for $|b| + |k| \geq 1$ we can get the following corollary.

Corollary 3.2.6. *Suppose that the Julia set of f is not contained in a circle in $\widehat{\mathbb{C}}$. Then, for any $\varepsilon > 0$, there exist $C_\varepsilon > 0$ and $\rho_\varepsilon \in (0,1)$ such that for all $b \in \mathbb{R}$ and all $k \in \mathbb{Z}$ with $|b| + |k| > 1$ we have that*

$$\|\widehat{\mathcal{L}}^n_{(b,k)}\|_{C^1} \leq C_\varepsilon (|b| + |k|)^{1+\varepsilon} \rho_\varepsilon^n,$$

for all $n \in \mathbb{N}$. In particular, $\mathrm{spr}(\widehat{\mathcal{L}}_{(b,k)}) < \rho_\varepsilon < 1$.

Now that we have stated the required bounds on the C^1 norm of our transfer operators we proceed to bound the sums

$$Z_n(s,k) := \sum_{f^n(z)=z} e^{sR^n(z)+ik\theta^n(z)},$$

for $s = \xi + ib$ and $k \in \mathbb{Z}$. This next result follows essentially from Ruelle's work in [Rue90], except that we require explicit dependence on b and k. A proof can be found in the appendix of [Nau05] and a more rigorous approach in [Wri12] without the dependence on $k \in \mathbb{Z}$, which appeared as Proposition 6.1 in [OW17].

In the statement below, χ_j is the characteristic function of U_j for each $1 \leq j \leq N$ and recall that γ is the expansion rate given in (2.1.1). (Note that, since U is the *disjoint* union of the sets U_j, for each such j we have that $\chi_j \in C^1(U)$.)

Proposition 3.2.7. *Fix an arbitrary $b_0 > 0$. There exists $x_j \in P_j$, for $1 \leq j \leq N$, such that for any $\eta > 0$, there exists $C_\eta > 0$ such that for all $n \geq 2$ and any $k \in \mathbb{Z}$*

$$\left| Z_n(\xi+ib,k) - \sum_{j=1}^{N} \mathcal{L}^n_{(\xi+ib,k)}(\chi_j)(x_j) \right| \leq C_\eta(|b|+|k|) \sum_{p=2}^{n} \|\mathcal{L}^{n-p}_{(\xi+ib,k)}\|_{C^1} \left(\gamma^{-1} e^{\eta+\Pr(\xi R)}\right)^p$$

for all $|b|+|k| > b_0$.

We are now ready to prove the decay estimates that will enable us to prove Theorem 3.1.1 in the next section. Fixing $\varepsilon > 0$ then by Corollary 3.2.6 and Proposition 3.2.7, we get that for all $|b|+|k| > 1$,

$$\begin{aligned} |Z_n(\xi+ib,k)| &\leq \left| Z_n(\xi+ib,k) - \sum_{j=1}^{N} \mathcal{L}^n_{(\xi+ib,k)}(\chi_j)(x_j) \right| + N\, C_\varepsilon(|b|+|k|)^{1+\varepsilon} \left(\rho_\varepsilon e^{\Pr(\xi R)}\right)^n \\ &\leq C_\eta\, C_\varepsilon(|b|+|k|)^{2+\varepsilon} \left(\rho_\varepsilon e^{\Pr(\xi R)}\right)^n \sum_{p=2}^{n} \left(\frac{e^\eta}{\gamma\rho_\varepsilon}\right)^p + N\, C_\varepsilon(|b|+|k|)^{1+\varepsilon} \left(\rho_\varepsilon e^{\Pr(\xi R)}\right)^n \end{aligned}$$

We note that it is possible to choose $1 > \rho_\varepsilon > 1/\gamma$. Provided η is small enough such that $e^\eta/\gamma\rho_\varepsilon < 1$ we get that for some $C > 0$

$$|Z_n(\xi+ib,k)| \leq C\,(|b|+|k|)^{2+\varepsilon} \left(\rho_\varepsilon e^{\Pr(\xi R)}\right)^n. \tag{3.2.3}$$

Finally, we will also need a more elementary result to bound the sums $Z_n(\xi + ib, 0)$ for small $b \in \mathbb{R}$. These estimates can be derived as in the symbolic case in [PP90].

Lemma 3.2.8. *Let $K \subset \mathbb{R}$ be a compact set. There exists $\varepsilon > 0$ such that for each $n \in \mathbb{N}$ and some $\beta \in (0,1)$ we have that*

1. *for $b \in K \setminus (-\varepsilon, \varepsilon)$ we can bound $Z_n(\xi + ib, 0) = O(\beta^n e^{H(\alpha)n})$ and*
2. *for $b \in (-\varepsilon, \varepsilon)$ we have*

$$Z_n(\xi + ib, 0) = e^{n \operatorname{Pr}((\xi+ib)R)} + O(\beta^n e^{H(\alpha)n}).$$

Proof. For part (1), we use the fact that, Oh and Winter proved that if J is not contained in a circle then R satisfies the non-lattice property [OW17, Corollary 6.2], i.e. that it is not cohomologous to any function of the form $a + bu$, with $a, b \in \mathbb{R}$ and $u : J \to \mathbb{Z}$. Since R is non-lattice we have that $\operatorname{spr}(\mathcal{L}_{(\xi+ib,0)}) < e^{\operatorname{Pr}(\xi R)}$ for $b \neq 0$, with a uniform bound on $K \setminus (-\varepsilon, \varepsilon)$, and Proposition 3.2.7. Recall that $\operatorname{Pr}(\xi R) = H(\alpha)$. Then part (2) follows from the spectral gap in the Ruelle–Perron–Frobenius theorem, which is uniform over an interval $(-\varepsilon, \varepsilon)$. □

In the rest of this section we adapt Oh and Winter's arguments to prove our decay estimates stated in Theorem 3.2.5.

3.2.3 The Non-Local Integrability Condition

This technical property, whose precise definition is given below, is essential to apply a Dolgopyat-type argument. It was shown to hold true in [OW17, Theorem 3.4] for the distortion function of f, when f is not conjugate to $z^{\pm d}$ for any $d \in \mathbb{N}$. In particular, it is satisfied when the Julia set of f is not inside a circle in $\widehat{\mathbb{C}}$.

Consider a multi-index of length n, $\mathrm{I} = (i_n, \dots i_1) \in \{1, \dots, N\}^n$. We call I admissible if

$$M_{i_n, i_{n-1}} = \cdots = M_{i_2, i_1} = 1,$$

and for such an I we write

$$g_{\mathrm{I}} := g_{i_n, i_{n-1}} \circ \ldots \circ g_{i_2, i_1} : U_{i_1} \to U_{i_n}.$$

Furthermore, for $n \in \mathbb{N}$ and an admissible sequence $\underline{\zeta} = (\ldots, \zeta_{-2}, \zeta_{-1}, \zeta_0)$ with local inverses $g_{\zeta_{-k},\zeta_{-(k+1)}} : U_{\zeta_{-k}} \to U_{\zeta_{-(k+1)}}$ for each $k \geq 0$, we denote by

$$g_{\underline{\zeta}}^n := g_{(\zeta_{-n},\ldots,\zeta_{-1},\zeta_0)} : U_{\zeta_0} \to U_{\zeta_{-n}},$$

the local inverse of f^n defined on U_{ζ_0} corresponding to $\underline{\zeta}$. For any $x \in U_{\zeta_0}$, consider the sum $\sum_{n=1}^m r^n(g_{\underline{\zeta}}^n(x))$. This sum always diverges as $m \to \infty$ since f is hyperbolic and hence

$$\begin{aligned}\sum_{n=1}^m r^n(g_{\underline{\zeta}}^n(x)) &= \sum_{n=1}^m \sum_{i=0}^{n-1} \log|f'(f^i(g_{\underline{\zeta}}^n(x)))| = \sum_{n=1}^m \log|(f^n)'(g_{\underline{\zeta}}^n(x))| \\ &\geq \sum_{n=1}^m \log C + n\log\gamma = \frac{m(m+1)}{2}\log\gamma + m\log C \to \infty.\end{aligned}$$

Again since f is hyperbolic the local inverses $g_{\underline{\zeta}}^n$ are contracting. Using also the fact that r is Lipschitz we get that the series $\sum_{n=1}^\infty r(g_{\underline{\zeta}}^n(x)) - r(g_{\underline{\zeta}}^n(y))$ converges for any pair $x, y \in U_{\zeta_0}$. We set

$$r_\infty(\underline{\zeta}, x, y) := \sum_{n=1}^\infty r(g_{\underline{\zeta}}^n(x)) - r(g_{\underline{\zeta}}^n(y)).$$

Definition 3.2.9. [Non-Local-Integrability] We say that the distortion function r satisfies the non-local integrability property (NLI) if there exists $j \in \{1, \ldots, N\}$, points x_0, $x_1 \in P_j$, and admissible sequences $\underline{\zeta} = (\ldots, \zeta_{-2}, \zeta_{-1}, j)$, $\underline{\hat{\zeta}} = (\ldots, \hat{\zeta}_{-2}, \hat{\zeta}_{-1}, j)$ with the property that the gradient of

$$\hat{r}(x) := r_\infty(\underline{\zeta}, x, x_0) - r_\infty(\underline{\hat{\zeta}}, x, x_0)$$

is non-zero at x_1.

Note that the existence of a single choice of $x_0 \in P_j$, implies that it must hold for all choices of $x_0 \in P_j$. In fact it will be more convenient to use the following reformulation of the (NLI) property [OW17, Lemma 3.2].

Lemma 3.2.10 (Non-Local-Integrability II)**.** *Suppose that r satisfies the (NLI) property with respect to the sequences $\underline{\zeta}, \underline{\hat{\zeta}}$ and the points $x_0, x_1 \in P_j$. Then, there exists an open neighbourhood U_0 of x_1 and constants $\delta_2 \in (0,1)$ and $m \in \mathbb{N}$ such that the following hold: for any $n \geq m$, the map*

$$(\tilde{r}, \tilde{\theta}) := (r^n \circ g_{\underline{\zeta}}^n - r^n \circ g_{\underline{\hat{\zeta}}}^n,\, \theta^n \circ g_{\underline{\zeta}}^n - \theta^n \circ g_{\underline{\hat{\zeta}}}^n) : U_0 \to \mathbb{R} \times \mathbb{R}/2\pi\mathbb{Z}$$

is a local diffeomorphism satisfying $\|(\tilde{r},\tilde{\theta})\|_{C^2} < \frac{1}{2\delta_2}$ *and* $\inf_{u\in U_0} |\nabla(\tilde{r},\tilde{\theta})(u)\cdot v| \geq \delta_2|v|$ *for all* $v \in \mathbb{R}^2$.

Remark 3.2.11. In the lemma above, x_1 can be chosen to be any point of P_j with at most finitely many exceptions. This is because of the fact that the collection of critical points for $\hat{r}$ is either discrete or everything, since $\hat{r}$ is the real part of a holomorphic function. Thus, if the (NLI) property holds for some x_1 in P_j it holds for almost every x_1 in P_j.

Theorem 3.2.12. *For a hyperbolic rational function* $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ *of degree at least two, the distortion function* $r = \log|f'|$ *on* J *satisfies the (NLI) property if and only if* f *is not conjugate to* $f(z) = z^{\pm d}$ *for all* $d \in \mathbb{N}$.

3.2.4 The Non-Concentration Principle

The non-concentration principle is a property of the Julia set that arises from the complexity of its geometry and it will be an important ingredient in our approach. Recall that we assumed that the Julia set of f is not contained in a circle in $\hat{\mathbb{C}}$.

Notation 3.2.13. We denote cylinders of length $n \in \mathbb{N}$ by

$$\mathrm{C}([i_0,\ldots,i_{n-1}]) := \{x \in J : f^j(x) \in P_{i_j} \text{ for } 0 \leq j \leq n-1\}.$$

Note that we can regard cylinders either as subsets of J or as subsets of $P_{i_0} \subset U_{i_0} \subset U$.

Definition 3.2.14 (The Non-Concentration Property)**.** The Julia set J has the non-concentration property (NCP) if, for each cylinder C of J, there exists $0 < \delta_1 < 1$ such that for all $x \in C$, all $w \in \mathbb{S}^1$ and all $\varepsilon \in (0,1)$

$$B_\varepsilon(x) \cap \{y \in C : |\langle y-x, w\rangle| > \delta_1\varepsilon\} \neq \varnothing,$$

where $\langle a+bi, c+di\rangle = ac+bd$ for $a,b,c,d \in \mathbb{R}$.

Oh and Winter showed [OW17, Theorem 4.3] that the non-concentration principle holds for Julia sets which are not contained in a circle in $\hat{\mathbb{C}}$.

3.2.5 Construction of Dolgopyat operators

The rest of this section is devoted to the proof of Theorem 3.2.5. We start by fixing some constants all of which are positive real numbers. Recalling that f is hyperbolic and the Julia set of f is a compact set we fix constants

$$0 < c_0 < 1 < \kappa_1 < 2 < \kappa_2,$$

so that for each $z \in J$ and $n \in \mathbb{N}$ we have

$$c_0\, \kappa_1^n \leq |(f^n)'(z)| \leq \kappa_2^n.$$

By the (NLI) property of r (Theorem 3.2.12) and Remark 3.2.11, we may choose constants as in Lemma 3.2.10. Namely, we fix a partition element P_j, points $x_0, x_1 \in P_j$, admissible sequences $\underline{\zeta}, \underline{\hat{\zeta}}$, a neighbourhood U_0 of x_1 and $\delta_2 > 0$ satisfying the conditions of Lemma 3.2.10. For simplicity we can assume that the x_1 and U_0 described above satisfy that U_0 is an open disc and

$$x_1 \in P_1 \text{ and } U_0 \subset U_1 \text{ with } \overline{U_0} \cap \overline{U_j} = \varnothing \text{ for all } j \neq 1.$$

Using the topological mixing properties of f on its Julia set we can find $m^* \in \mathbb{N}$ large enough such that for each $1 \leq i \leq N$, there exists a length $m^* + 1$ cylinder X_i contained in U_0 such that

$$f^{m^*} X_i = P_i.$$

Fix such cylinders $X_1, \ldots, X_N$ and let $\delta_1 \in (0, 1)$ be a constant with respect to which they all satisfy the (NCP) as in Definition 3.2.14. Denote the minimal doubling constant from definition 3.2.4 for any ν_j by $C_3 > 1$. Let

$$A_0 > \frac{32}{c_0(\kappa_1 - 1)} \max\{||r||_{C^1}, ||\psi||_{C^1}, ||\theta||_{C^1}\} + \frac{1}{c_0} + \frac{2}{\delta_2},$$

$$E \geq 2A_0 + 1,$$

and

$$\delta_3 \leq \frac{\delta_1 \delta_2}{4E}.$$

Choose $m_0 \in \mathbb{N}$ large enough such that the (NLI) condition from Lemma 3.2.10 holds and also

$$4(E+1) < \kappa_1^{m_0} \text{ and } 160E < c_0\delta_1\delta_2\kappa_1^{m_0}.$$

We write

$$v_1 := g_{\underline{\zeta}}^{m_0} \text{ and } v_2 := g_{\underline{\hat{\zeta}}}^{m_0},$$

and note that they satisfy the conclusion of Lemma 3.2.10. We write

$$M = m_0 + m^*.$$

Choose

$$\varepsilon_1 \leq \min\left\{\frac{\log 2}{20E}, \frac{1}{160E}, \frac{c_0 \log 2}{200\kappa_2^{m^*}E}, \frac{\delta_1\delta_2^2}{100}\right\}.$$

Additionally, we assume that ε_1 is less than one tenth the distance from U_0 to the complement of U_1, that $\varepsilon_1\kappa_2^{m^*}$ is less than the minimum distance from any P_i to the complement U_i^c and that $2\varepsilon_1$ is less than the distance from U_0 to any U_j with $2 \leq j \leq N$. Then choose

$$\eta < \frac{1}{4N}\min\left\{\frac{c_0\varepsilon_1\delta_3}{\kappa_2^{m_0}}, , \frac{\delta_1^2\delta_2^2\varepsilon_1^2}{512}, 1\right\}.$$

Finally, let $C_1 := \exp\left(\log C_3 \log_2 \frac{200\kappa_2^{m^*}}{c_0^2\delta_3\kappa_1^{m^*}}\right)$ and choose

$$0 < \varepsilon_2 \text{ satisfying that } \left(1 - \frac{\eta e^{-MA_0}}{8C_1}\right) \leq (1-\varepsilon_2)^2.$$

Consider

$$(\tilde{r}, \tilde{\theta}) := (r^{m_0} \circ g_{\underline{\zeta}}^{m_0} - r^{m_0} \circ g_{\underline{\hat{\zeta}}}^{m_0}, \theta^{m_0} \circ g_{\underline{\zeta}}^{m_0} - \theta^{m_0} \circ g_{\underline{\hat{\zeta}}}^{m_0})$$

as in Lemma 3.2.10. Recalling the choice of A_0, we have that

$$||\tilde{r}||_{C^1} < \frac{A_0}{8} \quad \text{and} \quad ||\tilde{\theta}||_{C^1} < \frac{A_0}{8}.$$

Indeed we can bound the supremum norm of $\tilde{r}$ as follows

$$\begin{aligned}
\|\tilde{r}\|_\infty &= \left\| r^{m_0} \circ g_{\underline{\zeta}}^{m_0} - r^{m_0} \circ g_{\underline{\hat{\zeta}}}^{m_0} \right\|_\infty = \left\| \sum_{i=0}^{m_0-1} r \circ g_{\underline{\zeta}}^{m_0-i} - r \circ g_{\underline{\hat{\zeta}}}^{m_0-i} \right\|_\infty \\
&\leq \|r\|_{C^1} \sum_{i=0}^{m_0-1} \left\| g_{\underline{\zeta}}^{m_0-i} - g_{\underline{\hat{\zeta}}}^{m_0-i} \right\|_\infty \leq \|r\|_{C^1} \sum_{i=0}^{m_0-1} \frac{|U_0|}{|(f^{m_0-i})'|_\infty} \\
&\leq \frac{\|r\|_{C^1}}{c_0} \sum_{i=1}^{\infty} \frac{1}{\kappa_1}^i \leq \frac{\|r\|_{C^1}}{c_0(\kappa_1 - 1)} \leq A_0/16
\end{aligned}$$

We can also bound the derivative by:

$$\begin{aligned}
\|\tilde{r}'\|_\infty &= \left\| (r^{m_0} \circ g_{\underline{\zeta}}^{m_0} - r^{m_0} \circ g_{\underline{\hat{\zeta}}}^{m_0})' \right\|_\infty = \left\| \sum_{i=0}^{m_0-1} (r \circ g_{\underline{\zeta}}^{m_0-i} - r \circ g_{\underline{\hat{\zeta}}}^{m_0-i})' \right\|_\infty \\
&\leq \|r\|_{C^1} \sum_{i=0}^{m_0-1} \left\| (g_{\underline{\zeta}}^{m_0-i} - g_{\underline{\hat{\zeta}}}^{m_0-i})' \right\|_\infty \leq \|r\|_{C^1} \sum_{i=0}^{m_0-1} \frac{2}{|(f^{m_0-i})'|_\infty} \\
&\leq \frac{2\,\|r\|_{C^1}}{c_0(\kappa_1 - 1)} \leq A_0/16.
\end{aligned}$$

The bounds for $\left\|\tilde{\theta}\right\|_{C^1}$ follow in a similar manner.

Recall the length $m^* + 1$ sub-cylinders $X_1, \ldots, X_m$ of P_1 and write $\tilde{\varepsilon} = \varepsilon_1/(|b| + |k|)$. For each $1 \leq i \leq N$, consider a cover of $U_0 \cap X_i$ by finitely many balls $B_{50\tilde{\varepsilon}}(x_r^i)$, $r = 1, \ldots, r_0 = r_0(i)$ with $x_r^i \in U_0 \cap X_i$ and $B_{10\tilde{\varepsilon}}(x_r^i)$ pairwise disjoint; this is provided by a Vitali covering argument. For each x_r^i, we consider the gradient vector at x_r^i given by

$$w_r^i = b\nabla\tilde{r}(x_r^i) + k\nabla\tilde{\theta}(x_r^i),$$

and its normalisation

$$\hat{w}_r^i = \frac{w_r^i}{|w_r^i|}.$$

Note that the (NLI) condition implies that

$$|w_r^i| > \frac{\delta_2(|b| + |k|)}{2}.$$

Using the non-concentration property we choose, for each x_r^i, a partner point $y_r^i \in B_{5\tilde{\varepsilon}}(x_r^i) \cap X_i$ with

$$|\langle y_r^i - x_r^i, \hat{w}_r^i \rangle| > 5\delta_1\tilde{\varepsilon}.$$

For each point $z \in \mathbb{C}$ and each $\varepsilon > 0$, we choose a smooth cut-off function $\psi_{z,\varepsilon}$ taking the value zero on the exterior of the $B_\varepsilon(z)$ and the value one on $B_{\varepsilon/2}(z)$. We may assume that

$$||\psi_{z,\varepsilon}||_{C^1} \leq \frac{4}{\varepsilon}.$$

For each $j = 1, 2$, we will associate $v_j(x_r^i)$ with $(j, 1, r, i)$ and $v_j(y_r^i)$ with $(j, 2, r, i)$, so that we parametrise the set

$$\{v_j(x_r^i), v_j(y_r^i) : 1 \leq j \leq 2, 1 \leq r \leq r_0, 1 \leq i \leq N\},$$

by $\{1, 2\} \times \{1, 2\} \times \{1, \ldots, r_0\} \times \{1, \ldots, N\}$.

Note that by [OW17, Lemma 2.2] we have that $v_1(U_1) \cap v_2(U_1) = \varnothing$. Then, for a subset $\Lambda \subset \{1, 2\} \times \{1, 2\} \times \{1, \ldots, r_0\} \times \{1, \ldots, N\}$, we define the function β_Λ on U as

$$\begin{cases} 1 - \eta \left(\sum_{v_1(x_r^i) \in \Lambda} \psi_{x_r^i, 2\delta_3 \tilde{\varepsilon}} \circ f^{m_0} + \sum_{v_1(y_r^i) \in \Lambda} \psi_{y_r^i, 2\delta_3 \tilde{\varepsilon}} \circ f^{m_0} \right) & \text{on } v_1(U_1), \\ 1 - \eta \left(\sum_{v_2(x_r^i) \in \Lambda} \psi_{x_r^i, 2\delta_3 \tilde{\varepsilon}} \circ f^{m_0} + \sum_{v_2(y_r^i) \in \Lambda} \psi_{y_r^i, 2\delta_3 \tilde{\varepsilon}} \circ f^{m_0} \right) & \text{on } v_2(U_1), \\ 1 & \text{elsewhere.} \end{cases}$$

Definition 3.2.15. We will say that $\Lambda \subset \{1, 2\} \times \{1, 2\} \times \{1, \ldots, r_0\} \times \{1, \ldots, N\}$ is *full* if for every $1 \leq i \leq N$ and $1 \leq r \leq r_0$, there is $j \in \{1, 2\}$ such that $v_j(x_r^i)$ or $v_j(y_r^i)$ belongs to Λ or equivalently if there exist $1 \leq j, k \leq 2$ such that $(j, k, r, i) \in \Lambda$. We write $\mathcal{F}$ for the collection of all full subsets.

Fullness implies that for each $i \in \{1, \ldots, N\}$ the set of points x_r^i, y_r^i indicated by Λ forms a $100\tilde{\varepsilon}$ net for X_i.

Definition 3.2.16. Recall $M := m_0 + m^*$. For each $\Lambda \in \mathcal{F}$, we define the Dolgopyat operator $\mathcal{M}_\Lambda$ on $C^1(U)$ by

$$\mathcal{M}_\Lambda h := \widehat{\mathcal{L}}_{(b,0)}^M (h \, \beta_\Lambda).$$

Definition 3.2.17. For a positive real B, we write $K_B(U)$ for the cone set of all positive functions $h \in C^1(U)$ satisfying

$$|h'(u)| \leq B \, h(u) \quad \text{for all } u \in U,$$

that is the functions with logarithmic derivative bounded by B.

The following lemma is a traditional Lasota-Yorke type inequality [LY73] that appeared in [OW17, Lemma 5.1].

Lemma 3.2.18. *Fixing $B > 0$,*

- *if $H \in K_B(U)$, then*

$$\left|\left(\widehat{\mathcal{L}}_{(b,0)}^m H\right)'(x)\right| \leq A_0 \left(1 + \frac{B}{\kappa_1^m}\right) |\widehat{\mathcal{L}}_{(b,0)}^m H(x)|,$$

  for all $x \in U$ and any $m \geq 0$,

- *if $h \in C^1(U)$ and $H \in C^1(U, \mathbb{R})$ satisfy*

$$|h(x)| < H(x) \text{ and } |h'(x)| \leq B\, H(x) \quad \text{for all } x \in U,$$

  then

$$\left|\left(\widehat{\mathcal{L}}_{(b,k)}^m h\right)'(x)\right| \leq A_0 \left(\frac{B}{\kappa_1^m}(\widehat{\mathcal{L}}_{(b,0)}^m H)(x) + (|b| + |k| + 1)(\widehat{\mathcal{L}}_{(b,0)}^m |h|)(x)\right),$$

  for all $x \in U$ and any $m \geq 0$.

3.2.6 Proof of Theorem 3.2.5

We are now ready to state two main technical theorems. The first one provides us with some important properties of the Dolgopyat operators whereas the second one will allow us to bound iterates of the Ruelle transfer operators with the use of Dolgopyat operators. With these tools in hand, we will prove Theorem 3.2.5 using an iterative argument.

Theorem 3.2.19. *Fix $\Lambda \in \mathcal{F}$. If $H \in K_{E(|b|+|k|)}(U)$, then*

1. *$\mathcal{M}_\Lambda H \in K_{E(|b|+|k|)}(U)$ and*
2. *$||\mathcal{M}_\Lambda H||_{L^2(\mu)} \leq (1 - \varepsilon_2)||H||_{L^2(\mu)}$.*

Theorem 3.2.20. *For every $h \in C^1(U)$ and $H \in K_{E(|b|+|k|)}(U)$ satisfying*

$$|h| \leq H \text{ and } |h'| \leq E(|b| + |k|)H \quad \text{pointwise on } U,$$

there is a choice of $\Lambda \in \mathcal{F}$ such that

$$\left|\widehat{\mathcal{L}}_{(b,k)}^{M} h\right| \leq \mathcal{M}_{\Lambda} H,$$

and

$$\left|\left(\widehat{\mathcal{L}}_{(b,k)}^{M} h\right)'\right| \leq E(|b|+|k|)\mathcal{M}_{\Lambda} H,$$

both hold pointwise on U.

The proofs of Theorems 3.2.19 and 3.2.20 follow in the same way as the proofs of Theorems 5.6 and 5.7 in [OW17]. The deduction of Theorem 3.2.5 from Theorems 3.2.19 and 3.2.20 is now standard as in [Nau05, Section 5]. Let $h \in C^1(U)$. Combining Theorems 3.2.19 and 3.2.20 we inductively choose functions

$$H_i \in K_{E(|b|+|k|)}(U) \subset C^1(U, \mathbb{R}),$$

with the following properties

1. H_0 is the constant function $||h||_{(|b|+|k|)}$,
2. $H_{i+1} = \mathcal{M}_{\Lambda_i} H_i$ for some $\Lambda_i \in \mathcal{F}$,
3. $\left|\widehat{\mathcal{L}}_{(b,k)}^{iM} h\right| \leq H_i$ pointwise, and
4. $\left|\left(\widehat{\mathcal{L}}_{(b,k)}^{iM} h\right)'\right| \leq E(|b|+|k|)H_i$ pointwise.

From these we get using Theorem 3.2.19 that

$$\left\|\widehat{\mathcal{L}}_{(b,k)}^{iM} h\right\|_{L^2(\mu)} \leq (1-\varepsilon_2)^i ||h||_{(|b|+|k|)},$$

for any $h \in C^1(U)$ and any $i \geq 0$. For general $n = dM + r$ with $0 \leq r \leq M-1$, we then have

$$\begin{aligned}\left\|\widehat{\mathcal{L}}_{(b,k)}^{n} h\right\|_{L^2(\mu)} = \left\|\widehat{\mathcal{L}}_{(b,k)}^{dM} \widehat{\mathcal{L}}_{(b,k)}^{r} h\right\|_{L^2(\mu)} &\leq (1-\varepsilon_2)^d \left\|\widehat{\mathcal{L}}_{(b,k)}^{r} h\right\|_{(|b|+|k|)} \\ &\leq (1-\varepsilon)^n \frac{\left(\left\|\widehat{\mathcal{L}}_{(b,k)}\right\|_{(|b|+|k|)} + 1\right)^M}{(1-\varepsilon)^M} \|h\|_{(|b|+|k|)},\end{aligned}$$

where we have chosen $\varepsilon > 0$ such that $(1-\varepsilon)^M \geq (1-\varepsilon_2)$. The Lasota-Yorke type inequalities from Lemma 3.2.18 imply that there is a uniform bound on $\left\|\widehat{\mathcal{L}}_{(b,k)}\right\|_{(|b|+|k|)}$ independent of $b \in \mathbb{R}$ and $k \in \mathbb{Z}$ and so we are done.

3.3 Proof of Theorem 3.1.1

Throughout this section we fix a hyperbolic rational map $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ of degree at least two. We suppose that its Julia set is not contained inside a circle in $\widehat{\mathbb{C}}$ and we fix α a constant in the interior of $\mathcal{I}_f$. We set $\xi = \xi(\alpha)$ to be the unique real number given by Lemma 2.2.2. Let $K \subset \mathbb{R}$ be a compact set, let $(I_n)_{n=1}^\infty$ be a sequence of intervals in K and let $(S_n)_{n=1}^\infty$ be a sequence of arcs in $\mathbb{S}^1$. For convenience we parametrise $\mathbb{S}^1 \cong \mathbb{R}/\mathbb{Z}$ as $\left[-\frac{1}{2}, \frac{1}{2}\right]$ and assume that the sequence of arcs $(S_n)_{n=1}^\infty$ is contained inside a fixed reference arc $S = [-\frac{\kappa}{2}, \frac{\kappa}{2}]$ of length $\kappa < 1$.

For each $n \in \mathbb{N}$ we denote by p_n the midpoint of the interval I_n and by ϑ_n the midpoint of the arc S_n. Denote also their lengths by $\ell_n = \ell(I_n)$ and $\kappa_n = \nu(S_n)$. Furthermore, suppose that $(\ell_n^{-1})_{n=1}^\infty$ and $(\kappa_n^{-1})_{n=1}^\infty$ have sub-exponential growth. Then we can write

$$\begin{aligned}\pi(n, \alpha, I_n, S_n) &= \sum_{\tau \in \mathcal{P}_n} \mathbb{1}_{I_n} \left(\log|\lambda(\tau)| - n\alpha\right) \mathbb{1}_{S_n}\left(\hat{\lambda}(\tau)\right)\\ &= \sum_{\tau \in \mathcal{P}_n} \mathbb{1}_{[-\frac{1}{2},\frac{1}{2}]}\left(\ell_n^{-1}\left(\log|\lambda(\tau)| - n\alpha - p_n\right)\right) \mathbb{1}_{[-\frac{\kappa}{2},\frac{\kappa}{2}]}\left(\frac{\kappa}{\kappa_n}\left(\hat{\lambda}(\tau) - \vartheta_n\right)\right).\end{aligned}$$

3.3.1 Some auxiliary estimates

We fix $\phi \in C^4(\mathbb{R}, \mathbb{R}_{\geq 0})$ compactly supported and $\psi \in C^4(\mathbb{S}^1, \mathbb{R}_{\geq 0})$ and consider the following auxiliary counting number

$$\pi_{\phi,\psi}(n) := \sum_{\tau \in \mathcal{P}_n} \phi\left(\ell_n^{-1}\left(\log|\lambda(\tau)| - n\alpha - p_n\right)\right) \psi\left(\frac{\kappa}{\kappa_n}\left(\hat{\lambda}(\tau) - \vartheta_n\right)\right).$$

We study the asymptotic behaviour of $\pi_{\phi,\psi}(n)$ to deduce our result using an approximation argument in the next subsection.

We begin by changing the summation over $\mathcal{P}_n$, that is primitive periodic orbits of length n, to a sum over the set of fixed points of the iterated map f^n. Clearly, a primitive periodic orbit corresponds to n distinct points in this set. However this set also contains points belonging in primitive periodic orbits of shorter lengths. In the following lemma we bound the error from these shorter primitive orbits. Recall the distortion function r and the rotation function θ from Definition (2.2.1) and that $R(z) = r(z) - \alpha$ for $z \in J$. We define

$$\tilde{\pi}_{\phi,\psi}(n) := \frac{1}{n} \sum_{f^n(z)=z} \phi\left(\ell_n^{-1}\left(R^n(z) - p_n\right)\right) \psi\left(\frac{\kappa}{\kappa_n}(\theta^n(z) - \vartheta_n)\right). \tag{3.3.1}$$

Lemma 3.3.1. *For all $\eta > 0$ we have that*

$$\pi_{\phi,\psi}(n) = \tilde{\pi}_{\phi,\psi}(n) + O\left(e^{(H(\alpha)+\eta)n/2}\right).$$

Proof. Call a fixed point z of the iterated map f^n non-primitive when there exists q, a proper divisor of n, such that $f^q(z) = z$. We can then get the following bound

$$\begin{aligned}
\tilde{\pi}_{\phi,\psi}(n) - \pi_{\phi,\psi}(n) =& \frac{1}{n} \sum_{\substack{f^n(z)=z \\ \text{non-primitive}}} \phi\left(\ell_n^{-1}\left(R^n(z) - p_n\right)\right) \psi\left(\frac{\kappa}{\kappa_n}(\theta^n(z) - \vartheta_n)\right) \\
=& O\left(\frac{\|\psi\|_\infty}{n} \sum_{\substack{q|n \\ q\le n/2}} \sum_{f^q(z)=z} \phi\left(\ell_n^{-1}(R^n(z) - p_n)\right)\right) \\
=& O\left(\frac{1}{n} \sum_{\substack{q|n \\ q\le n/2}} \sum_{f^q(z)=z} \frac{\phi\left(\ell_n^{-1}(R^n(z) - p_n\right)}{e^{\xi R^q(z)}} e^{\xi R^q(z)}\right).
\end{aligned}$$

We are only interested in periodic points which satisfy $\ell_n^{-1}\left(R^n(z) - p_n\right) \in \operatorname{supp}\phi$ that is $R^n(z) \in p_n + \ell_n \operatorname{supp}\phi$. Recalling that the intervals I_n were chosen inside a compact set K we conclude that for such a periodic point the absolute value of $R^n(z)$ is bounded. Therefore for a non-primitive periodic point x, satisfying $f^q(z) = z$ for q as above, we get that $R^q(z) = \frac{q}{n} R^n(z)$ and thus $e^{\xi R^q(z)}$ is bounded from below. From this we conclude using Lemma 3.2.8 that for any $\eta > 0$,

$$\begin{aligned}
&\frac{1}{n} \sum_{\substack{q|n \\ q\le n/2}} \sum_{f^q(z)=z} \frac{\phi\left(\ell_n^{-1}(R^n(z) - p_n\right)}{e^{\xi R^q(z)}} e^{\xi R^q(z)} = O\left(\frac{\|\phi\|_\infty}{n} \sum_{\substack{q|n \\ q\le n/2}} \sum_{f^q(z)=z} e^{\xi R^q(z)}\right) \\
=& O\left(\frac{\|\phi\|_\infty}{n} \sum_{q\le n/2} Z_q(\xi, 0)\right) = O\left(\frac{1}{n} \sum_{q\le n/2} e^{\left(\Pr(\xi R)+\eta\right)q}\right) = O\left(e^{\left(H(\alpha)+\eta\right)n/2}\right).
\end{aligned}$$

□

Setting

$$\phi_n(x) := \phi(\ell_n^{-1}(x - p_n))e^{-\xi(x-p_n)}$$

we note that $\phi_n \in C^4(\mathbb{R}, \mathbb{R}_{\geq 0})$ and ϕ_n is compactly supported. Similarly, set

$$\psi_n(x) := \psi\left(\frac{\kappa}{\kappa_n}(x - \vartheta_n)\right).$$

Using the above notation we have that

$$\tilde{\pi}_{\phi,\psi}(n) = \frac{1}{n}\sum_{f^n(z)=z} \phi_n\left(R^n(z)\right)\psi_n\left(\theta^n(z)\right)e^{\xi(R^n(z)-p_n)}.$$

Proposition 3.3.2.

$$\tilde{\pi}_{\phi,\psi}(n) \sim e^{-\xi p_n}\frac{\int_{\mathbb{R}}\phi_n \int_{\mathbb{S}^1}\psi_n}{\sigma_\alpha\sqrt{2\pi}}\frac{e^{H(\alpha)n}}{n^{3/2}}, \qquad \textit{as } n \to \infty.$$

To prove this proposition we consider

$$A(n) := \left|\frac{e^{\xi p_n}\sigma_\alpha\sqrt{2\pi n^3}}{e^{H(\alpha)n}}\tilde{\pi}_{\phi,\psi}(n) - \int_{\mathbb{R}}\phi_n\int_{\mathbb{S}^1}\psi_n\right|,$$

and show that $A(n) \to 0$ as $n \to \infty$. The following proposition provides us with an initial bound. Using Fourier inversion and Fourier expansion we get

$$\phi_n(x)e^{\xi(x-p_n)} = e^{-\xi p_n}\int_{-\infty}^{\infty}\hat{\phi}_n(t)e^{(\xi+2\pi i t)x}\,dt \qquad \text{and} \tag{3.3.2}$$

$$\psi_n(x) = \sum_{k\in\mathbb{Z}} c_{n,k}\,e^{2\pi i k x}. \tag{3.3.3}$$

Proposition 3.3.3.

$$A(n) \leq \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left|\sum_{k\in\mathbb{Z}}\frac{c_{n,k}}{e^{H(\alpha)n}}\hat{\phi}_n\left(\frac{t}{2\pi\sigma_\alpha\sqrt{n}}\right)Z_n\left(\xi + \frac{it}{\sigma_\alpha\sqrt{n}}, k\right) - e^{-\frac{t^2}{2}}\int_{\mathbb{R}}\phi_n\int_{\mathbb{S}^1}\psi_n\right|\,dt.$$

Proof. Using (3.3.2) and (3.3.3) we can get

$$\frac{e^{\xi p_n}\sigma_\alpha\sqrt{2\pi n^3}}{e^{H(\alpha)n}}\tilde{\pi}_{\phi,\psi}(n) = \frac{\sigma_\alpha\sqrt{2\pi n}}{e^{H(\alpha)n}}\sum_{f^n(z)=z}\int_{-\infty}^{\infty}\hat{\phi}_n(t)e^{(\xi+2\pi it)R^n(z)}\,dt\sum_{k\in\mathbb{Z}}c_{n,k}\,e^{2\pi ik\theta^n(z)}$$
$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\sum_{k\in\mathbb{Z}}\frac{c_{n,k}}{e^{H(\alpha)n}}\hat{\phi}_n\left(\frac{t}{2\pi\sigma_\alpha\sqrt{n}}\right)\sum_{f^n(z)=z}e^{\left(\xi+\frac{it}{\sigma_\alpha\sqrt{n}}\right)R^n(z)+2\pi ik\theta^n(z)}\,dt$$
$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\sum_{k\in\mathbb{Z}}\frac{c_{n,k}}{e^{H(\alpha)n}}\,\hat{\phi}_n\left(\frac{t}{2\pi\sigma_\alpha\sqrt{n}}\right)Z_n\left(\xi+\frac{it}{\sigma_\alpha\sqrt{n}},k\right)\,dt.$$

In addition, recalling that $\int_{-\infty}^{\infty}e^{-t^2/2}\,dt=\sqrt{2\pi}$ we get,

$$\sqrt{2\pi}A(n)=\left|\int_{-\infty}^{\infty}\sum_{k\in\mathbb{Z}}\frac{c_{n,k}}{e^{H(\alpha)n}}\,\hat{\phi}_n\left(\frac{t}{2\pi\sigma_\alpha\sqrt{n}}\right)Z_n\left(\xi+\frac{it}{\sigma_\alpha\sqrt{n}},k\right)-e^{-\frac{t^2}{2}}\int_{\mathbb{R}}\phi_n\int_{\mathbb{S}^1}\psi_n\,dt\right|$$
$$\leq\int_{-\infty}^{\infty}\left|\sum_{k\in\mathbb{Z}}\frac{c_{n,k}}{e^{H(\alpha)n}}\,\hat{\phi}_n\left(\frac{t}{2\pi\sigma_\alpha\sqrt{n}}\right)Z_n\left(\xi+\frac{it}{\sigma_\alpha\sqrt{n}},k\right)-e^{-\frac{t^2}{2}}\int_{\mathbb{R}}\phi_n\int_{\mathbb{S}^1}\psi_n\right|\,dt.$$

□

Consider now the following three quantities

$$A_1(n):=\int_{-\varepsilon\sigma_\alpha\sqrt{n}}^{\varepsilon\sigma_\alpha\sqrt{n}}\left|\sum_{k\in\mathbb{Z}}\frac{c_{n,k}}{e^{H(\alpha)n}}\,\hat{\phi}_n\left(\frac{t}{2\pi\sigma_\alpha\sqrt{n}}\right)Z_n\left(\xi+\frac{it}{\sigma_\alpha\sqrt{n}},k\right)-e^{-\frac{t^2}{2}}\int_{\mathbb{R}}\phi_n\int_{\mathbb{S}^1}\psi_n\right|\,dt,$$
$$A_2(n):=\int_{|t|\geq\varepsilon\sigma_\alpha\sqrt{n}}\left|\sum_{k\in\mathbb{Z}}\frac{c_{n,k}}{e^{H(\alpha)n}}\,\hat{\phi}_n\left(\frac{t}{2\pi\sigma_\alpha\sqrt{n}}\right)Z_n\left(\xi+\frac{it}{\sigma_\alpha\sqrt{n}},k\right)\right|\,dt,$$
$$A_3(n):=\int_{|t|\geq\varepsilon\sigma_\alpha\sqrt{n}}\left|e^{-\frac{t^2}{2}}\int_{\mathbb{R}}\phi_n\int_{\mathbb{S}^1}\psi_n\right|\,dt,$$

with $\varepsilon>0$ small enough as in Lemmas 3.2.3 and 3.2.8. It then follows from Proposition 3.3.3 that

$$A(n)\leq\frac{1}{\sqrt{2\pi}}\Big[A_1(n)+A_2(n)+A_3(n)\Big].$$

We hence bound these three quantities separately to show that $\lim_{n\to\infty}A(n)=0$. To obtain these bounds we first recall a standard result from Fourier Analysis.

Lemma 3.3.4. *If $\psi\in C^4(\mathbb{S}^1,\mathbb{R})$ has Fourier coefficients $(c_k)_{k\in\mathbb{Z}}$ then $c_0=\int_{\mathbb{S}^1}\psi$ and uniformly for $\psi\in C^4(\mathbb{S}^1,\mathbb{R})$ and $k\neq 0$*

$$c_k=O(\|\psi\|_{C^4}|k|^{-4}).$$

If $\phi \in C^4(\mathbb{R},\mathbb{R})$ is compactly supported and $\hat{\phi}$ is its Fourier transform then $\hat{\phi}(0) = \int_{\mathbb{R}} \phi$ and uniformly for $\phi \in C^4(\mathbb{R},\mathbb{R})$ we have that for $u \in \mathbb{R}\setminus\{0\}$

$$\hat{\phi}(u) = O(\|\phi\|_{C^4}|u|^{-4}).$$

These bounds follow by repeated applications of integration by parts. Now since

$$\psi_n^{(q)}(x) = \left(\frac{\kappa}{\kappa_n}\right)^q \psi^{(q)}\left(\frac{\kappa}{\kappa_n}(x-\vartheta_n)\right)$$

there exists a constant $C > 0$ such that for all $n, |k| \geq 1$

$$|c_{n,k}| \leq C\kappa_n^{-4}|k|^{-4}\|\psi\|_{C^4}. \tag{3.3.4}$$

Similarly, there exists $C > 0$ such that for $n \in \mathbb{N}$ and $u \in \mathbb{R}\setminus\{0\}$

$$|\hat{\phi}_n(u)| \leq C\ell_n^{-4}|u|^{-4}\|\phi\|_{C^4}. \tag{3.3.5}$$

Proposition 3.3.5. $\lim_{n\to\infty} A_1(n) = 0$.

Proof. We can use inequality (3.2.3) to bound $Z_n\left(\xi + \frac{it}{\sigma_\alpha\sqrt{n}}, k\right)$ for $k \neq 0$. Therefore fixing $\eta \in (0,1)$ and recalling the bounds for the Fourier coefficients from (3.3.4) we get

$$\int_{-\varepsilon\sigma_\alpha\sqrt{n}}^{\varepsilon\sigma_\alpha\sqrt{n}} \left| \sum_{k\neq 0} \frac{c_{n,k}}{e^{H(\alpha)n}}\, \hat{\phi}_n\left(\frac{t}{2\pi\sigma_\alpha\sqrt{n}}\right)\, Z_n\left(\xi + \frac{it}{\sigma_\alpha\sqrt{n}}, k\right)\right| dt$$
$$= O\left(\int_{-\varepsilon\sigma_\alpha\sqrt{n}}^{\varepsilon\sigma_\alpha\sqrt{n}} \sum_{k\neq 0} \kappa_n^{-4}|k|^{-4}\|\hat{\phi}_n\|_\infty \left(\frac{|t|}{\sigma_\alpha\sqrt{n}} + |k|\right)^{2+\eta} \rho_\eta^n\, dt\right) = O\left(\kappa_n^{-4}\|\hat{\phi}_n\|_\infty\, \rho_\eta^n\right),$$

for some $\rho_\eta \in (0,1)$. Since ϕ is compactly supported and $p_n \in K$ we can uniformly bound $\hat{\phi}_n$ for all $n \in \mathbb{N}$. Further, recalling that the sequence $(\kappa_n^{-1})_{n=1}^\infty$ is of sub-exponential growth we get that this error tends to zero as $n \to \infty$. Therefore, we are now left to bound

$$\int_{-\varepsilon\sigma_\alpha\sqrt{n}}^{\varepsilon\sigma_\alpha\sqrt{n}} \left| \frac{c_{n,0}}{e^{H(\alpha)n}} \hat{\phi}_n\left(\frac{t}{2\pi\sigma_\alpha\sqrt{n}}\right) Z_n\left(\xi + \frac{it}{\sigma_\alpha\sqrt{n}}, 0\right) - e^{-\frac{t^2}{2}} \int_{\mathbb{R}} \phi_n \int_{\mathbb{S}^1} \psi_n \right| dt.$$

Using part (2) from Lemma 3.2.8 we get that for some $\beta \in (0,1)$

$$\int_{\mathbb{S}^1} \psi_n \int_{-\varepsilon\sigma_\alpha\sqrt{n}}^{\varepsilon\sigma_\alpha\sqrt{n}} \left| \hat{\phi}_n \left(\frac{t}{2\pi\sigma_\alpha\sqrt{n}} \right) e^{n\left(\Pr\left(\left(\xi + \frac{it}{\sigma_\alpha\sqrt{n}}\right)R\right) - H(\alpha)\right)} - e^{-\frac{t^2}{2}} \int_{\mathbb{R}} \phi_n \right| dt + O\left(\beta^n\right),$$

On the domain of integration, we see that as $n \to \infty$

1. $e^{n\left(\Pr\left(\left(\xi + \frac{it}{\sigma_\alpha\sqrt{n}}\right)R\right) - H(\alpha)\right)} \to e^{-t^2/2}$ by Lemma 3.2.3,
2. $\hat{\phi}_n\left(\frac{t}{2\pi\sigma_\alpha\sqrt{n}}\right) \to \hat{\phi}_n(0) = \int_{\mathbb{R}} \phi_n$ by continuity.

Furthermore, for large n we have the bound $e^{n\left(\Pr\left(\left(\xi + \frac{it}{\sigma_\alpha\sqrt{n}}\right)R\right) - H(\alpha)\right)} \leq e^{-t^2/4}$ and so

$$\left| e^{n\left(\Pr\left(\left(\xi + \frac{it}{\sigma_\alpha\sqrt{n}}\right)R\right) - H(\alpha)\right)} - e^{-t^2/2} \right| \leq 2e^{-t^2/4}.$$

Finally, since $\hat{\phi}_n$ is uniformly bounded, we can apply the Dominated Convergence Theorem to get that $\lim_{n\to\infty} A_1(n) = 0$.

□

Proposition 3.3.6. $\lim_{n\to\infty} A_2(n) = 0$.

Proof.

$$A_2(n) \leq \sum_{k\in\mathbb{Z}} \frac{|c_{n,k}|}{e^{H(\alpha)n}} \int_{|t|\geq\varepsilon\sigma_\alpha\sqrt{n}} \left| \hat{\phi}_n \left(\frac{t}{2\pi\sigma_\alpha\sqrt{n}} \right) Z_n \left(\xi + \frac{it}{\sigma_\alpha\sqrt{n}}, k \right) \right| dt.$$

Firstly, we use the bounds from (3.3.4) and (3.3.5). In addition, for $k \neq 0$ we use inequality (3.2.3) to get that a fixed $\eta \in (0,1)$ there exists $\rho_\eta \in (0,1)$ such that

$$\begin{aligned}
&\sum_{k\neq 0} \frac{|c_{n,k}|}{e^{H(\alpha)n}} \int_{|t|\geq\varepsilon\sigma_\alpha\sqrt{n}} \left| \hat{\phi}_n \left(\frac{t}{2\pi\sigma_\alpha\sqrt{n}} \right) Z_n \left(\xi + \frac{it}{\sigma_\alpha\sqrt{n}}, k \right) \right| dt \\
=&O\left(\sum_{k\neq 0} \kappa_n^{-4}|k|^{-4} \int_{|t|\geq\varepsilon\sigma_\alpha\sqrt{n}} \left| \ell_n^{-4} \left(\frac{t}{2\pi\sigma_\alpha\sqrt{n}} \right)^{-4} \left(\left| \frac{t}{\sigma_\alpha\sqrt{n}} \right| + |k| \right)^{2+\eta} \rho_\eta^n \right| dt \right) \\
=&O\left(\frac{n^2\rho_\eta^n}{\kappa_n^4\ell_n^4} \sum_{k\neq 0} \int_{|t|\geq\varepsilon\sigma_\alpha\sqrt{n}} \frac{\left(|t/\sigma_\alpha\sqrt{n}| + |k|\right)^{2+\eta}}{t^4k^4} dt \right) = O\left(\frac{n^2}{\kappa_n^4\ell_n^4} \rho_\eta^n \right).
\end{aligned}$$

On the other hand, for $k=0$ we get using part (2) of Lemma 3.2.8 that for some $\beta \in (0,1)$

$$
\begin{aligned}
&\frac{|c_{n,0}|}{e^{H(\alpha)n}} \int_{|t|\geq \varepsilon\sigma_\alpha\sqrt{n}}^{|t|\leq \sigma_\alpha\sqrt{n}} \left| \hat{\phi}_n \left(\frac{t}{2\pi\sigma_\alpha\sqrt{n}} \right) Z_n \left(\xi + \frac{it}{\sigma_\alpha\sqrt{n}}, 0 \right) \right| \, dt \\
=&O\left(|c_{n,0}| \|\hat{\phi}_n\|_\infty \beta^n \right) = O(|c_{n,0}|\beta^n),
\end{aligned}
$$

since $\hat{\phi}_n$ is uniformly bounded across all $n \in \mathbb{N}$. We can also uniformly bound $|c_{n,0}|$ since

$$
c_{n,0} = \int_{\mathbb{S}^1} \psi_n \leq \|\psi\|_\infty .
$$

Finally, as above we can use inequality (3.2.3) to bound the remaining

$$
\begin{aligned}
&\frac{|c_{n,0}|}{e^{H(\alpha)n}} \int_{|t|\geq \sigma_\alpha\sqrt{n}} \left| \hat{\phi}_n \left(\frac{t}{2\pi\sigma_\alpha\sqrt{n}} \right) Z_n \left(\xi + \frac{it}{\sigma_\alpha\sqrt{n}}, 0 \right) \right| \, dt \\
=&O\left(\int_{|t|\geq \sigma_\alpha\sqrt{n}} \ell_n^{-4} \left(\frac{t}{2\pi\sigma_\alpha\sqrt{n}} \right)^{-4} \left| \frac{t}{\sigma_\alpha\sqrt{n}} \right|^{2+\eta} \rho_\eta^n \, dt \right) \\
=&O\left(\frac{n^2\rho_\eta^n}{\ell_n^4} \int_{|t|\geq \sigma_\alpha\sqrt{n}} |t|^{\eta-2} \, dt \right) = O\left(\frac{n^2}{\ell_n^4} \rho_\eta^n \right).
\end{aligned}
$$

Combining the three bounds obtained above and recalling that the sequences $(\ell_n^{-1})_{n=1}^\infty$ and $(\kappa_n^{-1})_{n=1}^\infty$ are of sub-exponential growth we obtain that $\lim_{n\to\infty} A_2(n) = 0$.

□

Finally, it is clear that $\lim_{n\to\infty} A_3(n) = 0$. This completes the proof of Proposition 3.3.2.

3.3.2 Approximation argument

Here we show how the previous auxiliary estimates provide us with the proof of Theorem 3.1.1 through an approximation argument. By Proposition 3.3.2 and Lemma 3.3.1 we have that for all compactly supported $\phi \in C^4(\mathbb{R}, \mathbb{R})$ and all $\psi \in C^4(\mathbb{S}^1, \mathbb{R})$

$$
\pi_{\phi,\psi}(n) \sim e^{-\xi p_n} \frac{\int \phi_n \int \psi_n}{\sigma_\alpha \sqrt{2\pi}} \frac{e^{H(\alpha)n}}{n^{3/2}}, \qquad \text{as } n \to \infty. \tag{3.3.6}
$$

Fixing $\eta > 0$ we wish to construct compactly supported $\phi \in C^4(\mathbb{R}, \mathbb{R})$ and $\psi \in C^4(\mathbb{S}^1, \mathbb{R})$ satisfying the following:

$$\mathbb{1}_{[-\frac{1}{2},\frac{1}{2}]} \leq \phi \leq 1+\eta, \ \ \mathrm{supp}(\phi) \subset \left[-\frac{1+\eta}{2}, \frac{1+\eta}{2}\right] \ \text{ and } \ \int_{\mathbb{R}} \phi \leq 1+\eta,$$
$$\mathbb{1}_{[-\frac{\kappa}{2},\frac{\kappa}{2}]} \leq \psi \leq 1+\eta, \ \ \mathrm{supp}(\psi) \subset \left[-\frac{\kappa+\eta}{2}, \frac{\kappa+\eta}{2}\right] \ \text{ and } \ \int_{\mathbb{S}^1} \psi \leq \kappa+\eta.$$

A smooth function $\Phi : \mathbb{R} \to \mathbb{R}_{\geq 0}$ is called a positive mollifier, if it satisfies the following properties:

1. it is compactly supported,
2. $\int_{\mathbb{R}} \Phi = 1$,
3. $\lim_{\varepsilon \to 0} \Phi_\varepsilon(x) = \lim_{\varepsilon\to 0} \varepsilon^{-1}\Phi(x/\varepsilon) = \delta(x)$ where $\delta(x)$ is the Dirac delta function.

Let $\gamma_1, ..., \gamma_4 > 0$ and set $G = (1+\gamma_1)\mathbb{1}_{[-\frac{1}{2}-\gamma_2, \frac{1}{2}+\gamma_2]}$ and $H = (1+\gamma_3)\mathbb{1}_{[-\frac{\kappa}{2}-\gamma_4, \frac{\kappa}{2}+\gamma_4]}$. Then for sufficiently small $\varepsilon, \gamma_1, ..., \gamma_4 > 0$ the functions

$$\phi = G * \Phi_\varepsilon \quad \text{and} \quad \psi = H * \Phi_e,$$

satisfy all the required properties. Note that since $\kappa < 1$ and the constants ε, γ_4 were chosen sufficiently small it is harmless to assume that ψ is defined on $\mathbb{R}$ rather than $\mathbb{S}^1$. Using (3.3.6) and the properties above we can deduce that

$$\begin{aligned}
&\limsup_{n\to\infty} \frac{\sigma_\alpha\sqrt{2\pi n^3}}{e^{H(\alpha)n}} \pi(n, \alpha, I_n, S_n) \\
&= \limsup_{n\to\infty} \frac{\sigma_\alpha\sqrt{2\pi n^3}}{e^{H(\alpha)n}} \sum_{\tau\in\mathcal{P}_n} \mathbb{1}_{[-\frac{1}{2},\frac{1}{2}]}\left(\ell_n^{-1}(\log|\lambda(\tau)| - n\alpha - p_n)\right) \mathbb{1}_{[-\frac{\kappa}{2},\frac{\kappa}{2}]}\left(\frac{\kappa}{\kappa_n}(\hat{\lambda}(\tau) - \vartheta_n)\right) \\
&\leq \limsup_{n\to\infty} \frac{\sigma_\alpha\sqrt{2\pi n^3}}{e^{H(\alpha)n}} \sum_{\tau\in\mathcal{P}_n} \phi\left(\ell_n^{-1}(\log|\lambda(\tau)| - n\alpha - p_n)\right) \psi\left(\frac{\kappa}{\kappa_n}(\hat{\lambda}(\tau) - \vartheta_n)\right) \\
&= \limsup_{n\to\infty} e^{-\xi p_n} \int_{\mathbb{R}} \phi_n \int_{\mathbb{S}^1} \psi_n.
\end{aligned}$$

We have

$$\int_{\mathbb{R}} \psi_n = \int_{\mathbb{R}} \psi\left(\frac{\kappa}{\kappa_n}(y - \vartheta_n)\right) dy = \int_{\mathbb{R}} \psi\left(\frac{\kappa}{\kappa_n} y\right) dy = \frac{\kappa_n}{\kappa}\int_{\mathbb{R}} \psi \leq \kappa_n + \frac{\eta}{\kappa} = \nu(S_n) + O(\eta).$$

Similarly,

$$\int_{\mathbb{R}} \phi_n = \int_{\mathbb{R}} \phi(\ell_n^{-1}(x-p_n))e^{-\xi(x-p_n)}\,dx = \ell_n \int_{\mathbb{R}} \phi(u)e^{-\xi\ell_n u}\,du$$
$$= \ell_n \int_{[-\frac{1+\eta}{2},\frac{1+\eta}{2}]} \phi(u)e^{-\xi\ell_n u}\,du \leq \ell_n \int_{[-\frac{1}{2},\frac{1}{2}]} \phi(u)e^{-\xi\ell_n u}\,du + \eta(1+\eta)e^{2(1+|\xi|)|K|}|K|.$$

$$\ell_n \int_{[-\frac{1}{2},\frac{1}{2}]} \phi(u)e^{-\xi\ell_n u}\,du \leq \ell_n \int_{[-\frac{1}{2},\frac{1}{2}]} (1+\eta)e^{-\xi\ell_n u}\,du \leq e^{\xi p_n} \int_{I_n} e^{-\xi u}\,du + \eta e^{(1+|\xi|)|K|}|K|.$$

Therefore,

$$e^{-\xi p_n} \int_{\mathbb{R}} \phi_n \int_{\mathbb{S}^1} \psi_n \leq \nu(S_n) \int_{I_n} e^{-\xi u}\,du + O(\eta).$$

Similarly, one can show that

$$\liminf_{n\to\infty} \frac{\sigma_\alpha\sqrt{2\pi n^3}}{e^{H(\alpha)n}}\pi(n,\alpha,I_n,S_n) \geq \liminf_{n\to\infty}\left(\nu(S_n)\int_{I_n} e^{-\xi x}\,dx\right) + O(\eta).$$

Since the choice of $\eta > 0$ was arbitrary we get the result.

Assuming $\lim_{n\to\infty}\ell_n = 0$ the derivation of the asymptotic formula (3.1.2) from (3.1.1) is immediate. In particular, assuming that the sequence $(p_n)_{n=1}^{\infty}$ are the midpoints of our target intervals is harmless as the limit is the same. The asymptotic formula (3.1.3) corresponding to choosing the measure of maximal entropy follows in a similar manner. By the definition of the pressure function $\mu_{\max}$ is the equilibrium state of ξR for $\xi = 0$. Then, the proof follows in the same way as above.

3.4 Proof of Theorem 3.1.3

We follow a similar approach to the previous section. We start by fixing a hyperbolic rational map $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ of degree at least two and suppose that its Julia set is not contained inside a circle in $\widehat{\mathbb{C}}$. We fix α in the interior of $\mathcal{I}_f$ and set $\xi = \xi(\alpha)$ to be the unique real number and $\mu_\alpha = \mu_{\xi r}$ the probability measure given by Lemma 2.2.2. Let $K \subset \mathbb{R}$ be a compact set, let $(I_n)_{n=1}^{\infty}$ be a sequence of intervals in K and let $(S_n)_{n=1}^{\infty}$ be a sequence of arcs in $\mathbb{S}^1$. For convenience we parametrise $\mathbb{S}^1 \cong \mathbb{R}/\mathbb{Z}$ as $\left[-\frac{1}{2},\frac{1}{2}\right]$ and assume that the sequence of arcs $(S_n)_{n=1}^{\infty}$ is contained inside a fixed reference arc $S = [-\frac{\kappa}{2},\frac{\kappa}{2}]$ of length $\kappa < 1$.

For each $n \in \mathbb{N}$ we denote by p_n the midpoint of the interval I_n and by ϑ_n the midpoint of the arc S_n. Denote also their lengths by $\ell_n = \ell(I_n)$ and $\kappa_n = \nu(S_n)$. Furthermore, suppose that $(\ell_n^{-1})_{n=1}^{\infty}$ and $(\kappa_n^{-1})_{n=1}^{\infty}$ have sub-exponential growth. Recalling that we defined $R = r - \alpha$ we can write

$$\mu_\alpha \{z \in J \;:\; r^n(z) - n\alpha \in I_n \quad \text{and} \quad \theta^n(z) \in S_n\} = \int_J \mathbb{1}_{I_n}(R^n(z))\, \mathbb{1}_{S_n}(\theta^n(z))\, d\mu_\alpha(z)$$
$$= \int_J \mathbb{1}_{[-\frac{1}{2},\frac{1}{2}]}\left(\ell_n^{-1}(R^n(z) - p_n)\right) \mathbb{1}_{[-\frac{\kappa}{2},\frac{\kappa}{2}]}\left(\frac{\kappa}{\kappa_n}(\theta^n(z) - \vartheta_n)\right) d\mu_\alpha(z).$$

3.4.1 Some auxiliary estimates

We fix $\phi \in C^3(\mathbb{R}, \mathbb{R}_{\geq 0})$ compactly supported and $\psi \in C^3(\mathbb{S}^1, \mathbb{R}_{\geq 0})$ and consider the auxiliary counting number

$$\Pi_{\phi,\psi}(n) := \int_J \phi\left(\ell_n^{-1}(R^n(z) - p_n)\right) \psi\left(\frac{\kappa}{\kappa_n}(\theta^n(z) - \vartheta_n)\right) d\mu_\alpha(z).$$

For convenience we use similar notation as before and write

$$\phi_n(x) = \phi(\ell^{-1}(x - p_n)) \qquad \psi_n(x) = \psi\left(\frac{\kappa}{\kappa_n}(x - \vartheta_n)\right)$$

so that

$$\Pi_{\phi,\psi}(n) = \int_J \phi_n(R^n(z))\psi_n(\theta^n(z))\, d\mu_\alpha(z).$$

We rewrite the counting number $\Pi_{\phi,\psi}(n)$ using the Fourier inverse of ϕ_n and the Fourier expansion of ψ_n and use Fubini's theorem to obtain that

$$\Pi_{\phi,\psi}(n) = \int_J \int_{\mathbb{R}} \hat{\phi}_n(t)\, e^{2\pi i t R^n(z)}\, dt \sum_{k \in \mathbb{Z}} c_{n,k}\, e^{2\pi i k \theta^n(z)}\, d\mu_\alpha(z)$$
$$= \sum_{k \in \mathbb{Z}} c_{n,k} \int_{\mathbb{R}} \hat{\phi}_n(t) \int_J e^{2\pi i (t R^n(z) + k\theta^n(z))}\, d\mu_\alpha(z)\, dt.$$

As we have seen in Lemma 2.2.2 there exists a unique real number $\xi = \xi_\alpha$ such that the measure μ_α is the unique equilibrium measure of ξr, and hence also the unique equilibrium state of ξR since they differ only by a constant. Using the Ruelle–Perron–Frobenius Theorem we get that in fact μ_α is the unique probability measure on J such that

$$\mathcal{L}_{\xi R}{}^{*}\mu_\alpha = e^{\Pr(\xi R)}\mu_\alpha.$$

Therefore we get that

$$\begin{aligned}\int_J e^{2\pi i(tR^n(z)+k\theta^n(z))}\,d\mu_\alpha(z) &= e^{-n\,\mathrm{Pr}(\xi R)}\int_J \mathcal{L}^n_{\xi R}(e^{2\pi i(tR^n(z)+k\theta^n(z))})\,d\mu_\alpha(z)\\ &= e^{-n\,\mathrm{Pr}(\xi R)}\int_J \mathcal{L}^n_{(\xi+2\pi it)R+2\pi ik\theta}(\mathbb{1})(z)\,d\mu_\alpha(z).\end{aligned}$$

For $k\neq 0$ we use Corollary 3.2.6. We have that for a fixed $\varepsilon\in(0,1)$

$$\|\mathcal{L}^n_{(\xi+2\pi it,k)}\|_{C^1}\le C_\varepsilon(2\pi|t|+|k|)^{1+\varepsilon}(\rho_\varepsilon e^{\mathrm{Pr}(\xi R)})^n,$$

for some $\rho_\varepsilon\in(0,1)$. This bound combined with the bounds for the Fourier transform and the Fourier coefficients from Lemma 3.3.4 give that

$$\begin{aligned}&\sum_{|k|\ge 1} c_{n,k}\int_{\mathbb{R}}\frac{\hat{\phi}_n(t)}{e^{\mathrm{Pr}(\xi R)n}}\int_J \mathcal{L}^n_{(\xi+2\pi it,k)}(\mathbb{1})(z)\,d\mu_\alpha(z)\,dt\\ =&O\left(\sum_{|k|\ge 1}\frac{\|\psi_n\|_{C^3}}{k^3}\int_{\mathbb{R}}\frac{\|\phi_n\|_{C^3}}{t^3}(|t|+|k|)^{1+\varepsilon}\rho_\varepsilon^n\,dt\right)\\ =&O\left(\|\psi\|_{C^3}\,\kappa_n^{-3}\|\phi\|_{C^3}\,\ell_n^{-3}\rho_\varepsilon^n\sum_{|k|\ge 1}\int_{\mathbb{R}}\frac{(|t|+|k|)^{1+\varepsilon}}{|t|^3+|k|^3}\,dt\right)\\ =&O(\kappa_n^{-3}\ell_n^{-3}\rho_\varepsilon^n).\end{aligned}$$

Given that we assumed that $(\ell_n^{-1})_{n=1}^\infty$ and $(\kappa_n^{-1})_{n=1}^\infty$ are of sub-exponential growth this bound goes to zero exponentially fast as $n\to\infty$. We now proceed to treat the case of $k=0$. For $t\ge 1$ we use the same bounds as above and get another bound that tends to zero exponentially with n in the following way

$$\int_{\mathbb{S}^1}\psi_n\int_{|t|\ge 1}\frac{\hat{\phi}_n(t)}{e^{\mathrm{Pr}(\xi R)n}}\int_J \mathcal{L}^n_{(\xi+2\pi it,0)}(\mathbb{1})(z)\,d\mu_\alpha(z)\,dt = O\left(\|\psi_n\|_\infty\rho_\varepsilon^n\int_{|t|\ge 1}\frac{\|\phi_n\|_{C^3}}{|t|^{3-\varepsilon}}\,dt\right).$$

Finally we are left to study the quantity

$$\int_{\mathbb{S}^1}\psi_n\int_{|t|<1}\frac{\hat{\phi}_n(t)}{e^{\mathrm{Pr}(\xi R)n}}\int_J \mathcal{L}^n_{(\xi+2\pi it,0)}(\mathbb{1})\,d\mu_\alpha\,dt.$$

As mentioned in the proof of Lemma 3.2.8 for $t\neq 0$ we can use the fact that, Oh and Winter proved that if J is not contained in a circle then R satisfies the non-lattice property [OW17, Corollary 6.2], i.e. that it is not cohomologous to any function of the form $a+bu$, with $a, b \in \mathbb{R}$ and $u : J \to \mathbb{Z}$. Since R is non-lattice we have that $\operatorname{spr}(\mathcal{L}_{(\xi+2\pi it,0)}) < e^{\Pr(\xi R)}$ for $t \neq 0$, with a uniform bound for $t \in [-1,1] \setminus (-\varepsilon', \varepsilon')$ for any $\varepsilon' \in (0,1)$. This gives the following bound

$$\int_{\mathbb{S}^1} \psi_n \int_{\varepsilon'<|t|<1} \frac{\hat{\phi}_n(t)}{e^{\Pr(\xi R)n}} \int_J \mathcal{L}^n_{(\xi+2\pi it,0)}(\mathbb{1})\, d\mu_\alpha\, dt = O\left(\|\psi_n\|_\infty \|\hat{\phi}_n\|_\infty \beta^n\right),$$

for some $\beta \in (0,1)$.

We want to choose $\varepsilon' > 0$ small enough so that we can use a standard perturbation theory result that guarantees that for $t \in (-\varepsilon', \varepsilon')$ the transfer operator $\mathcal{L}_{(\xi+2\pi it,0)}$ has the unique maximal eigenvalue given by $e^{\Pr((\xi+2\pi it)R)}$. Moreover, we wish that ε' is small enough so that we can apply Lemma 3.2.3 to get that for $t \in (-\varepsilon', \varepsilon')$ we have that for some $C_{\varepsilon'} > 0$

$$|\Pr((\xi + 2\pi it)R)/\Pr(\xi R) - 1 + 2\pi^2\sigma_a^2 t^2| \leq C_{\varepsilon'}|t|^3.$$

For technical reasons that appear in the proof of Proposition 3.4.3 we may need to further adjust our choice of $\varepsilon' \in (0,1)$. We choose $0 < \varepsilon \leq \varepsilon'$ to make sure that two extra conditions hold. We need to assume that

$$\varepsilon < \min\left\{\frac{1}{2^{100}(\sigma_a + 1)^{10}}, \frac{1}{10\sigma_\alpha C_{\varepsilon'}^{10}}\right\}.$$

As above we use the Ruelle–Perron–Frobenius Theorem to get the following bound

$$\int_{\mathbb{S}^1} \psi_n \int_{\varepsilon<|t|<\varepsilon'} \frac{\hat{\phi}_n(t)}{e^{\Pr(\xi R)n}} \int_J \mathcal{L}^n_{(\xi+2\pi it,0)}(\mathbb{1})\, d\mu_\alpha\, dt = O\left(\|\psi_n\|_\infty \|\hat{\phi}_n\|_\infty \beta_1^n\right),$$

for some $\beta_1 \in (0,1)$.

Finally using the uniform spectral gap for $t \in [-\varepsilon, \varepsilon]$ provided by the Ruelle–Perron–Frobenius Theorem we get

$$\begin{aligned}
&\int_{\mathbb{S}^1} \psi_n \int_{|t|<\varepsilon} \frac{\hat{\phi}_n(t)}{e^{\Pr(\xi R)n}} \int_J \mathcal{L}^n_{(\xi+2\pi it,0)}(\mathbb{1})\, d\mu_\alpha \\
&= \int_{\mathbb{S}^1} \psi_n \int_{|t|<\varepsilon} \hat{\phi}_n(t) \left(e^{(\Pr((\xi+2\pi it)R)-\Pr(\xi R))n)} \operatorname{Proj}_t(\mathbb{1}) + O(\beta^n)\right) dt,
\end{aligned}$$

for some $\beta \in (0,1)$ where $\operatorname{Proj}_t : C^1(U,\mathbb{R}) \to C^1(U,\mathbb{R})$ is the projection corresponding to the maximal eigenvalue $e^{\Pr(\xi+2\pi it)R}$ guaranteed from Theorem 3.2.1. Perturbation theory shows that the projection depends analytically on the parameters and in fact

$$\mathrm{Proj}_t(\mathbb{1}) = \mathbb{1} + O(|t|).$$

At this point notice that the following relationship holds between the sequence of Fourier transforms

$$\begin{aligned}\hat{\phi}_n(t) =& \int_{\mathbb{R}} \phi_n(-x) e^{2\pi itx}\, dx = \int_{\mathbb{R}} \phi(\ell_n^{-1}(-x - p_n)) e^{2\pi itx}\, dx \\ =& \ell_n \int_{\mathbb{R}} \phi(-x) e^{2\pi it(x\ell_n - p_n)}\, dx = e^{-2\pi itp_n}\, \ell_n\, \hat{\phi}(t\ell_n).\end{aligned}$$

Recalling that the sequence of intervals $(I_n)_{n=1}^{\infty}$ was chosen inside a compact set we get that $(\ell_n)_{n=1}^{\infty}$ is bounded from above and $(p_n)_{n=1}^{\infty}$ lies in a compact set. Thus using the Mean Value Theorem we get that

$$\hat{\phi}_n(t) = \hat{\phi}_n(0) + O(|t| \|\hat{\phi}_n'\|_\infty) = (1 + O(|t|)) \int_{\mathbb{R}} \phi_n,$$

where the implied constant is uniform for all $n \in \mathbb{N}$. Together with Lemma 3.2.3 we get that

$$\begin{aligned}&\int_{\mathbb{S}^1} \psi_n \int_{|t|<\varepsilon} \hat{\phi}_n(t) \left(e^{(\Pr(\xi R + 2\pi it) - \Pr(\xi R))n} (1 + O(|t|)) \right) dt \\ =& \int_{\mathbb{S}^1} \psi_n \int_{\mathbb{R}} \phi_n \int_{|t|<\varepsilon} \left(1 - 2(\pi\sigma_\alpha t)^2 + O(|t|^3)\right)^n (1 + O(|t|))\, dt.\end{aligned}$$

We now consider the following proposition. This was proved in [PS94] but a minor error appeared in their calculation. We fix this error with Proposition 3.4.3.

Proposition 3.4.1. *As $n \to \infty$ we have that*

$$\int_{|t|<\varepsilon} \left(1 - 2(\pi\sigma_\alpha t)^2 + O(|t|^3)\right)^n (1 + O(|t|))\, dt \sim \frac{1}{\sigma_\alpha \sqrt{2\pi n}}.$$

Firstly note that we can do the following simplifications using symmetry and a change of variables

$$\begin{aligned}
&\int_{|t|<\varepsilon}\left(1-2(\pi\sigma_\alpha t)^2+O(|t|^3)\right)^n(1+O(|t|))\,dt\\
=&2\int_{t=0}^{\varepsilon}\left(1-2(\pi\sigma_\alpha t)^2+O(|t|^3)\right)^n(1+O(|t|))\,dt\\
=&\frac{\sqrt{2}}{\pi\sigma_\alpha}\int_{r=0}^{\pi\sigma_\alpha\varepsilon\sqrt{2}}\left(1-r^2+O(|r|^3)\right)^n(1+O(|r|))\,dr.
\end{aligned}$$

Recall that $C_{\varepsilon'}>0$ is the implied constant in the big-O notation $O(|r|^3)$ above and for simplicity write $a=\pi\sigma_\alpha\varepsilon\sqrt{2}$. Then we have that

$$\begin{aligned}
&\int_{r=0}^{a}\left(1-r^2+O(r^3)\right)^n(1+O(|r|))\,dr\;-\;\int_{r=0}^{a}(1-r^2)^n\,dr\\
=&O\left(\int_{r=0}^{a}(1-r^2)^n r\,dr+\sum_{i=1}^{n}\binom{n}{i}\int_{r=0}^{a}(1-r^2)^{n-i}C_{\varepsilon'}^i r^{3i}\,dr\right). \qquad (3.4.1)
\end{aligned}$$

In the following two propositions we show that this difference above decays asymptotically like n^{-1} whereas the principal term

$$\int_{r=0}^{\pi\sigma_\alpha\varepsilon\sqrt{2}}\left(1-r^2\right)^n\,dr,$$

decays asymptotically like $n^{-1/2}$.

Proposition 3.4.2. *As $n\to\infty$ we have*

$$\int_0^a(1-r^2)^n\,dr\;\sim\;\frac{\sqrt{\pi}}{2\sqrt{n}}.$$

Proof. We begin by comparing our integral to the one integrating over the full unit interval and observe that although $a\ll 1$ this difference decays exponentially fast with n since

$$\left|\int_{r=0}^{a}\left(1-r^2\right)^n\,dr-\int_{r=0}^{1}\left(1-r^2\right)^n\,dr\right|=\int_{r=a}^{1}\left(1-r^2\right)^n\,dr\le(1-a^2)^n.$$

Since the difference above decays exponentially fast we study instead the integral over the whole unit interval. We do a final change of variables and get that

$$\int_{r=0}^{1}(1-r^2)^n\,dr=\frac{1}{2}\int_{u=0}^{1}(1-u)^n u^{-1/2}\,du.$$

We use the standard result [Ste73, p.236] that for $n, m \in \mathbb{N}$ we have that

$$\int_0^1 (1-u)^n \, u^{m/2-1} \, dr = \frac{\Gamma(n+1)\,\Gamma(m/2)}{\Gamma(n+1+m/2)},$$

where Γ denotes the gamma function. This implies in particular that

$$\int_0^1 (1-r^2)^n \, dr = \frac{\Gamma(n+1)\Gamma(1/2)}{2\,\Gamma(n+1+1/2)} = \frac{\sqrt{\pi}}{2} \frac{\Gamma(n+1)}{\Gamma(n+1+1/2)} \sim \frac{\sqrt{\pi}}{2\sqrt{n}} \qquad \text{as } n \to \infty,$$

using the well-known result that $\lim_{n\to\infty} \frac{\Gamma(n)n^k}{\Gamma(n+k)} = 1$ for all real numbers k. This completes the proof of this proposition.

□

Finally, we bound the difference in (3.4.1) in a different way to [PS94].

Proposition 3.4.3.

$$\int_{r=0}^a (1-r^2)^n r \, dr + \sum_{i=1}^n \binom{n}{i} \int_{r=0}^a (1-r^2)^{n-i} C_{\varepsilon'}^i r^{3i} \, dr \;\leq\; \frac{C_{\varepsilon'}+3}{n}$$

Proof. We start with an easy observation. We can bound the first term by the following

$$\int_{r=0}^a \left(1-r^2\right)^n r \, dr \leq \int_{r=0}^1 \left(1-r^2\right)^n r \, dr = \frac{1}{2}\int_0^1 (1-u)^n \, du \leq \int_0^1 e^{-nu} \, du \;\leq\; \frac{1}{n}.$$

We now proceed to bound the remaining sum above. For simplicity we write $C = C_{\varepsilon'}$ and we begin by making a change of variables. We get

$$\sum_{i=1}^n \binom{n}{i} \int_{r=0}^a (1-r^2)^{n-i} C^i r^{3i} \, dr \;=\; \frac{1}{2} \sum_{i=1}^n \binom{n}{i} \int_{u=0}^{a^2} (1-u)^{n-i} C^i u^{(3i-1)/2} \, du.$$

We use an easy calculation on the first summand before we bound the rest. Observe that

$$\begin{aligned}
&\frac{Cn}{2}\int_{u=0}^{a^2} (1-u)^{n-1} u \, du = \frac{C}{2}\int_{u=0}^{a^2} -u \, d\left((1-u)^n\right) \\
&= -\frac{C}{2}(1-a^2)^n a^2 + \frac{C}{2}\int_{u=0}^{a^2} (1-u)^n \, du \\
&\leq -\frac{C(1-a^2)^{n+1}}{2(n+1)} + \frac{C}{2(n+1)} \leq \frac{C}{n}.
\end{aligned}$$

We then proceed by bounding the remaining sum in the following way

$$
\begin{aligned}
&\frac{1}{2}\sum_{i=2}^{n}\binom{n}{i}\int_{u=0}^{a^2}(1-u)^{n-i}C^i u^{(3i-1)/2}\,du\\
\leq&\frac{1}{2}\sum_{i=2}^{n}\frac{n(n-1)\dots(n-i+1)}{i!}\int_{u=0}^{a^2}(1-u)^n\left(\frac{C}{1-u}\right)^i u^{11i/10}\,du\\
\leq&\frac{1}{2}\int_{u=0}^{a^2}(1-u)^n\sum_{i=2}^{n}\frac{\left(nCu^{11/10}/(1-u)\right)^i}{i!}\,du\\
\leq&\frac{1}{2}\int_{u=0}^{a^2}(1-u)^n\sum_{i=0}^{\infty}\frac{\left(nCu^{11/10}/(1-u)\right)^i}{i!}\,du\\
\leq&\frac{1}{2}\int_{u=0}^{a^2}(1-u)^n e^{nCu^{11/10}/(1-u)}\,du\\
\leq&\frac{1}{2}\int_{u=0}^{a^2}e^{-nu}e^{nCu^{11/10}/(1-u)}\,du.
\end{aligned}
$$

At this point we recall our choice of $\varepsilon > 0$ was small enough so that two things happen. Firstly we have that

$$Cu^{1/20} \leq Ca^{1/10} \leq 1,$$

and secondly that

$$\frac{u^{1/20}}{1-u} \leq \frac{a^{1/10}}{1-a^2} \leq 1.$$

We can thus complete the proof by using the bounds above to obtain that

$$
\begin{aligned}
&\frac{1}{2}\int_{u=0}^{a^2}e^{-nu}e^{nCu^{11/10}/(1-u)}\,du \leq \int_{u=0}^{a^2}e^{-nu}e^{nu/2}\,du\\
=&\int_{u=0}^{a^2}e^{-nu/2}\,du = -\frac{2}{n}\left(e^{-na^2/2}-1\right) \leq \frac{2}{n}.
\end{aligned}
$$

Finally, combining the three obtained bounds we get that

$$\int_{r=0}^{a}(1-r^2)^n r\,dr + \sum_{i=1}^{n}\binom{n}{i}\int_{r=0}^{\pi\sigma_\alpha\varepsilon\sqrt{2}}(1-r^2)^{n-i}C^i r^{3i}\,dr \;\leq\; \frac{C+3}{n}$$

which completes the proof of this proposition. □

Proof of Proposition 3.4.1. Using equation (3.4.1) and the bounds from Propositions 3.4.2 and Proposition 3.4.3 we get the result. □

3.4.2 Approximation argument

Here we show how the previous auxiliary estimates provide us with the proof of Theorem 3.1.3 through a similar approximation argument as before. In the last section we have shown that for all compactly supported $\phi \in C^3(\mathbb{R}, \mathbb{R})$ and all $\psi \in C^3(\mathbb{S}^1, \mathbb{R})$ we have that

$$\Pi_{\phi,\psi}(n) \sim \frac{\int_{\mathbb{R}} \phi_n \int_{\mathbb{S}^1} \psi_n}{\sigma_\alpha \sqrt{2\pi\, n}}, \qquad \text{as } n \to \infty. \tag{3.4.2}$$

Fixing $\eta > 0$ we wish to construct compactly supported $\phi \in C^3(\mathbb{R}, \mathbb{R})$ and $\psi \in C^3(\mathbb{S}^1, \mathbb{R})$ satisfying the following:

$$\begin{aligned} &\mathbb{1}_{[-\frac{1}{2},\frac{1}{2}]} \le \phi \le 1+\eta, \ \ \operatorname{supp}(\phi) \subset \left[-\frac{1+\eta}{2}, \frac{1+\eta}{2}\right] \ \text{ and } \ \int_{\mathbb{R}} \phi \le 1+\eta, \\ &\mathbb{1}_{[-\frac{\kappa}{2},\frac{\kappa}{2}]} \le \psi \le 1+\eta, \ \ \operatorname{supp}(\psi) \subset \left[-\frac{\kappa+\eta}{2}, \frac{\kappa+\eta}{2}\right] \ \text{ and } \ \int_{\mathbb{S}^1} \psi \le \kappa+\eta. \end{aligned}$$

A smooth function $\Phi : \mathbb{R} \to \mathbb{R}_{\ge 0}$ is called a positive mollifier, if it satisfies the following properties:

1. it is compactly supported,
2. $\int_{\mathbb{R}} \Phi = 1$,
3. $\lim_{\varepsilon\to 0} \Phi_\varepsilon(x) := \lim_{\varepsilon\to 0} \varepsilon^{-1}\Phi(x/\varepsilon) = \delta(x)$ where $\delta(x)$ is the Dirac delta function.

Let $\gamma_1, ..., \gamma_4 > 0$ and set $G = (1+\gamma_1)\mathbb{1}_{[-\frac{1}{2}-\gamma_2, \frac{1}{2}+\gamma_2]}$ and $H = (1+\gamma_3)\mathbb{1}_{[-\frac{\kappa}{2}-\gamma_4, \frac{\kappa}{2}+\gamma_4]}$. Then for sufficiently small $\varepsilon, \gamma_1, ..., \gamma_4 > 0$ the functions

$$\phi = G * \Phi_\varepsilon \quad \text{and} \quad \psi = H * \Phi_e,$$

satisfy all the required properties. Note that since $\kappa < 1$ and the constants ε, γ_4 were chosen sufficiently small it is harmless to assume that ψ is defined on $\mathbb{R}$ rather than $\mathbb{S}^1$.

Using (3.4.2) and the properties above we can deduce that

$$\begin{aligned}
&\limsup_{n\to\infty} \quad \sigma_\alpha\sqrt{2\pi n}\,\Pi(n,\alpha,I_n,S_n) \\
&= \limsup_{n\to\infty} \quad \sigma_\alpha\sqrt{2\pi n} \int_J \mathbb{1}_{[-\frac{1}{2},\frac{1}{2}]}\left(\ell_n^{-1}(R^n(z)-p_n)\right)\mathbb{1}_{[-\frac{\kappa}{2},\frac{\kappa}{2}]}\left(\frac{\kappa}{\kappa_n}(\theta^n(z)-\vartheta_n)\right)\,d\mu_\alpha(z) \\
&\le \limsup_{n\to\infty} \quad \sigma_\alpha\sqrt{2\pi n} \int_J \phi\left(\ell_n^{-1}(R^n(z)-p_n)\right)\psi\left(\frac{\kappa}{\kappa_n}(\theta^n(z)-\vartheta_n)\right)\,d\mu_\alpha(z) \\
&= \limsup_{n\to\infty} \quad \int_{\mathbb{R}}\phi_n\int_{\mathbb{S}^1}\psi_n.
\end{aligned}$$

We have

$$\int_{\mathbb{R}}\psi_n = \int_{\mathbb{R}}\psi\left(\frac{\kappa}{\kappa_n}(y-\vartheta_n)\right)\,dy = \int_{\mathbb{R}}\psi\left(\frac{\kappa}{\kappa_n}y\right)\,dy = \frac{\kappa_n}{\kappa}\int_{\mathbb{R}}\psi \le \kappa_n + \frac{\eta}{\kappa} = \nu(S_n)+O(\eta).$$

Similarly,

$$\int_{\mathbb{R}}\phi_n = \int_{\mathbb{R}}\phi(\ell_n^{-1}(x-p_n))\,dx = \ell_n\int_{\mathbb{R}}\phi(u)\,du \le \ell_n+\eta = l(I_n)+\eta.$$

Thus we obtain that

$$\limsup_{n\to\infty} \quad \sigma_\alpha\sqrt{2\pi n}\,\Pi(n,\alpha,I_n,S_n) \le \limsup_{n\to\infty} \ (\nu(S_n)\ell(I_n))+O(\eta).$$

Similarly, one can show that

$$\liminf_{n\to\infty} \quad \sigma_\alpha\sqrt{2\pi n}\,\Pi(n,\alpha,I_n,S_n) \ge \liminf_{n\to\infty} \ (\nu(S_n)\ell(I_n))+O(\eta).$$

Since the choice of $\eta > 0$ was arbitrary we get the result.

Chapter 4

The geodesic and frame flow on convex-cocompact hyperbolic manifolds

4.1 Introduction and definitions

Hyperbolic Geometry Let $\mathbb{H}^N$ be the N-dimensional hyperbolic space for $N \geq 3$, i.e. the unique complete simply connected N-dimensional Riemannian manifold with constant negative sectional curvature. We will denote by $\langle ., . \rangle$ and $\|.\|$ the inner product and norm respectively on any tangent space of $\mathbb{H}^N$ induced by the hyperbolic metric. We will use the Poincaré ball model for the N-dimensional hyperbolic space. Let $\mathbb{D}^N$ denote the unit ball in $\mathbb{R}^N$; equipped with the Poincaré metric denoted by d where

$$|ds| = \frac{|dz|}{1 - |z|^2},$$

this gives us a model for the N-dimensional hyperbolic space. Let $\mathbb{S}^{N-1}$ be the unit sphere in $\mathbb{R}^N$ and hence the boundary of $(\mathbb{D}^N, d)$. Let $G = \mathrm{Isom}^+(\mathbb{H}^N)$ denote the group of orientation preserving isometries of $(\mathbb{D}^N, d)$. It then follows that G is isomorphic to the identity component of $\mathrm{SO}(1, N)$ which we denote by $\mathrm{SO}(N, 1)^o$ and is the set

$$\mathrm{SO}(N, 1)^o := \{A \in \mathrm{GL}(N, \mathbb{R}) \ : \ A_{(0,0)} > 0, \ A^T J_{1,N} A = J_{1,N}\},$$

where $J_{1,N} = \mathrm{diag}(-1, 1, \dots, 1)$. An introduction and a systematic discussion of hyperbolic geometry can be found in the following classical textbooks [And99, Bea83, BH99].

Kleinian groups We call a discrete subgroup of the group of orientation preserving isometries of $\mathbb{H}^N$ a *Kleinian group*. When a Kleinian group $\Gamma \leq G$ is torsion-free the quotient $X = \Gamma\backslash\mathbb{H}^N$ is a hyperbolic manifold of dimension N. Further, we call a Kleinian group Γ *convex-cocompact* if there exists a closed, convex invariant set $\mathcal{C} \subset \mathbb{H}^N$ such that the action of Γ on $\mathcal{C}$ is cocompact. Viewing G as a real algebraic group we call a subset $S \subseteq G$ *Zariski-dense* if S is not contained in any proper real algebraic subgroup of G. For the rest of this chapter we fix Γ to be a torsion-free, convex-cocompact and Zariski-dense Kleinian subgroup of G and consider the quotient hyperbolic manifold $X = \Gamma\backslash\mathbb{H}^N$. Note that since we assume that Γ is a torsion-free and Zariski-dense Kleinian group, Γ is not an elementary group, i.e. it is not virtually cyclic.

Fix an arbitrary point $x_0 \in \mathbb{H}^N$.

Limit set The *limit set* of Γ, denoted by Λ_Γ, is the set of accumulation points of Γ orbits of x_0, i.e. $x \in \Lambda_\Gamma \subset \mathbb{S}^N$ if there exists a sequence $(\gamma_n)_{n\in\mathbb{N}}$ in Γ such that $\lim_{n\to\infty} \gamma_n\, x_0 = x$.

Critical exponent Consider the Poincaré series of Γ

$$P_\Gamma(s) = \sum_{\gamma\in\Gamma} e^{-s\, d(x_0, \gamma\, x_0)}.$$

We call the abscissa of convergence of this series the critical exponent of Γ and denote it by δ_Γ. In our case, $\delta_\Gamma \in (0, N-1]$ and it coincides with the Hausdorff dimension of Λ_Γ. Moreover, the limit set and the critical exponent are in fact independent of the choice of $x_0 \in \mathbb{H}^N$.

It was proved recently in [Hou21] that all convex-cocompact groups $\Gamma \leq \mathrm{Isom}^+(\mathbb{H}^3)$ whose limit sets in $\mathbb{S}^2$ have Hausdorff dimension strictly less than one are in fact Schottky groups, as defined below.

Classical Schottky groups Let $S_1, \dots, S_{2p}$ be $2p$ non-intersecting $N-1$ dimensional Euclidean spheres in $\mathbb{H}^N$ that meet the boundary $\mathbb{S}^{N-1}$ at right angles. Further, assume that the spheres are pairwise exterior, that is the centre of each sphere is not included in the interior of any other sphere. For each pair of spheres $\{S_i, S_{2p+1-i}\}$ where $i = 1, \dots, p$

there exist isometries $g_i^{\pm 1} \in G$ that map one sphere onto the other and reversely. Note that since each isometry g_i is orientation preserving it maps the exterior of S_i to the interior of S_{2p+1-i} and vice-versa. A group generated from a symmetric set of isometries obtained as above is called a *classical Schottky group* of rank p. Further, we call the region in $\mathbb{H}^N$ exterior to all spheres $S_1, \ldots, S_{2p}$ a *fundamental domain* and denote it by $\mathcal{F}$. The limit set of a classical Schottky group has a particularly nice structure as it is a Cantor set in the boundary of $\mathbb{H}^N$. In three dimensions if we choose a Schottky group of rank at least two with centres for the spheres not lying in a unique plane we obtain a prototypical example of a torsion-free, convex-cocompact and Zariski-dense Kleinian subgroup of $\mathrm{Isom}^+(\mathbb{H}^3)$.

We now recall the definition of the *geodesic flow*.

The geodesic flow and the non-wandering set Let X be a smooth Riemannian manifold of dimension N with negative sectional curvatures. Let $T^1(X)$ denote the unit-tangent bundle, that is $T^1(X) = \{(x, v) \in TX : \|v\|_x = 1\}$, where $\|.\|_x$ is the norm induced by the Riemannian structure on T_xX. The *geodesic flow* $\phi_t : T^1(X) \to T^1(X)$ is defined as follows. Given $(x, v) \in T^1(X)$, there is a unique unit-speed geodesic $\gamma : \mathbb{R} \to X$ with $\gamma(0) = x$ and $\gamma'(0) = v$. We then define $\phi_t(x, v) = (\gamma(t), \gamma'(t))$.

The *non-wandering set* Ω of the geodesic flow, also known as the *convex core* of $X = \Gamma\backslash\mathbb{H}^N$ is the smallest convex subset of $T^1(X)$ containing all closed geodesics. Since $\Gamma \leq G$ was chosen to be convex-cocompact the convex core of $\Gamma\backslash\mathbb{H}^N$ is compact.

A generalised Poincaré series For a Hölder function $F : T^1(X) \to \mathbb{R}$ we can define the generalised Poincaré series $P_{\Gamma,F}$. Fix two distinct points $x_0, y_0 \in \mathbb{H}^N$. We write

$$\int_{x_0}^{y_0} F := \int_0^{d(x_0,y_0)} F(\phi_t(v))\, dt,$$

where $v \in T^1_{x_0}(X)$ such that $\phi_{d(x_0,y_0)}(v) \in T^1_{y_0}(X)$. This vector is unique when $x_0 \neq y_0$. The Poincaré series for (Γ, F) is the map $P_{\Gamma,F} = P_{\Gamma,F,x_0,y_0} : \mathbb{R} \to [0, \infty]$ defined by

$$P_{\Gamma,F}(s) := \sum_{\gamma \in \Gamma} e^{\int_{x_0}^{\gamma y_0}(F-s)}.$$

Then the critical exponent of (Γ, F) denoted by $\delta_{\Gamma,F} \in \mathbb{R}$ [PPS15, Propositions 3.9,3.11 and 5.11] is defined by

$$\delta_{\Gamma,F} := \liminf_{n\to\infty} \frac{1}{n} \log \sum_{\substack{\gamma\in\Gamma \\ n-1<d(x_0,\gamma y_0)<n}} e^{\int_{x_0}^{\gamma y_0} F}.$$

In fact $\delta_{\Gamma,F}$ is independent of the points $x_0, y_0 \in \mathbb{H}^N$. Also, observe that for $F = 0$ we recover the standard Poincaré series of Γ and the critical exponent of Γ.

Closed geodesics in hyperbolic manifolds Let ϕ_t be a C^1 flow on a smooth connected Riemann manifold M and $\Omega \subset M$ be an ϕ-invariant compact set. We say $\phi_t : \Omega \to \Omega$ is hyperbolic if

1. there exists a $D\phi_t$-invariant continuous splitting of the tangent bundle

$$T_\Omega M = E^0 \oplus E^s \oplus E^u,$$

where E^0 is the line bundle tangent to the non-singular flow and where there exist constants $C, c > 0$ such that

$$\|D\phi_t v\| \le Ce^{-ct}\|v\| \quad \text{for all} \quad v \in E^s \text{ and } t > 0 \quad \text{and}$$
$$\|D\phi_{-t} v\| \le Ce^{-ct}\|v\| \quad \text{for all} \quad v \in E^u \text{ and } t > 0,$$

2. the periodic orbits of Ω are dense and Ω is not a single point,

3. Ω contains a dense orbit and

4. there exists an open set $U \supset \Omega$ such that $\Omega = \bigcap_{t=-\infty}^{\infty} \phi_t(U)$.

We say that ϕ_t is mixing if for all non-empty open sets $U, V \subset M$ we have that $U \cap \phi_t(V) \neq \varnothing$, for all sufficiently large t. The geodesic flow on $T^1(X)$ is a topologically mixing hyperbolic flow [Dal99]. The mixing of the geodesic flow is equivalent to the fact that the set of lengths $\{l(\gamma) : \gamma \in P\}$ is not contained in a discrete subgroup of $\mathbb{R}$ [Bab02].

There is a natural one-to-one correspondence between periodic orbits for ϕ_t and closed geodesics in $T^1(X)$, with the least period being equal to the length of the closed geodesic. Our notation will not distinguish between the two sets of objects.

Another object of interest in the study of differentiable manifolds equipped with a metric are the orthonormal frames. An oriented orthonormal N-frame at a point $x \in X$ is an ordered, orthonormal basis for the N-dimensional space $T_x X$.

The frame flow Consider now the bundle of oriented orthonormal N-frames on X and denote it by $F(X)$. This produces a fiber bundle $\pi_1 : F(X) \to T^1(X)$ where the natural projection π_1 sends a frame onto its first vector. Over the N-dimensional manifold $T^1(X)$ we get a fiber bundle where the associated structure group $\mathrm{SO}(N-1)$ acts on fibers by rotating the frames, keeping the first vector fixed. Therefore, we can identify the fibers with $\mathrm{SO}(N-1)$. The frame flow Φ_t acts on frames by moving their first vectors by the geodesic flow and moving the other vectors by parallel translation along the geodesic defined by the first vector. Thus, $\pi_1 \circ \Phi_t = \phi_t \circ \pi_1$ for each $t \in \mathbb{R}$.

A reparametrisation for the geodesic and the frame flows We fix an arbitrary point $x_0 \in \mathbb{H}^N$ and an arbitrary frame $F_0 \in F(\mathbb{H}^N)$ based at x_0 with first coordinate $v_0 \in T^1(\mathbb{H}^N)$. The group G acts freely and transitively on $F(\mathbb{H}^N)$ hence we can identify $F(\mathbb{H}^N)$ with G. Consider the stabiliser subgroups

$$K = \mathrm{Stab}_G(x_0) \quad \text{and} \quad M = \mathrm{Stab}_G(v_0) < K.$$

Note that $K \cong \mathrm{SO}(N)$ and it is a maximal compact subgroup of G and $M \cong \mathrm{SO}(N-1)$. Our base hyperbolic manifold is

$$X = \Gamma \backslash \mathbb{H}^N \cong \Gamma \backslash G / K,$$

its unit tangent bundle is

$$T^1(X) \cong \Gamma \backslash G / M,$$

and its frame bundle is

$$F(X) \cong \Gamma \backslash G,$$

which is a principal $\mathrm{SO}(N)$-bundle over X and a principal $\mathrm{SO}(N-1)$-bundle over $T^1(X)$. It is convenient to parametrise the geodesic and the frame flow, so let us introduce the following notation. There is a one parameter subgroup of semi-simple elements

$$A = \{a_t : t \in \mathbb{R}\} < G,$$

whose centraliser is given by

$$C_G(A) = AM,$$

and its elements are parametrised such that their canonical right action on $\Gamma\backslash G/M$ and $\Gamma\backslash G$ corresponds to the geodesic flow and the frame flow respectively.

We choose a left G-invariant and right K-invariant Riemannian metric on G [Sas58, Mok78] which descends down to the previous hyperbolic metric on $\mathbb{H}^N \cong G/K$, and again use the notations $\langle .,. \rangle$, $\|.\|$ and d on G and any of its quotient spaces. As before, let Ω be the non-wandering set which is a compact A-invariant subset of $\Gamma\backslash G/M$ since Γ is convex-cocompact. Further, set $H^+ < G$ and $H^- < G$ to be the expanding and contracting horospherical subgroups of G, that is

$$H^{\pm} = \left\{ h^{\pm} \in G : \lim_{t \to \pm\infty} d(e\,,\, a_t\, h^{\pm}\, a_{-t}) = 0 \right\} \cong \mathbb{R}^{N-1}. \tag{4.1.1}$$

Patterson densities Before presenting the family of Patterson densities for a pair (Γ, F) we define the *Gibbs cocycle* of (Γ, F). There exists a well-defined map

$$C_F : \mathbb{S}^{N-1} \times \mathbb{H}^N \times \mathbb{H}^N \to \mathbb{R},$$

for the potential F given by

$$(\xi, x, y) \mapsto C_{F,\xi}(x, y) = \lim_{t \to \infty} \int_y^{\xi(t)} F - \int_x^{\xi(t)} F,$$

where $\xi : \mathbb{R} \to \mathbb{H}^N$ is any geodesic such that $\lim_{t\to\infty} \xi(t) = \xi$. Note that when $F = -1$ the Gibbs cocycle equals the more well-known *Busemann cocycle* $\beta_\xi(x, y)$ defined by

$$\beta_\xi(x, y) = \lim_{t \to \infty} (d(\xi(t), y) - d(\xi(t), x)),$$

where again $\xi : \mathbb{R} \to \mathbb{H}^N$ is any geodesic such that $\lim_{t\to\infty} \xi(t) = \xi$. Hence for every $s \in \mathbb{R}$ we have that

$$C_{F-s,\xi}(x, y) = C_{F,\xi}(x, y) - s\beta_\xi(x, y).$$

We allow tangent vector arguments for the Gibbs cocycles as well in which case we will use their basepoints in the definition. The Gibbs cocycle satisfies the following three useful properties: for all $\xi \in \mathbb{S}^{N-1}$, $\gamma \in \Gamma$ and $x, y, z \in \mathbb{H}^N$

- $C_{F,\xi}(x,z) = C_{F,\xi}(x,y) + C_{F,\xi}(y,z)$,
- $C_{F,\xi}(x,z) = -C_{F,\xi}(z,x)$ and
- $C_{F,\gamma\xi}(\gamma x, \gamma z) = C_{F,\xi}(x,z)$.

Now let $\{\mu_x : x \in \mathbb{H}^N\}$ denote the *Patterson densities* of (Γ, F) [Pat76, Sul79, PPS15], that is the set of finite Borel measures on $\mathbb{S}^{N-1}$ supported on Λ_Γ such that

1. $\gamma_* \mu_x = \mu_{\gamma x}$ for all $\gamma \in \Gamma$ and $x \in \mathbb{H}^N$ and
2. $\frac{d\mu_x}{d\mu_y}(\xi) = e^{-C_{F-\delta_{\Gamma,F},\xi}(x,y)}$ for all $\xi \in \mathbb{S}^{N-1}$ and $x, y \in \mathbb{H}^N$.

Bowen–Margulis–Sullivan measure For all $u \in T^1(\mathbb{H}^N)$, let u^+ and u^- denote its forward and backward limit points. Using the Hopf parametrisation via the homeomorphism

$$G/M \cong \mathrm{T}^1(\mathbb{H}^N) \to \{(u^+, u^-) \in \mathbb{S}^N \times \mathbb{S}^N : u^+ \neq u^-\} \times \mathbb{R}$$

given by

$$u \mapsto (u^+, u^-, t = \beta_{u^-}(x_0, u)),$$

we define the *Bowen–Margulis–Sullivan (BMS) measure* m on G/M [Mar04, Bow71, Kai90, PPS15] by

$$d\mathsf{m}(u) = e^{C_{F-\delta_{\Gamma,F},u^+}(x_0,u) + C_{F-\delta_{\Gamma,F},u^-}(x_0,u)}\, d\mu_{x_0}(u^+)\, d\mu_{x_0}(u^-)\, dt.$$

Note that this definition only depends on Γ and not on the choice of reference point $x_0 \in \mathbb{H}^n$. Moreover, m is left Γ-invariant. We now define induced measures on other spaces, all of which we call the BMS measures and denote by m by abuse of notation. By left Γ-invariance, m descends to a measure on $\Gamma \backslash G/M$. We normalise it to a probability measure so that $\mathsf{m}(\Gamma \backslash G/M) = 1$. Since M is compact, we can then use the probability Haar measure on M to lift m to a right M-invariant measure on $\Gamma \backslash G$. It can be checked that the BMS measures are invariant with respect to the geodesic flow or the frame flow as appropriate, that is they are right A-invariant. We denote the right A-invariant subset $\Omega = \mathrm{supp}(\mathsf{m}) \subset \Gamma\backslash G/M$ which is compact since Γ is convex-cocompact.

4.2 Markov sections and symbolic dynamics

4.2.1 Markov sections

In this subsection we recall how we can use a Markov section on the non-wandering set

$$\Omega \subset T^1(X) \cong \Gamma\backslash G/M,$$

to obtain a symbolic coding for the geodesic flow on Ω. The existence of such a Markov section was shown by Bowen and Ratner in [Bow70, Rat73b].

We denote the leaves of the strong unstable and strong stable foliations though a point $x \in T^1(X)$ by

$$W^{\mathrm{su}}(x) := \left\{ y \in T^1(X) \ : \ \lim_{t\to-\infty} d(xa_t, ya_t) = 0 \right\} \qquad \text{and}$$
$$W^{\mathrm{ss}}(x) := \left\{ y \in T^1(X) \ : \ \lim_{t\to\infty} d(xa_t, ya_t) = 0 \right\},$$

respectively. Further, we denote by

$$W^{\mathrm{su}}_{\epsilon}(x) : = \{ y \in W^{\mathrm{su}}(x) \ : \ \text{for all } t \le 0,\ d(xa_t, ya_t) < \varepsilon \} \quad \text{and}$$
$$W^{\mathrm{ss}}_{\epsilon}(x) : = \{ y \in W^{\mathrm{ss}}(x) \ : \ \text{for all } t \ge 0,\ d(xa_t, ya_t) < \varepsilon \},$$

the open balls in $W^{\mathrm{su}}(x)$, $W^{\mathrm{ss}}(x)$ respectively, centred at x and of radius $\epsilon > 0$. Recall that the weak unstable and stable foliations are given by

$$W^{\mathrm{wu}}(x) := \bigcup_{t\in\mathbb{R}} W^{\mathrm{su}}(x)a_t \quad \text{and} \quad W^{\mathrm{ws}}(x) := \bigcup_{t\in\mathbb{R}} W^{\mathrm{ss}}(x)a_t.$$

The hyperbolicity of the geodesic flow restricted on its non-wandering set Ω provides a constant $C_{hyp} > 0$ such that for all $x \in T^1(X)$ and all $t \ge 0$

$$d_{\mathrm{su}}(ua_{-t}, va_{-t}) \le C_{hyp} e^{-t} d_{\mathrm{su}}(u,v) \quad \text{if } u, v \in W^{\mathrm{su}}(x) \quad \textit{and}$$
$$d_{\mathrm{ss}}(ua_t, va_t) \le C_{hyp} e^{-t} d_{\mathrm{ss}}(u,v) \quad \text{if } u, v \in W^{\mathrm{ss}}(x).$$

Here d_{su}, d_{ss} denote the induced Riemannian metrics on $W^{\mathrm{su}}(x)$, $W^{\mathrm{ss}}(x)$ respectively. We now recall the Bowen bracket notation. There exist $\epsilon_0, \epsilon_0' > 0$ such that for all $x \in T^1(X)$, $u \in W^{\mathrm{wu}}_{\epsilon_0}(x)$ and $s \in W^{\mathrm{ss}}_{\epsilon_0}(x)$, there exists a unique intersection denoted by

$$[u, s] = W^{\mathrm{ss}}_{\epsilon_0'}(u) \cap W^{\mathrm{wu}}_{\epsilon_0'}(s), \tag{4.2.1}$$

and moreover, $[\cdot, \cdot]$ defines a homeomorphism from $W^{\mathrm{wu}}_{\epsilon_0}(x) \times W^{\mathrm{ss}}_{\epsilon_0}(x)$ onto its image [Rat73b]. We call a subset $U \subset W^{\mathrm{su}}_{\epsilon_0}(x) \cap \Omega$ and a subset $S \subset W^{\mathrm{ss}}_{\epsilon_0}(x) \cap \Omega$ for some $x \in \Omega$ *proper* if $U = \overline{\mathrm{int}^{\mathrm{su}}(U)}^{\mathrm{su}}$ and $S = \overline{\mathrm{int}^{\mathrm{ss}}(S)}^{\mathrm{ss}}$, where the interiors and closures are taken in the topology of $W^{\mathrm{su}}(x)$ and $W^{\mathrm{ss}}(x)$ respectively. We will often drop the superscripts henceforth and include them whenever further clarity in notation is required. For any $\delta > 0$ and proper sets $U \subset W^{\mathrm{su}}_{\epsilon_0}(x) \cap \Omega$ and $S \subset W^{\mathrm{ss}}_{\epsilon_0}(x) \cap \Omega$ containing some $x \in \Omega$, we call

$$R = [U, S] = \{[u, s] \in \Omega : u \in U, s \in S\} \subset \Omega,$$

a *rectangle of size* δ if $\mathrm{diam}_{d_{\mathrm{su}}}(U), \mathrm{diam}_{d_{\mathrm{ss}}}(S) \leq \delta$, and we call x the *centre* of R. For any rectangle $R = [U, S]$, we generalise the notation and define $[v_1, v_2] = [u_1, s_2]$ for all $v_1 = [u_1, s_1] \in R$ and $v_2 = [u_2, s_2] \in R$.

Complete set of rectangles Let $\delta > 0$ and $m \in \mathbb{N}$. A set

$$\mathcal{R} = \{R_1, R_2, \dots, R_m\} = \Big\{[U_1, S_1], [U_2, S_2], \dots, [U_m, S_m]\Big\},$$

consisting of rectangles in Ω is called a *complete set of rectangles of size* δ if

1. $R_j \cap R_k = \varnothing \quad$ for all $1 \leq j, k \leq m$ with $j \neq k$,
2. $\mathrm{diam}_{d_{\mathrm{su}}}(U_j), \mathrm{diam}_{d_{\mathrm{ss}}}(S_j) \leq \delta \quad$ for all $1 \leq j \leq m$,
3. $\Omega = \bigcup_{j=1}^{m} \bigcup_{t \in [0,\delta]} R_j a_t$.

Henceforth, we fix

$$0 < \delta < \min\{1, \epsilon_0, \epsilon_0', \mathrm{inj}(T^1(X))\}, \tag{4.2.2}$$

where $\mathrm{inj}(T^1(X))$ denotes the injectivity radius of $T^1(X)$ and where ϵ_0 and ϵ_0' are from (4.2.1). We also fix $\mathcal{R} = \{R_1, R_2, \dots, R_m\} = \{[U_1, S_1], [U_2, S_2], \dots, [U_m, S_m]\}$ to be a complete set of rectangles of size δ in Ω. We set

$$R = \bigsqcup_{j=1}^{m} R_j \quad \text{and} \quad U = \bigsqcup_{j=1}^{m} U_j.$$

We introduce the distance function d on U defined by

$$d(u,v) = \begin{cases} d_{\mathrm{su}}(u,v) & \text{if } u, v \in U_j \text{ for some } 1 \leq j \leq m, \\ 1 & \text{otherwise.} \end{cases}$$

We will use d_{su} whenever further clarity is required. Denote by $\tau : R \to \mathbb{R}^+$ the Poincaré first return time map defined by

$$\tau(x) = \inf\{t > 0 : xa_t \in R\} \qquad \text{for all } x \in R.$$

Note that τ is *constant* on $[u, S_j]$ for all $u \in U_j$ and $1 \leq j \leq m$ (see Lemma 4.3.1). So abusing notation we can by collapsing the stable leaves assume that the Poincaré first return time map is defined on U and consider it as the map $\tau : U \to \mathbb{R}^+$.

Let $P : R \to R$ be the Poincaré first return map defined by

$$P(x) = xa_{\tau(x)} \qquad \text{for all } x \in R.$$

Again by abusing notation we will also write $P : U \to U$ for the projection of the Poincaré first return map on the unstable leaves so that

$$P(u) = u\, a_{\tau(u)} \qquad \text{for all } u \in U.$$

Define the *cores*

$$\hat{R} = \{x \in R : P^k(x) \in \mathrm{int}(R) \text{ for all } k \in \mathbb{Z}\} \quad \text{and}$$
$$\hat{U} = \{u \in U : P^k(u) \in \mathrm{int}(U) \text{ for all } k \in \mathbb{Z}_{\geq 0}\},$$

which are residual subsets of R and U respectively.

Markov section We call a complete set of rectangles $\mathcal{R} = \{R_1, \dots, R_m\}$ of size $\delta > 0$ a *Markov section* if they satisfy the *Markov property*, that is for all $1 \le j,\ k \le m$ such that

$$\operatorname{int}(R_j) \cap P^{-1}(\operatorname{int}(R_k)) \neq \varnothing,$$

and all x in this intersection we have that

$$[\operatorname{int}(U_k), P(x)] \subset P([\operatorname{int}(U_j), x]) \qquad \text{and} \qquad P([x, \operatorname{int}(S_j)]) \subset [P(x), \operatorname{int}(S_k)].$$

The existence of Markov sections of arbitrarily small size for hyperbolic flows was proved by Bowen and Ratner [Bow70, Rat73b]. Thus, for the rest of this section we fix a Markov section $\mathcal{R} = \{R_1, \dots, R_m\}$ of size $\delta > 0$ for the non-wandering set Ω.

4.2.2 Symbolic dynamics

Let $\mathcal{A} = \{1, 2, \dots, m\}$ be the *alphabet* for the coding corresponding to the Markov section considered above. Define the $m \times m$ *transition matrix* T by

$$T_{(j,k)} = \begin{cases} 1 & \text{if } \operatorname{int}(R_j) \cap P^{-1}(\operatorname{int}(R_k)) \neq \varnothing, \\ 0 & \text{otherwise.} \end{cases} \qquad \text{for all } 1 \le j,\ k \le m.$$

The transition matrix T is *topologically mixing* [Rat73b, Theorem 4.3], that is there exists $n \in \mathbb{N}$ such that all the entries of T^n are positive. This definition is equivalent to the one in [Rat73b] in the setting of Markov sections. Define the spaces of bi-infinite and infinite *admissible sequences* by

$$\Sigma = \{(\dots, x_{-1}, x_0, x_1, \dots) \in \mathcal{A}^{\mathbb{Z}} : T_{(x_j, x_{j+1})} = 1 \text{ for all } j \in \mathbb{Z}\} \quad \text{and}$$
$$\Sigma^+ = \{(x_0, x_1, \dots) \in \mathcal{A}^{\mathbb{Z}_{\ge 0}} : T_{(x_j, x_{j+1})} = 1 \text{ for all } j \in \mathbb{Z}_{\ge 0}\},$$

respectively and let σ denote the shift map on Σ or Σ^+. We will use the term *admissible words* for finite sequences whenever all transitions are allowed from T. For any $\theta \in (0, 1)$, we can endow Σ with the distance function d_θ defined by $d_\theta(x, y) = \theta^{\inf\{|j| \in \mathbb{Z}_{\ge 0} : x_j \neq y_j\}}$ for all $x, y \in \Sigma$. We can similarly endow Σ^+ with a distance function which we also denote by d_θ.

For all $k \in \mathbb{Z}_{\geq 0}$ and for all admissible words $w = (w_0, w_1, \dots, w_{k-1})$, we define the corresponding *cylinder* to be

$$C[w] = \{u \in U : P^j(u) \in \operatorname{int}(U_{w_j}) \text{ for all } 0 \leq j \leq k-1\}$$

with *length* $\operatorname{len}(C[w]) = k$.

Although $P : U \to U$ and $\tau : U \to \mathbb{R}^+$ are not even continuous, we note that for all admissible pairs (j,k), the restricted maps $P|_{C[j,k]} : C[j,k] \to \operatorname{int}(U_k)$, $(P|_{C[j,k]})^{-1} : \operatorname{int}(U_k) \to C[j,k]$, and $\tau|_{C[j,k]} : C[j,k] \to \mathbb{R}$ are Lipschitz in our setting.

There exist natural continuous surjections $\zeta : \Sigma \to R$ and $\zeta^+ : \Sigma^+ \to U$ defined by

$$\zeta(x) \in \bigcap_{j=-\infty}^{\infty} \overline{P^{-j}(\operatorname{int}(R_{x_j}))} \quad \text{for all } x \in \Sigma \quad \text{and}$$
$$\zeta^+(x) \in \bigcap_{j=0}^{\infty} \overline{P^{-j}(\operatorname{int}(U_{x_j}))} \quad \text{for all } x \in \Sigma^+.$$

Set $\hat{\Sigma} = \zeta^{-1}(\hat{R})$ and $\hat{\Sigma}^+ = (\zeta^+)^{-1}(\hat{U})$. Then restricting our projection maps to $\zeta|_{\hat{\Sigma}} : \hat{\Sigma} \to \hat{R}$ and $\zeta^+|_{\hat{\Sigma}^+} : \hat{\Sigma}^+ \to \hat{U}$ we obtain continuous projections that are bijective and satisfy

$$\zeta|_{\hat{\Sigma}} \circ \sigma|_{\hat{\Sigma}} = P|_{\hat{R}} \circ \zeta|_{\hat{\Sigma}} \qquad \text{and} \qquad \zeta^+|_{\hat{\Sigma}^+} \circ \sigma|_{\hat{\Sigma}^+} = P|_{\hat{U}} \circ \zeta^+|_{\hat{\Sigma}^+}.$$

For $\theta \in (0,1)$ sufficiently close to 1, the maps ζ and ζ^+ are Lipschitz [Bow73, Lemma 2.2] with some Lipschitz constant $C_\theta > 0$. We now fix θ to be any such constant. Let $C^{\operatorname{Lip}(d_\theta)}(\Sigma, \mathbb{R})$ denote the space of Lipschitz functions $f : \Sigma \to \mathbb{R}$. We use similar notations for domain space Σ^+ or target space $\mathbb{C}$.

Since $(\tau \circ \zeta)|_{\hat{\Sigma}}$ and $(\tau \circ \zeta^+)|_{\hat{\Sigma}^+}$ are Lipschitz, there exist unique Lipschitz extensions $\tau_\Sigma : \Sigma \to \mathbb{R}^+$ and $\tau_{\Sigma^+} : \Sigma^+ \to \mathbb{R}^+$ respectively. Note that the resulting maps are distinct from $\tau \circ \zeta$ and $\tau \circ \zeta^+$ because they may differ precisely on $x \in \Sigma$ for which $\zeta(x) \in \partial(C)$ and $x \in \Sigma^+$ for which $\zeta^+(x) \in \partial(C)$ respectively, for some cylinder $C \subset U$ with $\operatorname{len}(C) = 1$. Then the previous properties extend to

$$\zeta(\sigma(x)) = \zeta(x) a_{\tau_\Sigma(x)} \quad \text{for all } x \in \Sigma \quad \text{and}$$
$$\zeta^+(\sigma(x)) = (\zeta^+(x) a_{\tau_{\Sigma^+}(x)}) \quad \text{for all } x \in \Sigma^+.$$

Word length for closed geodesics Recall that $\mathcal{G}$ denotes the set of primitive closed geodesics which are in one-to-one correspondence with the primitive periodic orbits of the geodesic flow. Every periodic orbit $\gamma \in \mathcal{G}$ for the geodesic flow a_t corresponds to a periodic orbit $\{u, P(u), \ldots, P^{n-1}(u)\}$ with $P^n(u) = u$ for the Poincaré first return map $P : U \to U$. In fact, this correspondence is possibly non-unique when the periodic orbit γ is passing through the boundaries of the rectangles of our Markov section $\mathcal{R}$. If $\{u, P(u), \ldots, P^{n-1}(u)\}$ is unique then we define

$$|\gamma|_{\mathcal{R}} = n,$$

that is the period of u. Otherwise, we choose $|\gamma|_{\mathcal{R}}$ to be equal to the smallest period of all the P-orbits corresponding to γ. Crucially, we have the identity

$$l(\gamma) = \tau^n(u) := \sum_{i=0}^{n-1} \tau(P^i(u)),$$

where $\{u, P(u), \ldots, P^{n-1}(u)\}$ is any P-orbit corresponding to γ with period equal to $|\gamma|_{\mathcal{R}}$.

Asymptotic counting By the Perron-Frobenius Theorem, the transition matrix T has a positive eigenvalue $\lambda > 1$, with all the other eigenvalues having strictly smaller modulus. Furthermore, λ is related to the topological entropy $h(\sigma)$ of $\sigma : \Sigma^+ \to \Sigma^+$ since $h(\sigma) = \log \lambda$. The number of periodic points of period n of σ is given by

$$\# \operatorname{Fix}_n(\sigma) := \{x \in \Sigma \ : \ \sigma^n(x) = x\} = \operatorname{trace}(T^n) = \lambda^n + O((\theta_0 \lambda)^n),$$

where $0 < \theta_0 < 1$. Given any σ-invariant probability measure ν on Σ^+, we may define its entropy $h_\sigma(\nu)$. This always satisfies $h_\sigma(\nu) \leq h(\sigma)$ and there is a unique σ-invariant probability measure μ_0, called the measure of maximal entropy, for which $h(\mu_0) = h(\sigma)$.

In particular, the topological entropy of $P : U \to U$ satisfies $h(P) = h(\sigma) = \log \lambda$. The topological entropy gives the exponential growth rate of periodic points for P. More precisely, if we write $\operatorname{Fix}_n(P) = \{u \in U \ : \ P^n(u) = u\}$ then there exists $0 < \theta_1 < 1$ such that

$$\# \operatorname{Fix}_n(P) = \lambda^n + O((\theta_1 \lambda)^n). \tag{4.2.3}$$

This next result is due to Bowen [Bow73].

Lemma 4.2.1. *There exists* $0 < \theta_2 < 1$ *such that*

$$\#\{\gamma \in \mathcal{G} \ : \ |\gamma|_{\mathcal{R}} = n\} = \frac{\#\operatorname{Fix}_n(P)}{n} + O((\theta_2\lambda)^n). \tag{4.2.4}$$

The difference between counting closed geodesics of word length n and the number of periodic orbits of length n described above does not cause a problem for our analysis. This follows from [Bow73, Theorem 6.1]. In particular, using (4.2.3) and (4.2.4) we obtain that

$$\#\{\gamma \in \mathcal{G} \ : \ |\gamma|_{\mathcal{R}} = n\} \sim \frac{\lambda^n}{n}, \qquad \text{as } n \to \infty.$$

4.3 Holonomy and representation theory

In this section, we define the holonomy map which is required in addition to the Markov section to proceed with our proof. Since the holonomy map will be defined below as an M-valued function, we will naturally for our purposes need to also consider $L^2(M, \mathbb{C})$ and so in addition we introduce the required representation theory.

4.3.1 Definition of the Holonomy

Unlike the case of the geodesic flow, we do not have a Markov section available for the frame flow. Instead, we consider the Markov section $\mathcal{R}$ chosen for $\Omega \subset T^1(X)$ the non-wandering set of the geodesic flow and then choose a smooth section F on R for the frame bundle $F(X)$ over $T^1(X)$. Let w_j be the centre of R_j for all $j \in \mathcal{A}$. Below we define a smooth section

$$F : \bigsqcup_{j=1}^{m} [W^{su}_{\varepsilon_0}(w_j), W^{ss}_{\varepsilon_0}(w_j)] \to F(X),$$

where without loss of generality we assume ε_0 is sufficiently small so that the union is indeed a disjoint union.

To construct a smooth section we start by choosing arbitrary frames

$$F(w_j) \in F(X),$$

based at the tangent vectors $w_j \in T^1(X)$ for all $j \in \mathcal{A}$. (Recall that for each $j \in \mathcal{A}$ we fixed w_j to be the centre of the rectangle R_j of our fixed Markov section for Ω.) Then we extend the section F by requiring that for all $j \in \mathcal{A}$ and $u_1, u_2 \in W^{su}_{\varepsilon_0}(w_j)$, the frames $F(u_1)$ and $F(u_2)$ are backwards asymptotic, that is

$$\lim_{t \to -\infty} d(F(u_1)\, a_t\,,\, F(u_2)\, a_t) = 0.$$

Recalling the definitions of the expanding and contacting horospherical subgroups from (4.1.1) we must have

$$F(u_1) = F(u_2)h^+,$$

for some unique $h^+ \in H^+$. We complete the construction by further requiring that for all $j \in \mathcal{A}$, $u \in W^{su}_{\varepsilon_0}(w_j)$, and $s_1, s_2 \in W^{ss}_{\varepsilon_0}(w_j)$, we have that the frames $F([u, s_1])$ and $F([u, s_2])$ are forwards asymptotic. In particular, this implies that since

$$\lim_{t \to +\infty} d(F([u, s_1])a_t,\, F([u, s_2])a_t) = 0,$$

we must have

$$F([u, s_2]) = F([u, s_1])h^-,$$

for some unique $h^- \in H^-$.

Holonomy The holonomy is a map $\theta : R \to M$ such that for all $x \in R$, we have

$$F(x)\ a_{\tau(x)} = F(P(x))\, \theta(x)^{-1}.$$

Just as τ is constant on the strong stable leaves of the rectangles, we show below that the same is true for θ. This allows us to work solely on the union of unstable leaves U.

Lemma 4.3.1. *For all $j \in \mathcal{A}$ and all $u \in U_j$ the Poincaré first return time map $\tau : R \to \mathbb{R}^+$ and the holonomy map $\theta : R \to M$ are constant on $[u, S_j]$.*

Proof. Let $j \in \mathcal{A}$ and $u \in U_j$. Let $s_1, s_2 \in S_j$ and set $u_i = [u, s_i]$ for $i = 1, 2$. We have that

$$\begin{aligned} d(P(u_1)\, a_t, P(u_2)\, a_t) &= d(u_1\, a_{t+\tau(u_1)}, u_2\, a_{t+\tau(u_2)}) \\ &\geq d(u_1\, a_{t+\tau(u_1)}, u_1\, a_{t+\tau(u_2)}) - d(u_1\, a_{t+\tau(u_2)}, u_2\, a_{t+\tau(u_2)}), \end{aligned}$$

for all $t \in \mathbb{R}$. Now since $u_1, u_2 \in W^{ss}_{\varepsilon_0'}(u)$ are in the same strong stable leaf we get that

$$\lim_{t\to\infty} d(u_1\, a_{t+\tau(u_2)}, u_2\, a_{t+\tau(u_2)}) = 0.$$

The structure of the Markov partition gives that $P(u_i) \in W^{ss}_{\varepsilon_0'}(P(u))$ for $i = 1, 2$ and so we also obtain that

$$\lim_{t\to\infty} d(P(u_1)\, a_t, P(u_2)\, a_t) = 0.$$

It then follows from the inequality above that $\tau(u_1) = \tau(u_2)$.

For the holonomy θ, recall that by construction of the smooth section we have that $F(u_2) = F(u_1)h^-$ for some $h^- \in H^-$. From the definition of the holonomy map, we have

$$F(P(u_1)) = F(u_1)\, a_{\tau(u_1)}\theta(u_1) \text{ and } F(P(u_2)) = F(u_2)\, a_{\tau(u_2)}\theta(u_2) = F(u_1)h^-\, a_{\tau(u_1)}\theta(u_2),$$

since $\tau(u_1) = \tau(u_2)$. We then have that

$$\begin{aligned} d\left(F(P(u_1))\, a_t, F(P(u_2))\, a_t\right) &= d\left(F(u_1)\, a_{\tau(u_1)+t}\theta(u_1), F(u_1)h^-\, a_{\tau(u_1)+t}\theta(u_2)\right) \\ &\geq d\left(F(u_1)a_{\tau(u_1)+t}\theta(u_1), F(u_1)\, a_{\tau(u_1)+t}\theta(u_2)\right) \\ &\qquad - d\left(F(u_1)a_{\tau(u_1)+t}\theta(u_2), F(u_1)h^- a_{\tau(u_1)+t}\theta(u_2)\right), \end{aligned}$$

for all $t \geq 0$. Now since $h^- \in H^-$ is an element of the contracting horospherical group we have that

$$\lim_{t\to\infty} d(F(u_1)\, a_{\tau(u_1)+t}\theta(u_2), F(u_1)h^-\, a_{\tau(u_1)+t}\theta(u_2)) = 0.$$

Moreover, the structure of the Markov partition implies that $P(u_1), P(u_2) \in W^{ss}_{\varepsilon_0'}(P(u))$. Our choice of the smooth section then gives that the frames $F(P(u_1))$ and $F(P(u_2))$ are forwards asymptotic, that is

$$\lim_{t\to\infty} d(F(P(u_1))\, a_t, F(P(u_2))a_t) = 0.$$

It thus follows from the inequality above that $\theta(u_1) = \theta(u_2)$.

$\square$

Let $R^* \subset F(X)$ be the subset of frames over the set $R = \bigsqcup_{j=1}^m R_j$ and similarly define U^*. Via the section $F : \bigsqcup_{j=1}^m \left[W^{su}_{\varepsilon_0}(w_j), W^{ss}_{\varepsilon_0}(w_j)\right] \to F(X)$, we have the natural identifications

$R^* \cong R \times M$ and $U^* \cong U \times M$. Specifically, given $(x, m) \in R \times M$ we associate it to the frame based at $x \in R$ in the following way

$$F(x)\, g\, m\, g^{-1} \in R^*,$$

where g is the unique element in G that sends the frame $F(x)$ to the fixed reference frame F_0. In the following chapter we will need to deal with the function space $C(U^*, \mathbb{C})$. It is convenient to consider this function space as a subspace of $C(U, L^2(M, \mathbb{C}))$ in the following way

$$C(U^*, \mathbb{C}) \cong C(U \times M, \mathbb{C}) \cong C(U, C(M, \mathbb{C})) \subset C(U, L^2(M, \mathbb{C})).$$

4.3.2 Representation Theory

In this subsection we recall some standard results and definitions on the representation theory of compact Lie groups that we will use in our analysis. The aim is to present how we can obtain a Fourier decomposition of square integrable functions on compact Lie groups using the Peter–Weyl Theorem.

Unitary representations of compact Lie groups

Let G be a compact Lie group and let $L^2(G) = L^2(G, \mathbb{C})$ denote the space of (equivalences classes of) square integrable functions from G to $\mathbb{C}$, where integration is with respect to the Haar measure. Recall that $L^2(G)$ is universal is the sense that, up to isometric isomorphism, there is a unique (separable) complex Hilbert space of each dimension $d \in \mathbb{N} \cup \{\infty\}$. For a complex Hilbert space H, $U(H)$ will denote the space of unitary operators on H. If $H \cong \mathbb{C}^d$ (d finite) then we write $U(\mathbb{C}^d) = U(d)$ (i.e. $d \times d$ unitary matrices once we fix a basis). A unitary representation of G is a continuous homomorphism from G to $U(G)$ for some Hilbert space H.

The left regular representation of G is

$$\lambda : G \to U(L^2(M, \mathbb{C})),$$

defined by

$$\lambda(g)(\phi)(x) = \phi(g^{-1}x),$$

for all $g \in G$, $\phi \in L^2(G, \mathbb{C})$ and $x \in G$. As we shall describe, the left regular representation λ may be decomposed into a sum of irreducible finite dimensional unitary representations.

We call a unitary representation $\pi : G \to U(H)$ *irreducible* if H has no closed subspace invariant under $\pi(G)$. Further, we call two unitary representations $\pi_1 : G \to U(H_1)$ and $\pi_2 : G \to U(H_2)$ *unitarily equivalent* if there exists a unitary operator $\Psi : H_1 \to H_2$ such that for all $g \in G$

$$\pi_1(g) = \Psi \, \pi_2(g) \, \Psi^*,$$

where Ψ^* is the adjoint operator of Ψ. We define the *unitary dual* of G, denoted by $\widehat{G}$ to be the set of equivalence classes of irreducible unitary representations and by a slight abuse of notation denote the equivalence classes again by π.

For a *finite dimensional* irreducible representation $\pi : G \to U(H_\pi)$ we write $\dim(\pi)$ for the dimension of H_π. Denote the trivial irreducible representation by $1 \in \widehat{G}$ and observe that it is one-dimensional.

In fact, the Peter–Weyl Theorem [PW27] guarantees that every irreducible representation of G is finite dimensional. Moreover, we obtain an orthogonal Hilbert space decomposition of the set of square integrable functions given by

$$L^2(G, \mathbb{C}) \cong \widehat{\bigoplus}_{\pi \in \widehat{G}} H_\pi^{\oplus \dim(\pi)}, \tag{4.3.1}$$

where $H^{\oplus \dim(\pi)} = \underbrace{H_\pi \oplus \cdots \oplus H_\pi}_{\dim(\pi) \text{ times}}$ and $\widehat{\bigoplus}$ denotes the closure of the infinite direct sum. This decomposition corresponds to the decomposition of the left regular unitary representation given by

$$\lambda = \widehat{\bigoplus}_{\pi \in \widehat{G}} \pi^{\oplus \dim(\pi)},$$

where $\pi^{\oplus \dim(\pi)} = \underbrace{\pi \oplus \cdots \oplus \pi}_{\dim(\pi) \text{ times}}$ maps G onto $U(H_\pi^{\oplus \dim(\pi)})$.

Matrix coefficients and class functions

As above let $\pi : G \to U(H_\pi)$ be an irreducible unitary representation for G. Denote the inner product of H_π by $\langle .,. \rangle_\pi$ and its dimension by $\dim(\pi)$. (Recall that $\dim(\pi) \in \mathbb{N}$.)

We fix an orthonormal basis $\{e_1^\pi, \ldots, e_{\dim(\pi)}^\pi\}$ for H_π and we write, for each group element $g \in G$, $\pi(g) \in U(H_\pi)$ as a unitary matrix of dimension $\dim(\pi)$. The (i, j)-coordinate

$\phi_{ij}^{(\pi)}(g)$ of this matrix with respect to the basis is given by

$$\phi_{ij}^{(\pi)}(g) = \langle \pi(g)e_i^\pi, e_j^\pi \rangle_\pi.$$

We call these functions $\phi_{ij}^\pi : G \to \mathbb{C}$ *matrix coefficients*. They are continuous functions and hence in $L^2(G, \mathbb{C})$.

For each equivalence class π, we think of

$$\operatorname{span}\{\phi_{ij}^\pi \,:\, 1 \leq i, j \leq \dim(\pi)\},$$

as the finite dimensional subspace of $L^2(G, \mathbb{C})$ corresponding to $H_\pi^{\oplus \dim(\pi)}$ through the isomorphism in (4.3.1).

In particular, the Peter–Weyl Theorem [PW27] gives that the set

$$\bigcup_{\pi \in \widehat{G}} \left\{ \sqrt{\dim(\pi)}\, \phi_{ij}^\pi \,:\, 1 \leq i, j \leq \dim(\pi) \right\},$$

is an orthonormal basis for $L^2(G, \mathbb{C})$.

Further the function $\chi_\pi : G \to \mathbb{C}$ given by

$$\chi_\pi(g) = \operatorname{tr}(\pi(g)),$$

is called the *character* of π. It is well defined as, it only depends on the equivalence class of π and crucially it does not depend on the choice of basis for H_π. In terms of matrix coefficients fixing any basis for H_π we have that

$$\chi_\pi(g) = \operatorname{tr}(\pi(g)) = \sum_{i=1}^{\dim(\pi)} \phi_{ii}^\pi(g).$$

A function $f : G \to \mathbb{C}$ is called a *class function* if it is constant on conjugacy classes of G, that is

$$f(gxg^{-1}) = f(x) \quad \text{for all } x, g \in G.$$

Since conjugate matrices have the same trace it is clear that characters are class functions. Denote the closed subspace of $L^2(G)$ consisting of class functions by $L^2_C(G)$. Specifically, the Peter–Weyl Theorem [PW27] gives that the set

$$\{\chi_\pi \ : \ \pi \in \widehat{G}\},$$

is an orthonormal basis for $L^2_C(G)$. Thus we can expand a class function $f \in L^2_C(G, \mathbb{C})$ in the form

$$f = \sum_{\pi \in \widehat{G}} \langle f, \chi_\pi \rangle_{L^2} \ \chi_\pi,$$

where $\langle f, \chi_\pi \rangle_{L^2} = \int_G f(g)\, \overline{\chi_\pi(g)}\, dg$ and dg is the Haar probability measure of G and the convergence is in the L^2-norm.

Finally, let $C(G) \subset L^2(G)$ denote the space of continuous functions from G to $\mathbb{C}$ equipped with the uniform norm. As before, let $C_C(G)$ denote the closed subspace of $C(G)$ consisting of continuous class functions. We have the following result [PW27].

Theorem 4.3.2. *The set*

$$\operatorname{span} \left\{ \chi_\pi \ : \ \pi \in \widehat{G} \right\},$$

is uniformly dense in $C_C(G)$.

The result above will be particularly helpful in the final chapter where we will use the characters of unitary irreducible representations of M to approximate the indicator functions of the target sets for the holonomies which are class functions but not continuous. These target sets for the holonomies will be introduced in the following subsection.

Tensored unitary representation

To prove our results in the following chapter we will consider a family of transfer operators depending on a complex parameter $s \in \mathbb{C}$ and on irreducible unitary representations $\lambda \in \widehat{M}$. It turns out that the spectral radii of transfer operators in this family corresponding to complex parameters $s \in \mathbb{C}$ with large imaginary values or non-trivial irreducible unitary representations $\lambda \in \widehat{M}$ will enjoy some nice decay bounds which we discuss in the next chapter. For this reason it will be convenient to combine these two parameters of this family of transfer operators into a single parameter.

We thus consider for all $b \in \mathbb{R}$ and $\lambda \in \widehat{M}$ the tensored unitary representation

$$\lambda_b : AM \to U(H_\lambda),$$

by

$$\lambda_b(a_t m)(z) = e^{-ibt}\lambda(m)(z) \quad \text{for all } z \in H_\lambda,\ t \in \mathbb{R},\ \text{and } m \in M.$$

We introduce some notations related to Lie algebras. We denote Lie algebras corresponding to Lie groups by the corresponding Fraktur letters, e.g.,

$$\mathfrak{a} = T_e(A),\ \mathfrak{m} = T_e(M),\ \mathfrak{h}^+ = T_e(H^+) \quad \text{and} \quad \mathfrak{h}^- = T_e(H^-).$$

For any left regular unitary representation $\lambda : M \to U(H)$ for some Hilbert space H , we denote the differential at $e \in M$ by $d\lambda = (d\lambda)_e : \mathfrak{m} \to \mathfrak{u}(H)$, and define the norm

$$\|\lambda\| = \sup_{\substack{z \in \mathfrak{m} \\ \text{such that } \|z\|=1}} \|d\lambda(z)\|_{\text{op}},$$

and similarly for any tensored unitary representation $\lambda : AM \to U(H)$.

Remark 4.3.3. The norms remain the same if we replace H_λ with $H_\lambda^{\oplus \dim(\lambda)}$ since the M-action is identical across all components.

The following lemma appeared in [SW21, Lemma 4.3] and records some useful properties of the norms defned above.

Lemma 4.3.4. *For all $b \in \mathbb{R}$ and $\lambda \in \widehat{M}$, we have that*

$$\sup_{a\in A, m \in M} \sup_{\substack{z \in T_{am}(AM) \\ \textit{such that } \|z\|=1}} \|(d\lambda_b)_{am}(z)\|_{\text{op}} = \|\lambda_b\|,$$

and

$$\max(|b|, \|\lambda\|) \leq \|\lambda_b\| \leq |b| + \|\lambda\|.$$

We complete this subsection by fixing some constants and with the following definition. For $\beta > 0$ set

$$\widehat{M}_\beta = \{(b, \lambda) \in \mathbb{R} \times \widehat{M} : |b| > \beta \text{ or } \lambda \neq 1\}.$$

This set contains pairs of real numbers b, for our complex parameter $s = a + ib \in \mathbb{C}$, and irreducible unitary representations $\lambda \in \widehat{M}$. The pairs contained in a set $\widehat{M}_\beta$ for large enough $\beta > 0$ will be exactly the pairs for which we will use Dolgopyat's method, as adapted by Sarkar and Winter in [SW21], to obtain some decay estimates for the spectral radii of transfer operators corresponding to these pairs of parameters.

Finally, we fix some related constants. Observe that since M is constant

$$\delta_{\widehat{M}} := \inf_{b\in\mathbb{R},\, \lambda\in\widehat{M}} \|\lambda_b\| = \inf_{\lambda\in\widehat{M}} \|\lambda\| > 0.$$

Furthermore, we can deduce that from Lemma 4.3.4 that

$$\inf_{(b,\lambda)\in\widehat{M}_1} \|\lambda_b\| \geq \min\left(1, \delta_{\widehat{M}}\right).$$

Hence we fix $\delta_{1,\widehat{M}} = \min\left(1, \delta_{\widehat{M}}\right) > 0$. We complete this subsection with the following lemma which will only be used in the Appendix.

Lemma 4.3.5. *There exists $\delta > 0$ such that for all $b \in \mathbb{R}$, $\lambda \in \widehat{M}$ and $\omega \in H_\lambda^{\oplus \dim(\lambda)}$ with $\|\omega\|_2 = 1$, there exists $z \in \mathfrak{a} \oplus \mathfrak{m}$ with $\|z\| = 1$ such that $\|d\lambda_b(z)(\omega)\|_2 \geq \delta\|\lambda_b\|$.*

Fix $\varepsilon_1 > 0$ to be the δ provided by Lemma 4.3.5.

4.3.3 Target sets for the holonomies

To each closed geodesic we can associate a holonomy element h_γ by parallel transport which corresponds to a conjugacy class in $M \cong \mathrm{SO}(N-1)$. More precisely, let $\gamma \in \mathcal{G}$ be a periodic orbit of the geodesic flow. Given our Markov section $\mathcal{R}$ we can associate to γ a periodic orbit of the Poincaré first return map $P : U \to U$, say $p = \{u, \dots, P^{n-1}(u)\}$. Fixing a point $u \in p$ in this periodic orbit we have from the definition of the holonomy map that

$$F(u)\, a_{\tau^n(u)}\theta^n(u) = F(P^n(u)) = F(u).$$

If we choose a different point in $u' \in p$ then the corresponding group element in M given by the Birkhoff product of the holonomy map $\theta^n(u')$ is conjugate to $\theta^n(u)$. We thus define the holonomy of $\gamma \in \mathcal{G}$ to be the conjugacy class $[\theta^n(u)]$ consisting of elements in M. We now proceed to define the target arcs for these holonomies. To do that we use the isomorphism

$$\iota : M \to \mathrm{SO}(N-1).$$

Since ι is an isomorphism between Lie groups it is real analytic. Moreover, consider the map

$$\underline{E} : \mathrm{SO}(N-1) \to \mathbb{C}^{N-1}/\,\mathrm{Sym}(N-1),$$

that sends an orthogonal matrix in $\mathrm{SO}(N-1)$ to the set of its eigenvalues. For each matrix in $\mathrm{SO}(N-1)$ the set of its eigenvalues consists of conjugate pairs in $\mathbb{S}^1$ plus the eigenvalue 1 when N is even (and so $N-1$ is odd). To ease notation we parametrise $\mathbb{S}^1$ by $\left[-\frac{1}{2}, \frac{1}{2}\right]$ and set $r = \lfloor \frac{N-1}{2} \rfloor$ to be the rank of $\mathrm{SO}(N-1)$. Without loss of generality for each $A \in \mathrm{SO}(N-1)$, we parametrise the set $\underline{E}(A)$ by the vector $(e_1, \dots, e_r)$ where $-\frac{1}{2} \leq e_1 \leq \cdots \leq e_r \leq \frac{1}{2}$ and $\{e^{\pm 2\pi i e_j} \,:\, 1 \leq j \leq r\}$ is the set of eigenvalues of A (excluding the eigenvalue 1 possibly, which is guaranteed when N is even).

The isomorphism ι sends conjugate elements in M to similar matrices in $\mathrm{SO}(N-1)$. Given that similar matrices have the same eigenvalues it follows that the composition of functions $E := \underline{E} \circ \iota : M \to \left[-\frac{1}{2}, \frac{1}{2}\right]^r$ is constant on each conjugacy class of M. However, this map is not even continuous at points that map close to $\{\pm\frac{1}{2}\}$. To avoid this technicality we fix an arbitrary constant $\kappa \in (0,1)$ and consider the map $E_\kappa := E|_{E^{-1}(-\kappa/2,\,\kappa/2)^r}$ restricted to the pre-image of the hypercube $(-\kappa/2, \kappa/2)^r$.

We consider a target vector $\underline{\vartheta} = (\vartheta_1, \dots, \vartheta_r)$ of rotation angles satisfying

$$-\frac{\kappa}{2} < \vartheta_1 < \cdots < \vartheta_r < \frac{\kappa}{2}.$$

Let $(k_n)_{n=1}^{\infty}$ be a sequence of real numbers in $(0,1)$ such that for all $n \in \mathbb{N}$ we have that

$$-\frac{\kappa}{2} < \vartheta_1 - \frac{k_n}{2} < \vartheta_1 + \frac{k_n}{2} < \cdots < \vartheta_r - \frac{k_n}{2} < \vartheta_r + \frac{k_n}{2} < \frac{\kappa}{2}.$$

We use these sequences to parametrise the sequence of sets

$$A'_n = \left(\vartheta_1 - \frac{k_n}{2}, \vartheta_1 + \frac{k_n}{2}\right) \times \cdots \times \left(\vartheta_r - \frac{k_n}{2}, \vartheta_r + \frac{k_n}{2}\right) \subseteq \left(-\frac{\kappa}{2}, \frac{\kappa}{2}\right)^r.$$

We can parametrise a sequence of sets $(A_n)_{n=1}^{\infty}$ containing full conjugacy classes of M in the following way. For each $n \in \mathbb{N}$ we set

$$A_n = \{m \in M \,:\, E(m) \in A'_n\}\,, \tag{4.3.2}$$

and we consider this sequence of sets as our possibly shrinking targets for the holonomies. Observe that the target sets $\left(\vartheta_i - \frac{k_n}{2}, \vartheta_i + \frac{k_n}{2}\right)$ for $1 \leq i \leq r$ were chosen to be pairwise disjoint. This implies that if a matrix $A \in \mathrm{SO}(N-1)$ has one eigenvalue in each of these sets then all its eigenvalues are simple. The Perturbation Theorem 3.2.2 then implies that our map E is analytic around A. In particular, it is smooth which is sufficient for our purposes.

4.4 Pressure function and equilibrium states

Before introducing the pressure function we discuss some quantities that appeared in Theorem 1.2.5. We begin by imposing a restriction on $\alpha \in \mathbb{R}$ and, to do this, define the set

$$\mathcal{I} := \left\{ \int \tau \, d\mu \, : \, \mu \in \mathcal{M}(U) \right\}, \tag{4.4.1}$$

where $\mathcal{M}(U)$ is the set of P-invariant Borel probability measures on U. This set of measures is convex and compact with respect to the weak* topology. Hence, the image of $\mathcal{M}(U)$ onto the reals under the continuous projection

$$\mu \to \int \tau \, d\mu,$$

is a closed interval, which we denote by $\mathcal{I}$. Since the geodesic flow is mixing, we show in Lemma 4.4.1 that $\mathcal{I}$ has non-empty interior.

For $\alpha \in \operatorname{int} \mathcal{I}$ we define

$$H(\alpha) := \sup \left\{ h_\mu(P) \, : \, \mu \in \mathcal{M}(U) \text{ with } \int \tau \, d\mu = \alpha \right\},$$

where $h_\mu(P)$ denotes the measure-theoretic entropy of $P : U \to U$ with respect to μ. There is a unique $\mu_\alpha \in \mathcal{M}(U)$ that realises this supremum above and a unique real number $a = a(\alpha)$ such that

$$h_{\mu_\alpha}(P) + a \int \tau \, d\mu_\alpha = \sup \left\{ h_\mu(P) + a \int \tau \, d\mu \, : \, \mu \in \mathcal{M}(U) \right\}.$$

We also define the variance of $\tau - \alpha$ by

$$\sigma_\alpha^2 := \lim_{n\to\infty} \frac{1}{n} \int (\tau^n - n\alpha)^2 \, d\mu_\alpha,$$

where $\tau^n(u)$ denotes the n-th Birkhoff sum or τ at $u \in U$. We show in the next subsection that the limit exists and $\sigma_\alpha^2 > 0$ along with proving some of the statements made above.

4.4.1 Pressure

For $f \in C(U, \mathbb{R})$, called the *potential*, the *pressure* is defined by

$$\Pr(f) = \sup \left\{ \int f \, d\mu + h_\mu(P) \; : \; \mu \in \mathcal{M}(U) \right\},$$

where again $\mathcal{M}(U)$ is the set of P-invariant Borel probability measures on U and $h_\mu(P)$ is the measure theoretic entropy of $P : U \to U$ with respect to μ.

For all Hölder functions f on U, there is in fact a unique P-invariant Borel probability measure μ_f on U which attains the supremum above called the *equilibrium state* of f [Bow08, Theorems 2.17 and 2.20] and it satisfies $\mu_f(\hat{U}) = 1$ [Che02, Corollary 3.2].

Given two functions $f, g \in C(U, \mathbb{R})$ we have the inequality

$$|\Pr(f) - \Pr(g)| \leq \|f - g\|_\infty. \tag{4.4.2}$$

Two functions f and g in $C(U, \mathbb{R})$ are called cohomologous if there exists a continuous function $h : U \to \mathbb{R}$ such that $f - g = h \circ P - h$. For Hölder functions f, g on U, $\mu_f = \mu_g$ if and only if $f - g$ is cohomologous to a constant. If f and g are Hölder continuous then the function $t \to Pr(tf + g)$ is real analytic and

$$\left.\frac{d \Pr(tf + g)}{dt}\right|_{t=0} = \int f \, d\mu_g, \tag{4.4.3}$$

$$\left.\frac{d^2 \Pr(tf + g)}{dt^2}\right|_{t=0} = \lim_{n\to\infty} \frac{1}{n} \int \left(f^n(x) - n \int f \, d\mu_g \right)^2 d\mu_g, \tag{4.4.4}$$

see [PP90, Propositions 4.10 and 4.11] and [Rue04]. Furthermore, as in [PP90, Proposition 4.12], if g is not cohomologous to a constant then $t \to \Pr(tf + g)$ is strictly convex and

$$\left.\frac{d^2 \Pr(tf + g)}{dt^2}\right|_{t=0} > 0. \tag{4.4.5}$$

(The references provided are for the symbolic case but all proofs follow from the spectral gap property which will appear in the following chapter.) We have the following result.

Lemma 4.4.1. *$\mathcal{I}$ is not a singleton. Further, for each $\alpha \in \mathrm{int}(\mathcal{I})$, there is a unique $a = a(\alpha) \in \mathbb{R}$ such that*

$$H(\alpha) = h_{\mu_{a\tau}}(P) \quad \textit{and} \quad \int \tau \, d\mu_{a\tau} = \alpha.$$

Proof. Similarly to the proof of Lemma 2.2.2 we can use Theorem 2.2.3 (Livšic's Theorem) to show that if $\mathcal{I}$ consists of a single point $c \in \mathbb{R}$ then the Poincaré first return time map $\tau : U \to \mathbb{R}^+$ is cohomologous to c. The contradiction in this case follows from the topological mixing of the geodesic flow. By a result of Babillot in [Bab02] the topological mixing of the geodesic flow is equivalent to the fact that the length spectrum of the geodesic flow, and so in particular the Birkhoff's sums of $\tau : U \to \mathbb{R}^+$, are not contained in a discrete subset of $\mathbb{R}$. So τ cannot be cohomologous to a constant. This shows that $\mathcal{I}$ has a non-empty interior. Moreover, since the geodesic flow is mixing and as we discussed above τ is not cohomologous to any constant we get that the function

$$\mathfrak{p} : \mathbb{R} \to \mathbb{R} \qquad \text{defined by} \qquad \mathfrak{p}(t) = \Pr(t\tau),$$

is strictly convex. Now consider the set

$$\mathcal{D} := \{\mathfrak{p}'(a) : a \in \mathbb{R}\} = \left\{ \int \tau \, d\mu_{a\tau} : a \in \mathbb{R} \right\} \subset \mathcal{I}.$$

Since $\mathfrak{p}$ is strictly convex, $\mathcal{D}$ is an open interval. By the definition of pressure, for all $\mu \in \mathcal{M}(U)$,

$$\mathfrak{p}(t) \geq h_\mu(P) + t \int \tau \, d\mu.$$

In particular, the graph of the convex function $\mathfrak{p}$ lies above a line with slope $\int \tau \, d\mu$ (possibly touching it tangentially) and so $\int \tau \, d\mu \in \overline{\mathcal{D}}$. Thus, since μ is arbitrary, $\mathrm{int}(\mathcal{I}) \subset \overline{\mathcal{D}}$, and so we have $\mathcal{D} = \mathrm{int}(\mathcal{I})$. Thus, for $\alpha \in \mathrm{int}(\mathcal{I})$, there is a unique $a = a(\alpha) \in \mathbb{R}$ with

$$\alpha = \mathfrak{p}'(a) = \int \tau \, d\mu_{a\tau}.$$

Since the map $\mu \to h_\mu(P)$ is upper semi-continuous [New89], the supremum in

$$H(\alpha) = \sup\left\{ h_\mu(P) \,:\, \mu \in \mathcal{M}(U) \text{ with } \int \tau \, d\mu = \alpha \right\},$$

is attained. Since $\mu_{a\tau}$ is the equilibrium state for $a\tau$, we have, for any $\mu \in \mathcal{M}(U)$ with $\mu \neq \mu_{a\tau}$,

$$h_{\mu_{a\tau}}(P) + a \int \tau \, d\mu_{a\tau} > h_\mu(P) + a \int \tau \, d\mu.$$

In particular, if $\int \tau \, d\mu = \alpha$ then $h_{\mu_{a\tau}}(P) > h_\mu(P)$. Therefore, $\mu_{a\tau}$ is the unique measure with the desired properties. $\square$

For the rest of this paper we fix a real number $\alpha \in \operatorname{int} \mathcal{I}$ and set $a = a(\alpha)$ to be the unique real number given from Lemma 4.4.1. Setting $\mu_\alpha = \mu_{a\tau}$, we have the measure whose existence is claimed in the beginning of this section. Furthermore,

$$\sigma_\alpha^2 = \lim_{n\to\infty} \frac{1}{n} \int (\tau^n - n\alpha)^2 \, d\mu_\alpha = \mathfrak{p}''(a) > 0,$$

where we have used that $\mu_{a\tau} = \mu_{a(\tau-\alpha)}$.

Chapter 5

Statistics for closed geodesics on convex-cocompact hyperbolic manifolds

5.1 Statement of results

We begin this chapter by presenting our results in full generality. In the following sections we proceed by providing the proofs of our theorems. Let Γ be a convex-cocompact, Zariski-dense and torsion-free discrete subgroup of $G = \mathrm{Isom}^+(\mathbb{H}^N)$ and consider the quotient hyperbolic manifold $X = \Gamma\backslash\mathbb{H}^N$. Fix a Markov partition $\mathcal{R}$ for the non-wondering subset $\Omega \subseteq T^1(X)$ and denote by $P : U \to U$ the Poincaré first return map defined on the union of unstable leaves. To each closed geodesic γ in Ω we assigned a word length with respect to $|.|_{\mathcal{R}}$ that was given as the least period of a P-orbit included in γ. Further, recall that we fixed a unique real number $\alpha \in \mathrm{int}\,\mathcal{I}$. Using Lemma 4.4.1 we have unique corresponding real numbers $H(\alpha) > 0$ and $a = a(\alpha)$ together with a corresponding probability measure for U denoted by μ_α which in fact is the $a\tau$-equilibrium state.

We fix an arbitrary constant $\kappa \in (0, 1)$. We also fix $(k_n)_{n=1}^{\infty}$ a sequence of real numbers in $(0, 1)$ and a target vector $\underline{\vartheta} = (\vartheta_1, \ldots, \vartheta_k)$ with $-\frac{\kappa}{2} < \vartheta_1 < \cdots < \vartheta_r < \frac{\kappa}{2}$ where $r = \lfloor \frac{N-1}{2} \rfloor$ is the rank of M. We use these fixed constants to parametrise a sequence of target sets $A_n \subseteq M$ for the holonomies as described in Section 4.3.3.

We also fix a sequence of intervals $(I_n)_{n=1}^{\infty}$ inside a compact subset K of $\mathbb{R}$. Recall that we call a sequence $(s_n)_{n=1}^{\infty}$ of sub-exponential growth if $\limsup_{n\to\infty} |\log s_n|/n = 0$. Writing

$$\pi_{\mathcal{R}}(n,\alpha,I_n,A_n) := \#\left\{|\gamma|_{\mathcal{R}} = n \ : \ l(\gamma) - n\alpha \in I_n \ \text{ and } \ h_\gamma \in A_n\right\},$$

we have the following theorem.

Theorem 5.1.1. *Let Γ be a convex-cocompact, Zariski-dense, torsion-free discrete subgroup of orientation preserving isometries of $\mathbb{H}^N$. Let $K \subset \mathbb{R}$ be a compact set and let $(I_n)_{n=1}^{\infty}$ be a sequence of intervals in K. Let $(A_n)_{n=1}^{\infty}$ be a sequence of sets consisting of conjugacy classes in M as above. Furthermore, suppose that $(\ell(I_n)^{-1})_{n=1}^{\infty}$ and $(k_n^{-1})_{n=1}^{\infty}$ have sub-exponential growth. Then, for each $\alpha \in \mathrm{int}(\mathcal{I})$ there exists $a \in \mathbb{R}$, $\sigma_\alpha > 0$ and $H(\alpha) > 0$ such that*

$$\pi_{\mathcal{R}}(n,\alpha,I_n,A_n) \sim \frac{k_n^r}{\sigma_\alpha\sqrt{2\pi}} \int_{I_n} e^{-ax}\,dx\, \frac{e^{H(\alpha)n}}{n^{3/2}}, \qquad \text{as } n \to \infty. \tag{5.1.1}$$

In particular, if in addition we have that $\lim_{n\to\infty} \ell(I_n) = 0$ and $p_n \in I_n$ is arbitrary then

$$\pi_{\mathcal{R}}(n,\alpha,I_n,A_n) \sim \frac{k_n^r\,\ell(I_n)e^{-ap_n}}{\sigma_\alpha\sqrt{2\pi}}\,\frac{e^{H(\alpha)n}}{n^{3/2}}, \qquad \text{as } n \to \infty. \tag{5.1.2}$$

Corollary 5.1.2. *If $\alpha = \int \tau \, d\mu_{\max}$, where $\mu_{\max}$ is the measure of maximal entropy for the Poincaré first return map P on the union of unstable leaves U then*

$$\pi_{\mathcal{R}}(n,\alpha,I_n,A_n) \sim \frac{k_n^r\,\ell(I_n)}{\sigma_\alpha\sqrt{2\pi}}\,\frac{e^{hn}}{n^{3/2}}, \qquad \text{as } n \to \infty. \tag{5.1.3}$$

where h is the topological entropy of $P : U \to U$.

We proceed to prove our results in the following sections.

5.2 Transfer operators with holonomy

In this section, we define the transfer operators with holonomy and then present the main technical theorems regarding their spectral bounds. We start with some preparation.

5.2.1 Modified constructions using the smooth structure on G

In order to deduce our technical results later on we will need the smooth properties of the Lie group G. Unfortunately, these properties are not available on the union of unstable leaves U which is usually a fractal set. To overcome this technicality, we will choose an appropriately enlarged open set $\tilde{U}$ of the strong unstable foliation containing U. Since the strong unstable foliation is smooth, $\tilde{U} \subset T^1(X)$ would then be a smooth submanifold and provide a smooth structure at our disposal. Now that we are considering the enlarged open set $\tilde{U}$ we will need to extend P to a map on $\tilde{U}$. To achieve this, we first extend the local inverses of P in the following sense.

For all $j \in \mathcal{A}$ denote by w_j the centre of the rectangle R_j. Using arguments from [Rue89, Lemma 1.2] for sufficiently small neighbourhoods, and increasing δ if necessary while ensuring that equation (4.2.2) still holds, there exist open sets

$$\tilde{U}_j \supset U_j \quad \text{such that} \quad \overline{\tilde{U}_j^{su}} \subset W^{su}_{\varepsilon_0}(w_j),$$

with $\operatorname{diam} d_{su}(\tilde{U}_j) \leq \delta$ for all $j \in \mathcal{A}$ such that for all admissible pairs (j,k), we can naturally extend the inverse $(P|_{C[j,k]})^{-1} : \operatorname{int}(U_k) \to C[j,k]$, to a smooth injective map $P^{-(j,k)} : \tilde{U}_k \to \tilde{U}_j$. More specifically, assuming that ε_0 and δ are sufficiently small, without loss of generality, taking any $u_0 \in U_j$ such that $P(u_0) \in U_k$, we can define $P^{-(j,k)}(u)$ to be the unique intersection

$$P^{-(j,k)}(u) = \left(\bigcup_{t \in (-\tau(u_0) - \inf(\tau),\, -\tau(u_0) + \inf(\tau))} W^{ss}_{\varepsilon_0}(u)\, a_t \right) \cap W^{su}_{\varepsilon_0}(w_j) \qquad \text{for all } u \in \tilde{U}_k.$$

We define

$$\tilde{U} = \bigsqcup_{j=1}^{m} \tilde{U}_j,$$

and note that we can extend any probability measure ν on U to a probability measure on $\tilde{U}$ by setting

$$\nu(B) = \nu(B \cap U),$$

for all Borel sets $B \subset \tilde{U}$.

Let $j \in \mathbb{Z}_{\geq 0}$ and $\omega = (\omega_0, \omega_1, \dots, \omega_j)$ be an admissible word. Define $P^{-\omega} = \mathrm{Id}_{\tilde{U}_{\omega_0}}$ if $j = 0$ whereas if $j > 0$ set

$$P^{-\omega} = P^{-(\omega_0,\omega_1)} \circ P^{-(\omega_1,\omega_2)} \circ \cdots \circ P^{(\omega_{j-1},\omega_j)} : \tilde{U}_{\omega_j} \to \tilde{U}_{\omega_0}.$$

Define the cylinder $\tilde{C}[\omega] = P^{-\omega}(\tilde{U}_{\omega_j}) \supset C[\omega]$. Define the smooth maps

$$P^{\omega} = (P^{-\omega})^{-1} : \tilde{C}[\omega] \to \tilde{U}_{\omega_j}.$$

These maps are sufficient for our purposes in defining transfer operators. For convenience we define

$$\tilde{R}_j = [\tilde{U}_j, S_j] \quad \text{for all } j \in \mathcal{A}.$$

We define more extended maps. Let (j,k) be an admissible pair with respect to the transition matrix T. The maps $\tau|_{C[j,k]}$ and $\theta|_{C[j,k]}$ naturally extend to smooth maps $\tau^{(j,k)} : \tilde{C}[j,k] \to \mathbb{R}^+$ and $\theta^{(j,k)} : \tilde{C}[j,k] \to M$ as follows. In light of the above definition of $P^{-(j,k)}$, using the same notation and writing $v = P^{(j,k)}(u)$, we define

$$\tau^{(j,k)}(u) \in (\tau(u_0) - \inf(\tau), \tau(u_0) + \inf(\tau)),$$

uniquely such that

$$W^{ss}_{\varepsilon_0}(v)\, a_{-\tau^{(j,k)}(u)} \cap W^{su}_{\varepsilon_0}(w_k) \neq \varnothing \qquad \text{for all } u \in \tilde{C}[j,k].$$

Similar to before $\theta^{(j,k)}(u)$ is such that

$$F(u)\, a_{\tau^{(j,k)}(u)} = F(v)\theta^{(j,k)}(u)^{-1} \qquad \text{for all } u \in \tilde{C}[j,k].$$

Now for all $k \in \mathbb{N}$ and admissible words $\omega = (\omega_0, \omega_1, \ldots, \omega_k)$, we define the smooth maps $\tau^{\omega} : \tilde{C}[\omega] \to \mathbb{R}$, $\theta^{\omega} : \tilde{C}[\omega] \to M$ and $\Phi^{\omega} : \tilde{C}[\omega] \to AM$ by

$$\begin{aligned}
\tau^{\omega}(u) &= \sum_{j=0}^{k-1} \tau^{(\omega_j,\omega_{j+1})}(P^{(\omega_0,\omega_1,\ldots,\omega_j)}(u)),\\
\theta^{\omega}(u) &= \prod_{j=0}^{k-1} \theta^{(\omega_j,\omega_{j+1})}(P^{(\omega_0,\omega_1,\ldots,\omega_j)}(u)) \text{ and}\\
\Phi^{\omega}(u) &= a_{\tau_{\omega}(u)}\theta^{\omega}(u) = \prod_{j=0}^{k-1} \Phi^{(\omega_j,\omega_{j+1})}(P^{(\omega_0,\omega_1,\ldots,\omega_j)}(u)),
\end{aligned}$$

for all $u \in \tilde{C}[\omega]$, where the terms of the products are to be in ascending order from left to right. For all admissible words $\omega = (\omega_0)$, we define $\tau^{\omega}(u) = 0$ and $\theta^{\omega}(u) = \Phi^{\omega}(u) = e \in AM$ for all $u \in \tilde{C}[\omega]$.

Remark 5.2.1. Note that for all $u \in U$, there is a corresponding unique admissible sequence in Σ^+ and hence we can instead use the notations $\tau^k(u)$, $\theta^k(u)$ and $\Phi^k(u)$ for all $k \in \mathbb{Z}_{\geq 0}$.

The following lemma is derived from the hyperbolicity of the geodesic flow.

Lemma 5.2.2. *There exist constants* $0 < c_0 < 1 < \kappa_1 < 2 < \kappa_2$ *such that for all* $j \in \mathbb{N}$ *and admissible words* $\omega = (\omega_0, \omega_1, \dots, \omega_j)$, *we have*

$$c_0\, \kappa_1^j \leq \sup_{u \in \tilde{U}} \|(dP^w)_u\|_{\mathrm{op}} \leq \kappa_2^j.$$

We fix constants $c_0 \in (0,1)$ and $\kappa_2 > 2 > \kappa_1 > 1$ as above for the rest of this chapter and use these inequalities without further comments.

5.2.2 Transfer operators with holonomy

Recall the definition of the closed interval $\mathcal{I}$ from (4.4.1). For the rest of this chapter we fix a real number $\alpha \in \operatorname{int} \mathcal{I}$ and set $a = a(\alpha)$ to be the unique real number given by Lemma 4.4.1. For the purposes of this chapter it will suffice to study a family of transfer operators with complex parameters $s = a + ib$, twisted by irreducible unitary representations $\lambda \in \widehat{M}$. We use the convention that sums over words are actually sums over admissible words, throughout the rest of this chapter.

Transfer operator with holonomy For all $s = a + ib \in \mathbb{C}$ and $\lambda \in \widehat{M}$, the transfer operator with holonomy $\tilde{\mathcal{L}}_{s,\lambda} : C(\tilde{U}, H_\lambda^{\oplus \dim(\lambda)}) \to C(\tilde{U}, H_\lambda^{\oplus \dim(\lambda)})$ is defined by

$$\tilde{\mathcal{L}}_{s,\lambda}(H)(u) = \sum_{\substack{(j,k) \\ v = P^{-(j,k)}(u)}} e^{s\tau^{(j,k)}(v)}\, \lambda(\theta^{(j,k)}(v)^{-1})\, H(v),$$

for all $u \in \tilde{U}$ and $H \in C\left(\tilde{U}, H_\lambda^{\oplus \dim(\lambda)}\right)$.

When $\lambda \in \widehat{M}$ is trivial we simply write $\tilde{\mathcal{L}}_s = \tilde{\mathcal{L}}_{s,1}$ and call it the transfer operator. For any $\lambda \in \widehat{M}$, denote by $|_U : C(\tilde{U}, H_\lambda^{\oplus \dim(\lambda)}) \to C(U, H_\lambda^{\oplus \dim(\lambda)})$ the restriction map. Then for all $\lambda \in \widehat{M}$, we also define the transfer operator with holonomy

$$\mathcal{L}_{s,\lambda} = |_U \circ \tilde{\mathcal{L}}_{s,\lambda} \circ (|_U)^{-1} :\ C(U, H_\lambda^{\oplus \dim(\lambda)}) \to C(U, H_\lambda^{\oplus \dim(\lambda)}),$$

where $(|_U)^{-1}$ denotes taking any continuous pre-image using the Tietze Extension Theorem and denote the transfer operator by $\mathcal{L}_s = \mathcal{L}_{s,1}$.

Remark 5.2.3. Let $s \in \mathbb{C}$ and $\lambda \in \widehat{M}$. Then $\tilde{\mathcal{L}}_{s,\lambda}$ preserves $C^k(\tilde{U}, H_\lambda^{\oplus \dim(\lambda)})$ for all $k \in \mathbb{Z}_{\geq 0}$ and $\mathcal{L}_{s,\lambda}$ preserves $C^{\mathrm{Lip}(d)}(U, H_\lambda^{\oplus \dim(\lambda)})$. Here we regard the target space as a real vector space.

We recall the Ruelle–Perron–Frobenius Theorem along with the theory of Gibbs measures in this setting [Bow08, PP90].

Theorem 5.2.4. *For all $a \in \mathbb{R}$, the transfer operator $\mathcal{L}_a : C(U, \mathbb{C}) \to C(U, \mathbb{C})$ and its dual $\mathcal{L}_a^* : C(U, \mathbb{C})^* \to C(U, \mathbb{C})^*$ have eigenvectors with the following properties. There exist a unique strictly positive function $\psi_a \in C^{\mathrm{Lip}(d)}(U, \mathbb{R})$ and a unique Borel probability measure ν_a on U such that*

1. *$\mathcal{L}_a(\psi_a) = e^{\Pr(a\tau)}\psi_a$,*
2. *$\mathcal{L}_a^*(\nu_a) = e^{\Pr(a\tau)}\nu_a$,*
3. *the eigenvalue $e^{\Pr(a\tau)}$ is maximal and simple while the rest of the spectrum of $\mathcal{L}_a|_{C^{\mathrm{Lip}(d)}(U,\mathbb{C})}$ is contained in a disk of radius strictly less than $e^{\Pr(a\tau)}$ and*
4. *$\nu_a(\psi_a) = 1$ and the Borel probability measure μ_a defined by $d\mu_a = \psi_a d\nu_a$ is P-invariant and is the the $a\tau$-equilibrium state on U.*

Analytic extension of the pressure In the previous chapter we defined the pressure of a real valued function using a variational principle. Here, we use the Ruelle–Perron–Frobenius Theorem to extend this definition. Consider the function $a \to e^{\Pr(a\tau)}$, $a \in \mathbb{R}$. We view $e^{\Pr(a\tau)}$ as the simple maximal positive eigenvalue of the operator $\mathcal{L}_{a\tau}$ and show that $e^{\Pr(a\tau)}$ can be analytically extended to a neighbourhood of the real line using the Perturbation Theorem (Theorem 3.2.2). The Perturbation Theorem and the spectral gap property for the transfer operator $\mathcal{L}_{a\tau} : C^1(U, \mathbb{R}) \to C^1(U, \mathbb{R})$ guaranteed from the Ruelle–Perron–Frobenius together with (4.4.3) and (4.4.4) from Chapter 4 give the following result [PP90, Proposition 4.7]. Recall that using Lemma 4.4.1 we fixed real values $a = a(\alpha)$ and $\sigma_\alpha > 0$ where $\alpha \in \mathrm{int}(\mathcal{I})$.

Corollary 5.2.5. *The function* $t \to e^{\Pr((a+it)\tau)}$ *is analytic and for some* $\varepsilon > 0$ *we can write for each* $t \in [-\varepsilon, \varepsilon]$

$$e^{\Pr((a+it)\tau)} = e^{\Pr(a\tau)} \left(1 + i\alpha\, t - \frac{\sigma_\alpha^2\, t^2}{2} + O(|t|^3)\right),$$

where the implied constant is uniform on $[-\varepsilon, \varepsilon]$.

5.3 Decay estimates

In light of Theorem 5.2.4, it is convenient to normalise the transfer operators defined above. Set $\lambda_a = e^{\Pr(a\tau)}$ which is the maximal simple eigenvalue of $\mathcal{L}_a$. Consider the corresponding eigenvector, that is the unique positive function $\psi_a \in C^{\mathrm{Lip}(d)}(U, \mathbb{C})$ and the unique probability measure ν_a on U with $\nu_a(\psi_a) = 1$ such that

$$\mathcal{L}_a(\psi_a) = \lambda_a \psi_a \quad \text{and} \quad \mathcal{L}_a^*(\nu_a) = \lambda_a \nu_a,$$

provided by Theorem 5.2.4. Note that $d\mu_{a\tau} = \psi_a d\nu_a$. We can extend the eigenvector $\psi_a \in C^{\mathrm{Lip}(d)}(U, \mathbb{R})$ to an eigenvector $\psi_a \in C^\infty(\tilde{U}, \mathbb{R})$ with bounded derivatives for $\tilde{\mathcal{L}}_a$ using [SW21, Theorem A.2]. For all admissible pairs (j, k), we define the smooth map $f^{(j,k)} : \tilde{U}_j \to \mathbb{R}$ by

$$f^{(j,k)} = a\tau^{(j,k)} + \log(\psi_a) - \log(\psi_a \circ P^{(j,k)}) - \log(\lambda_a).$$

For all $k \in \mathbb{N}$ and admissible words $\omega = (\omega_0, \omega_1, \ldots, \omega_k)$, we define the smooth map $f^\omega : \tilde{C}[\omega] \to \mathbb{R}$ by

$$f^\omega(u) = \begin{cases} 0 & k = 0, \\ \sum_{j=0}^{k-1} f^{(\omega_j, \omega_{j+1})}(P^{(\omega_0, \omega_1, \ldots, \omega_j)}(u)) & \text{otherwise,} \end{cases} \qquad \text{for all } u \in \tilde{C}[\omega].$$

As before, for all $u \in U$, we can also use the notation $f^k(u)$ for any $k \in \mathbb{N}$.

We now define the normalised transfer operator with holonomy. Let $s = a + ib \in \mathbb{C}$ and $\lambda \in \widehat{M}$. We define $\tilde{\mathcal{N}}_{s,\lambda} : C(\tilde{U}, H_\lambda^{\oplus \dim(\lambda)}) \to C(\tilde{U}, H_\lambda^{\oplus \dim(\lambda)})$ by

$$\tilde{\mathcal{N}}_{s,\lambda}(H)(u) = \sum_{\substack{(j,k) \\ v = P^{-(j,k)}(u)}} e^{(f^{(j,k)} + ib\tau^{(j,k)})(v)} \lambda(\theta^{(j,k)}(v)^{-1}) H(v),$$

for all $u \in \tilde{U}$ and $H \in C(\tilde{U}, H_\lambda^{\oplus \dim(\lambda)})$. Using the tensored unitary representation notation we write

$$\tilde{\mathcal{N}}_{s,\lambda}(H)(u) = \sum_{\substack{(j,k) \\ v=P^{-(j,k)}(u)}} e^{f^{(j,k)}(v)} \lambda_b(\Phi^{(j,k)}(v)^{-1}) H(v),$$

for all $u \in \tilde{U}$ and $H \in C(\tilde{U}, H_\lambda^{\oplus \dim(\lambda)})$. Further, for all $k \in \mathbb{N}$, its k-th iteration is

$$\tilde{\mathcal{N}}_{s,\lambda}^k(H)(u) = \sum_{\substack{\omega=(\omega_0,\dots,\omega_k) \\ v=\tau^{-\omega}(u)}} e^{f^{\omega}(v)} \lambda_b(\Phi^{\omega}(v)^{-1})\, H(v),$$

for all $u \in \tilde{U}$ and $H \in C(\tilde{U}, H_\lambda^{\oplus \dim(\lambda)})$. Again, we denote the normalised transfer operator by $\tilde{N}_s = \tilde{N}_{s,1}$. Again, using the restriction map $|_U$, we get corresponding normalised operators with holonomy $\mathcal{N}_{s,\lambda} : C(U, H_\lambda^{\oplus \dim(\lambda)}) \to C(U, H_\lambda^{\oplus \dim(\lambda)})$ and the normalised transfer operator $\mathcal{N}_s : C(U, \mathbb{C}) \to C(U, \mathbb{C})$. Observe that with this normalisation, we have that

$$\mathcal{N}_a^*(\mu_{a\tau}) = \mu_{a\tau}.$$

We fix some related constants. Fix

$$\overline{\tau} = \max_{(j,k)} \sup_{u \in \tilde{C}[j,k]} \tau^{(j,k)}(u), \qquad \underline{\tau} = \min_{(j,k)} \inf_{u \in \tilde{C}[j,k]} \tau^{(j,k)}(u) \qquad \text{and}$$

$$T_0 > \max\left(\max_{(j,k)} \|\tau^{(j,k)}\|_{C^1}, \max_{(j,k)} \|\theta^{(j,k)}\|_{C^1}, \max_{(j,k)} \sup \|f^{(j,k)}\|_{C^1} \right),$$

which is possible by [PS16, Lemma 4.1].

5.3.1 Spectral bounds with holonomy

The goal of this subsection is to present all the decay estimates necessary for our proofs. We first introduce some norms and semi-norms. Let $\lambda \in \widehat{M}$ be a unitary irreducible representation of M and let $H \in C(U, H_\lambda^{\oplus \dim(\lambda)})$. We will denote by $\|H\| \in C(U, \mathbb{R})$ the function defined by

$$\|H\|(u) = \|H(u)\|_2 \quad \text{for all } u \in U,$$

and if $\lambda = 1$, we will denote by $|H| \in C(U, \mathbb{R})$ the function defined by

$$|H|(u) = |H(u)| \in \mathbb{R} \quad \text{for all } u \in U.$$

We define

$$\|H\|_\infty = \sup \|H\|.$$

We use similar notations if the domain is $\tilde{U}$. We define the Lipschitz semi-norm and the Lipschitz norm by

$$\mathrm{Lip}_d(H) = \sup_{u_1 \neq u_2 \in U} \frac{\|H(u_1) - H(u_2)\|_2}{d(u_1, u_2)} \qquad \text{and} \qquad \|H\|_{\mathrm{Lip(d)}} = \|H\|_\infty + \mathrm{Lip}_d(H),$$

respectively. Since we will mostly use the C^1 norm, we avoid defining the C^k norm for a general $k \in \mathbb{N}$. Let Y be a Riemannian manifold and $H \in C^1(\tilde{U}, Y)$. We define the C^1 semi-norm and the C^1 norm by

$$|H|_{C^1} = \sup_{u \in \tilde{U}} \|(dH)_u\|_{\mathrm{op}} \quad \text{and} \quad \|H\|_{C^1} = \|H\|_\infty + |H|_{C^1},$$

respectively. In fact, as we will see later our transfer operators with holonomy are not uniformly bounded in the C^1 norm. Therefore, we also define a family of useful norms by

$$\|H\|_{1,t} = \|H\|_\infty + \frac{|H|_{C^1}}{\max(1,t)} \qquad \text{for } t \geq 0,$$

which we will use to bound the iterates of the transfer operators with holonomy for certain parameters. Henceforth, by differentiable function spaces on $\tilde{U}$ or its derived suspension spaces, such as $C^1(\tilde{U}, Y)$, we will always mean the space of C^1 functions whose C^1 norm is bounded. For all $\lambda \in \widehat{M}$, we will work with the Banach spaces $C^{\mathrm{Lip}(d)}\left(U, H_\lambda^{\oplus \dim(\lambda)}\right)$ and $C^1\left(\tilde{U}, H_\lambda^{\oplus \dim(\lambda)}\right)$.

Now we can state the main technical theorem [SW21, Theorem 5.3] regarding the spectral bounds of transfer operators with holonomy. Recall the set

$$\widehat{M}_1 = \left\{(b, \lambda) \in \mathbb{R} \times \widehat{M} : |b| > 1 \text{ or } \lambda \neq 1\right\}.$$

Theorem 5.3.1. *There exists $\eta > 0$ and $C > 0$ such that for all $s = a + ib \in \mathbb{C}$ if $(b, \lambda) \in \widehat{M}_1$, then for all $n \in \mathbb{N}$ and $H \in C^1\left(\tilde{U}, H_\lambda^{\oplus \dim(\lambda)}\right)$ we have*

$$\int_{\tilde{U}} \|\tilde{\mathcal{N}}_{s,\lambda}^n(H)\|^2 \, d\mu_{a\tau} \leq C e^{-\eta n} \|H\|_{1,\|\lambda_b\|}.$$

We note that $\|\cdot\|_{C^1} \leq (1+\|\lambda_b\|)\|\cdot\|_{1,\|\lambda_b\|}$. The following corollary is an estimate that appeared first in [Dol98] and was later used in many other papers [Nau05, OW17, SS22, PS98b]. We provide a proof in our situation in the following section.

Corollary 5.3.2. *For any $\varepsilon > 0$, there exist $C_\varepsilon > 0$ and $\varepsilon' \in (0,1)$ such that for all $s = a+ib \in \mathbb{C}$ and all unitary irreducible representations $\lambda \in \widehat{M}$ with $(b,\lambda) \in \widehat{M}_1$ we have that*

$$\|\mathcal{N}^n_{s,\lambda}\|_{C^1} \leq C_\varepsilon \left(1+\|\lambda_b\|\right)^{1+\varepsilon} e^{-n\varepsilon'},$$

for all $n \in \mathbb{N}$. Particularly,

$$\|\mathcal{L}^n_{s,\lambda}\|_{C^1} \leq C_\varepsilon \left(1+\|\lambda_b\|\right)^{1+\varepsilon} e^{(\mathrm{Pr}(a\tau)-\varepsilon')n}.$$

5.3.2 Reduction to technical theorem about Dolgopyat operators

We reduce the proof of Theorem 5.3.1 to proving Theorem 5.3.4 [SW21, Theorem 5.4] which captures the mechanism of Dolgopyat's method in our setting. Similar theorems have appeared in [Dol98, Sto11, OW16, SW21, OW17, Gou09]. The main difference with previous works is that here we need to deal with the holonomies.

For $B > 0$ we define the cone set of functions

$$K_B(\tilde{U}) = \left\{ h \in C^1(\tilde{U},\mathbb{R}) \ : \ h > 0 \text{ and } \|(dh)_u\|_{\mathrm{op}} \leq Bh(u) \text{ for all } u \in \tilde{U} \right\}.$$

Remark 5.3.3. It is useful to note that we can easily derive the equivalent log-Lipschitz characterisation given by $K_B(\tilde{U}) = \left\{ h \in C^1(\tilde{U},\mathbb{R}) \ : \ h > 0 \text{ and } |\log h|_{C^1} \leq B \right\}$.

Theorem 5.3.4. *For $\beta > 0$ there exist $p \in \mathbb{N}$, $\eta \in (0,1)$, $E > \max\left(1, \frac{1}{\beta}, \frac{1}{\delta_{\widehat{M}}}\right)$ and a set of operators*

$$\left\{ \mathcal{D}^H_J : C^1(\tilde{U},\mathbb{R}) \to C^1(\tilde{U},\mathbb{R}) \ : \ H \in C(\tilde{U}, H_\lambda^{\oplus \dim(\lambda)}), J \in \mathcal{J}(b,\lambda), \text{ for some } (b,\lambda) \in \widehat{M}_\beta \right\},$$

where $\mathcal{J}(b,\lambda)$ is some finite set for all $(b,\lambda) \in \widehat{M}_\beta$, such that

1. *for all $H \in C(\tilde{U}, H_\lambda^{\oplus \dim(\lambda)})$, $J \in \mathcal{J}(b,\lambda)$, and $(b,\lambda) \in \widehat{M}_\beta$ we have*

$$\mathcal{D}^H_J(K_{E\|\lambda_b\|}(\tilde{U})) \subset K_{E\|\lambda_b\|}(\tilde{U}), \tag{5.3.1}$$

2. *for all* $h \in K_{E\|\lambda_b\|}(\tilde{U})$, $H \in C(\tilde{U}, H_\lambda^{\oplus \dim(\lambda)})$, $J \in \mathcal{J}(b,\lambda)$, *and* $(b,\lambda) \in \widehat{M}_\beta$ *we have*

$$\left\|\mathcal{D}_J^H(h)\right\|_2 \leq \eta \|h\|_2 \tag{5.3.2}$$

3. *setting* $s = a+ib \in \mathbb{C}$ *if* $(b,\lambda) \in \widehat{M}_\beta$, $H \in C(\tilde{U}, H_\lambda^{\oplus \dim(\lambda)})$ *and* $h \in K_{E\|\lambda_b\|}(\tilde{U})$ *satisfy*

$$\|H(u)\|_2 \leq h(u) \quad \textit{and} \quad \|(dH)_u\|_{\text{op}} \leq E\|\lambda_b\| h(u) \textit{ for all } u \in \tilde{U}, \tag{5.3.3}$$

then there exists $J \in \mathcal{J}(b,\lambda)$ *such that for all* $u \in \tilde{U}$ *we have that*

$$\|\tilde{\mathcal{N}}_{s,\lambda}^p(H)(u)\|_2 \leq \mathcal{D}_J^H(h)(u) \quad \textit{and} \tag{5.3.4}$$

$$\left\|\left(d\tilde{\mathcal{N}}_{s,\lambda}^p(H)\right)_u\right\|_{\text{op}} \leq E\|\lambda_b\| \mathcal{D}_J^H(h)(u). \tag{5.3.5}$$

A sketch of the proof for the theorem above appears in the appendix.

Proof that Theorem 5.3.4 implies Theorem 5.3.1. Fix $p \in \mathbb{N}$, $\beta > 0, E > 0$ to be the constants from Theorem 5.3.4 and $\tilde{\eta} \in (0,1)$ to be the η from Theorem 5.3.4. Fix

$$B = \sup_{\lambda \in \widehat{M}} \|\tilde{\mathcal{N}}_{s,\lambda}\|_{\text{op}} \leq \|\tilde{\mathcal{N}}_s\|_{\text{op}} \leq m e^{T_0},$$

viewing the transfer operators as operators on the spaces $L^2\left(\tilde{U}, C(\tilde{U}, H_\lambda^{\oplus \dim(\lambda)})\right)$ and $L^2\left(\tilde{U}, \mathbb{R}\right)$, respectively. Fix also

$$\eta = \frac{-\log(\tilde{\eta})}{p} \qquad \text{and} \qquad C = B^p \tilde{\eta}^{-1}.$$

Let $s = a+ib \in \mathbb{C}$ and suppose that $(b,\lambda) \in \widehat{M}_\beta$. Let $k \in \mathbb{N}$ and $H \in C(\tilde{U}, H_\lambda^{\oplus \dim(\lambda)})$. The theorem is trivial if $H = 0$, so suppose that $H \neq 0$. First set $h_0 \in K_{E\|\lambda_b\|}(\tilde{U})$ to be the positive constant function defined by

$$h_0(u) = \|H\|_{1,\|\lambda_b\|} \quad \text{for all } u \in \tilde{U}.$$

Then H and h_0 satisfy property (5.3.3) from Theorem 5.3.4. Thus, given $h_j \in K_{E\|\lambda_b\|}(\tilde{U})$ for any $j \geq 0$, Theorem 5.3.4 provides a $J_j \in \mathcal{J}(b)$ and we inductively obtain

$$h_{j+1} = \mathcal{D}^H_{J_j}(h_j) \in K_{E\|\lambda_b\|}(\tilde{U}).$$

Then $\|\tilde{\mathcal{N}}^{jp}_{s,\lambda}(H)(u)\|_2 \leq h_j(u)$ for all $u \in \tilde{U}$ and hence

$$\|\tilde{\mathcal{N}}^{jp}_{s,\lambda}(H)\|_2 \leq \|h_j\|_2 \leq \tilde{\eta}^j \|h_0\|_2 = \tilde{\eta}^j \|H\|_{1,\|\lambda_b\|} \quad \text{for all } j \in \mathbb{Z}_{\geq 0}.$$

Then writing $k = jp + l$ for some $j \in \mathbb{Z}_{\geq 0}$ and $0 \leq l < p$, we have

$$\|\tilde{\mathcal{N}}^{k}_{s,\lambda}(H)\|_2 \leq B^l \|\tilde{\mathcal{N}}^{jp}_{s,\lambda}(H)\|_2 \leq B^l \tilde{\eta}^j \|H\|_{1,\|\lambda_b\|} \leq C e^{-\eta k} \|H\|_{1,\|\lambda_b\|}.$$

□

Now that the required bounds on the C^1 norm of our transfer operators with holonomy have been presented (Corollary 5.3.2) we proceed to bound the sums

$$Z_n(s,\lambda) := \sum_{P^n(u)=u} e^{a\tau^n(u)} \lambda_b(\Phi^n(u)^{-1}),$$

for $s = a + ib \in \mathbb{C}$ and $\lambda \in \widehat{M}$. We divide our weighted sums by $e^{\mathrm{Pr}(a\tau)n}$ to obtain the normalised sums

$$\hat{Z}_n(s,\lambda) := \frac{Z_n(s,\lambda)}{e^{\mathrm{Pr}(a\tau)n}},$$

observing that

$$\begin{aligned}\hat{Z}_n(s,\lambda) &= \sum_{P^n(u)=u} e^{a\tau^n(u) + (\log\psi_a - \log\psi_a \circ P)^n(u) - \mathrm{Pr}(a\tau)n} \lambda_b(\Phi^n(u)^{-1}) \\ &= \sum_{P^n(u)=u} e^{f^n(u)} \lambda_b(\Phi^n(u)^{-1}).\end{aligned}$$

This next result follows essentially from Ruelle's work in [Rue90], except that we require explicit dependence on $\|\lambda_b\|$. We provide a proof in the next section following the rigorous approach of [Wri12]. In the statement below, χ_j is the characteristic function of U_j and κ_1 is the contraction rate given in (5.2.2). (Note that, since U is the *disjoint* union of the sets U_j, for each such j we have that $\chi_j \in C^1(U, \mathbb{R})$.)

Proposition 5.3.5. *Fix $\beta > 0$. For $1 \leq j \leq m$ there exists $u_j \in U_j$ such that for any $\eta > 0$, there exists C_η, $W > 0$ such that for any $\lambda \in \widehat{M}$ and $s = a + ib \in \mathbb{C}$ with $(b, \lambda) \in \widehat{M}_\beta$ we have*

$$\left\| \hat{Z}_n(s,\lambda) - \sum_{j=1}^{m} \mathcal{N}_{s,\lambda}^n(\chi_j)(u_j) \right\| \leq C_\eta (1 + \|\lambda_b\|)^W \sum_{l=2}^{n} \|\mathcal{N}_{s,\lambda}^{n-l}\|_{C^1} \left(\frac{e^\eta}{\kappa_1}\right)^l$$

for all $n \in \mathbb{N}$.

We are now ready to prove the decay estimates that will give us the proof of Theorem 5.1.1 in the next section. Fixing $\varepsilon > 0$ then by Corollary 5.3.2 and Proposition 5.3.5, we get that for all $s = a + ib \in \mathbb{C}$ and $\lambda \in \widehat{M}$ with $(b, \lambda) \in \widehat{M}_1$

$$\begin{aligned} \left\| \hat{Z}_n(s,\lambda) \right\| &\leq \left\| \hat{Z}_n(a+ib,\lambda) - \sum_{j=1}^{m} \mathcal{N}_{s,\lambda}^n(\chi_j)(u_j) \right\| + m\, C_\varepsilon (1 + \|\lambda_b\|)^{1+\varepsilon} e^{-\varepsilon' n} \\ &\leq C_\eta C_\varepsilon (1 + \|\lambda_b\|)^{W+1+\varepsilon} e^{-\varepsilon' n} \sum_{p=2}^{n} \left(\frac{e^{\eta+\varepsilon'}}{\kappa_1}\right)^p + m\, C_\varepsilon (1 + \|\lambda_b\|)^{1+\varepsilon} e^{-\varepsilon' n}. \end{aligned}$$

We note that it is possible to choose $1 < e^{\varepsilon'} < \kappa_1$. Provided η is small enough such that $e^{\eta+\varepsilon'}/\kappa_1 < 1$ we get that for some $C > 0$

$$\|Z_n(s,\lambda)\| \leq C(1 + \|\lambda_b\|)^{W+1+\varepsilon} e^{(\Pr(a\tau)-\varepsilon')n}. \tag{5.3.6}$$

Finally, we will also need a more elementary result to bound the sums $Z_n(s,1)$ for small $\mathrm{Im}(s) \in \mathbb{R}$. These estimates can be derived as in the symbolic case in [PP90].

Lemma 5.3.6. *There exists $\varepsilon > 0$ such that for each $n \in \mathbb{N}$ and some $\varepsilon'' > 0$ we have that*

1. *for $\mathrm{Im}(s) \in [-1,1] \setminus (-\varepsilon, \varepsilon)$ we can bound $Z_n(s,1) = O(e^{(\Pr(\mathrm{Re}(s)\tau)-\varepsilon'')n})$ and*

2. *for $\mathrm{Im}(s) \in (-\varepsilon, \varepsilon)$ we have*

$$Z_n(s,1) = e^{\Pr(s\tau)n} + O(e^{(\Pr(\mathrm{Re}(s)\tau)-\varepsilon'')n}).$$

Proof. For part 1., we use the fact that since the geodesic flow is mixing the length spectrum of the geodesic flow on X is not contained in discrete subgroup of $\mathbb{R}$ [Bab02]. In particular, this implies that the Poincaré first return time map $\tau : U \to \mathbb{R}^+$ is non-lattice and so we have that the spectral radius of our operator satisfies $\mathrm{spr}(\mathcal{L}_{s,1}) < e^{P(\mathrm{Re}(s)\tau)}$ for $\mathrm{Im}(s) \neq 0$, with a uniform bound on $[-1,1] \setminus (-\varepsilon,\varepsilon)$, and Proposition 5.3.5. Part 2. follows from the spectral gap in the Ruelle–Perron–Frobenius Theorem, which is uniform over an interval $(-\varepsilon,\varepsilon)$ and Proposition 5.3.5. □

5.4 Two useful Lemmas

5.4.1 Proof of Dolgopyat's L^2-argument

In this subsection we prove Corollary 5.3.2. We recall a standard argument, see [Dol98, Corollary 2] and [Nau05, Section 5.1], which is used to convert the decay estimates on the L^2 norm (with respect to $\mu_\alpha = \mu_{a\tau}$) of our normalised transfer operators with holonomy $\mathcal{N}_{s,\lambda}$ to decay estimates for the modified $\|\cdot\|_{1,\|\lambda_b\|}$ norm. Consider the complex parameter $s = a + ib$ and an irreducible unitary representation $\lambda \in \widehat{M}$ such that $(b,\lambda) \in \widehat{M}_1$; recall that at the end of section 4.3 we fixed constants $\delta_{\widehat{M}}, \delta_{1,\widehat{M}} > 0$ such that

$$\|\lambda_b\| \geq \min\{|b|, \delta_{\widehat{M}}\} \geq \min\{1, \delta_{\widehat{M}}\} = \delta_{1,\widehat{M}} > 0.$$

Let $C_0, \eta > 0$ be the constants from Theorem 5.3.1 so that for all $n \in \mathbb{N}$ and $H \in C^1\left(U, H_\lambda^{\oplus \dim(\lambda)}\right)$ with $\|H\|_{1,\|\lambda_b\|} \leq 1$ we have

$$\int_U \left\|\mathcal{N}_{s,\lambda}^n(H)(u)\right\|^2 d\mu_\alpha \leq C_0\, e^{-\eta n}.$$

Let $C_1 \gg 1$ to be chosen later and set $n = \lfloor C_1 \log(1 + \|\lambda_b\|)\rfloor$. Fix an arbitrary function $H \in C^{\mathrm{Lip}(d)}\left(U, H_\lambda^{\oplus \dim(\lambda)}\right)$ with $\|H\|_{1,\|\lambda_b\|} \leq 1$. Since λ is unitary we have

$$\begin{aligned}\left\|\mathcal{N}_{s,\lambda}^{2n}(H)(u)\right\| &= \left\|\mathcal{N}_{s,\lambda}^n\left(\mathcal{N}_{s,\lambda}^n(H)\right)(u)\right\| = \left\|\sum_{P^n(v)=u} e^{f^n(v)+ib\tau^n(v)} \lambda(\theta^n(v)^{-1})\, \mathcal{N}_{s,\lambda}^n(H)(v)\right\| \\ &\leq \sum_{P^n(v)=u} e^{f^n(v)} \left\|\mathcal{N}_{s,\lambda}^n(H)\right\|(v) = \mathcal{N}_a^n\left(\left\|\mathcal{N}_{s,\lambda}^n(H)\right\|\right)(u).\end{aligned}$$

Now since $\mathcal{N}_a$ is normalised we get by convexity that

$$\begin{aligned}\left(\mathcal{N}_a^n\left(\left\|\mathcal{N}_{s,\lambda}^n(H)\right\|\right)(u)\right)^2 &= \left(\sum_{P^n(v)=u} e^{f^n(v)} \left\|\mathcal{N}_{s,\lambda}^n(H)\right\|(v)\right)^2 \\ &\leq \sum_{P^n(v)=u} e^{f^n(v)} \left\|\mathcal{N}_{s,\lambda}^n(H)\right\|^2(v) = \mathcal{N}_a^n\left(\left\|\mathcal{N}_{s,\lambda}^n(H)\right\|^2\right)(u).\end{aligned}$$

Our aim is to bound the C^1 norm of the normalised transfer operator with holonomy $\mathcal{N}_{s,\lambda}$. We start by bounding the supremum norm using the Ruelle–Perron–Frobenius Theorem

$$\begin{aligned}\left\|\mathcal{N}_{s,\lambda}^{2n}(H)\right\|_\infty^2 &\leq \left\|\mathcal{N}_a^n\left(\left\|\mathcal{N}_{s,\lambda}^n(H)\right\|^2\right)\right\|_\infty \\ &\leq \int_U \left\|\mathcal{N}_{s,\lambda}^n(H)(u)\right\|^2 \, d\mu_\alpha + C_2\, e^{-n\varepsilon}\left|\mathcal{N}_{s,\lambda}^n(H)\right|_{\mathrm{Lip}(d)}.\end{aligned}$$

We can bound the first summand using Theorem 5.3.1 whereas for the second observe that

$$\left|\mathcal{N}_{s,\lambda}^n(H)\right|_{\mathrm{Lip}(d)} \leq \sup_{u\in\tilde{U}} \|(d\mathcal{N}_{s,\lambda}^n H)_u\|_{\mathrm{op}} \leq 2A_0(1+\|\lambda_b\|),$$

where the last inequality is proved in the next page. Therefore

$$\begin{aligned}\left\|\mathcal{N}_{s,\lambda}^{2n}(H)\right\|_\infty^2 &\leq C_0\, e^{-\eta n} + C_3\, e^{-\varepsilon n}(1+\|\lambda_b\|) \\ &\leq \frac{C_0}{(\|\lambda_b\|+1)^{C_1|\log\eta|}} + \frac{C_3}{(\|\lambda_b\|+1)^{C_1|\log\varepsilon|-1}}.\end{aligned}$$

We finally choose $C_1 \gg 1$ to be large enough so that $C_1|\log\varepsilon| > 2$. Then since $\|\lambda_b\| \geq \delta_{1,\widehat{M}} > 0$ and perhaps assuming C_1 is chosen large enough there exists $\beta > 0$ small enough so that

$$\left\|\mathcal{N}_{s,\lambda}^{2n}(H)\right\|_\infty \leq \frac{1}{(\|\lambda_b\|+1)^\beta}.$$

We now wish to bound the quantity $\frac{\left|\mathcal{N}_{s,\lambda}^{2n}H\right|_{C^1}}{\max\{1,\|\lambda_b\|\}}$. To do that we will use a Lasota–Yorke type inequality [LY73]. Recall the definition of the cone set

$$K_B(U) := \left\{h \in C^1(U,\mathbb{C}) \ : \ h>0, \ |\log h|_{C^1} \leq B\right\},$$

and fix

$$A_0 > \max\left\{10, \frac{4T_0}{c_0(\kappa_2-1)}, \frac{2T_0}{\delta_{1,\widehat{M}}c_0(\kappa_2-1)}, \frac{1}{c_0}\right\},$$

where c_0, κ_2 come from Lemma 5.2.2. We will use Lemma A.2.3 twice.

Firstly note that the fact that $\|H\|_{1,\|\lambda_b\|} \leq 1$ implies that $\|(dH)_u\|_{\mathrm{op}} \leq \max\{1, \|\lambda_b\|\}$ and so using (A.2.3) we get that

$$\left\|\left(d\mathcal{N}_{s,\lambda}^n H\right)_u\right\|_{\mathrm{op}} \leq A_0 \max\{1, \|\lambda_b\|\}\left(\frac{1}{\kappa_2^n}\mathcal{N}_a^n(\mathbb{1})(u) + \mathcal{N}_a^n\|H\|(u)\right)$$
$$\leq A_0 \max\{1, \|\lambda_b\|\}\left(\frac{1}{\kappa_2^n} + 1\right) \leq 2A_0 \max\{1, \|\lambda_b\|\}.$$

Therefore we can use (A.2.3) again together with Cauchy–Schwarz to obtain

$$\left\|\left(d\mathcal{N}_{s,\lambda}^{2n}(H)\right)_u\right\|_{\mathrm{op}} = \left\|\left(d\mathcal{N}_{s,\lambda}^n(\mathcal{N}_{s,\lambda}^n(H))\right)_u\right\|_{\mathrm{op}} \left\|\left(d\mathcal{N}_{s,\lambda}^n(H)\right)_u\right\|_{\mathrm{op}}$$
$$\leq 2A_0^2 \max\{1, \|\lambda_b\|\}^2 \left(\frac{1}{\kappa_2{}^n} + \mathcal{N}_a^n\|\mathcal{N}_{s,\lambda}^n(H)\|(u)\right)$$
$$\leq 2A_0^2 \max\{1, \|\lambda_b\|\}^2 \left(\frac{1}{\kappa_2{}^n} + \sqrt{\mathcal{N}_a^n\|\mathcal{N}_{s,\lambda}^n(H)\|^2(u)}\right)$$
$$\leq 2A_0^2 \max\{1, \|\lambda_b\|\}^2 \left(\frac{1}{\kappa_2{}^n} + \sqrt{\int_U \left\|\mathcal{N}_{s,\lambda}^n(H)(u)\right\|^2 d\mu_\alpha + C_2\, e^{-n\varepsilon}\left|\mathcal{N}_{s,\lambda}^n(H)\right|_{\mathrm{Lip}(d)}}\right),$$

for all $u \in U$. Then similarly to before, perhaps after increasing the constant C_1, we can find $\beta' > 0$ such that

$$\left\|\mathcal{N}_{s,\lambda}^{2n} H\right\|_{1,\|\lambda_b\|} = \left\|\mathcal{N}_{s,\lambda}^{2n} H\right\|_\infty + \frac{\left|\mathcal{N}_{s,\lambda}^{2n} H\right|_{C^1}}{\max\{1, \|\lambda_b\|\}} \leq \frac{1}{(\|\lambda_b\| + 1)^{\beta'}}.$$

To finish the proof for any $k \geq 1$ write $k = 2nd + r$ with $d, r \in \mathbb{N}$ such that $0 \leq r \leq 2n-1$. Using the Lasota–Yorke type inequality (A.2.3) and since $\mathcal{N}_a$ is normalised we can bound $\left\|\mathcal{N}_{s,\lambda}^r\right\|_{1,\|\lambda_b\|}$ by $M > 0$ which is uniform in $0 \leq r \leq 2n-1$ for any $(b, \lambda) \in \widehat{M}_1$. Then

$$\left\|\mathcal{N}_{s,\lambda}^k H\right\|_{1,\|\lambda_b\|} \leq M\left(\frac{1}{(\|\lambda_b\| + 1)^{\beta'}}\right)^d \leq M(\|\lambda_b\| + 1)^{\beta'}\varepsilon_{\beta'}^k,$$

where $0 < \varepsilon_{\beta'} < 1$. Since the previous estimates are valid for all $\beta' > 0$ small enough, by using the fact that $\|.\|_{C^1} \leq (1 + \|\lambda_b\|)\|.\|_{1,\|\lambda_b\|}$, we get the result.

5.4.2 Proof of Ruelle's Lemma

In this subsection we prove Proposition 5.3.5 known as Ruelle Lemma which essentially first appeared in [Rue90] and was later used in [PS98b, Nau05, OW17, SS22]. We follow the rigorous approach of [Wri12]. Let $\xi > 0$. We start by fixing a complex number $s = a + bi$ and an irreducible representation $\lambda \in \widehat{M}$ such that $(b, \lambda) \in \widehat{M}_\xi$.

For all admissible words $\alpha = (\alpha_0, \ldots, \alpha_{n-1})$ of length $n \geq 2$, we denote by χ_α the $C^1(U, \mathbb{R})$ cut-off function such that $\chi_\alpha = 1$ on the cylinder set $C[\alpha]$ and $\chi_\alpha = 0$ on the other cylinders of length n. Such a cut-off function clearly exists by Urysohn's Lemma since $|\alpha| = |\beta|$ and $\alpha \neq \beta$ implies that $\mathrm{dist}(C[\alpha], C[\beta]) > 0$.

Given two admissible words $\alpha = (\alpha_0, \ldots, \alpha_p)$ and $\beta = (\beta_0, \ldots, \beta_q)$, we denote the concatenation of these words by $\alpha \vee \beta = (\alpha_0, \ldots \alpha_p, \beta_0, \ldots, \beta_q)$, whenever it makes sense, that is when $T_{(\alpha_p, \beta_0)} = 1$.

For $1 \leq j \leq m$ fix arbitrary points $u_j \in U_j$. For all $n \geq 2$ and all admissible words $\alpha = (\alpha_0, \ldots, \alpha_{n-1})$ of length n, we denote by u_α the unique fixed point of P^n in $C[\alpha]$ if it exists, otherwise we choose $u_\alpha \in C[\alpha] \cap U$ such that $u_\alpha \notin P(U_{\alpha_{n-1}})$. Indeed, if $C[\alpha] \cap U \subset P(U_{\alpha_{n-1}})$, then by the Markov property we have $T_{(\alpha_{n-1}, \alpha_0)} = 1$ and $C[\alpha]$ would contain a periodic point of period n. Notice that for words of length one we already fixed points $u_j \in U_j$ for each $1 \leq j \leq m$. For $n \geq 2$ and a word α of length n our choice of u_α implies that

$$\mathcal{N}^n_{s,\lambda}(\chi_\alpha)(u_\alpha) = \begin{cases} e^{f^n(u_\alpha)} \lambda_b(\Phi^n(u_\alpha)^{-1}) & \text{if } u_\alpha \text{ is periodic,} \\ 0 & \text{otherwise.} \end{cases}$$

Observe that

$$\hat{Z}_n(s, \lambda) = \sum_{P^n(u) = u} e^{f^n(u)} \lambda_b(\Phi^n(u)^{-1}) = \sum_{\substack{\bar{\alpha} \text{ admissible} \\ |\alpha| = n}} \mathcal{N}^n_{s,\lambda}(\chi_\alpha)(u_\alpha) = \sum_{|\alpha| = n} \mathcal{N}^n_{s,\lambda}(\chi_\alpha)(u_\alpha), \tag{5.4.1}$$

where $\bar{\alpha} = \alpha \vee \alpha \vee \cdots$. Therefore noting that

$$\sum_{j=1}^m \mathcal{N}^n_{s,\lambda}(\chi_j)(u_j) = \sum_{|\beta| = 1} \mathcal{N}^n_{s,\lambda}(\chi_\beta)(u_\beta)$$

we get

$$\left\| \hat{Z}_n(s, \lambda) - \sum_{j=1}^m \mathcal{N}^n_{s,\lambda}(\chi_j)(u_j) \right\| \leq \sum_{l=2}^n \left\| \sum_{|\beta| = l} \mathcal{N}^n_{s,\lambda}(\chi_\beta)(u_\beta) - \sum_{|\alpha| = l-1} \mathcal{N}^n_{s,\lambda}(\chi_\alpha)(u_\alpha) \right\|.$$

Note that for $u \in U$ and and admissible word $\alpha = (\alpha_0, \ldots, \alpha_{l-1})$ we have

$$\chi_\alpha(u) = \sum_{T_{(\alpha_{l-1},i)}=1} \chi_{\alpha\vee i}(u),$$

and hence

$$\sum_{|\alpha|=l-1} \mathcal{N}^n_{s,\lambda}(\chi_\alpha)(u_\alpha) = \sum_{|\alpha|=l-1} \sum_{T_{(\alpha_{l-2},i)}=1} \mathcal{N}^n_{s,\lambda}(\chi_{\alpha\vee i})(u_\alpha) = \sum_{|\beta|=l} \mathcal{N}^n_{s,\lambda}(\chi_\beta)(u_{\hat{\beta}}),$$

where $\hat{\beta}$ is the word β with its last symbol removed. Therefore,

$$\begin{aligned} \left\| \hat{Z}_n(s,\lambda) - \sum_{j=1}^m \mathcal{N}^n_{s,\lambda}(\chi_j)(u_j) \right\| &\le \sum_{l=2}^n \left\| \sum_{|\beta|=l} \mathcal{N}^n_{s,\lambda}(\chi_\beta)(u_\beta - u_{\hat{\beta}}) \right\| \\ &\le \sum_{l=2}^n \left\| \mathcal{N}^{n-l}_{s,\lambda} \right\|_{C^1} \sum_{|\beta|=l} \left\| \mathcal{N}^l_{s,\lambda}(\chi_\beta) \right\|_{\mathrm{Lip}(d)} d(u_\beta, u_{\hat{\beta}}), \qquad (5.4.2) \end{aligned}$$

since $\mathcal{N}^l_{s,\lambda}(\chi_\beta)$ is Lipschitz. We now proceed to bound the three terms on the right hand side separately.

Claim 5.4.1. *For some constant $C > 0$ we have $d(u_\beta, u_{\hat{\beta}}) \le C/\kappa_1^l$.*

Proof. Since $u_\beta, u_{\hat{\beta}} \in C[\hat{\beta}] \subset P^{-(l-2)} U_{\beta_{l-2}}$ we get from Lemma 5.2.2 that

$$d(u_\beta, u_{\hat{\beta}}) \le C/\kappa_1^l.$$

□

For fixed $l \ge 2$ and an admissible word $\beta = (\beta_0, \ldots, \beta_{l-1})$ we fix $y_\beta \in U \cap P(U_{\beta_{l-1}})$. We will see later how to choose these points. Set $z_\beta = P^{-\beta}(y_\beta)$.

Lemma 5.4.2. *For $2 \le l \le n$ and a word β of length l there exist constants $C, W > 0$ so that*

$$\|\mathcal{N}^l_{s,\lambda}(\chi_\beta)\|_{\mathrm{Lip}(d)} \le C(\|\lambda_b\| + 1)^W e^{f^l(z_\beta)}.$$

Proof. For $u \in U$ we have that

$$\mathcal{N}^l_{s,\lambda}(\chi_\beta)(u) = \begin{cases} e^{f^l(P^{-\beta}(u))} \lambda_b(\Phi^l(P^{-\beta}(u))^{-1}) & \text{if } u \in P(U_{\beta_{l-1}}), \\ 0 & \text{otherwise.} \end{cases}$$

Recall that

$$\|\mathcal{N}^l_{s,\lambda}(\chi_\beta)\|_{\mathrm{Lip}(d)} = \|\mathcal{N}^l_{s,\lambda}(\chi_\beta)\|_\infty + \mathrm{Lip}_d\left(\mathcal{N}^l_{s,\lambda}(\chi_\beta)\right).$$

Firstly, we produce a bound for $\|\mathcal{N}^l_{s,\lambda}(\chi_\beta)\|_\infty$ in terms of $e^{f^l(z_\beta)}$. Let $x \in P(U_{\beta_{l-1}})$. Since $P^{-\beta}(x)$, $z_\beta \in C[\beta]$ we have that

$$\begin{aligned}\|\mathcal{N}^l_{s,\lambda}(\chi_\beta)(x)\|_2 = \|e^{f^l(P^{-\beta}(x))}\lambda_b(\Phi^l(P^{-\beta}(x))^{-1})\|_2 = e^{f^l(P^{-\beta}(x))}\\ \leq e^{|f^l(P^{-\beta}(x))-f^l(z_\beta)|}e^{f^l(z_\beta)} \leq C\, e^{f^l(z_\beta)},\end{aligned}$$

where the last inequality follows from the Lipschitz properties of τ and ψ_a and the fact that the diameter of each U_j is bounded.

Next we produce a bound for $\mathrm{Lip}_d\left(\mathcal{N}^l_{s,\lambda}(\chi_\beta)\right)$. Given that $\mathcal{N}^l_{s,\lambda}(\chi_\beta)(x)$ takes non-zero values only for $x \in P(U_{\beta_{l-1}})$ we fix $u, v \in P(U_{\beta_{l-1}})$ and bound the following difference

$$\begin{aligned}&\left\|\mathcal{N}^l_{s,\lambda}(\chi_\beta)(u) - \mathcal{N}^l_{s,\lambda}(\chi_\beta)(v)\right\|_2\\ &= \left\|e^{f^l(P^{-\beta}(u))-f^l(P^{-\beta}(v))}\lambda_b(\Phi^l(P^{-\beta}(u))^{-1}) - \lambda_b(\Phi^l(P^{-\beta}(v))^{-1})\right\|_2 \; e^{f^l(P^{-\beta}v)}\\ &\leq \left(\dim(\lambda)\left|e^{f^l(P^{-\beta}u)-f^l(P^{-\beta}v)} - 1\right| + \left\|\lambda_b(\Phi^\beta(P^{-\beta}u)^{-1}) - \lambda_b(\Phi^\beta(P^{-\beta}v)^{-1})\right\|_2\right)\; e^{f^l(P^{-\beta}v)}.\end{aligned}$$

Recall that the Weyl's dimension formula guarantees the existence of constants $C, W > 0$ such that $\dim(\lambda) \leq C\|\lambda\|^W$ [Sug71, (1.17)]. Further since f^l is Lipschitz it is a straightforward calculation that

$$\begin{aligned}\left|e^{f^l(P^{-\beta}(u))-f^l(P^{-\beta}(v))} - 1\right| \leq \left|f^l(P^{-\beta}(u)) - f^l(P^{-\beta}(v))\right| e^{f^l(P^{-\beta}(u))-f^l(P^{-\beta}(v))}\\ \leq A_0\, d(u,v)e^{A_0|U|}.\end{aligned}$$

Recall further that $\lambda_b(a_t m)(z) = e^{-ibt}\lambda(m)(z)$ for $z \in H_\lambda^{\oplus \dim(\lambda)}$ and $\Phi^l(u) = a_{\tau^l(u)}\theta^l(u)$. Using the Lipschitz properties of τ and θ and recalling the choice of A_0 we have

$$\left\|\lambda_b(\Phi^\beta(P^{-\beta}u)^{-1}) - \lambda_b(\Phi^\beta(P^{-\beta}v)^{-1})\right\|_2 \leq A_0\|\lambda_b\| d(u,v).$$

Therefore we get $\left\|\mathcal{N}^l_{s,\lambda}(\chi_\beta)\right\|_{\mathrm{Lip}(d)} \leq C(\|\lambda_b\|+1)^W e^{f^l(z_\beta)}$. □

Recall that we can choose $y_\beta \in U \cap P(U_{\beta_{l-1}})$, and hence $z_\beta = P^{-\beta}(y_\beta)$ however we like. Since $C[\beta]$ may not have an l-periodic point, we cannot simply choose $P^\beta(z_\beta) = z_\beta$.

However, $C[\beta]$ must have a periodic point of some higher order, because the periodic points of P are dense in U.

We let $p(\beta)$ be the smallest integer such that $C[\beta]$ has a $p(\beta)$ periodic point z_β i.e. $P^{p(\beta)}(z_\beta) = z_\beta$. Define $p(l)$ for any $l \geq 2$ to be the smallest integer such that for all words β of length l, there exists $z_\beta \in C[\beta]$ with $P^{p(l)}(z_\beta) = z_\beta$. Equivalently, we define

$$p(l) = \mathrm{lcm}\{p(\beta) : |\beta| = l\},$$

where lcm is the lowest common multiple of the set. Now if for any admissible word β of length l we choose $z_\beta \in C[\beta]$ so that $P^{p(l)}(z_\beta) = z_\beta$, then we have

$$\sum_{|\beta|=l} e^{f^{p(l)}(z_\beta)} \leq \sum_{P^{p(l)}(z)=z} e^{f^{p(l)}(z)}.$$

Finally we bound $e^{f^l(z_\beta)}$ in terms of $e^{f^{p(l)}(z_\beta)}$ to get the estimate we need. For any $l \geq 2$, we have

$$p(l) \leq l + r, \tag{5.4.3}$$

for some integer constant r that depends only on the matrix T. This clearly follows from the irreducibility of T. For each admissible word β of length l we have

$$|f^{p(l)}(z_\beta) - f^l(z_\beta)| \leq \sum_{i=l}^{p(l)-1} f(P^i(z_\beta)) \leq r\|f\|_\infty.$$

So in fact we get the following bound

$$\begin{aligned} \sum_{|\beta|=l} e^{f^l(z_\beta)} &\leq \sum_{|\beta|=l} e^{f^{p(l)}(z_\beta)} \left| e^{f^l(z_\beta) - f^{p(l)}(z_\beta))} \right| \\ &\leq \sum_{|\beta|=l} e^{f^{p(l)}(z_\beta)} e^{r\|f\|_\infty} \\ &\leq e^{r\|f\|_\infty} \sum_{P^{p(l)}(z)=z} e^{f^{p(l)}(z)} \end{aligned} \tag{5.4.4}$$

Claim 5.4.3. *For each $\varepsilon > 0$ we can find $C_\varepsilon > 0$ such that*

$$\sum_{|\beta|=l} e^{f^l(z_\beta)} \leq C_\varepsilon \, e^{l\varepsilon}. \tag{5.4.5}$$

Proof. Using Bowen's result mentioned in (4.2.3) together with the fact that the transfer operator $\mathcal{N}_{a,1}$ is normalised we obtain that

$$\lim_{\lambda \to \infty} \frac{1}{p(l)} \log \sum_{P^{p(l)}(z)=z} e^{f^{p(l)}(z)} = 1.$$

Thus for any $\varepsilon > 0$ we have that for large enough l we get that $\sum_{P^{p(l)}(z)=z} e^{f^{p(l)}(z)} \leq e^{\varepsilon p(l)}$. Therefore there exists a constant $C_\varepsilon > 0$ such that for any $l \geq 2$ we get

$$\sum_{P^{p(l)}(z)=z} e^{f^{p(l)}(z)} \leq C_\varepsilon \, e^{\varepsilon \, p(l)}.$$

This bound in combination with the two bounds obtained in (5.4.3) and (5.4.4) completes the proof of this claim. □

Combining the bounds obtained in Lemma 5.4.2 and Claim 5.4.3 we obtain that for each $\varepsilon > 0$ there exists a constant $C_\varepsilon > 0$ such that for all $l \geq 2$ we have

$$\sum_{|\beta|=l} \left\| \mathcal{N}_{s,\lambda}^l(\chi_\beta) \right\|_{\mathrm{Lip}(d)} \leq C_\varepsilon (\|\lambda_b\| + 1)^W e^{l\varepsilon}. \tag{5.4.6}$$

In particular, it follows from (5.4.2) and Claim 5.4.1 that for any $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$\left\| \hat{Z}_n(s,\lambda) - \sum_{j=1}^{m} \mathcal{N}_{s,\lambda}^n(\chi_j)(u_j) \right\| \leq C_\varepsilon (1 + \|\lambda_b\|)^W \sum_{l=2}^{n} \|\mathcal{N}_{s,\lambda}^{n-l}\|_{C^1} \left(\frac{e^\varepsilon}{\kappa_1} \right)^l.$$

5.5 Proof of Theorem 5.1.1

Recall we fixed a constant α in the interior of $\mathcal{I}$ and set $a = a(\alpha)$ to be the unique real number provided by Lemma 4.4.1. Let $K \subset \mathbb{R}$ be a compact set and let $(I_n)_{n=1}^\infty$ be a sequence of intervals in K. For each $n \in \mathbb{N}$ we denote by p_n the midpoint of the interval I_n and by $\ell_n = \ell(I_n)$ its length. In addition, let $\underline{\vartheta} = (\vartheta_1, \ldots, \vartheta_r)$ be a vector of rotation angles with $-\frac{\kappa}{2} < \vartheta_1 < \cdots < \vartheta_r < \frac{\kappa}{2}$ where $r = \lfloor \frac{N-1}{2} \rfloor$ and $\kappa \in (0,1)$ is arbitrary. Then let A_n be a sequence of target sets for the holonomies as defined in (4.3.2) with corresponding centres given by $\underline{\vartheta}$ and corresponding lengths given by the sequence $(\kappa_n)_{n=1}^{\infty}$.

Furthermore, suppose that $(\ell_n^{-1})_{n=1}^{\infty}$ and $(\kappa_n^{-1})_{n=1}^{\infty}$ have sub-exponential growth. Then we write

$$\begin{aligned}\pi_{\mathcal{R}}(n, a, I_n, A_n) &= \sum_{|\gamma|_{\mathcal{R}}=n} \mathbb{1}_{I_n}\left(l(\gamma) - n\alpha\right) \mathbb{1}_{A_n}\left(E(h_\gamma)\right) \\ &= \sum_{|\gamma|_{\mathcal{R}}=n} \mathbb{1}_{[-\frac{1}{2},\frac{1}{2}]}\left(\ell_n^{-1}\left(l(\gamma) - n\alpha - p_n\right)\right) \mathbb{1}_{[-\frac{\kappa}{2},\frac{\kappa}{2}]^r}\left(\frac{\kappa}{\kappa_n}(E(h_\gamma) - \underline{\vartheta})\right).\end{aligned}$$

5.5.1 Some auxiliary estimates

We fix a compactly supported function $\phi \in C^\infty(\mathbb{R}, \mathbb{R}_{\geq 0})$ and a compactly supported function $\psi \in C^\infty(\mathbb{R}^r, \mathbb{R}_{\geq 0})$ with $\mathrm{supp}(\psi) \subset \left(-\frac{1}{2}, \frac{1}{2}\right)^r$ and set

$$\pi_{\phi,\psi,\mathcal{R}}(n) := \sum_{|\gamma|_{\mathcal{R}}=n} \phi(\ell_n^{-1}\left(l(\gamma) - n\alpha - p_n\right))\psi\left(\frac{\kappa}{\kappa_n}(E(h_\gamma) - \underline{\vartheta})\right).$$

We study the asymptotic behaviour of $\pi_{\phi,\psi,\mathcal{R}}(n)$ to deduce our result using an approximation argument in the next subsection. We define

$$\phi_n(x) := \phi(\ell_n^{-1}\left(x - p_n\right))e^{-a(x-p_n)} \quad \text{and} \quad \psi_n(h_\gamma) := \psi\left(\frac{\kappa}{\kappa_n}(E(h_\gamma) - \underline{\vartheta})\right),$$

so that

$$\pi_{\phi,\psi,\mathcal{R}}(n) = \sum_{|\gamma|_{\mathcal{R}}=n} \phi_n(l(\gamma) - n\alpha)\, \psi_n\left(h_\gamma\right)\, e^{a(l(\gamma)-n\alpha-p_n)}.$$

Note that for all $n \in \mathbb{N}$, $\phi_n \in C^\infty(\mathbb{R}, \mathbb{R})$ is compactly supported and $\psi_n \in C^\infty(M, \mathbb{R})$ is a smooth class function. We start by changing the summation over $|\gamma|_{\mathcal{R}} = n$, that is primitive closed geodesics of word length n, to a sum over periodic points of $P: U \to U$ of period length n. This set also contains words corresponding to non-primitive closed geodesics. In the following lemma we bound the error from these non-primitive closed geodesics. Setting

$$\tilde{\pi}_{\phi,\psi,\mathcal{R}}(n) := \frac{1}{n} \sum_{P^n(u)=u} \phi_n(\tau^n(u) - n\alpha)\, \psi_n\left(\theta^n(u)\right)\, e^{a(\tau^n(u)-n\alpha-p_n)}, \tag{5.5.1}$$

we get the following relationship between the two counting numbers.

Lemma 5.5.1. *For all $\eta > 0$ we have that*

$$\pi_{\phi,\psi,\mathcal{R}}(n) = \tilde{\pi}_{\phi,\psi,\mathcal{R}}(n) + O\left(e^{(H(\alpha)+\eta)n/2}\right).$$

Proof. For a closed geodesic $\gamma \in \mathcal{G}$ of word length n and a corresponding P-orbit $\{u, P(u), \ldots, P^{n-1}(u)\}$ we have the identities

$$\begin{aligned} l(\gamma) &= \tau^n(u) = \tau(u) + \cdots + \tau(P^{n-1}(u)) \quad \text{and} \\ h_\gamma &\ni \theta^n(u) = \prod_{i=0}^{n-1} \theta(P^i(u)), \end{aligned}$$

where the product is in ascending order from left to right. It suffices to bound the summands corresponding to non-primitive τ-orbits. Call a fixed point u of the iterated map P^n non-primitive when there exists q, a proper divisor of n such that $P^q(u) = u$. We thus have

$$\begin{aligned} \tilde{\pi}_{\phi,\psi}(n) - \pi_{\phi,\psi}(n) =& \frac{1}{n} \sum_{\substack{P^n(u)=u \\ \text{non-primitive}}} \phi(\ell^{-1}(\tau^n(u) - n\alpha - p_n))\psi_n(\theta^n(u)) \\ =& O\left(\frac{\|\psi_n\|_\infty}{n} \sum_{\substack{q|n \\ q\leq n/2}} \sum_{P^q(u)=u} \phi(\ell^{-1}(\tau^n(u) - n\alpha - p_n))\right. \\ =& O\left(\frac{\|\psi\|_\infty}{n} \sum_{\substack{q|n \\ q\leq n/2}} \sum_{P^q(u)=u} \frac{\phi(\ell^{-1}(\tau^n(u) - n\alpha - p_n))}{e^{a(\tau^q(u)-q\alpha)}} e^{a(\tau^q(u)-q\alpha)}\right). \end{aligned}$$

We are only interested in periodic points which satisfy $\ell_n^{-1}(\tau^n(u) - n\alpha - p_n) \in \operatorname{supp}\phi$ that is when $\tau^n(u) - n\alpha \in p_n + \ell_n \operatorname{supp}\phi$. Recalling that the intervals I_n were chosen inside a compact subset of $\mathbb{R}$ we conclude that for such a periodic point the absolute value of $\tau^n(u) - n\alpha$ is bounded independently of n and u. Therefore for a non-primitive periodic point u, satisfying $P^q(u) = u$ for q as above, we get that $\tau^q(u) - q\alpha = \frac{q}{n}(\tau^n(u) - n\alpha)$ and thus $e^{a(\tau^q(u)-q\alpha)}$ is bounded from below. From this we conclude using Lemma 5.3.6 that for any $\eta > 0$,

$$
\begin{aligned}
&\frac{1}{n}\sum_{\substack{q|n\\ q\le n/2}}\sum_{P^q(u)=u}\frac{\phi(\ell^{-1}(\tau^n(u)-n\alpha-p_n))}{e^{a(\tau^q(u)-q\alpha)}}e^{a(\tau^q(u)-q\alpha)}\\
=&O\left(\frac{\|\phi\|_\infty}{n}\sum_{\substack{q|n\\ q\le n/2}}\sum_{P^q(u)=u}e^{a(\tau^q(u)-q\alpha)}\right)\\
=&O\left(\frac{\|\phi\|_\infty}{n}\sum_{q\le n/2}e^{-aq\alpha}Z_q(a,1)\right)=O\left(\frac{\|\phi\|_\infty}{n}\sum_{q\le n/2}e^{\left(\eta+\mathrm{Pr}(a\tau)-a\alpha\right)q}\right)\\
=&O\left(\|\phi\|_\infty\, e^{\left(\eta+H(\alpha)\right)n/2}\right).
\end{aligned}
$$

□

The proof of our theorem will follow from the next proposition.

Proposition 5.5.2.

$$\tilde{\pi}_{\phi,\psi,\mathcal{R}}(n)\sim e^{-ap_n}\frac{\int_{\mathbb{R}}\phi_n\int_M\psi_n}{\sigma_\alpha\sqrt{2\pi}}\frac{e^{H(\alpha)n}}{n^{3/2}},\qquad as\ n\to\infty.$$

To prove this proposition we consider

$$\Pi(n):=\left|\frac{\sigma_\alpha\sqrt{2\pi n^3}}{e^{H(\alpha)n-ap_n}}\tilde{\pi}_{\phi,\psi,\mathcal{R}}(n)-\int_{\mathbb{R}}\phi_n\int_M\psi_n\right|,$$

and show that $\Pi(n)\to 0$ as $n\to\infty$. The following proposition provides us with an initial bound. We use Fourier inversion for the sequence of functions $\phi_n\in C^\infty(\mathbb{R},\mathbb{R})$ and write

$$\phi_n(x)=\int_{-\infty}^{\infty}\hat{\phi}_n(t)\,e^{2\pi itx}\,dt. \tag{5.5.2}$$

Recall that the functions $\psi_n : M\to\mathbb{R}$ are smooth class functions. We therefore consider the character of each unitary irreducible representation $\lambda\in\widehat{M}$ and denote it by

$$\chi_\lambda(m):=\mathrm{tr}(\lambda(m)).$$

Using the Peter–Weyl Theorem we can express the sequence of functions ψ_n as follows

$$\psi_n(m) = \sum_{\lambda\in\widehat{M}} \langle \psi_n, \chi_\lambda \rangle_{L^2}\, \chi_\lambda(m), \tag{5.5.3}$$

where $\langle \psi_n, \chi_\lambda\rangle_{L^2} = \int_M \psi_n(m)\chi_\lambda(m^{-1})\, dm$ and dm is the Haar probability measure on M.

Proposition 5.5.3. *We can bound $\Pi(n)$ above by the following expression*

$$\int_{-\infty}^{\infty} \left| \sum_{\lambda\in\widehat{M}} \frac{\langle \psi_n, \chi_\lambda\rangle_{L^2}}{e^{\Pr(a\tau)n}}\, \hat{\phi}_n\left(\frac{t}{2\pi\sigma\sqrt{n}}\right) \operatorname{tr}\, Z_n\left(a + \frac{it}{\sigma\sqrt{n}}, \lambda\right) - e^{-\frac{t^2}{2}+\frac{it\alpha\sqrt{n}}{\sigma\alpha}} \int_{\mathbb{R}} \phi_n \int_M \psi_n \right| dt.$$

Proof. Using (5.5.2) and (5.5.3) we get

$$\begin{aligned}
&\frac{\sigma_\alpha\sqrt{2\pi n^3}}{e^{H(\alpha)n - ap_n}}\, \tilde{\pi}_{\phi,\psi,\mathcal{R}}(n) \\
&= \frac{\sigma_\alpha\sqrt{2\pi n}}{e^{H(\alpha)n}} \sum_{P^n(u)=u} \int_{-\infty}^{\infty} \hat{\phi}_n(t) e^{(a+2\pi it)(\tau^n(u) - n\alpha)}\, dt \sum_{\lambda\in\widehat{M}} \langle \psi_n, \chi_\lambda\rangle_{L^2}\, \chi_\lambda(\theta^n(u)) \\
&= \frac{e^{-(H(\alpha)+a\alpha)n}}{\sqrt{2\pi}} \sum_{P^n(u)=u} \int_{-\infty}^{\infty} e^{\frac{-it\sqrt{n}\alpha}{\sigma_\alpha}}\, \hat{\phi}_n\left(\frac{t}{2\pi\sigma_\alpha\sqrt{n}}\right) e^{\left(a+\frac{it}{\sigma_\alpha\sqrt{n}}\right)\tau^n(u)}\, dt \\
&\qquad \sum_{\lambda\in\widehat{M}} \langle \psi_n, \chi_\lambda\rangle_{L^2}\, \operatorname{tr}(\lambda(\theta^n(x))).
\end{aligned}$$

The smoothness of the functions ϕ_n and ψ_n allows us to change the order of summation using Fubini's theorem. A more precise statement will appear later with the decay estimates for the Fourier coefficients of these functions. We have

$$\begin{aligned}
&\frac{\sigma_\alpha\sqrt{2\pi n^3}}{e^{H(\alpha)n - ap_n}}\tilde{\pi}_{\phi,\psi,\mathcal{R}}(n) \\
&= \frac{e^{-\Pr(a\tau)n}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sum_{\lambda\in\widehat{M}} \frac{\langle \psi_n, \chi_\lambda\rangle_{L^2}}{e^{it\alpha\sqrt{n}/\sigma_\alpha}}\, \hat{\phi}_n\left(\frac{t}{2\pi\sigma_\alpha\sqrt{n}}\right) \sum_{P^n(u)=u} e^{\left(a+\frac{it}{\sigma_\alpha\sqrt{n}}\right)\tau^n(x)} \operatorname{tr}(\lambda(\theta^n(x)))\, dt \\
&= \int_{-\infty}^{\infty} \sum_{\lambda\in\widehat{M}} \frac{\langle \psi_n, \chi_\lambda\rangle_{L^2}\, e^{\frac{-it\alpha\sqrt{n}}{\sigma_\alpha}}}{e^{\Pr(a\tau)n}\sqrt{2\pi}}\, \hat{\phi}_n\left(\frac{t}{2\pi\sigma_\alpha\sqrt{n}}\right) \operatorname{tr}\, Z_n\left(a + \frac{it}{\sigma_\alpha\sqrt{n}}, \lambda\right) dt.
\end{aligned}$$

Additionally, recalling that $\int_{-\infty}^{\infty} e^{-t^2/2}\, dt = \sqrt{2\pi}$ we get that $\sqrt{2\pi}\,\Pi(n)$ is equal to

$$\left| \int_{-\infty}^{\infty} \sum_{\lambda\in\widehat{M}} \frac{\langle \psi_n, \chi_\lambda\rangle_{L^2}}{e^{\Pr(a\tau)n + \frac{it\alpha\sqrt{n}}{\sigma_\alpha}}}\, \hat{\phi}_n\left(\frac{t}{2\pi\sigma_\alpha\sqrt{n}}\right) \operatorname{tr}\, Z_n\left(a + \frac{it}{\sigma_\alpha\sqrt{n}}, \lambda\right) - e^{-\frac{t^2}{2}} \int_{\mathbb{R}} \phi_n \int_M \psi_n\, dt \right|.$$

Finally an application of the triangle inequality proves the result. ∎

Consider now the following three quantities.

$$
\begin{aligned}
\Pi_1(n) &:= \int_{-\varepsilon\sigma\sqrt{n}}^{\varepsilon\sigma\sqrt{n}} \left| \sum_{\lambda\in\widehat{M}} \frac{\langle \psi_n, \chi_\lambda\rangle_{L^2}}{e^{\Pr(a\tau)n}} \hat{\phi}_n\left(\frac{t}{2\pi\sigma_\alpha\sqrt{n}}\right) \operatorname{tr} Z_n\left(a + \frac{it}{\sigma_\alpha\sqrt{n}}, \lambda\right) \right. \\
&\qquad\qquad\qquad\qquad\qquad\qquad \left. - e^{-\frac{t^2}{2} + \frac{it\alpha\sqrt{n}}{\sigma_\alpha}} \int_{\mathbb{R}} \phi_n \int_M \psi_n \right| dt, \\
\Pi_2(n) &:= \int_{|t|\geq\varepsilon\sigma\sqrt{n}} \left| \sum_{\lambda\in\widehat{M}} \frac{\langle \psi_n, \chi_\lambda\rangle_{L^2}}{e^{\Pr(a\tau)n}} \hat{\phi}_n\left(\frac{t}{2\pi\sigma_\alpha\sqrt{n}}\right) \operatorname{tr} Z_n\left(a + \frac{it}{\sigma_\alpha\sqrt{n}}, \lambda\right) \right| dt, \\
\Pi_3(n) &:= \int_{|t|\geq\varepsilon\sigma\sqrt{n}} \left| e^{-\frac{t^2}{2}} \int_{\mathbb{R}} \phi_n \int_M \psi_n \right| dt,
\end{aligned}
$$

with $\varepsilon > 0$ small enough as in Lemmas 5.2.5 and 5.3.6. It then follows from Proposition 5.5.3 that

$$\Pi(n) \leq \Pi_1(n) + \Pi_2(n) + \Pi_3(n).$$

We hence bound these three quantities separately to show that $\lim_{n\to\infty} \Pi(n) = 0$. To obtain these bounds we first recall a standard result from Fourier Analysis.

Lemma 5.5.4. *For a compactly supported function $\phi \in C^\infty(\mathbb{R}, \mathbb{R})$ with Fourier transform $\hat{\phi}$ we have that $\hat{\phi}(0) = \int_{\mathbb{R}} \phi$ and uniformly for $\phi \in C^\infty(\mathbb{R}, \mathbb{R})$ we have that for each $h \in \mathbb{N}$*

$$\hat{\phi}(u) = O(\|\phi\|_{C^h} |u|^{-h}).$$

This bound follows by repeated applications of integration by parts. In particular, note that there exists $C > 0$ such that for n, $h \in \mathbb{N}$ and $u \in \mathbb{R} \setminus \{0\}$

$$|\hat{\phi}_n(u)| \leq C\, \ell_n^{-h} |u|^{-h} \|\phi\|_{C^h}. \tag{5.5.4}$$

Similarly, we can obtain bounds for the Fourier coefficients $\langle \psi_n, \chi_\lambda\rangle_{L^2}$. We have that

$$\langle \psi_n, \chi_\lambda\rangle_{L^2} = \operatorname{tr}\langle \psi_n, \lambda\rangle_{L^2},$$

and so setting

$$\mathcal{F}_\psi(\lambda) = \langle \psi, \lambda\rangle_{L^2},$$

we observe that the Cauchy–Schwarz inequality gives the following

$$\langle \psi_n, \chi_\lambda \rangle_{L^2} \leq \sqrt{\dim(\lambda)}\, \|\mathcal{F}_{\psi_n}(\lambda)\|. \tag{5.5.5}$$

The following two results will be useful in our analysis.

Proposition 5.5.5 ([Sug71], Theorem 3)**.** *A function $\psi : M \to \mathbb{C}$ is smooth if and only if its Fourier coefficients $\mathcal{F}_\psi(\lambda)$ decay rapidly, i.e. for every $h \in \mathbb{N}$ we have*

$$\lim_{\|\lambda\| \to \infty} \|\lambda\|^h \|\mathcal{F}_\psi(\lambda)\| = 0. \tag{5.5.6}$$

In particular, using equation (1.18) in [Sug71] and the two bounds above we get that for every $h \in \mathbb{N}$

$$\langle \psi_n, \chi_\lambda \rangle_{L^2} = O(\|\psi\|_{C^h} \sqrt{\dim(\lambda)}\, \|\lambda\|^{-h}), \tag{5.5.7}$$

where the implied constant is uniform for $\psi \in C^\infty(M, \mathbb{R})$.

Lemma 5.5.6 ([Sug71], Lemma 1.3)**.** *The series $\sum_{\lambda \neq 1} \|\lambda\|^{-h}$ converges for h strictly larger than the rank of M.*

We are now ready to prove the following proposition.

Proposition 5.5.7. $\lim_{n\to\infty} \Pi_1(n) = 0$.

Proof. Let $\lambda \in \widehat{M}$ be non-trivial. Using the Cauchy–Schwarz inequality we get that $\operatorname{tr} Z_n(s, \lambda) \leq \sqrt{\dim(\lambda)}\, \|Z_n(s, \lambda)\|$. Together with the bounds from (5.3.6) given a fixed $\eta \in (0, 1)$ we get that

$$\operatorname{tr} Z_n\left(a + \frac{it}{\sigma_\alpha \sqrt{n}}, \lambda\right) = O\left(\sqrt{\dim(\lambda)} \left(\frac{|t|}{\sigma\sqrt{n}} + \|\lambda\|\right)^{W+1+\eta} e^{(\Pr(a\tau) - \varepsilon')n}\right).$$

Therefore combining the above bound with the bounds for the Fourier coefficients from (5.5.4) and (5.5.7) we get

$$\int_{-\varepsilon\sigma\sqrt{n}}^{\varepsilon\sigma\sqrt{n}} \left| \sum_{\lambda\neq 1} \frac{\langle \psi_n, \chi_\lambda \rangle_{L^2}}{e^{\Pr(a\tau)n}} \hat{\phi}_n \left(\frac{t}{2\pi\sigma\sqrt{n}} \right) \operatorname{tr} Z_n \left(a + \frac{it}{\sigma\sqrt{n}}, \lambda \right) \right| dt$$
$$= O\left(\|\hat{\phi}_n\|_\infty \|\psi_n\|_{C^h} e^{-\varepsilon' n} \int_{-\varepsilon\sigma\sqrt{n}}^{\varepsilon\sigma\sqrt{n}} \sum_{\lambda\neq 1} \frac{\dim(\lambda)}{\|\lambda\|^h} \left(\frac{|t|}{\sigma\sqrt{n}} + \|\lambda\| \right)^{W+1+\eta} dt \right),$$

for some $\varepsilon' \in (0,1)$. Weyl's dimension formula ensures that there exist uniform constants C, $W > 0$ such that for any non-trivial $\lambda \in \widehat{M}$ we have that $\dim(\lambda) \leq C\|\lambda\|^W$ [Sug71, (2.9)]. Combining this with Lemma 5.5.6 we get that provided h is large enough the error above is equal to $O\left(\|\psi_n\|_{C^h} \|\hat{\phi}_n\|_\infty e^{-\varepsilon' n}\right)$. Since ϕ is compactly supported and $p_n \in K$ we can uniformly bound $\hat{\phi}_n$ for all $n \in \mathbb{N}$. Further, the sequence $\|\psi_n\|_{C^h}$ is of sub-exponential growth since $(\kappa_n^{-1})_{n=1}^\infty$ is. We therefore get that this error tends to zero as $n \to \infty$. We are now left to bound

$$\int_{-\varepsilon\sigma\sqrt{n}}^{\varepsilon\sigma\sqrt{n}} \left| \frac{\int_M \psi_n}{e^{\Pr(a\tau)n}} \hat{\phi}_n \left(\frac{t}{2\pi\sigma\sqrt{n}} \right) \operatorname{tr} Z_n \left(a + \frac{it}{\sigma_\alpha\sqrt{n}}, 1 \right) - e^{-\frac{t^2}{2} + \frac{it\alpha\sqrt{n}}{\sigma\alpha}} \int_{\mathbb{R}} \phi_n \int_M \psi_n \right| dt.$$

Using part (2) from Lemma 5.3.6 we get that for some $\varepsilon'' \in (0,1)$, up to an error bounded by $O\left(e^{-\varepsilon'' n}\right)$ we are left to bound

$$\int_M \psi_n \int_{-\varepsilon\sigma_\alpha\sqrt{n}}^{\varepsilon\sigma\sqrt{n}} \left| \hat{\phi}_n \left(\frac{t}{2\pi\sigma\sqrt{n}} \right) e^{\left(\Pr\left((a+\frac{it}{\sigma_\alpha\sqrt{n}})\tau\right) - \Pr(a\tau)\right)n} - e^{-\frac{t^2}{2} + \frac{it\alpha\sqrt{n}}{\sigma\alpha}} \int_{\mathbb{R}} \phi_n \right| dt.$$

On the domain of integration, we see that as $n \to \infty$

1. $e^{\left(\Pr\left(\left(a+\frac{it}{\sigma_\alpha\sqrt{n}}\right)\tau\right) - \Pr(a\tau) - \frac{it\alpha}{\sigma_\alpha\sqrt{n}}\right)n} \to e^{-t^2/2}$ by Lemma 5.2.5,
2. $\hat{\phi}_n \left(\frac{t}{2\pi\sigma\sqrt{n}} \right) \to \hat{\phi}_n(0) = \int_{\mathbb{R}} \phi_n$ by continuity.

Furthermore, for large n we have the bound $e^{\left(\Pr\left(\left(a+\frac{it}{\sigma_\alpha\sqrt{n}}\right)\tau\right) - \Pr(a\tau)\right)n} \leq e^{-t^2/4}$ and so

$$\left| e^{n\left(\Pr\left(\left(a+\frac{it}{\sigma_\alpha\sqrt{n}}\right)\tau\right) - \Pr(a\tau) + \frac{it\alpha}{\sigma_\alpha\sqrt{n}}\right)} - e^{-t^2/2} \right| \leq 2e^{-t^2/4}.$$

Finally, since $\hat{\phi}_n$ is uniformly bounded, we can apply the Dominated Convergence Theorem to get that $\lim_{n\to\infty} \Pi_1(n) = 0$. □

Proposition 5.5.8. $\lim_{n\to\infty} \Pi_2(n) = 0$.

Proof.

$$\Pi_2(n) \le \sum_{\lambda\in\widehat{M}} \frac{\langle \psi_n, \chi_\lambda\rangle_{L^2}}{e^{\Pr(a\tau)n}} \int_{|t|\ge\varepsilon\sigma\sqrt{n}} \left| \hat{\phi}_n\left(\frac{t}{2\pi\sigma_\alpha\sqrt{n}}\right) \operatorname{tr} Z_n\left(a + \frac{it}{\sigma_\alpha\sqrt{n}}, \lambda\right)\right| dt,$$

For non-trivial $\lambda \in \widehat{M}$ and a fixed $\eta \in (0,1)$ we use inequality (5.3.6) along with Cauchy–Schwarz to bound $\left|\operatorname{tr} Z_n\left(a + \frac{it}{\sigma_\alpha\sqrt{n}}, \lambda\right)\right|$. In addition using the bounds from (5.5.4) and (5.5.7) we get

$$\begin{aligned}
&\sum_{\lambda\ne 1} \frac{\langle \psi_n, \chi_\lambda\rangle_{L^2}}{e^{\Pr(a\tau)n}} \int_{|t|\ge\varepsilon\sigma\sqrt{n}} \left| \hat{\phi}_n\left(\frac{t}{2\pi\sigma_\alpha\sqrt{n}}\right) \operatorname{tr} Z_n\left(a + \frac{it}{\sigma_\alpha\sqrt{n}}, \lambda\right)\right| dt \\
=&O\left(\sum_{\lambda\ne 1} \frac{\|\psi_n\|_{C^h}}{\|\lambda\|^h} \int_{|t|\ge\varepsilon\sigma\sqrt{n}} \left| \left(\frac{t\,\ell_n}{2\pi\sigma\sqrt{n}}\right)^{-(W+3)} \dim(\lambda) \left(\left|\frac{t}{\sigma\sqrt{n}}\right| + \|\lambda\|\right)^{W+1+\eta} e^{-\varepsilon'' n}\right| dt\right) \\
=&O\left(\frac{n^{W+3}\|\psi_n\|_{C^h}}{\ell_n^{W+3}\, e^{\varepsilon'' n}} \sum_{\lambda\ne 1} \int_{|t|\ge\varepsilon\sigma\sqrt{n}} \frac{\dim(\lambda)\big(\,|t/\sigma\sqrt{n}| + \|\lambda\|\big)^{W+1+\eta}}{t^{W+3}\|\lambda\|^h}\, dt\right) \\
=&O\left(\frac{n^{W+3}\|\psi\|_{C^h}}{\ell_n^{W+3}\,\kappa_n^h}\, e^{-\varepsilon'' n}\right),
\end{aligned}$$

for some $\varepsilon'' > 0$, provided $h \in \mathbb{N}$ is large enough as in Proposition 5.5.7. On the other hand, for $\lambda = 1$ we use two separate bounds. Firstly we use part (2) of Lemma 5.3.6 to get that for some $\varepsilon' > 0$

$$\frac{\int_M \psi_n}{e^{\Pr(a\tau)n}} \int_{|t|\ge\varepsilon\sigma_\alpha\sqrt{n}}^{|t|\le\sigma_\alpha\sqrt{n}} \left| \hat{\phi}_n\left(\frac{t}{2\pi\sigma_\alpha\sqrt{n}}\right) \operatorname{tr} Z_n\left(a + \frac{it}{\sigma_\alpha\sqrt{n}}, 1\right)\right| dt = O\left(\|\hat{\phi}_n\|_\infty\, e^{-\varepsilon' n} \int_M \psi_n\right).$$

Finally, as above we can use inequality (5.3.6) to bound the remaining by,

$$\begin{aligned}
&\frac{\int \psi_n}{e^{\Pr(a\tau)n}} \int_{|t|\ge\sigma_\alpha\sqrt{n}} \left| \hat{\phi}_n\left(\frac{t}{2\pi\sigma_\alpha\sqrt{n}}\right) \operatorname{tr} Z_n\left(a + \frac{it}{\sigma_\alpha\sqrt{n}}, 1\right)\right| dt \\
=&O\left(\int_{|t|\ge\sigma_\alpha\sqrt{n}} \left(\frac{t\,\ell_n}{2\pi\sigma\sqrt{n}}\right)^{-(W+3)} \left|\frac{t}{\sigma\sqrt{n}}\right|^{W+1+\eta} e^{-\varepsilon'' n}\, dt\right) \\
=&O\left(\frac{n^{W+3}\, e^{-\varepsilon'' n}}{\ell_n^{W+3}} \int_{|t|\ge\sigma\sqrt{n}} |t|^{\eta-2}\, dt\right) = O\left(\frac{n^{W+3}}{\ell_n^{W+3}} e^{-\varepsilon'' n}\right).
\end{aligned}$$

Combining the three bounds obtained above and recalling that the sequences $(\ell_n^{-1})_{n=1}^\infty$ and $(\kappa_n^{-1})_{n=1}^\infty$ are of sub-exponential growth we obtain that $\lim_{n\to\infty} \Pi_2(n) = 0$.

□

Finally, since as we discussed above $\int_{\mathbb{R}} \phi_n$ and $\int_M \psi_n$ are uniformly bounded it is clear that $\lim_{n\to\infty} \Pi_3(n) = 0$. This completes the proof of Proposition 5.5.2.

5.5.2 Approximation argument

Here we show how the previous auxiliary estimates provide us with the proof of Theorem 5.1.1 through an approximation argument. By Proposition 5.5.2 and Lemma 5.5.1 we have that for all compactly supported $\phi \in C^\infty(\mathbb{R},\mathbb{R})$ and all $\psi \in C^\infty(\mathbb{R}^r,\mathbb{R})$ with $\mathrm{supp}(\psi) \subset \left(-\frac{1}{2},\frac{1}{2}\right)^r$ we have

$$\pi_{\phi,\psi,\mathcal{R}}(n) \sim e^{-ap_n} \frac{\int_{\mathbb{R}} \phi_n \int_M \psi_n}{\sigma\sqrt{2\pi}} \frac{e^{H(\alpha)n}}{n^{3/2}}, \qquad \text{as } n \to \infty. \tag{5.5.8}$$

Fixing $\eta \in (0, \frac{1-\kappa}{2})$ we wish to construct compactly supported $\phi \in C^\infty(\mathbb{R},\mathbb{R})$ and $\psi \in C^\infty(\mathbb{R}^r,\mathbb{R})$ satisfying the following:

$$\begin{aligned}
&\mathbb{1}_{[-\frac{1}{2},\frac{1}{2}]} \le \phi \le 1+\eta, \ \ \mathrm{supp}(\phi) \subset \left[-\frac{1+\eta}{2}, \frac{1+\eta}{2}\right] \ \text{ and } \int_{\mathbb{R}} \phi \le 1+\eta,\\
&\mathbb{1}_{[-\frac{\kappa}{2},\frac{\kappa}{2}]^r} \le \psi \le 1+\eta, \ \ \mathrm{supp}(\psi) \subset \left[-\frac{\kappa+\eta}{2}, \frac{\kappa+\eta}{2}\right]^r \ \text{ and } \int_{\mathbb{R}^r} \psi \le \kappa^r+\eta.
\end{aligned}$$

A smooth function $\Phi_n : \mathbb{R}^n \to \mathbb{R}_{\ge 0}$ is called a positive mollifier, if it satisfies the following properties:

1. it is compactly supported,
2. $\int_{\mathbb{R}^n} \Phi_n = 1$,
3. $\lim_{\varepsilon\to 0} \Phi_{n,\varepsilon}(x) := \lim_{\varepsilon\to 0} \varepsilon^{-1}\Phi_n(x/\varepsilon) = \delta(x)$ where $\delta(x)$ is the Dirac delta function.

Let $\gamma_1,\dots,\gamma_4 > 0$ and set $G = (1+\gamma_1)\mathbb{1}_{[-\frac{1}{2}-\gamma_2,\frac{1}{2}+\gamma_2]}$ and $H = (1+\gamma_3)\mathbb{1}_{[-\frac{\kappa}{2}-\gamma_4,\frac{\kappa}{2}+\gamma_4]^r}$. Then for sufficiently small $\varepsilon,\gamma_1,\dots,\gamma_4 > 0$ the functions

$$\phi = G * \Phi_{1,\varepsilon} \qquad \text{and} \qquad \psi = H * \Phi_{r,\varepsilon}\,,$$

satisfy all the required properties. Note that since $\kappa < 1$ and provided that the constants ε, γ_4 were chosen sufficiently small it is harmless to assume that $\mathrm{supp}(\psi) \subset \left(-\frac{1}{2},\frac{1}{2}\right)^r$.

Using (5.5.8) and the properties above we can deduce that

$$\begin{aligned}
&\limsup_{n\to\infty} \frac{\sigma_\alpha\sqrt{2\pi n^3}}{e^{H(\alpha)n}} \pi_{\mathcal{R}}(n,\alpha,I_n,A_n) \\
&= \limsup_{n\to\infty} \frac{\sigma_\alpha\sqrt{2\pi n^3}}{e^{H(\alpha)n}} \sum_{|\gamma|_{\mathcal{R}}=n} \mathbb{1}_{[-\frac{1}{2},\frac{1}{2}]}\left(\ell_n^{-1}(l(\gamma)-n\alpha-p_n)\right) \mathbb{1}_{[-\frac{\kappa}{2},\frac{\kappa}{2}]^r}\left(\frac{\kappa}{\kappa_n}(E(h_\gamma)-\underline{\vartheta})\right) \\
&\le \limsup_{n\to\infty} \frac{\sigma\sqrt{2\pi n^3}}{e^{H(\alpha)n}} \sum_{|\gamma|_{\mathcal{R}}=n} \phi\left(\ell_n^{-1}(l(\gamma)-n\alpha-p_n)\right)\psi\left(\frac{\kappa}{\kappa_n}(E(h_\gamma)-\underline{\vartheta})\right) \\
&= \limsup_{n\to\infty} e^{-ap_n}\int_{\mathbb{R}}\phi_n\int_M\psi_n.
\end{aligned}$$

We have

$$\begin{aligned}
\int_M \psi_n &= \int_M \psi\left(\frac{\kappa}{\kappa_n}(E(m)-\underline{\vartheta})\right)dm = \int_M \psi\left(\frac{\kappa}{\kappa_n}E(m)\right)dm \\
&= \frac{\kappa_n^r}{\kappa^r}\int_M \psi(E(m))\,dm = \frac{\kappa_n^r}{\kappa^r}\int_{\mathbb{R}^r}\psi \le \kappa_n^r + \frac{\eta}{\kappa^r} = \nu(A_n)+O(\eta).
\end{aligned}$$

Similarly,

$$\begin{aligned}
\int_{\mathbb{R}}\phi_n &= \int_{\mathbb{R}}\phi(\ell_n^{-1}(x-p_n))e^{-a(x-p_n)}\,dx = \ell_n\int_{\mathbb{R}}\phi(u)e^{-a\ell_n u}\,du \\
&= \ell_n\int_{[-\frac{1+\eta}{2},\frac{1+\eta}{2}]}\phi(u)e^{-a\ell_n u}\,du \le \ell_n\int_{[-\frac{1}{2},\frac{1}{2}]}\phi(u)e^{-a\ell_n u}\,du + \eta(1+\eta)e^{2(1+|a|)|K|}|K|.
\end{aligned}$$

$$\begin{aligned}
\ell_n\int_{[-\frac{1}{2},\frac{1}{2}]}\phi(u)e^{-a\ell_n u}\,du &\le \ell_n\int_{[-\frac{1}{2},\frac{1}{2}]}(1+\eta)e^{-a\ell_n u}\,du \\
&\le e^{ap_n}\int_{I_n}e^{-au}\,du + \eta e^{(1+|a|)|K|}|K|.
\end{aligned}$$

Therefore,

$$e^{-ap_n}\int_{\mathbb{R}}\phi_n\int_M\psi_n \le \nu(S_n)\int_{I_n}e^{-au}\,du + O(\eta).$$

Similarly, one can show that

$$\liminf_{n\to\infty}\frac{\sigma_\alpha\sqrt{2\pi n^3}}{e^{H(\alpha)n}}\pi(n,\alpha,I_n,A_n) \ge \liminf_{n\to\infty}\left(\nu(A_n)\int_{I_n}e^{-ax}\,dx\right) + O(\eta).$$

Since the choice of $\eta > 0$ was arbitrary we get the result.

Dolgopyat-type estimates

In this appendix we present the ideas required from sections $6-9$ of [SW21] to provide a sketch proof for Theorem 5.3.4. This theorem is motivated by Dolgopyat's ideas presented in his thesis, and later in [Dol98], where combining geometric considerations with ideas from thermodynamic formalism, he introduced a strategy to obtaining sufficiently good bounds on the iterates of transfer operators depending on parameters in a non-compact region. Particularly, in [Dol98] using these methods he proved the exponential decay of correlations of the geodesic flow on the unit tangent bundle of negatively curved compact surfaces. His extended methods assert the exponential mixing of Anosov flows on negatively curved compact manifolds. Following Dolgopyat's work many others attempted to extend his results by adapting his arguments. Notably, as mentioned previously in the introduction, Naud adapted Dolgopyat's ideas in [Nau05] for the setup of the geodesic flow in a negatively curved convex-cocompact surface. Stoyanov in [Sto11] proved the exponential mixing of contact Anosov flows on higher dimensional compact manifolds of variable negative curvature using a symbolic method, under some geometric and regularity conditions. Moreover, in [AGY06] the authors used Dolgopyat–type arguments to obtain the exponential mixing of the Teichmüller flow in the moduli space of Abelian differentials but crucially they only obtained their results by working with John domains, perhaps suggesting the need for extra structure in the Markov partition for the Dolgopyat approach to be extended in higher dimensions. More recently, in [BW20] there was an attempt to prove the exponential mixing of Anosov flows while reducing the required C^1 regularity of both the stable and unstable leaves, as it appeared in Dolgopyat's work in [Dol98], to that of only the stable leaves.

Our approach will closely follow that of [SW21] which is essentially following a Dolgopyat–type approach. This approach avoids dealing with some issues arising from the possibly complicated structure of the rectangles in the Markov section, for example the assumption of connectivity of the open neighbourhoods $\tilde{U}_j$ which in lower dimensional cases are just open intervals (see the proof of Lemma A.4.6 for more details). The only real difference in our approach is that we need to consider other equilibrium states to the one they considered in [SW21]. However each equilibrium state we use is still doubling and satisfies all the required properties needed in order to proceed with our proof in the same way as in [SW21]. We start by presenting the two main technical ideas that allow us to use the Dolgopyat arguments.

A.1 Local non-integrability condition and non-concentration property

We will need a non-integrability type condition *(LNIC)* to run the Dolgopyat argument. The appropriate formulation in our setting is presented in the following subsection and proved in Proposition A.1.6. In addition the presence of holonomies also requires Proposition A.1.7 which we call the *non-concentration property (NCP).*

A.1.1 Local non-integrability condition

First, we will define a function related to Brin–Pesin [Bri82] moves which will be needed for the LNIC in our setting. We fix unique isometric lifts

$$\overline{\tilde{R}}_j = [\overline{\tilde{U}}_j, \overline{S}_j] \subset \mathrm{T}^1(\mathbb{H}^N) \quad \text{of } \tilde{R}_j \text{ for all } j \in \mathcal{A}.$$

Define

$$\overline{\tilde{R}} = \bigsqcup_{j\in\mathcal{A}} \overline{\tilde{R}}_j \qquad \text{and} \qquad \overline{\tilde{U}} = \bigsqcup_{j\in\mathcal{A}} \overline{\tilde{U}}_j.$$

For all $u \in \tilde{R}$, let $\overline{u} \in \overline{\tilde{R}}$ denote the unique lift in $\overline{\tilde{R}}$. We then lift the section F to $\overline{F} : \bigsqcup_{\gamma\in\Gamma} \gamma\overline{\tilde{R}} \to \mathrm{F}(\mathbb{H}^n)$ in the natural way.

Definition A.1.1 (Associated sequence of frames)**.** Let $z_1 \in \tilde{R}_1$ be the centre. Consider some sequence of tangent vectors $(z_1, z_2, z_3, z_4, z_5) \in (\tilde{R}_1)^5$ such that $z_2 \in S_1$, $z_4 \in \tilde{U}_1$ and $z_3 = [z_4, z_2]$. Its lift to the universal cover is $(\overline{z}_1, \overline{z}_2, \overline{z}_3, \overline{z}_4, \overline{z}_5) \in (\overline{\tilde{R}}_1)^5 \subset \mathrm{T}^1(\mathbb{H}^n)^5 \cong (G/M)^5$. We define an *associated sequence of frames* to be the sequence $(g_1, g_2, \dots, g_5) \in \mathrm{F}(\mathbb{H}^n)^5 \cong G^5$ by "moving the frame $\overline{F}(\overline{z}_1)$ only along the strong unstable and strong stable directions" corresponding to the path represented by the sequence $(\overline{z}_1, \overline{z}_2, \overline{z}_3, \overline{z}_4, \overline{z}_5)$. Recalling the definition of $\underline{\tau}$ from Chapter 4 we have

$$g_1 = \overline{F}(\overline{z}_1),$$
$$g_2 = \overline{F}(\overline{z}_2) \in g_1 H^- \text{ such that } g_2 M = \overline{z}_2 \in \mathrm{T}^1(\mathbb{H}^n) \cong G/M,$$
$$g_3 \in g_2 H^+ \text{ such that } g_3 a_t M = \overline{z}_3 \in \mathrm{T}^1(\mathbb{H}^n) \cong G/M \text{ for some } t \in (-\underline{\tau}, \underline{\tau}),$$
$$g_4 \in g_3 H^- \text{ such that } g_4 a_t M = \overline{z}_4 \in \mathrm{T}^1(\mathbb{H}^n) \cong G/M \text{ for some } t \in (-\underline{\tau}, \underline{\tau}),$$
$$g_5 \in g_4 H^+ \text{ such that } g_5 a_t M = \overline{z}_5 = \overline{z}_1 \in \mathrm{T}^1(\mathbb{H}^n) \cong G/M \text{ for some } t \in (-\underline{\tau}, \underline{\tau}).$$

Remark A.1.2. Using properties of the strong unstable and strong stable leaves, we see that $t \in (-\underline{\tau}, \underline{\tau})$ must be the same throughout the sequence in the definition above.

We continue using the notation in the above definition. Define the open set

$$H_1^+ = \{h^+ \in H^+ : F(z_1)h^+ \in F(\tilde{U}_1)\} \subset H^+,$$

and the compact set

$$H_1^- = \{h^- \in H^- : F(z_1)h^- \in F(S_1)\} \subset H^-.$$

Now, if the above sequence $(z_1, z_2, z_3, z_4, z_5)$ is corresponding to some $h^+ \in H_1^+$ and some $h^- \in H_1^-$ such that $F(z_4) = F(z_1)h^+$ and $F(z_2) = F(z_1)h^-$ respectively, then we can define the map

$$\Xi : H_1^+ \times H_1^- \to AM \quad \text{by} \quad \Xi(h^+, h^-) = {g_5}^{-1} g_1 \in AM.$$

To view it as a function of only the first coordinate for a fixed $h^- \in H_1^-$, we write $\Xi_{h^-} : H_1^+ \to AM$.

Let $z_1 \in \tilde{R}_1$ be the centre of the rectangle. Let $j \in \mathbb{N}$ and $\omega = (\omega_0, \omega_2, \dots, \omega_{j-1}, 1)$ be an admissible word. By following the definitions, there exists an element which we denote by $h_\omega \in H_1^-$ such that

$$F(\mathcal{P}^j(P^{-\omega}(z_1))) = F(z_1)h_\omega.$$

This is well-defined since $P^{-\omega}(z_1) \in \mathtt{C}[\omega] \subset U$.

In order to derive the LNIC in Proposition A.1.6, we first start with a few useful lemmas regarding $\Xi : H_1^+ \times H_1^- \to AM$ [SW21, Lemmas 6.2–6.4].

Lemma A.1.3. *Let $j \in \mathbb{N}$, $\omega = (\omega_0, \omega_1, \ldots, \omega_{j-1}, 1)$ be an admissible sequence and $h^- = h_\omega \in H_1^-$. Let $u \in \tilde{U}_1$ and $h^+ \in H_1^+$ such that $F(u) = F(z_1)h^+$ where $z_1 \in \tilde{R}_1$ is the centre of the rectangle. Then, we have*

$$\Xi(h^+, h^-) = \Phi^\omega(P^{-\omega}(z_1))^{-1}\Phi^\omega(P^{-\omega}(u)).$$

Recall from definitions that $e \in H_1^+$ where $H_1^+ \subset H^+$ is an open subset and hence $\mathrm{T}_e(H_1^+) = \mathrm{T}_e(H^+) = \mathfrak{h}^+$. Note that $AMH^+H^- \subset G$ is an open dense subset and hence we have the vector space decomposition $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{m} \oplus \mathfrak{h}^+ \oplus \mathfrak{h}^-$. Denote the projection onto $\mathfrak{a} \oplus \mathfrak{m}$ with respect to this decomposition by $\pi : \mathfrak{g} \to \mathfrak{a} \oplus \mathfrak{m}$. We then have Lemma A.1.4 where ϵ_0 is as in section 4.2.1 and set

$$H^-_{1,\epsilon_0} = \left\{h^- \in H^- : F(z_1)h^- \in F\left(W^{\mathrm{ss}}_{\epsilon_0}(z_1)\right)\right\},$$

where $z_1 \in \tilde{R}_1$ is the centre of the rectangle.

Lemma A.1.4. *For all $h^- \in H_1^-$, we have*

$$(d\Xi_{h^-})_e(\omega) = \pi(\mathrm{Ad}_{h^-}((df_{h^-})_e(\omega))),$$

for all $\omega \in \mathfrak{h}^+$ where $f_{h^-} : H_1^+ \to H^+$ is a diffeomorphism onto its image which is also smooth in $h^- \in H^-_{1,\epsilon_0}$ and satisfies $f_e = \mathrm{Id}_{H_1^+}$. Moreover, the image $(d\Xi_{h^-})_e(\mathfrak{h}^+) \subset \mathfrak{a} \oplus \mathfrak{m}$ is the projection $(d\Xi_{h^-})_e(\mathfrak{h}^+) = \pi(\mathrm{Ad}_{h^-}(\mathfrak{h}^+))$.

Throughout this appendix it is often convenient to use the upper half space model

$$\mathbb{H}^N \cong \{(x_1, x_2, \ldots, x_N) \in \mathbb{R}^N : x_N > 0\},$$

with boundary at infinity

$$\partial_\infty(\mathbb{H}^N) \cong \{(x_1, x_2, \ldots, x_N) \in \mathbb{R}^N : x_N = 0\} \cup \{\infty\} \cong \mathbb{R}^{N-1} \cup \{\infty\}.$$

We also use the isometry

$$\mathrm{T}(\mathbb{H}^N) \cong \mathbb{H}^N \times \mathbb{R}^N.$$

Let $(e_1, e_2, \ldots, e_N)$ be the standard basis on $\mathbb{R}^N$. We assume without loss of generality that the identifications are made such that the reference vector is $v_0 = (e_N, e_N)$ and the reference frame is $F_0 = ((e_N, e_1), (e_N, e_2), \ldots, (e_N, e_N))$ where the first entries of the tangent vectors are their basepoints. Let d_{E} denote the Euclidean distance. Let $B_\epsilon^{\mathrm{E}}(x) \subset \mathbb{R}^{N-1}$ denote the open Euclidean ball of radius $\epsilon > 0$ centred at $x \in \mathbb{R}^{N-1}$.

Lemma A.1.5. *There exist $h_1^-, h_2^-, \ldots, h_{j_{\mathrm{m}}}^- \in H_1^-$ for some $j_{\mathrm{m}} \in \mathbb{N}$ and $\delta > 0$ such that if $\eta_1^-, \eta_2^-, \ldots, \eta_{j_{\mathrm{m}}}^- \in H_1^-$ with $d_{H^-}(\eta_j^-, h_j^-) \leq \delta$ for all $1 \leq j \leq j_{\mathrm{m}}$, then*

$$\sum_{j=1}^{j_{\mathrm{m}}} \pi\left(\mathrm{Ad}_{\eta_j^-}(\mathfrak{h}^+)\right) = \mathfrak{a} \oplus \mathfrak{m}.$$

We now fix $j_{\mathrm{m}} \in \mathbb{N}$ as in Lemma A.1.5 for the rest of the appendix. The following proposition is the required LNIC in our setting [SW21, Proposition 6.5].

Proposition A.1.6 (LNIC)**.** *There exist $\epsilon \in (0, 1)$, $m_0 \in \mathbb{N}$, $j_{\mathrm{m}} \in \mathbb{N}$, and an open subset $U_0 \subset \tilde{U}_1$ containing the centre $z_1 \in \tilde{R}_1$ such that for all $m \geq m_0$, there exist sections $v_j = P^{-\omega_j} : \tilde{U}_1 \to \tilde{U}_{\omega_{j,0}}$ for some admissible sequences $\omega_j = (\omega_{j,0}, \omega_{j,1}, \ldots, \omega_{j,m-1}, 1)$ for all $0 \leq j \leq j_{\mathrm{m}}$ such that for all $u \in U_0$ and $\omega \in \mathfrak{a} \oplus \mathfrak{m}$ with $\|\omega\| = 1$, there exists a $1 \leq j \leq j_{\mathrm{m}}$ and $Z \in \mathrm{T}_u(\tilde{U}_1)$ with $\|Z\| = 1$ such that*

$$|\langle (d\,\mathrm{BP}_{j,u})_u(Z), \omega \rangle| \geq \epsilon,$$

where we define $\mathrm{BP}_j : \tilde{U}_1 \times \tilde{U}_1 \to AM$ by

$$\mathrm{BP}_j(u', u) = \Phi^{\omega_0}(v_0(u))^{-1}\Phi^{\omega_0}(v_0(u'))\Phi^{\omega_j}(v_j(u'))^{-1}\Phi^{\omega_j}(v_j(u)),$$

and we write $\mathrm{BP}_{j,u} : \tilde{U}_1 \to AM$ when we view it as a function of only the first coordinate, for all $u', u \in \tilde{U}_1$ and $1 \leq j \leq j_{\mathrm{m}}$. Moreover, $v_0(U_0), v_1(U_0), \ldots, v_{j_{\mathrm{m}}}(U_0)$ are mutually disjoint.

For the rest of this appendix fix $\varepsilon_2 \in (0, 1)$, $m_0 \in \mathbb{N}$, $j_{\mathrm{m}} \in \mathbb{N}$, and the open subset $U_0 \subset \tilde{U}_1$ containing the centre $z_1 \in \tilde{R}_1$ to be the ϵ, m_0, j_{m}, and U_0 provided by Proposition A.1.6.

A.1.2 Non-concentration property

In the upper half space model, applying an appropriate isometry, we can assume that the vectors in $\tilde{\tilde{U}}_1$ have direction $\pi_2(\tilde{\tilde{U}}_1) = -e_N$ and their basepoints lie on the hyperplane $\langle \pi_1(\tilde{\tilde{U}}_1), e_N \rangle = 1$. In the rest of this appendix we will often view the limit set as $\Lambda_\Gamma \subset \mathbb{R}^{N-1} \cup \{\infty\}$. The following proposition is the required NCP [SW21, Proposition 6.6].

Proposition A.1.7 (NCP)**.** *There exists $\delta \in (0,1]$ such that for all $\epsilon \in (0,1)$, $w \in \mathbb{R}^{N-1}$ with $\|w\| = 1$, and $x \in \Lambda_\Gamma \cap \mathbb{R}^{N-1}$, there exists $y \in \Lambda_\Gamma \cap B^{\mathrm{E}}_\epsilon(x)$ such that $|\langle y - x, w \rangle| \geq \epsilon\delta$.*

Fix $\varepsilon_3 > 0$ to be the δ provided by Proposition A.1.7 henceforth.

A.2 Preliminary lemmas and constants

In this section, we cover some more lemmas and then fix many constants which are needed to construct the Dolgopyat operators requiered in Theorem 5.3.4.

Let $\Psi_1 : \tilde{\tilde{U}}_1 \to \mathbb{R}^{N-1}$ be the diffeomorphism defined by $\Psi_1(u) = u^+$ for all $u \in \tilde{\tilde{U}}_1$. Let $\Psi_2 : \tilde{\tilde{U}}_1 \to \tilde{U}_1$ be the isometry obtained from the covering map. Define the diffeomorphism

$$\Psi : \Psi_1(\tilde{\tilde{U}}_1) \to \tilde{U}_1 \quad \text{by} \quad \Psi(x) = \Psi_2({\Psi_1}^{-1}(x)) \qquad \text{for all } x \in \Psi_1(\tilde{\tilde{U}}_1).$$

Then ${(d\Psi)_x}^*$ is invertible for all $x \in \Psi_1(\tilde{\tilde{U}}_1)$ and hence by continuity, we can fix $\delta_\Psi > 0$ such that

$$\inf_{x \in \Psi_1(\tilde{\tilde{U}}_1)} \inf_{\|w\|=1} \|{(d\Psi)_x}^*(w)\| \geq \delta_\Psi.$$

We also fix $C_\Psi > 1$ such that

$$\frac{1}{C_\Psi} d_{\mathrm{E}}(x,y) \leq d(\Psi(x), \Psi(y)) \leq C_\Psi \, d_{\mathrm{E}}(x,y), \qquad \text{for all } x, y \in \Psi_1(\tilde{\tilde{U}}_1).$$

We now introduce a technical lemma [SW21, Lemma 7.1]. For all $x \in \tilde{U}_1$ set

$$\check{x} = \Psi^{-1}(x).$$

Let $1 \leq j \leq j_{\mathrm{m}}$, $x, y \in \tilde{U}_1$ and

$$z = (\check{x}, \check{y} - \check{x}) \in \mathrm{T}_{\check{x}}(\mathbb{R}^{N-1}) \quad \text{such that} \quad \{\check{x} + tz \in \mathbb{R}^{N-1} : t \in [0,1]\} \subset \Psi^{-1}(\tilde{U}_1).$$

Define the curve

$$\varphi^{\mathrm{BP}}_{j,x,z} : [0,1] \to AM \quad \text{by} \quad \varphi^{\mathrm{BP}}_{j,x,z}(t) = \mathrm{BP}_{j,x}(\Psi(\check{x} + tz)) \quad \text{for all } t \in [0,1].$$

Note that the curve has endpoints $\varphi^{\mathrm{BP}}_{j,x,z}(0) = e$ and $\varphi^{\mathrm{BP}}_{j,x,z}(1) = \mathrm{BP}_{j,x}(y) = \mathrm{BP}_j(y,x)$. There exists $\delta_0 > 0$ such that any pair of points in $B^{AM}_{\delta_0}(e) \subset AM$ has a unique geodesic through them. Fix

$$C_{\mathrm{BP},\Psi} = \sup_{\substack{x,y \in \tilde{U}_1,\\ 1 \le j \le j_{\mathrm{m}}}} \|d(\mathrm{BP}_{j,x} \circ \Psi)_{\check{y}}\|_{\mathrm{op}}.$$

Lemma A.2.1. *There exists $C > 0$ such that for all $1 \le j \le j_{\mathrm{m}}$ and $x, y \in \tilde{U}_1$ with $d(x,y) < \frac{\delta_0}{C_\Psi C_{\mathrm{BP},\Psi}}$ such that $\{\check{x} + tz \in \mathbb{R}^{N-1} : t \in [0,1]\} \subset \Psi^{-1}(\tilde{U}_1)$, we have*

$$d_{AM}\left(\exp(Z), \varphi^{\mathrm{BP}}_{j,x,z}(1)\right) \le C d(x,y)^2,$$

where $z = (\check{x}, \check{y} - \check{x}) \in \mathrm{T}_{\check{x}}(\mathbb{R}^{N-1})$, and $Z = d(\mathrm{BP}_{j,x} \circ \Psi)_{\check{x}}(z)$.

Fix $C_{\exp,\mathrm{BP}} > 0$ to be the C provided by Lemma A.2.1.

Remark A.2.2. Choosing a smaller open set if necessary, we can assume without loss of generality that $U_0 \subset \tilde{U}_1$ was chosen such that $\Psi^{-1}(U_0) \subset \mathbb{R}^{N-1}$ is a convex open subset so that Lemma A.2.1 applies for our purposes.

Recall that Dolgopyat's method can be successfully carried out when the derivative of λ_b is large, which motivated the definition of $\widehat{M}_\beta$ for all $\beta > 0$. This criterion is ultimately manifested in Lemma A.2.3 [SW21, Lemma 7.3] which is a Lasota–Yorke type inequality [LY73].

Lemma A.2.3. *There exists $A_0 > 0$ such that for all $s = a + ib \in \mathbb{C}$ with $(b, \lambda) \in \widehat{M}_1$ and all $k \in \mathbb{N}$, we have*

1. *if $h \in K_B(\tilde{U})$ for some $B > 0$, then we have $\tilde{\mathcal{L}}^k_a(h) \in K_{B'}(\tilde{U})$ where*

$$B' = A_0 \left(\frac{B}{{\kappa_2}^k} + 1\right),$$

2. *if* $H \in C^1\left(\tilde{U}, {H_\lambda}^{\oplus \dim(\lambda)}\right)$ *and* $h \in C^1(\tilde{U}, \mathbb{R})$ *satisfy* $\|(dH)_u\|_{\mathrm{op}} \leq Bh(u)$ *for all* $u \in \tilde{U}$*, for some* $B > 0$*, then we have*

$$\left\|\left(d\tilde{\mathcal{N}}_{s,\lambda}^{k}(H)\right)_u\right\|_{\mathrm{op}} \leq A_0\left(\frac{B}{{\kappa_2}^k}\tilde{\mathcal{L}}_a^k(h)(u) + \|\lambda_b\|\,\tilde{\mathcal{L}}_a^k\|H\|(u)\right) \quad \textit{for all } u \in \tilde{U}.$$

Fix $A_0 > 0$ provided by Lemma A.2.3. Fix a sufficiently large $m_1 \in \mathbb{N}$ and cylinders

$$\mathtt{C}_1, \mathtt{C}_2, \ldots, \mathtt{C}_m \subset U_0 \cap U_1,$$

with $\mathrm{len}(\mathtt{C}_k) = m_1$ such that

$$\overline{\mathtt{C}}_k \subseteq U \quad \text{and} \quad P^{m_1}(\mathtt{C}_k) = \mathrm{int}(U_k) \qquad \text{for all } k \in \mathcal{A}.$$

Let the corresponding sections be

$$\mathtt{v}_k : \tilde{U}_k \to \tilde{U}_1 \qquad \text{for all } k \in \mathcal{A}.$$

Fix $C_{Vit} = \min_{k\in\mathcal{A}} d(\overline{\mathtt{C}}_k, \partial(U))$. We defer the definition of $C_\phi > 0$, which only depends on the Markov section $\mathcal{R}$, until subsection A.4.2 where it is needed. Now, fix positive constants

$$\beta = 1, \tag{A.2.1}$$

$$E > \frac{2A_0}{\delta_{1,\widehat{M}}}; \tag{A.2.2}$$

$$\delta_1 < \frac{\varepsilon_1\varepsilon_2\varepsilon_3\delta_\Psi}{14C_\Psi}; \tag{A.2.3}$$

$$\epsilon_1 < \min\left(C_{Vit}, \frac{2\delta_0\delta_{1,\widehat{M}}}{C_\Psi C_{\mathrm{BP},\Psi}}, \frac{4\delta_1}{(C_\Psi C_{\mathrm{BP},\Psi})^2}, \frac{4\delta_1\delta_{1,\widehat{M}}}{C_{\exp,\mathrm{BP}}}, \frac{1}{\delta_1}, \frac{c_0 C_{hyp} C_\phi \delta e^\delta}{5{\kappa_1}^{m_1}{C_\Psi}^2\delta_{1,\widehat{M}}}\right); \tag{A.2.4}$$

$$\epsilon_2 < \min\left(\frac{\varepsilon_3\epsilon_1}{4m{C_\Psi}^2}, \frac{\delta_1\epsilon_1}{4m(A_0+\delta_1)}\right); \tag{A.2.5}$$

$$\epsilon_3 = \frac{c_0{\kappa_2}^{m_1}\epsilon_2}{2}; \tag{A.2.6}$$

$$\epsilon_4 = 10{c_0}^{-1}{\kappa_1}^{m_1}{C_\Psi}^2\epsilon_1; \tag{A.2.7}$$

$$m_2 > m_0 \text{ such that } {\kappa_2}^{m_2} > \max\left(8A_0, \frac{4Em\epsilon_2}{c_0\log(2)}, \frac{32Em\epsilon_2}{c_0}, \frac{4E}{c_0\delta_1}\right); \tag{A.2.8}$$

$$\mu < \min\left(\frac{E\epsilon_2}{2m}, \frac{1}{4m}, \frac{1}{16\cdot 16e^{2m_2T_0}\cdot m}\arccos\left(1-\frac{(\delta_1\epsilon_1)^2}{2}\right)^2\right). \tag{A.2.9}$$

Set

$$M = m_1 + m_2.$$

Fix admissible sequences

$$\omega_j = (\omega_{j,0}, \omega_{j,1}, \dots, \omega_{j,m_2-1}, 1),$$

and corresponding sections

$$v_j = P^{-\omega_j} : \tilde{U}_1 \to \tilde{U}_{\omega_{j,0}},$$

provided by Proposition A.1.6 for all $0 \leq j \leq j_{\mathrm{m}}$. Fix corresponding maps $\mathrm{BP}_j : \tilde{U}_1 \times \tilde{U}_1 \to AM$ provided by Proposition A.1.6 for all $1 \leq j \leq j_{\mathrm{m}}$.

A.3 Construction of Dolgopyat operators

Now we have the tools to construct the Dolgopyat operators required to prove Theorem 5.3.4.

Let $(b, \lambda) \in \widehat{M}_\beta$ and $k \in \mathcal{A}$. We can use the map Ψ and the Vitali covering lemma on $\mathbb{R}^{N-1}$ to choose a finite subset $\{x_{k,r,1}^{(b,\lambda)} \in \mathtt{C}_k : r \in \{1, 2, \dots, r_k^{(b,\lambda)}\}\} \subset \mathtt{C}_k$ for some $r_k^{(b,\lambda)} \in \mathbb{N}$ and corresponding open balls

$$C_{k,r}^{(b,\lambda)} = W^{\mathrm{su}}_{\epsilon_1/\|\lambda_b\|}(x_{k,r,1}^{(b,\lambda)}) \quad \text{and} \quad \hat{C}_{k,r}^{(b,\lambda)} = W^{\mathrm{su}}_{5{C_\Psi}^2\epsilon_1/\|\lambda_b\|}(x_{k,r,1}^{(b,\lambda)}),$$

for all $1 \leq r \leq r_k^{(b,\lambda)}$ such that

$$C_{k,r}^{(b,\lambda)} \cap C_{k,r'}^{(b,\lambda)} = \varnothing \quad \text{for all } 1 \leq r, r' \leq r_k^{(b,\lambda)} \text{ with } r \neq r' \text{ and } \mathtt{C}_k \subset \bigcup_{r=1}^{r_k^{(b,\lambda)}} \hat{C}_{k,r}^{(b,\lambda)}.$$

Define

$$\check{x}_{k,r,1}^{(b,\lambda)} = \Psi^{-1}(x_{k,r,1}^{(b,\lambda)}) \quad \text{for all } \ 1 \leq r \leq r_k^{(b,\lambda)}.$$

We have the following lemma [SW21, Lemma 8.1].

Lemma A.3.1. *For all $(b, \lambda) \in \widehat{M}_\beta$, $\omega \in {H_\lambda}^{\oplus \dim(\lambda)}$ with $\|\omega\|_2 = 1$, $k \in \mathcal{A}$, and $1 \leq r \leq r_k^{(b,\lambda)}$, there exists*

$$1 \leq j \leq j_{\mathrm{m}} \quad \textit{and} \quad \check{x}_2 \in \Lambda_\Gamma \cap \left(B^{\mathrm{E}}_{s_1}(\check{x}_{k,r,1}^{(b,\lambda)}) \setminus B^{\mathrm{E}}_{s_2}(\check{x}_{k,r,1}^{(b,\lambda)})\right),$$

such that

$$\left\| d\lambda_b \left(d(\mathrm{BP}_{j, x_{k,r,1}^{(b,\lambda)}} \circ \Psi)_{\check{x}_{k,r,1}^{(b,\lambda)}}(z) \right)(\omega) \right\|_2 \geq 7\delta_1 \epsilon_1,$$

where $s_1 = \frac{\epsilon_1}{2C_\Psi \|\lambda_b\|}$, $s_2 = \frac{\varepsilon_3 \epsilon_1}{2C_\Psi \|\lambda_b\|}$, *and* $z = (\check{x}_{k,r,1}^{(b,\lambda)}, \check{x}_2 - \check{x}_{k,r,1}^{(b,\lambda)}) \in \mathrm{T}_{\check{x}_{k,r,1}^{(b,\lambda)}}(\mathbb{R}^{N-1})$.

Let $(b, \lambda) \in \widehat{M}_\beta$, $H \in C^1\left(\tilde{U}, H_\lambda{}^{\oplus \dim(\lambda)}\right)$, $k \in \mathcal{A}$, and $1 \leq r \leq r_k^{(b,\lambda)}$. Corresponding to

$$\omega = \frac{\lambda_b(\Phi^{\omega_0}(v_0(x_{k,r,1}^{(b,\lambda)}))^{-1}) H(v_0(x_{k,r,1}^{(b,\lambda)}))}{\|H(v_0(x_{k,r,1}^{(b,\lambda)}))\|_2} \in H_\lambda{}^{\oplus \dim(\lambda)},$$

denote $j_{k,r}^{(b,\lambda),H}$ and $x_{k,r,2}^{(b,\lambda),H}$ to be the j and $\Psi(\check{x}_2) \in W_{s_1}^{\mathrm{su}}(x_{k,r,1}^{(b,\lambda)}) \setminus W_{s_2}^{\mathrm{su}}(x_{k,r,1}^{(b,\lambda)})$ provided by Lemma A.3.1, where $s_1 = \frac{\epsilon_1}{2\|\lambda_b\|}$ and $s_2 = \frac{\varepsilon_3 \epsilon_1}{2{C_\Psi}^2 \|\lambda_b\|}$. Define

$$\begin{aligned}
D_{k,r,1}^{(b,\lambda)} &= W_{\epsilon_2/\|\lambda_b\|}^{\mathrm{su}}(x_{k,r,1}^{(b,\lambda)}) \subset C_{k,r}^{(b,\lambda)}; & D_{k,r,2}^{(b,\lambda),H} &= W_{\epsilon_2/\|\lambda_b\|}^{\mathrm{su}}(x_{k,r,2}^{(b,\lambda),H}) \subset C_{k,r}^{(b,\lambda)}; \\
\not{D}_{k,r,1}^{(b,\lambda)} &= W_{\frac{\epsilon_2}{2\|\lambda_b\|}}^{\mathrm{su}}(x_{k,r,1}^{(b,\lambda)}) \subset C_{k,r}^{(b,\lambda)}; & \not{D}_{k,r,2}^{(b,\lambda),H} &= W_{\frac{\epsilon_2}{2\|\lambda_b\|}}^{\mathrm{su}}(x_{k,r,2}^{(b,\lambda),H}) \subset C_{k,r}^{(b,\lambda)}; \\
\hat{D}_{k,r,1}^{(b,\lambda)} &= W_{2m\epsilon_2/\|\lambda_b\|}^{\mathrm{su}}(x_{k,r,1}^{(b,\lambda)}) \subset C_{k,r}^{(b,\lambda)}; & \hat{D}_{k,r,2}^{(b,\lambda),H} &= W_{2m\epsilon_2/\|\lambda_b\|}^{\mathrm{su}}(x_{k,r,2}^{(b,\lambda),H}) \subset C_{k,r}^{(b,\lambda)}.
\end{aligned}$$

Denote $\psi_{k,r,1}^{(b,\lambda)}, \psi_{k,r,2}^{(b,\lambda),H} \in C^\infty(\tilde{U}, \mathbb{R})$ to be smooth cut-off functions with

$$\mathrm{supp}\,(\psi_{k,r,1}^{(b,\lambda)}) = \overline{D_{k,r,1}^{(b,\lambda)}} \cap \tilde{U} \qquad \text{and} \qquad \mathrm{supp}\,(\psi_{k,r,2}^{(b,\lambda),H}) = \overline{D_{k,r,2}^{(b,\lambda),H}} \cap \tilde{U},$$

such that they attain the maximum values

$$\psi_{k,r,1}^{(b,\lambda)}\Big|_{\overline{\not{D}_{k,r,1}^{(b,\lambda)}} \cap \tilde{U}} = \psi_{k,r,2}^{(b,\lambda),H}\Big|_{\overline{\not{D}_{k,r,2}^{(b,\lambda),H}} \cap \tilde{U}} = 1,$$

and the minimum values

$$\psi_{k,r,1}^{(b,\lambda)}\Big|_{\tilde{U} \setminus D_{k,r,1}^{(b,\lambda)}} = \psi_{k,r,2}^{(b,\lambda),H}\Big|_{\tilde{U} \setminus D_{k,r,2}^{(b,\lambda),H}} = 0,$$

and we can further assume that

$$\left|\psi_{k,r,1}^{(b,\lambda)}\right|_{C^1}, \left|\psi_{k,r,2}^{(b,\lambda),H}\right|_{C^1} \leq \frac{4\|\lambda_b\|}{\epsilon_2}.$$

It can be checked that

$$D_{k,r_1,p_1}^{(b,\lambda)} \cap D_{k,r_2,p_2}^{(b,\lambda),H} = \varnothing,$$

for all $(r_1,p_1),(r_2,p_2) \in \{1,2,\ldots,r_k^{(b,\lambda)}\} \times \{1,2\}$ with $(r_1,p_1) \neq (r_2,p_2)$ and $k \in \mathcal{A}$. Define

$$\Xi_1(b,\lambda) = \{(k,r) \in \mathbb{Z}^2 : k \in \mathcal{A}, r \in \{1,2,\ldots,r_k^{(b,\lambda)}\}\} \quad \text{and} \quad \Xi_2 = \{1,2\} \times \{1,2\}.$$

Define $\Xi(b,\lambda) = \Xi_1(b,\lambda) \times \Xi_2$. For all $(k,r,p,l) \in \Xi(b,\lambda)$, denoting $j_{k,r}^{(b,\lambda),H}$ by j for convenience, we define the function $\tilde{\psi}_{(k,r,p,l)}^{(b,\lambda),H} \in C^\infty(\tilde{U},\mathbb{R})$ by

$$\tilde{\psi}_{(k,r,p,l)}^{(b,\lambda),H} = \begin{cases} \mathbb{1}_{\tilde{\mathtt{C}}[\omega_0]} \cdot \left(\psi_{k,r,1}^{(b,\lambda)} \circ P^{\omega_0}\right), & p=1, l=1 \\ \mathbb{1}_{\tilde{\mathtt{C}}[\omega_j]} \cdot \left(\psi_{k,r,1}^{(b,\lambda)} \circ P^{\omega_j}\right), & p=1, l=2 \\ \mathbb{1}_{\tilde{\mathtt{C}}[\omega_0]} \cdot \left(\psi_{k,r,2}^{(b,\lambda),H} \circ P^{\omega_0}\right), & p=2, l=1 \\ \mathbb{1}_{\tilde{\mathtt{C}}[\omega_j]} \cdot \left(\psi_{k,r,2}^{(b,\lambda),H} \circ P^{\omega_j}\right), & p=2, l=2, \end{cases}$$

where using P^{ω_0} and P^{ω_j} are indeed justified because of the indicator functions

$$\mathbb{1}_{\tilde{\mathtt{C}}[\omega_0]} = \mathbb{1}_{v_0(\tilde{U}_1)} \quad \text{and} \quad \mathbb{1}_{\tilde{\mathtt{C}}[\omega_j]} = \mathbb{1}_{v_j(\tilde{U}_1)}.$$

For all subsets $J \subset \Xi(b,\lambda)$, we define

$$\beta_J^H = \mathbb{1}_{\tilde{U}} - \mu \sum_{(k,r,p,l) \in J} \tilde{\psi}_{(k,r,p,l)}^{(b,\lambda),H} \in C^\infty(\tilde{U},\mathbb{R}).$$

Remark A.3.2. We will often include the superscript H even when there is no dependence on it for a more uniform notation to simplify exposition.

The following lemma appeared in [SW21, Lemma 8.2]

Lemma A.3.3. *Let* $(b,\lambda) \in \widehat{M}_\beta$, $H \in C^1\left(\tilde{U}, H_\lambda{}^{\oplus \dim(\lambda)}\right)$, *and* $J \subset \Xi(b,\lambda)$. *Then any connected component of*

$$\bigcup \left\{ D_{k,r,p}^{(b,\lambda),H} : (k,r,p,l) \in J \text{ for some } l \in \{1,2\} \right\}$$

is a union of at most m *terms and hence contained in* $\hat{D}_{k,r,p}^{(b,\lambda),H}$ *for any* $(k,r,p,l) \in J$ *corresponding to one of those terms.*

Using the lemma above and the (LNIC) we obtain the following corollary [SW21, corollary 8.3].

Corollary A.3.4. *Let* $(b,\lambda) \in \widehat{M}_\beta$, $H \in C^1\left(\tilde{U}, H_\lambda{}^{\oplus \dim(\lambda)}\right)$, *and* $J \subset \Xi(b,\lambda)$. *Then we have*

$$1 - m\mu \leq \beta_J^H \leq 1 \quad \textit{and} \quad \left|\beta_J^H\right|_{C^1} \leq \frac{4m\mu\|\lambda_b\|}{\epsilon_2}.$$

We are now ready to define the *Dolgopyat operators*. Recall the positive constant $M = m_1 + m_2$. For all $s = a + ib \in \mathbb{C}$ such that $(b,\lambda) \in \widehat{M}_\beta$, we define for all $J \subset \Xi(b,\lambda)$ and $H \in C^1\left(\tilde{U}, H_\lambda{}^{\oplus \dim(\lambda)}\right)$, the Dolgopyat operator

$$\mathcal{D}_J^H : C^1(\tilde{U}, \mathbb{R}) \to C^1(\tilde{U}, \mathbb{R}) \qquad \text{by} \qquad \mathcal{D}_J^H(h) = \tilde{\mathcal{N}}_a^M(\beta_J^H h) \quad \text{for all } h \in C^1(\tilde{U}, \mathbb{R}).$$

Definition A.3.5 (Dense)**.** For all $(b,\lambda) \in \widehat{M}_\beta$, a subset $J \subset \Xi(b,\lambda)$ is said to be *dense* if for all $(k,r) \in \Xi_1(b,\lambda)$, there exists $(p,l) \in \Xi_2$ such that $(k,r,p,l) \in J$.

For all $(b,\lambda) \in \widehat{M}_\beta$, define

$$\mathcal{J}(b,\lambda) = \{J \subset \Xi(b,\lambda) : J \text{ is dense}\}.$$

A.4 Proof of Theorem 5.3.4

We devote this section to the proof of Theorem 5.3.4. We do this by proving all the properties in the theorem in the following subsections.

For this section recall that we already fixed $\beta = 1$.

A.4.1 Proof of properties (5.3.1) and (5.3.5) in Theorem 5.3.4

The following two lemmas appeared in [SW21, subsection 9.1].

Lemma A.4.1. *For all* $s = a + ib \in \mathbb{C}$ *such that* $(b,\lambda) \in \widehat{M}_\beta$, *then for all* $J \in \mathcal{J}(b,\lambda)$ *and* $H \in C^1\left(\tilde{U}, H_\lambda{}^{\oplus \dim(\lambda)}\right)$, *we have*

$$\mathcal{D}_J^H(K_{E\|\lambda_b\|}(\tilde{U})) \subset K_{E\|\lambda_b\|}(\tilde{U}).$$

Proof. Let $s = a + ib \in \mathbb{C}$ and suppose $(b, \lambda) \in \widehat{M}_\beta$. Let $J \in \mathcal{J}(b, \lambda)$ and $H \in C^1\left(\tilde{U}, H_\lambda{}^{\oplus \dim(\lambda)}\right)$. Let $h \in K_{E\|\lambda_b\|}(\tilde{U})$ and $u \in \tilde{U}$. Corollary A.3.4 and the choice of μ in A.2.9 give

$$\begin{aligned}
\left\|d\left(\beta_J^H h\right)_u\right\|_{\mathrm{op}} &= \left\|\left(d\beta_J^H\right)_u\right\|_{\mathrm{op}} \cdot h(u) + \beta_J^H(u) \cdot \|(dh)_u\|_{\mathrm{op}} \\
&\leq \frac{4m\mu\|\lambda_b\|}{\epsilon_2} h(u) + E\|\lambda_b\| h(u) \\
&\leq (2E + E)\|\lambda_b\| h(u) \cdot \frac{\beta_J^H(u)}{1 - m\mu} \\
&\leq 4E\|\lambda_b\|(\beta_J^H h)(u).
\end{aligned}$$

So $\beta_J^H h \in K_{4E\|\lambda_b\|}(\tilde{U})$. Now applying Lemma A.2.3, we have

$$\begin{aligned}
\left\|\left(d\mathcal{D}_J^H(h)\right)_u\right\|_{\mathrm{op}} &= \left\|\left(d\tilde{\mathcal{N}}_a^M(\beta_J^H h)\right)_u\right\|_{\mathrm{op}} \\
&\leq A_0 \left(\frac{4E\|\lambda_b\|}{\kappa_2{}^m} + 1\right) \tilde{\mathcal{N}}_a^M(\beta_J^H h)(u) \\
&\leq A_0 \left(\frac{4E\|\lambda_b\|}{8A_0} + \frac{E\|\lambda_b\|}{2A_0}\right) \tilde{\mathcal{N}}_a^M(\beta_J^H h)(u) \\
&= E\|\lambda_b\| \mathcal{D}_J^H(h)(u).
\end{aligned}$$

□

Lemma A.4.2. *For all $s = a + ib \in \mathbb{C}$ such that $(b, \lambda) \in \widehat{M}_\beta$, if $H \in C^1\left(\tilde{U}, H_\lambda{}^{\oplus \dim(\lambda)}\right)$ and $h \in C^1(\tilde{U}, \mathbb{R})$ satisfy property (5.3.3) in Theorem 5.3.4, then for all $J \in \mathcal{J}(b, \lambda)$ we have*

$$\left\|\left(d\tilde{\mathcal{N}}_{s,\lambda}^M(H)\right)_u\right\|_{\mathrm{op}} \leq E\|\lambda_b\| \mathcal{D}_J^H(h)(u) \qquad \textit{for all } u \in \tilde{U}.$$

Proof. Let $s = a + ib \in \mathbb{C}$ and suppose $(b, \lambda) \in \widehat{M}_\beta$. Suppose $H \in C^1\left(\tilde{U}, H_\lambda{}^{\oplus \dim(\lambda)}\right)$ and $h \in B(\tilde{U}, \mathbb{R})$ satisfy property 5.3.3 in Theorem 5.3.4. Let $J \in \mathcal{J}(b, \lambda)$ and $u \in \tilde{U}$. Applying

Lemma A.2.3, we have

$$\begin{aligned}
\left\|\left(d\tilde{\mathcal{N}}_{s,\lambda}^{M}(H)\right)_u\right\|_{\mathrm{op}} &\leq A_0\left(\frac{E\|\lambda_b\|}{\kappa 2^m}\tilde{\mathcal{N}}_a^M(h)(u) + \|\lambda_b\|\tilde{\mathcal{N}}_a^M\|H\|(u)\right)\\
&\leq A_0\left(\frac{E}{8A_0} + \frac{E}{2A_0}\right)\|\lambda_b\|\tilde{\mathcal{N}}_a^M(h)(u)\\
&\leq \left(\frac{E}{8(1-m\mu)} + \frac{E}{2(1-m\mu)}\right)\|\lambda_b\|\tilde{\mathcal{N}}_a^M(\beta_J^H h)(u)\\
&\leq \left(\frac{E}{6} + \frac{2E}{3}\right)\|\lambda_b\|\mathcal{D}_J^H(h)(u)\\
&\leq E\|\lambda_b\|\mathcal{D}_J^H(h)(u).
\end{aligned}$$

□

A.4.2 Proof of property (5.3.2) in Theorem 5.3.4

Recall the constants from (A.2.6) and (A.2.7) and note that $\epsilon_4 > 80\epsilon_3$. Let $(b,\lambda) \in \widehat{M}_\beta$ and $H \in C^1\left(\tilde{U}, H_\lambda{}^{\oplus \dim(\lambda)}\right)$. For all $k \in \mathcal{A}$ and $1 \leq r \leq r_k^{(b,\lambda)}$, define the open sets

$$\begin{aligned}
Z_{k,r,1}^{(b,\lambda)} &= W^{\mathrm{su}}_{\epsilon_3/\|\lambda_b\|}(P^{m_1}(x_{k,r,1}^{(b,\lambda)})) \cap \tilde{U}_k,\\
Z_{k,r,2}^{(b,\lambda),H} &= W^{\mathrm{su}}_{\epsilon_3/\|\lambda_b\|}(P^{m_1}(x_{k,r,2}^{(b,\lambda),H})) \cap \tilde{U}_k,
\end{aligned}$$

which then satisfy $\mathtt{v}_k(Z_{k,r,1}^{(b,\lambda)}) \subset \not{D}_{k,r,1}^{(b,\lambda)}$ and $\mathtt{v}_k(Z_{k,r,2}^{(b,\lambda),H}) \subset \not{D}_{k,r,2}^{(b,\lambda),H}$. We need to first prove the crucial Corollary A.4.5.

We begin with definitions for this subsection. For all $w \in \mathrm{T}^1(X)$, the Patterson density induces the measure $\mu_{W^{\mathrm{su}}(w)}$ on the leaf $W^{\mathrm{su}}(w)$ defined by

$$d\mu_{W^{\mathrm{su}}(w)}(u) = e^{C_{a\tau-\delta_{\Gamma,a\tau},(\overline{u})^+}(x_0,\overline{u})}\, d\mu_{x_0}((\overline{u})^+).$$

Let $k \in \mathcal{A}$ and $w_k \in R_k$ be the centres. Then, we have

$$\frac{d\left(\nu_\alpha|_{\tilde{U}_k}\right)}{d(\mu_{W^{\mathrm{su}}(w_k)}|_{\tilde{U}_k})}(u) = C\int_{[u,S_k]} e^{C_{a\tau-\delta_{\Gamma,a\tau},[\overline{u},\overline{s}]^-}(x_0,[\overline{u},\overline{s}])}\, d\mu_{x_0}([\overline{u},\overline{s}]^-),$$

for all $u \in \tilde{U}_k$, for some $C > 0$. In particular, by positivity and continuity of the integrand, there exists $C_P > 0$ such that

$$\frac{1}{C_P} \leq \frac{d\left(\nu_\alpha|_{\tilde{U}_k}\right)}{d(\mu_{W^{\mathrm{su}}(w_k)}|_{\tilde{U}_k})} \leq C_P.$$

Recall the trajectory isomorphism ψ from [Rat73b, Definition 1.1]. We define another map for each $w \in [W^{\mathrm{su}}_{\epsilon_0}(w_k), W^{\mathrm{ss}}_{\epsilon_0}(w_k)]$ by

$$\phi_w : U_k \to W^{\mathrm{su}}(w) \quad \text{by} \quad \phi_w(u) = {\psi_w}^{-1}([u, w]) \quad \text{for all } u \in U_k.$$

The maps ϕ_w are Lipschitz and smooth in $w \in [W^{\mathrm{su}}_{\epsilon_0}(w_k), W^{\mathrm{ss}}_{\epsilon_0}(w_k)]$, and hence there exists

$$C_\phi = \max_{k \in \mathcal{A}} \sup_{w \in R_k} \mathrm{Lip}_d(\phi_w).$$

The proofs of the following lemmas and corollaries is identical to [SW21, subsection 9.2].

Lemma A.4.3. *For all $j \in \mathcal{A}$, let $w_j \in R_j$ be the centres. There exists $C > 0$ such that for all $j \in \mathcal{A}$, $u \in U_j$, and $\epsilon \in (0, 2C_{hyp}C_\phi \delta e^\delta)$, we have*

$$\nu_\alpha(W^{\mathrm{su}}_\epsilon(u) \cap \tilde{U}_j) \geq C\mu_{W^{\mathrm{su}}(w_j)}(W^{\mathrm{su}}_\epsilon(u)).$$

Proof. Let $j \in \mathcal{A}$, $\varepsilon \in (0, 2C_{hyp}C_\phi\delta e^\delta)$ and $u \in U_j$. Set $\omega_0 = j$ and $\omega = (\omega_0, \ldots, \omega_l)$ with $l \in \mathbb{N}$ such that $u \in \overline{\mathtt{C}[\omega]}$ and

$$2C_{hyp}C_\phi\delta \leq e^t\varepsilon \leq 2C_{hyp}C_\phi\delta e^\delta,$$

where $t = \tau^\omega(u)$. Write $k = \omega_l$ so that

$$u' = ua_t \in R_k.$$

Also note that

$$\overline{\mathtt{C}[\omega]} = P^{-\omega}(U_k) = \phi_{u'}(U_k)a_{-t}.$$

Since $\mathrm{diam}_{d_{su}}(\phi_{u'}(U_k)) \leq C_\phi\,\delta$ we have that

$$\overline{\mathtt{C}[\omega]} \subset U_j \cap W^{su}_{C_{hyp}e^{-t}C_\phi\delta}(u) \subset U_j \cap W^{su}_\varepsilon(u).$$

Noting that $W^{su}(w_j) = W^{su}(u)$ we obtain

$$
\begin{aligned}
\nu_\alpha(W^{su}_\varepsilon(u) \cap U_j) \geq& \nu_\alpha(\overline{\mathtt{C}[\omega]}) \geq \frac{1}{C_P}\mu_{W^{su}(u)}(\overline{\mathtt{C}[\omega]}) \\
=& \frac{1}{C_P}\, e^{C_{a\tau-\delta_{\Gamma,a\tau},(\overline{u})^+}(x_0,\overline{u}) - C_{a\tau-\delta_{\Gamma,a\tau},(\overline{u})^+}(x_0,\overline{u}a_t)} \mu_{W^{su}(u')}(\overline{\mathtt{C}[\omega]}a_t) \\
=& \frac{1}{C_P}\, e^{C_{a\tau-\delta_{\Gamma,a\tau},(\overline{u})^+}(x_0,\overline{u}) - C_{a\tau-\delta_{\Gamma,a\tau},(\overline{u})^+}(x_0,\overline{u}a_t)} \mu_{W^{su}(u')}(U_k) \\
\geq& \frac{C_2}{C_P}\, e^{C_{a\tau-\delta_{\Gamma,a\tau},(\overline{u})^+}(x_0,\overline{u}) - C_{a\tau-\delta_{\Gamma,a\tau},(\overline{u})^+}(x_0,\overline{u}a_t)},
\end{aligned}
$$

where $C_2 = \min_{k\in\mathcal{A}} \mu_{W^{su}(u')}(U_k)$.

Finally,

$$
\begin{aligned}
\mu_{W^{su}(u)}(W^{su}_\varepsilon(u)) =& e^{C_{a\tau-\delta_{\Gamma,a\tau},(\overline{u})^+}(x_0,\overline{u}) - C_{a\tau-\delta_{\Gamma,a\tau},(\overline{u})^+}(x_0,\overline{u}a_t)} \mu_{W^{su}(u')}(W^{su}_\varepsilon(u)a_t) \\
\leq& e^{C_{a\tau-\delta_{\Gamma,a\tau},(\overline{u})^+}(x_0,\overline{u}) - C_{a\tau-\delta_{\Gamma,a\tau},(\overline{u})^+}(x_0,\overline{u}a_t)} \mu_{W^{su}(u')}(W^{su}_{2C_{hyp}^2 C_\phi \delta e^\delta}(u')) \\
\leq& C_3\, e^{C_{a\tau-\delta_{\Gamma,a\tau},(\overline{u})^+}(x_0,\overline{u}) - C_{a\tau-\delta_{\Gamma,a\tau},(\overline{u})^+}(x_0,\overline{u}a_t)},
\end{aligned}
$$

where $C_3 = \sup_{u'\in R} \mu_{W^{su}(u')}(W^{su}_{2C_{hyp}^2 C_\phi \delta e^\delta}(u'))$. Setting $C = \frac{C_2}{C_P\, C_3}$ we are done.

$\square$

Corollary A.4.4. *The measure ν_α satisfies the* doubling/Federer property*, i.e., there exists $C > 0$ such that for all $k \in \mathcal{A}$, $u \in U_k$, and $\epsilon \in (0, 2C_{hyp}C_\phi\delta e^\delta)$, we have*

$$\nu_\alpha(W^{\mathrm{su}}_{2\epsilon}(u) \cap \tilde{U}_k) \leq C\nu_\alpha(W^{\mathrm{su}}_\epsilon(u) \cap \tilde{U}_k).$$

Proof. By [PPS15, Proposition 3.12], we know that $\mu_{W^{\mathrm{su}}(w_k)}$ satisfies the doubling property for all $k \in \mathcal{A}$. Fix $C_1 > 0$ to be an upper bound for the corresponding doubling constants for all $k \in \mathcal{A}$. Fix $C_2 > 0$ to be the constant from Lemma A.4.3. Fix $C = \frac{C_1 C_P}{C_2}$. Let $k \in \mathcal{A}$, $u \in U_k$, and $\epsilon \in (0, 2C_{hyp}C_\phi\hat{\delta}e^{\hat{\delta}})$. We have

$$
\begin{aligned}
\nu_\alpha(W^{\mathrm{su}}_{2\epsilon}(u) \cap \tilde{U}_k) &\leq C_P\, \mu_{W^{\mathrm{su}}(w_k)}(W^{\mathrm{su}}_{2\epsilon}(u)) \leq C_1\, C_P\, \mu_{W^{\mathrm{su}}(w_k)}(W^{\mathrm{su}}_\epsilon(u)) \\
&\leq \frac{C_1\, C_P}{C_2}\, \nu_\alpha(W^{\mathrm{su}}_\epsilon(u) \cap \tilde{U}_k) = C\, \nu_\alpha(W^{\mathrm{su}}_\epsilon(u) \cap \tilde{U}_k).
\end{aligned}
$$

$\square$

Corollary A.4.5. *There exists $C > 1$ such that for all $(b,\lambda) \in \widehat{M}_\beta$, $k \in \mathcal{A}$, and $u \in U_k$, we have*

$$\nu_\alpha\left(W^{\mathrm{su}}_{\epsilon_4/\|\lambda_b\|}(u) \cap \tilde{U}_k\right) \le C\,\nu_\alpha\left(W^{\mathrm{su}}_{\epsilon_3/\|\lambda_b\|}(u) \cap \tilde{U}_k\right).$$

For all $(b,\lambda) \in \widehat{M}_\beta$, $H \in C^1\left(\tilde{U}, H_\lambda{}^{\oplus \dim(\lambda)}\right)$, and $J \in \mathcal{J}(b,\lambda)$, define the set

$$W_J^H = \bigcup_{(k,r,p,l)\in J} Z_{k,r,p}^{(b,\lambda),H}.$$

Lemma A.4.6. *There exists $\eta \in (0,1)$ such that for all $(b,\lambda) \in \widehat{M}_\beta$, $J \in \mathcal{J}(b,\lambda)$, $H \in C^1\left(\tilde{U}, H_\lambda{}^{\oplus \dim(\lambda)}\right)$, and $h \in K_{2E\|\lambda_b\|}(\tilde{U})$, we have*

$$\int_{W_J^H} h\, d\nu_\alpha \ge \eta \int_{\tilde{U}} h\, d\nu_\alpha.$$

Proof. Fix C to be the one provided by Corollary A.4.5 and $\eta = (Ce^{4E\epsilon_4})^{-1} \in (0,1)$. Let $(b,\lambda) \in \widehat{M}_\beta$, $J \in \mathcal{J}(b,\lambda)$, $H \in C^1\left(\tilde{U}, H_\lambda{}^{\oplus \dim(\lambda)}\right)$, and $h \in K_{E\|\lambda_b\|}(\tilde{U})$. Denote $\epsilon_j' = \frac{\epsilon_j}{\|\lambda_b\|}$ and $W_{j,k}(u) = W^{\mathrm{su}}_{\epsilon_j'}(u) \cap \tilde{U}_k$ for all $u \in \tilde{U}_k$, $k \in \mathcal{A}$, and $j \in \{3,4\}$. Define

$$P_k = \{P^{m_1}(x_{k,r,p}^{(b,\lambda),H}) \in U : (k,r,p,l) \in J \text{ for some } l \in \{1,2\}\}.$$

Since $\{\hat{C}_{k,r}^{(b,\lambda)} \subset W^{\mathrm{su}}(w_1) : 1 \le r \le r_k^{(b,\lambda)}\}$, where $w_1 \in R_1$ is the centre, covers C_k for all $k \in \mathcal{A}$ and $J \subset \Xi(b,\lambda)$ is dense, so $\{W^{\mathrm{su}}_{\epsilon_4'}(x) \subset \tilde{U}_k : x \in P_k\}$ covers $\mathrm{int}(U_k)$ for all $k \in \mathcal{A}$. Let $l_x = \inf_{u \in W_{4,k}(x)} h(u)$ and $L_x = \sup_{u\in W_{4,k}(x)} h(u)$ for all $x \in P_k$ and $k \in \mathcal{A}$. Using $|\log \circ h|_{C^1} \le 2E\|\lambda_b\|$, we can derive using the mean value theorem that $L_x \le l_x e^{2E\|\lambda_b\| \operatorname{diam}_d\left(W^{\mathrm{su}}_{\epsilon_4'}(x)\right)} = l_x e^{4E\epsilon_4}$. In order to use the mean value theorem to obtain this last bound we need to assert that the neighbourhoods $\tilde{U}_j$ are connected. Assuming this and using Corollary A.4.5, we have

$$\begin{aligned}
\int_{\tilde{U}} h(u)\,d\nu_\alpha(u) &= \sum_{k\in\mathcal{A}}\sum_{x\in P_k}\int_{W_{4,k}(x)} h(u)\,d\nu_\alpha(u)\\
&\leq \sum_{k\in\mathcal{A}}\sum_{x\in P_k} L_x\cdot\nu_\alpha(W_{4,k}(x))\\
&\leq Ce^{4E\epsilon_4}\sum_{k\in\mathcal{A}}\sum_{x\in P_k} l_x\cdot\nu_\alpha(W_{3,k}(x))\\
&\leq Ce^{4E\epsilon_4}\sum_{k\in\mathcal{A}}\sum_{x\in P_k}\int_{W_{3,k}(x)} h(u)\,d\nu_\alpha(u)\\
&\leq \frac{1}{\eta}\int_{W_J^H} h(u)\,d\nu_\alpha(u).
\end{aligned}$$

□

Lemma A.4.7. *There exist $\eta\in(0,1)$ such that for all $s=a+ib\in\mathbb{C}$ with $(b,\lambda)\in\widehat{M}_\beta$, then for all $J\in\mathcal{J}(b,\lambda)$, $H\in C^1\left(\tilde{U},{H_\lambda}^{\oplus\dim(\lambda)}\right)$, and $h\in K_{E\|\lambda_b\|}(\tilde{U})$, we have*

$$\left\|\mathcal{D}_J^H(h)\right\|_2\leq\eta\|h\|_2.$$

Proof. Fix $\eta'\in(0,1)$ to be the η provided by Lemma A.4.6 and set

$$\eta=\sqrt{1-\eta'\mu e^{-MT_0}}\in(0,1).$$

Let $s=a+ib\in\mathbb{C}$ and suppose $(b,\lambda)\in\widehat{M}_\beta$. Let $J\in\mathcal{J}(b,\lambda)$, $H\in C^1\left(\tilde{U},{H_\lambda}^{\oplus\dim(\lambda)}\right)$, and $h\in K_{E\|\lambda_b\|}(\tilde{U})$. We have the estimate $\mathcal{D}_J^H(h)^2\leq\mathcal{N}_J^H(h)^2$ since by the Cauchy–Schwarz inequality, we have

$$\mathcal{D}_J^H(h)^2=\tilde{\mathcal{N}}_a^M\left(\beta_J^H h\right)^2\leq\tilde{\mathcal{N}}_a^M\left((\beta_J^H)^2\right)\tilde{\mathcal{N}}_a^M(h^2),$$

since $\tilde{\mathcal{N}}_a$ is normalised. Observe that $h^2\in K_{2E\|\lambda_b\|}(\tilde{U})$. Then Lemma A.2.3 gives $\tilde{\mathcal{N}}_a^M(h^2)\in K_{B'}(\tilde{U})$ where $B'=A_0\left(\frac{2E|b|}{\kappa_2{}^m}+1\right)\leq A_0\left(\frac{2E|b|}{8A_0}+\frac{E|b|}{2A_0}\right)\leq 2E|b|$. So $\tilde{\mathcal{N}}_a^M(h^2)\in K_{2E\|\lambda_b\|}(\tilde{U})$. Now, Lemma A.4.6 gives $\int_{W_J^H}\tilde{\mathcal{N}}_a^M(h^2)\,d\nu_\alpha\geq\eta'\int_{\tilde{U}}\tilde{\mathcal{N}}_a^M(h^2)\,d\nu_\alpha$. Note that

$$\tilde{\mathcal{N}}_a^M\left((\beta_J^H)^2\right)(u)\leq\tilde{\mathcal{N}}_a^M\left(\mathbb{1}_{\tilde{U}}-\mu\tilde{\psi}_{(k,r,p,l)}^{(b,\lambda),H}\right)(u)\leq 1-\mu e^{-MT_0}$$

for all $u \in W_J^H$ by choosing any $(k, r, p, l) \in J$. So putting everything together and using $\tilde{\mathcal{N}}_a^*(\nu_\alpha) = \nu_\alpha$, we have

$$\begin{aligned}
\int_{\tilde{U}} \mathcal{D}_J^H(h)^2\, d\nu_\alpha \leq & \left(\int_{W_J^H} \tilde{\mathcal{N}}_a^M\left((\beta_J^H)^2\right)\tilde{\mathcal{N}}_a^M(h^2)\, d\nu_\alpha + \int_{\tilde{U}\setminus W_J^H} \tilde{\mathcal{N}}_a^M\left((\beta_J^H)^2\right)\tilde{\mathcal{N}}_a^M(h^2)\, d\nu_\alpha \right) \\
\leq & \left((1-\mu e^{-MT_0}) \int_{W_J^H} \tilde{\mathcal{N}}_a^M(h^2)\, d\nu_\alpha + \int_{\tilde{U}\setminus W_J^H} \tilde{\mathcal{N}}_a^M(h^2)\, d\nu_\alpha \right) \\
= & \left(\int_{\tilde{U}} \tilde{\mathcal{N}}_a^M(h^2)\, d\nu_\alpha - \mu e^{-MT_0} \int_{W_J^H} \tilde{\mathcal{N}}_a^M(h^2)\, d\nu_\alpha \right) \\
\leq & (1-\eta'\mu e^{-MT_0}) \int_{\tilde{U}} \tilde{\mathcal{N}}_a^M(h^2)\, d\nu_\alpha \\
= & \eta^2 \int_{\tilde{U}} h^2\, d\nu_\alpha.
\end{aligned}$$

$\square$

A.4.3 Proof of property (5.3.4) in Theorem 5.3.4

Now, for all $s = a + ib \in \mathbb{C}$ with $(b, \lambda) \in \widehat{M}_\beta$, then for all $H \in C^1\left(\tilde{U}, H_\lambda{}^{\oplus \dim(\lambda)}\right)$, $h \in K_{E\|\lambda_b\|}(\tilde{U})$, and $1 \leq j \leq j_{\mathrm{m}}$, we define the functions

$$\chi_1^j[s, \lambda, H, h],\ \chi_2^j[s, \lambda, H, h] : \tilde{U}_1 \to \mathbb{C} \quad \text{by}$$

$$\chi_1^j[s,\lambda, H, h](u) = \frac{\left\| e^{f^{\omega_0}(v_0(u))}\lambda_b(\Phi^{\omega_0}(v_0(u))^{-1})H(v_0(u)) + e^{f^{\omega_j}(v_j(u))}\lambda_b(\Phi^{\omega_j}(v_j(u))^{-1})H(v_j(u)) \right\|_2}{(1-m\mu)e^{f^{\omega_0}(v_0(u))}h(v_0(u)) + e^{f^{\omega_j}(v_j(u))}h(v_j(u))},$$

and

$$\chi_2^j[s,\lambda, H, h](u) = \frac{\left\| e^{f^{\omega_0}(v_0(u))}\lambda_b(\Phi^{\omega_0}(v_0(u))^{-1})H(v_0(u)) + e^{f^{\omega_j}(v_j(u))}\lambda_b(\Phi^{\omega_j}(v_j(u))^{-1})H(v_j(u)) \right\|_2}{e^{f^{\omega_0}(v_0(u))}h(v_0(u)) + (1-m\mu)e^{f^{\omega_j}(v_j(u))}h(v_j(u))},$$

for all $u \in \tilde{U}_1$.

The next four lemmas complete the proof of Theorem 5.3.4 and their proofs can be found in [SW21, subsection 9.3].

Lemma A.4.8. *Let $(b, \lambda) \in \widehat{M}_\beta$. Suppose that $H \in C^1\left(\tilde{U}, H_\lambda{}^{\oplus \dim(\lambda)}\right)$ and $h \in K_{E\|\lambda_b\|}(\tilde{U})$ satisfy property (5.3.3) in Theorem 5.3.4. Then for all $(k, r, p, l) \in \Xi(b, \lambda)$, denoting 0 by j if $l = 1$ and $j_{k,r}^{(b,\lambda),H}$ by j if $l = 2$, we have*

$$\frac{1}{2} \leq \frac{h(v_j(u))}{h(v_j(u'))} \leq 2 \qquad \text{for all } u, u' \in \hat{D}_{k,r,p}^{(b,\lambda),H}$$

and also either of the alternatives

1. $\|H(v_j(u))\|_2 \leq \frac{3}{4}h(v_j(u))$ *for all* $u \in \hat{D}_{k,r,p}^{(b,\lambda),H}$,
2. $\|H(v_j(u))\|_2 \geq \frac{1}{4}h(v_j(u))$ *for all* $u \in \hat{D}_{k,r,p}^{(b,\lambda),H}$.

Proof. Let $(b, \lambda) \in \widehat{M}_\beta$. Suppose that $H \in C^1\left(\tilde{U}, H_\lambda{}^{\oplus \dim(\lambda)}\right)$ and $h \in K_{E\|\lambda_b\|}(\tilde{U})$ satisfy property 5.3.3 in Theorem 5.3.4. Let $(k, r, p, l) \in \Xi(b, \lambda)$. We show the first inequality. Let $u, u' \in \hat{D}_{k,r,p}^{(b,\lambda),H}$. Since $|\log \circ h|_{C^1} \leq E\|\lambda_b\|$, we have

$$\begin{aligned}
|\log(h(v_j(u))) - \log(h(v_j(u')))| &\leq |\log \circ h|_{C^1} \cdot |v_j|_{C^1} \cdot d(u, u') \\
&\leq E\|\lambda_b\| \cdot \frac{1}{c_0\kappa_2{}^{m_2}} \cdot \operatorname{diam}_d\left(\hat{D}_{k,r,p}^{(b,\lambda),H}\right) \leq \frac{4EN\epsilon_2}{c_0\kappa_2{}^{m_2}} \\
&\leq \log(2).
\end{aligned}$$

Hence $\left|\log\left(\frac{h(v_j(u))}{h(v_j(u'))}\right)\right| \leq \log(2)$ which implies the first inequality.

Now we show the alternatives. If $\|H(v_j(u))\|_2 \geq \frac{1}{4}h(v_j(u))$ for all $u \in \hat{D}_{k,r,p}^{(b,\lambda),H}$, then we are done. Otherwise, there exists $u_0 \in \hat{D}_{k,r,p}^{(b,\lambda),H}$ such that $\|H(v_j(u_0))\|_2 \leq \frac{1}{4}h(v_j(u_0))$. Let $u \in \hat{D}_{k,r,p}^{(b,\lambda),H}$, $D = d(u_0, u) \leq \operatorname{diam}_d\left(\hat{D}_{k,r,p}^{(b,\lambda),H}\right) = \frac{4N\epsilon_2}{\|\lambda_b\|}$, and $\gamma : [0, D] \to \tilde{U}_1$ be a unit speed geodesic from u_0 to u. Note that $H(v_j(u)) = H(v_j(u_0)) + \int_0^D (H \circ v_j \circ \gamma)'(t)\, dt$. Then using the first proven inequality, we have

$$\begin{aligned}
\|H(v_j(u))\|_2 &\leq \|H(v_j(u_0))\|_2 + \int_0^D \|(dH)_{v_j(\gamma(t))}\|_{\mathrm{op}} |v_j|_{C^1}\, dt \\
&\leq \frac{1}{4}h(v_j(u_0)) + \int_0^D E\|\lambda_b\| h(v_j(\gamma(t))) \cdot \frac{1}{c_0\kappa_2{}^{m_2}}\, dt \\
&\leq \frac{1}{2}h(v_j(u)) + \frac{E\|\lambda_b\|}{c_0\kappa_2{}^{m_2}} \int_0^D 2h(v_j(\gamma(D)))\, dt \\
&\leq \left(\frac{1}{2} + \frac{8EN\epsilon_2}{c_0\kappa_2{}^{m_2}}\right) h(v_j(u)) \leq \frac{3}{4}h(v_j(u)).
\end{aligned}$$

$\square$

For any $k \geq 2$, let $\Theta : (\mathbb{R}^k \setminus \{0\}) \times (\mathbb{R}^k \setminus \{0\}) \to [0, \pi]$ be the map which gives the angle defined by $\Theta(w_1, w_2) = \arccos\left(\frac{\langle w_1, w_2 \rangle}{\|w_1\| \cdot \|w_2\|}\right)$ for all $w_1, w_2 \in \mathbb{R}^k \setminus \{0\}$, where we use the standard inner product and norm. The following lemma can be proven by elementary trigonometry.

Lemma A.4.9. *Let $k \geq 2$. If $w_1, w_2 \in \mathbb{R}^k \setminus \{0\}$ such that $\Theta(w_1, w_2) \geq \omega$ and $\frac{\|w_1\|}{\|w_2\|} \leq L$ for some $\omega \in [0, \pi]$ and $L \geq 1$, then we have*

$$\|w_1 + w_2\| \leq \left(1 - \frac{\omega^2}{16L}\right) \|w_1\| + \|w_2\|.$$

Lemma A.4.10. *Let $s = a + ib \in \mathbb{C}$ with $(b, \lambda) \in \widehat{M}_\beta$. Suppose $H \in C^1\left(\tilde{U}, H_\lambda{}^{\oplus \dim(\lambda)}\right)$ and $h \in K_{E\|\lambda_b\|}(\tilde{U})$ satisfy property (5.3.3) in Theorem 5.3.4. For all $(k, r) \in \Xi_1(b, \lambda)$, denoting $j_{k,r}^{(b,\lambda),H}$ by j, there exists $(p, l) \in \Xi_2$ such that $\chi_{j,l}^{[s,\lambda,H,h]}(u) \leq 1$ for all $u \in \hat{D}_{k,r,p}^{(b,\lambda),H}$.*

Proof. Let $s = a + ib \in \mathbb{C}$ and suppose $(b, \lambda) \in \widehat{M}_\beta$. Suppose $H \in C^1\left(\tilde{U}, H_\lambda{}^{\oplus \dim(\lambda)}\right)$ and $h \in K_{E\|\lambda_b\|}(\tilde{U})$ satisfy property 5.3.3 in Theorem 5.3.4. Let $(k, r) \in \Xi_1(b, \lambda)$. Denote $j_{k,r}^{b,\lambda,H}$ by j, $x_{k,r,1}^{(b,\lambda)}$ by x_1, $x_{k,r,2}^{(b,\lambda),H}$ by x_2, and $\hat{D}_{k,r,p}^{(b,\lambda),H}$ by $\hat{D}_p$. Now, suppose case 1 in Lemma A.4.8 holds for $(k, r, p, l) \in \Xi(b, \lambda)$ for some $(p, l) \in \Xi_2$. Then it is easy to check that $\chi_l^j[\xi, \lambda, H, h](u) \leq 1$ for all $u \in \hat{D}_p$. Otherwise, case 2 in Lemma A.4.8 holds for $(k, r, 1, 1), (k, r, 1, 2), (k, r, 2, 1), (k, r, 2, 2) \in \Xi(b, \lambda)$. We would like to use Lemma A.4.9 but first we need to establish bounds on relative angle and relative size. We start with the former. Define $\hat{H}_\ell(u) = \frac{H(v_\ell(u))}{\|H(v_\ell(u))\|_2}$ and $\phi_\ell(u) = \Phi^{\omega_\ell}(v_\ell(u))$ for all $u \in \tilde{U}_1$ and $\ell \in \{0, j\}$. Let $D = 2\dim(\lambda)^2$ and define the map $\varphi : \mathbb{R}^D \setminus \{0\} \to \mathbb{R}^D$ by $\varphi(w) = \frac{w}{\|w\|}$ for all $w \in \mathbb{R}^D \setminus \{0\}$, where we use the standard inner product and norm on $\mathbb{R}^D$. Then we note that $\|(d\varphi)_w\|_{\text{op}} = \frac{1}{\|w\|}$ for all $w \in R^D$. We can write $\hat{H}_\ell = \varphi \circ H \circ v_\ell$ using the isomorphism $H_\lambda^{\oplus \dim(\lambda)} \cong \mathbb{R}^D$ of real vector spaces. Then using Lemma 5.2.2, we calculate that

$$\begin{aligned}
\|(d\hat{H}_\ell)_u\|_{\text{op}} &\leq \|(d\varphi)_{H(v_\ell(u))}\|_{\text{op}} \|(dH)_{v_\ell(u)}\|_{\text{op}} \|(dv_\ell)_u\|_{\text{op}} \\
&\leq \frac{1}{\|H(v_\ell(u))\|_2} \cdot E\|\lambda_b\| h(v_\ell(u)) \cdot \frac{1}{c_0 \kappa_2{}^{m_2}} \\
&\leq \frac{4E\|\lambda_b\|}{c_0 \kappa_2{}^{m_2}} \leq \delta_1 \|\lambda_b\|
\end{aligned}$$

for all $u \in \hat{D}_p$, $\ell \in \{0, j\}$, and $p \in \{1, 2\}$. In other words, $\hat{H}_0$ and $\hat{H}_j$ are Lipschitz on $\hat{D}_p$ with Lipschitz constant $\delta_1 \|\lambda_b\|$ for all $p \in \{1, 2\}$. Define

$$\left.\begin{aligned} V_\ell(u) &= e^{f^{\omega_l}(v_\ell(u))} \lambda_b(\phi_\ell(u)^{-1}) H(v_\ell(u)); \\ \hat{V}_\ell(u) &= \frac{V_\ell(u)}{\|V_\ell(u)\|_2} = \lambda_b(\phi_\ell(u)^{-1}) \hat{H}_\ell(u); \end{aligned}\right. \qquad \text{for all } u \in \tilde{U}_1 \text{ and } \ell \in \{0, j\}.$$

Since $\hat{H}_0$ and $\hat{H}_j$ are Lipschitz and $d(x_1, x_2) \leq \frac{\epsilon_1}{2\|\lambda_b\|}$, we have

$$\begin{aligned}
&\left\|\hat{V}_0(x_2) - \hat{V}_j(x_2)\right\|_2 \\
&= \|\lambda_b(\phi_0(x_2)^{-1}) \hat{H}_0(x_2) - \lambda_b(\phi_j(x_2)^{-1}) \hat{H}_j(x_2)\|_2 \\
&= \|\lambda_b(\phi_j(x_2)\phi_0(x_2)^{-1}) \hat{H}_0(x_2) - \hat{H}_j(x_2)\|_2 \\
&\geq \|\lambda_b(\phi_j(x_2)\phi_0(x_2)^{-1}) \hat{H}_0(x_1) - \hat{H}_j(x_1)\|_2 \\
&\quad - \|\lambda_b(\phi_j(x_2)\phi_0(x_2)^{-1}) \hat{H}_0(x_2) - \lambda_b(\phi_j(x_2)\phi_0(x_2)^{-1}) \hat{H}_0(x_1)\|_2 \\
&\quad - \|\hat{H}_j(x_2) - \hat{H}_j(x_1)\|_2 \\
&= \|\lambda_b(\phi_j(x_2)\phi_0(x_2)^{-1}) \hat{H}_0(x_1) - \hat{H}_j(x_1)\|_2 - \|\hat{H}_0(x_2) - \hat{H}_0(x_1)\|_2 \\
&\quad - \|\hat{H}_j(x_2) - \hat{H}_j(x_1)\|_2 \\
&\geq \|\lambda_b(\phi_j(x_2)\phi_0(x_2)^{-1}) \hat{H}_0(x_1) - \lambda_b(\phi_j(x_1)\phi_0(x_1)^{-1}) \hat{H}_0(x_1)\|_2 \\
&\quad - \|\lambda_b(\phi_j(x_1)\phi_0(x_1)^{-1}) \hat{H}_0(x_1) - \hat{H}_j(x_1)\|_2 - \delta_1\epsilon_1 \\
&= \|\lambda_b(\phi_0(x_1)^{-1}) \hat{H}_0(x_1) - \lambda_b(\phi_0(x_1)^{-1}\phi_0(x_2)\phi_j(x_2)^{-1}\phi_j(x_1)\phi_0(x_1)^{-1}) \hat{H}_0(x_1)\|_2 \\
&\quad - \|\lambda_b(\phi_0(x_1)^{-1}) \hat{H}_0(x_1) - \lambda_b(\phi_j(x_1)^{-1}) \hat{H}_j(x_1)\|_2 - \delta_1\epsilon_1 \\
&\geq \|\lambda_b(\phi_0(x_1)^{-1}) \hat{H}_0(x_1) - \lambda_b(\mathrm{BP}_j(x_2, x_1))\lambda_b(\phi_0(x_1)^{-1}) \hat{H}_0(x_1)\|_2 \\
&\quad - \left\|\hat{V}_0(x_1) - \hat{V}_j(x_1)\right\|_2 - \delta_1\epsilon_1.
\end{aligned}$$

Denote $\omega = \lambda_b(\phi_0(x)^{-1}) \hat{H}_0(x)$ and $Z = d(\mathrm{BP}_{j,x_1} \circ \Psi)_{\check{x}_1}(z)$ where $z = (\check{x}_1, \check{x}_2 - \check{x}_1) \in \mathrm{T}_{\check{x}_1}(\mathbb{R}^{n-1})$. Recall the curve $\varphi^{\mathrm{BP}}_{j,x_1,z} : [0, 1] \to AM$ defined by $\varphi^{\mathrm{BP}}_{j,x_1,z}(t) = \mathrm{BP}_{j,x_1}(\Psi(x_1 + tz))$ for all $t \in [0, 1]$. Recall that ${\varphi^{\mathrm{BP}}_{j,x_1,z}}'(0) = Z$ and $\varphi^{\mathrm{BP}}_{j,x_1,z}(0) = \mathrm{BP}_{j,x_1}(x_1) = e$ and $\varphi^{\mathrm{BP}}_{j,x_1,z}(1) = \mathrm{BP}_{j,x_1}(x_2) = \mathrm{BP}_j(x_2, x_1)$. Continuing to bound the first term above, we apply Lemmas

A.3.1 and A.2.1 to get

$$\begin{aligned}
&\|\omega - \lambda_b(\mathrm{BP}_j(x_2, x_1))(\omega)\|_2 \\
&\geq \|\omega - \lambda_b(\exp(Z))(\omega)\|_2 - \left\|\lambda_b(\exp(Z))(\omega) - \lambda_b\left(\varphi^{\mathrm{BP}}_{j,x_1,z}(1)\right)(\omega)\right\|_2 \\
&\geq \|\omega - \exp(d\lambda_b(Z))(\omega)\|_2 - \|\lambda_b\| \cdot d_{AM}\left(\exp(Z), \varphi^{\mathrm{BP}}_{j,x_1,z}(1)\right) \\
&\geq \|d\lambda_b(Z)(\omega)\|_2 - \|\lambda_b\|^2\|Z\|^2 - \|\lambda_b\| \cdot d_{AM}\left(\exp(Z), \varphi^{\mathrm{BP}}_{j,x_1,z}(1)\right) \\
&\geq \|d\lambda_b(Z)(\omega)\|_2 - \|\lambda_b\|^2(C_{\mathrm{BP},\Psi}C_\Psi)^2 d(x_1, x_2)^2 - C_{\exp,\mathrm{BP}} \cdot \|\lambda_b\| \cdot d(x_1, x_2)^2 \\
&\geq 7\delta_1\epsilon_1 - \delta_1\epsilon_1 - \delta_1\epsilon_1 \geq 5\delta_1\epsilon_1.
\end{aligned}$$

Hence, we have

$$\left\|\hat{V}_0(x_1) - \hat{V}_j(x_1)\right\|_2 + \left\|\hat{V}_0(x_2) - \hat{V}_j(x_2)\right\|_2 \geq 4\delta_1\epsilon_1.$$

Then $\left\|\hat{V}_0(x_p) - \hat{V}_j(x_p)\right\|_2 \geq 2\delta_1\epsilon_1$ for some $p \in \{1, 2\}$. Recalling estimates from A.2.3 and that $\hat{H}_\ell$ is Lipschitz, we have

$$\begin{aligned}
&\left\|\hat{V}_\ell(x_p) - \hat{V}_\ell(u)\right\|_2 \\
&= \|(\lambda_b(\phi_\ell(x_p)^{-1}) - \lambda_b(\phi_\ell(u)^{-1}))\hat{H}_\ell(x_p) + \lambda_b(\phi_\ell(u)^{-1})(\hat{H}_\ell(x_p) - \hat{H}_\ell(u))\|_2 \\
&\leq \|(\lambda_b(\phi_\ell(x_p)^{-1}) - \lambda_b(\phi_\ell(u)^{-1}))\hat{H}_\ell(x_p)\|_2 + \|\hat{H}_\ell(x_p) - \hat{H}_\ell(u)\|_2 \\
&\leq A_0\|\lambda_b\|d(x_p, u) + \delta_1\|\lambda_b\|d(x_p, u) \\
&\leq (A_0 + \delta_1)\|\lambda_b\| \cdot \frac{2N\epsilon_2}{\|\lambda_b\|} = 2N\epsilon_2(A_0 + \delta_1) \leq \frac{\delta_1\epsilon_1}{2}
\end{aligned}$$

for all $u \in \hat{D}_p$ and $\ell \in \{0, j\}$. Hence $\left\|\hat{V}_0(u) - \hat{V}_j(u)\right\|_2 \geq \delta_1\epsilon_1 \in (0, 1)$ for all $u \in \hat{D}_p$. Then using the cosine law, the required bound for relative angle is

$$\Theta(V_0(u), V_j(u)) = \Theta(\hat{V}_0(u), \hat{V}_j(u)) \geq \arccos\left(1 - \frac{(\delta_1\epsilon_1)^2}{2}\right) \in (0, \pi).$$

For the bound on relative size, let $(\ell, \ell') \in \{(0, j), (j, 0)\}$ such that $h(v_\ell(u_0)) \leq h(v_{\ell'}(u_0))$ for some $u_0 \in \hat{D}_p$. Let $l = 1$ if $(\ell, \ell') = (0, j)$ and $l = 2$ if $(\ell, \ell') = (0, j)$. Recalling that λ_b

is a unitary representation, by A.4.8, we have

$$\begin{aligned}\frac{\|V_\ell(u)\|_2}{\|V_{\ell'}(u)\|_2} &= \frac{e^{f^{\omega_\ell}(v_\ell(u))}\|H(v_\ell(u))\|_2}{e^{f^{\omega_{\ell'}}(v_{\ell'}(u))}\|H(v_{\ell'}(u))\|_2} \leq \frac{4e^{f^{\omega_\ell}(v_\ell(u))-f^{\omega_{\ell'}}(v_{\ell'}(u))}h(v_\ell(u))}{h(v_{\ell'}(u))} \\ &\leq \frac{16e^{2m_2T_0}h(v_\ell(u_0))}{h(v_{\ell'}(u_0))} \leq 16e^{2m_2T_0}\end{aligned}$$

for all $u \in \hat{D}_p$, which is the required bound on relative size. Now using A.4.9, A.2.9, and $\|H\| \leq h$ on $\|V_\ell(u) + V_{\ell'}(u)\|_2$ gives $\chi_l^j[\xi, \lambda, H, h](u) \leq 1$ for all $u \in \hat{D}_p$. $\square$

Lemma A.4.11. *For all $s = a + ib \in \mathbb{C}$ with $(b, \lambda) \in \widehat{M}_\beta$, if $H \in C^1\left(\tilde{U}, H_\lambda{}^{\oplus \dim(\lambda)}\right)$ and $h \in K_{E\|\lambda_b\|}(\tilde{U})$ satisfy property (5.3.3) in Theorem 5.3.4, then there exists $J \in \mathcal{J}(b, \lambda)$ such that*

$$\left\|\tilde{\mathcal{N}}_{s,\lambda}^{M}(H)(u)\right\|_2 \leq \mathcal{D}_J^H(h)(u) \qquad \textit{for all } u \in \tilde{U}.$$

Proof. Let $s = a + ib \in \mathbb{C}$ and suppose $(b, \lambda) \in \widehat{M}_\beta$. Suppose $H \in C^1\left(\tilde{U}, H_\lambda{}^{\oplus \dim(\lambda)}\right)$ and $h \in K_{E\|\lambda_b\|}(\tilde{U})$ satisfy property 5.3.3 in Theorem 5.3.4. We drop superscripts (b, λ) and H to simply notation. For all $(k, r) \in \Xi_1(b, \lambda)$, there exists $(p_{k,r}, l_{k,r}) \in \Xi_2$ as guaranteed by Lemma A.4.10. Let $J_0 = \{(k, r, p_{k,r}, l_{k,r}) \in \Xi(b, \lambda) : (k, r) \in \Xi_1(b, \lambda)\} \subset \Xi(b, \lambda)$ which is then dense by construction and so $J_0 \in \mathcal{J}(b, \lambda)$. Now, we make necessary modifications to J_0 to define $J \in \mathcal{J}(b, \lambda)$. Recall the definitions from the proof of Lemma A.3.3. For all equivalence classes $[D_{k,r,p}] \in D_{\cup}^{\mathrm{conn}}$, we do the following. Choose any representative, say $D_{k,r,p} \in [D_{k,r,p}]$ and make the modification $j_{k',r'} = j_{k,r}$ and $l_{k',r'} = l_{k,r}$ for all $(k', r') \in \Xi_1(b, \lambda)$ with $D_{k',r',p'} \in [D_{k,r,p}]$ for some $p' \in \{1, 2\}$. Define $J \in \mathcal{J}(b, \lambda)$ by $J = \{(k, r, p_{k,r}, l_{k,r}) \in \Xi(b, \lambda) : (k, r) \in \Xi_1(b, \lambda)\} \subset \Xi(b, \lambda)$. Now let $u \in \tilde{U}$. If $u \notin D_{k,r,p}$ for all $(k, r, p, l) \in J$, then $\beta_J^H(v) = 1$ for all branches $v = P^{-\omega}(u)$ where ω is an admissible sequence with $\mathrm{len}(\omega) = m_2$. Hence $\left\|\tilde{\mathcal{N}}_{s,\lambda}^{m_2}(H)(u)\right\|_2 \leq \tilde{\mathcal{N}}_a^{m_2}(\beta_J^H h)(u)$ follows trivially from definitions. Otherwise, by construction, there exist $(k, r), (k_0, r_0) \in \Xi_1(b, \lambda)$ such that $u \in D_{k,r,p_{k,r}} \in [D_{k_0,r_0,p_{k_0,r_0}}]$ corresponding to $(k, r, p_{k,r}, l_{k,r}) \in J$, and such that $j_{k',r'} = j_{k_0,r_0}$ and $l_{k',r'} = l_{k_0,r_0}$ for all $D_{k',r',p_{k',r'}} \in [D_{k_0,r_0,p_{k_0,r_0}}]$. Denote j_{k_0,r_0} by j_0 and l_{k_0,r_0} by l_0. Let $(\ell, \ell') = (0, j_0)$ if $l_0 = 1$ and $(\ell, \ell') = (j_0, 0)$ if $l_0 = 2$. Then by construction of J, we have $\chi_{l_0}^{j_0}[\xi, \lambda, H, h](u) \leq 1$, $\beta_J^H(v_\ell(u)) \geq 1 - m\mu$, and $\beta_J^H(v_j(u)) = 1$ for all

$0 \leq j \leq j_{\mathrm{m}}$ with $j \neq \ell$. Hence, we compute that

$$\begin{aligned}
&\|\tilde{\mathcal{N}}_{s,\lambda}^{m_2}(H)(u)\|_2 \\
&= \Big\| \sum_{\substack{\omega:\mathrm{len}(\omega)=m_2 \\ v=P^{-\omega}(u)}} e^{f^{\omega}(v)} \lambda_b(\Phi^{\omega}(v)^{-1}) H(v) \Big\|_2 \\
&\leq \sum_{\substack{\omega:\mathrm{len}(\omega)=m_2 \\ v=P^{-\omega}(u)\notin\{v_0(u),v_{j_0}(u)\}}} \Big\| e^{f^{\omega}(v)} \lambda_b(\Phi^{\omega}(v)^{-1}) H(v) \Big\|_2 \\
&\quad + \Big\| e^{f^{\omega_\ell}(v_\ell(u))} \lambda_b(\Phi^{\omega_\ell}(v_\ell(u))^{-1}) H(v_\ell(u)) \\
&\quad + e^{f^{\omega_{\ell'}}(v_{\ell'}(u))} \lambda_b(\Phi^{\omega_{\ell'}}(v_{\ell'}(u))^{-1}) H(v_{\ell'}(u)) \Big\|_2 \\
&\leq \sum_{\substack{\omega:\mathrm{len}(\omega)=m_2 \\ v=P^{-\omega}(u)\notin\{v_0(u),v_{j_0}(u)\}}} e^{f^{\omega}(v)} h(v) + \Big((1-N\mu) e^{f^{\omega_\ell}(v_\ell(u))} h(v_\ell(u)) \\
&\quad + e^{f^{\omega_{\ell'}}(v_{\ell'}(u))} h(v_{\ell'}(u)) \Big) \\
&\leq \tilde{\mathcal{N}}_a^{m_2}(\beta_J^H h)(u).
\end{aligned}$$

Thus, we have

$$\begin{aligned}
\|\tilde{\mathcal{N}}_{s,\lambda}^{m}(H)(u)\|_2 &\leq \Big\| \Big(\tilde{\mathcal{N}}_{s,\lambda}^{m_1} \circ \tilde{\mathcal{N}}_{s,\lambda}^{m_2} \Big)(H)(u) \Big\|_2 \leq \tilde{\mathcal{N}}_a^{m_1} \|\tilde{\mathcal{N}}_{s,\lambda}^{m_2}(H)\|(u) \\
&\leq \tilde{\mathcal{N}}_a^{m_1} \Big(\tilde{\mathcal{N}}_a^{m_2}(\beta_J^H h) \Big)(u) = \tilde{\mathcal{N}}_a^{m}(\beta_J^H h)(u) = \mathcal{D}_J^H(h)(u)
\end{aligned}$$

for all $u \in \tilde{U}$. $\square$

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