HUISKEN-YAU-TYPE UNIQUENESS FOR AREA-CONSTRAINED WILLMORE SPHERES

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ABSTRACT. Let (M,g) be a Riemannian 3-manifold that is asymptotic to Schwarzschild. We study the existence of large area-constrained Willmore spheres $\Sigma\subset M$ with non-negative Hawking mass and inner radius ρ dominated by the area radius λ. If the scalar curvature of (M,g) is non-negative, we show that no such surfaces with $\log\lambda \ll \rho$ exist. This answers a question of G. Huisken.

1. Introduction

Let (M,g) be a connected, complete Riemannian 3-manifold. Let $\Sigma\subset M$ be a closed, two-sided surface with area element $\mathrm{d}\mu$, outward normal ν, and mean curvature H with respect to ν. The Hawking mass of Σ is

$$m_H(\Sigma)=\sqrt{\frac{|\Sigma|}{16\,\pi}}\left(1-\frac{1}{16\,\pi}\int_\Sigma H^2\,\mathrm{d}\mu\right).$$

In the case where (M,g) arises as maximal initial data for the Einstein field equations, the Hawking mass has been proposed as a quasi-local measure for the strength of the gravitational field; see [12].

Recall that time-symmetric initial data for a Schwarzschild black hole with mass $m>0$ are given by

$$\left(\left\{x\in\mathbb{R}^3:|x|>\frac{m}{2}\right\},\left(1+\frac{m}{2\,|x|}\right)^4\bar g\right) \tag{1}$$

where

$$\bar g=\sum_{i=1}^3 dx^i\otimes dx^i$$

is the Euclidean metric on $\mathbb{R}^3$. A special class of general initial data consists of those asymptotic to Schwarzschild. Given a non-negative integer k, we say that (M,g) is C^k-asymptotic to Schwarzschild with mass $m>0$ if there is a non-empty compact set $K\subset M$ such that the end $M\backslash K$ is diffeomorphic to $\{x\in\mathbb{R}^3\,:\,|x|>1\}$ and, in this special chart, there holds, as $x\to\infty$,

$$g=\left(1+\frac{m}{2\,|x|}\right)^4\bar g+\sigma\qquad\text{where}\qquad\partial_J\sigma=O(|x|^{-2-|J|})$$

for every multi-index J with $|J|\le k$. Given $r>1$, we define $B_r\subset M$ to be the compact domain whose boundary corresponds to $S_r(0)$ in this chart. Moreover, given a closed, two-sided surface $\Sigma\subset M$, we define the area-radius $\lambda(\Sigma)>0$ and inner radius $\rho(\Sigma)$ of Σ by

$$4\,\pi\,\lambda(\Sigma)^2=|\Sigma|\qquad\text{and}\qquad\rho(\Sigma)=\sup\{r>1\,:\,B_r\cap\Sigma=\emptyset\}.$$

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The Hawking mass $m_H(\Sigma)$ provides useful information on the strength of the gravitational field if the surface Σ is either a stable constant mean curvature sphere with large enclosed volume or, alternatively, an area-constrained Willmore sphere with large area; see also [2, p. 2348]. Stable constant mean curvature surfaces are stable critical points of the area functional under a volume constraint and therefore candidates to have least perimeter among surfaces of the same enclosed volume. S.-T. Yau and D. Christodoulou [5] have observed that the Hawking mass of stable constant mean curvature spheres is non-negative if (M, g) has non-negative scalar curvature. Note that, in this context, the scalar curvature provides a lower bound for the energy density of the initial data set. Stable constant mean curvature spheres have since been studied extensively in the context of mathematical relativity; see for example the recent survey in [9, §1, Appendix G, and Appendix H].

Recall that $\Sigma \subset M$ is an area-constrained Willmore surface if there is a number $\kappa \in \mathbb{R}$ such that

$$\Delta H + (|\mathring{h}|^2 + \mathrm{Ric}(\nu, \nu) + \kappa)\, H = 0. \tag{2}$$

Here, Δ is the non-positive Laplace-Beltrami operator with respect to the induced metric on Σ, $\mathring{h}$ the traceless part of the second fundamental form h of Σ, and Ric the Ricci curvature of (M, g). Note that (2) is the Euler-Lagrange equation of the Willmore energy

$$\int_\Sigma H^2 \,\mathrm{d}\mu \tag{3}$$

with respect to an area constraint and κ the corresponding Lagrange parameter. Area-constrained Willmore surfaces are therefore candidates to have largest Hawking mass among all closed surfaces of the same area. As observed in [10, p. 4], large area-constrained Willmore spheres capture information on the asymptotic distribution of scalar curvature that large stable constant mean curvature spheres are impervious to.

Existence and uniqueness of large-area constrained Willmore surfaces. In the recent papers [8, 10], the first-named author and the second-named author have studied the existence, uniqueness, and physical properties of large area-constrained Willmore spheres. We recall the following result; see Figure 1.

Theorem 1 ([10, Theorem 6]). *Let (M, g) be C^4-asymptotic to Schwarzschild with mass $m > 0$ and suppose that*

$$\sum_{i=1}^{3} x^i\, \partial_i(|x|^2\, R) \leq 0 \tag{4}$$

outside a compact set. There exists $\kappa_0 > 0$ and a family

$$\{\Sigma(\kappa) \,:\, \kappa \in (0, \kappa_0)\} \tag{5}$$

of area-constrained Willmore spheres $\Sigma(\kappa) \subset M$ where $\Sigma(\kappa)$ satisfies (2) with parameter κ. The family (5) sweeps out the complement of a compact set in M. Moreover, there holds $m_H(\Sigma(\kappa)) \geq 0$ for each $\kappa \in (0, \kappa_0)$.

Moreover, given $\delta > 0$, there exists $\lambda > 1$ and a compact set $K \subset M$ with the following property. If $\Sigma \subset M \setminus K$ is an area-constrained Willmore sphere with $m_H(\Sigma) \geq 0$ and $|\Sigma| > 4\,\pi\,\lambda$, then either $\Sigma = \Sigma(\kappa)$ for some $\kappa \in (0, \kappa_0)$ or $\rho(\Sigma) < \delta\, \lambda(\Sigma)$.

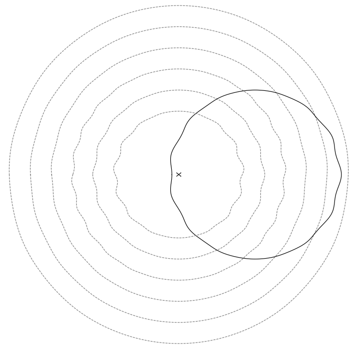

FIGURE 1. An illustration of the asymptotic family (5) by area-constrained Willmore spheres. The cross marks the origin in the asymptotically flat chart. The solid black line indicates a potential area-constrained Willmore sphere $\Sigma \subset M$ with $m_H(\Sigma) \geq 0$ and $\rho(\Sigma) < \delta\, \lambda(\Sigma)$.

Remark 2.

i) *Note that* (4) *implies that* $R \geq 0$ *at infinity.*

ii) *T. Lamm, the third-named author, and the fourth-named author have previously proved the existence of an asymptotic foliation by large-area constrained Willmore spheres if* (M, g) *is a so-called small perturbation of Schwarzschild; see* [19, Theorem 1 and Theorem 2].

iii) *In* $\mathbb{R}^3$*, round spheres are Willmore surfaces and the only closed surfaces with non-negative Hawking mass; see* [23, (3)].

iv) *S. Brendle* [3] *has shown that the spheres of symmetry are the only closed, embedded constant mean curvature surfaces in spatial Schwarzschild* (1)*. It is not known if these are also the only area-constrained Willmore spheres; see also* [16, Remark 1.5 and Theorem 1.6].

As discussed for example in [8, p. 4], the assumptions that the surfaces considered have non-negative Hawking mass, be large, and be disjoint from a certain bounded set appear to be essential for a characterization result such as Theorem 1 to hold. By contrast, as we explain below, we conjecture that the alternative $\rho(\Sigma) < \delta\, \lambda(\Sigma)$ in the conclusion of Theorem 1 does not actually arise. In fact, in this paper, we improve the uniqueness result in Theorem 1 by ruling out the existence of certain large area-constrained Willmore spheres whose respective inner radius is small compared to their area radius.

Theorem 3. *Let* (M, g) *be* C^4*-asymptotic to Schwarzschild and suppose that, as* $x \to \infty$

$$R \geq -o(|x|^{-4}). \tag{6}$$

There are $\delta > 0$ *and* $\lambda > 1$ *with the following property.*

There is no area-constrained Willmore sphere $\Sigma \subset M$ *with*

- $m_H(\Sigma) \geq 0$,
- $|\Sigma| > 4\,\pi\,\lambda^2$,
- $\rho(\Sigma) < \delta\, \lambda(\Sigma)$,
- $\log \lambda(\Sigma) < \delta\, \rho(\Sigma)$.

Remark 4.

- *i) The conclusion of Theorem 3 can fail if the assumption* (6) *is dropped; see* [10, Theorem 11].
- *ii) Note that* $\log \lambda(\Sigma) < \lambda(\Sigma)^{1/s}$ *for every* $s > 1$. *Analytically, the result established by G. Huisken and S.-T. Yau in* [15, Theorem 5.1] *on the uniqueness of large stable constant mean curvature spheres corresponds to the case where* $1 < s < 2$.
- *iii) The assumption* $\log \lambda(\Sigma) < \delta\, \rho(\Sigma)$ *is essential to obtain* (46) *from* (40) *and seems to be optimal for the method employed in this paper. Specifically, note that* (40) *is the best possible estimate based on Lemma 23.*
- *iv) The assumptions of Theorem 3 imply that* $\Sigma \cap B_2 = \emptyset$. *Note that, unlike in* [15] *and* [21], *we do not assume that* Σ *encloses* B_2.

Combining Theorem 3 with Theorem 1, we obtain the following corollary.

Corollary 5. *Let* (M, g) *be* C^4*-asymptotic to Schwarzschild. Suppose that*

$$\sum_{i=1}^{3} x^i\, \partial_i(|x|^2\, R) \leq 0$$

outside a compact set. Let $\{\Sigma_i\}_{i=1}^{\infty}$ *be a sequence of area-constrained Willmore spheres* $\Sigma_i \subset M$ *with*

- $m_H(\Sigma) \geq 0$,
- $\lim_{i\to\infty} \rho(\Sigma_i) = \infty$,
- $\lim_{i\to\infty} \lambda(\Sigma_i) = \infty$,
- Σ_i *is not part of the foliation* (5).

There holds $\rho(\Sigma_i) = O(\log \lambda(\Sigma_i))$.

Outline of related results. We say that a sequence $\{\Sigma_i\}_{i=1}^{\infty}$ of spheres $\Sigma_i \subset M$ with

$$\lim_{i\to\infty} \rho(\Sigma_i) = \infty \qquad \text{and} \qquad \rho(\Sigma_i) = o(\lambda(\Sigma_i)) \tag{7}$$

is slowly divergent. As with large stable constant mean curvature spheres, a substantial obstacle towards establishing the uniqueness of large area-constrained Willmore spheres with non-negative Hawking mass in Riemannian 3-manifolds asymptotic to Schwarzschild is to rule out the possibility of a slowly divergent sequence $\{\Sigma_i\}_{i=1}^{\infty}$ of area-constrained Willmore spheres $\Sigma_i \subset M$ with non-negative Hawking mass. The main difficulty in understanding the geometry of the spheres Σ_i owes to the fact that unrefined curvature estimates generally do not yield global analytic control. In fact, there holds, as $i \to \infty$,

$$h(\Sigma_i) = O(\lambda(\Sigma_i)^{-1}) + O(\rho(\Sigma_i)^{-1}\, |x|^{-1}); \tag{8}$$

see Proposition 24. If for example $\rho(\Sigma_i) = o(\lambda(\Sigma_i)^{1/2})$, estimate (8) fails to bound the sequence $\{\lambda(\Sigma_i)^{-1}\, \Sigma_i\}_{i=1}^{\infty}$ in C^2. If for example $\rho(\Sigma_i) = o(\log \lambda(\Sigma_i))$, (8) even fails to bound the sequence $\{\lambda(\Sigma_i)^{-1}\, \Sigma_i\}_{i=1}^{\infty}$ in C^1; see Figure 2.

G. Huisken and S.-T. Yau [15] have shown that there are no slowly divergent sequences $\{\Sigma_i\}_{i=1}^{\infty}$ of stable constant mean curvature spheres $\Sigma_i \subset M$ that enclose B_2 with $\lambda(\Sigma_i) = O(\rho(\Sigma_i)^s)$ where $1 < s < 2$. In this case, an estimate similar to (8) provides uniform estimates in C^2. These estimates are sufficient to conclude their argument based on analyzing a certain flux integral related to the

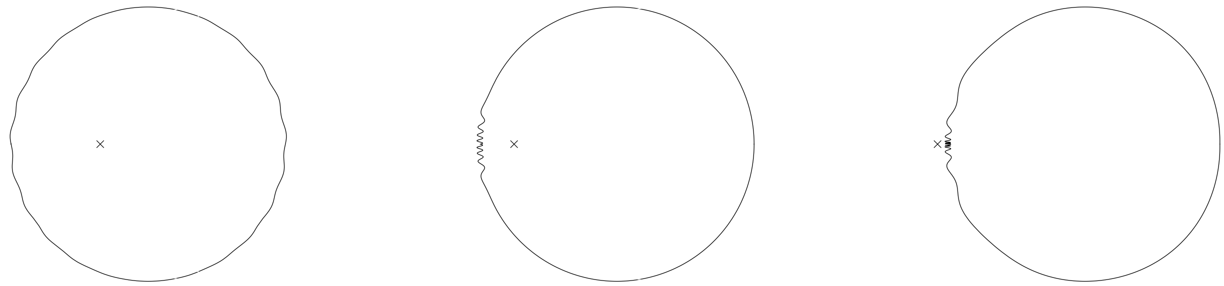

FIGURE 2. An illustration of a slowly divergent sequence $\{\Sigma_i\}_{i=1}^\infty$ of area-constrained Willmore spheres $\Sigma_i \subset M$ on the scale of the area radius $\lambda(\Sigma_i)$. The cross marks the origin in the asymptotically flat chart. Away from the origin, the surfaces Σ_i converge uniformly to a round sphere.

variation of the area functional with respect to a translation. Following the same strategy, J. Qing and G. Tian [21] have shown that the assumption $\lambda(\Sigma_i) = O(\rho(\Sigma_i)^s)$, $1 < s < 2$, can be dropped. To overcome the potential loss of C^1-control, they carry out a delicate asymptotic analysis based on the observation that the Gauss maps $\{\nu(\Sigma_i)\}_{i=1}^\infty$ form a sequence of almost harmonic maps. Finally, O. Chodosh and the first-named author [4] have shown that the assumption that Σ_i encloses B_2 can be dropped if the scalar curvature of (M, g) is non-negative. Their method is based on an analysis of the Hawking mass of Σ_i. To obtain the required analytic control, they combine the Christodoulou-Yau estimate [5, p. 13]

$$\frac{2}{3}\int_\Sigma (|\mathring{h}|^2 + R)\,\mathrm{d}\mu \leq 16\,\pi - \int_\Sigma H^2\,\mathrm{d}\mu, \tag{9}$$

valid for every stable constant mean curvature sphere $\Sigma \subset M$, with global methods developed by G. Huisken and T. Ilmanen [14]. We also refer to the papers of L.-H. Huang [13], of S. Ma [20], and of the first-named author and second-named author [9] on slowly divergent sequences of large stable constant mean curvature spheres in general asymptotically flat Riemannian 3-manifolds.

When studying slowly divergent sequences (7) of area-constrained Willmore spheres with non-negative Hawking mass, additional difficulties arise. On the one hand, the absence of an estimate comparable to (9) renders the curvature estimates for large area-constrained Willmore spheres less powerful than those for large stable constant mean curvature spheres. Moreover, the fourth-order nature of the area-constrained Willmore equation (2) poses additional analytical challenges. On the other hand, the variation of the Willmore energy (3) with respect to a translation is of a smaller scale than that of the area functional, at least when Σ encloses B_2. Consequently, more precise analytic control is needed. In fact, we are not aware of any previous positive results on the non-existence of slowly divergent sequences of area-constrained Willmore spheres with non-negative Hawking mass. By contrast, in [10, Theorem 11], the first-named author and the second-named author have shown that such sequences may exist if the scalar curvature of (M, g) is allowed to change sign.

Outline of the proof of Theorem 3. By scaling, we may assume that $m = 2$, that is,

$$g = (1 + |x|^{-1})^4\,\bar{g} + \sigma.$$

We use a bar to indicate that a geometric quantity has been computed with respect to the Euclidean background metric $\bar g$. Likewise, we use a tilde to indicate that the Schwarzschild metric

$$\tilde g = (1 + |x|^{-1})^4\, \bar g$$

with mass $m = 2$ has been used in the computation.

Let $s > 1$. Assume that $\{\Sigma_i\}_{i=1}^\infty$ is a sequence of area-constrained Willmore spheres $\Sigma_i \subset M$ with $m_H(\Sigma_i) \geq 0$ and

$$\lim_{i\to\infty} \rho(\Sigma_i) = \infty, \qquad \rho(\Sigma_i) = o(\lambda(\Sigma_i)), \qquad \log \lambda(\Sigma_i) = o(\rho(\Sigma_i)). \tag{10}$$

Let $a \in \mathbb{R}^3$. To prove Theorem 3, we expand the variation of the Willmore energy of Σ_i with respect to a translation in direction a given by

$$0 = -\int_{\Sigma_i} g(a, \nu)\, \Big[\Delta H + (|\mathring{h}|^2 + \operatorname{Ric}(\nu,\nu) + \kappa)\, H\Big]\, \mathrm{d}\mu. \tag{11}$$

Contrary to the variation of the area functional with respect to a translation, we expect the right-hand side of (11) to be small independently of whether Σ_i encloses the origin or not; see (51). Consequently, precise analytic control is needed to expand the terms on the right-hand side of (11) with appropriate error control.

To this end, we first adapt the localized curvature estimates proved by E. Kuwert and R. Schätzle [18] for Willmore surfaces in $\mathbb{R}^3$ to the setting of large area-constrained Willmore spheres in Riemannian 3-manifolds asymptotic to Schwarzschild; see Appendices A and B. In conjunction with (10), it follows that Σ_i is the radial graph of a function u_i over a large coordinate sphere $S_i = S_{\lambda_i}(\lambda_i\, \xi_i)$ where $\lambda_i > 1$ and $\xi_i \in \mathbb{R}^3$; see Lemma 7. The resulting estimates in Lemma 10 below are still too coarse for our purposes. To overcome this, we first prove explicit estimates for the Laplace operator of a round sphere based on Green's function methods; see Lemma 11. Second, we observe that the quotient of $H(\Sigma_i)$ and the potential function of the spatial Schwarzschild manifold satisfies an equation slightly more useful than the area-constrained Willmore equation (72); see Lemma 73. We then use the explicit estimates for the Laplace operator to investigate this equation to obtain the sharp estimate

$$H(\Sigma_i) = (2 + o(1))\, \lambda(\Sigma_i)^{-1} - 4\, \lambda(\Sigma_i)^{-1}\, |x|^{-1} + o(\lambda(\Sigma_i)^{-1}\, \rho(\Sigma_i)^{-1}); \tag{12}$$

see Lemma 13. We note that this procedure requires assumption (10) in an essential way; see Remark 4.

With the estimate (12) at hand, we compute that

$$\begin{aligned} 0 &= -\int_{\Sigma_i} g(\xi_i, \nu)\, \Big[\Delta H + (|\mathring{h}|^2 + \operatorname{Ric}(\nu,\nu) + \kappa)\, H\Big]\, \mathrm{d}\mu \\ &= 8\,\pi\, \lambda(\Sigma_i)^{-1}\, \rho(\Sigma_i)^{-2} - \lambda(\Sigma_i)^{-1} \int_{S_i} \bar g(\xi_i, \bar\nu)\, R\, \mathrm{d}\bar\mu - o(\lambda(\Sigma_i)^{-1}\, \rho(\Sigma_i)^{-2}); \end{aligned}$$

see (52). Using that $R \geq -o(|x|^{-4})$, it follows that

$$0 \geq 8\,\pi\, \lambda(\Sigma_i)^{-1}\, \rho(\Sigma_i)^{-2} - o(\lambda(\Sigma_i)^{-1}\, \rho(\Sigma_i)^{-2}).$$

This is a contradiction for sufficiently large i.

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2. Asymptotic analysis of large area-constrained Willmore spheres

We assume that g is a Riemannian metric on $\mathbb{R}^3$ such that, as $x \to \infty$,

$$g = \left(1 + |x|^{-1}\right)^4 \bar{g} + \sigma \qquad \text{where} \qquad \partial_J \sigma = O(|x|^{-2-|J|})$$

for every multi-index J with $|J| \leq 4$.

Let $\xi \in \mathbb{R}^3$ and $\lambda > 0$. Given $u \in C^\infty(S_\lambda(\lambda\,\xi))$, we define the map

$$\Phi^u_{\xi,\lambda} \,:\, S_\lambda(\lambda\,\xi) \to \mathbb{R}^3 \qquad \text{given by} \qquad \Phi^u_{\xi,\lambda}(x) = x + u(x)\,(\lambda^{-1}\,x - \xi).$$

We denote by

$$\Sigma_{\xi,\lambda}(u) = \Phi^u_{\xi,\lambda}(S_\lambda(\lambda\,\xi)) \tag{13}$$

the Euclidean graph of u over $S_\lambda(\lambda\,\xi)$. We tacitly identify functions defined on $\Sigma_{\xi,\lambda}(u)$ with functions defined on $S_\lambda(\lambda\,\xi)$ by precomposition with $\Phi^u_{\xi,\lambda}$.

We consider a sequence $\{\Sigma_i\}_{i=1}^\infty$ of area-constrained Willmore spheres $\Sigma_i \subset \mathbb{R}^3$ with

$$m_H(\Sigma_i) \geq 0, \qquad \lim_{i\to\infty} \rho(\Sigma_i) = \infty, \qquad \rho(\Sigma_i) = o(\lambda(\Sigma_i)). \tag{14}$$

We assume that, as $i \to \infty$,

$$\log \lambda(\Sigma_i) = o(\rho(\Sigma_i)). \tag{15}$$

The goal of this section is to study the shape of Σ_i as $i \to \infty$. More precisely, we show that Σ_i is a graph over a nearby coordinate sphere, provided i is sufficiently large.

We abbreviate $\rho_i = \rho(\Sigma_i)$ and $\lambda_i = \lambda(\Sigma_i)$.

Passing to a subsequence, we may assume that either Σ_i encloses B_2 for every i or that the bounded region enclosed by Σ_i is disjoint from B_2 for every i. Let $x_i \in \Sigma_i \cap S_{\rho_i}(0)$. Passing to a further subsequence if necessary, we may assume that there is $\xi \in \mathbb{R}^3$ with $|\xi| = 1$ such that

$$\lim_{i\to\infty} \rho_i^{-1}\, x_i = -\xi. \tag{16}$$

Note that, by (14), (55), and Lemma 21,

$$\sup_{x\in\Sigma_i} |x| = O(\lambda_i). \tag{17}$$

Lemma 6. *If Σ_i encloses B_2 for every i, the surfaces $\lambda_i^{-1}\,\Sigma_i$ converge to $S_1(\xi)$ in C^1. If the bounded region enclosed by Σ_i is disjoint from B_2 for every i, the surfaces $\lambda_i^{-1}\,\Sigma_i$ converge to $S_1(-\xi)$ in C^1.*

Proof. This is similar to [9, Lemma 19]. We repeat the argument for the reader's convenience.

We first assume that Σ_i encloses B_2 for every i.

We may assume that $\xi = e_3$. Let $a_i \in \mathbb{R}^3$ with $|a_i| = 1$ and $a_i \perp x_i, e_3$. Let $R_i \in SO(3)$ be the unique rotation with $R(a_i) = a_i$ and $R(x_i) = |x_i|\, e_3$. By (16), $\lim_{i\to\infty} R_i = \text{Id}$.

Let $\gamma_i > 0$ be the largest radius such that there is a smooth function $u_i : \{y \in \mathbb{R}^2 : |y| \leq \gamma_i\} \to \mathbb{R}$ with

- $|(\bar\nabla u_i)(y)| \leq 1$
- $(y, \rho_i + u_i(y)) \in R_i(\Sigma_i)$

for all $y \in \mathbb{R}^2$ with $|y| \leq \gamma_i$. Clearly, $\gamma_i > 0$, $(\bar\nabla u_i)(0) = 0$, and $u_i(0) = 0$. It follows that

$$|y| + \rho_i \leq 3\,|(y, \rho_i + u_i(y))| \leq 6\,(|y| + \rho_i) \tag{18}$$

and

$$|(\bar\nabla^2 u_i)(y)| \leq 8\,|\bar h(R_i(\Sigma_i))((y, \rho_i + u_i(y)))| \tag{19}$$

for every $y \in \mathbb{R}^2$ with $|y| \leq \gamma_i$. Moreover, by Corollary 28,

$$\bar h(R_i(\Sigma_i)) = \lambda_i^{-1}\,\bar g|_{R_i(\Sigma_i)} + O(\lambda_i^{-1/2} + \rho_i^{-1})\,|x|^{-1}. \tag{20}$$

Combining (19), (20), and (18), we have

$$|(\bar\nabla^2 u_i)|_y| \leq 8\,\lambda_i^{-1} + O(\lambda_i^{-1/2} + \rho_i^{-1})\,(|y| + \rho_i)^{-1}.$$

Integrating and using (18), (17), (14), and (15),

$$|(\bar\nabla u_i)|_y| \leq 8\,|y|\,\lambda_i^{-1} + O(\lambda_i^{-1/2} + \rho_i^{-1})\,\log(\rho_i^{-1}\,\lambda_i) = 8\,|y|\,\lambda_i^{-1} + o(1). \tag{21}$$

It follows that $16\,\gamma_i \geq \lambda_i$ for all i sufficiently large. (21) also shows that, given $\epsilon > 0$, there is $\delta > 0$ such that

$$|\nu(R_i(\Sigma_i)) - e_3| \leq \epsilon \qquad \text{on} \qquad \big\{(y, \rho_i + u_i(y)) : y \in \mathbb{R}^2 \text{ with } \lambda_i^{-1}\,|y| \leq \delta\big\}.$$

According to Proposition 25, $\lambda_i^{-1}\,R_i(\Sigma_i)$ converges to $S_1(\tilde\xi)$ in C^2 locally in $\mathbb{R}^3 \setminus \{0\}$ where $\tilde\xi \in \mathbb{R}^3$; see also [21, Lemma 3.1] and [4, Proposition 2.2]. The preceding argument shows that $\tilde\xi = \xi$ and that the convergence is in C^1 in $\mathbb{R}^3$.

This finishes the proof in the case where each Σ_i encloses B_2. The case where B_2 is disjoint from the bounded region enclosed by Σ_i for every i requires only formal modifications. $\square$

If Σ_i encloses B_2, we define

$$\xi_i = (\lambda_i^{-1} - \rho_i^{-1})\,x_i.$$

If the bounded region enclosed by Σ_i is disjoint from B_2, we define

$$\xi_i = (\lambda_i^{-1} + \rho_i^{-1})\,x_i.$$

Note that, in either case,

$$|1 - |\xi_i|| = \lambda_i^{-1}\,\rho_i \qquad \text{and} \qquad x_i = \lambda_i(1 - |\xi_i|^{-1})\,\xi_i \in S_{\lambda_i}(\lambda_i\,\xi_i);$$

see Figure 3.

The following lemma is a direct consequence of Lemma 6.

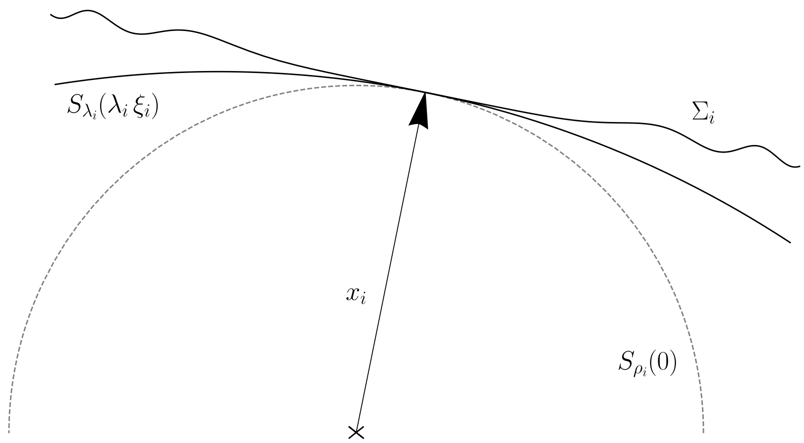

FIGURE 3. An illustration of Σ_i and $S_{\lambda_i}(\lambda_i\,\xi_i)$. The cross marks the origin in the asymptotically flat chart. Here, Σ_i encloses B_2.

Lemma 7. *For all sufficiently large i, there are $u_i \in C^\infty(S_{\lambda_i}(\lambda_i\,\xi_i))$ with the following properties.*

- $\Sigma_i = \Sigma_{\xi_i,\lambda_i}(u_i)$
- $u_i(x_i) = 0$
- $(\bar\nabla u_i)(x_i) = 0$
- $\bar\nabla u_i = o(1)$

We abbreviate $S_i = S_{\lambda_i}(\lambda_i\,\xi_i)$ and $\Phi_i = \Phi^{u_i}_{\xi_i,\lambda_i}$.

Remark 8. *It follows from Lemma 7 that $u_i(x) = o(|x|)$ and $\Phi_i(x) = x + o(|x|)$.*

To proceed, we need the following technical lemma.

Lemma 9. *Let $c \geq 1$ and $\beta : [0,1] \to \mathbb{R}$ be a non-negative, measurable function with*

$$\int_0^1 \beta(s)\,\mathrm{d}s \leq \frac{1}{16}\,c^{-2}\,(1+2\,c)^{-1}\,\exp(-2\,c).$$

Suppose that $\alpha : [0,1] \to \mathbb{R}$ be a differentiable function with absolutely continuous derivative such that $\alpha(0) = \alpha'(0) = 0$ and

$$|\alpha''| \leq c^2\,|\alpha| + c^2\,(\alpha')^2 + \beta$$

almost everywhere. Then

$$|\alpha'| \leq 4\,(1+2\,c)\,\exp(2\,c)\int_0^1 \beta(s)\,\mathrm{d}s.$$

Proof. We may and will assume that $\beta > 0$. The general case follows by approximation.

Let $\omega : [0,1] \to \mathbb{R}$ be given by

$$\omega(t) = 4\,t\,\exp(2\,c\,t)\int_0^1 \beta(s)\,\mathrm{d}s.$$

We claim that $|\alpha'(t)| < \omega'(t)$ on $[0,1]$. To see this, note that

$$4\int_0^1 \beta(s)\,\mathrm{d}s < \omega' \leq 4\,(1+2\,c)\,\exp(2\,c)\int_0^1 \beta(s)\,\mathrm{d}s \leq \frac{1}{4}\,c^{-2} \tag{22}$$

and

$$0 \leq 4\,c^2\,\omega < \omega''. \tag{23}$$

Suppose that there is $t_0 \in (0,1]$ with $|\alpha'(t_0)| = \omega'(t_0)$ and such that $|\alpha'(t)| < \omega'(t)$ on $[0,t_0)$. It follows that $|\alpha(t)| < \omega(t)$ on $[0,t_0)$. Consequently,

$$\omega'(t_0) = |\alpha'(t_0)| \leq \int_0^{t_0} |\alpha''(s)|\,\mathrm{d}s \leq c^2 \int_0^{t_0} \omega(s)\,\mathrm{d}s + c^2 \int_0^{t_0} \omega'(s)^2\,\mathrm{d}s + \int_0^1 \beta(s)\,\mathrm{d}s. \tag{24}$$

By (23), $\omega'(t) \leq \omega'(t_0)$ on $[0,t_0)$. Using this, (22), and (23), we have

$$c^2 \int_0^{t_0} \omega(s)\,\mathrm{d}s + c^2 \int_0^{t} \omega'(s)^2\,\mathrm{d}s \leq \frac{1}{4} \int_0^{t_0} \omega''(s)\,\mathrm{d}s + \frac{1}{4} \int_0^{t_0} \omega'(s)\,\mathrm{d}s < \frac{1}{2}\,\omega'(t_0).$$

In conjunction with (24), we conclude that

$$\omega'(t_0) \leq 2 \int_0^1 \beta(s)\,\mathrm{d}s.$$

This is incompatible with (22).

It follows that $|\alpha'(t)| < \omega'(t)$ on $[0,1]$. The assertion now follows from (22). □

Lemma 10. *There holds*

$$|x|^{-1}\,|u_i| + |\bar\nabla u_i| + |x|\,|\bar\nabla^2 u_i| = O((\log(\rho_i^{-1}\,\lambda_i))^{1/2}\,(\lambda_i^{-1/2} + \rho_i^{-1})).$$

Proof. Let $x \in S_i$ and $\gamma : [0,1] \to S_i$ be a minimizing geodesic with $\gamma(0) = x_i$ and $\gamma(1) = x$. Note that $|\dot\gamma| \leq \pi\,\lambda_i$. Given an integer ℓ, let

$$S_{i,\ell} = \{z \in S_i : 2^{\ell-1}\,\rho_i \leq |z| < 2^{\ell}\,\rho_i\}.$$

Note that

$$B_{1/2\,|z|}(z) \subset S_{i,\ell-1} \cup S_{i,\ell} \cup S_{i,\ell+1}$$

for every $z \in S_{i,\ell}$ and that

$$\int_{\gamma \cap S_{i,\ell}} |z|^{-1}\,\mathrm{d}\bar\mu = O(1)$$

uniformly for all ℓ.

Let $z \in S_i$. By Lemma 41 and Proposition 24,

$$(\bar h(\Sigma_i) - \lambda_i^{-1}\,\bar g|_{\Sigma_i})(\Phi_i(z)) = (h(\Sigma_i) - \lambda_i^{-1}\,g|_{\Sigma_i})(\Phi_i(z)) + O(|\Phi_i(z)|^{-2}).$$

Using Proposition 35 and (60), we have

$$\begin{aligned}(h(\Sigma_i) - \lambda_i^{-1}\,g|_{\Sigma_i})(\Phi_i(z)) = {}& O(|\Phi_i(z)|^{-1}) \left(\int_{\Sigma_i \cap B_{1/4\,|\Phi_i(z)|}(\Phi_i(z))} |h - \lambda_i^{-1}\,g|_{\Sigma_i}|^2\,\mathrm{d}\mu \right)^{1/2} \\ & + O(|\Phi_i(z)|^{-2}) + O((\lambda_i^{-1/2} + \rho_i^{-1})\,\lambda_i^{-1}).\end{aligned}$$

Using Remark and Lemma 42, we conclude that

$$\begin{aligned}(\bar\nabla^2 u_i)(z) = {}& O(|z|^{-1}) \left(\int_{S_i \cap B_{1/2\,|z|}(z)} |h(\Sigma_i) - \lambda_i^{-1}\,g|_{\Sigma_i}|^2\,\mathrm{d}\mu \right)^{1/2} \\ & + O(|z|^{-2}) + O((\lambda_i^{-1/2} + \rho_i^{-1})\,\lambda_i^{-1}) + O(\lambda_i^{-2}\,|u_i(z)|) + O(\lambda_i^{-1}\,|\bar\nabla u_i(z)|^2).\end{aligned} \tag{25}$$

We have

$$\int_\gamma |z|^{-2} + \int_\gamma (\lambda_i^{-1/2} + \rho_i^{-1})\, \lambda_i^{-1} = O(\rho_i^{-1}) + O(\lambda_i^{-1/2} + \rho_i^{-1}) = O(\lambda_i^{-1/2} + \rho_i^{-1}). \tag{26}$$

Let $k = \lceil (\log 2)^{-1} \log(\rho_i^{-1} |x|) \rceil$ and note that $k = O(\log(\rho_i^{-1} \lambda_i))$. We have

$$\begin{aligned}
&\int_\gamma |z|^{-1} \left(\int_{S_i \cap B_{1/2\,|z|}(z)} |h(\Sigma_i) - \lambda_i^{-1} g|_{\Sigma_i}|^2 \,\mathrm{d}\mu \right)^{1/2} \\
&= O(1) \sum_{\ell=1}^{k} \int_{\gamma \cap S_{i,\ell}} |z|^{-1} \,\mathrm{d}\bar\mu(z) \left(\int_{S_{i,\ell-1} \cup S_{i,\ell} \cup S_{i,\ell+1}} |h(\Sigma_i) - \lambda_i^{-1} g|_{\Sigma_i}|^2 \,\mathrm{d}\mu \right)^{1/2} \\
&= O(1) \sqrt{k} \left(\sum_{\ell=1}^{k} \int_{S_{i,\ell-1} \cup S_{i,\ell} \cup S_{i,\ell+1}} |h(\Sigma_i) - \lambda_i^{-1} g|_{\Sigma_i}|^2 \,\mathrm{d}\mu \right)^{1/2} \qquad (27)\\
&= O(1) \,(\log(\rho_i^{-1} \lambda_i))^{1/2} \left(\int_{\Sigma_i} |h(\Sigma_i) - \lambda_i^{-1} g|_{\Sigma_i}|^2 \,\mathrm{d}\mu \right)^{1/2} \\
&= O(1) \,(\log(\rho_i^{-1} \lambda_i))^{1/2} \,(\lambda_i^{-1/2} + \rho_i^{-1}).
\end{aligned}$$

We have used Lemma 23 in the last equation.

Let

$$\alpha : [0,1] \to \mathbb{R} \qquad \text{be given by} \qquad \alpha(s) = \int_0^s |(\bar\nabla u_i)(\gamma(t))| \,\mathrm{d}t.$$

By Lemma 7, $\alpha'(0) = 0$ and $|u_i(\gamma(s))| \leq |\dot\gamma(s)|\, \alpha(s)$ for all $s \in [0,1]$. Moreover, there holds $|\alpha''(s)| \leq |\dot\gamma(s)|\, |(\bar\nabla^2 u_i)(\gamma(s))|$ whenever $\alpha''(s)$ exists. Using Lemma 9 and (25-27), we obtain

$$\bar\nabla u_i = O((\log(\rho_i^{-1} \lambda_i))^{1/2} \,(\lambda_i^{-1/2} + \rho_i^{-1})).$$

Integrating and using Lemma 7, we have

$$|x|^{-1}\, u_i = O((\log(\rho_i^{-1} \lambda_i))^{1/2} \,(\lambda_i^{-1/2} + \rho_i^{-1})).$$

Returning to (25) and using Lemma 23, we have

$$|x|\, \bar\nabla^2 u_i = O((\log(\rho_i^{-1} \lambda_i))^{1/2} \,(\lambda_i^{-1/2} + \rho_i^{-1})).$$

The assertion follows. □

3. Asymptotic analysis of the mean curvature

As in Section 2, we assume that g is a Riemannian metric on $\mathbb{R}^3$ such that, as $x \to \infty$,

$$g = \left(1 + |x|^{-1}\right)^4 \bar g + \sigma \qquad \text{where} \qquad \partial_J \sigma = O(|x|^{-2-|J|}) \tag{28}$$

for every multi-index J with $|J| \leq 4$. Let $\{\Sigma_i\}_{i=1}^\infty$ be a sequence of area-constrained Willmore spheres $\Sigma_i \subset \mathbb{R}^3$ with

$$m_H(\Sigma_i) \geq 0, \qquad \lim_{i \to \infty} \rho(\Sigma_i) = \infty, \qquad \rho(\Sigma_i) = o(\lambda(\Sigma_i)) \tag{29}$$

and assume that, as $i \to \infty$,

$$\log \lambda(\Sigma_i) = o(\rho(\Sigma_i)). \tag{30}$$

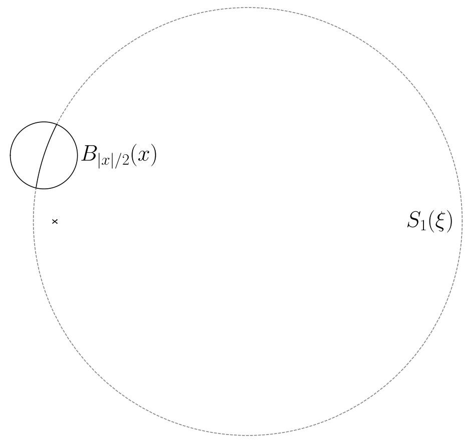

FIGURE 4. An illustration of the partition (32) for $|\xi| \approx 1$ and $|x| \approx |1-\xi|$. The cross marks the origin of $\mathbb{R}^3$. The gradient of the Green's function G is large within the part of $S_1(\xi)$ illustrated by the black line while f may be large within the part of $S_1(\xi)$ illustrated by the dashed, gray line.

As before, we abbreviate $\lambda_i = \lambda(\Sigma_i)$ and $\rho_i = \rho(\Sigma_i)$. Recall from Lemma 7 that, for all i large, $\Sigma_i = \Sigma_{\xi_i,\lambda_i}(u_i)$ is the Euclidean graph of a function u_i over the sphere $S_i = S_{\lambda_i}(\lambda_i\,\xi_i)$. Moreover, recall that $\Phi_i = \Phi^{u_i}_{\xi_i,\lambda_i}$.

The goal of this section is to obtain an improved estimate for the mean curvature $H(\Sigma_i)$. To this end, we investigate the area-constrained Willmore equation (2).

Given $\xi \in \mathbb{R}^3$ and $\lambda > 0$, let $\Lambda_0(S_\lambda(\lambda\,\xi))$ be the space of constant functions on $S_\lambda(\lambda\,\xi)$ and $\Lambda_0(S_\lambda(\lambda\,\xi))^\perp$ be its orthogonal complement in $C^\infty(S_\lambda(\lambda\,\xi))$ with respect to the Euclidean L^2-inner product. We use $\operatorname{proj}_{\Lambda_0(S_\lambda(\lambda\,\xi))}$ and $\operatorname{proj}_{\Lambda_0(S_\lambda(\lambda\,\xi))^\perp}$ to denote the L^2-projection onto these spaces.

We need the following gradient estimate for the Laplace operator.

Lemma 11. *There is a constant $c > 0$ with the following property. Let $\xi \in \mathbb{R}^3$ and $\lambda > 0$. Suppose that $u,\, f \in \Lambda_0(S_\lambda(\lambda\,\xi))^\perp$ are such that $\bar\Delta u = f$. Then*

$$\sup_{x\in S_\lambda(\lambda\,\xi)} |x|\,|\bar\nabla u(x)| \leq c\left(\int_{S_\lambda(\lambda\,\xi)} |f|\,\mathrm{d}\bar\mu + \sup_{x\in S_\lambda(\lambda\,\xi)} |x|^2\,|f|\right).$$

Proof. By scaling, we may assume that $\lambda = 1$ and

$$\int_{S_1(\xi)} |f|\,\mathrm{d}\bar\mu + \sup_{x\in S_1(\xi)} |x|^2\,|f| = 1.$$

Recall from [1, §3.3] that the Green's function of $\bar\Delta : \Lambda_0(S_1(0))^\perp \to \Lambda_0(S_1(0))^\perp$ is given by

$$G(x,y) = \frac{1}{2\,\pi}\,\log|x-y|.$$

It follows that

$$(\bar\nabla u)(x) = \int_{S_1(\xi)} (\bar\nabla G)(x,y)\,f(y)\,\mathrm{d}\bar\mu(y) \tag{31}$$

where differentiation is with respect to x. Note that

$$(\bar{D}G)(x,y) = O(1)\,|x-y|^{-1}$$

for all $x, y \in \mathbb{R}^3$ with $x \neq y$.

Let $x \in S_1(\xi)$. We may assume that $x \neq 0$. We estimate the integral (31) over the regions

$$\{y \in S_1(\xi)\,:\,2\,|y-x| \geq |x|\} \qquad \text{and} \qquad \{y \in S_1(\xi)\,:\,2\,|y-x| \leq |x|\} \tag{32}$$

separately; see Figure 4. We have

$$\begin{aligned}\int_{\{y\in S_1(\xi)\,:\,2\,|y-x|\geq|x|\}} & |x-y|^{-1}\,|f|\,\mathrm{d}\bar{\mu}(y)\\ &\leq O(1)\,|x|^{-1}\int_{S_1(\xi)}|f|\,\mathrm{d}\bar{\mu}(y)\\ &\leq O(1)\,|x|^{-1}.\end{aligned}$$

Likewise,

$$\begin{aligned}\int_{\{y\in S_1(\xi)\,:\,2\,|y-x|\leq|x|\}} & |x-y|^{-1}\,|f|\,\mathrm{d}\bar{\mu}(y)\\ &\leq \int_{\{y\in S_1(\xi)\,:\,2\,|y-x|\leq|x|\}}|x-y|^{-1}\,|y|^{-2}\,\mathrm{d}\bar{\mu}(y)\\ &\leq O(1)\,|x|^{-2}\int_{\{y\in S_1(\xi)\,:\,2\,|y-x|\leq|x|\}}|x-y|^{-1}\,\mathrm{d}\bar{\mu}(y)\\ &\leq O(1)\,|x|^{-1}.\end{aligned}$$

The assertion follows from these estimates. □

Remark 12. *Let $\xi \in \mathbb{R}^3$ with $1/2 < |\xi| < 1$ or $1 < |\xi| < 3/2$ and $\lambda > 0$. By Lemma 36, there is a constant $c > 0$ with the following properties:*

- $\displaystyle\int_{S_\lambda(\lambda\,\xi)}|f|\,\mathrm{d}\bar{\mu} \leq c\,\lambda^2 \sup_{x\in S_\lambda(\lambda\,\xi)}|f|$
- $\displaystyle\int_{S_\lambda(\lambda\,\xi)}|f|\,\mathrm{d}\bar{\mu} \leq c\,|\log|1-|\xi|||\sup_{x\in S_\lambda(\lambda\,\xi)}|x|^2\,|f|$
- $\displaystyle\int_{S_\lambda(\lambda\,\xi)}|f|\,\mathrm{d}\bar{\mu} \leq c\,|1-|\xi||^{-1}\,\lambda^{-1}\sup_{x\in S_\lambda(\lambda\,\xi)}|x|^3\,|f|$

Lemma 13. *There holds, as $i \to \infty$,*

$$\mathrm{proj}_{\Lambda_0} H(\Sigma_i) = \frac{1}{4\,\pi}\,\lambda_i^{-2}\int_{S_i}H(\Sigma_i)\,\mathrm{d}\bar{\mu} = 2\,\lambda_i^{-1} + o(\lambda_i^{-1})$$

and

$$\kappa(\Sigma_i) = o(\lambda_i^{-2}\,\rho_i^{-1}).$$

Moreover,

$$\mathrm{proj}_{\Lambda_0(S_i)^\perp} H(\Sigma_i) = -4\,\lambda_i^{-1}\,|x|^{-1} + o(\lambda_i^{-1}\,\rho_i^{-1}).$$

Proof. By Proposition 24 and Remark 8,

$$H(\Sigma_i) = 2\,\lambda_i^{-1} + O((\lambda_i^{-1/2} + \rho_i^{-1})\,|x|^{-1}).$$

Using Lemma 36, we have

$$\int_{S_i} H(\Sigma_i)\,\mathrm{d}\bar\mu = 8\,\pi\,\lambda_i + O(1)\,(\lambda_i^{-1/2}+\rho_i^{-1})\int_{S_i}|x|^{-1}\,\mathrm{d}\bar\mu = 8\,\pi\,\lambda_i + O((\lambda_i^{-1/2}+\rho_i^{-1})\,\lambda_i).$$

Consequently,

$$\operatorname{proj}_{\Lambda_0(S_i)} H(\Sigma_i) = 2\,\lambda_i^{-1} + O((\lambda_i^{-1/2}+\rho_i^{-1})\,\lambda_i^{-1}). \tag{33}$$

We define the function $F_i:\Sigma_i\to\mathbb{R}$ by $F_i = N^{-1}\,H(\Sigma_i)$ where

$$N:\mathbb{R}^3\setminus\{0\}\to\mathbb{R}\qquad \text{is given by}\qquad N(x) = (1+|x|^{-1})^{-1}\,(1-|x|^{-1})$$

is the potential function of Schwarzschild; see (73). By Remark 8,

$$F_i = (1+2\,|x|^{-1}+o(|x|^{-1}))\,H(\Sigma_i). \tag{34}$$

Repeating the argument that led to (33), we obtain

$$\operatorname{proj}_{\Lambda_0(S_i)} F_i = 2\,\lambda_i^{-1} + o(\lambda_i^{-1}). \tag{35}$$

By Lemma 44, Lemma 39, Lemma 41, and Proposition 24, we have

$$\begin{aligned}\Delta_{\Sigma_i}F_i = &-(|\mathring{h}(\Sigma_i)|^2+\kappa(\Sigma_i))F_i + O(1)\,(|x|^{-4}+\lambda_i^{-1}\,|x|^{-2}+(\lambda_i^{-1/2}+\rho_i^{-1})\,|x|^{-3})\,|F_i|\\ &+O(|x|^{-3})\,|x|\,|\nabla F_i|.\end{aligned} \tag{36}$$

Using Lemma 41, Proposition 24, and Remark 8, we have

$$\bar\Delta_{\Sigma_i}F_i = (1+O(|x|^{-1}))\,\Delta_{\Sigma_i}F_i + O(|x|^{-3})\,(|x|\,|\bar\nabla F_i| + |x|^2\,|\bar\nabla^2 F_i|).$$

In conjunction with (36) and Proposition 24, we conclude that

$$\begin{aligned}\bar\Delta_{\Sigma_i}F_i = &-(|\mathring{h}(\Sigma_i)|^2+\kappa(\Sigma_i))\,F_i + O(1)\,(\lambda_i^{-1}\,|x|^{-2}+(\lambda_i^{-1/2}+\rho_i^{-1})\,|x|^{-3})\,|F_i|\\ &+O(|x|^{-3})\,(|x|\,|\bar\nabla F_i|+|x|^2\,|\bar\nabla^2F_i|).\end{aligned} \tag{37}$$

Using Lemma 43, Lemma 10, and (17), we have

$$\begin{aligned}\bar\Delta_{\Sigma_i}F_i = &\,(1+o(1))\,\bar\Delta_{S_i}F_i\\ &+O(1)\,\log(\rho_i^{-1}\,\lambda_i)\,(\lambda_i^{-1/2}+\rho_i^{-1})^2\,|x|^{-2}\,(|x|\,|\bar\nabla F_i|+|x|^2\,|\bar\nabla^2F_i|).\end{aligned}$$

In conjunction with (37), (29), and (30), we obtain

$$\begin{aligned}\bar\Delta_{S_i}F_i = &\,O(1)\,(|\mathring{h}(\Sigma_i)|^2+|\kappa(\Sigma_i)|)\,|F_i|\\ &+O(1)\,(\lambda_i^{-1}\,|x|^{-2}+(\lambda_i^{-1/2}+\rho_i^{-1})\,|x|^{-3})\,|F_i|\\ &+O(1)\,(\lambda_i^{-1/2}+\rho_i^{-1})\,|x|^{-2}\,(|x|\,|\bar\nabla F_i|+|x|^2\,|\bar\nabla^2F_i|).\end{aligned} \tag{38}$$

According to Proposition 24 and Remark 8,

$$\begin{aligned}&\sup_{x\in S_i}|x|^2\,(|\mathring{h}(\Sigma_i)|^2+|\kappa(\Sigma_i)|+\lambda_i^{-1}\,|x|^{-2}+(\lambda_i^{-1/2}+\rho_i^{-1})\,|x|^{-3})\\ &\qquad= O(1)\,((\lambda_i^{-1/2}+\rho_i^{-1})^2+\lambda_i^2\,|\kappa(\Sigma_i)|).\end{aligned}$$

Using Lemma 41, Remark 8, and Lemma 7, we have

$$\mathrm{d}\mu(\Sigma_i) = (1+o(1))\,\mathrm{d}\bar\mu(\Sigma_i) = (1+o(1))\,\mathrm{d}\bar\mu(S_i). \tag{39}$$

Using this, Lemma 23, and Lemma 36, we see that

$$\begin{aligned}&\int_{S_i} (|\mathring{h}(\Sigma_i)|^2 + |\kappa(\Sigma_i)| + \lambda_i^{-1}\,|x|^{-2} + (\lambda_i^{-1/2}+\rho_i^{-1})\,|x|^{-3})\,\mathrm{d}\bar\mu \\ &\qquad = O(1)\,((\lambda_i^{-1/2}+\rho_i^{-1})^2 + \lambda_i^2\,|\kappa(\Sigma_i)| + \log(\rho_i^{-1}\,\lambda_i)\,\lambda_i^{-1}).\end{aligned}$$

Likewise, we have

$$\sup_{x\in S_i} |x|^2\,(\lambda_i^{-1/2}+\rho_i^{-1})\,|x|^{-2} = \lambda_i^{-1/2}+\rho_i^{-1}$$

and, using Lemma 36,

$$\int_{S_i} (\lambda_i^{-1/2}+\rho_i^{-1})\,|x|^{-2}\,\mathrm{d}\bar\mu = O(1)\log(\rho_i^{-1}\,\lambda_i)\,(\lambda_i^{-1/2}+\rho_i^{-1}).$$

Using (38) and Lemma 11, we conclude that

$$\begin{aligned}&\sup_{x\in S_i} |x|\,|\bar\nabla F_i| \\ &\quad = O(1)\,((\lambda_i^{-1/2}+\rho_i^{-1})^2 + \lambda_i^2\,|\kappa(\Sigma_i)| + \log(\rho_i^{-1}\,\lambda_i)\,\lambda_i^{-1})\,\sup_{x\in S_i}|F_i| \\ &\qquad + O(1)\,\log(\rho_i^{-1}\,\lambda_i)\,(\lambda_i^{-1/2}+\rho_i^{-1})\,(\sup\nolimits_{x\in S_i}|x|\,|\bar\nabla F_i| + \sup\nolimits_{x\in S_i}|x|^2\,|\bar\nabla^2 F_i|).\end{aligned} \tag{40}$$

By standard elliptic theory,

$$\sup_{x\in S_i}|x|^2\,|\bar\nabla^2 F_i| = O(1)\,\sup_{x\in S_i}|x|\,|\bar\nabla F_i| + O((\lambda_i^{-1/2}+\rho_i^{-1})^2 + \lambda_i^2\,|\kappa(\Sigma_i)|)\,\sup_{x\in S_i}|F_i|; \tag{41}$$

see Remark 14. Using (29) and (30) and absorbing, we conclude that

$$\begin{aligned}&\sup_{x\in S_i}|x|\,|\bar\nabla F_i| + \sup_{x\in S_i}|x|^2\,|\bar\nabla F_i|^2 \\ &\qquad = O(1)\,((\lambda_i^{-1/2}+\rho_i^{-1})^2 + \lambda_i^2\,|\kappa(\Sigma_i)| + \log(\rho_i^{-1}\,\lambda_i)\,\lambda_i^{-1})\,\sup_{x\in S_i}|F_i|.\end{aligned} \tag{42}$$

Note that there is $z\in S_i$ with $(\operatorname{proj}_{\Lambda_0(S_i)^\perp} F_i)(z) = 0$. Integrating, we find

$$\sup_{x\in S_i}|\operatorname{proj}_{\Lambda_0(S_i)^\perp} F_i| = O(\log(\rho_i^{-1}\,\lambda_i))\,\sup_{x\in S_i}|x|\,|\bar\nabla F_i|.$$

Using (60), (29), and (30), we have

$$(\lambda_i^{-1/2}+\rho_i^{-1})^2 + \lambda_i^2\,|\kappa(\Sigma_i)| + \log(\rho_i^{-1}\,\lambda_i)\,\lambda_i^{-1} = o((\log(\rho_i^{-1}\,\lambda_i))^{-1}).$$

Returning to (42) and absorbing, we obtain

$$\begin{aligned}&(\log(\rho_i^{-1}\,\lambda_i))^{-1}\,\sup_{x\in S_i}|\operatorname{proj}_{\Lambda_0(S_i)^\perp} F_i| + \sup_{x\in S_i}|x|\,|\bar\nabla F_i| + \sup_{x\in S_i}|x|^2\,|\bar\nabla F_i|^2 \\ &\qquad = O(1)\,((\lambda_i^{-1/2}+\rho_i^{-1})^2 + \lambda_i^2\,|\kappa(\Sigma_i)| + \log(\rho_i^{-1}\,\lambda_i)\,\lambda_i^{-1})\,|\operatorname{proj}_{\Lambda_0(S_i)} F_i|. \\ &\qquad = O(1)\,((\lambda_i^{-1/2}+\rho_i^{-1})^2 + \lambda_i^2\,|\kappa(\Sigma_i)| + \log(\rho_i^{-1}\,\lambda_i)\,\lambda_i^{-1})\,\lambda_i^{-1}.\end{aligned} \tag{43}$$

We have used (35) in the last equation. In particular, using (60),

$$F_i = \operatorname{proj}_{\Lambda_0(S_i)} F_i + \operatorname{proj}_{\Lambda_0(S_i)^\perp} F_i = O(\lambda_i^{-1}). \tag{44}$$

Using (37), (43), (44), (29), and (30), we have

$$\begin{aligned}\bar\Delta_{\Sigma_i} F_i = & -\kappa(\Sigma_i)\, F_i + O(\lambda_i\,|x|^{-3}\,|\kappa(\Sigma_i)|) + O(\lambda_i^{-1}\,|\mathring{h}(\Sigma_i)|^2) \\ & + O(\lambda_i^{-2}\,|x|^{-2}) + O((\lambda_i^{-1/2} + \rho_i^{-1})\,\lambda_i^{-1}\,|x|^{-3}).\end{aligned}$$

Integrating and using (35), (39), Lemma 36, and Lemma 23, we obtain

$$\kappa(\Sigma_i) = O(\lambda_i^{-2}\,\rho_i^{-2}) + O(\log(\rho_i^{-1}\,\lambda_i)\,\lambda_i^{-3}). \tag{45}$$

Returning to (43), we conclude that

$$\operatorname{proj}_{\Lambda_0(S_i)^\perp} F_i = o(\lambda_i^{-1}\,\rho_i^{-1}). \tag{46}$$

Moreover, by (45), $\kappa(\Sigma_i) = o(\lambda_i^{-2}\,\rho_i^{-1})$. We have used (29) and (30) in both of these estimates.

By (34), (44), and (46), we have

$$\begin{aligned}\operatorname{proj}_{\Lambda_0(S_i)^\perp} H(\Sigma_i) &= \operatorname{proj}_{\Lambda_0(S_i)^\perp} F_i + \operatorname{proj}_{\Lambda_0(S_i)^\perp} O(\rho_i^{-1})\, H(\Sigma_i) \\ &= o(\lambda_i^{-1}\,\rho_i^{-1}) + O(\rho_i^{-1}) \sup_{x\in S_i} |H(\Sigma_i)| \\ &= o(\lambda_i^{-1}\,\rho_i^{-1}) + O(\rho_i^{-1}) \sup_{x\in S_i} |F_i| \\ &= o(\lambda_i^{-1}).\end{aligned}$$

In conjunction with (33) and (34), we obtain

$$F_i = H(\Sigma_i) + 4\,|x|^{-1}\,\lambda_i^{-1} + o(\rho_i^{-1}\,\lambda_i^{-1}).$$

By Lemma 36,

$$\operatorname{proj}_{\Lambda_0(S_i)} |x|^{-1} = O(\lambda_i^{-1}) = o(\rho_i^{-1}).$$

Using (46), we conclude that

$$\operatorname{proj}_{\Lambda_0(S_i)^\perp} H(\Sigma_i) = -4\,|x|^{-1}\,\lambda_i^{-1} + o(\lambda_i^{-1}\,\rho_i^{-1}).$$

The assertion follows. □

Remark 14. *We provide additional details on how to obtain* (41).

Let $z_i \in S_i$ and $a_i \in T_{z_i} S_i$ with $|a_i| = 1$. The estimates below are independent of these choices.

Let $X_i = a_i^\top$. By interior L^4-estimates as in [11, Theorem 9.11] *and the Sobolev embedding theorem, using also Lemma 41, Lemma 43, Proposition 24, and Lemma 10, we have*

$$\begin{aligned}&|z_i|^2\,|a_i \lrcorner (\bar\nabla^2 F_i)(z_i)| \\ &\quad = O(1) \sup_{x\in S_i} |x|\,|\bar\nabla F_i| + O(1)\,|z_i|^{5/2} \left(\int_{S_i \cap B_{1/2\,|z_i|}(z_i)} (\Delta \nabla_{X_i} F_i)^4 \,\mathrm{d}\bar\mu \right)^{1/4}.\end{aligned}$$

Note that

$$\Delta \nabla_{X_i} F_i = \nabla_{X_i} \Delta F_i + K^\Sigma\, g(X_i, \nabla F_i) + 2\, g(\nabla X_i, \nabla(\nabla F_i)) + g(\operatorname{tr} \nabla^2 X_i, \nabla F_i)$$

where K^Σ is the Gauss curvature of Σ. Using this, Lemma 44, (28), (60)*, and Corollary 28, we obtain*

$$\Delta \nabla_{X_i} F = O(|x|^{-3})\,(\lambda_i^{-1/2} + \rho_i^{-1})^2\,|F_i| + O(|x|^{-3})\,(|x|\,|\bar\nabla F_i| + |x|^2\,|\bar\nabla^2 F_i|).$$

Consequently,

$$\begin{aligned}|z_i|^{5/2}&\left(\int_{S_i\cap B_{1/2\,|z_i|}(z_i)} (\Delta\nabla_{X_i}F_i)^4\,\mathrm{d}\bar\mu\right)^{1/4}\\ &= O((\lambda_i^{-1/2}+\rho_i^{-1})^2)\,\sup_{x\in S_i}|F_i| + O(1)\,\sup_{x\in S_i}|x|\,|\bar\nabla F_i|\\ &\qquad + O(1)\,|z_i|^{3/2}\left(\int_{S_i\cap B_{1/2\,|z_i|}(z_i)} |\bar\nabla^2 F_i|^4\,\mathrm{d}\bar\mu\right)^{1/4}.\end{aligned}$$

Finally, by (36) *and Proposition 24,*

$$\Delta F_i = O(|x|^{-2})\,((\lambda_i^{-1/2}+\rho_i^{-1})^2 + \lambda_i^2\,|\kappa(\Sigma_i)|)\,|F_i| + O(|x|^{-3})\,|x|\,|\bar\nabla F_i|.$$

By interior L^4-estimates, we conclude that

$$\begin{aligned}|z_i|^{3/2}&\left(\int_{S_i\cap B_{1/2\,|z_i|}(z_i)} |\bar\nabla^2 F_i|^4\,\mathrm{d}\bar\mu\right)^{1/4}\\ &= O(1)\,\sup_{x\in S_i}|x|\,|\bar\nabla F_i| + O(1)((\lambda_i^{-1/2}+\rho_i^{-1})^2 + \lambda_i^2\,|\kappa(\Sigma_i)|)\,\sup_{x\in S_i}|F_i|.\end{aligned}$$

4. Variations of the Willmore energy by translations

As in Section 2, we assume that g is a Riemannian metric on $\mathbb{R}^3$ such that, as $x\to\infty$,

$$g = \left(1+|x|^{-1}\right)^4\,\bar g + \sigma \qquad \text{where} \qquad \partial_J\sigma = O(|x|^{-2-|J|}) \tag{47}$$

for every multi-index J with $|J|\le 4$. Let $\{\Sigma_i\}_{i=1}^\infty$ be a sequence of area-constrained Willmore spheres $\Sigma_i\subset\mathbb{R}^3$ with

$$m_H(\Sigma_i)\ge 0, \qquad \lim_{i\to\infty}\rho(\Sigma_i)=\infty, \qquad \rho(\Sigma_i)=o(\lambda(\Sigma_i)) \tag{48}$$

and assume that, as $i\to\infty$,

$$\log\lambda(\Sigma_i) = o(\rho(\Sigma_i)). \tag{49}$$

As before, we abbreviate $\lambda_i=\lambda(\Sigma_i)$ and $\rho_i=\rho(\Sigma_i)$. Recall from Lemma 7 that, for all i large, $\Sigma_i=\Sigma_{\xi_i,\lambda_i}(u_i)$ is the Euclidean graph of a function u_i over the sphere $S_i=S_{\lambda_i}(\lambda_i\,\xi_i)$ where $\xi_i\in\mathbb{R}^3$ satisfies

$$|1-|\xi_i|| = \lambda_i^{-1}\,\rho_i. \tag{50}$$

In this section, we compute an expansion of the variation of the Willmore energy of Σ_i with respect to a translation in direction ξ_i.

For the statement of the next lemma, we define the form

$$\zeta_i:\Gamma(T\Sigma_i)\times\Gamma(T\Sigma_i)\to C^\infty(\mathbb{R}) \qquad \text{given by} \qquad \zeta_i(X,Y)=g(D_X\xi_i,Y).$$

Lemma 15. *There holds*

$$\begin{aligned}\int_{\Sigma_i} & g(\xi_i,\nu)\,\big[\Delta H+|\mathring{h}|^2\,H+\operatorname{Ric}(\nu,\nu)\,H\big]\,\mathrm{d}\mu\\ &=\int_{\Sigma_i}\big[g(\operatorname{tr}_{\Sigma_i}D^2\xi_i,\nu)+\operatorname{Ric}(\xi_i,\nu)\big]\,H\,\mathrm{d}\mu\\ &\qquad+\frac{1}{2}\int_{\Sigma_i}[\operatorname{div}_{\Sigma_i}\xi_i-2\,g(D_\nu\xi_i,\nu)]\,H^2\,\mathrm{d}\mu+2\int_{\Sigma_i}g(\zeta_i,\mathring{h})\,H\,\mathrm{d}\mu.\end{aligned}$$

Proof. Note that

$$\Delta(g(\xi_i,\nu))=g(\operatorname{tr}_{\Sigma_i}D^2\xi_i,\nu)-g(D_\nu\xi_i,\nu)\,H+2\,g(\zeta_i,h)+(\operatorname{div}_{\Sigma_i}h)(\xi_i^\top)-g(\xi_i,\nu)\,|h|^2.$$

Integrating by parts and using the trace of the Gauss-Codazzi equation,

$$\operatorname{div}_{\Sigma_i}h=\nabla H+\nu\lrcorner\ \operatorname{Rc},$$

we obtain

$$\begin{aligned}\int_{\Sigma_i}g(\xi_i,\nu)\,\Delta H\,\mathrm{d}\mu=\int_{\Sigma_i}&\big[g(\operatorname{tr}_{\Sigma_i}D^2\xi_i,\nu)\,H-g(D_\nu\xi_i,\nu)\,H^2+2\,g(\zeta_i,h)\,H\\ &+H\,g\,(\xi_i,\nabla H)+\operatorname{Ric}(\xi_i^\top,\nu)\,H-g(\xi_i,\nu)\,|h|^2\,H\big]\,\mathrm{d}\mu.\end{aligned}$$

Note that

$$\int_{\Sigma_i}H\,g(\xi_i,\nabla H)\,\mathrm{d}\mu=\frac{1}{2}\int_{\Sigma_i}g(\xi_i,\nabla H^2)\,\mathrm{d}\mu=\frac{1}{2}\int_{\Sigma_i}[g(\xi_i,\nu)\,H^3-(\operatorname{div}_{\Sigma_i}\xi_i)\,H^2]\,\mathrm{d}\mu$$

where we have integrated by parts in the second equality. Using the decomposition

$$h=\frac{1}{2}\,H\,g|_{\Sigma_i}+\mathring{h},$$

we see that

$$2\,g(\zeta_i,h)=(\operatorname{div}_{\Sigma_i}\xi_i)\,H+2\,g(\zeta_i,\mathring{h})$$

and

$$2\,g(\xi_i,\nu)\,|h|^2=g(\xi_i,\nu)\,H^2+2\,g(\xi_i,\nu)\,|\mathring{h}|^2.$$

Using that $\xi_i=\xi_i^\top+\xi_i^\perp$, we obtain

$$\operatorname{Ric}(\xi_i^\top,\nu)=\operatorname{Ric}(\xi_i,\nu)-g(\xi_i,\nu)\ \operatorname{Ric}(\nu,\nu).$$

The assertion follows from these identities.

□

Lemma 16. *There holds, as $i\to\infty$,*

$$\begin{aligned}\int_{\Sigma_i}&\big[g(\operatorname{tr}_{\Sigma_i}D^2\xi_i,\nu)+\operatorname{Ric}(\xi_i,\nu)\big]\,H\,\mathrm{d}\mu\\ &=-8\,\pi\,\lambda_i^{-1}\,\rho_i^{-2}+\lambda_i^{-1}\int_{S_i}\bar{g}(\xi_i,\bar{\nu})\,R\,\mathrm{d}\bar{\mu}+o(\lambda_i^{-1}\,\rho_i^{-2}).\end{aligned}$$

Proof. Using Lemma 39, Lemma 40, and Lemma 41, we have

$$\begin{aligned}
&\big[\tilde g(\tilde{\mathrm{tr}}_{\Sigma_i}\tilde D^2\xi_i,\tilde\nu)+\tilde{\mathrm{Ric}}(\xi_i,\tilde\nu)\big]\,\mathrm{d}\tilde\mu\\
&\qquad=\bigg[4\,|x|^{-3}\,\bar g(\xi_i,\bar\nu)-12\,|x|^{-5}\,\bar g(x,\xi_i)\,\bar g(x,\bar\nu)+4\,|x|^{-4}\,\bar g(\xi_i,\bar\nu)-8\,|x|^{-6}\,\bar g(x,\xi_i)\,\bar g(x,\bar\nu)\bigg]\mathrm{d}\bar\mu.
\end{aligned}$$

Using also Lemma 38, we conclude that

$$\begin{aligned}
&\big[g(\mathrm{tr}_{\Sigma_i} D^2\xi_i,\nu)+\mathrm{Ric}(\xi_i,\nu)\big]\,\mathrm{d}\mu\\
&\quad=\bigg[4\,|x|^{-3}\,\bar g(\xi_i,\bar\nu)-12\,|x|^{-5}\,\bar g(x,\xi_i)\,\bar g(x,\bar\nu)+4\,|x|^{-4}\,\bar g(\xi_i,\bar\nu)-8\,|x|^{-6}\,\bar g(x,\xi_i)\,\bar g(x,\bar\nu)\\
&\qquad+\frac12\sum_{j=1}^3\big[(\bar D^2_{\xi_i,e_j}\sigma)(e_j,\bar\nu)-(\bar D^2_{\xi_i,\bar\nu}\sigma)(e_j,e_j)\big]\\
&\qquad+\frac12\sum_{j=1}^3(\bar D^2_{\xi_i,e_j}\sigma)(e_j,\bar\nu)-(\bar D^2_{\xi_i,\bar\nu}\sigma)(\bar\nu,\bar\nu)\\
&\qquad+O(|x|^{-5})\bigg]\mathrm{d}\bar\mu.
\end{aligned}$$

By the divergence theorem,

$$\int_{\Sigma_i}\big[|x|^{-3}\,\bar g(\xi_i,\bar\nu)-3\,|x|^{-5}\,\bar g(x,\xi_i)\,\bar g(x,\bar\nu)\big]\,\mathrm{d}\bar\mu=0. \tag{51}$$

Note that this holds independently of whether Σ_i encloses the origin or not. In conjunction with Lemma 7, Lemma 13, and Lemma 19, we conclude that

$$\begin{aligned}
&\int_{\Sigma_i}\big[g(\mathrm{tr}_{\Sigma_i} D^2\xi_i,\nu)+\mathrm{Ric}(\xi_i,\nu)\big]\,H\,\mathrm{d}\mu\\
&\quad=-8\,\lambda_i^{-1}\int_{S_i}\big[|x|^{-4}\,\bar g(\xi_i,\bar\nu)-4\,|x|^{-6}\,\bar g(x,\xi_i)\,\bar g(x,\bar\nu)\big]\,\mathrm{d}\bar\mu\\
&\qquad+\lambda_i^{-1}\int_{S_i}\sum_{j=1}^3\big[(\bar D^2_{\xi_i,e_j}\sigma)(e_j,\bar\nu)-(\bar D^2_{\xi_i,\bar\nu}\sigma)(e_j,e_j)\big]\,\mathrm{d}\bar\mu\\
&\qquad+\lambda_i^{-1}\int_{S_i}\bigg[\sum_{j=1}^3(\bar D^2_{\xi_i,e_j}\sigma)(e_j,\bar\nu)-(\bar D^2_{\xi_i,\bar\nu}\sigma)(\bar\nu,\bar\nu)\bigg]\,\mathrm{d}\bar\mu\\
&\qquad+o(\lambda_i^{-1}\,\rho_i^{-2}).
\end{aligned}$$

Using Lemma 37 and Lemma 36, we have

$$\lambda_i^{-1}\int_{S_i}\big[|x|^{-4}\,\bar g(\xi_i,\bar\nu)-4\,|x|^{-6}\,\bar g(x,\xi_i)\,\bar g(x,\bar\nu)\big]\,\mathrm{d}\bar\mu=\pi\,\lambda_i^{-3}\,(1-|\xi_i|)^{-2}+o(\lambda_i^{-3}\,(1-|\xi_i|)^{-2}).$$

Using (50), we conclude that

$$\lambda_i^{-1}\int_{S_i}\big[|x|^{-4}\,\bar g(\xi_i,\bar\nu)-4\,|x|^{-6}\,\bar g(x,\xi_i)\,\bar g(x,\bar\nu)\big]\,\mathrm{d}\bar\mu=\pi\,\lambda_i^{-1}\,\rho_i^{-2}+o(\lambda_i^{-1}\,\rho_i^{-2}).$$

On the one hand, applying the divergence theorem, commuting derivatives, and applying the divergence theorem again, we obtain

$$\int_{S_i}\sum_{j=1}^{3}\left[(\bar D^2_{\xi_i,e_j}\sigma)(e_j,\bar\nu)-(\bar D^2_{\xi_i,\bar\nu}\sigma)(e_j,e_j)\right]\mathrm{d}\bar\mu=\int_{S_i}[\bar{\operatorname{div}}\,\bar{\operatorname{div}}\,\sigma-\bar\Delta\,\bar{\operatorname{tr}}\,\sigma]\,\bar g(\xi_i,\bar\nu)\,\mathrm{d}\bar\mu.$$

On the other hand, note that

$$\bar{\operatorname{div}}_{S_i}(\nu\lrcorner\,\bar D_{\xi_i}\sigma)=\sum_{j=1}^{3}(\bar D^2_{\xi_i,e_j}\sigma)(e_j,\bar\nu)-(\bar D^2_{\xi_i,\bar\nu}\sigma)(\bar\nu,\bar\nu)+\lambda_i^{-1}\,\bar D_{\xi_i}\operatorname{tr}\sigma-3\,\lambda_i^{-1}(\bar D_{\xi_i}\sigma)(\bar\nu,\bar\nu).$$

Consequently,

$$\int_{S_i}\left[\sum_{j=1}^{3}(\bar D^2_{\xi_i,e_j}\sigma)(e_j,\bar\nu)-(\bar D^2_{\xi_i,\bar\nu}\sigma)(\bar\nu,\bar\nu)\right]\mathrm{d}\bar\mu=O(\lambda_i^{-1}\,\rho_i^{-1}).$$

In conjunction with Lemma 39, Lemma 19, and (48), we conclude that

$$\begin{aligned}\lambda_i^{-1}\int_{S_i}&\left[\sum_{j=1}^{3}\left[(\bar D^2_{\xi_i,e_j}\sigma)(e_j,\bar\nu)-(\bar D^2_{\xi_i,\bar\nu}\sigma)(e_j,e_j)\right]+\sum_{j=1}^{3}(\bar D^2_{\xi_i,e_j}\sigma)(e_j,\bar\nu)-(\bar D^2_{\xi_i,\bar\nu}\sigma)(\bar\nu,\bar\nu)\right]\mathrm{d}\bar\mu\\&=\lambda_i^{-1}\int_{S_i}\bar g(\xi_i,\bar\nu)\,R\,\mathrm{d}\bar\mu+o(\lambda_i^{-1}\,\rho_i^{-2}).\end{aligned}$$

The assertion follows from these estimates. ☐

Lemma 17. *There holds, as $i\to\infty$,*

$$\int_{\Sigma_i}g(\zeta_i,\mathring{h})\,H\,\mathrm{d}\mu=o(\lambda_i^{-1}\,\rho_i^{-2})$$

and

$$\int_{\Sigma_i}[\operatorname{div}_{\Sigma_i}\xi_i-2\,g(D_\nu\xi_i,\nu)]\,H^2\,\mathrm{d}\mu=o(\lambda_i^{-1}\,\rho_i^{-2}).$$

Proof. Using (47), Lemma 38, Lemma 41, Proposition 24, Lemma 13, and Lemma 19, we have

$$\int_{\Sigma_i}g(\zeta_i,\mathring{h})\,H\,\mathrm{d}\mu=2\,\lambda_i^{-1}\int_{\Sigma_i}\tilde g(\tilde\zeta_i,\mathring{\tilde h})\,\mathrm{d}\tilde\mu+o(\lambda_i^{-1}\,\rho_i^{-2})$$

and, using also (48),

$$\begin{aligned}\int_{\Sigma_i}&[\operatorname{div}_{\Sigma_i}\xi_i-2\,g(D_\nu\xi_i,\nu)]\,H^2\,\mathrm{d}\mu\\&=4\,\lambda_i^{-2}\int_{\Sigma_i}\left[\tilde{\operatorname{div}}_{\Sigma_i}\xi_i-2\,\tilde g(\tilde D_{\tilde\nu}\xi_i,\tilde\nu)\right]\mathrm{d}\tilde\mu+o(\lambda_i^{-1}\,\rho_i^{-2}).\end{aligned}$$

Here, we recall that a tilde indicates that a geometric quantity is computed with respect to the Schwarzschild background metric with mass 2.

By Lemma 38,

$$\tilde\zeta_i=2\,(1+|x|^{-1})^{-1}\,|x|^{-3}\,\bar g(x,\xi_i)\,\tilde g|_{\Sigma_i}.$$

Consequently, $\tilde g(\tilde\zeta_i,\mathring{\tilde h})=0$. Using Lemma 38 again, we have

$$\tilde{\operatorname{div}}_{\Sigma_i}\xi_i=2\,\tilde g(\tilde D_{\tilde\nu}\xi_i,\tilde\nu)=4\,(1+|x|^{-1})^{-1}\,\bar g(x,\xi_i).$$

The assertion follows. ☐

Lemma 18. *There holds*

$$\kappa(\Sigma_i)\int_{\Sigma_i} g(\xi_i,\nu)\,H\,\mathrm{d}\mu = o(\lambda_i^{-1}\,\rho_i^{-2}).$$

Proof. Using Lemma 13, Lemma 41, and (47), we have

$$\int_{\Sigma_i} g(\xi_i,\nu)\,H\,\mathrm{d}\mu = \operatorname{proj}_{\Lambda_0(S_i)} H(\Sigma_i)\int_{\Sigma_i}\bar g(\xi_i,\bar\nu)\,\mathrm{d}\bar\mu + O(\lambda_i\,\rho_i^{-1}).$$

Note that, by the divergence theorem,

$$\int_{\Sigma_i}\bar g(\xi_i,\bar\nu)\,\mathrm{d}\bar\mu = 0.$$

The assertion follows from this and Lemma 13. □

5. Proof of Theorem 3

Suppose, for a contradiction, that there exists a sequence $\{\Sigma_i\}_{i=1}^\infty$ of area-constrained Willmore spheres $\Sigma_i\subset M$ such that (48) and (49) hold. Assembling Lemma 15, Lemma 16, Lemma 17, and Lemma 18, we have

$$\begin{aligned} 0 &= -\int_{\Sigma_i} g(\xi_i,\nu)\,[\Delta H + (|\mathring{h}|^2 + \operatorname{Ric}(\nu,\nu) + \kappa(\Sigma_i))\,H]\,\mathrm{d}\mu \\ &= 8\,\pi\,\lambda_i^{-1}\,\rho_i^{-2} - \lambda_i^{-1}\int_{S_i}\bar g(\xi_i,\bar\nu)\,R\,\mathrm{d}\bar\mu + o(\lambda_i^{-1}\,\rho_i^{-2}). \end{aligned} \tag{52}$$

Using $R\geq -o(|x|^{-4})$ and Lemma 19, we find that

$$-\int_{S_i}\bar g(\xi_i,\bar\nu)\,R\,\mathrm{d}\bar\mu \geq -\int_{\{x\in S_i\,:\,\bar g(\xi_i,\bar\nu)\geq 0\}}\bar g(\xi_i,\bar\nu)\,R\,\mathrm{d}\bar\mu - o(\rho_i^{-2}).$$

Moreover, using (47) and (48),

$$\int_{\{x\in S_i\,:\,\bar g(\xi_i,\bar\nu)\geq 0\}}\bar g(\xi_i,\bar\nu)\,R\,\mathrm{d}\bar\mu = O(\lambda_i^{-2}) = o(\rho_i^{-2}).$$

These estimates are incompatible with (52).

A. Pointwise curvature estimates

In this section, we prove explicit pointwise estimates for large area-constrained Willmore spheres with non-negative Hawking mass. To this end, we combine the integral curvature estimates from Appendix B with the integral estimate on the second fundamental form proven in Lemma 23.

We assume that g is a Riemannian metric on $\mathbb{R}^3$ such that, as $x\to\infty$,

$$g = \left(1+|x|^{-1}\right)^4\,\bar g + \sigma \qquad \text{where} \qquad \partial_J\sigma + O(|x|^{-2-|J|})$$

for every multi-index J with $|J|\leq 4$.

Let $\{\Sigma_i\}_{i=1}^\infty$ be a sequence of area-constrained Willmore spheres $\Sigma_i\subset\mathbb{R}^3$ with

$$\int_{\Sigma_i} H^2\,\mathrm{d}\mu \leq 16\,\pi, \qquad \lim_{i\to\infty}\rho(\Sigma_i)=\infty, \qquad \rho(\Sigma_i) = O(\lambda(\Sigma_i)). \tag{53}$$

We abbreviate $\rho_i=\rho(\Sigma_i)$ and $\lambda_i=\lambda(\Sigma_i)$.

We recall the following decay estimate.

Lemma 19 ([15, Lemma 5.2])**.** *For every $q > 2$ there is a constant $c(q) > 0$ such that for every closed surface $\Sigma \subset \mathbb{R}^3 \setminus \{0\}$,*

$$\rho(\Sigma)^{q-2} \int_\Sigma |x|^{-q} \, \mathrm{d}\bar\mu \le c(q) \int_\Sigma \bar H^2 \, \mathrm{d}\bar\mu.$$

The estimates in the following lemma are stated in [22] except for the explicit constants. We revisit the proof in [22] and compute explicit constants below.

Lemma 20 ([22, Lemma 1.1 and (1.3)])**.** *Let $\Sigma \subset \mathbb{R}^3$ be a closed surface. Given $x \in \Sigma$ and $r > 0$, we have*

$$r^{-2} \, |\Sigma \cap B_r(x)| \le \frac{5}{12} \int_\Sigma \bar H^2 \, \mathrm{d}\bar\mu. \tag{54}$$

Moreover

$$\sup\{|y-z|^2 : y, z \in \Sigma\} \le \frac{225}{16} \, |\Sigma|_{\bar g} \int_\Sigma \bar H^2 \, \mathrm{d}\bar\mu. \tag{55}$$

Proof. Let $x \in \Sigma$. Recall from [22, (1.1)] that for all $0 < r \le t$,

$$\begin{aligned} & r^{-2} \, |\Sigma \cap B_r(x)|_{\bar g} \\ & \quad \le t^{-2} \, |\Sigma \cap B_t(x)|_{\bar g} + \frac{1}{16} \int_{\Sigma \cap B_t(x)} \bar H^2 \, \mathrm{d}\bar\mu \\ & \qquad + \frac{1}{2} \, t^{-2} \int_{\Sigma \cap B_t(x)} \bar H \, \bar g(z - x, \bar\nu) \, \mathrm{d}\bar\mu(z) - \frac{1}{2} \, r^{-2} \int_{\Sigma \cap B_r(x)} \bar H \, \bar g(z - x, \bar\nu) \, \mathrm{d}\bar\mu(z). \end{aligned} \tag{56}$$

To show (55), we obtain, using the estimates

$$\left| \int_{\Sigma \cap B_r(x)} \bar H \, \bar g(z - x, \bar\nu) \, \mathrm{d}\bar\mu(z) \right| \le \frac{1}{2} \, r^2 \int_{\Sigma \cap B_r(x)} \bar H^2 \, \mathrm{d}\bar\mu + \frac{1}{2} \, |\Sigma \cap B_r(x)|$$

and

$$\left| \int_{\Sigma \cap B_t(x)} \bar H \, \bar g(z - x, \bar\nu) \, \mathrm{d}\bar\mu(z) \right| \le \frac{1}{2} \, t^2 \int_{\Sigma \cap B_t(x)} \bar H^2 \, \mathrm{d}\bar\mu + \frac{1}{2} \, |\Sigma \cap B_t(x)|,$$

that

$$r^{-2} \, |\Sigma \cap B_r(x)|_{\bar g} \le 3 \, t^{-2} \, |\Sigma \cap B_t(x)|_{\bar g} + \frac{3}{4} \int_{\Sigma \cap B_t(x)} \bar H^2 \, \mathrm{d}\bar\mu. \tag{57}$$

Revisiting the proof of [22, Lemma 1.1] and using (57) instead of [22, (1.3)], we obtain (55).

To obtain (54), we let $t \to \infty$ in (56). □

Lemma 21. *There holds, as $i \to \infty$:*

- $\displaystyle\int_{\Sigma_i} \bar H^2 \, \mathrm{d}\bar\mu = 16 \, \pi + O(\rho_i^{-1})$
- $\displaystyle\int_{\Sigma_i} |h|^2 \, \mathrm{d}\mu = O(1)$
- $\displaystyle\int_{\Sigma_i} |\bar h|_{\bar g}^2 \, \mathrm{d}\bar\mu = O(1)$

Proof. Clearly,

$$\int_{\Sigma_i} \bar{H}^2 \, \mathrm{d}\bar{\mu} \geq 16\,\pi.$$

Integrating the Gauss equation and using the Gauss-Bonnet theorem, we have

$$\int_{\Sigma_i} H^2 \, \mathrm{d}\mu = 16\,\pi + 2 \int_{\Sigma_i} |\mathring{h}|^2 \, \mathrm{d}\mu + 4 \int_{\Sigma_i} \left(\mathrm{Rc}(\nu,\nu) - \frac{1}{2} R \right) \mathrm{d}\mu.$$

Using Lemma 19 and (53), we obtain

$$\int_{\Sigma_i} |h|^2 \, \mathrm{d}\mu \leq 8\,\pi + O(\rho_i^{-1}) \int_{\Sigma_i} \bar{H}^2 \, \mathrm{d}\bar{\mu}. \tag{58}$$

By Lemma 41,

$$\bar{H}^2 \, \mathrm{d}\bar{\mu} = H^2 \, \mathrm{d}\mu + O(|x|^{-1} \, |h|^2) \, \mathrm{d}\mu + O(|x|^{-3}) \, \mathrm{d}\bar{\mu}$$

Using Lemma 19 and (58), we obtain

$$\int_{\Sigma_i} \bar{H}^2 \, \mathrm{d}\bar{\mu} \leq 16\,\pi + O(\rho_i^{-1}) + O(\rho_i^{-1}) \int_{\Sigma_i} \bar{H}^2 \, \mathrm{d}\bar{\mu}.$$

The assertion follows from these estimates. □

Lemma 22. *There holds*

$$\int_{\Sigma_i} (\bar{H} - 2\,\lambda_i^{-1})^2 \, \mathrm{d}\bar{\mu} = O(1) \int_{\Sigma_i} |\mathring{\bar{h}}|_{\bar{g}}^2 \, \mathrm{d}\bar{\mu} + O(\rho_i^{-2}).$$

Proof. By [7, (38)],

$$\int_{\Sigma_i} (\bar{H} - 2\,\bar{\lambda}_i^{-1})^2 \, \mathrm{d}\bar{\mu} = O(1) \int_{\Sigma_i} |\mathring{\bar{h}}|_{\bar{g}}^2 \, \mathrm{d}\bar{\mu}.$$

From Lemma 41 we obtain $\lambda_i = (1 + O(\rho_i^{-1}))\, \bar{\lambda}(\Sigma_i)$ from which the assertion follows. □

For the next lemma, note that

$$\int_{\Sigma_i} |h - \lambda_i^{-1} \, g|_{\Sigma_i}|^2 \, \mathrm{d}\mu = \int_{\Sigma_i} (H - 2\,\lambda_i^{-1})^2 \, \mathrm{d}\mu + \int_{\Sigma_i} |\mathring{h}|^2 \, \mathrm{d}\mu.$$

Lemma 23. *There holds*

$$\int_{\Sigma_i} |h - \lambda_i^{-1} \, g|_{\Sigma_i}|^2 \, \mathrm{d}\mu = O((\lambda_i^{-1/2} + \rho_i^{-1})^2).$$

Proof. Using Lemma 41, Lemma 19, and Lemma 21, we have

$$\int_{\Sigma_i} H^2 \, \mathrm{d}\mu = \int_{\Sigma_i} \tilde{H}^2 \, \mathrm{d}\tilde{\mu} + O(\rho_i^{-2})$$

and

$$\int_{\Sigma_i} \tilde{H}^2 \, \mathrm{d}\tilde{\mu} = \int_{\Sigma_i} [\bar{H}^2 - 8\,(1 + |x|^{-1})^{-1} \, |x|^{-3} \, \bar{g}(x, \bar{\nu}) \, \bar{H}] \, \mathrm{d}\bar{\mu} + O(\rho_i^{-2}).$$

By the Gauss-Bonnet theorem,

$$\int_{\Sigma_i} \bar{H}^2 \, \mathrm{d}\bar{\mu} = 16\,\pi + 2 \int_{\Sigma} |\mathring{\bar{h}}|_{\bar{g}}^2 \, \mathrm{d}\bar{\mu}.$$

Moreover, by Lemma 19 and Lemma 22, we have

$$\left| \int_{\Sigma_i} (1 + |x|^{-1})^{-1} \, |x|^{-3} \, \bar{g}(x, \bar{\nu}) \, \bar{H} \, \mathrm{d}\bar{\mu} \right|$$

$$
\begin{aligned}
&= \left| \int_{\Sigma_i} |x|^{-3}\, \bar g(x,\bar\nu)\, \bar H \,\mathrm{d}\bar\mu \right| + O(\rho_i^{-2}) \\
&\leq 2\,\lambda_i^{-1} \left| \int_{\Sigma_i} |x|^{-3}\, \bar g(x,\bar\nu) \,\mathrm{d}\bar\mu \right| + \frac{1}{8} \int_{\Sigma_i} |\mathring{\bar h}|_{\bar g}^2 \,\mathrm{d}\bar\mu + O(\rho_i^{-2}).
\end{aligned}
$$

Note that $\bar{\operatorname{div}}(|x|^{-3}\, x) = 0$. Using the divergence theorem, we find that

$$
\int_{\Sigma_i} |x|^{-3}\, \bar g(x,\bar\nu) \,\mathrm{d}\bar\mu = 4\,\pi
$$

if Σ_i encloses B_2 and

$$
\int_{\Sigma_i} |x|^{-3}\, \bar g(x,\bar\nu) \,\mathrm{d}\bar\mu = 0
$$

otherwise. In conjunction with (53), these estimates imply that

$$
\int_{\Sigma_i} |\mathring{\bar h}|_{\bar g}^2 \,\mathrm{d}\bar\mu = O(\lambda_i^{-1}) + O(\rho_i^{-2}).
$$

Using this, Lemma 41, Lemma 19, and Lemma 21, we conclude that

$$
\begin{aligned}
\int_{\Sigma_i} |\mathring{h}|^2 \,\mathrm{d}\mu &= \int_{\Sigma_i} |\mathring{\bar h}|_{\bar g}^2 \,\mathrm{d}\bar\mu + O(\rho_i^{-1}) \int_{\Sigma_i} |\mathring{\bar h}|_{\bar g}^2 \,\mathrm{d}\bar\mu + O(\rho_i^{-2}) \\
&= O(\lambda_i^{-1}) + O(\rho_i^{-2}).
\end{aligned}
$$

Likewise, by Lemma 41, Lemma 22, Lemma 21, and Lemma 19, we have

$$
\begin{aligned}
\int_{\Sigma_i} (H - 2\,\lambda_i^{-1})^2 \,\mathrm{d}\mu &\leq \int_{\Sigma_i} (\bar H - 2\,\lambda_i^{-1})^2 \,\mathrm{d}\bar\mu + O(\rho_i^{-2}) \\
&\leq O(1) \int_{\Sigma_i} |\mathring{\bar h}|^2 \,\mathrm{d}\bar\mu + O(\rho_i^{-2}) \\
&\leq O(\lambda_i^{-1}) + O(\rho_i^{-2}).
\end{aligned}
$$

The assertion follows. □

Proposition 24. *There holds, as $i \to \infty$,*

(59) $$h - \lambda_i^{-1}\, g|_{\Sigma_i} = O((\lambda_i^{-1/2} + \rho_i^{-1})\, |x|^{-1})$$

and

(60) $$\kappa(\Sigma_i) = O((\lambda_i^{-1/2} + \rho_i^{-1})\, \lambda_i^{-2}).$$

Proof. We choose $\psi \in C^\infty(\mathbb{R})$ with

- $0 \leq \psi \leq 1$,
- $\psi(s) = 1$ if $s \geq 1/2$,
- $\psi(s) = 0$ if $s \leq 1/4$.

We define $\gamma_i \in C^\infty(\mathbb{R}^3)$ by

$$
\gamma_i(x) = \psi(\lambda_i^{-1}\, |x|).
$$

Using Lemma 41 and Lemma 23, we have

$$
\int_{\Sigma_i} \gamma_i\, H \,\mathrm{d}\mu = 2\,\lambda_i^{-1} \int_{\Sigma_i} \gamma_i \,\mathrm{d}\bar\mu + o(\lambda_i).
$$

Choosing $x_i \in \Sigma_i$ with $|x_i| = \rho_i$, applying (54) with $2\,r = \lambda_i + \rho_i$, and using Lemma 21, we have

$$\int_{\Sigma_i} \gamma_i \, \mathrm{d}\bar{\mu} \geq \frac{7\,\pi}{3} \lambda_i^2 + o(\lambda_i^2).$$

It follows that

$$4\,\pi\,\lambda_i \leq \int_{\Sigma_i} \gamma_i \, H \, \mathrm{d}\mu \leq \int_{\Sigma_i} |H| \, \mathrm{d}\mu \leq 16\,\pi\,\lambda_i \tag{61}$$

for all sufficiently large i. Using (2), we have

$$-\kappa(\Sigma_i) \int_{\Sigma_i} \gamma_i \, H \, \mathrm{d}\mu = \int_{\Sigma_i} (\Delta \gamma_i) \, H \, \mathrm{d}\mu + \int_{\Sigma_i} \gamma_i \, H \, |\mathring{h}|^2 \, \mathrm{d}\mu + \int_{\Sigma_i} \gamma_i \, \operatorname{Rc}(\nu, \nu) \, H \, \mathrm{d}\mu.$$

Using (61), we have

$$|\kappa(\Sigma_i)| \int_{\Sigma_i} \gamma_i \, H \, \mathrm{d}\mu \geq 4\,\pi\,\lambda_i \, |\kappa(\Sigma_i)|.$$

Note that

$$\nabla^2 \gamma_i = O(\lambda_i^{-1} \, |h|) + O(\lambda_i^{-2}).$$

In conjunction with Lemma 23 and Lemma 21, we obtain that

$$\int_{\Sigma_i} (\Delta \gamma_i) \, H \, \mathrm{d}\mu = \int_{\Sigma_i} (\Delta \gamma_i) \, (H - 2\,\lambda_i^{-1}) \, \mathrm{d}\mu = O((\lambda_i^{-1/2} + \rho_i^{-1}) \, \lambda_i^{-1}).$$

By Proposition 35 and Lemma 23, we have

$$|\mathring{h}|^2 = O((\lambda_i^{-1/2} + \rho_i^{-1})^2 \, \lambda_i^{-2}) + o(\kappa(\Sigma_i))$$

on $\Sigma_i \cap \operatorname{spt}(\gamma_i)$. In conjunction with (61), we obtain

$$\int_{\Sigma_i} \gamma_i \, H \, |\mathring{h}|^2 \, \mathrm{d}\mu = O((\lambda_i^{-1/2} + \rho_i^{-1})^2 \, \lambda_i^{-1}) + o(\kappa(\Sigma_i) \, \lambda_i).$$

Likewise, (61) gives

$$\int_{\Sigma_i} \gamma_i \, \operatorname{Rc}(\nu, \nu) \, H \, \mathrm{d}\mu = O(\lambda_i^{-2}).$$

(60) follows from these estimates. Now, (59) follows from Proposition 35 and Lemma 23. □

Proposition 25. *A subsequence of* $\{\lambda_i^{-1} \, \Sigma_i\}_{i=1}^\infty$ *converges to a round sphere in* C^2 *locally in* $\mathbb{R}^3 \setminus \{0\}$.

Proof. Let $\hat{\Sigma}_i = \lambda_i^{-1} \, \Sigma_i$. By Proposition 28 and Lemma 41, we have

$$\bar{h}(\hat{\Sigma}_i) - \bar{g}|_{\hat{\Sigma}_i} = o(1) \tag{62}$$

locally uniformly in $\mathbb{R}^3 \setminus \{0\}$. Let $x_i \in \hat{\Sigma}_i$ with

$$|x_i| = \sup\{|y| : y \in \hat{\Sigma}_i\}.$$

Using (55) and (53), it follows that there is $x \in \mathbb{R}^3$ such that, passing to a subsequence,

$$\lim_{i \to \infty} x_i = x.$$

By (54), $x \neq 0$. Given $\delta \in (0,1/2)$, let $\hat\Sigma_i^\delta$ be the connected component of $\hat\Sigma_i \setminus B_\delta(0)$ containing x_i. Using (62), it follows that $\hat\Sigma_i^\delta$ converges to $S_1((1-|x|^{-1})\,x) \setminus B_\delta(0)$ in C^2. In particular,

$$\int_{\hat\Sigma^i_\delta} \bar H^2 \,\mathrm{d}\bar\mu \geq 16\,\pi - 4\,\pi\,\delta^2 - o(1).$$

If $\hat\Sigma_i \setminus B_{2\,\delta}(0)$ has more than one component for infinitely many i, we may apply the same argument to the second component to conclude that

$$\liminf_{i\to\infty} \int_{\hat\Sigma_i \setminus B_\delta(0)} \bar H^2 \,\mathrm{d}\bar\mu \geq 32\,\pi - 8\,\pi\,\delta^2.$$

This estimate is incompatible with Lemma 21.

The assertion now follows from taking a suitable diagonal subsequence. □

Proposition 26. *Suppose that $\rho_i = o(\lambda_i)$. A subsequence of $\{\rho_i^{-1}\,\Sigma_i\}_{i=1}^\infty$ converges to a flat plane with unit distance to the origin in C^2 locally in $\mathbb{R}^3$ as $i \to \infty$.*

Proof. Let $\hat\Sigma_i = \rho_i^{-1}\,\Sigma_i$. By Proposition 28 and Lemma 41, we have

$$\bar h(\hat\Sigma_i) = o(1) \tag{63}$$

locally uniformly in $\mathbb{R}^3$. Let $x_i \in \hat\Sigma_i$ with $|x_i| = 1$. Given $r > 0$, let $\hat\Sigma_i^r$ be the connected component of $\hat\Sigma_i \cap B_r(0)$ containing x_i. Using (63), it follows that, passing to a subsequence, $\hat\Sigma_i^{2\,r}$ converges to a bounded subset of a flat plane with unit distance to the origin in C^2. If $\hat\Sigma_i \cap B_r(0)$ has more than one connected component for infinitely many i, then, passing to a further subsequence, $\hat\Sigma_i \cap B_{2\,r}(0)$ has a second component that passes through $B_r(0)$ and converges to a bounded subset of a flat plane. In particular,

$$\liminf_{i\to\infty} r^{-2}\,|\hat\Sigma_i \cap B_{2\,r}(0)|_{\bar g} > 4\,\pi.$$

Applying (56) with $x = x_i$ and letting $t \to \infty$, we conclude that

$$\liminf_{i\to\infty} \int_{\hat\Sigma_i} \bar H^2 \,\mathrm{d}\bar\mu > 16\pi.$$

This estimate is incompatible with Lemma 21.

The assertion now follows from taking a suitable diagonal subsequence. □

Proposition 27. *Suppose that $\rho_i = o(\lambda_i)$ and that $\{x_i\}_{i=1}^\infty$ is a sequence of points $x_i \in \Sigma_i$ with $\rho_i = o(|x_i|)$ and $x_i = o(\lambda_i)$. A subsequence of $|x_i|^{-1}\,\Sigma_i$ converges to a flat plane passing through the origin in C^2 locally in $\mathbb{R}^3 \setminus \{0\}$ as $i \to \infty$.*

Proof. The proof is similar to that of Proposition 26. We omit the formal modifications. □

Corollary 28. *There holds*

$$|h - \lambda_i^{-1}\,g|_{\Sigma_i}| + |x|\,|\nabla h| = O((\lambda_i^{-1/2} + \rho_i^{-1})\,|x|^{-1}).$$

Likewise,

$$|\bar h - \lambda_i^{-1}\,\bar g|_{\Sigma_i}|_{\bar g} + |x|\,|\bar\nabla \bar h|_{\bar g} = O((\lambda_i^{-1/2} + \rho_i^{-1})\,|x|^{-1}).$$

Proof. Let $x_i \in \Sigma_i$ be such that

$$|x_i|^2 \, |(\bar\nabla h)(x_i)| = \sup_{x\in\Sigma_i} |x|^2 \, |\bar\nabla h|.$$

By (55) and Lemma 21, $x_i = O(\lambda_i)$. Using either Proposition 25 if $\lambda_i = O(|x_i|)$, Proposition 26 if $x_i = O(\rho_i)$, or Proposition 27 if $x_i = o(\lambda_i)$ and $\rho_i = o(|x_i|)$, it follows that $|x_i|^{-1}\,(\Sigma_i \cap B_{3/4\,|x_i|}(x_i))$ converges either to a subset of a round sphere of radius at least 1 or to a bounded subset of a flat plane in C^2. In particular, the geometry of $|x_i|^{-1}\,(\Sigma_i \cap B_{3/4\,|x_i|}(x_i))$ is uniformly bounded.

By interior L^4-estimates as in [11, Theorem 9.11] applied to the area-constrained Willmore equation (2) and by the Sobolev embedding theorem, using also Lemma 41, and Proposition 24, we obtain

$$|x_i|^{5/2} \left(\int_{\Sigma_i \cap B_{1/2\,|x_i|}(x_i)} |\bar\nabla^2 H|^4 \, \mathrm{d}\bar\mu \right)^{1/4} + |x_i|^2 \, |(\bar\nabla H)(x_i)| = O(\lambda_i^{-1/2} + \rho_i^{-1}). \tag{64}$$

Applying the same argument to (66), using also (64), we conclude that

$$|x_i|^2 \, |(\bar\nabla \mathring{h})(x_i)| = O(\lambda_i^{-1/2} + \rho_i^{-1}).$$

The assertion follows from this and Lemma 41. □

B. Integral curvature estimates

In [18], E. Kuwert and R. Schätzle have established integral curvature estimates for Euclidean Willmore surfaces with small traceless second fundamental form. In this section, we adapt their method to establish integral curvature estimates for large area-constrained Willmore spheres with non-negative Hawking mass in Riemannian 3-manifolds that are asymptotic to Schwarzschild.

In short, we use integration by parts, the area-constrained Willmore equation, and Lemma 23 to prove local $W^{2,2}$-bounds for the second fundamental form h. In conjunction with the Sobolev inequality, we obtain an L^∞-estimate for h. Compared to [18], additional curvature terms owing to the non-flat background arise and are addressed.

We assume that g is a Riemannian metric on $\mathbb{R}^3$ such that, as $x \to \infty$,

$$g = \left(1 + |x|^{-1}\right)^4 \bar g + \sigma \qquad \text{where} \qquad \partial_J \sigma = O(|x|^{-2-|J|})$$

for every multi-index J with $|J| \leq 4$.

Let $\{\Sigma_i\}_{i=1}^\infty$ be a sequence of area-constrained Willmore spheres $\Sigma_i \subset \mathbb{R}^3$ with

$$\int_{\Sigma_i} H^2 \, \mathrm{d}\mu \leq 16\,\pi, \qquad \lim_{i\to\infty} \rho(\Sigma_i) = \infty, \qquad \rho(\Sigma_i) = O(\lambda(\Sigma_i)).$$

We abbreviate $\rho_i = \rho(\Sigma_i)$ and $\lambda_i = \lambda(\Sigma_i)$.

Let X, Y, Z be vector fields tangent to Σ and Rm the Riemann curvature tensor of (M, g). We recall the Gauss-Codazzi equation

$$(\nabla_X h)(Y, Z) = (\nabla_Y h)(X, Z) + \mathrm{Rm}(X, Y, \nu, Z)$$

and its trace

$$\mathrm{div}_{\Sigma_i} h = \nabla H + \nu \lrcorner \, \mathrm{Rc}\,. \tag{65}$$

For the statement of the following Simons-type identities, note that the contraction of the divergence is with respect to the first entry. Given a covariant tensor T, we use the convention

$$X \lrcorner (\nabla T) = (\nabla_X T).$$

Lemma 29 ([17, Lemma 3.2])**.** *There holds, on Σ_i,*

$$\Delta \mathring{h} = \mathring{\nabla}^2 H + \frac{1}{2} H^2 \mathring{h} + \mathring{h} * \mathring{h} * \mathring{h} + O(|x|^{-3} |h|) + O(|x|^{-4}), \tag{66}$$

$$\operatorname{div}_{\Sigma_i} \nabla^2 H = \nabla \Delta H + \frac{1}{4} H^2 \nabla H + \mathring{h} * \mathring{h} * \nabla H + O(|x|^{-3} |\nabla H|), \tag{67}$$

$$\operatorname{div}_{\Sigma_i} \nabla^2 \mathring{h} = \nabla \Delta \mathring{h} + h * h * \nabla \mathring{h} + O(|x|^{-3} |\mathring{h}| |h|) + O(|x|^{-4} |\mathring{h}|) + O(|x|^{-3} |\nabla \mathring{h}|). \tag{68}$$

Let $\psi \in C^\infty(\mathbb{R})$ with

- $0 \leq \psi \leq 1$,
- $\psi(1) = 1$,
- $\psi(s) = 0$ if $s < 3/4$ or $s > 5/4$,
- $|\psi'| \leq \frac{9}{2}$.

We fix $x \in \mathbb{R}^3$ and define $\eta \in C^\infty(\mathbb{R}^3)$ by

$$\eta(z) = \psi(|z| \, |x|^{-1}). \tag{69}$$

Note that, for all sufficiently large x,

$$|D\eta| \leq 5 \, |x|^{-1}. \tag{70}$$

Moreover, by Lemma 41, (54), and Lemma 21, we have, uniformly for all $x \in \Sigma_i$,

$$|\Sigma_i \cap \operatorname{spt}(\eta)| = O(|x|^{-2}). \tag{71}$$

The following lemma is an adaptation of [18, Lemma 2.2].

Lemma 30. *There holds, uniformly for all $x \in \Sigma_i$,*

$$\begin{aligned} \int_{\Sigma_i} \eta^2 \, |\nabla \mathring{h}|^2 \, \mathrm{d}\mu \leq \frac{1}{2} \int_{\Sigma_i} \eta^2 \, H^2 \, |\mathring{h}|^2 \, \mathrm{d}\mu + O(1) \int_{\Sigma_i} \eta^2 \, |\mathring{h}|^4 \, \mathrm{d} \\ + O(|x|^{-2}) \int_{\Sigma_i \cap \operatorname{spt}(\eta)} |h - \lambda_i^{-1} \, g|_{\Sigma_i}|^2 \, \mathrm{d}\mu \\ + O(|x|^{-4}) + O\big(|x|^2 \, \kappa(\Sigma_i)^2\big). \end{aligned}$$

Proof. Multiplying (66) by $\eta^2 \, \mathring{h}$, integrating by parts, and using (65), (70), and (71), we obtain

$$\begin{aligned} &\int_{\Sigma_i} \eta^2 \left[|\nabla \mathring{h}|^2 + \frac{1}{2} H^2 \, |\mathring{h}|^2 \right] \mathrm{d}\mu \\ &= \frac{1}{2} \int_{\Sigma_i} \eta^2 \, |\nabla H|^2 \, \mathrm{d}\mu + O(|x|^{-1}) \int_{\Sigma_i} \eta \, |\mathring{h}| \, |\nabla \mathring{h}| \, \mathrm{d}\mu + O(1) \int_{\Sigma_i} \eta^2 \, |\mathring{h}|^4 \, \mathrm{d}\mu \\ &\quad + O(|x|^{-4}) \int_{\Sigma_i} \eta \, |\mathring{h}| \, \mathrm{d}\mu + O(|x|^{-3}) \int_{\Sigma_i} \eta^2 \, |\mathring{h}| \, |h| \, \mathrm{d}\mu \\ &\quad + O(|x|^{-3}) \int_{\Sigma_i} \eta^2 \, |\nabla \mathring{h}| \, \mathrm{d}\mu + O(|x|^{-6}) \int_{\Sigma_i} \eta^2 \, \mathrm{d}\mu. \end{aligned}$$

Note that

$$O(|x|^{-1}) \int_{\Sigma_i} \eta\, |\mathring{h}|\, |\nabla \mathring{h}| \,\mathrm{d}\mu + O(|x|^{-4}) \int_{\Sigma_i} \eta\, |\mathring{h}| \,\mathrm{d}\mu + O(|x|^{-3}) \int_{\Sigma_i} \eta^2\, |\nabla \mathring{h}| \,\mathrm{d}\mu$$
$$\leq \frac{1}{2} \int_{\Sigma_i} \eta^2\, |\nabla \mathring{h}|^2 \,\mathrm{d}\mu + O(|x|^{-2}) \int_{\Sigma_i \cap \operatorname{spt}(\eta)} |\mathring{h}|^2 \,\mathrm{d}\mu + O(|x|^{-6}) \int_{\Sigma_i} \eta^2 \,\mathrm{d}\mu.$$

Likewise, using Remark 21,

$$O(|x|^{-3}) \int_{\Sigma_i} \eta^2\, |\mathring{h}|\, |h| \,\mathrm{d}\mu \leq O(|x|^{-2}) \int_{\Sigma_i \cap \operatorname{spt}(\eta)} |\mathring{h}|^2 \,\mathrm{d}\mu + O(|x|^{-4}).$$

Integrating by parts we have

$$\int_{\Sigma_i} \eta^2\, |\nabla H|^2 \,\mathrm{d}\mu = - \int_{\Sigma_i} \eta^2\, (H - 2\,\lambda_i^{-1})\, \Delta H \,\mathrm{d}\mu - 2 \int_{\Sigma_i} \eta\, (H - 2\,\lambda_i^{-1})\, g(\nabla \eta, \nabla H) \,\mathrm{d}\mu.$$

Using (70), we estimate

$$- 2 \int_{\Sigma_i} \eta\, (H - 2\,\lambda_i^{-1})\, g(\nabla \eta, \nabla H) \,\mathrm{d}\mu$$
$$\leq \frac{1}{2} \int_{\Sigma_i} \eta^2\, |\nabla H|^2 \,\mathrm{d}\mu + O\left(|x|^{-2}\right) \int_{\Sigma_i \cap \operatorname{spt}(\eta)} (H - 2\,\lambda_i^{-1})^2 \,\mathrm{d}\mu.$$

Using (2), we have

$$- \int_{\Sigma_i} \eta^2\, (H - 2\,\lambda_i^{-1})\, \Delta H \,\mathrm{d}\mu = \int_{\Sigma_i} \eta^2\, (H - 2\,\lambda_i^{-1})\, H\, |\mathring{h}|^2 \,\mathrm{d}\mu$$
$$+ O(|x|^{-3} + |\kappa(\Sigma_i)|) \int_{\Sigma_i} \eta^2\, |H - 2\,\lambda_i^{-1}|\, |H| \,\mathrm{d}\mu\Big).$$

Note that

$$\int_{\Sigma_i} \eta^2\, (H - 2\,\lambda_i^{-1})\, H\, |\mathring{h}|^2 \,\mathrm{d}\mu$$
$$= \int_{\Sigma_i} \eta^2\, H^2\, |\mathring{h}|^2 \,\mathrm{d}\mu + \int_{\Sigma_i} \eta^2\, [\lambda_i^{-1}\, (H - 2\,\lambda_i^{-1}) + 2\,\lambda_i^{-2}]\, |\mathring{h}|^2 \,\mathrm{d}\mu$$

and

$$\int_{\Sigma_i} \eta^2\, [\lambda_i^{-1}\, (H - 2\,\lambda_i^{-1}) + 2\,\lambda_i^{-2}]\, |\mathring{h}|^2 \,\mathrm{d}\mu$$
$$\leq O(1) \int_{\Sigma_i} \eta^2\, |\mathring{h}|^4 \,\mathrm{d}\mu + O(|x|^{-2}) \int_{\Sigma_i \cap \operatorname{spt}(\eta)} |\mathring{h}|^2 \,\mathrm{d}\mu$$
$$+ O(|x|^{-2}) \int_{\Sigma_i \cap \operatorname{spt}(\eta)} (H - 2\,\lambda_i^{-1})^2 \,\mathrm{d}\mu.$$

Moreover, using Remark 21,

$$O(|x|^{-3} + |\kappa(\Sigma_i)|) \int_{\Sigma_i} \eta^2\, |H - 2\,\lambda_i^{-1}|\, |H| \,\mathrm{d}\mu$$
$$\leq O(|x|^{-4}) + O(|x|^2\, \kappa(\Sigma_i)^2) + O(|x|^{-2}) \int_{\Sigma_i \cap \operatorname{spt}(\eta)} (H - 2\,\lambda_i^{-1})^2 \,\mathrm{d}\mu.$$

The assertion follows from these estimates and (54). □

Using the trace of the Gauss-Codazzi equation (65), we obtain the following corollary.

Corollary 31. *There holds, uniformly for all $x \in \Sigma_i$,*

$$\begin{aligned}\int_{\Sigma_i} \eta^2\, |\nabla H|^2 \,\mathrm{d}\mu \le 4 \int_{\Sigma_i} \eta^2\, H^2\, |\mathring{h}|^2 \,\mathrm{d}\mu + O(1) \int_{\Sigma_i} \eta^2\, |\mathring{h}|^4 \,\mathrm{d}\mu \\ + O(|x|^{-2}) \int_{\Sigma_i \cap \operatorname{spt}(\eta)} |h - \lambda_i^{-1}\, g|_{\Sigma_i}|^2 \,\mathrm{d}\mu \\ + O(|x|^{-4}) + O(|x|^2\, \kappa(\Sigma_i)^2).\end{aligned}$$

The next two lemmas follow [18, Lemma 2.3].

Lemma 32. *There holds, uniformly for all $x \in \Sigma_i$,*

$$\begin{aligned}&\int_{\Sigma_i} \eta^4\, H^2\, |\nabla \mathring{h}|^2 \,\mathrm{d}\mu + \int_{\Sigma_i} \eta^4\, H^4\, |\mathring{h}|^2 \,\mathrm{d}\mu \\ &\quad\le 2 \int_{\Sigma_i} \eta^4\, H^2\, |\nabla H|^2 \,\mathrm{d}\mu + O(1) \int_{\Sigma_i} \eta^4\, H^2\, |\mathring{h}|^4 \,\mathrm{d}\mu + O(1) \int_{\Sigma_i} \eta^4\, |\mathring{h}|^2\, |\nabla \mathring{h}|^2 \,\mathrm{d}\mu \\ &\qquad + O(|x|^{-4}) \int_{\Sigma_i \cap \operatorname{spt}(\eta)} |\mathring{h}|^2 \,\mathrm{d}\mu + O(|x|^{-6}).\end{aligned}$$

Proof. Multiplying (66) by $\eta^4\, H^2\, \mathring{h}$, integrating by parts, and using (65) and (70), we have

$$\begin{aligned}&\int_{\Sigma_i} \eta^4\, H^2\, |\nabla \mathring{h}|^2 \,\mathrm{d}\mu + \frac{1}{2} \int_{\Sigma} \eta^4\, H^4\, |\mathring{h}|^2 \,\mathrm{d}\mu \\ &\quad\le \frac{1}{2} \int_{\Sigma} \eta^4\, H^2\, |\nabla H|^2 \,\mathrm{d}\mu + O(1) \int_{\Sigma_i} \eta^4\, H^2\, |\mathring{h}|^4 \,\mathrm{d}\mu \\ &\qquad + O(|x|^{-1}) \int_{\Sigma_i} \eta^3\, H^2\, |\mathring{h}|\, |\nabla \mathring{h}| \,\mathrm{d}\mu + O(|x|^{-1}) \int_{\Sigma_i} \eta^3\, H^2\, |\mathring{h}|\, |\nabla H| \,\mathrm{d}\mu \\ &\qquad + O(1) \int_{\Sigma_i} \eta^4\, |H|\, |\mathring{h}|\, |\nabla H|\, |\nabla \mathring{h}| \,\mathrm{d}\mu + O(1) \int_{\Sigma_i} \eta^4\, |H|\, |\mathring{h}|\, |\nabla H|^2 \,\mathrm{d}\mu \\ &\qquad + O(|x|^{-3}) \int_{\Sigma_i} \eta^4\, H^2\, |\nabla H| \,\mathrm{d}\mu + O(|x|^{-3}) \int_{\Sigma_i} \eta^4\, H^2\, |\mathring{h}|\, |h| \,\mathrm{d}\mu \\ &\qquad + O(|x|^{-4}) \int_{\Sigma_i} \eta^4\, H^2\, |\mathring{h}| \,\mathrm{d}\mu.\end{aligned}$$

Note that

$$\begin{aligned}&O(|x|^{-1}) \int_{\Sigma_i} \eta^3\, H^2\, |\mathring{h}|\, |\nabla \mathring{h}| \,\mathrm{d}\mu + O(|x|^{-1}) \int_{\Sigma_i} \eta^3\, H^2\, |\mathring{h}|\, |\nabla H| \,\mathrm{d}\mu \\ &\quad\le \frac{1}{16} \int_{\Sigma_i} \eta^4\, H^2\, |\nabla H|^2 \,\mathrm{d}\mu + \frac{1}{16} \int_{\Sigma_i} \eta^4\, H^2\, |\nabla \mathring{h}|^2 \,\mathrm{d}\mu + \frac{1}{16} \int_{\Sigma_i} \eta^4\, H^4\, |\mathring{h}|^2 \,\mathrm{d}\mu \\ &\qquad + O(|x|^{-4}) \int_{\Sigma_i \cap \operatorname{spt}(\eta)} |\mathring{h}|^2 \,\mathrm{d}\mu.\end{aligned}$$

Likewise,

$$O(1) \int_{\Sigma_i} \eta^4\, H\, |\mathring{h}|\, |\nabla H|\, |\nabla \mathring{h}| \,\mathrm{d}\mu \le \frac{1}{32} \int_{\Sigma_i} \eta^4\, H^2\, |\nabla H|^2 \,\mathrm{d}\mu + O(1) \int_{\Sigma_i} \eta^4\, |\mathring{h}|^2\, |\nabla \mathring{h}|^2 \,\mathrm{d}\mu.$$

Moreover, using (65),

$$\begin{aligned}&O(1)\int_{\Sigma_i}\eta^4\,|H|\,|\mathring{h}|\,|\nabla H|^2\,\mathrm{d}\mu\\&\qquad\le\frac{1}{32}\int_{\Sigma_i}\eta^4\,H^2\,|\nabla H|^2\,\mathrm{d}\mu+O(1)\int_{\Sigma_i}\eta^4\,|\mathring{h}|^2\,|\nabla\mathring{h}|^2\,\mathrm{d}\mu+O(|x|^{-6})\int_{\Sigma_i\cap\operatorname{spt}(\eta)}|\mathring{h}|^2\,\mathrm{d}\mu.\end{aligned}$$

Finally, using Remark 21,

$$\begin{aligned}&O(|x|^{-3})\int_{\Sigma_i}\eta^4\,H^2\,|\nabla H|\,\mathrm{d}\mu+O(|x|^{-3})\int_{\Sigma_i}\eta^4\,H^2\,|\mathring{h}|\,|h|\,\mathrm{d}\mu+O(|x|^{-4})\int_{\Sigma_i}\eta^4\,H^2\,|\mathring{h}|\,\mathrm{d}\mu\\&\qquad\le\frac{1}{16}\int_{\Sigma_i}\eta^4\,H^2\,|\nabla H|^2\,\mathrm{d}\mu+\frac{1}{16}\int_{\Sigma_i}\eta^4\,H^4\,|\mathring{h}|^2\,\mathrm{d}\mu+O(|x|^{-6})+O(|x|^{-8})\int_{\Sigma_i}\eta^4\,\mathrm{d}\mu.\end{aligned}$$

The assertion follows from these estimates and (54). ☐

Lemma 33. *There holds, uniformly for all $x\in\Sigma_i$,*

$$\begin{aligned}&\int_{\Sigma_i}\eta^4\,[|\nabla^2H|^2+|h|^2\,|\nabla h|^2+|h|^4\,|\mathring{h}|^2]\,\mathrm{d}\mu\\&\qquad\le O(1)\int_{\Sigma_i}\eta^4\,|\mathring{h}|^2\,|\nabla\mathring{h}|^2\,\mathrm{d}\mu+O(1)\int_{\Sigma_i}\eta^4\,|\mathring{h}|^6\,\mathrm{d}\mu\\&\qquad\quad+O(|x|^{-4})\int_{\Sigma_i\cap\operatorname{spt}(\eta)}|h-\lambda_i^{-1}\,g|_{\Sigma_i}|^2\,\mathrm{d}\mu\\&\qquad\quad+O(|x|^{-6})+O(\kappa(\Sigma_i)^2).\end{aligned}$$

Proof. Multiplying (67) by $\eta^4\,\nabla H$, integrating by parts, and using (65) and (70), we obtain

$$\begin{aligned}&\int_{\Sigma_i}\eta^4\,\left[|\nabla^2H|^2+\frac{1}{4}\,H^2\,|\nabla H|^2\right]\mathrm{d}\mu\\&\quad\le\int_{\Sigma_i}\eta^4\,(\Delta H)^2\,\mathrm{d}\mu+40\,|x|^{-1}\int_{\Sigma_i}\eta^3\,|\nabla H|\,|\nabla^2H|\,\mathrm{d}\mu\\&\qquad+O(1)\int_{\Sigma_i}\eta^4\,|\mathring{h}|^2\,|\nabla\mathring{h}|^2\,\mathrm{d}\mu+O(|x|^{-6})\int_{\Sigma_i}\eta^4\,|\mathring{h}|^2\,\mathrm{d}\mu+O(|x|^{-3})\int_{\Sigma_i}\eta^4\,|\nabla H|^2\,\mathrm{d}\mu.\end{aligned}$$

Using (2) and Remark 21, we conclude that

$$\begin{aligned}&\int_{\Sigma_i}\eta^4\,\left[|\nabla^2H|^2+\frac{1}{4}\,H^2\,|\nabla H|^2\right]\mathrm{d}\mu\\&\quad\le\int_{\Sigma}\eta^4\,H^2\,|\mathring{h}|^4\,\mathrm{d}\mu+\frac{1}{2}\int_{\Sigma_i}\eta^4\,|\nabla^2H|^2\,\mathrm{d}\mu+10^3\,|x|^{-2}\int_{\Sigma_i}\eta^4\,|\nabla H|^2\,\mathrm{d}\mu\\&\qquad+O(|x|^{-6})+O(\kappa(\Sigma_i)^2)+O(1)\int_{\Sigma_i}\eta^4\,|\mathring{h}|^2\,|\nabla\mathring{h}|^2\,\mathrm{d}\mu+O(|x|^{-6})\int_{\Sigma_i}\eta^4\,|\mathring{h}|^2\,\mathrm{d}\mu.\end{aligned}$$

Note that

$$\int_{\Sigma}\eta^4\,H^2\,|\mathring{h}|^4\,\mathrm{d}\mu\le\frac{1}{32}\int_{\Sigma}\eta^4\,H^4\,|\mathring{h}|^2\,\mathrm{d}\mu+O(1)\int_{\Sigma}\eta^4\,|\mathring{h}|^6\,\mathrm{d}\mu.$$

The assertion follows from these estimates, Corollary 31, the trivial estimate

$$\int_{\Sigma_i}\eta^2\,H^2\,|\mathring{h}|^2\,\mathrm{d}\mu\le\frac{1}{128}\,10^{-3}\,|x|^2\int_{\Sigma_i}\eta^4\,H^4\,|\mathring{h}|^2\,\mathrm{d}\mu+O(|x|^{-2})\int_{\Sigma_i\cap\operatorname{spt}(\eta)}|\mathring{h}|^2\,\mathrm{d}\mu,$$

Lemma 32, and (65). □

The following lemma is an adaptation of [18, Proposition 2.4].

Lemma 34. *There holds, uniformly for all $x \in \Sigma_i$,*

$$\begin{aligned}\int_{\Sigma_i} \eta^4\,[|\nabla^2 h|^2 + |h|^2\,|\nabla h|^2 + |h|^4\,|\mathring{h}|^2]\,\mathrm{d}\mu \\ \leq O(|x|^{-4}) \int_{\Sigma_i \cap \operatorname{spt}(\eta)} |h - \lambda_i^{-1}\, g|_{\Sigma_i}|^2\,\mathrm{d}\mu \\ + O(|x|^{-6}) + O(\kappa(\Sigma_i)^2).\end{aligned}$$

Proof. Multiplying (68) by $\eta^4\,\nabla\mathring{h}$, integrating by parts, and using Remark 21 and (70), we find that

$$\begin{aligned}\int_{\Sigma_i} \eta^4 |\nabla^2\mathring{h}|^2\,\mathrm{d}\mu \leq \int_{\Sigma_i} \eta^4\,|\Delta\mathring{h}|^2\,\mathrm{d}\mu + \frac{1}{4}\int_{\Sigma_i} \eta^4\,|\nabla^2\mathring{h}|^2\,\mathrm{d}\mu + O(|x|^{-2})\int_{\Sigma_i} |\nabla\mathring{h}|^2\,\mathrm{d}\mu \\ + O(1)\int_{\Sigma_i} \eta^4\,|h|^2\,|\nabla h|^2\,\mathrm{d}\mu + O(|x|^{-6}).\end{aligned}$$

Using (66) and Remark 21, we find

$$\begin{aligned}\int_{\Sigma_i} \eta^4\,|\Delta\mathring{h}|^2\,\mathrm{d}\mu \leq \int_{\Sigma_i} \eta^4\,|\nabla^2 H|^2\,\mathrm{d}\mu + O(1)\int_{\Sigma_i} \eta^4\,H^4\,|\mathring{h}|^2\,\mathrm{d}\mu + O(1)\int_{\Sigma_i} \eta^4\,|\mathring{h}|^6\,\mathrm{d}\mu \\ + O(|x|^{-6}) + O(|x|^{-8})\int_{\Sigma_i} \eta^2\,\mathrm{d}\mu.\end{aligned}$$

Assembling these estimates and using Lemma 33, Lemma 30, and (71), we have

$$\begin{aligned}\int_{\Sigma_i} \eta^4\,[|\nabla^2 h|^2 + |h|^2\,|\nabla h|^2 + |h|^4\,|\mathring{h}|^2]\,\mathrm{d}\mu \\ \leq O(1)\int_{\Sigma_i} \eta^4\,|\mathring{h}|^2\,|\nabla\mathring{h}|^2\,\mathrm{d}\mu + O(1)\int_{\Sigma_i} \eta^4\,|\mathring{h}|^6\,\mathrm{d}\mu \\ + O(|x|^{-4})\int_{\Sigma_i \cap \operatorname{spt}(\eta)} |h - \lambda_i^{-1}\, g|_{\Sigma_i}|^2\,\mathrm{d}\mu \\ + O(|x|^{-6}) + O(\kappa(\Sigma_i)^2).\end{aligned}$$

The argument now concludes as in [18, Lemma 2.5 and Proposition 2.6], using Lemma 23 and the Michael-Simon Sobolev inequality in the form [15, Proposition 5.4]. □

Proposition 35. *There holds, uniformly for all $x \in \Sigma_i$,*

$$\begin{aligned}|h - \lambda_i^{-1}\, g|_{\Sigma_i}|^4 = O(|x|^{-4}) \Big(\int_{\Sigma_i \cap B_{1/4\,|x|}(x)} |h - \lambda_i^{-1}\, g|_{\Sigma_i}|^2\,\mathrm{d}\mu\Big)^2 \\ + O(|x|^{-8}) + O(\kappa(\Sigma_i)^2)\int_{\Sigma_i \cap B_{1/4\,|x|}(x)} |h - \lambda_i^{-1}\, g|_{\Sigma_i}|^2\,\mathrm{d}\mu.\end{aligned}$$

Proof. Repeating the argument that lead to [18, Lemma 2.8] using [15, Proposition 5.4], we find that

$$|\eta^2\,\mathring{h}|^4_{L^\infty(\Sigma_i)} \leq O(1)\int_{\Sigma_i \cap \operatorname{spt}(\eta)} |\mathring{h}|^2\,\mathrm{d}\mu\,\Big[\int_{\Sigma_i} \eta^8\,[|\nabla^2\mathring{h}|^2 + H^4\,|\mathring{h}|^2]\,\mathrm{d}\mu + |x|^{-4}\int_{\Sigma_i \cap \operatorname{spt}(\eta)} |\mathring{h}|^2\,\mathrm{d}\mu\Big]$$

and

$$|\eta^2\,(H-2\,\lambda_i^{-1})|^4_{L^\infty(\Sigma_i)} \leq O(1)\int_{\Sigma_i\cap\operatorname{spt}(\eta)}|\mathring{h}|^2\,\mathrm{d}\mu\,\bigg[\int_{\Sigma_i}\eta^8\,[|\nabla^2 H|^2+H^4\,(H-2\,\lambda_i^{-1})^2]\,\mathrm{d}\mu + |x|^{-4}\int_{\Sigma_i\cap\operatorname{spt}(\eta)}(H-2\,\lambda_i^{-1})^2\,\mathrm{d}\mu\bigg].$$

By Lemma 23,

$$\int_{\Sigma_i}\eta^8\,H^4\,(H-2\,\lambda_i^{-1})^2\,\mathrm{d}\mu \leq\; o(1)\,|\eta^2(H-2\,\lambda_i^{-1})|^4_{L^\infty(\Sigma_i)}+O(\lambda_i^{-4})\int_{\Sigma_i\cap\operatorname{spt}(\eta)}(H-2\,\lambda_i^{-1})^2\,\mathrm{d}\mu.$$

By (69),

$$\operatorname{spt}(\eta)\subset\cap B_{1/4\,|x|}(x).$$

The assertion follows from these estimates and Lemma 34. □

C. Geometric identities on round spheres

Lemma 36. *Let $\xi\in\mathbb{R}^3$. The following hold if $|\xi|<1$:*

- $\int_{S_1(\xi)}|x|^{-1}\,\mathrm{d}\bar\mu = 4\,\pi$
- $\int_{S_1(\xi)}|x|^{-3}\,\mathrm{d}\bar\mu = 4\,\pi\,(1-|\xi|^2)^{-1}$
- $\int_{S_1(\xi)}|x|^{-5}\,\mathrm{d}\bar\mu = \frac{4\,\pi}{3}\,(3+|\xi|^2)\,(1-|\xi|^2)^{-3}$

The following hold if $|\xi|>1$:

- $\int_{S_1(\xi)}|x|^{-1}\,\mathrm{d}\bar\mu = 4\,\pi\,|\xi|^{-1}$
- $\int_{S_1(\xi)}|x|^{-3}\,\mathrm{d}\bar\mu = 4\,\pi\,|\xi|^{-1}\,(|\xi|^2-1)^{-1}$
- $\int_{S_1(\xi)}|x|^{-5}\,\mathrm{d}\bar\mu = \frac{4\,\pi}{3}\,|\xi|^{-1}\,(1+3\,|\xi|^2)\,(|\xi|^2-1)^{-3}$

The following hold if $|\xi|\neq 1$:

- $\int_{S_1(\xi)}|x|^{-2}\,\mathrm{d}\bar\mu = 2\,\pi\,|\xi|^{-1}\,\log\frac{1+|\xi|}{|1-|\xi||}$
- $\int_{S_1(\xi)}|x|^{-4}\,\mathrm{d}\bar\mu = 4\,\pi\,(1-|\xi|^2)^{-2}$
- $\int_{S_1(\xi)}|x|^{-6}\,\mathrm{d}\bar\mu = 4\,\pi\,(1+|\xi|^2)\,(1-|\xi|^2)^{-4}$

Lemma 37. *Let $\xi\in\mathbb{R}^3$. The following identities hold on $S_1(\xi)$:*

- $2\,\bar g(x,\bar\nu) = |x|^2+1-|\xi|^2$
- $2\,\bar g(x,\xi) = |x|^2+|\xi|^2-1$

- $2\,\bar g(\xi,\bar\nu) = |x|^2 - 1 - |\xi|^2$

D. Geometric expansions for perturbations of the Euclidean metric

In this section, we collect some expansions that relate geometric quantities computed with respect to different background metrics.

We assume that g is a Riemannian metric on $\mathbb{R}^3$ such that, as $x \to \infty$,

$$g = \left(1 + |x|^{-1}\right)^4 \bar g + \sigma \qquad \text{where} \qquad \partial_J \sigma = O(|x|^{-2-|J|})$$

for every multi-index J with $|J| \leq 4$. We denote by $\tilde g = (1 + |x|^{-1})^4\, \bar g$ the Schwarzschild metric of mass $m = 2$. We use a bar for geometric quantities pertaining to $\bar g$ and a tilde for quantities pertaining to $\tilde g$.

Lemma 38. *Let $\xi \in \mathbb{R}^3$ and $i,\, j \in \{1,\,2,\,3\}$. There holds*

$$\tilde D_{e_i}\xi = 2\,(1+|x|^{-1})^{-1}\,|x|^{-3}\,\left(\bar g(\xi, e_i)\,x - \bar g(\xi, x)\,e_i - \bar g(e_i, x)\,\xi\right).$$

Moreover, as $x \to \infty$,

$$D_{e_i}\xi - \tilde D_{e_i}\xi = O(|x|^{-3}),$$

$$D^2_{e_i,e_j}\xi - \tilde D^2_{e_i,e_j}\xi = \frac{1}{2}\sum_{k=1}^{3}\left[(\bar D^2_{e_i,\xi}\sigma)(e_j,e_k) + (\bar D^2_{e_i,e_j}\sigma)(\xi,e_k) - (\bar D^2_{e_i,e_k}\sigma)(\xi,e_j)\right] e_k + O(|x|^{-5}).$$

Lemma 39 ([10, Lemma 37])**.** *There holds*

$$\tilde{\operatorname{Ric}}(e_i,e_j) = 2\,(1+|x|^{-1})^{-2}\,|x|^{-3}\left[\bar g(e_i,e_j) - 3\,|x|^{-2}\,\bar g(e_i,x)\,\bar g(e_j,x)\right].$$

Moreover, as $x \to \infty$,

$$\begin{aligned}
&\operatorname{Ric}(e_i,e_j) - \tilde{\operatorname{Ric}}(e_i,e_j)\\
&\quad = \frac{1}{2}\sum_{k=1}^{3}\left[(\bar D^2_{e_k,e_i}\sigma)(e_k,e_j) + (\bar D^2_{e_k,e_j}\sigma)(e_k,e_i) - (\bar D^2_{e_k,e_k}\sigma)(e_i,e_j) - (\bar D^2_{e_i,e_j}\sigma)(e_k,e_k)\right]\\
&\qquad + O(|x|^{-5})
\end{aligned}$$

and

$$R = \sum_{i,j=1}^{3}\left[(\bar D^2_{e_i,e_j}\sigma)(e_i,e_j) - (\bar D^2_{e_i,e_i}\sigma)(e_j,e_j)\right] + O(|x|^{-5}).$$

Lemma 40. *Let $\xi \in \mathbb{R}^3$. There holds, as $x \to \infty$,*

$$\bar g(\tilde D^2_{e_1,e_1}\xi + \tilde D^2_{e_2,e_2}\xi, e_3) = \tilde{\operatorname{Ric}}(\xi,e_3) + 4\,|x|^{-4}\,\bar g(\xi,e_3) - 8\,|x|^{-6}\,\bar g(\xi,x)\,\bar g(e_3,x) + O(|x|^{-5}).$$

Proof. This follows from Lemma 38 and Lemma 39. □

Lemma 41. *Let $\{\Sigma_i\}_{i=1}^\infty$ be a sequence of surfaces $\Sigma_i \subset \mathbb{R}^3$ with $\lim_{i\to\infty} \rho(\Sigma_i) = \infty$. The following expansions hold:*

- $\tilde{\nu} = (1 + |x|^{-1})^{-2}\, \bar{\nu}$
- $\tilde{H} = (1 + |x|^{-1})^{-2}\, \bar{H} - 4\,(1 + |x|^{-1})^{-3}\, |x|^{-3}\, \bar{g}(x, \bar{\nu})$
- $\mathring{\tilde{h}} = (1 + |x|^{-1})^{-2}\, \mathring{\bar{h}}$
- $\tilde{\nabla}\tilde{h} = \bar{\nabla}\bar{h} + O(|x|^{-3}) + O(|\bar{h}|\, |x|^{-2}) + O(|\bar{\nabla}\bar{h}|\, |x|^{-1})$
- $\mathrm{d}\tilde{\mu} = (1 + |x|^{-1})^4\, \mathrm{d}\bar{\mu}$

and

- $\nu = \tilde{\nu} + O(|x|^{-2})$
- $H = \tilde{H} + O(|x|^{-3}) + O(|\bar{h}|\, |x|^{-2})$
- $\mathring{h} = \mathring{\tilde{h}} + O(|x|^{-3}) + O(|\bar{h}|\, |x|^{-2})$
- $\nabla h = \tilde{\nabla}\tilde{h} + O(|x|^{-4}) + O(|\bar{h}|\, |x|^{-3}) + O(|\bar{\nabla}\bar{h}|\, |x|^{-2})$
- $\mathrm{d}\mu = [1 + O(|x|^{-2})]\, \mathrm{d}\tilde{\mu}$

Moreover, if $\{u_i\}_{i=1}^\infty$ is a sequence of functions $u_i \in C^\infty(\Sigma_i)$, then

- $\tilde{\nabla} u_i = (1 + |x|^{-1})^{-2}\, \bar{\nabla} u_i$
- $\tilde{\Delta} u_i = (1 + |x|^{-1})^{-4}\, \bar{\Delta} u_i$

and

- $\nabla u_i = \tilde{\nabla} u_i + O(|x|^{-2}\, |\bar{\nabla} u_i|)$
- $\Delta u_i = \tilde{\Delta} u_i + O(|x|^{-2}\, |\bar{\nabla}^2 u_i|) + O(|x|^{-3}\, |\bar{\nabla} u_i|) + O(|\bar{h}|\, |x|^{-2}\, |\bar{\nabla} u_i|)$

E. Geometric expansions for graphs over Euclidean spheres

In this section, we collect some geometric identities for graphs over Euclidean spheres.

Let $\xi \in \mathbb{R}^3$, $\lambda > 0$, and $u \in C^\infty(S_\lambda(\lambda\,\xi))$. Recall from (13) that, $\Sigma_{\xi,\lambda}(u)$ denotes the Euclidean graph of u over $S_\lambda(\lambda\,\xi)$.

Lemma 42. *There holds*

- $\bar{g}|_{\Sigma_{\xi,\lambda}(u)} = (1 + \lambda^{-1}\, u)^2\, \bar{g}|_{S_\lambda(\lambda\xi)} + du \otimes du$
- $\bar{\nu}(\Sigma_{\xi,\lambda}(u)) = ((1 + \lambda^{-1}\, u)^2 + |\bar{\nabla} u|^2)^{-1/2}\, ((1 + \lambda^{-1}\, u)\, \nu(S_\lambda(\lambda\xi)) - \bar{\nabla} u)$
- $\bar{h}(\Sigma_{\xi,\lambda}(u)) = ((1 + \lambda^{-1}\, u)^2 + |\bar{\nabla} u|^2)^{-1/2}$ $(\lambda^{-1}\, (1 + \lambda^{-1}\, u)^2\, \bar{g}|_{S_\lambda(\lambda\xi)} + 2\,\lambda^{-1}\, du \otimes du - (1 + \lambda^{-1}\, u)\, \bar{\nabla}^2 u)$

Lemma 43. *Suppose that*

$$\lambda^{-1}\, |u| + |\bar{\nabla} u| \leq 1. \tag{72}$$

There holds, for all $f \in C^\infty(S_\lambda(\lambda\,\xi))$,

$\bar{\Delta}_{\Sigma_{\xi,\lambda}}(u) f = (1 - 2\,\lambda^{-1}\, u)\, \bar{\Delta}_{S_\lambda}(\lambda\,\xi) f$

$$+\,O(1)\,|\bar\nabla f|\,|\bar\nabla u|\,(\lambda^{-2}\,|u|+\lambda^{-1}\,|\bar\nabla u|+|\bar\nabla^2 u|)+O(1)\,|\bar\nabla^2 f|\,(\lambda^{-2}\,u^2+|\bar\nabla u|^2).$$

Proof. We may assume that $\lambda=1$ and $\xi=0$.

By direct computation,

$$\frac{d}{ds}\Big|_{s=0}\bar\Delta_{\Sigma_{0,1}}(s\,u)f=-\,2\,u\,\bar\Delta_{S_1(0)}f$$

and

$$\frac{d^2}{ds^2}\Delta_{\Sigma_{0,1}}(s\,u)f=O(1)\,|\bar\nabla f|\,|\bar\nabla u|\,(|u|+|\bar\nabla u|+|\bar\nabla^2 u|)+O(1)\,|\bar\nabla^2 f|\,(u^2+|\bar\nabla u|^2).$$

uniformly for all $s\in[0,1]$.

The assertion follows from these estimates and Taylor's theorem. □

F. The potential function

In this section, we collect some facts about the potential function of the spatial Schwarzschild manifold.

We assume that g is a Riemannian metric on $\mathbb{R}^3$. We denote by $\tilde g=(1+|x|^{-1})^4\,\bar g$ the Schwarzschild metric of mass $m=2$. We use a bar for geometric quantities pertaining to $\bar g$ and a tilde for quantities pertaining to $\tilde g$.

Recall from [6, §2] that the potential function $N:\mathbb{R}^3\setminus\{0\}\to\mathbb{R}$ of the spatial Schwarzschild manifold is given by

$$N(x)=(1+|x|^{-1})^{-1}\,(1-|x|^{-1}). \tag{73}$$

Moreover, recall that N satisfies the static metric equation

$$\tilde D^2 N=N\,\tilde{\mathrm{Rc}}. \tag{74}$$

Let $\Sigma\subset M$ be a closed, two-sided surface with outward normal ν, mean curvature H with respect to ν, second fundamental form h, and non-positive Laplace-Beltrami operator Δ such that

$$\Delta H+(|\mathring{h}|^2+\mathrm{Rc}(\nu,\nu)+\kappa)\,H=0$$

for some $\kappa\in\mathbb{R}$.

Lemma 44. *Let $F:\Sigma\to\mathbb{R}$ be given by $F=N^{-1}\,H(\Sigma)$ and suppose that $X\in\Gamma(T\Sigma)$. There holds*

$$\begin{aligned}\Delta F=&-(|\mathring{h}|^2+\kappa+\mathrm{Rc}(\nu,\nu)-\tilde{\mathrm{Rc}}(\tilde\nu,\tilde\nu)+N^{-1}\,\Delta N-N^{-1}\,\tilde\Delta N-\tilde g(\tilde\nu,\tilde DN)\,\tilde H)\,F\\&-2\,N^{-1}\,g(\nabla F,\nabla N).\end{aligned} \tag{75}$$

Proof. We have

$$\Delta F=-N^{-1}\,F\,\Delta N+2\,N^{-2}\,F\,g(\nabla N,\nabla N)-2\,N^{-2}\,g(\nabla H,\nabla N)+N^{-1}\,\Delta H.$$

Using (74) and that $\tilde R=0$, we have

$$\tilde\Delta N=-N^{-1}\,\tilde{\mathrm{Rc}}(\tilde\nu,\tilde\nu)-\tilde g(\tilde\nu,\tilde DN)\,\tilde H.$$

Moreover, note that

$$\nabla H=N\,\nabla F+F\,\nabla N.$$

(75) follows from these identities. □

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