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Joint density of the stable process and its supremum: Regularity and upper bounds

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This article uses a combination of three ideas from simulation to establish a nearly optimal polynomial upper bound for the joint density of the stable process and its associated supremum at a fixed time on the entire support of the joint law. The representation of the concave majorant of the stable process and the Chambers-Mallows-Stuck representation for stable laws are used to define an approximation of the random vector of interest. An interpolation technique using multilevel Monte Carlo is applied to accelerate the approximation, allowing us to establish the infinite differentiability of the joint density as well as nearly optimal polynomial upper bounds for the joint mixed derivatives of any order.

Keywords: Joint density bounds; stable supremum

1. Introduction

Let $(X_t)_{t\geq 0}$ be a non-monotonic α-stable process started at zero, $\mathbb{P}(X_0 = 0) = 1$, with $\alpha \in (0,2)$ and positivity parameter

$$\rho := \mathbb{P}(X_1 > 0) \in [1 - 1/\alpha, 1/\alpha] \cap (0,1). \tag{1.1}$$

For any fixed $T > 0$, denote by $\overline{X}_T := \sup_{s\in[0,T]} X_s$ its supremum over the time interval $[0,T]$. Our main result, Theorem 1 below, provides the regularity and upper bounds for the joint density of $(X_T, \overline{X}_T)$ and its derivatives of any order. These explicit polynomial bounds, valid on the entire support set of the joint law, are nearly optimal. For a detailed explanation, see the discussion following Theorem 1.

The joint law of $(X_T, \overline{X}_T)$ arises in the scaling limit of many stochastic models, including queues with heavy-tailed workloads (see [11, §5.2] and the references therein). In such cases, the bounds in Theorem 1 are necessary to construct asymptotic confidence intervals. Moreover, in some prediction problems (e.g. [2]), regularity of the density of $\overline{X}_T - X_T$, established in Theorem 1, is important.

Our approach to this problem is rooted in recent advances in simulation used to build an efficient approximation of the law of $(X_T, \overline{X}_T)$. More precisely, we use the representation of the concave majorant of a stable process, recently applied in [19] to construct a geometrically convergent simulation algorithm for sampling from the law of $(X_T, \overline{X}_T)$. In order to analyse the regularity of this joint law, we express the stable random variables arising in the concave majorant representation of the supremum $\overline{X}_T$ via the classical Chambers-Mallows-Stuck representation. This approach in studying the regularity and upper bounds of the densities of the joint law differs from the probabilistic and analytical techniques applied in the literature so far (see [20] for a presentation of our results and techniques).

In general, it is well-known that the properties of an approximation do not necessarily persist in the limit (see [3] for a comprehensive study in the case of the central limit theorem). In our case, in order to establish regularity and achieve nearly optimal upper bounds of the limit law, we accelerate the convergence of the approximation procedure using ideas behind the multilevel Monte Carlo method. This method has been successfully applied in Monte Carlo estimation (see [15] and the references in [14]) to reduce the computational complexity of the algorithm for a pre-specified level of accuracy. In theoretical terms, we apply the multilevel idea as an interpolation methodology (we have not been able to find multilevel Monte Carlo methods used for this purpose in the literature). Other interpolation techniques applied to stochastic equations are found in [1], see also the references therein. In fact, the authors in [1] use a different interpolation technique to obtain qualitative properties using approximation methods. In the examples they treat, it is hard to tell if they achieve optimal results. In our case, the near optimality is due to the geometrical convergence of the approximation of the joint law based on the concave majorant, see [19].

To the best of our knowledge, only the regularity of the density of the marginals of $(X_T,\overline{X}_T)$ has been considered so far. The first component X_T follows a stable law, which is very well understood (see e.g. [29] and the references therein) and whose density has the following asymptotic behavior

$$\mathbb{P}(X_T\in dx)\overset{x\to\infty}{\approx} Tx^{-\alpha-1}dx;\quad \mathbb{P}(X_T\in dx)\overset{x\to 0}{\approx} T^{-1/\alpha}dx. \tag{1.2}$$

Even though its law has been the focus of a number of papers over the past seven decades (starting with Darling [10], Heyde [21] and Bingham [5]), far less information is available about the density of the second component, $\overline{X}_T$, which is a functional of the path of a stable process. Most of the results about the law of the supremum of a Lévy process rely on the Wiener-Hopf factorisation and/or the equivalence with laws related to excursions of reflected processes [8,9]. For example, in [7], the author obtains explicit formulae for the supremum in the spectrally negative stable and symmetric Cauchy cases. The smoothness of the density of the supremum $\overline{X}_T$ is known, see e.g. [27, Thm 2.4 & Rem. 2.14].

The papers [12,13] study the asymptotic behaviour of the density of the supremum at infinity and at zero. In [13], the authors rely on local times and excursion theory, the Wiener-Hopf factorisation and a distributional connection between stable suprema and stable meanders. Power series expansions of the density of $\overline{X}_T$ have been established in [22,23] in some particular situations. Since stable processes are self-similar and Markov, results in [27] can be used to deduce the asymptotic behaviour of the density (and its derivatives) of $\overline{X}_T$, see the paragraph following Corollary 2 below.

In short, the proofs of the results obtained so far in the literature rely on excursion theory or the Wiener-Hopf factorisation. These methods exploit the independence of $\overline{X}_{\mathrm{e}}$ and $\overline{X}_{\mathrm{e}}-X_{\mathrm{e}}$ over an independent exponential time horizon e. The dependence of all of the above methods on a number of specific analytical identities for the law of $\overline{X}_T$ makes them hard to generalise in order to study the law of $(X_T,\overline{X}_T)$.

A result closer to our study appears in [13,27]:

$$\mathbb{P}(\overline{X}_T\in dy)\overset{y\to\infty}{\approx} Ty^{-\alpha-1}dy;\quad \mathbb{P}(\overline{X}_T\in dy)\overset{y\to 0}{\approx} T^{-\rho}y^{\alpha\rho-1}dy. \tag{1.3}$$

Taking into consideration the asymptotics in (1.2)–(1.3), it is natural that the asymptotics for the law of $(X_T,\overline{X}_T)$ are determined by four sub-domains in the support $O:=\{(x,y)\in\mathbb{R}^2: y>\max\{x,0\}\}$. Our upper bound on the joint density and its derivatives, illustrated in Figure 1 below, is close to optimal in the sense that we obtain such a result for any α' arbitrarily close to α featuring in (1.2)–(1.3). The reason why we are unable to obtain the result for the choice $\alpha'=\alpha$ is technical and due to the use of moments to bound tail behaviours in the spirit of Markov's and Chebyshev's inequalities.

Malliavin calculus is a long developed subject in the area of stochastic analysis of jump processes. The ultimate goal of the general theory is to obtain an infinite dimensional calculus with the view of investigating random quantities generated by the jump process and, in particular, the regularity of the law of path functionals of the process (see e.g. [4,26] for a general reference). Notably, these theoretical developments in Malliavin calculus have fallen short of the problem of the regularity of the density of $\overline{X}_T$, because the supremum of a jump process (as a random variable) appears not to depend smoothly on the underlying jumps. An exception is the result in [6], where the authors rely on the Lipschitz property of the supremum functional to prove the existence of a density for the supremum of a jump process in a general class, using the so-called lent-particle method. However, since $\overline{X}_T$ is not a smooth functional of the path, it is unclear how to apply these methods to analyse the regularity and behavior of the density near the boundary of its support.

The approach used in this article does not fall in any of the above categories of Malliavin Calculus, nor does it rely on any results from Malliavin Calculus of jump processes. More precisely, we do not use infinite dimensional objects but only study limits of finite collections of random variables, arising in the noise used in our representation of the law of $(X_T, \overline{X}_T)$. Our main underlying idea is to exploit the geometrically convergent approximation of the random vector of interest, establish the required properties of the densities for the approximate vectors and prove that these properties persist in the limit. In this sense, our approach is both self-contained and elementary.

More specifically, we establish a probabilistic representation for the joint density of $(X_T, \overline{X}_T)$ and its derivatives in Theorem 7 below, based on a telescoping sum of successive approximations analogous to the multilevel method (cf. [15]). The telescoping sum formula for the density and its derivatives is based on an elementary integration-by-parts formula for successive finite dimensional approximations of $(X_T, \overline{X}_T)$. These approximations do *not* use the path of the stable process $(X_t)_{t\in[0,T]}$ directly as would be the case in Malliavin Calculus for processes with jumps. Instead, the concave majorant of $(X_t)_{t\in[0,T]}$, given in [28, Thm 1], is used to represent $(X_T, \overline{X}_T)$ as an infinite series [18,19]. The terms in this series are the increments of the stable process over macroscopic (but geometrically small) time steps given by an independent stick-breaking process on $[0,T]$ (for more details, see Section 3.1). We then build our elementary finite-dimensional integration-by-parts formulae for the partial sum approximations of $(X_T, \overline{X}_T)$ using the scaling property of stable increments and their Chambers-Mallows-Stuck representation [31], which in the non-Cauchy case $\alpha \neq 1$, amounts to a semi-linear function of independent uniform and exponential variables, Section 3.1.

1.1. Organisation

The remainder of the paper is organised as follows. In Section 2 we present Theorem 1, the main result of the paper, and some applications of these results. Subsection 3.1 introduces the technical notation for the proofs and Subsection 3.2 establishes the integration-by-parts formula (Ibpf). In Section 4, we give the proof of our main result, Theorem 1 and an important technical Proposition 8, which gives all the bounds needed in order to be applied in the Ibpf formula obtained. The proof of our main result uses the interpolation method in the sense that the approximation method based on the convex majorant converges geometrically fast while the density bounds explode polynomially. Combining these two characteristics one obtains the almost optimal bounds.

We close the article with some technical appendices which prove the important technical Proposition 8. The proof of this proposition is composed of algebraic inequalities which are obtained in Subsection 5.2. The upper bounds are products of powers of basic random variables. After the proof we give also a heuristic interpretation of a basic interpolation technique used in the estimation of the moments. Finally, the moment estimates are obtained in Subsection 5.3. Throughout the article we concentrate on the case $\alpha \neq 1$ leaving the special Cauchy case, $\alpha = 1$ as well as estimates on the moments of a stick-breaking process to the supplement in [16].

Section 6 concludes the paper, remarking on our techniques and methodology as well as possible extensions.

2. Main result and applications

Recall the open set $O = \{(x,y) \in \mathbb{R}^2 : y > \max\{x,0\}\}$ defined above. We now state our main result:

Theorem 1. *Assume that $\alpha \in (0,2)$. Let $F(x,y) := \mathbb{P}(X_T \le x, \overline{X}_T \le y)$ be the distribution function of $(X_T,\overline{X}_T)$. The joint density of F exists and is infinitely differentiable on the open set O. Moreover, for any fixed $n,m \ge 1$ and $\alpha' \in [0,\alpha)$ there is some $C > 0$ such that for all $(x,y) \in O$ and $T > 0$, we have*

$$|\partial_x^n \partial_y^m F(x,y)| \le C y^{-m}(y-x)^{1-n-m}(2y-x)^{m-1}\min\{f_{\alpha'}^{00}(x,y), f_{\alpha'}^{01}(x,y), f_{\alpha'}^{10}(x,y), f_{\alpha'}^{11}(x,y)\}, \quad (2.1)$$

where $f_{\alpha'}^{ij}(x,y) := T^{\frac{\alpha'}{\alpha}(i(2-\rho)+j(1+\rho)-1)}(y-x)^{\alpha'(1-\rho)-i\alpha'(2-\rho)}y^{\alpha'\rho-j\alpha'(1+\rho)}$ for $i,j \in \{0,1\}$.

Theorem 1 presents a bound on the mixed derivatives of the joint density of $(X_T,\overline{X}_T)$. The decay of the bound as y tends to either infinity or zero is almost sharp in the following sense: if one sets $n = 1$ and $\alpha' = \alpha$ in (2.1) (cf. Figure 1 below) and integrates out x over $\mathbb{R}$, the decay of the obtained bound matches the actual asymptotic behaviour of the density of $\overline{X}_T$ known from the literature [13,22,23]. That is, marginals of the above bounds match the estimates in (1.2) and (1.3). In fact, the bound in Corollary 2 below is established in this way. The constant C in (2.1) can be made explicit. Instead of giving a formula for C, which would be lengthy and suboptimal (cf. Remark 4(i) below), we point out that $(\alpha - \alpha')C$ remains bounded as $\alpha' \uparrow \alpha$. An alternative way to understand the optimality property is through a change of variables in equation (3.1) which will be proven in Section 4.

Theorem 1 above suggests that the asymptotic behaviour of the joint density at (x,y) of $(X_T,\overline{X}_T)$ as $T \to 0$ is proportional to $T^2(y-x)^{-\alpha-1}y^{-\alpha-1}$, see Figure 1. This is corroborated by the results in [7,9] as we now explain. Recall from [7, Thm 6] that the density of $(\overline{X}_T, \overline{X}_T - X_T)$ satisfies

$$\mathbb{P}(\overline{X}_T \in dx, \overline{X}_T - X_T \in dy) = dx\,dy \int_0^1 q_{sT}^*(x) q_{(1-s)T}(y) T\,ds, \quad (2.2)$$

where q_t^* (resp. q_t) is the entrance density of the excursion measure of the reflected process of X (resp. $-X$). If X has jumps of both signs, then [9, Thm 3.1] and [7, Ex. 3], imply that, as $T \to 0$, the quantities $q_{sT}^*(x)/(T^\rho s^\rho x^{-\alpha-1})$ and $q_{(1-s)T}^*(x)/(T^{1-\rho}s^{1-\rho}y^{-\alpha-1})$ have positive finite limits that depend neither on s nor (x,y). Thus the integral on the right-hand side of (2.2) is proportional to $T^2x^{-\alpha-1}y^{-\alpha-1}$ as predicted the bound in Theorem 1 (see also (3.1) below).

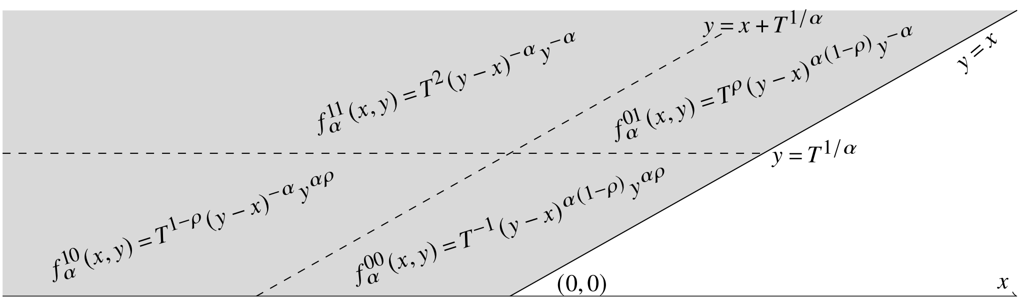

Figure 1. The set $O = \{(x,y) \in \mathbb{R}^2 : y > \max\{x,0\}\}$ (shaded in the figure) is the support of the joint density of $(X_T,\overline{X}_T)$. According to Theorem 1, the support can be partitioned into 4 sub-regions according to which of the functions f_α^{ij}, $i,j \in \{0,1\}$, is the smallest in the (optimal) case $\alpha' = \alpha$.

Setting $n = 1$ and explicitly integrating in y over $(0,\infty)$ yields the following bounds.

Corollary 2. *Let $\alpha \in (0,2)$ and define $\tau_y := \inf\{t > 0 : X_t > y\}$, $y > 0$. Then the distribution functions $F(y) := \mathbb{P}(\overline{X}_T \le y)$ and $\mathbb{P}(\tau_y \le T)$ are infinitely smooth on $(0,\infty)$ and, for every $\alpha' \in [0,\alpha)$ and $n \ge 1$, there exists some constant $C > 0$ such that for all $y > 0$ and $T > 0$, we have*

$$|\partial_y^n F(y)| \le C y^{-n} \min\left\{T^{\frac{\alpha'}{\alpha}} y^{-\alpha'}, T^{-\frac{\alpha'}{\alpha}\rho} y^{\alpha'\rho}\right\}, \quad |\partial_T^n \mathbb{P}(\tau_y \le T)| \le C T^{-\frac{1}{\alpha}-n} \min\left\{T^{\frac{\alpha'}{\alpha}} y^{-\alpha'}, 1\right\}.$$

It has been pointed out to us [30] that the bound in Corollary 2 for $\alpha' = \alpha$ can be obtained from the literature. By studying the Mellin transform of $\overline{X}_T$ [27, Thm 2.4] (via a distributional identity linking $\overline{X}_T$ to an exponential integral arising in the Lamperti representation of self-similar Markov processes [27, Rem. 2.14]), one obtains the asymptotic behaviour in (1.3). Similar bounds can be obtained for the derivatives of the density, implying Corollary 2.

Other consequences of our main Theorem 1 can also be derived. For example, the following result reveals an interplay between the final value X_T and the supremum $\overline{X}_T$.

Corollary 3. *Assume that $\alpha \in (0,2)$ and let $y \ge T^{1/\alpha}$, $x \le 0$. Then for any $\alpha' \in (0,\alpha)$*

$$\mathbb{P}(X_T \le x, \tau_y < T) \le C T^{2\frac{\alpha}{\alpha'}} y^{-\alpha'} \min\{y^{-\alpha'}, (-x)^{-\alpha'}\}$$

Proof. The inequalities are obtained by direct integration of the bound in Theorem 1 as follows:

$$\mathbb{P}(X_T \le x, \overline{X}_T > y) \le C T^{2\frac{\alpha}{\alpha'}} \int_{L/(-x)}^{\infty} w^{-1-\alpha'}(1+w)^{-\alpha'} dw. \qquad \square$$

Our methods apply to the Brownian motion case $\alpha = 2$, but the result does not reveal new information since the density of $(X_T, \overline{X}_T)$ is known explicitly. Furthermore, in (1.1) we exclude boundary cases $\rho \in \{0,1\}$ as in those cases the monotonicity of paths of X implies $\overline{X}_T = X_T$ (resp. $\overline{X}_T = X_0$) a.s. if $\rho = 1$ (resp. $\rho = 0$).

We conclude this section with a remark on potential alternative approaches, based on analytical methods rooted in the Wiener-Hopf factorisation, to the problem of controlling partial derivatives of any order of the distribution function of the vector $(X_T, \overline{X}_T)$.

Remark 1. It is natural to enquire if similar results to those established here can be obtained by analytic means. Let us comment on some of the steps one would have to take to obtain such results. The Wiener-Hopf factorisation gives a representation for the characteristic exponent of $(\overline{X}_{\mathrm{e}}, \overline{X}_{\mathrm{e}} - X_{\mathrm{e}})$, where e is an independent and exponentially distributed random variable. Fourier and Laplace inversions (in space and time, respectively) could be applied to such an expression to describe the law of $(\overline{X}_T, \overline{X}_T - X_T)$. Similar ideas (together with local-time and excursion theories) were used to obtain asymptotic behaviour of the density at infinity and at zero of the marginal law $\overline{X}_T$ [12,13] as well as to derive, for particular choices of parameters α and ρ, double power-series expansions of the density of $\overline{X}_T$ [22,23]. The restrictions on the parameters are such that, in particular if α is rational, then the series expansions converge only for finitely many values of ρ [23], suggesting that obtaining information about the distribution of $\overline{X}_T$ via the Wiener-Hopf factorisation is highly non-trivial.

Alternatively, a representation of $\partial_x F(x,y) = \mathbb{P}(X_T \in dx, \overline{X}_T \le y)$ in the special case $\alpha > 1$ can be obtained by [24, Thm 1] as follows: note that $\partial_x F(x,y) = \mathbb{P}(y - X_T \in y - dx, \inf_{t\in[0,T]}(y - X_t) \ge 0) = \widehat{\mathbb{P}}_y(\widehat{X}_T \in y - dx, \inf_{t\in[0,T]} \widehat{X}_t \ge 0)$, where the law $\widehat{\mathbb{P}}_y(\cdot)$ of the Lévy process $\widehat{X} := y - X$, started at y, is α-stable with positivity parameter $\widehat{\mathbb{P}}_y(\widehat{X}_1 - y \ge 0) = 1 - \rho$. In [24], the authors describe the density

$\widehat{\mathbb{P}}_y(\widehat{X}_T \in y - dx, \inf_{t\in[0,T]} \widehat{X}_t \geq 0)$ in terms of the *double sine function* S_2, defined via the functional equations

$$S_2(z+1;\alpha) = \tfrac{1}{2}S_2(z;\alpha)/\sin(\pi z/\alpha), \quad S_2(z+\alpha;\alpha) = \tfrac{1}{2}S_2(z;\alpha)/\sin(\pi z),$$

with the normalising condition $S_2((1+\alpha)/2;\alpha) = 1$ and the completely monotone function

$$G(x) := \int_0^\infty e^{-zx} z^{(\alpha\rho-1)/2} |S_2(1+\alpha+\alpha(1-\rho)/2 + i\alpha(\log z)/(2\pi);\alpha)|^2 dz.$$

The density $\widehat{\mathbb{P}}_y(\widehat{X}_T \in y - dx, \inf_{t\in[0,T]} \widehat{X}_t \geq 0)$ is an integral of the product of functions

$$\phi(x;\rho) := e^{x\cos(\pi\rho)} \sin(x\sin(\pi\rho) + \pi\rho(1-\alpha(1-\rho))/2) + \frac{\sqrt{\alpha}}{4\pi} S_2(-\alpha(1-\rho))G(x).$$

More precisely, [24, Thm 1] yields the following identity valid when either $\alpha > 1$ or $\rho = 1/2$:

$$\partial_x F(x,y) = \frac{2}{\pi}\int_0^\infty e^{-Tt^\alpha} \phi(yt;1-\rho)\phi((y-x)t;\rho)dt.$$

In order to apply this identity to obtain asymptotic bounds on the derivatives $|\partial_x^n \partial_y^m F(x,y)|$ for arbitrary $n,m \in \mathbb{N}$ when $\alpha > 1$ or $\rho = 1/2$, it would be necessary to employ Leibniz product rule repeatedly and control the sum of the products of the derivatives of ϕ that arise under the integral. This approach, suggested by the referee, appears plausible but evidently nontrivial.

We do not pursue these ideas further in this paper as our aim here is to show the utility of a set of probabilistic techniques different from those commonly used in the literature. Moreover, we stress that, as will become clear from the differential structure introduced in Section 3.2 below, our arguments use differentiation with respect to *a part* of the randomness of the stable laws *only*. In particular, the trigonometric functions appearing in the structure of stable laws are not differentiated. In this sense, our technique is difficult to compare directly with the potential approach described in the present remark.

3. Tools: Approximation method and sequential Ibpf

In this section, we describe the elements and notation used to describe the approximation to $(X_T, \overline{X}_T)$ as well as the Ibpf that this approximation generates.

3.1. Approximation method for $(X_T, \overline{X}_T)$

Throughout the article, we fix $T > 0$ and we will use a decomposition of the random variable X_T based on $X_+ := \overline{X}_T$ which denotes the supremum of $(X_t)_{t\in[0,T]}$ and its reflected process $X_- := X_+ - X_T$. Henceforth, we work with (X_+, X_-), supported on $\mathbb{R}_+^2$, instead of working with $(X_T, \overline{X}_T)$. The $\pm$ notation is useful in order to write dual formulas that are valid for both signs.

In fact, the proof of Theorem 1 studies the equivalent pair (X_+, X_-), instead of $(X_T, \overline{X}_T)$, and shows the following: let $\widetilde{F}(x,y) := \mathbb{P}(X_+ \leq x, X_- \leq y)$, then for any $\alpha' \in [0,\alpha)$ and $n,m \geq 1$ there exists some constant $C > 0$ such that for any $T,x,y > 0$ we have

$$|\partial_x^n \partial_y^m \widetilde{F}(x,y)| \leq C x^{-n} y^{-m} \min\left\{T^{\frac{\alpha'}{\alpha}} x^{-\alpha'}, T^{-\frac{\alpha'}{\alpha}\rho} x^{\alpha'\rho}\right\} \min\left\{T^{\frac{\alpha'}{\alpha}} y^{-\alpha'}, T^{-\frac{\alpha'}{\alpha}(1-\rho)} y^{\alpha'(1-\rho)}\right\}. \quad (3.1)$$

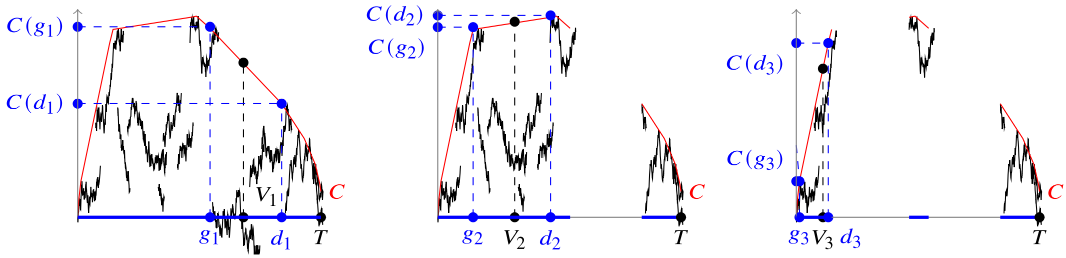

Figure 2. Randomly selecting three faces of the concave majorant C of X (the smallest concave function dominating the path of X) in a size-biased way. The total length of the thick blue segment(s) on the abscissa equal the stick remainders $L_0 = T$, $L_1 = T - \ell_1$ and $L_2 = T - \ell_1 - \ell_2$, respectively, where $\ell_1 = d_1 - g_1$ and $\ell_2 = d_2 - g_2$. The independent random variables V_1, V_2, V_3 are uniform on the sets $[0,T]$, $[0,T] \setminus (g_1, d_1)$, $[0,T] \setminus \bigcup_{i=1}^{2} (g_i, d_i)$, respectively. The interval (g_i, d_i) is the face of C containing V_i. By [19, §4.1], this procedure yields a stick-breaking process ℓ and, conditionally given ℓ, the increments $C(d_i) - C(g_i)$ are independent with the same law as X_t at $t = \ell_i$, i.e., $C(d_i) - C(g_i) \overset{d}{=} \ell_i^{1/\alpha} S_i$.

For this reason, we will use in many formulas multiple $\pm$ and $\mp$ signs. It is assumed that the signs match, i.e., all $\pm$ are $+$ (resp. $-$) and all $\mp$ are $-$ (resp. $+$) simultaneously. For example, $A_\pm = \mp B_\mp$ if and only if $A_+ = -B_-$ and $A_- = +B_+$. Additionally, we use the notation $[x]^+ = \max\{x, 0\}$ and $[x]^- = \max\{-x, 0\}$. We stress that if the brackets are not present, then the notation refers to a different object. For example, $X_{\pm,n}$ denote the approximations for $X_\pm$ respectively and $\mathcal{D}_n^\pm$ are the associated derivative operators to be defined below. Finally, we denote $x \wedge y = \min\{x, y\}$ and $x \vee y = \max\{x, y\}$.

We will use an approximation method for the pair $(X_T, \overline{X}_T)$ used in [19, §4.1] (see also [18, Eq. (2.2)], [28, Thm 1] and [17, Thm 2.1]) which is based on the concave majorant of X, see Figure 2.

The procedure constructs a random sequence of disjoint sub-intervals of the time interval $[0,T]$ that cover it geometrically fast. This is called a stick-breaking process: $\ell = (\ell_i)_{i \ge 1}$ on the interval $[0,T]$. That is, based on the i.i.d. uniform random variables $U_i \sim \mathrm{U}(0,1)$, define $L_0 := T$ and for each $i \in \mathbb{N}$, $L_i := L_{i-1} U_k$ and $\ell_i = L_{i-1} - L_i = L_{i-1}(1 - U_i) = T(1 - U_i) \prod_{j=1}^{i-1} U_j$. It is easy to see that $\sum_{i=1}^\infty \ell_i = T$ and $\mathbb{E}[\ell_i^p] = T^p (1+p)^{-i}$ for any $p > 0$. That is, the convergence of the total length of the sequence of disjoint intervals $\bigcup_{j=1}^{i} [L_{j-1}, L_j]$ to T is geometrically fast.

Consider an independent i.i.d. sequence of stable random variables $(S_i)_{i \ge 1}$ with parameters (α, ρ) (i.e. $S_i \overset{d}{=} X_1$). When $\alpha \ne 1$, the Chambers-Mallows-Stuck representation of these stable random variables (see [31]) is $S_i = E_i^{1-1/\alpha} G_i$ and $G_i = g(V_i)$, $i \in \mathbb{N}$, for i.i.d. exponential random variables $(E_i)_{i \ge 1}$ with unit mean independent of the i.i.d. $\mathrm{U}(-\frac{\pi}{2}, \frac{\pi}{2})$ random variables $(V_i)_{i \ge 1}$ and function

$$g(x) := \frac{\sin\big(\alpha(x+\omega)\big)}{\cos^{1/\alpha}(x) \cos^{1-1/\alpha}\big((1-\alpha)x - \alpha\omega\big)}, \qquad x \in \Big(-\frac{\pi}{2}, \frac{\pi}{2}\Big), \tag{3.2}$$

where $\omega := \pi(\rho - \frac{1}{2})$. Note that indeed $\mathbb{P}(S_i > 0) = \rho$. We assume that all the above random variables are defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. These random elements and the coupling in [19, §4.1] provide an almost sure representation for $(X_T, \overline{X}_T)$:

$$\overline{X}_T = X_+ \qquad \text{and} \qquad X_T = X_+ - X_-, \qquad \text{where} \qquad X_\pm := \sum_{i=1}^\infty \ell_i^{1/\alpha} [S_i]^\pm. \tag{3.3}$$

The series in the definitions of X_+ and X_- have non-negative terms and converge almost surely by the equalities in (3.3). Note again, that the convergence in the above infinite sum is "geometrically fast".

As stated in the Introduction, we base our finite dimensional integration by parts formulas using only the exponential random variables E_i. In order to build approximations of the above random variables on finite dimensional spaces with smooth laws, we truncate the infinite sums up to the n-th term. With this in mind and in order to preserve the existence of densities, we replace the remainder with $a_n\eta_\pm$ as follows: let $(a_n)_{n\in\mathbb{N}}$ be a positive and strictly decreasing sequence defined as $a_n := T^{1/\alpha}\kappa^n$ with $\kappa \in (0,1)$. Therefore $a_n \downarrow 0$ as $n \to \infty$. The random variables $\eta_\pm$ are exponentially distributed with unit mean independent of each other and of every other random variable. With these elements we define the n-th approximation to $\chi = (X_+, X_-)$ as $\chi_n = (X_{+,n}, X_{-,n})$, $n \in \mathbb{N}$ given by (with convention $X_{\pm,0} := 0$)

$$X_{\pm,n} := \sum_{i=1}^{n} \ell_i^{1/\alpha}[S_i]^\pm + a_n\eta_\pm^{1-1/\alpha} = \sum_{i=1}^{n} \ell_i^{1/\alpha} E_i^{1-1/\alpha}[G_i]^\pm + a_n\eta_\pm^{1-1/\alpha}. \tag{3.4}$$

We introduce the following assumption, valid throughout the paper, and crucial to obtain good positive and negative moment estimates for $X_{\pm,n}$ (see Lemma 11 below).

Assumption (A-κ). The constant $\kappa \in (0,1)$ in $a_n = T^{1/\alpha}\kappa^n$ satisfies $\kappa^\alpha \geq \rho \vee (1-\rho)$.

For any $m \in \mathbb{N}$, $n \in \mathbb{N} \cup \{\infty\}$ and $A \subset \mathbb{R}^m$, let $C_b^n(A)$ be the set of bounded and n-times continuously differentiable functions $f : \mathbb{R}^m \to \mathbb{R}$ on the open set A and whose derivatives of order at most n are all bounded. Furthermore for $f \in C_b^1(\mathbb{R}^2)$ we denote the partial derivatives with respect to the first and second component by $\partial_+ f$ and $\partial_- f$, respectively.

3.2. Sequential integration by parts formulae via a multilevel method

In order to state the finite dimensional Ibpf, we will use a derivative operator notation with respect to a set of random variables. Thus, for any random variable $F = f(\vartheta, \mathcal{K})$, where f is differentiable in the first component and the random variable ϑ is independent of the random element $\mathcal{K}$, the derivative $\partial_\vartheta[\cdot]$ is well-defined and given by $\partial_\vartheta[F] = \partial_\vartheta f(\vartheta, \mathcal{K})$. Recall that the random variables $\{E_i, U_i, V_i, \eta_\pm; i \in \mathbb{N}\}$ are independent (i.e. the joint law is a product measure), making the derivatives in the following lemma well-defined.

Lemma 4. *For any $m \in \mathbb{N}$, define the differential operators*

$$\mathcal{D}_m^\pm := \eta_\pm \partial_{\eta_\pm} + \sum_{i=1}^{m} E_i \mathbb{1}_{\{[G_i]^\pm > 0\}} \partial_{E_i}. \tag{3.5}$$

Then for any function $f : \mathbb{R}_+ \to \mathbb{R}_+$ and $p \in \mathbb{R} \setminus \{0\}$ we have

$$\begin{aligned} E_i \partial_{E_i}[X_{\pm,n}] &= (1-1/\alpha)\ell_i^{1/\alpha} E_i^{1-1/\alpha}[G_i]^\pm \mathbb{1}_{\{i \leq n\}}, & & k \in \mathbb{N}, \\ \mathcal{D}_m^\pm X_{\pm,n}^p &= (1-1/\alpha) p X_{\pm,n}^p, \qquad \mathcal{D}_m^\pm\big[f(X_{\mp,n})\big] = 0, & & m \geq n \geq 1. \end{aligned} \tag{3.6}$$

Proof. The first two identities follow easily. For the third identity, note that $X_{\pm,n} > 0$ a.s. and thus, its reciprocal and any of its powers are always well defined real numbers. The other identities follow from the first one and the corresponding formula for $\eta_\pm \partial_{\eta_\pm}[X_{\pm,n}]$. ☐

Remark 2. (i) The identity $\mathcal{D}_m^{\pm} X_{\pm,n}^p = (1-1/\alpha)pX_{\pm,n}^p$, $m \ge n \ge 1$, in (3.6) reveals a regenerative property of $X_{\pm,n}$ with respect to the operator $\mathcal{D}_m^{\pm}$ (like the fact that in classical calculus the derivative of the exponential function is itself). In fact, this is the main motivation behind the definition of $\mathcal{D}_m^{\pm}$. This regenerative structure relies on the particular dependence of $X_{\pm,n}$ on S_i and E_i, $i \in \{1,\dots,n\}$.
(ii) The indicators $1_{\{[G_i]^{\pm}>0\}}$ in the definition of $\mathcal{D}_m^{\pm}$ ensure that when applied to $f(\chi_n)$, only one of the partial derivatives of f appears due to (3.6) (see (3.8) below).

Now, we introduce the space of smooth random variables. Given any metric space S, define the space of real-valued bounded and continuous functions on $(0,\infty)^m \times S$ that are C_b^{∞} in its first m components

$$\mathbb{S}_{\infty}((0,\infty)^m, S) := \Big\{\phi : (0,\infty)^m \times S \to \mathbb{R};\ \ \phi \text{ is continuous},\ \ \phi(\cdot,s) \in C_b^{\infty}((0,\infty)^m;\mathbb{R}),\ \forall s \in S\Big\}.$$

Define $\mathbb{S}_m(\Omega) := \{\Phi \in L^0(\Omega) : \exists \phi(\cdot,\vartheta) \in \mathbb{S}_{\infty}((0,\infty)^{3m+2}, S) \text{ and } \Phi = \phi(\mathcal{E}_m, \mathcal{U}_m, \mathcal{V}_m, \eta_+, \eta_-, \vartheta)\}$, where $\mathcal{E}_m := (E_1,\dots,E_m)$, $\mathcal{U}_m := (U_1,\dots,U_m)$, $\mathcal{V}_m := (V_1,\dots,V_m)$ and ϑ is any random element in some metric space S independent of $(\mathcal{E}_m, \mathcal{U}_m, \mathcal{V}_m, \eta_+, \eta_-)$. For instance, if the random variable Φ is a function of $(\mathcal{E}_\infty, \mathcal{U}_\infty, \mathcal{V}_\infty)$, we say that $\Phi \in \mathbb{S}_m(\Omega)$ if the property defining this set is satisfied with $\vartheta = ((E_{m+1}, E_{m+2}, \dots), (U_{m+1}, U_{m+2}, \dots), (V_{m+1}, V_{m+2}, \dots))$ representing all the random variables with indices larger than m. We describe now the following finite dimensional Ibpf for a fixed approximation parameter n. Recall that $\chi_n = (X_{+,n}, X_{-,n})$ for $n \in \mathbb{N}$.

Proposition 5. *Fix $n, m \in \mathbb{N}$ with $m \ge n$. Then for any $\Phi \in \mathbb{S}_m(\Omega)$ and $f \in C_b^1((\varepsilon,\infty)^2)$,*

$$\begin{aligned} &\mathbb{E}[\partial_{\pm} f(\chi_n)\Phi] = \mathbb{E}[f(\chi_n) H_{n,m}^{\pm}(\Phi)], \quad \textit{where} \\ &H_{n,m}^{\pm}(\Phi) := \frac{1}{X_{\pm,n}} \frac{\alpha}{\alpha-1}\Big(\Big(\eta_{\pm} - \frac{1}{\alpha} + \sum_{i=1}^{m}(E_i - 1)\mathbb{1}_{\{[G_i]^{\pm}>0\}}\Big)\Phi - \mathcal{D}_m^{\pm}[\Phi]\Big) \in \mathbb{S}_m(\Omega). \end{aligned} \tag{3.7}$$

Proof. Note that $[x]^{\pm} > 0$ if and only if $\pm x > 0$. The chain rule for derivatives and (3.6) yield

$$\mathcal{D}_m^{\pm}[f(\chi_n)] = \partial_{\pm} f(\chi_n) \mathcal{D}_m^{\pm}[X_{\pm,n}] = (1 - 1/\alpha)\partial_{\pm} f(\chi_n) X_{\pm,n}. \tag{3.8}$$

Denote $\widetilde{\partial}_{\vartheta}[Y] := Y - \partial_{\vartheta}[Y]$. Let η be an exponential random variable with unit mean. If $\Lambda_i := h_i(\eta)$ for some $h_i \in \mathbb{S}_{\infty}((0,\infty);\mathbb{R})$, $i \in \{1,2\}$, then the classical Ibpf (with respect to the density of η) gives

$$\mathbb{E}[\Lambda_1 \eta \partial_{\eta}[\Lambda_2]] = \mathbb{E}[\partial_{\eta}[\Lambda_1\Lambda_2\eta] - \Lambda_2 \partial_{\eta}[\Lambda_1\eta]] = \mathbb{E}[\Lambda_1\Lambda_2\eta - \Lambda_2\partial_{\eta}[\Lambda_1\eta]] = \mathbb{E}[\Lambda_2 \widetilde{\partial}_{\eta}[\Lambda_1\eta]]. \tag{3.9}$$

Integration by parts with respect to $\eta_{\pm}$ and E_k for each $i \le n$ gives, by (3.6), (3.8) and (3.9),

$$\begin{aligned} &\mathbb{E}[\partial_{\pm} f(\chi_n)\Phi | \mathcal{F}_{-E}] = \frac{\alpha}{\alpha-1}\mathbb{E}\left[\frac{\Phi}{X_{\pm,n}} \mathcal{D}_m^{\pm}[f(\chi_n)] \middle| \mathcal{F}_{-E}\right] \\ &= \frac{\alpha}{\alpha-1}\mathbb{E}\left[f(\chi_n)\left(\widetilde{\partial}_{\eta_{\pm}}\left[\frac{\Phi\eta_{\pm}}{X_{\pm,n}}\right] + \sum_{i=1}^{m} \widetilde{\partial}_{E_i}\left[\frac{\Phi E_i \mathbb{1}_{\{[G_i]^{\pm}>0\}}}{X_{\pm,n}}\right]\right)\middle| \mathcal{F}_{-E}\right] = \mathbb{E}[f(\chi_n) H_{n,m}^{\pm}(\Phi) | \mathcal{F}_{-E}]. \end{aligned} \tag{3.10}$$

Above we denoted by $\mathcal{F}_{-E}$ the σ-algebra generated by all but the random variables η_+, η_- and E_i, $i \in \mathbb{N}$ which are used in the integration-by-parts. Taking expectations in (3.10) completes the proof. □

Remark 3. (i) Observe that the role of ε in the previous result is to ensure that the expectation on the right-hand side in (3.7) is finite (by making the quotient $f(\chi_n)/X_{\pm,n}$ bounded).

(ii) Recall that exponential laws are discontinuous at zero. Still, in the above Ibpf, these boundary terms do no appear. This is due to the factors $E_i\partial_{E_i}$ and $\eta_\pm\partial_{\eta_\pm}$ which appear in the definition of $\mathcal{D}_m^\pm$ in (3.5). In exchange, one has $X_{\pm,n}$ in the denominator of the expression for $H_{n,m}^\pm(\Phi)$.

As $H_{n,m}^\pm(\Phi)\in\mathbb{S}_m(\Omega)$ for any $\Phi\in\mathbb{S}_m(\Omega)$, $m\ge n$, we inductively define the sequence of operators $\{H_{n,m}^{\pm,k}(\cdot)\}_{k\in\mathbb{N}}$ for every $n,m\in\mathbb{N}$ such that $m\ge n$ as

$$H_{n,m}^{\pm,k+1}(\Phi):=H_{n,m}^\pm(H_{n,m}^{\pm,k}(\Phi))\qquad \text{for } k\ge 0, \text{ where}\qquad H_{n,m}^{\pm,0}(\Phi):=\Phi.$$

Let us state some basic properties of the weights $H_{n,m}^\pm(\Phi)$.

Lemma 6. *The operators $H_{n,m}^{\pm,k}(\cdot)$ and $H_{n,m}^{\mp,j}(\cdot)$ commute. Moreover, if $\alpha\ne 1$ and Φ does not depend on $\mathcal{E}_m$ or $\eta_\pm$, then $\mathcal{D}_m^\pm[\Phi]=0$ and hence $H_{n,m}^\pm(\Phi)=H_{n,m}^\pm(1)\Phi$.*

These iterated operators are useful in order to define the multiple Ibpf formulas for the limit random variables in combination with the so-called Multi level Monte Carlo method which can be interpreted as an interpolation formula which uses approximations in order to describe the behavior of the limit. This is done in the next result.

Theorem 7. *Let $\Phi\in\mathbb{S}_n(\Omega)$ for all $n\in\mathbb{N}$. For any $n\ge 1$, $k_+,k_-\ge 0$ and $f\in C_b^{k_++k_-}([\varepsilon,\infty)^2)$ we have*

$$\mathbb{E}\left[\partial_+^{k_+}\partial_-^{k_-}f(\chi)\Phi\right]=\mathbb{E}\left[\langle f,\Phi\rangle_n^{k_+,k_-}\right] \tag{3.11}$$

$$\begin{aligned}\langle f,\Phi\rangle_n^{k_+,k_-}:=&f(\chi_n)H_{n,n}^{+,k_+}\left(H_{n,n}^{-,k_-}(\Phi)\right)\\&+\sum_{i=n}^{\infty}\left(f(\chi_{i+1})H_{i+1,i+1}^{+,k_+}\left(H_{i+1,i+1}^{-,k_-}(\Phi)\right)-f(\chi_i)H_{i,i+1}^{+,k_+}\left(H_{i,i+1}^{-,k_-}(\Phi)\right)\right).\end{aligned} \tag{3.12}$$

Proof. Note that $\mathbb{E}[\widetilde{f}(\chi_n)]\to\mathbb{E}[\widetilde{f}(\chi)]$ as $n\to\infty$ for any bounded and continuous function $\widetilde{f}$ since $\chi_n\to\chi$ a.s. Recall that $\partial_+^{k_+}\partial_-^{k_-}f$ is continuous and bounded. By telescoping we find

$$\mathbb{E}[\partial_+^{k_+}\partial_-^{k_-}f(\chi)\Phi]=\mathbb{E}[\partial_+^{k_+}\partial_-^{k_-}f(\chi_n)\Phi]+\mathbb{E}\left[\sum_{i=n}^{\infty}(\partial_+^{k_+}\partial_-^{k_-}f(\chi_{i+1})-\partial_+^{k_+}\partial_-^{k_-}f(\chi_i))\Phi\right].$$

The first term equals $\mathbb{E}[f(\chi_n)H_{n,n}^{+,k_+}(H_{n,n}^{-,k_-}(\Phi))]$ by Proposition 5. Applying Proposition 5 again shows that each term in the above sum equals its corresponding term in (3.12), yielding (3.11). □

It is clear that iterations of $H_{n,m}^\pm$ have long and complex explicit expressions. In particular, the remaining goal is to find proper bounds for the iterated operators which appear in formula (3.11).

In order to complete the arguments for our main proofs we will need that the infinite sum appearing in (3.11) converges absolutely. Furthermore, bounding this sum becomes important in obtaining upper bounds for the joint density and its derivatives. This is all done at once in the next proposition. Its proof is technical but only uses basic algebra and moments of the random variables in involved in Section 3.1.

We are interested in the explicit decay rate of the terms in the sum of Theorem 7 for a special class of functions f related to the distribution of χ. This description will then be used to finally prove Theorem 1. More precisely, given some measurable and bounded $h:\mathbb{R}_+^2\to\mathbb{R}$ and $x_+,x_->0$, we will consider the function f given by

$$f(x,y):=\int_0^x\int_0^y h(x',y')\mathbb{1}_{\{x'>x_+,y'>x_-\}}dy'dx',\qquad x,y\in\mathbb{R}_+. \tag{3.13}$$

We are interested in such class of functions since the particular choice $h=1$ yields $\mathbb{E}[\partial_+\partial_- f(\chi)]=\mathbb{P}(X_+>x_+,X_->x_-)$. Note also that for a general h, the inequality $|f(x,y)|\le\|h\|_\infty xy$ holds for any $x,y\in\mathbb{R}_+$, where $\|h\|_\infty:=\sup_{x,y\in\mathbb{R}_+}|h(x,y)|$. We denote by $\mathcal{A}(K,x_+,x_-)$, $K>0$, the class of functions f satisfying (3.13) for some measurable function $h:\mathbb{R}_+^2\to\mathbb{R}$ with $\|h\|_\infty\le K$.

We denote the random variables arising in $\langle f,\Phi\rangle_n^{k_+,k_-}$ of Theorem 7 by

$$\begin{aligned}\Theta^f_{n,m}&\equiv\Theta^{f,\Phi}_{n,m}(k_+,k_-):=f(\chi_n)H^{+,k_+}_{n,m}\big(H^{-,k_-}_{n,m}(\Phi)\big),\quad\text{for } m\ge n,\text{ and}\\ \widetilde{\Theta}^f_n&\equiv\widetilde{\Theta}^{f,\Phi}_n(k_+,k_-):=\Theta^f_{n+1,n+1}(k_+,k_-)-\Theta^f_{n,n+1}(k_+,k_-).\end{aligned} \tag{3.14}$$

We will drop Φ and or (k_+,k_-) from the notation if it is well understood from the context. The following key result provides bounds on moments.

Proposition 8. *Let $\kappa\in(0,1)$ be as in Assumption (A-κ). Fix any $p\ge1$, $k_\pm\ge2$ and $\alpha'\in[0,\alpha)$. Given some $\phi\in C_b^{k_++k_-}(\mathbb{R}^2)$, define $\Phi:=\phi(\chi)$. Let $\Theta^f_{n,m}$ and $\widetilde{\Theta}^f_n$ be given by (3.14), then the following hold.*
(a) *For $s:=p\wedge\alpha'$ there is a constant $C>0$ such that for any $K,T,x_+,x_->0$ and $m\ge n$:*

$$\mathbb{E}\Big[\sup_{f\in\mathcal{A}(K,x_+,x_-)}|\widetilde{\Theta}^f_n|^p\Big]\le CK^p\frac{T^{2\frac{\alpha'}{\alpha}}\big((1+\frac{s}{\alpha})^{-n}+\kappa^{ns}\big)n^{p'}}{x_+^{p(k_+-1)+\alpha'}x_-^{p(k_--1)+\alpha'}}, \tag{3.15}$$

$$\mathbb{E}\Big[\sup_{f\in\mathcal{A}(K,x_+,x_-)}|\Theta^f_{n,m}|^p\Big]\le CK^p\frac{T^{2\frac{\alpha'}{\alpha}}m^{p'}}{x_+^{p(k_+-1)+\alpha'}x_-^{p(k_--1)+\alpha'}}, \tag{3.16}$$

where $p'=p(k_++k_-)+[\alpha'-1]^++[\alpha'-s-1]^+$.
(b) *Consider any $u\in(0,(\alpha-\alpha')(\rho\wedge(1-\rho))/p)$ and let $p'=p(k_++k_-)$, then for some $C>0$ and all $K,T,x_+,x_->0$ and $m\ge n$, the following inequalities hold*

$$\mathbb{E}\Big[\sup_{f\in\mathcal{A}(K,x_+,x_-)}|\widetilde{\Theta}^f_n|^p\Big]\le CK^p\frac{T^{-\frac{\alpha'}{\alpha}}\big((1+\frac{pu}{\alpha})^{-n}+\kappa^{npu}\big)n^{p'}}{x_+^{p(k_+-1)-\alpha'\rho}x_-^{p(k_--1)-\alpha'(1-\rho)}}, \tag{3.17}$$

$$\mathbb{E}\Big[\sup_{f\in\mathcal{A}(K,x_+,x_-)}|\Theta^f_{n,m}|^p\Big]\le CK^p\frac{T^{-\frac{\alpha'}{\alpha}}m^{p'}}{x_+^{p(k_+-1)-\alpha'\rho}x_-^{p(k_--1)-\alpha'(1-\rho)}}. \tag{3.18}$$

(c) *Consider any $u\in(0,(\alpha-\alpha')(\rho\wedge(1-\rho))/p)$ and let $p'=p(k_++k_-)$, then for some $C>0$ and all $K,T,x_+,x_->0$ and $m\ge n$, the following inequalities hold*

$$\mathbb{E}\Big[\sup_{f\in\mathcal{A}(K,x_+,x_-)}|\widetilde{\Theta}^f_n|^p\Big]\le CK^p\frac{\big((1+\frac{pu}{\alpha})^{-n}+\kappa^{npu}\big)n^{p'}}{x_+^{p(k_+-1)}x_-^{p(k_--1)}}\min\Big\{\frac{T^{\frac{\alpha'}{\alpha}(1-\rho)}}{x_+^{-\alpha'\rho}x_-^{\alpha'}},\frac{T^{\frac{\alpha'}{\alpha}\rho}}{x_+^{\alpha'}x_-^{-\alpha'(1-\rho)}}\Big\}, \tag{3.19}$$

$$\mathbb{E}\Big[\sup_{f\in\mathcal{A}(K,x_+,x_-)}|\Theta^f_{n,m}|^p\Big]\le CK^p\frac{m^{p'}}{x_+^{p(k_+-1)}x_-^{p(k_--1)}}\min\left\{\frac{T^{\frac{\alpha'}{\alpha}(1-\rho)}}{x_+^{-\alpha'\rho}x_-^{\alpha'}},\frac{T^{\frac{\alpha'}{\alpha}\rho}}{x_+^{\alpha'}x_-^{-\alpha'(1-\rho)}}\right\}. \tag{3.20}$$

Remark 4. (i) Clearly the above inequalities imply the absolute convergence of the sum in (3.11).

(ii) We will use part (a) when x_+ and x_- both take large values, part (b) when they are both small and part (c) for the mixed case in which x_+ is small and x_- is large or vice versa, cf. Figure 1.

The proof of this key technical result is given in Subsections 5.2 and 5.3. With these preparations, now we are ready to give the proof of our main result.

4. Proof of Theorem 1

In the present subsection we will prove Theorem 1. We will follow the structure presented in the proof of Theorem 2.1.4 in [25]. In fact, consider a test function $f\in C_b^\infty(\mathbb{R}^2)$ then a similar representation as (3.13) gives for $F(x,y):=[x-x_+]^+[y-x_-]^+$

$$f(x_+,x_-)=\int_{\mathbb{R}^2_+}F(x',y')\partial_+\partial_-f(x',y')dy'dx'.$$

Next, using Theorem 7 and Fubini theorem with $\hat F(x',y')=[x'-X_+]^+[y'-X_-]^+$, we obtain

$$\mathbb{E}[\partial_+\partial_-f(\chi)]=\mathbb{E}\left[\langle f,1\rangle_n^{1,1}\right]=\int_{\mathbb{R}^2_+}\partial_+\partial_-f(x',y')\mathbb{E}\left[\langle\hat F(x',y'),1\rangle_n^{1,1}\right]dy'dx'.$$

This readily implies that the density of χ at (x',y') exists and can be expressed as $\mathbb{E}\big[\langle\hat F(x',y'),1\rangle_n^{1,1}\big]$.

In a similar fashion, one considers for $k_+,k_-\ge 1$

$$\mathbb{E}\left[\partial_+^{k_+}\partial_-^{k_-}f(\chi)\right]=\mathbb{E}\left[\langle f,1\rangle_n^{k_+,k_-}\right]=\int_{\mathbb{R}^2_+}\partial_+\partial_-f(x',y')\mathbb{E}\left[\langle\hat F(x',y'),1\rangle_n^{k_+,k_-}\right]dy'dx'.$$

From here, one obtains the regularity of the law of χ. The next step, is to obtain the upper bound for $\mathbb{E}\left[\langle\hat F(x_+,x_-),1\rangle_n^{k_+,k_-}\right]$. That is, our goal is to prove

$$\begin{aligned}\left|\mathbb{E}\big[\langle\hat F(x_+,x_-),1\rangle_n^{k_+,k_-}\big]\right|&\le Cx_+^{-k_-}x_-^{-k_+}\\&\times\min\big\{T^{2\frac{\alpha'}{\alpha}}x_+^{-\alpha'}x_-^{-\alpha'},T^{\frac{\alpha'}{\alpha}\rho}x_+^{-\alpha'}x_-^{\alpha'(1-\rho)},T^{\frac{\alpha'}{\alpha}(1-\rho)}x_+^{\alpha'\rho}x_-^{-\alpha'},T^{-\frac{\alpha'}{\alpha}}x_+^{\alpha'\rho}x_-^{\alpha'(1-\rho)}\big\}.\end{aligned} \tag{4.1}$$

In fact, the above bounds follow from Proposition 8 (a)–(c) (with $p=1$). We use part (a) when x_+ and x_- both take large values, part (b) when they are both small and part (c) for the mixed case in which x_+ is small and x_- is large or vice versa, cf. Figure 1. Each application of Proposition 8 yields summable upper bounds on the summands of the series defined by $\langle\hat F(x_+,x_-),1\rangle_n^{k_+,k_-}$. The minimum in (4.1) is the smallest sum of these upper bounds as a function of (x_+,x_-) and T.

Observe that the derivatives of F in Theorem 1 can be expressed in terms of the derivatives of $G(x_+,x_-):=\mathbb{P}(X_+>x_+,X_->x_-)$ as follows: the linear transformation $(X_T,\overline{X}_T)\mapsto(\overline{X}_T,\overline{X}_T-X_T)$

yields $\partial_x\partial_y F(x,y)=\partial_+\partial_- G(y,y-x)$ for $y>x\vee 0$. Therefore, (4.1) gives (2.1) as follows:

$$
\begin{aligned}
|\partial_x^n\partial_y^m F(x,y)| &= \left|\sum_{i=0}^{m-1}\binom{m-1}{i}\partial_+^{m-i}\partial_-^{n+i}G(y,y-x)\right| \le \sum_{i=0}^{m-1}\binom{m-1}{i}|\partial_+^{m-i}\partial_-^{n+i}G(y,y-x)| \\
&\le \sum_{i=0}^{m-1}\binom{m-1}{i} Cy^{i-m}(y-x)^{-n-i}\min\{f_{\alpha'}^{00}(x,y),f_{\alpha'}^{01}(x,y),f_{\alpha'}^{10}(x,y),f_{\alpha'}^{11}(x,y)\} \\
&= Cy^{-m}(y-x)^{1-n-m}(2y-x)^{m-1}\min\{f_{\alpha'}^{00}(x,y),f_{\alpha'}^{01}(x,y),f_{\alpha'}^{10}(x,y),f_{\alpha'}^{11}(x,y)\}. \quad \square
\end{aligned}
$$

5. Technical results

In this section, we study the upper bounds in the technical Proposition 8. It is the key result in order to obtain Theorem 1.

5.1. Upper bounds on the Ibpf

We start with some basic properties for the operator H which are useful for bounding $\Theta_{n,m}^f$ and $\widetilde{\Theta}_n^f$. For any $m\in\mathbb{N}$, define

$$
\Sigma_m^{\pm} := \eta_\pm + \sum_{i=1}^m E_i \mathbb{1}_{\{[G_i]^\pm>0\}} \qquad \text{and} \qquad \sigma_m^{\pm} := 1+\sum_{i=1}^m \mathbb{1}_{\{[G_i]^\pm>0\}}.
$$

This notation allows us to simplify (3.7). In fact, for any $\Phi\in\mathbb{S}_m(\Omega)$, we have

$$
H_{n,m}^{\pm}(\Phi) = \frac{\alpha/(\alpha-1)}{X_{\pm,n}}\left(\left(\Sigma_m^\pm - \sigma_m^\pm + 1 - \frac{1}{\alpha}\right)\Phi - \mathcal{D}_m^\pm[\Phi]\right), \quad \mathcal{D}_m^\pm[\Sigma_m^\pm]=\Sigma_m^\pm, \quad \mathcal{D}_m^\pm[\sigma_m^\pm]=0. \tag{5.1}
$$

Lemma 9. *Fix any $k_\pm\ge 0$ and suppose $\Phi:=\phi(\chi)$ for some $\phi\in C_b^{k_++k_-}(A)$ with $A\subset\mathbb{R}_+^2$. Then for any $m>n$, we have*

$$
H_{n,m}^{+,k_+}(H_{n,m}^{-,k_-}(\Phi))X_{+,n}^{k_+}X_{-,n}^{k_-} = H_{n+1,m}^{+,k_+}(H_{n+1,m}^{-,k_-}(\Phi))X_{+,n+1}^{k_+}X_{-,n+1}^{k_-}. \tag{5.2}
$$

Moreover, if we set

$$
Z_m := \Sigma_{+,m}+\Sigma_{-,m} = \eta_+ + \eta_- + \sum_{i=1}^m E_i, \quad m\in\mathbb{N}, \tag{5.3}
$$

then there is a bivariate polynomial $p_{k_+,k_-}^\phi(\cdot,\cdot)$ of degree at most k_++k_- whose coefficients do not depend on n or m, such that

$$
|H_{n,m}^{+,k_+}(H_{n,m}^{-,k_-}(\Phi))X_{+,n}^{k_+}X_{-,n}^{k_-}| \le \mathbb{1}_{\{\chi\in A\}}p_{k_+,k_-}^\phi(Z_m,m), \qquad \textit{for all } m\ge n. \tag{5.4}
$$

Proof. The proof is simple: we only need to expand the formula for $H_{n,m}^{+,k_+}(H_{n,m}^{-,k_-}(\Phi))$ and then uniformly bound all the derivatives of ϕ by the same constant.

Recalling that $\mathcal{D}_m^\pm[(\Sigma_m^\pm, X_{\pm,n}^{-p})] = (\Sigma_m^\pm, (1/\alpha - 1)pX_{\pm,n}^{-p})$ and $\mathcal{D}_m^\mp[(\Sigma_m^\pm, \sigma_m^\pm, \sigma_m^\mp, X_{\pm,n}^{-p})] = 0$ for $p > 0$, we deduce that an iteration of (5.1) yields $X_{+,n}^{-k_+} X_{-,n}^{-k_-}$ multiplied by a polynomial of degree k_+ in Σ_m^+ and σ_m^+. Its coefficients are themselves polynomials of degree k_- in Σ_m^- and σ_m^- multiplied by a linear combination of the derivatives $\partial_+^{j_+}\partial_-^{j_-}\phi(\chi)$ for $j_\pm \le k_\pm$. This directly implies (5.2). Since those derivatives are bounded and we have the a.s. bounds $\Sigma_m^\pm \le Z_m$ and $\sigma_m^\pm \le m$, we may bound the entire expression by a constant (independent of n and m) multiplied by a polynomial of degree $k_+ + k_-$ in Z_m and m. This completes the proof in this case. □

5.2. Proof of Proposition 8, Part I: Interpolation inequalities

As we stated previously the proof of the technical Proposition 8 is self-contained and it is divided in two parts. In a first part, we mainly use basic inequalities which will depend on powers of $X_{\pm,n}$, Z_m, ℓ_n, $\eta_\pm$ and $\Delta_{\pm,n} := X_{\pm,n} - X_{\pm,n-1}$. Taking expectations on these inequalities will bring us to consider moments properties which are studied later in Subsection 5.3. We assume those results and give the proof of this Proposition here.

Proof of Proposition 8. In the estimates that follow, we make repeated use of the following inequality

$$\left|\sum_{i=1}^k x_i\right|^q \le k^{[q-1]^+} \sum_{i=1}^k |x_i|^q, \qquad \text{for any } q > 0 \text{ and } x_i \in \mathbb{R}, \tag{5.5}$$

which follows from the concavity of $x \mapsto x^q$ if $q \le 1$ and Jensen's inequality if $q > 1$. Moreover, we frequently apply the following basic interpolating inequalities: $\mathbb{1}_{\{y>x\}} \le y^\nu x^{-\nu}$ for all $\nu \ge 0$ where we interpret the upper bound as 1 if $\nu = 0$. Also, if $y, z \ge 0$ then for all $r \in [0,1]$

$$y \wedge z \le y^r z^{1-r} \quad \text{and} \quad y \vee z \ge y^r z^{1-r}. \tag{5.6}$$

Define $(m_{\pm,n}, M_{\pm,n}) := (X_{\pm,n} \wedge X_{\pm,n+1}, X_{\pm,n} \vee X_{\pm,n+1})$ then $m_{\pm,n} = X_{\pm,n+1} \wedge X_{\pm,n} \ge \kappa X_{\pm,n}$ since $X_{\pm,n+1} \ge \kappa X_{\pm,n}$. Similarly, $M_{\pm,n} \le \kappa^{-1} X_{\pm,n+1}$.

Part (a). We will proceed in three steps. Step I) is also used in the proofs of (b) and (c).

I) Recall the definition Z_m in (5.3) and consider the polynomial p_{k_+,k_-}^ϕ from Lemma 9. According to Lemma 9 with $A = \mathbb{R}_+^2$, we have for $\widetilde{f}(x,y) := f(x,y)/(x^{k_+} y^{k_-})$

$$\begin{aligned} \left|\widetilde{\Theta}_n^f\right|^p &= \left|f(\chi_{n+1}) H_{n+1,n+1}^{+,k_+}\big(H_{n+1,n+1}^{-,k_-}(\Phi)\big) - f(\chi_n) H_{n,n+1}^{+,k_+}\big(H_{n,n+1}^{-,k_-}(\Phi)\big)\right|^p \\ &= \left|\frac{f(\chi_{n+1})}{X_{+,n+1}^{k_+} X_{-,n+1}^{k_-}} - \frac{f(\chi_n)}{X_{+,n}^{k_+} X_{-,n}^{k_-}}\right|^p \left|H_{n,n+1}^{+,k_+}\big(H_{n,n+1}^{-,k_-}(\Phi)\big) X_{+,n}^{k_+} X_{-,n}^{k_-}\right|^p \\ &\le \left|\widetilde{f}(\chi_{n+1}) - \widetilde{f}(\chi_n)\right|^p p_{k_+,k_-}^\phi (Z_{n+1}, n+1)^p. \end{aligned} \tag{5.7}$$

The goal for the rest of the proof is to provide algebraic inequalities for the above expression which depend explicitly on powers of $\Delta_{\pm,n}$, $X_{\pm,n}$ and Z_{n+1}. Through these expressions, we will later show that, in expectation, the first factor in the last line of (5.7) decays geometrically in n while the second factor has polynomial growth in n.

II) Next, we obtain an upper bound for the modulus of continuity of the map $\widetilde{f}$ which appears in (5.7) and where f is given in (3.13). This map is absolutely continuous with respect to Lebesgue measure and thus a.e. differentiable with

$$|\partial_+ \widetilde{f}(x,y)| = \mathbb{1}_{\{x>x_+,y>x_-\}}\left|\frac{\partial_+ f(x,y)}{x^{k_+}y^{k_-}} - \frac{k_+ f(x,y)}{x^{k_++1}y^{k_-}}\right| \le \mathbb{1}_{\{x>x_+,y>x_-\}} c_1 x^{-k_+} y^{1-k_-},$$

$$|\partial_- \widetilde{f}(x,y)| = \mathbb{1}_{\{x>x_+,y>x_-\}}\left|\frac{\partial_- f(x,y)}{x^{k_+}y^{k_-}} - \frac{k_- f(x,y)}{x^{k_+}y^{k_-+1}}\right| \le \mathbb{1}_{\{x>x_+,y>x_-\}} c_1 x^{1-k_+} y^{-k_-},$$

where $c_1 := (k_+ + 1)(k_- + 1)\|h\|_\infty$. Then, for any $x, x', y, y' \in \mathbb{R}_+$, denote $(m_x, M_x) := (x \wedge x', x \vee x')$ and $(m_y, M_y) := (y \wedge y', y \vee y')$ and observe:

$$\begin{aligned} |\widetilde{f}(x,y) - \widetilde{f}(x',y')| &= \left|\int_{x'}^{x} \partial_+ \widetilde{f}(z,y)dz + \int_{y'}^{y} \partial_+ \widetilde{f}(x',z)dz\right| \\ &\le \frac{\mathbb{1}_{\{M_x>x_+,M_y>x_-\}} c_1|x-x'|}{(m_x \vee x_+)^{k_+}(m_y \vee x_-)^{k_--1}} + \frac{\mathbb{1}_{\{M_x>x_+,M_y>x_-\}} c_1|y-y'|}{(m_x \vee x_+)^{k_+-1}(m_y \vee x_-)^{k_-}} \qquad (5.8) \\ &\le \mathbb{1}_{\{M_x>x_+,M_y>x_-\}} \frac{c_1}{x_+^{k_+}x_-^{k_-}}\left(|x-x'|x_- + |y-y'|x_+\right). \qquad (5.9) \end{aligned}$$

Note that in the inequality (5.8), we have used that $k_+, k_- \ge 2$ and that the support of g is contained in $[x_+, \infty) \times [x_-, \infty)$. Moreover, since f in (3.13) satisfies $|f(x,y)| \le \|h\|_\infty xy$, we have $|\widetilde{f}(x,y)| \le \|h\|_\infty x^{1-k_+}y^{1-k_-}$. Hence, for any $x, x', y, y' \in \mathbb{R}_+$ we have

$$\begin{aligned} |\widetilde{f}(x,y) - \widetilde{f}(x',y')| &\le \mathbb{1}_{\{M_x>x_+,M_y>x_-\}} \sup_{z>m_x,w>m_y} |\widetilde{f}(z,w)| \\ &\le \mathbb{1}_{\{M_x>x_+,M_y>x_-\}} c_2 (m_x \vee x_+)^{1-k_+}(m_y \vee x_-)^{1-k_-}, \end{aligned} \qquad (5.10)$$

where $c_2 := 2\|h\|_\infty$. Typically, each maximum in a denominator is lower bounded via (5.6).

III) Now, with the above bound we will show that the upper bound for $\widetilde{\Theta}_n^f$ depends on moments of basic random variables. Recall that $s = p \wedge \alpha'$. Applying (5.5) (with $q = s/p$) and (5.6) (with $r = s/p$) to the minimum of the two bounds obtained in (5.9) and (5.10) in the form $(5.9)^{s/p}(5.10)^{1-s/p}$ and using $x_+ \le m_x \vee x_+$ and $x_- \le m_y \vee x_-$ yields: for any $x, x', y, y' \in \mathbb{R}_+$ the following inequality holds,

$$|\widetilde{f}(x,y) - \widetilde{f}(x',y')| \le \frac{\mathbb{1}_{\{M_x>x_+,M_y>x_-\}} c_3^{1/p}}{x_+^{k_+-1+s/p} x_-^{k_--1+s/p}}\left(|x-x'|^{s/p}x_-^{s/p} + |y-y'|^{s/p}x_+^{s/p}\right),$$

where $c_3 := c_1^s c_2^{p-s}$. In what follows, this interpolation method is used in all cases with different combinations of power parameters.

Then (5.5) gives

$$|\widetilde{f}(\chi_{n+1}) - \widetilde{f}(\chi_n)|^p \le \frac{\mathbb{1}_{\{M_{+,n}>x_+,M_{-,n}>x_-\}} 2^{p-1} c_3}{x_+^{p(k_+-1)+s} x_-^{p(k_--1)+s}}\left(|\Delta_{+,n+1}|^s x_-^s + |\Delta_{-,n+1}|^s x_+^s\right).$$

Applying the inequality $\mathbb{1}_{\{M_{\pm,n}>x_\pm\}} \le x_\pm^{-v} M_{\pm,n}^v$, for $v=\alpha'-s\ge 0$ and (5.7) we obtain

$$|\widetilde{\Theta}_n^f|^p \le \frac{2^{p-1}c_3 p^\phi_{k_+,k_-}(Z_{n+1},n+1)^p}{x_+^{p(k_+-1)+\alpha'}x_-^{p(k_--1)+\alpha'}}\big(|\Delta_{+,n+1}|^s M_{+,n}^{\alpha'-s}M_{-,n}^{\alpha'}+|\Delta_{-,n+1}|^s M_{-,n}^{\alpha'-s}M_{+,n}^{\alpha'}\big)$$
$$\le \frac{2^{p-1}c_3 p^\phi_{k_+,k_-}(Z_{n+1},n+1)^p}{\kappa^{2\alpha'-s}x_+^{p(k_+-1)+\alpha'}x_-^{p(k_--1)+\alpha'}}\big(|\Delta_{+,n+1}|^s X_{+,n+1}^{\alpha'-s}X_{-,n+1}^{\alpha'}+|\Delta_{-,n+1}|^s X_{-,n+1}^{\alpha'-s}X_{+,n+1}^{\alpha'}\big),$$

where the second inequality follows from $M_{\pm,n}\le\kappa^{-1}X_{\pm,n+1}$. Finally, as $\alpha'<\alpha$, Lemma 10 gives (3.15).

To prove the second statement in (a), we proceed as before. We start by using the inequality $|\widetilde{f}(\chi_n)|^p \le \mathbb{1}_{\{X_{+,n}>x_+,X_{-,n}>x_-\}}\|h\|_\infty^p x_+^{p(1-k_+)}x_-^{p(1-k_-)}$ and the bound $\mathbb{1}_{\{X_{\pm,n}>x_\pm\}}\le X_{\pm,n}^{\alpha'}x_\pm^{-\alpha'}$. An application of Lemma 10 then yields (3.16).

Part (b). Let $c_4 := 2^{1-1/p}c_1^u c_2^{1-u}$ where $u\in[0,1]$ is given in the statement. Applying (5.6) (with $r=u$) and (5.5) (with $q=p$) to the minimum of (5.8) and (5.10) in the form $(5.8)^u(5.10)^{1-u}$ yields

$$|\widetilde{f}(x,y)-\widetilde{f}(x',y')|^p \le \mathbb{1}_{\{M_x>x_+,M_y>x_-\}}c_4^p\frac{|x-x'|^{pu}/(m_x\vee x_+)^{pu}+|y-y'|^{pu}/(m_y\vee x_-)^{pu}}{(m_x\vee x_+)^{p(k_+-1)}(m_y\vee x_-)^{p(k_--1)}}$$
$$\le \mathbb{1}_{\{M_x>x_+,M_y>x_-\}}c_4^p\frac{|x-x'|^{pu}/m_x^{pu}+|y-y'|^{pu}/m_y^{pu}}{(m_x\vee x_+)^{p(k_+-1)}(m_y\vee x_-)^{p(k_--1)}}. \tag{5.11}$$

By (5.6) we have $m_x\vee x_+\ge m_x^r x_+^{1-r}$ and $m_y\vee x_-\ge m_y^{r'}x_-^{1-r'}$ for any $r,r'\in[0,1]$. Since $\alpha'<\alpha\le 1/[\rho\vee(1-\rho)]$, we choose $r=\alpha'\rho/[p(k_+-1)]$ and $r'=\alpha'(1-\rho)/[p(k_--1)]$. Applying these interpolating inequalities to (5.11) and combining them with (5.7) gives

$$\big|\widetilde{\Theta}_n^f\big|^p \le \frac{c_4^p p^\phi_{k_+,k_-}(Z_{n+1},n+1)^p}{x_+^{p(k_+-1)-\alpha'\rho}x_-^{p(k_--1)-\alpha'(1-\rho)}}\left(\frac{|\Delta_{+,n+1}|^{pu}}{m_{+,n}^{\alpha'\rho+pu}m_{-,n}^{\alpha'(1-\rho)}}+\frac{|\Delta_{-,n+1}|^{pu}}{m_{+,n}^{\alpha'\rho}m_{-,n}^{\alpha'(1-\rho)+pu}}\right),$$
$$\le \frac{c_4^p p^\phi_{k_+,k_-}(Z_{n+1},n+1)^p}{\kappa^{\alpha'+pu}x_+^{p(k_+-1)-\alpha'\rho}x_-^{p(k_--1)-\alpha'(1-\rho)}}\left(\frac{|\Delta_{+,n+1}|^{pu}}{X_{+,n}^{\alpha'\rho+pu}X_{-,n}^{\alpha'(1-\rho)}}+\frac{|\Delta_{-,n+1}|^{pu}}{X_{+,n}^{\alpha'\rho}X_{-,n}^{\alpha'(1-\rho)+pu}}\right),$$

where we used the restriction that $m_{\pm,n}\ge\kappa X_{\pm,n}$. Moreover, as $u\in(0,(\alpha-\alpha')(\rho\wedge(1-\rho))/p)$, we have $\alpha'\rho+pu<\alpha\rho$ and $\alpha'(1-\rho)+pu<\alpha(1-\rho)$. Hence, applying Lemma 12 gives (3.17).

The proof of (3.18) is analogous to that of (3.17). Indeed, using (5.6) and the inequality $|\widetilde{f}(\chi_n)|\le \|h\|_\infty(X_{+,n}\vee x_+)^{1-k_+}(X_{-,n}\vee x_-)^{1-k_-}$ we obtain

$$|\widetilde{f}(\chi_n)|^p \le \|h\|_\infty^p x_+^{p(1-k_+)+\alpha'\rho}x_-^{p(1-k_-)+\alpha'(1-\rho)}X_{+,n}^{-\alpha'\rho}X_{-,n}^{-\alpha'(1-\rho)}.$$

The inequality (3.18) then follows from Lemma 12, completing the proof of (b).

Part (c). We will only prove the bound for the first argument of the minimum in the right hand side of (3.19) and (3.20); the other case is analogous. We proceed as in (b): using (5.6), (5.7), (5.11) but instead of the interpolating inequality using r, we use the bound $\mathbb{1}_{\{M_{+,n}>x_+\}}\le M_{+,n}^v x_+^{-v}$, for any

$v = \alpha' - pu, \alpha' \geq 0$), to obtain

$$\left|\widetilde{\Theta}_n^f\right|^p \leq \frac{c_4^p p_{k_+,k_-}^{\phi}(Z_{n+1}, n+1)^p}{x_+^{p(k_+-1)+\alpha'} x_-^{p(k_--1)-\alpha'(1-\rho)}} \left(\frac{|\Delta_{+,n+1}|^{pu} M_{+,n}^{\alpha'-pu}}{m_{-,n}^{\alpha'(1-\rho)}} + \frac{|\Delta_{-,n+1}|^{pu} M_{+,n}^{\alpha'}}{m_{-,n}^{\alpha'(1-\rho)+pu}} \right)$$

$$\leq \frac{c_4^p p_{k_+,k_-}^{\phi}(Z_{n+1}, n+1)^p}{\kappa^{\alpha'(2-\rho)+pu} x_+^{p(k_+-1)+\alpha'} x_-^{p(k_--1)-\alpha'(1-\rho)}} \left(\frac{|\Delta_{+,n+1}|^{pu} X_{+,n+1}^{\alpha'-pu}}{X_{-,n}^{\alpha'(1-\rho)}} + \frac{|\Delta_{-,n+1}|^{pu} X_{+,n+1}^{\alpha'}}{X_{-,n}^{\alpha'(1-\rho)+pu}} \right),$$

where we used the fact that $M_{\pm,n} \leq \kappa^{-1} X_{\pm,n+1}$ and $m_{\pm,n} \geq \kappa X_{\pm,n}$. Lemma 12 then implies (3.19).

Using the inequality $|\widetilde{f}(\chi_n)| \leq \mathbb{1}_{\{X_{+,n}>x_+, X_{-,n}>x_-\}} \|h\|_\infty (X_{+,n} \vee x_+)^{1-k_+} (X_{-,n} \vee x_-)^{1-k_-}$, (5.6) and the bound $\mathbb{1}_{\{X_{+,n}>x_+\}} \leq X_{+,n}^{\alpha'} x_+^{-\alpha'}$, we obtain

$$\left|\Theta_{n,m}^f\right|^p \leq \frac{\|h\|_\infty^p p_{k_+,k_-}^{\phi}(Z_m, m)^p}{x_+^{p(k_+-1)+\alpha'} x_-^{p(k_--1)-\alpha'(1-\rho)}} \frac{X_{+,n}^{\alpha'}}{X_{-,n}^{\alpha'(1-\rho)}}.$$

which yields (3.20) by Lemma 12, completing the proof of the proposition. □

Remark 5. Analyzing the above proof, we can see the interpolation method at work here. In fact, to interpret the estimates of Proposition 8, one may say that all polynomial terms in n arise due to the polynomial growth of $H_{n,m}^{\pm;k_\pm}$ (see (5.4) in Lemma 9), through the term $p_{k_+,k_-}^{\phi}(Z_m, m)^p$ which appears in the upper bounds. On the other hand, the geometrically decreasing terms are produced by the exponentially fast decay of the differences $\Delta_{\pm,n} := X_{\pm,n} - X_{\pm,n-1}$ in $\widetilde{\Theta}_n^f$. We stress that another achievement of the interpolation method is that the moment estimates of Proposition 8 hold for any $p \geq 1$.

5.3. Proof of Proposition 8, Part II: The moment bounds

In this section, we state explicit moment estimates for the quantities that appear in the weights of the multiple Ibpf of Theorem 7. These bounds were the last step in the proof of Proposition 8 above. The proofs in this section, are independent of everything that have preceded them. In order to obtain near optimal bounds in Theorem 1, we first study the growth of the moments of $X_{\pm,n}^p$ for p arbitrarily close to α in Lemmas 10, 11 and 12. Since the α-moment of the stable law does not exist, the bounds in these lemmas cannot be obtained e.g. via Hölder's inequality. Their proofs consist of a direct, but very careful, analysis of the corresponding expectations.

There are two types of bounds according to whether they involve positive or negative moments of $X_{\pm,n}$. They correspond to the behavior at infinity or at zero in the estimates that we obtain in Theorem 1 as can be deduced from the proof of Proposition 8. Throughout the present section we use the notation from Subsection 3.1. In particular, recall the definition of Z_m in (5.3) and Assumption (A-κ): $\kappa^\alpha \in [\rho \vee (1-\rho), 1)$. Explicit constants in the results in this section can be recovered from the proofs.

We begin by recalling the Mellin transform of a stable random variable (see [32, Thm 2.6.3])

$$\mathbb{E}[S_1^p \mathbb{1}_{\{S_1>0\}}] = \rho \frac{\Gamma(1+p)\Gamma(1-p/\alpha)}{\Gamma(1+p\rho)\Gamma(1-p\rho)}, \qquad p \in (-1, \alpha).$$

When $\alpha \neq 1$, by the independence $E_i \perp\!\!\!\perp G_i$ we deduce that, for any $p \in [0, \alpha)$,

$$\mathbb{E}[G_i^p \mathbb{1}_{\{G_i>0\}}] = \frac{\mathbb{E}[S_i^p \mathbb{1}_{\{S_i>0\}}]}{\mathbb{E}\big[E_i^{p(1-1/\alpha)}\big]} = \frac{\rho\Gamma(1+p)\Gamma(1-p/\alpha)}{\Gamma(1+p\rho)\Gamma(1-p\rho)\Gamma(p(1-1/\alpha)+1)}. \tag{5.12}$$

Finally, we recall that $\mathbb{E}[E_1^p] = \Gamma(1+p)$ is finite if and only if $p > -1$.

5.4. Positive moments

Lemma 10. *Let $p,q,s \geq 0$ satisfy $q \leq p < \alpha$. Then, there exists a constant $C > 0$ such that for any $m \geq n$ and $T > 0$ we have*

$$\mathbb{E}\big[X_{\pm,n}^{p-q} X_{\mp,n}^{p} |\Delta_{\pm,n}|^q Z_m^s\big] \leq CT^{2\frac{p}{\alpha}}\Big(\big(1+\tfrac{q}{\alpha}\big)^{-n} + \kappa^{qn}\Big) m^{[p-1]^+ + [p-q-1]^+ + s}. \tag{5.13}$$

Remark 6. (i) Note that the exponent in $E^{1-1/\alpha}$ appearing in $X_{\pm,n}$ changes sign when $\alpha \in (0,1)$ and $\alpha \in (1,2)$. For this reason, most of the proofs for estimating the above bounds of moments will have to be done in three separate cases: $\alpha \in (0,1)$, $\alpha = 1$ and $\alpha \in (1,2)$. This makes the proofs slightly long because some inequalities change depending on the above cases.

(ii) Note that due to the scaling property of the stick breaking process and a_n the factor of $T^{2\frac{p}{\alpha}}$ is easily obtained. In fact, we will assume, without loss of generality, in all proofs in this section that $T = 1$. In the Lemma statements, we have left the dependence on T and in some major points of the proof too. In a first reading, one may assume always that $T = 1$.

(iii) We consider in all proofs only one combination of $\pm$ signs. The other case follows *mutatis mutandis*.

Proof of Lemma 10. We first make a number of reductions that simplify the proof. We will assume $p,q > 0$. The remaining cases (when at least one of the two parameters is zero) follow similarly by ignoring the corresponding terms in the calculations.

Let $c = 2^{[p-1]^+ + [p-q-1]^+ + [q-1]^+}$ and use (5.5) to obtain

$$X_{+,n}^{p-q} X_{-,n}^{p} |\Delta_{+,n}|^q \leq c\Bigg(\bigg(\sum_{i=1}^{n} \ell_i^{1/\alpha}[S_i]^+\bigg)^{p-q} + a_n^{p-q}\eta_+^{p-q}\Bigg)\Bigg(\bigg(\sum_{i=1}^{n} \ell_i^{1/\alpha}[S_i]^-\bigg)^{p} + a_n^{p}\eta_-^{p}\Bigg) \times \big((\ell_n^{1/\alpha}[S_n]^+)^q + a_{n-1}^q \eta_+^q\big). \tag{5.14}$$

Our goal is now to provide an upper bound for the expectation of the right hand side of the above inequality multiplied by Z_n^s. This leads to eight terms which must be treated individually to show that their expectations decay exponentially at least as a polynomial (in n) multiple of a_{n-1}^q or $\mathbb{E}[\ell_n^{q/\alpha}] = (1+q/\alpha)^{-n}$. We treat the hardest term in (5.14); which contains the product of sums of $[S_i]^\pm$. The other terms are easier to treat as we remark at the end of the proof. Therefore we will consider, for $r \in \{0,q\}$

$$A := \mathbb{E}\Bigg[\bigg(\sum_{i=1}^{n} \ell_i^{1/\alpha}[G_i]^+ c_i\bigg)^{p-q}\bigg(\sum_{j=1}^{n} \ell_j^{1/\alpha}[G_j]^- c_j\bigg)^{p}(\ell_n^{1/\alpha}[G_n]^+ c_n)^r\bigg|\mathcal{F}_{-E}\Bigg] Z_m^s, \qquad r \in \{0,q\}, \tag{5.15}$$

where $c_i = E_i^{1-1/\alpha}$ and $\mathcal{F}_{-E} = \sigma(\ell_i, G_i; i \in \mathbb{N})$. We estimate (5.15) in steps:

I) In this step we separate the expectation in (5.15) using the independence of G, E and ℓ. Let $r \in \{0,q\}$ and $p' := [p-1]^+ + [p-q-1]^+$ and fix any positive constants $(c_i)_{i\in\mathbb{N}}$. Applying (5.5) yields

$$\mathbb{E}\Bigg[\bigg(\sum_{i=1}^{n} \ell_i^{1/\alpha}[G_i]^+ c_i\bigg)^{p-q}\bigg(\sum_{j=1}^{n} \ell_j^{1/\alpha}[G_j]^- c_j\bigg)^{p}(\ell_n^{1/\alpha}[G_n]^+ c_n)^r\bigg|\mathcal{F}_{-E}\Bigg] \tag{5.16}$$

$$\leq n^{p'}\mathbb{E}\left[\sum_{i=1}^{n}\sum_{j=1}^{n}\ell_i^{\frac{p-q}{\alpha}}\ell_n^{\frac{r}{\alpha}}\ell_j^{\frac{p}{\alpha}}([G_i]^+c_i)^{p-q}([G_j]^-c_j)^p([G_n]^+c_n)^r\middle|\mathcal{F}_{-E}\right]$$

$$\leq 2n^{p'}\sum_{j=1}^{n-1}\sum_{i=j+1}^{n}\mathbb{E}\left[\ell_i^{\frac{p-q}{\alpha}}\ell_n^{\frac{r}{\alpha}}\ell_j^{\frac{p}{\alpha}}\right]\mathbb{E}\left[([G_i]^+)^{p-q}([G_n]^+)^r\right]\mathbb{E}\left[([G_j]^-)^p\right]c_i^{p-q}c_j^p c_n^r,$$

Note that the cases $j\in\{i,n\}$ do not appear because $[x]^+[x]^-=0$. The above expression is a linear combination of monomials in c_i, c_n and c_j. We will analyse and bound the coefficients.

The last two expectations within the sum on the right side of the above inequality can be computed using (5.12) and the value of their product only depends on whether $i=n$ or not. In fact, for $r\in\{0,q\}$

$$\mathbb{E}\left[([G_i]^+)^{p-q}([G_n]^+)^r\right]\leq\max\{\mathbb{E}[([G_1]^+)^p],\mathbb{E}[([G_1]^+)^{p-q}]\mathbb{E}[([G_1]^+)^q],\mathbb{E}[([G_1]^+)^{p-q}]\},$$

which can be bounded by an explicit constant using (5.12).

II) Now, we obtain an important part of the bound in (5.13) which is due to the stick breaking process. That is, an application of Lemma 13(b) in [16] yields the existence of some $c'>0$ independent of j, i and n such that for $\theta=\frac{\alpha+p+r}{\alpha+2p+r}<1$, we have $\mathbb{E}[\ell_i^{(p-q)/\alpha}\ell_j^{p/\alpha}\ell_n^{r/\alpha}]\leq c'\theta^{i+j}(1+r/\alpha)^{-n}$.

III) Now, we estimate the moments of the remaining random variables E_i which appear in the coefficients c_i. By the previous steps and (5.16), we deduce that for some constant $c''>0$ independent of j, i and n, we have

$$\mathbb{E}[A]\leq c''n^{p'}\left(1+\tfrac{r}{\alpha}\right)^{-n}\sum_{j=1}^{n-1}\sum_{i=j+1}^{n}\theta^{i+j}\mathbb{E}\left[E_i^{(1-1/\alpha)(p-q)}E_j^{(1-1/\alpha)p}E_n^{(1-1/\alpha)r}Z_m^s\right].$$

Next, we will show that the expectation on the right side in the above inequality is bounded by a multiple of m^s. As the term θ^{i+j} vanishes geometrically fast, we would then obtain

$$\mathbb{E}[A]\leq c'''n^{p'}m^s\left(1+\tfrac{r}{\alpha}\right)^{-n}. \tag{5.17}$$

To prove (5.17), observe that Z_n in (5.3) is a Gamma distributed random variable, hence $\mathbb{E}\left[Z_n^s\right]=\Gamma(n+s+2)/(n+1)!$. Using the two-sided bounds in Stirling's formula we see that this expression is bounded by a multiple of m^s. In fact, a similar upper bound holds for $\mathbb{E}[Z_m^sE_i^{r_1}E_j^{r_2}E_n^{r_3}]$ with $r_1=(1-1/\alpha)(p-q)$, $r_2=(1-1/\alpha)p$ and $r_3=(1-1/\alpha)r$. Note that $r_1,r_2,r_3>-1$ in the case $i<n$ and $r_1+r_3>-1$, $r_2>-1$ in the case that $i=n$. Furthermore, even in the case $\alpha\in(0,1)$, our hypotheses on p and q ensure that r_1, r_2 and r_3 satisfy these conditions. Indeed, for instance, when the indices n,i,j are different and $n\geq 4$, we can decompose Z_m into 4 terms according to the index of E within Z_m which may equal one of the indices n,i,j so that, by (5.5),

$$\begin{aligned}\mathbb{E}\left[Z_n^sE_i^{r_1}E_n^{r_2}E_j^{r_3}\right]=4^{[s-1]^+}\Big(&\mathbb{E}[E_i^{s+r_1}]\mathbb{E}[E_n^{r_2}]\mathbb{E}[E_j^{r_3}]+\mathbb{E}[E_i^{r_1}]\mathbb{E}[E_n^{s+r_2}]\mathbb{E}[E_j^{r_3}]\\&+\mathbb{E}[E_i^{r_1}]\mathbb{E}[E_n^{r_2}]\mathbb{E}[E_j^{s+r_3}]+\mathbb{E}[E_i^{r_1}]\mathbb{E}[E_n^{r_2}]\mathbb{E}[E_j^{r_3}]\mathbb{E}[Z_{m-3}^s]\Big).\end{aligned} \tag{5.18}$$

The quantity in (5.18) grows as a constant multiple of m^s (via the s-moment of Z_{m-3}), implying (5.17).

Finally, to bound other terms in (5.14), we repeat the above arguments. This is slightly easier because:

1. The variables η_+ and η_- are independent of the sequence $(\ell_i,S_i)_{i\in\mathbb{N}}$.
2. Hence, when taking expectations, the variables η_+ and η_- will factorise by independence. These variables are multiplied by powers of $a_n=\kappa^n$ and satisfy $\mathbb{E}[\eta_\pm^r]=\Gamma(1+r)$ for $r>-1$ so their estimation is easier.

3. The final bound also uses the inequality $a_n \le a_{n-1}$, a consequence of Assumption (A-κ).

Putting the above arguments together completes the proof of Lemma 10, since all eight terms decay as fast as a_n^q or $(1+q/\alpha)^{-n}$ and $n \le m$. □

5.5. Negative and mixed moments: Proofs of Lemmas 11 and 12

The ideas of the proof of Lemma 10 can be used again for negative moments but an additional idea is required in order to use similar techniques. This is provided by the following equality:

$$\lambda^{-p} = \Gamma(p)^{-1} \int_0^\infty x^{p-1} e^{-\lambda x} dx, \text{ for } p > 0, \tag{5.19}$$

which expresses the negative power λ^{-p} using an exponential expression which in our case leads to the study of the Laplace transform of the respective random variable whose inverse moment we want to estimate. The use of this technique leads to the following results.

Lemma 11. *Recall that $\alpha \neq 1$ and the Assumption (A-κ) are in force. Below, we assume that $r \ge 0$ and $u, v, w \ge 0$ be arbitrary.*

(a) *Fix any $p \in (0, \alpha\rho)$, $q \in [0, \alpha(1-\rho))$. There exists a positive constant C such that for any $j, n \in \mathbb{N}$ and $T > 0$, the following bound holds:*

$$\mathbb{E}\left[\frac{\ell_{n+1}^r E_j^u \eta_+^v \eta_-^w}{X_{+,n}^p X_{-,n}^q}\right] \le C T^{r - \frac{p+q}{\alpha}} (1+r)^{-n}. \tag{5.20}$$

(b) *Fix any $p \in (0, \alpha\rho)$, $q \in (0, \alpha)$. There exists a positive constant C such that for any $j, n \in \mathbb{N}$ and $T > 0$, the following bound holds:*

$$\mathbb{E}\left[\frac{X_{-,n}^q \ell_{n+1}^r E_j^u \eta_+^v \eta_-^w}{X_{+,n}^p}\right] \le C T^{r + \frac{q-p}{\alpha}} (1+r)^{-n}.$$

Lemma 12. *Let $p, q, r, s \ge 0$ satisfy $p \in [0, \alpha\rho)$, $q \in [0, \alpha(1-\rho))$ and $r \in [0, \alpha)$. There exists a constant $C > 0$ such that for any $m \ge n$ and $T > 0$ we have*

$$\mathbb{E}\left[\frac{|\Delta_{\pm,n+1}|^r Z_m^s}{X_{+,n}^p X_{-,n}^q}\right] \le C T^{(2r-p-q)/\alpha} \left(\left(1 + \frac{r}{\alpha}\right)^{-n} + \kappa^{nr}\right) m^{s'},$$

where $s' = \mathbb{1}_{\{s>0\}}(s \vee 1)$. Similarly, for any $p \in [0, \alpha)$, $r \in [0, p]$ and $q \in [0, \alpha(1-\rho))$, there is some $C > 0$ such that for all $m \ge n$ and $T > 0$

$$\mathbb{E}\left[\frac{|\Delta_{+,n+1}|^r X_{+,n+1}^{p-r} Z_m^s}{X_{-,n}^q}\right] \le C T^{(p-q)/\alpha} \left(\left(1 + \frac{r}{\alpha}\right)^{-n} + \kappa^{nr}\right) m^{s'}.$$

Proof of Lemma 11. Recall that we assume without loss of generality that $T = 1$. The identity (5.19) will be used with $\lambda = X_{+,n}$ (and later with $\lambda = X_{-,n}$). The resulting expression will be bounded by separately integrating the variables $G_1, \dots, G_n$ and η_+, η_-, then and $E_1, \dots, E_n$ and finally $\ell_1, \dots, \ell_{n+1}$ as in the proof of Lemma 10. These bounds require preliminary calculations for the expressions arising in the inequalities developed below, so we begin with those. Let $\zeta = 1 - 1/\alpha$, c, δ and γ be as defined in

Lemma 15 of [16]. These are constants that appear in the bounds for the Laplace transform of random variables used in the Chambers-Mallows-Stuck representation of stable laws.

Proof of (a)*, part 1: The case* $q=0$. We first provide an explicit upper bound for $\mathbb{E}\big[\ell_{n+1}^r E_j^u \eta_+^v X_{+,n}^{-p}\big]$ (the case $q=0$ and $w=0$ in Lemma 11(a)) using the following: let $b_\rho := 1/(\gamma\alpha\rho) \ge 1$ and

$$
\begin{aligned}
d_s &:= 2^s \max\{1, s^s e^{-s}, \Gamma(s+1)\}, && \text{for } s \ge 0,\\
P(x,p,q) &:= \frac{((x\wedge 1)^{-p}-1)}{p} + \frac{x^{-q}(x\wedge 1)^{q-p}}{q-p}, && \text{for } x,p,q>0, q>p,\\
Q_p(r,u) &:= \frac{\alpha u(1+r)-up}{p(1-p)(\alpha u(1+r)-p)(1-p/\alpha)}, && \text{for } u\in(0,1],\ p\in(0,\min\{\alpha u,1\}),\ r\ge 0.\\
d_u' &:= \max\{\mathbb{E}[E_j^u], \mathbb{E}[E_j^u]\mathbb{E}[E_k^{-\zeta}], \mathbb{E}[E_j^{u-\zeta}]\}.
\end{aligned}
\tag{5.21}
$$

Using these definitions, it is enough to prove that for $p\in(0,\alpha\rho)$, $r,u,v\ge 0$ and $j,n\in\mathbb{N}$ it holds

$$
\mathbb{E}\left[\frac{\ell_{n+1}^r E_j^u \eta_+^v}{X_{+,n}^p}\right] \le \frac{T^{r-\frac{p}{\alpha}} b_\rho c d_u' d_v}{\Gamma(p)(1+r)^n}\left(\frac{1}{p} + Q_p(r,\rho) + (1-\rho)^n P\big(T^{-\frac{1}{\alpha}} a_n, p, \delta\big)\right). \tag{5.22}
$$

The special case of (5.20) with $q=0$, $w>0$ follows from the independence $\eta_- \perp\!\!\!\perp (\eta_+, E_j, \ell_{n+1}, X_{+,n})$, (1.1), (5.22) and Assumption (A-κ). In fact, all the terms within the parentheses in (5.22) are readily bounded by a constant except $(1-\rho)^n\kappa^{-pn}$ which is bounded due to Assumption (A-κ).

For the proof of (5.22), recall that $X_{+,n} = \sum_{i=1}^n \ell_i^{1/\alpha} E_i^\zeta [G_i]^+ + a_n \eta_+^\zeta$ with $\zeta = 1-1/\alpha$.

Fix $p\in(0,\alpha\rho)$. A change of variables applied to the definition of the Gamma function gives

$$
\Gamma(p) X_{+,n}^{-p} = \int_0^\infty x^{p-1} e^{-xX_{+,n}}\,dx \le 1/p + J_{+,p}, \quad \text{where} \quad J_{\pm,p} := \int_1^\infty x^{p-1} e^{-xX_{\pm,n}}\,dx. \tag{5.23}
$$

Next, we bound the conditional expectation $\mathbb{E}\big[\eta_+^v J_{+,p}\big|\mathcal{G}\big]$, where $\mathcal{G} := \sigma(\ell_k, E_k; k\in\mathbb{N})$. By Lemma 15 in [16] (with parameter $x\ell_k^{1/\alpha}E_k^\zeta$), we have

$$
\begin{aligned}
\mathbb{E}\big[\eta_+^v J_{+,p}\big|\mathcal{G}\big] &= \int_1^\infty x^{p-1}\mathbb{E}\big[\eta_+^v e^{-xa_n\eta_+^\zeta}\big|\mathcal{G}\big]\prod_{k=1}^n \mathbb{E}\big[e^{-x\ell_i^{1/\alpha}E_i^\zeta[G_i]^+}\big|\mathcal{G}\big]\,dx\\
&\le c d_v \int_1^\infty x^{p-1}\min\{1,(a_n x)^{-\delta}\}\prod_{k=1}^n\left(1-\rho+\rho\min\left\{1, \frac{b_\rho E_k^{-\zeta}}{\ell_k^{1/\alpha} x}\right\}\right)dx.
\end{aligned}
\tag{5.24}
$$

Using that $\ell_{n+1} = (1-U_{n+1})U_n \dots U_{k+1} L_k$ and $L_k \le L_{k-1}$ (see Subsection 3.1), then for any measurable function $g\ge 0$ and $k\le n$, we have

$$
\mathbb{E}[\ell_{n+1}^r g(\ell_k)] = (1+r)^{k-n-1}\mathbb{E}[L_k^r g(\ell_k)] \le (1+r)^{k-n-1}\mathbb{E}[L_{k-1}^r g(\ell_k)]. \tag{5.25}
$$

Moreover, we have $\mathbb{E}[E_j^u \min\{1, E_k^{-\zeta} y\}] \le d_u' \min\{1,y\}$ by definition (5.21). In fact, if $y>1$, then $d_u' \ge \mathbb{E}[E_j^u]$ and if $y<1$, then $d_u' y \ge \mathbb{E}[E_j^u E_k^{-\zeta} y]$.

Since the factors in the product of (5.24) are in $[0,1]$, the first inequality in Lemma 14 of [16], (5.25) and $b_\rho \ge 1$ yields

$$
\mathbb{E}\left[\ell_{n+1}^r E_j^u \prod_{k=1}^n \left(1-\rho+\rho\min\left\{1, b_\rho E_k^{-\zeta}\ell_k^{-1/\alpha}/x\right\}\right)\right]
$$

$$\le (1-\rho)^n\mathbb{E}[\ell_{n+1}^r]\mathbb{E}[E_j^u] + \sum_{k=1}^{n}\rho(1-\rho)^{k-1}\mathbb{E}\big[\ell_{n+1}^r E_j^u \min\big\{1, b_\rho E_k^{-\zeta}\ell_k^{-1/\alpha}/x\big\}\big]$$
$$\le \frac{(1-\rho)^n d_u'}{(1+r)^{n+1}} + \frac{d_u'}{(1+r)^n}\sum_{k=1}^{\infty}\rho(1-\rho)^{k-1}(1+r)^{k-1}\mathbb{E}\big[L_{k-1}^r \min\big\{1, b_\rho \ell_k^{-1/\alpha}/x\big\}\big]$$
$$\le d_u'(1+r)^{-n}\big((1-\rho)^n + b_\rho A_\rho(x)\big),$$

where we define $A_\rho(x) := \sum_{k=1}^{\infty}\rho(1-\rho)^{k-1}(1+r)^{k-1}\mathbb{E}[L_{k-1}^r \min\{1,\ell_k^{-1/\alpha}/x\}]$ for $x>0$.

Hence, the inequality $\mathbb{E}[\ell_{n+1}^r E_j^u \eta_+^v X_{+,n}^{-p}]\Gamma(p) \le d_u' d_v (1+r)^{-n}(1/p + cK)$ holds for

$$K := \int_1^{\infty} x^{p-1}\min\big\{1,(a_n x)^{-\delta}\big\}((1-\rho)^n + b_\rho A_\rho(x))dx.$$

Next, we apply Lemma 15(c) in [16] to find a formula for $\int_1^\infty x^{p-1}A_\rho(x)dx$. Note that $p<\alpha\rho$ implies $\frac{(1-\rho)(1+r)}{1+r-p/\alpha} = \frac{1+r-\rho(1+r)}{1+r-p/\alpha} < 1$, so Fubini's theorem and Lemmas 15(c) and 13(c) in [16] yield

$$\begin{aligned}\int_1^\infty x^{p-1}A_\rho(x)dx &= \sum_{k=1}^{\infty}\rho[(1-\rho)(1+r)]^{k-1}\mathbb{E}\bigg[L_{k-1}^r\int_1^\infty x^{p-1}\min\{1,(\ell_k^{1/\alpha}x)^{-1}\}dx\bigg]\\ &= \sum_{k=1}^{\infty}\rho[(1-\rho)(1+r)]^{k-1}\mathbb{E}\bigg[L_{k-1}^r\bigg(\frac{\ell_k^{-p/\alpha}}{p(1-p)} - \frac{1}{p}\bigg)\bigg]\\ &= \sum_{k=1}^{\infty}\rho[(1-\rho)(1+r)]^{k-1}\bigg(\frac{(1+r-p/\alpha)^{1-k}}{p(1-p)(1-p/\alpha)} - \frac{(1+r)^{1-k}}{p}\bigg) = Q_p(r,\rho).\end{aligned} \tag{5.26}$$

Thus (5.22) follows from (5.26) and Lemma 15(c) in [16] since for any $p<\alpha\rho$ and $n\in\mathbb{N}$ we have

$$K \le \int_1^\infty x^{p-1}\big((1-\rho)^n\min\{1,(a_n x)^{-\delta}\} + b_\rho A_\rho(x)\big)dx \le b_\rho\big[(1-\rho)^n P\big(a_n,p,\delta\big) + Q_p(r,\rho)\big].$$

Proof of (a)*, part 2. The case* $q\in(0,\alpha(1-\rho))$. The general case of (5.20) for $q>0$ follows similarly but with lengthier expressions. Recall that $B(\cdot,\cdot)$ denotes the beta function and define for any $u\in(0,1]$, $p\in(0,\alpha u\wedge 1)$, $q\in(0,\alpha\wedge 1)$ and $r\ge 0$,

$$R_{p,q}(r,u) := \frac{(\Gamma(1/\alpha)\vee 1)B\big(1+r-p/\alpha, 1-q/\alpha\big)u(1-u)(1+r)^2(1+r-p/\alpha)}{pq(1-p)(1-q)(1-p/\alpha)(u(1+r)-p/\alpha)}.$$

Fix $p\in(0,\alpha\rho)$, $q\in(0,\alpha(1-\rho))$ and $r,u,v,w\ge 0$. We will prove that for all $j,n\in\mathbb{N}$, we have

$$\begin{aligned}\mathbb{E}\bigg[\frac{\ell_{n+1}^r E_j^u \eta_+^v \eta_-^w}{X_{+,n}^p X_{-,n}^q}\bigg] \le \frac{T^{r-\frac{p+q}{\alpha}} b_\rho b_{1-\rho} c^2 d_u' d_v d_w}{\Gamma(p)\Gamma(q)(1+r)^n}\Big[&(1-\rho)^n P\big(T^{-\frac{1}{\alpha}}a_n,p,\delta\big)/q + \rho^n P\big(T^{-\frac{1}{\alpha}}a_n,q,\delta\big)/p\\ &+ \big((1-\rho)^n+\rho^n\big)P\big(T^{-\frac{1}{\alpha}}a_n,p,\delta\big)P\big(T^{-\frac{1}{\alpha}}a_n,q,\delta\big)\\ &+ 1/(pq) + Q_p(r,\rho)/q + Q_q(r,1-\rho)/p + R_{p,q}(r,\rho) + R_{q,p}(r,1-\rho)\Big],\end{aligned}$$

Once this bound is proven the final result follows as in part 1, by (1.1) and Assumption (A-κ). Indeed, $((1-\rho)^n+\rho^n)P(T^{-1/\alpha}a_n,p,\delta)P(T^{-1/\alpha}a_n,q,\delta) \le (1/p+1/(\delta-p))(1/q+1/(\delta-q))2\kappa^{n\alpha-n(p+q)}$, by

Assumption (A-κ), which is bounded for $n \in \mathbb{N}$ because $p+q<\alpha$. Therefore we just need to prove the above inequality.

Applying (5.23) twice, we get $\Gamma(p)\Gamma(q)X_{+,n}^{-p}X_{-,n}^{-q} \le 1/(pq) + J_{-,q}/p + J_{+,p}/q + J_{+,p}J_{-,q}$. It remains to multiply the previous inequality by $\ell_n^r E_j^u \eta_+^v \eta_+^w$ and take expectations. The first term on the right side yields $\mathbb{E}[\ell_{n+1}^r E_j^u \eta_+^v \eta_-^w]/(pq) \le (1+r)^{-n-1} d_u' d_v d_w/(pq)$. The second and third terms are bounded as in the special case $q=0$, since η_+ (resp. η_-) is independent of $X_{-,n}$ (resp. $X_{+,n}$).

It remains to bound $\mathbb{E}[\ell_{n+1}^r E_j^u \eta_+^v \eta_-^w J_{+,p} J_{-,q}]$. Applying Lemma 15(b) of [16] twice,

$$\mathbb{E}[e^{-x[G_1]^+ - y[G_1]^-}] \le \Upsilon(x,y) := \rho \min\{1, b_\rho/x\} + (1-\rho)\min\{1, b_{1-\rho}/y\} \le 1.$$

Write $\mathbb{E}[\ell_{n+1}^r E_j^u \eta_+^v \eta_-^w J_{+,p} J_{-,q} | \sigma(\ell_k, E_k; k \in \mathbb{N})] = \int_1^\infty \int_1^\infty x^{p-1} y^{q-1} S(x,y)\,dy\,dx$, where

$$\begin{aligned}
S(x,y) &:= \ell_{n+1}^r E_j^u \mathbb{E}\big[\eta_+^v \eta_-^w e^{-xX_{+,n} - yX_{-,n}} \big| \sigma(\ell_k, E_k; k \in \mathbb{N})\big] \\
&= \ell_{n+1}^r E_j^u \mathbb{E}\Big[\eta_+^v e^{-x a_n \eta_+^\zeta}\Big] \mathbb{E}\Big[\eta_-^v e^{-y a_n \eta_-^\zeta}\Big] \prod_{k=1}^n \mathbb{E}\Big[e^{-\ell_k^{1/\alpha} E_k^\zeta (x[G_k]^+ - y[G_k]^-)} \Big| \sigma(\ell_k, E_k; k \in \mathbb{N})\Big] \\
&\le c^2 d_v d_w \min\{1, (a_n x)^{-\delta}\} \min\{1, (a_n y)^{-\delta}\} \ell_{n+1}^r E_j^u \prod_{k=1}^n \Upsilon(E_k^\zeta \ell_k^{1/\alpha} x, E_k^\zeta \ell_k^{1/\alpha} y),
\end{aligned}$$

and the inequality follows from Lemma 15(a) in [16]. Use Lemma 14 in [16] and (5.25) on the inequality $\mathbb{E}[E_j^u \min\{1, E_k^{-\zeta} x\} \min\{1, E_1^{-\zeta} y\}] \le d_u'(\Gamma(\frac{1}{\alpha}) \vee 1) \min\{1,x\} \min\{1,y\}$ for $k \ge 2$ to get

$$\begin{aligned}
&\mathbb{E}\Big[\ell_{n+1}^r E_j^u \prod_{k=1}^n \Upsilon(E_k^\zeta \ell_k^{1/\alpha} x, E_k^\zeta \ell_k^{1/\alpha} y)\Big] \\
&\quad \le ((1-\rho)^n + \rho^n)\mathbb{E}[\ell_{n+1}^r]\mathbb{E}[E_j^u] + \sum_{k=2}^n \rho(1-\rho)^{k-1} \mathbb{E}\Big[\ell_{n+1}^r E_j^u \min\Big\{1, \frac{b_\rho E_k^{-\zeta}}{\ell_k^{1/\alpha} x}\Big\} \min\Big\{1, \frac{b_{1-\rho} E_1^{-\zeta}}{\ell_1^{1/\alpha} y}\Big\}\Big] \\
&\qquad + \sum_{k=2}^n (1-\rho)\rho^{k-1} \mathbb{E}\Big[\ell_{n+1}^r E_j^u \min\Big\{1, \frac{b_\rho E_1^{-\zeta}}{\ell_1^{1/\alpha} x}\Big\} \min\Big\{1, \frac{b_{1-\rho} E_k^{-\zeta}}{\ell_k^{1/\alpha} y}\Big\}\Big] \\
&\quad \le b_\rho b_{1-\rho} d_u' (1+r)^{-n} [(1-\rho)^n + \rho^n + (\Gamma(1/\alpha) \vee 1)(B_\rho(x,y) + B_{1-\rho}(y,x))],
\end{aligned}$$

where $B_s(x,y) := \sum_{k=2}^\infty s(1-s)^{k-1}(1+r)^k \mathbb{E}\big[L_{k-1}^r \min\big\{1, \ell_k^{-1/\alpha}/x\big\} \min\big\{1, \ell_1^{-1/\alpha}/y\big\}\big]$ for $x,y>0$.

Next we bound some integrals of B_s. Recall that $p+q<\alpha$ and $\mathbb{E}[U^r(1-U)^s] = B(r+1, s+1)$. Thus an application of Fubini's theorem, Lemmas 13(c) and 15(c) in [16] yields

$$\begin{aligned}
&\int_1^\infty \int_1^\infty x^{p-1} y^{q-1} B_\rho(x,y)\,dy\,dx \\
&\quad = \sum_{k=2}^\infty \frac{\rho(1-\rho)^{k-1}}{(1+r)^{-k}} \mathbb{E}\Big[L_{k-1}^r \int_1^\infty \int_1^\infty x^{p-1} y^{q-1} \min\big\{1, \ell_k^{-1/\alpha}/x\big\} \min\big\{1, \ell_1^{-1/\alpha}/y\big\}\,dy\,dx\Big] \\
&\quad = \sum_{k=2}^\infty \frac{\rho(1-\rho)^{k-1}}{(1+r)^{-k}} \mathbb{E}\Big[L_{k-1}^r \Big(\frac{\ell_k^{-p/\alpha}}{p(1-p)} - \frac{1}{p}\Big)\Big(\frac{\ell_1^{-q/\alpha}}{q(1-q)} - \frac{1}{q}\Big)\Big]
\end{aligned}$$

$$\le \sum_{k=2}^{\infty} \frac{\rho(1-\rho)^{k-1}}{(1+r)^{-k}} \mathbb{E}\left[\frac{L_{k-1}^{r} \ell_k^{-p/\alpha} \ell_1^{-q/\alpha}}{pq(1-p)(1-q)}\right] = R_{p,q}(r,\rho)/(\Gamma(1/\alpha) \vee 1).$$

Putting all the above arguments together, the following inequalities imply part (a):

$$\begin{aligned}\mathbb{E}[\ell_{n+1}^{r} E_j^u \eta_+^v \eta_-^w J_{+,p} J_{-,q}] &\le \frac{b_\rho b_{1-\rho} d_u' d_v d_w}{(1+r)^n/c^2} \int_1^\infty \int_1^\infty x^{p-1} y^{q-1} \min\{1,(a_n x)^{-\delta}\} \min\{1,(a_n y)^{-\delta}\} \\ &\quad \times \left((1-\rho)^n + \rho^n + (\Gamma(1/\alpha) \vee 1)\left(B_\rho(x,y) + B_{1-\rho}(y,x)\right)\right) dy dx \\ &\le \frac{b_\rho b_{1-\rho} c^2 d_u' d_v d_w}{(1+r)^n} \left[\left((1-\rho)^n + \rho^n\right) P(a_n,p,\delta) P(a_n,q,\delta) + R_{p,q}(r,\rho) + R_{q,p}(r,1-\rho)\right].\end{aligned}$$

Proof of (b). Again, we use a slightly different combination of some of the previously explained ideas. We begin using (5.5) and the equivalent of (5.23) to obtain

$$\frac{\Gamma(p) X_{-,n}^q}{2^{(q-1)^+} X_{+,n}^p} \le \frac{\Gamma(p) a_n^q \eta_-^{q\zeta}}{X_{+,n}^p} + (X_{-,n} - a_n \eta_-^\zeta)^q \left(\frac{1}{p} + \int_1^\infty x^{p-1} e^{-x X_{+,n}} dx\right). \tag{5.27}$$

It remains to multiply the above expression by $\ell_{n+1}^r E_j^u \eta_+^v \eta_-^w$ and take expectations.

The first term in (5.27) can be bounded as in part (a). The second term $(X_{-,n} - a_n \eta_-^{1-1/\alpha})^q/p$ in (5.27) can be handled as in Lemma 10 (see (5.14) and (5.17)). Indeed, we have

$$\left(X_{-,n} - a_n \eta_-^\zeta\right)^q \ell_{n+1}^r E_j^u \eta_+^v \eta_-^w = \left(\sum_{k=1}^{n} \ell_k^{\frac{1}{\alpha}} E_k^\zeta [G_k]^- \ell_{n+1}^{\frac{r}{q}} E_j^{\frac{u}{q}}\right)^q \eta_+^v \eta_-^w. \tag{5.28}$$

The expected value of (5.28) may be bounded via L^q-seminorms: denote by $\|\vartheta\|_q = \mathbb{E}[\vartheta^q]^{1/q'}$ the L^q-seminorm of ϑ where $q' = q \vee 1$ (which is a true norm if $q \ge 1$). Let $g_q = \mathbb{E}[([G_k]^-)^q]$ and $h_u = \max\{\Gamma(1+u+q\zeta), \Gamma(1+q\zeta)\Gamma(1+u)\}$; observe that when $\alpha < 1$, we have $q\zeta > \alpha - 1 > -1$. Then the triangle inequality and the independence gives

$$\begin{aligned}\left\|\sum_{k=1}^{n} \ell_k^{\frac{1}{\alpha}} E_k^\zeta [G_k]^- \ell_{n+1}^{\frac{r}{q}} E_j^{\frac{u}{q}}\right\|_q^{q'} &\le \left(\sum_{k=1}^{n} \left\|\ell_k^{\frac{1}{\alpha}} \ell_{n+1}^{\frac{r}{q}}\right\|_q \left\|E_k^\zeta E_j^{\frac{u}{q}}\right\|_q \|[G_k]^-\|_q\right)^{q'} \\ &\le \frac{h_u g_q B(1+\frac{q}{\alpha}, 1+r)}{(1+r)^n} \left(\sum_{k=1}^{n} \left(\frac{1+r+\frac{q}{\alpha}}{1+r}\right)^{(1-k)/q'}\right)^{q'} \le \frac{h_u g_q}{(1+r)^n} \frac{B(1+\frac{q}{\alpha}, 1+r)(1+r+\frac{q}{\alpha})}{\left((1+r+\frac{q}{\alpha})^{1/q'} - (1+r)^{1/q'}\right)^{q'}},\end{aligned}$$

which completes the bound on the second term in (5.27) once one notes that η_+ and η_- are independent from the other variables and $\mathbb{E}[\eta_+^v \eta_-^w] = \Gamma(v+1)\Gamma(w+1)$.

The third term in (5.27) may be bounded as follows. Set $s = q/2 < \alpha/2 \le 1$, then use (5.5) to obtain

$$\begin{aligned}&\mathbb{E}\left[\left(\sum_{k=1}^{n} \ell_k^{\frac{1}{\alpha}} E_k^\zeta [G_k]^-\right)^q \ell_{n+1}^r E_j^u \eta_+^v \eta_-^w \int_1^\infty x^{p-1} e^{-x X_{+,n}} dx\right] \\ &\quad \le \int_1^\infty x^{p-1} \mathbb{E}\left[\left(\sum_{k=1}^{n} \ell_k^{\frac{s}{\alpha}} E_k^{s\zeta} ([G_k]^-)^s\right)^2 \ell_{n+1}^r E_j^u e^{-x X_{+,n}} \eta_+^v \eta_-^w\right] dx\end{aligned}$$

$$= 2\int_1^\infty x^{p-1}\sum_{k=1}^n\sum_{i=k+1}^n \mathbb{E}\Big[\ell_k^{\frac{s}{\alpha}}E_k^{s\zeta}([G_k]^-)^s\ell_i^{\frac{s}{\alpha}}E_i^{s\zeta}([G_i]^-)^s\ell_{n+1}^r E_j^u e^{-xX_{+,n}}\eta_+^v\eta_-^w\Big]\,dx$$

$$+\int_1^\infty x^{p-1}\sum_{k=1}^n \mathbb{E}\Big[\ell_k^{\frac{q}{\alpha}}E_k^{q\zeta}([G_k]^-)^q\ell_{n+1}^r E_j^u e^{-xX_{+,n}}\eta_+^v\eta_-^w\Big]\,dx.$$

The previous expression can be dealt with as in (a) and (b). That is, first we average with respect to $(G_n)_{n\in\mathbb{N}}$, using that $\mathbb{E}[([G_k]^-)^s e^{-x[G_k]^+}] = \mathbb{E}[([G_k]^-)^s]$. In particular, one uses Lemma 15(b) in [16] for the terms containing exponentials of G and (5.12) for the terms which do not contain exponentials of G in order to obtain a similar estimate as in (5.24). For the terms which contain $\eta_\pm$, one uses Lemma 15(a) in [16]. Next, one takes expectations with respect to $(E_n)_{n\in\mathbb{N}}$. As in the proof of Lemma 11 (a), one defines the appropriate d'_u which will bound all the required powers of E. Finally, as in steps II) and III) of the proof of Lemma 10, we take the expectations for $(\ell_n)_{n\in\mathbb{N}}$ using Lemma 13(b) in [16]. Each term in the first sum can be bounded by $C'\theta_1^{i+k}(1+r)^{-n}$ for some $C'>0$, $\theta_1\in(0,1)$ (independent of i,k,n) and all $k<i\le n$, whereas each term in the second sum can be bounded by some $C''\theta_2^k(1+r)^{-n}(1+(1-\rho)^n P(a_n,p,\delta))$ for some $C''>0$, $\theta_2\in(0,1)$ (independent of i,k,n) and all $k\le n$. The claim of part (b) then follows, completing the proof. □

Proof of Lemma 12. We will prove the case $s>0$, as the case $s=0$ is very similar. The result is a consequence of Lemma 11(a). Since $[S_k]^+[S_k]^-=0$, observe that using (5.5)

$$Z_m^s \le (m+2)^{[s-1]^+}\left(\eta_+^s+\eta_-^s+\sum_{k=1}^m E_k^s\right) \quad\text{and}\quad \Delta_{n+1}^+ = \ell_{n+1}^{1/\alpha}[S_{n+1}]^+ + (a_{n+1}-a_n)\eta_+^\zeta.$$

Recall that $S_{n+1}=E_{n+1}^\zeta G_{n+1}$, where G_{n+1} and E_{n+1} are independent of each other and of every other random variable in the expectations of the statement. Similarly, $(E_{n+2},\dots,E_m)$ is independent of every other random variable in the expectations of the statement. An application of Lemma 11(a) (and (5.12)) gives the claim if one uses hypothesis (A-κ). The second claim follows similarly using Lemma 11(b). In particular, note that the restriction on r in the case (a) is due to the r-th moment of G_{n+1} while in the case (b), the restriction on r ensures that the power $p-r$ of $X_{+,n+1}$ is non-negative. □

Remark 7. Note that in the above results the parameters for the negative moments can not achieve their upper limit. This is the main reason for not being able to achieve $\alpha'=\alpha$ in Theorem 1.

6. Final remarks

In this section, we gathered some extra technical comments that may be useful for other developments. (i) Our claim for nearly-optimal bound is not proven in two particular situations. That is, in the special case where the stable process is of infinite variation and has only negative jumps (i.e. $\alpha\rho=1$), $\overline{X}_T$ has exponential moments and therefore our bound is suboptimal for large y. However, the optimality of the bound is retained in a neighborhood of 0. Although we do not provide the details here, our methods could be applied to obtain the corresponding exponential bound for the density as $y\to\infty$ in this special case, one may use the techniques in the proof of Proposition 8 (a) and (c) to obtain exponential bounds in x_+. In those cases, we would show that the densities and all their derivatives decay faster than any polynomial x_+^{-p}, $p>0$, as $x_+\to\infty$. In the other extreme, when the infinite variation process has only positive jumps (i.e. $\alpha(1-\rho)=1$), analogous remarks apply. (ii) We stress that the constant C in Proposition 8 is independent of n and $x_\pm>0$ and that $(\alpha-\alpha')C$ is bounded as $\alpha'\to\alpha$.

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Supplementary Material

Supplement to "Joint density of the stable process and its supremum: Regularity and upper bounds" (DOI: 10.3150/23-BEJ1590SUPP; .pdf). This supplement contains estimates on the moments of a stick-breaking process as well as the special Cauchy case $\alpha = 1$.

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