TALL F_σ SUBIDEALS OF TALL ANALYTIC IDEALS

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Abstract. Answering a question of Hrušák, we show that every analytic tall ideal on ω contains an F_σ tall ideal. We also give an example of an F_σ tall ideal without a Borel selector.

Recall that an (non-trivial) *ideal* $\mathcal{I}$ on ω is a collection of subsets of ω with the following properties for every $A, B \subset \omega$:

(1) if $A, B \in \mathcal{I}$ then $A \cup B \in \mathcal{I}$,
(2) if $A \in \mathcal{I}$ and $B \subset A$ then $B \in \mathcal{I}$,
(3) if A is finite, then $A \in \mathcal{I}$,
(4) $\omega \notin \mathcal{I}$.

Identifying subsets of ω with their characteristic functions, an ideal can be considered to be a subset of 2^ω, and as such, can have definability properties, i.e., being Borel, analytic etc. Recall that a set $S \subset 2^\omega$ is analytic if it is a continuous image of the Baire space.

Analytic ideals are remarkably well-behaving, and play a central role in the study of ideals on ω (see, e.g., [9], [5], [7]).

One of the properties of ideals that is enjoyed by most natural examples is called tallness. A family H of subsets of ω is *tall* if for every $x \in [\omega]^\omega$ there exists a $y \in [x]^\omega$ with $y \in H$. Here $[S]^\omega$ ($[S]^{<\omega}$) denotes the collection of countably infinite (finite) subsets of a set S. This notion is fundamental, for example, in the study of *Katětov order* on ideals, see [2]. Recall that an ideal $\mathcal{I}$ is *Katětov below* an ideal $\mathcal{J}$ if there is a function $f : \omega \to \omega$ such that $f^{-1}(A) \in \mathcal{J}$ whenever $A \in \mathcal{I}$.

It has been asked by Hrušák [2, Question 5.6], whether every analytic tall ideal contains an F_σ tall ideal. It is easy to see that this question is equivalent to the seemingly weaker problem whether tall F_σ ideals are dense among tall analytic ideals in the Katětov order. We give a simple, affirmative answer to these questions.

The first author was supported by Leverhulme Research Project Grant RPG-2018-424. The second author was supported by the National Research, Development and Innovation Office – NKFIH, grant no. 113047 and FWF Grants P29999 and M2779.

Theorem 0.1. *Let $\mathcal{I}$ be an analytic tall ideal on ω. Then there exists an F_σ tall ideal $\mathcal{J} \subset \mathcal{I}$. In particular, tall F_σ ideals are dense among tall analytic ideals in the Katětov order.*

Proof. First we find a closed tall set $H \subset \mathcal{I}$.

Let us denote by INC the collection of the strictly increasing $\omega \to \omega$ functions, note that this is a Polish space with the topology inherited from ω^ω. For $x \in [\omega]^\omega$ we will denote by $Enum(x)$ the increasing enumeration of the elements of x.

Let

$$\mathcal{I}' = \{Enum(x) : x \in \mathcal{I} \cap [\omega]^\omega\}.$$

Clearly, $\mathcal{I}'$ is analytic. By standard descriptive set theoretic facts (see, [3, Exercise 14.3]) we can pick a closed set $F \subset INC \times INC$ such that $\mathcal{I}'$ is the projection of F to the first coordinate. Let us define a map $\Psi : F \to 2^\omega$ by letting

$$\Psi(f,g)(n) = 1 \iff n \in \operatorname{ran}(f \circ g),$$

for every $n \in \omega$. Clearly, Ψ is continuous and $\Psi(f,g) \in [\omega]^\omega$ for every $(f,g) \in F$.

We claim that the set $H = \overline{\Psi(F)}$ works. First note that H is tall: indeed, if $x \in [\omega]^\omega$ then there exists $y \in [x]^\omega$ with $y \in \mathcal{I}$, in turn there exists some $g \in INC$ such that $(Enum(y), g) \in F$. So, $\Psi(Enum(y), g) \subset \operatorname{ran}(Enum(y)) = y \subset x$.

Second, we show that $H \subset \mathcal{I}$. Towards a contradiction, assume that $x \in H \setminus \mathcal{I}$. Then, by $[\omega]^{<\omega} \subset \mathcal{I}$ we have that x is infinite. Pick a sequence $(f_n, g_n)_{n\in\omega}$ of elements of F such that $\Psi(f_n, g_n) \to x$. Let $h = Enum(x)$. Then, for any $k \in \omega$ there exists an n_k, such that $(f_m \circ g_m)(k) = h(k)$ holds for $m \geq n_k$. In particular, using monotonicity, we get that $g_m(k) \leq h(k)$ and $f_m(k) \leq (f_m \circ g_m)(k) \leq h(k)$, for every $m \geq n_k$. But then there is $(f,g) \in INC \times INC$ and a subsequence such that $(f_{n_l}, g_{n_l})_{l\in\omega} \to (f,g)$. By the continuity of Ψ, and F being closed we have $\Psi(f,g) = x$ and, as $f \in \mathcal{I}'$, $x \subset \operatorname{ran}(f) \in \mathcal{I}$, a contradiction.

In order to finish the proof of the theorem we just need the following easy observation:

(*) If $H \subset 2^\omega$ is σ-compact, then so is the ideal generated by H.

Indeed, if $S \subset 2^\omega$ is compact, then so are the sets $\{x : (\exists y \in S)(x \subset y)\}$ and $\{x \cup y : x, y \in S\}$ (the latter one is by the continuity of the $\cup$ operation). Thus, the analogous statement holds for σ-compact sets, as well.

So, the ideal generated by $H \cup [\omega]^{<\omega}$ is σ-compact, tall, and contained in $\mathcal{I}$.

□

Let us point out that the idea behind the above argument has already appeared in [6] and [8].

In what follows, we construct two F_σ tall ideals without Borel selectors (recall that a *selector* for a tall ideal on ω is a map $\phi : [\omega]^\omega \to \mathcal{I}$ such that for each $S \in [\omega]^\omega$ we have $\phi(S) \in [S]^\omega \cap \mathcal{I}$). The existence of such an object has been proved by Uzcategui and the first author [1], without giving an explicit example.

Our first example is based on Theorem 0.1 and will use effective descriptive set theory, while the second example will be completely elementary. An argument for including the first one is that it uses a peculiar analytic σ-ideal, which seems to be new.

We will define an ideal on $2^{<\omega}$ (a countable set). Note that $2^{<\omega}$ is endowed with a natural tree structure, and the branches of the tree determine a continuum size family of pairwise almost disjoint subsets of $2^{<\omega}$.

For $f \in 2^\omega$ let us denote by f' the set $\{f \restriction n : n \in \omega\} \subseteq 2^{<\omega}$. In the definition below we will use notions of effective descriptive set theory (see, e.g., [4]). In order to do that we have to fix a recursive bijection identifying ω with $2^{<\omega}$, and hence endowing $\mathcal{P}(2^{<\omega})$ with a recursive Polish structure.

Let $S \in A_0$ iff $S \subset 2^{<\omega}$ and S is an antichain, and let $S \in A_1$ iff

$$\exists f \in 2^\omega \ (S \subset f' \wedge \forall T \in \Delta^1_1(f')(|S \cap T| < \infty \vee |S \setminus T| < \infty)).$$

Roughly speaking, we collect those subsets of a given f' which cannot be split into two infinite pieces by a real computable from f'.

By the Spector-Gandy theorem, the set A_1 is analytic, and, clearly, the set A_0 is closed. It is not hard to check that the ideal $\mathcal{I}$ generated by $A_0 \cup A_1$ is analytic, tall and admits no Borel selector. Now, an application of Theorem 0.1 to $\mathcal{I}$ yields an F_σ tall ideal without a Borel selector.

Finally, let us construct the second, more concrete example. The ideal is defined on $2^{<\omega}$. Again, let $S \in A_0$ iff $S \subset 2^{<\omega}$ is an antichain. Moreover, fix a closed set $F \subset 2^\omega \times \omega^\omega$ that projects onto 2^ω and admits no Borel unformization (see, [3, Section 18] or [4, 4D.11]). Let $S \in B$ if and only if

$$\exists f \in 2^\omega \ (S \subset f' \wedge \exists g \in \omega^\omega \ ((f, g) \in F \wedge g \leq Enum(S))),$$

where we abuse the notation and write $Enum(S)$ for the enumeration of indices $n \in \mathbb{N}$ such that $f \restriction n \in S$, and $\leq$ stands for the pointwise ordering of functions from ω^ω.

Finally, let $\mathcal{J}$ be the ideal generated by $A_0 \cup B$.

Proposition 0.2. *$\mathcal{J}$ is an F_σ tall ideal that doesn't admit a Borel selector.*

Proof. We first check that the set $A_0 \cup B \cup [2^{<\omega}]^{<\omega}$ is σ-compact in $\mathcal{P}(2^{<\omega})$. This is sufficient to show that $\mathcal{J}$ is F_σ by $(*)$ above. The argument will be rather similar to the proof of Theorem 0.1.

Since A_0 is closed, it is enough to show that $\overline{B} \subseteq B \cup [2^{<\omega}]^{<\omega}$. Let $x_n \to x$ be a convergent sequence where $x_n \in B$, and suppose that $x \notin [2^{<\omega}]^{<\omega}$. Pick witnesses $(f_n, g_n) \in 2^\omega \times \omega^\omega$ to x_n, that is, $(f_n, g_n) \in F$ and $g_n \leq Enum(x_n)$. It is easy to see that there is an $f \in 2^\omega$ such that $f_n \to f$ and $x \subseteq f'$. Similarly, one gets $Enum(x_n) \to Enum(x)$. Hence, there must be $g \in \omega^\omega$ and a subsequence $g_{n_l} \to g$ in ω^ω becuase $g_{n_l} \leq Enum(x_n)$. So, we have $(f_{n_l}, g_{n_l}) \to (f, g)$ and $g \leq Enum(x)$. Consequently, $x \in B$ because F is closed.

To see that $\mathcal{J}$ is tall it is enough to realize that every $x \in [2^{<\omega}]^\omega$ either satisfies $|f' \cap x| = \omega$ for some $f \in 2^\omega$ or can be covered by finitely many antichains by König's theorem. In the former case one can easily find an infinte subset of x that dominates some g where $(f, g) \in F$.

Suppose for a contradiction that $\mathcal{J}$ admits a Borel selector S. Let $K = A_0 \cup \overline{B}$. It follows from the second paragraph above that K is closed, $K \subseteq \mathcal{J}$, K generates $\mathcal{J}$ and $B = \overline{B} \cap [2^{<\omega}]^\omega$. By [1, Proposition 4.5] we may assume that $S(x) \in K$ for every $x \in [2^{<\omega}]^\omega$. In particular, $S(f') \in B$ for every $f \in 2^\omega$. Define

$$G = \{(f, g) \in F : g \leq Enum(S(f'))\}.$$

Then $G \subseteq F$ is a Borel set that has σ-compact vertical sections and projects to 2^ω by the definition of B. Such a set has a Borel uniformization by [3, Theorem 18.18], contradicting the choice of F.

□

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