Geometric flows without boundary data at infinity

by

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Contents

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Declarations

The majority of this thesis is a compilation of the author's original work from the preprints [Pea22b] & [Pea22a].

In order to complement the original material in this thesis, in §1 we give an overview of some geometric flows and cite the relevant theorems from the literature. In §2 we give an overview of parabolic Hölder spaces, using notation taken from a 2020 lecture course given by Peter Topping at Warwick. We also summarise a result due to Angenent on zeros of parabolic equations and its application to curve shortening flow [Ang88], [Ang91]. Towards the end of §2 we include a brief exposition of Brakke flows. For this we mainly follow a 2021 lecture course given by Felix Schulze at Warwick, which includes results due to Brakke, Ecker, Huisken, Ilmanen, Neves, Schulze and White.

In §4 we expand upon a construction originally due to Ilmanen in [Ilm92]. In §5, we extend a result of Chen from [CZ06], as well as give variations of results from Topping & Yin in [TY21].

We give extensive references within the text for all background material which we required.

I declare that to the best of my knowledge, the material contained in this thesis is original and my own work except where otherwise indicated. This thesis has not been submitted for a degree at any other university.

Abstract

The intention of this thesis is to explore non-compact objects evolving under geometric flows without any prescribed boundary control. In particular, we focus on Ricci flow and curve shortening flow defined on open manifolds.

We consider the uniqueness of the Cauchy problem for smooth properly embedded curves evolving under curve shortening flow. We construct an example to show that the class of smooth properly embedded solutions is too large to expect uniqueness, and in turn introduce a new suitable subclass for which we conjecture uniqueness when our ambient space is flat. We then show that even within this subclass of solutions, we can have non-uniqueness of the Cauchy problem in general ambient surfaces. Finally, we give a partial characterisation of those ambient surfaces which exhibit this non-uniqueness.

To complement these results, we also consider low dimensional Ricci flow spacetimes. We prove that, for a complete $(2 + 1)$-dimensional Ricci flow spacetime, the spatial-slice at any later time must contain (under the flow of the time vector field) the spatial-slice at any earlier time. We then prove, after imposing a necessary regularity condition on such a complete Ricci flow spacetime, that all of its spatial-slices must agree with one another, and our Ricci flow spacetime is isomorphic to a classical Ricci flow on a fixed ambient surface.

Chapter 1

Introduction

As the title of the thesis suggests, we will be investigating properties of geometric flows. As far as the author is aware, there is currently no formal definition of a geometric flow; we instead motivate geometric flows via the following example, which can be considered as the first formally defined geometric flow, and which commenced this field of research.

In their groundbreaking work [ES64], James Eels and Joseph Sampson introduced the *harmonic map heat flow*. Given a continuously differentiable map between smooth Riemannian manifolds $(M, g) \xrightarrow{f} (X, \overline{g})$, we can define its *Dirichlet energy*

$$E(f) := \int_M |df|^2 d\mu_g,$$

where $d\mu_g$ denotes the volume form of the metric g, and $df \in \Gamma(T^*M \otimes f^*(TX))$, so that in local coordinates

$$|df|^2 = g^{ij} \frac{\partial f^\alpha}{\partial x^i} \frac{\partial f^\beta}{\partial x^j} (\overline{g}_{\alpha\beta} \circ f).$$

Critical points of this functional are known as *harmonic maps*, which solve the Euler-Lagrange equation

$$\tau(f) := \mathrm{Tr}(\nabla df) = 0 \in \Gamma(f^*(TX)),$$

where $\tau(f)$ is known as the *tension field*, which can be represented in local coordinates by

$$\tau(f) = g^{ij} \left(\frac{\partial^2 f^\alpha}{\partial x^i \partial x^j} - \Gamma^k_{ij} \frac{\partial f^\alpha}{\partial x^k} + \overline{\Gamma}^\alpha_{\beta\gamma} \frac{\partial f^\beta}{\partial x^i} \frac{\partial f^\gamma}{\partial x^j} \right) \frac{\partial}{\partial y^\alpha}.$$

In order to construct solutions to such an elliptic problem, Eels and Sampson studied the corresponding parabolic problem, whereby they searched instead for a family of maps evolving in time under the gradient flow of this functional.

Given a smooth map $f : M \times (0, \infty) \to X$, we say it is a solution to the harmonic map heat flow if

$$\frac{\partial f}{\partial t}(x,t) = \tau(f)(x,t), \quad \forall (x,t) \in M \times (0,\infty). \tag{1.0.1}$$

The work of Eels and Sampson, in combination with a later refinement of PHILIP HARTMAN[1], shows that, under reasonable hypotheses on these manifolds, there exists a solution to harmonic map heat flow starting from any continuously differentiable initial data.

Theorem 1.0.1 (Eels-Sampson-Hartman, [ES64], [Har67]). *Suppose (M,g) is a closed Riemannian manifold and $(X,\overline{g})$ is a closed Riemannian manifold with non-positive sectional curvature. Then for any continuously differentiable map $(M,g) \xrightarrow{f} (X,\overline{g})$, there exists a solution $f : M \times (0,\infty) \to X$ to (1.0.1) such that $f(\cdot,t)$ converges in C^1-norm to f as $t \searrow 0$. Furthermore, $f(\cdot,t)$ converges smoothly on M to a harmonic function $(M,g) \xrightarrow{f_\infty} (X,\overline{g})$ as $t \nearrow \infty$.*

With this example in mind, one may describe a geometric flow as the gradient flow associated to a functional whose critical points have some geometric interpretation. The primary focus of this thesis will be two different geometric flows, namely mean curvature flow and Ricci flow. We begin with an overview of mean curvature flow.

1.0.1 Mean curvature flow

Given a smooth embedding of a topological manifold within a Riemannian manifold $M^k \overset{f}{\hookrightarrow} (X^{n+k},\overline{g})$, consider the k-dimensional volume of M under this embedding

$$\mathcal{A}^k(f) := \int_M d\mu_{f^*(\overline{g})}.$$

Critical points of this functional are known as *minimal surfaces*, and solve the Euler-Lagrange equation

$$\vec{H} = \mathrm{Tr}(\mathrm{I\!I}) = 0 \in \Gamma(\nu M),$$

where $\vec{H}$ denotes the mean curvature vector, and $\mathrm{I\!I} \in \Gamma(\mathrm{Sym}^2(T^*M) \otimes \nu M)$ is the second fundamental form, given by

$$\mathrm{I\!I}(v,w) = (\nabla_v w)^{\perp}, \quad \forall v,w \in \Gamma(T^*M).$$

In his doctoral thesis [Bra78], KENNETH BRAKKE introduced the geometric flow corresponding to the gradient flow of this functional, now known as *mean curvature flow*[2]. Given a smooth map $f : M \times (0,T) \to X$, for some $T > 0$, we say that f is a solution to mean curvature flow if

[1] The original work of Eels and Sampson showed that along a divergent sequence of times you could extract a harmonic limit. As well showing uniform convergence in time, Hartman also removed the need for an embedding assumption on the target manifold.

[2] Due to finite time singularities forming, Brakke wanted to allow the topology of the underlying space M to change under the flow. Instead, he considered a weak formulation of this flow where now our objects are given by Radon measures on the corresponding Grassmanian bundle of the ambient space, now known as *Brakke flow* (see §2).

f satisfies

$$\frac{\partial f}{\partial t}(x,t) = \vec{H}(x,t), \quad \forall (x,t) \in M \times (0,T), \tag{1.0.2}$$

where $\vec{H}(x,t)$ denotes the mean curvature vector of the image of $f(\cdot,t) : M \to X$ at the point $f(x,t) \in X$.

Much like harmonic map heat flow, this equation depends on an embedding of M within an ambient space X. Such a flow is referred to as an *extrinsic* geometric flow.

Although one can consider mean curvature flow for manifolds of any dimension and co-dimension, we restrict our attention in this thesis to the case M is one dimensional, with co-dimension 1. In this setting, our mean curvature vector can be written as $\kappa\nu$, where ν is a choice of unit normal, and κ is the geodesic curvature of the curve with respect to this choice of unit normal.

In the case M is closed ($M \cong S^1$), we have the following long-time existence and uniqueness result of Matthew Grayson, which extends the work of Michael Gage and Richard Hamilton on convex curves in the plane.

Theorem 1.0.2 (Gage-Hamilton-Grayson, [GH86], [Gra89]). *Let $(X^2, \overline{g})$ be a smooth Riemannian surface which is convex at infinity*[3]*. Given a smooth curve $\eta_0 : S^1 \to X$, there exists $T > 0$, and a unique continuous function $\eta : S^1 \times [0,T) \to X$ with initial data $\eta(\cdot, 0) = \eta_0$ such that*

- *(i) η is a solution to (1.0.2) on $S^1 \times (0,T)$;*
- *(ii) If $T < \infty$, $\eta(\cdot,t)$ converges uniformly to a point in X as $t \nearrow T$;*
- *(iii) If $T = \infty$, then the curvature of $\eta(\cdot,t)$ converges smoothly to zero as $t \nearrow \infty$.*

1.0.2 Ricci flow

Ricci flow was first introduced by Richard Hamilton in his influential paper [Ham82] as a way of deforming positively curved metrics on 3-manifolds to ones of constant curvature.

Given a smooth manifold M^n equipped with a smooth family of Riemannian metrics $g(t)$ for $t \in (0,T)$, we say that $g(t)$ is a Ricci flow on $M \times (0,T)$, if the metrics solve the equation

$$\frac{\partial g}{\partial t}(t) = -2 \operatorname{Ric} g(t), \tag{1.0.3}$$

at every point in $M \times (0,T)$.

Unlike for the case of harmonic map heat flow or mean curvature flow, Ricci flow depends only on intrinsic geometric properties of the metric without reference to an ambient space X. We refer to such a flow as an *intrinsic* geometric flow.

[3] X is convex at infinity if the convex hull of any compact set is compact

Although Hamilton's original inspiration for Ricci flow was from the gradient flow of the total scalar curvature functional[4]

$$Y(g) := \int_M R_g d\mu_g,$$

Ricci flow was not initially known upon its conception to be a gradient flow. It was only much later in the seminal paper [Per02] of Grisha Perelman that Ricci flow was shown to be (in the case M is closed) the gradient flow of the $\mathcal{F}$-functional

$$\mathcal{F}(g,f) := \int_M \left(R_g + |\nabla f|^2\right) e^{-f} d\mu_g, \quad \forall g \in \Gamma(\mathrm{Sym}^2(T^*M)),\ \forall f \in C^\infty(M),$$

when f evolves via the conjugate heat equation on M with weighted volume form $e^{-f}d\mu_{g(t)}$

$$\frac{\partial f}{\partial t} = -\Delta f + |\nabla f|^2 - R_{g(t)}.$$

Critical points of this functional are *gradient Ricci solitons* (g, f, λ) satisfying

$$\mathrm{Ric}(g) + \nabla^2 f = \lambda \cdot g, \tag{1.0.4}$$

for some $\lambda \in \mathbb{R}$. Setting $f \equiv 0$ in the above equation (1.0.4), we see that these are a generalisation of the concept of an Einstein manifold.

For the case of a closed manifold, we have the following existence and uniqueness result from Hamilton's original paper, the proof of which was later simplified by Dennis DeTurck.

Theorem 1.0.3 (Hamilton-DeTurck, [Ham82, Theorem 14], [DeT83])**.** *Suppose* (M^n, g) *is a closed Riemannian manifold. Then there exists* $T \in (0, \infty]$ *and a unique Ricci flow* $g(t)$ *on* $M \times (0,T)$*, such that* $g(t)$ *converges smoothly to* g *as* $t \searrow 0$*. Moreover, if* $T < \infty$*, then*

$$\lim_{t \nearrow T} \|Rm_{g(t)}\|_{L^\infty(M)} = \infty.$$

1.1 Motivation

For the flows mentioned so far, as they are given by weakly-parabolic equations, we have suitable existence & uniqueness theory under the assumption our domain is closed. If we drop this assumption on our domain, existence and uniqueness questions now become much more subtle. For example, consider the prototypical parabolic equation: given a Riemannian manifold (M^n, g), a smooth function $u : M \times (0,T) \to \mathbb{R}$ solves the *heat equation* if

$$\frac{\partial u}{\partial t} = \Delta u, \quad \forall (x,t) \in M \times (0,T),$$

[4]The flow of this functional is a different geometric flow known as the Yamabe flow.

where Δ is the Laplace-Beltrami operator of (M, g). Even for such a simple parabolic equation, we no longer have uniqueness of the Cauchy problem when M is open. The following result is due to ANDREY TYCHONOFF.

Theorem 1.1.1 (Tychonoff, [Tyc35]). *There exists a continuous non-zero function $u : \mathbb{R} \times [0, \infty) \to \mathbb{R}$, which is smooth on $\mathbb{R} \times (0, \infty)$, and solves the heat equation*

$$\frac{\partial u}{\partial t}(x,t) = \frac{\partial^2 u}{\partial x^2}(x,t), \quad \forall (x,t) \in \mathbb{R} \times (0, \infty),$$

with zero initial data $u(\cdot, 0) \equiv 0$.

This begs the following question which lies at the heart of this thesis:

What can be said about existence and uniqueness of our geometric flows if our domain M is an open manifold?

Let (M^n, g) be a Riemannian manifold with M open. Under certain restrictions on our initial data g, we have the following existence result for Ricci flow due to WAN-XIONG SHI.

Theorem 1.1.2 (Shi, [Shi89b, Theorem 1.1]). *If (M^n, g) is a complete Riemannian manifold with bounded curvature tensor*

$$\|Rm_g\|_{L^\infty(M)} \leq C < \infty,$$

there exists $T(n, C) > 0$ and a solution to Ricci flow $g(t)$ on $M \times (0, T)$ such that $g(t)$ converges locally smoothly to g as $t \searrow 0$.

Moreover, due to the work of BING-LONG CHEN and XI-PING ZHU, such a Ricci flow is unique within the class of solutions with bounded curvature.

Theorem 1.1.3 (Chen-Zhu, [CZ06, Theorem 1.1]). *Let (M^n, g) be a complete Riemannian manifold with bounded curvature tensor*

$$\|Rm_g\|_{L^\infty(M)} \leq C < \infty,$$

Suppose $g_1(t), g_2(t)$ are two solutions to Ricci flow on $M \times (0, T)$ with bounded curvature

$$\|Rm_{g_i(t)}\|_{L^\infty(M)} \leq C < \infty, \quad \forall t \in (0, T),$$

and such that $g_i(t)$ converges locally smoothly to g as $t \searrow 0$. Then $g_1(t) = g_2(t)$, for all $t \in (0, T)$.

If we drop the bounded curvature restrictions, do we still expect an existence and uniqueness result for Ricci flow? In the 2-dimensional case, this question has been fully resolved by the work of GREGOR GIESEN and PETER TOPPING.

Theorem 1.1.4 (Giesen-Topping, [GT11], [Top15]). *Let (M^2, g) be a smooth Riemannian sur-*

face, not necessarily complete. Then there exists $T \in (0, \infty]$ depending on the volume and conformal type of the initial data, and a unique instantaneously complete Ricci flow $g(t)$ on $M \times (0, T)$ such that $g(0)$ converges locally smoothly to g as $t \searrow 0$.

Since the Ricci flow equation (1.0.3) is local, we could instead ask for a family of metrics defined only on an open subset of $M \times (0, T)$, and which solve equation (1.0.3) on this open subset.

Definition 1.1.5. Given a smooth manifold M^n, we say that $\mathcal{M}$ is a spacetime in the ambient space $M \times (0, T)$ if

- $\mathcal{M}$ is an open subset of $M \times (0, T)$ equipped with the product topology.
- The spatial slice of $\mathcal{M}$ at time $t \in (0, T)$,

$$\mathcal{M}_t := \{x \in M : (x, t) \in \mathcal{M}\},$$

  is non-empty, for all $t \in (0, T)$.

Definition 1.1.6. Let $\mathcal{M}$ be a spacetime in $M \times (0, T)$. A Ricci flow on $\mathcal{M}$ is a smooth family of Riemannian metrics $g(t)$ on $\mathcal{M}_t$, for each $t \in (0, T)$, satisfying equation (1.0.3) at every point in $\mathcal{M}$. We call such a Ricci flow complete if $g(t)$ is a complete metric on the spatial slice $\mathcal{M}_t$, for all $t \in (0, T)$.

The following should be considered a simple motivating question:

Which spacetimes $\mathcal{M}$ in a given ambient space $M \times (0, T)$ admit complete Ricci flows?

Example 1.1.7. Suppose that M^2 is a surface. Let $N \subseteq M$ be a fixed subsurface. If $\mathcal{M}$ is chosen so that $\mathcal{M}_t = N$ for every $t \in (0, T)$, then $\mathcal{M}$ does admit a complete Ricci flow. One way to see this would be to fix a metric g on N (with large enough volume depending on the conformal type of N) and use the existence result of Giesen & Topping [GT11] to find an instantaneously complete Ricci flow on $N \times (0, T)$ starting from this initial data. Topologically, these spacetimes looks like a cylinder.

A priori, we may guess that any spacetime that admits a complete Ricci flow should look like a cylinder. Unlike for closed manifolds however, the topology of open manifolds can change within a complete Ricci flow without encountering a singularity.

Example 1.1.8. Fix $t_0 > 0$ and a point $p \in \mathbb{T}^2 := S^1 \times S^1$ in the torus. We choose our ambient space to be $\mathbb{T}^2 \times (0, \infty)$, and our spacetime to be the open subset $\mathcal{M} := \mathbb{T}^2 \times (0, \infty) \setminus \{p\} \times (0, t_0]$. Our spatial slices are

$$\mathcal{M}_t := \begin{cases} \mathbb{T}^2 \setminus \{p\} & : \forall t \in (0, t_0] \\ \mathbb{T}^2 & : \forall t > t_0 \end{cases}.$$

If we take $g(t)$ to be a homothetically expanding complete hyperbolic metric on $\mathcal{M}_t$ for $t \in (0, t_0]$, we can then cap off the hyperbolic cusp at p after time t_0, to give the complete contracting cusp

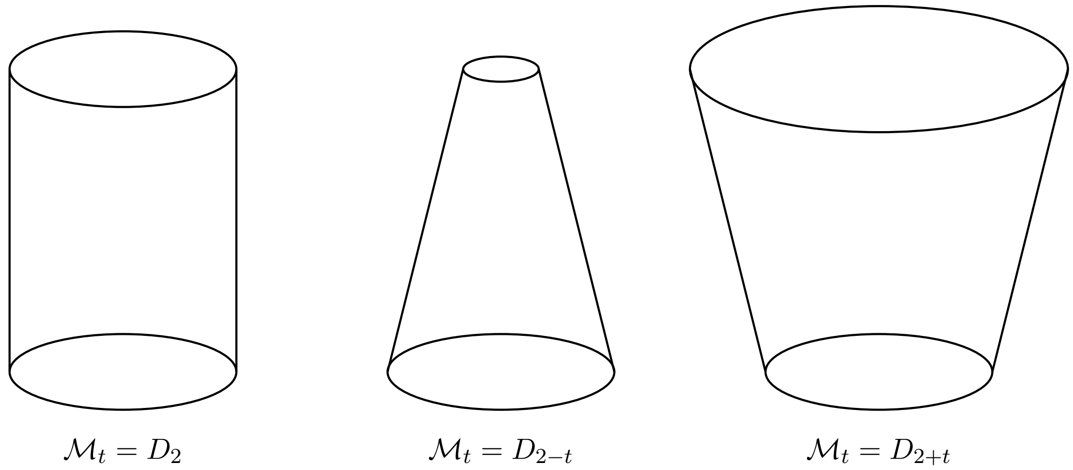

Figure 1.1: An illustration of spacetimes within $\mathbb{C} \times (0, 1)$.

Ricci flow $g(t)$ on $\mathcal{M}_t$, for $t \in (t_0, \infty)$, first constructed in [Top12, Theorem 1.2].

Example 1.1.9. Consider $\mathcal{M} \subseteq \mathbb{C} \times (0, 1)$ to be some subset of the ambient space whose spatial slice varies in time. For example, we could choose $\mathcal{M}_t = D_{2+t}$, the disk centred at the origin of radius $2 + t$ (with respect to the Euclidean metric), for all $t \in (0, 1)$. Can we find a complete Ricci flow on this spacetime $\mathcal{M}$? What about the case that $\mathcal{M}_t = D_{2-t}$, for all $t \in (0, 1)$?

We shall show that the conical spacetimes mentioned in Example 1.1.9 do *not* admit complete Ricci flows. See Corollary 1.3.7.

Remark 1.1.10. Unlike what we have considered so far, there are multiple ways in the literature to formulate a Ricci flow on a changing underlying manifold without reference to an ambient space. In this thesis, we will use the language of a Ricci flow spacetime. By using this formulation, it provides immediate context for our results. However, when working with this more general definition, one should always keep in mind the easy to visualise examples discussed above, where our spacetime is an open subset of a larger ambient space. Nothing is lost by doing so, as we will ultimately reduce our more general Ricci flow spacetime to one within an ambient space by virtue of an embedding lemma (see §5.2).

Results in Ricci flow usually have analogues in mean curvature flow and visa versa. Let us analyse the results of Giesen & Topping in Ricci flow mentioned in Theorem 1.1.4 more closely.

Given any smooth connected Riemannian surface, we can always find a Ricci flow starting from this initial data [GT11]. If the surface is open, there can be multiple Ricci flows starting from the same initial data. In order for the problem to be well-posed, Topping imposes a geometric hypothesis (completeness) on the solutions at each time. That is, if we restrict ourselves to the class of instantaneously complete Ricci flows, there now exists a unique Ricci flow starting from any initial data [Top15].

Due to this result in low dimensional Ricci Flow, we may search for the corresponding result in low dimensional mean curvature flow, prompting the following question:

What are the correct geometric hypotheses to impose on open solutions to curve shortening flow so that the initial-value problem is well-posed?

1.2 Set-up

In contrast to the 2-dimensional case (Theorem 1.1.4), in higher dimensions there are smooth manifolds equipped with smooth metrics from which we do not expect to be able to start the Ricci flow.[5] However, imposing curvature bounds on our initial metric leads to the following well-known short-time existence conjecture in 3-dimensions.

Remark 1.2.1. It is unclear who the conjecture is originally attributed to, although the conjecture arises naturally from the work of Shi in [Shi89a].

Conjecture 1.2.2. *Let* (M^3, g_0) *be a* 3*-dimensional complete Riemannian manifold with non-negative Ricci curvature* $\mathrm{Ric}(g_0) \geq 0$*. Then there exists* $T > 0$ *and a smooth family of complete metrics* $g(t)$ *on* $M \times (0, T)$ *such that* $g(t)$ *converges to* g_0 *locally smoothly as* $t \searrow 0$ *and* $g(t)$ *is a solution to the Ricci flow equation (1.0.3) on* $M \times (0, T)$*.*

A partial resolution to this conjecture was given by Yi Lai in [Lai20] using the language of Ricci flow spacetimes. The following definition was first introduced by Bruce Kleiner & John Lott in [KL17].

Definition 1.2.3 (Kleiner-Lott, [KL17, Definition 1.2])**.** A *Ricci flow spacetime* is a tuple $(\mathcal{M}^{n+1}, \mathfrak{t}, \partial_t, g)$ where

- $\mathcal{M}$ is a smooth and connected $(n+1)$-manifold (without boundary).
- The time function $\mathfrak{t}$ is a smooth submersion $\mathfrak{t} : \mathcal{M} \twoheadrightarrow I$, for some open interval $I \subseteq \mathbb{R}$.
- ∂_t is a smooth vector field on $\mathcal{M}$ satisfying $\partial_t(\mathfrak{t}) \equiv 1$.
- g is a smooth inner product on the bundle $T\mathcal{M}^{spat} := \ker(d\mathfrak{t})$, such that its restriction $g(t)$ to the time slice $\mathcal{M}_t = \mathfrak{t}^{-1}(t)$ is a Riemannian metric, for all $t \in I$.
- g and ∂_t satisfy the Ricci flow spacetime equation $\mathcal{L}_{\partial_t} g = -2\,\mathrm{Ric}(g)$.

A Ricci flow spacetime is said to be complete if $g(t)$ is a complete Riemannian metric on $\mathcal{M}_t$, for every $t \in I$.

Example 1.2.4. Suppose M^n is a connected smooth manifold. An important class of easy to visualise examples of Ricci flow spacetimes are ones lying inside the cylinder $M^n \times \mathbb{R}$, mentioned above in §1.1. More precisely, suppose $\mathcal{M}^{n+1}$ is an open subset of $M^n \times I$ (equipped with the product topology), $\mathfrak{t}$ is the restriction of the standard projection $t : M^n \times I \to I$, and ∂_t is

[5] For example, consider $S^2 \times \mathbb{R}$ equipped with a metric so that it looks geometrically like a countable collection of 3-spheres connected by thinner and thinner necks of increasing length [Top20].

the restriction of $\frac{\partial}{\partial t}$. Since we specify that the time function is a submersion, each spatial slice $\mathcal{M}_t \subseteq M$ is a non-empty open subset of M, for $t \in I$. The Ricci flow spacetime equation then corresponds to the metrics $g(t)$ locally satisfy the usual Ricci flow equation (1.0.3). We say that we have a Ricci flow spacetime $(\mathcal{M}^{n+1}, g)$ in the ambient space $M^n \times I$. If $\mathcal{M}^{n+1}$ is equal to the entire ambient space, then we say it is a *cylindrical* spacetime and denote it by $(M^n \times I, g)$.

Definition 1.2.5. A pair of Ricci flow spacetimes $(\mathcal{M}^{n+1}, \mathfrak{t}, \partial_t, g)$, $(\mathcal{N}^{n+1}, \mathfrak{s}, \partial_s, G)$ are *isomorphic* if $\mathfrak{t}(\mathcal{M}) = I = \mathfrak{s}(\mathcal{N})$, and there exists a smooth diffeomorphism $\Phi : \mathcal{M} \to \mathcal{N}$, such that

(i) For each $t \in I$, the restriction $\Phi : \mathcal{M}_t \to \mathcal{N}_t$ is a diffeomorphism;

(ii) $\mathfrak{t} = \mathfrak{s} \circ \Phi$;

(iii) $\Phi_*(\partial_t) = \partial_s$;

(iv) $g(t) = \Phi^*(G(t))$.

In this case, we write $(\mathcal{M}^{n+1}, \mathfrak{t}, \partial_t, g) \cong (\mathcal{N}^{n+1}, \mathfrak{s}, \partial_s, G)$.

Returning to the conjecture, in [Lai20], Lai constructed a Ricci flow spacetime $(\mathcal{M}^{3+1}, \mathfrak{t}, \partial_t, g)$ containing within it a Ricci flow on $M \times (0, T)$. That is, after restricting to a suitable subset of $\mathcal{M}$, the corresponding Ricci flow spacetime is isomorphic to a cylindrical spacetime $(M^3 \times (0, T), g)$. Moreover, $g(t)$ converges locally smoothly to g_0 as $t \searrow 0$.

Although this Ricci flow $g(t)$ on $M \times (0, T)$ may not necessarily be complete, it lies within the larger Ricci flow spacetime $(\mathcal{M}^{3+1}, \mathfrak{t}, \partial_t, g)$ which does satisfy a completeness like property.[6] We are therefore motivated to ask what the structure of such a Ricci flow spacetime can be. For example, if one could show that the Ricci flow spacetime $(\mathcal{M}^{3+1}, \mathfrak{t}, \partial_t, g)$ constructed by Lai was in fact a complete and cylindrical spacetime itself, this would lead to a full resolution of the short-time existence conjecture stated above, motivating the question:

When is a complete Ricci flow spacetime necessarily cylindrical?

In order to say something about the structure of $\mathcal{M}$, we need to look at how points inside different time-slices are associated to one another under the flow of the vector field ∂_t. Recall the following definition also taken from [KL17].

Definition 1.2.6 (Kleiner-Lott, [KL17, Definition 1.11])**.** Let $(\mathcal{M}, \mathfrak{t}, \partial_t, g)$ be a Ricci flow spacetime. The *worldline* of a point $x \in \mathcal{M}$ is the maximal integral curve $I_x \to \mathcal{M}$, $t \mapsto x(t)$ of ∂_t that passes through x at time $\mathfrak{t}(x) \in I_x$.

More generally, for a subset $U \subseteq \mathcal{M}_s$ of some time-slice ($s \in I$), for each $t \in I$ we set

$$U(t) := \{x(t) \in \mathcal{M}_t : x \in U, t \in I_x\},$$

[6] More precisely, the spacetime is forward 0-complete and weakly backward 0-complete. See [Lai20] for the precise definitions.

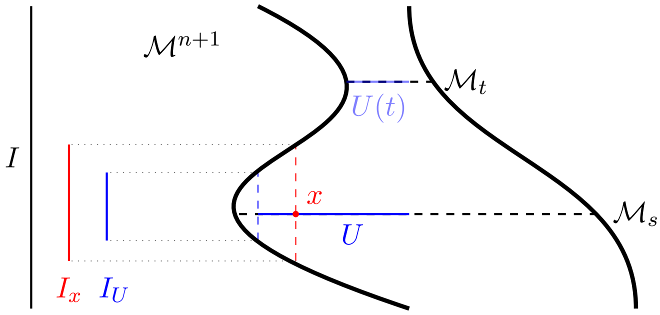

Figure 1.2: An illustration of Definition 1.2.6

and denote those times where the flow lines exist for all points in U by $I_U := \bigcap_{x \in U} I_x$.

Example 1.2.7. Suppose $(\mathcal{M}^{n+1}, g)$ is a Ricci flow spacetime in the ambient space $M^n \times I$. Then, for any point $(x_0, t_0) \in \mathcal{M} \subseteq M \times I$, the interval $I_{(x_0,t_0)}$ is the connected component of $\mathcal{M} \cap (\{x_0\} \times I)$ containing t_0, and $x_0(t) = (x_0, t)$ for each $t \in I_{(x_0,t_0)}$. Similarly, given a subset $U \subseteq M$, we have that $U(t) = \mathcal{M} \cap (U \times \{t\})$, for any $t \in I$. In the special case $(M^n \times I, g)$ is cylindrical, $I_{(x_0,t_0)} = I$ for every $(x_0, t_0) \in M \times I$, and $U(t) = U \times \{t\}$ for every $U \subseteq M$.

Given a complete Riemannian surface $(X^2, \overline{g})$ and a smooth map $\gamma : \mathbb{R} \times (0, T) \to X$ such that $\gamma(\cdot, t)$ is a smoothly embedded curve at each time $t \in (0, T)$, we say that the family of curves $\gamma(\cdot, t)$ evolves under curve shortening flow (CSF) if

$$\langle \partial_t \gamma, \nu \rangle_{\overline{g}} = \kappa, \tag{1.2.1}$$

at every point in $\mathbb{R} \times (0, T)$, where ν is a choice of unit normal vector to the curve, $\kappa := \langle \nabla_\tau \tau, \nu \rangle_{\overline{g}}$ is the geodesic curvature of the curve with respect to ν, and τ is the unit tangent vector to the curve.

If there exists a continuous extension $\gamma : \mathbb{R} \times [0, T) \to X$ of our map, we say that $\gamma(\cdot, t)$ is a solution to CSF with initial data $\gamma(\cdot, 0)$.

Remark 1.2.8. Instead of defining CSF via equation (1.2.1), we could instead require γ to move only in the normal direction, and solve the equation

$$\partial_t \gamma = \kappa \nu, \tag{1.2.2}$$

at each point of $\mathbb{R} \times (0, T)$. Note that every solution to (1.2.2) is a solution to (1.2.1). However, unlike equation (1.2.1), equation (1.2.2) does not allow any tangential component of the vector $\partial_t \gamma$. In many applications, such as when discussing graphical solutions, it is useful to allow our solution to have tangential motion. As such, we make the choice of using equation (1.2.1). In the case of closed curves $\eta : S^1 \times (0, T) \to X$, if η solves (1.2.1) at every point in $S^1 \times (0, T)$, then we can always find a suitable time-dependent family of diffeomorphisms $\{\phi_t : S^1 \to S^1\}_{t \in (0,T)}$, such that the new map $\tilde{\eta} : S^1 \times (0,T) \to X$, defined by

$$\tilde{\eta}(x,t) := \eta(\phi_t(x),t), \quad \forall (x,t) \in S^1 \times (0,T),$$

is a solution to (1.2.2). To see this, consider the unique time-dependent vector field Y_t on S^1, so that $(\eta(\cdot,t))_*(Y_t) = -(\partial_t \eta)^\top$, the negative of the tangential projection of $\partial_t \eta$. Let ϕ_t be the flow of Y_t with $\phi_{t_0} = \mathrm{id}_{S^1}$, for some arbitrarily chosen $t_0 \in (0,T)$. Then

$$\begin{aligned}\partial_t \tilde{\eta}(x,t) &= \partial_t \eta(\phi_t(x),t) + \nabla \eta(\phi_t(x),t) \circ \dot{\phi}_t(x) \\ &= \kappa\nu + (\partial_t \eta(\phi_t(x),t))^\top + [(\eta(\cdot,t))_*(Y_t)](\phi_t(x),t) = \kappa\nu.\end{aligned}$$

Since each $\phi_t : S^1 \to S^1$ is a diffeomorphism, $\mathrm{Im}\,(\eta(\cdot,t)) = \mathrm{Im}\,(\tilde{\eta}(\cdot,t))$ for every $t \in (0,T)$, and it is therefore unnecessary to distinguish between these two equations in the closed case. However, returning to the case of proper curves $\gamma : \mathbb{R} \times [0,T] \to X$, it is no longer true that solutions to (1.2.1) can be re-parameterised to solutions of (1.2.2) without changing their image. It is therefore important to distinguish these two equations from one another.

We have already seen existence and uniqueness in the case our curve is closed (Theorem 1.0.2). In fact, in the plane, the following existence and uniqueness result was shown by JOSEPH LAUER after dropping the regularity of the initial data to a finite length Jordan curve.

Theorem 1.2.9 (Lauer, [Lau13, Theorem 11.1])**.** *Let $J \subseteq \mathbb{R}^2$ be a finite length Jordan curve. Then there exists a continuous parameterisation $\eta_0 : S^1 \to \mathbb{R}^2$ of J so that the following is true: there exists $T > 0$, and a unique continuous function $\eta : S^1 \times [0,T) \to \mathbb{R}^2$ with initial data $\eta(\cdot,0) = \eta_0$ such that*

(i) η is a solution to CSF on $S^1 \times (0,T)$;

(ii) the length of $\mathrm{Im}\,(\eta(\cdot,t))$ converges to the length of J as $t \searrow 0$.

Despite these results in the closed case, similar fundamental questions regarding existence and uniqueness in the non-closed case remain open.

Given any properly embedded Lipschitz curve in the plane, a result due to KAI-SENG CHOU & XI-PING ZHU shows that there always exists a properly embedded solution to curve shortening flow starting from this initial data.

Theorem 1.2.10 (Chou-Zhu, [CZ98])**.** *Let $\gamma_0 : \mathbb{R} \to \mathbb{R}^2$ be a properly embedded, locally Lipschitz curve. Then there exists $T \in (0,\infty]$ and a continuous map $\gamma : \mathbb{R} \times [0,T) \to \mathbb{R}^2$ such that*

(i) γ is smooth and solves (1.2.1) on $\mathbb{R} \times (0,T)$;

(ii) $\gamma(\cdot,t)$ is a smooth properly embedded curve at each time $t \in (0,T)$;

(iii) $\gamma(\cdot,0) = \gamma_0$.

Suppose the initial data $\gamma_0 : \mathbb{R} \to \mathbb{R}^2$ is a properly embedded, locally Lipschitz curve with a graphical representation. That is, there exists $u_0 : \mathbb{R} \to \mathbb{R}$ such that, in Cartesian coordinates on $\mathbb{R}^2$,

$$\gamma_0(x) = (x, u_0(x)), \quad \forall x \in \mathbb{R}.$$

Due to earlier work of Klaus Ecker & Gerhard Huisken, we know the existence of a solution which is also an entire graph at every time.

Theorem 1.2.11 (Ecker-Huisken, [EH89], [EH91])**.** *Given any locally Lipschitz entire graph $u_0 : \mathbb{R} \to \mathbb{R}$, there exists a continuous function $u : \mathbb{R} \times [0, \infty) \to \mathbb{R}$ with initial data $u(\cdot, 0) = u_0$ such that*

(i) The graph of $u(\cdot, t)$ is an entire locally Lipschitz graph;

(ii) The map $\mathbb{R} \times (0, \infty) \ni (x, t) \mapsto (x, u(x, t))$ is smooth and solves equation (1.2.1).

Moreover, within the class of entire graphical solutions, Panagiota Daskalopoulos and Mariel Saez proved uniqueness.

Theorem 1.2.12 (Daskalopoulos-Saez, [DS21, Theorem 1.1])**.** *Fix $T > 0$ and let $u_i : \mathbb{R} \times [0, T) \to \mathbb{R}$ be two continuous functions such that*

(i) $u_1(\cdot, 0) = u_2(\cdot, 0)$, with image a locally Lipschitz graph.

(ii) The maps $\mathbb{R} \times (0, T) \ni (x, t) \mapsto (x, u_i(x, t))$ are smooth and solve equation (1.2.1).

Then $u_1 \equiv u_2$ on $\mathbb{R} \times [0, T)$.

Given some smooth, properly embedded curve $\gamma(\cdot, 0)$ in X, it is natural to ask whether solutions to (1.2.1) starting from this initial data exist and if they are unique within a particular class.

1.3 Main results

Although we were motivated by $(3+1)$-dimensional Ricci flow spacetimes, we shall consider the $(2+1)$-dimensional case as a concrete starting point. The following theorem shows that, for a complete $(2+1)$-dimensional Ricci flow spacetime, worldlines always persist until the final time.

Theorem 1.3.1. *Let $(\mathcal{M}^{2+1}, \mathfrak{t}, \partial_t, g)$ be a complete Ricci flow spacetime with $I = (0, T)$. Then $\mathcal{M}$ is expanding. That is, the vanishing times*

$$T_x := \sup I_x = T, \quad \forall x \in \mathcal{M}.$$

As seen in Example 1.1.8, we cannot expect a general complete Ricci flow spacetime to be cylindrical. This leads us to formulate the following regularity hypotheses that we shall impose on our spacetimes.

Definition 1.3.2. Let $(\mathcal{M}^{n+1}, \mathfrak{t}, \partial_t, g)$ be a Ricci flow spacetime. For any $s, t \in I$, define the *temporal closure* of the time slice $\mathcal{M}_s$ at time t to be

$$\overline{\mathcal{M}_s}(t) := \{x \in \mathcal{M}_t : s \in \overline{I_x}\} \supseteq \mathcal{M}_s(t).$$

We say that the Ricci flow spacetime is *continuous* if

$$(\overline{\mathcal{M}_s}(t))^\circ = \mathcal{M}_s(t), \quad \forall s, t \in I.$$

This condition prevents cusps from being capped off as in Example 1.1.8 (see Lemma 5.2.5). For the next condition, we need the definition of a time-preserving path from [KL17].

Definition 1.3.3 (Kleiner-Lott, [KL17, Definition 1.11])**.** Let $(\mathcal{M}, \mathfrak{t}, \partial_t, g)$ be a Ricci flow spacetime. Suppose $J \subseteq I$ is an interval and let $\eta : J \to \mathcal{M}$ be a smooth path. We say that η is *time-preserving* if

$$\mathfrak{t} \circ \eta(t) = t, \quad \forall t \in J.$$

Given a point in spacetime, we may ask how far backwards in time we can see from that point. More precisely, consider every time-preserving path ending at this point, and then take the infimum of their starting times.

Definition 1.3.4. Let $(\mathcal{M}, \mathfrak{t}, \partial_t, g)$ be a Ricci flow spacetime. Define the *hindsight function* $\mathfrak{h} : \mathcal{M} \to \overline{I}$ by

$$\mathfrak{h}(x) := \inf\{s \in I : \text{there exists a time-preserving path } \eta : [s, \mathfrak{t}(x)] \to \mathcal{M} \text{ with } \eta \circ \mathfrak{t}(x) = x\}.$$

The following condition, in conjunction with continuity, rules out any auxiliary data from being added at positive time.

Definition 1.3.5. Let $(\mathcal{M}, \mathfrak{t}, \partial_t, g)$ be a Ricci flow spacetime with $\inf I = 0$. $\mathcal{M}$ is said to be *initially determined* if $\mathfrak{h} \equiv 0$, where $\mathfrak{h}$ denotes the hindsight function (see Definition 1.3.4).

With these extra hypotheses, complete $(2+1)$-dimensional spacetimes are in fact cylindrical.

Theorem 1.3.6. *Let $(\mathcal{M}^{2+1}, \mathfrak{t}, \partial_t, g)$ be a complete, continuous and initially determined Ricci flow spacetime with $I = (0, T)$. Then $I_x = (0, T)$, for every $x \in \mathcal{M}$.*

Corollary 1.3.7. *Let $(\mathcal{M}^{2+1}, \mathfrak{t}, \partial_t, g)$ be a complete, continuous and initially determined Ricci flow spacetime with $I = (0, T)$. Then there exists a connected smooth surface M^2 such that $(\mathcal{M}^{2+1}, \mathfrak{t}, \partial_t, g)$ is isomorphic to a complete Ricci flow on $M \times (0, T)$. That is, there exists a smooth diffeomorphism $\Phi : \mathcal{M} \to M \times (0, T)$ such that*

(i) For each $t \in (0, T)$, the restriction $\Phi : \mathcal{M}_t \to M \times \{t\}$ is a diffeomorphism;

(ii) $\mathfrak{t} = t \circ \Phi$, where $t : M \times (0, T) \to (0, T)$ denotes the standard projection map;

(iii) $\Phi_*(\partial_t) = \frac{\partial}{\partial t}$*;*

(iv) $\Phi_*(g(t))$ *is a complete Ricci flow on* $M \times (0, T)$*.*

Although we have an existence theorem for solutions to curve shortening flow within the class of properly embedded solutions, it turns out that this class is too large to expect a uniqueness theorem to hold.

Example 1.3.8. There exists a continuous map $\gamma : \mathbb{R} \times [0, \infty) \to \mathbb{R}^2$ such that

- γ is smooth and solves (1.2.1) on $\mathbb{R} \times (0, \infty)$.
- $\gamma(\cdot, t) : \mathbb{R} \to \mathbb{R}^2$ is a proper embedding, for every $t \geq 0$.
- $\operatorname{Im}(\gamma(\cdot, 0))$ is an entire locally Lipschitz graph over the x-axis, but $\operatorname{Im}(\gamma(\cdot, t))$ is not a graph over the x-axis, for any $t > 0$.

In conjunction with the existence result of Ecker-Huisken (Theorem 1.2.11), this example shows that the Cauchy problem for properly embedded curves is ill-posed. In light of this, we introduce the following sub-class of solutions.

Definition 1.3.9. Let $(X, \overline{g})$ be a complete Riemannian surface and $T \in (0, \infty)$. We say that $\gamma : \mathbb{R} \times [0, T] \to X$ is a *uniformly proper solution* to CSF (in X) if

i) $\gamma : \mathbb{R} \times [0, T] \to X$ is a continuous proper map.

ii) $\gamma(\cdot, t) : \mathbb{R} \to X$ is a smooth proper embedding $\forall t \in (0, T]$.

iii) γ is smooth and solves (1.2.1) on $\mathbb{R} \times (0, T)$.

Remark 1.3.10. We restrict ourselves to solutions that are proper as a map on space-time to avoid tangential re-parameterisations which get arbitrarily bad as t goes to zero. We shall see that Example 1.3.8 is not a uniformly proper solution, which shows that the family of curves being uniformly proper (in time) is a necessary condition to impose on a class of solutions in which you expect uniqueness. Moreover, requiring our solution to be uniformly proper is also sufficient for the usual avoidance principle with closed curves to hold (see Theorem 4.2.4).

Within the class of uniformly proper solutions to CSF, the only solution starting from a properly embedded geodesic in the plane is the static solution.

Example 1.3.11. Consider any uniformly proper solution $\gamma : \mathbb{R} \times [0, T] \to \mathbb{R}^2$ to CSF, with initial data $\gamma(\cdot, 0)$ a parameterisation of the x-axis. Closed circles evolve homothetically under CSF, with radius at time t given by the equation $R(t) = \sqrt{R(0)^2 - 2t}$. Fix $\epsilon > 0$ and choose $R(0)$ sufficiently large so that

$$R(0) - R(T) \leq \left(R(T) + \frac{T}{R(T)} \right) - R(T) = \frac{T}{\sqrt{R(0)^2 - 2T}} < \epsilon.$$

Consider the family of circles, all of radius $R(0)$, centred at each point along the lines $\{y = \pm(R(0) + \epsilon)\}$. For any one of these circles, it is initially disjoint from the x-axis, and by Theorem 4.2.4, our uniformly proper solution γ must avoid these shrinking circle as they simultaneously evolve under CSF. In particular, for any $t \in [0, T]$, we have trapped the image of our solution $\gamma(\cdot, t)$ within the 2ϵ-tubular neighbourhood of the x-axis. Thus, our solution must coincide with the static solution.

Definition 1.3.12. Let $(X^2, \overline{g})$ be a complete Riemannian surface. We say that *CSF is unique* on $(X, \overline{g})$ if, for any pair of uniformly proper solutions $\gamma_i : \mathbb{R} \times [0, T_i] \to X$ to CSF (see Definition 1.3.9) with the same initial data

$$\gamma_1(x, 0) = \gamma_2(x, 0), \quad \forall x \in \mathbb{R},$$

then their images agree wherever they are both defined

$$\mathrm{Im}(\gamma_1(\cdot, t)) = \mathrm{Im}(\gamma_2(\cdot, t)), \quad \forall t \in [0, T],$$

with $T := \min\{T_1, T_2\}$.

Tom Ilmanen remarks that for surfaces with no lower curvature bound, it is possible for curves to bloom at infinity and rush inwards under CSF [Ilm94, Remark 3.6]. Drawing parallels with the heat equation, this property is analogous to stochastic incompleteness, whereby heat is allowed to instantly escape at infinity. Returning our attention to Example 1.3.11, we see that for surfaces that allow curves to bloom at infinity, our geometric proof of uniqueness now fails.

Example 1.3.13. Let $\overline{g} = dx^2 + e^{2\phi(x)}dy^2$ be a complete metric on the plane $\mathbb{R}^2$. Consider any uniformly proper solution starting from the x-axis. Since the x-axis is still a geodesic with respect to this metric, we would like to show our solution is the static solution by the same reasoning as in Example 1.3.11. However, we suppose that $\overline{g}$ is chosen in such a way that vertical lines *bloom at infinity*. That is, there exists a smooth function $x : (0, \infty) \to \mathbb{R}$ such that $\lim_{t \searrow 0} x(t) = \infty$, with the property that the vertical lines $\{x = x(t)\}$ are evolving under CSF in $(\mathbb{R}^2, \overline{g})$. Since these vertical lines define a uniformly proper solution (see §4.2), by Theorem 4.2.4, every closed curve moving under CSF is instantly pulled in from infinity, and we no longer have control of the non-closed solution at infinity. See §4.3 for more details.

We demonstrate non-uniqueness by constructing a geodesic line within such a surface that can start moving (without any forcing term) under curve shortening flow.

Theorem 1.3.14. *There exists a smooth, complete metric $\overline{g} = dx^2 + e^{2\phi(x)}dy^2$ on the plane and a uniformly proper solution $\gamma : \mathbb{R} \times [0, 1] \to \mathbb{R}^2$ of CSF in $(\mathbb{R}^2, \overline{g})$ with initial condition $\gamma(\cdot, 0)$ a parameterisation of the x-axis, such that for all $t > 0$, the curve $\mathrm{Im}(\gamma(\cdot, t))$ is not the x-axis.*

Corollary 1.3.15. *There exists a Riemannian surface $(X^2, \overline{g})$ on which CSF is not unique (see Definition 1.3.12).*

Within the class of rotationally symmetric metrics on the plane, we are able to formulate a more precise notion of what it means for a metric to allow curves to bloom at infinity. Given the usual action of the orthogonal group $O(2)$ on $\mathbb{R}^2$, consider a complete smooth $O(2)$-invariant metric $\overline{g}$ on the plane. In polar coordinates (r, θ), the metric has the form

$$\overline{g} = dr^2 + e^{2\phi(r)} d\theta^2 \tag{1.3.1}$$

for some smooth warping function $\phi : (0, \infty) \to \mathbb{R}$. Under equation (1.2.1), the radii of the circles $\partial B_R := \{(R, \theta) : \theta \in S^1\}$ solve the ODE

$$\frac{\partial R}{\partial t}(t) = -\frac{\partial \phi}{\partial r}(R(t)). \tag{1.3.2}$$

We characterise such a metric to allow blooming at infinity if solutions to this ODE can come in from infinity in finite time:

Definition 1.3.16. Consider the plane $(\mathbb{R}^2, \overline{g})$ equipped with a complete smooth $O(2)$-invariant metric, so that in polar coordinates it has the form $\overline{g} = dr^2 + e^{2\phi(r)} d\theta^2$ as in (1.3.1). We say that g *allows blooming at infinity* if there exists $T \in (0, \infty)$ and a solution $R : (0, T) \to (0, \infty)$ to the ODE (1.3.2) such that $R(t) \to \infty$ as $t \searrow 0$. If no such solution exists, we say that g does not allow blooming at infinity.

Within the class of smooth complete $O(2)$-invariant metrics which have non-positive curvature, we prove that if a metric does not allow blooming at infinity, then with respect to this metric we have uniqueness for uniformly proper solutions to CSF which start from a radial geodesic.

Theorem 1.3.17. *Consider a complete smooth $O(2)$-invariant metric $\overline{g}$ with non-positive curvature on the plane. Let $\gamma : \mathbb{R} \times [0, T] \to \mathbb{R}^2$ be a uniformly proper solution to CSF starting from the x-axis. If $\overline{g}$ does not allow blooming at infinity then γ is the static solution to CSF.*

Chapter 2

Background

In this chapter, we introduce relevant notation, as well as give a brief overview of many of the results from the literature we utilise in this thesis.

Notation:

We write $a \lesssim b$ if there exists a constant $C > 0$ such that $a \leq C \cdot b$. If the constant C depends explicity on certain values $\alpha_1, \ldots, \alpha_k$, we write this as $a \lesssim_{\alpha_1,\ldots,\alpha_k} b$. We also establish the following conventions for function spaces. Given a space $F(X)$ of functions $u : X \to \mathbb{R}$ on some topological space X, we define the space $F_{loc}(X)$ to be those functions $v : X \to \mathbb{R}$ satisfying the following: for every $x \in X$, there is a neighbourhood $x \in U \subseteq X$ such that v restricted to U is in $F(U)$. Moreover, we define the space

$$F_c(X) := \{u \in F(X) : \operatorname{supp}(u) \Subset X\}.$$

2.1 Parabolic Hölder spaces

Given a subset $\Omega \subseteq \mathbb{R}^n$, let $C^0(\Omega)$ denote the Banach space of bounded real-valued continuous functions on Ω. For $\alpha \in (0, 1]$, define the α-Hölder semi-norm to be

$$[u]_{\alpha,\Omega} := \sup_{x \neq y \in \Omega} \frac{|u(x) - u(y)|}{|x - y|^\alpha},$$

and hence the space of α-Hölder continuous functions by

$$C^{0,\alpha}(\Omega) := \{u \in C^0(\Omega) : \|u\|_{C^{0,\alpha}(\Omega)} < \infty\},$$

where

$$\|u\|_{C^{0,\alpha}(\Omega)} := \|u\|_{L^\infty(\Omega)} + [u]_{\alpha,\Omega}.$$

Remark 2.1.1. In the case $\Omega \Subset \mathbb{R}$ is an open interval say, then any α-Hölder continuous function has a unique extension to a α-Hölder continuous function on the closure $\overline{\Omega}$. As such, the space we call $C^{0,\alpha}(\Omega)$ is often denoted by $C^{0,\alpha}(\overline{\Omega})$ in the literature, and what we call $C^{0,\alpha}_{loc}(\Omega)$ is often denoted by $C^{0,\alpha}(\Omega)$.

Given any multi-index $a := (a_1, \dots, a_n) \in (\mathbb{N}_0)^n$, we denote the partial derivative of index a by

$$D^a u(x) := \frac{\partial^{a_1}}{\partial x_1^{a_1}} \circ \cdots \circ \frac{\partial^{a_n}}{\partial x_n^{a_n}} u(x).$$

We call $|a| = a_1 + \cdots + a_n$ the order of the partial derivative. Given $k \in \mathbb{N}$, define the space of continuously differentiable functions of order k, denoted by $C^k(\Omega)$, to be

$$C^k(\Omega) := \{u : \Omega \to \mathbb{R} : D^a u \in C^0(\Omega),\ \forall a \in (\mathbb{N}_0)^n \text{ with } |a| \leq k\}.$$

This is a Banach space with respect to the norm

$$\|u\|_{C^k(\Omega)} := \sum_{|a| \leq k} \|D^a u\|_{L^\infty(\Omega)}.$$

Define the Hölder space

$$C^{k,\alpha}(\Omega) := \{u \in C^k(\Omega) : \|u\|_{C^{k,\alpha}(\Omega)} < \infty\},$$

where

$$\|u\|_{C^{k,\alpha}(\Omega)} := \sum_{|a| \leq k} \|D^a u\|_{C^{0,\alpha}(\Omega)}.$$

Consider now $\Omega_T := \Omega \times (0, T) \subseteq \mathbb{R}^n \times \mathbb{R}$. Define the parabolic distance on Ω_T to be

$$d_P((x,t),(y,s)) := \max\{|x-y|, |t-s|^{\frac{1}{2}}\}, \quad \forall (x,t),(y,s) \in \Omega \times (0,T),$$

and the parabolic α-Hölder semi-norm to be

$$[u]_{P^\alpha, \Omega_T} := \sup_{(x,t) \neq (y,s) \in \Omega_T} \frac{|u((x,t)) - u((y,s))|}{d_P((x,t),(y,s))^\alpha}.$$

Using this semi-norm instead, we can define the space of parabolic α-Hölder continuous functions

$$P^{0,\alpha}(\Omega_T) := \{u \in C^0(\Omega_T) : \|u\|_{P^{0,\alpha}(\Omega_T)} < \infty\},$$

where

$$\|u\|_{P^{0,\alpha}(\Omega_T)} := \|u\|_{L^\infty(\Omega_T)} + [u]_{P^\alpha, \Omega_T}.$$

Given a multi-index $a = (a_0, a_1, \dots, a_n) \in (\mathbb{N}_0)^{n+1}$, we define the parabolic partial derivative of index a by

$$D^a u(x,t) := \frac{\partial^{a_0}}{\partial t^{a_0}} \circ \frac{\partial^{a_1}}{\partial x_1^{a_1}} \circ \cdots \circ \frac{\partial^{a_n}}{\partial x_n^{a_n}} u(x,t).$$

We call $|a|_P = 2a_0 + a_1 + \cdots + a_n$ the parabolic order of the partial derivative. Define the space

$$P^k(\Omega_T) := \{u : \Omega_T \to \mathbb{R} : D^a u \in C^0(\Omega_T),\ \forall a \in (\mathbb{N}_0)^{n+1} \text{ with } |a|_P \leq k\}.$$

This is a Banach space with respect to the norm

$$\|u\|_{P^k(\Omega_T)} := \sum_{|a|_P \leq k} \|D^a u\|_{L^\infty(\Omega_T)}.$$

Define the parabolic Hölder space

$$P^{k,\alpha}(\Omega_T) := \{u \in P^k(\Omega_T) : \|u\|_{P^{k,\alpha}(\Omega_T)} < \infty\},$$

where

$$\|u\|_{P^{k,\alpha}(\Omega_T)} := \sum_{|a|_P \leq k} \|D^a u\|_{P^{0,\alpha}(\Omega)}.$$

For any $X \subseteq \mathbb{R}^n$, let $C^\infty(X) := \bigcap_k C^k(X)$ denote the space of smooth functions. Note that, in the case our domain $X = \Omega_T$ admits a parabolic distance, then $C^\infty(\Omega_T) = \bigcap_k P^k(\Omega_T)$ also.

2.2 Monotonicity of zeroes and the intersection principle

Let $\Omega := (a_1, a_2) \Subset \mathbb{R}$, $\Omega_T := \Omega \times (0,T)$, and $\Gamma_T := (\Omega \times \{0\}) \cup (\partial\Omega \times [0,T))$. Consider the linear operator

$$\mathcal{L}(u) := u_t - A(x,t)u_{xx} + B(x,t)u_x + C(x,t)u, \tag{2.2.1}$$

with $A \in P^2(\Omega_T)$, $B \in C^1(\Omega_T)$ $C \in C^0(\Omega_T)$, where $A(x,t) \geq \lambda > 0$, for some $\lambda > 0$.

In his 1988 paper [Ang88], Sigurd Angenent describes the zero set of a solution to such a linear parabolic equation at positive times. In particular, for non-zero solutions on a compact parabolic rectangle with conducive auxiliary data (see Definition 2.2.1 below), the number of zeros is finite and decreasing at positive times:

Given $\alpha \in (0,1]$ and a function $u \in P^{2,\alpha}(\Omega_T)$, we define its zero set

$$Z := \{(x,t) \in \overline{\Omega} \times [0,T) : u(x,t) = 0\},$$

and the zero set at time $t \in [0,T)$ to be

$$Z_t := \{x \in \overline{\Omega} : (x,t) \in Z\}.$$

For any zero $(x,t) \in Z$, we say it is a simple zero if $u_x(x,t) \neq 0$, and a multiple zero if $u_x(x,t) = 0$. We also define a function which counts the number of zeros at each time

$$z : [0,T) \to \mathbb{N}_0 \cup \{\infty\}, \quad z(t) := |Z_t|.$$

The following definition specifies suitable behaviour for solutions on the parabolic walls of our domain for which the following monotonicity theorem will hold. We thus refer to such behaviour as being *conducive*.

Definition 2.2.1 (Conducive auxiliary behaviour, [Ang88])**.** We say that a function $u \in P^{2,\alpha}(\Omega_T)$ is *conducive* if, for each $i \in \{1,2\}$, either:

(i) $u(a_i,t) \neq 0 \quad \forall t \in [0,T)$.

(ii) $u(a_i,t) = 0 \quad \forall t \in [0,T)$.

The following theorem is taken directly from Angenent's paper.

Theorem 2.2.2 ([Ang88, Theorems C & D])**.** *Fix $\alpha \in (0,1]$ and let $u \in P^{2,\alpha}(\Omega_T)$ be a non-zero conducive solution to $\mathcal{L}(u) \equiv 0$ on Ω_T. Then for $t \in (0,T)$, $z(t)$ is finite. Moreover, if $x \in Z_t$ is a multiple zero of u, then*

$$z(t_2) < z(t_1), \quad \text{for each} \quad t_1 < t < t_2.$$

Although Angenent's original result works for positive times, if all of the initial zeros are simple, we may extend the monotonicity all the way up to the initial time:

Lemma 2.2.3. *Fix $\alpha \in (0,1]$ and let $u \in P^{2,\alpha}(\Omega_T)$ be a non-zero conducive solution to $\mathcal{L}(u) \equiv 0$ on Ω_T. Suppose that 0 is a regular value of $u(\cdot,0) : \overline{\Omega} \to \mathbb{R}$. Then $[0,T) \ni t \mapsto z(t)$ is decreasing.*

Remark 2.2.4. It is actually possible to drop the assumption that 0 is a regular value of $u(\cdot,0)$ in this lemma by modifying Angenent's original argument. However, it will not be necessary to do so for our purposes, and so we omit this added complication.

Proof. We first use the Whitney extension theorem (Theorem B.0.4), to extend $u \in C^1(\mathbb{R}^2)$. By our assumptions, we can apply the implicit function theorem at all zeros $x \in Z_0$ to deduce that, for some sufficiently small open neighbourhood $U \ni Z_0$, and some sufficiently small time $t_0 > 0$, the number of zeros of $u(\cdot,t) : U \to \mathbb{R}$ for $t \in (-t_0,t_0)$ is constant. Moreover, on the compact complement $\Omega \setminus U$, we have that u_0 is non-zero. Therefore, after shrinking t_0 possibly, we can conclude that every zero of $u(\cdot,t)$ is contained in U for $t \in (-t_0,t_0)$, and hence $z(t)$ is constant for $t \in [0,t_0)$. This result now follows from Angenent's original result, Theorem 2.2.2. □

This theorem is extremely useful for comparing how many times two solutions intersect one another. Consider the quasi-linear parabolic operator

$$\mathcal{Q}(u) := u_t - A(x, u, u_x)u_{xx} + B(x, u, u_x), \tag{2.2.2}$$

with $A(x, z, p), B(x, z, p) \in C^\infty(\Omega \times \mathbb{R} \times \mathbb{R})$ and $A(x, z, p) > 0$ on $\Omega \times \mathbb{R} \times \mathbb{R}$.

Given two solutions $u_1, u_2 \in C^0(\overline{\Omega} \times [0, T)) \cap C^\infty(\Omega_T)$ of $\mathcal{Q} \equiv 0$, we define their intersection number at time t to be

$$I(t) := |\{x \in \overline{\Omega} : u_1(x, t) = u_2(x, t)\}|, \quad \forall t > 0.$$

We now apply Theorem 2.2.2 and Lemma 2.2.3 to the difference $u_1 - u_2$.

Theorem 2.2.5 (Graphical intersection principle)**.** *Let $u_1, u_2 \in C^\infty(\overline{\Omega} \times [0, T))$ be two distinct solutions to $\mathcal{Q} \equiv 0$ on Ω_T, where their difference $u_1 - u_2$ is conducive. Then $I(t)$ is decreasing and finite, for all $t \in (0, T)$. Moreover, if $u_1(x, t) = u_2(x, t)$ and $\frac{\partial u_1}{\partial x}(x, t) = \frac{\partial u_2}{\partial x}(x, t)$, then*

$$I(t_2) < I(t_1), \quad \text{for each} \quad t_1 < t < t_2.$$

Finally, if 0 is a regular value of $u_1(\cdot, 0) - u_2(\cdot, 0) : \overline{\Omega} \to \mathbb{R}$, then $[0, T) \ni t \mapsto I(t)$ is decreasing.

Proof. Setting $v := u_1 - u_0$ and $u(s) := su_1 + (1 - s)u_0$, we have the differential equation

$$\begin{aligned}
v_t &= A(x, u_1, (u_1)_x)(u_1)_{xx} - B(x, u_1, (u_1)_x) - A(x, u_0, (u_0)_x)(u_0)_{xx} + B(x, u_0, (u_0)_x) \\
&= \int_0^1 \frac{\partial}{\partial s} \left[A(x, u(s), (u(s))_x)(u(s))_{xx} - B(x, u(s), (u(s))_x)\right] ds \\
&= \left[\int_0^1 A(x, u(s), (u(s))_x) ds\right] v_{xx} \\
&+ \left[\int_0^1 \left(\frac{\partial A}{\partial p}(x, u(s), (u(s))_x)(u(s))_{xx} - \frac{\partial B}{\partial p}(x, u(s), (u(s))_x)\right) ds\right] v_x \\
&+ \left[\int_0^1 \left(\frac{\partial A}{\partial z}(x, u(s), (u(s))_x)(u(s))_{xx} - \frac{\partial B}{\partial z}(x, u(s), (u(s))_x)\right) ds\right] v \\
&:= \tilde{A}(x, t)v_{xx} - \tilde{B}(x, t)v_x - \tilde{C}(x, t)v
\end{aligned}$$

In particular, define the linear parabolic operator

$$\widetilde{\mathcal{L}}(u) := u_t - \tilde{A}(x, t)u_{xx} + \tilde{B}(x, t)u_x + \tilde{C}(x, t)u.$$

Note that $\tilde{A}, \tilde{B}, \tilde{C} \in C^\infty(\Omega_T)$ and so there exists some lower bound $\tilde{C} \geq -C_0$ on Ω_T. Moreover, since our original solutions $u_1, u_2 \in C^\infty(\Omega_T)$, the C^1-norms of $u(s)$ are uniformly bounded for $s \in [0, 1]$. By compactness, $A(x, u(s), (u(s))_x)$ is bounded below by a fixed positive constant

$\lambda > 0$, and hence

$$\tilde{A}(x,t) = \int_0^1 A(x, u(s), u(s)_x) ds \geq \int_0^1 \lambda \, ds = \lambda, \quad \forall (x,t) \in \Omega_T.$$

This means $\widetilde{\mathcal{L}}$ is strictly parabolic on Ω_T. We can then apply Theorem 2.2.2 and Lemma 2.2.3 to v, which yields the result. $\square$

By considering two solutions to CSF, around any intersection point, we can write them locally as graphs which solve a suitable quasi-linear parabolic equation. Applying the previous theorem gives the following intersection principle, also originally due to Angenent.

Theorem 2.2.6 (Intersection principle, [Ang91, Theorem 1.3])**.** *Let M_1, M_2 be a pair of compact (possibly with boundary) 1-dimensional manifolds. Let $\eta_i : M_i \times [0,T] \to X$ be two continuous maps which solve CSF (1.2.1) on $M_i^\circ \times (0,T)$. Suppose we have the boundary conditions*

$$\partial(\mathrm{Im}\,(\eta_1(\cdot,t))) \cap \mathrm{Im}\,(\eta_2(\cdot,t)) = \mathrm{Im}\,(\eta_1(\cdot,t)) \cap \partial(\mathrm{Im}\,(\eta_2(\cdot,t))) = \emptyset, \quad \forall t \in [0,T].$$

Then the intersection number between the solutions

$$\iota(t) := |\{(x_1,x_2) \in M_1 \times M_2 : \eta_1(x_1,t) = \eta_2(x_2,t)\}|,$$

is finite and decreasing for $t \in (0,T]$.

As a corollary to this, we can quickly deduce the following version of the avoidance principle.

Corollary 2.2.7. *[Avoidance principle] Let M_1, M_2 be a pair of compact (possibly with boundary) 1-dimensional manifolds. Let $\eta_i : M_i \times [0,T] \to X$ be two continuous maps which solve CSF (1.2.1) on $M_i^\circ \times (0,T)$. Suppose we have the boundary conditions*

$$\partial(\mathrm{Im}\,(\eta_1(\cdot,t))) \cap \mathrm{Im}\,(\eta_2(\cdot,t)) = \mathrm{Im}\,(\eta_1(\cdot,t)) \cap \partial(\mathrm{Im}\,(\eta_2(\cdot,t))) = \emptyset, \quad \forall t \in [0,T].$$

Moreover, suppose that the solutions are initially disjoint

$$\mathrm{Im}(\eta_1(\cdot,0)) \cap \mathrm{Im}(\eta_2(\cdot,0)) = \emptyset.$$

Then the solutions remain disjoint at all later times

$$\mathrm{Im}(\eta_1(\cdot,t)) \cap \mathrm{Im}(\eta_2(\cdot,t)) = \emptyset, \quad \forall t \in [0,T].$$

Proof. If $\overline{d}$ denotes the distance function on $(X, \overline{g})$, then the function $d : M_1 \times M_2 \times [0,T] \to [0,\infty)$,

$$d(x_1,x_2,t) := \overline{d}(\eta_1(x_1,t), \eta_2(x_2,t)),$$

is continuous. Since $M_1 \times M_2$ is compact, the function $D : [0,T] \to [0,\infty)$,

$$D(t) := \inf_{(x_1,x_2)\in M_1\times M_2} D(x_1,x_2,t), \quad \forall t \in [0,T],$$

is also continuous. $D(0) > 0$ since our solutions are initially disjoint. By continuity, D remains positive for a small time. In particular, $\iota(t) = 0$ for some arbitrarily small positive times, which by the intersection principle (Theorem 2.2.6), implies $\iota \equiv 0$. □

Sometimes, we may want to apply the avoidance principle with one of the solutions replaced with a subsolution or a supersolution to the equation. In order to do this, we need some ordering of the solutions that respects this distinction. As such, we state a suitable avoidance principle for graphical solutions, where this ordering can be stated easily, and then apply this theorem locally on a case by case basis depending on the geometry of the situation.

Theorem 2.2.8 (Graphical avoidance principle)**.** *Let $\mathcal{Q}$ be a quasi-linear operator as in (2.2.2), and fix $u_0, u_1 \in P^{2,\alpha}(\Omega_T)$ for some $\alpha \in (0,1]$. Suppose $\mathcal{Q}u_0 \le 0$ and $\mathcal{Q}u_1 \ge 0$ on Ω_T. If $u_0 \le u_1$ on Γ_T, then $u_0 \le u_1$ on Ω_T.*

Proof. Setting $v := u_1 - u_0$ and $u(s) := su_1 + (1-s)u_0$, we repeat the construction of the strictly linear operator $\widetilde{L}$ from the proof of Theorem 2.2.5, and deduce that $\widetilde{L}(v) \ge 0$ on Ω_T. Therefore, by the maximum principle (Theorem A.0.1), we have

$$v(x,t) \ge \min\{0, \inf_{\Gamma_T}\left(ve^{C_0(T-t)}\right)\} \ge 0, \quad \forall (x,t) \in \Omega_T.$$ □

2.3 Ricci flow spacetimes

Let $(\mathcal{M}^{n+1}, \mathfrak{t}, \partial_t, g)$ be a Ricci flow spacetime. Since ∂_t is a smooth vector field on $\mathcal{M}$, we can consider its maximal flow $\{(x,t) \in \mathcal{M} \times I : t \in I_x\} \to \mathcal{M}$, mapping $(x,t) \mapsto x(t)$. The following are standard properties of the flow (e.g see [Lee13])

(a) The set of points we can flow for time t, $\mathcal{M}^{(t)} := \{x \in \mathcal{M} : \mathfrak{t}(x) + t \in I_x\}$, is an open subset of $\mathcal{M}$, for every $t \in \mathbb{R}$.

(b) The map $\mathcal{M}^{(t)} \to \mathcal{M}^{(-t)}$, sending $x \mapsto x(\mathfrak{t}(x) + t)$, is a smooth diffeomorphism.

For any $s,t \in I$, consider flowing all of the points in the spatial slice at time s into the spatial slice at time t. This is $\mathcal{M}_s(t)$ in the notation from Definition 1.2.6. Similarly, consider $\mathcal{M}_t(s)$, which we get from flowing all of the points in the spatial slice at time t into the spatial slice at time s. Since $\mathcal{M}_s(t) = \mathcal{M}_t \cap \mathcal{M}^{(s-t)}$, (a) tells us that $\mathcal{M}_s(t)$ is open in $\mathcal{M}_t$. Similarly $\mathcal{M}_t(s)$ is open in $\mathcal{M}_s$. Then by (b), we have the smooth diffeomorphism $\mathcal{M}_t(s) \to \mathcal{M}_s(t)$, mapping $x \mapsto x(t)$. In particular, can conclude that the flow of ∂_t preserves open subsets within our spatial slices:

Lemma 2.3.1. *If $U \subseteq \mathcal{M}_s$ is open for some $s \in I$, then $U(t) \subseteq \mathcal{M}_t$ is open, $\forall t \in I$.*

Proof. $U \cap \mathcal{M}_t(s)$ is open in $\mathcal{M}_s$, whose image under the diffeomorphism $\mathcal{M}_t(s) \to \mathcal{M}_s(t)$, $x \mapsto x(t)$, is $U(t)$. Since $\mathcal{M}_s(t)$ is open in $\mathcal{M}_t$, the result follows. $\square$

Fix an open subset of a time slice, $U \subseteq \mathcal{M}_s$, for some $s \in I$. Given an interval $J \subseteq I$, we define the parabolic cylinder

$$U(J) := \bigcup_{t \in J} U(t) \subseteq \mathcal{M}. \tag{2.3.1}$$

For $x \in \mathcal{M}$ and $r > 0$, consider the special case where U is the ball centred at x of radius r

$$B(x, r) := B_{g(\mathfrak{t}(x))}(x, r) \subseteq \mathcal{M}_{\mathfrak{t}(x)}.$$

In the language of equation (2.3.1), we set

$$C(x, r) := [B(x, r)](\mathfrak{t}(x) - r^2, \mathfrak{t}(x) + r^2),$$

to be the parabolic cylinder centred at x of radius r.

Given a general parabolic cylinder $U(J)$, we say that it is unscathed if $J \subseteq I_U$ (see Definition 1.2.6). Since $\mathfrak{t}$ is a smooth submersion, for any $x \in \mathcal{M}$, there exists smooth coordinates $(x_1, \cdots, x_n, \mathfrak{t})$ locally around x, so that $\partial_t \cdot x_i \equiv 0$ for $i \in \{1, \ldots, n\}$. In particular, we can choose $r > 0$ sufficiently small so that the parabolic cylinder $C(x, r)$ is unscathed.

Lemma 2.3.2. *Fix $s \in I$. For any open subset $U \subseteq \mathcal{M}_s$, and any open sub-interval $J \subseteq I$, the parabolic cylinder $U(J)$ is open in $\mathcal{M}$.*

Proof. Fix $x \in U(J)$. For r sufficiently small, the parabolic cylinder $C(x, r)$ is unscathed and open in $\mathcal{M}$. Since $U(\mathfrak{t}(x))$ is open in $\mathcal{M}_{\mathfrak{t}(x)}$ and J is open in I, shrinking r if necessary, $C(x, r) \subseteq U(J)$. $\square$

Lemma 2.3.3. *Fix $s \in I$. Suppose $K \Subset \mathcal{M}_s$. Then there exists an open sub-interval $J \subseteq I$ containing s, such that $K(J)$ is unscathed.*

Proof. Cover K by sufficiently small, unscathed parabolic cylinders centred at points in K, and use the compactness of K. $\square$

Consider now the map $\Psi : U(J) \subseteq \mathcal{M} \to U \times J$, given by the inverse of the flow lines of ∂_t

$$\Psi(x) := (x(s), \mathfrak{t}(x)), \quad \forall x \in U(J).$$

This is a diffeomorphism onto it's image $\Psi(U(J)) = \{(x,t) \in U \times J : t \in I_x\}$. Note that $(\Psi)_*(\partial_t) = \frac{\partial}{\partial t}$, and the push forward of the metric satisfies the usual Ricci flow equation

$$\frac{\partial}{\partial t}\Psi_*(g) = \Psi_*(\mathcal{L}_{\partial_t} g) = \Psi_*(-2\operatorname{Ric} g) = -2\operatorname{Ric}\Psi_*(g), \quad \text{on } \Psi(U(J)).$$

We call such a map, $\Psi : U(J) \subseteq \mathcal{M} \to U \times J$, *cylindrical coordinates* on $U(J)$.

2.4 Brakke flows

The following is a collection of results regarding Brakke flows from the literature that we will use later. We begin with some standard definitions (e.g see [Ilm94]).

Definition 2.4.1. Given a manifold X^n, let $G(k,X) \xrightarrow{\pi} X$ denote the Grassmanian k-plane bundle over X. A *k-dimensional varifold V* in X is a Radon measure on $G(k,X)$.

Note that, for a k-dimensional varifold V in X, the push-forward measure $\mu := \pi_*(V)$ is a Radon measure on X. Since we want to consider Brakke flows in X, we need to equip it with a smooth metric $(X, \overline{g})$. By Nash's embedding theorem, we can find an embedding $X^n \hookrightarrow \mathbb{R}^N$. Therefore, for $x \in X$ and $\lambda > 0$, we can consider the re-scaled Radon measure on X

$$\mu_{x,\lambda}(B) := \lambda^{-k}\mu(\lambda \cdot B + x), \quad \forall B \in \mathcal{B}.$$

Given $P \in \pi^{-1}(x)$ and $m \in \mathbb{N}$, we say that P is the tangent plane of μ at x with multiplicity m if

$$\mu_{x,\lambda} \rightharpoonup m \cdot \mathcal{H}^k|_P,$$

as $\lambda \searrow 0$, where $\mathcal{H}^k$ denotes the k-dimensional Hausdorff measure.

Definition 2.4.2. A Radon measure μ on X is said to be *integer k-rectifiable* if, for μ almost every point $x \in X$, there exists a tangent plane P of μ at x, with some multiplicity $m \in \mathbb{N}$. Given a varifold V, if its push-forward $\pi_*(V)$ is an integer k-rectifiable Radon measure on X, then we say that V is a k-dimensional integral varifold.

Definition 2.4.3 (Brakke, [Bra78])**.** A *k-dimensional integral Brakke flow* in $(X^n, \overline{g})$ is a 1-parameter family of Radon measures on X, $[0,T] \ni t \mapsto \mu_t$, so that

1. For almost every $t \in [0,T]$, there exists a k-dimensional integral varifold $V(t)$ in X, such that $\mu(t) = \pi_*(V(t))$. Moreover, the first variation of $V(t)$ satisfies

$$\delta V_t(Y) = -\int_X \langle H_t, Y\rangle \, d\mu_t, \quad \forall Y \in \Gamma_0(TX),$$

for some $H_t \in \mathscr{L}^2_{loc}(TX, \mu_t)$.

2. If $f \in C^1_c(X^n \times [0,T])$ with $f \geq 0$, then

$$\int_X f(\cdot,T)d\mu_T - \int_X f(\cdot,0)d\mu_0 \leq \int_0^T \int_X \left(-|H_t|^2 f + H_t \cdot \nabla f + \partial_t f\right) d\mu_t dt. \tag{2.4.1}$$

One major benefit of working with this weaker notion of solution is that they enjoy the following compactness theorem.

Theorem 2.4.4 (Ilmanen, [Ilm94]). *Suppose that $[0,T] \ni t \mapsto \mu^n_t$ is a sequence of integral Brakke flows in $(X^n, \overline{g})$. Assume that we have locally uniform volume bounds:*

$$\sup_{n\in\mathbb{N}} \sup_{t\in[0,T]} \mu^n_t(K) \leq C_K < \infty, \quad \forall K \Subset X.$$

Then, after passing to a subsequence, we have

1. $\mu^n_t \rightharpoonup \mu_t$, *for every* $t \in [0,T]$;
2. $[0,T] \ni t \mapsto \mu_t$ *is an integral Brakke flow;*
3. *For almost every $t \in [0,T]$, after possibly passing to a further subsequence which depends on t, the associated varifolds converge weakly $V^n(t) \rightharpoonup V(t)$.*

The following theorem gives a local area bound for Brakke flows in general ambient spaces. Although the statement and proof of this theorem presented here are original, I believe similar results are likely to exist in the literature.

Lemma 2.4.5 (Local area bound). *Let $[0,T] \ni t \mapsto \mu_t$ be an k-dimensional Brakke flow in $(X^n, \overline{g})$. Given $x_0 \in X$ and $\rho > 2$, choose $\sigma \geq 0$ such that* $\sec_{\overline{g}} \geq -\sigma$, *in* $\overline{B_{\overline{g}}(x_0, 2\rho)}$. *Then, for any $t \in [t_0, t_0 + \frac{\rho^2}{2(k+1)+4k\rho\sqrt{\sigma}}] \subseteq [0,T]$ we have*

$$\mu_t\left(B(x_0,\rho)\right) + \int_{t_0}^t \int_{B(x_0,\rho)} |H_\tau|^2 d\mu_\tau d\tau \leq 8\mu_{t_0}\left(B(x_0,2\rho)\right). \tag{2.4.2}$$

Proof. Consider the metric $g_\sigma := dr^2 + \sigma^{-1}\sinh(\sqrt{\sigma}r)g_{S^{n-1}}$ on $[0,\infty) \times S^{n-1}$, of constant sectional curvature $-\sigma$. If d_σ denotes the distance function from $\{r = 0\}$, then by a direct calculation (e.g see [Pet06, Section 2.3]) we have

$$\nabla^2 d_\sigma(r) = \sqrt{\sigma}\coth(\sqrt{\sigma}r)(g_\sigma - dr^2).$$

In particular

$$\nabla^2 d_\sigma(r)(X,X) \leq \sqrt{\sigma}, \quad \forall r \geq 1,$$

where X is any unit tangent vector (with respect to g_σ). Let d denote the distance function in $(X, \overline{g})$ from x_0. Working within $B_{\overline{g}}(x_0, 2\rho) \setminus B_{\overline{g}}(x_0, 1)$, we can apply the Hessian comparison theorem to get a lower bound

$$\Box d = -\sum_{i=1}^{k} \nabla^2 d(e_i, e_i) \geq -\sum_{i=1}^{k} \nabla^2 d_\sigma(\tilde{e}_i, \tilde{e}_i) \geq -k\sqrt{\sigma},$$

and hence

$$\Box d^2 = 2d \cdot \Box d - 2k \geq -(2k + 4k\rho\sqrt{\sigma}).$$

We now smooth out the distance function on $B(x_0, 1)$ to get a lower bound within the entire ball $B(x_0, 2\rho)$. Define $\chi : \mathbb{R} \to \mathbb{R}$ by

$$\chi(x) := \begin{cases} 1 & : x \leq 1 \\ x & : x \geq 2 \end{cases},$$

with $\chi' \in [0, 1]$ and $\chi'' \in [0, 2]$. Choosing $C := 2(k + 1) + 4k\rho\sqrt{\sigma}$, we then have

$$\Box \left(\chi(d^2)\right) = \chi'(d^2) \cdot \Box d^2 - \chi''(d^2) \cdot |\nabla d|^2 \geq -C,$$

inside the entire ball $B(x_0, 2\rho)$, and hence

$$\Box \left(\chi(d^2) + C \cdot (t - t_0)\right) \geq 0.$$

Therefore, the cut-off function

$$f(x, t) := \left(1 - \frac{\chi(d^2) + C \cdot (t - t_0)}{4\rho^2}\right)_+^3,$$

satisfies $\Box f \leq 0$ in the region $B(x_0, 2\rho) \times (t_0, t_0 + \rho^2/C)$. By the defining property of a Brakke flow, we have

$$\begin{aligned}
\int f(\cdot, t) d\mu_t - \int f(\cdot, t_0) d\mu_{t_0} &\leq \int_{t_0}^{t} \int \left(-f|H_\tau|^2 + \langle H_\tau, \nabla f\rangle + \frac{\partial f}{\partial \tau}\right) d\mu_\tau d\tau \\
&= \int_{t_0}^{t} \int \left(-f|H_\tau|^2 + \frac{df}{d\tau}\right) d\mu_\tau d\tau \\
&= \int_{t_0}^{t} \int -f|H_\tau|^2 d\mu_\tau d\tau.
\end{aligned}$$

For any point (x, t) with $\chi(d^2) \leq \rho$ and $t \in [t_0, t_0 + \rho^2/C]$, we have

$$1/8 = \left(1 - \frac{\rho^2 + \rho^2}{4\rho^2}\right)_+^3 \leq f(x, t),$$

and hence

$$\begin{aligned}\mu_t\left(B(x_0,\rho)\right)+\int_{t_0}^{t}\int_{B(x_0,\rho)}|H_\tau|^2 d\mu_\tau d\tau &\leq 8\left(\int f(\cdot,t)d\mu_t+\int_{t_0}^{t}\int f|H_\tau|^2 d\mu_\tau d\tau.\right)\\ &\leq 8\int f(\cdot,t_0)d\mu_{t_0}\\ &\leq 8\mu_{t_0}\left(B(x_0,2\rho)\right). \qquad \square\end{aligned}$$

Remark 2.4.6. Note that in the special case that $(X,\overline{g})$ is Euclidean space, then we can choose f to be a simpler cut-off function (see equation (2.4.4)) and $C=2k$. We then recover the local area bound in Euclidean space due to Ecker.

Lemma 2.4.7 (Ecker, [Eck12, Proposition 4.9])**.** *Let $[0,T]\ni t\mapsto \mu_t$ be an k-dimensional Brakke flow in $\mathbb{R}^N$. Given any $\rho>0$, for $x_0\in\mathbb{R}^N$ and $t\in[t_0,t_0+\frac{\rho^2}{2k}]\subseteq[0,T]$, we have*

$$\mu_t\left(B(x_0,\rho)\right)+\int_{t_0}^{t}\int_{B(x_0,\rho)}|H_\tau|^2 d\mu_\tau d\tau\leq 8\mu_{t_0}\left(B(x_0,2\rho)\right). \tag{2.4.3}$$

For the remainder of this section we restrict ourselves to the case of a Brakke flow in Euclidean space. Given any point in space-time $X_0:=(x_0,t_0)\in\mathbb{R}^N\times(0,\infty)$, let $\Phi_{X_0}:\mathbb{R}^N\times(-\infty,t_0)\to(0,\infty)$ be a re-scaled (depending on k) solution to the conjugate heat equation on $\mathbb{R}^N$ starting from the Dirac measure centred at the point X_0:

$$\Phi_{X_0}(x,t):=\left(4\pi(t_0-t)\right)^{-k/2}e^{-\frac{|x-x_0|^2}{4(t_0-t)}},\quad \forall(x,t)\in\mathbb{R}^N\times(-\infty,t_0).$$

The following definition was introduced by Tobias Colding and William Minicozzi.

Definition 2.4.8 (Colding-Minicozzi, [CM12])**.** Given a Radon measure μ on $\mathbb{R}^N$, and a point in space-time $X_0=(x_0,t_0)\in\mathbb{R}^N\times(0,\infty)$, let

$$\lambda_k(\mu,X_0):=\int\Phi_{X_0}(x,0)\ d\mu(x).$$

Define the *k-entropy* of μ to be

$$\lambda_k(\mu):=\sup_{X_0\in\mathbb{R}^N\times(0,\infty)}\lambda_k(\mu,X_0).$$

A related concept to the entropy of a measure is the area ratio of a measure, which will be used later in the statement of a pseudolocality result.

Definition 2.4.9. Given a Radon measure μ on $\mathbb{R}^N$, a point $x\in\mathbb{R}^N$ and $r>0$, let

$$\Lambda_k(\mu,x,r):=\frac{\mu(B(x,r))}{\omega_k r^k},$$

where ω_k denotes the measure of the unit ball in $\mathbb{R}^k$. Define the *k-area ratio* of μ to be

$$\Lambda_k(\mu) := \sup_{x\in\mathbb{R}^N, r>0} \Lambda_k(\mu, x, r).$$

Using integration by parts, one can show that the k-entropy and the k-area ratio of a Radon measure are comparable. A reference for this can be found in recent work of Brian White.

Theorem 2.4.10 (White, [Whi19, Theorem 9.1])**.** *There exists $c_k > 0$ such that, for any Radon measure μ on $\mathbb{R}^N$,*

$$c_k \cdot \Lambda_k(\mu) \leq \lambda_k(\mu) \leq \Lambda_k(\mu).$$

Remark 2.4.11. Of course, these two values could be infinite for a given Radon measure, but the proof given by White shows that if either one is infinite, then the other must also be infinite, and the inequality still holds.

For any $R > 0$, define the cut-off function

$$f_R(x,t) := \left(1 - \frac{|x|^2 + 2kt}{R^2}\right)_+^3, \quad \forall (x,t) \in \mathbb{R}^N \times (-\infty, \infty), \tag{2.4.4}$$

and for any point $X_0 := (x_0, t_0) \in \mathbb{R}^N \times (0, \infty)$, the re-centred function $f_{R,X_0}(x,t) := f_R(x - x_0, t - t_0)$.

Definition 2.4.12. Let $[0,T] \ni t \mapsto \mu_t$ be a k-dimensional Brakke flow in $\mathbb{R}^N$. For any point $X_0 := (x_0, t_0) \in \mathbb{R}^N \times (0, T]$ and $R > 0$, define the *R-local Gaussian density ratios* at the point X_0 to be

$$\Theta^R(\mu_t, X_0, r) := \int_{\mathbb{R}^N} \Phi_{X_0}(x, t_0 - r^2) \cdot f_{R,X_0}(x, t_0 - r^2) d\mu_{t_0 - r^2}(x), \quad \forall r \in (0, \sqrt{t_0}\,]. \tag{2.4.5}$$

Remark 2.4.13. If we knew that μ_t has finite entropy, we could remove the cut-off function in equation (2.4.5), and instead define the *Gaussian density ratios* at the point X_0 by

$$\Theta(\mu_t, X_0, r) := \int_X \Phi_{X_0}(x, t_0 - r^2) d\mu_{t_0 - r^2}(x), \quad \forall r \in (0, \sqrt{t_0}\,].$$

However, for a general Brakke flow, we cannot be sure that this integral is well-defined, and so we consider the local version instead.

The following monotonicity formula was originally discovered by Gerhard Huisken in the setting of mean curvature flows, before being extended to weak flows by Tom Ilmanen and localised by Klaus Ecker.

Theorem 2.4.14 (Huisken-Ilmanen-Ecker, [Hui90], [Ilm95], [Eck12])**.** *Let $[0,T] \ni t \mapsto \mu_t$ be a*

k-dimensional Brakke flow in $\mathbb{R}^N$. Fix $X_0 := (x_0, t_0) \in \mathbb{R}^N \times (0, T]$ and $R > 0$. Then the map

$$(0, \sqrt{t_0}\ ] \ni r \mapsto \Theta^R(\mu_t, X_0, r),$$

is increasing in r. Moreover, the limiting value

$$\Theta(\mu, X) := \lim_{r \searrow 0} \Theta^R(\mu_t, X_0, r),$$

is independent of R, known as the Gaussian density of the Brakke flow at X_0.

Corollary 2.4.15. *Let $[0, T] \ni t \mapsto \mu_t$ be a k-dimensional Brakke flow in $\mathbb{R}^N$. Then the k-entropy $[0, T] \ni t \mapsto \lambda_k(\mu_t)$ is a decreasing function. Thus*

$$\sup_{t \in [0,T]} \Lambda_k(\mu_t) \lesssim_k \lambda_k(\mu_0).$$

Given a Brakke flow $(0, T) \ni t \mapsto \mu_t$ in $\mathbb{R}^N$, and a point in spacetime $X \in \mathbb{R}^2 \times (0, T)$, we say that X is a regular point if for some small space-time neighbourhood $U \ni X$, the Brakke flow corresponds to a smooth mean curvature flow inside of U. For the following class of Brakke flows, we can use the Gaussian density to infer regularity of the flow about a point.

Definition 2.4.16. A k-dimensional Brakke flow $[0, T) \ni t \mapsto \mu_t$ in $\mathbb{R}^N$ is *unit-regular* if, for every space-time point $X \in \mathbb{R}^N \times (0, T)$ with unit Gaussian density $\Theta(\mu_t, X) = 1$, X is a regular point.

This class of Brakke flows is useful due to the following result.

Theorem 2.4.17 (Schulze-White, [SW16]). *The class of unit-regular Brakke flows is closed under weak convergence of Brakke flows. Moreover, there exists $\epsilon_{gap}(k, N) > 0$ such that, if $[0, T) \ni t \mapsto \mu_t$ is a unit-regular k-dimensional Brakke flow in $\mathbb{R}^N$, and $\theta(\mu_t, X) < 1 + \epsilon_{gap}$ for some space-time point $X \in \mathbb{R}^N \times (0, T)$, then X is a regular point.*

Given a point $(x_0, y_0) \in \mathbb{R}^k \times \mathbb{R}^{N-k}$ and $r > 0$, define the box

$$I_r(x_0, y_0) := \{(x, y) \in \mathbb{R}^N : |x - x_0| < r, |y - y_0| < r\} = B_r^k(x_0) \times B_r^{N-k}(y_0).$$

The following pseudolocality result is due to Tom Ilmanen, André Neves & Felix Schulze.

Theorem 2.4.18 (Ilmanen-Neves-Schulze, [INS19, Theorem 1.5, Remarks 1.6]). *Let $[0, T) \ni t \mapsto \mu_t$ be a unit regular k-dimensional Brakke flow in $\mathbb{R}^N$, with k-area ratios bounded by D. Let $M_t := \operatorname{supp}(\mu_t)$. Then for any $\eta > 0$, there exists $\epsilon, \delta > 0$ depending on n, k, η and D such that, if $(x_0, y_0) \in M_0$ and $M_0 \cap I_1((x_0, y_0))$ can be written as a graph over $B_1^k(x_0)$ with Lipschitz constant less than ϵ, then for each $t \in (0, \delta^2)$, $M_t \cap I_\delta(x_0, y_0)$ can be written as a Lipschitz graph over $B_\delta^k(x_0)$, with Lipschitz constant less than η, and height bounded by $\eta\delta$.*

Chapter 3

Graphical curve shortening flow with respect to warped metrics

Here we introduce some results about the long-time existence and regularity of graphical solutions to curve shortening flow with respect to different metrics on the plane. Later, we shall use these solutions as approximations to proper solutions, as well as solutions to foliate regions of our space-time with.

3.1 Graphical formulas

We begin by fixing a metric $\overline{g} := dx^2 + e^{2\phi(x)}dy^2$ on the plane for some smooth $\phi : \mathbb{R} \to \mathbb{R}$, where (x, y) are the standard cartesian coordinates. We will consider graphical solutions to CSF with respect to this metric. That is, we suppose we have a curve satisfying CSF such that you can either write $x(y, t)$ as a function of y and t, or $y(x, t)$ as a function of x and t. Since ϕ is independent of y, a solution to CSF remains a solution after a translation along the y-axis. In the case $x(y, t)$, translating along the y-axis corresponds to a horizontal translations of the graph, and in the case $y(x, t)$, a vertical translation. As such, we refer to these cases as *horizontal* or *vertical* graphs respectively. It is a routine calculation to show that the geodesic curvature κ of our curve is given by

$$\text{For a horizontal graph } x(y,t):\ \kappa = \frac{\phi' e^{\phi}(e^{2\phi} + 2x_y^2) - e^{\phi} x_{yy}}{(e^{2\phi} + x_y^2)^{\frac{3}{2}}} \tag{3.1.1}$$

$$\text{For a vertical graph } y(x,t):\ \kappa = \frac{\phi' e^{\phi} y_x (y_x^2 e^{2\phi} + 2) + e^{\phi} y_{xx}}{(1 + e^{2\phi} y_x^2)^{\frac{3}{2}}} \tag{3.1.2}$$

Substituting into (1.2.1) gives the graphical formulations for CSF on $(\mathbb{R}^2, \overline{g})$

$$x_t = \frac{x_{yy}}{e^{2\phi} + x_y^2} - \phi'(x)\left(1 + \frac{x_y^2}{e^{2\phi} + x_y^2}\right), \tag{3.1.3}$$

which in divergence form is

$$x_t = \frac{\partial}{\partial y}\left(e^{-\phi}\tan^{-1}(x_y e^{-\phi})\right) + \phi'(x)\left(x_y e^{-\phi}\tan^{-1}(x_y e^{-\phi}) - 1\right).$$

$$y_t = \frac{y_{xx}}{1+e^{2\phi}y_x^2} + y_x\phi'(x)\left(1+\frac{1}{1+e^{2\phi}y_x^2}\right), \tag{3.1.4}$$

which in divergence form is

$$y_t = \frac{\partial}{\partial x}\left(e^{-\phi}\tan^{-1}(y_x e^{\phi})\right) + \phi'(x)\left(y_x + e^{-\phi}\tan^{-1}(y_x e^{\phi})\right).$$

Since we refer to these PDEs frequently throughout the rest of the thesis, we introduce the notation

$$\mu(x,p) := \frac{1}{1+p^2e^{2\phi(x)}} \in (0,1], \quad \nu(x,p) := \frac{1}{e^{2\phi(x)}+p^2} \in (0, e^{-2\phi(x)}].$$

With this notation, we have the quasi-linear operators

$$\mathcal{H}(x) := x_t - \nu(x,x_y)x_{yy} + \phi'(x)(1+\nu(x,x_y)x_y^2),$$
$$\mathcal{V}(y) := y_t - \mu(x,y_x)y_{xx} - \phi'(x)(1+\mu(x,y_x))y_x,$$

so that equations (3.1.3) and (3.1.4) become $\mathcal{H}=0$ and $\mathcal{V}=0$ respectively.

We can consider what the geodesics in our space now look like. Since geodesics are invariant under the flow, they are useful barriers. Setting $\kappa = 0$ in equation (3.1.2) yields the first order ODE

$$(y_x e^{\phi})_x + \phi'(x)\cdot(y_x e^{\phi})\cdot(1+y_x^2e^{2\phi}) = 0.$$

Given an interval $\Omega \ni 0$, we can solve this equation over Ω to give solutions

$$y_x = \frac{m}{e^{\phi}\sqrt{e^{2\phi}-m^2}},$$

for each constant m with $|m| < \inf_\Omega e^{\phi}$. Thus, we can parameterise the geodesics with a vertical graphical representation over Ω, by

$$\{\sigma_{m,h} : \mathbb{R} \to \mathbb{R} \mid |m| < \inf_{\Omega} e^{\phi}, h \in \mathbb{R}\},$$

where

$$\sigma_{m,h}(x) := h + \int_0^x \frac{m}{e^{\phi(s)}\sqrt{e^{2\phi(s)}-m^2}}ds, \quad \forall x \in \Omega.$$

We note that $\sigma_{0,h}$ parameterises the horizontal line $\{y=h\}$. For $m \neq 0$ however, $\sigma'_{m,h} \neq 0$ everywhere and the corresponding geodesic also has a horizontal graphical representation $\eta_{m,h} := (\sigma_{m,h})^{-1}$.

3.2 Existence and uniqueness for non-strictly parabolic equations

Recall the quasi-linear parabolic operator we considered in §2, equation (2.2.2)

$$\mathcal{Q}(u) := u_t - A(x, u, u_x)u_{xx} + B(x, u, u_x),$$

with $A(x, z, p), B(x, z, p) \in C^\infty(\overline{\Omega} \times \mathbb{R} \times \mathbb{R})$ and $A(x, z, p) > 0$ on $\overline{\Omega} \times \mathbb{R} \times \mathbb{R}$. In one spatial dimension, we can always write our operator $\mathcal{Q}$ in divergence form

$$\mathcal{Q}(u) := u_t - \frac{\partial}{\partial x}\left(a(x, u, u_x)\right) + b(x, u, u_x), \tag{3.2.1}$$

with $A(x, z, p) = \frac{\partial a}{\partial p}(x, z, p)$ and $b(x, z, p) = B(x, z, p) + \frac{\partial a}{\partial x}(x, z, p) + \frac{\partial a}{\partial z}(x, z, p) \cdot p$.

Fix $T < \infty$. Given $u_0 \in C^{2,\alpha}(\Omega)$, and $\psi \in P^{2,\alpha}(\Omega_T)$ satisfying the compatibility conditions of order 0 ($\psi = u_0$ on $\partial\Omega \times \{0\}$), we consider the Dirichlet problem

$$\begin{cases} \mathcal{Q}(u) = 0 & \text{in } \Omega_T \\ u = u_0 & \text{on } \overline{\Omega} \times \{0\} \\ u = \psi & \text{on } \{a, b\} \times [0, T) \end{cases} \tag{3.2.2}$$

For each $s \in (0, T]$, we can restrict to the shorter time Dirichlet problem

$$(D_s) := \begin{cases} \mathcal{Q}(u) = 0 & \text{in } \Omega_s \\ u = u_0 & \text{on } \overline{\Omega} \times \{0\} \\ u = \psi & \text{on } \{a, b\} \times [0, s) \end{cases}$$

The following theorem can be found in [LSU88]. The proof uses the existence theory for linear operators and the Leray-Schauder principle to interpolate between the quasi-linear operator and the standard linear heat equation.

Theorem 3.2.1 (Existence and uniqueness for strictly parabolic operators, [LSU88, Chapter V, Theorem 6.1]). *Suppose that for each $M > 0$ the coefficients of $\mathcal{Q}$ from (2.2.2) and (3.2.1) satisfy*

(i) $B(x, z, 0) \geq 0, \quad \forall (x, z) \in \overline{\Omega} \times \mathbb{R}$.

(ii) $A \lesssim 1$, $|a|, |\frac{\partial a}{\partial z}| \lesssim (1 + |p|)$, $|\frac{\partial a}{\partial x}|, |b| \lesssim (1 + |p|)^2, \quad \forall (x, z, p) \in \overline{\Omega} \times [-M, M] \times \mathbb{R}$.

(iii) $1 \lesssim A(x, z, p), \quad \forall (x, z, p) \in \overline{\Omega} \times [-M, M] \times \mathbb{R}$.

Then there exists a unique solution $u \in P^{2,\alpha}(\Omega_T)$ to the Dirichlet problem (D_T).

Unfortunately, our operators $\mathcal{H}$ and $\mathcal{V}$ are not strictly parabolic (they fail to satisfy (iii)), since both $\nu(\cdot, p)$ and $\mu(\cdot, p)$ tend to zero as p gets large. So, suppose now that our operator $\mathcal{Q}$ only satisifies criteria (i) and (ii). We instead have the following short-time existence and uniqueness theorem.

Theorem 3.2.2. *Suppose that for each $M > 0$ the coefficients of $\mathcal{Q}$ from (2.2.2) and (3.2.1) satisfy*

(i) $B(x, z, 0) \geq 0, \quad \forall (x, z) \in \overline{\Omega} \times \mathbb{R}.$

(ii) $A \lesssim 1,\ |a|, |\frac{\partial a}{\partial z}| \lesssim (1 + |p|), |\frac{\partial a}{\partial x}|, |b| \lesssim (1 + |p|)^2, \quad \forall (x, z, p) \in \overline{\Omega} \times [-M, M] \times \mathbb{R}.$

Then, there exists $s \in (0, T]$ and a unique $u \in P^{2,\alpha}(\Omega_s)$ such that u solves the Dirichlet problem (D_s).

Proof. We employ a standard trick of modifying the coefficients of the operator for sufficiently large values of z and p.

Choose $C := 2 \cdot |u_0|_{C^1(\Omega)} < \infty$ and let χ be any smooth bump function supported on $[-2, 2]$ and equal to 1 on $[-1, 1]$. We define a new coefficient

$$\tilde{a}(x, z, p) := a(x, z, 0) + \int_0^p \chi\left(\frac{s}{C}\right) \frac{\partial a}{\partial p}(x, z, s) + \left(1 - \chi\left(\frac{s}{C}\right)\right)\, ds,$$

a new quasi-linear operator

$$\widetilde{\mathcal{Q}}(u) := u_t - \frac{\partial}{\partial x}\left(\tilde{a}(x, u, u_x)\right) + b(x, u, u_x),$$

and a class of new Dirichlet problems

$$(\widetilde{D_s}) := \begin{cases} \widetilde{\mathcal{Q}}(u) = 0 & \text{in } \Omega_s \\ u = u_0 & \text{on } \overline{\Omega} \times \{0\} \\ u = \psi & \text{on } \{a, b\} \times [0, s] \end{cases}$$

Observe the following:

- For any $p \leq C$, $\tilde{a}(x, z, p) = a(x, z, p)$, which means $\tilde{B}(x, z, 0) = B(x, z, 0)$ and $\widetilde{Q}$ satisfies (i).
- We have that

$$\tilde{A}(x, z, p) = \frac{\partial \tilde{a}}{\partial p}(x, z, p) = \chi\left(\frac{p}{C}\right) \frac{\partial a}{\partial p}(x, z, p) + \left(1 - \chi\left(\frac{p}{C}\right)\right) > 0,$$

  and so

$$|\tilde{A}| \leq |\frac{\partial \tilde{a}}{\partial p}| + 1 \lesssim 1.$$

- Using that a is smooth, we have that

$$
\begin{aligned}
|\tilde{a}(x,z,p)| &\leq \sup_{p\in[-C,C]} |a(x,z,\cdot)| + \int_{-2C}^{2C} \left(|\frac{\partial a}{\partial p}|(x,z,p) \right) ds + |p| \\
&= \|a\|_{L^\infty(\overline{\Omega}\times[-M,M]\times[-2C,2C])} + \|\frac{\partial a}{\partial p}\|_{L^1(\overline{\Omega}\times[-M,M]\times[-2C,2C])} + |p| \lesssim (1+|p|)
\end{aligned}
$$

- Similarly, from the equations

$$
\begin{aligned}
\frac{\partial \tilde{a}}{\partial z}(x,z,p) &= \frac{\partial a}{\partial z}(x,z,0) + \int_0^p \chi\left(\frac{s}{C}\right) \frac{\partial^2 a}{\partial z \partial p}(x,z,s) ds, \\
\frac{\partial \tilde{a}}{\partial x}(x,z,p) &= \frac{\partial a}{\partial x}(x,z,0) + \int_0^p \chi\left(\frac{s}{C}\right) \frac{\partial^2 a}{\partial x \partial p}(x,z,s) ds,
\end{aligned}
$$

  we can use a being smooth to deduce that $\widetilde{\mathcal{Q}}$ satisfies (ii).

Finally, since $\tilde{A} \equiv 1$ outside of a compact set, $\widetilde{\mathcal{Q}}$ satisfies (iii). By Theorem 3.2.1, there exists $\tilde{u} \in P^{2,\alpha}(\Omega_T)$ solving $(\widetilde{D_T})$. Moreover, by the continuity of $\tilde{u}$ and $\tilde{u}_x$, there exists $s \in (0,T]$ such that

$$
|\tilde{u}(\cdot,t)|_{C^1(\Omega)} \leq C, \quad \forall t \in [0,s].
$$

In particular, since $\widetilde{\mathcal{Q}} = \mathcal{Q}$ on $\overline{\Omega} \times \mathbb{R} \times [-C,C]$, we have that $\tilde{u} \in P^{2,\alpha}(\Omega_s)$ solves (D_s). Finally, if $u_1, u_2 \in P^{2,\alpha}(\Omega_s)$ are solutions to (D_s), then by applying the avoidance principle (Theorem 2.2.8) to their difference, we have $u_1 = u_2$ on Ω_s. □

Corollary 3.2.3. *Suppose that for each $M > 0$ the coefficients of $\mathcal{Q}$ from (2.2.2) and (3.2.1) satisfy*

(i) $B(x,z,0) \geq 0, \quad \forall (x,z) \in \overline{\Omega} \times \mathbb{R}$.

(ii) $A \lesssim 1$, $|a|, |\frac{\partial a}{\partial z}| \lesssim (1+|p|), |\frac{\partial a}{\partial x}|, |b| \lesssim (1+|p|)^2, \quad \forall (x,z,p) \in \overline{\Omega} \times [-M,M] \times \mathbb{R}$.

Then there exists a unique pair $\tau \in (0,T]$ and $u : \overline{\Omega} \times [0,\tau) \to \mathbb{R}$ such that

(A) u solves (D_τ).

(B) $u \in P^{2,\alpha}(\Omega_s)$, $\forall s \in (0,\tau)$.

(C) If $\tau < T$, then $u \notin P^{2,\alpha}(\Omega_\tau)$ and $\limsup_{s\to\tau} |u(\cdot,s)|_{C^1(\Omega)} = \infty$.

Proof. By the previous theorem

$$
\tau := \sup\{s \in (0,T] : \exists u \in P^{2,\alpha}(\Omega_s) \text{ such that } u \text{ solves } (D_s)\},
$$

is well defined. By uniqueness the solutions agree on overlaps, and give a well defined, unique function $u : \overline{\Omega} \times [0,\tau) \to \mathbb{R}$ satisfying properties (A) and (B). For $\tau < T$, assume that $u \in P^{2,\alpha}(\Omega_\tau)$. Then $u(\cdot,\tau) \in C^{2,\alpha}(\Omega)$ and we can reapply Theorem 3.2.2 to get a solution

$\hat{u} \in P^{2,\alpha}(\Omega_{\tau,\tau+\epsilon})$ for some $\epsilon > 0$. By virtue of the PDE that they solve, u and $\hat{u}$ piece together to give $u \in P^{2,\alpha}(\Omega_{\tau+\epsilon})$ solving $(D_{\tau+\epsilon})$, contradicting the definition of τ. So $u \notin P^{2,\alpha}(\Omega_\tau)$.

Finally, assume that $\limsup_{s\to\tau} |u(\cdot,s)|_{C^1(\Omega)} = \frac{C}{2} < \infty$ for some positive constant $C > 0$. Consider the Dirichlet problem $(\widetilde{D_\tau})$ defined in the proof of Thoerem 3.2.2. By Theorem 3.2.1 there exists a unique $\tilde{u} \in P^{2,\alpha}(\Omega_\tau)$ solving $(\widetilde{D_\tau})$. Since $\widetilde{\mathcal{Q}} = \mathcal{Q}$ on $\overline{\Omega} \times \mathbb{R} \times [-C, C]$, u also solves $(\widetilde{D_s})$ for $s \in (0,\tau)$. Therefore, by the uniqueness of solutions, $\tilde{u}$ is an extension of u in $P^{2,\alpha}(\Omega_\tau)$, contradicting what we have just previously shown. □

3.3 Curve shortening flow as a Dirichlet problem

We consider the following Dirichlet problems, the first of which is for vertical graphs.

Vertical graphs

Let $\Omega \Subset \mathbb{R}$ and choose some initial data $Y : \Omega \to \mathbb{R}$. We trivially extend Y to a function on $\Omega \times [0,\infty)$ by making it constant in time, so that using it for auxillary data will correspond to fixing the endpoints of the arc. For each $s \in (0,\infty)$, consider the Dirichlet problem for vertical graphs

$$V(s) := \begin{cases} \mathcal{V}(y) = 0 & \text{in } \Omega_s \\ y = Y & \text{on } \Gamma_s \end{cases} \tag{3.3.1}$$

In order to apply our existence and uniqueness theorem, we must check the following bounds on the coefficients of the operator $\mathcal{V}$:

- When $p = 0$, $-\phi'(x)(1 + \mu(x,p)) \cdot p = 0$, for all $(x,y) \in \overline{\Omega} \times \mathbb{R}$.
- Given a fixed positive constant $M > 0$, for any $(x,y,p) \in \overline{\Omega} \times [-M,M] \times \mathbb{R}$, we have the inequalities

$$\begin{gathered}
\mu(x,p) \le 1, \\
|e^{-\phi(x)} \arctan(pe^{\phi(x)})| \le \frac{\pi}{2} \cdot |e^{-\phi(x)}| \lesssim 1, \\
\frac{\partial}{\partial y}\left(e^{-\phi(x)} \arctan(pe^{\phi(x)})\right) = 0, \\
|\frac{\partial}{\partial x}\left(e^{-\phi(x)} \arctan(pe^{\phi(x)})\right)| \le \frac{\pi}{2} \cdot |\phi'(x)| \cdot |e^{-\phi(x)}| + |p| \cdot |\phi'(x)| \cdot \mu(x,p) \lesssim (1+|p|), \\
|-\phi'(x)\left(p + e^{-\phi(x)} \arctan(pe^{\phi(x)})\right)| \le |\phi'(x)| \cdot |p| + |\phi'(x)| \cdot |e^{-\phi(x)}| \cdot \frac{\pi}{2} \lesssim (1+|p|).
\end{gathered}$$

Therefore, under the assumption that our initial data $Y \in P^{2,\alpha}(\Omega)$ for some $\alpha \in (0,1]$, we can apply Corollary 3.2.3 to our Dirichlet problem, to deduce that there exists $T \in (0,\infty]$ and a unique maximal solution $y : \overline{\Omega} \times [0,T) \to \mathbb{R}$ satisfying

(A) $y \in P^{2,\alpha}(\Omega_s)$, $\forall s \in (0,T)$.

(B) y solves $V(s)$, $\forall s \in (0,T)$.

(C) If $T < \infty$, then $\limsup_{s \to T} |y(\cdot, s)|_{C^1(\Omega)} = \infty$.

After applying interior Schuader estimates (Theorem A.0.4), we also have that $y \in C^\infty_{loc}(\Omega \times (0,T))$.

Horizontal graphs

Given $c \in \mathbb{R}$, consider the unique maximal solution to equation (3.1.3) starting with constant initial condition $x \equiv c$. We note that this solution remains constant in y at all later times, and so the corresponding curves will always be straight lines parallel to the y-axis. We denote this solution by $c(t)$. From the equation $\mathcal{H} = 0$, we see that $c : [0, T') \to \mathbb{R}$ is the maximal solution to the ODE

$$\frac{\partial c}{\partial t} = -\phi'(c(t)), \quad c(0) \equiv c \in \mathbb{R}.$$

Given some initial data $c \in \mathbb{R}$, we say that $c(t)$ is *immortal* if our solution exists for all positive times $c : [0, \infty) \to \mathbb{R}$.

Unlike for vertical graphs where we keep the endpoints fixed, we will instead use these solutions $c(t)$ for the auxiliary data. For each $c \in \mathbb{R}$ with $c(t)$ immortal, and $s \in (0, \infty)$, consider the Dirichlet problem for horizontal graphs

$$H_c(s) := \begin{cases} \mathcal{H}(x) = 0 & \text{in } (0,1) \times (0,s) \\ x = c & \text{on } [0,1] \times \{0\} \\ x(0,t) = c(t), \ x(1,t) = c & \forall t \in (0,s). \end{cases} \tag{3.3.2}$$

so that on the parabolic wall $\{y = 1\}$ the endpoint of the curve is fixed, but on the parabolic wall $\{y = 0\}$ the endpoint is moving at the same rate as the constant solution with this value.

Again, we must check the following bounds on the coefficients of the operator $\mathcal{H}$:

- When $p = 0$, $\phi'(x)(1 + \nu(x,p)p^2) = \phi'(x)$, for all $(y, x) \in [0,1] \times \mathbb{R}$.
- Given a fixed positive constant $M > 0$, for any $(y, x, p) \in [0,1] \times [-M, M] \times \mathbb{R}$, we have

the inequalities

$$\nu(x,p) \leq e^{-2\phi(x)} \lesssim 1,$$
$$|e^{-\phi(x)} \arctan(pe^{-\phi(x)})| \leq \frac{\pi}{2} \cdot |e^{-\phi(x)}| \lesssim 1,$$
$$|\frac{\partial}{\partial x}\left(e^{-\phi(x)} \arctan(pe^{-\phi(x)})\right)| = |-e^{-\phi(x)}\phi'(x) \arctan(pe^{-\phi(x)}) + \nu(x,p)\phi'(x)p|$$
$$\leq |\phi'(x)| \cdot |e^{-\phi(x)}| \cdot \frac{\pi}{2} + |e^{-2\phi(x)}| \cdot |\phi'(x)| \cdot |p| \lesssim (1+|p|),$$
$$\frac{\partial}{\partial y}\left(e^{-\phi(x)} \arctan(pe^{-\phi(x)})\right) = 0,$$
$$|-\phi'(x)\left(pe^{-\phi(x)} \arctan(pe^{-\phi(x)}) - 1\right)| \leq |\phi'(x)| \left(|p| \cdot |e^{-\phi(x)}| \cdot \frac{\pi}{2} + 1\right) \lesssim (1+|p|).$$

Therefore, under the assumption that $\phi' \geq 0$, we can apply Corollary 3.2.3 to our Dirichlet problem, for any $c \in \mathbb{R}$ with $c(t)$ immortal, to deduce that there exists $T_c \in (0,\infty]$ and a unique maximal solution $g_c : [0,1] \times [0,T_c) \to [0,\infty)$ satisfying,

(A) $g_c \in P^{2,1}((0,1) \times (0,s))$, $\forall s \in (0,T_c)$.

(B) g_c solves $H_c(s)$, $\forall s \in (0,T_c)$.

(C) If $T_c < \infty$ then $\limsup_{s\to T_c} |g_c(\cdot,s)|_{C^1([0,1])} = \infty$.

Again, using interior Schuader estimates (Theorem A.0.4), we also have that $g_c \in C^\infty_{loc}((0,1) \times (0,T_c))$.

We shall now show that, under reasonable assumptions, each of the maximal solutions we mentioned above is immortal. This is not obvious a priori; intuitively, solutions to the Dirichlet problem converge towards a geodesic between the endpoints, which need not be graphical. In order to show that the solutions are immortal, it suffices to show that the solutions and their gradients cannot blow up in finite time, as otherwise this would contradict conclusion (C) from Corollary 3.2.3

3.3.1 Preservation of monotonicity and long-time existence

The following subsection shows that, if our initial data is monotonic, then a solution to the Dirichlet problems must remain monotonic. Moreover, under additional assumptions on the parabolic boundary, we can show that our maximal solutions are immortal.

Vertical graphs

Theorem 3.3.1. *Let $y : \overline{\Omega} \times [0,T) \to \mathbb{R}$ be the maximal solution to the Dirichlet problems $V(\cdot)$ constructed in §3.3. Suppose that we chose increasing initial data $Y \in C^\infty(\Omega)$. Then $y(\cdot,t)$ is increasing for all $t \in [0,T)$. Moreover, if y has bounded gradient on the parabolic boundary Γ_T, then $T = \infty$ and y is immortal.*

Proof. Since our solution is smooth on the interior of our domain, we can differentiate (3.1.4) to get the evolution equation for the gradient $v := \frac{\partial y}{\partial x}$ on $\Omega \times (0, T)$

$$v_t = \mu(x,v)v_{xx} + \phi'(x)(1+\mu(x,v))v_x + \phi''(x)(1+\mu(x,v))v - 2\mu(x,v)^2 e^{2\phi}(v_x + \phi'(x)v)^2 v. \quad (3.3.3)$$

By the maximum principle, $v \geq 0$ on Γ_T. To show that v is non-negative everywhere, we use the following standard argument. We first note that we can bound the coefficient $\phi''(x)(1 + \mu(x,v)) < M$, for some constant $M > 0$. Fix $\epsilon > 0$ and consider the function

$$f(t) := -\epsilon \cdot e^{Mt}, \quad \forall t \in [0, T).$$

Choose $t_0 > 0$ maximal such that $v(\cdot, t) \geq f(t)$, for every $t \in [0, t_0)$. If $t_0 < T$, then there exists $x_0 \in \Omega$ such that $v(x_0, t_0) = f(t_0)$, and at this point

$$v_t(x_0, t_0) \leq f'(t_0), \quad v_x(x_0, t_0) = 0, \quad v_{xx}(x_0, t_0) \geq 0.$$

In particular, at this point we deduce the contradiction

$$Mf(t_0) = f'(t_0) \geq v_t(x_0, t_0) \geq \phi''(x)(1 + \mu(x,v))f(t_0) > Mf(t_0).$$

Therefore, taking ϵ to zero, we have that $v \geq 0$ on all of $\Omega \times (0, T)$, and $y(\cdot, t)$ is increasing for each $t \in [0, T)$. Suppose now that $T < \infty$. Using $v \geq 0$ in equation (3.3.3), we now that have v satisfies the differential inequality

$$v_t - \mu(x,v)v_{xx} - \phi'(x)(1 + \mu(x,v))v_x - \phi''(x)(1 + \mu(x,v))v \leq 0. \quad (3.3.4)$$

By the maximum principle (Lemma A.0.1), we deduce that

$$0 \leq v \lesssim \sup_{\Gamma_T} v < \infty \quad (3.3.5)$$

Since

$$|y| \leq \sup_{\Gamma_T} y = \sup_{\Omega} Y < \infty,$$

we therefore deduce the following contradiction to Corollary 3.2.3

$$\limsup_{s \to T} |y(\cdot, s)|_{C^1(\Omega)} \lesssim \sup_{\Omega} Y + \sup_{\Gamma_T} v < \infty. \qquad \square$$

Horizontal graphs

Recall that, in order to apply Theorem 3.2.2 to our Dirichlet problem, we had to assume that $\phi' \geq 0$. Under this assumption, the solutions $c(t)$ are always decreasing in t. We therefore expect our solution to be increasing at each time.

Theorem 3.3.2. *Fix $c \in \mathbb{R}$, with the maximal solution $c(t)$ from §3.3 immortal and bounded* *below. Assume that both ϕ' and ϕ'' are non-negative, and let $g_c : [0,1] \times [0,T_c) \to [0,\infty)$ be the maximal solution to the Dirichlet problems $H_c(\cdot)$ constructed in §3.3. Then $g_c(\cdot,t)$ is increasing for all $t \in [0,T_c)$. Moreover, if g_c has bounded gradient on the parabolic boundary Γ_{T_c}, then $T_c = \infty$ and g_c is immortal.*

Proof. Since our solution is smooth on the interior of our domain, we can differentiate (3.1.3) to get the evolution equation for the gradient $w := \frac{\partial g_c}{\partial y}$ on $(0,1) \times (0,T_c)$

$$w_t = \nu(x,w)w_{yy} + \Big(2\phi'(x)^2 e^{2\phi}\nu(x,w) - \phi''(x)(1+\nu(x,w)w^2)\Big)\, w - 2\nu(x,w)^2(w_y + \phi'(x)e^{2\phi})^2 w. \tag{3.3.6}$$

By the maximum principle, $w \geq 0$ on Γ_{T_c}. To show that w is non-negative everywhere, we repeat the argument from the proof of Theorem 3.3.1. We first note that we can bound the coefficient $2\phi'(x)^2 e^{2\phi}\nu(x,w) \leq M$, for some constant $M > 0$. Fix $\epsilon > 0$ and consider the function

$$f(t) := -\epsilon \cdot e^{Mt}, \quad \forall t \in [0,T).$$

Choose $t_0 > 0$ maximal such that $w(\cdot,t) \geq f(t)$, for every $t \in [0,t_0)$. If $t_0 < T_c$, then there exists $y_0 \in (0,1)$ such that $w(y_0,t_0) = f(t_0)$, and at this point

$$w_t(x_0,t_0) \leq f'(t_0), \quad w_x(x_0,t_0) = 0, \quad w_{xx}(x_0,t_0) \geq 0.$$

In particular, at this point we deduce the contradiction

$$Mf(t_0) = f'(t_0) \geq w_t(x_0,t_0) \geq 2\phi'(x)^2 e^{2\phi}\nu(x,w)f(t_0) > Mf(t_0).$$

Therefore, taking ϵ to zero, we have that $w \geq 0$ on all of $(0,1) \times (0,T_c)$. Suppose now that $T_c < \infty$. Substituting this back into equation (3.3.6), we now have that w satisfies the differential inequality

$$w_t - \nu(x,w)w_{yy} - 2\phi'(x)^2 e^{2\phi}\nu(x,w)w \leq 0. \tag{3.3.7}$$

We again apply the maximum principle (Lemma A.0.1), to deduce that

$$w \lesssim \sup_{\Gamma_{T_c}} w, \tag{3.3.8}$$

and hence arrive at the following contradiction

$$\limsup_{s \to T_c} |g_c(\cdot,s)|_{C^1([0,1])} \lesssim \sup_{t \in [0,T_c)} |c(t)| + \sup_{\Gamma_{T_c}} w < \infty. \qquad \square$$

3.3.2 Strict monotonicity at positive times

By using the intersection principle of Angenent (Theorem 2.2.5), we can show that our immortal solutions are strictly increasing at any positive time.

Vertical graphs

Proposition 3.3.3. *Let $y : \overline{\Omega} \times [0,T) \to \mathbb{R}$ be the maximal solution to the Dirichlet problems $V(\cdot)$ constructed in §3.3, with initial data $Y \in C^\infty(\overline{\Omega})$ increasing and non-constant. Suppose $y \in C^\infty(\Omega_s)$ for all $s \in (0,T)$. Then $y(\cdot,t)$ is strictly increasing for each $t \in (0,T)$. Moreover, the gradient is strictly positive away from the initial time*

$$\frac{\partial y}{\partial x}(x,t) > 0, \quad \forall (x,t) \in \overline{\Omega} \times (0,T).$$

Proof. For any $h \in \mathrm{Im}\,(Y)$, we can apply the the intersection principle (Theorem 2.2.5) to the function y and the constant solution h, to get that the intersection number between them is decreasing and finite for positive time. By the intermediate value theorem, there is always at least one intersection point. Fix $t \in (0,T)$. By Theorem 3.3.1, $y(\cdot,t)$ is increasing. So, if $|y(\cdot,t)^{-1}(h)| > 1$, $y(\cdot,t)^{-1}(h)$ has positive measure, contradicting the fact that there are only finitely many intersections between h and $y(\cdot,t)$ (Theorem 2.2.5). We conclude that $|y(\cdot,t)^{-1}(h)| = 1$, and hence this single intersection point between $y(\cdot,t)$ and h is transverse. A transverse intersection point implies $\frac{\partial y}{\partial x}$ is positive at this point. □

Horizontal graphs

If we assume that $c(t)$ is in fact *strictly* decreasing, the following proposition tells us that each of the maximal solutions $g_c(\cdot,t)$ are strictly increasing at positive times. This will be crucial later when using these solutions are leaves in a foliation.

Proposition 3.3.4. *Fix $c \in \mathbb{R}$, with the maximal solution $c(t)$ from §3.3 immortal and bounded below. Assume that ϕ'' is non-negative and ϕ' is strictly positive. Let $g_c : [0,1] \times [0,T_c) \to [0,\infty)$ be the maximal solution to the Dirichlet problems $H_c(\cdot)$ constructed in §3.3. Then $g_c(\cdot,t)$ is strictly increasing for each $t \in (0,T_c)$. Moreover, the gradient is strictly positive away from the initial time*

$$\frac{\partial g_c}{\partial y}(y,t) > 0, \quad \forall (y,t) \in [0,1) \times (0,T_c).$$

Proof. We repeat the proof of Proposition 3.3.3, but use solutions $c(t)$ in the place of constant solutions. Fix $t_0 \in (0,T_c)$. By Theorem 3.3.2, $g_c(\cdot,t_0)$ is increasing, and so its image is $[c(t_0),c]$. By the strong maximum principle, $g_c(y,t_0)$ must lie in the region $[c(t_0),c)$ for $y \in [0,1)$. Note that this region $[c(t_0),c)$ can be described as $\{c(s) : s \in (0,t_0]\}$. Fix $s \in (0,t_0]$, and choose $y_0 \in (0,1)$ sufficiently large such that $g_c(y_0,t) > c(\frac{s}{2})$ for all $t \in [0,s]$. Note that compatibility conditions of all orders are satisfied at the point $\{0\} \times \{0\}$. Therefore, by the global Schauder estimates (Theorem A.0.3), we have that $g_c \in C^\infty([0,y_0] \times [0,t'])$, for any $t' < T_c$. We can therefore apply the intersection principle (Theorem 2.2.5) to the solutions $c(\frac{s}{2}+t)$ and $g_c(y,t+t_0-\frac{s}{2})$ defined for $(y,t) \in [0,y_0] \times [0,T_c-(t_0-\frac{s}{2}))$, to deduce that the line $c(s)$ and $g_c(\cdot,t_0)$ intersect at a single transverse point, and so $\frac{\partial g_c}{\partial y}(\cdot,t_0)$ is positive over $[0,1)$. □

Chapter 4

The evolution of proper curves under curve shortening flow

In this next chapter, we utilise the results of the previous chapter to discuss proper solutions to our flow. The following is a brief outline.

- §4.1 gives a detailed construction of Example 1.3.8. To do so, we utilise ideas originally due to Tom Ilmanen in [Ilm92], and find a properly embedded solution to CSF converging backwards in time to a cusp in the plane. We then choose a suitable parameterisation of this solution so that it remains properly embedded, but 'loses' some of its initial data as in Remark 1.2.8.

- In §4.2 we prove that uniformly proper solutions are Brakke flows all the way up to the initial time, and hence satisfy the avoidance principle with closed curves.

- In §4.3, we prove Theorem 1.3.14. To do this we use a sequence of solutions to the Dirichlet problems from §3 defined over an exhaustion of the real line. We then employ a foliation argument to prove local uniform gradient bounds on this sequence and extract a smooth proper limit which starts from the x-axis. Finally, we construct a barrier which moves in from infinity in finite time, and which pushes our solution away from the x-axis instantaneously.

- In §4.4 we give a proof of Theorem 1.3.17. The idea of the proof is a modified version of the barrier argument seen in Example 1.3.11. We show that at an arbitrarily large time and for arbitrarily thin convex neighbourhoods of our geodesic, we can find closed solutions to CSF which not only exist until this time, but also lies arbitrarily far out. The result then follows from the avoidance principle proven in §4.2.

4.1 Non-uniqueness of properly embedded solutions

The aim of this subsection is to construct the following example.

Example 4.1.1. There exists a continuous map $\gamma : \mathbb{R} \times [0, \infty) \to \mathbb{R}^2$ such that

- γ is smooth and solves (1.2.1) on $\mathbb{R} \times (0, \infty)$.
- $\gamma(\cdot, t) : \mathbb{R} \to \mathbb{R}^2$ is a proper embedding, for every $t \geq 0$.
- $\mathrm{Im}\,(\gamma(\cdot, 0))$ is an entire locally Lipschitz graph over the x-axis, but $\mathrm{Im}\,(\gamma(\cdot, t))$ is not a graph over the x-axis, for any $t > 0$.

Many of the ideas used in this construction originate from the work of Ilmanen. The following contains within it a careful exposition of some of the details from the sketch proof in [Ilm92, Example 7.3].

4.1.1 Constructing compact approximations

Let us define $u : \mathbb{R} \to (0, \infty)$ to be the function

$$u(x) := \min\{1, (1 + cx)^{-2}\}, \quad \forall x \in \mathbb{R},$$

where $c > 0$ is some small positive constant to be determined later.

Since the area enclosed within the cusp that forms between the graphs of $\pm u$ is finite in a neighbourhood of spatial infinity, we expect there to be a smooth solution starting from the graphs of $\pm u$, but which rushes inwards instantaneously from spatial infinity, and hence at each positive time will be connected.

To construct this solution, we approximate our initial data from the inside by compact curves. That is, for each $n \in \mathbb{N}$, consider the Jordan curve given by the union of the curves:

- $\{(x, u(x)) \in \mathbb{R}^2 : |x| \leq n\}$, the graph of u over $[-n, n]$;
- $\{(x, -u(x)) \in \mathbb{R}^2 : |x| \leq n\}$, the graph of $-u$ over $[-n, n]$;
- $\{(n, y) \in \mathbb{R}^2 : -u(n) \leq y \leq u(n)\}$, the vertical line joining $(n, u(n))$ and $(n, u(-n))$;
- $\{(-n, y) \in \mathbb{R}^2 : -1 \leq y \leq 1\}$, the vertical line joining $(-n, 1)$ and $(-n, -1)$.

Since these are finite length Jordan curves, we can flow them under CSF using the existence result of Lauer (Theorem 1.2.9). That is, for each $n \in \mathbb{N}$, there exists $T_n > 0$ and a continuous function $\gamma_n : S^1 \times [0, T_n) \to \mathbb{R}^2$ such that

(i) the image of the curve at time zero, $\mathrm{Im}\,(\gamma_n(\cdot, 0))$, is the Jordan curve given above;

(ii) γ_n is a smooth solution to CSF (1.2.2) on $S^1 \times (0, T_n)$;

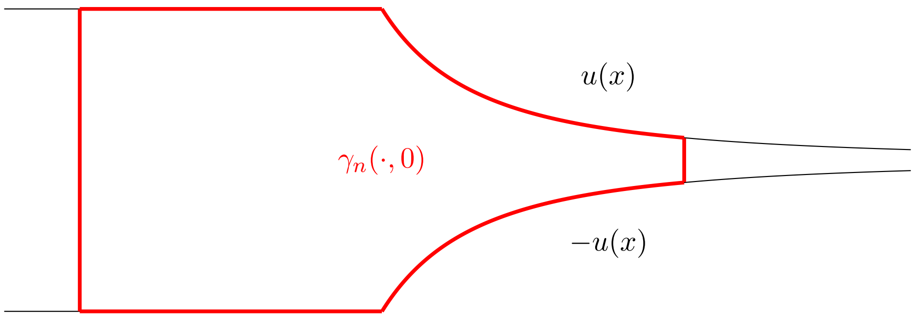

Figure 4.1: Initial data for our approximations

(iii) if $L_n(t)$ denotes the length of $\mathrm{Im}\,(\gamma_n(\cdot,t))$ for each $t \in [0,T_n)$, then $\lim_{t\searrow 0} L_n(t) = L_n(0) > n$.

Moreover, $T_n \geq \frac{n}{\pi}$ by Gauss-Bonnet [Cha06, Theorem V.2.7]. To begin, we construct suitable barriers to our approximations as in [Ilm92, Example 7.3].

Define the function $q : [-1/(2c),\infty) \times [0,\infty) \to (0,\infty)$ by $q(x,t) := \frac{e^{24ct}}{(1+cx)^2}$. By a direct calculation, we have that

$$\begin{aligned} q_t - \frac{q_{xx}}{1+q_x^2} &= q \cdot \left(24c - \frac{6c(1+cx)^4}{(1+cx)^6 + 4c^2e^{48ct}}\right) \\ &\geq q \cdot \left(24c - \frac{6c}{(1+cx)^2}\right) \geq 0, \end{aligned}$$

and $q(x,t)$ is a super solution to the graphical CSF equation within the half-plane $\{(x,y) \in \mathbb{R}^2 : x > -1/(2c)\}$. Since the horizontal lines $\{y = y_0\}$ can be parameterised as static uniformly proper solutions to CSF (see §4.2 for a full justification of this step), we can apply the avoidance principle with closed curves to deduce that any approximation γ_n remains disjoint from any horizontal line with $|y_0| > 1$. We conclude that each of the curves $\gamma_n(\cdot,t)$ remains bounded between the lines $\{y = \pm 1\}$, for all $t \geq 0$. Fix $n \in \mathbb{N}$. Repeating the same argument but with horizontal lines, we conclude that $\gamma_n(\cdot,t)$ remains bounded between the lines $\{x = \pm n\}$. For $\epsilon > 0$, consider now the graph of the function $q+\epsilon$ for $(x,t) \in [-1/(2c), n+1/2] \times [0,\infty)$. Since $\mathrm{Im}\,(\gamma_n(\cdot,0))$ is initially disjoint from the graph of $q(\cdot,0)+\epsilon$, $\mathrm{Im}\,(\gamma_n(\cdot,t))$ can only intersect the graph at times $t > 0$. Suppose they do intersect and let t_0 be the infimum of those times. By the continuity of the solutions, we deduce that $t_0 > 0$, and that at t_0, the graph and $\mathrm{Im}\,(\gamma_n(\cdot,t_0))$ must intersect tangentially. Let $x_0 \in (-1/(2c), n]$ be a point where $q(x_0,t_0)+\epsilon$ intersects $\mathrm{Im}\,(\gamma_n(\cdot,t_0))$. Note that x_0 cannot be equal to $-1/(2c)$, as $q(-1/(2c),t_0) > 1$. By viewing $\gamma_n(\cdot,t_0)$ locally as a graph over the x-axis near this intersection point, we get a contradiction to the avoidance principle (Theorem 2.2.8). Therefore, taking ϵ to zero, we can deduce that our approximations are contained within the following cusp:

$$\mathrm{Im}\,(\gamma_n(\cdot,t)) \subseteq C(t) := \{(x,y) \in \mathbb{R}^2 : |y| \leq \min\{1, q(x,t)\}\}, \quad \forall n \in \mathbb{N}.$$

We now use a different barrier to show that at any positive time, all of the approximations have

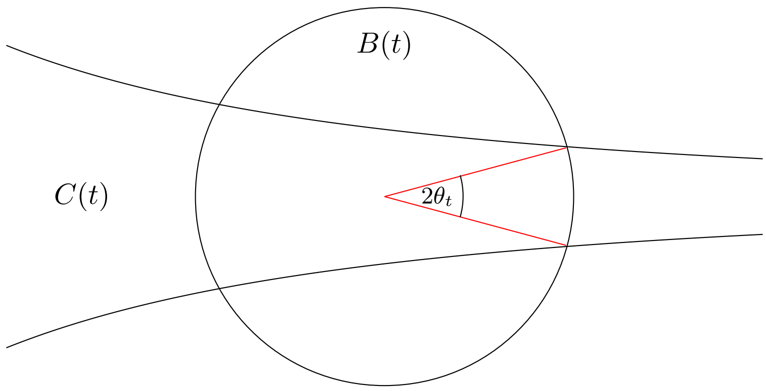

Figure 4.2: The intersection angle θ_t

been pulled in a uniform amount from the tip of the cusp. For $t > 0$, consider the balls $B(t)$ centred at $(t^{-2}, 0)$ of radius $\frac{t^3}{4}$. We see that for $\tau_c > 0$ sufficiently small, we have

$$\frac{t^3/4}{q(t^{-2},t)} = \frac{e^{-24ct}}{4} \cdot \left(t^{\frac{3}{2}} + ct^{\frac{-1}{2}}\right)^2 > \frac{c^2 e^{-24ct}}{4t} \geq \sqrt{2}, \quad \forall t \in (0, \tau_c].$$

In particular, we see that $C(t) \setminus B(t)$ is disconnected, for every $t \in (0, \tau_c]$. As such, define $\theta_t \in (0, \frac{\pi}{2})$ to be the unique value such that

$$\frac{t^3}{4} \cdot \sin\theta_t = q\left(t^{-2} + \frac{t^3}{4} \cdot \cos\theta_t, t\right), \quad \forall t \in (0, \tau_0].$$

Geometrically, θ_t is half of the angle of the cap $C(t) \setminus B(t)$, measured from the center of $B(t)$. As $q(\cdot, t)$ is decreasing, we have

$$\sin\theta_t = \frac{q\left(t^{-2} + t^3/4 \cdot \cos\theta_t, t\right)}{t^3/4} \leq \frac{q\left(t^{-2}, t\right)}{t^3/4} < \frac{1}{\sqrt{2}},$$

which implies that $\theta_t < \frac{\pi}{4}$, or $\cos\theta_t > \frac{1}{\sqrt{2}}$ for all $t \in (0, \tau_c]$. Consider the moving arc

$$\{(t^{-2} + t^3/4\cos(\theta), \sin(\theta)) : t > 0, \theta \in [-\pi/4, \pi/4]\}.$$

Since

$$|\partial_t \cdot t^{-2}| = \frac{2}{t^3} < \frac{\cos(\theta)}{t^3/4}, \quad \forall t \in (0, \tau_c],\ \forall \theta \in [-\pi/4, \pi/4],$$

we see that, if we consider the moving arc as a graph over the y-axis, it would be a supersolution to graphical CSF. Since the region of the moving arc that lies within the cusp $C(t)$ is parameterised by $\theta \in [-\theta_t, \theta_t] \subseteq (-\pi/4, \pi/4)$, we can apply the avoidance principle (Theorem 2.2.8) as we did above for the graph of q, to deduce that our approximations $\gamma_n(\cdot, t)$ remain disjoint with this moving arc, at all positive times. Choosing $\alpha_c := \tau_c^{-2} + \tau_c^3/4$, we see that our solutions now pull in from infinity instantly. That is

$$\mathrm{Im}\left(\gamma_n(\cdot, t)\right) \subseteq \Omega(t) := \{(x, y) \in C(t) : x \leq \max\{\alpha_c, \left(1 + t^{-2}\right)\}\}, \quad \forall n \in \mathbb{N}.$$

Finally, we must find a region laying within $\Omega(t)$ that the approximations avoid. For each $n \in \mathbb{N}$, let A_n be the rectangle $[-n, 0] \times [-1, 1]$. Note that the area enclosed inside of A_n is $2n$. Therefore, using Lauer's Theorem 1.2.9 and Hamilton-Gage-Grayson Theorem 1.0.2, we can flow the boundary of A_n under CSF, which gives us a continuous function $\eta_n : S^1 \times [0, \frac{n}{\pi}) \to \mathbb{R}^2$ such that

- $\eta_n(\cdot, 0)$ is a parameterisation of ∂A_n,
- η_n is a smooth solution to CSF (1.2.2) on $S^1 \times (0, \frac{n}{\pi})$,
- $\eta_n(\cdot, t)$ is a convex curve, for all $t \in (0, \frac{n}{\pi})$.

Given any $t_0 > 0$, choose n sufficiently large such that $\frac{n}{\pi} >> t_0$. Consider the two points laying in the intersection $\{x = -n/2\} \cap \mathrm{Im}\,(\eta_n(\cdot, t_0))$. If one of these point is $(-n/2, y)$ for some $y \in (0, 1)$, by symmetry, the other must be $(-n/2, -y)$. Using convexity of our solution, the region $([-n, 0] \times [y, 1]) \cup ([-n, 0] \times [-1, -y])$ must be disjoint from the region inside of the curve $\mathrm{Im}\,(\eta_n(\cdot, t_0))$. By Gauss-Bonnet Theroem C.0.1, the area lost under the flow is $2\pi t_0$. Therefore, $y \geq 1 - \frac{\pi t_0}{n}$. Similarly, consider the two points laying in the intersection $\{y = 0\} \cap \mathrm{Im}\,(\eta_n(\cdot, t_0))$. If one of these points is $(-x, 0)$ for some $x \in (0, n/2)$, then the other must be $(x - n, 0)$. By convexity and Gauss-Bonnet, $x \leq \frac{\pi t_0}{2}$.

For each $t > 0$ and $n \in \mathbb{N}$, we choose the four points

- $q_1(n, t) := (-n/2, 1 - \frac{\pi t}{n})$,
- $q_2(n, t) := (-n/2, \frac{\pi t}{n} - 1)$,
- $q_3(n, t) := (-\frac{\pi t}{2}, 0)$,
- $q_4(n, t) := (\frac{\pi t}{2} - n, 0)$.

It follows from the convexity of $\eta_n(\cdot, t)$ that the convex hull of the four points

$$\{q_i(n, t) : i = 1, \ldots, 4\},$$

is contained within the region bounded by the curve $\eta_n(\cdot, t)$, for every sufficiently large $n \in \mathbb{N}$ (n depending on t).

4.1.2 Extracting a limiting Brakke flow

For each of the smooth approximations γ_n, consider the associated Brakke flow $[0, T_n) \ni t \mapsto \mu_t^n$. Let $\tilde{M}_0$ be the union of the Jordan curves $\mathrm{Im}\,(\gamma_n(\cdot, 0))$ for $n \in \mathbb{N}$, and consider $\tilde{\mu}_0$ the 1-dimensional Hausdorff measure restricted to $\tilde{M}_0$. Note that $\mu_0^n \leq \tilde{\mu}_0$, for all $n \in \mathbb{N}$.

Fix $m \in \mathbb{N}$ and choose n sufficiently large so that $T_n > m$, and hence μ_t^n is defined on the time

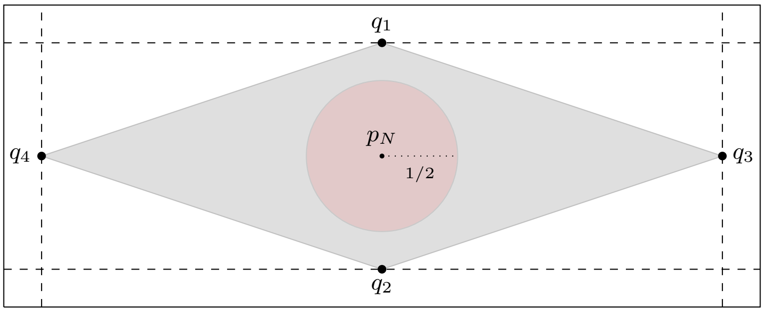

Figure 4.3: The convex hull of the points $q_1, \dots, q_4$

interval $[0, m]$. For such an n, applying the local area bounds from Lemma 2.4.7 we have

$$\mu_t^n(B(x_0, \rho)) \le 8 \cdot \mu_0^n(B(x_0, 2\rho)) \le 8 \cdot \tilde{\mu}_0(B(x_0, 2\rho)),$$

for all $t \in [0, m]$ and $x_0 \in \mathbb{R}^2$. In particular, we have that

$$\sup_{t\in[0,m]} \sup_{n\in\mathbb{N}} \mu_t^n(K) < \infty, \quad \forall K \Subset \mathbb{R}^2.$$

By the compactness theorem for Brakke flows (Theorem 2.4.4), we can extract a Brakke flow limit over $[0, m]$. Repeating for all $m \in \mathbb{N}$ and using a diagonal argument, we have a eternal limiting Brakke flow $[0, \infty) \ni t \mapsto \mu_t$, with $M_t = \operatorname{supp}(\mu_t)$. This Brakke flow has the following properties:

- By the weak convergence of the measures $\mu_0^n \rightharpoonup \mu_0$, we have that M_0 is the union of the graphs of $\pm u$.
- Since the approximations are smooth, μ_t is a unit-regular Brakke flow by Theorem 2.4.17.
- μ_t has finite 1-entropy, and hence bounded 1-area ratios by Corollary 2.4.15.

To see why μ_t has finite 1-entropy, we only need to check that M_0 has finite entropy (Corollary 2.4.15). Recall that the 1-entropy of M_0 is

$$\begin{aligned}\lambda_1(M_0) &:= \sup_{r>0, z_0\in\mathbb{R}^2} \frac{1}{\sqrt{4\pi r^2}} \int_{M_0} e^{-\frac{|z-z_0|^2}{4r^2}} d\mathcal{H}^1(z) \\ &= \sup_{r>0, z_0\in\mathbb{R}^2} \frac{1}{\sqrt{4\pi r^2}} \int_{\mathbb{R}} \left(e^{-\frac{|(x,u(x))-z_0|^2}{4r^2}} + e^{-\frac{|(x,-u(x))-z_0|^2}{4r^2}} \right) \sqrt{1+u'(x)^2}\, dx\end{aligned}$$

Choosing $z_0 = (x_0, y_0) \in \mathbb{R}^2$, we have the primitive upper bound

$$\lambda_1(M_0) \le \sup_{r>0, x_0\in\mathbb{R}} \frac{2}{\sqrt{4\pi r^2}} \int_{\mathbb{R}} e^{-\frac{(x-x_0)^2}{4r^2}} \sqrt{1+u'(x)^2}\, dx.$$

Since $|u'| \leq 2c$, we deduce that

$$\lambda_1(M_0) \leq 2\sqrt{1+4c^2} < \infty.$$

4.1.3 Local uniform gradient bounds

We now want to show regularity of this limit. Much like in [Ilm94, Example 7.3], we want to show uniform gradient bounds on our sequence of approximations via a foliation argument. Unlike in the case Ilmanen considers however, our approximations are not contained uniformly within a compact region. We modify his argument to work here as follows:

Fix $K \times [\tau, T] \Subset \mathbb{R} \times (0, \infty)$. We shall show local C^1-bounds for our approximations within this compact region of space-time. To do this, we first choose $N \in \mathbb{N}$ sufficiently large to ensure that, for any $n \geq N$:

(i) $T_n > 2T$.

(ii) If $p_n := (-n/2, 0) \in \mathbb{R}^2$, then

$$\operatorname{Im}(\gamma_n(\cdot, t)) \subseteq C(t) \setminus B(p_N, 1/2), \quad \forall t \in [0, 2T].$$

(iii) K is contained within the horizontal wedge of size $\pi/4$ about the point p_N:

$$K \subseteq \{p_N + (r\cos\theta, r\sin\theta) : r > 0,\ \theta \in (-\pi/8, \pi/8)\}.$$

Note that (i) follows from the fact $T_n \geq \frac{n}{\pi}$. To see (ii), we note that for sufficiently large n, $B(p_N, 1/2)$ is contained within the convex hull of the points $\{q_i(n,t) : i = 1, \ldots, 4\}$ for every $t \leq 2T$. Finally, (iii) is a consequence of the fact that the collection $(\{p_n + (r\cos\theta, r\sin\theta) : r > 0,\ \theta \in (-\pi/8, \pi/8)\})_{n\in\mathbb{N}}$, form an open cover of the plane.

If we now restrict our attention to the half space $H_N := \{(x, y) \in \mathbb{R}^n : x \geq -\frac{N}{2}\}$, the distance between the point p_N and any point in $H_N \cap C(t)$ is bounded above for all $t \in [\tau/3, 2T]$. Therefore, there exists $R > 0$ such that, for any $t \in [\tau/3, 2T]$

$$\operatorname{Im}(\gamma_n(\cdot, t)) \cap H_N \subseteq B(p_N, R) \setminus B(p_N, 1/2), \quad \forall n \geq N.$$

It suffices to show uniform C^1-bounds for the approximations with $n \geq N$. So, without loss of generality, lets fix $n \geq N$. We first show that the region enclosed by $\gamma_n(\cdot, t)$ is star-shaped about p_N for $t \in [0, 2T]$. To see this, consider all of the half-lines eminating from the point p_N. Since the image of the approximation γ_n is bounded and avoids the ball of radius $1/2$ about this point, we can apply the intersection principle (Theorem 2.2.6) to deduce that each of these half-lines intersects $\operatorname{Im}(\gamma_n(\cdot, t))$ exactly once, for all $t \in [0, 2T]$, and hence $\gamma_n(\cdot, t)$ bounds a star-shaped region centred at p_N.

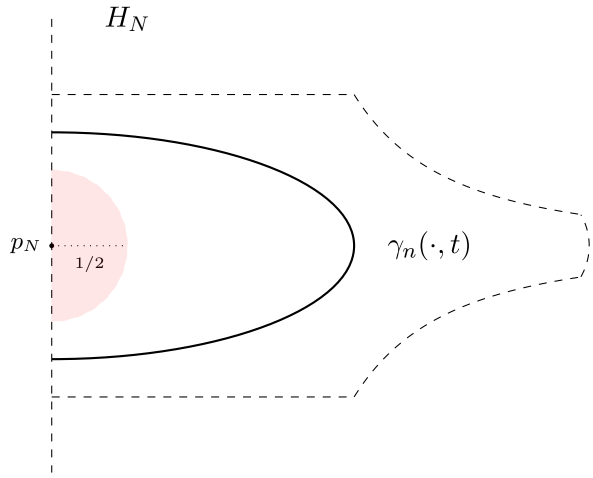

Figure 4.4: Star-shaped approximation in H_N

Since our approximation is star-shaped about p_N, we may parameterise its image within the half-space H_N as a graph in polar coordinates about the point p_N. That is, there exists a smooth function $r_n : [-\pi/2, \pi/2] \times [\tau/3, 2T] \to [\frac{1}{2}, R]$ such that

$$\mathrm{Im}\,(\gamma_n(\cdot, t)) \cap H_N = \{p_N + (r_n(\theta, t)\cos\theta, r_n(\theta, t)\sin\theta) : \theta \in [-\pi/2, \pi/2]\}, \quad \forall t \in [\tau/3, 2T].$$

Since our approximation solves CSF, we know that r_n is a solution to the equation

$$r_t = \frac{r_{\theta\theta}}{r^2 + r_\theta^2} - \frac{1}{r}\left(1 + \frac{r_\theta^2}{r^2 + r_\theta^2}\right). \tag{4.1.1}$$

(For a derivation of this equation, substitute $\phi(x) = \log(x)$ into equation (3.1.3)). We now use the same foliation as Ilmanen in [Ilm94, Example 7.3], but centred about the point p_N instead of the origin.

Given our choice of R above, define a function $f_0 : [1/4, 2R] \to \mathbb{R}$ so that

- $f_0'', f_0 < 0$ on $[1/4, 1/2)$;
- $f_0'', f_0 > 0$ on $(R, 2R]$;
- $f_0 \equiv 0$ on $[1/2, R]$;
- $f_0(1/4) = -1/4$ and $f_0(2R) = 2R$.

Solving the Dirichlet problems $V(\cdot)$ from §3 with the flat metric, we can flow the graph of f under CSF with fixed endpoints to give an eternal solution $f : [1/4, 2R] \times [0, \infty) \to \mathbb{R}$, with $f(\cdot, 0) = f_0$. Consider also the linear graphs $x \mapsto mx$, for $m \in [-1, 1]$. These are static solutions to graphical CSF. Note that for m non-zero, the intersection number between the graphs of u_m and f_0 is one, and hence by the intersection principle Theorem 2.2.6, there is always exactly one intersection point at any positive time. For the case $m = 0$, we see that

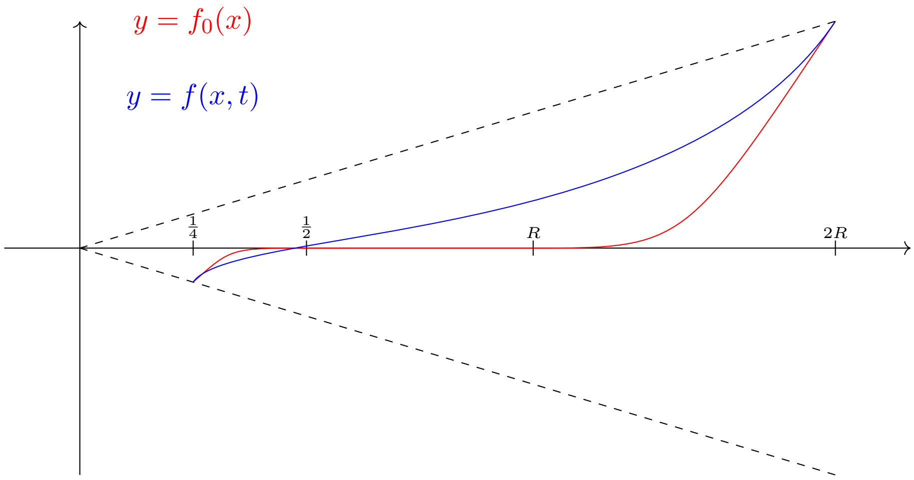

Figure 4.5: The graph of our leaf $f(\cdot, t)$

there is a single intersection point by the fact that f becomes strictly monotone at positive times (Proposition 3.3.3). So, for each $m \in (-1,1)$, let $x_m(t) \in (1/4, 2R)$ denote the unique input such that $m \cdot x_m(t) = f(x_m(t), t)$. Since the intersection is transverse, we conclude that $f'_m(x_m(t), t) > m$, for any $m \in (-1,1)$ and $t \in (0,T]$.

Note that, for any time t, the intersection point $x_m(t) \to 2R$ as $m \to 1$, and $x_m(t) \to 1/4$ as $m \to -1$. This means we can find $m_0 \in (0,1)$ such that, for any $t \in [0, 2T]$, $x_m(t) \in [1/2, R]$ only if $m \in [-m_0, m_0]$. Since the angle between the tangent to the graph of f_m and the line $x \mapsto mx$ is given by

$$v(m,t) := \arctan\left(\frac{f'_m(x_m(t),t) - m}{1 + m f'_m(x_m(t),t)}\right) > 0,$$

by compactness, there exists $v_0 > 0$ such that $v(m,t) \geq v_0$ for any $(m,t) \in [-m_0, m_0] \times [\tau/3, 2T]$.

Consider the arc in the plane moving under CSF given by the graph of this function f. That is, let

$$F(t) := \{p_N + (x, f(x,t)) \in \mathbb{R}^2 : x \in [1/2, R]\}, \quad \forall t \geq 0.$$

Away from time zero, we can switch gauge and view this arc as a graphical solution in polar coordinates about p_N. That is, there exists a function

$$\rho : [-\pi/4, \pi/4] \times (0, \infty] \to (0, \infty),$$

such that

$$F(t) = \{p_N + (\rho(\theta,t)\sin\theta, \rho(\theta,t)\cos\theta) : \theta \in [-\pi/4, \pi/4]\}, \quad \forall t > 0.$$

We can reflect this arc in the x-axis, and then rotate these arcs about the point p_N, to give the family of arcs

$$F^{\pm}_{\alpha}(t) = \{p_N + (\rho(\alpha \pm \theta, t)\sin\theta, \rho(\alpha \pm \theta, t)\cos\theta) : \theta \in [-\pi/4, \pi/4]\}, \quad \forall \alpha \in [-\pi/4, \pi/4], \ \forall t > 0.$$

In particular, the two subfamilies of arcs

$$\mathcal{F}^{\pm}(t) := \{F^{\pm}_{\alpha}(t) : \alpha \in [-\pi/4, \pi/4]\},$$

each foliate the region $\{p_N + (r\cos\theta, r\sin\theta) : 1/2 \le r \le R,\ |\theta| \le \pi/4\}$, at every time $t > 0$.

Returning to our approximation, for each $\theta \in [-\pi/2, \pi/2]$, we know that $r_n(\theta, \tau/3) \in (1/2, R)$. Therefore, its image intersects exactly once with each of the arcs $F^{\pm}_{\alpha}(0)$. Applying the intersection principle (Theorem 2.2.6), the functions $r_n(\cdot, t)$ and $\rho(\alpha \pm \cdot, t - \tau/3)$ intersect at a single transverse point for all $t \in [2\tau/3, 2T]$.

Recall, we showed that at every point in the region

$$\{p_N + (r\cos\theta, r\sin\theta) : 1/2 \le r \le R, |\theta| \le \pi/4\},$$

each of the arcs in the foliations $\mathcal{F}^{\pm}(t)$ has its tangent vector form an angle of magnitude at least v_0 with the radial vector $\partial_r = \cos\theta \cdot \partial_x + \sin\theta \cdot \partial_y$, for every $t \in [2\tau/3, 2T]$. Therefore, using that r_n is bounded above, we can deduce a uniform gradient bound for the function r_n restricted to $[-\pi/4, \pi/4] \times [2\tau/3, 2T]$. To see why, the unit tangent vector of our curve at a given time is

$$\frac{(r_n)_\theta \partial_r + \partial_\theta}{\sqrt{(r_n)_\theta^2 + r_n^2}},$$

where $\partial_\theta = -\sin\theta \cdot \partial_x + \cos\theta \cdot \partial_y$. So, if the angle between this and the vectors $\pm\partial_r$ is at least $v_0 > 0$, we deduce that

$$\frac{|(r_n)_\theta|}{\sqrt{(r_n)_\theta^2 + (r_n)^2}} \le \cos(v_0), \quad \text{or} \quad |(r_n)_\theta| \le R \cdot \cot(v_0).$$

Instead of using the functions r_n, consider $u_n := 1/r_n$, which take values in $[1/R, 2]$. Since

$$|(u_n)_\theta| = \frac{|(r_n)_\theta|}{r_n^2} \le R^2 \cdot |(r_n)_\theta|,$$

we also have uniform gradient bounds on the sequence of functions u_n for $n \ge N$, over the same space-time region. Substituting into equation (4.1.1), we see that each of the u_n solve the equation

$$u_t = \left(\frac{u^4}{u^2 + u_\theta^2}\right)(u_{\theta\theta} + u). \tag{4.1.2}$$

Using the uniform gradient bounds in equation (4.1.2), we can apply De Giorgi-Nash-Moser (Theorem A.0.2) to this equation to deduce uniform parabolic Hölder bounds on the space-time domain $[-\pi/4, \pi/4] \times [2\tau/3, 2T]$. Combining this with interior Schauder estimates (Theorem A.0.4), we have uniform $P^{k,\alpha}$-bounds on the space-time domain $[-\pi/8, \pi/8] \times [\tau, T]$, for every $k \in \mathbb{N}$. In particular, by Arzela-Ascoli, we can extract a subsequence which converges smoothly over this region.

Returning to our geometric picture, after passing to a subsequence, we have smooth convergence of the approximations γ_n within the space-time region $K \times [\tau, T] \subseteq \mathbb{R}^2 \times (0, \infty)$, to a family of smooth connected embedded arcs. Covering our entire space-time $\mathbb{R}^2 \times (0, \infty)$ by a countable exhaustion of compact regions and applying a diagonal argument, we can deduce that a subsequence of the approximations converge locally smoothly on all of $\mathbb{R}^2 \times (0, \infty)$ to a family of smooth connected and embedded curves. That is, M_t is a smooth embedded copy of $\mathbb{R}$, for all $t > 0$.

4.1.4 Parameterising our solution

We now attempt to find a parameterisation of our smooth solution with the desired properties. To do this, we will show that for arbitrarily small times, our solution can be seen as a Lipschitz graph over an arbitrarily large region of the x-axis.

Set $\eta = 1$ and choose the constant $c > 0$ used in the definition of u to be sufficiently small such that u has Lipschitz constant less than the ϵ from the pseudolocality result of Ilmanen-Neves-Schulze (Theorem 2.4.18). In particular, for every $n \in \mathbb{N}$, there exists $\alpha_n > 0$ sufficiently small such that, for every $x \in [-n, n]$, $M_0 \cap I_{\alpha_n}((x, -u(x))$ is a Lipschitz graph over $(x - \alpha_n, x + \alpha_n)$, with Lipschitz constant bounded by ϵ.

Parabolically rescaling our flow, applying the pseudolocality result of Ilmanen-Neves-Schulze (Theorem 2.4.18), and parabolically rescaling our flow back down, we deduce that, for every $x \in [-n, n]$ and every $t \in (0, \alpha_n^2 \cdot \delta^2)$, $M_t \cap I_{\alpha_n \delta}(x, -u(x))$ can be written as a Lipschitz graph over $(x_0 - \delta\alpha_n, x_0 + \delta\alpha_n)$, with Lipschitz constant less than 1, and height bounded by $\alpha_n \delta$. In particular, we have a Lipschitz (in space) function $v_n : (-n - \alpha_n, n + \alpha_n) \times (0, \alpha_n \delta^2) \to \mathbb{R}$ such that $M_t \cap \big(\cup_{x \in [-n,n]} I_{\alpha_n \delta}(x, -u(x))\big)$ is given by the graph of $v_n(\cdot, t)$, for all $t \in (0, \alpha_n^2 \delta^2)$.

Repeating for all $n \in \mathbb{N}$, we can glue all of the functions v_n together to give a well-defined Lipschitz (in space) function $v : U \to \mathbb{R}$, where

$$U := \bigcup_{n \in \mathbb{N}} \big((-n - \alpha_n \times n + \alpha_n) \times (0, \alpha_n \delta^2)\big) \subseteq \mathbb{R} \times (0, \infty).$$

Then, for any $t > 0$,

$$M_t \cap \left(\bigcup_{n : t < \alpha_n^2 \delta^2} \ \bigcup_{x \in [-n,n]} I_{\alpha_n \delta}(x, -u(x)) \right),$$

is given by the graph of $v(x, t)$, for all $(x, t) \in U$.

For each $t > 0$, we choose a parameterisation $\gamma(\cdot, t) : \mathbb{R} \to \mathbb{R}^2$ of M_t such that, for each $(x, t) \in U$, $\gamma(x, t) = (x, v(x, t))$. Since our solution is smooth, we can do this in a smooth consistent way as t varies, giving a smooth map $\gamma : \mathbb{R} \times (0, \infty) \to \mathbb{R}^2$ such that $\mathrm{Im}\,(\gamma(\cdot, t)) = M_t$, for all $t > 0$, and $\gamma(x, t) = (x, v(x, t))$, for all $(x, t) \in U$. Moreover, by modifying the parameterisation outside of a compact region for each t, we can ensure that each $\gamma(\cdot, t) : \mathbb{R} \to \mathbb{R}^2$ is a proper map.

Finally, for any $x_0 \in \mathbb{R}$, we can choose n sufficiently large such that $(x,t) \in U$ for all $x \in (x_0 - \alpha_n, x_0 + \alpha_n)$ and $t \in (0, \alpha_n^2\delta^2)$. In particular, for any such (x,t), we have that $|v(x,t) + u(x)| < \alpha_n\delta$. Taking $n \to \infty$, we deduce that γ can be extended continuously up to time zero by setting $\gamma(x,0) := (x, -u(x))$, for all $x \in \mathbb{R}$.

4.1.5 A variation of the previous construction

One could modify the previous construction to get something even more striking. That is, a continuous map $\gamma : \mathbb{R} \times [0,\infty) \to \mathbb{R}^2$ such that

- $\gamma(\cdot,t) : \mathbb{R} \to \mathbb{R}^2$ is a smooth proper embedding, for every $t \geq 0$.
- γ is smooth and solves (1.2.1) on $\mathbb{R} \times (0,\infty)$.
- $\mathrm{Im}\,(\gamma(\cdot,0))$ is the x-axis, but $\mathrm{Im}\,(\gamma(\cdot,t))$ isn't the x-axis, for any $t > 0$.

Unfortunately, to do this is a substantially more difficult than the previous case. This is because our approximations are no longer star-shaped, rendering the foliation trick of Angenent to deduce gradient bounds obsolete. We instead sketch the outline of how one would go about such a construction.

Define u as before, but now, take our approximations γ_n to be the closed solutions to CSF, starting from the Jordan curves given by the union of the arcs

- $\{(x,0) \in \mathbb{R}^2 : |x| \leq n\} = [-n,n] \times \{0\}$,
- $\{(x,u(x)) \in \mathbb{R}^2 : |x| \leq n\} = \mathrm{graph}(u)$ over $[-n,n]$,
- $\{(n,y) \in \mathbb{R}^2 : 0 \leq y \leq u(n)\} = \{n\} \times [0, \frac{1}{(1+cn)^2}]$,
- $\{(-n,y) \in \mathbb{R}^2 : 0 \leq y \leq u(-n)\} = \{-n\} \times [0,1]$.

Extracting a limit of these closed flows, we have a Brakke flow $[0,\infty) \ni t \mapsto \mu_t$, with μ_0 corresponding to the varifold given by the union of the graph of u and the x-axis. Using the exact same barriers, we have that the approximations are contained in the region

$$\mathrm{Im}\,(\gamma_n(\cdot,t)) \subseteq \{(x,y) \in \mathbb{R}^2 : x \leq \max\{\alpha_c, \left(1+t^{-2}\right)\},\ 0 \leq y \leq \min\{1, q(x,t)\}\}, \quad \forall n \in \mathbb{N}.$$

As mentioned above, our approximations are not star-shaped. In order to deduce regularity, we instead use the pseudolocality result of Ilmanen-Neves-Schulze (Theorem 2.4.18) from §2 to deduce regularity on most of space-time. Applying this result around all points on the x-axis and graph of u, we have that M_t is smooth in a region of space-time which gets thinner and thinner in time around $t = 0$ as the x-coordinate gets larger and larger. To show that the solution is smooth for uniform time, we use a blow-up argument.

Suppose that, for any $t_0 > 0$, our solution is not a smooth solution in $\mathbb{R}^2 \times (0,t_0)$. Then there exists a sequence (z_n, t_n) in the support of our Brakke flow with $t_n \searrow 0$, such that our flow is not regular at (z_n, t_n). Let $z_n = (x_n, y_n)$. Due to Theorem 2.4.18, we know that $(1 + cx_n)^{-4} \lesssim t_n$, and thus $x_n \to \infty$ as $n \to \infty$. We perform a blow-up along this sequence of points at scales $\lambda_n > 0$, which we will later choose depending on the of values of y_n.

Consider the Brakke flow $[-\lambda_n^2 t_n, \infty) \ni t \mapsto \mu_t^n$ defined by

$$\mu_t^n(B) := \lambda_n \cdot \mu_{t_n + \lambda_n^{-2} t}(x_n + \lambda_n^{-1} \cdot B), \quad \forall B \in \mathcal{B}.$$

At time $-\lambda_n^2 t_n$, the Brakke flow is given by the varifold

$$\{y = -\lambda_n y_n\} \cup \operatorname{graph}(\lambda_n \cdot (u(x_n + \cdot) - y_n)).$$

By our barrier q, we see that

$$0 \le y_n(1 + cx_n)^2 \le e^{24ct_n}, \quad \forall n \in \mathbb{N}.$$

Since the right hand side converges to 1 as $n \to \infty$, we extract a convergent subsequence

$$\alpha := \lim_{n\to\infty} y_n(1 + cx_n)^2 \in [0, 1].$$

We split our analysis into three cases:

1. If $\alpha = 0$, perform our blow-up at scales $\lambda_n := \frac{1+cx_n}{\sqrt{y_n}} > 0$. Note that

$$\begin{aligned}
\lambda_n^2 \cdot t_n &= \frac{t_n(1 + cx_n)^4}{y_n(1 + cx_n)^2} \ge \frac{1}{100(y_n(1 + cx_n)^2)} \to \infty,\\
\lambda_n \cdot y_n &= \sqrt{y_n(1 + cx_n)^2} \to 0,\\
\lambda_n \cdot u(x_n) &= \frac{1}{\sqrt{y_n(1 + cx_n)^2}} \to \infty.
\end{aligned}$$

   Extracting a limit, we show have the eternal unit-regular Brakke flow which is just the static x-axis, and hence smooth.

2. If $\alpha = 1$, perform a blow-up at scales $\lambda_n := \frac{1+cx_n}{\sqrt{q(x_n,t_n)-y_n}} > 0$. We see that

$$\limsup_{n\to\infty}(q(x_n, t_n) - y_n)(1 + cx_n)^2 = \limsup_{n\to\infty} q(x_n, t_n)(1 + cx_n)^2 - 1 = \limsup_{n\to\infty} e^{24ct_n} - 1 = 0,$$

   and thus

$$\lim_{n\to\infty}(q(x_n, t_n) - y_n)(1 + cx_n)^2 = 0.$$

Using this we have that

$$\lambda_n^2 \cdot t_n = \frac{t_n(1+cx_n)^4}{(q(x_n,t_n)-y_n)(1+cx_n)^2} \geq \frac{1}{100((q(x_n,t_n)-y_n)(1+cx_n)^2)} \to \infty,$$
$$\lambda_n \cdot y_n = \frac{y_n(1+cx_n)^2}{\sqrt{(q(x_n,t_n)-y_n)(1+cx_n)^2}} \to \infty,$$
$$\lambda_n \cdot (u(x_n)-y_n) \leq \lambda_n \cdot (q(x_n,t_n)-y_n) = \sqrt{(q(x_n,t_n)-y_n)(1+cx_n)^2} \to 0.$$

Extracting a limit, we should again obtain a Brakke flow which is the static line and so smooth.

3. If $\alpha \in (0,1)$, we perform a blow-up at scales $\lambda_n = (1+cx_n)^2 > 0$. Note that

$$\lambda_n \cdot y_n \to \alpha, \quad \lambda_n \cdot (u(x_n)-y_n) \to 1-\alpha.$$

If $\lambda_n^2 t_n$ doesn't diverge, after passing to a subsequence, it converges to some positive value $T_0 > 0$. Extracting a limit would give a unit-regular Brakke flow starting from the parallel lines $\{y = -\alpha\}, \{y = 1-\alpha\}$ at time $-T_0$. Hence the Brakke flow has to be two static lines. But then there is not a point of non-zero density at the origin, which gives a contradiction. Thus $\lambda_n^2 t_n \to \infty$ and instead we extract a limit which looks, intuitively, like it is coming from the parallel lines $\{y = -\alpha, 1-\alpha\}$ at time $-\infty$. If one could show that this Brakke flow is the smooth Grim Reaper soliton, this would conclude the argument.

4.2 Uniformly proper solutions

Recall the following subclass of solutions to CSF within the class of properly embedded smooth solutions.

Definition 4.2.1. Let $(X, \overline{g})$ be a complete Riemannian surface and $T \in (0,\infty)$. We say that $\gamma : \mathbb{R} \times [0,T] \to X$ is a *uniformly proper solution* to CSF (in X) if

i) $\gamma : \mathbb{R} \times [0,T] \to X$ is a continuous proper map.

ii) $\gamma(\cdot,t) : \mathbb{R} \to X$ is a smooth proper embedding $\forall t \in (0,T]$.

iii) γ is smooth and solves (1.2.1) on $\mathbb{R} \times (0,T)$.

By working within a class of solutions that are proper as maps on space-time, we avoid tangential re-parameterisations which get arbitrarily bad as t goes to zero. In particular, Example 1.3.8 from the previous subsection is *not* a uniformly proper solution. To begin, we show that working within this class of solutions is sufficient to deduce many of the usual properties that closed solutions to mean curvature flow exhibit, such as the avoidance principle with closed solutions, as well as the local monotonicity formula (Theorem 2.4.14) when the ambient space is Euclidean.

4.2.1 Properties of uniformly proper solutions

Given any smooth properly embedded solution $\gamma : \mathbb{R} \times (0,T) \to X$ to CSF, we can consider the associated Brakke flow $(0,T) \ni t \mapsto \mu_t$, where the measures μ_t correspond to the varifolds $\mathrm{Im}\,(\gamma(\cdot,t))$ for each $t \in (0,T)$. Using the local area bounds for Brakke flows (Lemma 2.4.5), there is a well-defined Radon measure $\mu^- := \lim_{t \searrow 0} \mu_t$ on X.

Recall the smooth properly embedded solution to CSF we constructed in Example 1.3.8. Even though the convergence backwards in time to the initial data is locally smooth, looking at this solution as a Brakke flow, the limiting measure μ^- at time zero would be given by the Hausdorff measure restricted to the the graphs of $\pm u$. This means that the Radon measure given by the initial data μ_0 does *not* agree with the limiting measure μ^-. The following lemma shows that for a uniformly proper solution, if the solution converges to the initial data in a locally smooth way, then this limiting measure μ^- does in fact agree with the measure μ_0 given by the initial data $\mathrm{Im}\,(\gamma(\cdot,0))$, and hence our solution gives a well-defined Brakke flow all the way up to time zero.

Lemma 4.2.2. *Suppose $\gamma : \mathbb{R} \times [0,T] \to X^2$ is a uniformly proper solution to CSF with $\gamma(\cdot,t) \to \gamma(\cdot,0)$ in $C^\infty_{loc}(\mathbb{R},X)$ as $t \searrow 0$. For each $t \in [0,T]$, let μ_t denote the measure corresponding to $\gamma(\cdot,t)$. That is*

$$\mu_t(B) = \mathcal{H}^1(\mathrm{Im}\,(\gamma(\cdot,t)) \cap B), \quad \forall B \in \mathcal{B},\ \forall t \in [0,T],$$

where $\mathcal{B}$ denotes the Borel sets in X. Then $[0,T] \ni t \mapsto \mu_t$ is a well-defined Brakke flow in $X^2 \times [0,T]$.

Proof. It is clear that for each $t \in (0,T]$, $\mathrm{supp}(\mu_t)$ is smooth, and hence an integral 1-dimensional varifold in X. Fix $f \in C^0_c(X)$. As γ is uniformly proper, there exists $L > 0$ such that $\gamma^{-1}(\mathrm{supp}\,(f)) \subseteq [-L,L] \times [0,T]$. Since $\gamma(\cdot,t) \to \gamma(\cdot,0)$ in $C^\infty([-L,L],X)$ as $t \searrow 0$, we can conclude that

$$\int f d\mu_t = \int_{\gamma([-L,L],t)} f d\mu_t \to \int_{\gamma([-L,L],0)} f d\mu_0 = \int f d\mu_0.$$

We have shown that $\mu_t \rightharpoonup \mu_0$ as $t \searrow 0$. In particular, μ_0 is a well-defined radon measure on X. Fixing $f \in C^1_c(X \times [0,T])$ with $f \geq 0$, we now need to show that

$$\int_X f(\cdot,T) d\mu_T - \int_X f(\cdot,0) d\mu_0 \leq \int_0^T \int_X \left(-k^2 f + k\nu \cdot \nabla f + \partial_t f\right) d\mu_t dt. \tag{4.2.1}$$

To this end, choose $x_0 \in X$ and $\rho > 0$ such that $\mathrm{supp}(f) \subseteq B(x_0,\rho) \times [0,T]$. Choosing T_0 sufficiently small, we can apply the local area bounds (2.4.2) on $B_{2\rho} \times (0,T_0)$ to give

$$\mu_t(B_\rho) + \int_0^t \int_{B_\rho} k^2 d\mu_t dt \leq 8\mu_0(B_{2\rho}) < \infty, \quad \forall t \in [0,T_0].$$

Therefore, we have that

$$\int_0^{T_0}\int k^2 f d\mu_t dt \le \left(\int_0^{T_0}\int_{B_\rho} k^2 d\mu_t dt\right)\|f\|_{L^\infty} \le 8\mu_0(B_{2\rho})\|f\|_{L^\infty} < \infty,$$

$$\int_0^{T_0}\int |k\nu\cdot\nabla f| d\mu_t dt \le \left(\int_0^{T_0}\int_{B_\rho} k^2 d\mu_t dt\right)^{\frac{1}{2}}\left(\int_0^{T_0}\int |\nabla f|^2 d\mu_t dt\right)^{\frac{1}{2}} < \infty,$$

$$\int_0^{T_0}\int |\partial_t f| d\mu_t dt \le \left(\int_0^{T_0}\int_{B_\rho} d\mu_t dt\right)\|\partial_t f\|_{L^\infty} \le 8T_0\mu_0(B_{2\rho})\|\partial_t f\|_{L^\infty} < \infty,$$

which implies that the function $-k^2 f + k\nu\cdot\nabla f + \partial_t f$ is integrable on $X\times[0,T]$. By the fact that the flow is smooth away from time zero, we have that

$$\int f(\cdot,T)d\mu_T - \int f(\cdot,\epsilon)d\mu_\epsilon = \int_\epsilon^T\int \left(-k^2 f + k\nu\cdot\nabla f + \partial_t f\right) d\mu_t dt,$$

for any $\epsilon > 0$. Taking $\epsilon \searrow 0$, equation (4.2.1) follows from the dominated convergence theorem. $\square$

As a corollary to the previous lemma, we see that in Euclidean space, uniformly proper solutions satisfy the local monotonicity formula (Theorem 2.4.14) all the way up to the initial time.

Corollary 4.2.3. *Suppose $\gamma : \mathbb{R}\times[0,T] \to \mathbb{R}^2$ is a uniformly proper solution to CSF with $\gamma(\cdot,t)\to\gamma(\cdot,0)$ in $C^\infty_{loc}(\mathbb{R},\mathbb{R}^2)$ as $t\searrow 0$. Then for any point $X_0 = (x_0,t_0)\in\mathbb{R}^2\times(0,T]$ and any $R>0$, the R-local Gaussian density ratio*

$$(0,\sqrt{t_0}] \ni r \mapsto \Theta^R(\gamma, X_0, r),$$

as defined in (2.4.5) is increasing, and therefore we have the upper bound on the Gaussian density

$$\Theta(\gamma, X_0) \le \Theta^R(\gamma, X_0, \sqrt{t_0}),$$

where the right-hand side depends only on the initial data.

Given any smooth properly embedded solution to CSF $\gamma : \mathbb{R}\times(0,T) \to X$, we can ask if our solution satisfies the avoidance principle with any closed solution to CSF. That is, if $\eta : S^1\times[0,T]\to X$ is any closed solution to CSF, whose image is initially disjoint from $\mathrm{Im}\,(\gamma(\cdot,0))$, then the image of η at any later time t must remain disjoint from the image of γ at that same time t. Without the uniformly proper hypothesis on γ, the avoidance principle with closed curves will not hold in general (consider the curve constructed in Example 1.3.8 and any closed solution η that is initially disjoint from the graph of $-u$, but intersects the graph of u at every time $t\in[0,1]$). The following lemma shows that for a uniformly proper solution, we can conclude the avoidance principle with closed curves.

Theorem 4.2.4. *Let $\gamma : \mathbb{R}\times[0,T] \to X$ be a uniformly proper solution to CSF and η :*

$S^1 \times [0,T] \to X$ a closed solution to CSF. If the curves are initially disjoint, then they remain disjoint.

$$\mathrm{Im}(\eta(\cdot,0)) \cap \mathrm{Im}(\gamma(\cdot,0)) = \emptyset \implies \mathrm{Im}(\eta(\cdot,t)) \cap \mathrm{Im}(\gamma(\cdot,t)) = \emptyset, \quad \forall t \in [0,T].$$

Proof. Since η is continuous, $\mathrm{Im}\,(\eta) \Subset X$. Since γ is uniformly proper, there exists $K \Subset \mathbb{R}$ such that $\gamma^{-1}(\mathrm{Im}\,(\eta)) \subseteq K \times [0,T]$. Choose $n \in \mathbb{N}$ such that $K \Subset [-n,n]$. We only need to check that the restriction of our uniformly proper solution $\gamma : [-n,n] \times [0,T] \to X$ remains disjoint from the solution η. By our choice of n, we have that

$$\gamma(\pm n, t) \cap \mathrm{Im}\,(\eta) = \emptyset, \quad \forall t \in [0,T].$$

Therefore, the result follows from the usual avoidance principle, Corollary 2.2.7. $\square$

We may ask if there are any conditions under which a properly embedded solution can be seen to be uniformly proper. One possible answer to this question is if our solution can be expressed with respect to some time-independent gauge:

Suppose we have a smooth proper map $\mathcal{F} : \mathbb{R}^2 \to X$ from the plane to our surface. Let $\gamma : \mathbb{R} \times [0,T] \to X$ be a smooth properly embedded solution to CSF. We say that γ has time-independent gauge $\mathcal{F}$ if these exists a continuous function $u : \mathbb{R} \times [0,T] \to \mathbb{R}$ such that

$$\gamma(x,t) = \mathcal{F}(x, u(x,t)), \quad \forall (x,t) \in \mathbb{R} \times [0,T].$$

Lemma 4.2.5. *Let $\mathcal{F} : \mathbb{R}^2 \to X$ be a smooth proper map and $\gamma : \mathbb{R} \times [0,T] \to X$ a properly embedded solution to CSF with time-independent gauge $\mathcal{F}$. Then γ is a uniformly proper solution.*

Proof. For any $K \Subset X$, there exists $I \Subset \mathbb{R}$ such that

$$\mathcal{F}^{-1}(K) \subseteq I \times \mathbb{R}.$$

It is then easy to see that

$$\begin{aligned}
\gamma^{-1}(K) &= \{(x,t) \in \mathbb{R} \times [0,T] : \mathcal{F}(x, u(x,t)) \in K\} \\
&\subseteq \{x \in \mathbb{R} : \exists y \in \mathbb{R}, \text{ with } \mathcal{F}(x,y) \in K\} \times [0,T] \\
&\subseteq I \times [0,T].
\end{aligned}$$

$\square$

Example 4.2.6. In the case that $X = \mathbb{R}^2$ and $\mathcal{F} : \mathbb{R}^2 \to \mathbb{R}^2$ is just the identity function, solutions with time-independent gauge $\mathcal{F}$ are just graphical solutions over the x-axis.

4.2.2 A uniqueness conjecture

We discussed the result of Daskalopoulos & Saez in §1 that showed uniqueness within the class of entire graphical solutions in the plane (Theorem 1.2.12). Due to Lemma 4.2.5, we could view the following conjecture as one way to extend their uniqueness result.

Conjecture 4.2.7. *CSF is unique on the flat plane (see Definition 1.3.12).*

Despite this conjecture potentially ruling out non-uniqueness in the flat plane, we show that for other ambient surfaces, CSF can be non-unique.

4.3 Non-uniqueness of curve shortening flow

The aim of this subsection is to construct a uniformly proper solution to CSF (with respect to a suitably metric on the plane) starting from the x-axis, which moves away from the x-axis instantaneously under CSF. If we choose our metric to be of the form $g := dx^2 + e^{2\phi(x)}dy^2$ as in §3, then the x-axis is a geodesic, and so there is also the uniformly proper solution starting from the x-axis which remains static.

4.3.1 Choosing our ambient metric

Lemma 4.3.1. *There exists a smooth function $\phi : \mathbb{R} \to \mathbb{R}$ such that*

1. *ϕ is an even function, which is increasing on $(0, \infty)$.*
2. *$\phi(x) = 0$ for all $x \in [0, 1]$.*
3. *$\phi'(x) > 0$ and ϕ' is strictly increasing on $(1, \infty)$.*
4. *$\phi'(x) < \frac{1}{2}$ for all $x \in (1, \frac{3}{2})$.*
5. *$\phi'(x) = x^2$ for all $x \geq 2$.*

The construction of ϕ, and hence our metric, is just an exercise in choosing suitable bump functions.

Proof. For $x \in \mathbb{R}$, define the following smooth bump functions

$$f_1(x) := \begin{cases} 0 : x \leq 0 \\ e^{-\frac{1}{x}} : x > 0 \end{cases}, \quad f_2(x) := \frac{f_1(x)}{f_1(x) + f_1(\frac{1}{4} - x)}, \quad f_3(x) := \frac{f_2(x-1) + 8f_2\left(x - \frac{7}{4}\right)}{9}.$$

For all $x > 0$, we then define

$$\phi(x) := \int_0^x y^2 \cdot f_3(y)\, dy.$$

Since $f_3(y) = 0$ for any $y \leq 1$, $\phi(x) = 0$ for any $x \in (0, 1]$, and hence we can extend ϕ to a smooth even function on all of $\mathbb{R}$. For any $x > 0$, $\phi'(x) = x^2 \cdot f_3(x)$. Therefore

- As $f_3(x) > 0$ for $x > 1$, $\phi'(x) > 0$ for any $x > 1$.
- As $f_3(x) = 1$ for $x \geq 2$, $\phi'(x) = x^2$ for any $x \geq 2$.
- As $f_3(x) \leq \frac{1}{9}$ for $x \leq \frac{3}{2}$, $\phi'(x) \leq \frac{x^2}{9} < \frac{1}{2}$ for any $x \in (1, \frac{3}{2})$. $\square$

Other than the growth rate at infinity, our choices for ϕ are not crucial, but instead help reduce the technicality of our arguments. The last condition however is essential. The rapid growth of ϕ for large x is what allows curves to *bloom at infinity* as discussed in Example 1.3.13. Given $c \in \mathbb{R}$, consider the unique maximal solution to $\mathcal{H} = 0$ with initial condition c, denoted $c(t)$, that we defined in §3. Recall, $c(t)$ solves the ODE

$$\frac{\partial c}{\partial t} = -\phi'(c(t)), \quad c(0) \equiv c \in \mathbb{R}.$$

Suppose our initial condition $c > 0$. By our choice of ϕ, $c(t)$ is decreasing and immortal. Moreover, for large c and small t, the ODE that $c(t)$ solves is

$$\frac{\partial c}{\partial t} = -c(t)^2,$$

with explicit solution $c(t) = (c^{-1} + t)^{-1}$. Taking $c \to \infty$, for small positive times $t \in (0, \epsilon]$, we have the solution $\zeta(t) = t^{-1}$ to the equation $\mathcal{H}(\zeta) = 0$. Geometrically, this means that lines parallel to the y-axis fly in from infinity in finite time under CSF. In particular, consider any closed solution $\eta : S^1 \times [0, T] \to \mathbb{R}^2$ to CSF with respect to this ambient metric. Since $\mathrm{Im}\,(\eta(\cdot, 0)) \Subset \mathbb{R}^2$, we can find $t_0 \in (0, \epsilon]$ sufficiently small, such that the line $\{y = \zeta(t_0)\}$ lies to the right of $\mathrm{Im}\,(\eta(\cdot, 0))$. Note that, the solution $x : \mathbb{R} \times [t_0, \epsilon] \to \mathbb{R}^2$, with $x(y, t) \equiv \zeta(t)$ is a graphical solution to CSF with respect to the metric g, and hence a uniformly proper solution. Therefore, by Theorem 4.2.4, we deduce that $\mathrm{Im}\,(\eta(\cdot, \epsilon))$ lies to the left of the vertical line $\{y = \zeta(\epsilon - t_0)\}$, and hence to the left of the vertical line $\{y = \zeta(\epsilon)\}$. Consider the *level set flow* starting from the x-axis: the maximal region in plane that avoids all closed solutions which are initially disjoint from the x-axis. Repeating the above argument for all η, we have that, for all $\epsilon > 0$ arbitrarily small, the region $\{x \geq \zeta(\epsilon)\}$ is contained within this level set flow at time ϵ, and the level set flow starting from the x-axis instantly *fattens*. We now show that within this fattening, there does exist a smooth non-zero graphical solution to CSF with the x-axis as initial data.

Theorem 4.3.2. *With respect to this choice of $\phi : \mathbb{R} \to \mathbb{R}$, there exists a continuous function $y : \mathbb{R} \times [0, \infty) \to [-1, 1]$ such that*

(i) $y(\cdot, 0) \equiv 0$ *on* $\mathbb{R}$.

(ii) $y(\cdot, t)$ *is an increasing odd function,* $\forall t \in (0, \infty)$.

(iii) y *is smooth and satisfies* $\mathcal{V}(y) = 0$ *on* $\mathbb{R} \times (0, \infty)$.

(iv) $y(\cdot,t)$ instantly peels away at infinity:

$$\forall \epsilon, t > 0,\ \exists x_0 > 0 \quad \text{such that} \quad y(x,t) > 1-\epsilon, \quad \forall x > x_0.$$

4.3.2 Constructing compact graphical approximations

Lets find a sequence of solutions to the Dirichlet problems $V(\cdot)$ from §3 on larger and larger compact subsets of the real line. We start by defining our initial data.

Fix $\chi : [0,1] \to [0,1]$ a smooth, decreasing cut off function such that $\chi \equiv 1$ on $[0, \frac{1}{4})$, $\chi \equiv 0$ on $(\frac{3}{4}, 1]$ and $\chi' > -4$ on $[0,1]$. For each $n \in \mathbb{N}$, define the function $Y_n : [-n,n] \to [-1,1]$ to be the unique odd function such that

$$Y_n(x) := \begin{cases} 0 & : x \in [0, n-1] \\ \chi(x+1-n)) & : x \in [n-1, n]. \end{cases}$$

For each $n \in \mathbb{N}$ and $s \in (0,\infty)$, use Y_n as the auxiliary data in the Dirichlet problem $V(s)$ from §3. Then there exists $T_n \in (0,\infty]$ and a continuous function $y_n : [-n,n] \times [0,T_n) \to \mathbb{R}$ with the following properties:

(i) y_n solves the Dirichlet problem $V(s)$, with auxiliary data Y_n, $\forall s \in (0,T_n)$;

(ii) $y_n \in P^{2,1}([-n,n] \times [0,s])$, $\forall s \in (0,T_n)$;

(iii) $y_n \in C^\infty_{loc}((-n,n) \times (0,T_n))$.

Note that Y_n vanishes identically in a neighbourhood of the parabolic walls, and so the compatibility conditions of all orders are satisfied. Therefore, using the global Schauder estimates (Theorem A.0.3) and bootstrapping, we can improve (ii) to

(ii') $y_n \in C^\infty([-n,n] \times [0,s])$, $\forall s \in (0,T_n)$.

Moreover, Proposition 3.3.3 then also applies, and for every $t > 0$, $y_n(\cdot,t)$ is strictly increasing. Finally, by the symmetries of Y_n, ϕ and $\mathcal{V}$, $y_n(\cdot,t)$ is an odd function for all $t \in (0,T_n)$.

Lemma 4.3.3. *For each $n \in \mathbb{N}$, let $y_n : [-n,n] \times [0,T_n) \to \mathbb{R}$ be the solution to the Dirichlet problem mentioned above. Then for each $t \in [0,T_n)$, the graph of $x \mapsto y_n(x,t)$ is contained in the parallelogram*

$$\{(x,y) \in [-n,n] \times [-1,1] : 1 + 4(x-n) \le y \le -1 + 4(x+n)\}\}. \tag{4.3.1}$$

Proof. By the maximum principle, we have the upper and lower barriers ± 1. Using that $\phi'(x) \ge 0$ for $x \ge 0$ and $\mu(x,p) > 0$ for any $p \in \mathbb{R}$, we have that

$$\mathcal{V}(4x) = -4\phi'(x)(1 + \mu(4)) \le 0.$$

Since $0 \le \frac{\partial y_n}{\partial x}(\cdot, 0) \le 4$, we have that

$$1 + 4(x - n) \le y_n(x, 0), \quad \forall x \in [0, n].$$

As $y_n(\cdot, t)$ is an odd function for each $t \ge 0$, we have that

$$\begin{aligned} 1 + 4(0 - n) &\le 0 = y_n(0, \cdot) \\ 1 + 4(n - n) &= 1 = y_n(n, \cdot). \end{aligned}$$

Hence $1 + 4(x - n)$ is a lower barrier to y_n over $[0, n]$. By symmetry, $-1 + 4(x + n)$ is an upper barrier over $[-n, 0]$. $\square$

Note that, Lemma 4.3.3 implies y_n has gradient bounded by 4 on the parabolic boundary Γ_{T_n}. Therefore, the 2nd part of Theorem 3.3.1 applies and each y_n is immortal.

Recall from the discussion in §4.3.1, that for any $c > 1$, we have the immortal, strictly decreasing solution $c(t)$ to $\mathcal{H} = 0$ starting from the constant initial condition c. Note that $c(t)$ is bounded below by 1. Since $\phi'(x) \ge 0$ whenever $x > 0$, for any $c > 1$ and $s \in (0, \infty)$, we can solve the Dirichlet problem $H_c(s)$ from §3, and hence there exists a $T_c \in (0, \infty]$ and a continuous function $g_c : [0, 1] \times [0, T_c) \to [0, \infty)$ satisfying,

(i) g_c solves $H_c(s)$, $\forall s \in (0, T_c)$;

(ii) $g_c \in P^{2,1}((0, 1) \times (0, s))$, $\forall s \in (0, T_c)$;

(iii) $g_c \in C^\infty_{loc}((0, 1) \times (0, T_c)$.

Lemma 4.3.4. *For each $c > 1$, let $g_c : [0, 1] \times [0, T_c) \to [0, \infty)$ be the solution to the Dirichlet problem mentioned above. Then there exists a constant $m \in (0, 1)$ depending on c such that, for each $t \in [0, T_c)$, the graph of $y \mapsto g_c(y, t)$ is contained in the region*

$$\{(x, y) \in [c(t), c] \times [0, 1] : \eta_{m,0}(y) \le x \le c(t)(1 - y) + cy\}, \tag{4.3.2}$$

where $\eta_{m,0}$ refers to the horizontal geodesic constructed in §3.

Proof. Using that $\phi'(x) \ge 0$ is increasing for $x > 0$, $c(t) > 0$ is decreasing, and $\nu(x, p) > 0$ for any $p \in \mathbb{R}$, we have that

$$\begin{aligned} \mathcal{H}(c(t)(1 - y) + cy) &\ge c'(t)(1 - y) + \phi'(c(t)(1 - y) + cy) \\ &\ge \phi'(c(t))(y - 1) + \phi'(c(t)) \ge 0. \end{aligned}$$

So $c(t)(1 - y) + cy$ is a supersolution. Moreover

$$c(0)(1 - y) + cy = c, \quad c(t)(1 - 0) + c \cdot 0 = c(t), \quad c(t)(1 - 1) + c \cdot 1 = c.$$

Hence $c(t)(1-y)+cy$ is an upper barrier to g_c. For a lower barrier to g_c, choose $m \in (0,1)$ such that $\sigma_{m,0}(c) = 1$. Since $m \neq 0$, $\sigma_{m,0}$ is invertible, and we have the horizontal graph $\eta_{m,0}$ which is an increasing geodesic with $\eta_{m,0}(0) = 0$ and $\eta_{m,0}(1) = c$. $\square$

Note that, Lemma 4.3.4 implies g_c has gradient bounded by the maximum of $c-1$ and $\eta'_{m,0}(1)$ on the parabolic boundary Γ_{T_c}. Therefore, Theorem 3.3.2 applies and each g_c is immortal. Furthermore, Lemma 4.3.4 implies $\frac{\partial g_c}{\partial y}(1,t) > 0$ for every $t > 0$. In combination with Proposition 3.3.4, $g_c(\cdot,t)$ is has strictly positive gradient for every $t > 0$.

4.3.3 Extracting an entire solution

We currently have a sequence of continuous functions

$$y_n : [-n,n] \times [0,\infty) \to [-1,1], \quad \forall n \in \mathbb{N},$$

such that

(i) y_n is smooth on $(-n,n) \times (0,\infty)$, with $\mathcal{V}(y_n) \equiv 0$ over this region.

(ii) $y_n(\cdot,t)$ is a strictly increasing odd function with positive gradient for any $t > 0$.

(iii) $y_n(x,0) = 0$ for $x \in [1-n, n-1]$, and $y_n(\pm n, t) = \pm n$ for $t \geq 0$.

(iv) $y_n(x,t)$ is a decreasing sequence in n for any $(x,t) \in [0,n] \times [0,\infty)$.

To see (iv) we note that, for $m \leq n$

$$\begin{aligned} y_n(x,0) &\leq y_m(x,0), \quad \forall x \in [0,m], \\ y_n(0,t) &= 0 = y_m(0,t), \quad \forall t > 0, \\ y_n(m,t) &\leq 1 = y_m(m,t), \quad \forall t > 0. \end{aligned}$$

Monotonicity then follows from the avoidance principle (Theorem 2.2.8).

These properties allow us to do several things:

- By (ii), $y_n(\cdot,t)$ is invertible for each $n \in \mathbb{N}$ and $t > 0$. Thus, we can change gauge and consider the curves as horizontal graphs

  $$x_n : [-1,1] \times (0,\infty) \to [0,n], \quad x_n(\cdot,t) := y_n(\cdot,t)^{-1}, \ \forall n \in \mathbb{N}.$$

  Moreover, by (i), the horizontal graphs x_n are smooth and satisfy $\mathcal{H}(x_n) \equiv 0$ on $(-1,1) \times (0,\infty)$.

- (iv) allows us to take a limit of this sequence to get a function $y : \mathbb{R} \times [0,\infty) \to [-1,1]$,

  $$y(x,t) := \lim_{n\to\infty} y_n(x,t), \quad \forall (x,t) \in \mathbb{R} \times [0,\infty).$$

Moreover, by (iii), $y(\cdot, 0) \equiv 0$ on $\mathbb{R}$, and by (ii), $y(\cdot, t)$ is an increasing odd function for any $t > 0$.

We want to show that y is smooth and solves $\mathcal{V}(y) = 0$ on $\mathbb{R} \times (0, \infty)$. This will follow if we can show that the convergence of the sequence y_n to y is locally smooth on $\mathbb{R} \times (0, \infty)$. We also want to show that y is continuous on $\mathbb{R} \times [0, \infty)$. However, once we have shown local smooth convergence on $\mathbb{R} \times (0, \infty)$, continuity on $\mathbb{R} \times [0, \infty)$ follows from the monotonicity of each term in the sequence and the monotonicity of the limit.

Lemma 4.3.5. *If y_n converges to y locally smoothly on $\mathbb{R} \times (0, \infty)$, then $y \in C(\mathbb{R} \times [0, \infty))$.*

Proof. Since $y(\cdot, 0)$ is an odd function, it suffices to show that y is continuous at any point $(x_0, 0)$ with $x_0 > 0$. Fix $\epsilon > 0$ and choose n large enough so that $y_n(x_0, 0) = 0$. Since y_n is continuous at $(x_0, 0)$, there exists some $\delta \in (0, x_0)$ such that

$$|y_n(x, t)| \leq \epsilon, \quad \forall (x, t) \in (x_0 - \delta, x_0 + \delta) \times [0, \delta).$$

As $y_n(x, t)$ is an increasing function in x for any fixed t

$$0 = y_n(0, t) \leq y_n(x, t) < \epsilon \quad \forall (x, t) \in (x_0 - \delta, x_0 + \delta) \times [0, \delta).$$

Finally, as $y_n(x, t)$ is a decreasing sequence in n

$$0 \leq y(x, t) < \epsilon \quad \forall (x, t) \in (x_0 - \delta, x_0 + \delta) \times [0, \delta).$$ □

4.3.4 Local regularity

The goal of this next section is to show the following theorem.

Theorem 4.3.6 (Local gradient bounds)**.** *Fix $k, T > 0$. Then there exists $M_1(k, T) > 0$ such that*

$$|y_n(\cdot, t)|_{C^1([-k,k])} \leq M_1, \quad \forall t \in [0, T], \quad \forall n > k + 1.$$

That is, on any compact region of space time, our sequence has a uniform spatial C^1-bound.

The strategy we employ to achieve this is to foliate our region of space-time with curves of controlled gradient, and then show that at any time, each solution y_n intersects each foliating curve only once. To begin, we shall show a uniform gradient bound at all times for our solutions over some compact subset of space strictly containing $[-1, 1]$.

Lemma 4.3.7 (Local gradient bounds on a small spatial neighbourhood)**.** *There exists $a > 1$ and $M_1 > 0$ such that*

$$0 \leq \frac{\partial y_n}{\partial x}(x, t) \leq M_1 \quad \forall x \in [0, a], \ \forall t \in [0, T], \ \forall n > 2.$$

In particular

$$|y_n(\cdot,t)|_{C^1([-a,a])} \leq M_1, \quad \forall t \in [0,T], \quad \forall n > 2.$$

The last line of this lemma rephrases it as the special case of Theorem 4.3.6 for $k \leq a$. To prove Lemma 4.3.7, we shall foliate $[-2,2] \times \mathbb{R}$ with the geodesics mentioned in §3.1. As such, the proof will require the following property of the geodesics.

Claim. *For each $m \in (0,1)$, let $\sigma_{m,0}$ denote the graphical geodesic constructed in §3.1. For m sufficiently close to* 1

$$\sigma_{m,0}(2) > \sigma_{m,0}(1) + 1 > 2.$$

Proof of claim. Since $\phi'(x) < \frac{1}{2}$ for $x \in (1, \frac{3}{2})$, we note that $\phi(1+s) \leq \frac{s}{2}$ for $s \in (0, \frac{1}{2})$. For each $m \in (0,1)$

$$\sigma_{m,0}(2) = \sigma_{m,0}(1) + \int_1^2 \frac{m}{e^{\phi(s)}\sqrt{e^{2\phi(s)} - m^2}}\, ds > \sigma_{m,0}(1) + \int_0^{\frac{1}{2}} \frac{m}{e^{\frac{s}{2}}\sqrt{e^s - m^2}}\, ds$$

Since $\int_0^{\frac{1}{2}} \frac{1}{e^{\frac{s}{2}}\sqrt{e^s-1}}\, ds > 1$, for m sufficiently close to 1, $\sigma_{m,0}(2) > \sigma_{m,0}(1) + 1$. We get the last inequality for free as the gradient of $\sigma_{m,0}$ is decreasing. □

Proof of Lemma 4.3.7. By the previous claim there exists $m \in (0,1)$ and $\epsilon > 0$ such that $\sigma_{m,0}(2) = \sigma_{m,0}(1) + 1 + 2\epsilon$. Fix $a > 1$ such that $\sigma_{m,0}(a) = \sigma_{m,0}(1) + \epsilon$. We consider the geodesics $\sigma_{m,h} : \mathbb{R} \to \mathbb{R}$ for $|h| \leq \sigma_{m,0}(1) + \epsilon$. Note that, for any such h, it is a regular value of $(y_n(\cdot,0) - \sigma_{m,h}(\cdot))$ over $[-2,2]$. In particular, since

$$\sigma_{m,h}(2) \geq 1 + \epsilon > 1 \geq y_n(2,t),$$
$$\sigma_{m,h}(-2) \leq -1 - \epsilon < -1 \leq y_n(-2,t),$$

by intersection principle (Lemma 2.2.5), at each time $t \in [0,T]$, the curves $\sigma_{m,h}(\cdot)$ and $y_n(\cdot,t)$ intersect at a single point over $[-2,2]$.

Finally, as the region $[0,a] \times [0,1]$ is entirely foliated by the curves $\sigma_{m,h}$ for $|h| \leq \sigma_{m,0}(1) + \epsilon$, we have

$$0 \leq \frac{\partial y_n}{\partial x}(x,t) \leq \sigma'_{m,h}(x) \leq \sup_{[-2,2]} \sigma'_{m,0}, \quad \forall (x,t) \in [0,a] \times [0,T].$$ □

Next we shall show a uniform gradient bound over $[-k,k]$ for a short amount of time.

Lemma 4.3.8 (Local gradient bounds for a short time)**.** *Fix $k > 0$. Then there exists $\tau > 0$ and $M_1(k) > 0$ such that*

$$0 \leq \frac{\partial y_n}{\partial x}(x,t) \leq M_1, \quad \forall x \in [0,k],\ \forall t \in [0,\tau],\ \forall n > k+1.$$

In particular

$$|y_n(\cdot,t)|_{C^1([-k,k])} \le M_1, \quad \forall t \in [0,\tau], \quad \forall n > k+1.$$

The last line of this lemma again rephrases it as the special case of Theorem 4.3.6 for for $T \le \tau$. To prove Lemma 4.3.8 we foliate $[-k,k] \times \mathbb{R}$ for a short amount of time. We will not use geodesics to foliate, as their gradient becomes too shallow far out. Instead, we use the Dirichlet problems $V(\cdot)$ from §3.1 to construct an immortal, smooth solution to $\mathcal{V} \equiv 0$ from a suitable initial condition, and use vertical translations of this solution as a foliation. To be more precise, lets solve the Dirichlet problems $V(\cdot)$ with initial data given by the map $x \mapsto 4x$ on the slightly larger domain $[-(k+1), k+1]$. By the same reasoning as in §3.1, we have a continuous function $F : [0, k+1] \times [0,\infty) \to \mathbb{R}$ such that

- F is smooth and solves $\mathcal{V}(F) = 0$ on $(0, k+1) \times (0,\infty)$;
- $F(x,0) = 4x$, $F(k+1,t) = 4(k+1)$, $F(0,t) = 0$;
- $F(\cdot,t)$ is increasing for each $t > 0$.

By the vertical translation invariance of (3.1.4), for each time $t > 0$, we can foliate with the solutions

$$\mathcal{F}(t) := \{F(\cdot,t) - h : h \in \mathbb{R}\}.$$

We are concerned with which curves in our foliation $\mathcal{F}(t)$ intersect with our solution $y_n(\cdot,t)$. Define

$$H(x,t) := \operatorname{Im}\left(F(\cdot,t) - y_n(\cdot,t)|_{[0,x]}\right), \quad \forall x \in [0,k+1],$$

so that a curve $F(\cdot,t) - h \in \mathcal{F}(t)$ intersects $y_n(\cdot,t)$ over $[0,x]$ if and only if $h \in H(x,t)$. The following proposition bounds the size of $H(x,t)$.

Proposition 4.3.9. *For each $k > 0$, there exists a constant $A_k > 0$ such that*

$$H(x,t) \subseteq \left[0, 4xe^{A_k t}\right], \quad \forall (x,t) \in [0,k+1] \times (0,\infty). \tag{4.3.3}$$

Proof. Fix $A_k > 0$ to be determined later. We showed in §3.1 that $\mathcal{V}(4x) \le 0$. By a similar calculation

$$\mathcal{V}(4xe^{A_k t}) = 4e^{A_k t}\left(A_k x - \phi'(x)(1 + \mu(x, 4e^{A_k t}))\right) \ge 4e^{A_k t}\left(A_k x - 2\phi'(x)\right).$$

Note, for $x \in [0,1]$, $\phi'(x) = 0$ so $A_k x - 2\phi'(x) \ge 0$, and for $x \in [1, k+1]$, if we choose $A_k := 2(k+1)^2 = 2\phi'(k+1)$, then $A_k x - 2\phi'(x) \ge A_k - 2\phi'(x) \ge 0$. So $\mathcal{V}(4xe^{A_k t}) \ge 0$ over $[0, k+1]$ and by the avoidance principle (Lemma 2.2.8)

$$4x \le F(x,t) \le 4xe^{A_k t}, \quad \forall (x,t) \in [0,k+1] \times (0,\infty).$$

So for any $t \in [0,T]$, over $[0,x]$ we have

$$-1 \leq F(\cdot,t) - y_n(\cdot,t) \leq 4xe^{A_k t}$$

To finish the lemma, we note that for $h < 0$

$$F(x,0) - h > 4x \geq y_n(x,0), \quad F(0,t) - h > 0 = y_n(0,t), \quad F(k+1,t) - h > 1 = y_n(k+1,t).$$

So by the avoidance principle (Lemma 2.2.8), $h \notin H(x,t)$. □

Proof of Lemma 4.3.8. Choose $\tau := \frac{1}{A_k}\log(1+\frac{1}{4k}) > 0$ so that equation (4.3.3) becomes

$$H(k,t) \subseteq [0, 4k+1], \quad \forall t \in [0,\tau]. \tag{4.3.4}$$

Fix $(x_0,t_0) \in [0,k] \times [0,\tau]$. Since $\mathcal{F}(t_0)$ is a foliation, $\exists\ h_0 \in H(k,t_0)$ such that

$$F(x_0,t_0) - h_0 = y_n(x_0,t_0).$$

Since $\frac{\partial F}{\partial x}(x,0) = 4 > \frac{\partial y_n}{\partial x}(x,0)$, There is a single zero of $F - h_0 - y_n$ at time $t = 0$. By equation (4.3.3), for any $t \in [0,\tau]$

$$\begin{aligned} F(0,t) - h_0 &= -h_0 \leq 0 = y_n(0,t), \\ F(k+1,t) - h_0 &= 4(k+1) - h_0 \geq 3 > y_n(k+1,t). \end{aligned}$$

Therefore, at time t_0 the curves intersect transversely at x_0 by the intersection principle (Lemma 2.2.5), giving

$$0 \leq \frac{\partial y_n}{\partial x}(x_0,t_0) \leq \frac{\partial F}{\partial x}(x_0,t_0) \leq \sup_{[0,k]\times[0,\tau]} \left(\frac{\partial F}{\partial x}\right)$$ □

We are now ready to prove Theorem 4.3.6. Here we shall use the family of curves g_c that we constructed in § 3.1 to foliate our space. Although this foliation doesn't cover all of space-time, the regions it misses are covered by Lemma 4.3.7 and Lemma 4.3.8.

Proof of Theorem 4.3.6. Take $a > 1$ and $\tau > 0$ as in Lemma 4.3.8 and Lemma 4.3.7. To prove the theorem, it suffices to show that for any $(x^*,t^*) \in [a,k] \times [\tau,T]$, we can find a gradient bound for y_n at (x^*,t^*).

We begin by switching gauge for our solutions. View the vertical graphs y_n as the horizontal graphs $x_n : [0,1] \times (0,\infty) \to [0,n]$. By the intermediate value theorem and the monotonicity of $x_n(\cdot,t^*)$, $\exists\ y^* \in (0,1)$ such that $x_n(y^*,t^*) = x^*$. Choose $\tilde{\tau} > 0$ sufficiently small so that it is both less than τ and so that $g_{k+1}(\tilde{\tau}) \geq c_{k+1}(\tilde{\tau}) \geq k$. This implies that the region $\{(x,y) \in$

$[a, k] \times [0, 1]\}$ is folliated by the curves

$$\mathcal{G} := \{(g_c(y, \tilde{\tau}), y) : y \in [0, 1],\ c \in [a, k+1]\}.$$

In particular, there exists $c^* \in [a, k+1]$ such that

$$g_{c^*}(y^*, \tilde{\tau}) = x^* = x_n(y^*, t^*).$$

Consider the intersection number of the curves $g_{c^*}(\cdot, t)$ and $x_n(\cdot, t + (t^* - \tilde{\tau}))$. As $y_n(0, \cdot) = 0$ and $y_n(n, \cdot) = 1$, we have

$$x_n(0, t) = 0, \quad x_n(1, t) = n, \quad \forall t > 0.$$

In particular, as $g_{c^*}(\cdot, 0) = c^* \in (0, n)$ and $x_n(\cdot, t^* - \tilde{\tau})$ is strictly increasing, the curves initially intersect only once at a transverse point. Moreover, for any $t \geq 0$

$$\begin{aligned} g_{c^*}(0, t) &= c^*(t) > 0 = x_n(0, t + (t^* - \tilde{\tau})), \\ g_{c^*}(1, t) &= c^* < n = x_n(1, t + (t^* - \tilde{\tau})), \end{aligned}$$

and by the intersection principle (Lemma 2.2.5), the curves always intersect exactly once. Therefore

$$\frac{\partial x_n}{\partial y}(y^*, t^*) \geq \frac{\partial g_{c^*}}{\partial y}(y^*, \tilde{\tau}).$$

By the smooth dependence on auxiliary conditions for solutions to the Dirichlet problems $H_c(\cdot)$, the map

$$G : [0, 1] \times (1, \infty) \times (0, \infty) \to (0, \infty), \quad G(y, c, t) := \frac{\partial g_c}{\partial y}(y, t),$$

is continuous. Hence for any $X \Subset (1, \infty) \times (0, \infty)$, there exists $\epsilon(X) > 0$ such that

$$G([0, 1] \times X) \geq \epsilon > 0. \tag{4.3.5}$$

In particular, choosing $X := [a, k+1] \times [\tau, T]$ in equation (4.3.5), there exists $\epsilon > 0$ such that

$$\frac{\partial x_n}{\partial y}(y^*, t^*) \geq \frac{\partial g_{c^*}}{\partial y}(y^*, \tilde{\tau}) = G(y^*, c^*, \tilde{\tau}) \geq \epsilon.$$

That is

$$0 \leq \frac{\partial y_n}{\partial x}(x^*, t^*) \leq \frac{1}{\epsilon}.$$

□

We can now bootstrap to get uniform higher order bounds locally on our sequence.

Theorem 4.3.10 (Local bounds)**.** *Fix $j \in \mathbb{N}$ and $K \Subset \mathbb{R} \times (0, \infty)$. Then there exists $M_j > 0$ such that*

$$|y_n|_{P^j(K)} \leq M_j, \quad \forall n \in \mathbb{N}.$$

Proof. There exists some $\epsilon > 0$ such that $K_{2\epsilon} \subseteq \mathbb{R} \times (0, \infty)$, where $K_{2\epsilon}$ denotes the 2ϵ-fattening of K. We begin by substituting y_n into the coefficients of $\mathcal{V}$. By Theorem 4.3.6, our operator $\mathcal{L}_n$ is then a strictly parabolic linear operator. Moreover, on $K_{2\epsilon}$, these coefficients are uniformly bounded in $L^\infty(K_{2\epsilon})$. Thus, we can apply De Giorgi-Nash-Moser (Theorem A.0.2) to conclude that our sequence of solutions y_n are uniformly bounded in $P^{0,\alpha}(K_\epsilon)$ for some $\alpha \in (0, 1)$. Using interior Schauder estimates (Theorem A.0.4) the result follows. $\square$

As a consequence of Theorem 4.3.10, we now have local uniform P^j-bounds for our sequence y_n. Using Arzela-Ascoli and monotonicity, the sequence y_n converges to y in $C^\infty_{\text{loc}}(\mathbb{R} \times (0, \infty))$. Hence y is smooth with $\mathcal{V}(y) = \lim_{n\to\infty} \mathcal{V}(y_n) = 0$ on $\mathbb{R} \times (0, \infty)$.

4.3.5 Long term behaviour

We currently have a continuous function $y : \mathbb{R} \times [0, \infty) \to [-1, 1]$ such that

(i) $y(\cdot, 0) \equiv 0$ on $\mathbb{R}$.

(ii) $y(\cdot, t)$ is an increasing odd function $\forall t \in (0, \infty)$.

(iii) y is smooth and satisfies $\mathcal{V}(y) = 0$ on $\mathbb{R} \times (0, \infty)$.

The final step in the proof of Theorem 4.3.2 is to show that our solution does not remain equal to zero as we flow forwards in time. To do this we construct a suitable barrier. In particular, we find a graphical solution that acts as a barrier to our sequence of solutions y_n.

Let $\zeta : (0, \infty) \to (1, \infty)$ be the solution to the ODE

$$\frac{\partial}{\partial t}\zeta(t) = -\phi'(\zeta(t)), \tag{4.3.6}$$

such that $\zeta(t) \to \infty$ as $t \searrow 0$. As discussed earlier in §4.3.1, $\zeta(t) = t^{-1}$ for small $t > 0$. Consider the barrier function $b : (0, \infty) \times (0, \infty) \to (0, \infty)$ given by $b(y, t) := t + \zeta(t) + \frac{1}{\log(1+y)}$. We shall show that as a horizontal graph, this is a supersolution to equation (3.1.3). Since $\phi(b(y, t)) > 0$, we have the upperbound

$$\frac{b_{yy}}{b_y^2 + e^{2\phi(b)}} = \frac{2\log(1 + y) + \log(1 + y)^2}{1 + e^{2\phi(b)}(1 + y)^2} \le 1.$$

Moreover, since $\phi'(x)$ is increasing, we have that $\phi'(b(y, t)) + \dot{\zeta}(t) = \phi'(b(y, t)) - \phi'(\zeta(t)) \ge 0$. Using the above inequalities and substituting b into $\mathcal{H}$ gives

$$\mathcal{H}(b) = 1 + \dot{\zeta}(t) - \frac{b_{yy}}{b_y^2 + e^{2\phi}} + \phi'(b(y, t))\left(1 + \frac{b_y^2}{b_y^2 + e^{2\phi}}\right) \ge 0.$$

So b is a supersolution to (3.1.3). Switching gauge, $(x, t) \mapsto \exp\left(\frac{1}{x-(t+\zeta(t))}\right) - 1$ is a supersolution to (3.1.4) in the region $U := \{(x, t) \in (t + \zeta(t), \infty) \times (0, \infty)\}$. In particular, using $\mathcal{V}(-y) =$

$-\mathcal{V}(y)$ and the vertical translation invariance of $\mathcal{V}$, we have that the graph of the function $u(x,t) := 2 - \exp\left(\frac{1}{x-(t+\zeta(t))}\right)$ is a subsolution to (3.1.4) in the region U. We can then modify u to get a subsolution $\overline{u}$ in the barrier sense, defined on all of $(0,\infty)\times(0,\infty)$, by setting

$$\overline{u}(x,t) := \begin{cases} -1 & : (x,t) \notin U \\ \max\{-1, u(x,t)\} & : (x,t) \in U \end{cases}$$

This $\overline{u}$ is our barrier. The following lemma shows that this barrier does indeed push our solution y away from zero.

Lemma 4.3.11.

$$\overline{u}(x,t) \leq y(x,t), \quad \forall (x,t) \in \mathbb{R}\times[0,\infty).$$

Proof. Fix $n \in \mathbb{N}$. On the parabolic boundary of the region where y_n is defined

$$\overline{u}(\cdot,0) = -1 \leq y_n(\cdot,0), \quad \overline{u}(-n,\cdot) = -1 = y_n(0,\cdot), \quad \overline{u}(n,\cdot) < 1 = y_n(n,\cdot).$$

By the avoidance principle (Lemma 2.2.8)

$$\overline{u}(x,t) \leq y_n(x,t), \quad \forall (x,t) \in [-n,n]\times[0,\infty), \quad \forall n \in \mathbb{N}.$$

The result follows from the convergence of y_n to y. □

For any $t > 0$ and $\epsilon \in (0,1)$, setting $x_0 := t + \zeta(t) + \frac{1}{\log(1+\epsilon)}$, we have

$$1 - \epsilon \leq \overline{u}(x,t) \leq y(x,t), \quad \forall x > x_0.$$

This concludes the proof of Theorem 4.3.2.

4.4 Rotationally symmetric Hadamard surfaces

For the final section, we consider metrics of the form $g = dr^2 + e^{2\phi(r)}d\theta^2$ as in (1.3.1) which are complete smooth $O(2)$-invariant metrics on the plane with non-positive curvature. As any such g is complete, smooth and $O(2)$-invariant, we have the previous analytic definition of g blooming at infinity (see Definition 1.3.16). Moreover, under the additional assumption that the curvature is non-positive, we can show that an equivalent geometric formulation for g blooming at infinity is that all closed solutions to CSF become extinct within a finite uniform time. We shall first make this statement precise, before using it to prove Theorem 1.3.17.

Given a region $U \subseteq M$ within our surface, we want to quantify the maximal existence time for closed solutions to CSF which initially lie within U.

Definition 4.4.1. For any subset $U \subseteq M$, let $\mathcal{C}(U)$ denote the class of smooth closed solutions to equation (1.2.1)

$$\eta : S^1 \times [0,T) \to M,$$

such that $\eta(\cdot,0) \subseteq U$. For such a solution η, we say that its existence time is T. Define the existence time of the subset U to be the supremum of all such existence times

$$\tau(U) = \sup\{T \in (0,\infty) : T \text{ is the existence time for some } \eta \in \mathcal{C}(U)\}.$$

The following lemma provides an upperbound for the existence time of a closed solution to CSF inside a simply connected negatively curved space. It is a simple application of Gauss-Bonnet, Theorem C.0.1.

Lemma 4.4.2. *Let $(\mathbb{R}^2, g)$ be a Hadamard surface (non-positive curvature) and $\eta_i : S^1 \times [0,T_i) \to \mathbb{R}^2$ be a family of closed disjoint solutions to CSF for $i \in \{1,\dots,k\}$. Suppose $\eta : S^1 \times [0,T) \to \mathbb{R}^2$ is a maximal closed solution to CSF such that the region enclosed by $\mathrm{Im}\,(\eta(\cdot,0))$ contains all of the other curves $\bigcup_{i=1}^k \mathrm{Im}\,(\eta_i(\cdot,0))$. Then*

$$T \leq \frac{\alpha}{2\pi} + \sum_{i=1}^k T_i,$$

where $\alpha \geq 0$ is the initial area discrepancy.

Proof. Let $\Gamma(t)$, $\Gamma_i(t)$ denote the regions enclosed by the curves $\mathrm{Im}\,(\eta(\cdot,t))$ and $\mathrm{Im}\,(\eta_i(\cdot,t))$ respectively. Without loss of generality $0 =: T_{k+1} < T_k \leq \cdots \leq T_1$. Then by the avoidance principle for closed curves, for each $m \in \{1,\dots,k\}$ we have

$$\bigcup_{i=1}^m \Gamma_i(t) \subseteq \Gamma(t), \quad \forall t \in (T_{m+1}, T_m).$$

Let $A(t)$, $A_i(t)$ denote the areas of $\Gamma(t)$, $\Gamma_i(t)$ respectively, so that the initial area discrepancy $\alpha := A(0) - \sum\limits_{i=1}^k A_i(0)$. For $t \in (T_{m+1}, T_m)$ we apply Gauss-Bonnet to give

$$\frac{\partial A}{\partial t} = -2\pi + \int_{\Gamma(t)} K dA \leq -2\pi + \sum_{i=1}^m \int_{\Gamma_i(t)} K dA = \sum_{i=1}^m \frac{\partial A_i}{\partial t} + 2\pi(m-1).$$

Integrating, we have for each $m \in \{1,\dots,k\}$

$$A(T_m) - A(T_{m+1}) \leq \sum_{i=1}^m (A_i(T_m) - A_i(T_{m+1})) + 2\pi(m-1)(T_m - T_{m+1}). \tag{4.4.1}$$

Summing (4.4.1) over $m \in \{1, \dots, k\}$

$$A(T_1) \leq A(0) - \sum_{i=1}^{k} A_i(0) + 2\pi \sum_{i=2}^{k} T_i = \alpha + 2\pi \sum_{i=2}^{k} T_i.$$

Applying Gauss-Bonnet once more

$$T \leq T_1 + \frac{A(T_1)}{2\pi} = \frac{\alpha}{2\pi} + \sum_{i=1}^{k} T_i. \qquad \square$$

The following lemma shows equivalent formulations for such a metric to allow blooming at infinity.

Lemma 4.4.3. *Let $g = dr^2 + e^{2\phi(r)} d\theta^2$ as in (1.3.1) be a complete smooth $O(2)$-invariant metric on the plane with non-positive curvature. For each $m \in \mathbb{N}$, let $S_m := \{(r, \theta) : r \geq 0,\ \theta \in [0, \frac{2\pi}{m}]\} \subset \mathbb{R}^2$. The following conditions are equivalent.*

1. *g allows blooming at infinity. That is, there exists $T \in (0, \infty)$ and a solution $R : (0, T) \to (0, \infty)$ to the ODE (1.3.2) such that $R(t) \to \infty$ as $t \searrow 0$.*

2. *For any $t > 0$, there exists $R(t) \in (0, \infty)$ such that, for any $m \in \mathbb{N}$ and $\eta \in \mathcal{C}(S_m)$, we have that $\eta(\cdot, t) \subseteq \overline{B_{R(t)}}$.*

3. *For any $m \in \mathbb{N}$, the existence time $\tau(S_m) < \infty$ (see Definition 4.4.1).*

Proof. We shall show $1 \implies 2 \implies 3 \implies 1$.

($1 \implies 2$) Let $\eta \in \mathcal{C}(S_m)$. By compactness, for any $\epsilon > 0$, there exists some $\delta \in (0, \epsilon)$ such that $\eta(\cdot, \epsilon) \subseteq B_{R(\delta)}$. By the avoidance principle for closed curves, $\eta(\cdot, t) \subseteq B_{R(t-\epsilon+\delta)}$. Letting $\epsilon \searrow 0$ yields the result.

($2 \implies 3$) Fix $\eta \in \mathcal{C}(S_m)$. Either the existence time of η is less than 1, or by our assumption, there exists some $R > 0$ independent of η such that $\eta(\cdot, 1) \subseteq \overline{B_R}$. Using that the curvature is non-positive and Gauss-Bonnet, we have that $T \leq 1 + \frac{|B_R|}{2\pi m}$ and hence $\tau(S_m) \leq 1 + \frac{|B_R|}{2\pi m} < \infty$.

($3 \implies 1$) Fix $r > 0$ and consider the region $B_r \cap S_m$. We can flow the boundary of this region under CSF to get a solution $\eta : S^1 \times [0, T) \to S_m$ with existence time $T \leq \tau(S_m)$. Also consider the maximal solution $r(t) : (0, T_0) \to (0, \infty)$ to the ODE (1.3.2), starting from $r(0) = r$. We note that under the usual $O(2)$-action on the plane, the rotated curves $(\frac{2\pi j}{m} \cdot \eta)$ for $j \in \{0, 1, \dots, m-1\}$ are disjoint, and completely fill the region B_r. Therefore, by Lemma 4.4.2

$$T_0 \leq m \cdot T \leq m \cdot \tau(S_m).$$

In particular, taking $r \nearrow \infty$ gives $\tau(\mathbb{R}^2) \leq m \cdot \tau(S_m) < \infty$. Now consider the sequence of maximal solutions $R_n : [-T_n, 0) \to (0, \infty)$ to the ODE (1.3.2) with $R_n(-T_n) = n$, for all $n \in \mathbb{N}$. Note that the T_n are strictly increasing and bounded above by $\tau(\mathbb{R}^2)$, so they converge to some finite limit T. Taking the limit of the sequence R_n in n and reparametrising gives a solution $R : (0, T) \to (0, \infty)$ to the ODE (1.3.2), with $R(t) \to \infty$ as $t \searrow 0$. □

Proof of Theorem 1.3.17

Theorem 1.3.17. *Consider a complete smooth $O(2)$-invariant metric $\overline{g}$ with non-positive curvature on the plane. Let $\gamma : \mathbb{R} \times [0, T] \to \mathbb{R}^2$ be a uniformly proper solution to CSF starting from the x-axis. If $\overline{g}$ does not allow blooming at infinity then γ is the static solution to CSF.*

Proof of Theorem 1.3.17. Let $\gamma : \mathbb{R} \times [0, T] \to \mathbb{R}^2$ be any uniformly proper solution to CSF starting from the x-axis. Fix $m \in \mathbb{N}$ and $r > 0$. Using Lemma 4.4.3 there exists $\eta_r \in \mathcal{C}(S_m)$ with existence time greater than T and a point $x \in S^1$ such that $\eta_r(x, T)$ lies outside the ball B_r centred at the origin radius r. Consider now the rotated slice $(\pi - \frac{2\pi}{m}) \cdot S_m = \{(r, \theta) : r \geq 0,\ \theta \in [\pi - \frac{2\pi}{m}, \pi]\}$, and the convex region $\Omega_m := S_m \cup (\pi - \frac{2\pi}{m}) \cdot S_m \cup B_{\frac{1}{m}}$. We currently have a smooth closed curve $\eta_r(\cdot, 0) \subseteq S_m$. We choose $\widehat{\eta}_r \in \mathcal{C}(\Omega_m)$ such that $\widehat{\eta}_r(\cdot, 0)$ is a smooth closed curve in Ω_m enclosing both $\eta_r(\cdot, 0)$ and its rotated image $(\pi - \frac{2\pi}{m}) \cdot \eta_r(\cdot, 0) \subseteq (\pi - \frac{2\pi}{m}) \cdot S_m$. By the avoidance principle for closed curves, the existence time $\widehat{T}$ of $\widehat{\eta}_r$ is greater than T and there exists points $x, y \in S^1$ such that both $\widehat{\eta}_r(x, T)$ and $\widehat{\eta}_r(y, T)$ lie outside of B_r, but with $\widehat{\eta}_r(x, T) \in S_m$ and $\widehat{\eta}_r(y, T) \in (\pi - \frac{2\pi}{m}) \cdot S_m$.

For each $r > 0$, we now apply the avoidance principle (Theorem 4.2.4) to the closed curve $\widehat{\eta}_r$ and the uniformly proper solution γ, as well as to the rotated closed curve $\pi \cdot \widehat{\eta}_r \in \mathcal{C}(\pi \cdot \Omega_m)$ and γ to deduce that

$$\operatorname{Im} \gamma(\cdot, t) \subseteq \Omega_m \cup \pi \cdot \Omega_m, \quad \forall m \in \mathbb{N},\ \forall t \in [0, T].$$

Taking $m \to \infty$ gives

$$\operatorname{Im} \gamma(\cdot, t) \subseteq \bigcap_{m \in \mathbb{N}} (\Omega_m \cup \pi \cdot \Omega_m) = \{(r, \theta) : r \geq 0, \theta \in \{0, \pi\}\}, \quad \forall t \in [0, T].$$

We have shown that $\operatorname{Im} \gamma(\cdot, t)$ is the x-axis for each $t \in [0, T]$. □

In light of the previous theorems, we tentatively make the following uniqueness conjecture, claiming that within our special class of metrics, the only obstruction to uniqueness under curve shortening flow starting from any initial data is precisely blooming at infinity.

Conjecture 4.4.4. *Let $(\mathbb{R}^2, \overline{g})$ be the plane equipped with a complete smooth $O(2)$-invariant metric with non-positive curvature. Then CSF is unique on $(\mathbb{R}^2, g)$ (see Definition 1.3.12) iff $\overline{g}$ does not allow blooming at infinity (see Definition 1.3.16).*

Chapter 5

Complete (2+1)-dimensional Ricci flow spacetimes

For the final chapter of this thesis, we return our attention to Ricci flow. The following is a brief outline of its content.

- In §5.1, we extend a lower scalar curvature bound originally due to Bing-long Chen in [Che09] to Ricci flow spacetimes, which allows us, via a Harnack estimate, to show that Ricci flow spacetimes with connected spatial slices are expanding. We then decompose any Ricci flow spacetime into the union of such spacetimes to show Theorem 1.3.1.
- In §5.2, we show that expanding Ricci flow spacetimes can be embedded within a larger ambient space. With the use of the ambient space, we can now define a global conformal structure on our spacetime, allowing us to significantly simplify Theorem 1.3.6.
- In §5.3, we further reduce Theorem 1.3.6 to complete and conformal Ricci flows on spacetimes within the unit disk. We then formulate a geometric condition equivalent to continuity, which we use to then prove a comparison principle for Ricci flows on such a spacetime.
- Taking any non-atomic Radon measure on a Riemann surface, a recent result of Peter Topping & Hao Yin [TY21] allows us to start the Ricci flow from this measure. In §5.4, we show that the converse is true, and that any complete conformal Ricci flow on a surface must start weakly from such a measure.
- In §5.5, we introduce the idea of an initial time blow-up. Taking any Ricci flow starting from a Radon measure, by looking at larger and larger parabolic rescalings of such a flow away from the support of the initial measure, in the limit at time zero, we have a hyperbolic metric. Finally, we combine this with the comparison principle from §5.3 to show Theorem 1.3.6.

5.1 Complete Ricci flow spacetimes are expanding

The aim of this section is to prove Theorem 1.3.1. The key idea in the proof is Lemma 5.1.7, which states that the vanishing times of points is locally constant within each spatial slice of our spacetime. To show Lemma 5.1.7 we require bounds on our metric along the worldlines. This takes the form of a Harnack estimate.

5.1.1 A simple Harnack estimate for spacetimes

This short subsection introduces a simple Harnack estimate for complete Ricci flow spacetimes. Recall that if $\inf I = 0$ and the hindsight function $\mathfrak{h} \equiv 0$, then we say that the spacetime $\mathcal{M}$ is initially determined.

The following is Chen's lower scalar curvature bound adapted to Ricci flow spacetimes. The proof of Lemma 5.1.1 presented here is a modification of the original proof given by Chen, where now the basepoint of the balls is allowed to vary smoothly in time.

Lemma 5.1.1 (*Variation of* Chen, [Che09, Proposition 2.1])**.** *$\forall \delta \in (0, \frac{2}{n})$, $\exists\ C(\delta) > 0$ with the following property. Let $(\mathcal{M}^{n+1}, \mathfrak{t}, \partial_t, g)$ be a Ricci flow spacetime. Fix $[t_1, t_2] \subseteq I$ and $\eta : [t_1, t_2] \to \mathcal{M}$ a time-preserving path. Let $\pi : T\mathcal{M} \to T\mathcal{M}^{spat}$ denote the spatial projection, and suppose there are constants $r_0, K > 0$ and $A \geq 2 + 24(n-1)r_0^{-2}t_2$ such that*

- *$B_{g(t)}(\eta(t), Ar_0) \Subset \mathcal{M}_t$, for every $t \in [t_1, t_2]$;*
- *$\mathrm{Ric}(g(t)) \leq (n-1)r_0^{-2}$ on $B_{g(t)}(\eta(t), r_0)$, for every $t \in [t_1, t_2]$;*
- *$R_{g(t_1)} \geq -K$ on $B_{g(t_1)}(\eta(t_1), Ar_0)$;*
- *$|\pi \circ \eta'(t)|_{g(t)} \leq r_0^{-1}$, for every $t \in [t_1, t_2]$.*

Then for each $t \in [t_1, t_2]$ and $x \in B_{g(t)}(\eta(t), \frac{3Ar_0}{4})$, we have

$$R_{g(t)}(x) \geq \min\{-\frac{1}{(\frac{2}{n} - \delta)(t - t_1) + \frac{1}{K}}, -\frac{C}{A^2 r_0^2}\}.$$

Proof. Fix a smooth decreasing function $\phi : \mathbb{R} \to \mathbb{R}$ with $\phi \equiv 1$ on $(-\infty, \frac{7}{8}]$ and $\phi \equiv 0$ on $[1, \infty)$. For points $x, y \in \mathcal{M}$ in the same time slice $s = \mathfrak{t}(x) = \mathfrak{t}(y)$, let $d_g(x, y) := d_{g(s)}(x, y)$ denote the distance between x and y in $(\mathcal{M}_s, g(s))$. Define $u : \mathcal{M}_{[t_1, t_2]} \to \mathbb{R}$ by

$$u(x) := \phi\left(\frac{d_g(x, \eta \circ \mathfrak{t}(x)) + 3(n-1)r_0^{-1}\mathfrak{t}(x)}{Ar_0}\right) \cdot R_{g(\mathfrak{t}(x))}(x)$$

In the region where d_g is smooth, a direct calculation yields

$$\Box u = \frac{R\phi'}{Ar_0} \cdot \left(\Box d_g + \pi \circ \eta' \cdot \nabla d_g + 3(n-1)r_0^{-1}\right) - 2\nabla\phi \cdot \nabla R - \frac{R\phi''}{A^2 r_0^2} + 2\phi|\mathrm{Ric}|^2, \qquad (5.1.1)$$

where $\square := \partial_t - \Delta_g$ denotes the heat operator on $\mathcal{M}$, and ∇d_g denotes the gradient of the function $\mathcal{M}_{\mathfrak{t}(x)} \to \mathbb{R}$, $y \mapsto d_g(x,y)$, evaluated at the point $y = \eta \circ \mathfrak{t}(x)$.

Fix $x \in \mathcal{M}_{(t_1,t_2)}$. If the distance $d_g(x, \eta \circ \mathfrak{t}(x)) \leq r_0$, then $\frac{d_g(x,\eta\circ\mathfrak{t}(x))+3(n-1)r_0^{-1}\mathfrak{t}(x)}{Ar_0} \leq \frac{5}{8}$, and hence $\phi'\left(\frac{d_g(x,\eta\circ\mathfrak{t}(x))+3(n-1)r_0^{-1}\mathfrak{t}(x)}{Ar_0}\right) \cdot R_{g(\mathfrak{t}(x))}(x) = 0$. Otherwise, the distance $d_g(x, \eta \circ \mathfrak{t}(x)) > r_0$, and we have the lower bound (in the barrier sense) from [Per02, Lemma 8.3]

$$\square d_g(x, \eta \circ \mathfrak{t}(x)) + \pi \circ \eta'(\mathfrak{t}(x)) \cdot \nabla d_g(x, \eta \circ \mathfrak{t}(x)) \geq -\frac{8(n-1)}{3r_0}.$$

In particular, on all of $\mathcal{M}_{(t_1,t_2)}$ we have the lower bound (in the barrier sense)

$$\square d_g + \pi \circ \eta' \cdot \nabla d_g + 3(n-1)r_0^{-1} \geq 0. \tag{5.1.2}$$

For each $s \in [t_1, t_2]$ define

$$u_0(s) := \inf_{x \in \mathcal{M}_s} u(x).$$

Assume for some fixed $t_0 \in (t_1, t_2)$ that $u_0(t_0) < 0$. Then, since u is continuous on the compact set $\overline{B_{g(t_0)}(\eta(t_0), Ar_0)}$ and vanishes outside of it, the minimum is attained at some point $x_0 \in \mathcal{M}_{t_0}$. By continuity, for any point y in a sufficiently small neighbourhood of x_0, $R_{g(\mathfrak{t}(y))}(y) < 0$, and so $\phi'\left(\frac{d_g(y,\eta\circ\mathfrak{t}(y))+3(n-1)r_0^{-1}\mathfrak{t}(y)}{Ar_0}\right) \cdot R_{g(\mathfrak{t}(y))}(y) \geq 0$. Combining this with (5.1.2), we have

$$R\phi' \cdot \left(\square d_g + \pi \circ \eta' \cdot \nabla d_g + 3(n-1)r_0^{-1}\right) \geq 0,$$

in a neighbourhood x_0. Therefore, in a spacetime neighbourhood of x_0, equation (5.1.1) implies

$$\square u \geq -2\nabla\phi \cdot \nabla R - \frac{R\phi''}{A^2 r_0^2} + \frac{2}{n}\phi R^2, \tag{5.1.3}$$

in the barrier sense. Moreover, using that $\phi > 0$ near x_0, within a possibly smaller neighbourhood of x_0, at smooth points of d_g, we have

$$\nabla R = \frac{\nabla u}{\phi} - \frac{R}{\phi}\nabla\phi, \quad |\nabla\phi|^2 = \frac{(\phi')^2}{(Ar_0)^2}.$$

Therefore, on this small neighbourhood of x_0, the differential inequality

$$\square u \geq -\frac{2}{\phi}\nabla\phi \cdot \nabla u + \frac{1}{(Ar_0)^2}\left(\frac{2(\phi')^2}{\phi} - \phi''\right)R + \frac{2}{n}\phi R^2, \tag{5.1.4}$$

holds in the barrier sense. With our choice of ϕ, we can ensure that we have the bound $|\frac{2(\phi')^2}{\phi} - \phi''| \leq C'\phi^{\frac{1}{2}}$, for some $C' > 0$ depending only on ϕ. Using Peter-Paul, we also have the inequality

$$\frac{C' R\phi^{\frac{1}{2}}}{(Ar_0)^2} \leq \frac{2}{\delta}\left(\frac{C'}{(Ar_0)^2}\right)^2 + \frac{\delta}{2}\phi R^2 = \frac{\delta}{2}\left(\left(\frac{C}{(Ar_0)^2}\right)^2 + \phi R^2\right),$$

where C now depends on δ. Therefore, we can simplify equation (5.1.4) to the differential inequality

$$\Box u \geq -\frac{2}{\phi}\nabla\phi\cdot\nabla u + \left(\frac{2}{n}-\frac{\delta}{2}\right)\phi R^2 - \frac{\delta}{2}\left(\frac{C}{(Ar_0)^2}\right)^2. \tag{5.1.5}$$

Applying the maximum principle, we conclude that

$$\begin{aligned}\liminf_{h\searrow 0}\frac{u_0(s+h)-u_0(s)}{h} &\geq \left(\frac{2}{n}-\frac{\delta}{2}\right)\phi R^2 - \frac{\delta}{2}\left(\frac{C}{(Ar_0)^2}\right)^2 \\ &\geq \left(\frac{2}{n}-\delta\right)u_0^2(s) + \frac{\delta}{2}\left(u_0(s)^2 - \left(\frac{C}{(Ar_0)^2}\right)^2\right).\end{aligned}$$

Integrating this up, we have the desired inequality. □

As a corollary to Lemma 5.1.1, we also have a lower scalar curvature bound for complete Ricci flow spacetimes (see [Che09, Corollary 2.3] for the comparable result on cylindrical spacetimes).

Corollary 5.1.2. *Let $(\mathcal{M}^{n+1}, \mathfrak{t}, \partial_t, g)$ be a complete Ricci flow spacetime. Then for any $x \in \mathcal{M}$ we have*

$$R_{g(\mathfrak{t}(x))}(x) \geq \frac{-n}{2(\mathfrak{t}(x)-\mathfrak{h}(x))}.$$

If $\mathcal{M}$ is initially determined, this simplifies to

$$R_{g(\mathfrak{t}(x))}(x) \geq \frac{-n}{2\mathfrak{t}(x)}.$$

Proof. Fix $x \in \mathcal{M}$ and choose $s \in (\mathfrak{h}(x), \mathfrak{t}(x))$. Then by the definition of $\mathfrak{h}$ we can find a time-preserving path $\eta : [s, \mathfrak{t}(x)] \to \mathcal{M}$ such that $\eta \circ \mathfrak{t}(x) = x$. For r_0 sufficiently small, $\mathrm{Ric}(g(t)) \leq (n-1)r_0^{-2}$ on $B_{g(t)}(\eta(t), r_0)$ for every $t \in [s, \mathfrak{t}(x)]$, and $|\pi\circ\eta'(t)|_{g(t)} \leq r_0^{-1}$ for every $t \in [s, \mathfrak{t}(x)]$. For any $A > 0$, and any $t \in [s, \mathfrak{t}(x)]$, $B_{g(t)}(\eta(t), Ar_0) \Subset \mathcal{M}_t$ follows from the completeness of the metric $g(t)$. By compactness, there exists some lower bound $R_{g(s)} \geq -K$ on $B_{g(s)}(\eta(s), Ar_0)$. So, for each $\delta \in (0, \frac{2}{n})$, choosing A sufficiently large, Lemma 5.1.1 implies the existence of $C > 0$ such that

$$R_{g(\mathfrak{t}(x))}(x) \geq \min\{-\frac{1}{(\frac{2}{n}-\delta)(\mathfrak{t}(x)-s)+\frac{1}{K}}, -\frac{C}{Ar_0^2}\} \geq \min\{-\frac{1}{(\frac{2}{n}-\delta)(\mathfrak{t}(x)-s)}, -\frac{C}{Ar_0^2}\}. \tag{5.1.6}$$

Taking A sufficiently large, we conclude that $R_{g(\mathfrak{t}(x))}(x) \geq -\frac{1}{(\frac{2}{n}-\delta)(\mathfrak{t}(x)-s)}$. The corollary is finished by taking $\delta \searrow 0$ and $s \searrow \mathfrak{h}(x)$. □

Recall that in dimension $n = 2$, Ricci flow preserves the conformal class of the metric. Within a Ricci flow spacetime $(\mathcal{M}^{2+1}, \mathfrak{t}, \partial_t, g)$, consider cylindrical coordinates $\Psi : U(I) \subseteq \mathcal{M} \to U \times I$, for some open set $U \subseteq \mathcal{M}_s$, $s \in I$.

Shrinking U if necessary, we can assume that U, equipped with the metric $\Psi_*(g(s))$, admits isothermal coordinates (x, y), so that $\Psi_*(g(s))$ is conformally equivalent to $(dx^2 + dy^2)$ on

$U \times \{s\}$. Since $\Psi_*(g)$ solves the usual Ricci flow equation on $\Psi(U(I)) \subseteq U \times I$, we see that

$$\Psi_*(g) = u(dx^2 + dy^2) \quad \text{on } \Psi(U(I)), \tag{5.1.7}$$

where $u : \Psi(U(I)) \to (0, \infty)$ is a smooth function satisfying the logarithmic fast diffusion equation (LFDE)

$$\frac{\partial u}{\partial t} = \Delta \log u. \tag{5.1.8}$$

Working locally, we can combine the lower scalar curvature bound with the LFDE to give the following simple Harnack estimate.

Definition 5.1.3. Given a smooth manifold M, a subset $\Gamma \subseteq M$ is a *smooth arc* in M if it has a smooth and regular parameterisation $\gamma : J \to M$, for some interval $J \subseteq \mathbb{R}$.

$$\Gamma = \{\gamma(s) : s \in J\}.$$

If we can take $J = [0, 1]$, we say that Γ is a *compact smooth arc.*

Lemma 5.1.4. *Let $(\mathcal{M}^{2+1}, \mathfrak{t}, \partial_t, g)$ be a complete Ricci flow spacetime with $I = (0, T)$. For any $x_0 \in \mathcal{M}_{t_0}$, choose $r > 0$ sufficiently small such that the ball $B := B(x_0, r) \Subset \mathcal{M}_{t_0}$ admits isothermal coordinates. Let Γ be a smooth arc in B. Then*

$$\ell_g(\Gamma(t)) \leq \sqrt{\frac{t - \mathfrak{h}(x_0)}{t_0 - \mathfrak{h}(x_0)}} \cdot \ell_g(\Gamma), \quad \forall t \in [t_0, T),$$

where $\ell_g(\Gamma(t))$ denotes the length of $\Gamma(t) \subseteq \mathcal{M}_t$ with respect to the metric $g(t)$. If $\mathcal{M}$ is initially determined, this simplifies to

$$\ell_g(\Gamma(t)) \leq \sqrt{\frac{t}{t_0}} \cdot \ell_g(\Gamma), \quad \forall t \in [t_0, T).$$

Proof. As we did above, there exist cylindrical coordinates $\Psi : B(I) \subseteq \mathcal{M} \to B \times I$, so that $\Psi_*(g)$ satisfies (5.1.7) for some $u : \Psi(B(I)) \to (0, \infty)$ solving the LFDE (5.1.8). Since

$$\Delta \log u = -u \cdot R_{\Psi_* g} = -u \cdot \Psi_*(R_g),$$

we can use Corollary 5.1.2 to bound the time derivative of our conformal factor

$$\frac{\partial u}{\partial t}(z,t) = -u(z,t) \cdot R_g(\Psi^{-1}(z,t)) \leq \frac{u(z,t)}{t - \mathfrak{h} \circ \Psi^{-1}(z,t)} = \frac{u(z,t)}{t - \mathfrak{h}(x_0)}, \quad \forall (z,t) \in \Psi(B(I)).$$

Note that $(z,t) \in \Psi(B(I))$ iff $[t_0, t] \subseteq I_z$. So we can integrate the above to get

$$u(z,t) \leq \left(\frac{t - \mathfrak{h}(x_0)}{t_0 - \mathfrak{h}(x_0)}\right) u(z, t_0), \quad \forall (z,t) \in \Psi(B(I)).$$

Let Γ be the image of the smooth map $\gamma : J \to B$. Recall, the set $\Gamma(t)$ denotes the collection of points $\{x(t) \in \mathcal{M}_t : x \in \Gamma, t \in I_x\}$ (see Definition 1.2.6). We shall consider those points in Γ which persist until time t,

$$[\Gamma(t)](s) = \{x \in \Gamma : t \in I_x\} \subseteq \Gamma,$$

and let

$$J_t := \gamma^{-1}([\Gamma(t)](s)) \subseteq J.$$

As $[\Gamma(t)](s)$ is open in Γ, J_t is open in J. Since $\Psi(\Gamma(t)) = [\Gamma(t)](s) \times \{t\}$, we have

$$\ell_g(\Gamma(t)) = \ell_{\Psi_*(g)}([\Gamma(t)](s) \times \{t\}) = \int_{J_t} u(\gamma(s), t)^{\frac{1}{2}} |\gamma'(s)| \, ds,$$

where $|\cdot|$ is the size of a vector with respect to our local isothermal coordinates. In particular

$$\begin{aligned}
\ell_g(\Gamma(t)) &= \int_{J_t} u(\gamma(s), t)^{\frac{1}{2}} |\gamma'(s)| \, ds \\
&\leq \sqrt{\frac{t - \mathfrak{h}(x_0)}{t_0 - \mathfrak{h}(x_0)}} \cdot \int_{J_t} u(\gamma(s), t_0)^{\frac{1}{2}} |\gamma'(s)| \, ds \\
&\leq \sqrt{\frac{t - \mathfrak{h}(x_0)}{t_0 - \mathfrak{h}(x_0)}} \cdot \int_{J} u(\gamma(s), t_0)^{\frac{1}{2}} |\gamma'(s)| \, ds \\
&= \sqrt{\frac{t - \mathfrak{h}(x_0)}{t_0 - \mathfrak{h}(x_0)}} \cdot \ell_g(\Gamma).
\end{aligned}$$

□

Remark 5.1.5. In higher dimensions, the same reasoning applied to the evolution equation for the volume form

$$\mathcal{L}_{\partial_t} dV_g = -R_g dV_g,$$

gives an analogous inequality, and hence a local upper bound on volume growth.

By piecing together the above lemma locally, we have the same result for any smooth arc within a spatial slice.

Lemma 5.1.6. *Let $(\mathcal{M}^{2+1}, \mathfrak{t}, \partial_t, g)$ be a complete Ricci flow spacetime with $I = (0, T)$. Suppose Γ is a smooth arc in $\mathcal{M}_{t_0}$. Then*

$$\ell_g(\Gamma(t)) \leq \sqrt{\frac{t - \mathfrak{h}(x_0)}{t_0 - \mathfrak{h}(x_0)}} \cdot \ell_g(\Gamma), \quad \forall t \in [t_0, T).$$

where $\ell_g(\Gamma(t))$ denotes the length of $\Gamma(t)$ with respect to the metric $g(t)$. If $\mathcal{M}$ is initially determined, this simplifies to

$$\ell_g(\Gamma(t)) \leq \sqrt{\frac{t}{t_0}} \cdot \ell_g(\Gamma), \quad \forall t \in [t_0, T).$$

Proof. We aim to apply the Harnack estimate to small balls covering Γ. Let Γ be the image of the smooth map $\gamma : J \to B$. For each $s \in J$, choose $r_s > 0$ sufficiently small such that the ball $B_s := B(\gamma(s), r_s)$ satisfies the hypothesis of Lemma 5.1.4. Consider the open cover $\gamma^{-1}(B_s)$ of J. As $\mathbb{R}$ is locally compact, write J as a union of compact intervals K_i for $i \in \mathbb{N}$, overlapping only at their endpoints. Applying the Lebesgue number lemma to each K_i, we can find a finite number of compact intervals $J_{i,l}$ such that $K_i = \cup_l J_{i,l}$, with the collection $J_{i,l}$ overlapping only at their endpoints, and with the additional property that $J_{i,l} \subseteq \gamma^{-1}(B_{s_{i,l}})$, for some $s_{i,l} \in J$. Then, applying Lemma 5.1.4 to each of the balls $B_{s_{i,l}}$, we conclude that

$$\begin{aligned}\ell_g(\Gamma(t_1)) &= \sum_{i,l} \ell_g(\Gamma(t_1) \cap [\gamma(J_{i,l})](t_1)) \\ &\leq \sqrt{\frac{t_1 - \mathfrak{h}(x_0)}{t_0 - \mathfrak{h}(x_0)}} \sum_{i,l} \ell_g(\Gamma \cap \gamma(J_{i,j})) \\ &= \sqrt{\frac{t_1 - \mathfrak{h}(x_0)}{t_0 - \mathfrak{h}(x_0)}} \cdot \ell_g(\Gamma).\end{aligned}$$

□

5.1.2 Vanishing times are locally constant

Let $\Gamma \Subset \mathcal{M}_s$ be a compact smooth arc within the spatial slice at time $s \in I$. Recall, in Definition 1.2.6, we defined the interval $I_\Gamma = \cap_{x\in\Gamma} I_x$. Since Γ is compactly contained in $\mathcal{M}_s$, s is in the interior of I_Γ by Lemma 2.3.2. Let $T_\Gamma = \sup I_\Gamma > s$, be the vanishing time of the arc. The following lemma shows that along a compact smooth arc Γ, the vanishing times of all of the points within the arc are the same.

Lemma 5.1.7. *Let $(\mathcal{M}^{2+1}, \mathfrak{t}, \partial_t, g)$ be a complete Ricci flow spacetime with $I = (0, T)$. Fix $s \in I$ and let $\Gamma \Subset \mathcal{M}_s$ be a compact smooth arc. Then*

$$T_x = T_\Gamma, \quad \forall x \in \Gamma.$$

Proof. We can assume that $T_\Gamma < T$, otherwise we would have $T_x = T_\Gamma = T$, for any $x \in \Gamma$. Let Γ be the image of the smooth map $\gamma : [0,1] \to \mathcal{M}_s$. Consider those points in Γ which persist past the vanishing time of Γ, $[\Gamma(T_\Gamma)](s) \subset \Gamma$. As in Lemma 5.1.4, we look at the preimage of this subset under our parameterisation

$$J' := \gamma^{-1}([\Gamma(T_\Gamma)](s)) \subseteq [0,1].$$

We note that J' is open in $[0,1]$. If J' is non-empty, choose $J'' \subseteq J'$ to be a non-empty connected component. We use Lemma 5.1.4 to show that J'' contains its infimum and supremum.

Claim. $\inf J'', \sup J'' \in J''$.

Proof of claim. Let $s := \sup J''$. Choose an increasing sequence $s_j \in J''$ such that $s_j \nearrow s$, and hence $\gamma(s_j) \to \gamma(s)$ in $\mathcal{M}_s$, as $j \to \infty$. Since the metric is smooth,

$$\ell_g(\gamma([s_j, s])) \to 0, \quad \text{as } j \to \infty. \tag{5.1.9}$$

In particular, by the Harnack estimate from Lemma 5.1.4, the sequence $[\gamma(s_j)](T_\Gamma)$ is Cauchy in $\mathcal{M}_{T_\Gamma}$. As $g(T_\Gamma)$ is complete, there exists a limit $z \in \mathcal{M}_{T_\Gamma}$. Choose $r > 0$ such that the parabolic cylinder $C(z, r)$ is unscathed. In particular, since g is smooth, for some $\tau \in (T_\Gamma - r^2, T_\Gamma) \cap (s, T_\Gamma)$, we deduce that $[\gamma(s_j)](\tau) \to z(\tau)$ as $j \to \infty$. Finally, applying the Harnack estimate (5.1.4) again to equation (5.1.9), we see that $[\gamma(s_j)](\tau) \to [\gamma(s)](\tau)$ as $j \to \infty$. So $[\gamma(s)](\tau) = z(\tau)$, or $z = [\gamma(s)](T_\Gamma)$, which implies $s \in J''$. The argument for $\inf J''$ is the same. □

By the above claim, J'' must be the entire interval $[0, 1]$, and $[\Gamma(T_\Gamma)](s) = \Gamma$. This gives a contradiction to the value of T_Γ as $\Gamma(T_\Gamma) \Subset \mathcal{M}_{T_\Gamma}$. Therefore J' is empty, and all points in Γ vanish at time T_Γ. □

Corollary 5.1.8. *Let $(\mathcal{M}^{2+1}, \mathfrak{t}, \partial_t, g)$ be a complete Ricci flow spacetime with $I = (0, T)$. Suppose $U \subseteq \mathcal{M}_s$ is connected. Then $T_x = T_y$, for all $x, y \in U$.*

5.1.3 Spatially-connected spacetimes are expanding

Definition 5.1.9. Let $(\mathcal{M}, \mathfrak{t}, \partial_t, g)$ be a connected Ricci flow spacetime. We say that $\mathcal{M}$ is spatially-connected if the time slices $\mathcal{M}_t$ are connected, for all $t \in I$.

Under the additional assumption that $\mathcal{M}$ is spatially-connected, we have a well-defined vanishing time for each spatial slice. If our spacetime was not expanding, then $\mathcal{M}_s(t) = \emptyset$ for some $s < t \in I$. We can now use this to contradict our assumption that our spacetime is connected.

Theorem 5.1.10. *Let $(\mathcal{M}^{2+1}, \mathfrak{t}, \partial_t, g)$ be a complete, spatially-connected Ricci flow spacetime with $I = (0, T)$. Then it is expanding. That is, the vanishing times*

$$T_x = T, \quad \forall x \in \mathcal{M}.$$

Proof. By Corollary 5.1.8, each spatial slice $\mathcal{M}_s$ has extinction time $T_s \in (s, T]$ for each $s \in (0, T)$. That is, $T_x = T_s$ for all $x \in \mathcal{M}_s$. If $T_s < T$ for some $s \in (0, T)$, then $T_t = T_s$ for all $t \in (s, T_s)$. Choosing any point $x \in \mathcal{M}_{T_s}$, we can pick $\delta > 0$ sufficiently small so that $T_s \pm \delta \in I_x$. This leads to the obvious contradiction

$$T_s + \delta \leq T_x = T_{x(T_s - \delta)} = T_s.$$

□

5.1.4 Decomposing a spacetime

For the final part of this section, we show that any complete Ricci flow spacetime can be decomposed into a collection of complete and spatially-connected Ricci flow spacetimes.

Definition 5.1.11. Let $(\mathcal{M}^{n+1}, \mathfrak{t}, \partial_t, g)$ be a Ricci flow spacetime and let $\eta : J \to \mathcal{M}$ be a time-preserving path. Define $\mathcal{M}^\eta \subseteq \mathcal{M}$ by setting $\mathcal{M}^\eta_t$ to be the connected component of $\mathcal{M}_t$ containing $\eta(t)$, for each $t \in J$.

Lemma 5.1.12. *Let $(\mathcal{M}^{n+1}, \mathfrak{t}, \partial_t, g)$ be a complete Ricci flow spacetime with $I = (0, T)$. Fix $(t_0, t_1) \subseteq (0, T)$. If $\eta : (t_0, t_1) \to \mathcal{M}$ a time-preserving path, then the restriction of $\mathfrak{t}$, ∂_t and g to $\mathcal{M}^\eta$ is a complete and spatially-connected Ricci flow spacetime.*

Proof. In order to show that $(\mathcal{M}^\eta, \mathfrak{t}, \partial_t, g)$ is a Ricci flow spacetime, it suffices to prove that $\mathcal{M}^\eta$ is an open subset of $\mathcal{M}$. Fix $s \in (t_0, t_1)$ and $x \in \mathcal{M}^\eta_s$. Since $g(s)$ is complete, $B := B(x, 1) \Subset \mathcal{M}^\eta_s$. Since $\mathcal{M}^\eta_s$ is path-connected, there exists a continuous path $\rho : [0, 1] \to \mathcal{M}^\eta_s$ from $x = \rho(0)$ to $\eta(s) = \rho(1)$. Using the continuity of η, there exists $\delta > 0$ such that $(s - \delta, s + \delta) \subseteq (t_0, t_1)$, with $t_0 \in I_{\eta(t)}$ for any $t \in (s - \delta, s + \delta)$. As such, let

$$K := [\eta((s - \delta, s + \delta))](s) \cup B \cup \rho([0, 1]) \Subset \mathcal{M}^\eta_s.$$

By compactness, after possibly shrinking δ, we can assume that the parabolic cylinder $K((s - \delta, s + \delta))$ is unscathed. In particular, the unscathed parabolic cylinder $B((s - \delta, s + \delta))$ lies within $\mathcal{M}^\eta$. This shows that $\mathcal{M}^\eta$ is open in $\mathcal{M}$. Finally, we note that $\mathcal{M}^\eta_t$ is a closed subset of the complete space $(\mathcal{M}_t, g(t))$ for each $t \in (t_0, t_1)$, so the Ricci flow spacetime $(\mathcal{M}^\eta, \mathfrak{t}, \partial_t, g)$ is also complete. □

The following lemma shows that, if we have two time-preserving paths starting from the same point, then the corresponding spacetimes we construct from them agree.

Lemma 5.1.13. *Let $(\mathcal{M}^{2+1}, \mathfrak{t}, \partial_t, g)$ be a complete Ricci flow spacetime with $I = (0, T)$. Let $\eta'_i : [t_0, t_1) \to \mathcal{M}$, $i = 1, 2$, be two time-preserving paths such that $\eta'_1(t_0) = \eta'_2(t_0)$. Consider the restrictions on the interior $\eta_i := \eta'_i|_{(t_0, t_1)}$. Then $\mathcal{M}^{\eta_1} = \mathcal{M}^{\eta_2}$.*

Proof. Let $x_0 := \eta'_1(t_0) = \eta'_2(t_0)$, and denote the domain of the worldline of x_0 within the spacetime $\mathcal{M}^{\eta_i}$ by $I^{\{i\}} \subseteq (t_0, t_1)$, for $i = 1, 2$.

Choose $r > 0$ sufficiently small such that the parabolic cylinder $C(x_0, r)$ is unscathed. From this, we can conclude that $\eta_1(s)$ is path-connected to $x_0(s)$ in $\mathcal{M}_s$, for any $s \in (t_0, t_0 + r^2)$. In particular, $x_0(s) \in \mathcal{M}^{\eta_1}_s$ for any $s \in (t_0, t_0 + r^2)$. As $\mathcal{M}^{\eta_1}$ is complete and spatially-connected, by Theorem 5.1.10, it is expanding. Thus, $I^{\{1\}} = (t_0, t_1)$.

Repeating the same argument with $i = 2$, we have $I^{\{2\}} = (t_0, t_1)$ also. So, for any $t \in (t_0, t_1)$, $\mathcal{M}^{\eta_1}_t$ and $\mathcal{M}^{\eta_2}_t$ are connected components of $\mathcal{M}_t$, both containing $x_0(t)$. Hence they agree. □

Although our spacetime $\mathcal{M}$ is path-connected, it is not necessarily path-connected with time-preserving paths. The following argument shows that any two points in $\mathcal{M}$ can be connected by a finite number of concatenated time-preserving paths with alternating orientations.

5.1.5 Topology of spacetimes

Let $(\mathcal{M}, \mathfrak{t}, \partial_t, g)$ be a Ricci flow spacetime. Given any point $x \in \mathcal{M}$, we define the set U_x^- to be the collection of all points at earlier times which can be connected to x by a time-preserving path:

$$U_x^- := \{y \in \mathcal{M} : \mathfrak{t}(y) < \mathfrak{t}(x), \exists \eta : [\mathfrak{t}(y), \mathfrak{t}(x)] \to \mathcal{M},\ \eta \circ \mathfrak{t}(y) = y, \eta \circ \mathfrak{t}(x) = x, \text{time-preserving}\}.$$

Similarily, we define U_x^+ to be the points at later times connected to x by time preserving paths:

$$U_x^+ := \{y \in \mathcal{M} : \mathfrak{t}(x) < \mathfrak{t}(y), \exists \eta : [\mathfrak{t}(x), \mathfrak{t}(y)] \to \mathcal{M}, \eta \circ \mathfrak{t}(x) = x, \eta \circ \mathfrak{t}(y) = y, \text{time-preserving}\}.$$

Since every point in our spacetime admits a small unscathed parabolic cylinder around it, we deduce that the sets $U_x^\pm$ are both non-empty and open in $\mathcal{M}$.

Fixing $x \in \mathcal{M}$, we can now union open sets of the form defined above in an iterative way to see which points in $\mathcal{M}$ can be joined to x by a string of time-preserving paths. Define $U_1^+ := U_x^+$ and $U_1^- := \bigcup_{y \in U_1^+} U_y^-$. For each $m \in \mathbb{N}$, we recursively set

$$U_m^+ := \bigcup_{y \in U_{m-1}^-} U_y^+, \quad U_m^- := \bigcup_{y \in U_m^+} U_y^-.$$

Finally, let $U^- := \cup_{m \in \mathbb{N}} U_m^- \subseteq \mathcal{M}$. The following lemma shows that spacetimes are path-connected by strings of time-preserving paths.

Lemma 5.1.14. *Let $(\mathcal{M}, \mathfrak{t}, \partial_t, g)$ be a Ricci flow spacetime. Fix $x \in \mathcal{M}$ and let U^- be the open subset of $\mathcal{M}$ defined above. Then $\mathcal{M} = U^-$.*

Proof. Since U^- is open and non-empty, it suffices to show that U^- is closed in $\mathcal{M}$. As such, consider a sequence $x_n \in U^-$ such that $x_n \to x_\infty \in \mathcal{M}$. For $r > 0$ sufficiently small, the parabolic cylinder $C(x_\infty, r)$ is unscathed in $\mathcal{M}$. For n sufficiently large, x_n lies inside this parabolic ball. Then, for this large value of n we have that either

$$\begin{aligned}
\mathfrak{t}(x_n) > \mathfrak{t}(x_\infty) &\implies x_\infty \in U_{x_n}^-, \\
\mathfrak{t}(x_n) \leq \mathfrak{t}(x_\infty) &\implies \exists\ y \in U_{x_n}^+ \text{ with } x_\infty \in U_y^-.
\end{aligned}$$

So, as $x_n \in U^-$, there is some $m \in \mathbb{N}$ such that $x_n \in U_m^-$, and hence by the above, $x_\infty \in U_{m+1}^- \subseteq U^-$ as required. □

Corollary 5.1.15. *Let $(\mathcal{M}^{n+1}, \mathfrak{t}, \partial_t, g)$ be a Ricci flow spacetime. Suppose $x, x' \in \mathcal{M}$. Then there exists $m \in \mathbb{N}$, a collection of points $x_0, \ldots, x_m, y_1, \ldots, y_m \in \mathcal{M}$, and a collection of time-dependent paths $\eta_i : [\mathfrak{t}(x_{i-1}), \mathfrak{t}(y_i)] \to \mathcal{M}$, $\gamma_i : [\mathfrak{t}(x_i), \mathfrak{t}(y_i)] \to \mathcal{M}$ for $i = 1, \ldots, m$, such that*

- *$x_0 = x$, and $x_m = x'$.*

- $\eta_i \circ \mathfrak{t}(x_{i-1}) = x_{i-1}$ *and* $\eta_i \circ \mathfrak{t}(y_i) = y_i$*, for each* $i \in \{1, \ldots, m\}$.
- $\gamma_i \circ \mathfrak{t}(x_i) = x_i$ *and* $\gamma_i \circ \mathfrak{t}(y_i) = y_i$*, for each* $i \in \{1, \ldots, m\}$.

In light of Lemma 5.1.13 and Theorem 5.1.10, we can immediately improve this corollary for complete spacetimes.

Lemma 5.1.16. *Let* $(\mathcal{M}^{2+1}, \mathfrak{t}, \partial_t, g)$ *be a complete Ricci flow spacetime with* $I = (0, T)$*. Suppose* $x, x' \in \mathcal{M}$*. Then there exists* $y \in \mathcal{M}$ *and time-dependent paths* $\eta : [\mathfrak{t}(x), \mathfrak{t}(y)] \to \mathcal{M}$*,* $\gamma : [\mathfrak{t}(x'), \mathfrak{t}(y)] \to \mathcal{M}$ *such that* $\eta(\mathfrak{t}(x)) = x$*,* $\gamma(\mathfrak{t}(x')) = x'$*, and* $\eta(\mathfrak{t}(y)) = \gamma(\mathfrak{t}(y)) = y$*. That is, in the language of Corollary 5.1.15, we can always choose* $m = 1$*.*

Proof. Suppose $m \in \mathbb{N}$ is chosen to be the smallest possible value so that the paths from Corollary 5.1.15 exist, and assume $m > 1$. Without loss of generality, we can assume $\mathfrak{t}(y_1) \leq \mathfrak{t}(y_2)$. Choose $r > 0$ sufficiently small so that the parabolic cylinders $C(y_1, r)$ and $C(\eta_2 \circ \mathfrak{t}(y_1), r)$ are unscathed. Since $\gamma_1 : [\mathfrak{t}(x_1), \mathfrak{t}(y_1)] \to \mathcal{M}$ and $\eta_2 : [\mathfrak{t}(x_1), \mathfrak{t}(y_2)] \to \mathcal{M}$ are such that $\gamma_1 \circ \mathfrak{t}(x_1) = \eta_2 \circ \mathfrak{t}(x_1) = x_1$, by Lemma 5.1.13, after restricting these paths to $(\mathfrak{t}(x_1), \mathfrak{t}(y_1))$, we have $\mathcal{M}^{\gamma_1} = \mathcal{M}^{\eta_2}$. In particular, we can find a time-preserving path from a point in $C(y_1, r)$ to a point in $C(\eta_2 \circ \mathfrak{t}(y_1), r)$. We then modify the end of the path $\eta_1 : [\mathfrak{t}(x_0), \mathfrak{t}(y_1)] \to \mathcal{M}$ and the start of the path $\eta_2 : [\mathfrak{t}(y_1), \mathfrak{t}(y_2)] \to \mathcal{M}$ to connect up with our time-preserving path between the parabolic cylinders, to give a new time-preserving path $\eta : [\mathfrak{t}(x_0), \mathfrak{t}(y_2)] \to \mathcal{M}$ such that $\eta(\mathfrak{t}(x_0)) = x_0$ and $\eta(\mathfrak{t}(y_2) = y_2$. This shows that we can reduce the value of m by at least one, which is a contradiction. Therefore $m = 1$. □

Equipped with the above lemma, we can complete the proof that connected spacetimes are expanding.

Theorem 1.3.1. *Let* $(\mathcal{M}^{2+1}, \mathfrak{t}, \partial_t, g)$ *be a complete Ricci flow spacetime with* $I = (0, T)$*. Then* $\mathcal{M}$ *is expanding. That is, the vanishing times*

$$T_x := \sup I_x = T, \quad \forall x \in \mathcal{M}.$$

Proof of Theorem 1.3.1. Suppose there exists $x \in \mathcal{M}$ such that its extinction time $T_x < T$. Choose any $x' \in \mathcal{M}$ such that $\mathfrak{t}(x') > T_x$. Applying Lemma 5.1.16 to the pair of points $x, x' \in \mathcal{M}$, there exists $T' \in [\mathfrak{t}(x'), T)$ and a time-preserving path $\eta : [\mathfrak{t}(x), T'] \to \mathcal{M}$ such that $\eta(\mathfrak{t}(x)) = x$. Restricting η to the interior $(\mathfrak{t}(x), T')$ and applying Lemma 5.1.12, we have that $\mathcal{M}^\eta$ is a complete and spatially-connected Ricci flow spacetime. Note that for small $r > 0$, the parabolic cylinder $C(x, r)$ is unscathed, and hence $x(s) \in \mathcal{M}^\eta$ for $s \in (\mathfrak{t}(x), \mathfrak{t}(x) + r^2)$. Applying Theorem 5.1.10, $\mathcal{M}^\eta$ is expanding. Therefore, inside of $\mathcal{M}^\eta$, the extinction time of the point x is T', from which we can deduce that inside of the spacetime $\mathcal{M}$, the point x has extinction time $T_x \geq T' > T_x$. This is a contradiction. □

5.2 Embedding spacetimes within an ambient space

As we shall see later, an ambient space for a Ricci flow spacetime can be a useful tool. In the case that our Ricci flow spacetime is expanding, we can define a map which looks like a global cylindrical coordinate chart. This will give an embedding (up to some non-final time) of our spacetime into an ambient space.

Lemma 5.2.1. *Let $(\mathcal{M}^{n+1}, \mathfrak{t}, \partial_t, g)$ be an expanding Ricci flow spacetime with $I = (0, T)$. For each $\tau \in (0, T)$, there exists a smooth map*

$$\Phi : \mathcal{M}_{(0,\tau)} \to \mathcal{M}_\tau \times (0, \tau),$$

such that

(i) *Φ is a diffeomorphism onto its image.*

(ii) *For each $t \in (0, \tau)$, restricting to the spatial slice $\Phi : \mathcal{M}_t \to \mathcal{M}_\tau \times \{t\}$ is a diffeomorphism onto its image.*

(iii) *If $t : M \times (0, \tau) \to (0, \tau)$ denotes the standard projection, then $\mathfrak{t} = t \circ \Phi$ and $\Phi_*(\partial_t) = \frac{\partial}{\partial t}$.*

(iv) *$\Phi_*(g)$ is a solution to the usual Ricci flow equation (1.0.3) on $\Phi(\mathcal{M}_{(0,\tau)})$.*

Proof. Since $\mathcal{M}$ is expanding, for any $s \in (0, \tau)$, we have the smooth embedding

$$\mathcal{M}_s \hookrightarrow \mathcal{M}_\tau, \quad x \mapsto x(\tau).$$

Define the map $\Phi : \mathcal{M}_{(0,\tau)} \hookrightarrow \mathcal{M}_\tau \times (0, \tau)$ by

$$\Phi(x) := (x(\tau), \mathfrak{t}(x)).$$

Φ is smooth, with smooth inverse $(p, t) \mapsto p(t)$. In particular, conditions (i) and (ii) are satisfied. Directly from the definition of Φ, we see that $t \circ \Phi = \mathfrak{t}$. Moreover, since worldlines are integral curves of ∂_t, the identity $\partial_t \cdot \mathfrak{t} \equiv 1$ implies $\Phi_* \partial_t = \frac{\partial}{\partial t}$. Finally we have the equality

$$\frac{\partial}{\partial t}\Phi_*(g) = \mathcal{L}_{\Phi_*(\partial_t)}\Phi_*(g) = \Phi_*(\mathcal{L}_{\partial_t} g) = \Phi_*(-2\,\mathrm{Ric}(g)) = -2\,\mathrm{Ric}(\Phi_*(g)). \qquad \square$$

5.2.1 Existence of an ambient space for expanding spacetimes

The previous lemma shows that for an expanding Ricci flow spacetime, we have an embedding locally in time. The following theorem extends this to all times.

Theorem 5.2.2. *Let $(\mathcal{M}^{n+1}, \mathfrak{t}, \partial_t, g)$ be an expanding Ricci flow spacetime with $I = (0, T)$. Then there exists a smooth connected manifold M^n and a smooth map*

$$\Phi : \mathcal{M} \hookrightarrow M \times (0, T),$$

such that

(i) Φ *is a diffeomorphism onto its image;*

(ii) *For each* $t \in (0,T)$, *restricting to the spatial slice* $\Phi : \mathcal{M}_t \hookrightarrow M \times \{t\}$ *is a diffeomorphism onto its image;*

(iii) *If* $t : M \times (0,T) \to (0,T)$ *denotes the standard projection, then* $\mathfrak{t} = t \circ \Phi$ *and* $\Phi_*(\partial_t) = \frac{\partial}{\partial t}$*;*

(iv) $\Phi_*(g)$ *is a solution to the usual Ricci flow equation (1.0.3) on* $\Phi(\mathcal{M})$.

That is, $(\mathcal{M}^{n+1}, \mathfrak{t}, \partial_t, g)$ *is isomorphic to a spacetime inside the ambient space* $M \times (0,T)$.

Proof. We first construct M. Choose any increasing sequence $t_i \nearrow T$ in $(0,T)$, and for $i \leq j$, consider the embeddings $\mathcal{M}_{t_i} \hookrightarrow \mathcal{M}_{t_j}$, $x \mapsto x(t_j)$. Choose M to be the direct limit

$$M = \lim_{\to} \mathcal{M}_{t_i} := \bigsqcup_{i \in \mathbb{N}} \mathcal{M}_{t_i} \big/ \sim \ , \tag{5.2.1}$$

where $x \sim y$ iff $x(t) = y$ for some $t \in I$, and with canonical maps $f_i : \mathcal{M}_{t_i} \to M$, $x \mapsto [x]$. It is straight forward to show that M is connected, and can be equipped with a smooth atlas so that the canonical maps $f_i : \mathcal{M}_{t_i} \hookrightarrow M$ are smooth embeddings (see Lemma C.0.4 for details). For each $i \in \mathbb{N}$, we combine the map we get by choosing $t_i \in (0,T)$ in Lemma 5.2.1 and the canonical map f_i to get the well-defined map $\Phi^i : \mathcal{M}_{(0,t_i)} \to M \times (0, t_i)$

$$\Phi^i(x) := (f_i \circ x(t_i), \mathfrak{t}(x)).$$

Suppose $i \leq j$ and $x \in \mathcal{M}_{(0,t_i)}$. Since $x(t_j) = (x(t_i))(t_j)$, we have that

$$f_j \circ x(t_j) = f_i \circ x(t_i),$$

and Φ^j is an extension of the function Φ^i. Therefore, we can piece the functions $\{\Phi^i : i \in \mathbb{N}\}$ together, giving the well-defined function $\Phi : \mathcal{M} \to M \times (0,T)$. The properties of Φ follow from the properties of the embeddings in Lemma 5.2.1. $\square$

Due to Theorem 1.3.1, we can apply Theorem 5.2.2 to any complete $(2+1)$-dimensional Ricci flow spacetime. The following corollary shall be used implicitly from now on.

Corollary 5.2.3. *Let* $(\mathcal{M}^{2+1}, \mathfrak{t}, \partial_t, g)$ *be a complete Ricci flow spacetime with* $I = (0,T)$. *Then there exists a connected smooth ambient surface* M^2 *such that our spacetime is isomorphic to a complete Ricci flow spacetime* $(\mathcal{M}^{2+1}, g)$ *in* $M^2 \times (0,T)$. *Moreover,* M *is chosen as small as possible. That is, up to isomorphism, we can assume that*

(i) $\mathcal{M}$ *is an open subset of* $M \times (0,T)$ *equipped with the product topology;*

(ii) $\mathfrak{t}$ *is the restriction of the standard projection map* $t : M \times (0,T) \to (0,T)$ *to* $\mathcal{M}$*;*

(iii) ∂_t *is the restriction of the vector field* $\frac{\partial}{\partial t}$ *to* $\mathcal{M}$*;*

(iv) g *solves the Ricci flow equation (1.0.3) on* $\mathcal{M}$*;*

(v) $M = \bigcup_{t\in(0,T)} \mathcal{M}_t$ *and* $\mathcal{M}$ *is expanding:* $\mathcal{M}_{t_1} \subseteq \mathcal{M}_{t_2}$ *for every* $0 < t_1 \leq t_2 < T$*.*

5.2.2 Continuity within an ambient space

Now that we have an ambient space, we have the following simplification for the definition of a spacetime being continuous.

Lemma 5.2.4. *Let* $(\mathcal{M}^{2+1}, g)$ *be a complete Ricci flow spacetime in* $M \times (0,T)$*. Then the spacetime is continuous (see Definition 1.3.2) iff*

$$\mathcal{M}_s = \left(\bigcap_{t>s} \mathcal{M}_t\right)^{\circ} \subseteq M, \quad \forall s \in (0,T). \tag{5.2.2}$$

Proof. Since $\mathcal{M}$ is expanding, for $t \leq s$, we have the continuity criteria for free:

$$\mathcal{M}_s(t) = \mathcal{M}_t = \left(\overline{\mathcal{M}_s}(t)\right)^{\circ},$$

and so we only need to consider the case $s < t$. Using again that $\mathcal{M}$ is expanding, as subsets of M, we see that $\mathcal{M}_s(t) = \mathcal{M}_s$, and

$$\overline{\mathcal{M}_s}(t) = \{x \in M : (s,t] \subseteq I_x\} = \bigcap_{t>s} \mathcal{M}_t.$$

Since each spatial slice is open in M, our original definition of continuity reduces to (5.2.2). □

The following lemma shows that for complete and continuous spacetimes, isolated punctures cannot be added to the spatial slices (see Example 1.1.8).

Lemma 5.2.5. *Suppose* $(\mathcal{M}^{2+1}, g)$ *is a complete and continuous Ricci flow spacetime in* $M \times (0,T)$*. Consider a point* $x \in M \setminus \mathcal{M}_s$ *laying outside of the spatial slice of* $\mathcal{M}$ *at some time* $s \in (0,T)$*. Suppose* x *is an isolated point of* $M \setminus \mathcal{M}_s$*. Then* x *is never in* $\mathcal{M}$*. That is,* $(x,t) \notin \mathcal{M}$*, for all* $t \in (0,T)$*.*

Proof. Since x is an isolated point of the complement, there exists an open neighbourhood $x \in U \subseteq M$ such that the punctured neighbourhood $U \setminus \{x\}$ is contained in $\mathcal{M}_s$. As $\mathcal{M}$ is expanding, $U \setminus \{x\} \subseteq \mathcal{M}_t$, for all $t \in [s,T)$. If the lemma is false, we have a well defined first time t_0 that x enters our spacetime

$$t_0 := \inf\{t \in (s,T) : x \in \mathcal{M}_t\} \geq s.$$

Note that $U \subseteq \mathcal{M}_t$ for every $t \in (t_0, T)$, which by continuity implies

$$U \subseteq \left(\bigcap_{t>t_0} \mathcal{M}_t \right)^{\circ} = \mathcal{M}_{t_0}.$$

In particular, $p \in \mathcal{M}_{t_0}$. This contradicts the definition of t_0 as $\mathcal{M}$ is open in $M \times (0,T)$. $\square$

Corollary 5.2.6. *Suppose $(\mathcal{M}^{2+1}, g)$ is a complete and continuous spacetime in $M \times (0,T)$. Then for any times $0 < t_1 < t_2 < T$, $\mathcal{M}_{t_2} \setminus \mathcal{M}_{t_1}$ contains no isolated points.*

5.2.3 Lifting a spacetime

Let $(\mathcal{M}^{2+1}, g)$ be a complete Ricci flow spacetime inside the ambient space $M \times (0,T)$. Suppose $p : X \to M$ is a covering map from a connected surface X. We define the lifted spacetime $(\mathcal{M}', g')$ inside of $X \times (0,T)$ via:

$$\mathcal{M}'_t := p^{-1}(\mathcal{M}_t), \quad g'(t) := p^*(g(t)), \quad \forall t \in (0,T). \tag{5.2.3}$$

In the following lemma, we first show that what we constructed above is in fact a well-defined Ricci flow spacetime. Moreover, we show that the lifted spacetime inherits continuity and being initially determined from the original spacetime.

Lemma 5.2.7. *If $(\mathcal{M}^{2+1}, g)$ is a complete Ricci flow spacetime inside the ambient space $M \times (0,T)$, and $p : X \to M$ is a covering map, with X a connected surface, then $(\mathcal{M}', g')$ as defined in equation (5.2.3) is a complete Ricci flow spacetime in $X \times (0,T)$. Furthermore, if $(\mathcal{M}, g)$ is continuous (initially determined), then $(\mathcal{M}', g')$ is continuous (initially determined).*

Proof. We must first show that $(\mathcal{M}', g')$ is a spacetime in $X \times (0,T)$. For any $(x,t) \in \mathcal{M}'$, since $\mathcal{M}$ is open inside $M \times (0,T)$, there exists $U \subseteq M$ open and $\delta > 0$ such that $(p(x),t) \in U \times (t-\delta, t+\delta) \subseteq \mathcal{M}$, and hence $(x,t) \in p^{-1}(U) \times (t-\delta, t+\delta) \subseteq \mathcal{M}'$. Since p is continuous, this neighbourhood is open in $X \times (0,T)$, and so $\mathcal{M}' \subseteq X \times (0,T)$ is open. It is also clear that

$$X = p^{-1}(M) = p^{-1}\Big(\bigcup_{t \in (0,T)} \mathcal{M}_t\Big) = \bigcup_{t \in (0,T)} p^{-1}(\mathcal{M}_t) = \bigcup_{t \in (0,T)} \mathcal{M}'_t,$$

and $\mathcal{M}'$ is expanding: $\mathcal{M}'_{t_1} = p^{-1}(\mathcal{M}_{t_1}) \subseteq p^{-1}(\mathcal{M}_{t_2}) = \mathcal{M}'_{t_2}$ for every $0 < t_1 \leq t_2 < T$.

For any $(x_0, t_0), (x_1, t_1) \in \mathcal{M}'$, there exists a continuous path $\gamma : [0,1] \to X$ such that $\gamma(0) = x_0$ and $\gamma(1) = x_1$. Since $\gamma([0,1]) \subseteq X$ is compact and the set $\{\mathcal{M}'_t : t \in (0,T)\}$ form a nested open cover of X, there exists some $\tau \in (0,T)$ such that $t_0, t_1 \leq \tau$, and $\gamma : [0,1] \to \mathcal{M}_\tau$. By concatenating γ with the worldlines of x_0 and x_1, we see that $\mathcal{M}'$ is also connected. Directly from the definition of g', we also see that $(\mathcal{M}', g')$ is complete. This finishes the first part of the statement.

If $\mathcal{M}$ is continuous, since p is a local homeomorphism, for any $s \in (0,T)$ we have

$$\left(\bigcap_{t>s}\mathcal{M}'_t\right)^\circ = \left(p^{-1}\left(\bigcap_{t>s}\mathcal{M}_t\right)\right)^\circ = p^{-1}\left(\left(\bigcap_{t>s}\mathcal{M}_t\right)^\circ\right) = p^{-1}(\mathcal{M}_s) = \mathcal{M}'_s,$$

and $\mathcal{M}'$ is also continuous. Similarly, suppose $\mathcal{M}$ is initially determined. For any $x \in \mathcal{M}'$ and $\epsilon > 0$, there exists a smooth time-preserving path $\eta : [\epsilon, \mathfrak{t}(x)] \to \mathcal{M}$, such that $\eta \circ \mathfrak{t}(x) = p(x)$. As the restriction $p : \mathcal{M}' \to \mathcal{M}$ is a covering map, there exists a unique lift of η to a smooth time-preserving path $\eta' : [\epsilon, \mathfrak{t}(x)] \to \mathcal{M}'$, such that $\eta \circ \mathfrak{t}(x) = x$. Taking $\epsilon \searrow 0$, we conclude that $\mathfrak{h}(x) = 0$ for any $x \in \mathcal{M}'$, and $\mathcal{M}'$ is also initially determined. $\square$

Recall that we are aiming to show Theorem 1.3.6. Suppose $(\mathcal{M}^{2+1}, g)$ is a complete, continuous and initially determined spacetime inside $M \times (0,T)$. Since M is connected, it has connected universal cover $X \to M$. Lemma 5.2.7 then tells us that the lifted spacetime is also complete, continuous and initially determined. Moreover, if the lifted spacetime is cylindrical, $\mathcal{M}' = X \times (0,T)$, then the original spacetime would also be cylindrical, $\mathcal{M} = M \times (0,T)$, since the covering map is surjective onto M. Therefore, to prove Theorem 1.3.6, it suffices to consider the case when M is simply connected.

5.2.4 Conformal structures

Let $(\mathcal{M}^{2+1}, g)$ be a complete Ricci flow spacetime inside the ambient space $M \times (0,T)$. From the previous subsection, we may assume that M is simply connected.

Given a conformal structure on M, each spatial slice $\mathcal{M}_t$ for $t \in (0,T)$ inherits a conformal structure as a subset of M. Our Ricci flow $g(t)$ on $\mathcal{M}$ is then said to be conformal if this inherited conformal structure on $\mathcal{M}_t$ agrees with $g(t)$ for all $t \in (0,T)$. Fix $0 < t_1 < t_2 < T$ and consider the conformal structures on $\mathcal{M}_{t_i}$ determined by the metrics $g(t_i)$, for $i = 1, 2$. Recall that on an orientable surface, a choice of metric and hence its corresponding conformal class is equivalent to a choice of complex structure on the surface. Since our ambient surface M is simply connected, it is orientable, and hence each spatial slice $\mathcal{M}_t$ is orientable too. As such, the conformal structure on $\mathcal{M}_{t_1}$ defines a complex structure. Let U be a complex coordinate chart on $\mathcal{M}_{t_1}$. Writing our Ricci flow locally as in (5.1.7), we see that U is also a complex chart on $\mathcal{M}_{t_2}$, and therefore the complex structure on $\mathcal{M}_{t_1}$ agrees with the complex structure it inherits when viewed as a subspace of $\mathcal{M}_{t_2}$. For each $t \in (0,T)$, consider the complex coordinate charts defined on $\mathcal{M}_t$ by the metric $g(t)$. We have shown that taking the union of all such charts gives a well-defined complex atlas on M, and with respect to this conformal structure, $g(t)$ is a conformal Ricci flow.

Lemma 5.2.8. *Let $(\mathcal{M}^{2+1}, g)$ be a complete Ricci flow spacetime in $M^2 \times (0,T)$, with $M = \cup_{t\in(0,T)}\mathcal{M}_t$. If M is orientable, then there exists a conformal structure on M such that $g(t)$ is a conformal Ricci flow.*

In light of this, we can assume that our ambient space is a Riemann surface whose conformal structure is compatible with the Ricci flow $g(t)$. Combined with the assumption that M is simply connected and the uniformisation theorem, we have reduced Theorem 1.3.6 to proving the following.

Theorem 5.2.9. *Suppose $(\mathcal{M}^{2+1}, g)$ is a complete, continuous and initially determined spacetime in $M^2 \times (0,T)$ with $M = \cup_{t\in(0,T)}\mathcal{M}_t$, and where M^2 is either the disk, plane or sphere equipped with their standard conformal structures. Suppose further that $g(t)$ is a conformal Ricci flow on $\mathcal{M}$. Then $\mathcal{M} = M \times (0,T)$.*

5.3 Spacetimes in the disk

In the previous section, we reduced our theorem to the special case that our Ricci flow is conformal on a spacetime lying in either the disk, plane or sphere. We can simply this further to the case that the spacetime lies in the disk.

5.3.1 Reduction to spacetimes in the disk

Definition 5.3.1 (Hyperbolic surface)**.** Given any (possibly disconnected) Riemann surface N, we say that N is hyperbolic if each of its connected components has universal cover the disk D, or equivalently, if N admits a smooth conformal complete metric of constant curvature -1.

The following is a simple application of the uniformisation theorem.

Lemma 5.3.2. *Let M be the sphere S^2 or the complex plane $\mathbb{C}$ equipped with its standard conformal structure. Consider a Riemann surface $A \subsetneq M$ given by some open subset. If the (non-empty) complement of A in M contains no isolated points, then A is hyperbolic.*

Proof. Since we can identify the once punctured sphere with the plane, it suffices to consider the case $M = \mathbb{C}$. Consider a connected component $\tilde{A}$ of A. If the complement of A (and hence $\tilde{A}$) in $\mathbb{C}$ has no isolated points, then $\mathbb{C} \setminus \tilde{A}$ contains at least 2 points. As a consequence of the Uniformisation Theorem C.0.2, $\tilde{A}$ is covered by either the sphere, plane or disk. Since $\tilde{A}$ is not compact, it cannot be covered by the sphere. Lets suppose it was covered by the plane for a contradiction. Note that the Deck group of this covering is discrete as a subspace of the homeomorphism group of $\mathbb{C}$, and acts properly discontinuously. By the Galois correspondence, $\tilde{A}$ is isomorphic to the plane quotiented by these deck transformations. However, the only said quotients are the plane itself, the punctured plane, or a torus. This is a contradiction. ☐

Theorem 5.3.3. *Suppose $(\mathcal{M}^{2+1}, g)$ is a complete, continuous and initially determined spacetime in $D\times(0,T)$ with $D = \cup_{t\in(0,T)}\mathcal{M}_t$, and where D is the disk equipped with its hyperbolic conformal structure. Suppose further that g is a conformal Ricci flow on $\mathcal{M}$. Then $\mathcal{M} = D\times(0,T)$.*

Proof (Theorem 5.3.3 $\implies$ Theorem 5.2.9).
Suppose that the theorem fails and there exists $0 < t_1 < t_2 < T$ such that $\mathcal{M}_{t_1} \subsetneq \mathcal{M}_{t_2}$. By our assumption, M is either the plane or the sphere. By Corollary 5.2.6, $\mathcal{M}_{t_2} \setminus \mathcal{M}_{t_1} \subseteq M \setminus \mathcal{M}_{t_1}$ contains no isolated points, and so $\mathcal{M}_{t_1}$ must be hyperbolic by Lemma 5.3.2. $\mathcal{M}_{t_2}$ cannot be hyperbolic, as otherwise we could lift each connected component of the spacetime $\mathcal{M}_{(0,t_2)}$ into $D \times (0, t_2)$ and apply Theorem 5.3.3 to conclude its lift is cylindrical, which would then imply that $\mathcal{M}_{(0,t_2)}$ is cylindrical, contradicting our assumption that $\mathcal{M}_{t_1} \subsetneq \mathcal{M}_{t_2}$. Since $\mathcal{M}_{t_2}$ is a non-hyperbolic subset of M, we can again apply Lemma 5.3.2 and Corollary 5.2.6 to deduce that $\mathcal{M}_{t_2} = M$. Consider the first time at which the spatial slices are not hyperbolic:

$$t_0 := \inf\{t \in [t_1, t_2] : \mathcal{M}_t \text{ is not hyperbolic}\}.$$

Using Corollary 5.2.6 yet again, we have that $\mathcal{M}_t = M$ for all $t > t_0$, and hence by continuity, $\mathcal{M}_{t_0} = M$. Finally, we note that each connected component of $\mathcal{M}_{(0,t_0)}$ can be lifted into $D \times (0, t_0)$. Applying the same reasoning as before, $\mathcal{M}_{(0,t_0)}$ must be cylindrical, and $\mathcal{M}_t = \mathcal{M}_{t_1}$ for all $t \in (t_1, t_0)$. From this we deduce the contradiction

$$\mathcal{M}_{t_2} = M = \mathcal{M}_{t_0} = \bigcup_{t<t_0} \mathcal{M}_t = \mathcal{M}_{t_1}. \qquad \square$$

5.3.2 Comparison principle for spacetimes

From now on, we may assume that $(\mathcal{M}^{2+1}, g)$ is a complete, connected, continuous and initially determined spacetime in $D \times (0, T)$ with $D = \cup_{t\in(0,T)}\mathcal{M}_t$, and that the Ricci flow is conformal with respect to the standard conformal structure on D. In particular, we can write our metrics $g(t)$ in the form

$$g(t) = v(z, t)|dz|^2,$$

where $|dz|^2$ denotes the standard flat metric on the disk, and $v : \mathcal{M} \to (0, \infty)$ is some smooth solution to the LFDE (5.1.8).

Using the conformal structure on each spatial slice, we see that each slice admits a unique complete hyperbolic metric $h(t)$ (by Lemma 5.3.2). Writing $h(t) = H(z, t)|dz|^2$, since $\mathcal{M}$ is expanding, the Schwarz lemma C.0.3 gives the inequality $h(t) \leq h(s)$ for any $0 < s < t < T$. In particular, the conformal factors $H(\cdot, t)$ are decreasing in t.

Since the metrics $g(t)$ are complete on $\mathcal{M}_t$ and have the lower scalar curvature bound from Corollary 5.1.2, we can use the Schwarz lemma C.0.3 again to get

$$v(z, t) \geq 2t \cdot H(z, t), \quad \forall (z, t) \in \mathcal{M}, \tag{5.3.1}$$

where $H : \mathcal{M} \to (0, \infty)$ is the conformal factor of the complete hyperbolic metrics $h(t)$ on $\mathcal{M}$ defined above.

We want a comparison principle between our solution v to the LFDE, and any other solution u to the LFDE on $\mathcal{M}$. In order to do so, we want a lower bound on v that behaves like a proper function. We define the correct notion of proper in the parabolic setting below.

Definition 5.3.4. Let $(\mathcal{M}, \mathfrak{t}, \partial_t, g)$ be a Ricci flow spacetime with $I = (0, T)$. A function $f : \mathcal{M} \to \mathbb{R}$ is *parabolically proper* if

$$\mathcal{M}_{(\epsilon, T-\epsilon)} \cap f^{-1}(I) \Subset \mathcal{M}, \quad \forall I \Subset \mathbb{R},\ \forall \epsilon > 0.$$

The following lemma shows that, for a complete spacetime in the disk, the conformal factors H being a parabolically proper map is an equivalent way to characterise our spacetime being continuous. In particular, due to equation (5.3.1), H is an appropriate lower bound to take.

Lemma 5.3.5. *Let $(\mathcal{M}^{2+1}, g)$ be a complete Ricci flow spacetime in $D \times (0, T)$, and let $H : \mathcal{M} \to (0, \infty)$ be the conformal factors of the complete hyperbolic metrics $h(t)$ on $\mathcal{M}$. Then $\mathcal{M}$ is continuous iff $H : \mathcal{M} \to (0, \infty)$ is parabolically proper (see Definition 5.3.4).*

Proof. If $\mathcal{M}$ is not continuous, there exists $s \in (0, T)$ and a point $p \in \left(\bigcap_{t>s} \mathcal{M}_s\right)^\circ \setminus \mathcal{M}_s$. In particular, with respect to the background metric $|dz|^2$, there exists $r > 0$ such that the ball $B(p, r) \subseteq \mathcal{M}_t$, for every $t > s$. By the Schwarz lemma C.0.3, $H(p, t) \leq \frac{4}{r^2}$, for every $t > s$. That is, for any sequence of times $t_n \searrow s$, and for any $\epsilon > 0$ sufficiently small, we have the sequence of points $(p, t_n) \in \mathcal{M}_{(\epsilon, T-\epsilon)}$ such that $(p, t_n) \to (p, s) \in \partial\mathcal{M}$, but $H(p, t_n)$ is uniformly bounded for all n. That is, H isn't parabolically proper.

Conversely, suppose $\mathcal{M}$ is continuous and fix $\epsilon > 0$. Let $(z_n, t_n) \in \mathcal{M}_{(\epsilon, T-\epsilon)}$ be any sequence such that $(z_n, t_n) \to (z, s) \in \partial\mathcal{M}$. If we can show that the sequence $H(z_n, t_n)$ diverges then we are done. Passing to a subsequence, we may assume that the sequence of times t_n is monotone. Suppose $t_n \nearrow s$. As $\mathcal{M}$ is expanding, $z_n \in \mathcal{M}_s$ with $H(z_n, s) \leq H(z_n, t_n)$ for every n. In particular, $z_n \in \mathcal{M}_s$ with $z_n \to z \in \partial\mathcal{M}_s$. As $H(\cdot, s) : \mathcal{M}_s \to (0, \infty)$ is a proper map, $H(z_n, s)$ is unbounded, and hence so is $H(z_n, t_n)$. Instead we assume that $t_n \searrow s$. For any fixed $m \in \mathbb{N}$, using that $\mathcal{M}$ is expanding, we have that $z_n \in \mathcal{M}_{t_m}$ with $H(z_n, t_m) \leq H(z_n, t_n)$ for any $n \geq m$. So, for each $m \in \mathbb{N}$, we have a sequence $z_n \in \mathcal{M}_{t_m}$ with $z_n \to z$. If $z \in \partial\mathcal{M}_{t_m}$, then using again that $H(\cdot, t_m) : \mathcal{M}_{t_m} \to (0, \infty)$ is a proper map, we would deduce that $H(z_n, t_m)$ diverges, so $H(z_n, t_n)$ diverges. This leads us to assume that $z \in \mathcal{M}_{t_m}$ for every $m \in \mathbb{N}$. By the continuity of each $H(\cdot, t_m) : \mathcal{M}_{t_m} \to (0, \infty)$, we have

$$H(z, t_m) = \lim_{n\to\infty} H(z_n, t_m) \leq \limsup_{n\to\infty} H(z_n, t_n),$$

and hence

$$\limsup_{n\to\infty} H(z, t_n) \leq \limsup_{n\to\infty} H(z_n, t_n).$$

From the above equation, it suffices to show that $H(z, t_n) \nearrow \infty$ as $n \to \infty$. Since $\mathcal{M}$ is continuous, $z \in \left(\bigcap_{t>s} \mathcal{M}_t\right) \setminus \mathcal{M}_s = \left(\bigcap_{t>s} \mathcal{M}_t\right) \setminus \left(\bigcap_{t>s} \mathcal{M}_t\right)^\circ$. So there exists $\epsilon_n \searrow 0$ such that the ball of radius ϵ_n centred at z is contained in our spatial slice, $B(z, \epsilon_n) \subseteq \mathcal{M}_{t_n}$, but some point in the boundary of the ball $b_n \in \partial B(z, \epsilon_n)$ isn't in our spatial slice $b_n \notin \mathcal{M}_{t_n}$. Note that all of the spatial slices are contained within the disk and hence at each time t_n, each spatial slice is contained within the punctured ball of radius 2 centred at b_n

$$\mathcal{M}_{t_n} \subseteq B^\times(b_n, 2), \quad \forall n \in \mathbb{N}.$$

Consider the complete hyperbolic metric on this punctured ball $B^\times(b_n, 2)$

$$h_n^\times(z) := \frac{1}{|z - b_n|^2(-\log(2|z - b_n|))^2}|dz|^2.$$

By the Schwarz lemma C.0.3, $h(t_n) \geq h_n^\times$ on $\mathcal{M}_{t_n}$. In particular, we have that

$$H(z, t_n) \geq \frac{1}{\epsilon_n^2(-\log(2\epsilon_n))^2}, \quad \forall n \in \mathbb{N},$$

and thus $H(z, t_n) \nearrow \infty$ as $n \to \infty$, concluding the proof. □

The follow is a standard comparison principle for solutions to LFDE with regularity up to the boundary. Note that we must account for the fact our spacetime is not necessarily cylindrical.

Lemma 5.3.6 (Direct comparison principle)**.** *Let $\mathcal{M}$ be a compactly contained, open subset of $D \times (0, T)$, and let $u, v \in C^{2,1}(\overline{\mathcal{M}})$ be solutions to the LFDE (5.1.8) with $u, v > 0$. If $v > u$ on the parabolic boundary $\partial_P \mathcal{M}$, then $v \geq u$ on $\mathcal{M}$.*

Proof. We modify the argument used by Giesen in [Gie12, Theorem 2.3.1].

By compactness, we can choose $\delta > 0$ such that $v \geq u + \delta$ on $\partial_P \mathcal{M}$. For any $\epsilon > 0$, consider

$$\mathcal{M}(\epsilon) := \{(z, t) \in D \times (0, T) : (z, \epsilon^{-1} \log(1 + \epsilon t)) \in \mathcal{M}\},$$

and the modified function

$$v_\epsilon(z, t) := (1 + \epsilon t) \cdot v(z, \epsilon^{-1} \log(1 + \epsilon t)), \quad \forall (z, t) \in \overline{\mathcal{M}(\epsilon)}.$$

This modification makes v_ϵ a strict supersolution to the LFDE on $\mathcal{M}(\epsilon)$:

$$(\partial_t v_\epsilon - \Delta \log(v_\epsilon))(z, t) = \epsilon \cdot v(z, \epsilon^{-1} \log(1 + \epsilon t)) > 0, \quad \forall (z, t) \in \mathcal{M}(\epsilon).$$

Since u and v are continuous on $\overline{\mathcal{M}}$ compact, they are uniformly continuous. In particular, from the inequality

$$0 < t - \epsilon^{-1} \log(1 + \epsilon t) \leq T - \epsilon^{-1} \log(1 + \epsilon T), \quad \forall t \in (0, T),$$

we see that $|t - \epsilon^{-1} \log(1 + \epsilon t)|$ converges to zero uniformly in $t \in (0, T)$ as $\epsilon \searrow 0$. Thus, for ϵ

sufficiently small, $|v(z,t)-v(z,\epsilon^{-1}\log(1+\epsilon t))|<\delta/2$, and $|u(z,t)-u(z,\epsilon^{-1}\log(1+\epsilon t))|<\delta/2$, whenever $(z,t),(z,\epsilon^{-1}\log(1+\epsilon t))\in\overline{\mathcal{M}}$.

We now consider those points lying in both $\mathcal{M}$ and the shifted spacetime $\mathcal{M}(\epsilon)$.

Claim. *Define $M(\epsilon)=\mathcal{M}\cap\mathcal{M}(\epsilon)$. For $\epsilon>0$ sufficiently small, $v_\epsilon\geq u$ on $\partial_P(M(\epsilon))$.*

Proof of Claim. Fix $(z,t)\in\partial_P(M(\epsilon))$. Note that $\partial_P(M(\epsilon))\subseteq(\partial_P\mathcal{M})\cup(\partial_P(\mathcal{M}(\epsilon)))$. This splits our analysis into two cases:

(I) $(z,t)\in\partial_P\mathcal{M}$. In this case we have the inequality

$$\begin{aligned}v_\epsilon(z,t)&\geq v(z,t)-|v_\epsilon(z,t)-v(z,t)|\\&\geq u(z,t)+\delta-|\epsilon t|\cdot|v(z,\epsilon^{-1}\log(1+\epsilon t))|-|v(z,\epsilon^{-1}\log(1+\epsilon t))-v(z,t)|\\&\geq u(z,t)+\delta-\epsilon\cdot T\cdot\|v\|_\infty-\frac{\delta}{2}\geq u(z,t),\end{aligned}$$

for any ϵ sufficiently small.

(II) $(z,t)\in\partial_P(\mathcal{M}(\epsilon))$. Note that, this implies $(z,\epsilon^{-1}\log(1+\epsilon t))\in\partial_P\mathcal{M}$, and therefore we have the inequality

$$\begin{aligned}v_\epsilon(z,t)&\geq v(z,\epsilon^{-1}\log(1+\epsilon t))\\&\geq u(z,\epsilon^{-1}\log(1+\epsilon t))+\delta\\&\geq u(z,t)+\delta-|u(z,\epsilon^{-1}\log(1+\epsilon t))-u(z,t)|\\&\geq u(z,t)+\delta-\frac{\delta}{2}\geq u(z,t),\end{aligned}$$

for any ϵ sufficiently small. □

Claim. *$v_\epsilon\geq u$ on $M(\epsilon)$.*

Proof of claim. Let M_t denote $M(\epsilon)\cap(D\times\{t\})$ for each $t\in(0,T)$. If the claim doesn't hold, then for $n\in\mathbb{N}$ sufficiently large, the times

$$t_n:=\inf\{t\in(0,T):\min_{\overline{M_t}}(v_\epsilon-u)(z,t)\leq-1/n\}\in(0,T),$$

are well-defined. As $\overline{M_{t_n}}$ is compact, there exists $z_n\in\overline{M_{t_n}}$ such that $(v_\epsilon-u)(z_n,t_n)=-1/n$. By the previous claim, $(v_\epsilon-u)\geq 0$ on the parabolic boundary $\partial_P(M(\epsilon))$, and hence $z_n\in M_{t_n}$. Since the point (z_n,t_n) is in the interior, we have the inequalities

$$(v_\epsilon-u)=-1/n,\quad \Delta(v_\epsilon-u)\geq 0,\quad \partial_t(v_\epsilon-u)\leq 0,$$

and therefore

$$\begin{aligned}
0 &< \epsilon v(z_n, \epsilon^{-1}\log(1+t_n\epsilon)) \\
&\le (\partial_t v_\epsilon - \Delta\log(v_\epsilon))(z_n,t_n) - (\partial_t u - \Delta\log(u))(z_n,t_n) \\
&\le \Delta\log(u) - \Delta\log(v_\epsilon) \\
&= \left(\frac{\Delta u}{u} - \frac{|\nabla u|^2}{u^2}\right) - \left(\frac{\Delta v_\epsilon}{v_\epsilon} - \frac{|\nabla v_\epsilon|^2}{v_\epsilon^2}\right) \\
&= \frac{1}{v_\epsilon}\Delta(u-v_\epsilon) + (\frac{1}{u} - \frac{1}{v_\epsilon})\Delta u + |\nabla u|^2\left(\frac{1}{v_\epsilon^2} - \frac{1}{u^2}\right) \\
&\le \frac{1}{n}\left(\frac{|\Delta u|}{uv_\epsilon} + |\nabla u|^2\left(\frac{(u+v_\epsilon)}{u^2v_\epsilon^2}\right)\right).
\end{aligned}$$

Since u and v are uniformly bounded away from zero, and $u, v \in C^{2,1}(\overline{\mathcal{M}})$, there exists $C < \infty$, independent of n, such that

$$0 < \epsilon \le \frac{1}{v(z_n, \epsilon^{-1}\log(1+t_n\epsilon))} \cdot \left(\frac{1}{n}\left(\frac{|\Delta u|}{uv_\epsilon} + |\nabla u|^2\left(\frac{(u+v_\epsilon)}{u^2v_\epsilon^2}\right)\right)\right) \le \frac{C}{n}.$$

Taking $n \to \infty$, this is gives a contradiction. ☐

Fix $(z,t) \in \mathcal{M}$. Recall that $|t - \epsilon^{-1}\log(1+\epsilon t)|$ converges to zero uniformly in $t \in (0,T)$ as $\epsilon \searrow 0$. Therefore, as $\mathcal{M}$ is open, for any ϵ sufficiently small, $(z,t) \in M(\epsilon)$. Moreover, by continuity, $v_\epsilon(z,t)$ converges to $v(z,t)$ as $\epsilon \searrow 0$, and we can conclude that $v \ge u$ on all of $\mathcal{M}$. ☐

Using Lemma 5.3.5, we now have a comparison principle for solutions to the LFDE corresponding to complete Ricci flows on spacetimes in the disk.

Lemma 5.3.7 (Comparison principle)**.** *Let $(\mathcal{M}^{2+1}, g)$ be a complete, continuous and initially determined spacetime in $D \times (0,T)$. Let $u \in C^{2,1}(\mathcal{M})$ be a bounded solution to the LFDE with $u > 0$, and $v \in C^{2,1}(\mathcal{M})$ be the solution to the LFDE such that $g(t) = v(z,t) \cdot |dz|^2$. If for some $s \in (0,T)$ we have $v > u$ on $\mathcal{M}_{(0,s)}$, then $v \ge u$ on $\mathcal{M}$.*

Proof. Let $C := \sup_{\mathcal{M}} u < \infty$ and $\epsilon \in (0,s)$. Consider the region

$$\mathcal{M}' := \{(x,t) \in \mathcal{M} : t \in (\epsilon, T-\epsilon), H(x,t) < \frac{C+1}{2\epsilon}\}.$$

Since $\mathcal{M}$ is continuous, by Lemma 5.3.5, H is parabolically proper, and hence $\mathcal{M}' \Subset \mathcal{M}$. We note that $u, v \in C^{2,1}(\overline{\mathcal{M}'})$. On the region $\mathcal{M}_{(\epsilon,T-\epsilon)} \setminus \mathcal{M}'$, by the Schwarz lemma C.0.3, we have

$$v(z,t) \ge 2t \cdot H(z,t) \ge 2\epsilon \cdot H(z,t) \ge C+1 > u(z,t).$$

Since $\epsilon \in (0,s)$, we conclude that $v > u$ on the parabolic boundary $\partial_P \mathcal{M}'$. Applying the usual comparison principle for the LFDE to u and v on the region $\mathcal{M}'$ (Lemma 5.3.6), we deduce that $v \ge u$ on $\mathcal{M}_{(0,T-\epsilon)}$. To finish the proof take $\epsilon \searrow 0$. ☐

5.4 Complete conformal Ricci flows start weakly from a Radon measure

We now aim to find suitable lower barriers for our Ricci flow. In order to do so, we must make a small diversion into Ricci flows on Riemann surfaces starting from measures.

Given a connected Riemann surface M, denote the collection of all non-atomic, non-zero Radon measures on M by $\mathcal{R}(M)$. Due to the work of Topping & Yin in [TY21], these are precisely those measures which can be smoothed out using Ricci flow.

Theorem 5.4.1 (Topping-Yin, [TY21, Theorem 1.2]). *Suppose M is a connected Riemann surface and $\mu \in \mathcal{R}(M)$ is any non-zero, non-atomic Radon measure on M. Define*

$$T := \begin{cases} \frac{\mu(M)}{4\pi} & : \text{if } M = \mathbb{C} \\ \frac{\mu(M)}{8\pi} & : \text{if } M = S^2 \\ \infty & : \text{otherwise} \end{cases} \tag{5.4.1}$$

Then there exists a smooth complete conformal Ricci flow $g(t)$ on $M \times (0,T)$ starting weakly from μ. That is, the Riemannian volume measure $\mu_{g(t)} \rightharpoonup \mu$ as $t \searrow 0$.

We now present a converse to this theorem, which states that complete conformal Ricci flows always have some weak limit backwards in time.

Theorem 5.4.2. *Let $(M^2, g(t)_{t\in(0,T)})$ be a connected Riemann surface admitting a smooth complete conformal Ricci flow. Then there exists a radon measure μ on M such that $g(t)$ starts weakly from μ. That is*

$$d\mu_{g(t)} \rightharpoonup \mu, \quad \text{as } t \searrow 0.$$

Proof of Theorem 5.4.2

Given a Riemann surface M, we say that a compactly contained neighbourhood $U \Subset M$ is a conformal ball in M if for some $p \in M$ and $r > 0$, there exists a local complex coordinate z about p such that $U = B(p,r)$, the ball centred at p of radius r with respect to the locally defined metric $|dz|^2$.

The following is a result of Topping and Yin [TY21, Lemma 3.1] which stops complete Ricci flows in the plane from losing volume too quickly within a conformal ball. We make minor adjustments to this argument so that it works in the sphere, and for weak initial data.

Theorem 5.4.3 (*Variation of* Topping-Yin, [TY21, Lemma 3.1]). *Let $g(t)$ be a smooth instantaneously complete conformal Ricci flow on $S^2 \times (0,T)$ starting weakly from $\mu \in \mathcal{R}(S^2)$. Fix $0 < r < R < \infty$ and suppose $B_r \subseteq B_R$ are concentric conformal balls in the sphere. Then*

$$\mu(B_r) \leq \mathrm{Vol}_{g(t)}(B_R) + Ct, \quad \forall t \in \left[0, T \wedge \frac{1}{8\pi}\mathrm{Vol}_{g(0)}(B_r)\right),$$

where $C := \frac{8\pi}{1-(\frac{r}{R})^2}$ *is the explicit constant depending only on the ratio* r/R.

Proof. We first deal with the case $g(t)$ is smooth up to time zero. For each $n \in \mathbb{N}$ choose $r_n \in (r, R)$ such that $r_n \searrow r$ and define a new smooth metric g_n on S^2 such that

- $g_n \equiv g(0)$ on B_r.
- $g_n \leq g(0)$ on S^2.
- $g_n \leq \frac{1}{n} H_{r_n}$ on $S^2 \setminus B_{r_n}$.

where H_r denotes the complete hyperbolic metric on $S^2 \setminus B_r$. Let $g_n(t)$ be the instantaneously complete Ricci flow on S^2 starting from g_n. Since $\mathrm{Vol}_{g_n}(S^2) \geq \mathrm{Vol}_{g(0)}(B_r) =: v_0$, by Gauss-Bonnet, we have that each of the $g_n(t)$ exist for $t \in (0, \frac{v_0}{8\pi})$. By the maximum principle, we have that $g_n(t) \leq g(t)$ on $S^2 \times (0, T \wedge \frac{v_0}{8\pi})$. In particular, we have that for each $t \in (0, T \wedge \frac{v_0}{8\pi})$,

$$\begin{aligned}\mathrm{Vol}_{g(t)}(B_R) \geq \mathrm{Vol}_{g_n(t)}(B_R) &= \mathrm{Vol}_{g_n(t)}(S^2) - \mathrm{Vol}_{g_n(t)}(S^2 \setminus B_R) \\ &\geq v_0 - 8\pi t - (\frac{1}{n} + 2t)\mathrm{Vol}_{H_{r_n}}(S^2 \setminus B_R).\end{aligned}$$

By a direct calculation we have that

$$\mathrm{Vol}_{H_r}(S^2 \setminus B_R) = \frac{4\pi r^2}{R^2 - r^2},$$

and so taking $n \to \infty$ we have that, for each $t \in (0, \frac{v_0}{8\pi} \wedge T - t_0)$,

$$\mathrm{Vol}_{g(t)}(B_R) \geq \mathrm{Vol}_{g(0)}(B_r) - t(8\pi + 2\mathrm{Vol}_{H_r}(\mathbb{C} \setminus B_R) = \mathrm{Vol}_{g(0)}(B_r) - \frac{8\pi t}{1 - (\frac{r}{R})^2}. \tag{5.4.2}$$

This deals with the case g is initially smooth. For the general case of weak initial data, fix $\epsilon > 0$. Since μ is Radon, there exists $K \Subset B_r$ such that $\mu(K) \geq \mu(B_r) - \epsilon$. Choosing a test function f with support in B_r and equal to 1 on K, we have for δ sufficiently small

$$\mathrm{Vol}_{g(\delta)}(B_r) \geq \int f d\mu_{g(\delta)} \geq \int f d\mu - \epsilon \geq \mu(K) - \epsilon \geq \mu(B_r) - 2\epsilon. \tag{5.4.3}$$

For any fixed $t \in (0, T \wedge \frac{\mu(B_r)}{8\pi})$, we can choose $\delta \in (0, T)$ sufficiently small such that equation (5.4.3) holds as well as $t \in (\delta, T \wedge \frac{\mathrm{Vol}_{g(\delta)}(B_r)}{8\pi})$. Applying (5.4.2) to $g(t)$ on $S^2 \times [\delta, T)$ gives

$$\mu(B_r) \leq \mathrm{Vol}_{g(\delta)}(B_r) + 2\epsilon \leq \mathrm{Vol}_{g(t)}(B_R) + Ct + 2\epsilon.$$

Taking $\epsilon \searrow 0$ finishes the proof. $\square$

Although this lemma only deals with Ricci flows on the sphere, it is actually strong enough to show uniform volume bounds on conformal balls for any smooth complete conformal Ricci flow on any Riemann surface.

Lemma 5.4.4. *Let $(M^2, g(t)_{t\in(0,T)})$ be a connected Riemann surface admitting a smooth complete conformal Ricci flow with $T < \infty$. Then for any conformal ball $B \Subset M$, we have that*

$$\sup_{t\in(0,T)} \mathrm{Vol}_{g(t)}(B) < \infty.$$

Proof. If the lemma is false then there exists a conformal ball B and a sequence $t_n \in (0, T)$ such that $\mathrm{Vol}_{g(t_n)}(B) \to \infty$ as $n \to \infty$. Note that by the monotonicity of $t \mapsto u(\cdot, t)/t$ we have that

$$\mathrm{Vol}_{g(t)}(B) \le \left(\frac{t}{s}\right) \mathrm{Vol}_{g(s)}(B),$$

for any $0 < s < t < T$. So without loss of generality, after passing to a subsequence, we may assume that t_n is decreasing and null. Choose a slightly larger concentric conformal ball $B \subset B' \Subset M$. We first show the theorem in the case M is simply connected. By the uniformisation theorem, M is either the disk, the plane or the sphere. In all of these cases, we can find a biholomorphic embedding of M into the sphere S^2. In particular, we can view $M \subseteq S^2$, and the conformal balls B and B' in M remain conformal balls in S^2. For each $n \in \mathbb{N}$ let G_n be a smooth metric on S^2 such that G_n agrees with $g(t_n)$ on B and $G_n \le g(t_n)$ on M. Fix $t_0 \in (0, T)$ such that $\mathrm{Vol}_{G_n}(S^2) > 8\pi t_0$. Let $G_n(t)$ be the complete conformal Ricci flow on $S^2 \times [t_n, t_n + t_0]$ starting from G_n at time t_n. By the maximum principle, we have that $G_n(t) \le g_n(t)$ on $M \times [t_n, t_n + t_0]$. Since for sufficiently large n we have that $8\pi t_0 < \mathrm{Vol}_{g(t_n)}(B)$, we can apply Lemma 5.4.3 to give

$$\mathrm{Vol}_{G_n}(B) \le \mathrm{Vol}_{G_n(t_0)}(B') + C(t_0 - t_n),$$

for some fixed constant C independent of n. In particular we have that

$$\mathrm{Vol}_{g(t_n)}(B) = \mathrm{Vol}_{G_n}(B) \le \mathrm{Vol}_{G_n(t_0)}(B') + C(t_0 - t_n) \le \mathrm{Vol}_{g(t_0)}(B') + Ct_0.$$

However this yields a contradiction, as taking $n \to \infty$ makes the left hand side diverge to infinity, whereas the right hand side is a finite constant. This covers the case M is simply connected. For the general case, pullback the Ricci flow to the universal cover. Since the covering map is a local biholomorpism, the pullbacked Ricci flow is smooth, complete and conformal on the universal cover, and sufficiently small conformal balls in M lift to conformal balls in the universal cover. Therefore, using the simply connected case, we have the result for sufficiently small conformal balls in a general surface. The full result follows by covering any conformal ball with sufficiently small conformal balls and from the fact that conformal balls are compactly contained in the original surface M. $\square$

The local uniform bounds on volume in Lemma 5.4.4 allow the extraction of a convergent subsequence of the volume forms as time goes to zero.

Lemma 5.4.5. *Let $(M^2, g(t)_{t\in(0,T)})$ be a connected Riemann surface admitting a smooth com-*

plete conformal Ricci flow. Then there exists a positive radon measure μ on M and a null sequence $t_n \searrow 0$ such that

$$d\mu_{g(t_n)} \rightharpoonup \mu, \quad \text{as } n \to \infty.$$

Proof. Let B be a conformal ball in M. Consider the family of radon measures $\mu_t := d\mu_{g(t)}|_B$ on B. By Riesz-Markov-Kakutani (Theorem B.0.2), the measures μ_t can be viewed as elements of $C_c^0(B)^*$, the dual space of the compactly contained continuous functions on B. Choosing any null sequence t_n, Lemma 5.4.4 then implies that the family μ_{t_n} are bounded in $C_c^0(B)^*$ and so by Banach-Alaoglu (Theorem B.0.3) there exists a convergent subsequence $\mu_{t_n} \rightharpoonup \mu$ to some positive Radon measure on B. Instead of doing this on a single conformal ball, choose instead a locally finite cover $(B_j)_{j\in\mathbb{N}}$ of M consisting of conformal balls. Combining the above with a diagonal argument, we deduce that there exists a family of positive radon measures μ_j on each of the conformal balls B_j and a null sequence $t_n \searrow 0$ such that $\mu_{g(t_n)}|_{B_j} \rightharpoonup \mu_j$ as $n \to \infty$, for all $j \in \mathbb{N}$. Furthermore, since the cover was chosen to be locally finite, the Radon measures μ_j piece together to give a well defined positive radon measure μ on M, with $\mu_{g(t_n)} \rightharpoonup \mu$ as $n \to \infty$. □

Now that we have constructed a candidate for our initial measure, we use another estimate of Topping and Yin to give an upper bound on volume growth with respect to this measure.

Lemma 5.4.6 (*Variation of* Topping-Yin, [TY21, Lemma 3.2])**.** *Let M be a Riemann surface and let $g(t)$ be a smooth complete conformal Ricci flow on $M \times (0,T)$. Fix $0 < r < R < \infty$ and suppose $B_r \subseteq B_R$ are concentric conformal balls in M. Suppose there exists a positive Radon measure μ on M and a null sequence $t_n \searrow 0$ such that*

$$d\mu_{g(t_n)} \rightharpoonup \mu, \quad \text{as } n \to \infty.$$

Then there exists a constant $\eta < \infty$ depending only on the ratio r/R such that

$$\mathrm{Vol}_{g(t)}(B_r) \leq \mu(\overline{B_R}) + \eta t, \quad \forall t \in (0,T)\,.$$

Proof. Fix $\epsilon > 0$. Then since μ is Radon, there exists some open subset U containing $\overline{B_R}$ such that $\mu(U) \leq \mu(\overline{B_R}) + \epsilon$. Choosing a test function f with support in U and equal to 1 on B_R, we have for n sufficiently large

$$\mathrm{Vol}_{g(t_n)}(B_R) \leq \int f d\mu_{g(t_n)} \leq \int f d\mu + \epsilon \leq \mu(B) + \epsilon \leq \mu(\overline{B_R}) + 2\epsilon.$$

Applying [TY21, Lemma 3.2] to the Ricci flow on $B_R \times [t_n, T)$, we have that

$$\mathrm{Vol}_{g(t)}(B_r) \leq \mathrm{Vol}_{g(t_n)}(B_R) + \eta(t - t_n) \leq \mu(\overline{B_R}) + \eta t + 2\epsilon. \qquad \square$$

Proof of Theorem 5.4.2. Fix a conformal ball B in M and choose some $f \in C_c^\infty(B)$. As in the proof of Lemma 5.4.4, choose a slightly larger conformal ball $B \subset B' \Subset M$. Our Ricci flow is given locally by $g(t) = u(z,t)|dz|^2$ on $B' \times (0,T)$, with u solving the LFDE. Note that, by the monotonicity of $\frac{u(\cdot,t)}{t}$, there exists some $\epsilon > 0$ such that $u(\cdot,t) \geq \epsilon t$ on $B \times (0,t_1]$. Using Lemma 5.4.6 and the same argument as in [TY21, Lemma 4.5] we have that

$$\begin{aligned}
|\partial_t \int f d\mu_{g(t)}| = |\int f\Delta \log(u) dz| = |\int \Delta f \log(u) dz| \\
&\leq \|\Delta f\|_\infty \int_B |\log(u)| dz \\
&\leq \|\Delta f\|_\infty \left(\mathrm{Vol}_{g(t)}(B) + |\log(\epsilon t)|\right) \\
&\leq \|\Delta f\|_\infty \left(\mu(\overline{B'}) + \eta t + |\log(\epsilon t)|\right),
\end{aligned}$$

for some fixed constant η independent of $t \in (0,t_1]$. In particular we have

$$\begin{aligned}
\limsup_{t\to 0} |\int f d\mu_{g(t)} - \int f d\mu| &\leq \limsup_{t\to 0} \liminf_{n\to\infty} |\int f d\mu_{g(t)} - \int f d\mu_{g(t_n)}| \\
&\leq \limsup_{t\to 0} \liminf_{n\to\infty} \int_{t_n}^t |\partial_s \int f d\mu_{g(s)}| ds \\
&\leq \limsup_{t\to 0} \int_0^t \|\Delta f\|_\infty \left(\mu(\overline{B'}) + \eta s + |\log(\epsilon s)|\right) ds = 0.
\end{aligned}$$

We have shown that, for any conformal ball B in M,

$$\lim_{t\searrow 0} \int f d\mu_{g(t)} = \int f d\mu, \quad \forall f \in C_c^\infty(B). \tag{5.4.4}$$

If we now take $f \in C_c^0(B)$, choose B'' a concentric conformal ball such that $B \subset B'' \subset B'$. Since f is uniformly continuous, for $\epsilon > 0$ sufficiently small, we can mollify f to give $f_\epsilon \in C_c^\infty(B'')$ with $\|f - f_\epsilon\|_\infty < \epsilon$. Using Lemma 5.4.6 again gives

$$\begin{aligned}
\limsup_{t\to 0} |\int f d\mu_{g(t)} - \int f d\mu| &\leq \limsup_{t\to 0} \left(\int |f - f_\epsilon| d\mu_{g(t)} + \int |f - f_\epsilon| d\mu\right) \\
&\leq \limsup_{t\to 0} \epsilon \cdot \left(\mathrm{Vol}_{g(t)}(B'') + \mu(B'')\right) \leq 2\epsilon\mu(\overline{B'}).
\end{aligned}$$

Taking $\epsilon \searrow 0$, we have improved (5.4.4) to

$$\lim_{t\searrow 0} \int f d\mu_{g(t)} = \int f d\mu, \quad \forall f \in C_c^0(B).$$

Finally, using a locally finite cover of M by conformal balls and a partition of unity we deduce

$$\lim_{t\searrow 0} \int f d\mu_{g(t)} = \int f d\mu, \quad \forall f \in C_c^0(M). \qquad \square$$

5.5 Initial time blow-ups

Let $(M, g(t))_{t\in(0,T)}$ be a Riemann surface equipped with a complete and conformal Ricci flow. Since the flow is complete, we have the lower scalar curvature bound $R_{g(t)} \geq -\frac{1}{t}$, and hence $t \mapsto \frac{g(t)}{t}$ is decreasing. It is then immediate that we have a well-defined (potentially infinite) limiting metric as time approaches zero.

Definition 5.5.1. Let $(M, g(t))_{t\in(0,T)}$ be a Riemann surface equipped with a complete and conformal Ricci flow. The *initial time blow-up* of $g(t)$ is the (possibly infinite) conformal metric $\widehat{g}$ on M, defined by

$$\widehat{g} := \lim_{t\searrow 0} \frac{g(t)}{2t}.$$

Remark 5.5.2. Suppose $\phi : N \to M$ is a local biholomorphism. Then if $g(t)$ is a complete and conformal Ricci flow on M with initial time blow-up $\widehat{g}$, we have that $\phi^*(g(t))$ is a complete and conformal Ricci flow on N, and its initial time blow-up is $\widehat{(\phi^* g)} = \phi^*(\widehat{g})$.

Given a measure $\mu \in \mathcal{R}(M)$, we now investigate what the initial time blow-up of a Ricci flow starting from this measure looks like. The following theorem shows that, away from the support of the initial measure, the initial time blow-up is a hyperbolic metric.

Theorem 5.5.3. *Let $g(t)$ be a complete and conformal Ricci flow on $M \times (0, T)$ starting weakly from some $\mu \in \mathcal{R}(M)$. Then the initial time blow-up $\widehat{g}$ of $g(t)$ is a well-defined smooth metric on $M \setminus \operatorname{supp}(\mu)$ with constant Gaussian curvature $K_{\widehat{g}} \equiv -1$.*

Proof of Theorem 5.5.3

One of the important steps to prove this theorem is to show that our initial time blow-up us finite away from the support of the initial measure μ. To do this, we need the following $L^1 - L^\infty$ smoothing result for solutions to Ricci flow by Topping & Yin [TY21, Theorem 2.1] generalised slightly for weak initial data.

Theorem 5.5.4 (*Variation of* Topping-Yin, [TY21, Theorem 2.1])**.** *Suppose $g(t) = u(z,t)|dz|^2$ is a smooth conformal Ricci flow on the ball $B_{3r} \times (0, T)$, for some $r > 0$. Suppose there exists a non-atomic, non-zero Radon measure $\mu \in \mathcal{R}(B_{3r})$ such that $g(t)$ starts weakly from μ. If $t \in (0, T)$ satisfies $t > \frac{\mu(\overline{B_{2r}})}{2\pi}$ then*

$$\sup_{B_r} u(t) \leq C_0 r^{-2} t,$$

where $C_0 < \infty$ is universal.

Proof. Since μ is Radon, for some $\delta > 0$, $\frac{\mu(B_{2r+\delta})}{2\pi} < t$. Choose a cut off function f such that

$f \equiv 1$ on $\overline{B_{2r}}$ and $\operatorname{supp}(f) \subseteq B_{2r+\delta}$. For $\epsilon > 0$ sufficiently small

$$\frac{\mathrm{Vol}_{g(\epsilon)}(B_{2r})}{2\pi} \leq \frac{\int f d\mu_{g(\epsilon)}}{2\pi} \leq \frac{\int f d\mu}{2\pi} + \left(t - \frac{\mu(B_{2r+\delta})}{2\pi}\right) \leq t.$$

Therefore applying the estimate in the smooth case [TY21, Theorem 2.1] to the Ricci flow on $B_{2r} \times (\epsilon, T)$ gives

$$\sup_{B_r} u(\epsilon + t) \leq C_0 r^{-2} t.$$

Since u is smooth away from zero, taking ϵ to zero gives the result. $\square$

The following lemma is a direct consequence of the previous result.

Lemma 5.5.5. *Let $(M, g(t))_{t\in(0,T)}$ be a Riemann surface equipped with a smooth complete conformal Ricci flow. Suppose there exists a non-atomic, non-zero Radon measure $\mu \in \mathcal{R}(M)$ such that $g(t)$ starts weakly from μ. Then the initial time blow-up $\widehat{g}$ of $g(t)$ is a well-defined smooth metric on $M \setminus \operatorname{supp}(\mu)$.*

Proof. For any point $p \in M \setminus \operatorname{supp}(\mu)$, choose a complex coordinate z on a neighbourhood of this point and choose $r > 0$ sufficiently small such that $B_{3r} \subseteq M \setminus \operatorname{supp}(\mu)$, where B_{3r} denotes the ball centred at p radius $3r$ with respect to the metric $|dz|^2$. Write $g(t) = u(t)|dz|^2$ on B_{3r}. Since $\mu(\overline{B_{2r}}) = 0$, the conclusion of Theorem 5.5.4 will apply at all positive times

$$\sup_{B_r} u(t) \leq C_0 r^{-2} t, \quad \forall t \in (0, T).$$

Hence

$$\widehat{u}(p) := \lim_{t\searrow 0} \frac{u(p,t)}{2t} \leq C_0 r^{-2} < \infty. \qquad \square$$

Proof of Theorem 5.5.3. Consider the parabolically rescaled Ricci flows $(M, g_m(t)_{t\in(0,mT)})$ given by

$$g_m(t) := m g(tm^{-1}), \quad \forall m \in \mathbb{N},$$

so that $g_1(t) \equiv g(t)$. We note that $g_m(t)$ is a complete and conformal Ricci flow starting weakly from the measure $m \cdot \mu \in \mathcal{R}(M)$. Given a local complex coordinate z on M, our metrics are given locally by $g_m(t) = u_m(z,t)|dz|^2$ with the conformal factors u_m satisfying the relation

$$u_m(z,t) := m u_1(z, tm^{-1}).$$

By the monotonicity of $t \mapsto t^{-1}u_1(\cdot, t)$, we have that $u_m(z,t)$ is an increasing sequence in m, for any fixed (z,t). In particular, we have the uniform lower bound $u_1(z,t)$ on our sequence of conformal factors. Moreover, we have local uniform upper bounds on our sequence u_m in $M \setminus \operatorname{supp}(\mu)$ by Lemma 5.5.5. Combining the lower and upper bounds with standard parabolic theory, we deduce local C^k-bounds on our sequence of conformal factors, for any $k \in \mathbb{N}$. Finally, by Arzela-Ascoli, our sequence of Ricci flows $g_m(t)$ converges locally smoothly to some limiting eternal Ricci flow $(M \setminus \operatorname{supp}\mu, g_\infty(t)_{t\in(0,\infty)})$. For any fixed $t>0$, we note that

$$g_\infty(t) := \lim_{m\to\infty} g_m(t) = \lim_{m\to\infty} 2t \cdot \frac{g(tm^{-1})}{2tm^{-1}} = 2t\cdot \widehat{g},$$

and so $g_\infty(t) = 2t\cdot\widehat{g}$. Substituting this into the Ricci flow equation

$$2\cdot\widehat{g} = \frac{\partial g_\infty}{\partial t} = -2\operatorname{Ric}(g_\infty) = -2K_{\widehat{g}}\cdot\widehat{g},$$

and hence $K_{\widehat{g}} \equiv -1$. $\square$

Example 5.5.6. In [TY21] the following expanding, non-gradient soliton $(\mathbb{C}, \frac{2}{1+x^2}|dz|^2, x\partial_x + y\partial_y)$ was constructed. The complete conformal Ricci flow associated to this soliton is $g(t) := \frac{2t}{t^2+x^2}|dz|^2$. As such, the initial time blow-up of this soliton is

$$\widehat{g} = \begin{cases} \frac{1}{x^2}|dz|^2 & : x\neq 0 \\ \infty & : x = 0 \end{cases}$$

We note that the soliton has weak initial data

$$\mu_{g(t)} \rightharpoonup \mu := 2\pi\mathcal{H}^1 \llcorner \{x=0\}, \quad t\searrow 0.$$

In particular, $\widehat{g}$ is the complete hyperbolic metric on the complement of the support of μ.

Example 5.5.7. Consider the expanding soliton $g(t) = u(z,t)|dz|^2$ on $\mathbb{C}$ starting weakly from the measure μ given by the Lebesgue measure restricted to the half space $\{x = Re(z) \geq 0\}$. By Lemma 5.5.3 we know that $\widehat{g}$ is a hyperbolic metric on $\{x<0\}$. By symmetry u depends only on the x-coordinate. Let $\phi_\lambda : \mathbb{C}\to\mathbb{C}$ denote the dilation $z\mapsto \lambda\cdot z$, for any $\lambda>0$. Note that both $\phi_\lambda^*(g(t))$ and $\lambda^2 g(\frac{t}{\lambda^2})$ are complete conformal Ricci flows, starting weakly from $\lambda\cdot\mu$. As μ is given by L^1_{loc}-data, a result of Topping and Yin [TY21, Theorem 1.4] implies that these two flows are the same. Setting $\lambda = t^{-1/2}$ and rearranging, we deduce that

$$g(t) = t\phi^*_{\frac{1}{\sqrt{t}}} g(1), \quad \forall t>0.$$

As such, we know that the conformal factor satisfies

$$u(x,t) = u(x\cdot t^{-\frac{1}{2}}, 1), \quad \forall (x,t)\in\mathbb{R}\times(0,\infty).$$

In particular, we have the lower bound

$$\widehat{u}(x) \geq \frac{u(x,|x|^2)}{2|x|^2} = \frac{u(-1,1)}{2|x|^2}, \quad \forall x<0,$$

and hence $\widehat{g}$ is the complete hyperbolic metric on the complement of the $\operatorname{supp}(\mu)$.

Proof of Theorem 5.3.3

We now have all of the necessary ingredients to prove Theorem 5.3.3, and hence Theorem 1.3.6.

Theorem 5.3.3. *Suppose $(\mathcal{M}^{2+1}, g)$ is a complete, continuous and initially determined space-time in $D\times(0,T)$ with $D = \cup_{t\in(0,T)}\mathcal{M}_t$, and where D is the disk equipped with its hyperbolic conformal structure. Suppose further that g is a conformal Ricci flow on $\mathcal{M}$. Then $\mathcal{M} = D\times(0,T)$.*

Proof. Fix $t_1 \in (0,T)$. By the Cantor-Bendixson theorem [Kec95], we can partition the set $\overline{D} \setminus \mathcal{M}_{t_1} = P \sqcup X$ into a perfect set P and a scattered set X. In particular we have the inclusions:

$$\mathcal{M}_{t_1} \subseteq \mathcal{M}_{t_2} \subseteq \mathcal{M}_{t_1} \sqcup P \sqcup X, \quad \forall t_2 \in (t_1, T). \tag{5.5.1}$$

Due to a result of Hebert & Lacey ([HL68], Lemma B.0.1), as P is a compact perfect subset of the plane, there exists a Radon measure $\mu \in \mathcal{R}(\mathbb{C})$ such that $\mathrm{supp}(\mu) = P$. Let $G(t) = u(z,t)\cdot|dz|^2$ be a complete conformal Ricci flow on $\mathbb{C}\times(0,T)$ starting weakly from $\mu \in \mathcal{R}(\mathbb{C})$ with initial time blow-up $\widehat{G}$. For any $\lambda > 0$, we note that the parabolically rescaled Ricci flow $G_\lambda(t) := \lambda G(\frac{t}{\lambda}) = \lambda u(z, \frac{t}{\lambda})|dz|^2$ is a complete conformal Ricci flow on $\mathbb{C}\times(0,\lambda T)$ starting weakly from $\lambda\mu \in \mathcal{R}(\mathbb{C})$, with the same initial time blow-up $\widehat{G}$.

Fix $t \in (0,t_1)$. By the monotonicity of $t \mapsto G_\lambda(t)/t$, $G_\lambda(t) \le 2t\cdot\widehat{G}$ within $\mathbb{C}\setminus P$. Since $\mathcal{M}$ is expanding, $\mathcal{M}_t \subseteq \mathcal{M}_{t_1} \subseteq \mathbb{C}\setminus P$, and $\widehat{G}$ is defined on all of $\mathcal{M}_t$. Applying Lemma 5.5.3 and the Schwarz Lemma C.0.3, we deduce that $\widehat{G} \le h(t)$ on $\mathcal{M}_t$. Note that, if we had equality between the metrics $\widehat{G}$ and $h(t)$ at any point inside of $\mathcal{M}_t$, then this would imply that the subharmonic function $\log(h(t)/\widehat{G})$ has a minimum in its interior, and hence by the elliptic maximum principle, $\widehat{G} \equiv h(t)$ in $\mathcal{M}_t$. This would contradict the fact that $\widehat{G}$ is defined on a larger domain. Therefore, we have the strict inequality $\widehat{G} < h(t)$ on $\mathcal{M}_t$. Finally, we can use equation (5.3.1) together with the previous inequalities to deduce that

$$\frac{G_\lambda(t)}{2t} \le \widehat{G} < h(t) \le \frac{g(t)}{2t}, \quad \forall t \in (0,t_1).$$

Therefore, $g > G_\lambda$ on $\mathcal{M}_{(0,t_1)}$. As G_λ is bounded on $\mathcal{M}$, we can then apply Lemma 5.3.7 to conclude that $g \ge G_\lambda$ on all of $\mathcal{M}$, for any $\lambda > 0$. Suppose at some later time $t_2 \in (t_1, T)$ we have that $\mathcal{M}_{t_2} \cap P \neq \emptyset$. That is, there exists some point $p \in P$ such that $(p, t_2) \in \mathcal{M}$. Since $\mathcal{M}$ is open, there exists some small $r > 0$ such that the conformal ball $B(p,r) \Subset \mathcal{M}_{t_2}$. By compactness, $\mathrm{Vol}_{g(t_2)}(B(p,r)) < \infty$. We now derive a contradiction by showing that the volume of this ball with respect to the metric $G_\lambda(t_2)$ blows up to infinity as $\lambda \to \infty$. Indeed, $\mu(B(p,r)) > 0$ by the definition of $p \in P = \mathrm{supp}(\mu)$. Using that the volume of a ball can't decrease too rapidly in a Ricci flow (see Lemma 5.4.3), we have that

$$\mathrm{Vol}_{G(t)}(B(p,r)) \ge \mu(B(p,r)) - Ct, \quad \forall t \in \left(0, T \wedge \frac{\mu(B(p,r))}{8\pi}\right),$$

which means for λ sufficiently large

$$\mathrm{Vol}_{G_\lambda(t_2)}(B(p,r)) = \lambda \cdot \mathrm{Vol}_{G(\frac{t_2}{\lambda})}(B(p,r)) \geq \lambda \cdot \left(\mu(B(p,r)) - \frac{Ct_2}{\lambda}\right) \geq \frac{\lambda}{2} \cdot \mu(B(p,r)).$$

Taking $\lambda \nearrow \infty$, this contradicts the volume of the ball with respect to $g(t_2)$ being finite. Therefore, we have shown that $\mathcal{M}_{t_2} \cap P = \emptyset$, which means that (5.5.1) reduces to

$$\mathcal{M}_{t_1} \subseteq \mathcal{M}_{t_2} \subseteq \mathcal{M}_{t_1} \sqcup X, \quad \forall t_2 \in (t_1, T).$$

Since $\partial D \subseteq \mathbb{C}$ is a perfect subset of the plane, we see that $X \subseteq D$. If X was non-empty, then because it is a scattered set, it must contain an isolated point, which would contradict Corollary 5.2.6. We can therefore conclude that X must be empty, and that $\mathcal{M}_{t_1} = \mathcal{M}_{t_2}$. $\square$

Appendix A

Parabolic equations

Recall the notation $\Omega := (a,b) \Subset \mathbb{R}$, $\Omega_T := \Omega \times (0,T)$, $\Gamma_T := (\Omega \times \{0\}) \cup (\partial\Omega \times [0,T))$. Consider the linear operator

$$\mathcal{L}(u) := u_t - A(x,t)u_{xx} + B(x,t)u_x + C(x,t)u, \tag{A.0.1}$$

where A, B, C are bounded functions on Ω_T, with $A(x,t) > 0$, and $C(x,t) \geq -C_0$.

Theorem A.0.1 (Maximum principle, [LSU88, Chaper II, Theorem 2.1])**.** *Fix $(x^*,t^*) \in \Omega_T$. If $u \in C^0(\overline{\Omega} \times [0,T)) \cap P^2_{loc}(\Omega_T)$ satisfies $\mathcal{L}(u) \leq 0$ on Ω_T, then*

$$u(x^*,t^*) \leq \max\{0, \sup_{\Gamma_{t^*}}(ue^{C_0(t^*-t)})\}.$$

Alternatively if $\mathcal{L}(u) \geq 0$ on Ω_T, then

$$u(x^*,t^*) \geq \min\{0, \inf_{\Gamma_{t^*}}(ue^{C_0(t^*-t)})\}$$

Theorem A.0.2 (De Giorgi-Nash-Moser, [LSU88, Chapter III, Theorem 10.1])**.** *Let $u \in C^0(\Omega_T) \cap P^2_{loc}(\Omega_T)$ be a solution of $\mathcal{L}u = 0$ on Ω_T such that the coefficients of $\mathcal{L}$ satisfy*

$$\|A\|_{L^\infty(\Omega_T)}, \|A^{-1}\|_{L^\infty(\Omega_T)}, \|B\|_{L^\infty(\Omega_T)}, \|C\|_{L^\infty(\Omega_T)} \lesssim 1.$$

Fix $K \Subset \Omega_T$. Then there exists

$$\alpha(\|A\|_{L^\infty(\Omega_T)}, \|A^{-1}\|_{L^\infty(\Omega_T)}) \in (0,1),$$

and a constant

$$C(K, \Omega_T, \|u\|_{L^\infty(\Omega_T)}, \|A\|_{L^\infty(\Omega_T)}, \|A^{-1}\|_{L^\infty(\Omega_T)}, \|B\|_{L^\infty(\Omega_T)}, \|C\|_{L^\infty(\Omega_T)}) > 0,$$

such that

$$\|u\|_{P^{0,\alpha}(K)} \leq C.$$

Given $u_0 \in C^{2,\alpha}(\Omega)$, $\psi \in P^{2,\alpha}(\Omega_T)$ and $f \in P^{0,\alpha}(\Omega_T)$, consider the Dirichlet problem

$$\begin{cases} \mathcal{L}(u) = f & \text{in } \Omega_T \\ u = u_0 & \text{on } \overline{\Omega} \times \{0\} \\ u = \psi & \text{on } \{a, b\} \times [0, T] \end{cases} \tag{A.0.2}$$

Theorem A.0.3 (Global Schauder estimate, [LSU88, Chapter IV, Theorem 5.2])**.** *Fix $\alpha \in (0,1]$ and $k \in \mathbb{N}_0$. Suppose $A, B, C, f \in P^{k,\alpha}(\Omega_T)$, $\psi \in P^{2+k,\alpha}(\Omega_T)$, and $u_0 \in C^{2+k,\alpha}(\Omega)$, with auxiliary data satisfying the compatibility conditions of orders $0, \ldots, k$. Then there exists a unique $u \in P^{2+k,\alpha}(\Omega_T)$ solving (A.0.2). Furthermore, there exists a constant $C(\Omega, \lambda, k, \alpha) > 0$ such that*

$$|u|_{P^{2+k,\alpha}(\Omega_T)} \leq C\left(|u_0|_{C^{2+k,\alpha}(\Omega)} + |\psi|_{P^{k+2,\alpha}(\Omega_T)} + |f|_{P^{k,\alpha}(\Omega_T)}\right).$$

Without the regularity at the boundary, if our coefficients and forcing term are regular, we can still deduce regularity of our solution on the interior

Theorem A.0.4 (Interior Schauder estimate, [LSU88, Chapter III, Theorem 12.1], [LSU88, Chapter IV, Theorem 10.1])**.** *Fix $\alpha \in (0,1]$ and $k \in \mathbb{N}_0$. Suppose $A, B, C, f \in P^{k,\alpha}(\Omega_T)$ and $u \in P^{2,\alpha}(\Omega_T)$ solving $\mathcal{L}(u) = f$ in Ω_T. Then $u \in P^{2+k,\alpha}_{loc}(\Omega_T)$. Moreover, for any $K \Subset \Omega_T$, there exists a constant $C(\Omega, \lambda, k, \alpha, K) > 0$ such that*

$$|u|_{P^{2+k,\alpha}(K)} \leq C\left(|u|_{P^{2,\alpha}(\Omega_T)} + |f|_{P^{k,\alpha}(\Omega_T)}\right).$$

Appendix B

Measure theory and analysis

Lemma B.0.1 (Hebert-Lacey, [HL68, Corollary 2.8]). *If X is a perfect and compact subset of a Riemann surface M, then there exists a non-atomic Radon measure $\mu \in \mathcal{R}(M)$ with* $\mathrm{supp}(\mu) = X$.

Theorem B.0.2 (Riesz-Markov-Kakutani, [Hal13, Section 56, Theorem D]). *Let X be a locally compact Hausdorff topological space. For every positive linear functional $\psi \in C_c^0(X)^*$, there exists a unique Radon measure μ on X such that*

$$\psi(f) = \int_X f d\mu, \quad \forall f \in C_c^0(X).$$

Theorem B.0.3 (Banach-Alaoglu, [Kes09, Theorem 5.2.1]). *Let X be a Banach space. Then the closed unit ball in the dual space X^* is weak*-compact.*

Theorem B.0.4 (Whitney, [Ste70, Chapter VI, Theorem 4]). *Let $F \subseteq \mathbb{R}^n$ be a closed set and $u \in C^{k,\alpha}(F)$, for some $k \in \mathbb{N}_0$ and $\alpha \in (0,1]$. Then there exists an extension $\widehat{u} \in C^{k,\alpha}(\mathbb{R}^n)$ so that*

$$\widehat{u}(x) = u(x), \quad \forall x \in F$$
$$\|\widehat{u}\|_{C^{k,\alpha}(\mathbb{R}^n)} \lesssim_{k,\alpha} \|u\|_{C^{k,\alpha}(F)}.$$

Appendix C

Geometry

Theorem C.0.1 (Gauss-Bonnet, [Cha06, Theorem V.2.7]). *Let (M^2, g) be a compact, orientable Riemannian surface with smooth boundary ∂M. Then*

$$\int_M K_g d\mu_g + \int_{\partial M} \kappa ds = 2\pi\chi(M),$$

where K_g denotes the Gaussian curvature of M, κ denotes the geodesic curvature of the boundary of M, $d\mu_g$ is the volume form of g, $ds = \iota_\nu d\mu_g$ is the arc-length form of the boundary, and $\chi(M)$ is the Euler-characteristic of M.

Theorem C.0.2 (Uniformisation Theorem, [Koe10]). *Every connected Riemann surface is conformally covered by either the sphere S^2, the plane $\mathbb{C}$ or the unit disk D.*

Lemma C.0.3 (Schwarz Lemma, [Yau73]). *Let (M_1, g_1), (M_2, g_2) be Riemannian surfaces. If*

1. *(M_1, g_1) is complete,*
2. *the Gaussian curvature $K(g_1) \geq -a_1$, for some $a_1 \geq 0$,*
3. *$K(g_2) \leq -a_2$, for some $a_2 > 0$,*

then any conformal map $f : M_1 \to M_2$ satisfies the inequality.

$$f^*(g_2) \leq \frac{a_1}{a_2} g_1.$$

The following lemma is used in Section 5.2 when constructing the ambient space for our spacetime. Although the proof of such an argument is standard, we include the details here.

Lemma C.0.4. *Let $\{M_i\}_{i\in\mathbb{N}}$ be a smooth family of manifolds which embed smoothly into one* *another $M_i \hookrightarrow M_j$, for $i \leq j$. Then the direct limit:*

$$M = \lim_{\rightarrow} M_i := \bigsqcup_{i \in \mathbb{N}} M_i \Big/ \sim \ , \tag{C.0.1}$$

where $x \sim y$ iff $x = y$ within an embedding, can be made into a smooth manifold. Moreover, the canonical maps $f_i : M_i \hookrightarrow M$, $x \mapsto [x]$ will be smooth embeddings.

Proof. Equip the set M with the final topology, so that $U \subseteq M$ is open iff $f_i^{-1}(U)$ is open, for all $i \in \mathbb{N}$. In particular, each f_i is now a continuous injection. Let ψ_{ij} denote the embedding $M_i \hookrightarrow M_j$. Note that

$$f_i = f_j \circ \psi_{ij}, \quad \forall i \leq j.$$

In particular, for any open subset $U \subseteq M_i$, we have

$$\begin{aligned} f_j^{-1} \circ f_i(U) = \psi_{ji}^{-1} \circ f_i^{-1} \circ f_i(U) = \psi_{ji}^{-1}(U), \quad \text{if } j < i \\ f_j^{-1} \circ f_i(U) = f_j^{-1} \circ f_j \circ \psi_{ij}(U) = \psi_{ij}(U), \quad \text{if } j \geq i. \end{aligned}$$

That is, $f_i(U)$ is open in M and f_i is an embedding.

Choosing a countable base $\{B_{ij} : j \in \mathbb{N}\}$ of M_i for each $i \in \mathbb{N}$, the collection

$$\mathcal{B} := \{f_i(B_{ij}) \subseteq M : i, j \in \mathbb{N}\},$$

is then a countable base of M. M being Hausdorff follows from the embedded subspaces M_i being Hausdorff. Finally, for each $j \in \mathbb{N}$, consider the atlas $\{c_\alpha : U_\alpha \subset M_j \hookrightarrow \mathbb{R}^n\}_{\alpha \in A_j}$ on M_j. Define the new collection of charts $\{\tilde{c}_\alpha : \tilde{U}_\alpha \subset M \hookrightarrow \mathbb{R}^n\}_{\alpha \in A_j}$ by

$$\tilde{U}_\alpha := f_j(U_\alpha), \quad \tilde{c}_\alpha := c_\alpha \circ f_j^{-1}.$$

The union over $j \in \mathbb{N}$ of all such charts gives a well defined smooth atlas on M. With respect to this smooth structure, the canonical maps are smooth. $\square$

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