Rates for maps and flows in a deterministic multidimensional weak invariance principle

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Contents

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Declarations

The content of this thesis is based on three papers, two of which are already published [42, 43]. The third one will be submitted solo-authored and represents the main content of this thesis: it is presented in Chapter 3. Chapters 4 and 5 present respectively the results of [43] and [42], published in collaboration with Dalia Terhesiu and Ian Melbourne.

It is crucial to remark that originally [42, 43] were submitted as a unique paper and then split for publication following the referee's suggestion. I was the main author of [43], and later put in some work to understand all the arguments of [42]. However, the results and proofs in [42] are not part of the author's main work and are presented in the thesis for completeness.

This thesis is submitted to the University of Warwick in support of my application for the degree of Doctor of Philosophy. It has been composed by myself and has not been submitted in any previous application for any degree.

Abstract

The work in this thesis concerns the branch of dynamics known as smooth ergodic theory. When a dynamical system and a probability measure are well-behaved, one can expect regular observables (typically Hölder continuous) to satisfy statistical properties that go beyond the classical ergodic theorem. Such results include the central limit theorem and its functional version, also called the weak invariance principle. The latter is analysed in the first (and main) part of this thesis, where we find rates of convergence to a Brownian motion in d-dimensions for discrete and continuous time systems. The proofs rely on a connection between dynamical system and martingale theory, via the martingale-coboundary decomposition introduced by Gordin [26].

The second part of the thesis presents results from two papers [42, 43] published by the author with Melbourne and Terhesiu, which discuss the decay of the transfer operator in continuous time. The chapter dedicated to [43] shows a restriction on the Banach spaces where such a transfer operator can have a spectral gap. The last chapter presents [42] and proves an exponential decay of the transfer operator in a strong norm for a class of nonuniformly expanding semiflows.

Chapter 1

Introduction

The study of chaos has always posed an interesting challenge to mathematicians. Rather than analysing each orbit of an evolving system, ergodic theory shifted its focus on qualitative properties that are true almost everywhere with respect to some invariant measure. The general philosophy that deterministic chaos shares many aspects with randomness gives a natural application to many standard probability results. It is also of great interest when statistical laws arise from a dynamical setting. This kind of analysis began in the 1970s with the work of Bowen, Ruelle, Sinai, and it still offers many interesting aspects to the modern researchers.

This thesis analyses how quickly some chaotic systems display randomness, somehow answering the question "How long does it take for an expanding system to generate Brownian motion?". For this purpose, Chapter 2 recaps general facts from dynamics and probability. Chapter 3 is the main content of this thesis, improving the existing rates of convergence for maps and presenting the first known results on this matter for flows and multidimensional observables. Chapter 4 shows a restriction on the decay in norm of a family of transfer operators, in contrast to what happens in discrete time. Finally, Chapter 5 provides an exponential decay in a Hölder norm for a particular class of observables on a suspension semiflow.

In Chapter 3, we focus on rates of convergence for a deterministic version of a classical result of Donsker [23], which states the convergence in distribution of a normalised i.i.d. random walk to a Brownian motion; we refer to this as the weak invariance principle (in brief WIP). The WIP was proved for various nonuniformly hyperbolic/expanding maps in [28] and for uniformly hyperbolic flows in [20]. More recent results for the WIP in a nonuniformly hyperbolic setting are [10, 27, 40, 46, 47].

Let $T\colon \Lambda \to \Lambda$ be a map on a bounded metric space Λ with an invariant measure μ and let $v\colon \Lambda \to \mathbb{R}^d$ be a regular observable. Define the sequence of continuous processes W_n as $W_n(k/n) = n^{-\frac{1}{2}} \sum_{j=0}^{k-1} v \circ T^j$, $0 \le k \le n$, and linear interpolation in $[0,1]$. We say that such a system satisfies the WIP if W_n converges in distribution as $n \to \infty$ to a Brownian motion W.

In the case of a measure-preserving flow $T_t\colon \Lambda \to \Lambda$, the sequence is defined as

$$W_n(t) = \frac{1}{\sqrt{n}} \int_0^{nt} v \circ T_s \, \mathrm{d}s, \quad t \in [0,1]. \tag{1.1}$$

This thesis analyses how well the laws of W_n on the space of continuous functions approximate the Wiener measure given by W.

For nonuniformly expanding/hyperbolic maps modelled by Young towers with exponential tails, Antoniou and Melbourne [5] proved a convergence rate of $\mathcal{O}(n^{-1/4+\delta})$ in the Prokhorov metric, while Liu and Wang [35] proved the same rates in the q-Wasserstein metric for $q > 1$, where δ gets smaller for bigger q. Their method is based on a generalisation [32] of the martingale-coboundary decomposition technique of Gordin [26], which allows to apply a martingale version of the Skorokhod embedding Theorem. It is known [11, 53] that such a method cannot yield better rates than $\mathcal{O}(n^{-1/4})$. Moreover, such a result is applicable to real-valued observables only, and we are not aware of a better method to get rates for the WIP in dimension one.

To our knowledge, the literature does not have any further results on the rates of convergence in the WIP for deterministic systems. Hence, our Chapter 3 gives the first multidimensional results for maps and, in particular, covers the case of nonuniformly expanding semiflows. When $d = 1$, we get a rate of $\mathcal{O}(n^{-1/4}(\log n)^{3/4})$ in Prokhorov for uniformly expanding maps and semiflows, improving the one of [5] in discrete time. For nonuniformly expanding semiflows, we recover the same rate of [5]. For $d \ge 1$ with exponential tails, we are able to achieve a rate of $\mathcal{O}(n^{-1/6+\delta})$ in the 1-Wasserstein metric, independently of the dimension (which yields $\mathcal{O}(n^{-1/12+\delta})$ in Prokhorov).

One of the main challenges in Chapter 3 was to find a way to adapt results from general martingale theory [18, 19, 33] to a continuous time setting. For maps, we could follow the same strategy of [5, 35] and rely on an advanced adaptation [32] of the martingale-coboundary decomposition introduced by Gordin [26]. However, for semiflows $T_t\colon \Lambda \to \Lambda$ it was necessary to generalise [32] to continuous time; this original work is found in Section 3.3.

It is important to mention that part of the author's PhD was dedicated (without success) to find a more direct proof of the results in continuous time of Chapter 3. Such a proof would have relied on ideas of a paper by Pène [52], where Berry-Esseen estimates for a billiard flow follow from the ones for the map. At a first glance, it seemed feasible to extend such a method to the WIP; yet, many difficulties came in the way. Even though some rates can be found, there are two main issues: (i) the rates are equal or worse than the ones obtained in Chapter 3 and (ii) the proof is more complex. Regarding (ii), the hardest part concerns passing the rates between two different measures, one of which is not invariant for the semiflow. Such an issue is sorted in [52] using methods that are not applicable to our setting. This thesis does not display the attempt to adapt [52], leaving it for a potential future research.

In Section 3.3 we start from $v\colon \Lambda \to \mathbb{R}^d$ Hölder continuous with mean 0. We find an extension $F_t\colon Y^\varphi \to Y^\varphi$ of $T_t\colon \Lambda \to \Lambda$ with semiconjugacy $\pi_\varphi\colon Y^\varphi \to \Lambda$, such that $T_t \circ \pi_\varphi = \pi_\varphi \circ F_t$. Here, $\varphi\colon Y \to [1,\infty)$ is a (possibly unbounded) return time of T_t to the set $Y \subset \Lambda$. We will show that

$$\psi = \int_0^1 (v \circ \pi_\varphi) \circ F_s \,\mathrm{d}s = m + \chi \circ F_1 - \chi, \tag{1.2}$$

for some functions $m, \chi\colon Y^\varphi \to \mathbb{R}^d$. We call (1.2) the *martingale-coboundary decomposition* of $v \circ \pi_\varphi$. We prove that $\mathbb{E}[m \circ F_n | F_{n+1}^{-1}\mathcal{B}] = 0$ for all $n \geq 1$, where $F_{n+1}^{-1}\mathcal{B}$ are pre-image σ-algebras in Y^φ. Such a property gives that $(m \circ F_n)_{n\geq 0}$ is a reversed martingale differences sequence, which in turn generates a sequence of martingales $M_n(k) = n^{-\frac{1}{2}} \sum_{j=1}^k m \circ F_{n-j}$, $0 \leq k \leq n$.

We see from (1.1) and (1.2) that for every $0 \leq k \leq n$,

$$W_n(k/n) \circ \pi_\varphi = \int_0^k (v \circ \pi_\varphi) \circ F_s \,\mathrm{d}s = \sum_{j=0}^{k-1} \psi \circ F_j = \sum_{j=0}^{k-1} m \circ F_j + \chi \circ F_k - \chi.$$

By the identity $M_n(k) = \sum_{j=0}^{n-1} m \circ F_j - \sum_{j=0}^{k-1} m \circ F_j$ and nice error bounds on the coboundary, statistical properties for W_n are equivalent to the ones of M_n, and we can pass rates found for the martingales to the sequence W_n.

The main advancement of [32] for maps, and of our Section 3.3 for flows, is a control over the squares of m. By mean of a secondary martingale-coboundary decomposition presented in Subsection 3.3.3, we decompose the new observable

$\breve{v} = \mathbb{E}[mm^T - \int mm^T | F_1^{-1}\mathcal{B}]$ similarly to (1.2), and then apply martingale-type inequalities to control the growth of its Birkhoff sums. More explicitly, for $p \in (2, \infty)$ we find a constant $C > 0$ (dependent on v) such that

$$\Big|\max_{1\le k\le n}\big|\sum_{j=0}^{k-1} \breve{v} \circ F_j\big|\Big|_{2(p-1)} \le Cn^{\frac{1}{2}},$$

for all $n \ge 1$. Such an estimate is later applied to find rates of convergence for M_n.

To introduce the content of Chapter 4, we remark that formula (1.2) is constructed by mean of the exponential decay of the transfer operator P in discrete time for a suitable Hölder norm $\| \; \|$. For uniformly expanding system, this decay is typically proved by establishing quasicompactness and a spectral gap for the associated transfer operator P. Such a spectral gap yields a decay rate $\|P^n v - \int v\| \le C_v e^{-an}$ for v Hölder, where a, C_v are positive constants. An immediate consequence of the decay of P^n is also the decay of correlations for Hölder observables. This approach has been extended to large classes of nonuniformly expanding dynamics with exponential [59] and subexponential decay of correlations [60].

In continuous time, decay of correlations for semiflows would follow if the transfer operator L_t for T_t showed an exponential (or summable) decay on a Hölder space. In addition, such a decay of would provide a direct proof of formula (1.2). Yet, this is not usually done for continuous time dynamical systems, since the standard techniques to get decay of correlations [21, 36, 50] bypass spectral gaps; see also [8] which proves exponential decay of correlations for billiard flows with a contact structure but does not establish a spectral gap. The only exceptions that we know of is Tsujii [54, 55] which provides a spectral gap in an anisotropic Banach space for (i) suspension semiflows over the doubling map and (ii) contact Anosov flows.

In Chapter 4 we obtain a restriction on the Banach spaces where the transfer operator can have a summable decay in Hölder norm. Let $\mathcal{C}^\eta(\Lambda)$ denote the space of η-Hölder continuous observables on Λ, for some $\eta \in (0,1)$. We prove the following theorem for a fixed $v \in L^\infty(\Lambda)$.

Theorem 1.1 ([43])**.** Let $\eta \in (\frac{1}{2}, 1)$. Suppose that $L_t v \in \mathcal{C}^\eta(\Lambda)$ for all $t > 0$ and that $\int_0^\infty \|L_t v\|_\eta \, dt < \infty$. Then $v_t = \int_0^t v \circ T_s \, \mathrm{d}s$ is a coboundary:

$$v_t = \chi \circ T_t - \chi \quad \text{for all } t \ge 0, \text{ a.e. on } \Lambda$$

where $\chi = \int_0^\infty L_t v \, dt \in \mathcal{C}^\eta(\Lambda)$. In particular, $\sup_{t\ge 0} |v_t|_\infty < \infty$.
Here, $|g|_\infty = \operatorname{ess\,sup}_\Lambda |g|$ and $\|g\|_\eta = |g|_\infty + \sup_{x\ne y} |g(x) - g(y)|/d(x,y)^\eta$.

As stated in [43], Theorem 1.1 implies that any Banach space admitting a spectral gap and embedded in $\mathcal{C}^\eta(\Lambda)$ for some $\eta > \frac{1}{2}$ is cohomologically trivial. However, for (non)uniformly expanding semiflows and (non)uniformly hyperbolic flows of the type in the aforementioned references, coboundaries are known to be exceedingly rare, see for example [15, Section 2.3.3]. Hence, Theorem 1.1 can be viewed as an "anti-spectral gap" result for such continuous time dynamical systems. Moreover, the uniform boundedness of v_t leads to trivial statistical properties. For example, under the assumptions of Theorem 1.1, $W_n(t) = n^{-\frac{1}{2}} \int_0^{nt} v \circ T_s \, \mathrm{d}s$ converges in distribution to the null process, with rates of $\mathcal{O}(n^{-1/2})$ in Prokhorov and any q-Wasserstein metrics. Any time that the WIP is not trivial, we cannot expect such hypotheses to hold.

The proof of Theorem 1.1 is incredibly straightforward, making Chapter 4 and paper [43] surprisingly concise. Assuming the flow to be Lipschitz a.e. on Λ, the Hölder property of χ on Λ implies a.s. Hölder continuity in $[0,1]$ of the sample paths of the martingales $M_n(t) = n^{-\frac{1}{2}} \sum_{j=1}^{\lfloor nt \rfloor} m \circ F_{n-j}$, $t \in [0,1]$. A direct argument proves that a non-constant martingale cannot have η-Hölder sample paths when $\eta > 1/2$. Hence, we prove that $m = 0$ a.e. in Λ and so v_t is just a coboundary.

In Chapter 5, we consider norm decay of transfer operators for uniformly and nonuniformly expanding semiflows modelled by a suspension over a Gibbs-Markov map with exponential tails. In spite of Theorem 1.1, it is still possible to control the Hölder norm of $L_t v$ for a large class of semiflows and observables v, and our Theorem 5.2 is the first in this direction. Such a result was published in collaboration with Melbourne and Terhesiu [43], giving the exponential decay of $L_t v$ in some strong norm.

The main ingredients of the proof are a Dolgopyat-type estimate [21] and operator renewal theory for semiflows [44] which enables consideration of the operator Laplace transform $\int_0^\infty e^{-st} L_t \, dt$. Hence, we get the decay of the correlation function from analyticity of Laplace transforms, bypassing spectral properties of L_t, see [21, 36, 50] . The observables v are required to be smooth in the flow direction and have a good support, that is $v = 0$ nearby the base and roof of the suspension. Yet, [43, Theorem 2.2] does not contradict Theorem 1.1, as the used norm does not give Hölder control of $L_t v$ in the flow direction when passing through points at the top of the suspension.

Chapter 2

Ergodic theory and probability

The current chapter presents general facts from dynamical systems and probability theory. These topics are widely known and can be found in general books on probability and dynamics such as [29, 30] and the lecture notes [38].

Notation We write interchangeably $a_n = \mathcal{O}(b_n)$ or $a_n \ll b_n$ for two sequences $a_n, b_n \geq 0$, if there exists a constant $C > 0$ and an integer $n_0 \geq 0$ such that $a_n \leq Cb_n$ for all $n \geq n_0$.
For $x \in \mathbb{R}^m$ and $J \in \mathbb{R}^{m\times n}$, write $|x| = (\sum_{i=1}^m x_i^2)^{1/2}$ and $|J| = (\sum_{i=1}^m \sum_{j=1}^n J_{i,j}^2)^{1/2}$.
For a function v, we use the standard notation $|v|_\infty = \operatorname{ess\,sup} |v|$, and for $p \in [1, \infty)$, we write $|v|_p = \left(\int |v|^p\right)^{\frac{1}{p}}$.
We write $T^{-n}\mathcal{B} = \{T^{-n}B : B \in \mathcal{B}\}$, $n \geq 1$, for the preimage σ-algebras given by a measurable map T and a σ-algebra $\mathcal{B}$.

2.1 Ergodic Theory

Let $(\Lambda, \mathcal{B}, \mu)$ be a probability space.

Definition 2.1. A (discrete time) dynamical system is a measurable map $T\colon \Lambda \to \Lambda$.

For $n \geq 0$, we refer to $T^n\colon \Lambda \to \Lambda$ as the map obtained by composing $n \geq 0$ times T, where $T^0 = \mathrm{Id}$. If we have a family of maps $T_t\colon \Lambda \to \Lambda$ for $t \in \mathbb{R}$ (or $t \geq 0$) such that $T_0 = \mathrm{Id}$ and $T_s \circ T_t = T_{s+t}$ for all s and t, then we call (Λ, T_t) a continuous time dynamical system and T_t is a *flow* (or *semiflow*).

Henceforth, the discrete or continuous nature of the system will be clear from context, and most definitions for discrete time can be extended naturally to the continuous case.

We say that a measurable T is *non-singular* if $\mu(T^{-1}B) = 0$ is equivalent to $\mu(B) = 0$ for all $B \in \mathcal{B}$. For such a map, the collections of null and positive measure sets are preserved under the iterations of T. Here follows a stronger condition on T.

Definition 2.2. A measurable map T is *measure-preserving* (or equivalently μ is *T-invariant*), if $\mu(T^{-1}B) = \mu(B)$ for every $B \in \mathcal{B}$. We call (Λ, T, μ) a measure-preserving system.

The following systems are assumed to be measure-preserving unless stated otherwise. A subset $B \subset \Lambda$ is said to be *invariant* if $T^{-1}B = B$. If B is invariant, so is $\Lambda \backslash B$; hence, we can partition the system given by T with $T|_B : B \to B$ and $T|_{\Lambda \backslash B} : \Lambda \backslash B \to \Lambda \backslash B$. These are examples of what we call *sub-systems* of T.

Definition 2.3. The measure μ (or the system T) is *ergodic* if every measurable T-invariant B has $\mu(B) \in \{0, 1\}$.

In other terms, a dynamical system T is ergodic if every of its sub-systems have either full or null measure.

Definition 2.4. Let $v \in L^1(\Lambda, \mathcal{B}, \mu)$ and let $\mathcal{A} \subset \mathcal{B}$ be a sub σ-algebra. The *conditional expectation* $\mathbb{E}[v|\mathcal{A}]$ is the unique element of $L^1(\Lambda, \mathcal{A}, \mu)$ such that $\int_A v \, \mathrm{d}\mu = \int_A \mathbb{E}[v|\mathcal{A}] \, \mathrm{d}\mu$ for all $A \in \mathcal{A}$.

Existence and uniqueness (almost everywhere) of the conditional expectation are shown by the Radon-Nikodym Theorem.

Example 2.5. Suppose that $\mathcal{A} = \sigma(A_1, A_2, \dots)$, where $\{A_i\} \subset \mathcal{B}$ is an (at most countable) partition of Λ with $\mu(A_i) > 0$ for every i. Then, it can be checked by the definition that

$$\mathbb{E}[v|\mathcal{A}] = \sum_{i=1}^{\infty} \frac{\mathbb{1}_{A_i}}{\mu(A_i)} \int_{A_i} v \, \mathrm{d}\mu, \quad \text{where} \quad \mathbb{1}_{A_i}(y) = \begin{cases} 1 & y \in A_i \\ 0 & y \in \Lambda \backslash A_i \end{cases}$$

It follows that $\mathbb{E}[v|\mathcal{A}]|_{A_i} = \mathbb{E}[v|A_i] = \int_{A_i} v \, \mathrm{d}\mu_i$ for every $i \geq 1$, where μ_i is the classical conditional probability measure, $\mu_i(B) = \mu(B \cap A_i)/\mu(A_i)$, $B \in \mathcal{B}$. Hence, Definition 2.4 is an extension of the standard expectation conditioned on an event.

Another important sub σ-algebra of $\mathcal{B}$, is the collection of measurable T-invariant sets $\mathcal{I} = \{B \in \mathcal{B} : T^{-1}B = B\}$.

Theorem 2.6 (Birkhoff's Ergodic Theorem)**.** For every $v \in L^1(\Lambda)$

$$\frac{1}{n}\sum_{j=0}^{n-1} v \circ T^j \longrightarrow \mathbb{E}[v|\mathcal{I}]$$

μ-a.e. as $n \to \infty$.

Unlike in Example 2.5, for a general μ we do not know an explicit formula for $\mathbb{E}[v|\mathcal{I}]$, however when μ is ergodic we do.

Corollary 2.7. If μ is ergodic, then for every $v \in L^1(\Lambda)$

$$\frac{1}{n}\sum_{j=0}^{n-1} v \circ T^j \longrightarrow \int_\Lambda v \, \mathrm{d}\mu,$$

μ-a.e. for $n \to \infty$.

Proof. By Theorem 2.6, it suffices to show that $\mathbb{E}[v|\mathcal{I}] = \int_\Lambda v \, \mathrm{d}\mu$, μ-a.e. Let B be invariant, so that by ergodicity $\mu(B)$ is either 0 or 1. If $\mu(B) = 0$, then we have $0 = \int_B \mathbb{E}[v|\mathcal{I}] \, \mathrm{d}\mu = \int_B (\int_\Lambda v \, \mathrm{d}\mu) \, \mathrm{d}\mu$. If instead $\mu(B) = 1$, we can finish by $\int_B \mathbb{E}[v|\mathcal{I}] \, \mathrm{d}\mu = \int_B v \, \mathrm{d}\mu = \int_\Lambda v \, \mathrm{d}\mu = \int_B (\int_\Lambda v \, \mathrm{d}\mu) \, \mathrm{d}\mu$ and the definition of conditional expectation. □

We often refer to a measurable function $v \colon \Lambda \to \mathbb{R}$ defined on a dynamical system as an *observable*. Such a function can be interpreted as the data collected (or observed) from a system that follows some dynamics.

Definition 2.8 (Koopman operator)**.** Define the operator $U \colon L^1(\Lambda) \to L^1(\Lambda)$ as $Uv = v \circ T$.

Proposition 2.9. The Koopman operator is linear and bounded in $L^p(\Lambda)$ for all $p \in [1, \infty)$. Moreover, $|Uv|_p = |v|_p$ for all $v \in L^p(\Lambda)$.

Proof. Linearity is straightforward. Let $v \in L^p(\Lambda)$; since T is measure-preserving, $|Uv|_p^p = \int |Uv|^p = \int |v|^p \circ T = |v|_p^p$ and the statement follows. □

Definition 2.10 (Transfer operator)**.** Define the (Ruelle-Perron-Frobenius) transfer operator $P \colon L^1(\Lambda) \to L^1(\Lambda)$, where Pv is the unique element of $L^1(\Lambda)$ satisfying the duality relation

$$\int_\Lambda v(w \circ T) \, \mathrm{d}\mu = \int_\Lambda (Pv) w \, \mathrm{d}\mu,$$

for every $v \in L^1(\Lambda)$ and $w \in L^\infty(\Lambda)$.

Proposition 2.11. We have that $|Pv|_p \le |v|_p$ for all $v \in L^p(\Lambda)$, $p \in [1,\infty]$.

Proof. For $v \in L^1$ define $w = \operatorname{sgn} Pv$. Hence,

$$|Pv|_1 = \int_\Lambda |Pv|\,\mathrm{d}\mu = \int_\Lambda (Pv)w\,\mathrm{d}\mu = \int_\Lambda v(w\circ T)\,\mathrm{d}\mu \le |w|_\infty |v|_1 = |v|_1.$$

For $p=\infty$, assume without loss that $|v|_\infty = 1$. Suppose by contradiction that there exists $v \in L^\infty$ such that $|Pv|_\infty > 1$. Hence, there exist $\varepsilon > 0$ and $A \subset \Lambda$ measurable, $\mu(A) > 0$, on which $Pv \ge 1+\varepsilon$. So,

$$(1+\varepsilon)\mu(A) \le \int_\Lambda (Pv)\mathbb{1}_A = \int_\Lambda v(\mathbb{1}_A \circ T) \le |v|_\infty |\mathbb{1}_A\circ T|_1 = \mu(A).$$

Therefore, $|Pv|_\infty \le |v|_\infty$.

Let now $p \in (1,\infty)$ and consider $q = p/(p-1)$ that is the conjugate exponent of p. Let $v \in L^\infty$ and write $w = |Pv|^{p-1}\operatorname{sgn}(Pv)$, which lies in L^∞ by what we have already proven. By Hölder's inequality and T-invariance,

$$\begin{aligned}|Pv|_p^p &= \int_\Lambda (Pv)w = \int_\Lambda v(w\circ T) \le |v|_p |w|_q \\ &= |v|_p\Big(\int_\Lambda \big(|Pv|^{p-1}\big)^{\frac{p}{p-1}}\Big)^{\frac{p-1}{p}} = |v|_p |Pv|_p^{p-1}.\end{aligned}$$

Hence, $|Pv|_p \le |v|_p$ for all $v \in L^\infty$. Since simple functions are dense in L^p, we finish by the bounded linear transformation theorem. □

Proposition 2.12. For every $v \in L^1(\Lambda)$,

(i) $\int_\Lambda Pv\,\mathrm{d}\mu = \int_\Lambda v\,\mathrm{d}\mu$;

(ii) $PUv = v$ and $UPv = \mathbb{E}[v|T^{-1}\mathcal{B}]$;

(iii) If T is invertible, then $Pv = v\circ T^{-1}$ and $UPv = v$.

Proof. By definition of P, $\int Pv = \int v(1\circ T) = \int v$, which proves (i).

Let $w \in L^\infty$. Since T is measure-preserving,

$$\int_\Lambda P(Uv)w\,\mathrm{d}\mu = \int_\Lambda (Uv)(w\circ T)\,\mathrm{d}\mu = \int_\Lambda (vw)\circ T\,\mathrm{d}\mu = \int_\Lambda vw\,\mathrm{d}\mu,$$

which proves the first identity in (ii). For the second part, let us check the conditions in Definition 2.4. The function Pv is integrable by definition. Let $B \in \mathcal{B}$ and note that $\mathbb{1}_B \circ T = \mathbb{1}_{T^{-1}B}$. So,

$$\begin{aligned}\int_{T^{-1}B} UPv &= \textstyle\int_\Lambda (Pv\circ T)\mathbb{1}_{T^{-1}B} = \int_\Lambda ((Pv)\mathbb{1}_B)\circ T \\ &= \int_\Lambda (Pv)\mathbb{1}_B = \int_\Lambda v(\mathbb{1}_B\circ T) = \int_{T^{-1}B} v.\end{aligned}$$

To prove that UPv is $T^{-1}\mathcal{B}$-measurable,

$$(UPv)^{-1}B = (Pv \circ T)^{-1}B = T^{-1}((Pv)^{-1}B) \in T^{-1}\mathcal{B},$$

because Pv is $\mathcal{B}$-measurable.

Assume now T invertible and let us show (iii). Let $w \in L^\infty(\Lambda)$. Hence,

$$\int (Pv)w = \int ((v \circ T^{-1})w) \circ T = \int (v \circ T^{-1})w,$$

which yields $Pv = v \circ T^{-1}$. Hence, $UPv = (v \circ T^{-1}) \circ T = v$. □

Remark 2.13. By Proposition 2.12(iii) and Proposition 2.9, if T is invertible, then the transfer operator P does not contract in any p-norm. A decay of $\|P^n v\|$ for an observable v in some strong norm $\| \ \|$ is desirable to prove statistical laws for T, see for example [8, 21, 49, 59, 60]. Hence, a non-invertible system is necessary for the direct application of such transfer operator methods.

Corollary 2.14. For every $v \in L^1(\Lambda)$ and $n \geq 1$ we have

$$\mathbb{E}[v \circ T^n | T^{-(n+1)}\mathcal{A}] = \mathbb{E}[v|T^{-1}\mathcal{A}] \circ T^n.$$

Proof. For any $n \geq 1$, the system $T^n \colon \Lambda \to \Lambda$ is still measure-preserving, with transfer operator P^n. We conclude by Proposition 2.12(ii) that

$$\mathbb{E}[v \circ T^n | T^{-(n+1)}\mathcal{A}] = U^{n+1}P^{n+1}(U^n v) = U^n(UPv) = \mathbb{E}[v|T^{-1}\mathcal{A}] \circ T^n. \qquad \square$$

2.2 Probability theory and martingales

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space and let (S, d_S) be a metric space with Borel σ-algebra $\mathcal{B}$.

Definition 2.15. A function $X \colon \Omega \to S$ is called *random element* of S if it is $\mathcal{A}/\mathcal{B}$-measurable.

We say random variable or vector when $S = \mathbb{R}^d$, and use the terms stochastic process or random function when S is a functional space. We say that a sequence of random elements X_n on $(\Omega, \mathcal{A}, \mathbb{P})$ converges in distribution to a random element X on $(\Omega', \mathcal{A}', \mathbb{Q})$ (denoted by| $X_n \to_d X$), if the sequence of laws of X_n converges weakly to the law of X, which means $\int_\Omega f(X_n)\,\mathrm{d}\mathbb{P} \to \int_{\Omega'} f(X)\,\mathrm{d}\mathbb{Q}$ for $n \to \infty$ and all $f \colon S \to \mathbb{R}$ continuous and bounded. Another possible notation is $X_{n*}\mathbb{P} \to_w X_*\mathbb{Q}$. If the random element Y has the same law as X, we write $Y =_d X$.

Definition 2.16. Let $d \geq 1$ and $\Sigma \in \mathbb{R}^{d\times d}$ be a positive semidefinite matrix. A d-dimensional *Brownian motion* on $[0,1]$ with mean $0 \in \mathbb{R}^d$ and covariance Σ is a continuous stochastic process $W = \{W(t) \in \mathbb{R}^d, t \in [0,1]\}$ that satisfies the following properties: (i) $\mathbb{P}(W(0) = 0) = 1$, (ii) for any partition $0 \leq t_1 < \cdots < t_k \leq 1$, the increments

$$\big(W(t_2) - W(t_1)\big), \ldots, \big(W(t_n) - W(t_{n-1})\big)$$

are stochastically independent, and (iii) for any $0 \leq s \leq t \leq 1$ the random variable $W(t) - W(s)$ is normally distributed in $\mathbb{R}^d$ with mean 0 and variance $\Sigma(t-s)$.

Brownian motion exists by Kolmogorov's existence Theorem for stochastic processes. The d-dimensional *Wiener measure* is the push-forward measure on $\mathcal{C}([0,1],\mathbb{R}^d)$ induced by any Brownian motion. Such a measure can be obtained as a limit distribution of a random walk via the next result [57, Theorem 4.3.5].

Theorem 2.17 (Donsker's weak Invariance Principle (WIP))**.** Let $(X_n)_{n\geq 0}$ be a sequence of i.i.d. random variables in $\mathbb{R}^d$ with mean 0 and covariance Σ, defined on the same probability space. Define the sequence of random functions W_n in $\mathcal{C}([0,1],\mathbb{R}^d)$[1] as

$$W_n(k/n) = \frac{1}{\sqrt{n}} \sum_{j=0}^{k-1} X_j, \qquad 0 \leq k \leq n,$$

and linear interpolation in $[0,1]$. Then, $W_n \to_d W$, where W is a d-dimensional Brownian motion on $[0,1]$ with mean 0 and covariance Σ. □

Remark 2.18 (Central Limit Theorem (CLT))**.** Under the same assumptions of Theorem 2.17, we have that $\frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} X_j \to_d \mathcal{N}$, where $\mathcal{N}$ is a d-dimensional centred Gaussian with covariance Σ.

We say that a system $(\Lambda, T\mu)$ satisfies the WIP for an observable $v \colon \Lambda \to \mathbb{R}^d$, if the thesis of Theorem 2.17 is true for the sequence $X_j = v \circ T^j$, $j \geq 0$.

The second part of this section is dedicated to a short introduction to discrete and continuous time martingales; see [29] for a reference. This type of stochastic processes is essential in Chapters 3 and 4 to apply probabilistic techniques to dynamics. Discrete time martingales are used in Chapter 3 to apply respectively results

[1][57, Theorem 4.3.5] defines W_n as a càdlàg process, however the statement is still true when W_n is continuous.

of [33] and [19]. Continuous time martingales play a role in two chapters: in Chapter 3 they are needed to adapt a result of Courbot [18] (see Proposition A.5), and in Chapter 4 we use the fact that a non-constant martingale cannot be $(1/2+\varepsilon)$-Hölder continuous.

Definition 2.19. A sequence of integrable $\mathbb{R}^d$-valued random variables $(M(n))_{n\geq 1}$, adapted to a non-decreasing filtration $(\mathcal{F}_n)_{n\geq 1}$ of σ-algebras, is a (discrete time) *martingale* if $\mathbb{E}[M(n+1)|\mathcal{F}_n] = M(n)$ for all $n \geq 1$.

The first example of such a sequence comes from the i.i.d. $(X_n)_{n\geq 0}$ in Theorem 2.17, being defined as $S_n = \sum_{j=0}^{n-1} X_j$, $n \geq 1$. It satisfies trivially the martingale property with respect to its natural filtration $\mathcal{F}_n = \sigma(S_1, \ldots, S_n)$.

The term "martingale" is also used for any finite family of integrable random variables $(M(k))_{1\leq k\leq n}$ adapted to a finite filtration $(\mathcal{F}_k)_{1\leq k\leq n}$, and satisfying the equation $\mathbb{E}[M(k+1)|\mathcal{F}_k] = M(k)$ for $1 \leq k < n$. That is because such a family can be extended constantly to a sequence that is a genuine martingale.

Definition 2.20. A sequence of integrable $\mathbb{R}^d$-valued random variables $(d_n)_{n\geq 0}$, together with the σ-algebras $(\mathcal{G}_n)_{n\geq 0}$, is called a *reversed martingale differences sequence* (in brief RMDS) if $\mathcal{G}_{n+1} \subseteq \mathcal{G}_n$, d_n is $\mathcal{G}_n$-measurable, and $\mathbb{E}[d_n|\mathcal{G}_{n+1}] = 0$ for all $n \geq 0$.

Proposition 2.21. Let $(\Lambda, T, \mathcal{B}, \mu)$ be a system with transfer operator P. If $v \in \ker P$, then the sequence $(v \circ T^n)_{n\geq 0}$ with $(T^{-n}\mathcal{B})_{n\geq 0}$ is an RMDS.

Proof. For $n \geq 0$, we have that $T^{-(n+1)}\mathcal{B} \subseteq T^{-n}\mathcal{B}$ and $v \circ T^n$ is $T^{-n}\mathcal{B}$-measurable. We conclude by Corollary 2.14 and Proposition 2.12(ii) that

$$\mathbb{E}[v \circ T^n|T^{-(n+1)}\mathcal{B}] = \mathbb{E}[v|T^{-1}\mathcal{B}] \circ T^n = (Pv) \circ T^{n+1} = 0. \qquad \square$$

Proposition 2.22. If $(d_n)_{n\geq 0}$ is an RMDS with σ-algebras $(\mathcal{G}_n)_{n\geq 0}$, then, for every $n \geq 1$ the process $M_n(k) = \sum_{j=1}^{k} d_{n-j}$, $1 \leq k \leq n$, with σ-algebras $\mathcal{F}_k = \mathcal{G}_{n-k}$, is a martingale.

Proof. Let $0 \leq k < n$. By the inclusions on $\mathcal{G}_j$, we get $\mathcal{F}_k \subseteq \mathcal{F}_{k+1}$. We have that $M_n(k)$ is integrable and $\mathcal{F}_k$-measurable, because it is a sum of integrable and $\mathcal{F}_k$-measurable random variables. We can finish by definition of an RMDS (Definition 2.20):

$$\mathbb{E}[M_n(k+1)|\mathcal{F}_k] = \sum_{j=1}^{k} d_{n-j} + \mathbb{E}[d_{n-k-1}|\mathcal{G}_{n-k}] = \sum_{j=1}^{k} d_{n-j} = M_n(k). \qquad \square$$

To conclude this subsection, we say that a function $f\colon [0,1] \to \mathbb{R}$ is *càdlàg* if it is right-continuous and all its left limits exist. We define a (continuous time) martingale following [29], to be an integrable càdlàg stochastic process $(M(t))_{t\in[0,1]}$, adapted to a filtration $(\mathcal{G}_t)_{t\in[0,1]}$, such that $\mathbb{E}[M(t)|\mathcal{G}_s] = M(s)$ for all $0 \le s \le t \le 1$.

Remark 2.23. Given a discrete time martingale $(M(n))_{n\ge1}$ with filtration $(\mathcal{F}_n)_{n\ge1}$, there is a natural way to construct a sequence of continuous time martingales: $t \mapsto M_n(\lfloor nt \rfloor)$, with σ-algebras $\mathcal{F}_{\lfloor nt \rfloor}$, for $n \ge 1$ and $t \in [0,1]$.

A martingale M is *square integrable* if $\sup_{t\in[0,1]} \mathbb{E}[|M(t)|^2] < \infty$. For such an M, [29, Theorem I.4.2] yields the existence of a real predictable process $\langle M \rangle$, such that $M^2 - \langle M \rangle$ is a martingale. The process $\langle M \rangle$ is called the *quadratic variation* of M and is unique up to *indistinguishability* that is, if another process X satisfies the same properties of $\langle M \rangle$, then $\mathbb{P}(X(t) = \langle M \rangle(t) \text{ for all } t \in [0,1]) = 1$.

2.3 Some dynamical systems

Definition 2.24. Let (Λ, d_Λ) be a metric space. For $\eta \in (0,1]$ and $d \ge 1$, a function $v\colon \Lambda \to \mathbb{R}^d$ is said to be η-Hölder continuous (and we write $v \in \mathcal{C}^\eta(\Lambda, \mathbb{R}^d)$) if there exists $C > 0$ such that

$$|v(x) - v(y)| \le C d_\Lambda(x,y)^\eta,$$

for all $x, y \in \Lambda$, $x \ne y$. We write $|v|_\eta = \sup_{x\ne y} |v(x) - v(y)|/d(x,y)^\eta$ and consider the norm $\|v\|_\eta = |v|_\infty + |v|_\eta$, which makes $\mathcal{C}^\eta(\Lambda, \mathbb{R}^d)$ a Banach space.

Example 2.25 (Doubling Map)**.** Define $T\colon [0,1] \to [0,1]$ as $Tx = 2x \pmod 1$. The Lebesgue measure $\mathcal{L}$ is T-invariant and ergodic. As shown in [38], for $v \in L^1$ we have $Pv(x) = 1/2\big(v(x/2) + v((x+1)/2)\big)$, $x \in [0,1]$. If $v \in \mathcal{C}^\eta([0,1], \mathbb{R})$ with $\int v \, \mathrm{d}\mathcal{L} = 0$, then $Pv \in \mathcal{C}^\eta([0,1], \mathbb{R})$ with mean 0, and $\|P^n v\|_\eta \le (2^\eta)^{-n} |v|_\eta$. The CLT and the WIP hold for Hölder observables, and can be proved by such a decay of P^n.

Example 2.26 (Gibbs-Markov maps)**.** Let μ be a Borel probability measure on a bounded metric space (Y, d_Y) and let $\{Y_j\}$ be an at most countable measurable partition of Y. Let $F : Y \to Y$ be a measure-preserving transformation such that Y restricts to a measure-theoretic bijection from Y_j onto Y for each j. Let $g = d\mu/(d\mu \circ Y)$ be the inverse Jacobian of F. We assume that there are the constants $\eta \in (0,1], \lambda > 1$, and $C > 0$ such that $d_Y(Fx, Fy) \ge \lambda d_Y(x,y)$ and

$|\log g(x) - \log g(y)| \le C d_\Lambda(Fy, Fy)^\eta$ for all $x, y \in Y_j$, $j \ge 1$. Then F is a (full-branch) Gibbs-Markov map as in [2] with ergodic (and mixing) invariant measure μ. We use this kind of systems in Subsections 3.1.2 and 3.1.3 to define nonuniformly expanding maps and flows, and in the Section 5.1 for the setup of Chapter 5.

By [2, Theorem 1.6], for every $v \in \mathcal{C}^\eta(Y, \mathbb{R}^d)$ with $\int v \, d\mu = 0$, there are $a, C > 0$ such that $\|P^n v\|_\eta \le C e^{-nk}$ for all $n \ge 1$. Such a result is used in [32, Section 2.2] and in Subsection 3.3.1 to construct the martingale-coboundary decomposition of a regular observable. As in Example 2.25, such e decay of P^n implies the CLT and WIP for Hölder observables.

Example 2.27 (Pomeau-Manneville intermittent maps)**.** Let $\gamma > 0$ and define $T\colon [0,1] \to [0,1]$ as

$$Tx = \begin{cases} x(1 + 2^\gamma x^\gamma) & x \in [0, 1/2) \\ 2x - 1 & x \in [1/2, 1]. \end{cases}$$

This dynamical system comes from [51] and was studied in [37]. If $\gamma = 0$, then T is the doubling map of Example 2.25. If $\gamma > 0$, this system is the prototypical example of nonuniformly expanding map defined in Subsection 3.1.1. The non-uniform expansion comes from the neutral fixed point at $x = 0$, as $\lim_{x\to 0} T'x = 1$. Such non-uniformity around 0 can be seen by the following fact: for any $N \ge 1$ there exists a neighbourhood U of 0 and $x \in U\backslash\{0\}$, such that $T^n x \in U$ for all $n \le N$ and $T^{N+1}x \notin U$. There is a unique ergodic absolutely continuous invariant measure μ for $\gamma \in (0, 1)$; moreover, Hölder continuous observables satisfy the CLT and WIP for $\gamma < 1/2$.

Example 2.28 (Suspension flow)**.** A classical way to construct a flow from a system (Y, F, μ), is to consider an integrable function $\varphi\colon Y \to [1, \infty)$ and define the suspension $Y^\varphi = \{(y, u) \in Y \times [0, \infty) : u \in [0, \varphi(y)]\}/\sim$ where $(y, \varphi(y)) \sim (Fy, 0)$. Define $\mu^\varphi = (\mu \times \text{Lebesgue})/\int_\Lambda \varphi \, d\mu$ which is a probability measure on Y^φ. The suspension flow $F_t\colon Y^\varphi \to Y^\varphi$ is defined as $F_t(y, u) = (y, u + t)$, $t \in \mathbb{R}$ (or $t \ge 0$), modulo identifications. We have that F_t preserves the measure μ^φ. Such a construction over a Gibbs-Markov map F is used in Subsection 3.1.3 and Section 5.1. We know by [45] and [47] that statistical laws for the suspension flow F_t follow from the base system. Hence, nice hyperbolic or expanding properties of F yield results as the CLT and the WIP for the flow as well.

2.4 Three metrics for probability measures

We conclude the first chapter of this thesis mentioning some metrics on the space of probability measure, which are essential to analyse rates of convergence in Chapter 3.

This section recalls the definitions of Wasserstein and Prokhorov metrics following [25], and the Ky Fan distance following [24]. Given a separable metric space (S, d_S) with Borel σ-algebra $\mathcal{B}$, we write $\mathcal{M}_1(S)$ for the set of Borel probability measures on S. Let $\mu, \nu \in \mathcal{M}_1(S)$, and let X, Y be random elements of S defined on the same probability space.

1-Wasserstein (or Kantorovich) metric

$$\mathcal{W}(\mu, \nu) = \sup_{\psi \in \mathrm{Lip}_1} | \textstyle\int_S \psi \, \mathrm{d}\mu - \int_S \psi \, \mathrm{d}\nu |,$$

where $\mathrm{Lip}_1 = \{\psi \colon S \to \mathbb{R} : |\psi(x) - \psi(y)| \leq d_S(x, y) \text{ for all } x, y \in S\}$.

Prokhorov (or Lévy-Prokhorov) metric

$$\Pi(\mu, \nu) = \inf\{\varepsilon > 0 : \mu(B) \leq \nu(B^\varepsilon) + \varepsilon \text{ for all } B \in \mathcal{B}\},$$

where $B^\varepsilon = \bigcup_{x \in B}\{y \in S \colon d_S(x, y) < \varepsilon\}$.

Ky Fan metric

$$\alpha(X, Y) = \inf\{\varepsilon > 0 : \mathbb{P}(d_S(X, Y) > \varepsilon) \leq \varepsilon\}.$$

If A and B are random elements with respectively laws μ and ν, we write $\Pi(A, B) = \Pi(\mu, \nu)$ and $\mathcal{W}(A, B) = \mathcal{W}(\mu, \nu)$.

Proposition 2.29. Let X, Y be random elements in S. Then

$$\Pi(X, Y) \leq \sqrt{\mathcal{W}(X, Y)}. \tag{2.1}$$

Let $q \in [1, \infty)$. If X and Y are defined on a common probability space, then

$$\Pi(X, Y) \leq \alpha(X, Y) \leq \begin{cases} |d_S(X, Y)|_q^{q/(q+1)} \\ |d_S(X, Y)|_\infty \end{cases} \tag{2.2}$$

Proof. The proofs of (2.1) and $\Pi(X,Y) \le \alpha(X,Y)$ are respectively in [25, Theorem 2] and [24, Theorem 11.3.5]. To prove the top inequality of (2.2), write $\varepsilon = |d_S(X,Y)|_q^{q/(q+1)}$. By Markov's inequality,

$$\mathbb{P}(d_S(X,Y) > \varepsilon) \le \varepsilon^{-q}|d_S(X,Y)|_q^q = \varepsilon^{-q}\varepsilon^{q+1} = \varepsilon.$$

Hence, $\alpha(X,Y) \le \varepsilon$. The bottom inequality of (2.2) follows by

$$\mathbb{P}(d_S(X,Y) > |d_S(X,Y)|_\infty) = 0 \le |d_S(X,Y)|_\infty. \qquad \square$$

For ν and a sequence $(\nu_n)_{n\ge1}$ in $\mathcal{M}_1(S)$, we have that $\mathcal{W}(\nu_n,\nu) \to 0$ implies $\nu_n \to_w \nu$. The distance Π metrizes weak convergence on $\mathcal{M}_1(S)$ and the same is true for $\mathcal{W}$ under the extra assumption $\mathrm{diam}(S) < \infty$ [25]. Finally, we recall that α metrizes convergence in probability [24, Theorem 9.2.2].

Chapter 3

Rates for the WIP

3.1 Setup and main results

3.1.1 Setup

Let (Λ, d_Λ) be a bounded metric space with a Borel probability measure ρ and suppose that $T\colon \Lambda \to \Lambda$ is a nonsingular map. Assume that ρ is ergodic.

Suppose that there exists a measurable $Y \subset \Lambda$ with $\rho(Y) > 0$, and let $\{Y_j\}$ be an at most countable measurable partition Y. Let $\tau\colon Y \to \mathbb{Z}^+$ be an integrable function with constant values $\tau_j \geq 1$ on partition elements Y_j. We assume that $T^{\tau(y)}y \in Y$ for all $y \in Y$ and define $F\colon Y \to Y$ as $F = T^\tau$.

The dynamical system (Λ, T, ρ) is said to be a *nonuniformly expanding map* if there are constants $\lambda > 1$, $\eta \in (0,1]$, $C \geq 1$, such that for each j and $x, y \in Y_j$,

(a) $F|_{Y_j} : Y_j \to Y$ is a measure-theoretic bijection;

(b) $d_\Lambda(Fx, Fy) \geq \lambda d_\Lambda(x, y)$;

(c) $d_\Lambda(T^\ell x, T^\ell y) \leq C d_\Lambda(Fx, Fy)$ for all $0 \leq \ell \leq \tau_j - 1$;

(d) $\zeta = d\rho|_Y \,/\, d\rho|_Y \circ F$ satisfies $|\log \zeta(x) - \log \zeta(y)| \leq C d_\Lambda(Fx, Fy)^\eta$.

We say that T is nonuniformly expanding of *order* $p \in [1, \infty]$ if the *return time* τ lies in $L^p(Y)$. It is a standard result that there exists a unique ρ-absolutely continuous ergodic (and mixing) T-invariant probability measure μ_Λ on Λ (see for example [32, Subsection 2.1]).

Example 3.1. Examples of such systems are given by the Pomeau-Manneville intermittent maps described in Example 2.27 for $\gamma \in (0,1)$. They are nonuniformly expanding of order p for every $p \in [1, 1/\gamma)$ (see [4, Subsection 2.5.2]).

Function space on Λ For $v : \Lambda \to \mathbb{R}^d$ and $\eta \in (0,1]$, define

$$\|v\|_\eta = |v|_\infty + |v|_\eta, \qquad |v|_\eta = \sup_{x,y\in\Lambda,\, x\neq y} \frac{|v(x)-v(y)|}{d_\Lambda(x,y)^\eta}.$$

Let $\mathcal{C}^\eta(\Lambda, \mathbb{R}^d)$ consist of observables $v : \Lambda \to \mathbb{R}^d$ with $\|v\|_\eta < \infty$. We say $v \in \mathcal{C}_0^\eta(\Lambda, \mathbb{R}^d)$ if $\int_\Lambda v \, \mathrm{d}\mu_\Lambda = 0$.

3.1.2 Rates for maps

Let $T\colon \Lambda \to \Lambda$ be nonuniformly expanding with ergodic measure μ. For $d \geq 1$ and $\eta \in (0,1)$, let $v \in \mathcal{C}_0^\eta(\Lambda, \mathbb{R}^d)$. Define the sequence $B_n\colon [0,1] \to \mathbb{R}^d$, $n \geq 1$, as

$$B_n(k/n) = \frac{1}{\sqrt{n}} \sum_{j=0}^{k-1} v \circ T^j,$$

for $0 \leq k \leq n$, and using linear interpolation in $[0,1]$. The process B_n is a random element in $\mathcal{C}([0,1], \mathbb{R}^d)$ defined on the probability space (Λ, μ). Note that the randomness of B_n comes exclusively from the initial point $y_0 \in \Lambda$, chosen according to μ.

Here follows a standard result for B_n, see for example [27, 32, 40].

Theorem 3.2. Let $T\colon \Lambda \to \Lambda$ be nonuniformly expanding of order 2 and suppose $v \in \mathcal{C}_0^\eta(\Lambda, \mathbb{R}^d)$.

(i) The matrix $\Sigma = \lim_{n\to\infty} n^{-1} \int_\Lambda (\sum_{j=0}^{n-1} v \circ T^j)(\sum_{j=0}^{n-1} v \circ T^j)^T \, \mathrm{d}\mu_\Lambda \in \mathbb{R}^{d\times d}$ exists and is positive semidefinite. Typically Σ is positive definite: there exists a closed subspace $\mathcal{C}_{deg}$ of $\mathcal{C}_0^\eta(\Lambda, \mathbb{R}^d)$ with infinite codimension, such that we have $\det(\Sigma) \neq 0$ if $v \notin \mathcal{C}_{deg}$.

(ii) The WIP holds: $B_n \to_d W$ in $\mathcal{C}([0,1], \mathbb{R}^d)$ on the probability space (Λ, μ), where W is a centred d-dimensional Brownian motion on $[0,1]$ with covariance Σ. □

The following theorems display rates in the WIP, where the order $p \in (2, \infty]$ influences the speed of convergence. These rates are stated in the 1-Wasserstein and Prokhorov metrics on $\mathcal{M}_1(S)$, where $S = \mathcal{C}([0,1], \mathbb{R}^d)$ with the uniform distance.

Theorem 3.3. Let $p \in (2,3)$, and suppose $v \in \mathcal{C}_0^\eta(\Lambda, \mathbb{R}^d)$ for $d \geq 1$. Then there is a constant $C > 0$ such that $\mathcal{W}(B_n, W) \leq Cn^{-\frac{p-2}{2p}}(\log n)^{\frac{p-1}{2p}}$ for all integers $n > 1$.

Remark 3.4. To our knowledge, the rates of Theorem 3.3 are the first in the dynamical system literature for multidimensional observables. They are likely not optimal, as one expects that they improve when p increases (as it happens for $d = 1$ in Theorem 3.6). Yet, the proof of Theorem 3.3 in Subsection 3.2.2 uses modern techniques by [19], which do not work for $p > 3$. In such cases, our rates become $\mathcal{O}(n^{-1/6+\delta})$ for any $\delta > 0$. If we consider the Pomeau-Manneville maps of Example 2.27, such a threshold is reached when $\gamma \in (0, 1/3)$.

Remark 3.5. For $d = 1$ and $p \geq 4$, [35, Theorem 3.5] gives $\mathcal{W}(B_n, W) \ll n^{-\frac{p-2}{4p}}$. Theorem 3.3 provides new rates for $d = 1$, $p \in (2,4)$ and, by Remark 3.4, it gives a better rate than [35] of order $\mathcal{O}(n^{-1/6+\delta})$ when $p \in [4, 6)$.

Theorem 3.6. Let $p \in (2, \infty]$, and suppose $v \in \mathcal{C}_0^\eta(\Lambda, \mathbb{R})$. Then there exists $C > 0$ such that

$$\Pi(B_n, W) \leq C \begin{cases} n^{-\frac{p-2}{4p}} & p \in (2, \infty), \quad (3.1) \\ n^{-1/4}(\log n)^{3/4} & p = \infty \quad (3.2) \end{cases}$$

for all integers $n > 1$.

Remark 3.7. The rates displayed in (3.1) are due to [5, Theorem 3.2], whereas the ones in (3.2) are proved in Subsection 3.2.4.

Remark 3.8. Using (2.1), Theorem 3.3 yields for $p \in (2,3)$ and every $d \geq 1$ that $\Pi(B_n, W) \ll n^{-\frac{p-2}{4p}}(\log n)^{\frac{p-1}{4p}}$. This result is only relevant for $d \geq 2$, as Theorem 3.6 gives better rates in $d = 1$.

Remark 3.9. Theorems 3.3 and 3.6 imply the corresponding rates for the CLT.

3.1.3 Rates for semiflows

Let (Λ, d_Λ) be a bounded metric space. Let $\{T_t \colon \Lambda \to \Lambda\}_{t \geq 0}$, be a family of maps with $T_0 = \mathrm{Id}$ and $T_{s+t} = T_s \circ T_t$, $s, t \geq 0$. Assume continuous dependence on initial condition, that is for any $K > 0$ there exists $C > 0$ such that, for all $t \in [0, K]$ and $x, y \in \Lambda$,

$$d_\Lambda(T_t x, T_t y) \leq C d_\Lambda(x, y). \quad (3.3)$$

Suppose also that the semiflow is Lipschitz continuous in time. Hence, there exists $L > 0$ such that, for all $t, s \geq 0$ and $x \in \Lambda$

$$d_\Lambda(T_t x, T_s x) \leq L|t - s|. \tag{3.4}$$

Let $\eta \in (0,1]$. Suppose that there exist a Borel subset $X \subset \Lambda$ and a function $r \in \mathcal{C}^\eta(X)$ with $\inf r \geq 1$ and $T_{r(x)}x \in X$ for all $x \in X$. Define $T\colon X \to X$ as $T = T_r$ and assume that it is nonuniformly expanding in the sense of Subsection 3.1.1. Some examples for such functions are the intermittent maps in Example 2.27. Hence, there exist a Borel probability measure ρ on X, a subset $Y \subset X$ with measurable partition $\{Y_j\}$, a return time $\tau \in L^1(Y)$, and a map $F = T^\tau\colon Y \to Y$ that satisfies conditions (a)-(d) of Subsection 3.1.1.

The dynamical system (Λ, T_t) is said to be a *nonuniformly expanding semiflow* of *order* $p \in [1, \infty]$ if (X, T, ρ) is nonuniformly expanding of order p in the sense of Subsection 3.1.2.

Let $g = \mathrm{d}\mu/(\mathrm{d}\mu \circ F)$ be the inverse Jacobian of F. There are $\eta \in (0,1]$ and $C > 0$ such that, for all $x, y \in Y_j$, $j \geq 1$, we have

$$g(y) \leq C\mu(Y_j), \qquad |g(x) - g(y)| \leq C\mu(Y_j) d_\Lambda(Fx, Fy)^\eta, \tag{3.5}$$

(see for example [2]). In particular, F is a (full-branch) Gibbs-Markov map as in [2]. So, there exists a unique ergodic (and mixing) probability measure μ that has bounded density with respect to $\rho|_Y$.

Let $\varphi\colon Y \to [1, \infty)$ be defined as $\varphi(y) = \sum_{j=0}^{\tau(y)-1} r(T^j y)$. Define the suspension space $Y^\varphi = \{(y, u) \in Y \times [0, \infty) : u \in [0, \varphi(y)]\}/\sim$ where $(y, \varphi(y)) \sim (Fy, 0)$. The suspension semiflow $F_t : Y^\varphi \to Y^\varphi$ is given by $F_t(y, u) = (y, u+t)$ computed modulo identifications. Then, the projection $\pi_\varphi\colon Y^\varphi \to \Lambda$ defined as $\pi_\varphi(y, u) = T_u y$, is a semiconjucacy from F_t to T_t. Define the ergodic F_t-invariant probability measure $\mu^\varphi = (\mu \times \text{Lebesgue})/\bar{\varphi}$, where $\bar{\varphi} = \int_Y \varphi \, d\mu$. Then, $\mu_\Lambda = (\pi_\varphi)_* \mu^\varphi$ is an ergodic T_t-invariant probability measure on Λ.

Denote with $L_t : L^1(Y^\varphi) \to L^1(Y^\varphi)$ the transfer operator for F_t, so we have $\int (L_t v) w \, \mathrm{d}\mu^\varphi = \int v(w \circ F_t) \, \mathrm{d}\mu^\varphi$ for all $v \in L^1$, $w \in L^\infty$, $t \geq 0$. Define the transfer operator $P : L^1(Y) \to L^1(Y)$ for F, so $\int (Pv) w \, \mathrm{d}\mu = \int v(w \circ F) \, \mathrm{d}\mu$ for all $v \in L^1$ and $w \in L^\infty$; recall that $|Pv|_q \leq |v|_q$ for all $q \in [1, \infty]$. Recall (see for example [2]) that $(Pv)(y) = \sum_j g(y_j) v(y_j)$ where y_j is the unique preimage of y under $F|_{Y_j}$.

Function space on Λ For $d \geq 1$ and $\eta \in (0,1]$, we use the notations $\mathcal{C}^\eta(\Lambda, \mathbb{R}^d)$ and $\mathcal{C}_0^\eta(\Lambda, \mathbb{R}^d)$ from Subsection 3.1.1, integrating with respect to μ_Λ to centre.

For $v \in \mathcal{C}_0^\eta(\Lambda, \mathbb{R}^d)$, define W_n as

$$W_n(t) = \frac{1}{\sqrt{n}} \int_0^{nt} v \circ T_s \, \mathrm{d}s, \tag{3.6}$$

for $n \geq 1$ and $t \in [0,1]$. The process W_n is a random element in $\mathcal{C}([0,1], \mathbb{R}^d)$, defined on the probability space (Λ, μ_Λ). The following result is a consequence of Theorem 3.2 passed to the suspension [31, 45, 47].

Theorem 3.10. Let $T_t \colon \Lambda \to \Lambda$ be nonuniformly expanding of order 2 and $v \in \mathcal{C}_0^\eta(\Lambda, \mathbb{R}^d)$.

(i) The matrix $\Sigma = \lim_{n\to\infty} n^{-1} \int_\Lambda (\int_0^n v \circ T_s \, \mathrm{d}s)(\int_0^n v \circ T_s \, \mathrm{d}s)^T \, \mathrm{d}\mu_\Lambda$ is positive semidefinite. Typically Σ is positive definite: there exists a closed subspace $\mathcal{C}_{deg}$ of $\mathcal{C}_0^\eta(\Lambda, \mathbb{R}^d)$ with infinite codimension, such that $\det(\Sigma) \neq 0$ if $v \notin \mathcal{C}_{deg}$.

(ii) The WIP holds: $W_n \to_d W$ in $\mathcal{C}([0,1], \mathbb{R}^d)$, where W is a d-dimensional centred Brownian motion with covariance Σ. □

The following theorems are the continuous time versions of Theorems 3.3 and 3.6.

Theorem 3.11. Let $p \in (2,3)$, and suppose $v \in \mathcal{C}_0^\eta(\Lambda, \mathbb{R}^d)$ for $d \geq 1$. Then there is a constant $C > 0$ such that $\mathcal{W}(W_n, W) \leq C n^{-\frac{p-2}{2p}} (\log n)^{\frac{p-1}{2p}}$ for all integers $n > 1$.

Theorem 3.12. Let $p \in (2, \infty]$ and suppose $v \in \mathcal{C}_0^\eta(\Lambda, \mathbb{R})$. Then there exists $C > 0$ such that

$$\Pi(W_n, W) \leq C \begin{cases} n^{-\frac{p-2}{4p}} & p \in (2, \infty), \quad (3.7) \\ n^{-1/4} (\log n)^{3/4} & p = \infty \quad (3.8) \end{cases}$$

for all integers $n > 1$.

Remark 3.13. To our knowledge, Theorems 3.11 and 3.12 are the first rates for the WIP in the dynamical systems literature for continuous time. Note that Theorem 3.11 implies rates in Π by the same argument of Remark 3.8.

The remaining of this chapter is organized as follows. Section 3.2 recalls techniques from [32] and proves the rates for maps. Section 3.3 presents two new decompositions and estimates in continuous time for regular observables, extending the work of [32]. Finally, Section 3.4 uses the new estimates to prove the rates for semiflows.

3.2 Discrete time rates

In the first part of this section, we recall results from [32] in order to apply [19, Theorem 2.3(2)] and we prove Theorem 3.3. Then, in Subsection 3.2.3 we derive new estimates that are used, together with [18, Lemma 3], to prove Theorem 3.6.

3.2.1 Approximation via martingales

We present here the relevant results from [32] to obtain a Gordin-type [26] reversed martingale differences sequence with a control over the sum of its squares.

Let $T\colon \Lambda \to \Lambda$ be nonuniformly expanding of order $p \in [2, \infty]$ with ergodic invariant measure μ_Λ. We call an *extension* of $(\Lambda, T, \mathcal{B}, \mu_\Lambda)$ any measure-preserving system $(\Delta, f, \mathcal{A}, \mu_\Delta)$ with a measure-preserving $\pi_\Delta\colon \Delta \to \Lambda$, such that $T \circ \pi_\Delta = \pi_\Delta \circ f$. Denote with $P : L^1(\Delta) \to L^1(\Delta)$ the transfer operator for f with respect to μ_Δ, which is characterised by $\int (Pv)w \, d\mu_\Delta = \int v(w \circ f) \, d\mu_\Delta$ for all $v \in L^1$, $w \in L^\infty$. By Proposition 2.12(ii), we have $P(v \circ f) = v$ and $(Pv) \circ f = \mathbb{E}[v|f^{-1}\mathcal{A}]$ for any integrable v.

Proposition 3.14. Let $p \in [2, \infty)$. There is an extension $f\colon \Delta \to \Delta$ of $T\colon \Lambda \to \Lambda$ such that for any $v \in \mathcal{C}_0^\eta(\Lambda, \mathbb{R}^d)$ there exist $m \in L^p(\Delta, \mathbb{R}^d)$ and $\chi \in L^{p-1}(\Delta, \mathbb{R}^d)$ satisfying

$$v \circ \pi_\Delta = m + \chi \circ f - \chi, \qquad Pm = 0. \tag{3.9}$$

Moreover, there exists $C > 0$ such that

$$|m|_p \le C\|v\|_\eta \quad \text{and} \quad \left|\max\nolimits_{1\le k\le n} |\chi \circ f^k - \chi|\right|_p \le C\|v\|_\eta n^{\frac{1}{p}}, \tag{3.10}$$

for all $n \ge 1$. If $p = \infty$, then $m, \chi \in L^\infty(\Delta)$ with estimates

$$|m|_\infty \le C\|v\|_\eta \qquad \text{and} \qquad |\chi|_\infty \le C\|v\|_\eta. \tag{3.11}$$

Proof. Equations (3.9) and (3.10) are [32, Propositions 2.4, 2.5, 2.7]. The estimates (3.11) come from $\tau \in L^\infty$, using the arguments displayed before [32, Proposition 2.4]. □

We call m the *martingale part* of v and χ its *coboundary part*. It is relevant to cite [32, Corollary 2.12] which gives the identity $\Sigma = \int_\Delta mm^T \, d\mu_\Delta$, where Σ is the matrix defined in Theorem 3.2(i).

Proposition 3.15. Let $p \in [2, \infty)$. There exists $C > 0$ such that

$$\big|\max_{1\le k\le n} |\sum_{j=0}^{k-1}(\mathbb{E}[mm^T - \Sigma | f^{-1}\mathcal{A}]) \circ f^j|\big|_p \le C\|v\|_\eta^2 n^{\frac{1}{2}},$$

for every $n \ge 1$.

Proof. Let $\breve{\Phi} = (P(mm^T)) \circ f - \int_\Delta mm^T \, d\mu_\Delta$. By Proposition 2.12(ii), we have $\breve{\Phi} = \mathbb{E}[mm^T - \Sigma | f^{-1}\mathcal{A}]$ and hence the result follows by [32, Corollary 3.2]. □

Proposition 3.16. Let $n \ge 1$ and $0 \le k \ne \ell \le n-1$ be integers. Then

$$\mathbb{E}[(m \circ f^k)(m \circ f^\ell)^T | f^{-n}\mathcal{A}] = 0.$$

Proof. Without loss suppose $k < \ell$. By Proposition 2.12(ii),

$$\begin{aligned}
\mathbb{E}[(m \circ f^k)(m \circ f^\ell)^T | f^{-n}\mathcal{A}] &= (P^n[(m \circ f^k)(m \circ f^\ell)^T]) \circ f^n \\
&= (P^{n-k}P^k[(m(m \circ f^{\ell-k})^T) \circ f^k]) \circ f^n \\
&= (P^{n-k}[m(m \circ f^{\ell-k})^T]) \circ f^n.
\end{aligned}$$

The proof is finished because $P[m(m \circ f^{\ell-k})^T] = (Pm)(m \circ f^{\ell-k-1})^T = 0$. □

The next theorem is an adaptation of [19, Theorem 2.3(2)] for an RMDS (see Definition 2.20), which is our main tool to prove multidimensional rates for the WIP.

Theorem 3.17 (Cuny, Dedecker, Merlevède)**.** Let $p \in (2, 3)$ and $d \ge 1$. Suppose that $(d_n)_{n\ge0}$ is a $\mathbb{R}^d$-valued stationary RMDS in L^p with σ-algebras $(\mathcal{G}_n)_{n\ge0}$. Let $M_n = \sum_{k=0}^{n-1} d_k$ for $n \ge 1$. Assume moreover that

$$\sum_{n=1}^{\infty} \frac{1}{n^{3-p/2}} \big|\mathbb{E}[M_n M_n^T | \mathcal{G}_n] - \mathbb{E}[M_n M_n^T]\big|_{p/2} < \infty. \tag{3.12}$$

Then, there is $C > 0$ and there exists a probability space supporting a sequence of random variables $(M_n^*)_{n\ge1}$ with the same joint distributions as $(M_n)_{n\ge1}$ and a sequence $(N_n)_{n\ge0}$ of iid $\mathbb{R}^d$-valued centred Gaussians with $\mathrm{Var}(N_0) = \mathbb{E}[d_0 d_0^T]$, such that for every integer $n > 1$,

$$\big|\max_{1\le k\le n} |M_k^* - \sum_{\ell=0}^{k-1} N_\ell|\big|_1 \le C n^{\frac{1}{p}} (\log n)^{\frac{p-1}{2p}}. \tag{3.13}$$

Proof. This proposition is a version of [19, Theorem 2.3(2)] for $p \in (2, 3)$. Such a theorem is stated for a martingale differences sequence, however [19, Remark 2.7] affirms that its thesis is true for reversed martingale differences sequences as well. To prove the sufficiency of condition (3.12), reason as in [19, Remark 2.4]. □

The last theorem of this subsection is a version of [18, Lemma 3] stated for a bounded RMDS. It will be used to prove one-dimensional rates in the WIP. See Appendix A for the details regarding [18] and general martingale theory.

Theorem 3.18 (Courbot)**.** Let $(d_n)_{n\geq 0}$ be a $\mathbb{R}$-valued bounded stationary RMDS with σ-algebras $(\mathcal{G}_n)_{n\geq 0}$. Consider W a real centred Brownian motion on $[0,1]$, with variance $\sigma^2 = \mathbb{E}[d_0^2]$. Define for $1 \leq k \leq n$ the process $M_n^c \colon [0,1] \to \mathbb{R}$ as $M_n^c(k/n) = n^{-\frac{1}{2}} \sum_{j=1}^k d_{n-j}$, using linear interpolation in $[0,1]$, and let us define $V_n(k) = n^{-1} \sum_{j=1}^k \mathbb{E}[d_{n-j}^2 | \mathcal{G}_{n-(j-1)}]$. Let

$$\kappa_n = \inf\big\{\varepsilon > 0 : \mathbb{P}\big(\max_{0\leq k\leq n} |V_n(k) - (k/n)\sigma^2| > \varepsilon\big) \leq \varepsilon\big\}, \tag{3.14}$$

$$\widetilde{\kappa}_n = \max\big\{\kappa_n |\log \kappa_n|^{-\frac{1}{2}}, n^{-\frac{1}{2}}\big\}. \tag{3.15}$$

Then, there exists $C > 0$ such that

$$\Pi(M_n^c, W) \leq C\widetilde{\kappa}_n^{1/2} |\log \widetilde{\kappa}_n|^{3/4}$$

for all $n \geq 1$ for which $\widetilde{\kappa}_n \in (0, \frac{1}{2})$. □

3.2.2 Proof of Theorem 3.3

For fixed $v \in \mathcal{C}_0^\eta(\Lambda, \mathbb{R}^d)$ with martingale part $m \in L^p(\Delta, \mathbb{R}^d)$, $p \in (2,\infty)$, define the sequence of processes $X_n \colon [0,1] \to \mathbb{R}^d$, $n \geq 1$,

$$X_n(k/n) = \frac{1}{\sqrt{n}} \sum_{j=0}^{k-1} m \circ f^j, \tag{3.16}$$

for $0 \leq k \leq n$, and using linear interpolation in $[0,1]$. Recall that the sequence B_n is defined as $B_n(k/n) = n^{-1/2} \sum_{j=0}^{k-1} v \circ T^j$ plus linear interpolation.

Remark 3.19. In spite of Theorem 3.3 being valid only for $p \in (2,3)$, we work with $p \in (2,\infty)$ where possible and restrict the range only when we apply Theorem 3.17

Lemma 3.20. There exists $C > 0$ such that $\mathcal{W}(B_n, X_n) \leq Cn^{-\frac{p-2}{2p}}$ for all $n \geq 1$.

Proof. By Proposition 3.14,

$$B_n(k/n) \circ \pi_\Delta - X_n(k/n) = n^{-\frac{1}{2}} \textstyle\sum_{j=0}^{k-1} (v \circ \pi_\Delta - m) \circ f^j = n^{-\frac{1}{2}} (\chi \circ f^k - \chi)$$

for $0 \le k \le n$. Since B_n and X_n are piecewise linear with the same interpolation nodes, equation (3.10) yields

$$\begin{aligned}\left|\sup_{t\in[0,1]}|B_n(t)\circ\pi_\Delta - X_n(t)|\right|_p &= \left|\sup_{t\in\{0,\frac{1}{n},\ldots,1\}}|B_n(t)\circ\pi_\Delta - X_n(t)|\right|_p \\ &= n^{-\frac{1}{2}}\left|\max_{1\le k\le n}|\chi\circ f^k - \chi|\right|_p \ll n^{-\frac{p-2}{2p}}.\end{aligned}$$

Since π_Δ is a semiconjugacy, for any $\psi \in \mathrm{Lip}_1$

$$\begin{aligned}\left|\int_\Lambda \psi(B_n)\,\mathrm{d}\mu_\Lambda - \int_\Delta \psi(X_n)\,\mathrm{d}\mu_\Delta\right| &\le \int_\Delta |\psi(B_n\circ\pi_\Delta) - \psi(X_n)|\,\mathrm{d}\mu_\Delta \\ &\le \left|\sup_{t\in[0,1]}|B_n(t)\circ\pi_\Delta - X_n(t)|\right|_p \ll n^{-\frac{p-2}{2p}},\end{aligned}$$

which completes the proof. □

Lemma 3.21. Let $\{\xi_n\}_{n\ge1}$ be a sequence of identically distributed real random variables, defined on the same probability space. If $a = \mathbb{E}[e^{\xi_1}] < \infty$, then we have that $\mathbb{E}[\max_{1\le k\le n}\xi_k] \le \log(na)$ for all $n \ge 1$.

Proof. We have that $e^{\max_{1\le k\le n}\xi_k} = \max_{1\le k\le n} e^{\xi_k} \le \sum_{k=1}^n e^{\xi_k}$. Since all ξ_k share the same distribution, $\mathbb{E}[e^{\max_{1\le k\le n}\xi_k}] \le \mathbb{E}[\sum_{k=1}^n e^{\xi_k}] = na$. Then by Jensen's inequality,

$$\mathbb{E}[\max_{1\le k\le n}\xi_k] \le \log\mathbb{E}[e^{\max_{1\le k\le n}\xi_k}] \le \log(na).$$ □

Lemma 3.22. Let W be a centred d-dimensional Brownian motion on $[0,1]$ with covariance Σ. Then $\mathbb{E}[e^{\sup_{t\in[0,1]}|W(t)|}] < \infty$.

Proof. Since Σ is symmetric and positive semidefinite, there exists an orthogonal $d \times d$ matrix P such that $P\Sigma P^T = \mathrm{diag}(\sigma_1^2,\ldots,\sigma_d^2)$, with $\sigma_i^2 \ge 0$. Then, PW is a centred Brownian motion with covariance $P\Sigma P^T$, and for all $1 \le i \le d$ the real-valued processes $(PW)_i$ are independent centred Brownian motions with variances σ_i^2. Let $\bar\sigma = \max_{1\le i\le d}\sigma_i^2$. If $\bar\sigma = 0$, then both PW and W are the constant zero process and the proof is finished. If $\bar\sigma > 0$, we use standard Gaussian estimates and get that for every $1 \le i \le d$ there exists $C_i > 0$ such that for all $s > 1$

$$\mathbb{P}\big(\sup_{t\in[0,1]}|(PW(t))_i| > s\big) \le C_i\exp\big(-s^2/(2\bar\sigma)\big).$$

Writing $\xi = \sup_{t\in[0,1]}|PW(t)|$, $\hat C = \sum_{i=1}^d C_i$, and $c = 2d^2\bar\sigma$, we get

$$\mathbb{P}(\xi > s) \le \sum_{i=1}^d \mathbb{P}\big(\sup_{t\in[0,1]}|PW_i(t)| > s/d\big) \le \hat C\exp(-s^2/c)$$

and

$$\mathbb{P}(e^\xi > s) = \mathbb{P}(\xi > \log s) \le \hat C\exp\big(-(\log s)^2/c\big).$$

Hence, by a change of variable $x = \log s$,

$$\mathbb{E}[e^{\xi}] = \int_0^1 \mathbb{P}(e^{\xi} > s)\,\mathrm{d}s + \int_1^{\infty} \mathbb{P}(e^{\xi} > s)\,\mathrm{d}s \leq 1 + \hat{C}\int_0^{\infty} e^{-x^2/c+x}\,\mathrm{d}x < \infty.$$

By orthogonality, $|P^T x| = |x|$ for all $x \in \mathbb{R}^d$. Hence,

$$|W(t)| = |P^T PW(t)| = |PW(t)|$$

for every $t \in [0,1]$. Therefore, $\mathbb{E}[e^{\sup_{t\in[0,1]}|W(t)|}] = \mathbb{E}[e^{\xi}] < \infty$. □

Proposition 3.23. Let W be a centred d-dimensional Brownian motion on $[0,1]$, and let $(N_n)_{n\geq 0}$ be a sequence of iid $\mathbb{R}^d$-valued centred Gaussians with variance $\mathrm{Var}(W(1))$. Define the sequence of processes $Y_n\colon [0,1] \to \mathbb{R}^d$ for $0 \leq k \leq n$ as $Y_n(k/n) = n^{-1/2}\sum_{j=0}^{k-1} N_j$ and with linear interpolation. Then, there exists $C > 0$ such that $\mathcal{W}(Y_n, W) \leq Cn^{-\frac{1}{2}}\log n$ for all integers $n > 1$.

Proof. Define the sequence $Y_n^*\colon [0,1] \to \mathbb{R}^d$ as $Y_n^*(k/n) = W(k/n)$ for $0 \leq k \leq n$, plus linear interpolation. We have that $Y_n =_d Y_n^*$ as continuous processes for all $n \geq 1$. So, for $\psi \in \mathrm{Lip}_1$,

$$|\mathbb{E}[\psi(Y_n)] - \mathbb{E}[\psi(W)]| = |\mathbb{E}[\psi(Y_n^*) - \psi(W)]| \leq \mathbb{E}[\sup_{t\in[0,1]}|Y_n^*(t) - W(t)|] \leq A_1 + A_2,$$

where

$$A_1 = \mathbb{E}[\sup_{t\in[0,1]}|Y_n^*(t) - W(\lfloor nt\rfloor/n)|] \quad\text{and}\quad A_2 = \mathbb{E}[\sup_{t\in[0,1]}|W(\lfloor nt\rfloor/n) - W(t)|].$$

Since

$$\begin{aligned} A_1 &= \mathbb{E}[\max_{1\leq k\leq n}|W(k/n) - W((k-1)/n)|] \\ &\leq \mathbb{E}[\max_{1\leq k\leq n}\sup_{t\in(\frac{k-1}{n},\frac{k}{n})}|W(t) - W((k-1)/n)|] = A_2, \end{aligned}$$

it is enough to estimate A_2. By the rescaling property, $\widehat{W}_n(t) = n^{\frac{1}{2}}W(t/n)$ is a centred Brownian motion on $[0,n]$ for every $n \geq 1$, with the same covariance as W. Let $(\xi_k)_{k\geq 1}$ be a identically distributed sequence of random variables with $\xi_1 =_d \sup_{t\in[0,1]}|W(t)|$. Then, for every $1 \leq k \leq n$,

$$\begin{aligned} \sup_{t\in(\frac{k-1}{n},\frac{k}{n})}|W(t) - W((k-1)/n)| &= n^{-\frac{1}{2}}\sup_{t\in(\frac{k-1}{n},\frac{k}{n})}|\widehat{W}_n(nt) - \widehat{W}_n(n(k-1))| \\ &= n^{-\frac{1}{2}}\sup_{t\in(k-1,k)}|\widehat{W}_n(t) - \widehat{W}_n(k-1)| \\ &=_d n^{-\frac{1}{2}}\xi_k. \end{aligned}$$

Lemma 3.22 yields that $\mathbb{E}[e^{\xi_1}] < \infty$, hence we can apply Lemma 3.21 getting that $A_2 = n^{-\frac{1}{2}}\mathbb{E}[\max_{1\leq k\leq n}\xi_k] \ll n^{-\frac{1}{2}}\log n$, which completes the proof. □

Proof of Theorem 3.3. Let $p \in (2,3)$ and let X_n be from (3.16). Consider W a Brownian motion on $[0,1]$ with mean 0 and covariance Σ, from Theorem 3.2(ii). Recall moreover that $\Sigma = \int_\Delta mm^T \, d\mu_\Delta$. By Lemma 3.20, it suffices to estimate $\mathcal{W}(X_n, W)$.

We claim that $M_n = \sum_{j=0}^{n-1} m \circ f^j$, $n \geq 1$, satisfies condition (3.12) on the probability space (Δ, μ_Δ). Since $Pm = 0$, Proposition 2.21 yields that $(m \circ f^n)_{n \geq 0}$ is an RMDS. It is in L^p by Proposition 3.14, and it is stationary because f^n is measure-preserving. In the following equation, the off-diagonal terms are zero by Proposition 3.16:

$$
\begin{aligned}
&\mathbb{E}[M_n M_n^T | f^{-n}\mathcal{A}] - \mathbb{E}[(M_n M_n^T] \\
&\qquad = \textstyle\sum_{k,\ell=0}^{n-1} \big(\mathbb{E}[(m \circ f^k)(m \circ f^\ell)^T | f^{-n}\mathcal{A}] - \mathbb{E}[(m \circ f^k)(m \circ f^\ell)^T] \big) \\
&\qquad = \textstyle\sum_{k=0}^{n-1} \big(\mathbb{E}[(mm^T) \circ f^k | f^{-n}\mathcal{A}] - \mathbb{E}[(mm^T) \circ f^k] \big) \\
&\qquad = \mathbb{E}\big[\textstyle\sum_{k=0}^{n-1} (mm^T - \Sigma) \circ f^k | f^{-n}\mathcal{A} \big].
\end{aligned}
$$

Using Proposition 3.15,

$$
\begin{aligned}
&\big| \mathbb{E}\big[\textstyle\sum_{k=0}^{n-1} (mm^T - \Sigma) \circ f^k | f^{-n}\mathcal{A} \big] \big|_{p/2} \\
&\qquad = \big| \mathbb{E}\big[\textstyle\sum_{k=0}^{n-1} \mathbb{E}[(mm^T - \Sigma) \circ f^k | f^{-k-1}\mathcal{A}] \big| f^{-n}\mathcal{A} \big] \big|_{p/2} \\
&\qquad \leq \big| \textstyle\sum_{k=0}^{n-1} \mathbb{E}[(mm^T - \Sigma) \circ f^k | f^{-k-1}\mathcal{A}] \big|_{p/2} \\
&\qquad = \big| \textstyle\sum_{k=0}^{n-1} \mathbb{E}[(mm^T - \Sigma) | f^{-1}\mathcal{A}] \circ f^k \big|_{p/2} \ll n^{\frac{1}{2}}.
\end{aligned}
$$

Hence for all $p \in (2,3)$ the series (3.12) converges, proving the claim.

By Theorem 3.17, there exists a probability space supporting a sequence $(M_n^*)_{n \geq 1}$ with the same joint distributions as $(M_n)_{n \geq 1}$ and a sequence $(N_n)_{n \geq 0}$ of iid $\mathbb{R}^d$-valued centred Gaussians with $\operatorname{Var}(N_0) = \mathbb{E}[mm^T] = \Sigma$, such that (3.13) holds.

Let Y_n be as in Proposition 3.23 and let $M_0^* = 0$. Define for $n \geq 1$ the process $X_n^* \colon [0,1] \to \mathbb{R}^d$ as $X_n^*(k/n) = n^{-\frac{1}{2}} M_k^*$ for $0 \leq k \leq n$, with linear interpolation. We have that $X_n^* =_d X_n$ as continuous processes. By Proposition 3.23, we have that $\mathcal{W}(X_n, W) \ll \mathcal{W}(X_n, Y_n) + n^{-\frac{1}{2}} \log n$. Using (3.13), we have that for all $\psi \in \mathrm{Lip}_1$,

$$
\begin{aligned}
\mathcal{W}(X_n, Y_n) &\leq \mathbb{E}[\psi(X_n^*) - \psi(Y_n)] \leq \mathbb{E}[\textstyle\sup_{t \in [0,1]} |X_n^*(t) - Y_n(t)|] \\
&= n^{-\frac{1}{2}} \big| \textstyle\max_{1 \leq k \leq n} |M_k^* - \sum_{\ell=0}^{k-1} N_\ell| \big|_1 \ll n^{-\frac{p-2}{2p}} (\log n)^{\frac{p-1}{2p}}.
\end{aligned}
$$

Hence $\mathcal{W}(X_n, W) \ll n^{-\frac{p-2}{2p}} (\log n)^{\frac{p-1}{2p}}$ and the proof is complete. □

3.2.3 Using bounded martingales

Let T be nonuniformly expanding of order ∞. For $v \in \mathcal{C}_0^\eta(\Lambda, \mathbb{R})$, we consider $m \in L^\infty(\Delta)$ from Proposition 3.14 and write $\breve{\Phi} = \mathbb{E}[m^2 | f^{-1}\mathcal{A}] - \sigma^2$, where by [32, Corollary 2.12], $\sigma^2 = \int_\Delta m^2 \, \mathrm{d}\mu_\Delta$. As pointed out before [32, Corollary 3.2], we can write $\breve{\Phi} = \breve{m} + \breve{\chi} \circ f - \breve{\chi}$ for $\breve{m}, \breve{\chi} \colon \Delta \to \mathbb{R}$ with $P\breve{m} = 0$, which we call the *secondary martingale-coboundary decomposition* of v. Since the return time τ from Subsection 3.1.1 lies in L^∞, [32, Proposition 3.1] and the arguments displayed before [32, Proposition 2.4] yield that there is $C > 0$ such that

$$|\breve{m}|_\infty \le C\|v\|_\eta^2 \qquad \text{and} \qquad |\breve{\chi}|_\infty \le C\|v\|_\eta^2. \tag{3.17}$$

If $g \colon \Delta \to \mathbb{R}$ and $n \ge 1$, we use the notation $g_n = \sum_{j=0}^{n-1} g \circ f^j$.

Proposition 3.24 (Azuma-Hoeffding inequality [58, pg 237])**.** Let $M(n) = \sum_{j=1}^n X_j$, $n \ge 1$, be a real-valued martingale with $X_j \in L^\infty$ for $j \ge 1$. Then

$$\mathbb{P}\Big(\max_{1\le k\le n} |M(k)| \ge x\Big) \le \exp\bigg\{\frac{-x^2/2}{\sum_{j=1}^n |X_j|_\infty^2}\bigg\},$$

for every $x > 0$ and $n \ge 1$. □

Proposition 3.25. Let $v \in \mathcal{C}_0^\eta(\Lambda, \mathbb{R})$. There exist $a, C > 0$ such that

$$\mu_\Delta\Big(\max_{1\le k\le n} |\textstyle\sum_{j=0}^{n-1} \breve{\Phi} \circ f^j| \ge x\Big) \le C\exp\Big\{-\frac{ax^2}{n}\Big\},$$

for every $x > 0$ and $n \ge 1$.

Proof. Let $\breve{\Phi} = \breve{m} + \breve{\chi} \circ f - \breve{\chi}$. For any $k \ge 1$ we get $\breve{\Phi}_k = \breve{m}_k + \breve{\chi} \circ f^k - \breve{\chi}$, and by (3.17) there exists $K > 0$ such that $\max_{1\le k\le n} |\breve{\Phi}_k| \le \max_{1\le k\le n} |\breve{m}_k| + K$. So,

$$\begin{aligned}\mu_\Delta(\max\nolimits_{1\le k\le n} |\breve{\Phi}_k| \ge x) &\le \mu_\Delta(\max\nolimits_{1\le k\le n} |\breve{m}_k| + K \ge x) \\ &\le \mu_\Delta(\max\nolimits_{1\le k\le n} |\breve{m}_k| \ge x/2) + \mu_\Delta(K \ge x/2).\end{aligned} \tag{3.18}$$

If $m = 0$, we have automatically $\mu_\Delta(\max_{1\le k\le n} |\breve{m}_k| \ge x/2) = 0$. If $m \ne 0$, we use that $P\breve{m}=0$ to get from Proposition 2.21 that $(\breve{m} \circ f^n)_{n\ge 0}$ is an RMDS in (Δ, μ_Δ). By Proposition 2.22, for every $n \ge 1$ the process $\breve{M}_n(k) = \sum_{j=1}^k \breve{m} \circ f^{n-j}$, $1 \le k \le n$ is a martingale. Since $\breve{m}_k = \breve{M}_n(n) - \breve{M}_n(n-k)$, using the Proposition 3.24 and (3.17), there is $c > 0$ such that

$$\begin{aligned}\mu_\Delta(\max\nolimits_{1\le k\le n} |\breve{m}_k| \ge x/2) &\le \mu_\Delta(\max\nolimits_{1\le k\le n} |\breve{M}_n(k)| \ge x/4) \\ &\le \exp\bigg\{\frac{-x^2/32}{\sum_{j=1}^n |\breve{m}|_\infty^2}\bigg\} = \exp\Big\{-\frac{cx^2}{n}\Big\}.\end{aligned}$$

Since $\mu_\Delta(K \geq x/2) = 1$ for $x \leq 2K$ and 0 otherwise,

$$\mu_\Delta(K \geq x/2) \leq \exp\{4K^2 - x^2\} \leq \exp\{4K^2\}\exp\{-x^2/n\}.$$

Conclude by applying these estimates to (3.18). □

3.2.4 Proof of Theorem 3.6 ($p = \infty$)

For fixed $v \in \mathcal{C}_0^\eta(\Lambda, \mathbb{R})$ with martingale part $m \in L^\infty(\Delta)$, define the sequence of processes $Y_n \colon [0,1] \to \mathbb{R}$, $n \geq 1$

$$Y_n(k/n) = \frac{1}{\sqrt{n}} \sum_{j=1}^{k} m \circ f^{n-j},$$

for $1 \leq k \leq n$, using linear interpolation in $[0,1]$. Following [31, Lemma 4.8], let $h \colon \mathcal{C}[0,1] \to \mathcal{C}[0,1]$ be the linear operator $(h\psi)(t) = \psi(1) - \psi(1-t)$.

Lemma 3.26. There exists $C > 0$ such that $\Pi(h \circ B_n, Y_n) \leq Cn^{-\frac{1}{2}}$ for all $n \geq 1$.

Proof. The process $h \circ B_n$ is piecewise linear in $[0,1]$ with interpolation nodes k/n for $0 \leq k \leq n$, attaining values $h \circ B_n(k/n) = \sum_{j=n-k}^{n-1} v \circ f^j$. By (3.9),

$$\begin{aligned} h \circ B_n(k/n) \circ \pi_\Delta - Y_n(k/n) &= n^{-\frac{1}{2}} \big(\textstyle\sum_{j=n-k}^{n-1} v \circ \pi_\Delta \circ f^j - \sum_{j=1}^{k} m \circ f^{n-j}\big) \\ &= n^{-\frac{1}{2}}((v \circ \pi_\Delta)_n - (v \circ \pi_\Delta)_{n-k} - (m_n - m_{n-k})) \\ &= n^{-\frac{1}{2}}(\chi \circ f^n - \chi \circ f^{n-k}). \end{aligned}$$

Since $h \circ B_n \circ \pi_\Delta$ and Y_n have the same interpolation nodes, we have by (3.11),

$$\big|\sup_{t\in[0,1]} |h \circ B_n(t) \circ \pi_\Delta - Y_n(t)|\big|_\infty \leq 2n^{-\frac{1}{2}}|\chi|_\infty \ll n^{-\frac{1}{2}}.$$

Using that π_Δ is a semiconjugacy and applying (2.2),

$$\Pi(h\circ B_n, Y_n) = \Pi(h\circ B_n \circ \pi_\Delta, Y_n) \leq \big|\sup_{t\in[0,1]} |h\circ B_n(t)\circ\pi_\Delta - Y_n(t)|\big|_\infty \ll n^{-\frac{1}{2}}.$$ □

Lemma 3.27. There exists $C > 0$ such that $\Pi(Y_n, W) \leq Cn^{-\frac{1}{4}}(\log n)^{\frac{3}{4}}$ for all integers $n > 1$.

Proof. Let $d_n = m \circ f^n$ for $n \geq 0$. Since $m \in \ker P$, d_n is a stationary RMDS on (Δ, μ_Δ) with σ-algebras $(f^{-n}\mathcal{A})_{n\geq 0}$, by Proposition 2.21. Equation (3.11) yields that the sequence d_n is bounded. We adopt the same notation of Theorem 3.18, noting that $\sigma^2 = \int_\Delta m^2 \, d\mu_\Delta$,

$$V_n(k) = n^{-1} \textstyle\sum_{j=1}^{k} \mathbb{E}[m^2 \circ f^{n-j} | f^{-n-(j-1)}\mathcal{A}] = n^{-1} \sum_{j=1}^{k} \mathbb{E}[m^2 | f^{-1}\mathcal{A}] \circ f^{n-j},$$

and that Y_n coincides with M_n^c. We claim that

$$\kappa_n \ll \sqrt{n^{-1}\log n}.$$

Assuming the claim true, let us evaluate $\widetilde{\kappa}_n$ from (3.15). Note that $x \mapsto x^2(\log x)^{-1}$ is decreasing for $x \in (0,1)$. Hence $x \mapsto x^2|\log x|^{-1}$ is increasing and so is the function $x \mapsto x|\log x|^{-\frac{1}{2}}$. Since $\kappa_n \ll \sqrt{n^{-1}\log n}$, we get that

$$\kappa_n|\log \kappa_n|^{-\frac{1}{2}} \ll \sqrt{\frac{\log n}{n|\log\log n - \log n|}} \ll \frac{1}{\sqrt{n}}.$$

By definition, $\widetilde{\kappa}_n \ll n^{-\frac{1}{2}}$ as well, and the statement follows from Theorem 3.18.

Let us now prove the claim. Writing $\breve{\Phi} = \mathbb{E}[m^2|f^{-1}\mathcal{A}] - \sigma^2$ and $\breve{\Phi}_k = \sum_{j=0}^{k-1} \breve{\Phi} \circ f^j$,

$$V_n(k) - (k/n)\sigma^2 = n^{-1}\textstyle\sum_{j=1}^k \breve{\Phi} \circ f^{n-j} = n^{-1}(\breve{\Phi}_n - \breve{\Phi}_{n-k}),$$

for every $n \geq 1$. So, $\max_{0\leq k\leq n} |V_n(k) - (k/n)\sigma^2| \leq 2n^{-1}\max_{1\leq k\leq n}|\breve{\Phi}_k|$. By Proposition 3.25, there are $a, C > 0$ such that

$$\mu_\Delta\big(\max_{0\leq k\leq n}|V_n(k) - (k/n)\sigma^2| \geq \varepsilon\big) \leq \mu_\Delta\big(\max_{1\leq k\leq n}|\breve{\Phi}_k| \geq n\varepsilon/2\big) \leq Ce^{-an\varepsilon^2},$$

for all $\varepsilon \geq 0$ and $n \geq 1$. Let now $\varepsilon_n = \sqrt{\log n/(an)}$. We have that $C \leq n\varepsilon_n$ for n large enough and

$$\mu_\Delta\big(\max_{0\leq k\leq n}|V_n(k) - (k/n)\sigma^2| > \varepsilon_n\big) \leq C\exp\{-an\varepsilon_n^2\} = C/n \leq \varepsilon_n.$$

By definition (3.14), $\kappa_n \ll \varepsilon_n \ll \sqrt{n^{-1}\log n}$, which proves the claim. □

Proposition 3.28. Let $Z(t)$, $t \in [0,1]$, be a $\mathbb{R}^d$-valued continuous process with $Z(0) = 0$ a.s. and let $W(t)$, $t \in [0,1]$, be a d-dimensional Brownian motion. Then we have that $\Pi(Z,W) \leq 2\Pi(h \circ Z, W)$.

Proof. We follow the proof of [5, Theorem 2.2]. It is easy to see that $h \circ W =_d W$, because (i) $t \mapsto h\circ W(t)$ is continuous, (ii) $h \circ W(0) = 0$, (iii) for fixed $0 \leq s \leq t \leq 1$ we have $h\circ W(t) - h \circ W(s) = W(1-s) - W(1-t) =_d \mathcal{N}(0,(t-s)\Sigma)$, and (iv) for $k \geq 1$ and any partition $0 \leq t_1 < \cdots < t_k \leq 1$, the increments

$$W(1-t_1) - W(1-t_2), W(1-t_2) - W(1-t_3), \ldots, W(1-t_{k-1}) - W(1-t_k)$$

are independent by the properties of W.

Note that $h(hf) = f$ if $f(0) = 0$, and the map $h\colon \mathcal{C}[0,1] \to \mathcal{C}[0,1]$ is Lipschitz with $\mathrm{Lip}(h) \leq 2$. We conclude by the Lipschitz mapping theorem [56, Theorem 3.2],

$$\Pi(Z,W) = \Pi(h(h\circ Z), h(h\circ W)) \leq 2\Pi(h\circ Z, h \circ W) = 2\Pi(h \circ Z, W). \qquad \square$$

Proof of Theorem 3.6 ($p = \infty$). Since $B_n(0) = 0$ for all $n \geq 1$, Proposition 3.28 yields

$$\Pi(B_n, W) \ll \Pi(h \circ B_n, W) \leq \Pi(h \circ B_n, Y_n) + \Pi(Y_n, W).$$

Apply Lemmas 3.26 and 3.27 to finish. □

3.3 Martingale-coboundary decompositions for semiflows

Let $T_t \colon \Lambda \to \Lambda$ be a nonuniformly expanding semiflow of order $p \in [2, \infty]$ as in Subsection 3.1.3, which is semiconjugated through π_φ to a suspension semiflow $F_t \colon Y^\varphi \to Y^\varphi$. We recall from Subsection 3.1.3 that $r \in \mathcal{C}^\eta(X)$ and $\tau \in L^p(Y)$ are the return functions for respectively the flow T_t and the map T. Next Proposition proves some properties of the map $\varphi \colon Y \to [1, \infty)$ that was defined as $\varphi(y) = \sum_{j=0}^{\tau(y)-1} r(T^j y)$.

Proposition 3.29. We have that $\varphi \in L^p(Y, \mu)$ and there is $C > 0$ such that

$$|\varphi(x) - \varphi(y)| \leq C(\inf_{Y_j} \varphi) d_\Lambda(Fx, Fy)^\eta, \tag{3.19}$$

for every $j \geq 1$ and $x, y \in Y_j$. If $p \in [2, \infty)$, then

$$\textstyle\sum_j \mu(Y_j)(\sup_{Y_j} \varphi^p) < \infty. \tag{3.20}$$

Proof. Since $\tau \in L^p(Y, \rho|_Y)$ and $d\mu/d\rho \in L^\infty(Y)$, it follows that $\tau \in L^p(Y, \mu)$. Hence,

$$\textstyle\int_Y |\varphi|^p \, \mathrm{d}\mu = \int_Y |\sum_{j=0}^{\tau-1} r \circ T^j|^p \, \mathrm{d}\mu \leq |r|_\infty^p \int_Y |\tau|^p \, \mathrm{d}\mu = |r|_\infty^p |\tau|_p^p < \infty.$$

Recall that τ is constant on partition elements. Using that $r \in \mathcal{C}^\eta(X)$ and point (c) from Subsection 3.1.1, there exists $C > 0$ such that for each $j \geq 1$ and $x, y \in Y_j$,

$$|\varphi(x) - \varphi(y)| \leq \textstyle\sum_{\ell=0}^{\tau(x)-1} |r(T^\ell x) - r(T^\ell y)| \leq C\tau(x)|r|_\eta d_\Lambda(Fx, Fy)^\eta.$$

By $\inf_X r \geq 1$ and the definition of φ, we get $\tau(x) \leq \varphi(y)$ for all $y \in Y_j$, which implies that $\tau(x) \leq (\inf_{Y_j} \varphi)$. Equation (3.19) follows.

By (3.19), we get $\sup_{Y_j} \varphi - \inf_{Y_j} \varphi \leq C \operatorname{diam}(Y)^\eta (\inf_{Y_j} \varphi)$. Hence, there exists $K > 0$ such that $\sup_{Y_j} \varphi \leq K \inf_{Y_j} \varphi$ for all $j \geq 1$. So,

$$\textstyle\sum_j \mu(Y_j)(\sup_{Y_j} \varphi^p) \leq K^p \sum_j \mu(Y_j)(\inf_{Y_j} \varphi^p) \leq K^p |\varphi|_p^p < \infty.$$ □

Lemma 3.30. There exists $C >$ such that

$$d_\Lambda(T_u x, T_u y) \le C(\inf_{Y_j}\varphi) d_\Lambda(Fx, Fy)^\eta$$

for all $j \ge 1$, $x, y \in Y_j$, and $u \le \min\{\varphi(x), \varphi(y)\}$.

Proof. For $m \ge 1$ and $g\colon X \to \mathbb{R}$, write $S_m g = \sum_{j=0}^{m-1} g \circ T^j$. For $t \ge 0$ and $z \in X$, define the *lap number* $N_t(z) = m \ge 0$ to be the unique integer such that $S_m r(z) \le t < S_{m+1} r(z)$. Let x, y, u be as in the statement, and let $r \in \mathcal{C}^\eta(X, \mathbb{R})$ from Subsection 3.1.3.

Write $n = N_u(x)$, and let $K = |r|_\infty$ for estimate (3.3). We can write that $u = S_n r(x) + E(x)$, where $E(x) \le r(T^n x) \le |r|_\infty$. Then, (3.3) yields

$$d_\Lambda(T_u x, T_u y) \ll d_\Lambda(T_{S_n r(x)} x, T_{S_n r(x)} y). \tag{3.21}$$

Using (3.4),

$$\begin{aligned} d_\Lambda(T_{S_n r(x)} x, T_{S_n r(x)} y) &\le d_\Lambda(T_{S_n r(x)} x, T_{S_n r(y)} y) + d_\Lambda(T_{S_n r(y)} y, T_{S_n r(x)} y) \\ &= d_\Lambda(T^n x, T^n y) + d_\Lambda(T_{S_n r(y)} y, T_{S_n r(x)} y) \\ &\ll d_\Lambda(T^n x, T^n y) + |S_n r(x) - S_n r(y)|. \end{aligned}$$

By our assumptions on x, y, u, we can apply point (c) of Subsection 3.1.1 to get

$$d_\Lambda(T^n x, T^n y) \ll d_\Lambda(Fx, Fy) \le \operatorname{diam}(\Lambda)^{1-\eta} d_\Lambda(Fx, Fy)^\eta.$$

Using Hölder continuity of r and again point (c),

$$|S_n r(y) - S_n r(x)| \le \textstyle\sum_{j=0}^{n-1} |r(T^j x) - r(T^j y)| \ll |r|_\eta n d_\Lambda(Fx, Fy)^\eta.$$

Since $u \le \min\{\varphi(x), \varphi(y)\}$ and $\inf\varphi \ge 1$, we have that $n \le \inf_{Y_j}\varphi$. Therefore, $|S_n r(y) - S_n r(x)| \ll |r|_\eta (\inf_{Y_j}\varphi) d_\Lambda(Fx, Fy)^\eta$. Apply these estimates to (3.21) to finish. □

Function space on Y Define the spaces $\mathcal{C}^\eta(Y, \mathbb{R}^d)$ and $\mathcal{C}_0^\eta(Y, \mathbb{R}^d)$ analogously to $\mathcal{C}^\eta(\Lambda, \mathbb{R}^d)$ and $\mathcal{C}_0^\eta(\Lambda, \mathbb{R}^d)$, integrating on Y by μ to centre.

Function space on Y^φ Let $\eta \in (0, 1]$, $d \ge 1$, and let $Y_j^\varphi = \{(y, u) \in Y^\varphi : y \in Y_j\}$. For $v : Y^\varphi \to \mathbb{R}^d$, define $|v|_\infty = \sup_{(y,u)\in Y^\varphi} |v(y, u)|$ and

$$\|v\|_\eta = |v|_\infty + |v|_\eta, \qquad |v|_\eta = \sup_{j\ge1} \sup_{(x,u),(y,u)\in Y_j^\varphi,\ x\ne y} \frac{|v(x, u) - v(y, u)|}{d_\Lambda(Fx, Fy)^\eta (\inf_{Y_j}\varphi)^{\sqrt{\eta}}}.$$

Let $\mathcal{F}^\eta(Y^\varphi, \mathbb{R}^d)$ consist of observables $v : Y^\varphi \to \mathbb{R}^d$ with $\|v\|_\eta < \infty$. We have that $\mathcal{F}^\eta(Y^\varphi, \mathbb{R}^d)$ is a Banach space, because it is a closed subspace of the functions on on Y^φ which are Hölder continuous in the y variable. Define

$$\mathcal{F}_0^\eta(Y^\varphi, \mathbb{R}^d) = \{v \in \mathcal{F}^\eta(Y^\varphi, \mathbb{R}^d) : \textstyle\int_{Y^\varphi} v \, \mathrm{d}\mu^\varphi = 0\}.$$

Proposition 3.31. Let $v \in \mathcal{C}_0^\eta(\Lambda, \mathbb{R}^d)$. Then $v \circ \pi_\varphi \in \mathcal{F}_0^{\eta^2}(Y^\varphi, \mathbb{R}^d)$ and there exists $C > 0$ such that $\|v \circ \pi_\varphi\|_{\eta^2} \le C\|v\|_\eta$.

Proof. We have clearly that $|v \circ \pi_\varphi|_\infty \le |v|_\infty$. Let $j \ge 1$ and $(x, u), (y, u) \in Y_j^\varphi$, such that $u \le \min\{\varphi(x), \varphi(y)\}$. By Hölder continuity of v,

$$|v \circ \pi_\varphi(x, u) - v \circ \pi_\varphi(y, u)| = |v(T_u x) - v(T_u y)| \le |v|_\eta d_\Lambda(T_u x, T_u y)^\eta.$$

By Lemma 3.30, there exists $C > 0$ such that

$$d_\Lambda(T_u x, T_u y)^\eta \le C(\inf_{Y_j} \varphi)^\eta d_\Lambda(Fx, Fy)^{\eta^2}$$

Hence, $|v \circ \pi_\varphi|_{\eta^2} \le C|v|_\eta$, giving that $\|v \circ \pi_\varphi\|_{\eta^2} \le C\|v\|_\eta$. To finish the proof, we see that $\int_{Y^\varphi}(v \circ \pi_\varphi) \, \mathrm{d}\mu^\varphi = \int_\Lambda v \, \mathrm{d}\mu_\Lambda = 0$. □

The remainder of this section deals with observables in $\mathcal{F}^\eta(Y^\varphi, \mathbb{R}^d)$. By Proposition 3.31, this approach is sufficient to obtain the same decompositions and estimates for elements of $\mathcal{C}^{\sqrt{\eta}}(\Lambda, \mathbb{R}^d)$, via the semiconjugacy π_φ.

We present in the following two new decompositions for an observable and the square of its martingale part, in the style of Gordin [26]. This follows and extends the approach of [32] to continuous time.

Notation For $n \ge 1$ and $g \colon Y^\varphi \to \mathbb{R}^d$, write $g_n = \sum_{j=0}^{n-1} g \circ F_j$.

3.3.1 Primary decomposition

Given $v \in \mathcal{F}^\eta(Y^\varphi, \mathbb{R}^d)$, define $v' \colon Y \to \mathbb{R}^d$ as $v'(y) = \int_0^{\varphi(y)} v(y, u) \, \mathrm{d}u$. Recall that $P \colon L^1(Y) \to L^1(Y)$ is the transfer operator for F defined in Subsection 3.1.3.

Proposition 3.32. There exists a constant $C > 0$ such that $\|Pv'\|_\eta \le C\|v\|_\eta$ for all $v \in \mathcal{F}^\eta(Y^\varphi, \mathbb{R}^d)$. We have that $v' \in L^p(Y, \mathbb{R}^d)$, and if $v \in \mathcal{F}_0^\eta(Y^\varphi, \mathbb{R}^d)$ then $Pv' \in \mathcal{C}_0^\eta(Y, \mathbb{R}^d)$.

Proof. Let $x, y \in Y_j$ and suppose without loss that $\varphi(x) \le \varphi(y)$. By (3.19),

$$\begin{aligned} |v'(x) - v'(y)| \le & \textstyle\int_0^{\varphi(x)} |v(x,u) - v(y,u)|\,\mathrm{d}u + \int_{\varphi(x)}^{\varphi(y)} |v(y,u)|\,\mathrm{d}u \\ \ll & (|v|_\eta (\inf_{Y_j} \varphi)^{\sqrt{\eta}})(\sup_{Y_j} \varphi) + |v|_\infty (\inf_{Y_j} \varphi) d_\Lambda(Fx, Fy)^\eta \\ \le & \|v\|_\eta (\sup_{Y_j} \varphi^{1+\sqrt{\eta}}) d_\Lambda(Fx, Fy)^\eta. \end{aligned} \tag{3.22}$$

Let now $x, y \in Y$, with preimages $x_j, y_j \in Y_j$ under F. Since $|v'| \le \varphi |v|_\infty$, we have that $|v'(x_j)| \le |v|_\infty (\sup_{Y_j} \varphi)$. Using (3.5), (3.22), and (3.20) with $p > 1$

$$\begin{aligned} |(Pv')(x) - (Pv')(y)| \le & \textstyle\sum_j |g(x_j) - g(y_j)||v'(x_j)| + \sum_j g(y_j)|v'(x_j) - v'(y_j)| \\ \ll & \|v\|_\eta \big(\textstyle\sum_j \mu(Y_j)(\sup_{Y_j} \varphi^{1+\sqrt{\eta}})\big) d_\Lambda(Fx_j, Fy_j)^\eta \ll \|v\|_\eta d_\Lambda(x,y)^\eta. \end{aligned}$$

Similarly, (3.20) yields also that $|Pv'|_\infty \ll |v|_\infty$, giving $\|Pv'\|_\eta \ll \|v\|_\eta$.

Using $|v'|_p \le |\varphi|_p |v|_\infty$, we see that $v' \in L^p(Y, \mathbb{R}^d)$. If moreover $v \in \mathcal{F}_0^\eta(Y^\varphi, \mathbb{R}^d)$, then

$$\textstyle\int_Y v'\,\mathrm{d}\mu = \int_Y \int_0^{\varphi(y)} v(y,u)\,\mathrm{d}u\,\mathrm{d}\mu = \bar{\varphi} \int_{Y^\varphi} v\,\mathrm{d}\mu^\varphi = 0.$$

Hence, $\int Pv'\,\mathrm{d}\mu = \int v'\,\mathrm{d}\mu = 0$ and so $Pv' \in \mathcal{C}_0^\eta(Y; \mathbb{R}^d)$. □

Define $\chi', m' \colon Y \to \mathbb{R}^d$ as follows:

$$\chi' = \textstyle\sum_{k=1}^\infty P^k v', \qquad m' = v' - \chi' \circ F + \chi'.$$

It is well known for Gibbs-Markov maps (see [2, Theorem 1.6]), that for every $w \in \mathcal{C}_0^\eta(Y, \mathbb{R}^d)$ there are $a, C > 0$ such that $\|P^k w\|_\eta \le C e^{-ak}$ for all $k \ge 1$. Since $Pv' \in \mathcal{C}^\eta(Y; \mathbb{R}^d)$, the series $\sum_{k=1}^\infty \|P^k v'\|_\eta = \sum_{k=0}^\infty \|P^k P v'\|_\eta$ converges. By completeness, $\chi' \in \mathcal{C}^\eta(Y; \mathbb{R}^d)$ and $Pm' = Pv' - \chi' + \sum_{k=2}^\infty P^k v' = 0$. We have that

$$\|\chi'\|_\eta \le \textstyle\sum_{k=0}^\infty \|P^k P v'\|_\eta \ll \|Pv'\|_\eta, \quad |m'|_p \le |\varphi|_p |v|_\infty + 2|\chi'|_\infty \ll |v|_\infty + \|Pv'\|_\eta. \tag{3.23}$$

Hence $m' \in L^p(Y, \mathbb{R}^d)$ and, by Proposition 3.32,

$$|m'|_p \ll \|v\|_\eta, \qquad \|\chi'\|_\eta \ll \|v\|_\eta. \tag{3.24}$$

Define $m, \chi \colon Y^\varphi \to \mathbb{R}^d$ by

$$\chi(y,u) = \chi'(y) + \int_0^u v(y,s)\,\mathrm{d}s, \qquad m(y,u) = \begin{cases} m'(y) & u \in [\varphi(y) - 1, \varphi(y)) \\ 0 & u \in [0, \varphi(y) - 1) \end{cases}. \tag{3.25}$$

Proposition 3.33. We have that $m \in L^p(Y^\varphi, \mathbb{R}^d)$ and $\chi \in L^{p-1}(Y^\varphi, \mathbb{R}^d)$, with the convention that $\infty - 1 = \infty$. Moreover, there exists $C > 0$ such that

$$|m|_p \le C\|v\|_\eta \qquad \text{and} \qquad |\chi|_{p-1} \le C\|v\|_\eta,$$

for all $v \in \mathcal{F}_0^\eta(Y^\varphi, \mathbb{R}^d)$.

Proof. First, suppose that $p = \infty$. Then, by (3.23) and (3.25),

$$|\chi|_\infty \le |\chi'|_\infty + |\varphi|_\infty |v|_\infty \ll |v|_\infty + \|Pv'\|_\eta, \qquad |m|_\infty = |m'|_\infty \ll |v|_\infty + \|Pv'\|_\eta. \tag{3.26}$$

By Proposition 3.32, $|v|_\infty + \|Pv'\|_\eta \ll \|v\|_\eta$ which concludes the first case.

Second, suppose that $p \in [2, \infty)$. By (3.23),

$$|\chi(y,u)| \le |\chi'|_\infty + u|v|_\infty \ll \varphi(y)(|v|_\infty + \|Pv'\|_\eta).$$

Hence,

$$|\chi|_{p-1} \ll (|v|_\infty + \|Pv'\|_\eta)(\textstyle\int_Y \int_0^\varphi |\varphi|^{p-1}\,\mathrm{d}s\,\mathrm{d}\mu)^{\frac{1}{p-1}} = (|v|_\infty + \|Pv'\|_\eta)|\varphi|_p^{\frac{p}{p-1}} < \infty.$$

Since $m' \in L^p(Y, \mathbb{R}^d)$, (3.25) and (3.23) yield

$$|m|_p^p \ll \textstyle\int_Y \int_0^\varphi |m'|^p \mathbb{1}_{\{\varphi - 1 \le u < \varphi\}}\,\mathrm{d}u\,\mathrm{d}\mu = |m'|_p^p \ll (|v|_\infty + \|Pv'\|_\eta)^p < \infty.$$

So,

$$|\chi|_{p-1} \ll |v|_\infty + \|Pv'\|_\eta \qquad \text{and} \qquad |m|_p \ll |v|_\infty + \|Pv'\|_\eta. \tag{3.27}$$

The statement follows by $|v|_\infty + \|Pv'\|_\eta \ll \|v\|_\eta$. ☐

Recall that $L_1 \colon L^1(Y^\varphi) \to L^1(Y^\varphi)$ is the transfer operator for the one-time map $F_1 \colon Y^\varphi \to Y^\varphi$ of the suspension semiflow F_t defined in Subsection 3.1.3. In the next proposition, we show how L_1 acts pointwise on integrable observables.

Proposition 3.34. Let $v \in L^1(Y^\varphi)$. Then

$$(L_1 v)(y,u) = \begin{cases} v(y, u-1) & u \in [1, \varphi(y)) \\ \sum_j g(y_j) v(y_j, u - 1 + \varphi(y_j)) & u \in [0, 1) \end{cases}$$

Proof. Let $w \in L^\infty(Y^\varphi)$. By definition of L_1 and μ^φ, and by the substitution $u \mapsto u + 1$,

$$\begin{aligned} \textstyle\int_{Y^\varphi} L_1(\mathbb{1}_{\{0 \le u < \varphi - 1\}} v) w \,\mathrm{d}\mu^\varphi &= \bar{\varphi}^{-1} \textstyle\int_Y \int_0^{\varphi(y)} \mathbb{1}_{\{0 \le u < \varphi(y) - 1\}} v(y,u) w(y, u+1)\,\mathrm{d}u\,\mathrm{d}\mu \\ &= \textstyle\int_{Y^\varphi} \mathbb{1}_{\{1 \le u < \varphi(y)\}} v(y, u-1) w(y,u)\,\mathrm{d}\mu^\varphi. \end{aligned} \tag{3.28}$$

Next, let us focus on $\mathbb{1}_{\{\varphi-1\le u<\varphi\}}v$. By the substitution $u\mapsto u+1-\varphi(y)$,

$$\begin{aligned}\int_{Y^\varphi} L_1(\mathbb{1}_{\{\varphi-1\le u<\varphi\}}v)w\,\mathrm{d}\mu^\varphi &= \bar\varphi^{-1}\int_Y\int_{\varphi(y)-1}^{\varphi(y)} v(y,u)w(Fy,u+1-\varphi(y))\,\mathrm{d}u\,\mathrm{d}\mu \\ &= \bar\varphi^{-1}\int_Y\int_0^1 v(y,u-1+\varphi(y))w(Fy,u)\,\mathrm{d}u\,\mathrm{d}\mu.\end{aligned}$$

Write $\tilde v_u(y)=v(y,u-1+\varphi(y))$ and $w^u(y)=w(y,u)$. Then,

$$\begin{aligned}\int_{Y^\varphi} L_1(\mathbb{1}_{\{\varphi-1\le u<\varphi\}}v)w\,\mathrm{d}\mu^\varphi &= \bar\varphi^{-1}\int_0^1\int_Y \tilde v_u(w^u\circ F)\,\mathrm{d}\mu\,\mathrm{d}u \\ &= \bar\varphi^{-1}\int_0^1\int_Y (P\tilde v_u)w^u\,\mathrm{d}\mu\,\mathrm{d}u \\ &= \int_{Y^\varphi}\mathbb{1}_{\{0\le u<1\}}(P\tilde v_u)w\,\mathrm{d}\mu^\varphi.\end{aligned} \tag{3.29}$$

We have by (3.28) and (3.29) that

$$\begin{aligned}(L_1v)(y,u) &= L_1(\mathbb{1}_{\{0\le u<\varphi-1\}}v+\mathbb{1}_{\{\varphi-1\le u<\varphi\}}v)(y,u) \\ &= \mathbb{1}_{\{1\le u<\varphi\}}v(y,u-1)+\mathbb{1}_{\{0\le u<1\}}(P\tilde v_u)(y).\end{aligned}$$

The proof is completed by the pointwise formula for P. □

Proposition 3.35. Let $v\in\mathcal{F}_0^\eta(Y^\varphi,\mathbb{R}^d)$ and let $\psi\colon Y^\varphi\to\mathbb{R}^d$ be $\psi=\int_0^1 v\circ F_s\,\mathrm{d}s$. Then $\psi=m+\chi\circ F_1-\chi$ and $m\in\ker L_1$.

Proof. Let $(y,u)\in Y^\varphi$ with $u\in[0,\varphi(y)-1)$. Then $F_1(y,u)=(y,u+1)$ and $\psi(y,u)=\int_u^{u+1}v(y,s)\,\mathrm{d}s$, so

$$\chi(F_1(y,u))-\chi(y,u)=\int_0^{u+1}v(y,s)\,\mathrm{d}s-\int_0^u v(y,s)\,\mathrm{d}s=\psi(y,u)=\psi(y,u)-m(y,u).$$

If $u\in[\varphi(y)-1,\varphi(y))$, then

$$\psi(y,u)=\int_0^{u+1-\varphi(y)}v(Fy,s)\,\mathrm{d}s+v'(y)-\int_0^u v(y,s)\,\mathrm{d}s.$$

We have that $F_1(y,u)=(Fy,u+1-\varphi(y))$. By definition, $v'-m'=\chi'\circ F-\chi'$ and $m(y,u)=m'(y)$, so

$$\begin{aligned}\chi(F_1(y,u))-\chi(y,u) &= \chi'(Fy)-\chi'(y)+\int_0^{u+1-\varphi(y)}v(Fy,s)\,\mathrm{d}s-\int_0^u v(y,s)\,\mathrm{d}s \\ &= v'(y)-m'(y)+\psi(y,u)-v'(y)=\psi(y,u)-m(y,u).\end{aligned}$$

Therefore $\psi=m+\chi\circ F_1-\chi$ on the whole of Y^φ.

We are left to prove that $m\in\ker L_1$ using the formula of Proposition 3.34. Let $y\in Y$. If $u\in[1,\varphi(y))$, then $u-1\in[0,\varphi(y)-1)$ and by definition of m,

$$(L_1m)(y,u)=m(y,u-1)=0.$$

If $u \in [0,1)$, then $u-1+\varphi(y_j) \in [\varphi(y_j)-1, \varphi(y_j))$ for all preimages y_j of y, and

$$(L_1 m)(y,u) = \textstyle\sum_j g(y_j) m(y_j, u-1+\varphi(y_j)) = (Pm')(y) = 0,$$

because $m' \in \ker P$. □

Following the terminology of Section 3.2, the new functions m and χ are called respectively the martingale and coboundary part of v. In view of Proposition 3.35, to estimate the Birkhoff sums of ψ in p-norm, it would be desirable to have $\chi \in L^p$. This is indeed true for $p=\infty$ by Proposition 3.33; however, in general χ lies in L^{p-1}. The next results sort out this problem for $p \in [2,\infty)$, showing by the ideas of [32] that $\chi \circ F_1 - \chi$ lies in L^p for all $n \geq 1$.

Proposition 3.36. $\max_{1\le k\le n} |\chi \circ F_k| = o(n^{1/p})$ a.e. in Y^φ.

Proof. We follow the proof of [32, Proposition 2.6]. Since $\varphi \in L^p(Y)$, we have by the ergodic theorem $\varphi \circ F^n = o(n^{1/p})$ a.e. on Y, and so $\max_{0\le k\le n} \varphi \circ F^k = o(n^{1/p})$ a.e.

By definition (3.25) and equation (3.24), $|\chi(y,u)| \le |\chi'|_\infty + u|v|_\infty \ll \varphi(y)\|v\|_\eta$. For any $(y,u) \in Y^\varphi$ and $n \ge 0$, there exists $j \in \{0,\dots,n\}$ and $u' \in [0, \varphi(F^n y))$ such that $F_n(y,u) = (F^j y, u')$. Hence, $|\chi(F_n(y,u))| \ll \|v\|_\eta \max_{0\le k\le n} \varphi(F^k y)$, and therefore $\max_{0\le k\le n} |\chi(F_k(y,u))| \ll \|v\|_\eta \max_{0\le k\le n} \varphi(F^k y) = o(n^{1/p})$ a.e. □

Proposition 3.37. There exists $C>0$ such that $|\max_{1\le k\le n}|\chi\circ F_k - \chi||_p \le C\|v\|_\eta n^{1/p}$ for all $n \ge 1$. Moreover,

$$\big|\max_{1\le k\le n}|\chi\circ F_k - \chi|\big|_p \le C\|v\|_\eta \big(n^{1/q} + n^{1/p} |1_{\{\varphi \ge n^{1/q}\}}\varphi|_p\big) \tag{3.30}$$

for all $n \ge 1$, $q \ge p$, and $v \in \mathcal{F}_0^\eta(Y^\varphi, \mathbb{R}^d)$.

Proof. We follow the proof of [32, Proposition 2.7]. Define $t_a = |1_{\{\varphi\ge a\}}\varphi|_p$, $a \ge 0$ and the families of sets for $n \ge 1$

$$A_n = \{(y,u) \in Y^\varphi : u \le \varphi(y) - n\}, \quad B_n = \{(y,u) \in Y^\varphi : n \le u \le \varphi(y)\}.$$

We have that $\mu^\varphi(A_n) = \mu^\varphi(B_n) = \bar\varphi^{-1} \int_{\{\varphi\ge n\}} \int_n^\varphi \mathrm{d}u\,\mathrm{d}\mu$. So,

$$n^{p-1}\mu^\varphi(A_n) \le \bar\varphi^{-1} \int_{\{\varphi\ge n\}} \int_n^\varphi u^{p-1}\,\mathrm{d}u\,\mathrm{d}\mu \le \bar\varphi^{-1} \int_{\{\varphi\ge n\}} \varphi^p\,\mathrm{d}\mu \le t_n^p.$$

If $(y,u)\in A_n$ and $k=1,\dots,n$, then $u+k\le\varphi(y)$. Hence, using Equation (3.25) we have $(\chi\circ F_k-\chi)(y,u)=\int_0^k v(y,u+s)\,\mathrm{d}s$ and $\mathbb{1}_{A_n}\max_{1\le k\le n}|\chi\circ F_k-\chi|\le n|v|_\infty$. Therefore,

$$\begin{aligned}\big|\mathbb{1}_{A_n}\max\nolimits_{1\le k\le n}|\chi\circ F_k-\chi|\big|_p &\le n|v|_\infty(\mu^\varphi(A_n))^{1/p}\\ &= n^{1/p}|v|_\infty(n^{p-1}\mu^\varphi(A_n))^{1/p}\le n^{1/p}|v|_\infty t_n.\end{aligned}\tag{3.31}$$

Write $K_v=|v|_\infty+\|Pv'\|_\eta$. By (3.23), $|\chi(y,u)|\le|\chi'|_\infty+u|v|_\infty\ll\varphi(y)K_v$ for all $(y,u)\in Y^\varphi$. Define $\varphi_a=\mathbb{1}_{\{\varphi\ge a\}}\varphi$ for $a\ge 0$. Since $\varphi^p\le a^p+\varphi_a^p$,

$$\begin{aligned}K_v^{-p}\max\nolimits_{1\le k\le n}|\chi\circ F_k(y,u)-\chi(y,u)|^p &\le 2^pK_v^{-p}\max\nolimits_{0\le k\le n}|\chi\circ F_k(y,u)|^p\\ &\ll \max\nolimits_{0\le k\le n}\varphi^p(F^ky)\le a^p+\textstyle\sum_{k=0}^n\varphi_a^p(F^ky)\end{aligned}\tag{3.32}$$

for all $(y,u)\in Y^\varphi$ and $a\ge 0$.

For any function $w\colon Y^\varphi\to\mathbb{R}$ of the form $w(y,u)=w_0(y)$ with $w_0\in L^1(Y)$,

$$\textstyle\int_{Y^\varphi\setminus A_n}|w|\,\mathrm{d}\mu^\varphi=\bar\varphi^{-1}\int_Y\min\{\varphi,n\}|w_0|\,\mathrm{d}\mu\le\int_Y\min\{\varphi,n\}|w_0|\,\mathrm{d}\mu.\tag{3.33}$$

Take $v\equiv 1$ in Proposition 3.32, which gives $v'=\varphi$ and $|P^k\varphi|_\infty\le|P\varphi|_\infty\ll 1$ for all $k\ge 1$, because P is a contraction. By equations (3.32) and (3.33),

$$\begin{aligned}K_v^{-p}\textstyle\int_{Y^\varphi\setminus A_n}\max_{1\le k\le n}|\chi\circ F_k-\chi|^p\,\mathrm{d}\mu^\varphi &\le a^p+\textstyle\sum_{k=0}^n\int_{Y^\varphi\setminus A_n}\varphi_a^p(F^ky)\,\mathrm{d}\mu^\varphi(y,u)\\ \le a^p+\textstyle\sum_{k=0}^n\int_Y\min\{\varphi,n\}(\varphi_a^p\circ F^k)\,\mathrm{d}\mu &\le a^p+n|\varphi_a^p|_1+\textstyle\sum_{k=1}^n|\varphi(\varphi_a^p\circ F^k)|_1\\ &\ll a^p+n|\varphi_a^p|_1+\textstyle\sum_{k=1}^n|(P^k\varphi)\varphi_a^p|_1\ll a^p+n|\varphi_a^p|_1=a^p+nt_a^p.\end{aligned}$$

So,

$$\big|\mathbb{1}_{Y^\varphi\setminus A_n}\max\nolimits_{1\le k\le n}|\chi\circ F_k-\chi|\big|_p\ll K_v(a^p+nt_a^p)^{1/p}\le K_v(a+n^{1/p}t_a)^{1/p}.\tag{3.34}$$

Let $q\ge p$ and let $a=n^{1/q}$. Since $t_n\le t_{n^{1/q}}$, we can apply (3.31) and (3.34) to get

$$\big|\max\nolimits_{1\le k\le n}|\chi\circ F_k-\chi|\big|_p\ll K_v(n^{1/q}+n^{1/p}|\mathbb{1}_{\{\varphi\ge n^{1/q}\}}\varphi|_p).$$

Since $t_{n^{1/q}}\le|\varphi|_p$ for all $n\ge 1$, we take $q=p$ to get

$$\big|\max\nolimits_{1\le k\le n}|\chi\circ F_k-\chi|\big|_p\ll K_vn^{1/p}.\tag{3.35}$$

Proposition 3.35 implies that $K_v\ll\|v\|_\eta$, which concludes the proof. □

Corollary 3.38. $|\max_{1\le k\le n}|\chi\circ F_k-\chi||_p=o(n^{1/p})$.

Proof. We follow the proof of [32, Corollary 2.8]. Using that $\varphi \in L^p(Y)$, we have $|\mathbb{1}_{\{\varphi \geq n^{1/q}\}}\varphi|_p \to 0$ by the monotone convergence theorem. Let $q > p$, then Proposition (3.37) yields for $n \to \infty$ that

$$n^{-1/p}\big|\max_{1\leq k\leq n}|\chi \circ F_k - \chi|\big|_p \ll n^{-\frac{q-p}{pq}} + |\mathbb{1}_{\{\varphi \geq n^{1/q}\}}\varphi|_p \longrightarrow 0. \qquad \square$$

Remark 3.39. The results displayed in this subsection hold for $p \in (1, 2)$ as well (dropping the regularity on χ in Proposition 3.33). Note that for such a p, the series in the proof of Proposition 3.32 may not converge for the given $\eta \in (0, 1]$. In such a case, we choose a new $\eta' \in (0, \eta)$ such that $1 + \sqrt{\eta'} \leq p$ in order to apply (3.20) and prove instead that $Pv \in \mathcal{F}_{\eta'}(Y^\varphi, \mathbb{R}^d)$. This would not be an obstruction because of the inclusion of the Hölder spaces and the independence of the previous results on the exponent η.

Remark 3.40. The method adopted in the current subsection requires only the conditions: *(a)* $\int v \, d\mu^\varphi = 0$, *(b)* $v \in L^\infty$, *(c)* $\|Pv'\|_\eta < \infty$. Hence, for any observable $v \colon Y^\varphi \to \mathbb{R}^d$ satisfying *(a)*–*(c)*, we can decompose $\psi = \int_0^1 v \circ F_s \, ds = m + \chi \circ F_1 - \chi$, for some $m, \chi \colon Y^\varphi \to \mathbb{R}^d$, $m \in \ker L_1$. If $p \in [2, \infty]$, then $m \in L^p$ and $\chi \in L^{p-1}$ as in Proposition 3.33. Write $v' = \int_0^\varphi v \, du$. We get as in (3.26) and (3.27) that

$$|m|_p \ll |v|_\infty + \|Pv'\|_\eta, \qquad |\chi|_{p-1} \ll |v|_\infty + \|Pv'\|_\eta. \tag{3.36}$$

For $p \in [2, \infty)$, we have that $|\max_{1\leq k\leq n}|\chi \circ F_k - \chi||_p = o(n^{1/p})$ as in Corollary 3.38, and

$$\big|\max_{1\leq k\leq n}|\chi \circ F_k - \chi|\big|_p \leq C(|v|_\infty + \|Pv'\|_\eta)n^{1/p} \tag{3.37}$$

for all $n \geq 1$ as in (3.35).

3.3.2 Key estimates

We recall here Rio's inequality [34] which, using ideas of [41], yields useful estimates for the martingale-coboundary decomposition of any $v \in \mathcal{F}_0^\eta(Y^\varphi, \mathbb{R}^d)$. Rio's inequality is stated from [48].

Proposition 3.41 (Rio's inequality)**.** Let $(X_n)_{n\geq 1}$ be a sequence of L^2 random variables adapted to an increasing filtration $(\mathcal{G}_n)_{n\geq 1}$. Let $q \geq 1$ and define for $1 \leq i \leq n$

$$b_{i,n} = \max_{i\leq u\leq n}|X_i \textstyle\sum_{j=i}^u \mathbb{E}[X_j|\mathcal{G}_i]|_q.$$

There exists a universal $C_q > 0$ such that

$$\mathbb{E}[\max_{1\le k\le n} |\sum_{j=1}^{k} X_j|^{2q}] \le C_q(\sum_{i=1}^{n} b_{i,n})^q,$$

for all $n \ge 1$. □

Proposition 3.42. Let $p \in [2,\infty)$. There exists $C > 0$ such that

$$\big|\max_{1\le k\le n} |\sum_{j=0}^{k-1} m \circ F_j|\big|_p \le C(|v|_\infty + \|Pv'\|_\eta)n^{\frac{1}{2}}, \tag{3.38}$$

and

$$\big|\max_{1\le k\le n} |\int_0^k v \circ F_s \,\mathrm{d}s|\big|_{2(p-1)} \le C(|v|_\infty + \|Pv'\|_\eta)n^{\frac{1}{2}}, \tag{3.39}$$

for all $n \ge 1$ and any $v\colon Y^\varphi \to \mathbb{R}^d$ satisfying (a)–(c) from Remark 3.40.

Proof. We follow the proof of [32, Corollary 2.10]. Let $K_v = |v|_\infty + \|Pv'\|_\eta$ and let $m \in L^p \cap \ker L_1$ be from Remark 3.40. By Proposition 2.21, $(m\circ F_n)_{n\ge 0}$ is an RMDS and $|m|_p \ll K_v$ from (3.36). By Burkholder's inequality [13],

$$\big|\max_{1\le k\le n} |\sum_{j=1}^{k} m \circ F_{n-j}|\big|_p \ll |m|_p n^{\frac{1}{2}} \ll K_v n^{\frac{1}{2}}.$$

Equation (3.38) follows because $m_k = \sum_{j=1}^{n} m \circ F_{n-j} - \sum_{j=1}^{n-k} m \circ F_{n-j}$.

Let $\psi = \int_0^1 v \circ F_s \,\mathrm{d}s = m + \chi \circ F_1 - \chi$ from Remark 3.40, and $X_j = \psi \circ F_{n-j}$ with filtration $\mathcal{G}_j = F_{n-j}^{-1}\mathcal{B}$, for $1 \le j \le n$. By Proposition 3.35 we have that $\mathbb{E}[m \circ F_{n-j}|\mathcal{G}_i] = \mathbb{E}[m|F_{j-i}^{-1}\mathcal{B}] \circ F_{n-j} = (L_{j-i}m) \circ F_{n-j+1} = 0$, for $i < j \le n$. So,

$$\sum_{j=i}^{u} \mathbb{E}[X_j|\mathcal{G}_i] = m \circ F_{n-i} + \mathbb{E}[\chi \circ F_{n+1-u}|\mathcal{G}_i] - \chi \circ F_{n-i}.$$

By (3.36), $\max_{1\le i\le u\le n} |\sum_{j=i}^{u} \mathbb{E}[X_j|\mathcal{G}_i]|_{p-1} \ll K_v$. Hence, by $|X_j|_\infty \le |\psi|_\infty \le |v|_\infty$,

$$\max_{1\le i\le u\le n} |X_i \sum_{j=i}^{u} \mathbb{E}[X_j|\mathcal{G}_i]|_{p-1} \le |v|_\infty \max_{1\le i\le u\le n} |\sum_{j=i}^{u} \mathbb{E}[X_j|\mathcal{G}_i]|_{p-1} \ll K_v^2.$$

Defining $b_{i,n}$ as in Proposition 3.41, we get $\max_{1\le i\le n} b_{i,n} \ll K_v^2$. By Proposition 3.41 with $q = p-1$,

$$\big|\max_{1\le k\le n} |\sum_{j=1}^{k} X_j|\big|_{2q} \le C_q^{\frac{1}{2q}} (\sum_{i=1}^{n} b_{i,n})^{\frac{1}{2}} \ll (n \max_{1\le i\le n} b_{i,n})^{\frac{1}{2}} \ll n^{\frac{1}{2}} K_v.$$

Equation (3.39) follows by $\int_0^k v \circ F_s \,\mathrm{d}s = \psi_k = \sum_{j=1}^{n} X_j - \sum_{j=1}^{n-k} X_j$. □

Corollary 3.43. The limit $\Sigma = \lim_{n\to\infty} n^{-1} \int_{Y^\varphi} (\int_0^n v \circ F_s \,\mathrm{d}s)(\int_0^n v \circ F_s \,\mathrm{d}s)^T \,\mathrm{d}\mu^\varphi$ exists in $\mathbb{R}^{d\times d}$ for any $v \in \mathcal{F}_0^\eta(Y^\varphi, \mathbb{R}^d)$. Moreover, $\Sigma = \int_{Y^\varphi} mm^T \,\mathrm{d}\mu^\varphi$.

Proof. We follow the proof of [32, Corollary 2.12]. Let $\psi = \int_0^1 v \circ F_s \,\mathrm{d}s$. By Proposition 3.35, $\int_0^n v \circ F_s \,\mathrm{d}s = \psi_n = m_n + \chi \circ F_n - \chi$. Since $m \in \ker L_1$,

$$\int_{Y^\varphi} (m \circ F_i)(m^T \circ F_j) \,\mathrm{d}\mu^\varphi = \int_{Y^\varphi} (m \circ F_{i-j}) m^T \,\mathrm{d}\mu^\varphi = \int_{Y^\varphi} m (L_{i-j} m^T) \,\mathrm{d}\mu^\varphi = 0,$$

for all integers $0 \le j < i$. Hence, $\int_{Y^\varphi} m_n m_n^T \,\mathrm{d}\mu^\varphi = n \int_{Y^\varphi} m m^T \,\mathrm{d}\mu^\varphi$.

Using that $|xx^T - yy^T| \le (|x| + |y|)|x - y|$ for all $x, y \in \mathbb{R}^d$, and the Cauchy-Schwarz inequality,

$$\begin{aligned}
\Big| n^{-1} \int_{Y^\varphi} \psi_n \psi_n^T \,\mathrm{d}\mu^\varphi - \int_{Y^\varphi} m m^T \,\mathrm{d}\mu^\varphi \Big| &= n^{-1} \Big| \int_{Y^\varphi} \psi_n \psi_n^T \,\mathrm{d}\mu^\varphi - \int_{Y^\varphi} m_n m_n^T \,\mathrm{d}\mu^\varphi \Big| \\
&\le n^{-1} |\psi_n \psi_n^T - m_n m_n^T|_1 \le n^{-1} |(|\psi_n| + |m_n|)|\psi_n - m_n||_1 \\
&\le n^{-1} (|\psi_n|_2 + |m_n|_2) |\psi_n - m_n|_2 \\
&= n^{-1} (|\psi_n|_2 + |m_n|_2) |\chi \circ F_n - \chi|_2.
\end{aligned}$$

Since $v \in \mathcal{F}_0^\eta(Y^\varphi, \mathbb{R}^d)$ satisfies (a)–(c) from Remark 3.40, equation (3.38) gives that $|m_n|_2 \ll n^{1/2}$, and (3.39) yields $|\psi_n|_2 \ll n^{1/2}$. By Corollary 3.38, we have that $|\chi \circ F_n - \chi|_2 = o(n^{1/2})$. So,

$$\Big| n^{-1} \int_{Y^\varphi} \psi_n \psi_n^T \,\mathrm{d}\mu^\varphi - \int_{Y^\varphi} m m^T \,\mathrm{d}\mu^\varphi \Big| \ll n^{-1/2} \|v\|_\eta |\chi \circ F_n - \chi|_2 \longrightarrow 0.$$

The latter proves simultaneously that Σ exists and is equal to $\int_{Y^\varphi} m m^T \,\mathrm{d}\mu^\varphi$. □

Remark 3.44. Following the same approach of [32, Corollary 2.13], it is possible to provide another proof of the WIP for $p \ge 2$.

Proposition 3.45. Let $p = \infty$ and $v \colon Y^\varphi \to \mathbb{R}^d$ satisfying (a)–(c) from Remark 3.40. There exist $a, C > 0$ such that

$$\mu^\varphi\Big(\max_{1 \le k \le n} |\textstyle\int_0^k v \circ F_j| \ge x \Big) \le C \exp\Big\{ -\frac{a x^2}{n} \Big\},$$

for all $n \ge 1$ and $x > 0$.

Proof. Let $\psi = \int_0^1 v \circ F_s \,\mathrm{d}s$. Remark 3.40 yields that $\psi = m + \chi \circ F_1 - \chi$, with $m \in L^\infty \cap \ker L_1$. Then, $(m \circ F_n)_{n \ge 1}$ is a bounded RMDS by Proposition 2.21, and we have $\int_0^n v \circ F_s \,\mathrm{d}s = \psi_n = m_n + \chi \circ F_n - \chi$ for $n \ge 1$. To conclude, reason as in the proof of Proposition 3.25, replacing $\breve{\Phi}$, $\breve{m}$, $\breve{\chi}$, f, $\mathcal{A}$ with ψ, m, χ, F_1, $\mathcal{B}$ respectively, and using estimates (3.36) instead of (3.17). □

3.3.3 Secondary decomposition

For $v \in \mathcal{F}_0^\eta(Y^\varphi, \mathbb{R}^d)$, consider m from (3.25). Let $U_F v = v \circ F$ be the Koopman operator for F, and $U_t v = v \circ F_t$, $t \geq 0$, be the family of Koopman operators relative to the semiflow.

Proposition 3.46. $(U_1 L_1(mm^T))(y,u) = \begin{cases} (U_F P(m'm'^T)(y) & u \in [\varphi(y)-1, \varphi(y)) \\ 0 & u \in [0, \varphi(y)-1) \end{cases}$

Proof. Let $(y,u) \in Y^\varphi$. By Proposition 3.34 and the definition of m, if $u \in [1, \varphi(y))$,

$$(L_1(mm^T))(y,u) = mm^T(y, u-1) = 0; \tag{3.40}$$

and if $u \in [0,1)$

$$(L_1(mm^T))(y,u) = \textstyle\sum_j g(y_j) mm^T(y_j, u-1+\varphi(y_j)) = (P(m'm'^T))(y). \tag{3.41}$$

Let us analyse $U_1 L_1(mm^T)$. If $(y,u) \in Y^\varphi$ is such that $u \in [0, \varphi(y)-1)$, then $u+1 \in [1, \varphi(y))$ and by (3.40) we get

$$(U_1 L_1(mm^T))(y,u) = (L_1(mm^T))(y, u+1) = 0.$$

If $u \in [\varphi(y)-1, \varphi(y))$, then $u+1-\varphi(y) \in [0,1)$ and (3.41) yields that

$$(U_1 L_1(mm^T))(y,u) = (L_1(mm^T))(Fy, u+1-\varphi(y)) = (P(m'm'^T))(Fy),$$

finishing the proof. □

Recall that $\Sigma = \int mm^T \, \mathrm{d}\mu^\varphi$ and define

$$\breve{v} = U_1 L_1(mm^T) - \Sigma = \mathbb{E}[mm^T - \Sigma | F_1^{-1}\mathcal{B}]. \tag{3.42}$$

Let $\breve{v}'(y) = \int_0^{\varphi(y)} \breve{v}(y,u) \, \mathrm{d}u$, $y \in Y$.

Proposition 3.47. There exists $C > 0$ such that

$$|\breve{v}|_\infty \leq C\|v\|_\eta^2 \qquad \text{and} \qquad \|P\breve{v}'\|_\eta \leq C\|v\|_\eta^2,$$

for all $v \in \mathcal{F}_0^\eta(Y^\varphi, \mathbb{R}^d)$. Furthermore, $\int_{Y^\varphi} \breve{v} \, \mathrm{d}\mu^\varphi = 0$.

Proof. Since $\int U_1 L_1(mm^T) \, \mathrm{d}\mu^\varphi = \int L_1(mm^T) \, \mathrm{d}\mu^\varphi = \Sigma$, it follows that $\breve{v}$ has mean 0.

By definition of m', we see that $|m'| \leq |v'| + 2|\chi'|_\infty$. Using $|v'| \leq \varphi|v|_\infty$ and $\|\chi'\|_\eta \ll \|v\|_\eta$ from (3.24), we get that $|m'| \ll \varphi\|v\|_\eta$ and $|m'm'^T| \leq \varphi^2\|v\|_\eta^2$. By (3.5) and (3.20) with $p = 2$,

$$|P(m'm'^T)(y)| \leq \textstyle\sum_j g(y_j)|m'm'^T(y_j)| \ll \sum_j \mu(Y_j)(\sup_{Y_j} \varphi^2) \; \|v\|_\eta^2 \ll \|v\|_\eta^2, \tag{3.43}$$

for all $y \in Y$. Moreover, Proposition 3.33 gives that that $|m|_2 \ll \|v\|_\eta$. By Proposition 3.46 and (3.43),

$$|\breve{v}|_\infty \le |U_F P(m'm'^T)|_\infty + \left|\int_{Y^\varphi} mm^T \, \mathrm{d}\mu^\varphi\right| \le |P(m'm'^T)|_\infty + |m|_2^2 \ll \|v\|_\eta^2.$$

Let us now show the second estimate. Proposition 3.46 yields

$$\breve{v}'(y) = \int_0^{\varphi(y)} (U_F P(m'm'^T)(y) \mathbb{1}_{\{\varphi(y)-1<u<\varphi(y)\}} - \Sigma) \, \mathrm{d}u = (U_F P(m'm'^T))(y) - \varphi(y)\Sigma.$$

The identity $PU_F = \mathrm{Id}_{L^1(Y)}$ implies that $P\breve{v}' = P(m'm'^T) - (P\varphi)\Sigma$. Therefore, to finish it suffices to show that $\|P(m'm'^T)\|_\eta \ll \|v\|_\eta^2$ and $\|(P\varphi)\Sigma\|_\eta \ll \|v\|_\eta^2$.

Let us focus on $(P\varphi)\Sigma$. Apply Proposition 3.32 with $v \equiv 1$ to get that $v' = \varphi$ and $\|P\varphi\|_\eta \ll 1$. Hence, $\|(P\varphi)\Sigma\|_\eta = \|P\varphi\|_\eta |\Sigma| \ll |m|_2^2 \ll \|v\|_\eta^2$.

Next, let us focus on $P(m'm'^T)$. We know by (3.43) that $|P(m'm'^T)|_\infty \ll \|v\|_\eta^2$. Let $x, y \in Y_j$. By definition of m', equation (3.22) and $\chi' \in \mathcal{C}^\eta(Y; \mathbb{R}^d)$, we get

$$\begin{aligned}|m'(x) - m'(y)| &\le |v'(x) - v'(y)| + |\chi'(Fx) - \chi'(Fy)| + |\chi'(x) - \chi'(y)| \\ &\ll \|v\|_\eta (\sup\nolimits_{Y_j} \varphi) \, d_\Lambda(Fx, Fy)^\eta + \|\chi'\|_\eta d_\Lambda(Fx, Fy)^\eta + \|\chi'\|_\eta d_\Lambda(x, y)^\eta.\end{aligned}$$

By point (b) of Subsection 3.1.1, $|m'(x) - m'(y)| \ll \|v\|_\eta (\sup_{Y_j} \varphi) d_\Lambda(Fx, Fy)^\eta$. Using again that $|m'| \le \varphi \|v\|_\eta$,

$$\begin{aligned}|m'(x)m'(x)^T - m'(y)m'(y)^T| &\le (|m'(x)| + |m'(y)|)|m'(x) - m'(y)| \\ &\ll \|v\|_\eta^2 (\sup\nolimits_{Y_j} \varphi^2) d_\Lambda(Fx, Fy)^\eta.\end{aligned}$$

Fix $x, y \in Y$ with preimages $x_j, y_j \in Y_j$ under F. By (3.5) and (3.20) with $p = 2$,

$$\begin{aligned}|(P(m'm'^T))(x) - (P(m'm'^T))(y)| &\le \textstyle\sum_j |g(x_j) - g(y_j)||(m'm'^T)(x_j)| \\ &\quad + \textstyle\sum_j g(y_j)|(m'm'^T)(x_j) - (m'm'^T)(y_j)| \\ &\ll \|v\|_\eta^2 \textstyle\sum_j \mu(Y_j)(\sup_{Y_j} \varphi^2) \, d_\Lambda(Fx_j, Fy_j)^\eta \\ &\ll \|v\|_\eta^2 d_\Lambda(x, y)^\eta.\end{aligned}$$

We conclude that $\|P(m'm'^T)\|_\eta \ll \|v\|_\eta^2$. □

Remark 3.48. In view of Remark 3.40 and Proposition 3.47, we can write

$$\breve{\psi} = \int_0^1 \breve{v} \circ F_s \, \mathrm{d}s = \breve{m} + \breve{\chi} \circ F_1 - \breve{\chi},$$

which is the *secondary martingale-decomposition* of $v \in \mathcal{F}_0^\eta(Y^\varphi, \mathbb{R}^d)$.

We show now that the Birkhoff sum and integral of $\breve{v}$ are close. For $n \geq 1$, define $S_k\varphi = \sum_{j=0}^{k-1} \varphi \circ F^j$. For $(y,u) \in Y^\varphi$ and $t > 0$, define the *lap number* $N_t(y,u) = n \geq 0$ to be the unique integer such that $S_n\varphi(y) \leq t + u < S_{n+1}\varphi(y)$.

Proposition 3.49. Let $p \in [2,\infty)$. There exists $C > 0$ such that

$$\big|\int_0^n \breve{v} \circ F_s \,\mathrm{d}s - \sum_{j=0}^{n-1} \breve{v} \circ F_j\big|_\infty \leq C\|v\|_\eta^2,$$

for every $n \geq 1$ and $v \in \mathcal{F}_0^\eta(Y^\varphi, \mathbb{R}^d)$.

Proof. Define $\alpha = U_F P(m'm'^T)$. Proposition 3.46 gives that $(U_1L_1(mm^T))(y,u) = \alpha(y)\ \mathbb{1}_{\{\varphi(y)-1 \leq u < \varphi(y)\}}$ for all $(y,u) \in Y^\varphi$. The integral $\int_0^n (U_1L_1(mm^T)) \circ F_s \,\mathrm{d}s$ sums α along an orbit under F, with an error given by

$$\big|\int_0^n (U_1L_1(mm^T))(F_s(y,u)) \,\mathrm{d}s - \sum_{j=0}^{N_{n-1}(y,u)} \alpha(F_j y)\big| \leq |\alpha(y)| + |\alpha(F^{N_n(y,u)}y)| \leq 2|\alpha|_\infty, \tag{3.44}$$

for all $n \geq 1$ and $(y,u) \in Y^\varphi$.

We find that every initial point $(y,u) \in Y^\varphi$ enters the strip $[\varphi - 1, \varphi)$ exactly once every lap. Still, the sum $\sum_{j=0}^{n-1} (U_1L_1(mm^T)) \circ F_j$ could miss the term $\alpha \circ F^{N_{n-1}}$, giving that for every $(y,u) \in Y^\varphi$ and all $n \geq 1$,

$$\big|\sum_{j=0}^{n-1} (U_1L_1(mm^T))(F_j(y,u)) - \sum_{j=0}^{N_{n-1}(y,u)} \alpha(F_j y)\big| \leq |\alpha(F^{N_{n-1}(y,u)}y)| \leq |\alpha|_\infty. \tag{3.45}$$

Both (3.44) and (3.45) can be restated with infinity norms, because the estimates are uniform in (y,u). Combine (3.44) and (3.45), noticing that the two terms $n\Sigma$ cancel out:

$$\begin{aligned}\big|\int_0^n \breve{v} \circ F_s \,\mathrm{d}s - \sum_{j=0}^{n-1} \breve{v} \circ F_j\big|_\infty &= \big|\int_0^n (U_1L_1(mm^T)) \circ F_s \,\mathrm{d}s - \sum_{j=0}^{n-1} (U_1L_1(mm^T)) \circ F_j\big|_\infty \\ &\leq 3|\alpha|_\infty \leq 3|P(m'm'^T)|_\infty \ll \|v\|_\eta^2,\end{aligned}$$

where the last inequality is true by (3.43). □

Corollary 3.50. Let $p \in [2,\infty)$. There exists $C > 0$ such that

$$\big|\max_{1 \leq k \leq n} |\sum_{j=0}^{k-1} \breve{v} \circ F_j|\big|_{2(p-1)} \leq C\|v\|_\eta^2 n^{\frac{1}{2}},$$

for all $n \geq 1$ and $v \in \mathcal{F}_0^\eta(Y^\varphi, \mathbb{R}^d)$.

Proof. By Proposition 3.47 and Remark 3.48, we can apply equation (3.39) to $\breve{v}$, getting $|\max_{1 \leq k \leq n} |\int_0^k \breve{v} \circ F_j||_{2(p-1)} \ll (|\breve{v}|_\infty + \|P\breve{v}'\|_\eta) n^{\frac{1}{2}} \ll \|v\|_\eta^2 n^{\frac{1}{2}}$. The statement follows by Proposition 3.49. □

Corollary 3.51. Let $p = \infty$ and $v \in \mathcal{F}_0^\eta(Y^\varphi, \mathbb{R})$. There exist $a, C > 0$ such that

$$\mu^\varphi\Big(\max_{1\le k\le n} |\textstyle\sum_{j=0}^{k-1} \breve{v} \circ F_j| \ge x\Big) \le C \exp\Big\{-\frac{a\varepsilon^2}{n}\Big\},$$

for all $x > 0$ and $n \ge 1$.

Proof. By Proposition 3.49, there exists $K > 0$ such that

$$\max_{1\le k\le n}\big|\textstyle\sum_{j=0}^{k-1} \breve{v} \circ F_j\big| \le \max_{1\le k\le n}\big|\int_0^k \breve{v} \circ F_j\big| + K.$$

Hence,

$$\mu^\varphi\Big(\max_{1\le k\le n} |\textstyle\sum_{j=0}^{k-1} \breve{v} \circ F_j| \ge x\Big) \le \mu^\varphi\Big(\max_{1\le k\le n} |\int_0^k v \circ F_j| \ge x/2\Big) + \mu^\varphi(K \ge x/2). \quad (3.46)$$

The first term of the right-hand side of (3.46) is sorted by Propositions 3.45, while the second term is treated as in (3.25). □

3.4 Continuous time rates

Let $T_t \colon \Lambda \to \Lambda$, be a nonuniformly expanding semiflow of order $p \in (2, \infty]$ and let $v \in \mathcal{C}_0^\eta(\Lambda, \mathbb{R}^d)$. For $t \in [0, 1]$ and $n \ge 1$, let $W_n(t) = n^{-\frac{1}{2}} \int_0^{nt} v \circ T_s \, \mathrm{d}s$ be as in (3.6), and let W be a d-dimensional Brownian motion with mean 0 and covariance matrix Σ as in Theorem 3.10. This section provides the proofs of Theorems 3.11 and Theorem 3.12, getting rates for $\mathcal{W}(W_n, W)$ when $d \ge 1$ and for $\Pi(W_n, W)$ when $d = 1$.

Let $F_t \colon Y^\varphi \to Y^\varphi$ be the suspension semiflow semiconjugated to T_t by the map $\pi_\varphi \colon Y^\varphi \to \Lambda$, as described in Subsection 3.1.3. Let $w = v \circ \pi_\varphi$ which lies in $\mathcal{F}_0^{\eta^2}(Y^\varphi, \mathbb{R}^d)$ by Proposition 3.31, and define the sequence of processes $\widehat{W}_n$ on (Y^φ, μ^φ) as $\widehat{W}_n = W_n \circ \pi_\varphi$ for $n \ge 1$. Hence,

$$\widehat{W}_n(t) = \frac{1}{\sqrt{n}} \int_0^{nt} w \circ F_s \, \mathrm{d}s, \quad (3.47)$$

for $t \in [0, 1]$. Since π_φ is measure-preserving, we have that (i) $\Sigma = \int_{Y^\varphi} mm^T \, \mathrm{d}\mu^\varphi$ by Corollary 3.43, where m is the martingale part of w and (ii) $W_n =_d \widehat{W}_n$ for all $n \ge 1$, so $\mathcal{W}(W_n, W) = \mathcal{W}(\widehat{W}_n, W)$ and $\Pi(W_n, W) = \Pi(\widehat{W}_n, W)$. Henceforth, we work with $w \in \mathcal{F}_0^\eta(Y^\varphi, \mathbb{R}^d)$ and prove rates for $\widehat{W}_n$.

We recall from Proposition 3.35 that there exist $m, \chi \colon Y^\varphi \to \mathbb{R}^d$ such that

$$\psi = \textstyle\int_0^1 w \circ F_s \, \mathrm{d}s = m + \chi \circ F_1 - \chi. \quad (3.48)$$

By Proposition 3.33, for $p \in (2, \infty]$ there exists $C > 0$ such that

$$|m|_p \le C\|w\|_\eta, \qquad |\chi|_{p-1} \le C\|w\|_\eta; \tag{3.49}$$

and by Proposition 3.37, for $p \in (2, \infty)$

$$\big|\max_{1\le k\le n} |\chi \circ F_k - \chi|\big|_p \le C\|w\|_\eta n^{1/p}. \tag{3.50}$$

Notation For $n \ge 1$ and $g\colon Y^\varphi \to \mathbb{R}^d$, write $g_n = \sum_{j=0}^{n-1} g \circ F_j$.

3.4.1 Proof of Theorem 3.11

For $p \in (2, \infty)$ and $d \ge 1$, let $w \in \mathcal{F}_0^\eta(Y^\varphi, \mathbb{R}^d)$ and consider its martingale part $m \in L^p(Y^\varphi, \mathbb{R}^d)$. Define the sequence of processes $X_n\colon [0, 1] \to \mathbb{R}^d$ as

$$X_n(k/n) = \frac{1}{\sqrt{n}} \sum_{j=0}^{k-1} m \circ F_j, \tag{3.51}$$

for $n \ge 1$, $0 \le k \le n$, and using linear interpolation in $[0, 1]$. See Remark 3.19 for a brief comment on the range of p.

Lemma 3.52. Let $\{\xi_n\}_{n\ge1}$ be a sequence of identically distributed real random variables, defined on the same probability space. If $\xi_1 \in L^q$ for some $q \in [1, \infty)$, then $|\max_{1\le k\le n} |\xi_k||_q \le n^{1/q}|\xi_1|_q$ for all $n \ge 1$.

Proof. We have that $(\max_{1\le k\le n} |\xi_k|)^q = \max_{1\le k\le n} |\xi_k|^q \le \sum_{k=1}^n |\xi_k|^q$. Since all ξ_k share the same distribution, $\mathbb{E}[(\max_{1\le k\le n} |\xi_k|)^q] \le \mathbb{E}[\sum_{k=1}^n |\xi_k|^q] = n\mathbb{E}[|\xi_1|^q]$. The statement follows. □

Recall the processes $\widehat{W}_n$ be from (3.47) and X_n be from (3.51).

Lemma 3.53. There exists $C > 0$ such that $\mathcal{W}(\widehat{W}_n, X_n) \le Cn^{-\frac{p-2}{2p}}$ for all $n \ge 1$.

Proof. Let $\psi = \int_0^1 w \circ F_s \, \mathrm{d}s$. By equation (3.48), $\psi_k = m_k + \chi \circ F_k - \chi$, $k \ge 1$, and

$$\widehat{W}_n(t) - X_n(t) = n^{-1/2}(\psi_{\lfloor nt\rfloor/n} - m_{\lfloor nt\rfloor/n}) + R_n(t) = n^{-1/2}(\chi \circ F_{\lfloor nt\rfloor/n} - \chi) + R_n(t)$$

for all $t \in [0, 1]$, where $R_n(t) = (\widehat{W}_n(t) - \widehat{W}_n(\lfloor nt\rfloor/n)) - (X_n(t) - X_n(\lfloor nt\rfloor/n))$. So,

$$n^{\frac{1}{2}}|R_n(t)| \le \big|\textstyle\int_{\lfloor nt\rfloor}^{nt} w \circ F_s \, \mathrm{d}s\big| + |m \circ F_{\lfloor nt\rfloor-1}| \le |w|_\infty + \max_{1\le k\le n} |m \circ F_{k-1}|.$$

By Lemma 3.52 and (3.49),

$$n^{-\frac{1}{2}}\big|\max_{1\le k\le n} |m \circ F_{k-1}|\big|_p \le n^{-\frac{1}{2}+\frac{1}{p}}|m|_p \ll n^{-\frac{p-2}{2p}}\|w\|_\eta.$$

Hence,

$$\left|\sup_{t\in[0,1]}|R_n(t)|\right|_p \le n^{-\frac{1}{2}}(|w|_\infty + \left|\max_{1\le k\le n}|m\circ F_{k-1}|\right|_p) \ll n^{-\frac{p-2}{2p}}\|w\|_\eta.$$

By the estimate on R_n and (3.50),

$$\left|\sup_{t\in[0,1]}|\widehat{W}_n(t) - X_n(t)|\right|_p \ll n^{-\frac{1}{2}}\left|\max_{1\le k\le n}|\chi\circ f^k - \chi|\right|_p + n^{-\frac{p-2}{2p}} \ll n^{-\frac{p-2}{2p}}.$$

We finish the proof showing that for any $\psi \in \mathrm{Lip}_1$,

$$\left|\int_{Y^\varphi}\psi(\widehat{W}_n)\,\mathrm{d}\mu^\varphi - \int_{Y^\varphi}\psi(X_n)\,\mathrm{d}\mu^\varphi\right| \le \left|\sup_{t\in[0,1]}|\widehat{W}_n(t) - X_n(t)|\right|_p \ll n^{-\frac{p-2}{2p}}. \qquad \square$$

Proof of Theorem 3.11. Let $p \in (2,3)$, $w \in \mathcal{F}_0^\eta(Y^\varphi, \mathbb{R}^d)$ and let $\widehat{W}_n$ be from (3.47). Using Lemma 3.53, it suffices to estimate $\mathcal{W}(X_n, W)$.

We claim that $M_n = \sum_{j=0}^{n-1} m\circ F_j$, $n \ge 1$, satisfies (3.12) on the probability space (Y^φ, μ^φ). Since $L_1 m = 0$, Proposition 2.21 yields that $(m\circ F_n)_{n\ge 0}$ is an RMDS. It is in L^p by (3.49), and is stationary because F_n is measure-preserving. Since $m \in \ker L_1$, we can follow the proof of Proposition 3.16 and get that $\mathbb{E}[(m\circ F_k)(m\circ F_\ell)^T | F_n^{-1}\mathcal{B}] = 0$ for all $0 \le k \ne \ell \le n-1$. Using the notation $\check{v} = \mathbb{E}[mm^T - \Sigma | F_1^{-1}\mathcal{B}]$ from (3.42), we apply Corollary 3.50 and reason as in the proof of Theorem 3.3 from Subsection 3.2.2 to prove the claim.

Since M_n satisfies condition (3.12), we can now apply Theorem 3.17 and follow the proof of Theorem 3.3 in Subsection 3.2.2 to finish. $\square$

3.4.2 Proof of Theorem 3.12 ($p = \infty$)

Let $p = \infty$ and $w \in \mathcal{F}_0^\eta(Y^\varphi, \mathbb{R})$, with martingale part $m \in L^\infty(Y^\varphi, \mathbb{R})$. Define the sequence of processes $Y_n\colon [0,1] \to \mathbb{R}$, $n \ge 1$

$$Y_n(k/n) = \frac{1}{\sqrt{n}}\sum_{j=1}^{k} m\circ F_{n-j},$$

for $1 \le k \le n$, using linear interpolation in $[0,1]$. Let $h\colon \mathcal{C}[0,1] \to \mathcal{C}[0,1]$ be the linear operator $(h\psi)(t) = \psi(1) - \psi(1-t)$. Let $\widehat{W}_n$ be from (3.47).

Lemma 3.54. There exists $C > 0$ such that $\Pi(h\circ\widehat{W}_n, Y_n) \le Cn^{-\frac{1}{2}}$ for all $n \ge 1$.

Proof. Let $\psi = \int_0^1 w\circ F_s\,\mathrm{d}s$. By equation (3.48),

$$\begin{aligned} h\circ\widehat{W}_n(t) - Y_n(t) &= n^{-\frac{1}{2}}\big(\textstyle\int_{n-\lfloor nt\rfloor}^{n} w\circ F_s\,\mathrm{d}s - \sum_{j=1}^{\lfloor nt\rfloor - 1} m\circ F_{n-j}\big) + R_n(t) \\ &= n^{-\frac{1}{2}}\big(\psi_n - \psi_{n-\lfloor nt\rfloor} - \big(m_n - m_{n-\lfloor nt\rfloor}\big)\big) + R_n(t) \\ &= n^{-\frac{1}{2}}\big(\chi\circ F_n - \chi\circ F_{n-\lfloor nt\rfloor}\big) + R_n(t) \end{aligned}$$

for every $t \in [0,1]$, where

$$R_n(t) = h \circ (\widehat{W}_n(t) - \widehat{W}_n(\lfloor nt \rfloor/n)) - (Y_n(t) - Y_n(\lfloor nt \rfloor/n))$$
$$= (\widehat{W}_n((1-\lfloor nt \rfloor)/n) - \widehat{W}_n(1-t)) - (Y_n(t) - Y_n(\lfloor nt \rfloor/n)).$$

So,

$$n^{\frac{1}{2}}|R_n(t)| \leq \big|\textstyle\int_{1-t}^{1-\lfloor nt \rfloor} w \circ F_s \, \mathrm{d}s\big| + |m \circ F_{n-\lfloor nt \rfloor -1}| \leq |w|_\infty + |m|_\infty,$$

and by (3.49), $\big|\sup_{t\in[0,1]} |R_n(t)|\big|_\infty \ll n^{-\frac{1}{2}}\|w\|_\eta$. Hence,

$$\big|\sup_{t\in[0,1]} |h \circ \widehat{W}_n(t) - Y_n(t)|\big|_\infty \ll n^{-\frac{1}{2}}(2|\chi|_\infty + \|w\|_\eta) \ll n^{-\frac{1}{2}}\|w\|_\eta.$$

We conclude by (2.2) that

$$\Pi(h \circ \widehat{W}_n, Y_n) \leq \big|\sup_{t\in[0,1]} |h \circ \widehat{W}_n(t) - Y_n(t)|\big|_\infty \ll n^{-\frac{1}{2}}.$$ □

Lemma 3.55. There exists $C > 0$ such that $\Pi(Y_n, W) \leq C n^{-\frac{1}{4}}(\log n)^{\frac{3}{4}}$ for all integers $n > 1$.

Proof. Let $d_n = m \circ F_n$ for $n \geq 0$, which is a stationary RMDS on (Y^φ, μ^φ) with the σ-algebras $(F_n^{-1}\mathcal{B})_{n\geq 0}$ by Proposition 2.21. Equation (3.49) yields that the sequence d_n is bounded. We adopt the same notation of Theorem 3.18, noting that Y_n coincides with M_n^c, $\sigma^2 = \int_{Y^\varphi} m^2 \, \mathrm{d}\mu^\varphi$, and

$$V_n(k) = n^{-1} \textstyle\sum_{j=1}^k \mathbb{E}[m^2 \circ F_{n-j} | F_{n-(j-1)}^{-1}\mathcal{B}] = n^{-1} \sum_{j=1}^k \mathbb{E}[m^2 | F^{-1}\mathcal{B}] \circ F_{n-j}.$$

Following the proof of Lemma 3.27, to finish it suffices to show $\kappa_n \ll \sqrt{n^{-1}\log n}$. Writing $\breve{v} = \mathbb{E}[m^2 | F_1^{-1}\mathcal{B}] - \sigma^2$ as in (3.42), we have that

$$V_n(k) - (k/n)\sigma^2 = n^{-1} \textstyle\sum_{j=1}^k \breve{v} \circ F_{n-j} = n^{-1}(\breve{v}_n - \breve{v}_{n-k}),$$

for every $n \geq 1$. So, $\max_{0\leq k\leq n} |V_n(k) - (k/n)\sigma^2| \leq 2n^{-1} \max_{1\leq k\leq n} |\breve{v}_k|$. By Corollary 3.51, there are $a, C > 0$ such that

$$\mu_\Delta\big(\max_{0\leq k\leq n} |V_n(k) - (k/n)\sigma^2| \geq \varepsilon\big) \leq \mu_\Delta\big(\max_{1\leq k\leq n} |\breve{v}_k| \geq n\varepsilon/2\big) \leq C e^{-an\varepsilon^2},$$

for all $\varepsilon \geq 0$ and $n \geq 1$. Hence, we prove $\kappa_n \ll \sqrt{n^{-1}\log n}$ as in Lemma 3.27. □

Proof of Theorem 3.12 ($p = \infty$). Let $w \in \mathcal{F}_0^\eta(Y^\varphi, \mathbb{R}^d)$ and let $\widehat{W}_n$ be from (3.47). Since $\widehat{W}_n(0) = 0$ for all $n \geq 1$, Proposition 3.28 yield

$$\Pi(\widehat{W}_n, W) \ll \Pi(h \circ \widehat{W}_n, W) \leq \Pi(h \circ \widehat{W}_n, Y_n) + \Pi(Y_n, W).$$

Conclude by Lemmas 3.54 and 3.55. □

3.4.3 Proof of Theorem 3.12 ($p \in (2,\infty)$)

In the current subsection, we use the new estimates in Section 3.3 to adapt the method in [5, Section 4] to the semiflow case. The following results are proven by the same techniques of [5]. Let $p \in (2,\infty)$ and $w \in \mathcal{F}_0^\eta(Y^\varphi,\mathbb{R})$, with the martingale part $m \in L^p(Y^\varphi,\mathbb{R})$. Consider $\sigma^2 = \int_{Y^\varphi} m^2 \,\mathrm{d}\mu^\varphi$ and define the RMDS $d_n = (m\circ F_n)/(n^{1/2}\sigma)$ with σ-algebras $\mathcal{G}_n = F_n^{-1}\mathcal{B}$. Then $(d_{n-j})_{0\le j\le n}$ with filtration $(\mathcal{G}_{n-j})_{0\le j\le n}$ is a martingale differences array. For $0 \le k \le n$, let

$$V_n(k) = \textstyle\sum_{j=1}^k \mathbb{E}[d_{n-j}^2|\mathcal{G}_{n-(j-1)}].$$

Define now a sequence of processes $X_n\colon [0,1] \to \mathbb{R}$, $n \ge 1$, as

$$X_n\left(\frac{V_n(k)}{V_n(n)}\right) = \sum_{j=1}^k d_{n-j}, \tag{3.52}$$

for $0 \le k \le n$, and linear interpolation in $[0,1]$. As stated in [5], the integer k in (3.52) is a random variable $k = k_n(t)\colon Y^\varphi \to \{0,\dots,n\}$, that satisfies the inequalities $V_n(k) \le tV_n(n) < V_n(k+1)$.

Proposition 3.56. There exists $C > 0$ such that $\left|\sup_{t\in[0,1]} |k_n(t) - \lfloor nt\rfloor|\right|_{2(p-1)} \le Cn^{\frac{1}{2}}$ for all $n \ge 1$.

Proof. The proof is carried as [5, Proposition 4.4]. The only fact left to show is that

$$\left|\max_{1\le k\le n} |V_n(k) - k/n|\right|_{2(p-1)} \ll n^{\frac{1}{2}}. \tag{3.53}$$

By Corollary 2.14,

$$\begin{aligned} V_n(k) - \frac{k}{n} &= \frac{1}{n\sigma^2}\sum_{j=1}^k \mathbb{E}[m^2\circ F_{n-j}|F_{n-(j-1)}^{-1}\mathcal{B}] - \frac{1}{n\sigma^2}\sum_{j=1}^k \sigma^2 \\ &= \frac{1}{n\sigma^2}\sum_{j=1}^k \big(\mathbb{E}[m^2 - \sigma^2|F_1^{-1}\mathcal{B}]\circ F_j\big), \end{aligned}$$

and can prove (3.53) by Corollary 3.50. □

Proposition 3.57. For $n \ge 1$ and $\psi = \int_0^1 v\circ F_s \,\mathrm{d}s$, define the new function $Z_n = \max_{0\le i,\ell\le\sqrt{n}} |\psi_l| \circ F_{i\lfloor\sqrt{n}\rfloor}$.

(a) $\left|\sum_{j=a}^{b-1} \psi\circ F_j\right| \le Z_n((b-a)(\sqrt{n}-1)^{-1} + 3)$ for all $0 \le a < b \le n$.

(b) $|Z_n|_{2(p-1)} \le C\|w\|_\eta n^{1/4+1/(4(p-1))}$ for all $n \ge 1$.

Proof. Part (a) is proven as in [5, Proposition 4.6], using ψ in place of v. Part (b) follows as well, using finally (3.39) to get $\big|\max_{1\le k\le n}|\psi_k|\big|_{2(p-1)} \ll n^{1/2}$. □

Let $h\colon \mathcal{C}[0,1]\to\mathcal{C}[0,1]$ be the linear operator $(h\psi)(t) = \psi(1)-\psi(1-t)$, and recall the definition of $\widehat{W}_n$ from (3.47).

Lemma 3.58. There exists $C>0$ such that $\Pi(h\circ\widehat{W}_n,\sigma X_n)\le Cn^{-\frac{p-2}{4p}}$ for all $n\ge 1$.

Proof. We follow the proof of [5, Lemma 4.7]. Define the piecewise constant process $V_n'(t) = n^{-1/2}\sum_{j=n-\lfloor nt\rfloor}^{n-k-1}\psi\circ F_j$, $t\in[0,1]$, where $\psi = \int_0^1 w\circ F_s\,\mathrm{d}s = m+\chi\circ F_1-\chi$ from equation (3.48), and $k=k_n(t)$ is the random variable from (3.52). By equation (3.48),

$$\begin{aligned}
h\circ\widehat{W}_n(t)-\sigma X_n(t) &= n^{-\frac12}\big(\textstyle\int_{n-\lfloor nt\rfloor}^{n} w\circ F_s\,\mathrm{d}s-\sum_{j=1}^{k} m\circ F_{n-j}\big)+R_n(t)\\
&= n^{-\frac12}(\psi_n-\psi_{n-\lfloor nt\rfloor}-(m_n-m_{n-k}))+R_n(t)\\
&= n^{-\frac12}(\psi_{n-k}-\psi_{n-\lfloor nt\rfloor}+\chi\circ F_n-\chi\circ F_{n-k})+R_n(t)\\
&= V_n'(t)+n^{-\frac12}(\chi\circ F_n-\chi\circ F_{n-k})+R_n(t),
\end{aligned} \tag{3.54}$$

for every $t\in[0,1]$, where $\big|\sup_{t\in[0,1]}|R_n(t)|\big|_p\le n^{-\frac12}\big(|w|_\infty+\big|\max_{1\le k\le n}|m\circ F_{k-1}|\big|_p\big)$. Reasoning as in the proof of Lemma 3.53, we get $\big|\sup_{t\in[0,1]}|R_n(t)|\big|_p\ll n^{-\frac{p-2}{2p}}\|w\|_\eta$. Using (3.50),

$$\begin{aligned}
n^{-\frac12}\big|\sup_{t\in[0,1]}|\chi\circ F_n-\chi\circ F_{n-k_n(t)}|\big|_p &= n^{-\frac12}\big|\sup_{t\in[0,1]}|\chi\circ F_{k_n(t)}-\chi|\big|_p\\
&= n^{-\frac12}\big|\max_{1\le k\le n}|\chi\circ F_k-\chi|\big|_p\ll n^{-\frac{p-2}{2p}}.
\end{aligned}$$

By Propositions 3.56 and 3.57, and by Cauchy-Schwarz,

$$\begin{aligned}
\big|\sup_{t\in[0,1]}|V'(t)|\big|_{p-1} &\le n^{-\frac12}\big|Z_n(n^{-\frac12}\sup_{t\in[0,1]}|\lfloor nt\rfloor-k_n(t)|+3)\big|_{p-1}\\
&\le n^{-\frac12}|Z_n|_{2(p-1)}(n^{-\frac12}\big|\sup_{t\in[0,1]}|\lfloor nt\rfloor-k_n(t)|\big|_{2(p-1)}+3)\\
&\ll n^{-\frac12}|Z_n|_{2(p-1)}\ll n^{-(\frac14-\frac{1}{4(p-1)})}=n^{-\frac14\frac{p-2}{p-1}}.
\end{aligned}$$

Applying these estimates to (3.54), $|\sup_{t\in[0,1]}|h\circ\widehat{W}_n(t)-\sigma X_n(t)||_{p-1}\ll n^{-\frac14\frac{p-2}{p-1}}$. Finish by applying the top inequality of (2.2) with $q=p-1$. □

We now state [33, Theorem 1] of Kubilius, as done in [5, Theorem 4.2].

Theorem 3.59 (Kubilius)**.** Let $\delta\in[0,3/4]\cup\{1\}$, and let B be a standard Brownian motion on $[0,1]$. There is a constant $C>0$ such that $\Pi(X_n,B)\le C\lambda|\log\lambda|$ where $\lambda=\lambda_1+\lambda_2$ and

$$\begin{aligned}
\lambda_1 &= \inf_{0\le\varepsilon\le1}\big\{\varepsilon^{\frac12}+\big(\textstyle\int\big(\sum_{j=1}^n|d_{n-j}|^{2+2\delta}\mathbb{1}_{\{|d_{n-j}|>\varepsilon\}}\big)\,\mathrm{d}\mu^\varphi\big)^{1/(3+2\delta)}\big\},\\
\lambda_2 &= \inf_{0\le\varepsilon\le1}\big\{\varepsilon+\mu^\varphi(|V_n(n)-1|>\varepsilon^2)\big\}.
\end{aligned}$$

□

Lemma 3.60. There exists $C > 0$ such that $\Pi(X_n, B) \le C n^{-\frac{p-2}{4p}}$ for all $n \ge 1$.

Proof. We follow the proof of [5, Lemma 4.3]. Let $\lambda = \lambda_1 + \lambda_2$ as in Theorem 3.59. It suffices to show that

$$\lambda_1 \ll n^{-r_1(p)} \qquad \text{and} \qquad \lambda_2 \ll n^{-\frac{p-1}{4p-3}},$$

where

$$r_1(p) = \begin{cases} \frac{p-2}{2p+2} & 2 < p \le \frac{7}{2} \\ \frac{p-2}{4p-5} & \frac{7}{2} < p < 4 \\ \frac{p-2}{4p-6} & p \ge 4 \end{cases}.$$

Assuming the claim, we have that $\lambda_1, \lambda_2 \ll n^{-r_1(p)}$, and Theorem 3.59 yields $\Pi(X_n, B) \ll n^{-r_1(p)} \log n$. The result follows because $r_1(p) > \frac{p-2}{4p}$.

Let us prove the claim. Choose $\delta \in [0, 3/4] \cup \{1\}$ greatest such that $2 + 2\delta \le p$. Reasoning as in [5] by Hölder's inequality, and then Markov's inequality,

$$\begin{aligned} \sigma^{2+2\delta} \int \big(\textstyle\sum_{j=1}^n |d_{n-j}|^{2+2\delta} \mathbb{1}_{\{|d_{n-j}| \ge \varepsilon\}}\big)\, \mathrm{d}\mu^\varphi &\le n^{-\delta} |m|_p^{2+2\delta} \mu^\varphi(|m| \ge \varepsilon\sigma n^{1/2})^{(p-2-2\delta)/p} \\ &\le n^{-\delta} |m|_p^{2+2\delta} \Big(\tfrac{|m|_p^p}{\varepsilon^p \sigma^p n^{p/2}}\Big)^{(p-2-2\delta)/p} \\ &= \sigma^{-(p-2-2\delta)} |m|_p^p \varepsilon^{-(p-2-2\delta)} n^{-(p-2)/2}. \end{aligned}$$

By (3.49),

$$\lambda_1 \ll \inf_{0 \le \varepsilon \le 1} \big\{\varepsilon^{\frac{1}{2}} + \varepsilon^{-\frac{p-2-2\delta}{3+2\delta}} n^{-\frac{p-2}{6+4\delta}}\big\} \le 2n^{-\frac{p-2}{4p-4\delta-2}} = 2n^{-r_1(p)}.$$

For λ_2, we use (3.53) to get by Markov's inequality

$$\mu^\varphi(|V_n(n) - 1| > \varepsilon^2) \ll \varepsilon^{-4(p-1)} n^{-(p-1)}.$$

Hence,

$$\lambda_2 \ll \inf_{0 \le \varepsilon \le 1} \{\varepsilon + \varepsilon^{-4(p-1)} n^{-(p-1)}\} \le 2n^{-\frac{p-1}{4p-3}}.$$ □

Proof of Theorem 3.12 ($p \in (2, \infty)$). Let $w \in \mathcal{F}_0^\eta(Y^\varphi, \mathbb{R}^d)$ and let $\widehat{W}_n$ be from (3.47). Since $\widehat{W}_n(0) = 0$ for all $n \ge 1$, Proposition 3.28 yields $\Pi(\widehat{W}_n, W) \ll \Pi(h \circ \widehat{W}_n, W)$. Using that $W =_d \sigma B$, we get

$$\Pi(\widehat{W}_n, W) \ll \Pi(h \circ \widehat{W}_n, \sigma B) \ll \Pi(h \circ \widehat{W}_n, \sigma X_n) + \Pi(\sigma X_n, \sigma B).$$

Conclude by Lemmas 3.58 and 3.60. □

Chapter 4

Nonexistence of a spectral gap in Hölder spaces

The current chapter displays paper [43] which was published, in collaboration with Melbourne and Terhesiu, in the Israel Journal of Mathematics. As stated in the abstract of [43], we show a natural restriction on the smoothness of spaces on which the transfer operator for a continuous dynamical system has a spectral gap. Such a space cannot be embedded in a Hölder space with Hölder exponent greater than $\frac{1}{2}$ unless it consists entirely of coboundaries.

4.1 Main result

Let (Λ, d) be a bounded metric space with Borel probability measure μ, and let $T_t : \Lambda \to \Lambda$ be a measure-preserving semiflow. We suppose that $t \to T_t$ is Lipschitz a.e. on Λ, that is there exists $L > 0$ such that $d(T_t x, T_s x) \le L|t-s|$ for all $t, s \ge 0$ and almost every $x \in \Lambda$. Let $L_t : L^1(\Lambda) \to L^1(\Lambda)$ denote the transfer operator corresponding to T_t (so $\int_\Lambda L_t v\, w\, d\mu = \int_\Lambda v\, w \circ T_t\, d\mu$ for all $v \in L^1(\Lambda)$, $w \in L^\infty(\Lambda)$, $t > 0$). Let $v \in L^\infty(\Lambda)$ and define $v_t = \int_0^t v \circ T_r \,\mathrm{d}r$ for $t \ge 0$.

Theorem 4.1. Let $\eta \in (\frac{1}{2}, 1)$. Suppose that $L_t v \in C^\eta(\Lambda)$ for all $t > 0$ and that $\int_0^\infty \|L_t v\|_\eta \,\mathrm{d}t < \infty$. Then v_t is a coboundary:

$$v_t = \chi \circ T_t - \chi \quad \text{for all } t \ge 0, \text{ a.e. on } \Lambda$$

where $\chi = \int_0^\infty L_t v \,\mathrm{d}t \in C^\eta(\Lambda)$. In particular, $\sup_{t \ge 0} |v_t|_\infty < \infty$.

Here, $|g|_\infty = \operatorname{ess\,sup}_\Lambda |g|$ and $\|g\|_\eta = |g|_\infty + \sup_{x \ne y} |g(x) - g(y)|/d(x, y)^\eta$.

Theorem 4.1 implies that any Banach space admitting a spectral gap and embedded in $C^\eta(\Lambda)$ for some $\eta > \frac{1}{2}$ is cohomologically trivial. However, for typical (non)uniformly expanding semiflows and (non)uniformly hyperbolic flows, coboundaries are known to be exceedingly rare, see for example [15, Section 2.3.3]. Hence, Theorem 4.1 can be viewed as an "anti-spectral gap" result for such continuous time dynamical systems.

4.2 Proof of Theorem 4.1

Let $v \in L^\infty(\Lambda)$, assume $L_t v \in C^\eta(\Lambda)$ for every $t > 0$, and $\int_0^\infty \|L_t v\|_\eta \, \mathrm{d}t < \infty$ where $\eta \in (\frac{1}{2}, 1)$. Following Gordin [26] we consider a martingale-coboundary decomposition. Define $\chi = \int_0^\infty L_t v \, \mathrm{d}t \in C^\eta(\Lambda)$, and

$$v_t = \int_0^t v \circ T_r \, \mathrm{d}r, \qquad m_t = v_t - \chi \circ T_t + \chi,$$

for $t \geq 0$. Let $\mathcal{B}$ denote the Borel σ-algebra on Λ.

Proposition 4.2. (i) $t \to m_t$ is C^η a.e. on Λ.

(ii) $\mathbb{E}(m_t | T_t^{-1}\mathcal{B}) = 0$ for all $t \geq 0$.

Proof. (i) For $0 \leq s \leq t \leq 1$ and $x \in \Lambda$,

$$\begin{aligned}|m_s(x) - m_t(x)| &\leq |v_s(x) - v_t(x)| + |\chi(T_s x) - \chi(T_t x)| \\ &\leq |s - t||v|_\infty + |\chi|_\eta \, d(T_s x, T_t x)^\eta.\end{aligned}$$

Since $t \mapsto T_t$ is a.e. Lipschitz, it follows that $t \mapsto m_t$ is a.e. C^η.

(ii) Let $U_t v = v \circ T_t$, and recall that $L_t U_t = I$ and $\mathbb{E}(\cdot | T_t^{-1}\mathcal{B}) = U_t L_t$. Then

$$\begin{aligned}L_t m_t = L_t(v_t - U_t\chi + \chi) &= \textstyle\int_0^t L_t U_r v \, \mathrm{d}r - \chi + \int_0^\infty L_t L_r v \, \mathrm{d}r \\ &= \textstyle\int_0^t L_{t-r} v \, \mathrm{d}r - \chi + \int_0^\infty L_{t+r} v \, \mathrm{d}r \\ &= \textstyle\int_0^t L_r v \, \mathrm{d}r - \chi + \int_t^\infty L_r v \, \mathrm{d}r = 0.\end{aligned}$$

Hence $\mathbb{E}(m_t | T_t^{-1}\mathcal{B}) = U_t L_t m_t = 0$. □

Proof. Theorem 4.1 Fix $T > 0$, and define

$$M_T(t) = m_T - m_{T-t} = m_t \circ T_{T-t}, \quad t \in [0, T].$$

Define the filtration $\mathcal{G}_{T,t} = T_{T-t}^{-1}\mathcal{B}$. It is immediate to see that $M_T(t) = m_t \circ T_{T-t}$ is $\mathcal{G}_{T,t}$-measurable. For $s < t$, $M_T(t) - M_T(s) = m_{T-s} - m_{T-t} = m_{t-s} \circ T_{T-t}$, so

$$\begin{aligned}\mathbb{E}(M_T(t) - M_T(s)|\mathcal{G}_{T,s}) &= \mathbb{E}(m_{t-s} \circ T_{T-t}|T_{T-s}^{-1}\mathcal{B}) \\ &= \mathbb{E}(m_{t-s}|T_{t-s}^{-1}\mathcal{B}) \circ T_{T-t} = 0\end{aligned}$$

by Proposition 4.2(ii). Hence M_T is a martingale for each $T > 0$. Next,

$$|M_T(t)|_\infty = |m_t \circ T_{T-t}|_\infty \leq |m_t|_\infty \leq |v_t|_\infty + 2|\chi|_\infty \leq T|v|_\infty + 2|\chi|_\infty.$$

Hence $M_T(t)$, $t \in [0, T]$, is a bounded martingale.

By Proposition 4.2(i), M_T has C^η sample paths. Since $\eta > \frac{1}{2}$, it follows from general martingale theory that $M_T \equiv 0$ a.e. Taking $t = T$, we obtain $m_T = 0$ a.e. Hence $v_T = \chi \circ T_t - \chi$ a.e. for all $T > 0$ as required.

For completeness, we include the argument that $M_T \equiv 0$ a.e. We require two standard properties of the quadratic variation process, written as $t \mapsto \langle M_T \rangle(t)$; a reference for these is [16, Theorem 4.1]. First, $\langle M_T \rangle(t)$ is the limit in probability as $n \to \infty$ of

$$S_n(t) = \sum_{j=1}^{n} (M_T(jt/n) - M_T((j-1)t/n))^2.$$

Second (noting that $M_T(0) = 0$),

$$\langle M_T \rangle(t) = M_T(t)^2 - 2\int_0^t M_T \, dM_T,$$

where the stochastic integral has expectation zero. In particular, $\mathbb{E}[\langle M_T \rangle] \equiv \mathbb{E}[M_T^2]$.

Since M_T has Hölder sample paths with exponent $\eta > \frac{1}{2}$, we have a.e. that

$$|S_n(t)| = \mathcal{O}(t^\eta n^{-(2\eta-1)}) \to 0 \quad \text{as } n \to \infty.$$

Hence $\langle M_T \rangle \equiv 0$ a.e. It follows that $\mathbb{E}[M_T^2] \equiv 0$ and so $M_T \equiv 0$ a.e. □

Chapter 5

Decays in norm of transfer operators

The current chapter displays paper [42] which was published, in collaboration with Melbourne and Terhesiu, in Studia Mathematica. As stated in the abstract of [42], we establish here exponential decay in Hölder norm of transfer operators applied to smooth observables of uniformly and nonuniformly expanding semiflows with exponential decay of correlations.

This chapter is organised as follows. In Section 5.1, we recall the setup for nonuniformly expanding semiflows with exponential decay of correlations and state our main result, Theorem 5.2, on decay in norm. In Section 5.2, we prove Theorem 5.2.

5.1 Setup and statement of the main result

In this section, we state our result on Hölder norm decay of transfer operators for uniformly and nonuniformly expanding semiflows.

Let (Y, d) be a bounded metric space with Borel probability measure μ and an at most countable measurable partition $\{Y_j\}$. Let $F : Y \to Y$ be a measure-preserving transformation such that F restricts to a measure-theoretic bijection from Y_j onto Y for each j. Let $g = d\mu/(d\mu \circ F)$ be the inverse Jacobian of F.

Fix $\eta \in (0, 1)$. Assume that there are constants $\lambda > 1$ and $C > 0$ such that $d(Fy, Fy') \geq \lambda d(y, y')$ and $|\log g(y) - \log g(y')| \leq Cd(Fy, Fy')^\eta$ for all $y, y' \in Y_j$, $j \geq 1$. In particular, F is a Gibbs-Markov map as in [2] (see also [1, 3]) with ergodic (and mixing) invariant measure μ.

Let $\varphi : Y \to [2, \infty)$ be a piecewise continuous roof function. We assume that there is a constant $C > 0$ such that

$$|\varphi(y) - \varphi(y')| \le C d(Fy, Fy')^\eta \tag{5.1}$$

for all $y, y' \in Y_j$, $j \ge 1$. Also, we assume exponential tails, namely that there exists $\delta_0 > 0$ such that

$$\textstyle\sum_j \mu(Y_j) e^{\delta_0 |1_{Y_j}\varphi|_\infty} < \infty. \tag{5.2}$$

Define the suspension $Y^\varphi = \{(y, u) \in Y \times [0, \infty) : u \in [0, \varphi(y)]\}/\sim$ where $(y, \varphi(y)) \sim (Fy, 0)$. The suspension semiflow $F_t : Y^\varphi \to Y^\varphi$ is given by $F_t(y, u) = (y, u+t)$ computed modulo identifications. We define the ergodic F_t-invariant probability measure $\mu^\varphi = (\mu \times \text{Lebesgue})/\bar\varphi$ where $\bar\varphi = \int_Y \varphi \, d\mu$.

Let $L_t : L^1(Y^\varphi) \to L^1(Y^\varphi)$ denote the transfer operator corresponding to F_t (so $\int_{Y^\varphi} L_t v \, w \, d\mu^\varphi = \int_{Y^\varphi} v \, w \circ F_t \, d\mu^\varphi$ for all $v \in L^1(Y^\varphi)$, $w \in L^\infty(Y^\varphi)$, $t > 0$) and let $P_0 : L^1(Y) \to L^1(Y)$ denote the transfer operator for F. Recall (see for example [2]) that $(P_0 v)(y) = \sum_j g(y_j) v(y_j)$ where y_j is the unique preimage of y under $F|Y_j$, and there is a constant $C > 0$ such that

$$|g(y)| \le C\mu(Y_j), \qquad |g(y) - g(y')| \le C\mu(Y_j) d(Fy, Fy')^\eta, \tag{5.3}$$

for all $y, y' \in Y_j$, $j \ge 1$.

Function space on Y^φ Let $Y_j^\varphi = \{(y, u) \in Y^\varphi : y \in Y_j\}$. Fix $\eta \in (0, 1]$, $\delta > 0$. For $v : Y^\varphi \to \mathbb{R}$, define $|v|_{\delta,\infty} = \sup_{(y,u)\in Y^\varphi} e^{-\delta u}|v(y, u)|$ and

$$\|v\|_{\delta,\eta} = |v|_{\delta,\infty} + |v|_{\delta,\eta}, \quad |v|_{\delta,\eta} = \sup_{j\ge1} \sup_{(y,u),(y',u)\in Y_j^\varphi,\, y\ne y'} e^{-\delta u} \frac{|v(y, u) - v(y', u)|}{d(y, y')^\eta}.$$

Then $\mathcal{F}_{\delta,\eta}(Y^\varphi)$ consists of observables $v : Y^\varphi \to \mathbb{R}$ with $\|v\|_{\delta,\eta} < \infty$.

Next, define $\partial_u v$ to be the partial derivative of v with respect to u at points $(y, u) \in Y^\varphi$ with $u \in (0, \varphi(y))$ and to be the appropriate one-sided partial derivative when $u \in \{0, \varphi(y)\}$. For $m \ge 0$, define $\mathcal{F}_{\delta,\eta,m}(Y^\varphi)$ to consist of observables $v \colon Y^\varphi \to \mathbb{R}$ such that $\partial_u^j v \in \mathcal{F}_{\delta,\eta}(Y^\varphi)$ for $j = 0, 1, \ldots, m$, with norm $\|v\|_{\delta,\eta,m} = \max_{j=0,\ldots,m} \|\partial_u^j v\|_{\delta,\eta}$.

Given $r > 0$, we consider the subset $\{(y, u) \in Y \times \mathbb{R} : u \in [r, \varphi(y) - r]\}$ viewed as a subset of Y^φ. We say that a function $v : Y^\varphi \to \mathbb{R}$ has *good support* if there exists $r > 0$ such that $\operatorname{supp} v \subset \{(y, u) \in Y \times \mathbb{R} : u \in [r, \varphi(y) - r]\}$.

For functions with good support, $\partial_u v$ coincides with the derivative in the flow direction $\partial_t v = \lim_{h\to0}(v \circ F_h - v)/h$.

Remark 5.1. It is standard to restrict to observables with good support when considering decay of correlations for semiflows, see for instance [22, 54].

Let

$$\mathcal{F}^0_{\delta,\eta,m}(Y^\varphi) = \{v \in \mathcal{F}_{\delta,\eta,m}(Y^\varphi) : \textstyle\int_{Y^\varphi} v \, d\mu^\varphi = 0\}.$$

We write $\mathcal{F}_{\delta,\eta}(Y^\varphi)$ and $\mathcal{F}^0_{\delta,\eta}(Y^\varphi)$ when $m = 0$.

Function space on Y For $v : Y \to \mathbb{R}$, define

$$\|v\|_\eta = |v|_\infty + |v|_\eta, \qquad |v|_\eta = \sup_{j\geq 1} \sup_{y,y' \in Y_j,\, y\neq y'} |v(y) - v(y')|/d(y,y')^\eta.$$

Let $\mathcal{F}_\eta(Y)$ consist of observables $v : Y \to \mathbb{R}$ with $\|v\|_\eta < \infty$.

Dolgopyat estimate Define the twisted transfer operators

$$\widehat{P}_0(s) : L^1(Y) \to L^1(Y), \qquad \widehat{P}_0(s)v = P_0(e^{-s\varphi}v).$$

We assume that there exists $\gamma \in (0,1)$, $\varepsilon > 0$, $m_0 \geq 0$, $A, D > 0$ such that

$$\|\widehat{P}_0(s)^n\|_{\mathcal{F}_\eta(Y)\mapsto\mathcal{F}_\eta(Y)} \leq |b|^{m_0}\gamma^n \tag{5.4}$$

for all $s = a+ib \in \mathbb{C}$ with $|a| < \varepsilon$, $|b| \geq D$ and all $n \geq A\log|b|$. Such an assumption holds in the settings of [6, 7, 9, 21].

Now we can state our main result on norm decay for L_t.

Theorem 5.2. Under these assumptions, there exists $\varepsilon > 0$, $m \geq 1$, $C > 0$ such that

$$\|L_t v\|_{\delta,\eta,1} \leq Ce^{-\varepsilon t}\|v\|_{\delta,\eta,m} \quad \text{for all } t > 0$$

for all $v \in \mathcal{F}^0_{\delta,\eta,m}(Y^\varphi)$ with good support.

Remark 5.3. Since the norm applied to v is stronger than the norm applied to $L_t v$, Theorem 5.2 does not imply a spectral gap for L_t. We note that the norm on $\mathcal{F}_{\delta,\eta,1}(Y^\varphi)$ gives no Hölder control in the flow direction when passing through points of the form $(y, \varphi(y))$. This lack of control is a barrier to mollification arguments of the type usually used to pass from smooth observables to Hölder observables. In fact, such arguments are doomed to fail at the operator level by [43, Theorem 1.1] (Theorem 4.1 in this thesis) when $\eta > \frac{1}{2}$ and hence seem unlikely for any η.

Remark 5.4. Usually, we can take $m_0 \in (0,1)$ in (5.4) in which case $m = 3$ suffices in Theorem 5.2.

There are numerous simplifications when $\{Y_j\}$ is a finite partition. In particular, conditions (5.1) and (5.2) are redundant and we can take $\delta = 0$.

Remark 5.5. At first glance, Theorem 5.2 has some similarities with [14, Theorem 1]. In particular, we mention formula (2.4) therein which takes the form $\|P_t\mu\|_{\mathcal{A}} \le C_\ell e^{-\ell t}\|Z\mu\|_{\mathcal{B}}$ where $Z = \partial_t$. However, $\|\ \|_{\mathcal{A}}$ corresponds to a "weak" norm which would just be the L^∞ norm in our setting. Moreover, the hypothesis in [14] that the operators $T_t : \mathcal{B} \to \mathcal{B}$ ($L_t : \mathcal{F}_{\delta,\eta,1}(Y^\varphi) \to \mathcal{F}_{\delta,\eta,1}(Y^\varphi)$ in our notation) are bounded looks to be unverifiable in our setting even for fixed t. On the other hand, the expansion in equation (2.3) of [14] is beyond our methods.

Remark 5.6. Numerous (non)uniformly hyperbolic flows are modelled (after inducing and quotienting along stable leaves) by "Gibbs-Markov semiflows" $F_t : Y^\varphi \to Y^\varphi$ of the type considered here with the exponential tail condition (5.2). These include basic sets for Axiom A flows, Lorentz gases with finite horizon, and Lorenz attractors (see for instance [39, Section 1.1]). Whenever the Dolgopyat estimate (5.4) is verified in such examples, as in [6, 7, 9, 21], Theorem 5.2 guarantees exponential decay for the norm of the transfer operator for the corresponding Gibbs-Markov semiflow.

5.2 Proof of Theorem 5.2

Our proof of norm decay is broken into three parts. In Subsection 5.2.1, we recall a continuous-time operator renewal equation [44] which enables estimates of Laplace transforms of transfer operators at the level of Y. In Subsection 5.2.2, we show how to pass to estimates of Laplace transforms of L_t. In Subsection 5.2.3, we invert the Laplace transform to obtain norm decay of L_t.

5.2.1 Operator renewal equation

Let $\widetilde{Y} = Y \times [0,1]$ and define

$$\widetilde{F} : \widetilde{Y} \to \widetilde{Y}, \qquad \widetilde{F}(y,u) = (Fy,u),$$

with transfer operator $\widetilde{P} : L^1(\widetilde{Y}) \to L^1(\widetilde{Y})$. Also, define

$$\widetilde{\varphi} : \widetilde{Y} \to [2, \infty), \quad \widetilde{\varphi}(y, u) = \varphi(y).$$

Define the twisted transfer operators

$$\widehat{P}(s) : L^1(\widetilde{Y}) \to L^1(\widetilde{Y}), \qquad \widehat{P}(s)v = \widetilde{P}(e^{-s\widetilde{\varphi}}v).$$

Let $\widetilde{Y}_j = Y_j \times [0, 1]$. For $v : \widetilde{Y} \to \mathbb{R}$, define

$$\|v\|_\eta = |v|_\infty + |v|_\eta, \qquad |v|_\eta = \sup_{j\geq 1} \sup_{(y,u),(y',u)\in\widetilde{Y}_j,\, y\neq y'} |v(y, u) - v(y', u)|/d(y, y')^\eta.$$

Let $\mathcal{F}_\eta(\widetilde{Y})$ consist of observables $v : \widetilde{Y} \to \mathbb{R}$ with $\|v\|_\eta < \infty$. Let

$$\mathcal{F}^0_\eta(\widetilde{Y}) = \{v \in \mathcal{F}_\eta(\widetilde{Y}) : \textstyle\int_{\widetilde{Y}} v\, d\tilde{\mu} = 0\}$$

where $\tilde{\mu} = \mu \times \mathrm{Leb}_{[0,1]}$.

Lemma 5.7. Write $s = a + ib \in \mathbb{C}$. There exists $\varepsilon > 0$, $m_1 \geq 0$, $C > 0$ such that

(a) $s \mapsto (I - \widehat{P}(s))^{-1} : \mathcal{F}^0_\eta(\widetilde{Y}) \to \mathcal{F}_\eta(\widetilde{Y})$ is analytic on $\{|a| < \varepsilon\}$;

(b) $s \mapsto (I - \widehat{P}(s))^{-1} : \mathcal{F}_\eta(\widetilde{Y}) \to \mathcal{F}_\eta(\widetilde{Y})$ is analytic on $\{|a| < \varepsilon\}$ except for a simple pole at $s = 0$;

(c) $\|(I - \widehat{P}(s))^{-1}\|_{\mathcal{F}_\eta(\widetilde{Y})\mapsto\mathcal{F}_\eta(\widetilde{Y})} \leq C|b|^{m_1}$ for $|a| \leq \varepsilon$, $|b| \geq 1$.

Proof. It suffices to verify these properties for $Z(s) = (I - \widehat{P}_0(s))^{-1}$ on Y. They immediately transfer to $(I - \widehat{P}(s))^{-1}$ on $\widetilde{Y}$ since $(\widehat{P}v)(y, u) = (\widehat{P}_0 v^u)(y)$ where $v^u(y) = v(y, u)$.

The arguments for passing from (5.4) to the desired properties for $Z(s)$ are standard. For completeness, we sketch these details now recalling arguments from [6]. Define $\mathcal{F}_\eta(Y)$ with norm $\|\ \|_\eta$ by restricting to $u = 0$ (this coincides with the usual Hölder space on Y). Let A, D, ε and m_0 be as in (5.4). Increase A and D so that $D > 1$ and $|b|^{m_0}\gamma^{[A \log |b|]} \leq \frac{1}{2}$ for $|b| \geq D$. Suppose that $|a| \leq \varepsilon$, $|b| \geq D$. Then $\|\widehat{P}_0(s)^{[A \log |b|]}\|_\eta \leq |b|^{m_0}\gamma^{[A \log |b|]} \leq \frac{1}{2}$ and $\|(I - \widehat{P}_0(s)^{[A \log |b|]})^{-1}\|_\eta \leq 2$.

As in [6, Proposition 2.5], we can shrink ε so that $s \to \widehat{P}_0(s)$ is continuous on $\mathcal{F}_\eta(Y)$ for $|a| \leq \varepsilon$. The simple eigenvalue 1 for $\widehat{P}_0(0) = P_0$ extends to a continuous family of simple eigenvalues $\lambda(s)$ for $|s| \leq \varepsilon$. We can choose ε so that $\frac{1}{2} < \lambda(a) < 2$

for $|a| \le \varepsilon$. By [6, Corollary 2.8], $\|\widehat{P}_0(s)^n\|_\eta \ll |b|\lambda(a)^n \le |b|2^n$ for all $n \ge 1$, $|a| \le \varepsilon$, $|b| \ge D$. Hence

$$\|Z(s)\|_\eta \le \big(1 + \|\widehat{P}_0(s)\|_\eta + \cdots + \|\widehat{P}_0(s)^{[A\log|b|]-1}\|_\eta\big)\|(I - \widehat{P}_0(s)^{[A\log|b|]})^{-1}\|_\eta$$
$$\ll (\log|b|)\,|b|\,2^{A\log|b|} \le |b|^{m_1},$$

with $m_1 = 1 + A\log 2$. This proves analyticity on the region $\{|a| < \varepsilon, |b| > D\}$ with the desired estimates for property (c) on this region.

For $|a| \le \varepsilon$, $|b| \le D$, we recall arguments from the proof of [6, Lemma 2.22] (where $\widehat{P}_0(s)$ is denoted Q_s). For ε sufficiently small, the part of spectrum of $\widehat{P}_0(s)$ that is close to 1 consists only of isolated eigenvalues. Also, the spectral radius of $\widehat{P}_0(s)$ is at most $\lambda(a)$ and $\lambda(a) < 1$ for $a \in [0, \varepsilon]$, so $s \mapsto Z(s)$ is analytic on $\{0 < a < \varepsilon\}$.

Suppose that $\widehat{P}_0(ib)v = v$ for some $v \in \mathcal{F}_\eta(Y)$, $b \ne 0$. Choose $q \ge 1$ such that $q|b| > D$. Since $\widehat{P}_0(s)$ is the L^2 adjoint of $v \mapsto e^{s\varphi}v \circ F$, we have $e^{ib\varphi}v \circ F = v$. Hence $e^{iqb\varphi}v^q \circ F = v^q$ and so $\widehat{P}_0(iqb)v^q = v^q$. But $\|Z(iqb)v^q\|_\eta < \infty$, so $v = 0$. Hence $1 \notin \operatorname{spec}\widehat{P}_0(ib)$ for all $b \ne 0$. It follows that for all $b \ne 0$ there exists an open set $U_b \subset \mathbb{C}$ containing ib such that $1 \notin \operatorname{spec}\widehat{P}_0(s)$ for all $s \in U_b$, and so $s \mapsto Z(s)$ is analytic on U_b.

Next, we recall that for s near to zero, $\lambda(s) = 1 + cs + O(s^2)$ where $c < 0$. Hence $s \mapsto Z(s)$ has a simple pole at zero. It follows that there exists $\varepsilon > 0$ such that $s \mapsto Z(s)$ is analytic on $\{|a| < \varepsilon, |b| < 2D\}$ except for a simple pole at $s = 0$. Combining this with the estimates on $\{|a| < \varepsilon, |b| \ge D\}$ we have proved properties (b) and (c) for $Z(s)$.

Finally, the spectral projection π corresponding to the eigenvalue $\lambda(0)= 1$ for $\widehat{P}_0(0) = P$ is given by $\pi v = \int_Y v\,d\mu$. Hence the pole disappears on restriction to observables of mean zero, proving property (a) for $Z(s)$. □

Next define

$$T_t v = 1_{\widetilde{Y}} L_t(1_{\widetilde{Y}} v), \qquad U_t v = 1_{\widetilde{Y}} L_t(1_{\{\widetilde{\varphi} > t\}} v)$$

and

$$\widehat{T}(s) = \int_0^\infty e^{-st} T_t\,dt, \qquad \widehat{U}(s) = \int_0^\infty e^{-st} U_t\,dt,$$

By [44, Theorem 3.3], we have the operator renewal equation

$$\widehat{T} = \widehat{U}(I - \widehat{P})^{-1}.$$

Proposition 5.8. There exists $\varepsilon > 0$, $C > 0$ such that $s \mapsto \widehat{U}(s) : \mathcal{F}_\eta(\widetilde{Y}) \to \mathcal{F}_\eta(\widetilde{Y})$ is analytic on $\{|a| < \varepsilon\}$ and $\|\widehat{U}(s)\|_{\mathcal{F}_\eta(\widetilde{Y}) \mapsto \mathcal{F}_\eta(\widetilde{Y})} \le C|s|$ for $|a| \le \varepsilon$.

Proof. By [44, Proposition 3.4],

$$(U_t v)(y,u) = \begin{cases} v(y, u-t)\mathbb{1}_{[t,1]}(u) & 0 \le t \le 1 \\ (\widetilde{P} v_t)(y,u) & t > 1 \end{cases}$$

where $v_t(y,u) = \mathbb{1}_{\{t<\varphi(y)<t+1-u\}} v(y, u-t+\varphi(y))$. Hence $\widehat{U}(s) = \widehat{U}_1(s) + \widehat{U}_2(s)$ where

$$(\widehat{U}_1(s)v)(y,u) = \int_0^u e^{-st} v(y,u-t)\,dt, \qquad \widehat{U}_2(s)v = \int_1^\infty e^{-st}\widetilde{P} v_t\,dt.$$

It is clear that $\|\widehat{U}_1(s)v\|_\eta \le e^\varepsilon \|v\|_\eta$. We focus attention on the second term

$$(\widehat{U}_2(s)v)(y,u) = \textstyle\sum_j g(y_j) \displaystyle\int_1^\infty e^{-st} v_t(y_j,u)\,dt = \textstyle\sum_j g(y_j) \widehat{V}(s)(y_j,u),$$

where $\widehat{V}(s)(y,u) = \int_u^1 e^{s(t-u-\varphi)} v(y,t)\,dt$. Clearly, $|\mathbb{1}_{Y_j}\widehat{V}(s)|_\infty \le e^{\varepsilon|\mathbb{1}_{Y_j}\varphi|_\infty}|v|_\infty$. Also,

$$\widehat{V}(s)(y,u) - \widehat{V}(s)(y',u) = I + J,$$

where

$$I = \int_u^1 (e^{s(t-u-\varphi(y))} - e^{s(t-u-\varphi(y'))}) v(y,t)\,dt,$$
$$J = \int_u^1 e^{s(t-u-\varphi(y'))}(v(y,t) - v(y',t))\,dt.$$

For $y, y' \in Y_j$,

$$|I| \le |v|_\infty \int_u^1 e^{\varepsilon(|\mathbb{1}_{Y_j}\varphi|_\infty + u - t)} |s||\varphi(y) - \varphi(y')|\,dt \ll |s||v|_\infty\, e^{\varepsilon|\mathbb{1}_{Y_j}\varphi|_\infty} d(Fy, Fy')^\eta$$

by (5.1), and

$$|J| \le \int_u^1 e^{\varepsilon(|\mathbb{1}_{Y_j}\varphi|_\infty + u - t)} |v(y,t) - v(y',t)|\,dt \le e^{\varepsilon|\mathbb{1}_{Y_j}\varphi|_\infty} |v|_\eta\, d(y,y')^\eta.$$

Hence $|\widehat{V}(s)(y,u) - \widehat{V}(s)(y',u)|_\eta \ll |s| e^{\varepsilon|\mathbb{1}_{Y_j}\varphi|_\infty}\|v\|_\eta\, d(Fy, Fy')^\eta$.

It follows from the estimates for $\mathbb{1}_{Y_j}\widehat{V}(s)$ together with (5.3) that

$$\|\widehat{U}_2(s)v\|_\eta \ll \textstyle\sum_j |s|\mu(Y_j) e^{\varepsilon|\mathbb{1}_{Y_j}\varphi|_\infty}\|v\|_\eta.$$

By (5.2), $\|\widehat{U}_2(s)v\|_\eta \ll |s|\|v\|_\eta$ for ε sufficiently small. Hence, we conclude that $\|\widehat{U}(s)v\|_\eta \ll |s|\|v\|_\eta$. □

5.2.2 From $\widehat{T}$ on $\widetilde{Y}$ to $\widehat{L}$ on Y^φ

Lemma 5.7 and Proposition 5.8 yield analyticity and estimates for $\widehat{T} = \widehat{U}(I - \widehat{P})^{-1}$ on $\widetilde{Y}$. In this subsection, we show how these properties are inherited by $\widehat{L}(s) = \int_0^\infty e^{-st} L_t\, dt$ on Y^φ. Recall that $\widetilde{Y} = Y \times [0,1]$ which we view as a subset of Y^φ.

Remark 5.9. The approach in this subsection is similar to that in [12, Section 5] but there are some important differences. The rationale behind the two step decomposition in Propositions 5.10 and 5.11 below is that the discreteness of the decomposition in Proposition 5.10 simplifies many formulas significantly. In particular, the previously problematic term E_t in [12] becomes elementary (and vanishes for large t when φ is bounded). The decomposition in Proposition 5.11 remains continuous to simplify the estimates in Proposition 5.14.

Since the setting in [12] is different (infinite ergodic theory, reinducing) we keep the exposition here self-contained even where the estimates coincide with those in [12].

Define

$$A_n : L^1(\widetilde{Y}) \to L^1(Y^\varphi), \qquad (A_n v)(y,u) = 1_{\{n \le u < n+1\}} (L_n v)(y,u),\ n \ge 0,$$
$$E_t : L^1(Y^\varphi) \to L^1(Y^\varphi), \qquad (E_t v)(y,u) = 1_{\{[t]+1 \le u \le \varphi(y)\}} (L_t v)(y,u),\ t > 0.$$

Proposition 5.10. $L_t = \sum_{j=0}^{[t]} A_j 1_{\widetilde{Y}} L_{t-j} + E_t$ for $t > 0$.

Proof. For $y \in Y$, $u \in (0, \varphi(y))$,

$$\begin{aligned}(L_t v)(y,u) &= \sum_{j=0}^{[t]} 1_{\{j \le u < j+1\}} (L_t v)(y,u) + 1_{\{[t]+1 \le u \le \varphi(y)\}} (L_t v)(y,u) \\ &= \sum_{j=0}^{[t]} (A_j L_{t-j} v)(y,u) + (E_t v)(y,u).\end{aligned}$$

Now use that $A_n = A_n 1_{\widetilde{Y}}$. □

Next, define

$$B_t : L^1(Y^\varphi) \to L^1(\widetilde{Y}), \qquad B_t v = 1_{\widetilde{Y}} L_t (1_{\Delta_t} v),$$
$$G_t : L^1(Y^\varphi) \to L^1(\widetilde{Y}), \qquad G_t v = B_t(\omega(t) v),$$
$$H_t : L^1(Y^\varphi) \to L^1(\widetilde{Y}), \qquad H_t v = 1_{\widetilde{Y}} L_t (1_{\Delta'_t} v),$$

for $t > 0$, where

$$\Delta_t = \{(y,u) \in Y^\varphi : \varphi(y) - t \le u < \varphi(y) - t + 1\}$$
$$\Delta_t' = \{(y,u) \in Y^\varphi : u < \varphi(y) - t\}, \qquad \omega(t)(y,u) = \varphi(y) - u - t + 1.$$

Proposition 5.11. $1_{\widetilde{Y}} L_t = \int_0^t T_{t-\tau} B_\tau \, d\tau + G_t + H_t$ for $t > 0$.

Proof. Let $y \in Y$, $u \in [0, \varphi(y)]$. Then

$$\begin{aligned}
\int_0^t 1_{\Delta_\tau}(y,u)\, d\tau &= \int_0^t 1_{\{\varphi(y)-u \le \tau \le \varphi(y)-u+1\}}\, d\tau \\
&= 1_{\{t \ge \varphi(y)-u+1\}} + 1_{\{\varphi(y)-u \le t < \varphi(y)-u+1\}}(t - \varphi(y) + u) \\
&= 1 - 1_{\{t < \varphi(y)-u+1\}} + 1_{\{\varphi(y)-u \le t < \varphi(y)-u+1\}}(t - \varphi(y) + u) \\
&= 1 - 1_{\Delta_t'}(y,u) + 1_{\Delta_t}(y,u)(t - \varphi(y) + u - 1).
\end{aligned}$$

Hence $\int_0^t 1_{\Delta_\tau}\, d\tau = 1 - 1_{\Delta_t}\omega(t) - 1_{\Delta_t'}$. It follows that

$$\begin{aligned}
\int_0^t T_{t-\tau} B_\tau v \, d\tau &= 1_{\widetilde{Y}} \int_0^t L_{t-\tau} 1_{\widetilde{Y}} B_\tau v \, d\tau = 1_{\widetilde{Y}} \int_0^t L_{t-\tau} B_\tau v \, d\tau \\
&= 1_{\widetilde{Y}} \int_0^t L_{t-\tau} L_\tau (1_{\Delta_\tau} v)\, d\tau = 1_{\widetilde{Y}} L_t \Big(\int_0^t 1_{\Delta_\tau} v \, d\tau \Big) \\
&= 1_{\widetilde{Y}} L_t v - G_t v - H_t v
\end{aligned}$$

as required. □

We have already defined the Laplace transforms $\widehat{L}(s)$ and $\widehat{T}(s)$ for $s = a + ib$ with $a > 0$. Similarly, define

$$\widehat{B}(s) = \int_0^\infty e^{-st} B_t \, dt, \qquad \widehat{E}(s) = \int_0^\infty e^{-st} E_t \, dt,$$
$$\widehat{G}(s) = \int_0^\infty e^{-st} G_t \, dt, \qquad \widehat{H}(s) = \int_0^\infty e^{-st} H_t \, dt.$$

Also, we define the discrete transform $\widehat{A}(s) = \sum_{n=0}^{\infty} e^{-sn} A_n$.

Corollary 5.12. $\widehat{L}(s) = \widehat{A}(s)\widehat{T}(s)\widehat{B}(s) + \widehat{A}(s)\widehat{G}(s) + \widehat{A}(s)\widehat{H}(s) + \widehat{E}(s)$ for $a > 0$.

Proof. By Proposition 5.10,

$$\begin{aligned}
\widehat{L}(s) - \widehat{E}(s) &= \int_0^\infty e^{-st} \sum_{j=0}^{[t]} A_j 1_{\widetilde{Y}} L_{t-j}\, dt = \sum_{j=0}^{\infty} e^{-sj} A_j 1_{\widetilde{Y}} \int_j^\infty e^{-s(t-j)} L_{t-j}\, dt \\
&= \widehat{A}(s) 1_{\widetilde{Y}} \int_0^\infty e^{-st} L_t \, dt = \widehat{A}(s) 1_{\widetilde{Y}} \widehat{L}(s).
\end{aligned}$$

Hence $\widehat{L} = \widehat{A} 1_{\widetilde{Y}} \widehat{L} + \widehat{E}$. In addition, by Proposition 5.11, $1_{\widetilde{Y}} \widehat{L} = \widehat{T}\widehat{B} + \widehat{G} + \widehat{H}$. □

Proposition 5.13. Let $\delta > \varepsilon > 0$. Then there is a constant $C > 0$ such that

(a) $\|\widehat{A}(s)\|_{\mathcal{F}_\eta(\widetilde{Y})\to\mathcal{F}_{\delta,\eta}(Y^\varphi)} \le 1$,

(b) $\|\widehat{E}(s)\|_{\mathcal{F}_{\delta,\eta}(Y^\varphi)\to\mathcal{F}_{\delta,\eta}(Y^\varphi)} \le C$,

(c) $\|\widehat{H}(s)\|_{\mathcal{F}_{\delta,\eta}(Y^\varphi)\to\mathcal{F}_\eta(\widetilde{Y})} \le e^\delta$,

for $|a| \le \varepsilon$.

Proof. (a) Let $v \in \mathcal{F}_\eta(\widetilde{Y})$. Let (y,u), $(y',u) \in Y_j^\varphi$, $j \ge 1$. Since $(A_n v)(y,u) = \mathbb{1}_{\{n\le u<n+1\}} v(y,u-n)$,

$$(\widehat{A}(s)v)(y,u) = \sum_{n=0}^\infty e^{-sn}\mathbb{1}_{\{n\le u<n+1\}} v(y,u-n) = e^{-s[u]}v(y,u-[u]).$$

Hence

$$|(\widehat{A}(s)v)(y,u)| \le e^{\varepsilon u}|v|_\infty, \quad |(\widehat{A}(s)v)(y,u) - (\widehat{A}(s)v)(y',u)| \le e^{\varepsilon u}|v|_\eta\, d(y,y')^\eta.$$

That is, $|\widehat{A}(s)v|_{\varepsilon,\infty} \le |v|_\infty$, $|\widehat{A}(s)v|_{\varepsilon,\eta} \le |v|_\eta$. Hence $\|\widehat{A}(s)v\|_{\delta,\eta} \le \|\widehat{A}(s)v\|_{\varepsilon,\eta} \le \|v\|_\eta$.

(b) We take $C = 1/(\delta-\varepsilon)$. Let $v \in \mathcal{F}_{\delta,\eta}(Y^\varphi)$. Let (y,u), $(y',u) \in Y_j^\varphi$, $j \ge 1$. Note that $(E_t v)(y,u) = \mathbb{1}_{\{[t]+1\le u\}} v(y,u-t)$, so

$$(\widehat{E}(s)v)(y,u) = \int_0^\infty e^{-st}\mathbb{1}_{\{[t]+1\le u\}} v(y,u-t)\,dt.$$

Hence

$$|(\widehat{E}(s)v)(y,u)| \le \int_0^\infty e^{\varepsilon t}|v|_{\delta,\infty}\, e^{\delta(u-t)}\,dt = C|v|_{\delta,\infty}\, e^{\delta u},$$

and

$$\begin{aligned}|(\widehat{E}(s)v)(y,u) - (\widehat{E}(s)v)(y',u)| &\le \int_0^\infty e^{\varepsilon t}|v|_{\delta,\eta}\, d(y,y')^\eta e^{\delta(u-t)}\,dt \\ &= Ce^{\delta u}|v|_{\delta,\eta}\, d(y,y')^\eta.\end{aligned}$$

That is, $|\widehat{E}(s)v|_{\delta,\infty} \le |v|_{\delta,\infty}$ and $|\widehat{E}(s)v|_{\delta,\eta} \le |v|_{\delta,\eta}$.

(c) Let $v \in \mathcal{F}_{\varepsilon,\eta}(Y^\varphi)$. Let (y,u), $(y',u) \in \widetilde{Y}_j$, $j \ge 1$. Then $(H_t v)(y,u) = \mathbb{1}_{\{t<u\}} v(y,u-t)$ and $(\widehat{H}(s)v)(y,u) = \int_0^u e^{-st} v(y,u-t)\,dt$. Hence,

$$\begin{aligned}&|\widehat{H}(s)v|_\infty \le e^\delta |v|_{\delta,\infty}, \\ &|(\widehat{H}(s)v)(y,u) - (\widehat{H}(s)v)(y',u)| \le e^\delta |v|_{\delta,\eta}\, d(y,y')^\eta.\end{aligned}$$

The result follows. ∎

Proposition 5.14. There exists $\delta > \varepsilon > 0$, $C > 0$ such that

$$\|\widehat{B}(s)\|_{\mathcal{F}_{\delta,\eta}(Y^\varphi)\to\mathcal{F}_\eta(\widetilde{Y})} \le C|s| \quad \text{and} \quad \|\widehat{G}(s)\|_{\mathcal{F}_{\delta,\eta}(Y^\varphi)\to\mathcal{F}_\eta(\widetilde{Y})} \le C|s|$$

for $|a| \le \varepsilon$.

Proof. Let $v \in L^1(Y^\varphi)$, $w \in L^\infty(\widetilde{Y})$. Using that $F_t(y,u) = (Fy, u+t-\varphi(y))$ for $(y,u) \in \Delta_t$,

$$\begin{aligned}
\int_{\widetilde{Y}} B_t v\, w\, d\tilde{\mu} &= \bar{\varphi}\int_{Y^\varphi} L_t(1_{\Delta_t} v)\, w\, d\mu^\varphi = \bar{\varphi}\int_{Y^\varphi} 1_{\Delta_t} v\, w \circ F_t\, d\mu^\varphi \\
&= \int_Y \int_0^{\varphi(y)} 1_{\{0\le u+t-\varphi(y)<1\}} v(y,u) w(Fy, u+t-\varphi)\, du\, d\mu \\
&= \int_Y \int_{t-\varphi(y)}^{t} 1_{\{0\le u<1\}} v(y, u+\varphi(y)-t) w(Fy,u)\, du\, d\mu \\
&= \int_{\widetilde{Y}} v_t\, w\circ \widetilde{F}\, d\tilde{\mu} = \int_{\widetilde{Y}} \widetilde{P} v_t\, w\, d\tilde{\mu}
\end{aligned}$$

where $v_t(y,u) = 1_{\{0<u+\varphi(y)-t<\varphi(y)\}} v(y, u+\varphi(y)-t)$.

Hence $B_t v = \widetilde{P} v_t$ and it follows immediately that $G_t v = \widetilde{P}(\omega(t)v)_t$. But

$$(\omega(t)v)_t(y,u) = 1_{\{0<u+\varphi(y)-t<\varphi(y)\}}(\omega(t)v)(y, u+\varphi(y)-t) = (1-u)v_t(y,u),$$

so $(G_t v)(y,u) = (1-u)(B_t v)(y,u)$.

Next, $\widehat{B}(s)v = \widetilde{P}\widehat{V}(s)$ where

$$\begin{aligned}
\widehat{V}(s)(y,u) = \int_0^\infty e^{-st} v_t(y,u)\, dt &= \int_u^{u+\varphi(y)} e^{-st} v(y, u+\varphi(y)-t)\, dt \\
&= \int_0^{\varphi(y)} e^{-s(\varphi(y)+u-t)} v(y,t)\, dt.
\end{aligned}$$

It is immediate that

$$(\widehat{G}(s)v)(y,u) = (1-u)(\widehat{B}(s)v)(y,u). \tag{5.5}$$

Suppose that $\delta > \varepsilon > 0$ are fixed. Let $v \in \mathcal{F}_{\delta,\eta}(Y^\varphi)$. Let (y,u), $(y',u) \in \widetilde{Y}_j$, $j \ge 1$. Then

$$|\widehat{V}(s)(y,u)| \le \int_0^{\varphi(y)} e^{-a\,(\varphi(y)+u-t)} |v|_{\delta,\infty}\, e^{\delta t}\, dt \ll e^{\delta\varphi(y)} |v|_{\delta,\infty}$$

and so $|1_{Y_j}\widehat{V}(s)|_\infty \ll e^{\delta |1_{Y_j}\varphi|_\infty} |v|_{\delta,\infty}$.

Next, suppose without loss that $\varphi(y') \le \varphi(y)$. Then

$$\widehat{V}(s)(y,u) - \widehat{V}(s)(y',u) = J_1 + J_2 + J_3$$

where

$$J_1 = \int_0^{\varphi(y)} (e^{-s(\varphi(y)+u-t)} - e^{-s(\varphi(y')+u-t)})v(y,t)\,dt,$$
$$J_2 = \int_0^{\varphi(y)} e^{-s(\varphi(y')+u-t)}(v(y,t) - v(y',t))\,dt,$$
$$J_3 = \int_{\varphi(y')}^{\varphi(y)} e^{-s(\varphi(y')+u-t)}v(y',t)\,dt.$$

For notational convenience we suppose that $a \in (-\varepsilon, 0)$ since the range $a \ge 0$ is simpler. Using (5.1),

$$\begin{aligned}|J_1| &\le \int_0^{\varphi(y)} e^{\varepsilon(|1_{Y_j}\varphi|_\infty+1-t)}|s||\varphi(y)-\varphi(y')||v|_{\delta,\infty}\,e^{\delta t}\,dt \\ &\ll |s|\varphi(y)e^{\delta|1_{Y_j}\varphi|_\infty}\,d(Fy,Fy')^\eta|v|_{\delta,\infty} \ll |s|e^{2\delta|1_{Y_j}\varphi|_\infty}\,d(Fy,Fy')^\eta|v|_{\delta,\infty}, \\ |J_2| &\le \int_0^{\varphi(y)} e^{\varepsilon(|1_{Y_j}\varphi|_\infty+1-t)}|v|_{\delta,\eta}\,e^{\delta t}d(y,y')^\eta\,dt \ll e^{\delta|1_{Y_j}\varphi|_\infty}\,d(y,y')^\eta|v|_{\delta,\eta}, \\ |J_3| &\le \int_{\varphi(y')}^{\varphi(y)} e^{\varepsilon(|1_{Y_j}\varphi|_\infty+1-t)}|v|_{\delta,\infty}\,e^{\delta t}\,dt \ll e^{2\delta|1_{Y_j}\varphi|_\infty}|v|_{\delta,\infty}\,d(Fy,Fy')^\eta.\end{aligned}$$

Hence

$$|\widehat{V}(s)(y,u) - \widehat{V}(s)(y,u)| \ll |s|e^{2\delta|1_{Y_j}\varphi|_\infty}\|v\|_{\delta,\eta}\,d(Fy,Fy')^\eta.$$

Now, for $(y,u) \in \widetilde{Y}$,

$$(\widehat{B}(s)v)(y,u) = (\widetilde{P}\widehat{V}(s))(y,u) = \textstyle\sum_j g(y_j)\widehat{V}(s)(y_j,u),$$

where y_j is the unique preimage of y under $F|Y_j$. It follows from the estimates for $\widehat{V}(s)$ together with (5.3) that

$$\|\widehat{B}(s)v\|_\eta \ll |s|\textstyle\sum_j \mu(Y_j)e^{2\delta|1_{Y_j}\varphi|_\infty}\|v\|_{\delta,\eta}.$$

Shrinking δ, the desired estimate for $\widehat{B}$ follows from (5.2). Finally, the estimate for $\widehat{G}$ follows from (5.5). □

Proposition 5.15. $\int_{\widetilde{Y}} \widehat{B}(0)v\,d\tilde{\mu} = \bar{\varphi}\int_{Y^\varphi} v\,d\mu^\varphi$ for $v \in L^1(Y^\varphi)$.

Proof. By the definition of $\widehat{B}$,

$$\begin{aligned}\int_{\widetilde{Y}} \widehat{B}(0)v\,d\tilde{\mu} &= \int_{\widetilde{Y}}\int_0^\infty L_t(1_{\Delta_t}v)\,dt\,d\tilde{\mu} = \bar{\varphi}\int_0^\infty\int_{Y^\varphi} L_t(1_{\Delta_t}v)\,d\mu^\varphi\,dt \\ &= \bar{\varphi}\int_0^\infty\int_{Y^\varphi} 1_{\Delta_t}v\,d\mu^\varphi\,dt = \bar{\varphi}\int_{Y^\varphi}\int_0^\infty 1_{\{\varphi-u<t<\varphi-u+1\}}v\,dt\,d\mu^\varphi \\ &= \bar{\varphi}\int_{Y^\varphi} v\,d\mu^\varphi,\end{aligned}$$

as required. □

Lemma 5.16. Write $s = a + ib \in \mathbb{C}$. There exists $\varepsilon > 0$, $\delta > 0$, $m_2 \geq 0$, $C > 0$ such that

(a) $s \mapsto \widehat{L}(s) : \mathcal{F}^0_{\delta,\eta}(Y^\varphi) \to \mathcal{F}_{\delta,\eta}(Y^\varphi)$ is analytic on $\{|a| < \varepsilon\}$;

(b) $s \mapsto \widehat{L}(s) : \mathcal{F}_{\delta,\eta}(Y^\varphi) \to \mathcal{F}_{\delta,\eta}(Y^\varphi)$ is analytic on $\{|a| < \varepsilon\}$ except for a simple pole at $s = 0$;

(c) $\|\widehat{L}(s)v\|_{\delta,\eta} \leq C|b|^{m_2}\|v\|_{\delta,\eta}$ for $|a| \leq \varepsilon$, $|b| \geq 1$, $v \in \mathcal{F}_{\delta,\eta}(Y^\varphi)$.

Proof. Recall that

$$\widehat{L} = \widehat{A}\widehat{T}\widehat{B} + \widehat{A}\widehat{G} + \widehat{A}\widehat{H} + \widehat{E}, \qquad \widehat{T} = \widehat{U}(I - \widehat{P})^{-1}$$

where $\widehat{U}$, $\widehat{A}$, $\widehat{B}$, $\widehat{G}$, $\widehat{H}$ and $\widehat{E}$ are analytic by Propositions 5.8, 5.13 and 5.14. Hence part (b) follows immediately from Lemma 5.7(b). Also, part (c) follows using Lemma 5.7(c).

By Proposition 5.15, $\widehat{B}(0)(\mathcal{F}^0_{\delta,\eta}(Y^\varphi)) \subset \mathcal{F}^0_\eta(\widetilde{Y})$. Hence the simple pole at $s = 0$ for $(I - \widehat{P})^{-1}\widehat{B}$ disappears on restriction to $\mathcal{F}^0_{\delta,\eta}(Y^\varphi)$ by Lemma 5.7(a). This proves part (a). □

5.2.3 Moving the contour of integration

Proposition 5.17. Let $m \geq 1$. Let $v \in \mathcal{F}_{\delta,\eta,m}(Y^\varphi)$ with good support. Then $\widehat{L}(s)v = \sum_{j=0}^{m-1}(-1)^j s^{-(j+1)}\partial_t^j v + (-1)^m s^{-m}\widehat{L}(s)\partial_t^m v$ for $a > 0$.

Proof. Recall that $\operatorname{supp} v \subset \{(y,u) \in Y^\varphi : u \in [r, \varphi(y) - r]\}$ for some $r > 0$. For $h \in [0, r]$, we can define $(\Psi_h v)(y,u) = v(y, u-h)$ and then $(\Psi_h v) \circ F_h = v$.

Let $w \in L^\infty(Y^\varphi)$ and write $\rho_{v,w}(t) = \int_{Y^\varphi} v\, w_t\, d\mu^\varphi$ where $w_t = w \circ F_t$. Then for $h \in [0, r]$,

$$\rho_{v,w}(t+h) = \int_{Y^\varphi} v\, w_t \circ F_h\, d\mu^\varphi = \int_{Y^\varphi} (\Psi_h v) \circ F_h\, w_t \circ F_h\, d\mu^\varphi = \int_{Y^\varphi} \Psi_h v\, w_t\, d\mu^\varphi.$$

Hence $h^{-1}(\rho_{v,w}(t+h) - \rho_{v,w}(t)) = \int_{Y^\varphi} h^{-1}(\Psi_h v - v)\, w_t\, d\mu^\varphi$ so

$$\rho'_{v,w}(t) = -\int_{Y^\varphi} \partial_t v\, w_t\, d\mu^\varphi = -\int_{Y^\varphi} \partial_t v\, w \circ F_t\, d\mu^\varphi = -\rho_{\partial_t v, w}(t).$$

Inductively, $\rho^{(j)}_{v,w}(t) = (-1)^j \rho_{\partial_t^j v, w}(t)$.

Now $\int_{Y^\varphi} \widehat{L}(s)v\, w\, d\mu^\varphi = \int_0^\infty e^{-st} \int_{Y^\varphi} L_t v\, w\, d\mu^\varphi\, dt = \int_0^\infty e^{-st} \rho_{v,w}(t)\, dt$, so repeatedly integrating by parts,

$$\begin{aligned}\int_{Y^\varphi} \widehat{L}(s)v\, w\, d\mu^\varphi &= \sum_{j=0}^{m-1} s^{-(j+1)} \rho_{v,w}^{(j)}(0) + s^{-m} \int_0^\infty e^{-st} \rho_{v,w}^{(m)}(t)\, dt \\ &= \sum_{j=0}^{m-1} (-1)^j s^{-(j+1)} \rho_{\partial_t^j v,w}(0) + (-1)^m s^{-m} \int_0^\infty e^{-st} \rho_{\partial_t^m v,w}(t)\, dt \\ &= \int_{Y^\varphi} \sum_{j=0}^{m-1} (-1)^j s^{-(j+1)} \partial_t^j v\, w\, d\mu^\varphi + (-1)^m s^{-m} \int_0^\infty e^{-st} \rho_{\partial_t^m v,w}(t)\, dt.\end{aligned}$$

Finally, $\int_0^\infty e^{-st} \rho_{\partial_t^m v,w}(t)\, dt = \int_{Y^\varphi} \widehat{L}(s) \partial_t^m v\, w\, d\mu^\varphi$ and the result follows since $w \in L^\infty(Y^\varphi)$ is arbitrary. □

We can now estimate $\|L_t v\|_{\delta,\eta}$.

Corollary 5.18. Under the assumptions of Theorem 5.2, there exists $\varepsilon > 0$, $m_3 \geq 1$, $C > 0$ such that

$$\|L_t v\|_{\delta,\eta} \leq C e^{-\varepsilon t} \|v\|_{\delta,\eta,m_3} \quad \text{for all } t > 0$$

for all $v \in \mathcal{F}^0_{\delta,\eta,m_3}(Y^\varphi)$ with good support.

Proof. Let $m_3 = m_2 + 2$. By Lemma 5.16(a), $\widehat{L}(s) : \mathcal{F}^0_{\delta,\eta,m_3}(Y^\varphi) \to \mathcal{F}_{\delta,\eta}(Y^\varphi)$ is analytic for $|a| \leq \varepsilon$. The alternative expression in Proposition 5.17 is also analytic on this region (the apparent singularity at $s = 0$ is removable by the equality with the analytic function $\widehat{L}$). Hence we can move the contour of integration to $s = -\varepsilon + ib$ when computing the inverse Laplace transform, to obtain

$$\begin{aligned}L_t v &= \int_{-\infty}^\infty e^{st} \Big(\sum_{j=0}^{m_3-1} (-1)^j s^{-(j+1)} \partial_t^j v + (-1)^{m_3} s^{-m_3} \widehat{L}(s) \partial_t^{m_3} v \Big)\, db \\ &= e^{-\varepsilon t} \sum_{j=0}^{m_3-1} (-1)^j \partial_t^j v \int_{-\infty}^\infty e^{ibt} s^{-(j+1)}\, db \\ &\qquad + (-1)^{m_3} e^{-\varepsilon t} \int_{-\infty}^\infty e^{ibt} s^{-m_3} \widehat{L}(s) \partial_t^{m_3} v\, db.\end{aligned}$$

The final term is estimated using Lemma 5.16(b,c):

$$\begin{aligned}&\Big\| \int_{-\infty}^\infty e^{ibt} s^{-m_3} \widehat{L}(s) \partial_t^{m_3} v\, db \Big\|_{\delta,\eta} \\ &\qquad\ll \int_{-\infty}^\infty (1+|b|)^{-(m_2+2)} (1+|b|)^{m_2} \|\partial_t^{m_3} v\|_{\delta,\eta}\, db \ll \|v\|_{\delta,\eta,m_3}.\end{aligned}$$

Clearly, the integrals $\int_{-\infty}^\infty e^{ibt} s^{-(j+1)}\, db$ converge absolutely for $j \geq 1$, while the integral for $j = 0$ converges as an improper Riemann integral. Hence altogether we obtain that $\|L_t v\|_{\delta,\eta} \ll e^{-\varepsilon t} \|v\|_{\delta,\eta,m_3}$. □

For the proof of Theorem 5.2, it remains to estimate $\|\partial_u L_t v\|_{\delta,\eta}$. Recall that the transfer operator P_0 for F has weight function g. We have the pointwise formula $(P_0^k v)(y) = \sum_{F^k y'=y} g_k(y')v(y')$ where $g_k = g \ldots g\circ F^{k-1}$. Let $\varphi_k = \sum_{j=0}^{k-1}\varphi\circ F^j$.

Proposition 5.19. Let $v \in L^1(Y^\varphi)$. Then for all $t > 0$, $(y,u)\in Y^\varphi$,

$$(L_t v)(y,u) = \sum_{k=0}^{[t/2]}\sum_{F^k y'=y} g_k(y')\mathbb{1}_{\{0\le u-t+\varphi_k(y')<\varphi(y')\}}v(y',u-t+\varphi_k(y')).$$

Proof. Recall that the roof function φ is bounded below by 2. The lap number $N_t(y,u)\in[0,t/2]\cap\mathbb{N}$ is the unique integer $k\ge 0$ such that $u+t-\varphi_k(y)\in[0,\varphi(F^k y))$. In particular, $F_t(y,u) = (F^{N_t(y,u)}y, u+t-\varphi_{N_t(y,u)}(y))$. For $w\in L^\infty(Y^\varphi)$,

$$\begin{aligned}\int_{Y^\varphi} L_t(\mathbb{1}_{\{N_t=k\}}v)\, w\, d\mu^\varphi &= \int_{Y^\varphi}\mathbb{1}_{\{N_t=k\}}v\, w\circ F_t\, d\mu^\varphi\\ &= \bar\varphi^{-1}\int_Y\int_0^{\varphi(y)}\mathbb{1}_{\{0\le u+t-\varphi_k(y)<\varphi(F^k y)\}}v(y,u)\, w(F^k y, u+t-\varphi_k(y))\, du\, d\mu\\ &= \bar\varphi^{-1}\int_Y\int_0^{\varphi(F^k y)}\mathbb{1}_{\{0\le u-t+\varphi_k(y)<\varphi(y)\}}v(y,u-t+\varphi_k(y))\, w(F^k y,u)\, du\, d\mu.\end{aligned}$$

Writing $v_{t,k}^u(y) = \mathbb{1}_{\{0\le u-t+\varphi_k(y)<\varphi(y)\}}v(y,u-t+\varphi_k(y))$ and $w^u(y) = w(y,u)$,

$$\begin{aligned}\int_{Y^\varphi} L_t(\mathbb{1}_{\{N_t=k\}}v)\, w\, d\mu^\varphi &= \bar\varphi^{-1}\int_0^\infty\int_Y \mathbb{1}_{\{u<\varphi\circ F^k\}}v_{t,k}^u\, w^u\circ F^k\, d\mu\, du\\ &= \bar\varphi^{-1}\int_0^\infty\int_Y\mathbb{1}_{\{u<\varphi\}}P_0^k v_{t,k}^u\, w^u\, d\mu\, du = \int_{Y^\varphi}(P_0^k v_{t,k}^u)(y)\, w(y,u)\, d\mu^\varphi.\end{aligned}$$

Hence,

$$(L_t v)(y,u) = \sum_{k=0}^{[t/2]}(L_t(\mathbb{1}_{\{N_t=k\}}v)(y,u) = \sum_{k=0}^{[t/2]}(P_0^k v_{t,k}^u)(y).$$

The result follows from the pointwise formula for P_0^k. □

Proof of Theorem 5.2. Let $m = m_3+1$. By Corollary 5.18, $\|L_t v\|_{\delta,\eta} \ll e^{-\varepsilon t}\|v\|_{\delta,\eta,m}$.

Recall that ∂_u denotes the ordinary derivative with respect to u at $0<u<\varphi(y)$ and denotes the appropriate one-sided derivative at $u=0$ and $u=\varphi(y)$. Since v has good support, the indicator functions in the right-hand side of the formula in Proposition 5.19 are constant on the support of v. It follows that $\partial_u L_t v = L_t(\partial_u v)$. By Corollary 5.18,

$$\|\partial_u L_t v\|_{\delta,\eta} = \|L_t(\partial_u v)\|_{\delta,\eta} \ll e^{-\varepsilon t}\|\partial_u v\|_{\delta,\eta,m_3}\le e^{-\varepsilon t}\|v\|_{\delta,\eta,m}.$$

Hence, $\|L_t v\|_{\delta,\eta,1}\ll e^{-\varepsilon t}\|v\|_{\delta,\eta,m}$ as required. □

Appendix A

Jump measures for martingales

In Subsections 3.2.4 and 3.4.2 we have shown a rate of $\mathcal{O}(n^{-1/4}(\log n)^{3/4})$ for real-valued observables defined on uniformly expanding maps and semiflows. The proofs in both discrete and continuous time rely on Theorem 3.18. This appendix presents notions from the general martingale theory, and explains how to get Theorem 3.18 from [18, Lemma 3]. Basic definitions about martingales can be found in Chapter 2. We use [29] as a reference, in particular Chapter II.1 for the content on jump measures, their compensators, and $\star$-processes.

Definition A.1. Let Y be a càdlàg process adapted to a filtration and defined on the probability space $(\Omega, \mathbb{P})$. The *jump measure* of Y is a family μ_Y of measures on $[0,1] \times \mathbb{R}$, indexed by $\omega \in \Omega$,

$$\mu_Y(\omega; \mathrm{d}t\, \mathrm{d}x) = \textstyle\sum_{s\in[0,1]} \mathbb{1}_{\{\Delta Y_s(\omega)\neq 0\}}\delta_{(s,\Delta Y_s(\omega))}(\mathrm{d}t, \mathrm{d}x),$$

where δ is the Dirac measure on $[0,1] \times \mathbb{R}$, and $\Delta Y_s = Y_s - \lim_{t\to s^-} Y_t$.

The sum in Definition A.1 is at most countable for all $\omega \in \Omega$, because any càdlàg function has at most countably many discontinuities. For any Borel-measurable $f\colon \mathbb{R} \to \mathbb{R}$, we define the $\star$-process of f with respect to μ_Y as

$$f \star \mu_Y(\omega, t) = \textstyle\int_0^t \int_{\mathbb{R}} f(x)\mu_Y(\omega; \mathrm{d}s, \mathrm{d}x), \qquad \omega \in \Omega,\ t \in [0,1], \tag{A.1}$$

when the integral exists finite, and ∞ otherwise.

Another important family of measures on $[0,1] \times \mathbb{R}$ is the *compensator* of μ_Y, denoted by ν_Y. Its characterisation can be found in [29, Theorem II.1.8]; we do not make it explicit because requires concepts that go beyond the purposes of this appendix. We know by [29, Theorem II.1.17(b)], that there exists a "good version" of the compensator of μ_Y, which is different from ν_Y at most on a null set of Ω. Let us remark that we can define the process $f \star \nu_Y$ as we did in (A.1) for μ_Y.

Proposition A.2. Let $f\colon \mathbb{R} \to \mathbb{R}$ such that $\mathbb{E}[|f| \star \nu_Y(1)] < \infty$. Suppose that there exists a predictable process A such that $f \star \mu_Y - A$ is a martingale. Then A is indistinguishable from $f \star \nu_Y$.

Proof. Apply [29, Theorem II.1.8(ii)] with $W = f$, using that any martingale is also a local martingale (which definition is omitted here). □

Lemma A.3. Let Y be a càdlàg process adapted to a filtration and with uniformly bounded jumps (that is $\beta_0 = |\sup_{t\in[0,1]} \Delta Y_t|_\infty < \infty$). Then for all measurable $f\colon \mathbb{R} \to \mathbb{R}$, the process $f(x)\mathbb{1}_{\{|x|>\beta\}} \star \tau$ is null, for $\tau = \mu_Y, \nu_Y$ and all $\beta \geq \beta_0$.

Proof. Let $\beta \geq \beta_0$. By Definition A.1, $\mu_Y(\omega; [0,1], \mathbb{R} \setminus [-\beta, \beta]) = 0$ for a.e. $\omega \in \Omega$. By (A.1), for a.e. $\omega \in \Omega$ and all $t \in [0,1]$, $f(x)\mathbb{1}_{\{|x|>\beta\}} \star \mu_Y(t) = 0$. The null process is a martingale, hence Proposition A.2 yields that $A = f(x)\mathbb{1}_{\{|x|>\beta\}} \star \nu_Y$ is the null process, too. □

For X, Y càdlàg processes defined on the same probability space, denote with

$$\alpha_U(X,Y) = \inf\{\varepsilon > 0 : \mathbb{P}(\sup_{t\in[0,1]} |X(t) - Y(t)| > \varepsilon) \leq \varepsilon\}.$$

Following Courbot [18], we call α_U the *uniform Ky Fan distance.*

Next proposition is an adaptation of [18, Lemma 3] for a bounded stationary RMDS. Such a result is stated in Courbot [18] for general continuous time martingales, however for our purposes it suffices to consider martingales constructed from an RMDS via Proposition 2.22 and Remark 2.23.

Remark A.4. The uniform Ky Fan distance satisfies the axioms to be a metric; however, despite of its name, α_U is not a Ky Fan distance in the sense of Section 2.4, because the càdlàg function space with the sup norm is not separable. Yet, if both X and Y have continuous sample paths, then $\alpha_U(X,Y)$ is the genuine Ky Fan distance of X and Y, defined by the metric space $(\mathcal{C}[0,1], \|\cdot\|_\infty)$.

Proposition A.5 (Courbot)**.** Let $(d_n)_{n\geq 0}$ be a real bounded stationary RMDS, with σ-algebras $(\mathcal{G}_n)_{n\geq 0}$. Define for $1 \leq k \leq n$ the process $M_n(t) = n^{-\frac{1}{2}} \sum_{j=1}^{\lfloor nt \rfloor} d_{n-j}$, $t \in [0,1]$, and $V_n(k) = n^{-1} \sum_{j=1}^{k} \mathbb{E}[d_{n-j}^2 | \mathcal{G}_{n-(j-1)}]$. Let κ_n and $\widetilde{\kappa}_n$be as in Theorem 3.18,

$$\kappa_n = \inf\big\{\varepsilon > 0 : \mathbb{P}\big(\max_{0\leq k\leq n} |V_n(k) - (k/n)\sigma^2| > \varepsilon\big) \leq \varepsilon\big\},$$
$$\widetilde{\kappa}_n = \max\big\{\kappa_n |\log \kappa_n|^{-\frac{1}{2}}, n^{-\frac{1}{2}}\big\}.$$

For every $n \geq 1$, there exist a probability space supporting a càdlàg process Z_n and a Brownian motion B with variance $\sigma^2 = \mathbb{E}[d_0]$, such that $M_n =_d Z_n$ in the Skorohod J_1 topology. Moreover, there exists $C > 0$ such that

$$\alpha_U(Z_n, B) \leq C\widetilde{\kappa}_n^{1/2}|\log \widetilde{\kappa}_n|^{3/4},$$

for all $n \geq 1$ for which $\widetilde{\kappa}_n \in (0, \frac{1}{2})$.

Proof. Let $L = |d_0|_\infty$. The processes M_n are continuous time martingales by Proposition 2.22 and Remark 2.23. They are square integrable because each M_n is bounded, $|\sup_{t\in[0,1]} |M_n(t)||_\infty \leq n^{1/2}L < \infty$. Using the angle brackets to denote the quadratic variation process, it can be checked that $\langle M_n\rangle(t) = V_n(\lfloor t \rfloor)$, $t \in [0,1]$. For any Brownian motion W with variance σ^2 it is known that $\langle W\rangle(t) = \sigma^2 t$; it follows that $\kappa_n = \alpha_U(\langle M_n\rangle, \langle W\rangle)$.

For $n \geq 1$ and $\beta > 0$, write as in [18]

$$M_n^\beta = M_n - x\mathbb{1}_{\{|x|>\beta\}} \star (\mu_n - \nu_n) \qquad \text{and} \qquad A_{n,\beta} = \alpha(|x|^2\mathbb{1}_{\{|x|>\beta\}} \star \nu_n, 0).$$

Here, μ_n is the jump measure of M_n, and ν_n is a good version of the compensator of μ_n. Since the jumps of M_n are bounded by $n^{-1/2}L$, Lemma A.3 yields that $M_n^\beta = M_n$ and $A_{n,\beta} = 0$ for all $\beta \geq n^{-1/2}L$.

Define

$$b_n = \max\{\kappa_n|\log \kappa_n|^{-\frac{1}{2}}, \inf\{\beta > 0 : \beta|\log \beta| \geq A_{n,\beta}\}\}.$$

Reasoning as in [17] and in the proof of [18, Lemma 3], for every $n \geq 1$ there exists a probability space supporting a Brownian motion B and a time change τ_n, such that $M_n = M_n^L =_d B \circ \tau_n$ as càdlàg processes. Write $Z_n = B \circ \tau_n$. The proof of [18, Lemma 3] yields that $\alpha_U(Z_n, B) \ll b_n^{1/2}|\log b_n|^{3/4}$, for b_n small enough.

We are left to show that $b_n \ll \widetilde{\kappa}_n$. Let $\beta_n = n^{-1/2}L$, so $\beta_n|\log \beta_n| \geq 0 = A_{n,\beta_n}$ for all $n \geq 1$. Then, $\inf\{\beta > 0 : \beta|\log \beta| \geq A_{n,\beta}\} \leq \beta_n$. We finish by

$$b_n \leq \max\{\kappa_n|\log \kappa_n|^{-\frac{1}{2}}, \beta_n\} \ll \max\{\kappa_n|\log \kappa_n|^{-\frac{1}{2}}, n^{-\frac{1}{2}}\} = \widetilde{\kappa}_n. \qquad \square$$

Proof of Theorem 3.18. Let $n \geq 1$ for which $\widetilde{\kappa}_n \in (0, \frac{1}{2})$, and define the sequence $a(n) = \widetilde{\kappa}_n^{1/2}|\log \widetilde{\kappa}_n|^{3/4}$. By Proposition A.5, there exist $Z_n =_d M_n$, $B =_d W$ in the J_1 topology, such that $\alpha_U(Z_n, B) \ll a(n)$. Since Z_n and M_n share the same law, Z_n is piecewise constant with jumps at the same places as M_n, with probability 1. Hence, if we define Z_n^c by linearly interpolating Z_n at k/n, $0 \leq k \leq n$, we get $M_n^c =_d Z_n^c$ as continuous processes.

Every jump of M_n is bounded by $n^{-1/2}|d_0|_\infty$, so

$$|\sup_{t\in[0,1]}|M_n^c(t)-M_n(t)||_\infty \le n^{-\frac{1}{2}}|d_0|_\infty.$$

Hence,

$$\begin{aligned}\mathbb{P}(\sup_{t\in[0,1]}|Z_n^c(t)-Z_n(t)|>n^{-\frac{1}{2}}C) =&\mathbb{P}(\sup_{t\in[0,1]}|M_n^c(t)-M_n(t)|>n^{-\frac{1}{2}}C)\\ =&0\le n^{-\frac{1}{2}}|d_0|_\infty.\end{aligned}$$

By definition, $\alpha_U(Z_n^c, Z_n) \ll n^{-\frac{1}{2}}$. By definition of $\widetilde{\kappa}_n$, we have $n^{-\frac{1}{2}} \ll a(n)$. Hence, $\alpha_U(Z_n^c, B) \le \alpha_U(Z_n^c, Z_n) + \alpha_U(Z_n, B) \ll n^{-\frac{1}{2}} + a(n) \ll a(n)$. Using Remark A.4 and equation (2.2), we can conclude with

$$\Pi(M_n^c, W) = \Pi(Z_n^c, B) \le \alpha_U(Z_n^c, B) \ll a(n). \qquad \square$$

Bibliography

[1] Aaronson, J. (1997). *An introduction to infinite ergodic theory*, volume 50 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI.

[2] Aaronson, J. and Denker, M. (2001). Local limit theorems for partial sums of stationary sequences generated by Gibbs-Markov maps. *Stoch. Dyn.*, 1(2):193–237.

[3] Aaronson, J., Denker, M., and Urbański, M. (1993). Ergodic theory for Markov fibred systems and parabolic rational maps. *Trans. Amer. Math. Soc.*, 337(2):495–548.

[4] Alves, J. F. (2020). *Nonuniformly hyperbolic attractors—geometric and probabilistic aspects*. Springer Monographs in Mathematics. Springer, Cham.

[5] Antoniou, M. and Melbourne, I. (2019). Rate of convergence in the weak invariance principle for deterministic systems. *Comm. Math. Phys.*, 369(3):1147–1165.

[6] Araújo, V. and Melbourne, I. (2016). Exponential decay of correlations for nonuniformly hyperbolic flows with a $C^{1+\alpha}$ stable foliation, including the classical Lorenz attractor. *Ann. Henri Poincaré*, 17(11):2975–3004.

[7] Avila, A., Gouëzel, S., and Yoccoz, J.-C. (2006). Exponential mixing for the Teichmüller flow. *Publ. Math. Inst. Hautes Études Sci.*, (104):143–211.

[8] Baladi, V., Demers, M. F., and Liverani, C. (2018). Exponential decay of correlations for finite horizon Sinai billiard flows. *Invent. Math.*, 211(1):39–177.

[9] Baladi, V. and Vallée, B. (2005). Exponential decay of correlations for surface semi-flows without finite Markov partitions. *Proc. Amer. Math. Soc.*, 133(3):865–874.

[10] Bálint, P. and Melbourne, I. (2018). Statistical properties for flows with unbounded roof function, including the Lorenz attractor. *J. Stat. Phys.*, 172(4):1101–1126.

[11] Borovkov, A. A. (1973). The rate of convergence in the invariance principle. *Teor. Verojatnost. i Primenen.*, 18:217–234.

[12] Bruin, H., Melbourne, I., and Terhesiu, D. (2019). Rates of mixing for non-Markov infinite measure semiflows. *Trans. Amer. Math. Soc.*, 371(10):7343–7386.

[13] Burkholder, D. L. (1973). Distribution function inequalities for martingales. *Ann. Probability*, 1:19–42.

[14] Butterley, O. (2016). A note on operator semigroups associated to chaotic flows. *Ergodic Theory Dynam. Systems*, 36(5):1396–1408.

[15] Chevyrev, I., Friz, P. K., Korepanov, A., Melbourne, I., and Zhang, H. (2019). Multiscale systems, homogenization, and rough paths. In *Probability and analysis in interacting physical systems*, volume 283 of *Springer Proc. Math. Stat.*, pages 17–48. Springer, Cham.

[16] Chung, K. L. and Williams, R. J. (2014). *Introduction to stochastic integration*. Modern Birkhäuser Classics. Birkhäuser/Springer, New York, second edition.

[17] Coquet, F., Mémin, J., and Vostrikova, L. (1994). Rate of convergence in the functional limit theorem for likelihood ratio processes. *Math. Methods Statist.*, 3(2):89–113.

[18] Courbot, B. (1999). Rates of convergence in the functional CLT for martingales. *C. R. Acad. Sci. Paris Sér. I Math.*, 328(6):509–513.

[19] Cuny, C., Dedecker, J., and Merlevède, F. (2021). Rates of convergence in invariance principles for random walks on linear groups via martingale methods. *Trans. Amer. Math. Soc.*, 374(1):137–174.

[20] Denker, M. and Philipp, W. (1984). Approximation by Brownian motion for Gibbs measures and flows under a function. *Ergodic Theory Dynam. Systems*, 4(4):541–552.

[21] Dolgopyat, D. (1998a). On decay of correlations in Anosov flows. *Ann. of Math. (2)*, 147(2):357–390.

[22] Dolgopyat, D. (1998b). Prevalence of rapid mixing in hyperbolic flows. *Ergodic Theory Dynam. Systems*, 18(5):1097–1114.

[23] Donsker, M. D. (1951). An invariance principle for certain probability limit theorems. *Mem. Amer. Math. Soc.*, 6:12.

[24] Dudley, R. M. (2002). *Real analysis and probability*, volume 74 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge. Revised reprint of the 1989 original.

[25] Gibbs, A. L. and Su, F. E. (2002). On choosing and bounding probability metrics. *Int. Stat. Rev.*, 70(3):419–435.

[26] Gordin, M. I. (1969). The central limit theorem for stationary processes. *Dokl. Akad. Nauk SSSR*, 188:739–741.

[27] Gouëzel, S. (2004). Central limit theorem and stable laws for intermittent maps. *Probab. Theory Related Fields*, 128(1):82–122.

[28] Hofbauer, F. and Keller, G. (1982). Ergodic properties of invariant measures for piecewise monotonic transformations. *Math. Z.*, 180(1):119–140.

[29] Jacod, J. and Shiryaev, A. N. (2003). *Limit theorems for stochastic processes*, volume 288 of *Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, second edition.

[30] Katok, A. and Hasselblatt, B. (1995). *Introduction to the modern theory of dynamical systems*, volume 54 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge. With a supplementary chapter by Katok and Leonardo Mendoza.

[31] Kelly, D. and Melbourne, I. (2016). Smooth approximation of stochastic differential equations. *Ann. Probab.*, 44(1):479–520.

[32] Korepanov, A., Kosloff, Z., and Melbourne, I. (2018). Martingale-coboundary decomposition for families of dynamical systems. *Ann. Inst. H. Poincaré C Anal. Non Linéaire*, 35(4):859–885.

[33] Kubilius, K. (1994). The rate of convergence in the invariance principle for martingale difference arrays. *Liet. Mat. Rink.*, 34(4):482–494.

[34] Kuelbs, J. and Philipp, W. (1980). Almost sure invariance principles for partial sums of mixing B-valued random variables. *Ann. Probab.*, 8(6):1003–1036.

[35] Liu, Z. and Wang, Z. (2023). Wasserstein convergence rate in the invariance principle for deterministic dynamical systems. *arXiv:2204.00263v2 [math.DS]*.

[36] Liverani, C. (2004). On contact Anosov flows. *Ann. of Math. (2)*, 159(3):1275–1312.

[37] Liverani, C., Saussol, B., and Vaienti, S. (1999). A probabilistic approach to intermittency. *Ergodic Theory Dynam. Systems*, 19(3):671–685.

[38] Melbourne, I. (2015). Fast-Slow Skew Product Systems and Convergence to Stochastic Differential Equations. *Lecture notes for LMS-CMI Research School Statistical Properties of Dynamical Systems at Loughborough.*

[39] Melbourne, I. (2018). Superpolynomial and polynomial mixing for semiflows and flows. *Nonlinearity*, 31(10):R268–R316.

[40] Melbourne, I. and Nicol, M. (2005). Almost sure invariance principle for nonuniformly hyperbolic systems. *Comm. Math. Phys.*, 260(1):131–146.

[41] Melbourne, I. and Nicol, M. (2008). Large deviations for nonuniformly hyperbolic systems. *Trans. Amer. Math. Soc.*, 360(12):6661–6676.

[42] Melbourne, I., Paviato, N., and Terhesiu, D. (2022a). Decay in norm of transfer operators for semiflows. *Studia Math.*, 266(2):149–166.

[43] Melbourne, I., Paviato, N., and Terhesiu, D. (2022b). Nonexistence of spectral gaps in Hölder spaces for continuous time dynamical systems. *Israel J. Math.*, 247(2):987–991.

[44] Melbourne, I. and Terhesiu, D. (2017). Operator renewal theory for continuous time dynamical systems with finite and infinite measure. *Monatsh. Math.*, 182(2):377–431.

[45] Melbourne, I. and Török, A. (2004). Statistical limit theorems for suspension flows. *Israel J. Math.*, 144:191–209.

[46] Melbourne, I. and Varandas, P. (2016). A note on statistical properties for nonuniformly hyperbolic systems with slow contraction and expansion. *Stoch. Dyn.*, 16(3):1660012, 13.

[47] Melbourne, I. and Zweimüller, R. (2015). Weak convergence to stable Lévy processes for nonuniformly hyperbolic dynamical systems. *Ann. Inst. Henri Poincaré Probab. Stat.*, 51(2):545–556.

[48] Merlevède, F., Peligrad, M., and Utev, S. (2006). Recent advances in invariance principles for stationary sequences. *Probab. Surv.*, 3:1–36.

[49] Parry, W. and Pollicott, M. (1990). *Zeta functions and the periodic orbit structure of hyperbolic dynamics.* Number 187-188.

[50] Pollicott, M. (1985). On the rate of mixing of Axiom A flows. *Invent. Math.*, 81(3):413–426.

[51] Pomeau, Y. and Manneville, P. (1980). Intermittent transition to turbulence in dissipative dynamical systems. *Commun. Math. Phys.*, 74(2):189–197.

[52] Pène, F. (2007). A Berry Esseen result for the billiard transformation. *hal-01101281.*

[53] Sawyer, S. (1972). Rates of convergence for some functionals in probability. *Ann. Math. Statist.*, 43:273–284.

[54] Tsujii, M. (2008). Decay of correlations in suspension semi-flows of angle-multiplying maps. *Ergodic Theory Dynam. Systems*, 28(1):291–317.

[55] Tsujii, M. (2010). Quasi-compactness of transfer operators for contact Anosov flows. *Nonlinearity*, 23(7):1495–1545.

[56] Whitt, W. (1974). Preservation of rates of convergence under mappings. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete*, 29:39–44.

[57] Whitt, W. (2002). *Stochastic-process limits: an introduction to stochastic - process limits and their application to queues.* Springer series in operations research. Springer, New York. OCLC: 56115454.

[58] Williams, D. (1991). *Probability with martingales.* Cambridge Mathematical Textbooks. Cambridge University Press, Cambridge.

[59] Young, L.-S. (1998). Statistical properties of dynamical systems with some hyperbolicity. *Ann. of Math. (2)*, 147(3):585–650.

[60] Young, L.-S. (1999). Recurrence times and rates of mixing. *Israel J. Math.*, 110:153–188.