

Optimal Markovian coupling for finite activity Lévy processes

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We study optimal Markovian couplings of Markov processes, where the optimality is understood in terms of minimization of concave transport costs between evaluations of the coupled processes at corresponding times. We provide explicit constructions of such optimal couplings for one-dimensional finite-activity Lévy processes (continuous-time random walks) whose jump distributions are unimodal but not necessarily symmetric. Remarkably, the optimal Markovian coupling does not depend on the specific concave transport cost. To this end, we combine McCann’s results on optimal transport and Rogers’ results on random walks with a novel uniformization construction that allows us to characterize all Markovian couplings of finite-activity Lévy processes. In particular, we show that the optimal Markovian coupling for finite-activity Lévy processes with non-symmetric unimodal Lévy measures has to allow for non-simultaneous jumps of the two coupled processes.

Keywords: Concave transport cost; continuous-time random walk; finite activity Lévy process; immersion coupling; Lévy process; Markovian coupling; maximal coupling; optimal coupling; simultaneous optimality; unimodal distribution; Wasserstein distance

1. Introduction

This paper considers Markovian couplings of a finite activity Lévy process (“continuous-time random walk” for which we always assume the holding times to be i.i.d. exponentials) X , started at 0 and $a > 0$ respectively. We show that if the jump distribution is unimodal then there is a unique Markovian coupling (X, Z) with $Z_0 = X_0 + a$, based on the anti-monotonic rearrangement construction described in [41], such that for any $t > 0$, the expected cost $\mathbb{E}[\phi(|X_t - Z_t|)]$ is simultaneously minimized for all bounded concave increasing cost functions $\phi : [0, \infty) \rightarrow [0, \infty)$. If the jumps are symmetric then this optimal Markovian coupling can be viewed as a reflection coupling. However, our arguments apply to all unimodal non-symmetric jump distributions, even as extreme as exponential (where all jumps are non-negative). In the latter case the optimal Markovian coupling involves the processes jumping independently till crossover, when they couple.

Recall that a *coupling* of two probability measures μ_1 and μ_2 , defined respectively on Polish measure spaces (E_1, \mathcal{E}_1) and (E_2, \mathcal{E}_2) , is a probability measure μ defined on the product space $(E_1 \times E_2, \mathcal{E}_1 \otimes \mathcal{E}_2)$ with marginals μ_1 and μ_2 . If we consider two canonical random variables X and Y on $(E_1, \mathcal{E}_1, \mu_1)$ and $(E_2, \mathcal{E}_2, \mu_2)$, respectively, then any coupling μ of μ_1 and μ_2 specifies a law of a random vector (X', Y') on $(E_1 \times E_2, \mathcal{E}_1 \otimes \mathcal{E}_2)$, which we also call a coupling of the random variables X and Y . These definitions extend naturally to any finite number of measures or random variables [47, Chapter 3, Section 3]. Furthermore, by considering random variables on $(E^I, \mathcal{E}^{\otimes I})$ for a time index set I , we can consider couplings of (E, \mathcal{E}) -valued stochastic processes indexed by I [47, Chapter 4].

The question “what is the best possible coupling?” clearly depends on context and the chosen notion of optimality. In the context of couplings of stochastic processes, the most widespread criterion of

optimality uses the notion of a *maximal coupling*, which is a coupling that minimizes the meeting time of the two coupled processes started from different initial positions [2,3,21,23,24,26,29,30,42]. Through the classical coupling inequality [36], the problem of minimizing the meeting time is closely related to the problem of obtaining sharp upper bounds on the total-variation distance between time-marginal distributions of the coupled processes.

In recent years, couplings have found numerous novel applications to the problem of quantifying convergence rates of Markov processes in other probabilistic metrics, in particular the Wasserstein distances. This has been an active area of research in stochastic analysis [18,19,37–39], computational statistics [4,20,40] and machine learning [7,11–13,27]. These applications focus on different optimality criteria for couplings, related to the optimal transport interpretation of the Wasserstein distances [8,22,34]. To see this, recall that the L^p -Wasserstein distance is defined for any $p \geq 1$, and for any two probability measures μ_1 and μ_2 on a measure space (E, \mathcal{E}) , by

$$W_{\text{dist},p}(\mu_1, \mu_2) = \left(\inf_{\pi \in \Pi(\mu_1, \mu_2)} \int_{E \times E} \text{dist}(x, y)^p \pi(\mathrm{d}x \mathrm{d}y) \right)^{1/p}, \quad (1)$$

where dist is a metric on (E, \mathcal{E}) and $\Pi(\mu_1, \mu_2)$ is the set of all couplings of μ_1 and μ_2 . By (1), any coupling $(X_t, Y_t)_{t \geq 0}$ of two stochastic processes $(X_t)_{t \geq 0}$ and $(Y_t)_{t \geq 0}$ on (E, \mathcal{E}) provides an upper bound on the Wasserstein distance between their time-marginal distributions $\mathcal{L}(X_t)$ and $\mathcal{L}(Y_t)$ for any $t \geq 0$:

$$W_{\text{dist},p}(\mathcal{L}(X_t), \mathcal{L}(Y_t)) \leq (\mathbb{E}[\text{dist}(X_t, Y_t)^p])^{1/p}.$$

Sharp upper bounds on such Wasserstein distances arise by minimizing the quantity $\mathbb{E}[\text{dist}(X_t, Y_t)^p]$ over all processes $(X_t, Y_t)_{t \geq 0}$ from a given class of couplings. General results guarantee existence of the optimal coupling for lower semi-continuous metrics dist ([31, Theorem 2.19] or [48, Chapter 4]); however these results are non-constructive and hence do not yield quantitative bounds on $W_{\text{dist},p}$. For a discussion on uniqueness of optimal couplings, see e.g. [48, Chapter 10]. In this paper we discuss explicit constructions of optimal couplings, forcing us to consider a more specific setting and in particular to focus on a specific class of couplings. A natural requirement for the choice of such a class is that both the coupled processes should respect an appropriate common filtration. Definition 2.1 below formalizes this by introducing the notions of *immersion couplings* and *Markovian couplings*. We focus on the case of $p = 1$ and a specific class of metrics $\text{dist}(x, y) = \phi(|x - y|)$ based on continuous concave cost functions $\phi : [0, \infty) \rightarrow [0, \infty)$. In particular our results cover metrics of the form $\text{dist}(x, y) = |x - y|^\gamma$ for $\gamma \in (0, 1)$. We would like to remark that one can naturally consider a related problem of finding sharp lower bounds on $W_{\text{dist},p}$, by considering a dual representation of Wasserstein distances [48, Chapter 6], which, however, is beyond the scope of the present paper.

The task of finding optimal immersion couplings is non-trivial, even for discrete-time processes, (in which case a general coupling can be specified by iteratively coupling the transition (jump) distributions of two processes step-by-step). Rogers [44] considered the problem of identifying the one-step optimal coupling for one-dimensional random walks with unimodal jump distributions, and characterised this in terms of a joint cumulative distribution function: he then showed the resulting random walk coupling was optimal (amongst Markovian couplings) over all time and for a wide range of bounded convex pay-off functions. The present paper uses a different characterisation to generalize [44] mildly to the case of one-dimensional Markov chains with non-symmetric unimodal jump distributions perhaps with atoms at the modes. This generalisation is useful when moving to continuous-time processes (Section 3).

For more general Markov processes with symmetric transition kernels, it seems to be established folklore that applying a symmetrized version of McCann's [41] anti-monotonic rearrangement (AMR) coupling is the preferred choice for concave costs, even in multi-dimensional isotropic settings

[11,18,20,38]. However, optimality has been established only for (one-dimensional) probability measures [41], not in the context of general stochastic processes. If the transition kernels are not symmetric then the general case becomes very complicated and no general optimality results have been considered. Without imposing any geometric conditions on transition probabilities, the only natural solution for a step-by-step coupling of stochastic processes seems to be to preserve the common mass of their jump measures and to couple the remaining mass independently: the *basic coupling* [10, Example 2.10]. Luo and Wang [37] proposed the *refined basic coupling*, preserving only half of the common mass of the jump measures, and coupling the remainder (the non-common mass) synchronously rather than independently, but imposing a different transformation on the second half of the common mass. This construction is typically non-optimal but permits useful upper bounds on the Wasserstein distances between time-marginals of various stochastic processes [28,33–35] without needing restrictive conditions on transition distributions. Determining the optimal coupling in general seems too ambitious a goal for now.

Even in settings where an optimal coupling is available for given probability measures, constructing an optimal coupling for stochastic processes having such measures as their jump distributions can be highly non-trivial, especially in continuous-time. A significant part of the problem is the need to work with general characterisations of all possible Markovian couplings of stochastic processes from a given class. Such characterisations have been used in coupling diffusions or birth-death processes [9], but have not yet been exploited for general pure jump Lévy processes [34, discussion in Section 4].

The present paper discusses explicit constructions of optimal Markovian couplings in the case of one-dimensional discrete-time Markov chains and continuous-time finite-activity Lévy processes whose jump distributions are unimodal (but not necessarily symmetric). To this end, Section 2 revisits results from [41] and [44] to provide a new construction of McCann’s AMR coupling, via the Skorokhod representation. Methods from [44] establish optimality for AMR, first for couplings of individual random variables and then for Markov chains (Theorem 2.11, which is our main optimality result in discrete time). Section 2 ends with Example 2.13 that illustrates the necessity of unimodality for the type of optimality results considered in the present paper. Section 3 then introduces a novel uniformization construction for immersion couplings of finite activity Lévy processes (Theorem 3.1), permitting application of modified arguments from Section 2 in the case of continuous time. The final result, Theorem 3.2, constructs an optimal coupling of finite-activity Lévy processes with a unimodal Lévy measure ν (i.e., with jumps distributed according to $\nu/\nu(\mathbb{R})$), using step-by-step application of the AMR coupling to the processes’ counterparts with jump measures $\frac{1}{2\nu(\mathbb{R})}\nu + \frac{1}{2}\delta_0$. This final observation leads to a conclusion that, for optimal couplings of finite-activity Lévy processes with non-symmetric unimodal Lévy measures, some non-zero jumps of the first coupled process may correspond to zero jumps of the second, and vice versa; so optimally coupled processes need not jump simultaneously.

2. Optimal Markovian coupling for Markov chains

As indicated in the introduction, we need to consider couplings which simultaneously respect a common filtration to which both random processes are adapted, but such that the coupling construction does not “look into the future” [6, Section 1]. We use the expressive and concise language of martingales:

Definition 2.1 (Immersion and Markovian couplings for random processes). *Consider two random processes X and Y taking values in a Polish space (E, \mathcal{E}) , defined on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and adapted to a common filtration of σ -algebras $\{\mathcal{F}_t : t \geq 0\}$. We suppose here and throughout that the filtration is regularized, in the sense that $\mathcal{F}_t = \bigcap_{s>t} \mathcal{F}_s$ and moreover for all $t \geq 0$ the σ -algebra \mathcal{F}_t contains all \mathbb{P} -null sets. We say that (X, Y) is an immersion coupling (synonyms: faithful*

coupling [45], co-adapted coupling [14, 15]) if the following holds: All martingales for the natural filtration of X are also martingales in the common filtration $\{\mathcal{F}_t : t \geq 0\}$, similarly all martingales for the natural filtration of Y are also martingales in the common filtration $\{\mathcal{F}_t : t \geq 0\}$. If additionally (X, Y) is Markovian in the common filtration then we say (X, Y) is a Markovian coupling.

Strictly we should speak of Markovian *immersion* couplings: however the abbreviated term “Markovian coupling” is now established in the literature. For optimal couplings the distinction between Markovian and immersion is not major: optimal immersion couplings will often be Markovian by virtue of maximizing the optimality criterion at each step of time (see the proof of Theorem 2.11 below). For further discussion see [1, 32].

When considering optimality, it is an advantage to restrict the attention to the classes of immersion or Markovian couplings, since these typically permit explicit constructions using tools such as stochastic calculus. For this reason such couplings are currently widely used for quantifying convergence rates of Markov processes in the stochastic analysis / machine learning literature (cf. the references in the introduction). In the following we seek couplings which are optimal among the class of immersion or Markovian couplings.

We work with the following notion.

Definition 2.2 (Optimal (immersion) Coupling). Consider an immersion coupling of two Markov processes X and Y taking values in a Polish space (E, \mathcal{E}) . We say that this is an optimal (immersion) coupling (for a specified loss function $\phi : E \times E \rightarrow [0, \infty)$ vanishing on the diagonal) if

$$\mathbb{E} \left[\phi(X(t+s), Y(t+s)) \mid X(s), Y(s) \right]$$

is minimized amongst all immersion couplings of X and Y for all $t, s \geq 0$ and all possible $X(s), Y(s)$. In a mild abuse of terminology, if the minimizing coupling is Markovian we call it an optimal Markovian coupling.

Generally the optimal coupling depends greatly on the specific function ϕ being considered. We consider the special case for which the Polish state space is the real line \mathbb{R} , and we require that optimality holds simultaneously for *all choices* of $\phi(x, y) = \phi(|x - y|)$ for concave continuous $\phi : [0, \infty) \rightarrow [0, \infty)$ vanishing at zero. Note that [41] considers this type of simultaneous convex-cost optimality for couplings of probability measures. Note also that $\phi(x, y) = |x - y|^p$ is a rather unsatisfactory cost when $p \geq 1$: if $p = 1$ then too many couplings are optimal [8, Theorem 3.8]; for $p > 1$ the optimal coupling is synchronous [22, Thm 3.7; 29, Thm 1.1]. The concave case ($0 < p < 1$) is more interesting.

The concept of an anti-monotonic rearrangement (AMR) coupling of probability measures [41] is fundamental for our optimality proofs. A useful tool for this is the Skorokhod representation. Note that we employ the Skorokhod representation to describe AMR, in order to be able to handle general cases where the coupled measures are not necessarily absolutely continuous. When the coupled measures have densities, AMR admits a simpler description, cf. Section 2.2.1.

2.1. Skorokhod representation

Recall that the Skorokhod representation generates a copy of a given scalar random variable X using its distribution function $F(x) = \mathbb{P}[X \leq x]$ and a Uniform(0, 1) random variable U . We use the fact that distribution functions are themselves càdlàg (continue à droite, limite à gauche), i.e., they are right-continuous with left limits. The functional inverse of F need not exist; adequate (though non-unique)

substitute inverses can be defined as follows [50, Section 3.12, for example]:

$$F^{-1,+}(z) = \inf\{x : F(x) > z\} = \sup\{x : F(x) \leq z\}, \quad (2)$$

$$F^{-1,-}(z) = \inf\{x : F(x) \geq z\} = \sup\{x : F(x) < z\}.$$

Set $X^+ = F^{-1,+}(U)$ and $X^- = F^{-1,-}(U)$. Then $\mathbb{P}[X^+ \neq X^-] = 0$ and $\mathbb{P}[X^- \leq x] = \mathbb{P}[X^+ \leq x] = F(x)$.

Consequently a copy of X can be generated using a substitute inverse of the cumulative distribution function F as above, and it does not matter which of the possible substitute inverses is employed. Indeed, if F is continuous then direct arguments show that $F(X^+) = F(X^-) = U$.

2.2. McCann's anti-monotonic rearrangement (AMR) coupling

McCann [41] discusses an *anti-monotonic rearrangement coupling* (AMR coupling) between two distribution functions F_1 and F_2 defined on the real line. If X and Y are the random variables implementing the coupling, then (conditional on $X \neq Y$) the random variables X and Y are antithetically coupled, in the sense that X and Y are negatively associated (the informal notion of “antithetic coupling” corresponds to the notion of “antithetic simulation” [25]).

We consider only the particular case, in which F_2 stochastically dominates F_1 :

$$F_1(x) \geq F_2(x) \quad \text{for all } x \in \mathbb{R}.$$

This stochastic domination is related to a condition of McCann [41, Proposition 2.12], where one of two measures is required to vanish on a connected set, while the other vanishes on its complement, after removing their common mass (the difference is that McCann works on the circle, while we work on the line). We further require that for each value $\ell \geq 0$ the super-level set

$$S_\ell = \{x \in \mathbb{R} : F_1(x) - F_2(x) \geq \ell\} \quad (3)$$

should be either connected (hence an interval) or empty: elementary properties of distribution functions show that the interval S_ℓ is non-increasing in ℓ and is bounded for positive ℓ . Note that the connectedness assumption will be satisfied for our main example, where F_1 corresponds to a (weakly) unimodal probability distribution, and F_2 is a translation of F_1 , cf. Section 2.3. It follows that we can find ζ such that $\zeta \in S_\ell$ for all $\ell < \ell_{\max} = \sup_{x \in \mathbb{R}} (F_1(x) - F_2(x))$. Indeed, since S_ℓ is connected and bounded, and must be non-empty if $\ell < \ell_{\max}$, removing S_ℓ from \mathbb{R} leaves a complement consisting of two (disjoint) connected components: $\mathbb{R} \setminus S_\ell = C_\ell^1 \sqcup C_\ell^2$. Both components, being connected, are half-infinite (closed or open) intervals. We define disjoint half-infinite intervals C^1 and C^2 by $C^1 = \cup_{\ell < \ell_{\max}} C_\ell^1$ and $C^2 = \cup_{\ell < \ell_{\max}} C_\ell^2$. Choose ζ to be the right end-point of C^1 : note that $C^1 \subseteq (-\infty, \zeta]$ and $C^2 \subseteq [\zeta, \infty)$. However, we note that equality cannot hold for both these inclusions, and may not hold in either.

The interval S_ℓ is non-increasing in ℓ , so the difference $F_1(x) - F_2(x)$ must be non-decreasing when $x < \zeta$ and non-increasing when $x \geq \zeta$. Note that F_1 and F_2 are càdlàg (so we can include $x = \zeta$ in the non-increasing range), but need not be continuous, so we cannot assume that $F_1(\zeta) - F_2(\zeta) = \ell_{\max}$.

It is now convenient to switch to the language of probability measures. Recall that if μ_1 and μ_2 are two non-negative measures then their *meet* is defined by

$$(\mu_1 \wedge \mu_2)(A) = \sup\{\mu_1(B) \wedge \mu_2(B) : B \subseteq A, B \in \mathcal{E}\}.$$

Consider the Hahn-Jordan decomposition for the signed measure $\mu_1 - \mu_2 = p\nu_1 - p\nu_2$, where μ_1 is the probability measure corresponding to F_1 and μ_2 is the probability measure corresponding to F_2 . Here

$1 - p$ is the total mass of $\mu_1 \wedge \mu_2$, while

$$p\nu_1 = \mu_1 - (\mu_1 \wedge \mu_2), \quad \text{and} \quad p\nu_2 = \mu_2 - (\mu_1 \wedge \mu_2).$$

Thus $p \in [0, 1]$ while ν_1 and ν_2 are probability measures of disjoint support. The stochastic domination together with connectedness of the super-level sets (3) implies that the non-negative measures $p\nu_1$ and $p\nu_2$ of the Hahn-Jordan decomposition are supported on the disjoint half-infinite intervals C^1 and C^2 .

Define a third probability measure ν^* by $(1 - p)\nu^* = \mu_2 \wedge \mu_1 = \mu_2 - p\nu_2 = \mu_1 - p\nu_1$. By stochastic domination,

$$(1 - p)\nu^*((-\infty, x]) = F_1(x) - F_2(x) \quad \text{for all } x \in \mathbb{R}.$$

McCann's AMR coupling is a maximal coupling of random variables X and Y with distribution functions F_1 and F_2 , implementing the representation implied by the Hahn-Jordan decomposition

$$\mu_1 = (1 - p)\nu^* + p\nu_1,$$

$$\mu_2 = (1 - p)\nu^* + p\nu_2$$

in an anti-monotonic manner, using a single Uniform(0, 1) random variable U as follows:

1. If $U \geq p$ then use the Skorokhod representation (and the notation of (2)) to generate

$$X = Y = (G^*)^{-1,+}((U - p)/(1 - p)), \quad (4)$$

where $(G^*)^{-1,+}$ is the substitute inverse of the distribution function G^* determined by $G^*(x) = \nu^*((-\infty, x])$ for all $x \in \mathbb{R}$;

2. If on the other hand $U < p$ then use the Skorokhod representation twice over to generate

$$X = (G_1)^{-1,+}(U/p), \quad (5)$$

$$Y = (G_2)^{-1,+}((p - U)/p),$$

where $(G_i)^{-1,+}$ for $i \in \{1, 2\}$ are the substitute inverses of the distribution functions G_i which are determined by $G_i(x) = \nu_i((-\infty, x])$ for all $x \in \mathbb{R}$;

Note that instead of $(G^*)^{-1,+}$, $(G_1)^{-1,+}$ and $(G_2)^{-1,+}$, we could have used $(G^*)^{-1,-}$, $(G_1)^{-1,-}$ and $(G_2)^{-1,-}$, respectively. Hence the AMR coupling depends on the choice of the substitute inverse used in the Skorokhod representation, as explained in Section 2.1. If $X \neq Y$ then (5) delivers an anti-monotonic relationship between X and Y via U : as U increases it is the case that X is non-decreasing and Y is non-increasing. However in general we cannot infer that Y is an anti-monotonic function of X ; if G_1 or G_2 have jumps then this may not be the case.

2.2.1. AMR for absolutely continuous distributions

In the case where F_1 and F_2 have densities (f_1 and f_2 respectively) then McCann's AMR coupling admits a more direct description. In that case F_1 and F_2 are continuous, while C_ℓ^1 and C_ℓ^2 are half-infinite closed intervals with ζ defined as above. Moreover we have $C^1 = (-\infty, \zeta)$ and $C^2 \subseteq (\zeta, \infty)$, while $F_1(\zeta) - F_2(\zeta) = \ell_{\max}$ by continuity of F_1 and F_2 .

In this case of densities, the coupled copy Y may then be constructed in terms of X and a different auxiliary random variable U , with U being Uniform(0, 1) independent of X . Set

$$Y = \begin{cases} \rho_{F_1, F_2}(X) & \text{when } U \geq (f_2(x) \wedge f_1(x))/f_1(x), \\ X & \text{when } U < (f_2(x) \wedge f_1(x))/f_1(x). \end{cases}$$

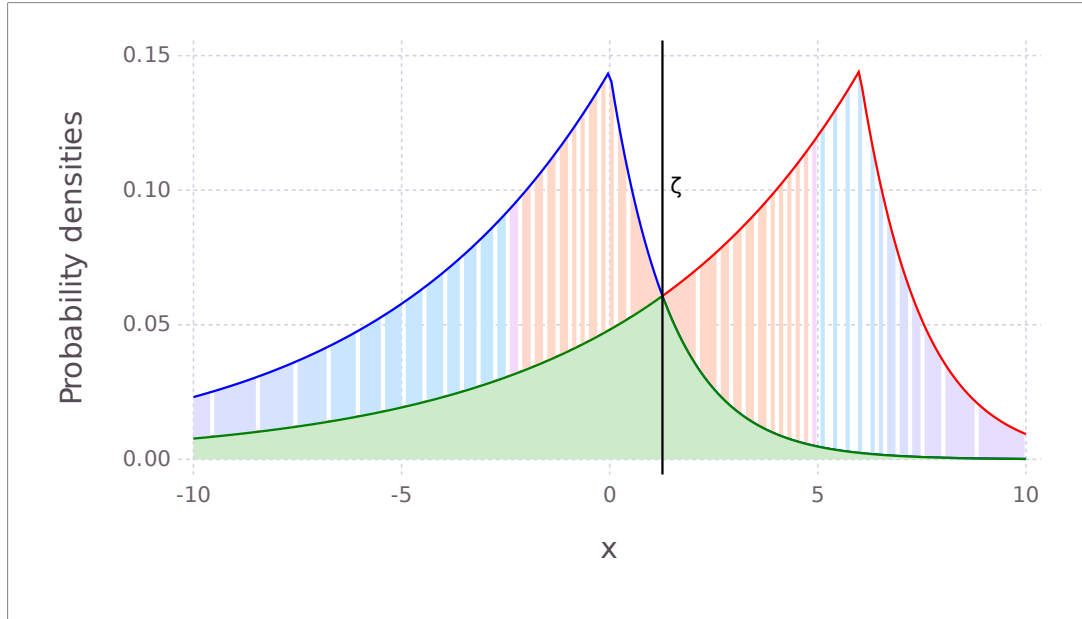


Figure 1: Anti-monotonic rearrangement. In the two uncoupled regions (left and right) the banding indicates the use of anti-monotonic rearrangement to map vertical strips in the left region to vertical strips in the right region and *vice versa*.

Here $\rho_{F_1, F_2} : C^1 \rightarrow C^2$ is the *anti-monotonic rearrangement function*, defined for $x \in C^1$ by

$$\rho_{F_1, F_2}(x) = \inf \left\{ \rho : \int_{\rho}^{\infty} (f_2(t) - f_1(t)) dt = \int_{-\infty}^x (f_1(t) - f_2(t)) dt \right\}. \quad (6)$$

Since F_1 and F_2 are continuous, it follows that

$$\zeta \in \{x \in \mathbb{R} : F_1(x) - F_2(x) = \ell_{\max}\} \neq \emptyset. \quad (7)$$

Moreover, if $\zeta \in \{x : F_1(x) - F_2(x) = \ell_{\max}\}$ then from (6) we may deduce that

$$\int_{-\infty}^{\zeta} (f_1(t) - f_2(t)) dt = F_1(\zeta) - F_2(\zeta) = (1 - F_2(\zeta)) - (1 - F_1(\zeta)) = \int_{\zeta}^{\infty} (f_2(t) - f_1(t)) dt.$$

Furthermore, $\int_u^{\zeta} (f_2(t) - f_1(t)) dt$ is always negative when $u < \zeta$, since then $u \in C_{\ell}^1$ for some $\ell < \ell_{\max}$. Therefore $\zeta = \rho_{F_1, F_2}(\zeta)$.

2.3. Unimodality and the measure-majorization of shifts

We plan to apply AMR to produce optimal Markovian couplings of two copies of a Markov chain $(X_k)_{k=0}^{\infty}$ started from two different initial points. To this end, we will need to couple transition probabilities of $(X_k)_{k=0}^{\infty}$ with their respective counterparts shifted by a for any $a > 0$. Note that this corresponds to coupling jump distributions of two processes with different starting points (separated by

$a > 0$). To prepare for this, we discuss a general definition of (weak) unimodality which makes no explicit reference to probability densities.

Theorem 2.3. *Suppose a distribution function F corresponds to a random variable X such that for all $x > 0$ there exists $a(x) \in (0, x)$ such that*

$$\begin{aligned} \mathbb{P}[X + x \in E] &\geq \mathbb{P}[X \in E] && \text{for measurable } E \subset [a(x), \infty), \\ \mathbb{P}[X - x \in E] &\geq \mathbb{P}[X \in E] && \text{for measurable } E \subset (-\infty, -a(x)]. \end{aligned}$$

Then the probability law corresponding to F can be expressed as a mixture of (a) a probability density f which is (weakly) unimodal at 0 and (b) a Dirac mass at 0.

Here “weakly unimodal” at $y \in \mathbb{R}$ means that there is a version of the density f which is non-decreasing on $(-\infty, y)$ and non-increasing on (y, ∞) .

It is convenient to prove a measure-theoretic version of Theorem 2.3. Consider two measures μ_1 and μ_2 defined on a σ -algebra \mathcal{A} . For any measurable $G \subset \mathbb{R}$, the notation $\mu_1|_G \geq \mu_2|_G$ means that $\mu_1(A) \geq \mu_2(A)$ for any $A \in \mathcal{A}$ such that $A \subset G$. For $x \in \mathbb{R}$ let δ_x denote the Dirac mass at x , so that the convolution measure $\delta_x * \nu$ is the shift of ν by x .

Theorem 2.4. *Suppose a non-negative measure ν on \mathbb{R} satisfies the following majorization relationship: for all $x > 0$ there exists $a(x) \in (0, x)$ such that*

$$\begin{aligned} (\delta_x * \nu)|_{[a(x), \infty)} &\geq \nu|_{[a(x), \infty)}, \\ (\delta_{-x} * \nu)|_{(-\infty, -a(x)]} &\geq \nu|_{(-\infty, -a(x)]}. \end{aligned} \tag{8}$$

Then ν has a density f on $\mathbb{R} \setminus \{0\}$, which can be chosen to be càdlàg non-decreasing on $(-\infty, 0)$ and càglàd non-increasing on $(0, \infty)$. We say that (the density component of) ν is weakly unimodal.

Thus the majorization condition (8) actually corresponds to weak unimodality of the density at 0. For weak unimodality at an arbitrary $y \in \mathbb{R}$, one would need to consider $a(x) \in (y, y + x)$ for $x > 0$, etc. The proof of Theorem 2.4 can be found in Section 4.

2.4. Optimal Wasserstein coupling for concave costs

In order to prove optimality for suitable AMR couplings we apply the results of Section 2.3 to probability distributions F and G on \mathbb{R} such that F is a mixture of a Dirac mass at 0 with an absolutely continuous distribution whose density is weakly unimodal at 0, while G is a right-shift of a (possibly different) distribution of the same form.

We further suppose that there exists $\zeta \in \mathbb{R}$ with the property that the function $F - G$ is strictly increasing on $(-\infty, \zeta]$ and strictly decreasing on $[\zeta, \infty)$, cf. (7). This is automatically true if F is strictly unimodal and G is a shift of F to the right.

As discussed at the beginning of Section 2, we are interested in finding couplings that minimize concave cost functions. However in this subsection it is convenient to focus on the equivalent problem of maximizing convex payoffs (see the discussion before Lemma 2.10 for further comments on this equivalence). To this end, we apply the methods from [44] to McCann’s AMR coupling defined in Section 2.2. Note that [44] considers a convex-monotone coupling defined in a completely different way, see (2) therein. However, [44, Lemma 1] combines with the optimality result below (which implies

uniqueness of the optimal coupling by Remark 2.7) to show that the AMR coupling is the same as the convex-monotone coupling if the distribution F is strictly unimodal and G is a right-shift of F . In general the two definitions are very different: AMR can be interpreted in many different ways in non-unimodal situations [41] while the convex-monotone coupling is well-defined in such cases.

By the definition of the AMR coupling (X, Y) from Section 2.2, for any $a < b \leq \zeta$

$$\mathbb{P} \left[(X, Y) \in (a, b]^2, X = Y \right] = \mathbb{P} [X \in (a, b], X = Y] = G(b) - G(a) \quad (9)$$

and for any $\zeta < a < b$

$$\mathbb{P} \left[(X, Y) \in (a, b]^2, X = Y \right] = \mathbb{P} [Y \in (a, b], X = Y] = F(b) - F(a). \quad (10)$$

Thus for any $a \leq \zeta < b$

$$\mathbb{P} \left[(X, Y) \in (a, b]^2, X = Y \right] = G(\zeta) - G(a) + F(b) - F(\zeta).$$

Now F and G are not necessarily diffuse, so (6) cannot be applied to define a mapping $\rho_{F,G}(x)$ for all $x \in \mathbb{R}$. However, for any $x < \zeta$ such that F does not have an atom at x (i.e., for any $x \neq 0$ in the setting of this section), we can define a real number $\underline{\text{AMR}}(x) \geq \zeta$ by

$$\underline{\text{AMR}}(x) = \inf \{ y > \zeta : F(x) - G(x) \geq F(y) - G(y) \}.$$

So for any a such that $a \leq \zeta < b$ and F does not have an atom at a , if $a < X < Y$ and $b > \underline{\text{AMR}}(a)$, then $Y \leq b$. Thus if $a \leq \zeta < b$ and $b > \underline{\text{AMR}}(a)$ then

$$\begin{aligned} \mathbb{P} \left[(X, Y) \in (a, b]^2, X < Y \right] &= \mathbb{P} [a < X \leq \zeta < Y \leq b] = \mathbb{P} [a < X \leq \zeta, X < Y] \\ &= (F(\zeta) - F(a)) - (G(\zeta) - G(a)). \end{aligned} \quad (11)$$

Similarly, for any a such that $a \leq \zeta < b$ and F does not have an atom at a , if $b \leq \underline{\text{AMR}}(a)$ then

$$\begin{aligned} \mathbb{P} \left[(X, Y) \in (a, b]^2, X < Y \right] &= \mathbb{P} [a < X \leq \zeta < Y \leq b] = \mathbb{P} [\zeta < Y \leq b, X < Y] \\ &= (G(b) - G(\zeta)) - (F(b) - F(\zeta)). \end{aligned} \quad (12)$$

Note that the corresponding probabilities are zero in all the other possible cases of a and b , so long as a and b respectively avoid the atoms of F and G .

We need the following technical result, whose proof is adapted from that of Lemma 1 in [44] and can be found in Section 4.

Lemma 2.5. *For any coupling of random variables X and Y and for any $c > 0$ we have*

$$\mathbb{E} [(c - |X - Y|)^+] = \int_{-\infty}^{\infty} \mathbb{P} \left[(X, Y) \in (t - c, t]^2 \right] dt.$$

As an immediate corollary, it follows that

$$\mathbb{E} [(c - |X - Y|)^+] \leq \int_{-\infty}^{\infty} (\mathbb{P} [t - c < X \leq t] \wedge \mathbb{P} [t - c < Y \leq t]) dt.$$

We will now show that the upper bound is actually obtained by the coupling by anti-monotone re-arrangement defined in Section 2.2, i.e., we will show that, since the AMR coupling (X, Y) satisfies (9)-(12), for Lebesgue-almost all t , then in this particular case

$$\mathbb{P} \left[(X, Y) \in (t - c, t]^2 \right] = \min\{F(t) - F(t - c), G(t) - G(t - c)\}. \quad (13)$$

In the sequel, condition (13) is needed only for Lebesgue-almost all $t \in \mathbb{R}$ since we are interested in values of the expressions in (13) after integration with respect to t . So it suffices to show that (13) holds only for t and c such that F does not have an atom at $t - c$ (so that (11) and (12) apply for $a = t - c$).

Consider the case of $t - c < \zeta \leq t$. Then we have

$$\begin{aligned} \mathbb{P} \left[(X, Y) \in (t - c, t]^2 \right] = \\ \mathbb{P} [t - c < X \leq \zeta, X = Y] + \mathbb{P} [\zeta < Y \leq t, X = Y] + \mathbb{P} [t - c < X \leq \zeta < Y \leq t]. \end{aligned}$$

Applying (9) and (10), the first two summands on the right add up to

$$G(\zeta) - G(t - c) + F(t) - F(\zeta).$$

If $t > \underline{\text{AMR}}(t - c)$, then we can apply (11) to show that the third summand equals

$$F(\zeta) - G(\zeta) + G(t - c) - F(t - c),$$

while if $t \leq \underline{\text{AMR}}(t - c)$ then by (12) the third summand is equal to

$$F(\zeta) - G(\zeta) + G(t) - F(t).$$

Note that if $t \leq \underline{\text{AMR}}(t - c)$ then $F(t) - F(t - c) \geq G(t) - G(t - c)$, while if $t > \underline{\text{AMR}}(t - c)$ then $F(t) - F(t - c) \leq G(t) - G(t - c)$, and hence addition yields (13). The other cases can be checked more directly, as the equivalent of the third summand is then zero.

We have proved the following result:

Lemma 2.6. *Let F and G be cumulative distribution functions. Suppose that there exists $\zeta \in \mathbb{R}$ with the property that the function $F - G$ is strictly increasing on $(-\infty, \zeta]$ and strictly decreasing on $[\zeta, \infty)$. Then for any $c > 0$, the AMR coupling (X, Y) is the optimal coupling in the sense of maximizing the payoff function $\mathbb{E}[(c - |X - Y|)^+]$.*

Remark 2.7. From the argument above, it is evident that any coupling (X, Y) satisfying (13) must also satisfy (9)-(12) and, as a consequence, must be defined via (4)-(5). This shows that the AMR coupling given by (4)-(5) is essentially the unique coupling (X, Y) maximizing the payoff function $\mathbb{E}[(c - |X - Y|)^+]$. This has to be understood in the sense that any coupling maximizing $\mathbb{E}[(c - |X - Y|)^+]$ must satisfy (4)-(5), however, conditions (4)-(5) themselves are only unique up to the choice of the Skorokhod representation as explained in Section 2.1. In particular, for F and G with atoms, there will be more than one “version” of AMR satisfying (4)-(5).

Intuitively, it is clear that any candidate for the optimal coupling should put the common mass together with the maximal probability, and hence identifying the optimal coupling for a given cost comes down to specifying what happens with the remaining mass. As explained in [41], the intuition for why AMR is the optimal coupling for minimizing concave costs (or maximizing convex payoffs)

is that for a concave cost, it is more efficient to have a transport map with one long and one short transport, rather than a transport map with two transports of average length.

From now on we will focus on the special case of $G = F_a$ for $a > 0$, where we define $F_a(x) = F(x - a)$ for all $x \in \mathbb{R}$. Notice that in this case the AMR coupling (X, Y) of random variables with distributions F and F_a depends only on X and a and hence we will denote it by $(X, \text{AMR}_a(X))$.

In order to proceed, we need the following lemma. Its proof is a reformulation of the proof of Lemma 2 in [44], and is included in Section 4 for completeness.

Lemma 2.8. *Fix a random variable X whose distribution is weakly unimodal. For any $a, c > 0$ consider the function ψ defined by*

$$\psi(a, c) = \sup_{Y: \mathcal{L}(Y) = \mathcal{L}(X+a)} \mathbb{E}[(c - |X - Y|)^+], \quad (14)$$

where $\mathcal{L}(Y)$ is the law of the random variable Y . Then it is the case that

$$\psi(a, c) + a = \psi(c, a) + c.$$

As a consequence [44, Lemma 2 and the following discussion] the function

$$a \mapsto \sup_{Y: \mathcal{L}(Y) = \mathcal{L}(X+a)} \mathbb{E}[(c - |X - Y|)^+]$$

is convex. However a more general result can be obtained (for the proof see Section 4):

Lemma 2.9. *Let $\varphi: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be a bounded continuous convex and non-increasing function. Then*

$$a \mapsto \sup_{Y: \mathcal{L}(Y) = \mathcal{L}(X+a)} \mathbb{E}[\varphi(|X - Y|)]$$

is bounded, continuous, convex, and non-increasing.

The representation $\mathbb{E}[\varphi(|X - Y|)] = \lim_{n \rightarrow \infty} \sum_{i=1}^n \mathbb{E}[\lambda_i^n (c_i^n - |X - Y|)^+]$ obtained in the proof (see (22)) permits extension of the optimality result for AMR (Lemma 2.6) to all bounded, continuous, convex, decreasing functions φ . Note further, that the optimization problem of maximizing the payoff $\mathbb{E}[\varphi(|X - Y|)]$ for such φ is equivalent to the problem of minimizing the cost $\mathbb{E}[\Phi(|X - Y|)]$ for bounded, continuous, concave, increasing functions Φ , and hence the AMR coupling solves this minimization problem for such Φ . This property of AMR extends to unbounded, continuous, concave, increasing costs Φ , so long as it is assumed that the random variable X in the optimization problem (and hence also Y) has finite first moment (or even a weaker condition $\mathbb{E}[\Phi(|X|)] < \infty$). Indeed, Φ specified above can be approximated by a sequence of bounded functions Φ^n (satisfying all the remaining properties of Φ), for which AMR solves the corresponding optimization problem by the discussion above. Then, due to the assumption on the finite moment of X , the monotone convergence theorem can again be applied to obtain

$$\mathbb{E}[\Phi(|X - Y|)] = \lim_{n \rightarrow \infty} \mathbb{E}[\Phi^n(|X - Y|)],$$

permitting the conclusion that AMR solves the optimization problem for Φ :

Lemma 2.10. *Let F and G be cumulative distribution functions such that there exists $\zeta \in \mathbb{R}$ so that the function $F - G$ is strictly increasing on $(-\infty, \zeta]$ and strictly decreasing on $[\zeta, \infty)$. Let Φ be a*

continuous, concave, increasing cost function (possibly unbounded). Finally, suppose $\mathbb{E}[\Phi(|X|)] < \infty$ if X is a random variable with distribution function F . Then the AMR coupling (X, Y) is the unique optimal coupling (in the sense of Remark 2.7) minimizing the payoff function $\mathbb{E}[\Phi(|X - Y|)]$.

We stress that the unimodality of the distribution of X is crucial for the existence of a single coupling optimal for all such costs (see Example 2.13).

In the sequel we extend this optimality result to cover suitable stochastic processes, namely random walks and finite-activity Lévy processes.

In the remainder of this section, let us focus on random walks with unimodal jumps. To this end, let ϕ be a convex, bounded, continuous, decreasing function as above and let $(X_k)_{k=0}^\infty$ be a Markov chain with a family of transition kernels $p = \{p(x, \cdot)\}_{x \in \mathbb{R}}$, where $p(x, A) = \mathbb{P}(X_n \in A | X_{n-1} = x)$ for all $n \geq 1$, i.e., the process $(X_k)_{k=0}^\infty$ is assumed to be time-homogeneous. Following Section 2.3, we assume that each measure $p(x, \cdot)$ for any $x \in \mathbb{R}$ is a mixture of a Dirac mass at 0 and a density which is strictly unimodal at 0. For any $a > 0$, any $n \geq 1$ and any $l \in \{0, \dots, n-1\}$ we define

$$\psi_{n,l}^\phi(a) = \sup \mathbb{E}[\phi(|X_n - Y_n|) : |X_l - Y_l| = a],$$

where the supremum is taken over all immersion couplings $(X_k, Y_k)_{k=l}^\infty$ of two copies of $(X_k)_{k=0}^\infty$ such that $|X_l - Y_l| = a$. Observe that, due to time-homogeneity and the fact that we work with immersion couplings, $\psi_{n,l}^\phi(a) = \psi_{n-l,0}^\phi(a) = \psi_{n-l}^\phi(a)$ depends only on the value of the difference $n-l$ rather than on the individual values of n and l . In other words, $\psi_{n,l}^\phi(a)$ is the supremum of the distance $\mathbb{E}[\phi(|X_{n-l} - Y_{n-l}|)]$ taken over all immersion couplings of Markov chains with the transition kernels p , such that $|X_0 - Y_0| = a$.

Theorem 2.11. *Let $(X_k)_{k=0}^\infty$ be a Markov chain with transition kernels p as specified above. Let $(Z_k^{l,a})_{k=l}^\infty$ be a Markov chain such that $|Z_l^{l,a} - X_l| = a$ and for all $k > l$ the random vector $(X_k, Z_k^{l,a})$, conditional on $(X_{k-1}, Z_{k-1}^{l,a})$, is an AMR coupling of random variables defined as in (4)-(5). Then, for any convex, bounded, continuous, decreasing function ϕ , any $n \geq 1$ and any $l \in \{0, \dots, n-1\}$,*

$$\psi_{n,l}^\phi(a) = \mathbb{E} \left[\phi(|X_n - Z_n^{l,a}|) \right].$$

Proof. By construction $(X, Z^{l,a})$ is an immersion coupling, and indeed a Markovian coupling.

The proof uses mathematical induction (following the ideas of dynamic programming and the Bellman principle): at level $n \geq 1$ the inductive hypothesis is that for any bounded convex continuous decreasing ϕ , and for any $m = 1, \dots, n$, the supremum $\psi_m^\phi(a)$ is realized using the iterated AMR coupling (X, Z) with $X_0 - Z_0 = a$: moreover ψ_m^ϕ itself is then also convex decreasing.

Consider an immersion coupling (X, Y) such that $Y_0 - X_0 = a > 0$ (the case $a < 0$ follows by exchanging rôles of X and Y). Consider the first level $n = 1$ of the inductive hypothesis. It follows from Lemmas 2.9 and 2.10 that, for any bounded convex continuous decreasing ϕ , the supremum $\psi_1^\phi(a)$ is realized using the single-jump AMR coupling (X, Z) with $X_0 - Z_0 = a$: moreover ψ_1^ϕ inherits the convex decreasing property of $\phi = \psi_0^\phi$. Thus case $n = 1$ of the induction is valid.

Suppose the inductive hypothesis is valid at level $n-1$. To prove level n it suffices to show that

$$\psi_n^\phi(a) = \sup \{ \mathbb{E}[\phi(|X_n - Y_n|)] ; |X_0 - Y_0| = a \text{ and } (X, Y) \text{ is an immersion coupling} \}$$

is maximized by taking $Y = Z^{0,a}$, and that ψ_n^ϕ is convex decreasing. For brevity we write $Z^{0,a} = Z$.

We argue as follows, First, use iterated conditional expectation and the definition of ψ_{n-1}^ϕ :

$$\mathbb{E}[\phi(|X_n - Y_n|)] = \mathbb{E}[\mathbb{E}[\phi(|X_n - Y_n|) | \mathcal{F}_1]] \leq \mathbb{E}\left[\psi_{n-1}^\phi(|X_1 - Y_1|)\right],$$

where $\{\mathcal{F}_n : n \geq 0\}$ is the common filtration from the definition of (X_n, Y_n) as an immersion coupling. The inequality above is a consequence of the definition of ψ_{n-1}^ϕ and the fact that immersion coupling respects conditional expectations. Now employ the inductive hypothesis at level 1 based on ψ_{n-1}^ϕ rather than ϕ :

$$\mathbb{E}\left[\psi_{n-1}^\phi(|X_1 - Y_1|)\right] \leq \mathbb{E}\left[\psi_{n-1}^\phi(|X_1 - Z_1|)\right].$$

Finally apply iterated conditional expectation and the inductive hypothesis at level $n - 1$:

$$\mathbb{E}\left[\psi_{n-1}^\phi(|X_1 - Z_1|)\right] = \mathbb{E}[\mathbb{E}[\phi(|X_n - Z_n|) | \mathcal{F}_1]] = \mathbb{E}[\phi(|X_n - Z_n|)].$$

Additionally

$$\psi_n^\phi(|X_0 - Z_0|) = \mathbb{E}[\phi(|X_n - Z_n|)] = \mathbb{E}\left[\psi_{n-1}^\phi(|X_1 - Z_1|)\right],$$

so the convex decreasing property of ψ_{n-1}^ϕ is inherited by ψ_n^ϕ . This completes the proof of the inductive step, and so the result follows. \square

By the equivalence of the maximization problem for convex payoffs and the minimization problem for concave costs, Theorem 2.11 solves the optimization problem described in Section 1 for Markov chains with unimodal transition probabilities (possibly with Dirac mass at the mode). An additional assumption of $\mathbb{E}[\Phi(|X_k|)] < \infty$ for all $k \geq 1$ is required for unbounded concave costs Φ (Lemma 2.10).

Remark 2.12. Note that our optimality result for the unimodal non-symmetric case is inherently one-dimensional: in the multi-dimensional setting it is not obvious even how to characterize the concept of unimodality, and, as we will demonstrate in Example 2.13, without unimodality it is impossible to obtain optimality results of the type discussed above. However, in the multi-dimensional unimodal symmetric absolutely continuous case, i.e., when $\mu(dx) = \mu(|x|)dx$, the relevant optimization problem is easily reduced to the one-dimensional setting (since we can assume without loss of generality that the measure μ and its shift by $a \in \mathbb{R}^d$ differ only in the first coordinate), and the AMR coupling reduces to the reflection coupling. Hence, as a consequence of our result, we can confirm the established folklore that the multidimensional reflection coupling is in fact the optimal coupling for concave costs in the rotationally symmetric setting (cf. the discussion in the introduction).

2.5. Example: necessity of unimodality

If unimodality fails then the optimal coupling may depend on the cost function.

Example 2.13. [Two-point equiprobable distribution] Suppose X has a two-point equiprobable distribution with probability masses of $1/2$ each, located at 0 and 1. Suppose Y has the same distribution, but shifted α units to the right for some $\alpha > 0$, thus with probability masses located at α and $1 + \alpha$. Consider two extreme forms of coupling:

1. Synchronous coupling, based on the transportation plan sending 0 to α and 1 to $1 + \alpha$.

2. *Anti-monotonic rearrangement (which is equivalent here to a reflection coupling, since the distributions are symmetric), based on the transportation plan sending 0 to $1 + \alpha$ and 1 to α ;*

Evaluate these couplings using a concave cost based on $c_\gamma(x, y) = |x - y|^\gamma$, with $\gamma \in (0, 1)$. Consider the difference of the cost under plan 2 minus cost under plan 1 as a function of α :

$$f(\alpha) = (1 + \alpha)^\gamma + |1 - \alpha|^\gamma - 2\alpha^\gamma.$$

It is an exercise to verify that the function $f(\alpha)$ is always positive for $\alpha \geq 1$, for all values of $\gamma \in (0, 1)$. Hence, for $\alpha \geq 1$, the cost under plan 1 is less than the cost under plan 2 and hence the synchronous coupling is the preferred choice. On the other hand, numerical solution of the equation $f(\alpha) = 0$ for $\alpha \in (0, 1)$ and $\gamma \in (0, 1)$ shows that the optimal strategy is always reflection if $\alpha \leq 0.7071 \dots$, whereas for $0.7071 \dots < \alpha < 1$ the optimal strategy depends on the choice of $\gamma \in (0, 1)$.

3. Optimal Markovian coupling for finite-activity Lévy processes

This section employs a uniformization procedure for finite-activity Lévy processes (*i.e.*, compound Poisson processes), which generalizes the uniformization procedure for finite state-space continuous-time Markov chains [46, Section 6.7]. It permits representation of any immersion coupling of finite-activity Lévy processes with a Lévy measure ν of mass $\nu(\mathbb{R}) < \infty$, in terms of a Poisson process with intensity $2\nu(\mathbb{R})$ supplying jump times for both the coupled processes. This in turn permits extension of the optimality result from the previous section from Markov chains to finite-activity Lévy processes.

Let $(X_t, Y_t)_{t \geq 0}$ be an immersion coupling of two copies of a real-valued compound Poisson process with a Lévy measure ν , adapted to a common filtration $(\mathcal{F}_t)_{t \geq 0}$. Then there exist Poisson processes $(N_t^1)_{t \geq 0}$ and $(N_t^2)_{t \geq 0}$ with intensity $\nu(\mathbb{R})$ and two sequences of i.i.d. random variables $(Z_i^1)_{i=1}^\infty, (Z_i^2)_{i=1}^\infty$ with $\mathcal{L}(Z_i^j) = \nu/\nu(\mathbb{R})$ for $i \in \mathbb{N}_+, j \in \{1, 2\}$, respectively independent of $(N_t^1)_{t \geq 0}$ and $(N_t^2)_{t \geq 0}$, with

$$X_t = X_0 + \sum_{i=1}^{N_t^1} Z_i^1 \quad \text{and} \quad Y_t = Y_0 + \sum_{i=1}^{N_t^2} Z_i^2. \quad (15)$$

Note that $(Z_i^1)_{i=1}^\infty$ and $(Z_i^2)_{i=1}^\infty$ are both i.i.d. sequences, but usually (except in the case of the independent coupling), the random variables Z_i^1 and Z_i^2 are dependent for any fixed $i \geq 1$. Moreover,

$$N_t^1 = \sum_{i=1}^{\infty} \mathbf{1}_{\{\tau_i^1 \leq t\}} \quad \text{and} \quad N_t^2 = \sum_{i=1}^{\infty} \mathbf{1}_{\{\tau_i^2 \leq t\}},$$

with sequences of random variables $(\tau_i^j)_{i=1}^\infty$ for $j \in \{1, 2\}$, such that $\mathcal{L}(\tau_i^j - \tau_{i-1}^j) = \text{Exp}(\nu(\mathbb{R}))$ for any $i \geq 2$, and all the random variables $\tau_i^j - \tau_{i-1}^j$ for $i \geq 2$ are mutually independent. We also write

$$X_t = X_0 + \sum_{i=1}^{\infty} \mathbf{1}_{\{\tau_i^1 \leq t\}} Z_i^1 \quad \text{and} \quad Y_t = Y_0 + \sum_{i=1}^{\infty} \mathbf{1}_{\{\tau_i^2 \leq t\}} Z_i^2.$$

If necessary, we enrich the underlying common filtration $(\mathcal{F}_t)_{t \geq 0}$ to ensure the existence of a Poisson process Ψ of rate $\nu(\mathbb{R})$ and $\text{Uniform}(0, 1)$ random variables M_1, M_2, \dots , independent of each other and of the finite activity Lévy processes, but appropriately adapted to the common filtration.

Theorem 3.1. *In the above situation, enriching the underlying common filtration as above if necessary, there exist a Poisson process $(N_t)_{t \geq 0}$ with intensity $2\nu(\mathbb{R})$ and two sequences of i.i.d. random variables $(\tilde{Z}_i^1)_{i=1}^\infty$ and $(\tilde{Z}_i^2)_{i=1}^\infty$ with $\mathcal{L}(\tilde{Z}_i^j) = \frac{1}{2}\delta_0 + \frac{1}{2\nu(\mathbb{R})}\nu$ such that*

$$X_t = X_0 + \sum_{i=1}^{N_t} \tilde{Z}_i^1 \quad \text{and} \quad Y_t = Y_0 + \sum_{i=1}^{N_t} \tilde{Z}_i^2. \quad (16)$$

Moreover, both $(\tilde{Z}_i^1)_{i=1}^\infty$ and $(\tilde{Z}_i^2)_{i=1}^\infty$ are independent of $(N_t)_{t \geq 0}$, though again the random variables \tilde{Z}_i^1 and \tilde{Z}_i^2 are typically dependent for any fixed $i \geq 1$.

Proof. We begin by constructing a suitable Poisson process $(N_t)_{t \geq 0}$ with intensity $2\nu(\mathbb{R})$ given by

$$N_t = \sum_{i=1}^{\infty} \mathbf{1}_{\{\tau_i \leq t\}},$$

where the sequence $(\tau_i)_{i=1}^\infty$ is such that the random variables $\tau_i - \tau_{i-1}$ are mutually independent with $\mathcal{L}(\tau_i - \tau_{i-1}) = \text{Exp}(2\nu(\mathbb{R}))$ for all $i \geq 2$. Moreover, it will be the case that

$$X_t = X_0 + \sum_{i=1}^{\infty} \mathbf{1}_{\{\tau_i \leq t\}} Z_i^1 \mathbf{1}_{\{\tau_i = \tau_i^1\}} \quad \text{and} \quad Y_t = Y_0 + \sum_{i=1}^{\infty} \mathbf{1}_{\{\tau_i \leq t\}} Z_i^2 \mathbf{1}_{\{\tau_i = \tau_i^2\}}, \quad (17)$$

so that $\tilde{Z}_i^1 = Z_i^1 \mathbf{1}_{\{\tau_i = \tau_i^1\}}$ and $\tilde{Z}_i^2 = Z_i^2 \mathbf{1}_{\{\tau_i = \tau_i^2\}}$ in the representation given by (16).

Let Ξ^i be Poisson point processes on $(0, \infty)$ counting the jumps of $(N_t^i)_{t \geq 0}$ for $i \in \{1, 2\}$:

$$\Xi^i(A) = \#\{t \in A : \Delta N_t^i \neq 0\}.$$

Let $\lambda = \nu(\mathbb{R})$ be the common intensity of $(N_t^i)_{t \geq 0}$ for $i \in \{1, 2\}$. Consider the point process Ξ defined (as a counting process) by

$$\Xi(A) = \#\{t \in A : \Delta N_t^1 \cdot \Delta N_t^2 \neq 0\},$$

counting the simultaneous jumps of $(N_t^1)_{t \geq 0}$ and $(N_t^2)_{t \geq 0}$. The counting process Ξ has an increasing càdlàg compensator $(\Lambda_t)_{t \geq 0}$ (see e.g. Chapter 7 of [17]). Since $\Xi_t \leq \Xi_t^1$ for all $t > 0$, it follows that $\Lambda_t \leq \lambda t$ for all $t > 0$.

Hence the compensator $(\Lambda_t)_{t \geq 0}$ is absolutely continuous w.r.t. the Lebesgue measure on \mathbb{R}_+ : so there exists a predictable process $(\eta_t)_{t \geq 0}$ with values in $[0, \lambda]$ such that $\Lambda_t = \int_0^t \eta_s \, ds$.

Recall the independent Poisson point process Ψ with intensity λ on $(0, \infty)$. Define a thinned version $\tilde{\Psi}$ of Ψ as follows. Suppose the points of Ψ are $0 < \tilde{\tau}_1 < \tilde{\tau}_2 < \dots$, deemed to be marked by the independent Uniform(0, 1) random variables M_1, M_2, \dots . Then set

$$\tilde{\Psi} = \{\tilde{\tau}_i : i \in \{1, 2, \dots\} \text{ and } M_i \leq \eta_{\tilde{\tau}_i} / \lambda\}.$$

Hence $\tilde{\Psi}$ has the compensator $(\Lambda_t^{\tilde{\Psi}})_{t \geq 0}$ given by $\Lambda_t^{\tilde{\Psi}} = \int_0^t \eta_s \, ds$, i.e., $\Lambda_t^{\tilde{\Psi}} = \Lambda_t$ for all $t \geq 0$.

Now define a counting process Υ by

$$\Upsilon = \Xi^1 + \Xi^2 - \Xi + \tilde{\Psi}.$$

Hence the compensator $(\Lambda_t^Y)_{t \geq 0}$ of Y is equal to

$$\Lambda_t^Y = \lambda t + \int_0^t (\lambda - \eta_s) ds + \int_0^t \eta_s ds = 2\lambda t.$$

Thus Y is a Poisson process with intensity 2λ , following the arguments of [5] or [49, Theorem 2.3]. Moreover, Y can be viewed as constructed as a sum of two point processes: Ξ^1 and $(\Xi^2 - \Xi) + \tilde{\Psi}$ which both have the compensator given by λt and hence are Poisson processes. Furthermore, Ξ^1 and $(\Xi^2 - \Xi) + \tilde{\Psi}$ clearly do not have simultaneous jumps and hence they must be independent, using the arguments of [43, Proposition XII-1.7] or [16, Proposition 5.3].

Set $N_t = Y([0, t])$. By construction this is a Poisson counting process with intensity $2\nu(\mathbb{R})$ for which (17) holds with a sequence of $(\tau_i)_{i=1}^\infty$ obtained by counting the points of Y . Moreover, due to the independence of Ξ^1 and $(\Xi^2 - \Xi) + \tilde{\Psi}$ (and, respectively, the independence of Ξ^2 and $(\Xi^1 - \Xi) + \tilde{\Psi}$), it follows that $\tilde{Z}_j^1 = Z_j^1 \mathbf{1}_{\{\tau_j = \tau_j^1\}}$ and $\tilde{Z}_j^2 = Z_j^2 \mathbf{1}_{\{\tau_j = \tau_j^2\}}$ for all $j \geq 1$ have the distribution $\frac{1}{2}\delta_0 + \frac{1}{2\nu(\mathbb{R})}\nu$, since for any $u \in \mathbb{R}$

$$\begin{aligned} \mathbb{E} \left[e^{i\langle u, \tilde{Z}_j^k \rangle} \right] &= \mathbb{E} \left[e^{i\langle u, Z_j^k \mathbf{1}_{\{\tau_j = \tau_j^k\}} \rangle} \mathbf{1}_{\{\tau_j = \tau_j^k\}} \right] + \mathbb{E} \left[e^{i\langle u, Z_j^k \mathbf{1}_{\{\tau_j = \tau_j^k\}} \rangle} \mathbf{1}_{\{\tau_j \neq \tau_j^k\}} \right] \\ &= \frac{1}{2} \mathbb{E} \left[e^{i\langle u, Z_j^k \rangle} \right] + \frac{1}{2} \mathbb{E} \left[e^{i\langle u, 0 \rangle} \right] = \frac{1}{2} \mathbb{E} \left[e^{i\langle u, Z_j^k \rangle} \right] + \frac{1}{2}, \end{aligned} \quad (18)$$

for all $j \geq 1$ and $k \in \{1, 2\}$, using the independence of Z_j^k from both τ_j and τ_j^k . Note that $\mathbb{P}(\tau_j = \tau_j^k) = \mathbb{P}(\tau_j \neq \tau_j^k) = 1/2$, arising directly from the construction of $(N_t)_{t \geq 0}$.

Note also that the mutual independence of $(\tilde{Z}_j^1)_{j=1}^\infty$ follows from the mutual independence of $(Z_j^1)_{j=1}^\infty$ combined with the mutual independence of the events $\{\tau_j = \tau_j^1\}$ for all $j \geq 1$.

To show that the process $(N_t)_{t \geq 0}$ and the event $\{\tau_j = \tau_j^1\}$ are independent for any $j \geq 1$, we first show that $\tilde{Z}_1^1 = Z_1^1 \mathbf{1}_{\{\tau_1 = \tau_1^1\}}$ is independent of τ_1 . Then the assertion follows easily by induction. Note that for any $u_1, u_2 \in \mathbb{R}$,

$$\begin{aligned} \mathbb{E} \left[e^{i\langle u_1, \tau_1 \rangle} e^{i\langle u_2, Z_1^1 \rangle} \right] &= \mathbb{E} \left[e^{i\langle u_1, \tau_1 \rangle} e^{i\langle u_2, Z_1^1 \rangle} \mathbf{1}_{\{\tau_1 = \tau_1^1\}} \right] + \mathbb{E} \left[e^{i\langle u_1, \tau_1 \rangle} e^{i\langle u_2, Z_1^1 \rangle} \mathbf{1}_{\{\tau_1 \neq \tau_1^1\}} \right] \\ &= 2 \times \mathbb{E} \left[e^{i\langle u_1, \tau_1 \rangle} e^{i\langle u_2, Z_1^1 \rangle} \mathbf{1}_{\{\tau_1 = \tau_1^1\}} \right], \end{aligned} \quad (19)$$

since $\mathbb{E} \left[e^{i\langle u_1, \tau_1 \rangle} e^{i\langle u_2, Z_1^1 \rangle} \mathbf{1}_{\{\tau_1 = \tau_1^1\}} \right] = \mathbb{E} \left[e^{i\langle u_1, \tau_1 \rangle} e^{i\langle u_2, Z_1^1 \rangle} \mathbf{1}_{\{\tau_1 \neq \tau_1^1\}} \right]$, following from the construction of $(N_t)_{t \geq 0}$ as a sum of two independent, identically distributed point processes. Hence for any $u_1, u_2 \in \mathbb{R}$

$$\begin{aligned} \mathbb{E} \left[e^{i\langle u_1, \tau_1 \rangle} e^{i\langle u_2, \tilde{Z}_1^1 \rangle} \right] &= \mathbb{E} \left[e^{i\langle u_1, \tau_1 \rangle} e^{i\langle u_2, Z_1^1 \rangle} \mathbf{1}_{\{\tau_1 = \tau_1^1\}} \right] + \mathbb{E} \left[e^{i\langle u_1, \tau_1 \rangle} e^{i\langle u_2, 0 \rangle} \mathbf{1}_{\{\tau_1 \neq \tau_1^1\}} \right] \\ &= \frac{1}{2} \mathbb{E} \left[e^{i\langle u_1, \tau_1 \rangle} e^{i\langle u_2, Z_1^1 \rangle} \right] + \mathbb{E} \left[e^{i\langle u_1, \tau_1 \rangle} \mathbf{1}_{\{\tau_1 \neq \tau_1^1\}} \right] \\ &= \frac{1}{2} \mathbb{E} \left[e^{i\langle u_1, \tau_1 \rangle} \right] \mathbb{E} \left[e^{i\langle u_2, Z_1^1 \rangle} \right] + \frac{1}{2} \mathbb{E} \left[e^{i\langle u_1, \tau_1 \rangle} \right] = \mathbb{E} \left[e^{i\langle u_1, \tau_1 \rangle} \right] \mathbb{E} \left[e^{i\langle u_2, \tilde{Z}_1^1 \rangle} \right], \end{aligned}$$

where the second step uses (19), the third step uses the independence of τ_1 and Z_1^1 for the first term and an argument analogous to (19) for the second term, and the last step uses (18) for $j = k = 1$. This implies the independence of \tilde{Z}_1^1 and τ_1 and completes the proof. \square

It is now possible to determine the optimal immersion coupling of two copies of a compound Poisson process with a Lévy measure ν having a strictly unimodal density. Based on Theorem 3.1, and possibly enriching the filtration using a marked Poisson process as in the statement of Theorem 3.1, any such coupling $(X_t, Y_t)_{t \geq 0}$ can be represented as in (16). This permits the interpretation of $(X_t, Y_t)_{t \geq 0}$ as a coupling of two processes always jumping simultaneously, albeit with a jump distribution which is different from the original, obtained by mixing the original with a Dirac mass at 0.

Given an immersion coupling $(X_t, Y_t)_{t \geq 0}$ as described above, with $|X_0 - Y_0| = a$, define a Markov chain $(\bar{X}_k)_{k=0}^\infty$ derived from the jumps of $(X_t)_{t \geq 0}$ by requiring that

$$\bar{X}_k = X_0 + \sum_{j=1}^k \tilde{Z}_j^1$$

for $k \geq 0$. Based on $(\bar{X}_n)_{n=0}^\infty$, construct a Markov chain $(\bar{Z}_k^{0,a})_{k=0}^\infty$ starting from Y_0 which is step-by-step AMR-coupled to $(\bar{X}_n)_{n=0}^\infty$, as in the proof of Theorem 2.11. Namely, set $\bar{Z}_0^{0,a} = Y_0$ (which implies $|\bar{X}_0 - \bar{Z}_0^{0,a}| = a$) and require that for any $k \geq 1$ the random vector $(X_k - X_{k-1}, \bar{Z}_k^{0,a} - \bar{Z}_{k-1}^{0,a})$ is an AMR coupling defined as in (4)-(5). Denote the jumps of $(\bar{Z}_k^{0,a})_{k=0}^\infty$ by

$$\Delta \bar{Z}_k^{0,a} = \bar{Z}_k^{0,a} - \bar{Z}_{k-1}^{0,a}$$

for $k \geq 1$. Finally, also define a compound Poisson process $(Z_t^{0,a})_{t \geq 0}$ based on these jumps, with the same driving Poisson process as for $(X_t)_{t \geq 0}$, i.e.,

$$Z_t^{0,a} = \bar{Z}_0^{0,a} + \sum_{k=1}^{N_t} \Delta \bar{Z}_k^{0,a}. \quad (20)$$

Note that $(Z_t^{0,a})_{t \geq 0}$ is in fact constructed from $(X_t)_{t \geq 0}$ by coupling all the jumps one-by-one using the AMR procedure. However, for our argument it is crucial that this procedure is applied to the extended jump distributions $\frac{1}{2\nu(\mathbb{R})}\nu + \frac{1}{2}\delta_0$ rather than just to $\frac{1}{\nu(\mathbb{R})}\nu$, cf. Figure 2. From this construction it is the case that $|X_0 - Z_0^{0,a}| = a$ and, since the random variables $\Delta \bar{Z}_k^{0,a}$ and \tilde{Z}_k^2 have the same law for each $k \geq 1$ (and $\bar{Z}_0^{0,a} = Y_0$), it follows that $(Z_t^{0,a})_{t \geq 0}$ and $(Y_t)_{t \geq 0}$ have the same finite-dimensional distributions.

We can now prove the optimality result for the AMR coupling of finite-activity Lévy processes. Note that it is expressed in terms of maximization of convex payoffs, but it equivalently solves the problem of minimization of concave costs, as explained in the discussion below the proof of Lemma 2.9.

Theorem 3.2. *Let $(X_t)_{t \geq 0}$ be a finite-activity Lévy process with a Lévy measure with strictly unimodal density and let $a > 0$. Let $(X_t, Z_t^{0,a})_{t \geq 0}$ be the AMR coupling defined by (20) and let $\phi : [0, \infty) \rightarrow [0, \infty)$ be a convex bounded continuous decreasing function. Then it is the case that*

$$\sup \{ \mathbb{E} [\phi(|X_t - Y_t|)] ; |X_0 - Y_0| = a \text{ and } (X, Y) \text{ is an immersion coupling} \} = \mathbb{E} \left[\phi(|X_t - Z_t^{0,a}|) \right].$$

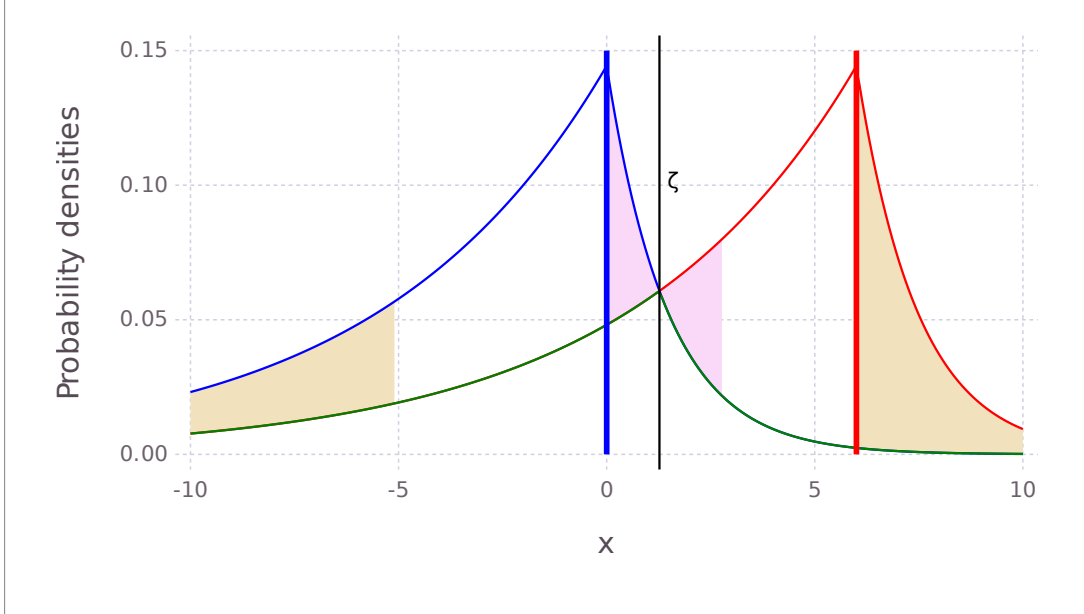


Figure 2: The anti-monotonic rearrangement in the case of a Lévy measure ν_1 with finite support, having a non-symmetric unimodal density f_1 with added Dirac mass (represented by the bold line) at the mode (corresponding to the process $(X_t)_{t \geq 0}$) and its copy ν_2 with density f_2 shifted to the right by $a > 0$ (corresponding to the process $(Z_t^{0,a})_{t \geq 0}$). The figure can be also interpreted as an illustration of the jump distribution of $(X_t)_{t \geq 0}$ (which is $\nu_1/\nu_1(\mathbb{R})$). Since the mass to the left of the mode of f_1 is larger than the mass to the right of the mode, some non-zero jumps of $(X_t)_{t \geq 0}$ to the left will be transformed into zero jumps of $(Z_t^{0,a})_{t \geq 0}$. The underlying intuition is that when a unimodal probability density is mixed with a probability mass at the mode so that the mass at the mode is not exceeded by the mass of the resulting sub-probability density, then anti-monotonic rearrangement of the result with its translates never extends across the modes. Thus the white subset of the left region is transported to the right atom, while the white subset of the right region is transported to the left atom.

Proof. For any convex, bounded, continuous, decreasing function ϕ and for any immersion coupling $(X_t, Y_t)_{t \geq 0}$ given by (16),

$$\begin{aligned}
 \mathbb{E}[\phi(|X_t - Y_t|)] &= \sum_{n=0}^{\infty} \mathbb{E} \left[\phi \left(\left| (X_0 - Y_0) + \sum_{j=1}^{N_t} (\tilde{Z}_j^1 - \tilde{Z}_j^2) \right| \right) \middle| N_t = n \right] \mathbb{P}[N_t = n] \\
 &= \sum_{n=0}^{\infty} \mathbb{E} \left[\phi \left(\left| (X_0 - Y_0) + \sum_{j=1}^n (\tilde{Z}_j^1 - \tilde{Z}_j^2) \right| \right) \right] \mathbb{P}[N_t = n] \\
 &\leq \sum_{n=0}^{\infty} \mathbb{E} \left[\phi(|\bar{X}_n - \bar{Z}_n^{0,a}|) \right] \mathbb{P}[N_t = n] = \mathbb{E} \left[\phi(|X_t - Z_t^{0,a}|) \right],
 \end{aligned}$$

where the inequality uses the optimality result for the discrete-time case (Theorem 2.11). Taking the supremum on the left hand side over all immersion couplings $(X_t, Y_t)_{t \geq 0}$ as described above, it follows that that $(X_t, Z_t^{0,a})_{t \geq 0}$ is the optimal Markovian coupling. \square

Finally, we stress that the optimal Markovian coupling for compound Poisson processes constructed above is unique. Indeed, if we take another optimal Markovian coupling $(\hat{X}_t, \hat{Y}_t)_{t \geq 0}$ given by

$$\hat{X}_t = X_0 + \sum_{i=1}^{\hat{N}_t} \hat{Z}_i^1 \quad \text{and} \quad \hat{Y}_t = Y_0 + \sum_{i=1}^{\hat{N}_t} \hat{Z}_i^2,$$

with a Poisson process $(\hat{N}_t)_{t \geq 0}$ and i.i.d. sequences $(\hat{Z}_i^1)_{i=1}^{\infty}$ and $(\hat{Z}_i^2)_{i=1}^{\infty}$ obtained in an analogous way as $(N_t)_{t \geq 0}$, $(\tilde{Z}_i^1)_{i=1}^{\infty}$ and $(\tilde{Z}_i^2)_{i=1}^{\infty}$ above, respectively, then by the argument in the proof of Theorem 3.2 and by the uniqueness of the optimal coupling for Markov chains, we conclude that for each $i \geq 1$ the random variables \hat{Z}_i^1 and \hat{Z}_i^2 have to be coupled via the AMR procedure. This gives uniqueness in law of the optimal Markovian coupling for continuous-time finite-activity Lévy processes.

To conclude, we briefly discuss two contrasting continuous-time examples.

Example 3.3. Consider the case of a finite activity Lévy process with symmetric jumps. The above arguments show the optimal coupling is a “reflection coupling”: the jumps of one process are the opposite of the jumps of the other till they both cross the midpoint between the starting points. At that crossover time the “left” process jumps to the location of the “right” process and coupling occurs. By arguments using the so-called recognition lemma [21, Lemma 20], this coupling is maximal amongst all couplings, immersion or not.

Example 3.4. Consider the case of a finite activity Lévy process with exponential jumps (hence the jumps are always positive). The above arguments show that, in the optimal coupling with starting points $x < y$, the two coupled processes jump independently according to independent Poisson processes, till the first time the “left” process jumps past the “right” process. At that crossover time the “right” process is forced to jump to the location of the “left” process. However, arguments using the so-called recognition lemma [21, Lemma 20] show that this coupling cannot be maximal.

Remark 3.5. As discussed in the introduction, in recent literature there have been plenty of other constructions of couplings of Lévy processes [34,35,37,38]. Using the notation from those papers, in the special setting where the Lévy measure ν has a density and hence we have access to the AMR transformation ρ_{F_1, F_2} given by (6), the idea for the AMR coupling can be described as

$$(x, y) \rightarrow \begin{cases} (x + z, y + z), & (\nu \wedge \delta_{y-x} * \nu)(dz) / \nu(\mathbb{R}) \\ (x + z, y + \rho_{F_1, F_2}(z)) & (\nu - \nu \wedge \delta_{y-x} * \nu)(dz) / \nu(\mathbb{R}), \end{cases}$$

where f_1 in (6) is the density of the probability measure $\nu / \nu(\mathbb{R})$, and f_2 is the density of its shift by $y - x$. However, while this description can be used to characterise the coupling in discrete time, when all the jumps happen simultaneously, it does not directly apply to the optimal coupling $(X_t, Z_t^{0,a})_{t \geq 0}$ in continuous time from Theorem 3.2, since that coupling applies the AMR construction to modified jump distributions with added δ_0 , thus allowing for non-simultaneous jumps of the coupled processes. Note that for jump distributions with atoms, we do not have access to the straightforward form of the AMR transformation (6) and we need to work with the Skorokhod representation. Note also that all the couplings from [34,35,37,38] only used simultaneous jumps.

Remark 3.6. Extending the optimality results presented in this paper to infinite activity Lévy processes is a challenging problem. In the infinity activity case, one no longer has access to the representation of the coupling via compound Poisson processes (16). However, jumps of Lévy processes can always be characterised by Poisson random measures and this fact has been recently used to construct couplings in the infinite activity case, see e.g. [38, Section 2.2] or [37, Section 2.2]. Employing such a construction in our case seems non-trivial though, even in the setting of absolutely continuous jump distributions, where the AMR transformation has a straightforward characterization given by (6) in Section 2.2.1. Note that when f_1 and f_2 in (6) correspond to densities of infinite Lévy measures, one would have to deal with the singularities arising due to their infinite mass. Hence the corresponding infinite activity AMR coupling would have to be constructed as a solution to a singular non-elliptic SDE with jumps, which creates numerous technical difficulties. Solving this problem falls beyond the scope of the present paper and is left for future work.

4. Proofs of auxiliary results

Proof of Theorem 2.4. It suffices to treat the case of $(0, \infty)$: the case of $(-\infty, 0)$ then follows by applying the symmetry of reflection in 0. Firstly we establish existence of a density for $\nu|_{(0, \infty)}$.

Fix $\varepsilon > 0$ and a set $A \subset [\varepsilon, \infty)$. Then choose a bounded interval $(0, H) \subset \mathbb{R}$ of positive Lebesgue measure such that $A \subset [a(x), \infty)$ whenever $x \in (0, H)$. We can do this since $0 < a(x) < x \rightarrow 0$ as $x \downarrow 0$.

Now choose a Uniform $(0, H)$ random variable U . Define a measure μ by $\mu(B) = \mathbb{E}[(\delta_U * \nu)(B)]$. Then μ has a density with respect to Lebesgue measure. For the density of U by $h = H^{-1}\mathbf{1}_{(0, H)}$, so

$$\begin{aligned} \mu(B) &= \int h(x)\nu(B-x)dx = \int h(x) \int_{B-x} \nu(dy)dx = \int \int h(x)\mathbf{1}_{B-x}(y)\nu(dy)dx \\ &= \int \int h(x)\mathbf{1}_{B-y}(x)dx\nu(dy) = \int \int_{B-y} h(x)dx\nu(dy). \end{aligned}$$

Hence $\mu(B) = 0$ for any Lebesgue null-set $B \subset \mathbb{R}$. Thus μ is an absolutely continuous measure.

Now observe that for any $x \in (0, H)$ we have $(\delta_x * \nu)|A \geq \nu|A$ (since $A \subset [a(x), \infty)$). Hence almost surely $(\delta_U * \nu)|A \geq \nu|A$. This implies that $\mu|A \geq \nu|A$ so long as $A \subset [a(x), \infty)$, hence that ν has a density on A . This holds for all $\varepsilon > 0$ and $A \subset [\varepsilon, \infty)$; therefore ν must have a density on all of $(0, \infty)$.

The unimodality condition (8) is symmetric under reflection, so ν has a density f on all of $\mathbb{R} \setminus \{0\}$.

We now establish a weakly decreasing property of (a càglàd version of) the density of $\nu|_{(0, \infty)}$.

Fix $z > 0$ and argue from condition (8) that

$$f(y-z) \geq f(y) \quad \text{for almost all } y > z > 0.$$

Indeed, if $f(y-z) < f(y)$ for $y \in A \subset (z, \infty)$ with $\text{Leb}(A) > 0$, then

$$(\delta_z * \nu)(A) = \int_A f(y-z)dy < \int_A f(y)dy = \nu(A),$$

which contradicts (8). So $f(u) \geq f(x)$ for almost all u, x with $u < x \in (0, \infty)$, and (for almost all $x > 0$)

$$f(x) \leq \bar{f}(x) = \text{ess inf}\{f(u) : 0 < u < x\}.$$

Evidently $\bar{f}(x)$ is non-increasing in x for $x > 0$. The proof is completed if we can show that $f(x) = \bar{f}(x)$ for almost all $x > 0$, as we can then deduce that \bar{f} is a càglàd non-increasing version of f on $(0, \infty)$. This final step is a little more delicate.

Consider a range $(a, a + T] \subset (0, \infty)$, and fix $\varepsilon > 0$. Consider the set $M_{a,T,\varepsilon} \subset (a, a + T]$ given by

$$M_{a,T,\varepsilon} = \{x \in (a, a + T] : f(x) < \bar{f}(x) - \varepsilon\},$$

and suppose $\text{Leb}(M_{a,T,\varepsilon}) > 0$. Divide $(a, a + T]$ into K disjoint half-open intervals of equal length T/K and suppose that L of these intersect with $M_{a,T,\varepsilon}$ in a set of positive measure; evidently $L \times T/K \geq \text{Leb}(M_{a,T,\varepsilon})$. For such an interval $I = (x, y]$ we know that $\text{Leb}(I \cap M_{a,T,\varepsilon}) > 0$ and therefore (by the properties of ess inf and the monotonicity of \bar{f})

$$\bar{f}(y) \leq \bar{f}(x-) - \varepsilon,$$

where $\bar{f}(x-) = \lim_{z \uparrow x} \bar{f}(z)$. Hence \bar{f} must decrease by at least ε on such an interval I .

Summing over all intervals in the above decomposition of $(a, a + T]$, we deduce

$$\bar{f}(a-) - \bar{f}(a + T) \geq L\varepsilon \geq \text{Leb}(M_{a,T,\varepsilon})(K/T)\varepsilon.$$

But the left-hand side is bounded above by $\bar{f}(a-)$, so we may deduce

$$\text{Leb}(M_{a,T,\varepsilon}) \leq \frac{\bar{f}(a-) T}{\varepsilon K}.$$

Since K can be chosen to be arbitrarily large, we deduce that $\text{Leb}(M_{a,T,\varepsilon}) = 0$. This holds for all $\varepsilon > 0$ and all $a, T > 0$. Hence $f(x) = \bar{f}(x)$ for almost all $x > 0$, as required to complete the proof. \square

Proof of Lemma 2.5. The statement follows by direct manipulation of the integral:

$$\begin{aligned} \int_{-\infty}^{\infty} \mathbb{P} \left[(X, Y) \in (t - c, t]^2 \right] dt &= \int_{-\infty}^{\infty} (\mathbb{P}[t - c < X \leq Y \leq t] + \mathbb{P}[t - c < Y < X \leq t]) dt \\ &= \mathbb{E} \left[\int_{-\infty}^{\infty} (\mathbf{1}_{\{Y \leq t < X + c, X \leq Y\}} + \mathbf{1}_{\{X \leq t < Y + c, Y < X\}}) dt \right] \\ &= \mathbb{E} \left[(X + c - Y) \mathbf{1}_{\{X \leq Y\}} + (Y + c - X) \mathbf{1}_{\{Y < X\}} \right] = \mathbb{E} \left[(c - |X - Y|)^+ \right]. \end{aligned}$$

\square

Proof of Lemma 2.8. We know that

$$\begin{aligned} \psi(a, c) &= \mathbb{E} \left[(c - |X - \text{AMR}_a(X)|)^+ \right] = \int_{-\infty}^{\infty} (F(t) - F(t - c)) \wedge (F_a(t) - F_a(t - c)) dt \\ &= \int_{-\infty}^{\infty} (F(t) - F(t - c)) \wedge (F(t - a) - F(t - c - a)) dt. \end{aligned}$$

As a consequence of the weak unimodality of the law of X , if the absolutely continuous part of F has a density f , then for a given $c > 0$ there exists a constant ζ such that $f(\zeta) = f(\zeta - c)$ and the function $t \mapsto F(t) - F(t - c)$ increases up to ζ and decreases afterwards. Hence, for Lebesgue-almost all a and $c > 0$, there exists $t_0 > \zeta$ such that

$$F(t_0) - F(t_0 - c) = F(t_0 - a) - F(t_0 - a - c). \quad (21)$$

and, for all $t \leq t_0$,

$$(F(t) - F(t - c)) \wedge (F(t - a) - F(t - c - a)) = F(t - a) - F(t - c - a),$$

whereas, for all $t > t_0$,

$$(F(t) - F(t - c)) \wedge (F(t - a) - F(t - c - a)) = F(t) - F(t - c).$$

Note that (21) may fail for some combinations of a and c since F is allowed to have an atom. However, as noted above, it is sufficient if (21) holds for Lebesgue-almost all a and c . Furthermore, observe that condition (21) is symmetric in a and c and hence if the roles of a and c in (14) are exchanged then the same $t_0 = t_0(a, c)$ is obtained. Letting $\bar{F}(t) = 1 - F(t)$ be the complementary distribution function,

$$\begin{aligned} \psi(a, c) &= \int_{-\infty}^{\infty} (F(t) - F(t - c)) \wedge (F(t - a) - F(t - c - a)) dt \\ &= \int_{-\infty}^{t_0} (F(t - a) - F(t - c - a)) dt + \int_{t_0}^{\infty} (F(t) - F(t - c)) dt \\ &= \int_{t_0 - c - a}^{t_0 - a} F(t) dt + \int_{t_0 - c}^{t_0} \bar{F}(t) dt \\ &= \int_{t_0 - c - a}^{t_0 - c} F(t) dt + \int_{t_0 - c}^{t_0 - a} F(t) dt + \int_{t_0 - c}^{t_0 - a} \bar{F}(t) dt + \int_{t_0 - a}^{t_0} \bar{F}(t) dt \\ &= \int_{t_0 - c - a}^{t_0 - c} F(t) dt + c - a + \int_{t_0 - a}^{t_0} \bar{F}(t) dt = \psi(c, a) + c - a, \end{aligned}$$

where the fourth equality is true when $a < c$, which can be assumed without loss of generality, and the last equality follows from the fact that $t_0(a, c) = t_0(c, a)$ as observed above. \square

Proof of Lemma 2.9. First observe that any bounded continuous convex and non-increasing function $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ can be represented as

$$\varphi(|x - y|) = \lim_{n \rightarrow \infty} \sum_{i=1}^n \lambda_i^n (c_i^n - |x - y|)^+,$$

with positive constants λ_i^n, c_i^n chosen to ensure that the sequence of functions $f_n(x, y) = \sum_{i=1}^n \lambda_i^n (c_i^n - |x - y|)^+$ is increasing. Hence

$$\mathbb{E}[\varphi(|X - Y|)] = \lim_{n \rightarrow \infty} \sum_{i=1}^n \mathbb{E}[\lambda_i^n (c_i^n - |X - Y|)^+] \quad (22)$$

(using the monotone convergence theorem). Moreover, for any pair of random variables (X, Y)

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n \mathbb{E}[\lambda_i^n (c_i^n - |X - Y|)^+] \leq \lim_{n \rightarrow \infty} \sum_{i=1}^n \sup_{Y: \mathcal{L}(Y) = \mathcal{L}(X+a)} \mathbb{E}[\lambda_i^n (c_i^n - |X - Y|)^+]$$

and hence

$$\begin{aligned}
& \sup_{Y: \mathcal{L}(Y)=\mathcal{L}(X+a)} \lim_{n \rightarrow \infty} \sum_{i=1}^n \mathbb{E} [\lambda_i^n (c_i^n - |X - Y|)^+] \leq \lim_{n \rightarrow \infty} \sum_{i=1}^n \sup_{Y: \mathcal{L}(Y)=\mathcal{L}(X+a)} \mathbb{E} [\lambda_i^n (c_i^n - |X - Y|)^+] \\
& = \lim_{n \rightarrow \infty} \sum_{i=1}^n \mathbb{E} [\lambda_i^n (c_i^n - |X - \text{AMR}(X)|)^+] = \mathbb{E} [\varphi(|X - \text{AMR}(X)|)] \\
& \leq \sup_{Y: \mathcal{L}(Y)=\mathcal{L}(X+a)} \mathbb{E} [\varphi(|X - Y|)] = \sup_{Y: \mathcal{L}(Y)=\mathcal{L}(X+a)} \lim_{n \rightarrow \infty} \sum_{i=1}^n \mathbb{E} [\lambda_i^n (c_i^n - |X - Y|)^+] ,
\end{aligned}$$

where in the first equality above Lemma 2.6 is applied to a distribution function F and its right shift $x \mapsto F(x - a)$, with the resulting optimal AMR coupling denoted by $(X, \text{AMR}(X))$. This shows that

$$\sup_{Y: \mathcal{L}(Y)=\mathcal{L}(X+a)} \lim_{n \rightarrow \infty} \sum_{i=1}^n \mathbb{E} [\lambda_i^n (c_i^n - |X - Y|)^+] = \lim_{n \rightarrow \infty} \sum_{i=1}^n \sup_{Y: \mathcal{L}(Y)=\mathcal{L}(X+a)} \mathbb{E} [\lambda_i^n (c_i^n - |X - Y|)^+] .$$

Combining this equality with Lemma 2.8, we conclude that the function

$$a \mapsto \sup_{Y: \mathcal{L}(Y)=\mathcal{L}(X+a)} \lim_{n \rightarrow \infty} \sum_{i=1}^n \mathbb{E} [\lambda_i^n (c_i^n - |X - Y|)^+]$$

is both bounded, continuous, and also convex, being a limit of a sequence of sums of bounded continuous convex functions. It is also non-increasing, since any bounded continuous convex function on $[0, \infty)$ must be non-increasing. Combining this with (22) concludes the proof. \square

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References

- [1] BANERJEE, S. and KENDALL, W. (2018). Coupling polynomial Stratonovich integrals: the two-dimensional Brownian case. *Electron. J. Probab.* **23** Paper No. 24, 43.
- [2] BANERJEE, S. and KENDALL, W. S. (2017). Rigidity for Markovian maximal couplings of elliptic diffusions. *Probab. Theory Related Fields* **168** 55–112.
- [3] BÖTTCHER, B. (2017). Markovian Maximal Coupling of Markov Processes. *arXiv e-prints* arXiv:1710.09654.
- [4] BOU-RABEE, N., EBERLE, A. and ZIMMER, R. (2020). Coupling and convergence for Hamiltonian Monte Carlo. *Ann. Appl. Probab.* **30** 1209–1250.
- [5] BRÉMAUD, P. (1975). An extension of Watanabe's theorem of characterization of Poisson processes over the positive real half line. *J. Appl. Probab.* **12** 396–399.

-
- [6] BURDZY, K. and KENDALL, W. S. (2000). Efficient Markovian couplings: examples and counterexamples. *Ann. Appl. Probab.* **10** 362–409.
- [7] CHAU, N. H., MOULINES, E., RÁSONYI, M., SABANIS, S. and ZHANG, Y. (2021). On stochastic gradient Langevin dynamics with dependent data streams: the fully nonconvex case. *SIAM J. Math. Data Sci.* **3** 959–986.
- [8] CHEN, M.-F. (1994). Optimal couplings and application to Riemannian geometry. In *Probability theory and mathematical statistics (Vilnius, 1993)* 121–142. TEV, Vilnius.
- [9] CHEN, M.-F. (1994). Optimal Markovian couplings and applications. *Acta Math. Sinica (N.S.)* **10** 260–275. A Chinese summary appears in *Acta Math. Sinica* **38** (1995), no. 4, 575.
- [10] CHEN, M.-F. (2005). *Eigenvalues, Inequalities, and Ergodic Theory. Probability and its Applications (New York)*. Springer-Verlag London, Ltd., London.
- [11] CHENG, X., CHATTERJI, N. S., ABBASI-YADKORI, Y., BARTLETT, P. L. and JORDAN, M. I. (2018). Sharp convergence rates for Langevin dynamics in the nonconvex setting. *arXiv e-prints* arXiv:1805.01648.
- [12] CHENG, X., CHATTERJI, N. S., BARTLETT, P. L. and JORDAN, M. I. (2018). Underdamped Langevin MCMC: A non-asymptotic analysis. In *Proceedings of the 31st Conference On Learning Theory (S. BUBECK, V. PERCHET and P. RIGOLLET, eds.)*. *Proceedings of Machine Learning Research* **75** 300–323. PMLR.
- [13] CHENG, X., YIN, D., BARTLETT, P. and JORDAN, M. (2020). Stochastic Gradient and Langevin Processes. In *Proceedings of the 37th International Conference on Machine Learning (H. D. III and A. SINGH, eds.)*. *Proceedings of Machine Learning Research* **119** 1810–1819. PMLR, Virtual.
- [14] CONNOR, S. (2013). Optimal coadapted coupling for a random walk on the hyper-complete graph. *J. Appl. Probab.* **50** 1117–1130.
- [15] CONNOR, S. and JACKA, S. (2008). Optimal co-adapted coupling for the symmetric random walk on the hypercube. *J. Appl. Probab.* **45** 703–713.
- [16] CONT, R. and TANKOV, P. (2004). *Financial Modelling with Jump Processes*. *Chapman & Hall/CRC Financial Mathematics Series*. Chapman & Hall/CRC, Boca Raton, FL.
- [17] DALEY, D. J. and VERE-JONES, D. (2003). *An Introduction to the Theory of Point Processes. Vol. I*, second ed. *Probability and its Applications (New York)*. Springer-Verlag, New York Elementary theory and methods.
- [18] EBERLE, A. (2016). Reflection couplings and contraction rates for diffusions. *Probab. Theory Related Fields* **166** 851–886.
- [19] EBERLE, A., GUILLIN, A. and ZIMMER, R. (2019). Quantitative Harris-type theorems for diffusions and McKean-Vlasov processes. *Trans. Amer. Math. Soc.* **371** 7135–7173.
- [20] EBERLE, A. and MAJKA, M. B. (2019). Quantitative contraction rates for Markov chains on general state spaces. *Electron. J. Probab.* **24** Paper No. 26, 36.
- [21] ERNST, P. A., KENDALL, W. S., ROBERTS, G. O. and ROSENTHAL, J. S. (2019). MEXIT: maximal uncoupling times for stochastic processes. *Stochastic Process. Appl.* **129** 355–380.
- [22] GANGBO, W. and MCCANN, R. J. (1996). The geometry of optimal transportation. *Acta Math.* **177** 113–161.
- [23] GOLDSTEIN, S. (1978/79). Maximal coupling. *Z. Wahrsch. Verw. Gebiete* **46** 193–204.
- [24] GRIFFEATH, D. (1974/75). A maximal coupling for Markov chains. *Z. Wahrsch. Verw. Gebiete* **31** 95–106.
- [25] HAMMERSLEY, J. M. and MORTON, K. W. (1956). A new Monte Carlo technique: antithetic variates. *Proc. Cambridge Philos. Soc.* **52** 449–475.
- [26] HSU, E. P. and STURM, K.-T. (2013). Maximal coupling of Euclidean Brownian motions. *Commun. Math. Stat.* **1** 93–104.
- [27] HU, K., REN, Z., ŠIŠKA, D. and SZPRUCH, L. U. (2021). Mean-field Langevin dynamics and energy landscape of neural networks. *Ann. Inst. Henri Poincaré Probab. Stat.* **57** 2043–2065.
- [28] HUANG, L.-J., MAJKA, M. B. and WANG, J. (2022). Strict Kantorovich contractions for Markov chains and Euler schemes with general noise. *Stochastic Process. Appl.* **151** 307–341.
- [29] JACKA, S. D. and MIJATOVIĆ, A. (2015). Coupling and tracking of regime-switching martingales. *Electron. J. Probab.* **20** no. 38, 39.
- [30] JACKA, S. D., MIJATOVIĆ, A. and ŠIRAJ, D. (2014). Mirror and synchronous couplings of geometric Brownian motions. *Stochastic Process. Appl.* **124** 1055–1069.
- [31] KELLERER, H. G. (1984). Duality theorems for marginal problems. *Z. Wahrsch. Verw. Gebiete* **67** 399–432.
-

- [32] KENDALL, W. S. (2015). Coupling, local times, immersions. *Bernoulli* **21** 1014–1046.
- [33] LIANG, M., MAJKA, M. B. and WANG, J. (2021). Exponential ergodicity for SDEs and McKean-Vlasov processes with Lévy noise. *Ann. Inst. Henri Poincaré Probab. Stat.* **57** 1665–1701.
- [34] LIANG, M., SCHILLING, R. L. and WANG, J. (2020). A unified approach to coupling SDEs driven by Lévy noise and some applications. *Bernoulli* **26** 664–693.
- [35] LIANG, M. and WANG, J. (2020). Gradient estimates and ergodicity for SDEs driven by multiplicative Lévy noises via coupling. *Stochastic Process. Appl.* **130** 3053–3094.
- [36] LINDVALL, T. (1992). *Lectures on the Coupling Method*. *Wiley Series in Probability and Mathematical Statistics: Probability and Mathematical Statistics*. John Wiley & Sons, Inc., New York A Wiley-Interscience Publication.
- [37] LUO, D. and WANG, J. (2019). Refined basic couplings and Wasserstein-type distances for SDEs with Lévy noises. *Stochastic Process. Appl.* **129** 3129–3173.
- [38] MAJKA, M. B. (2017). Coupling and exponential ergodicity for stochastic differential equations driven by Lévy processes. *Stochastic Process. Appl.* **127** 4083–4125.
- [39] MAJKA, M. B. (2019). Transportation inequalities for non-globally dissipative SDEs with jumps via Malliavin calculus and coupling. *Ann. Inst. Henri Poincaré Probab. Stat.* **55** 2019–2057.
- [40] MAJKA, M. B., MIJATOVIĆ, A. and SZPRUCH, L. (2020). Nonasymptotic bounds for sampling algorithms without log-concavity. *Ann. Appl. Probab.* **30** 1534–1581.
- [41] MCCANN, R. J. (1999). Exact solutions to the transportation problem on the line. *R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci.* **455** 1341–1380.
- [42] PITMAN, J. W. (1976). On coupling of Markov chains. *Z. Wahrsch. Verw. Gebiete* **35** 315–322.
- [43] REVUZ, D. and YOR, M. (1999). *Continuous Martingales and Brownian Motion*, third ed. *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]* **293**. Springer-Verlag, Berlin.
- [44] ROGERS, L. C. G. (1999). Fastest coupling of random walks. *J. London Math. Soc. (2)* **60** 630–640.
- [45] ROSENTHAL, J. S. (1997). Faithful couplings of Markov chains: now equals forever. *Adv. in Appl. Math.* **18** 372–381.
- [46] ROSS, S. M. (2014). *Introduction to Probability Models*, Eleventh ed. Elsevier/Academic Press, Amsterdam.
- [47] THORISSON, H. (2000). *Coupling, Stationarity, and Regeneration*. *Probability and its Applications (New York)*. Springer-Verlag, New York.
- [48] VILLANI, C. (2009). *Optimal Transport: Old and New*. Springer.
- [49] WATANABE, S. (1964). On discontinuous additive functionals and Lévy measures of a Markov process. *Jpn. J. Math.* **34** 53–70.
- [50] WILLIAMS, D. (1991). *Probability with Martingales*. *Cambridge Mathematical Textbooks*. Cambridge University Press, Cambridge.