ANCIENT SOLUTIONS AND TRANSLATORS OF LAGRANGIAN MEAN CURVATURE FLOW

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ABSTRACT

Suppose that $\mathcal{M}$ is an almost calibrated, exact, ancient solution of Lagrangian mean curvature flow in $\mathbf{C}^n$. We show that if $\mathcal{M}$ has a blow-down given by the static union of two Lagrangian subspaces with distinct Lagrangian angles that intersect along a line, then $\mathcal{M}$ is a translator. In particular in $\mathbf{C}^2$, all almost calibrated, exact, ancient solutions of Lagrangian mean curvature flow with entropy less than 3 are special Lagrangian, a union of planes, or translators.

1. Introduction

An important problem in complex and symplectic geometry is to find special Lagrangian submanifolds in Calabi–Yau manifolds. Szmoczyk [26] showed that the mean curvature flow preserves the class of Lagrangian submanifolds in Calabi–Yau manifolds, and so one can attempt to use the flow to deform any Lagrangian into a special Lagrangian. The Thomas–Yau conjecture [28], motivated by mirror symmetry [27], predicts that this is indeed possible, assuming that the initial Lagrangian satisfies a certain stability condition. More recently Joyce [20] formulated a detailed conjectural picture, relating singularity formation along the Lagrangian mean curvature flow to Bridgeland stability conditions.

To motivate our main result, recall that along the mean curvature flow of zero-Maslov Lagrangians, all tangent flows at singularities are given by unions of minimal Lagrangian cones, according to Neves [23]. In particular all such tangent flows are singular, or have higher multiplicity. In order to understand how such singularities form, it is therefore crucial to study a general class of ancient solutions of the flow, such as Type II blow-ups.

The simplest ancient solutions are those whose blow-down at $-\infty$ is special Lagrangian. In this case [22, Proposition 4.5] implies that the ancient solution itself is special Lagrangian, and in particular static. Our main result is the following, addressing the next simplest situation. See Section 2 for the basic definitions.

Theorem **1.1.** — *Let* $P_1, P_2 \subset \mathbf{C}^n$ *be Lagrangian subspaces which intersect along a line* ℓ *and have distinct Lagrangian angles. Let* $\mathcal{M}$ *be a smoothly immersed, ancient, Lagrangian Brakke flow in* $\mathbf{C}^n$ *with uniformly bounded area ratios. Assume further that* $\mathcal{M}$ *is exact and zero-Maslov with uniformly bounded variation of the Lagrangian angle. For* $n \geq 3$ *assume in addition that* $\mathcal{M}$ *is almost calibrated.*

If $\mathcal{M}$ *has a blow-down at* $-\infty$ *given by the static flow consisting of the union* $P_1 \cup P_2$, *then* $\mathcal{M}$ *is a translator.*

Remark **1.2.** — For the definition of a smoothly immersed Brakke flow, see Definition A.1. Note that a mean curvature flow $F : M^n \times I \to \mathbf{R}^{n+m}$ where $I \ni t \to F(\cdot, t)$ is a

https://doi.org/10.1007/s10240-023-00143-5

smooth family of proper immersions is an example of a smoothly immersed Brakke flow. The benefit of the notion of smoothly immersed Brakke flows is that it does not require a global parametrisation and thus the condition is preserved by local smooth convergence. In the Appendix we show that the weighted monotonicity formula naturally extends to this setting, allowing weights with polynomial growth.

Two possibilities for the translator in Theorem 1.1 are the static flows given by unions of translates of P_1 and P_2, and the non-trivial translators constructed by Joyce–Lee–Tsui [21], which play an important role in Joyce's conjectural picture [20]. It is an interesting question whether there are any more possibilities.

Combining Theorem 1.1 with the work in [22] shows the following. (Again, see Section 2 for the definitions.)

Corollary **1.3.** — *For $0 < T < \infty$, let $(L_t)_{0 \leq t < T}$ be a smooth, properly immersed, maximal, rational, almost calibrated Lagrangian mean curvature flow in $\mathbf{C}^2$ with entropy less than 3. Then any Type II blow-up at a singular point (x_0, T) is either special Lagrangian (and given in [22, Theorems 1.1 and 1.3]) or a non-trivial translator.*

This results follows from Theorem 1.1 since for a zero-Maslov ancient solution of the flow in $\mathbf{C}^2$ with entropy less than 3 the only possible blow-downs are a union of two planes. If the two planes have the same Lagrangian angle, then [22, Theorem 1.1 or Theorem 1.3] applies. If the two planes have different Lagrangian angle, then using [22, Proposition 4.1] the planes must meet along a line, and Theorem 1.1 applies. An analogous result holds also for Lagrangian mean curvature flow in a compact Calabi–Yau surface under a suitable rationality assumption, such as in Fukaya [16, Definition 2.2], at singularities with density less than 3.

In singularity analysis it is important to consider arbitrary blow-up limits of the flow, not just those that are smooth. Theorem 5.1 provides an extension of Theorem 1.1 in the case $n = 2$ to Brakke flows obtained as blow-up limits of smooth flows.

Recently there has been great progress in classifying ancient solutions of geometric flows such as Ricci flow and mean curvature flow (see e.g. [2–5, 10]). A crucial new difficulty in our work is that the blow-down $P_1 \cup P_2$ is singular along the line of intersection ℓ. As a result, an approach based on the analysis of the linearized operator on the blow-down faces substantial difficulties. An earlier result characterizing translators among eternal solutions to the mean curvature flow of hypersurfaces is due to Hamilton [17], relying on a differential Harnack estimate. It is not known if this approach can be extended to higher codimension flows. Our approach is completely different and relies on additional structure present in the Lagrangian setting. In particular, translators are characterized by the condition that one of the coordinate functions, w, is a linear combination of 1 and the Lagrangian angle θ, i.e. $w = a + b\theta$, see Proposition 2.10. The function w is an ancient solution of the heat equation along $\mathcal{M}$, and the basic idea of the proof is to obtain information about w through solutions of the heat equation on the possible blow-downs. This is related to work of Colding–Minicozzi [8] on mean curvature flow, and also to earlier works on harmonic functions [6, 11] and holomorphic functions [12].

To illustrate the basic ideas, suppose that $n=2$ and let w be a coordinate function vanishing on $\mathrm{P}_1\cup\mathrm{P}_2$. Define z so that on $\mathbf{C}^2$ we have $\nabla w=\mathrm{J}\nabla z$, i.e. the line ℓ is parallel to ∇z. Let x be a coordinate function vanishing on P_1, but not on P_2 and similarly y a coordinate vanishing on P_2 but not on P_1. The ancient solutions of the heat equation on $\mathrm{P}_1\cup\mathrm{P}_2$ with at most linear growth, allowing a different smooth solution on each plane, are spanned by 1, θ, x, y, z, $z\theta$ (see Lemma 3.6). Here θ is simply a different constant on each plane. At the same time, 1, θ and the coordinate functions x, y, z, w are ancient solutions of the heat equation on our ancient flow $\mathcal{M}$.

The first main step of the proof is to show that either $\mathcal{M}$ is a translator, or along a suitable sequence of scales $t\to-\infty$, the normalized projection of w orthogonal to x, y, z converges to $z\theta$ on the blow-down $\mathrm{P}_1\cup\mathrm{P}_2$ (see Proposition 3.12). The main technical difficulty at this stage is that the singular set given by the line ℓ has codimension one in $\mathrm{P}_1\cup\mathrm{P}_2$, and we need to exploit that the angle θ takes on different values on P_1 and P_2 in order to pass solutions of the heat equation along $\mathcal{M}$ to solutions on $\mathrm{P}_1\cup\mathrm{P}_2$ in the limit. This is the content of Proposition 3.7.

The proof of Theorem 1.1 is completed using Proposition 4.5, based on the idea that if along the flow w behaves like $z\theta$ at some scale, then the flow must break into two pieces, which roughly look like the two planes P_1, P_2 rotated in such a way that their intersections with the unit spheres are linked. Here the fact that θ is a different constant on each plane P_1, P_2 is crucial. This linking behaviour is used to show that the flow must have a point of density two, but the monotonicity formula then implies that the flow is a static union of planes.

2. Preliminaries

In this section we introduce various key definitions and notation that we shall require throughout the article. In particular, we introduce the set-up for our study.

2.1. *Lagrangians in* $\mathbf{C}^n$. — We first recall some basic definitions concerning Lagrangian submanifolds in $\mathbf{C}^n$.

Definition **2.1.** — *An oriented Lagrangian* L *in* $\mathbf{C}^n$ *is* zero-Maslov *if there exists a function* θ *on* L *(called the* Lagrangian angle*) so that*

$$\Omega|_{\mathrm{L}}=e^{i\theta}\,d\mathrm{Vol}_{\mathrm{L}},$$

where $\Omega=dz_1\wedge\cdots\wedge dz_n$ *is the standard holomorphic volume form on* $\mathbf{C}^n$ *and* $d\mathrm{Vol}_{\mathrm{L}}$ *is the Riemannian volume form of* L. *We then have* $\mathbf{H}=\mathrm{J}\nabla\theta$, *where* $\mathbf{H}$ *is the mean curvature vector of* L *and* J *is* *the complex structure on* $\mathbf{C}^n$*. We further say that* L *is* almost calibrated *if* θ *can be chosen so that*

$$\sup\theta - \inf\theta \leq \pi - \epsilon.$$

for some $\epsilon > 0$.

Definition **2.2.** — *An oriented Lagrangian* L *in* $\mathbf{C}^n$ *is* exact *if there exists a function* β *on* L *so that*

$$J\mathbf{x}^{\perp} = \nabla\beta,$$

where $\mathbf{x}^{\perp}$ *is the normal projection of the position vector* $\mathbf{x} \in \mathbf{C}^n$*. Equivalently,*

$$d\beta = \lambda|_{L},$$

where λ *is the Liouville form on* $\mathbf{C}^n$*, which is a* 1*-form on* $\mathbf{C}^n$ *so that* $\frac{1}{2}\lambda$ *is a primitive for the Kähler form* ω *on* $\mathbf{C}^n$*. The Lagrangian* L *is* rational *if the set* $\lambda(H_1(L, \mathbf{Z}))$ *is discrete in* $\mathbf{R}$*. An exact Lagrangian is clearly rational.*

2.2. *Spacetime track.* — Throughout we consider a smooth, zero-Maslov, exact, ancient solution to Lagrangian mean curvature flow (LMCF)

$$(-\infty, 0) \ni t \mapsto L_t \subset \mathbf{C}^n$$

which evolves with normal speed given by $\mathbf{H}$, with uniformly bounded variation of the Lagrangian angle. We assume that L_t has uniformly bounded area ratios, i.e. there exists $C > 0$ such that

$$\sup_{x,t} \mathcal{H}^n(L_t \cap B(\mathbf{x}, r)) \leq Cr^n \quad \text{for all } r > 0,$$

where $B(\mathbf{x}, r)$ is the Euclidean ball of radius r about $\mathbf{x} \in \mathbf{C}^n$. We call

$$\mathcal{M} := \{L_t \times \{t\} \mid t \in (-\infty, 0)\} \subset \mathbf{C}^n \times \mathbf{R}$$

the spacetime track of the flow, and write $\mathcal{M}(t) = L_t$.

Remark **2.3.** — For $n > 2$ we will additionally need to assume that the flow $\mathcal{M}$ is almost calibrated, so that one can apply the structure theory in [22] and [24].

Since our focus is on planes arising as blow-ups or blow-downs, it is useful to consider them as trivial static flows as follows.

Definition **2.4.** — *For a pair of n-dimensional planes* $P_1, P_2 \subset \mathbf{C}^n$*, we let* $\mathcal{M}_{P_1 \cup P_2}$ *denote the static flow corresponding to* $P_1 \cup P_2$.

2.3. *Rescalings.* — It will be useful to perform parabolic rescalings of our flows, so we shall introduce the following notation.

Definition **2.5.** — *For* $\lambda > 0$ *we shall denote the* parabolic rescaling

$$\mathcal{D}_\lambda : \mathbf{C}^n \times \mathbf{R} \to \mathbf{C}^n \times \mathbf{R}, \qquad (\mathbf{x}, t) \mapsto (\lambda \mathbf{x}, \lambda^2 t)\,.$$

Note that for a (Lagrangian) mean curvature flow $\mathcal{M}$, *we have that* $\mathcal{D}_\lambda \mathcal{M}$ *is again a (Lagrangian) mean curvature flow.*

It turns out to be helpful to consider a further rescaling, which turns self-similarly shrinking solutions into static points of the flow.

Definition **2.6.** — *The* rescaled flow *is*

$$\mathbf{R} \ni \tau \mapsto \mathrm{L}_\tau := e^{\frac{\tau}{2}} \mathcal{M}(-e^{-\tau}) = e^{\frac{\tau}{2}} \mathrm{L}_{-e^{-\tau}}$$

which evolves with normal speed

$$\mathbf{H} + \frac{\mathbf{x}^\perp}{2}\,. \tag{2.1}$$

We recall Huisken's monotonicity formula [18]:

$$\begin{aligned} \frac{d}{dt} \int_{\mathrm{L}_t} f \rho_{\mathbf{x}_0, t_0}\, d\mathcal{H}^n = & \int_{\mathrm{L}_t} (\partial_t f - \Delta f) \rho_{\mathbf{x}_0, t_0}\, d\mathcal{H}^n \\ & - \int_{\mathrm{L}_t} f \left| \mathbf{H} - \frac{(\mathbf{x} - \mathbf{x}_0)^\perp}{2(t - t_0)} \right|^2 \rho_{\mathbf{x}_0, t_0}\, d\mathcal{H}^n\,, \end{aligned} \tag{2.2}$$

for $t < t_0$, where f is a smooth function on L_t with (uniformly) compact support, and

$$\rho_{\mathbf{x}_0, t_0}(\mathbf{x}, t) = (4\pi(t_0 - t))^{-n/2} \exp\left(-\frac{|\mathbf{x} - \mathbf{x}_0|^2}{4(t_0 - t)} \right)$$

is the backwards heat kernel (centred at $(\mathbf{x}_0, t_0)$). For an extension to the non-compact setting (suitable for the current set-up) and functions f with at most polynomial growth see Proposition A.3. The density of a point $(\mathbf{x}_0, t_0)$ along the flow L_t is defined to be

$$\Theta(\mathbf{x}_0, t_0) = \lim_{t \nearrow t_0} \int_{\mathrm{L}_t} \rho_{\mathbf{x}_0, t_0}\, d\mathcal{H}^n.$$

Recall also the *entropy* $\mu(\mathrm{L})$ defined by Colding–Minicozzi [7]:

$$\mu(\mathrm{L}) = \sup_{\mathbf{x}_0 \in \mathbf{C}^n,\, r > 0} \frac{1}{(4\pi r)^{n/2}} \int_{\mathrm{L}} e^{-\frac{|\mathbf{x} - \mathbf{x}_0|^2}{4r}}\, \mathrm{d}\mathcal{H}^n.$$

By virtue of Huisken's monotonicity formula, $t \mapsto \mu(\mathrm{L}_t)$ is non-increasing along any n-dimensional mean curvature flow in $\mathbf{C}^n$.

2.4. *Set-up.* — We now describe the main set-up that we shall have throughout the article. In particular, this will be useful to fix notation.

We consider two oriented Lagrangian planes $\mathrm{P}_1, \mathrm{P}_2 \subset \mathbf{R}^{2n} = \mathbf{C}^n$ which intersect along an oriented (real) line ℓ through 0. Suppose further that P_1, P_2 have *distinct* Lagrangian angles, which we denote by $\overline{\theta}_1$, $\overline{\theta}_2$. (Note that this must be the case if $n = 2$.) Changing the Lagrangian angles by a fixed constant, we can in addition assume that $\overline{\theta}_1 = -\overline{\theta}_2$.

We assume that the unit vector in the direction of ℓ is given by $\mathbf{e}_z$, corresponding to the (real) coordinate z. We let $\mathbf{e}_w = \mathbf{J}\mathbf{e}_z$, corresponding to the real coordinate w, noting that $\mathbf{e}_w$ is necessarily orthogonal to $\mathrm{P}_1 \cup \mathrm{P}_2$. We will think of w as the "*height*" function, since $w = 0$ on $\mathrm{P}_1 \cup \mathrm{P}_2$. We choose coordinates $x_1, \ldots, x_{2n-2}$ such that $x_1, \ldots, x_{n-1}, w$ vanish along P_2 and $x_n, \ldots, x_{2n-2}, w$ vanish along P_1.

Our key assumption is that our ancient solution $\mathcal{M}$ to LMCF has a blow-down at $-\infty$ given by $\mathrm{P}_1 \cup \mathrm{P}_2$, i.e.

$$\mathcal{D}_{\lambda_i}(\mathcal{M}) \cap \{t < 0\} \rightharpoonup \mathcal{M}_{\mathrm{P}_1 \cup \mathrm{P}_2} \cap \{t < 0\} \tag{\textbf{2.3}}$$

for $\lambda_i \searrow 0$, where the convergence is in the sense of Brakke flows. We note that this is equivalent to the assumption that the sequence of smooth flows

$$(-\infty, 0) \ni t \mapsto \mathrm{L}^i_t = \lambda_i \mathrm{L}_{\lambda_i^{-2} t}$$

for $t < 0$ converges weakly to the (immersed) static flow $(-\infty, 0) \ni t \mapsto \mathrm{P}_1 \cup \mathrm{P}_2$.

At this point we make the following observation about blow-downs of $\mathcal{M}$.

Proposition **2.7.** — *Let $\mathcal{M}$ be an ancient solution to Lagrangian mean curvature flow as above satisfying* (2.3)*, so has blow-down at $-\infty$ given by* $\mathrm{P}_1 \cup \mathrm{P}_2$ *and for $n \geq 3$ is almost calibrated. Then all blow-downs at $-\infty$ of $\mathcal{M}$ are unions of two multiplicity one planes which meet along a subspace* L *of dimension $m \in \{1, \ldots, n-1\}$ and have Lagrangian angles $\overline{\theta}_1$, $\overline{\theta}_2$.*

Proof. — We have the set of angles $\{\overline{\theta}_1, \overline{\theta}_2\}$ for two multiplicity one planes in one blow-down and the set of angles is the same for any blow-down by [22, Theorem 3.1]. Therefore, since $\overline{\theta}_1 \neq \overline{\theta}_2$, and by (2.3) the (Gaussian) density at $-\infty$ is two, any blow-down has to consist of two distinct unit multiplicity planes with the given angles. Again using $\overline{\theta}_1 \neq \overline{\theta}_2$, the case that a blow-down is a pair of transverse planes is ruled out since in this case [22, Proposition 4.1] would force the ancient solution $\mathcal{M}$ to be the static flow consisting of the two transverse planes, which contradicts one blow-down being two planes meeting along a line. □

2.5. *Translators.* — There is a smooth, connected, zero-Maslov (in fact, almost calibrated), exact, ancient (in fact, eternal) solution to Lagrangian mean curvature flow in $\mathbf{C}^n$ whose blow-down at $-\infty$ is $\mathrm{P}_1 \cup \mathrm{P}_2$, which is a *translator* constructed by Joyce–Lee–Tsui [21]. We recall the definition of a translator as follows.

Definition **2.8.** — *A* translator *(in the* $\mathbf{e}_z$ *direction) is a solution of LMCF satisfying*

$$(\mathbf{2.4}) \qquad \mathbf{H} = \kappa \mathbf{e}_z^\perp$$

for some $\kappa \neq 0$ *at some (and therefore any) time. (In fact, we can rescale the translator and change its orientation so that any* $\kappa \neq 0$ *can be realised).*

Suppose we have a zero-Maslov translator satisfying (2.4). Since $\mathbf{H} = \mathrm{J}\nabla\theta$, where θ is the Lagrangian angle, $\mathbf{e}_z = -\mathrm{J}\mathbf{e}_w$ and thus $\mathbf{e}_z^\perp = -\mathrm{J}\mathbf{e}_w^{\mathrm{T}}$, we deduce that

$$(\mathbf{2.5}) \qquad \theta + \kappa w = c$$

for some $\kappa \neq 0$ and constant c (on each component of the translator).

Remark **2.9.** — It is worth noting that any Lagrangian plane P so that $\mathbf{e}_z$ is tangent to P will give a trivial example of a Lagrangian translator, since it will satisfy (2.4).

Using (2.5) we deduce the following result.

Proposition **2.10.** — *If the flow* $\mathcal{M}$ *satisfies* (2.3)*, so has blow-down at* $-\infty$ *given by* $\mathrm{P}_1 \cup \mathrm{P}_2$ *and for* $n \geq 3$ *is almost calibrated, then it is a translator in the* $\mathbf{e}_z$ *direction if and only if on each component of* $\mathcal{M}$ *the height satisfies*

$$(\mathbf{2.6}) \qquad w = a + b\theta$$

for some constants a *and* b*.*

Proof. — If $\mathcal{M}$ is a translator in the $\mathbf{e}_z$ direction then (2.6) is satisfied by (2.5).

We now suppose that (2.6) is satisfied on $\mathcal{M}$. If $b \neq 0$ we deduce that $\mathcal{M}$ is a translator by differentiating (2.6) along $\mathcal{M}(t)$ for each t, which yields $\mathbf{H} = b^{-1}\mathbf{e}_z^\perp$.

If $b = 0$ then w is constant on each component of $\mathcal{M}$ and so $\mathbf{e}_z$ is tangent to $\mathcal{M}$ as the flow is Lagrangian. Hence, $\mathcal{M}$ splits as $\mathcal{M}' \times \mathbf{R}$, where $\mathcal{M}'$ is an ancient solution to Lagrangian mean curvature flow in $\mathbf{C}^{n-1}$.

If $n \geq 3$, then $\mathcal{M}'$ is almost calibrated and by (2.3) has a blow-down at $-\infty$ given by $\mathrm{P}'_1 \cup \mathrm{P}'_2$ (where $\mathrm{P}_i = \mathrm{P}'_i \times \ell$). Note that P'_1, P'_2 are transverse, but have different Lagrangian angle. Then [22, Proposition 4.1] implies that $\mathcal{M}' = \mathcal{M}'_{\mathrm{P}'_1 \cup \mathrm{P}'_2}$ and thus $\mathcal{M} = \mathcal{M}_{\mathrm{P}_1 \cup \mathrm{P}_2}$.

If $n = 2$, then $\mathcal{M}'$ is an ancient solution γ to curve shortening flow in $\mathbf{R}^2$, which has a blow-down at $-\infty$ which is a pair of non-parallel lines. We now show that γ must in fact be the asymptotic lines.

Lemma **2.11.** — *Let $\gamma = (\gamma(t))_{-\infty<t<T}$ be an ancient smooth curve shortening flow in $\mathbf{R}^2$. Assume that a blow-down $\mathcal{D}_{\lambda_i}(\gamma)$ (for $\lambda_i \searrow 0$) is either a pair of unit density static lines $\ell_1 \cup \ell_2$ meeting at one point or a single unit density line. Then the flow γ is the static line(s).*

Proof. — The case where the blow-down is a single unit density line follows from the monotonicity formula, so we only consider the case of a pair of transverse lines in the blow-down.

If γ were almost calibrated, then the classification of almost calibrated ancient solutions to Lagrangian mean curvature flow in [22, Proposition 4.1] implies that γ must be the lines, since they have distinct angles.

Let $\gamma_i := \mathcal{D}_{\lambda_i}(\gamma)$. Using the right-hand side of the integrated monotonicity formula ((2.2) with $f = 1$) and Fatou's lemma we can pick a time $t < 0$ (and a subsequence in i) such that on $\gamma_i(t)$ the curvature of the curve is locally uniformly bounded in L^2. This implies locally uniform convergence in $C^{1,\alpha}$ from which it follows that, for i and R sufficiently large, $\gamma_i(t) \cap B_R(0)$ is given by the union of two small $C^{1,\alpha}$ graphs over $(\ell_1 \cup \ell_2)(t)$. Since the flow is smooth (and using the pseudolocality result [19, Theorem 1.5]), this description of the flow has to persist for a short time. We deduce that the flow has a point with Gaussian density two (where the two graphs intersect) and thus the flow is backwards self-similar around that point. Since we have assumed that one blow-down is $\ell_1 \cup \ell_2$, the result follows. $\square$

By Lemma 2.11 we deduce that each component of $\mathcal{M}$ is a plane which has $\mathbf{e}_z$ tangent to it, and hence $\mathcal{M}$ is trivially a translator. $\square$

3. The drift heat equation

It is well known that the functions $1, \theta$ and the coordinate functions x_i all satisfy the heat equation along the mean curvature flow. Along the rescaled flow we instead consider rescaled coordinate functions as follows.

Definition **3.1.** — *For any coordinate function x_i on $\mathbf{C}^n$ we have the* rescaled coordinate function

$$\tilde{x}_i = e^{-\tau/2} x_i$$

along the rescaled flow M_τ. In particular, we have the rescaled height $\tilde{w} = e^{-\tau/2} w$.

Using the above definition, the next result, which is key for our purposes, follows from a straightforward rescaling.

Lemma **3.2.** — *The functions* $1, \theta$ *and the rescaled coordinate functions* $\tilde{x}_i$ *satisfy the drift heat equation*

$$\frac{\partial f}{\partial \tau} = \mathcal{L}_0 f \tag{3.1}$$

along the rescaled flow, where

$$\mathcal{L}_0 f := \Delta f - \frac{1}{2}\langle \mathbf{x}, \nabla f\rangle \tag{3.2}$$

is the drift Laplacian.

Note that, when computing derivatives $\frac{\partial f}{\partial \tau}$, the rescaled flow has velocity $\mathbf{H} + \frac{1}{2}\mathbf{x}^{\perp}$.

We will compare solutions of the drift heat equation along the rescaled flow with solutions on the blow-downs. By Proposition 2.7 all possible blow-downs are unions $\mathrm{P}_1' \cup \mathrm{P}_2'$ of two n-dimensional subspaces of $\mathbf{C}^n$ meeting along a subspace of dimension less than n. We therefore study solutions of (3.1) on Euclidean spaces.

On an n-dimensional space $\mathrm{P} = \mathbf{R}^n$ we define the drift heat equation and drift Laplacian by (3.1) and (3.2). For a solution $f(\mathbf{x}, \tau)$ of the drift heat equation on P, we define the weighted norm $\|f\|_\tau$ by

$$\|f\|_\tau^2 = \int_{\mathrm{P}} f(\mathbf{x}, \tau)^2 e^{-|\mathbf{x}|^2/4}. \tag{3.3}$$

By [9, Theorem 0.6] the function $\log \|f\|_\tau$ is convex in τ, and it is linear if and only if f is homogeneous, i.e. $f(\mathbf{x}, \tau) = e^{-\lambda\tau} h(\mathbf{x})$, where h is an eigenfunction of $\mathcal{L}_0$ with eigenvalue λ. In this case $\log \|f\|_\tau^2 = -2\lambda\tau + \log \|h\|^2$ and we say that f has degree 2λ. The eigenfunctions of the Ornstein–Uhlenbeck operator $\mathcal{L} = \Delta - \mathbf{x} \cdot \nabla$ on Euclidean space are well-studied, see e.g. Bogachev [1, Chap. 1]. The eigenvalues of $\mathcal{L}$ are non-negative integers k, and the corresponding eigenfunctions are degree k homogeneous polynomials given by products of Hermite polynomials. If H_k is an eigenfunction of $\mathcal{L}$ with eigenvalue k, then the function $h_k(\mathbf{x}) = \mathrm{H}_k(\mathbf{x}/\sqrt{2})$ is an eigenfunction of $\mathcal{L}_0$ with eigenvalue $k/2$. This leads to the following.

Lemma **3.3.** — *Let* $\mathrm{P} = \mathbf{R}^n$ *and let* x_i *be coordinate functions on* P. *The eigenvalues of* $\mathcal{L}_0$ *on* P *are given by non-negative half integers, and so the homogeneous solutions of the drift heat equation on* P *have non-negative integer degrees. The homogeneous solutions with degree 0 are the constants, while those with degree 1 are spanned by the rescaled coordinate functions* $e^{-\tau/2}x_i$.

We will be interested in solutions to the drift heat equation on the blow-downs $\mathrm{P}_1' \cup \mathrm{P}_2'$, where P_j' are two distinct n-dimensional subspaces of $\mathbf{C}^n$. We define these as follows.

Definition **3.4.** — *A solution of the (drift) heat equation on* $P'_1 \cup P'_2$ *is a pair* $u = (u_1, u_2)$*, where* u_j *is a solution of the (drift) heat equation on* P'_j*. We define the weighted norm* $\|u\|_\tau$ *of* u *by* $\|u\|_\tau^2 = \|u_1\|_\tau^2 + \|u_2\|_\tau^2$.

We observe that the function θ, equal to the constant $\overline{\theta}_j$ on P'_j, is a solution of the (drift) heat equation on $P'_1 \cup P'_2$ in this sense. Note that we can see $u = (u_1, u_2)$ as one solution to the (drift) heat equation on the (immersed) shrinker $P'_1 \cup P'_2$, so we still have by [9, Theorem 0.6] that $\log \|u\|_\tau$ is convex, and it is linear if and only if u is homogeneous.

Lemma 3.3 implies the following.

Lemma **3.5.** — *On any blow-down* $P'_1 \cup P'_2$ *the homogeneous solutions of the drift heat equation have non-negative integer degrees.*

Recall our basic assumption that one blow-down is given by $P_1 \cup P_2$, where $P_1 \cap P_2 = \ell$ is a line. Recall the coordinates $x_1, \ldots, x_{2n-2}, z, w$ as chosen in Section 2.4, where the coordinate along ℓ is z and w is the height function vanishing along $P_1 \cup P_2$. We then have the following, which also uses the assumption that the Lagrangian angles of P_1 and P_2 are different.

Lemma **3.6.** — *The degree 0 solutions of the drift heat equation on* $P_1 \cup P_2$ *are spanned by* $1, \theta$*. The degree 1 solutions are spanned by* $e^{-\tau/2}x_1, \ldots, e^{-\tau/2}x_{2n-2}$ *and* $e^{-\tau/2}z$*,* $e^{-\tau/2}z\theta$*.*

Proof. — The degree 0 solutions on $P_1 \cup P_2$ are given by pairs (c_1, c_2) of constants. These are spanned by the functions $1, \theta$ since θ equals two distinct constants $\overline{\theta}_j$ on the subspaces P_j.

The degree 1 solutions on $P_1 \cup P_2$ are given by pairs (f_1, f_2) of linear functions on $\mathbf{C}^n$ restricted to the subspaces. According to our choice of coordinates in Section 2.4, f_1 is in the span of $x_1, \ldots, x_{n-1}, z$ and f_2 is in the span of $x_n, \ldots, x_{2n-2}, z$. Since $x_1, \ldots, x_{n-1}$ vanish on P_2, and $x_n, \ldots, x_{2n-2}$ vanish on P_1, the collection of functions $x_1, \ldots, x_{2n-2}$ on $P_1 \cup P_2$ define the pairs $(x_i, 0)$ and $(0, x_j)$, where $1 \leq i \leq n-1$ and $n \leq j \leq 2n-2$. At the same time z, $z\theta$ contain the pairs $(z, 0)$ and $(0, z)$ in their span (again since θ takes distinct values on P_1, P_2). ☐

3.1. *Limits of solutions of the heat equation.* — In this subsection we show that if we are given a solution u of the heat equation along the ancient mean curvature flow $\mathcal{M}$, and a sequence of rescalings of $\mathcal{M}$ converging to a blow-down $P'_1 \cup P'_2$ given by a union of distinct n-dimensional subspaces, then along a subsequence we can extract a normalized limit of u, determining a solution of the heat equation on both planes separately.

Standard methods allow us to extract limits on compact sets away from the intersection $E = P'_1 \cap P'_2$, and in the limit we obtain solutions of the heat equation on $P'_j \setminus E$ for $j = 1, 2$, which are in L^∞ across E. The main difficulty is that E may have codimension 1 in P'_j and codimension 1 sets are not removable for solutions of the heat equation. (Consider for instance the solution given by two different constants on the lower and upper half planes.) To overcome this issue it is crucial that the angle θ differs on the two subspaces (which we have by Proposition 2.7), while at the same time the space-time integral of $|\nabla\theta|^2$ converges to zero as we approach the blow-down. This allows us to show that the solutions that we obtain in the limit on $\mathrm{P}'_j \setminus \mathrm{E}$ are distributional solutions across E, and hence smooth.

To state the result, let L^i_t be a sequence of smooth solutions of LMCF in $\mathbf{C}^n$ defined for $t \in [-1, 0]$. We assume that the L^i_t have Euclidean area growth and uniformly bounded Lagrangian angles. We assume that $\mathrm{L}^i_t \rightharpoonup \mathrm{P}'_1 \cup \mathrm{P}'_2$ weakly as $i \to \infty$, where as above P'_j are n-dimensional subspaces meeting along a subspace E of dimension at most $n-1$. Here, as usual, we view $\mathrm{P}'_1 \cup \mathrm{P}'_2$ as a static flow. For the definition of functions with polynomial growth we refer the reader to Definition A.2.

Proposition **3.7.** — *In the setting above, for each i, let u_i be a solution to the heat equation on L^i_t for $t \in [-1, 0]$, with at most polynomial growth. Assume further that there is a uniform $\mathrm{C} > 0$ so that*

$$\int_{\mathrm{L}^i_{-1}} u_i^2 e^{-|\mathbf{x}|^2/4} < \mathrm{C}. \tag{3.4}$$

Then, after passing to a subsequence, we have $u_i \to \overline{u}$ where $\overline{u} = (\overline{u}_1, \overline{u}_2)$ is a solution of the heat equation on the union $\mathrm{P}'_1 \cup \mathrm{P}'_2$ for $t \in (-1, 0]$ in the sense of Definition 3.4. The convergence $u_i \to \overline{u}$ here means smooth convergence on compact subsets of $(-1, 0] \times \mathbf{C}^n \setminus \mathrm{E}$, i.e. on compact subsets away from $t = -1$ and away from the intersection $\mathrm{P}'_1 \cap \mathrm{P}'_2$.

Proof. — We have

$$(\partial_t - \Delta)u_i^2 = -2|\nabla u_i|^2.$$

We apply the monotonicity formula (2.2) (see Proposition A.3) to u_i^2 with different centers $(\mathbf{x}_0, t_0)$ in $\mathbf{C}^n \times (-1, 0]$. Using the uniform bound (3.4) we find that for any $\mathrm{R} > 0$ there is a constant $\mathrm{C}_\mathrm{R} > 0$ so that

$$\begin{aligned} &\sup_{\mathrm{B}_\mathrm{R}(0)\times[-1+\mathrm{R}^{-1},0]} |u_i| < \mathrm{C}_\mathrm{R}, \\ &\int_{-1+\mathrm{R}^{-1}}^0 \int_{\mathrm{B}_\mathrm{R}(0)\cap \mathrm{L}^i_t} |\nabla u_i|^2 < \mathrm{C}_\mathrm{R}. \end{aligned} \tag{3.5}$$

Let θ_i be the Lagrangian angle on L^i_{-1} and $\overline{\theta}_1 \neq \overline{\theta}_2$ the (constant) Lagrangian angles on $\mathrm{P}'_1, \mathrm{P}'_2$. As in [23, Theorem A], we have that for all $s \in (-1, 0)$, $f \in \mathrm{C}^2(\mathbf{R})$ and compactly supported smooth functions ϕ,

$$\lim_{i\to\infty} \int_{\mathrm{L}^i_s} f(\theta_i)\phi = \sum_{j=1}^{2} \int_{\mathrm{P}_j} f(\overline{\theta}_j)\phi.$$

Since $\overline{\theta}_1 \neq \overline{\theta}_2$, we can choose $f \in \mathrm{C}^2(\mathbf{R})$ such that $f(\overline{\theta}_1) = 1$ and $f(\overline{\theta}_2) = 0$, and we fix such a function f for the rest of the proof.

We also fix a smooth function χ compactly supported in $\mathrm{B}_\mathrm{R}(0) \times (-1, 0)$. Then we have

$$\begin{aligned} \frac{d}{dt}\int_{\mathrm{L}^i_t} f(\theta_i)u_i\chi = &\int_{\mathrm{L}^i_t} f'(\theta_i)(\Delta\theta_i)u_i\chi + \int_{\mathrm{L}^i_t} f(\theta_i)(\Delta u_i)\chi + \int_{\mathrm{L}^i_t} f(\theta_i)u_i\partial_t\chi \\ &- \int_{\mathrm{L}^i_t} f(\theta_i)u_i\langle \mathrm{J}\nabla\theta_i, \mathrm{D}\chi\rangle - \int_{\mathrm{L}^i_t} f(\theta_i)u_i\chi|\nabla\theta_i|^2, \end{aligned} \tag{3.6}$$

where D denotes the ambient derivative on Euclidean space, using the fact that both u_i and θ_i solve the heat equation on L^i_t and $\mathbf{H} = \mathrm{J}\nabla\theta_i$. Note that, since χ has compact support, we may use (3.5) and the fact that the (spacetime) L^2-norm of $|\mathbf{H}| = |\nabla\theta_i|$ goes to zero as $i \to \infty$ (see [23, Lemma 5.4]) to deduce that

$$\begin{aligned} &\int_{-1}^{0}\int_{\mathrm{L}^i_t} f'(\theta_i)(\Delta\theta_i)u_i\chi\,\mathrm{d}t - \int_{-1}^{0}\int_{\mathrm{L}^i_t} f(\theta_i)u_i\langle \mathrm{J}\nabla\theta_i, \mathrm{D}\chi\rangle\mathrm{d}t \\ &- \int_{-1}^{0}\int_{\mathrm{L}^i_t} f(\theta_i)u_i\chi|\nabla\theta_i|^2\mathrm{d}t \to 0 \quad \text{as } i \to \infty. \end{aligned}$$

Therefore, since χ has compact support in $\mathrm{B}_\mathrm{R}(0) \times (-1, 0)$, if we integrate (3.6) with respect to t on $[-1, 0]$, we have that

$$\int_{-1}^{0}\int_{\mathrm{L}^i_t} f(\theta_i)u_i\partial_t\chi\,\mathrm{d}t = -\int_{-1}^{0}\int_{\mathrm{L}^i_t} f(\theta_i)(\Delta u_i)\chi\,\mathrm{d}t + \epsilon_i, \tag{3.7}$$

where $\epsilon_i \to 0$ as $i \to \infty$. (Note that ϵ_i will depend on χ.) Integrating by parts on the right-hand side of (3.7) we get

$$\begin{aligned} \int_{-1}^{0}\int_{\mathrm{L}^i_t} f(\theta_i)u_i\partial_t\chi\,\mathrm{d}t = &\int_{-1}^{0}\int_{\mathrm{L}^i_t} f(\theta_i)\langle\nabla u_i, \nabla\chi\rangle\mathrm{d}t \\ &+ \int_{-1}^{0}\int_{\mathrm{L}^i_t} f'(\theta_i)\chi\langle\nabla u_i, \nabla\theta_i\rangle\mathrm{d}t + \epsilon_i. \end{aligned} \tag{3.8}$$

Again from (3.5) and the fact $\nabla\theta_i$ converges to zero in L^2 (in spacetime), we may absorb the second integral on the right-hand side of (3.8) into ϵ_i. We can then integrate by parts in the first term on the right-hand side of (3.8) and absorb another term involving $\nabla\theta_i$ by the same argument into ϵ_i to get

$$\int_{-1}^{0}\int_{\mathrm{L}^i_t} f(\theta_i)u_i\partial_t\chi\,\mathrm{d}t = -\int_{-1}^{0}\int_{\mathrm{L}^i_t} f(\theta_i)u_i\Delta\chi\,\mathrm{d}t + \epsilon_i. \tag{3.9}$$

Recall that $f(\overline{\theta}_1) = 1$ and $f(\overline{\theta}_2) = 0$. Since the u_i are uniformly bounded on the support of χ by (3.5), and we have good convergence away from the line $\ell = \mathrm{P}_1 \cap \mathrm{P}_2$, the contribution as we pass to the limit as $i \to \infty$ in (3.9) near the singular set ℓ is negligible. Therefore, we can pass to the limit in (3.9) along a subsequence, and get that the subsequential limit $\overline{u}_1$ of the u_i on P_1 satisfies

$$\int_{-1}^{0}\int_{\mathrm{P}_1} \overline{u}_1 \partial_t \chi \, \mathrm{d}t = -\int_{-1}^{0}\int_{\mathrm{P}_1} \overline{u}_1 \Delta \chi \, \mathrm{d}t.$$

This means that the limit $\overline{u}_1$ is a bounded distributional solution of the heat equation on P_1 so it follows that $\overline{u}_1$ is a classical solution on P_1.

Repeating the argument starting with the subsequence converging to $\overline{u}_1$ on P_1 and changing the choice of function f so that it takes the value 1 on $\overline{\theta}_2$ and 0 on $\overline{\theta}_1$ yields the result. □

Since the drift heat equation and usual heat equation are related by rescaling, one can apply Proposition 3.7 to sequences of solutions of the drift heat equation along rescaled mean curvature flows. In particular, suppose that we have a sequence of rescaled flows M^i_τ, for $\tau \in [-1, 0]$, converging to $\mathrm{P}'_1 \cup \mathrm{P}'_2$ weakly. Recall the weighted L^2-norm defined in (3.3) and let u_i be solutions of the drift heat equation on M^i_τ, with $\|u_i\|_{-1} \leq 1$, and such that the u_i have polynomial growth. Proposition 3.7 implies that after passing to a subsequence we have $u_i \to \overline{u}$, for a solution $\overline{u}$ of the drift heat equation on $\mathrm{P}'_1 \cup \mathrm{P}'_2$ for $\tau \in (-1, 0]$. We have the following additional information, saying that for $\tau > -1$ the weighted L^2-norms of the u_i cannot concentrate near E and near infinity.

Lemma **3.8.** — *Under the setup above we have*

$$\|\overline{u}\|_\tau = \lim_{i\to\infty} \|u_i\|_\tau \leq \liminf_{i\to\infty} \|u_i\|_{-1}, \tag{3.10}$$

for $\tau \in (-1, 0]$.

Proof. — The inequality $\|u_i\|_\tau \leq \|u_i\|_{-1}$ for $\tau > -1$ follows immediately from the monotonicity formula (2.2) and the observation that $(\partial_t - \Delta)u_i^2 \leq 0$.

For $r, \mathrm{R} > 0$ let us write $\mathrm{A}_{r,\mathrm{R}} = \mathrm{B}_\mathrm{R}(0) \setminus \mathrm{B}_r(\mathrm{E})$, where $\mathrm{E} = \mathrm{P}'_1 \cap \mathrm{P}'_2$ and $\mathrm{B}_r(\mathrm{E})$ denotes the r-neighbourhood of E. Let $\delta > 0$. From Proposition 3.7 we know that for any $r, \mathrm{R} > 0$, and $\tau \in [-1+\delta, 0]$ we have

$$\lim_{i\to\infty} \int_{\mathrm{M}^i_\tau \cap \mathrm{A}_{r,\mathrm{R}}} u_i^2 e^{-|\mathbf{x}|^2/4} = \int_{(\mathrm{P}'_1 \cup \mathrm{P}'_2)\cap \mathrm{A}_{r,\mathrm{R}}} \overline{u}^2 \, e^{-|\mathbf{x}|^2/4}.$$

To prove (3.10) it is enough to show that for any $\epsilon, \delta > 0$, there are $r, \mathrm{R} > 0$ such that for all i and $\tau \in [-1+\delta, 0]$ we have

$$\int_{\mathrm{M}^i_\tau \setminus \mathrm{A}_{r,\mathrm{R}}} u_i^2 \, e^{-|\mathbf{x}|^2/4} < \epsilon.$$

First, using the log Sobolev inequality due to Ecker [13, Theorem 3.4], we have a $p > 1$, depending on $\delta > 0$, such that

$$\left(\int_{M^i_\tau} |u_i|^{2p} e^{-|\mathbf{x}|^2/4}\right)^{1/p} < C,$$

for a uniform C, as long as $\tau \in [-1+\delta, 0]$. It follows using Hölder's inequality that, given $R > 0$, we have

$$\int_{M^i_\tau \setminus B_R(0)} |u_i|^2 \, e^{-|\mathbf{x}|^2/4} \leq C \left(\int_{M^i_\tau \setminus B_R(0)} e^{-|\mathbf{x}|^2/4}\right)^{1-1/p},$$

and so using the Euclidean area bounds for M^i_τ we can find an R (depending on δ, ϵ) such that

$$\int_{M^i_\tau \setminus B_R(0)} |u_i|^2 \, e^{-|\mathbf{x}|^2/4} \leq \frac{\epsilon}{2}, \tag{3.11}$$

for $\tau \in [-1+\delta, 0]$.

Viewing R (and δ) as fixed, the uniform bound in (3.5) implies that if r is sufficiently small (depending on ϵ, δ, R), then

$$\int_{M^i_\tau \cap B_R(0) \cap B_r(E)} |u_i|^2 e^{-|\mathbf{x}|^2/4} < \frac{\epsilon}{2}.$$

Combined with (3.11) this implies

$$\int_{M^i_\tau \setminus A_{r,R}} |u_i|^2 e^{-|\mathbf{x}|^2/4} < \epsilon,$$

as required. □

3.2. *Three annulus lemma.*

— A well-known method for controlling the growth of solutions of PDEs is the three annulus lemma, see for example [25]. In this subsection we prove a version of the three annulus lemma for solutions of the drift heat equation along the rescaled flow. We use an argument by contradiction, similar to Simon [25], based on the monotonicity of frequency shown by Colding–Minicozzi [9]. Related ideas are also applied in [8].

In this subsection we assume that L_τ is a rescaled Lagrangian mean curvature flow such that, along a sequence $\tau_i \to -\infty$, we have $L_{\tau_i} \rightharpoonup P_1 \cup P_2$. In addition we assume, as before, that the L_τ have uniformly bounded area ratios, uniformly bounded Lagrangian angle and are almost calibrated for $n \geq 3$.

Proposition **3.9.** — *For any $s \notin \mathbf{Z}$ there is a $T_0 = T_0(s) > 0$ with the following property. Suppose that u is a solution of the drift heat equation (3.1) on the rescaled flow M_τ for $\tau \in [-T -$*

$2, -\mathrm{T}]$ *with* $\mathrm{T} > \mathrm{T}_0$, *such that* u *has polynomial growth. If in addition we have that the weighted* L^2*-norm defined in* (3.3) *satisfies*

$$\|u\|_{-\mathrm{T}-1} \geq e^{s/2}\|u\|_{-\mathrm{T}},$$

then we also have

$$\|u\|_{-\mathrm{T}-2} \geq e^{s/2}\|u\|_{-\mathrm{T}-1}.$$

Proof. — We argue by contradiction. Suppose that there is a sequence of solutions u_i to (3.1) on intervals $[-\mathrm{T}_i - 2, -\mathrm{T}_i]$ with $\mathrm{T}_i \to \infty$, such that

$$\|u_i\|_{-\mathrm{T}_i-1} \geq e^{s/2}\|u_i\|_{-\mathrm{T}_i}, \tag{3.12}$$

but

$$\|u_i\|_{-\mathrm{T}_i-2} < e^{s/2}\|u_i\|_{-\mathrm{T}_i-1}. \tag{3.13}$$

By rescaling we can assume that $\|u_i\|_{-\mathrm{T}_i-1} = 1$ for all i. It follows from (3.13) that then $\|u_i\|_{-\mathrm{T}_i-2} < e^{s/2}$. We can apply Proposition 3.7 to time translations of the u_i, and along a subsequence we can extract a limit $\overline{u}$ satisfying the drift heat equation along a blow-down $\mathrm{P}_1' \cup \mathrm{P}_2'$ of the flow L_τ on the interval $(-2, 0]$. Using (3.10) we have

$$\|\overline{u}\|_{-1} = 1, \tag{3.14}$$

and at the same time the local uniform convergence of u_i to $\overline{u}$, together with (3.12) and (3.13), implies

$$\|\overline{u}\|_0 \leq e^{-s/2}, \qquad \|\overline{u}\|_\tau \leq e^{s/2} \quad \text{for all } \tau \in (-2, 0]. \tag{3.15}$$

By [9, Theorem 0.6] we know that $\log\|\overline{u}\|_\tau^2$ is convex. From (3.14) and (3.15) it follows that $\log\|\overline{u}\|_\tau^2$ is linear with slope s. By [9, Theorem 0.6] $\overline{u}$ must be homogeneous with degree s. By Lemma 3.5 the homogeneous solutions on any blow-down have integer degrees, so since $s \notin \mathbf{Z}$, we have a contradiction. ☐

We can use the three annulus lemma to extract the leading order behaviour of ancient solutions to the heat equation as follows.

Proposition **3.10.** — *Suppose that* u *is a non-zero solution of the drift heat equation along the rescaled flow* M_τ *for* $\tau \in (-\infty, 0]$, *with polynomial growth. Suppose that for some* $\mathrm{C}, d > 0$ *we have* $\|u\|_\tau^2 \leq \mathrm{C}e^{-d\tau}$ *for all* $\tau < 0$. *Let* $\tau_i \to -\infty$ *be integers. Up to choosing a subsequence we have the following. The translated (rescaled) flows* $\mathrm{L}_\tau^i = \mathrm{M}_{\tau-\tau_i}$ *converge weakly to a blow-down* $\mathrm{P}_1' \cup \mathrm{P}_2'$, *and the normalized translated solutions*

$$u_i(\tau) = \|u\|_{\tau_i}^{-1}\, u(\tau - \tau_i)$$

converge to a non-zero homogeneous solution $\overline{u}$ *of the drift heat equation on* $P_1' \cup P_2'$ *for* $\tau \in [-2, 0]$. *The convergence is locally smooth on* $[-2, 0]$ *away from* $P_1' \cap P_2'$, *and also in* L^2 *as in* (3.10).

Proof. — Let $s_0 > d$ for some $s_0 \notin \mathbf{Z}$. We claim that there is a $\tau_0 < 0$ such that we then have

$$\|u\|_{\tau-1} \leq e^{s_0/2}\|u\|_\tau \tag{3.16}$$

for all $\tau < \tau_0$. If this were not the case, then Proposition 3.9 would imply that in fact $\|u\|_{\tau-k} \geq e^{s_0/2}\|u\|_{\tau-k+1}$ for all integers $k > 0$ and some τ, but this would eventually contradict the assumption $\|u\|^2_{\tau-k} \leq Ce^{-d(\tau-k)}$.

The growth condition (3.16) together with the normalization of u_i implies that $\|u_i\|_{-3} \leq e^{3s_0/2}$. Using Proposition 3.7 we can extract a limit $\overline{u}$ along a subsequence on $P_1' \cup P_2'$. The convergence is locally smooth on $(-3, 0]$ away from $P_1' \cap P_2'$, and using Lemma 3.8 the convergence is in L^2 for $\tau \in [-2, 0]$ as required.

It remains to argue that $\overline{u}$ is homogeneous. For this note that Proposition 3.9 implies that for any $s \notin \mathbf{Z}$ one of the following must hold:

(1) $\|u\|_{\tau-1} \geq e^{s/2}\|u\|_\tau$ for all sufficiently negative integers τ,
(2) $\|u\|_{\tau-1} \leq e^{s/2}\|u\|_\tau$ for all sufficiently negative integers τ,

since if (1) holds for some sufficiently negative τ then it must hold for $\tau - k$ for all integers $k > 0$ by Proposition 3.9 as well. It follows that there is some $s_1 \in \mathbf{R}$ such that (1) holds for all $s < s_1$, and (2) holds for all $s > s_1$. We deduce that in the limit we have

$$\|\overline{u}\|_{-2} = e^{s_1/2}\|\overline{u}\|_{-1}, \quad \|\overline{u}\|_{-1} = e^{s_1/2}\|\overline{u}\|_0.$$

The convexity of $\log \|\overline{u}\|_\tau$ then implies that $\log \|\overline{u}\|_\tau$ is linear, from which it follows that $\overline{u}$ is homogeneous. ☐

We will also need the following variant of the three annulus lemma, similar to Donaldson–Sun [12, Proposition 3.11].

Proposition **3.11.** — *Let* $V_{\leq 1}$ *be the space of solutions of the drift heat equation along* M_τ *given by the span of* $1, \theta$ *and* $e^{-\tau/2}x_i$ *for the coordinate functions* x_i. *Let* $V \subset V_{\leq 1}$ *be any subspace and let* u *be a solution of the drift heat equation along* M_τ *with polynomial growth. Suppose that there is a constant* $C > 0$ *such that*

$$\|u\|^2_\tau \leq Ce^{-3\tau/2}, \quad \textit{for all } \tau < -1. \tag{3.17}$$

For any τ *let* $\Pi_\tau u := u - f$, *where* $f \in V$ *and* $u - f$ *is orthogonal to* V *at time* τ:

$$\langle u - f, g\rangle_\tau := \int_{M_\tau} (u-f)g\, e^{-|\mathbf{x}|^2/4} = 0, \quad \textit{for all } g \in V.$$

Given $s \notin \mathbf{Z}$, *there is a* $T_0 > 0$ *with the following property. If*

$$\|\Pi_{-T-1}u\|_{-T-1} \geq e^{s/2}\|\Pi_{-T}u\|_{-T}$$

for some $T > T_0$, *then*

$$\|\Pi_{-T-2}u\|_{-T-2} \geq e^{s/2}\|\Pi_{-T-1}u\|_{-T-1}.$$

Proof. — The proof is by contradiction, similar to that of Proposition 3.9. Suppose that we have a sequence $T_i \to \infty$ and corresponding u_i such that

$$\|\Pi_{-T_i-1}u_i\|_{-T_i-1} \geq e^{s/2}\|\Pi_{-T_i}u_i\|_{-T_i}, \tag{\textbf{3.18}}$$

and at the same time

$$\|\Pi_{-T_i-2}u_i\|_{-T_i-2} < e^{s/2}\|\Pi_{-T_i-1}u_i\|_{-T_i-1}. \tag{\textbf{3.19}}$$

Let $v_i = \Pi_{-T_i-2}u_i$, so that v_i is orthogonal to V at $\tau = -T_i - 2$. By scaling we can assume that $\|v_i\|_{-T_i-1} = 1$. It follows that $\|\Pi_{-T_i-1}v_i\|_{-T_i-1} \leq 1$, and so by (3.19) we have $\|v_i\|_{-T_i-2} \leq e^{s/2}$. We claim that for sufficiently large i we also have

$$\|v_i\|_{-T_i-3} \leq e^{4/5}\|v_i\|_{-T_i-2}. \tag{\textbf{3.20}}$$

If (3.20) did not hold, Proposition 3.9 would imply that for some constant $C > 0$ and for all integers $k > 3$ we would have $\|v_i\|_{-T_i-k} \geq C^{-1}e^{4k/5}$. At the same time $v_i = u_i - f_i$ for some $f_i \in V$, and since both u_i and f_i satisfy an estimate of the form (3.17), we get a contradiction. Thus (3.20) holds, and so we have a uniform bound $\|v_i\|_{-T_i-3} \leq e^{s/2+4/5}$.

Applying Proposition 3.7 we have that, along a subsequence and after time translations, the v_i converge to a limit solution $\overline{v}$ of the drift heat equation on a blow-down $P_1' \cup P_2'$ for $\tau \in (-3, 0]$. It follows using (3.10) that $\|\overline{v}\|_{-2} \leq e^{s/2}$ and $\|\overline{v}\|_{-1} = 1$.

We claim that we also have

$$\|\overline{v}\|_0 \leq e^{-s/2}, \tag{\textbf{3.21}}$$

in which case we will reach a contradiction just like in the proof of Proposition 3.9. Note that the new difficulty is that we only have the bound $\|\Pi_{-T_i}v_i\|_{-T_i} \leq e^{-s/2}$, and the norm of v_i can be larger than that of its projection $\Pi_{-T_i}v_i$.

To see that (3.21) holds we show that under our assumption that v_i is orthogonal to V at time $-T_i - 2$, we have that v_i is also approximately orthogonal to V at time $-T_i$. Let $g \in V$ and consider normalizations g_i of g such that $\|g_i\|_{-T_i} = 1$. By Proposition 3.10, after taking a further subsequence we can assume that the g_i converge to a homogeneous limit $\overline{g}$ on $P_1' \cup P_2'$, on the time interval $[-2, 0]$, satisfying the drift heat equation. We can apply the L^2-convergence (3.10) to $v_i \pm g_i$, together with our assumption $\langle v_i, g_i\rangle_{-T_i-2} = 0$ to find that $\langle \overline{v}, \overline{g}\rangle_{-2} = 0$. Since $\overline{g}$ is homogeneous, this implies that $\langle \overline{v}, \overline{g}\rangle_\tau = 0$ for all $\tau \in [-2, 0]$.

It follows from the L^2-convergence we have $\langle v_i, g_i\rangle_{-T_i} \to 0$. Since this applies to all $g \in V$, we find that

$$\lim_{i\to\infty} \frac{\|\Pi_{-T_i} v_i\|_{-T_i}}{\|v_i\|_{-T_i}} = 1,$$

and it follows that $\|\overline{v}\|_0 \leq e^{-s/2}$. This leads to a contradiction as discussed above. □

Let us use coordinates $x_1, \ldots, x_{2n-2}, z, w$ as in Section 2.4. Recall that a blow-down of our ancient rescaled flow M_τ along a sequence of scales $\tau_i \to -\infty$ is given by $P_1 \cup P_2$, where $P_1 \cap P_2 = \ell$ is a line, the coordinate w vanishes on $P_1 \cup P_2$ and $J\nabla z = \nabla w$. Without loss of generality we can assume that the τ_i are all integers. Let us write L^i_t for the corresponding sequence of flows for $t \in [-2, 0)$ converging weakly to $P_1 \cup P_2$. We then have the following dichotomy.

Proposition **3.12.** — *Either we have that $w = a + b\theta$ for some constants a, b along our flow L_t, so L_t is a translator, or up to choosing a subsequence of the τ_i we can find a sequence of linear functions $\phi_i \in \mathrm{Span}\{x_1, \ldots, x_{2n-2}, z\}$ with $\phi_i \to 0$ and a sequence $\sigma_i \to 0$ such that along the sequence L^i_t converging to $P_1 \cup P_2$ we have*

$$\sigma_i^{-1}(w - \phi_i) \to z\theta \quad \text{as } i \to \infty,$$

where the convergence is in L^2 and locally uniformly away from the line ℓ.

Proof. — Recall Definition 3.1 and let $V = \mathrm{Span}\{1, \theta, \tilde{x}_1, \ldots, \tilde{x}_{2n-2}, \tilde{z}\}$.

Suppose first that the rescaled height $\tilde{w}$ is in V. Note that $M_{\tau_i} \rightharpoonup P_1 \cup P_2$ and w vanishes on $P_1 \cup P_2$, but non-trivial linear combinations of $x_1, \ldots, x_{2n-2}, z$ do not vanish on $P_1 \cup P_2$. This implies that we must have $\tilde{w} = a + b\theta$ for constants a, b. By Proposition 2.10 the flow L_t is a translator.

Suppose now that $\tilde{w}$ is not in V. We apply Proposition 3.11 to $\tilde{w}$ along the flow with V as chosen. For any integer $k < 0$ let us write

$$\tilde{w}_k = \frac{\Pi_k \tilde{w}}{\|\Pi_k \tilde{w}\|_k}.$$

Note that by our assumption $\Pi_k \tilde{w} \neq 0$ for all k. Using Proposition 3.11 together with the argument in the proof of Proposition 3.10 we find that along a subsequence $k_i = \tau_i \to -\infty$, time translations of the $\tilde{w}_{k_i}$ converge to a homogeneous solution $\overline{w}$ of the drift heat equation on $P_1 \cup P_2$, which is orthogonal to the solutions $1, \theta, \tilde{x}_1, \ldots, \tilde{x}_{2n-2}, \tilde{z}$. At the same time the growth rate of $\overline{w}$ can be at most degree 1, so by Lemma 3.6 we must have $\overline{w} = ce^{-\tau/2} z\theta$ for a non-zero constant c.

To finish the proof we need to consider how the $\tilde{w}_k$ are related for different k. By definition we have

$$\tilde{w}_k = \gamma_k \tilde{w}_{k+1} + a_k + b_k\theta + F_k, \tag{3.22}$$

where γ_k, a_k, b_k are constants, and $F_k \in \mathrm{Span}\{\tilde{x}_1, \dots, \tilde{x}_{2n-2}, \tilde{z}\}$. Using Proposition 3.11, and arguing as in the proof of Proposition 3.10, we know that for any subsequence $k_j \to -\infty$ there is a further subsequence along which the $\tilde{w}_{k_j}$ (translated in time) converge to a homogeneous solution along some blow-down $P_1' \cup P_2'$, with degree 1 which is orthogonal to V. Since $\|\tilde{w}_k\|_k = 1$ for all k, it follows that $\gamma_k \to e^{-1/2}$ and $\|a_k\|_k, \|b_k\theta\|_k, \|F_k\|_k \to 0$ as $k \to -\infty$. Note that the norms $\|1\|_k$, $\|\theta\|_k$ are uniformly bounded away from 0 and ∞ for all k, using the fact that on all blow-downs $P_1' \cup P_2'$ the angle θ equals the same constants $\overline{\theta}_1, \overline{\theta}_2$ on the two subspaces P_1', P_2'. Therefore $a_k, b_k \to 0$.

Let us define the constants μ_k by $\mu_0 = 1$ and $\gamma_k = \mu_{k+1}/\mu_k$ for all sufficiently negative integers k. From (3.22) we have

$$\mu_k\tilde{w}_k = \mu_{k+1}\tilde{w}_{k+1} + \mu_k(a_k + b_k\theta) + \mu_k F_k,$$

and so

$$\tilde{w}_k = \mu_k^{-1}\tilde{w}_0 + \mu_k^{-1}\sum_{i=-1}^{k}\mu_i(a_i + b_i\theta) + \mu_k^{-1}\sum_{i=-1}^{k}\mu_i F_i. \tag{3.23}$$

Using that $\mu_{k+1}/\mu_k \to e^{-1/2}$ and $a_k, b_k \to 0$, it follows that

$$\left\| \mu_k^{-1}\sum_{i=-1}^{k}\mu_i(a_i + b_i\theta) \right\|_k \to 0. \tag{3.24}$$

At the same time, since $\tilde{w}_0$ is the normalized L^2-projection of w orthogonal to V (at time $\tau = 0$), we have $\tilde{w}_0 = c_0\tilde{w} + c_1 + c_2\theta + F$, where c_0, c_1, c_2 are constants with $c_0 \neq 0$ and $F \in \mathrm{Span}\{\tilde{x}_1, \dots, \tilde{x}_{2n-2}, \tilde{z}\}$. Using (3.23) and (3.24) we can write

$$\mu_k^{-1}c_0(\tilde{w} - \tilde{\phi}_k) = \tilde{w}_k + E_k,$$

where $\|E_k\|_k \to 0$ and $\tilde{\phi}_k$ is in the span of $\tilde{x}_1, \dots, \tilde{x}_{2n-2}, \tilde{z}$. Along our subsequence k_i we have $\tilde{w}_{k_i} \to \overline{w} = ce^{-\tau/2}z\theta$, and so as required we obtain a sequence L_t^i converging to $P_1 \cup P_2$, and $\sigma_i \neq 0$, $\phi_i \in \mathrm{Span}\{x_1, \dots, x_{2n-2}, z\}$ satisfying

$$\sigma_i^{-1}(w - \phi_i) \to z\theta.$$

It remains to show that $\sigma_i, \phi_i \to 0$. Note that since w vanishes on $P_1 \cup P_2$, on L_{-1}^i we have $\|w\|_{L_{-1}^i} \to 0$ as $i \to \infty$, while $\|x_j\|_{L_{-1}^i}$ and $\|z\|_{L_{-1}^i}$ are bounded away from 0 and ∞. It follows that if $\phi_i \not\to 0$, along a subsequence, then also $\sigma_i \not\to 0$ along this subsequence, and we would have $\sigma_i^{-1}\phi_i \to z\theta$ in L^2, but this contradicts the fact that on $P_1 \cup P_2$ the function $z\theta$ is L^2-orthogonal to $x_1, \dots, x_{2n-2}, z$. Therefore we must have $\phi_i \to 0$, which implies that $\|w - \phi_i\|_{L_{-1}^i} \to 0$ and so $\sigma_i \to 0$ as well. $\square$

In the next section we will show using a topological argument that the second alternative in Proposition 3.12 leads to a contradiction. This will complete the proof of our main result.

4. Linking argument

In this section we use a topological argument to rule out the second alternative in Proposition 3.12. Throughout this section we let $(-\infty, 0) \ni t \mapsto L_t \subset \mathbf{C}^n$ be a smooth, exact, ancient solution of LMCF with uniformly bounded area ratios and Lagrangian angle and which is almost calibrated for $n \geq 3$. Recall that for a positive sequence $\lambda_i \to 0$, we consider the sequence of parabolically rescaled flows

$$(-\infty, 0) \ni t \mapsto L^i_t = \lambda_i L_{\lambda_i^{-2} t}.$$

We assume that as $i \to \infty$ the flows $t \mapsto L^i_t$ converge weakly to the static flow $(-\infty, 0) \ni t \mapsto P_1 \cup P_2$, where P_1, P_2 are n-dimensional Lagrangian subspaces meeting along a line ℓ. We write $\overline{\theta}_j$ for the Lagrangian angles of P_j as before, where $\overline{\theta}_1 = -\overline{\theta}_2$.

Since the L^i_t are exact, they admit primitives β_i of the Liouville form as in Definition 2.2. We have the following (see Neves [23, Proposition 6.1]).

Lemma **4.1.** — *We can choose the primitives β_i along the flows L^i_t such that*

$$(\partial_t - \Delta)(\beta_i + 2t\theta_i) = 0.$$

Since $|\nabla \beta_i| = |\mathbf{x}^\perp|$ and L^i_{-1} converges to the union $P_1 \cup P_2$ locally smoothly away from ℓ, we have that $\beta_i|_{L^i_{-1}} \to \overline{\beta}_j$ as $i \to \infty$ locally smoothly on each plane P_j away from ℓ, for suitable constants $\overline{\beta}_j$. Similarly $\theta_i \to \overline{\theta}_j$ locally smoothly on P_j away from ℓ, as $i \to \infty$. Given this, we make the following definition.

Definition **4.2.** — *Since the L^i_t are exact, and almost calibrated for $n \geq 3$, by [24, Theorem 4.2] there exists a set $\mathcal{E} \subseteq (-2, 0)$ of measure zero so that whenever $s' \in (-2, 0) \setminus \mathcal{E}$, we have two distinct connected components $\Sigma^i_{1,s'}$, $\Sigma^i_{2,s'}$ of $B_3(0) \cap L^i_{s'}$ (after possibly passing to a subsequence) intersecting $B_2(0)$ and converging (as Radon measures) to the planes P_1, P_2 respectively in $B_2(0)$. Note that there might be more connected components of $B_3(0) \cap L^i_{s'}$, but the components $\Sigma^i_{1,s'}$, $\Sigma^i_{2,s'}$ are uniquely determined, and the remaining components converge to zero as Radon measures.*

Let $s_1 \in (-1/2, 0) \setminus \mathcal{E}$ so that $\overline{b}_1 \neq \overline{b}_2$, where

$$\overline{b}_j := \cos(\overline{\beta}_j - 2(1 + s_1)\overline{\theta}_j). \tag{4.1}$$

This is always possible since $\overline{\theta}_1 \neq \overline{\theta}_2$ and $\mathcal{E}$ has measure 0. We then let $\Sigma^i_j = \Sigma^i_{j,s_1} \cap B_2(0)$ in the notation above.

4.1. *Approximate solutions of the heat equation.* — We now prove our first key result, which provides solutions of the heat equation, which on the two components Σ^i_j, for $j = 1, 2$, approximate $\overline{b}_j z$ pointwise.

Proposition **4.3.** — *Recall the notation of Definition 4.2. Let*

$$B_i = \cos(\beta_i + 2(t - s_1)\theta_i) \tag{4.2}$$

and let h_i be the solution of the heat equation along L^i_t with polynomial growth (see Proposition A.5) such that at $t = -1$ we have $h_i = B_i z$. Then

$$\lim_{i\to\infty} \sup_{\Sigma^i_j \cap B_2(0)} |\overline{b}_j z - h_i| = 0 \quad \textit{for } j = 1, 2, \tag{4.3}$$

where $\overline{b}_j$ are the constants given in (4.1).

Remark **4.4.** — The idea of Proposition 4.3 is that $\overline{b}_j z$ defines a solution of the heat equation on the union $P_1 \cup P_2$, and we try to find solutions along the flows L^i_t which approximate it. Along the flows we do not have two components at each time converging to the two planes, so we cannot directly define a function like $\overline{b}_j z$. However, in the limit as $i \to \infty$, the functions B_i approximate the constants $\overline{b}_j$ on the two planes P_j. $B_i z$ only approximately satisfies the heat equation as $i \to \infty$ but should stay close to a genuine solution h_i with the same initial condition. In addition we have a good pointwise estimate for the difference between B_i and the constants $\overline{b}_j$ on the two components Σ^i_j at the specific time $t = s_1$, as in Neves [24, Theorem 4.2].

Proof. — Let

$$E_i = B_i z - h_i. \tag{4.4}$$

Our goal is to show that E_i is small as i becomes large. At $t = -1$ we have $E_i = 0$, so we compute the evolution of E_i. We have

$$\nabla(\beta_i + 2(t - s_1)\theta_i) = J(\mathbf{x}^\perp + 2(s_1 - t)\mathbf{H}),$$

and since $\beta_i + 2(t - s_1)\theta_i$ satisfies the heat equation we get

$$(\partial_t - \Delta)B_i = |\mathbf{x}^\perp + 2(s_1 - t)\mathbf{H}|^2 B_i.$$

Since $|B_i| \leq 1$, at $t = -1$ we have $|h_i| \leq (1 + |\mathbf{x}|^2)$. Using the maximum principle (see Ecker–Huisken [15, Corollary 1.1], which applies to subsolutions that satisfy the monotonicity formula) and the evolution equation $(\partial_t - \Delta)(1 + |\mathbf{x}|^2) = -2n$ we find that $|h_i| \leq C(1 + |\mathbf{x}|^2)$ for $t \in [-1, 0)$ for a dimensional constant $C > 0$. Below the constant

C may change from line to line but is independent of i, t. In particular we also have $|E_i| \leq C(1+|\mathbf{x}|^2)$.

Since z and h_i satisfy the heat equation along the flow L^i_t, we have the evolution equation

$$(\partial_t - \Delta)E_i = |\mathbf{x}^\perp + 2(s_1 - t)\mathbf{H}|^2 B_i z - 2\langle \nabla B_i, \nabla z\rangle.$$

We deduce that

$$|(\partial_t - \Delta)E_i| \leq |\mathbf{x}^\perp + 2(s_1 - t)\mathbf{H}|^2(1+|\mathbf{x}|^2) + 2|\mathbf{x}^\perp + 2(s_1 - t)\mathbf{H}|.$$

From this, together with the estimate $|E_i| \leq C(1+|\mathbf{x}|^2)$, we get

$$\begin{aligned}(\partial_t - \Delta)E_i^2 &\leq 2|E_i||(\partial_t - \Delta)E_i| - 2|\nabla E_i|^2 \\ &\leq 2|E_i||\mathbf{x}^\perp + 2(s_1 - t)\mathbf{H}|^2(1+|\mathbf{x}|^2) + 4|E_i|\,|\mathbf{x}^\perp + 2(s_1 - t)\mathbf{H}| \\ &\leq E_i^2 + C(|\mathbf{x}^\perp|^2 + (s_1 - t)^2|\mathbf{H}|^2)(1+|\mathbf{x}|^4), \end{aligned} \tag{4.5}$$

where we also used the estimate $4|E_i|b \leq E_i^2 + 4b^2$ to get the last line.

Using that θ_i satisfies the heat equation and $|\nabla\theta_i| = |\mathbf{H}|$, as well as $(\partial_t - \Delta)|\mathbf{x}|^4 \leq 0$, we also have

$$\begin{aligned}(\partial_t - \Delta)(1+|\mathbf{x}|^4)(t+1)\theta_i^2 &\leq (1+|\mathbf{x}|^4)\theta_i^2 - 2(1+|\mathbf{x}|^4)(t+1)|\mathbf{H}|^2 \\ &\quad - 4(t+1)\theta_i\langle \nabla\theta_i, \nabla|\mathbf{x}|^4\rangle \\ &\leq -(1+|\mathbf{x}|^4)(t+1)|\mathbf{H}|^2 + C(1+|\mathbf{x}|^4)\theta_i^2, \end{aligned}$$

for $t \in (-1, 0)$.

Let $\kappa > 0$ be small. Combining (4.5) with the previous inequality, for $t \in (-1, 0)$ we have

$$\begin{aligned}&(\partial_t - \Delta)\Big(e^{-t}E_i^2 + (1+|\mathbf{x}|^4)\kappa(t+1)\theta_i^2\Big) \\ &\quad\leq C(|\mathbf{x}^\perp|^2 + (s_1 - t)^2|\mathbf{H}|^2)(1+|\mathbf{x}|^4) \\ &\qquad + \kappa C(1+|\mathbf{x}|^4)\theta_i^2 - \kappa(t+1)(1+|\mathbf{x}|^4)|\mathbf{H}|^2. \end{aligned} \tag{4.6}$$

Suppose that $\mathbf{x}_0 \in B_2(0) \cap L^i_{s_1}$ and denote by $\rho_{\mathbf{x}_0, s_1}$ the backwards heat kernel centred at $(\mathbf{x}_0, s_1)$. For $t \in (-1, s_1)$ we have from (4.6), using the monotonicity formula (2.2), that

$$
\begin{aligned}
\textbf{(4.7)}\qquad & \frac{d}{dt}\int_{\mathrm{L}_t^i}(e^{-t}\mathrm{E}_i^2+(1+|\mathbf{x}|^4)\kappa(t+1)\theta_i^2)\rho_{\mathbf{x}_0,s_1} \\
&\le \int_{\mathrm{L}_t^i}\mathrm{C}(|\mathbf{x}^\perp|^2+(s_1-t)^2|\mathbf{H}|^2)(1+|\mathbf{x}|^4)\rho_{\mathbf{x}_0,s_1} \\
&\quad +\kappa\mathrm{C}\int_{\mathrm{L}_t^i}(1+|\mathbf{x}|^4)\theta_i^2\rho_{\mathbf{x}_0,s_1} \\
&\quad -\kappa(t+1)\int_{\mathrm{L}_t^i}|\mathbf{H}|^2(1+|\mathbf{x}|^4)\rho_{\mathbf{x}_0,s_1}.
\end{aligned}
$$

Integrating (4.7) with respect to t from -1 to s_1 yields:

$$
\begin{aligned}
\textbf{(4.8)}\qquad & e^{-s_1}\mathrm{E}_i^2(\mathbf{x}_0,s_1)+(1+|\mathbf{x}_0|^4)\kappa(s_1+1)\theta_i^2(\mathbf{x}_0,s_1) \\
&\le \int_{\mathrm{L}_{-1}^i}e^{-t}\mathrm{E}_i^2\rho_{\mathbf{x}_0,s_1}+\int_{-1}^{s_1}\int_{\mathrm{L}_t^i}\mathrm{C}(|\mathbf{x}^\perp|^2+(s_1-t)^2|\mathbf{H}|^2)(1+|\mathbf{x}|^4)\rho_{\mathbf{x}_0,s_1}\mathrm{d}t \\
&\quad +\kappa\mathrm{C}\int_{-1}^{s_1}\int_{\mathrm{L}_t^i}(1+|\mathbf{x}|^4)\theta_i^2\rho_{\mathbf{x}_0,s_1} \\
&\quad -\kappa\int_{-1}^{s_1}\int_{\mathrm{L}_t^i}(t+1)|\mathbf{H}|^2(1+|\mathbf{x}|^4)\rho_{\mathbf{x}_0,s_1}\mathrm{d}t.
\end{aligned}
$$

Note that $\mathrm{E}_i=0$ at $t=-1$ by the definition of h_i, and hence the first term on the right-hand side in (4.8) vanishes.

We now estimate the second term in (4.8). Note that for $t\in[-1,s_1-\kappa]$ we have

$$
(1+|\mathbf{x}|^4)\rho_{x_0,s_1}(\mathbf{x},t)\le \mathrm{C}_\kappa\rho_{0,0}(\mathbf{x},t)
$$

for some κ-dependent constant $\mathrm{C}_\kappa>0$, since $s_1<0$ and thus $\rho_{\mathbf{x}_0,s_1}$ will decay more rapidly at infinity than $\rho_{0,0}$ for any $t\in[-1,s_1-\kappa]$. Therefore,

$$
\begin{aligned}
\textbf{(4.9)}\qquad & \int_{-1}^{s_1-\kappa}\int_{\mathrm{L}_t^i}\mathrm{C}(|\mathbf{x}^\perp|^2+(s_1-t)^2|\mathbf{H}|^2)(1+|\mathbf{x}|^4)\rho_{\mathbf{x}_0,s_1}\mathrm{d}t \\
&\le \mathrm{C}_\kappa\int_{-1}^{0}\int_{\mathrm{L}_t^i}(|\mathbf{x}^\perp|^2+|\mathbf{H}|^2)\rho_{0,0}\mathrm{d}t.
\end{aligned}
$$

We now notice that

$$\begin{aligned}(\mathbf{4.10})\qquad &\int_{s_1-\kappa}^{s_1}\int_{\mathrm{L}_t^i} \mathrm{C}(s_1-t)^2|\mathbf{H}|^2(1+|\mathbf{x}|^4)\rho_{\mathbf{x}_0,s_1}\,\mathrm{d}t\\ &\le \mathrm{C}\kappa^2\int_{s_1-\kappa}^{s_1}\int_{\mathrm{L}_t^i}|\mathbf{H}|^2(1+|\mathbf{x}|^4)\rho_{\mathbf{x}_0,s_1}\,\mathrm{d}t\\ &\le \frac{\kappa}{2}\int_{s_1-\kappa}^{s_1}\int_{\mathrm{L}_t^i}(t+1)|\mathbf{H}|^2(1+|\mathbf{x}|^4)\rho_{\mathbf{x}_0,s_1}\,\mathrm{d}t,\end{aligned}$$

where $\mathrm{C}>0$ is a constant and κ is chosen sufficiently small that $(t+1)\ge 2\mathrm{C}\kappa$ for $t\in[s_1-\kappa,s_1]$. Equation (4.10) shows that the integral on the left-hand side of the inequality can be compensated for using the last term in (4.8).

Our remaining concern is

$$\begin{aligned}(\mathbf{4.11})\qquad \int_{s_1-\kappa}^{s_1}\int_{\mathrm{L}_t^i}|\mathbf{x}^\perp|^2(1+|\mathbf{x}|^4)\rho_{\mathbf{x}_0,s_1}\,\mathrm{d}t &\le \int_{s_1-\kappa}^{s_1}\int_{\mathrm{L}_t^i}|\mathbf{x}|^2(1+|\mathbf{x}|^4)\rho_{\mathbf{x}_0,s_1}\,\mathrm{d}t\\ &= \int_{s_1-\kappa}^{s_1}\int_{\mathrm{L}_t^i\cap \mathrm{B}_{\kappa^{-1/10}}(0)}|\mathbf{x}|^2(1+|\mathbf{x}|^4)\rho_{\mathbf{x}_0,s_1}\,\mathrm{d}t\\ &\quad+\int_{s_1-\kappa}^{s_1}\int_{\mathrm{L}_t^i\setminus \mathrm{B}_{\kappa^{-1/10}}(0)}|\mathbf{x}|^2(1+|\mathbf{x}|^4)\rho_{\mathbf{x}_0,s_1}\,\mathrm{d}t.\end{aligned}$$

The first integral can clearly be estimated as

$$\begin{aligned}(\mathbf{4.12})\qquad &\int_{s_1-\kappa}^{s_1}\int_{\mathrm{L}_t^i\cap \mathrm{B}_{\kappa^{-1/10}}(0)}|\mathbf{x}|^2(1+|\mathbf{x}|^4)\rho_{\mathbf{x}_0,s_1}\,\mathrm{d}t\\ &\le 2\kappa^{-6/10}\int_{s_1-\kappa}^{s_1}\int_{\mathrm{L}_t^i\cap \mathrm{B}_{\kappa^{-1/10}}(0)}\rho_{\mathbf{x}_0,s_1}\,\mathrm{d}t\le \mathrm{C}\kappa^{2/5}\end{aligned}$$

for some constant $\mathrm{C}>0$, using the uniform area bounds for L_t^i. Using the area bounds again for $t\in[-1,s_1]$, we can estimate our remaining spacetime integral by the integral over an n-plane P for κ sufficiently small:

$$\begin{aligned}(\mathbf{4.13})\qquad &\int_{s_1-\kappa}^{s_1}\int_{\mathrm{L}_t^i\setminus \mathrm{B}_{\kappa^{-1/10}}(0)}|\mathbf{x}|^2(1+|\mathbf{x}|^4)\rho_{\mathbf{x}_0,s_1}\,\mathrm{d}t\\ &\le \mathrm{C}_1\int_{-\kappa}^{0}\int_{\mathrm{P}\setminus \mathrm{B}_{\kappa^{-1/10}}(0)}|\mathbf{x}|^2(1+|\mathbf{x}|^4)\rho_{\mathbf{x}_0,0}\,\mathrm{d}t\\ &\le \mathrm{C}_2e^{-1/\kappa}\end{aligned}$$

for constants $\mathrm{C}_1,\mathrm{C}_2>0$.

Combining (4.9)–(4.13) shows that, for κ sufficiently small, we have

$$\begin{aligned}\textbf{(4.14)}\qquad &\int_{-1}^{s_1}\int_{\mathrm{L}_t^i} \mathrm{C}(|\mathbf{x}^\perp|^2+(s_1-t)^2|\mathbf{H}|^2)(1+|\mathbf{x}|^4)\rho_{\mathbf{x}_0,s_1}\mathrm{d}t\\ &\le \mathrm{C}_\kappa\int_{-1}^{0}\int_{\mathrm{L}_t^i}(|\mathbf{x}^\perp|^2+|\mathbf{H}|^2)\rho_{0,0}\mathrm{d}t\\ &\quad+\mathrm{C}\kappa^{2/5}+\frac{\kappa}{2}\int_{s_1-\kappa}^{s_1}\int_{\mathrm{L}_t^i}(t+1)|\mathbf{H}|^2(1+|\mathbf{x}|^4)\rho_{\mathbf{x}_0,s_1}\mathrm{d}t\end{aligned}$$

for some constant $\mathrm{C}>0$ and a constant $\mathrm{C}_\kappa>0$ depending on κ.

Noting also that θ_i^2 is uniformly bounded, we may therefore combine (4.8) and (4.14) to obtain

$$\textbf{(4.15)}\qquad \mathrm{E}_i^2(\mathbf{x}_0,s_1)\le \mathrm{C}_\kappa\int_{-1}^{0}\int_{\mathrm{L}_t^i}(|\mathbf{x}^\perp|^2+|\mathbf{H}|^2)\rho_{0,0}\mathrm{d}t+\mathrm{C}\kappa^{2/5},$$

if $\kappa>0$ is sufficiently small.

For fixed $\kappa>0$, the first term on the right-hand side of (4.15) converges to zero as $i\to\infty$, as in [23, Lemma 5.4]. It follows that for any $\kappa>0$ we can choose i sufficiently large so that

$$\mathrm{E}_i^2(\mathbf{x}_0,s_1)\le 2\mathrm{C}\kappa^{2/5}.$$

By definition of E_i in (4.4), and the fact that $\mathrm{B}_i=\cos(\beta_i)$ at $t=s_1$, we have that

$$\textbf{(4.16)}\qquad \lim_{i\to\infty}\sup_{\mathrm{B}_2(0)\cap\mathrm{L}_{s_1}^i}|\cos(\beta_i)z-h_i|=0.$$

As in [23, Lemma 7.3], we now use that the limiting behaviour of the functions B_i in (4.2) as $i\to\infty$ is t-independent. More precisely, for all ϕ with compact support in $\mathrm{B}_2(0)$ and $f\in\mathrm{C}^2(\mathbf{R})$ we have

$$\textbf{(4.17)}\qquad \lim_{i\to\infty}\int_{\mathrm{L}_{s_1}^i}f(\mathrm{B}_i)\phi\,d\mathcal{H}^n=\lim_{i\to\infty}\int_{\mathrm{L}_{-1}^i}f(\mathrm{B}_i)\phi\,d\mathcal{H}^n.$$

On $\mathrm{L}_{s_1}^i$ we have $\mathrm{B}_i=\cos(\beta_i)$ and so we have the pointwise bound $|\nabla\mathrm{B}_i|\le|\mathbf{x}^\perp|$. Using the Poincaré type inequality [23, Proposition A.1], we deduce that there are constants $\hat{b}_1$, $\hat{b}_2$ such that $\sup_{\Sigma_j^i\cap\mathrm{B}_2(0)}|\mathrm{B}_i-\hat{b}_j|\to 0$ as $i\to\infty$. At the same time from (4.17) we find that $\hat{b}_j=\overline{b}_j$ for the constants in (4.1), since on L_{-1}^i we have $\mathrm{B}_i=\cos(\beta_i-2(1+s_1)\theta_i)$. Note that by construction on L_{-1}^i we have $\beta_i\to\overline{\beta}_j$ and $\theta_i\to\overline{\theta}_j$ on the plane P_j, locally smoothly away from ℓ. It follows then from (4.16) that $\lim_{i\to\infty}\sup_{\Sigma_j^i\cap\mathrm{B}_2(0)}|\overline{b}_jz-h_i|=0$, as required. ☐

4.2. *The linking argument.* — Continuing the setup from the previous subsection, we now show that indeed the second possibility in Proposition 3.12 leads to a contradiction, if our flow $t \mapsto L_t$ is smooth and embedded.

Proposition **4.5.** — *Suppose that we have* $\phi_i \in \mathrm{Span}\{x_1, \ldots, x_{2n-2}, z\}$ *with* $\phi_i \to 0$ *and a sequence* $\lambda_i \to 0$ *such that along the sequence* L^i_t *we have*

$$u_i = \lambda_i^{-1}(w - \phi_i) \to z\theta \quad \text{on } P_1 \cup P_2 \text{ as } i \to \infty, \tag{4.18}$$

where the convergence is in L^2 *and locally uniform away from* ℓ. *Then for sufficiently large* i *the flow* L^i_t *is not embedded.*

Remark **4.6.** — Recall that Σ^i_j are the components of $B_2(0) \cap L^i_{s_1}$, as in Definition 4.2, and let us suppose for simplicity that $C^i_j = \Sigma^i_j \cap \partial B_1(0)$ are smooth $(n-1)$-dimensional submanifolds of the sphere (which can always be done by changing the radius of the ball slightly if necessary). The key to the argument is to show that the submanifolds C^i_j in the $(2n-1)$-sphere are linked for i sufficiently large, which implies that the Σ^i_j intersect in $B_2(0)$. Then $L^i_{s_1}$ cannot be embedded.

Proof. — Since θ equals the distinct constants $\overline{\theta}_1, \overline{\theta}_2$ on the planes P_1, P_2, by modifying the λ_i and adding multiples of z to the ϕ_i, we can assume for simplicity that

$$u_i \to \overline{b}_j z \quad \text{on } P_j \text{ as } i \to \infty,$$

where $\overline{b}_j$ are given in (4.1). The convergence is in L^2, and locally uniform away from the line ℓ. We also assume without loss of generality that $\lambda_i > 0$.

Recall the notation of Definition 4.2 and Proposition 4.3. At $t = -1$ we have $h_i = B_i z$ by definition, and the function B_i converges in L^2 and locally smoothly away from ℓ to the constants $\overline{b}_j$ on the two planes. It follows that at $t = -1$ we have $\|u_i - h_i\|_{L^2} = 0$. The monotonicity formula applied with points $(\mathbf{x}_0, s_1)$ for different $\mathbf{x}_0 \in B_2(0)$ then implies

$$\lim_{i\to\infty} \sup_{\Sigma^i_j} |u_i - h_i| = 0.$$

Applying Proposition 4.3 then yields

$$\lim_{i\to\infty} \sup_{\Sigma^i_j} |u_i - \overline{b}_j z| = 0.$$

We deduce that, given any $\epsilon > 0$, once i is sufficiently large we will have

$$|w - \phi_i - \lambda_i \overline{b}_j z| < \epsilon \lambda_i \quad \text{on } \Sigma^i_j. \tag{4.19}$$

Suppose without loss of generality that $\overline{b}_1 > \overline{b}_2$ and choose

$$0 < \epsilon < |\overline{b}_1 - \overline{b}_2|/100$$

in (4.19). Let

$$\bar{b}_0 = (\bar{b}_1 + \bar{b}_2)/2$$

and, recalling that $\phi_i \in \mathrm{Span}\{x_1, \dots, x_{2n-2}, z\}$, define half-spaces

$$\mathcal{H}^i_+ = \{(x_1, \dots, x_{2n-2}, z, w) \in \mathbf{C}^n : w > \phi_i + \lambda_i \bar{b}_0 z\},$$
$$\mathcal{H}^i_- = \{(x_1, \dots, x_{2n-2}, z, w) \in \mathbf{C}^n : w < \phi_i + \lambda_i \bar{b}_0 z\}.$$

The inequality (4.19) implies that, for all i sufficiently large,

$$\begin{aligned} &(\Sigma^i_1 \cap \{z > 1/2\}) \subseteq \mathcal{H}^i_+ \quad \text{and} \quad (\Sigma^i_2 \cap \{z > 1/2\}) \subseteq \mathcal{H}^i_-, \\ &(\Sigma^i_1 \cap \{z < -1/2\}) \subseteq \mathcal{H}^i_- \quad \text{and} \quad (\Sigma^i_2 \cap \{z < -1/2\}) \subseteq \mathcal{H}^i_+. \end{aligned} \tag{4.20}$$

In other words, the relative positions of the components Σ^i_j in terms of the halfspaces $\mathcal{H}^i_\pm$ must switch as we pass from $z > 1/2$ to $z < -1/2$.

We can choose $R = 1 + \delta$ for $\delta \geq 0$ small such that

$$C^i_j = \Sigma^i_j \cap \partial B_R(0) \tag{4.21}$$

are smooth. Our aim now is to show that the submanifolds C^i_j in $\partial B_R(0)$ are linked for sufficiently large i, which will imply that the Σ^i_j intersect in $B_R(0)$.

Consider the two points p_-, p_+ whose only non-zero entries are $\pm R$ in the z-component in coordinates $(x_1, \dots, x_{2n-2}, z, w)$ on $\mathbf{C}^n$. So p_-, p_+ lie on $\ell \cap \partial B_R(0)$ where $\ell = P_1 \cap P_2$. Since the Σ^i_j converge smoothly to P_j away from the singular line ℓ as $i \to \infty$, we can assume that outside of $B_{1/20}(p_\pm)$ the submanifolds C^i_j are smooth perturbations of $P_j \cap \partial B_R(0)$.

Any connected components of the C^i_j contained entirely inside $B_{1/10}(p_\pm)$ must lie in different half-spaces $\mathcal{H}^i_\pm$ by (4.20) for i sufficiently large, and so do not contribute to the linking number of the C^i_j. We may therefore discard these components, if there are any, and assume from now on that the C^i_j are connected.

For $j = 1, 2$, let $\tilde{P}^i_j$ be the graph of $w = \phi_i + \lambda_i \bar{b}_j z$ over P_j. Since $\phi_i, \lambda_i \to 0$, the $\tilde{P}^i_j$ are small perturbations of the P_j for i sufficiently large. Moreover, since $\bar{b}_1 \neq \bar{b}_2$ by (4.1) we have that the $\tilde{P}^i_j$ intersect transversely at the origin. Hence, the spheres

$$\tilde{C}^i_j = \tilde{P}^i_j \cap \partial B_R(0) \tag{4.22}$$

have linking number 1.

We now claim that, for i sufficiently large, the submanifolds C^i_j in (4.21) can be deformed to the $\tilde{C}^i_j$ in (4.22) without any crossings. Outside of the balls $B_{1/20}(p_\pm)$ this is clear since there the C^i_j are smooth perturbations of the $\tilde{C}^i_j$. At the same time, for i sufficiently large, inside the balls $B_{1/10}(p_\pm)$ the pairs of submanifolds $\{C_j^i, \tilde{C}_j^i\}$ are contained in disjoint half-spaces for $j = 1, 2$ by (4.20), so the submanifolds in each pair can be deformed to coincide without intersecting the submanifolds in the other pair.

We conclude that the linking number of the submanifolds C_j^i in (4.21) is therefore also 1 for sufficiently large i, which implies that the flow is not embedded. □

5. Proof of main theorem

We first show that combining the results from Section 3.2 and Section 4.2 yields a proof of the main theorem:

Proof of Theorem 1.1. — We can assume that the ancient flow $\mathcal{M}$ is defined for $t < 0$. We note that the assumption that the flow has a blow-down given by the static union of the planes $P_1 \cup P_2$ implies that the entropy is bounded above by 2. This implies that if the flow has an immersed point $(\mathbf{x}_0, t_0)$, then the monotonicity formula yields that the flow is backwards self-similar around $(\mathbf{x}_0, t_0)$, i.e. the flow is given by the static flow $(\mathcal{M}_{P_1 \cup P_2} + (\mathbf{x}_0, t_0)) \cap \{t < 0\}$.

We can thus assume that $\mathcal{M}$ is embedded. Combining Proposition 3.12 and Proposition 4.5 yields the statement. □

Since in many geometric applications it is essential to classify not only smooth ancient solutions to mean curvature flow, but also more general Brakke flows arising as limit flows, we also record the following extension of Theorem 1.1.

Theorem **5.1.** — *Let* $P_1, P_2 \subset \mathbf{C}^2$ *be Lagrangian subspaces which intersect along a line* ℓ *through* 0. *Let* $\mathcal{M}$ *be an ancient* 2*-dimensional Brakke flow which is the (weak) limit of smooth, zero-Maslov, exact Lagrangian mean curvature flows* $(L_t^i)_{-R_i^2 < t < 0}$ *defined on* $B(0, R_i) \subset \mathbf{C}^n$*, where* $R_i \to \infty$*, with uniformly bounded variation of the Lagrangian angle and uniformly bounded area ratios. If* $\mathcal{M}$ *has a blow-down at* $-\infty$ *given by the static flow consisting of the union of the planes* $P_1 \cup P_2$*, then* $\mathcal{M}$ *is a smooth translator.*

Proof. — We again assume that $\mathcal{M}$ is defined and non-vanishing for $t < 0$. Note that the assumptions imply that $\mathcal{M}$ has uniformly bounded area ratios and is unit regular, meaning that every point of Gaussian density one has a space-time neighbourhood where the flow is smooth. Furthermore, as in the proof of Theorem 1.1, it follows that the entropy is bounded above by 2. Assume now that there is a point $(\mathbf{x}_0, t_0)$ where the Gaussian density of $\mathcal{M}$ is 2. Then as above we see that $\mathcal{M} = (\mathcal{M}_{P_1 \cup P_2} + (\mathbf{x}_0, t_0)) \cap \{t < 0\}$ (using unit regularity to conclude that neither of the two planes can vanish before $t = 0$).

We can thus assume that all Gaussian density ratios of $\mathcal{M}$ are strictly less than 2. Neves structure theory [23] then implies that we obtain uniform local curvature bounds along the sequence $(L_t^i)_{-R_i^2 < t < 0}$, and thus the convergence is smooth. This yields that $\mathcal{M}$

is a smooth, ancient, zero-Maslov, exact Lagrangian mean curvature flow with uniformly bounded variation of the Lagrangian angle and uniformly bounded area ratios. Theorem 1.1 then implies the statement. ☐

Remark **5.2.** — In the previous theorem one can also allow that the flows $(\mathrm{L}^i_t)_{-\mathrm{R}_i^2<t<0}$ are defined on the Riemannian manifolds $(\mathrm{B}(p_i,\mathrm{R}_i),g_i)$ where $\mathrm{B}(p_i,\mathrm{R}_i)\subset\mathbf{R}^{2n}$ are geodesic balls with respect to g_i and g_i is a sequence of Calabi–Yau metrics on $\mathrm{B}(p,\mathrm{R}_i)$ converging smoothly to the standard Euclidean metric on $\mathbf{C}^n$.

Acknowledgements

This project grew out of discussions at the AIM workshop “Stability in mirror symmetry” in December 2020. We are grateful to the referees for many helpful suggestions which improved the exposition of the paper.

Funding information. — JDL and FS were partially supported by a Leverhulme Trust Research Project Grant RPG-2016-174. GSz was supported in part by NSF grant DMS-1906216.

Competing Interests

The authors declare no competing interests.

Publisher’s Note

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Appendix: Smoothly immersed Brakke flows

In this appendix we give a definition of smoothly immersed Brakke flows and record that the weighted version of Huisken's monotonicity formula, see [14, Theorem 4.13], can be extended to weights with polynomial growth. We recall Remark 1.2 for the relation with properly immersed mean curvature flows.

Definition **A.1** *(Smoothly immersed Brakke flows). — We say that an n-dimensional (integral) Brakke flow $\mathcal{M}$ in $\mathbf{R}^{n+m}$, defined on a time interval $\mathrm{I} \subset \mathbf{R}$, is* smoothly immersed *if around every point $(\mathbf{x}, t) \in \mathbf{R}^{n+m} \times \mathrm{I}$ in the support* $\operatorname{supp} \mathcal{M}$ *of the Brakke flow there exists an open space-time neighbourhood* U*, such that the flow can be represented by a smoothly immersed mean curvature flow in* U*. Note that this includes that the multiplicity $\theta(\mathbf{x}, t)$ agrees with the number of sheets passing through $(\mathbf{x}, t)$. If $n = m$ we say that $\mathcal{M}$ is in addition* Lagrangian *if the local immersions can be chosen to be Lagrangian.*

Consider an n-dimensional, smoothly immersed Brakke flow $\mathcal{M}$ in $\mathbf{R}^{n+m}$. We call a map f a *function* on $\mathcal{M}$ if it assigns to each $(\mathbf{x}, t) \in \operatorname{supp} \mathcal{M}$ an unordered $\theta(\mathbf{x}, t)$-tuple of real numbers. We say that such a function is *continuous, smooth, etc.* if locally it can be represented by a *continuous, smooth, etc.* function on a suitable local immersion representing $\mathcal{M}$.

Definition **A.2** *(Functions with polynomial growth). — We say that such a function f on $\mathcal{M}$ has* polynomial growth $d \in \mathbf{N}$ *if for every interval $\mathrm{J} \Subset \mathrm{I}$ there exists a constant $\mathrm{C_J}$ such that for all* $\mathrm{R} > 0$

$$\sup_{\operatorname{supp} \mathcal{M} \cap (\mathrm{B_R}(\mathbf{0}) \times \mathrm{J})} \|f(\mathbf{x}, t)\| \leq \mathrm{C_J}(1 + \mathrm{R}^d)\,, \tag{\textbf{A.1}}$$

where $\|f(\mathbf{x}, t)\|$ denotes the maximal possible absolute value of f at $(\mathbf{x}, t)$.

We recall that Ecker's weighted version of Huisken's monotonicity formula is given by (2.2) for a (sufficiently smooth) function f on $\mathcal{M}$ with (uniform) compact support. We can extend this as follows to a (sufficiently smooth) function with non-compact support. Note that in the following we will do all calculations implicitly on the local immersions representing the flow.

Proposition **A.3.** *— Let $\mathcal{M}$ be an n-dimensional, smoothly immersed Brakke flow $\mathcal{M}$ in $\mathbf{R}^{n+m}$ and let f be a smooth function on $\mathcal{M}$. Let $t_1, t_2 \in \mathrm{I}$ with $t_1 < t_2 < t_0$ and assume that*

$$\int_{t_1}^{t_2} \int \left(f^2 + \left|(\partial_t - \Delta) f\right|\right) \rho_{\mathbf{x}_0, t_0}\, d\mu_t\, dt < \infty\,,$$

as well as $f(\cdot, t_1) \in \mathrm{L}^1(\rho_{\mathbf{x}_0,t_0}(\cdot, t_1)\, d\mu_{t_1})$ *and* $f(\cdot, t_2) \in \mathrm{L}^1(\rho_{\mathbf{x}_0,t_0}(\cdot, t_2)\, d\mu_{t_2})$. *Then*

$$\begin{aligned}\int f\, \rho_{\mathbf{x}_0,t_0}\, d\mu_{t_2} \leq \int f\, \rho_{\mathbf{x}_0,t_0}\, d\mu_{t_1} &+ \int_{t_1}^{t_2} \int (\partial_t - \Delta) f\, \rho_{\mathbf{x}_0,t_0}\, d\mu_t\, dt \\ &- \int_{t_1}^{t_2} \int f \left| \mathbf{H} - \frac{(\mathbf{x} - \mathbf{x}_0)^\perp}{2(t - t_0)} \right|^2 \rho_{\mathbf{x}_0,t_0}\, d\mu_t\, dt\,.\end{aligned} \tag{\textbf{A.2}}$$

Proof. — Let η be an ambient cut-off function. We have

$$(\partial_t - \Delta)\,(\eta f) = \eta\,(\partial_t - \Delta) f + f\,(\partial_t - \Delta)\,\eta - 2\langle \nabla f, \nabla \eta \rangle$$

and thus after integration by parts

$$\begin{aligned}\int (\partial_t - \Delta)\,(\eta f)\, \rho\, d\mu_t = \int \eta\,(\partial_t - \Delta) f \rho\, d\mu_t \\ + \int f \left((\partial_t + \Delta)\,\eta + 2\langle \nabla \eta, \tfrac{\nabla \rho}{\rho} \rangle \right) \rho\, d\mu_t\,.\end{aligned} \tag{\textbf{A.3}}$$

Recall that for an ambient function η one has at $(\mathbf{x}, t) \in \operatorname{supp} \mathcal{M}$

$$\Delta_{\mathrm{M}_t^i} \eta = \mathrm{tr}_{\mathrm{T}_p \mathrm{M}_t^i} \mathrm{D}^2 \eta + \langle \mathrm{D}\eta, \mathbf{H}_{\mathrm{M}_t^i} \rangle\,, \tag{\textbf{A.4}}$$

where D is the standard ambient derivative and M_t^i is one of the sheets locally representing the flow. This implies (assuming η is independent of time)

$$\left(\partial_t + \Delta_{\mathrm{M}_t^i}\right) \eta = \mathrm{tr}_{\mathrm{T}_p \mathrm{M}_t^i} \mathrm{D}^2 \eta + 2\langle \mathrm{D}\eta, \mathbf{H}_{\mathrm{M}_t^i} \rangle\,.$$

Recall further that the assumption of bounded area ratios implies that

$$\int_{t_1}^{t_2} \int \left| \mathbf{H} - \frac{\mathbf{x}^\perp}{2t} \right|^2 \rho\, d\mu_t\, dt \leq \mathrm{C} < \infty,$$

where $\mathbf{H}(\mathbf{x}, t) = \sum_{i=1}^{\theta(\mathbf{x},t)} \mathbf{H}_{\mathrm{M}_t^i}(\mathbf{x}, t)$ is the varifold mean curvature and the M_t^i are the sheets passing through $(\mathbf{x}, t)$. Thus, again using bounded area ratios, we have

$$\int_{t_1}^{t_2} \int |\mathbf{H}|^2\, \rho\, d\mu_t\, dt \leq \mathrm{C}(t_2)\,.$$

The above gives

$$\begin{aligned}(\mathbf{A.5})\qquad &\left|\int f\left((\partial_t+\Delta)\,\eta+2\langle\nabla\eta,\tfrac{\nabla\rho}{\rho}\rangle\right)\rho\,d\mu_t\right|\\ &\leq C\int |f|\left(|D^2\eta|+|\nabla\eta|\tfrac{|\nabla\rho|}{\rho}+|D\eta||\mathbf{H}|\right)\rho\,d\mu_t\\ &\leq C\left(\int |f|^2\rho\,d\mu_t\right)^{\frac{1}{2}}\left(\int\left(|D^2\eta|^2+|\nabla\eta|^2\tfrac{|\nabla\rho|^2}{\rho^2}+|D\eta|^2|\mathbf{H}|^2\right)\rho\,d\mu_t\right)^{\frac{1}{2}}.\end{aligned}$$

We now choose φ to be a cutoff function which is equal to one on $B_1(0)$ and vanishes outside of $B_2(0)$ and let $\eta_R(\mathbf{x})=\varphi(\mathbf{x}/R)$. Integrating (A.3) from t_1 to t_2 with $\eta=\eta_R$ and letting $R\to\infty$ (using (A.5) and the space-time integral bound on $|\mathbf{H}|^2$) gives the result. ☐

Remark **A.4.** — For a smooth function f on $\mathcal{M}$ with polynomial growth such that $(\partial_t-\Delta)f$ also has polynomial growth the conditions of Proposition A.3 are satisfied.

We note that using polynomial barriers one obtains existence of caloric functions with polynomial growth given initial data of polynomial growth. Furthermore, the above monotonicity formula yields uniqueness. We record this in the following proposition. We write $\mathcal{M}_{t\geq t_0}$ for the restriction of the Brakke flow to $I\cap\{t\geq t_0\}$ and extend the definition of a function with polynomial growth on $\operatorname{supp}(\mathcal{M})\cap\{(\mathbf{x},t_0)\,|\,\mathbf{x}\in\mathbf{R}^{n+m}\}$ in the obvious way.

Proposition **A.5.** — *Let $\mathcal{M}$ be an n-dimensional, smoothly immersed Brakke flow in $\mathbf{R}^{n+m}$ and for $t_0\in I$ let f_0 be a smooth function on $\operatorname{supp}(\mathcal{M})\cap\{(\mathbf{x},t_0)\,|\,\mathbf{x}\in\mathbf{R}^{n+m}\}$ with polynomial growth. Then there exists a unique smooth function f on $\mathcal{M}_{t\geq t_0}$ of polynomial growth such that*

$$(\mathbf{A.6})\qquad (\partial_t-\Delta)f=0\quad\textit{and}\quad f|_{t=t_0}=f_0\,.$$

Proof. — We first note that Proposition A.3 together with Remark A.4 yields uniqueness as stated. For existence we have the following claim.

Claim. — *Given $R>0$ there exists a solution f_R to (A.6) on $(B_R(\mathbf{0})\times[t_0,t_0+R^2))\cap\operatorname{supp}\mathcal{M}$ such that for every $0<r\leq R$*

$$\sup_{\operatorname{supp}\mathcal{M}\cap(B_r(\mathbf{0})\times[t_0,t_1])}\|f_R\|\leq C_0e^{C_1(t_1-t_0)}(1+r^d)\,,$$

for some $C_1>0$ just depending on n, m and d.

Note that the claim does not specify any boundary values at the spatial boundary. From the claim, the existence follows, since interior higher order estimates imply that we can take a subsequential limit as $R \to \infty$ to obtain the stated solution f. The convergence of f to the initial data f_0 as $t \searrow t_0$ follows similarly from higher order interior estimates.

To prove the claim, let $R > 0$ be given. Note that since $\mathcal{M}$ is smooth there exists $K > 0$ such that the mean curvature $\mathbf{H}$ of the flow is bounded by $K > 0$ on $B_{4R}(\mathbf{0}) \times [t_0, t_0 + R^2]$. Since $\mathcal{M}$ is smoothly immersed, there exists an (open) n-manifold M and an immersion F_{t_0} such that $F_{t_0} : M \to \mathbf{R}^{n+m}$ smoothly parametrises $\mathcal{M}(t_0) \cap B_{3R}(\mathbf{0})$. Furthermore we can smoothly extend F_{t_0} to a standard (immersed) mean curvature flow F_t parametrising $\mathcal{M}$ for $t \in [t_0, t_0 + \delta]$ with $\delta := R/(2K)$: this follows from the bound on $\mathbf{H}$. Note further that $\mathcal{M}(t+\delta) \cap B_R(\mathbf{0}) \subset F_{t+\delta}(M)$. This again follows from the bound on $\mathbf{H}$ on $B_{4R}(\mathbf{0}) \times [t_0, t_0 + R^2]$.

Let f_0 be of polynomial growth of degree d such that for all $r > 0$

$$\sup_{(\mathbf{x}, t_0) \in \operatorname{supp} \mathcal{M} \cap (B_r(\mathbf{0}) \times \{t_0\})} \|f_0(\mathbf{x})\| \le C_0(1 + r^d)\,. \tag{A.7}$$

Choose $R' \in (2R, 3R)$ such that $U := F_{t_0}^{-1}(B_{R'}(\mathbf{0})) \subset M$ has smooth boundary. We can then construct a solution $\hat{f}$ to the heat equation (with respect to the induced metric g_t on M via F_t) with initial value f_0 and boundary value zero. Note that (A.4) together with $d \ge 2$ implies that there exists $C_1 > 0$ such that $C_0 e^{C_1(t-t_0)}(1 + |\mathbf{x}|^d)$ is a supersolution to the heat equation along the flow. Thus, by the maximum principle together with (A.7), as well as the assumptions on the boundary data, we have

$$|\hat{f}(x,t)| \le C_0 e^{C_1(t-t_0)}(1 + |F(x,t)|^d)\,,$$

for $(x,t) \in U \times [t_0, t_0 + \delta]$. Note that by restriction this yields a solution f to the heat equation with the claimed bounds on $\mathcal{M} \cap (B_R(\mathbf{0}) \times [t_0, t_0 + \delta])$. We can now repeat this process, starting at $t_0 + \delta$ where as initial data we take $\hat{f} \circ F_{t_0+\delta}^{-1}$ on $F_{t_0+\delta}(U) \subset \operatorname{supp} \mathcal{M}(t_0 + \delta) \cap B_{4R}(\mathbf{0})$ and zero on $(\operatorname{supp} \mathcal{M}(t_0 + \delta) \cap B_{4R}(\mathbf{0})) \setminus F_{t_0+\delta}(U)$. Repeating this process finitely many times yields the stated solution f_R. ☐

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Manuscrit reçu le 5 juillet 2022
Version révisée le 9 septembre 2023
Manuscrit accepté le 20 octobre 2023