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On Ultra-Sharp Queueing Bounds

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Abstract—We present a robust method to analyze a broad range of classical queueing models, e.g., the GI/G/1 queue with renewal arrivals, an AR/G/1 queue with alternating renewals (AR), as a special class of Semi-Markovian processes, and Markovian fluids queues. At the core of the method lies a standard change-of-measure argument to reverse the sign of the negative drift in the underlying random walks. Combined with a suitable representation of the overshoot, we obtain exact results in terms of series. Closed-form and computationally fast bounds follow by taking the series’ first terms, which are the dominant ones because of the positive drift under the new probability measure. The obtained bounds generalize the state-of-the-art class of martingale bounds and can be much sharper by orders of magnitude.

I. INTRODUCTION

The steady-state waiting time $W$ in the GI/G/1 queue is the maximal value of a random walk, i.e.,

$$W = \max_{n \geq 0} \{ X_1 + \cdots + X_n \}$$

whose expected increment $X_n = S_n - T_n$ satisfies $E[X_n] < 0$ for stability; $T_n$ are the (renewal) inter-arrival times and $S_n$ are the corresponding service times. In general, there do not exist exact and closed-form solutions for the distribution of $W$. Existing (exact) approaches involve inversion techniques and numerical methods, which are themselves very challenging due to dealing with Wiener-Hopf type of integral equations (e.g., Cohen [11]).

The SM/SM/1 queue generalizes the GI/G/1 queue by relaxing the underlying renewal assumption, i.e., the arrival and service processes follow some Semi-Markovian structures. Studying such models is not only driven by measurement studies (e.g., Crovella and Bestavros [12] or Mi et al. [20]) but also by the fundamentally different queueing behavior when compared to the baseline renewal models (e.g., Tin [28] or Patuwo et al. [22]). The standard analysis of SM/SM/1 models is computational in nature involving, e.g., transform methods (e.g., Çinlar [7] or Adan and Kulkarni [11]) or matrix analytical methods (e.g., Neuts [21] or Akar and Sohraby [3]).

The Markovian fluid queue is another classical model in which the arrival process is driven by a Markovian structure whereas the service is fluid. The Markov-Modulated Fluid model appeared in the seminal paper of Anick, Mitra, and Sondhi [4]: the arrival processes consists of several multiplexed On-Off sources and the queueing model can be exactly analyzed using matrix analysis and ordinary differential equations. The more general case requires however more advanced techniques, e.g., spectral decomposition (Akar and Sohraby [2]) or Wiener-Hopf factorization (Rogers [24]).

The numerical complexity of such techniques becomes however prohibitive when a large number of sources are multiplexed (Shroff and Schwartz [26]).

A robust alternative to computational techniques for such queueing models is Effective Bandwidth, which enables an asymptotically exact analysis, e.g.,

$$\mathbb{P}(W > \sigma) \approx \gamma_1 e^{-\theta \sigma},$$

for some asymptotic constant $\gamma_1$ and some asymptotic decay rate $\theta$; see the standard review on the topic by Kelly [17]. The asymptotically exact representation $f(\sigma) \approx g(\sigma)$ means that $\lim_{\sigma \to \infty} \frac{f(\sigma)}{g(\sigma)} = 1$.

Another robust analytical alternative is to represent the tail probability $\mathbb{P}(W > \sigma)$ in terms of rigorous stochastic bounds, i.e.,

$$\mathbb{P}(W > \sigma) \leq \gamma_2 e^{-\theta \sigma} \quad \forall \sigma \geq 0,$$

for some asymptotic constant $\gamma_2$. Such results can be obtained using martingale-based techniques starting with the work of Kingman [18] and stochastic network calculus (Jiang and Liu [16]), which is the probabilistic extension of the deterministic network calculus conceived by Cruz [13].

There are very few studies investigating the accuracy of effective bandwidth approximations or stochastic bounds obtained using either martingale techniques or stochastic network calculus. Using Markovian sources, Choudhury et al. [8] showed that effective bandwidth approximations can either overestimate or underestimate the corresponding exact results, in finite regimes, by orders of magnitude; the underlying reason is that while the asymptotic decay rate $\theta$ is exact, the corresponding asymptotic constant $\gamma_1$ can be extremely inaccurate, especially when many sources are multiplexed. For similar traffic sources, Ciucu et al. [10] showed that stochastic network calculus bounds overestimate exact results by orders of magnitude; moreover, related martingale bounds can be very sharp in heavy-traffic. The accuracy of martingale bounds does decay however at low utilizations or depending on the burstiness nature of the arrival processes (Poloczek and Ciucu [23], [9]).

In this paper we explore a new class of stochastic bounds for the three queueing models outlined above. The main idea rests on an exponential change-of-measure – a technique employed in several fields such as importance sampling or sequential hypothesis testing facing with similar technical problems as in queueing analysis. In our context, the crucial benefit of the exponential change-of-measure is reversing the sign of the drift in the underlying random walk: from negative in the original
(probability) space to positive in the transformed space. In this way, a fundamental technical difficulty in the original space can be addressed in a rather straightforward manner in the new space. Furthermore, using an elementary representation of the overshoot of a point process, we obtain exact results on \( \mathbb{P}(W > \sigma) \) in terms of an infinite sum with positive terms, each involving an integral; closed-form bounds could be obtained by solving for any number of integrals.

In their simplest form, the proposed bounds recover Kingman’s martingale-based bounds. They could be further arbitrarily sharpened, and yet retain a closed-form, but at the expense of solving for standard integrals. In other words, the proposed class of bounds is subject to an inherent tradeoff between simplicity/expressiveness and accuracy.

In the following, we first present the main ideas and results for the GI/G/1 renewal case, along with a numerical example illustrating the ability of the proposed bounds to gradually improve upon state-of-the-art martingale-based bounds. Then we present several extensions, first to the AR/G/1 queue, with an alternating renewal (AR) arrival process, and then to a Markovian fluid queue. We will particularly consider the cases of multiplexing both homogeneous and heterogeneous processes, and also the cases when the individual sources are either more or less bursty than Poisson; numerical evaluations against simulations confirm that the proposed bounds can be made arbitrarily sharp.

II. THE GI/G/1 QUEUE

First we give some background from Sequential Analysis [29], [27] on ‘change-of-measure’. Then, using an elementary observation on the overshoot of a renewal process, a new class of GI/G/1 bounds will be presented along with some numerical comparisons against the state-of-the-art.

Consider a stationary stochastic process \( \mathcal{X} = (X_n)_{n \geq 1} \), its natural filtration \( \mathcal{F}_n = \sigma(X_1, \ldots, X_n) \), and \( \mathcal{F} := \mathcal{F}_\infty \). For a probability space \((\Omega, \mathcal{F}, \mathbb{P})\) denote by \( \mathbb{P}_n \) the restriction of \( \mathbb{P} \) on \( \mathcal{F}_n \).

Assume for some \( \theta > 0 \) that the moment generating function (MGF) \( \phi(\theta) = \mathbb{E}[e^{\theta X}] \) is finite (\( X \) has the same law as \( X_1 \)). For each \( n \in \mathbb{N} \) and \( A \in \mathcal{F}_n \) define

\[
\mathbb{P}_{n,\theta}(A) := \mathbb{E}
\left[
\frac{e^{\theta(X_1+\cdots+X_n)}}{\phi(\theta)^n} 1_A
\right] = \int_A \frac{e^{\theta(x_1+\cdots+x_n)}}{\phi(\theta)^n} d\mathbb{P}_n(x),
\]

where \( 1_A \) denotes the indicator function. \( \mathbb{P}_{n,\theta} \) is a probability measure on \((\Omega, \mathcal{F}_n)\) and has the absolute continuity properties \( \mathbb{P}_{n,\theta} << \mathbb{P}_n \) and \( \mathbb{P}_n << \mathbb{P}_{n,\theta} \). The corresponding Radon-Nikodym derivatives are

\[
\frac{d\mathbb{P}_{n,\theta}}{d\mathbb{P}_n}(x) = \frac{e^{\theta(x_1+\cdots+x_n)}}{\phi(\theta)^n} \quad \text{and} \quad \frac{d\mathbb{P}_n}{d\mathbb{P}_{n,\theta}}(x) = \frac{\phi(\theta)^n}{e^{\theta(x_1+\cdots+x_n)}},
\]

respectively. Moreover, based on Kolmogorov’s Extension Theorem, there exists a probability measure \( \mathbb{P}_\theta \) on \((\Omega, \mathcal{F})\) such that \( \mathbb{P}_{n,\theta} \) is the restriction of \( \mathbb{P}_\theta \) on \( \mathcal{F}_n \); denote by \( \mathbb{E}_\theta \) the corresponding expectation.

A key result which we will use throughout is Wald’s Fundamental Identity (WI): for a stopping time \( T \) and a non-negative random variable \( Y \geq 0 \) which is prior to \( T \) (i.e., \( Y_{1{T=\infty}} \) is measurable \( \mathcal{F}_n \)) the following holds

\[
\mathbb{E}_\theta [Y_{1{T<\infty}}] = \mathbb{E}
\left[
Y e^{\theta(X_1+\cdots+X_T)} \phi(\theta)^{-T} 1_{T<\infty}
\right].
\]

Note that, in particular, if \( T = n \) and \( Y \) is measurable \( \mathcal{F}_n \), then

\[
\mathbb{E}_\theta [Y] = \mathbb{E}
\left[
Y e^{\theta(X_1+\cdots+X_n)} \phi(\theta)^{-n}
\right]
\]

and

\[
\mathbb{E}[Y] = \mathbb{E}_\theta
\left[
Y e^{-\theta(X_1+\cdots+X_n)} \phi(\theta)^n
\right].
\]

A. GI/G/1: State-of-the-Art Bounds

Consider now the GI/G/1 queue. The inter-arrival and service times are denoted by the iid (independent and identically distributed) sequences \((T_n)_n\) and \((S_n)_n\), respectively. We assume that the service times are light-tailed in the sense that there exists \( \theta > 0 \) such that \( \phi(\theta) = 1 \); note that such a condition depends on the inter-arrival times as well. Denote \( X_n := S_n - T_n \) and assume for stability that \( \mathbb{E}[X_1] < 0 \). In steady-state, the waiting time \( W \) of an arbitrary job is

\[
W = \max_{n \geq 1} \{ X_1 + \cdots + X_n \}
\]

We shall focus on the tail \( \mathbb{P}(W > \sigma) \), for some \( \sigma \geq 0 \), which is subject to the duality

\[
\mathbb{P}(W > \sigma) = \mathbb{P}(T < \infty),
\]

where \( T \) is the stopping time

\[
T := \min\{ n : X_1 + \cdots + X_n > \sigma \}
\]

By convention, the minimum of the empty set is denoted by \( \infty \) and sums with no terms are set to zero; an important remark is that \( \mathbb{P}(T = \infty) > 0 \) because \( \mathbb{E}[X_1] < 0 \).

Kingman [18] proposed a method to bound \( \mathbb{P}(W > \sigma) \) by first constructing the martingale

\[
M_n := e^{\theta(X_1+\cdots+X_n)}
\]

where \( \theta > 0 \) satisfies \( \phi(\theta) = 1 \). It then follows from the Optional Sampling Theorem (see Mehr and Ciucu [19] for more details) that

\[
1 = \mathbb{E}[M_0] = \mathbb{E}[M_{T\{T<\infty\}}] = \mathbb{E}
\left[
\phi^{X_1+\cdots+X_T} 1_{T<\infty}
\right]
\]

\[
\geq e^{\theta \sigma} \mathbb{P}(T < \infty) = e^{\theta \sigma} \mathbb{P}(W > \sigma),
\]

and consequently the Kingman bound

\[
\mathbb{P}(W > \sigma) \leq e^{-\theta \sigma}.
\]

The inequality in (4) can be refined as

\[
\mathbb{E}
\left[
\phi^{X_1+\cdots+X_T} 1_{T<\infty}
\right] \geq \inf_{x \geq 0} K(x) e^{\theta \sigma} \mathbb{P}(T < \infty),
\]

briefly, an integrable stochastic process \( M_n \) is a martingale if its conditional expected increment is zero, i.e., \( \mathbb{E}[M_{n+1} - M_n | \mathcal{F}_n] = 0 \) \( \forall n \), where \( \mathcal{F}_n = \sigma(M_1, \ldots, M_n) \).

Therefore, it is shown that

\[
\mathcal{X} = (X_n)_{n \geq 1}
\]

is a martingale if its conditional expected increment is zero, i.e., \( \mathbb{E}[M_{n+1} - M_n | \mathcal{F}_n] = 0 \) \( \forall n \), where \( \mathcal{F}_n = \sigma(M_1, \ldots, M_n) \).

Therefore, it is shown that
where $K(x) := \mathbb{E}[e^{\theta(X_1 - x)} \mid X_1 \geq x]$, which leads to the Ross [25] bound
\[ P(W > \sigma) \leq \frac{1}{\inf_{x \geq 0} K(x)} e^{-\theta \sigma}, \]
which is sharper than Kingman's bound because $K(x) \geq 1 \quad \forall x \geq 0$.

### B. A New Class of Bounds

The weakness of the Kingman and Ross bounds lies in the coarse treatment of the dependency inside $E[e^{\theta(X_1 + \cdots + X_T)}]_{T < \infty}$; in particular, Kingman's bound simply resorts to bounding the first term by the non-random quantity $e^{\theta \sigma}$.

A more effective method is to apply WFI from (2) with $Y = 1$, i.e.,
\[ \mathbb{E}[1_{T < \infty}] = \mathbb{E}_\theta [e^{-\theta(X_1 + \cdots + X_T)}]_{T < \infty} \quad . \quad (5) \]

The crucial observation is that $T < \infty$ a.s. on $(\Omega, \mathcal{F}, \mathbb{P}_\theta)$. Indeed, since $\phi(\theta)$ is convex and $\phi(0) = \phi(\theta) = 1$, it follows that $\phi'(\theta) = \mathbb{E}[X e^{\theta X}] > 0$ and hence
\[ \mathbb{E}_\theta[X] = \mathbb{E}[X e^{\theta X}] > 0 . \]

Thus $\mathbb{P}_\theta(T < \infty) = 1$ and (5) becomes
\[ \mathbb{P}(W > \sigma) = \mathbb{E}_\theta \left[ e^{-\theta(X_1 + \cdots + X_T)} \right] = e^{-\theta \sigma} \mathbb{E}_\theta \left[ e^{-\theta R_x} \right], \]
where
\[ R_x := X_1 + \cdots + X_T - \sigma \quad (6) \]
is the overshoot. Note that Kingman's bound (aka Lundberg inequality in Financial Mathematics) simply follows from $R_x \geq 0$. A standard approach to continue is to express the overshoot $R_x$ as the remaining lifetime relative to a new renewal process constructed in terms of the (positive) ladder heights of the original (and not necessarily positive) process $X_n$; finding the distribution of the ladder heights is however an open problem in the general case. The classical related result is the so-called Cramér-Lundberg approximation when $\sigma \to \infty$ [5].

Because our main target are non-asymptotic regimes (i.e., finite values of $\sigma$), we will rely on the following elementary expansion of the overshoot's tail
\[ \{R_x > x\} = \bigcup_{n \geq 1} \left\{ \sum_{i=1}^{n} X_i > \sigma + x, \max_{1 \leq k \leq n-1} \sum_{i=1}^{k} X_i \leq \sigma \right\} , \quad (7) \]
for any $x > 0$, in terms of a union of disjoint events; the disjointness follows from the fact that the events $\{\sum_{i=1}^{k} X_i \leq \sigma\}$ and $\{\sum_{i=1}^{k} X_i > \sigma + x\}$ are disjoint themselves for all $k \geq 1$ and $x > 0$. Using this expansion we obtain the following exact result on the tail $\mathbb{P}(W > \sigma)$.

**Theorem 1.** The waiting time distribution satisfies
\[ \mathbb{P}(W > \sigma) = e^{-\theta \sigma} \left( 1 - \sum_{n=1}^{\infty} g_n(\sigma) \right) \quad . \quad (8) \]

for all $\sigma > 0$, where
\[ g_n(\sigma) := \mathbb{E}_{\theta} \left[ e^{\theta \sum_{i=1}^{n} X_i - \theta \sigma} \right]_{1 \leq T = n} \forall n \geq 1 . \]

Note that, according to (3), $g_n(\sigma)$ can be rewritten as
\[ g_n(\sigma) := \mathbb{E}_{\theta} \left[ \left( 1 - e^{\theta(\sigma - \sum_{i=1}^{n} X_i)} \right) \right]_{1 \leq T = n} \forall n \geq 1 . \quad (9) \]

Since $g_n(\sigma) \geq 0$, upper bounds on $\mathbb{P}(W > \sigma)$ follow by taking any number of terms $g_n(\sigma)$ in the sum from (12). These can be written recursively for $n \geq 2$ as
\[ g_n(\sigma) = \mathbb{E}_{\theta} \left[ e^{\theta(\sigma - \sum_{i=1}^{n} X_i)} \left( 1_{\sigma - X_1} \right) \right] = e^{\theta X_1} 1_{X_1 \leq \sigma} \mathbb{E}_\theta \left[ \left( e^{\theta(\sum_{i=1}^{n} X_i + X_n - \sigma - X_1)} \right) \right] = e^{\theta X_1} 1_{X_1 \leq \sigma} g_{n-1}(\sigma - \sigma) \quad . \quad (10) \]
and $g_1(\sigma) = \mathbb{E}_\theta \left[ (e^{\theta X_1} - e^{\theta \sigma}) 1_{X_1 > \sigma} \right]$, where $A$ is the event $\{\sum_{i=2}^{n} X_i > \sigma, X_1, \max_{2 \leq k \leq n-1} \sum_{i=1}^{k} X_i \leq \sigma - X_1\}$.

**Proof.** Fixing $\sigma \geq 0$ we can write
\[ \mathbb{E}_\theta \left[ e^{-\theta R_x} \right] = \int_{0}^{\infty} \mathbb{P}_\theta (R_x > y) \, dy = 1 - \int_{0}^{\infty} \mathbb{P}_\theta \left( R_x > -\frac{\ln y}{\theta} \right) \, dy = 1 - \int_{0}^{\infty} \mathbb{P}_\theta (R_x > z) \theta e^{-\theta z} \, dz = 1 - \mathbb{P}_\theta (R_x > Z) = 1 - \sum_{n=1}^{\infty} g_n(\sigma) , \]
where $Z$ is an exponential random variable with parameter $\theta$, and independent of $\mathcal{F} := \mathcal{F}_\infty$, whereas the terms $g_n(\sigma)$ follow from the expansion (7) by first denoting for convenience $U := \sum_{i=1}^{n} X_i$ and $V := \max_{1 \leq k \leq n-1} \sum_{i=1}^{k} X_i$, i.e.,
\[ g_n(\sigma) = \mathbb{P}_\theta \left( \sum_{i=1}^{n} X_i > \sigma + Z, \max_{1 \leq k \leq n-1} \sum_{i=1}^{k} X_i \leq \sigma \right) = \mathbb{E}_{\theta} \left[ e^{\theta U} 1_{U > \sigma + X} \left( V \leq \sigma \right) \mid \mathcal{F}_n \right] = \mathbb{E} \left[ 1_{V \leq \sigma} \mathbb{E}_{\theta} \left[ e^{\theta U} 1_{U > \sigma + Z} \mid \mathcal{F}_n \right] \right] . \]

In the second line we rewrote $\mathbb{E}_\theta$ in terms of $\mathcal{E}$ according to (1), and in the last line we used the measurability of $1_{V \leq \sigma}$ with respect to $\mathcal{F}_n$. Denoting $U := \sum_{i=1}^{n} X_i$ we can expand the inner conditional expectation as
\[ \mathbb{E}_{\theta} \left[ e^{\theta U} 1_{U > \sigma + Z} \mid \mathcal{F}_n \right] = \mathbb{E}_{\theta} \left[ e^{\theta U} 1_{U > \sigma + Z} \mid U \right] = e^{\theta U} e^{\theta U} 1_{U > \sigma + Z} \mid U \]
for all $U > \sigma$.

Therefore
\[ g_n(\sigma) = \mathbb{E}_{\theta} \left[ e^{\theta \sum_{i=1}^{n} X_i - \theta \sigma} \right]_{1 \leq U > \sigma, V \leq \sigma} = \mathbb{E}_{\theta} \left[ e^{\theta \sum_{i=1}^{n} X_i - \theta \sigma} \right]_{1 \leq T = n} . \]

\[ \square \]
C. Example: \(M/D/1\)

For numerical illustration we consider the \(M/D/1\) queue with \(T_n \sim \text{Exp} (\lambda)\) and deterministic service time \(S\). For \(\sigma < S\), we obtain by elementary integration the following bound when using only the term \(g_1(\sigma)\) in the sum from (12)

\[
\mathbb{P}(W > \sigma) \leq 1 - \frac{\theta}{\lambda + \theta} e^{-\lambda (S - \sigma)},
\]

where \(\rho := \lambda S < 1\) denotes the utilization factor. In turn, by only using the first two terms, \(g_1(\sigma)\) and \(g_2(\sigma)\), we obtain

\[
\mathbb{P}(W > \sigma) \leq 1 - \left(1 + \theta Se^{-\theta S} - e^{-2\theta S}\right) e^{-\lambda (2S - \sigma)}.
\]

Similar bounds can be obtained for \(S \leq \sigma < 2S\); these are not shown here for brevity. Fig. 1 shows the two sets of bounds against the standard Ross bound and also simulations based on 10^7 independent runs; we note that \(M/D/1\) has an exact result, yet it is subject to significant numerical complications due to an underlying sum with nearly cancelling very-large terms (see Iversen and Staalhagen [15]).

![Fig. 1: Waiting-time CCDF for \(M/D/1\); \(\lambda = 0.02\), \(S = 5\), \(\rho = 0.1\).](image)

Besides the obvious improvement of Ross’ bound, the crucial observation is that the gradual improvements appear to decay exponentially. This is supported by the recursive representation of \(g_n(\sigma)\) from (10). Moreover, from the alternative representation from (9), the dominant terms in the sum \(\sum_n g_n(\sigma)\) are seemingly the first ones, given the positive drift \(\mathbb{E}_\sigma[X_1] > 0\). A fundamental open question is whether there exists \(k \geq 0\) such that \(\sum_{n \geq k} g_n(\sigma)\) is an analytic series; if so, then the first \(k\) terms from the sum would be largely sufficient to shed most of numerical inaccuracies in Ross’ bound.

III. THE AR/G/1 QUEUE

Here we extend the \(GI/G/1\) results to a queue with non-renewal arrivals. We consider the case of an alternating renewal (AR) process driven by a deterministic Markov chain \(a_n\) with two states 1 and 2, i.e.,

\[
\mathbb{P}(a_n = 3 - j \mid a_{n-1} = j) = 1 \quad \forall n \geq 1, \ j \in \{1, 2\}.
\]

In state \(j\), the inter-arrivals form an iid sequence \(T_n^{(j)}\) with the same law as \(T^{(j)}\); denote by \(T_n := T_n^{(a(n))}\) the inter-arrival time at state \(n\). The service times are denoted by the iid sequence \((S_n)_n\) with the same law as \(S\). Assume that \(\mathbb{P}(a_0 = 1) = 0.5\) and the stability condition \(\mathbb{E}[X_n] < 0\) where \(X_n := S_n - T_n\).

Denoting the MGfs \(\phi_j(\theta) = \mathbb{E}\left[e^{\theta (S - T^{(j)})}\right]\) for \(j \in \{1, 2\}\), the light-tailed condition of the service times is slightly more involved than in the \(GI/G/1\) case. Indeed, assume that there exist \(\theta_1\) such that \(\phi_j(\theta_j) = 1\) for \(j \in \{1, 2\}\). Assuming without loss of generality that \(\theta_1 \leq \theta_2\), we additionally assume that \(\phi_1(\theta_2) < \infty\). Using the continuity and convexity of \(\phi_j(\theta)\), along with \(\phi_1(0) = 0\), it then follows that there exists \(\theta \in [\theta_1, \theta_2]\) such that \(\phi_1(\theta)\phi_2(\theta) = 1\).

In the following, similarly as in the \(GI/G/1\) case, we seek a suitable change-of-measure to reverse the sign of the expected increment \(\mathbb{E}[X_n]\). Observe first that the process

\[
M_n := h_{a_n} e^{\theta (X_1 + \cdots + X_n)}
\]

is a martingale, where \(\theta > 0\) satisfies

\[
\sqrt{\mathbb{E}[e^{\theta (T^{(1)} + T^{(2)})}][e^{\theta S}]} = 1
\]

according to the assumption of light-tailed service times, and

\[
h_1 := 1 \quad \text{and} \quad h_2 := \sqrt{\frac{\mathbb{E}[e^{-\theta (T^{(1)})}]}{\mathbb{E}[e^{-\theta (T^{(2)})}]}}
\]

(note that \(h_2 \geq 1\) according to the earlier assumption that \(\theta_1 \leq \theta_2\). The proof follows immediately from

\[
\mathbb{E}\left[h_{a_n+1} e^{\theta (S_{n+1} - T_{n+1})} \mid a_n\right] = h_{a_n} \quad \forall n \geq 1.
\]

The change-of-measure

\[
\mathbb{P}_{n, \theta}(A) := \mathbb{E}\left[\frac{M_n}{\mathbb{E}[M_0]} 1_A\right]
\]

for \(A \in F_n = \sigma(a_1, X_1, \ldots, a_n, X_n)\) entails the same properties as in the renewal case from § II, such as the existence of \(\mathbb{P}_\theta\) on \((\Omega, \mathcal{F})\) and WFI

\[
\mathbb{E}_\theta [Y 1_{T < \infty}] = \mathbb{E} Y \frac{M_T}{\mathbb{E}[M_0]} 1_{T < \infty}
\]

for any stopping time \(T\) and any non-negative r.v. \(Y \geq 0\) prior to \(T\); a key reason is using a normalized martingale with \(\mathbb{E}\left[\frac{M_T}{\mathbb{E}[M_0]}\right] = 1\) (see Asmussen [5], pp. 358–359).

**Theorem 2.** Under the above setting, the stationary waiting-time distribution satisfies for all \(\sigma > 0\)

\[
\mathbb{P}(W > \sigma) = e^{-\sigma} \left(\frac{h_1 + h_2}{2} - \sum_{n=1}^{\infty} g_n(\sigma)\right),
\]

where \(g_n(\sigma) := \mathbb{E}g_n(\sigma, a_0)\) and

\[
g_n(\sigma, j) := \mathbb{E}\left[h_{a_n} e^{\theta (X_1 + \cdots + X_n)} - e^{\theta \sigma}\right] 1_{\{T = n\}} \mid a_0 = j
\]

for \(j \in \{1, 2\}\) and the standard stopping time

\(T := \inf \{n \geq 1 : X_1 + \cdots + X_n > \sigma\}\).

**Proof.** Fixing \(\sigma > 0\), WFI from (11) yields

\[
\mathbb{P}(W > \sigma) = \mathbb{E}[1_{T < \infty}] = e^{-\sigma} \mathbb{E}_\theta \left[\frac{e^{-\theta R_x}}{h_n} \right] \mathbb{E}[h_{a_n}].
\]
Since \( e^{-\theta R_e} \in [0,1] \), from the definition of \( \theta \), we can write as in the proof of Theorem 1

\[
e_\theta \left( \frac{e^{-\theta R_e}}{h_{\alpha T}} \right) = \int_0^1 \mathbb{P}_\theta \left( \frac{e^{-\theta R_e}}{h_{\alpha T}} > y \right) dy
\]

\[
= 1 - \sum_{n=1}^{\infty} \frac{g_n(\sigma)}{E[h_{\alpha n}]},
\]

where \( Z \) is an exponential random variable with parameter \( \theta \) independent of \( F := F_{\infty} \). Further, using the expansion from (7), and denoting for convenience \( U := \sum_{i=1}^n X_i \) and \( V := \max_{1 \leq k \leq n} \sum_{i=1}^k X_i \),

\[
g_n(\sigma) = \mathbb{P}_\theta \left( U > \left( Z - \frac{\ln h_{\alpha n}}{\theta} \right) + \sigma, V \leq \sigma \right) E[h(\alpha n)]
\]

\[
= \mathbb{E} \left[ h_{\alpha n} e^{\theta U - \sigma} \right] = \mathbb{E} \left[ h_{\alpha n} e^{\theta U} \left( 1 - \frac{1}{h_{\alpha n}} e^{\theta(\sigma - U)} \right) 1_{U > \sigma, V \leq \sigma} \right]
\]

\[
= \left( h_{\alpha n} e^{\theta U} - e^{\theta \sigma} \right) 1_{\{T=n\}}.
\]

Since \( g_n(\sigma) \geq 0 \), upper bounds on \( \mathbb{P}(W > \sigma) \) follow by taking any number of terms \( g_n(\sigma) \) in the sum from (12). In particular, dismissing all terms yields the Kingman bound

\[
\mathbb{P}(W > \sigma) \leq \frac{h_1 + h_2}{2} e^{-\theta \sigma}.
\]

The terms \( g_n(\sigma) \) can be written recursively for \( n \geq 2 \)

\[
g_{n}(\sigma,j) := \left( h_{\alpha j} e^{\theta(\sigma_1 + \cdots + \sigma_\alpha)} - e^{\theta \sigma} \right) 1_{\{T=n\}}|a_0 = j
\]

\[
= \mathbb{E} \left[ e^{\theta(S_1 - T_{1,j}^{(3-j)})} g_{n-1} \left( \sigma - S_1 + T_{1,j}^{(3-j)}, 3 - j \right) 1_{S_1 - T_{1,j}^{(3-j)} \leq \sigma} \right]
\]

and \( g_i(\sigma,j) = \mathbb{E} \left[ \left( h_{\alpha j} e^{\theta(\sigma_1 + \cdots + \sigma_\alpha)} - e^{\theta \sigma} \right) 1_{S_1 - T_{1,j}^{(3-j)} \geq \sigma} \right] \) for \( j \in \{1,2\} \).

For numerical evaluations we consider the extension of the \( M/D/1 \) queue, denoted here by \( MM/D/1 \), in which the alternating renewals are \( T^{(j)} \sim Exp(\lambda_j) \) for \( j \in \{1,2\} \) with \( \lambda_1 \geq \lambda_2 \), and deterministic service times \( S \). The condition on \( \theta \) becomes

\[
\sqrt{\frac{(\lambda_1 + \theta)(\lambda_2 + \theta)}{\lambda_1 \lambda_2}} = e^{\theta S}
\]

which can be solved numerically under the stability condition \( \rho := \frac{\lambda_1 + \lambda_2}{2} S < 1 \).

For \( \sigma < S \), Theorem 2 yields the following bound when using only \( g_1(\sigma) \):

\[
\mathbb{P}(W > \sigma) \leq 1 - \frac{1}{2} \left( 1 - e^{-\theta S} \right) \left( e^{-\lambda_1(S-\sigma)} - e^{-\lambda_2(S-\sigma)} \right)
\]

\[
+ \frac{1}{2} e^{-\theta S} \left( h_{\alpha 1} e^{-\lambda_1(S-\sigma)} - e^{-\lambda_2(S-\sigma)} \right),
\]

where \( h_2 = \sqrt{\frac{\lambda_1(\lambda_2 + \theta)}{\lambda_2(\lambda_1 + \theta)}} \).

Fig. 2 illustrates the accuracy of the bounds (see the caption for the parameters’ values); the expression of the bounds when accounting for \( g_2(\sigma) \) as well is not shown for brevity. Similar observations as in Fig. 1 can be drawn, i.e., the Kingman/Ross bounds overestimate by one order of magnitude, whereas the improvements when using \( g_1(\sigma) \) and \( g_2(\sigma) \) are drastic and almost match the simulations.

IV. MARKOVIAN FLUID QUEUES

Here we analyze a queueing model with Markovian arrivals and fluid service. First we present the main results and then we consider two examples with multiplexing homogeneous and heterogeneous sources.

Let \( a_n \) be a stationary and ergodic Markov chain with state space \( S = \{s_i \mid 1 \leq i \leq m\} \). Assume for simplicity that \( a_n \) is reversible and denote the stationary distribution \( \pi(i) = \mathbb{P}(a_n = s_i) \) and the transition matrix \( P(i,j) = \mathbb{P}(a_{n+1} = s_j \mid a_n = s_i) \).

Consider a queue with instantaneous arrivals \( a_n \) and constant service rate \( C \), and assume the stability condition \( \mathbb{E}[a_1] < C \). Because of the fluid service, we next focus on the stationary queue size which is subject to the convenient expression

\[
Q = \max_{n \geq 0} \{X_1 + \cdots + X_n\}
\]

where \( X_n := a_n - C \), and which is subject to the duality

\[
\mathbb{P}(Q > \sigma) = \mathbb{P}(T < \infty),
\]

where \( T \) is the stopping time

\[
T = \min\{n \geq 0 : X_1 + \cdots + X_n > \sigma\}.
\]

Similarly as in the \( GI/G/1 \) and \( AR/G/1 \) cases, we seek a suitable change-of-measure to reverse the sign of the expected increment \( \mathbb{E}[a_1 - C] \). For some \( \theta \geq 0 \), consider the transformed transition matrix \( P_\theta(i,j) := P(i,j) e^{\theta s_i} \) and denote by \( \lambda(\theta) \)
its spectral radius and by \(v = (v_1, \ldots, v_m)\) the corresponding non-negative right eigenvector, according to the Perron-Frobenius Theorem. From the stability condition, the equation \(\lambda(\theta) = e^{\theta C}\) has a unique solution and the process

\[ M_n = h_{an} e^{\theta(X_1 + \cdots + X_n)} \]

is an (arrival) martingale (see, e.g., Duffield [14]), where \(h_{sn} := v_i\).

We use the same change-of-measure as in the AR/G/1 case, i.e.,

\[ P_{n,\theta}(A) := E \left[ \frac{M_n}{E[M_0]} 1_A \right] \]

(14)

for \(A \in \mathcal{F}_n = \sigma(X_1, \ldots, X_n)\), entailing the same WFI from (11).

The next theorem provides an exact result on the distribution of \(Q\).

**Theorem 3.** The stationary queue size has the distribution

\[ P(Q > \sigma) = e^{-\theta \sigma} \left( \frac{E[h_{an}]}{H} - \sum_{n=1}^{\infty} g_n(\sigma) \right), \]

(15)

for all \(\sigma \geq 0\) where

\[ g_n(\sigma) := E \left[ \left( \frac{h_{an} e^{\theta \sum_{i=1}^{n} X_i} - e^{\theta \sigma}}{H} \right) 1_{\{T=n\}} \right], \]

\(T\) is the stopping time from (13) and

\[ H := \min h_{aT} = \min \{h_{an} : a_n > C\}. \]

**Proof.** We first need to prove the positivity of the increments \(E_\theta[X_1]\) under the constructed measure \(P_\theta;\) unlike in the GI/G/1 renewal case, the argument is slightly more compounded. Define first the auxiliary (new) measure

\[ \tilde{E}[Y] := \frac{E[h_{a1} Y]}{E[h_{a1}]} \]

for some non-negative random variable \(Y \geq 0\); note that \(\tilde{P}(A) = \tilde{E}(1_A)\) for some event \(A\). Because \(M_n\) is a martingale and \(a_n\) is stationary

\[ \tilde{E}[e^{\theta X_1}] = \frac{E[h_{a1} e^{\theta X_1}]}{E[h_{a1}]} = 1. \]

Then, using Jensen’s inequality for \(\tilde{\theta} < \theta \)

\[ \frac{E[h_{a1} e^{\theta X_1}]}{E[h_{a1}]} \leq \tilde{E}[e^{\tilde{\theta} X_1}] \leq \tilde{E}[e^{\theta X_1}] = 1 = \frac{E[h_{a1} e^{\theta X_1}]}{E[h_{a1}]} . \]

Since \(E[h_{a1} X_1 e^{\theta X_1}] < \infty\) and

\[ h_{a1} \frac{e^{\theta X_1} - e^{\theta X_i}}{\theta - \tilde{\theta}} \leq h_{a1} X_1 e^{\theta X_1} \]

from the Mean Value theorem, Fatou’s lemma implies that

\[ E_\theta[X_1] = E[h_{a1} X_1 e^{\theta X_1}] \geq \lim_{\tilde{\theta} \to \theta} \frac{E[h_{a1} e^{\theta X_1} - e^{\theta X_i}]}{\theta - \tilde{\theta}} \geq \lim_{\tilde{\theta} \to \theta} \tilde{E}[e^{\theta X_1}] - \tilde{E}[e^{\theta X_i}] \geq 0. \]

Therefore, the overshoot \(R_\sigma\) defined as in (6) exists and is non-negative.

For the remaining proof fix \(\sigma \geq 0\). Because \(H\) is non-random, applying WFI from (11) with \(Y = \frac{E[M_n]}{h_{aT}}\) yields

\[ F(Q > \sigma) = E[1_{T<\infty}] = e^{-\theta \sigma} E_\theta \left[ \frac{e^{-\theta R_\sigma H}}{h_{aT}} \right] E[h_{a0}] / H. \]

Since \(e^{-\theta R_\sigma H} / h_{aT} \in [0, 1]\) we can write

\[ E_\theta \left[ \frac{e^{-\theta R_\sigma H}}{h_{aT}} \right] = \int_0^1 \tilde{P}_\theta \left( \frac{e^{-\theta R_\sigma H}}{h_{aT}} > y \right) dy \]

\[ = 1 - \int_0^1 \tilde{P}_\theta \left( R_\sigma > \frac{\ln(H) - \ln(h_{aT})}{\tilde{\theta}} - \frac{\ln y}{\tilde{\theta}} \right) dy \]

\[ = 1 - \int_0^\infty \tilde{P}_\theta \left( R_\sigma > \frac{\ln(H) - \ln(h_{aT})}{\tilde{\theta}} + Z \right) \]

\[ = 1 + \sum_{n=1}^{\infty} g_n(\sigma) H / E[h_{a0}], \]

where \(Z\) is an exponential random variable with parameter \(\tilde{\theta}\) independent of \(\mathcal{F} := \mathcal{F}_\infty\) and, using the expansion from (7),

\[ g_n(\sigma) H = \tilde{E}_\theta \left[ U > \left( Z + \ln(h_{an}) - \frac{\ln(h_{an})}{\tilde{\theta}} \right) \right] + \sigma, \]

\[ V \leq \sigma \right] E[h_{a0}] \]

\[ = E \left[ \left( h_{a1} e^{\theta U} 1_{\{U > Z + \ln(h_{an}) - \ln(h_{an}) + 1, V \leq \sigma, U > \sigma \}} \right) \mid \mathcal{F}_n \right] \]

\[ = E \left[ h_{a1} e^{\theta U} \left( 1 - \frac{H}{h_{a1}} e^{\theta \sigma} \right) 1_{\{V \leq \sigma, U > \sigma \}} \right] \]

\[ = E \left[ \left( h_{a1} e^{\theta \sum_{i=1}^{n} X_i} - H e^{\theta \sigma} \right) 1_{\{T=n\}} \right] , \]

where \(U := \sum_{i=1}^{n} X_i\) and \(V := \max_{1 \leq k \leq n-1} \sum_{i=1}^{k} X_i\). In the first equality we used \(1_{U>(z)+\sigma} = 1_{U>z+\sigma} 1_{U>\sigma}\) for any real number \(z\).

Because \(g_n(\sigma) \geq 0\), upper bounds on \(P(W > \sigma)\) follow from Theorem 3 by taking any number of terms \(g_n(\sigma)\) in the sum from (15). These could be described recursively for \(n \geq 2\) in a slightly more complicated manner than in the GI/G/1 case due to necessary conditioning, i.e.,

\[ g_n(\sigma) = E \left[ E \left[ \left( h_{a1} e^{\theta \sum_{i=1}^{n} X_i} - e^{\theta \sigma} \right) 1_{\{T=n\}} \right] \mid X_1 \right] \]

\[ = E \left[ 1_{X_1 \leq \sigma} e^{\theta X_1} E \left[ \left( h_{a1} e^{\theta U} - e^{\theta \sigma} \right) \right] 1_{\{V \leq \sigma - X_1, U > \sigma - X_1 \}} \mid X_1 \right] \]

\[ = E \left[ 1_{X_1 \leq \sigma} e^{\theta X_1} g_{n-1}(\sigma - X_1, a_1) \right] , \]
where \( U := \sum_{i=2}^{n} X_i, \) \( V := \max_{2 \leq k \leq n-1} \sum_{i=1}^{k} X_i, \) and
\[
g_n(\sigma, s_i) = \mathbb{E}\left[ \frac{h_{a_n}}{H} e^{\sigma \sum_{i=1}^{n} X_i - e^{\sigma} } 1_{T=n} \mid a_0 = s_i \right]
\]
\[
= \sum_j P_{i,j} \mathbb{E}\left[ \frac{h_{a_n}}{H} e^{\sigma \sum_{i=1}^{n} X_i - e^{\sigma} } 1_{T=n} \mid a_1 = s_j \right]
\]
\[
= \sum_j P_{i,j} \mathbb{E}\left[ \frac{h_{a_n}}{H} e^{\sigma} X_{1} \left( \frac{h_{a_n}}{H} e^{\sigma U} - e^{\sigma} X_{1} \right) 1_{V \leq \sigma - X_i, U > \sigma - X_i} \mid a_1 = s_j \right]
\]
\[
= \sum_j P_{i,j} \mathbb{E}\left[ 1_{X_1 \leq \sigma} e^{\sigma} X_{1} g_{n-1}(\sigma - X_1, s_j) \mid a_1 = s_j \right]
\]
for all states \( s_i \in S. \) The initial conditions are
\[
g_1(\sigma, s_i) = \mathbb{E}\left[ \frac{h_{a_1}}{H} e^{\sigma} X_{1} - e^{\sigma} 1_{X_1 > \sigma} \mid a_0 = s_i \right].
\]

**A. Example 1: Homogeneous On-Off Processes**

As a particular case, consider an aggregate of (independent and homogeneous) \( N \) On-Off sources/processes \( a_n^{(s)} \) for \( s = 1, \ldots, N, \) each with state space \( \{0, 1\}. \) A convenient interpretation is that \( a_n^{(s)} = 0 \) if the source is ‘Off’ (or idle) at time \( n, \) and \( a_n^{(s)} = 1 \) if the source is ‘On’ and transmits \( R \) data units in a time unit. The probability transition matrix is
\[
P = \begin{pmatrix} 1 - \alpha & \alpha \\ \beta & 1 - \beta \end{pmatrix}.
\]
The stationary distribution is \( \pi_0 = \frac{\beta}{\alpha + \beta} \) and \( \pi_1 = \frac{\alpha}{\alpha + \beta} \) (the index 0 corresponds to the ‘Off’ state). Denoting by \( c \) the per-source capacity the utilization is \( \rho = \frac{\mathbb{E}[a_n^{(1)}]}{c} = \pi R \) and the overall capacity is \( C = N c. \)

Rather than working with the transition matrix for the aggregate process having the state space \( \{0, 1, \ldots, N\}, \) each corresponding to the number of sources in the ‘On’ state, we follow a computationally much simpler approach from [23]. Define the spectral radius \( \lambda(\theta) \) and the corresponding (non-negative) right eigenvector \( h = (h_0, h_1)^T \) for a single source only, relative to the transformed transition matrix \( P_{i,j} \) for \( i, j \in \{0, 1\}. \) Denoting by \( \theta > 0 \) the unique solution of \( \lambda(\theta) = e^{\theta c}, \) subject to the stability condition \( \mathbb{E}[a_n^{(1)}] < c, \) the arrival martingale is
\[
M_n = h_{a_n} e^{\theta(x_1 + \ldots + x_n)}
\]
where \( a_n = \sum_{s=1}^{N} a_n^{(s)}, \) \( h_{a_n} = \prod_{s=1}^{N} h_{a_n^{(s)}}, \) and \( X_n = a_n R - C. \)

Bounds on the tail probability \( \mathbb{P}(Q > \sigma) \) follow using the change-of-measure from (14) and applying Theorem 3. Interestingly, the behavior of \( \mathbb{P}(Q > \sigma) \) fundamentally differs on the value of \( \alpha + \beta \) relative to 1. Observe first that \( H \) from Theorem 3 is
\[
H = \inf h_{a_T} = \begin{cases} h_1^N & \text{if } h_1 \leq h_0 \\ h_{0}^{N-[C/R]} h_1^{[C/R]} & \text{if } h_1 > h_0 \end{cases}
\]
and \( \mathbb{E}[h_{a_T}] = \pi_0 h_0 + \pi_1 h_1, \) where \( T \) is the stopping time defined in (13).

The Kingman bound (see, e.g., Duffield [14]) is
\[
\mathbb{P}(Q > \sigma) \leq \frac{\mathbb{E}[h_{a_T}]^N}{H} e^{-\sigma}. \tag{17}
\]
In turn, a Theorem 3 bound when using only \( g_1(\sigma) \) becomes
\[
\mathbb{P}(Q > \sigma) \leq e^{-\sigma} \left( \frac{\mathbb{E}[h_{a_1}]}{H} \right)^N
- \sum_{i>0} \left( \frac{h_1 h_0^{N-i}}{H} e^{\theta(iR-C)} - e^{\theta \sigma} \right) \left( \frac{N}{i} \right) \pi_0^{N-i} \pi_1^i.
\]

The crucial observation is that, if \( h_1 > h_0 \) which is equivalent to \( \alpha + \beta < 1 \) (see Buffet and Duffield [6]), then
\[
\mathbb{P}(Q > \sigma) \approx \gamma N e^{-\sigma},
\]
for some \( \gamma < 1. \) In turn, if \( h_0 \geq h_1 \) which is equivalent to \( \alpha + \beta \geq 1 \) (6)] then
\[
\mathbb{P}(Q > \sigma) \approx \zeta N e^{-\sigma},
\]
but for some \( \zeta \geq 1. \) In the former regime, called *more bursty than Poisson*, the tail of the queue decays exponentially fast in the number of sources \( N, \) whereas in the latter regime, called *less bursty than Poisson*, the tail of the queue grows exponentially fast in the number of sources \( N. \) This behavior was conjectured by Choudhury et al. [8] and it is precisely captured by the Kingman bound from (17).

**Fig. 3:** Kingman and Theorem 3 bounds on the queue size \( \mathbb{P}(Q > \sigma) \) for \( N \) On-Off Markov processes, for both *more* and *less bursty* than Poisson regimes \( (N = 5, R = 1, \rho = 0.25) \)

(a) *more bursty:* \( \alpha = 0.1, \beta = 0.5 \) (b) *less bursty:* \( \alpha = 0.2, \beta = 0.9 \)

Fig. 3 illustrates the Kingman and Theorem 3 bounds, for several number of terms \( g_n(\sigma) \), against simulations for both the *more* and *less bursty* than Poisson regimes; the parameters’ values are given in the caption. (a) indicates that the Kingman bound overestimates by roughly one order of magnitude, whereas the bounds from Theorem 2 quickly become arbitrarily sharp, e.g., by using \( \sum_{n=1}^{5} g_n(\sigma) \) the match
with simulations is almost perfect. We mention that alternative effective bandwidth approximations or stochastic network calculus bounds overestimate themselves the Kingman bounds by several orders of magnitude (see, e.g., [8], [23]).

In turn, in the less-bursty regime from (b), the Kingman bound overestimates simulations by roughly three orders of magnitude; counterintuitively, the effective bandwidth approximation was shown to underestimate by several orders of magnitude [8]! The bounds from Theorem 3 are very sharp: using only $g_1(\sigma) + g_2(\sigma)$ the match with simulations over the first two ‘steps’ is almost perfect (we observe that simulations ‘stop’ around $\sigma = 1.4$ due to using only $10^7$ runs). The further behavior becomes apparent from using $g_1(\sigma) + g_3(\sigma) + g_3(\sigma)$; capturing the third ‘step’ of $P(Q > \sigma)$ using simulations would require about one year of running time.

**B. Example 2: Heterogenous On-Off Processes**

Here we generalize the results from §IV-A by considering $N_l$ On-Off sources of type $l = 1, 2$, all being independent. We will use the same notation except for an additional super(sub)script, e.g., the probability transition matrices are $P^{(l)},$ and the transition probabilities are $\alpha_l$ and $\beta_l.$ Without loss of generality we assume the same rate $R$ for both source types; different rates could be considered by suitably scaling the transition probabilities. The overall capacity is $C$ and assume the stability condition

$$N_1\pi_{1,1}R + N_2\pi_{2,1}R < C,$$

where $\pi_{1,1}$ is the ‘On’ stationary probability for a type $l$ source.

The key difficulty in the heterogeneous case is the construction of a martingale $M_n$ containing a single exponential, as the one from (16) in the homogeneous case. To do so, we split the capacity $C$ into $C = C_1 + C_2,$ where $C_1$ is the overall capacity allocated to the sources of type $l$. We parameterize $C_1 = wC$ where

$$\frac{N_1\pi_{1,1}R}{C} < w < 1 - \frac{N_2\pi_{2,1}R}{C},$$

such that a stability condition is satisfied for both source types.

As in the homogeneous case, for each source type $l$, denote by $\lambda_l(\theta)$ the spectral radius of the transformed transition matrix $P^{(l)}_\theta(i,j) = P^{(l)}_1 e^{\theta \beta_j}$ for $i,j \in \{0,1\}$. For each $w$ in the range above, denote by $\theta_{1,w}$ the unique solutions of

$$\lambda_l(\theta) = e^{\theta \beta_l},$$

for $l = 1, 2$; the unicity is guaranteed, as before, from stability. The crucial observation is that $\theta_{1,w} = 0$ when $w \rightarrow \frac{N_1\pi_{1,1}R}{C}$ as the utilization for the type 1 sources would approach 1; similarly, $\theta_{2,w} = 0$ at the other extreme $w \rightarrow 1 - \frac{N_2\pi_{2,1}R}{C}$ at which the utilization for the type 2 sources would approach 1. Therefore, from the continuity of the solutions $\theta$ in (18), depending on the (continuous) parameter $w$, it follows that there exists a value $w \in \left(\frac{N_1\pi_{1,1}R}{C}, 1 - \frac{N_2\pi_{2,1}R}{C}\right)$ such that

$$\theta := \theta_{1,w} = \theta_{2,w}.$$

For this value of $\theta$ we denote by $h_l = (h_{l,0}, h_{l,1})^T$ the (non-negative) right eigenvectors corresponding to the spectral radii $\lambda_l(\theta)$ for $l = 1, 2.$ We thus obtain the martingales

$$M_{1,n} = h_{1,1,a_n} e^{\theta(X_{1,1} + \cdots + X_{1,n})}$$

for $l = 1, 2,$ where $a_{l,n} := \sum_{s=1}^{N_{l_1}} h_{1,1,s},$ $h_{1,1,a_n} := \prod_{s=1}^{N_{l_1}} h_{1,1,s},$ and $X_{1,n} := a_{1,n}R - C_l.$ Furthermore, using the sources’ independence, we obtain the single martingale

$$M_n = h_{1,1,a_n,a_n} e^{\theta(X_{1,1} + \cdots + X_n)},$$

where $h_{1,1,a_n,a_n} := h_{1,1,a_n} h_{2,2,a_n}$ and $X_n := X_{1,n} + X_{2,n}.$

The rest follows as in the homogeneous case and Theorem 3. First, without loss of generality we assume that $h_{1,1} h_{1,0} \leq h_{2,2} h_{2,0}.$ Then $H = \inf h_{(1,1,1,1,1)},$ where $T$ is the stopping time from (13), can be expressed as

$$h_{N_1,1} h_{2,1} \quad \text{if } h_{1,1} h_{1,0} \leq h_{2,2} h_{2,0} < 1$$

$$h_{1,1} h_{2,2} h_{N_2,2} \quad \text{if } h_{1,1} h_{1,0} \leq 1 \leq h_{2,2} h_{2,0}$$

$$h_{1,1} h_{2,2} h_{N_1 - N_1 \wedge [C/R], 2} \quad \text{if } 1 < h_{1,1} h_{1,0} \leq h_{2,2} h_{2,0}$$

where $L := ([C/R] - N_1)_+, X_+ := \max(x, 0),$ and $x \wedge y := \min(x, y)$ for real numbers $x$ and $y.$

Observing that

$$E[h_{1,1,a_n,a_n}] = \left(\frac{\alpha_1 h_{1,1} + \beta_1 h_{1,0}}{\alpha_1 + \beta_1}\right)^{N_1} \left(\frac{\alpha_2 h_{2,1} + \beta_2 h_{2,0}}{\alpha_2 + \beta_2}\right)^{N_2}$$

it follows that the Kingman bound is

$$P(Q > \sigma) = \frac{E[h_{1,1,a_n,a_n}]}{H} e^{-\theta \sigma}$$

whereas the exact result is

$$P(Q > \sigma) = e^{-\theta \sigma} \left\{ \frac{E[h_{1,1,a_n,a_n}]}{H} - \sum_{k=1}^{\infty} g_k(\sigma) \right\}.$$
Fig. 4: Bounds on $\mathbb{P}(Q > \sigma)$ for $N_1$ less bursty than Poisson On-Off Markov processes and $N_2$ more bursty than Poisson On-Off Markov processes ($N_1 = 2, \alpha_1 = 0.2, \beta_1 = 0.9, N_2 = 2, \alpha_2 = 0.1, \beta_2 = 0.5, R = 1, \rho = 0.25$)

jointly taken after $\max_{1 \leq k \leq n-1} \left\{ \sum_{i=1}^{k} (i_i + \beta_j) - \frac{4C}{FR} \right\} \leq \frac{\sigma}{R}$ and $\sum_{i=1}^{n} (i_i + \beta_j) - \frac{4C}{FR} > \frac{\sigma}{R}$.

Fig. 4 illustrates the accuracy of the bounds by multiplexing the less bursty and more bursty than Poisson sources used in Figs. 3.(a-b). The inaccuracy of the Kingman bounds is roughly the average of the corresponding inaccuracies from Figs. 3.(a) and (b), due to multiplexing the same types of sources. The proposed bounds become very sharp with only the first four terms $g_n(\sigma)$, whereas simulations stop around $\sigma = 2.4$ due to restricting to $10^7$ runs only. Lastly, as an overarching conclusion, simple exponential approximations/bounds are not fit for purpose, given the true behavior of the exact results apparent from simulations; this is progressively captured by the $g_n(\sigma)$ terms.

V. CONCLUSIONS

We have proposed a new class of exact results for some of the most popular queueing models, which in their most general forms are only subject to computationally complex numerical methods or provably inaccurate approximations/bounds. The new results are expressed as an infinite series with closed-form terms. Remarkably, for numerical purposes, the first few terms are the dominant ones, in the sense of being sufficient to render ultra-sharp queueing bounds; the key reason lies in the reversal of the drift’s sign from the original queueing model. Several numerical examples confirm the ultra-sharpness of the new bounds, including the very challenging scenario with multiplexing heterogeneous Markovian sources, some of which being more bursty than Poisson and the others being less bursty than Poisson.

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