Microscopic Derivation of Gibbs Measures for the Focusing One Dimensional Nonlinear Schrödinger Equation

by

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Declarations

This thesis is submitted to the University of Warwick in support of my application for the degree of Doctor of Philosophy. It has been composed by myself and has not been submitted in any previous application for any degree. The work has been carried out by myself under the supervision of Vedran Sohinger.

Parts of this thesis have been published or submitted for publication by the author. Parts of Chapter 2 and Chapter 3, as well as Appendix A appear in an article co-authored with Vedran Sohinger, *A microscopic derivation of Gibbs measures for the 1D focusing cubic nonlinear Schrödinger equation*, due to appear in Communications in Partial Differential Equations [69]. Parts of Chapter 2 and Chapter 4 form a paper with Vedran Sohinger, *A microscopic derivation of Gibbs measures for the 1D focusing quintic nonlinear Schrödinger equation*, which has been submitted for publication [70].

Abstract

In this thesis, we give a mircoscopic derivation of Gibbs measures for the focusing cubic and quintic (nonlocal) nonlinear Schrödinger equations (NLS) on $\mathbb{T}$ from many body quantum Gibbs states. In the cubic case, this corresponds to taking a two-body interaction, whereas the quintic case corresponds to a three-body interaction. Since we are not making positivity assumptions on the interaction potential, it is necessary to truncate in the mass in the classical setting and the rescaled particle number in the quantum setting.

Our methods are based on the perturbative expansion developed in the work of Fröhlich, Knowles, Schlein, and Sohinger [29]. We obtain results in both the time independent and time dependent cases. These are the first known results in the focusing regime and for any quintic regime. In particular, we give the first microscopic derivation of time-dependent correlation functions for Gibbs measures corresponding to the quintic NLS, as studied in the work of Bourgain [9]. In the quintic case, we can only study a suitable nonlocal quintic NLS, preventing us from obtaining a derivation of the local NLS in the quintic case.

Chapter 1

Introduction

1.1 General introduction

The nonlinear Schrödinger equation (NLS) is a nonlinear partial differential equation which arises in the physics of quantum optics and Bose-Einstein condensation. It is also an example of a nonlinear dispersive equation, which intuitively means its solutions spread out in physical space unless boundary conditions are imposed. This phenomenon is called dispersion.

The well-posedness of the NLS is an interplay between the dispersive properties of the linear Schrödinger equation and the nonlinearity, with the sign playing an integral role. For a positive (defocusing) nonlinearity, one can use conservation of energy and mass, as well as the dispersion to show that the the NLS is well-posed for any sufficiently regular initial condition. For a non-positive (focusing) nonlinearity, blow up of solutions can occur for large initial conditions which are regular. For less regular initial conditions, even in the defocusing case, one cannot expect well-posedness for all inital conditions and the NLS is ill-posed. To get around this, one introduces an invariant (Gibbs) measure, which is supported at low regularity. One is then able to use the invariance of the measure to show that the NLS is globally well-posed for any initial condition in the support of the measure.

On the other hand, the NLS is an effective equation for a many-body quantum system satisfying the many-body linear Schrödinger equation. This means that, under certain conditions, as the number of particles goes to infinity, the solution of the many-body Schrödinger equation is well approximated by the solution of the NLS. This is made rigorous in the sense of reduced density matrices, see [35, 41, 78]. A natural question is what does the Gibbs measure for the NLS correspond to on the many-body side? This question has been well studied in the case of a defocusing (repulsive) two-body interaction, where it has been shown that the grand canonical ensemble (or quantum Gibbs state) converges to the Gibbs measure for the defocusing cubic NLS in the mean-field limit; see [29–31, 33, 34, 49–53, 80].

In this thesis, we study this question in the case of focusing (attractive) interactions in one dimension – in particular for two and three-body interactions. In this case, the (truncated) grand canonical ensemble corresponds to the (truncated) Gibbs measures for the appropriate focusing cubic and quintic NLS respectively.

1.2 The nonlinear Schrödinger equation

The general *nonlinear Schrödinger equation* (NLS) is given by

$$\begin{cases} i\partial_t u = -\Delta u + N(u), \\ u(x,0) = u_0 \in H^s(\mathbb{T}), \end{cases} \tag{1.1}$$

where $u : X \times \mathbb{R} \to \mathbb{C}$ is a function of time and space, N is a nonlinearity, and $H^s(X)$, defined in (1.14) below, denotes the regularity of the initial condition u_0. We denote by $x \in X$ and $t \in \mathbb{R}$ the spatial and time variables respectively. We will restrict our study to the case where $X := \mathbb{T} \equiv \mathbb{T}^1 \equiv \mathbb{R}/\mathbb{Z} \equiv [-1/2, 1/2)$. We will typically consider consider the following nonlinearities.

1. $N(u) := \int_T w(x-y)|u(y)|^2 u(x)$.
2. $N(u) := \int_{\mathbb{T}} dy\, dz\, w(x-y)w(y-z)w(z-x)|u(y)|^2|u(z)|^2 u(x)$. ,

where $w : \mathbb{T} \to \mathbb{R}$. Throughout we use $*$ to denote convolution with respect to the spatial variable, i.e. $u * v(x) := \int_{\mathbb{T}} u(y)v(x-y)$. When $w \in L^1(\mathbb{T})$, (1.1) is called the (one dimensional) Hartree equation. To differentiate between nonlinearities 1. and 2., we will describe the corresponding equations as the (one dimensional) cubic and quintic Hartree equations respectively. We note that although both nonlinearities depend on time, we will suppress the time from our notation throughout.

Taking $w = \pm\delta$, we recover the local NLS

$$\begin{cases} i\partial_t u = -\Delta u \pm |u|^{2k} u, \\ u(x,0) = u_0 \in H^s(\mathbb{T}), \end{cases} \tag{1.2}$$

which is so-called because its nonlinearity at (x,t) depends only on the value of $u(x,t)$, which is not true for nonlinearities 1. and 2.

Associated with (1.1) and (1.2) are two invariant quantities

$$\mathcal{N}(u) := \int_{\mathbb{T}} dx\, |u(x)|^2, \tag{1.3}$$

$$H(u) := \int_{\mathbb{T}} dx\, |\nabla u(x)|^2 + E_p(N), \tag{1.4}$$

where $\mathcal{N}$ is the *mass* and H is the *energy*, which we also call the *Hamiltonian.* We split the Hamiltonian into two terms, the *kinetic energy* and the *potential energy*, which we denote E_p. The potential energy is given by

1. $E_p(N) = \frac{1}{2}\int_{\mathbb{T}} dx\, dy\, |u(x)|^2 w(x-y)|u(y)|^2$,

2. $E_p(N) = \frac{1}{3}\int_{\mathbb{T}} dx\, dy\, dz |u(x)|^2|u(y)|^2|u(z)|^2 w(x-y)w(y-z)w(z-x)$,

for the nonlinearities 1. and 2. respectively. Putting $w = \pm\delta$, we recover the potential energy for the local NLS, namely

$$E_p = \pm\frac{1}{2}\int_{\mathbb{T}} |u(x)|^p dx,$$

for $p = 4$ or $p = 6$ respectively. The invariance of (1.3) and (1.4) for smooth solutions and potentials can be shown by differentiating under the integral sign. For solutions corresponding to initial conditions below the energy space, one uses a density argument and the persistence of regularity of solutions[1].

Where positivity assumptions are made on the interaction potential, we call (1.1) the defocusing problem. Standard positivity assumptions on the interaction potential are $w \geq 0$ pointwise, or that it is of positive type, meaning $\hat{w} \geq 0$ pointwise. These positivity assumptions mean that the potential energy is positive, which in general simplifies the analysis of the well-posedness properties of (1.1). In contrast, if we make no positivity assumptions on the interaction potential, the problem is called focusing.

1.3 Deterministic well-posedness of the NLS

The deterministic well-posedness of the NLS depends on an interplay between the conservation laws (1.3) and (1.4), the dispersive effects of the linear equation, which gives rise to *Stichartz estimates*, and the strength and sign of the nonlinearity. These estimates give bounds on the solution to the linear Schrödinger equation in terms of the initial condition u_0. In the periodic case, these estimates rely on techniques from

[1]This is only possible above a certain regularity of initial condition; see for example [55, 82]

analytic number theory. For a statement of the Strichartz results in the periodic setting, see Lemma 1.7.3 and the discussion after Remark 1.7.4. These estimates are captured by the *dispersive Sobolev spaces*, which are defined in Definition 1.7.5 below.

The Strichartz estimates are used to establish local well-posedness, and the conservation laws are used to show the solutions do not blow up and obtain global well-posedness. Even in the case of a positive nonlinearity, this approach only works up to a certain regularity of initial condition; see [36] for more details on finite time blow up. Precise statements of well-posedness results for various NLS equations can be found for example in [55, Sections 5 and 6] and [82, Section 4]. Moreover, the range of nonlinearities that can be dealt with can be understood with a scaling heuristic, for example see [82, Section 3]. Heuristically, the conservation laws are only defined for sufficiently regular functions, for example H^1 for (1.4). As such, we can only expect to get deterministic global well-posedness above a certain critical regularity. To get around this, we introduce an invariant measure, explained below.

Remark 1.3.1. We note here that the local cubic NLS in one spatial dimension is an integrable system, so has infinitely many conservation laws associated with it. We will only consider the NLS as the limit of the Hartree equation for suitable interaction potentials, so this integrability will not be used, and we make no further comments on it.

1.4 Gibbs measures

To consider the well-posedness of functions with low regularity, we need to introduce a probability measure on the space of initial conditions. To this end, for a defocusing potential, we consider the (heuristically) defined *Gibbs measure* on $L^2(\mathbb{T})$ given by

$$d\mathbb{P}_{\text{Gibbs}} := \frac{1}{z_{\text{Gibbs}}} \mathrm{e}^{-H(u)} du, \tag{1.5}$$

where z_{Gibbs} is a normalisation constant, H is the Hamiltonian defined in (1.4), and du is the (ill-defined) infinite dimensional Lebesgue measure. By drawing analogy to the finite dimensional case, Liouville's theorem suggests this measure should be invariant under the dynamics of the NLS. To make this rigorous, we will realise the Gibbs measure as a weighted *Wiener measure*, which is supported on functions of low regularity. The invariance of this measure acts as a replacement for the conservation laws, and implies probabilistic well-posedness results. The rigorous construction of these measures was first considered in the constructive field theory literature, for example [37, 58, 59, 75], and they were studied further in [5, 17, 18, 20, 39, 47, 56, 57, 65] and references within. Further details on the rigorous construction of such measures are given in Appendix A. The invariance of the measure was first proved by Bourgain in [9–11], with some preliminary results proved by Zhidkov [89]. This invariance has been extensively studied in the PDE community; we direct the reader to the expository works [19, 60, 64], and for further results [14–16, 24, 40, 65] and the references within. The idea of an invariant measure has been generalised to the study of quasi-invariant measures; see for example [27, 85] and the references therein.

In the case of a focusing potential, H is no longer positive definite, so formally one can have $z_{\mathrm{Gibbs}} = \infty$. Instead one considers the following modification of (1.5)

$$d\mathbb{P}^f_{\mathrm{Gibbs}} := \frac{1}{z^f_{\mathrm{Gibbs}}} \mathrm{e}^{-H(u)} f(\mathcal{N}) du, \tag{1.6}$$

where $f \in C_c^\infty(\mathbb{R})$ is a suitable cut off function, and z^f_{Gibbs} is a normalisation constant. Such measures were considered in [3, 9, 20, 25, 37, 84], and will be the main Gibbs measures discussed in this thesis.

Given the measure $d\mathbb{P}^f_{\mathrm{Gibbs}}$ as in (1.6) and a well defined time evolution S_t, for example as in (2.5) below, one can consider the corresponding *time-dependent correlation functions*. Namely, for $m \in \mathbb{N}^*$, times $t_1, \dots, t_m \in \mathbb{R}$, and functions $X^1, \dots, X^m \in C^\infty(\mathfrak{h})$, we define

$$\mathcal{Q}^f_{\mathrm{Gibbs}}(X^1, \dots, X^m; t_1, \dots, t_m) := \int d\mathbb{P}^f_{\mathrm{Gibbs}}(\varphi)\, X^1(S_{t_1}\varphi) \cdots X^m(S_{t_m}\varphi)\,, \tag{1.7}$$

which we call the *m-particle time-dependent correlation associated with H and X^j, t_j, $j = 1, \dots, m$.*

1.5 Many-body quantum mechanics

We can view (1.1) with a cubic non-local linearity as the *classical limit* of of a many-body quantum system. Namely, given $n \in \mathbb{N}$, we consider the *n-body Hamiltonian* given by

$$H^{(n)} := \sum_{j=1}^{n} -\Delta_j + \frac{\lambda}{2} \sum_{i \neq j}^{n} w(x_i - x_j), \tag{1.8}$$

where Δ_j denotes the Laplacian in the j^{th} component and $\lambda > 0$ is a coupling constant. The n-body Hamiltonian is defined on the *bosonic n-particle space* $L^2_{\mathrm{sym}}(\mathbb{T}^n)$.

This is the subset of $u \in L^2(\mathbb{T}^n)$ satisfying

$$u(x_1, \ldots, x_n) = u(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$$

for any permutation $\sigma \in S_n$. To make both terms in (1.8) have the same order, we take the *interaction strength* $\lambda > 0$ to be of order $\frac{1}{n}$. This scaling regime is known as the *mean-field regime*. The mean-field regime corresponds to short-range (weak) interactions, and is necessary to obtain a non-trivial limit as the number of particles tends to infinity.

We define the *n-body Schrödinger equation* as

$$i\partial_t \Psi_{n,t} = H^{(n)} \Psi_{n,t}, \tag{1.9}$$

where $H^{(n)}$ is as in (1.8). For suitably chosen initial conditions, one wants to compare the dynamics of (1.9) to the dynamics of (1.1) as $n \to \infty$. Rigorous results of this kind were first proved in [35, 41, 78], where the authors proved convergence in terms of reduced one-particle density matrices. For an expository view on these results, we direct the reader to [6, 72]

In this thesis, we study the relationship between the Gibbs measure associated with (1.1) and the *Gibbs states* of the (1.9), which are the equilibrium states of (1.9) at temperature $\tau > 0$. The Gibbs state at temperature $\tau > 0$ is defined as

$$P_\tau^{(n)} \equiv P_\tau^{f,(n)} := \mathrm{e}^{-H^{(n)}/\tau} f\left(\frac{n}{\tau}\right),$$

which acts on $L^2_{\mathrm{sym}}(\mathbb{T}^n)$, where we recall the cut-off function from (1.6). These states correspond to a cut-off *grand canonical ensemble* operator.

The main goal of this thesis is to show that one can obtain (1.7) as a *mean-field* limit of corresponding many-body quantum objects, which we henceforth refer to as a *microscopic derivation*. We do this in two steps.

(i) **Step 1:** Analysis of the *time-independent problem*, i.e. when $t_1 = \cdots = t_m = 0$.

(ii) **Step 2:** Analysis of the *time-dependent problem*. This is the general case.

The precise statements of our results can be found in Chapter 2.

For interaction potentials with positivity assumptions, grand canonical ensembles without cut-off were studied in [29–31, 33, 34, 49–53, 80]. An overview of these results can be found in [32].

For the quintic nonlinearity, we instead consider the n-body Hamiltonian defined by

$$H^{(n)} := \sum_{i=1}^{n} -\Delta_j + \frac{\lambda}{3} \sum_{\substack{i\neq j\neq k \\ i\neq k}}^{n} w(x_i - x_j)w(x_j - x_k)w(x_k - x_i), \tag{1.10}$$

where in this case $\lambda \sim \frac{1}{n^2}$. In this case, the classical limits of the cubic equation should be seen as a guiding heuristic. Some results for the quintic case were proved in [21, 22, 48, 88], however these are not directly related to the quintic equation studied in this thesis. For further details, we direct the reader to Section 2.3.

1.6 Notation and conventions

We adopt the convention that $\mathbb{N} = \{0, 1, 2, \ldots\}$ is the set of non-negative integers and we write $\mathbb{N}^* := \{1, 2, \ldots\}$. We write C to denote generic positive constants that can change line to line. If C depends on a set of parameters $\{a_1, \ldots, a_n\}$, we will write $C(a_1, \ldots, a_n)$. We also sometimes write $a \lesssim b$ to denote $a \leq Cb$, moreover we write $a \lesssim_{a_1,\ldots,a_n}$ is $a \leq C(a_1, \ldots, a_n)b$. Similarly we use $\sim$ to denote equality up to a constant. $C_c^\infty(U)$ will be used to denote the set of compactly supported smooth functions $f : U \to \mathbb{R}$.

We write $\mathbf{1}$ to denote the identity operator on a Hilbert space $\mathcal{H}$ and write $\mathcal{L}(\mathcal{H})$ for the space of bounded linear operators on $\mathcal{H}$. Moreover, if $\mathcal{H}$ is separable and $p \in [1, \infty]$, we define the Schatten space $\mathfrak{S}^p(\mathcal{H})$ to be the set of $\mathcal{A} \in \mathcal{L}(\mathcal{H})$ with $\|\mathcal{A}\|_{\mathfrak{S}^p(\mathcal{H})} < \infty$, where

$$\|\mathcal{A}\|_{\mathfrak{S}^p(\mathcal{H})} := \begin{cases} (\mathrm{Tr}|\mathcal{A}|^p)^{1/p} & \text{if } p < \infty \\ \sup \mathrm{spec}|\mathcal{A}| & \text{if } p = \infty. \end{cases} \tag{1.11}$$

Here $|\mathcal{A}| := \sqrt{\mathcal{A}^*\mathcal{A}}$ and spec denotes the spectrum of $\mathcal{A}$.

For $p \in \mathbb{N}$, we define $\mathfrak{h}^{(p)} := L^2_{\mathrm{sym}}(\mathbb{T}^p)$, and we note $\mathfrak{h} := \mathfrak{h}^{(1)} \equiv L^2_{\mathrm{sym}}(\mathbb{T}) \equiv L^2(\mathbb{T})$. We also write

$$\mathfrak{B}_p := \{\xi \in \mathfrak{S}^2(\mathfrak{h}^{(p)}) : \|\xi\|_{\mathfrak{S}^2(\mathfrak{h}^{(p)})} \leq 1\}, \tag{1.12}$$

$$\mathcal{C}_p := \mathfrak{B}_p \cup \mathbf{1}_p, \tag{1.13}$$

where $\mathbf{1}_p$ is the identity operator on $\mathfrak{h}^{(p)}$.

If ξ is a closed linear operator on $\mathfrak{h}^{(p)}$, we can identify it with its Schwartz kernel, which we write as $\xi(x_1, \ldots, x_p; y_1, \ldots, y_p)$; see for example [66, Corollary V.4.4].

For a set A, the indicator function on A is given by

$$\chi_A(x) := \begin{cases} 0 & \text{if } x \notin A \\ 1 & \text{if } x \in A. \end{cases}$$

Where clear, we also adopt the convention of dropping the space we are integrating over from our integrals.

1.7 Auxiliary results

Harmonic analysis

We take the convention that the *Fourier coefficients* for a function $u \in L^1(\mathbb{T})$ are given by

$$\hat{u}(k) := \int dx\, u(x) \mathrm{e}^{-2\pi ikx}.$$

Then we define the *Sobolev space of regularity* $s \in \mathbb{R}$ to be the space

$$H^s(\mathbb{T}) := \{u \in \mathcal{D}'(\mathbb{T}) : \sum_{k\in\mathbb{Z}} |\hat{u}(k)|^2 (1 + |k|^2)^s < \infty\}, \tag{1.14}$$

where $\mathcal{D}'(\mathbb{T})$ is the space of distributions with test functions taken in $C_c^\infty(\mathbb{T})$. Then

$$\|u\|_{H^s(\mathbb{T})}^2 := \sum_{k\in\mathbb{Z}} |\hat{u}(k)|^2 (1 + 2\pi|k|^2)^s \equiv \left\| \hat{u}(k)(1 + 2\pi|k|^2)^{s/2} \right\|_{\ell_k^2}^2 .$$

We also use the notation

$$H^{s-}(\mathbb{T}) := \bigcup_{s'<s} H^{s'}(\mathbb{T}).$$

The following embedding theorem for Sobolev spaces into L^p spaces holds; see for example [7].

Theorem 1.7.1 (Sobolev embedding theorem for $d = 1$)**.** *Let* $p \in (1, \infty]$, $s > 0$, *and* $\frac{1}{p} = \frac{1}{2} - s$. *Then*

$$\|u\|_{L^p(\mathbb{T})} \lesssim_{p,s} \|u\|_{H^s(\mathbb{T})}.$$

Finally, we will also need to use the theory of Fourier multipliers on the torus, so we need the following extension of the Mikhlin multiplier theorem. The statement and proof of the following theorem in full generality can be found in

[81, VII, Theorem 3.8]. Here we use the notation $(L^p(X), L^p(X))$ to denote the class of bounded operators from $L^p(X)$ to $L^p(X)$ which commute with translations.

Lemma 1.7.2 (Mikhlin multiplier theorem for $\mathbb{T}$)**.** *Let $p \in [1, \infty]$ and $T \in (L^p(\mathbb{R}), L^p(\mathbb{R}))$ be a Fourier multiplier operator. Let $\hat{u}$ be the multiplier corresponding to T and suppose that $\hat{u}$ is continuous at every point of $\mathbb{Z}$. For $k \in \mathbb{Z}$, let $\lambda(k) := \hat{u}(k)$. Then there is a unique periodised lattice operator $\tilde{T}$ defined by*

$$\tilde{T} f(x) \sim \sum_{k \in \mathbb{Z}} \lambda(k) \hat{f}(k) \mathrm{e}^{2\pi i k x}$$

such that $\tilde{T} \in (L^p(\mathbb{T}), L^p(\mathbb{T}))$ and $\|\tilde{T}\|_{L^p \to L^p} \leq \|T\|_{L^p \to L^p}$.

Mixed L^p spaces

Throughout, we will make use of so called "mixed L^p spaces." Formally, for $p, q \in [1, \infty]$, Banach spaces X, Y, and $u : Y \times X \to \mathbb{C}$, we define

$$\|u\|_{L^p_X L^q_Y} := \| \|u(\cdot, x)\|_{L^q_Y} \|_{L^p_X}. \tag{1.15}$$

In other words, for each $x \in X$, we compute the L^q norm of $y \mapsto u_x(y) := u(y, x)$, and then we compute the L^p norm of the function $x \mapsto \|u(x, y)\|_{L^q_Y}$. This is all formal because we are assuming that the quantity in (1.15) is well defined. This can be made rigorous using the theory of Bochner integrals and Bochner spaces, for example see [73]. Where $p = q$, we sometimes write $L^p_X L^q_Y \equiv L^p_{X,Y}$. One important duality result which holds is that, for $p, q \in (1, \infty)$ and Banach spaces X and Y

$$(L^p L^q(X \times Y))^* \cong L^{p'} L^{q'}(X \times Y),$$

where p' and q' denote the Hölder conjugate exponents of p and q respectively $(1/p + 1/p' = 1)$.

Strichartz estimates and $X^{s,b}$ spaces

A solution to the linear Schrödinger equation is a function $u : \mathbb{T} \times \mathbb{R} \to \mathbb{C}$ satisfying

$$\begin{cases} i\partial_t u = -\Delta u, \\ u(x, 0) = u_0(x) \in H^s. \end{cases} \tag{1.16}$$

Solutions to (1.16) can be written as $u(x,t) = e^{it\Delta}u_0(x)$, where

$$\mathrm{e}^{it\Delta}u_0(x) = \sum_k \hat{u}_0(k)\mathrm{e}^{-4\pi^2 ik^2 t + 2\pi ikx}.$$

We call $\mathrm{e}^{it\Delta}$ the *free Schrödinger kernel.* Solutions to the linear Schrödinger equation satisfy the following Strichartz estimates, as proven in [8].

Lemma 1.7.3 (Strichartz estimates for $\mathbb{T}$)**.** *Suppose that* $u_0 = \sum_k a_k \mathrm{e}^{2\pi ikx} \in L^2(\mathbb{T})$*. Let the Dirichlet projection* $P_N : L^2(\mathbb{T}) \to L^2(\mathbb{T})$ *be defined by* $P_N u_0(x) = \sum_{|k|<N} a_k \mathrm{e}^{2\pi ikx}$*. Suppose that* $u : \mathbb{T} \times \mathbb{R} \to \mathbb{C}$ *is a solution of the linear Schrödinger equation. Then*

1. $\|\mathrm{e}^{it\Delta}u_0\|_{L^4_t L^4_x} \lesssim \|u_0\|_{L^2_x}$.
2. *Suppose that* $\operatorname{supp}(\hat{u}) \subset \{k : |k| \leq N\}$ *(in other words* $a_k = 0$ *for any* $|k| > N$*). Then*

$$\|\mathrm{e}^{it\Delta}P_N u_0\|_{L^6_{t,x}} \lesssim N^{\varepsilon}\|P_N u_0\|_{L^2_x}.$$

3. *Under the same assumptions as in* (2)*, we have*

$$\|\mathrm{e}^{it\Delta}u_0\|_{L^q_{t,x}} \lesssim N^{\frac{1}{2}-\frac{3}{q}+\varepsilon}\|P_N u_0\|_{L^2_x},$$

for any $q \in (6, \infty]$*.*

Remark 1.7.4. We note that the ε loss of derivatives in 2. cannot be omitted. Indeed, in [8, (2.45)], Bourgain proved

$$\frac{1}{N^{1/2}}\|\mathrm{e}^{it\Delta}\sum_{n=0}^{N} e^{2\pi i(nx+n^2t)}\|_{L^6_{t,x}} \to \infty$$

as $N \to \infty$.

In n dimensions, for a function $f \in L^2(\mathbb{T}^n)$ with $P_N f = f$, we expect to find estimates of the form (since the pair $\left(\frac{2(n+2)}{n}, \frac{2(n+2)}{n}\right)$ is admissible in dimension n).

$$\|\mathrm{e}^{it\Delta}f\|_{L^q_t L^q_x} \lesssim \|f\|_{L^2_x}, \quad q < \frac{2(n+2)}{n}. \tag{1.17}$$

$$\|\mathrm{e}^{it\Delta}f\|_{L^q_t L^q_x} \lesssim N^{\varepsilon}\|f\|_{L^2_x}, \quad q = \frac{2(n+2)}{n}. \tag{1.18}$$

$$\|\mathrm{e}^{it\Delta}f\|_{L^q_t L^q_x} \lesssim N^{\frac{n}{2}-\frac{n+2}{q}+\varepsilon}\|f\|_{L^2_x}, \quad q > \frac{2(n+2)}{n}. \tag{1.19}$$

For $n = 1$, Bourgain proved these estimates in [8]. In the same paper, he also proved (1.17) for $n = 2, 3$, and he proved (1.18) and (1.19) for $n = 3$ under the assumption that $q \geq 4$. They were proven for all n, q under the assumption that the Fourier coefficients lie on a paraboloid and similar results proven for irrational tori in [12, Section 2].

When proving the well-posedness of dispersive PDEs, one makes use of the *dispersive Sobolev spaces*, or $X^{s,b}$ spaces. For a function $u : \mathbb{T} \times \mathbb{R} \to \mathbb{C}$, we define its spacetime Fourier transform to be

$$\tilde{u}(k, \eta) := \int_{\mathbb{R}} dt \int_{\mathbb{T}} u(x, t) \mathrm{e}^{-2\pi i x k - 2\pi i \eta t}.$$

Definition 1.7.5. Let $s, b \in \mathbb{R}$ and $u : \mathbb{T} \times \mathbb{R} \to \mathbb{C}$, we define

$$\|u\|_{X^{s,b}} := \left\|(1 + |2\pi k|)^s (1 + |\eta + 2\pi k^2|)^b \tilde{u}(k, \eta)\right\|_{\ell_k^2(\mathbb{Z}) L_\eta^2(\mathbb{R})} = \left\|\mathrm{e}^{it\Delta} u\right\|_{H_x^s(\mathbb{T}) H_x^b(\mathbb{R})}. \tag{1.20}$$

The final characterisation means that the $X^{s,b}$ norm heuristically measures how far a function is from being a solution of the linear Schrödinger equation. For an interval $I \subset \mathbb{R}$, the local $X^{s,b}$ space, denoted $X_I^{s,b}$ is defined as

$$\|u\|_{X_I^{s,b}} := \inf_v \|v\|_{X^{s,b}}, \tag{1.21}$$

where the infimum is taken over the set of functions $v : \mathbb{T} \times \mathbb{R} \to \mathbb{C}$ with $v|_{\mathbb{T} \times I} = u|_{\mathbb{T} \times I}$. We recall the following results about local $X^{s,b}$ spaces.

Lemma 1.7.6 (Properties of $X^{s,b}$ spaces)**.** *Let us fix $b = \frac{1}{2} + \varepsilon$ for $\varepsilon > 0$ small and $s \in \mathbb{R}$. The following estimates hold for all $t_0 \in \mathbb{R}$ and $\delta > 0$.*

(i) For all $u \in X^{s,b}_{[t_0, t_0+\delta]}$, we have $\|u\|_{L^\infty_{t \in [t_0, t_0+\delta]} H_x^s} \lesssim_b \|u\|_{X^{s,b}_{[t_0, t_0+\delta]}}$.

(ii) For all $\Phi \in H^s$, we have

$$\|\mathrm{e}^{\mathrm{i}(t - t_0)\Delta} \Phi\|_{X^{s,b}_{[t_0, t_0+\delta]}} \lesssim_b \|\Phi\|_{H^s}. \tag{1.22}$$

(iii) For all $F \in X^{s,b-1}_{[t_0, t_0+\delta]}$, we have

$$\left\| \int_{t_0}^t dt' \, \mathrm{e}^{\mathrm{i}(t - t')\Delta} F(\cdot, t') \right\|_{X^{s,b}_{[t_0, t_0+\delta]}} \lesssim_b \|F\|_{X^{s,b-1}_{[t_0, t_0+\delta]}}. \tag{1.23}$$

Proof of Lemma 1.7.6. The estimates above are standard. Claim (i) follows by

Sobolev embedding in the time variable, using the assumption that $b > \frac{1}{2}$. The particular form of the estimates given by (ii) and (iii) is proved in a self-contained way in [26, Lemma 3.10] and [26, Lemma 3.12] respectively[2]. In particular, the arguments from the proof of [26, Lemma 3.10] imply that for all $\Phi \in H^s$ and $\psi \in C_c^\infty(\mathbb{R})$, we have

$$\|\psi(t)\, \mathrm{e}^{\mathrm{i}t\Delta}\Phi\|_{X^{s,b}_{[t_0,t_0+\delta]}} \lesssim_{b,\psi} \|\Phi\|_{H^s}\,.$$

We now consider ψ which is equal to 1 on $[t_0, t_0+\delta]$ to deduce that

$$\|\mathrm{e}^{\mathrm{i}t\Delta}\Phi\|_{X^{s,b}_{[t_0,t_0+\delta]}} \lesssim_b \|\Phi\|_{H^s}\,. \tag{1.24}$$

The estimate (1.24) follows from (1.22) and the unitarity of $\mathrm{e}^{\mathrm{i}t\Delta}$.

The argument for (iii) is similar. When $t_0 = 0$, it follows from the proof of [26, Lemma 3.12]. More precisely, it suffices to show that for all nonnegative $\psi \in C_c^\infty(\mathbb{R})$, we have

$$\left\|\psi(t) \int_0^t dt'\, \mathrm{e}^{\mathrm{i}(t-t')\Delta}\, F(\cdot,t')\right\|_{X^{s,b}} \lesssim_{b,\psi} \|F\|_{X^{s,b-1}}\,. \tag{1.25}$$

The claim of (iii) then follows from (1.25) by taking infima over F as in (1.21). Using the fact that $\psi(t)$ and $\mathrm{e}^{\mathrm{i}t\Delta}$ commute, as well as (1.20), we have that the expression on the left-hand side of (1.25) equals

$$\left\|\psi(t) \int_0^t dt'\, \mathrm{e}^{-\mathrm{i}t'\Delta}\, F(\cdot,t')\right\|_{H^s_x H^b_t} \leq \left\|\psi(t) \int_0^t dt' \left\|\mathrm{e}^{-\mathrm{i}t'\Delta}\, F(\cdot,t')\right\|_{H^s_x}\right\|_{H^b_t}. \tag{1.26}$$

Using (1.20) and (1.26), we note that (1.25) follows from the following inequality.

$$\left\|\psi(t) \int_0^t dt'\, g(t')\right\|_{H^b_t} \lesssim_{b,\psi} \|g\|_{H^{b'}_t}\,. \tag{1.27}$$

The estimate (1.27) corresponds to [26, (3.18)], which is shown in the proof of [26, Lemma 3.12]. This shows (1.23) when $t_0 = 0$.

The proof of (1.23) for general t_0 follows by translation. In order to explain the last step in more detail, let us consider $F \in X^{s,b-1}_{[t_0,t_0+\delta]}$. We then consider $G(x,t) := F(x,t+t_0)$ and observe that

$$\|G\|_{X^{s,b-1}_{[0,\delta]}} = \|F\|_{X^{s,b-1}_{[t_0,t_0+\delta]}}\,. \tag{1.28}$$

[2]The results in [26] are proved for the Airy semigroup $W_t = \mathrm{e}^{-t\delta_x^3}$ on the real line. The arguments for the Schrödinger semigroup $\mathrm{e}^{\mathrm{i}t\Delta}$ on the torus follow analogously.

A direct calculation shows that

$$\int_{t_0}^{t} dt'\, \mathrm{e}^{\mathrm{i}(t-t')\Delta}\, F(t') = \int_0^{t-t_0} dt'\, \mathrm{e}^{\mathrm{i}(t-t_0-t')\Delta}\, G(t')\,, \tag{1.29}$$

from where we deduce

$$\left\| \int_{t_0}^{t} dt'\, \mathrm{e}^{\mathrm{i}(t-t')\Delta}\, F(\cdot,t') \right\|_{X^{s,b}_{[t_0,t_0+\delta]}} = \left\| \int_0^{t} dt'\, \mathrm{e}^{\mathrm{i}(t-t')\Delta}\, G(t') \right\|_{X^{s,b}_{[0,\delta]}}\,. \tag{1.30}$$

The result of claim (iii) then follows from (1.28)–(1.30) and (1.23) when $t_0 = 0$. □

Remark 1.7.7. We refer the reader to [30, Lemma 5.3], [30, Appendix A] for a self-contained summary of similar estimates in $X^{s,b}$ spaces, based on [43, 44]. We also refer the reader to [83, Section 2.6]. Note that, in contrast to [30, Lemma 5.3], in (ii)–(iii), we are working in local $X^{s,b}$ spaces instead of with a cut-off function ψ in the t variable.

We also have the following standard result, which states that $X^{s,b}$ and consequently local $X^{s,b}$ spaces interpolate nicely. We prove it for completeness, but the reader could also find a proof in [38]. We use it to interpolate between estimates for $X^{s,b}$ estimates.

Lemma 1.7.8. *Suppose $s, b \in \mathbb{R}$. Let $s = \theta s_1 + (1-\theta)s_2$ and $b = \theta b_1 + (1-\theta)b_2$. Then we have the following inequality.*

$$\|\cdot\|_{X^{s,b}} \leq \|\cdot\|^{\theta}_{X^{s_1,b_1}} \|\cdot\|^{1-\theta}_{X^{s_2,b_2}}.$$

Proof.

$$\begin{aligned}
\|u\|^2_{X^{s,b}} &= \Big\| |\tilde{u}(k,\eta)|^2 (1+|2\pi k|^2)^{2s} (1+\big|\eta+2\pi k^2\big|)^{2b}) \Big\|_{\ell^1_k(\mathbb{Z}) L^1_\eta(\mathbb{R})} \\
&= \Bigg\| \Big(|\tilde{u}(k,\eta)|^2 (1+|2\pi k|^2)^{2s_1} (1+\big|\eta+2\pi k^2\big|)^{2s_1} \Big)^{\theta} \\
&\quad \times \Big(|\tilde{u}(k,\eta)|^2 (1+|2\pi k|^2)^{2s_2} (1+\big|\eta+2\pi k^2\big|)^{2s_2} \Big)^{1-\theta} \Bigg\|_{\ell^1_k(\mathbb{Z}) L^1_\eta(\mathbb{R})} \\
&\leq \Big\| \Big(|\tilde{u}(k,\eta)|^2 (1+|2\pi k|^2)^{2s_1} (1+\big|\eta+2\pi k^2\big|)^{2s_1} \Big)^{\theta} \Big\|_{\ell^{\frac{1}{\theta}}_k(\mathbb{Z}) L^{\frac{1}{\theta}}_\eta(\mathbb{R})} \\
&\quad + \Big\| \Big(|\tilde{u}(k,\eta)|^2 (1+|2\pi k|^2)^{2s_2} (1+\big|\eta+2\pi k^2\big|)^{2s_2} \Big)^{1-\theta} \Big\|_{\ell^{\frac{1}{1-\theta}}_k(\mathbb{Z}) L^{\frac{1}{1-\theta}}_\eta(\mathbb{R})} \\
&= \|u\|^{2\theta}_{X^{s_1,b_1}} \|u\|^{2(1-\theta)}_{X^{s_2,b_2}},
\end{aligned}$$

where the inequality follows from Hölder's inequality. □

Results for Gibbs measures

In the construction of Gibbs measures, we will need the following result about finite dimensional Hamiltonian systems; see for example [4]. Recall we say that for a map $T : X \to X$, we say a measure μ is *invariant under* T if $d\mu(A) = d\mu(T^{-1}A)$, for any measurable set A.

Theorem 1.7.9 (Liouville's theorem)**.** *Let* $H := H(p_1, \ldots, p_n, q_1, \ldots, q_n)$ *be a Hamiltonian. For a finite dimensional Hamiltonian system with evolution equations*

$$\begin{cases} \dot{p}_j = \frac{\partial H}{\partial q_j} \\ \dot{q}_j = -\frac{\partial H}{\partial p_j} \end{cases}$$

and a non-negative smooth function g*,* $g(H)\, d\mathrm{Leb}$ *is invariant under the flow map,* S_t*.*

We will also use the following result to compute the expectations of Gaussian random variables. For an example of a self-contained introduction, see [31, Lemma 2.4].

Theorem 1.7.10 (Wick's theorem)**.** *Let* $(g_k)_{k=1}^n$ *be i.i.d. centred complex Gaussian random variables. Let* $(g_k)^*$ *denote either* g_k *or* $\overline{g_k}$*. Then*

$$\mathbb{E}\left[\prod_{j=1}^{n} (g_k)^{*_i}\right] = \sum_{\Pi \in \mathcal{M}(n)} \prod_{(i,j)\in\Pi} \mathbb{E}\left[(g_i)^{*_i} (g_j)^{*_j}\right], \tag{1.31}$$

where the sum is taken over all complete pairings of $\{1, \ldots, n\}$*, and where edges of* Π *are denoted by* (i, j) *with* $i < j$*.*

Remark 1.7.11. If g_1 and g_2 are independent centred complex random variables, we note that $\mathbb{E}[g_1 g_2] = \mathbb{E}[\overline{g_1 g_2}] = 0$. This observation allows us to simplify the sum on the right hand side of (1.31).

Feynman-Kac formula

In our analysis, we make use of the Feynman-Kac formula. To this end, let $T > 0$ and let Ω^T denote the space of continuous paths $\omega : [0, T] \to \mathbb{T}$. Given $t > 0$, we define

$$\psi^t(x) := \mathrm{e}^{t\Delta}(x) = \sum_{n\in\mathbb{Z}^d} (4\pi t)^{-1/2} \mathrm{e}^{-|x-n|^2/4t} \tag{1.32}$$

to be the periodic heat kernel on $\mathbb{T}$. For $x, \tilde{x} \in \mathbb{T}$, we characterise the Wiener measure $\mathbb{W}^T_{x,\tilde{x}}$ by its finite-dimensional distribution. Namely for $0 < t_1 < \ldots < t_n < T$ and $f : \mathbb{T}^n \to \mathbb{R}$ continuous

$$\int \mathbb{W}^T_{x,\tilde{x}}(d\omega) f(\omega(t_1), \ldots, \omega(t_n)) \\ = \int dx_1 \, \ldots \, dx_n \, \psi^{t_1}(x_1 - \tilde{x}) \psi^{t_2 - t_1}(x_2 - x_1) \ldots \\ \times \psi^{t_n - t_{n-1}}(x_n - x_{n-1}) \psi^{T - t_n}(x - x_n) f(x_1, \ldots, x_n) \, .$$

Then we have the following result, see for example [67, Theorem X.68].

Proposition 1.7.12 (Feynman-Kac Formula)**.** *Let $V : \mathbb{T} \to \mathbb{C}$ be continuous and bounded below. For $t > 0$*

$$\mathrm{e}^{t(\Delta - V)}(x; \tilde{x}) = \int \mathbb{W}^t_{x,\tilde{x}}(d\omega) \mathrm{e}^{-\int_0^t ds \, V(\omega(s))} \, .$$

1.8 Outline of thesis

The thesis consists of two main parts, namely the cubic and the quintic problems. In Chapter 2, we state the main results of the thesis, and explain the main differences in the analysis of the two problems. In Chapter 3, we deal with the cubic case, and in Chapter 4 we analyse the quintic problem. Appendix A is devoted to an exposition of the construction of the one dimensional Gibbs measure for the focusing 1D NLS and to a self-contained proof of the integrability of the weight function. Finally, Appendix B includes some basic computations within the second quantisation framework.

Chapter 2

Main Results

In this section we give the setup of our problem and a statement of the main results proved in the thesis. We begin by setting up the problem and stating our results for the cubic nonlinearity and then explain the differences for the quintic nonlinearity. In Section 2.3 we give an outline of the previously known results in the field.

2.1 Cubic problem

2.1.1 Classical problem

We fix the Hilbert space $\mathfrak{h} := L^2(\mathbb{T};\mathbb{C}) \equiv L^2(\mathbb{T})$. To define the Hamiltonian we work with, we first make the following assumption.

Assumption 2.1.1 (The interaction potential)**.** We consider an *interaction potential* which is of one of the following types.

(i) $w : \mathbb{T} \to \mathbb{R}$ is even and belongs to $L^1(\mathbb{T})$.

(ii) $w = -\delta$, where δ is the Dirac delta function.

Let us note that, in Assumption 2.1.1, we do not assume any conditions on the sign of w or the sign of $\hat{w}$ (pointwise almost everywhere).

With w as in Assumption 2.1.1, the Hamiltonian that we consider is given by

$$H(\varphi) := \int dx \, \left(|\nabla\varphi(x)|^2 + \kappa|\varphi(x)|^2\right) + \frac{1}{2}\int dx\, dy\, |\varphi(x)|^2\, w(x-y)|\varphi(y)|^2 \,. \quad (2.1)$$

In (2.1), and throughout the sequel, we fix $\kappa > 0$ to be the *(negative) chemical potential.* On the space of fields $\varphi : \mathbb{T} \to \mathbb{C}$, we consider a Poisson bracket defined by

$$\{\varphi(x),\overline{\varphi}(y)\} = i\delta(x-y), \quad \{\varphi(x),\varphi(y)\} = \{\overline{\varphi}(x),\overline{\varphi}(y)\} = 0\,. \tag{2.2}$$

We note that, by Assumption 2.1.1, the Hamiltonian (2.1) is not necessarily positive-definite. Hence, when studying the associated Gibbs measure, one has to use the modification given by (1.6), instead of (1.5). This setup was previously used in [9, 20, 47].

The Hamiltonian equation of motion associated with Hamiltonian (2.1) and Poisson bracket (2.2) is the time-dependent nonlocal *nonlinear Schrödinger equation* (NLS)

$$i\partial_t \varphi(x) = (-\Delta+\kappa)\varphi(x) + \int dy\, |\varphi(y)|^2\, w(x-y)\varphi(x)\,. \tag{2.3}$$

Here, we abbreviate the notation $\varphi(x) \equiv \varphi(x,t)$ with $\varphi : \mathbb{T}\times\mathbb{R} \to \mathbb{C}$. For $w \in L^1$, as in Assumption 2.1.1 (i), one usually refers to (2.3) as the *Hartree equation.* We will also consider the *focusing local cubic NLS*

$$i\partial_t \varphi(x) = (-\Delta+\kappa)\varphi(x) - |\varphi(x)|^2\varphi(x)\,, \tag{2.4}$$

which corresponds to (2.3) with $w = -\delta$, as in Assumption 2.1.1 (ii). We refer to[1] (2.3) and (2.4) as the *focusing cubic nonlinear Schrödinger equation (NLS).*

The arguments in [8] show that the focusing cubic NLS (2.3)–(2.4) is globally well-posed for initial data in $\mathfrak{h} \equiv L^2(\mathbb{T})$. In particular, there exists a well-defined solution map S_t that maps any initial data $\varphi_0 \in \mathfrak{h}$ to the solution at time t given by

$$\varphi(\cdot) \equiv \varphi(\cdot,t) := S_t\varphi_0(\cdot) \in \mathfrak{h}\,. \tag{2.5}$$

Moreover, $\|S_t\varphi_0\|_{\mathfrak{h}} = \|\varphi_0\|_{\mathfrak{h}}$.

The *one-particle space* on which we work is $\mathfrak{h} = L^2(\mathbb{T})$. We use the following convention for the scalar product.

$$\langle g_1, g_2\rangle_{\mathfrak{h}} := \int dx\, \overline{g_1}(x)\, g_2(x)\,.$$

We consider the *one-body Hamiltonian* given by

$$h := -\Delta + \kappa, \tag{2.6}$$

[1] When one has suitable positivity (in other words *defocusing*) assumptions on w, the analysis of the problem we are considering for (2.3) has already been done in [29]; see Section 2.3 below for an overview. Our main interest lies in the case when these assumptions are relaxed, which we refer to as the *focusing* regime.

where $\kappa > 0$ is as in (2.1). This is a positive self-adjoint densely defined operator on $\mathfrak{h}$. We can write h spectrally as

$$h := \sum_{k\in\mathbb{N}} \lambda_k u_k u_k^*, \tag{2.7}$$

where

$$\lambda_k := 4\pi^2|k|^2 + \kappa \tag{2.8}$$

are the eigenvalues of h and

$$u_k := \mathrm{e}^{2\pi ikx} \tag{2.9}$$

are the normalised eigenvalues of h on $\mathfrak{h}$. Since we are working on $\mathbb{T}$, we have

$$\mathrm{Tr}(h^{-1}) = \sum_{k\in\mathbb{N}} \frac{1}{4\pi^2|k|^2 + \kappa} < \infty, \tag{2.10}$$

where the trace is taken over $\mathfrak{h}$.

For each $k \in \mathbb{N}$, we define μ_k to be a standard complex Gaussian measure. In other words, $\mu_k := \frac{1}{\pi}\mathrm{e}^{-|z|^2}dz$, where dz is the Lebesgue measure on $\mathbb{C}$. Let $(\mathbb{C}^{\mathbb{N}}, \mathcal{G}, \mu)$ be the product probability space with

$$\mu := \bigotimes_{k\in\mathbb{N}} \mu_k. \tag{2.11}$$

We denote elements of the probability space $\mathbb{C}^{\mathbb{N}}$ by $\omega = (\omega_k)_{k\in\mathbb{N}}$. Let the *classical free field* $\varphi \equiv \varphi^\omega$ be defined by

$$\varphi := \sum_{k\in\mathbb{N}} \frac{\omega_k}{\sqrt{\lambda_k}} u_k. \tag{2.12}$$

Note that (2.10) implies (2.12) converges almost surely in $H^{\frac{1}{2}-\varepsilon}(\mathbb{T})$ for $\varepsilon > 0$ arbitrarily small.

Since μ is a Gaussian measure, it satisfies the following Wick theorem, which follows from Theorem 1.7.10.

Proposition 2.1.2. *Let φ be as in (2.12). Given $g \in H^{-\frac{1}{2}+\varepsilon}$ for $\varepsilon > 0$, we let $\varphi(g) := \langle g, \varphi\rangle$ and $\overline{\varphi}(g) := \langle \varphi, g\rangle$. Furthermore, we let $(\varphi)^*(g)$ denote either $\varphi(g)$ or $\overline{\varphi}(g)$. Then, given $n \in \mathbb{N}^*$ and $g_1, \ldots, g_n \in H^{-\frac{1}{2}+\varepsilon}$, we have*

$$\mathbb{E}_\mu\left[\prod_{i=1}^n (\varphi(g_i))^{*_i}\right] = \sum_{\Pi\in\mathcal{M}(n)} \prod_{(i,j)\in\Pi} \mathbb{E}_\mu\left[(\varphi(g_i))^{*_i}(\varphi(g_j))^{*_j}\right], \tag{2.13}$$

where the sum is taken over all complete pairings of $\{1, \dots, n\}$*, and where edges of* Π *are denoted by* (i,j) *with* $i < j$*.*

We note that, by gauge invariance, for all $(i,j) \in \Pi$

$$\mathbb{E}_\mu \left[\varphi(g_i)\varphi(g_j)\right] = \mathbb{E}_\mu \left[\overline{\varphi}(g_i)\,\overline{\varphi}(g_j)\right] = 0\,.$$

Therefore, each non-zero factor arising on the right-hand side of (2.13) can be computed using

$$\int d\mu\, \overline{\varphi}(\tilde{g})\varphi(g) = \langle g, h^{-1}\tilde{g}\rangle,$$

for $g, \tilde{g} \in H^{-\frac{1}{2}+\varepsilon}$. Here, the Green function h^{-1} is the covariance of μ. We note that, under a suitable pushforward, we can identify μ with a probability measure on H^s; see for example [29, Remark 1.3]. As in [29], we work directly with the measure μ as above and do not use this identification.

Remark 2.1.3. This is a fairly brief introduction to the Gibbs measure so that we can state our results. A longer introduction and details of the construction of the Gibbs measure in one spatial dimension is given in Appendix A.

Given $p \in \mathbb{N}^*$, the *p-particle space* $\mathfrak{h}^{(p)}$ is defined as the symmetric subspace of $\mathfrak{h}^{\otimes p}$, i.e. $u \in \mathfrak{h}^{(p)}$ if and only if for any permutation π,

$$u(x_{\pi(1)}, \dots, x_{\pi(p)}) = u(x_1, \dots, x_p)\,.$$

For ξ a closed linear operator on $\mathfrak{h}^{(p)}$ and φ as in (2.12), we define the random variable

$$\Theta(\xi) := \int dx_1 \dots dx_p\, dy_1 \dots dy_p\, \xi(x_1, \dots, x_p; y_1, \dots, y_p) \\ \times \overline{\varphi}(x_1) \dots \overline{\varphi}(x_p)\varphi(y_1) \dots \varphi(y_p)\,. \quad (2.14)$$

Recall we denote by $\mathcal{L}(\mathcal{H})$ the set of all bounded operators on a Hilbert Space $\mathcal{H}$. If $\xi \in \mathcal{L}(\mathfrak{h}^{(p)})$, then $\Theta(\xi)$ defined in (2.14) is almost surely well-defined, since $\varphi \in \mathfrak{h}$ almost surely.

Given w as in Assumption 2.1.1, we define the *classical interaction* as

$$\mathcal{W} := \frac{1}{2}\int dx\, dy\, |\varphi(x)|^2\, w(x-y)|\varphi(y)|^2\,. \quad (2.15)$$

The *free classical Hamiltonian* is given by

$$H_0 := \Theta(h) = \int dx\, dy\, \overline{\varphi}(x) h(x;y) \varphi(y)\,. \tag{2.16}$$

The *interacting classical Hamiltonian* is given by

$$H := H_0 + \mathcal{W}\,. \tag{2.17}$$

The *mass* is defined as

$$\mathcal{N} := \int dx\, |\varphi(x)|^2\,. \tag{2.18}$$

At this stage, we have to introduce the cut-off f that appears in (1.6). We now state the precise assumptions on f that we use in the sequel.

Assumption 2.1.4. In the cubic case, we fix $f \in C_c^\infty(\mathbb{R})$, which is not identically equal to zero such that $0 \leq f \leq 1$ and

$$\operatorname{supp}(f) \subset [-K, K]\,, \tag{2.19}$$

for some $K > 0$.

All of our estimates depend on K in (2.19), but we do not track this dependence explicitly.

We define the *classical state* $\rho^f(\cdot) \equiv \rho(\cdot)$ by

$$\rho(X) := \frac{\int d\mu\, X \mathrm{e}^{-\mathcal{W}} f(\mathcal{N})}{\int d\mu\, \mathrm{e}^{-\mathcal{W}} f(\mathcal{N})} \equiv \mathbb{E}_{\mathbb{P}^f_{\mathrm{Gibbs}}}(X)\,, \tag{2.20}$$

where X is a random variable. Let the *classical partition function* $z \equiv z^f_{\mathrm{Gibbs}}$ be defined as

$$z := \int d\mu\, \mathrm{e}^{-\mathcal{W}} f(\mathcal{N})\,. \tag{2.21}$$

Note that both ρ and z are well defined by Lemma 3.1.1 and Corollary 3.1.4 below. We characterise $\rho(\cdot)$ through its moments. Namely, we define the *classical p-particle correlation function* $\gamma_p \equiv \gamma_p^f$, which acts on $\mathfrak{h}^{(p)}$ through its kernel

$$\gamma_p(x_1, \ldots, x_p; y_1, \ldots, y_p) := \rho(\overline{\varphi}(y_1) \ldots \overline{\varphi}(y_p)\, \varphi(x_1) \ldots \varphi(x_p))\,. \tag{2.22}$$

For the time-dependent result, we will also need the following notion of *classical evolution.*

Definition 2.1.5. Let $p \in \mathbb{N}^*$ and $\xi \in \mathcal{L}(\mathfrak{h}^{(p)})$ be given. For $t \in \mathbb{R}$, we define the random variable

$$\Psi^t\Theta(\xi) := \int dx_1 \dots dx_p \, dy_1 \dots dy_p \, \xi(x_1, \dots, x_p; y_1 \dots y_p) \\ \times \overline{S_t\varphi}(x_1) \dots \overline{S_t\varphi}(x_p) S_t\varphi(x_p) \dots S_t\varphi(y_p), \quad (2.23)$$

where S_t is the flow map defined in (2.5). This is well defined since $\varphi \in \mathfrak{h}$ almost surely and S_t is norm preserving on $\mathfrak{h}$.

2.1.2 Quantum problem

We use the same conventions as in [29, Section 1.4]. For more details and motivation, we refer the reader to the aforementioned work. In the quantum setting, we work on the *bosonic Fock space*, which is defined as

$$\mathcal{F} \equiv \mathcal{F}(\mathfrak{h}) := \bigoplus_{p \in \mathbb{N}} \mathfrak{h}^{(p)}.$$

Let us denote vectors of $\mathcal{F}$ by $\Psi = (\Psi^{(p)})_{p \in \mathbb{N}}$. For $g \in \mathfrak{h}$, let $b^*(g)$ and $b(g)$ denote the bosonic creation and annihilation operators, defined respectively as

$$\left(b^*(g)\Psi\right)^{(p)}(x_1, \dots, x_p) := \frac{1}{\sqrt{p}} \sum_{i=1}^{p} g(x_i) \Psi^{(p-1)}(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_p), \quad (2.24)$$

$$\left(b(g)\Psi\right)^{(p)}(x_1, \dots, x_p) := \sqrt{p+1} \int dx \, \overline{g(x)} \Psi^{(p+1)}(x, x_1, \dots, x_p). \quad (2.25)$$

These are closed, densely-defined operators which are each other's adjoints; see for example [13]. The creation and annihilation operators satisfy the canonical commutation relations, i.e.

$$[b(g_1), b^*(g_2)] = \langle g_1, g_2 \rangle_{\mathfrak{h}}, \quad [b(g_1), b(g_2)] = [b^*(g_1), b^*(g_2)] = 0, \quad (2.26)$$

for all $g_1, g_2 \in \mathfrak{h}$. The computation of (2.26) is included in Appendix B.

We define the rescaled creation and annihilation operators

$$\varphi^*_\tau(g) := \tau^{-1/2} \, b^*(g), \quad \varphi_\tau(g) := \tau^{-1/2} \, b(g), \quad (2.27)$$

for $g \in \mathfrak{h}$. Here, we think of φ^*_τ and φ_τ as operator valued distributions, and we denote their distribution kernels by $\varphi^*_\tau(x)$ and $\varphi_\tau(x)$, respectively. Formally, $\varphi^*_\tau(x)$ and $\varphi_\tau(x)$ correspond to taking $g = \delta_x$ (the Dirac delta function centred at x) in

(2.27). In analogy to (2.12), we call φ_τ the *quantum field.*

As before, let ξ be a closed linear operator on $\mathfrak{h}^{(p)}$. The *lift* of ξ to $\mathcal{F}$ is defined by

$$\Theta_\tau(\xi) := \int dx_1 \dots dx_p \, dy_1 \dots dy_p \, \xi(x_1, \dots, x_p; y_1, \dots, y_p) \\ \times \varphi_\tau^*(x_1) \dots \varphi_\tau^*(x_p) \varphi_\tau(y_1) \dots \varphi_\tau(y_p) \,. \quad (2.28)$$

For $w \in L^\infty(\mathbb{T})$ real-valued and even, we define the *quantum interaction* as[2]

$$\mathcal{W}_\tau := \frac{1}{2} \Theta_\tau(W) = \frac{1}{2} \int dx \, dy \, \varphi_\tau^*(x) \varphi_\tau^*(y) w(x-y) \varphi_\tau(x) \varphi_\tau(y) \,. \quad (2.29)$$

Here W is the two particle operator on $\mathfrak{h}^{(2)}$ which acts by multiplication by $w(x_1 - x_2)$ for $w \in L^\infty$. We define the *free quantum Hamiltonian* as

$$H_{\tau,0} := \Theta_\tau(h) = \int dx \, dy \, \varphi_\tau^*(x) h(x;y) \varphi_\tau(y) \,, \quad (2.30)$$

where h is as in (2.6). We define the *interacting quantum Hamiltonian* as

$$H_\tau := H_{\tau,0} + \mathcal{W}_\tau \,.$$

We also define the rescaled particle number as

$$\mathcal{N}_\tau := \int dx \, \varphi_\tau^*(x) \varphi_\tau(x) \,. \quad (2.31)$$

In Appendix B.1 it is shown that (2.67) acts on the p^{th} sector of Fock space as multiplication by $\frac{p}{\tau}$.

The *(untruncated) grand canonical ensemble* is defined as

$$P_\tau := \mathrm{e}^{-H_\tau} \quad (2.32)$$

and the *(truncated) quantum state* $\rho_\tau^f(\cdot) \equiv \rho_\tau(\cdot)$ is defined as

$$\rho_\tau(\mathcal{A}) := \frac{\mathrm{Tr}(\mathcal{A} P_\tau f(\mathcal{N}_\tau))}{\mathrm{Tr}(P_\tau f(\mathcal{N}_\tau))} \,, \quad (2.33)$$

where the traces are taken over Fock space. Let the *quantum partition function* and

[2]In principle, we could consider w as in Assumption 2.1.1 in the quantum setting at the level of the definition. In practice, we take the interaction potential to be bounded; see Section 2.1.3 below for the precise statements.

the *free quantum partition function*, $Z_\tau \equiv Z_\tau^f$, $Z_{\tau,0}$ be defined respectively as

$$Z_\tau := \mathrm{Tr}(\mathrm{e}^{-H_\tau} f(\mathcal{N}_\tau)), \quad Z_{\tau,0} := \mathrm{Tr}(\mathrm{e}^{-H_{\tau,0}}) . \tag{2.34}$$

With $Z_\tau, Z_{\tau,0}$ as in (2.34), we define the *relative quantum partition function* $\mathcal{Z}_\tau \equiv \mathcal{Z}_\tau^f$ by

$$\mathcal{Z}_\tau := \frac{Z_\tau}{Z_{\tau,0}} . \tag{2.35}$$

In analogy to (2.22), we characterise the quantum state through its correlation functions. Namely, for $p \in \mathbb{N}^*$, we define the *quantum p-particle correlation function* $\gamma_{\tau,p}^f \equiv \gamma_{\tau,p}$, which acts on $\mathfrak{h}^{(p)}$ through its kernel

$$\gamma_{\tau,p}(x_1, \dots, x_p; y_1, \dots, y_p) := \rho_\tau(\varphi_\tau^*(y_1) \dots \varphi_\tau^*(y_p)\varphi_\tau(x_1) \dots \varphi_\tau(x_p)) . \tag{2.36}$$

For the time-dependent problem, we also define the *quantum time evolution* of an operator on Fock space.

Definition 2.1.6. Suppose $\mathcal{A} : \mathcal{F} \to \mathcal{F}$. Define the quantum time evolution of $\mathcal{A}$ as

$$\Psi_\tau^t \mathcal{A} := \mathrm{e}^{it\tau H_\tau} \, \mathcal{A} \, \mathrm{e}^{-it\tau H_\tau} .$$

Throughout the sequel, for a given quantity $Y = \rho, \mathcal{N}, H, \dots$, we will use the abbreviation $Y_\#$ to denote either Y_τ or Y.

2.1.3 Main results

We now state our results for the cubic problem. The first result that we prove concerns bounded interaction potentials.

Theorem 2.1.7 (Convergence for $w \in L^\infty(\mathbb{T})$). *Let $w \in L^\infty(\mathbb{T})$ be real-valued and even. Given $p \in \mathbb{N}^*$, we recall the quantities $\gamma_{\tau,p}$ and γ_p defined in* (2.36) *and* (2.22) *respectively. We then have*

$$\lim_{\tau \to \infty} \|\gamma_{\tau,p} - \gamma_p\|_{\mathfrak{S}^1(\mathfrak{h}^{(p)})} = 0 . \tag{2.37}$$

Moreover, recalling (2.21) *and* (2.35)*, we have*

$$\lim_{\tau \to \infty} \mathcal{Z}_\tau = z . \tag{2.38}$$

By applying an approximation argument, we prove results for $w \in L^1(\mathbb{T})$ and $w = -\delta$ as in Assumption 2.1.1. Throughout the sequel, any object with a superscript ε is the corresponding object defined by taking the interaction potential to be w^ε, which will be a suitable bounded approximation of w. In what follows, we always assume that all the approximating interaction potentials w^ε are *real-valued and even*, without mentioning this explicitly. We can now state the result for $L^1(\mathbb{T})$ interaction potentials.

Theorem 2.1.8 (Convergence for $w \in L^1(\mathbb{T})$). *Let w be as in Assumption 2.1.1 (i). Suppose that (w^ε) is a sequence of interaction potentials in $L^\infty(\mathbb{T})$ such that $w^\varepsilon \to w$ in $L^1(\mathbb{T})$. Then there exists a sequence (ε_τ) satisfying $\varepsilon_\tau \to 0$ as $\tau \to \infty$ such that for any $p \in \mathbb{N}^*$*

$$\lim_{\tau\to\infty} \|\gamma^{\varepsilon_\tau}_{\tau,p} - \gamma_p\|_{\mathfrak{S}^1(\mathfrak{h}^{(p)})} = 0 \tag{2.39}$$

and such that

$$\lim_{\tau\to\infty} \mathcal{Z}^{\varepsilon_\tau}_\tau = z\,. \tag{2.40}$$

Before considering $w = -\delta$ as in Assumption 2.1.1 (ii), we need to define the sequence more w^ε precisely. We fix $U : \mathbb{R} \to \mathbb{R}$ to be a continuous even function, with $\operatorname{supp} U \subset \mathbb{T}$ satisfying

$$\int_{\mathbb{R}} dx\, U(x) = \int_{\mathbb{T}} dx\, U(x) = -1\,. \tag{2.41}$$

For $\varepsilon \in (0,1)$, we define

$$w^\varepsilon := \frac{1}{\varepsilon} U\left(\frac{[x]}{\varepsilon}\right), \tag{2.42}$$

where $[x]$ is defined to be the unique element in $(x+\mathbb{Z})\cap\mathbb{T}$. In particular, $w^\varepsilon \in L^\infty(\mathbb{T})$ and w^ε converges to $-\delta$ weakly, with respect to continuous functions.

Theorem 2.1.9 (Convergence for $w = -\delta$). *With notation as in* (2.42)*, there exists a sequence (ε_τ) satisfying $\varepsilon_\tau \to 0$ as $\tau \to \infty$ such that for any $p \in \mathbb{N}^*$*

$$\lim_{\tau\to\infty} \|\gamma^{\varepsilon_\tau}_{\tau,p} - \gamma_p\|_{\mathfrak{S}^1(\mathfrak{h}^{(p)})} = 0 \tag{2.43}$$

and such that

$$\lim_{\tau\to\infty} \mathcal{Z}^{\varepsilon_\tau}_\tau = z\,. \tag{2.44}$$

Remark 2.1.10. We make the following observations about Theorems 2.1.7, 2.1.8, and 2.1.9.

1. For a pointwise almost everywhere non-negative, bounded, even interaction potential w, Theorem 2.1.7 holds without the need for a cut-off function f.

This is the content of [29, Theorem 1.8]. Moreover, by working with the non-normal ordered quantum interaction $\mathcal{W}_\tau'$ defined in (3.72), for a bounded, real-valued, even interaction potential w of positive type (i.e. $\hat{w}$ pointwise almost everywhere non-negative), the same proof as [29, Theorem 1.8] again shows that Theorem 2.1.7 holds without the need for a cut-off function f. We include the details of the proof of this claim in Section 3.4.1.

2. We conjecture that the results hold for f a characteristic function of an interval. The method that we apply in Lemma 3.2.10 of Section 3.2.5 requires suitable smoothness assumptions on f. This is a technical assumption.

3. For an individual $w \in L^\infty$, Theorem 2.1.7 holds with a cut-off function of the form $f(x) = \mathrm{e}^{-cx^2}$, for $c > 0$ sufficiently large depending on $\|w\|_{L^\infty}$. This is also proved by working with a non-normal ordered quantum interaction. The details are given in Section 3.4.2. We note this c cannot be chosen uniformly in the L^∞ norm of the interaction potential. So we cannot treat the unbounded interactions as in Theorems 2.1.8 and 2.1.9 using this kind of truncation.

4. One could consider the questions from Theorems 2.1.7 and 2.1.11 in the non-periodic setting when the spatial domain is $\mathbb{R}$ for the one-body Hamiltonian $h = -\Delta + \kappa + v$, where $v : \mathbb{R} \to [0, \infty)$ is a positive *one-body potential* such that h has compact resolvent and $\operatorname{Tr} h^{-1} < \infty$ holds (as in (2.10)). The analysis that we present in the periodic setting would carry through to this case, provided that we know that the time evolution S_t given in (2.5) is well-defined on the support of the Gibbs measure. We do not address this question further in the thesis.

5. By following the duality arguments in [29, Section 3.3], we can get the equivalents of equations (2.37), (2.39), and (2.43) in terms of ρ_τ and ρ. For more details when $w \in L^\infty$, see Corollary 3.2.13, Lemma 3.3.1, and Lemma 3.3.2 below. For the time-independent problem, we state the convergence as above in the trace class. For the time-dependent problem, we need to use the alternative formulation, which can be seen as a generalisation of the time-independent analysis. For more details, see Remark 2.1.14 below.

6. Our method works for more general interaction potentials. In particular, we can consider linear combinations of interaction potentials as in Assumption 2.1.1 (i) and (ii) with the same arguments.

Recalling the quantities ρ_τ and ρ defined as in (2.33) and (2.20) respectively, we now state the time-dependent results for the cubic problem.

Theorem 2.1.11 (Convergence for $w \in L^\infty(\mathbb{T})$)**.** *Let w be as in Theorem 2.1.7. Given $m \in \mathbb{N}^*$, $p_i \in \mathbb{N}^*$, $\xi^i \in \mathcal{L}(\mathfrak{h}^{(p_i)})$, and $t_i \in \mathbb{R}$, we have*

$$\lim_{\tau\to\infty} \rho_\tau \left(\Psi_\tau^{t_1}\Theta_\tau(\xi^1)\dots\Psi_\tau^{t_m}\Theta_\tau(\xi^m)\right) = \rho\left(\Psi^{t_1}\Theta(\xi^1)\dots\Psi^{t_m}\Theta(\xi^m)\right) .$$

Theorem 2.1.12 (Convergence for $w \in L^1(\mathbb{T})$)**.** *Let w, w^ε be as in the assumptions of Theorem 2.1.8. Then, there exists a sequence (ε_τ) satisfying $\varepsilon_\tau \to 0$ as $\tau \to \infty$ such that, given $m \in \mathbb{N}^*$, $p_i \in \mathbb{N}^*$, $\xi^i \in \mathcal{L}(\mathfrak{h}^{(p_i)})$, and $t_i \in \mathbb{R}$, we have*

$$\lim_{\tau\to\infty} \rho_\tau^{\varepsilon_\tau} \left(\Psi_\tau^{t_1}\Theta_\tau(\xi^1)\dots\Psi_\tau^{t_m}\Theta_\tau(\xi^m)\right) = \rho\left(\Psi^{t_1}\Theta(\xi^1)\dots\Psi^{t_m}\Theta(\xi^m)\right) .$$

Theorem 2.1.13 (Convergence for $w = -\delta$)**.** *Let w, w^ε be as in the assumptions of Theorem 2.1.9. Then, there exists a sequence (ε_τ) satisfying $\varepsilon_\tau \to 0$ as $\tau \to \infty$ such that, given $m \in \mathbb{N}^*$, $p_i \in \mathbb{N}^*$, $\xi^i \in \mathcal{L}(\mathfrak{h}^{(p_i)})$, and $t_i \in \mathbb{R}$, we have*

$$\lim_{\tau\to\infty} \rho_\tau^{\varepsilon_\tau} \left(\Psi_\tau^{t_1}\Theta_\tau(\xi^1)\dots\Psi_\tau^{t_m}\Theta_\tau(\xi^m)\right) = \rho\left(\Psi^{t_1}\Theta(\xi^1)\dots\Psi^{t_m}\Theta(\xi^m)\right) .$$

Remark 2.1.14. Theorems 2.1.11–2.1.13 can indeed be seen as generalisations of the results given in Theorems 2.1.7–2.1.9 respectively (the latter of which correspond to setting $m = 1$ and $t_1 = 0$). Namely, we use Remark 2.1.10 (3) above and note that the proofs show that the convergence is uniform in $\|\xi^1\| \leq 1$.

2.2 Quintic problem

2.2.1 Classical problem

The primary difference between the cubic and quintic classical cases is that we consider an equation with a nonlinearity which does not have a convolution structure. We begin by stating our assumptions on the interaction potential in the quintic model.

Assumption 2.2.1. Let $w : \Lambda \to \mathbb{R}$ be even and such that $w \in L^\infty(\Lambda)$.

Some of our analysis applies to more singular interaction potentials.

Assumption 2.2.2. Let $w : \Lambda \to \mathbb{R}$ be even and such that $w \in L^{\frac{3}{2}}(\Lambda)$.

In Assumptions 2.2.1–2.2.2 above, we make no condition on the (pointwise) sign of w or $\hat{w}$.

Convention: When working in the classical setting, we consider w as in Assumption 2.2.2. When working in the quantum setting, we will consider w as in Assumption 2.2.1. See Section 2.2.2 for more details.

For w as in Assumption 2.2.2, we study the following form of the *nonlinear Schrödinger equation (NLS)*.

$$\mathrm{i}\partial_t u + (\Delta - \kappa)u = \int dy\, dz\, w(x-y)\, w(y-z)\, w(z-x)\, |u(y)|^2\, |u(z)|^2\, u(x)\,. \quad (2.45)$$

We refer to (2.45) as the *quintic Hartree equation* or *nonlocal quintic NLS*. As is shown in Section 4.1 below, Assumption 2.2.2 is the natural setting for the interaction potential when studying (2.45); see Remark 2.2.8 below.

On the space of fields $u : \Lambda \to \mathbb{C}$, we consider the Poisson structure where the Poisson bracket is given by

$$\{u(x), \bar{u}(y)\} = \mathrm{i}\delta(x-y), \qquad \{u(x), u(y)\} = \{\bar{u}(x), \bar{u}(y)\} = 0\,. \quad (2.46)$$

Lemma 2.2.3. *With Poisson structure given by* (2.46)*, we have that* (2.45) *corresponds to the Hamiltonian equation of motion associated with Hamiltonian*

$$\begin{aligned} H(u) = &\int dx\, \left(|\nabla u(x)|^2 + \kappa |u(x)|^2\right) \\ &+ \frac{1}{3}\int dx\, dy\, dz\, w(x-y)\, w(y-z)\, w(z-x)\, |u(x)|^2\, |u(y)|^2\, |u(z)|^2\,. \end{aligned} \quad (2.47)$$

Proof. By a direct calculation using (2.46), we obtain that

$$\begin{aligned} &\{H, u\}(x) = \\ &\mathrm{i}(\Delta u(x) - \kappa u(x)) - \mathrm{i}\int dy\, dz\, w(x-y)\, w(y-z)\, w(z-x)\, |u(y)|^2\, |u(z)|^2\, u(x)\,. \end{aligned} \quad (2.48)$$

□

Using the framework of [8], we show in Section 4.1 that (2.45) is locally well-posed with w as in Assumption 2.2.2 and for initial data in $H^s(\Lambda)$ with $s > 0$; for details see Proposition 4.1.1 below. For the Gibbs measure given by

$$d\mathbb{P}^f_{\mathrm{Gibbs}}(u) := \frac{1}{z^f_{\mathrm{Gibbs}}}\, \mathrm{e}^{-H(u)}\, f(\|u\|^2_{L^2})\, du, \quad (2.49)$$

where $f \in C_0^\infty(\mathbb{R})$ is a suitable cut-off function and

$$z^f_{\mathrm{Gibbs}} = \int du\, \mathrm{e}^{-H(u)}\, f(\|u\|^2_{L^2}),$$

we show that (2.45) is globally well-posed for initial conditions in the support of (2.49). For a precise statement, see Proposition 4.1.2 below. In particular, there is a well-defined solution map S_t that maps any initial condition u_0 in the support of (2.49) to the solution of (2.45) at time t given by

$$u(\cdot) \equiv u(\cdot, t) := S_t(u_0)\,. \tag{2.50}$$

The map S_t preserves regularity in the sense that for $s \in (0, \frac{1}{2})$ and $u_0 \in H^s$, we have $S_t(u_0) \in H^s$.

The Wiener measure in the quintic case is defined in the same way as the cubic case; see (2.11) for more details. Similarly, the classical free field is given by (2.12), and the Wiener measure still satisfies the Wick theorem given in Proposition 2.1.2. The mass $\mathcal{N} \equiv \mathcal{N}^\omega = \|\varphi\|_{\mathfrak{h}}^2$ is also the same as the cubic case.

With w as in Assumption 2.2.2, the *classical interaction* $\mathcal{W} \equiv \mathcal{W}^\omega$ is given by

$$\mathcal{W} := \frac{1}{3} \int dx\, dy\, dz\, w(x-y)\, w(y-z)\, w(z-x)\, |\varphi(x)|^2\, |\varphi(y)|^2\, |\varphi(z)|^2\,. \tag{2.51}$$

Since $\varphi \in H^{\frac{1}{3}}$ μ-almost surely, by Lemma 4.1.3 below, it follows that $\mathcal{W}$ is finite μ-almost surely. We also define the *classical free Hamiltonian* $H_0 \equiv H_0^\omega$ as

$$H_0 := \int dx\, dy\, \overline{\varphi}(x)\, h(x;y)\, \varphi(y)\,, \tag{2.52}$$

where $h(x;y)$ is the kernel corresponding to (2.6), which is the same as in the cubic case. The *classical interacting Hamiltonian* $H \equiv H^\omega$ is given by

$$H := H_0 + \mathcal{W}\,. \tag{2.53}$$

Assumption 2.2.4. Whenever working in the quintic case, we fix $f \in C_0^\infty(\mathbb{R})$ a cut-off function, which is not identically zero, satisfying $0 \leq f \leq 1$, and

$$f(s) = 0 \text{ for } s > K\,, \tag{2.54}$$

where $K > 0$ is a sufficiently small positive constant.

Remark 2.2.5. The choice of K is dictated by Proposition 4.1.2 (i) below. Throughout, our estimates will depend on the K in (2.54), but we will not keep explicit track of this dependence.

With μ as in (2.11), $\mathcal{N}$ as in (2.18), $\mathcal{W}$ as in (2.51), and f as in Assumption

2.2.4, we now define the *classical Gibbs measure*

$$\mathbb{P}^f_{\mathrm{Gibbs}} := \frac{1}{z^f_{\mathrm{Gibbs}}} \, \mathrm{e}^{-\mathcal{W}} \, f(\mathcal{N}) \, \mu \,, \tag{2.55}$$

where the *classical partition function* $z \equiv z^f_{\mathrm{Gibbs}}$ is the normalisation constant

$$z := \int d\mu \, \mathrm{e}^{-\mathcal{W}} f(\mathcal{N}) \,. \tag{2.56}$$

We note that the probability measure $\mathbb{P}^f_{\mathrm{Gibbs}}$ in (2.55) and the classical partition function (2.56) are well-defined for sufficiently small K in (2.54) by Proposition 4.1.2 (i) below. In particular, this gives us a rigorous construction of (2.49) above.

The remainder of the classical model is defined analogously to the cubic case, though we include it for completeness.

For a random variable $X \equiv X^\omega$, the *classical state* $\rho \equiv \rho^f$ is given by

$$\rho(X) := \mathbb{E}_{\mathbb{P}^f_{\mathrm{Gibbs}}}(X) = \frac{\int d\mu \, X \mathrm{e}^{-\mathcal{W}} f(\mathcal{N})}{\int d\mu \, \mathrm{e}^{-\mathcal{W}} f(\mathcal{N})} \,. \tag{2.57}$$

We define the *classical p-particle correlation function* $\gamma_p \equiv \gamma_p^f$ as the operator on $\mathfrak{h}^{(p)}$ with kernel given by

$$\gamma_p(x_1, \ldots, x_p; y_1, \ldots, y_p) = \rho(\bar{\varphi}(y_1) \ldots \overline{\varphi}(y_p) \varphi(x_1) \ldots \varphi(x_p)) \,. \tag{2.58}$$

For a closed densely-defined linear operator ξ on $\mathfrak{h}^{(p)}$, we will consider the random variable $\Theta(\xi) \equiv \Theta^\omega(\xi)$ defined as

$$\Theta(\xi) := \int dx_1 \ldots dx_p \, dy_1 \ldots dy_p \, \xi(x_1, \ldots, x_p; y_1, \ldots, y_p) \\ \times \overline{\varphi}(x_1) \ldots \overline{\varphi}(x_p) \varphi(y_1) \ldots \varphi(y_p) \,. \tag{2.59}$$

Let us note that

$$H = \Theta(h) + \frac{1}{3} \Theta(W) \,, \tag{2.60}$$

where W is the operator corresponding to $p = 3$ and operator kernel

$$W(x_1, x_2, x_3; y_1, y_2, y_3) \\ := w(x_1 - x_2) \, w(x_2 - x_3) \, w(x_3 - x_1) \, \delta(x_1 - y_1) \, \delta(x_2 - y_2) \, \delta(x_3 - y_3) \,. \tag{2.61}$$

In particular, W acts as multiplication by $w(x_1 - x_2) w(x_2 - x_3) w(x_3 - x_1)$. Note that W is a linear operator on $\mathfrak{h}^{(3)}$ since by Assumption 2.2.2 we have that w is even. It is densely defined[3]. As was noted earlier, $\Theta(W) = 3\mathcal{W}$ is finite μ-almost surely.

With S_t defined as in (2.50) and ξ a closed densely-defined linear operator on $\mathfrak{h}^{(p)}$, we define for $t \in \mathbb{R}$ the random variable

$$\Psi^t\Theta(\xi) := \int dx_1 \dots dx_p \, dy_1 \dots dy_p \, \xi(x_1, \dots, x_p; y_1, \dots, y_p) \\ \times \overline{S_t\varphi}(x_1) \dots \overline{S_t\varphi}(x_p) S_t\varphi(y_1) \dots S_t\varphi(y_p) \,, \quad (2.62)$$

which corresponds to the time evolution of (2.59).

2.2.2 Quantum problem

The major difference between the cubic and quintic problems is we that we now consider the three-body interaction given in (2.63). When working in the quantum setting, we consider w as in Assumption 2.2.1 (i.e. $w \in L^\infty$ and even). This assumption is needed for technical reasons, as will be clear from the analysis in Section 4.2 below.

For w as in Assumption 2.2.1, the *quantum interaction* is defined as

$$\mathcal{W}_\tau := \frac{1}{3} \int dx \, dy \, dz \, w(x-y) \, w(y-z) \, w(z-x) \, \varphi_\tau^*(x)\varphi_\tau^*(y)\varphi_\tau^*(z)\varphi_\tau(x)\varphi_\tau(y)\varphi_\tau(z) \,. \quad (2.63)$$

For h as in (2.6) the *quantum free Hamiltonian* is defined as

$$H_{\tau,0} := \int dx \, dy \, \varphi_\tau^*(x) \, h(x,y) \, \varphi_\tau(y) \,. \quad (2.64)$$

Then the *quantum interacting Hamiltonian* is given by

$$H_\tau := H_{\tau,0} + \mathcal{W}_\tau \,. \quad (2.65)$$

Using (2.27) and (2.63)–(2.65), we have

$$H_\tau = \bigoplus_{n=0}^{\infty} H_\tau^{(n)} \,,$$

[3]It would be bounded if we were considering w as in Assumption 2.2.1.

where

$$H_\tau^{(n)} = \frac{1}{\tau} \sum_{i=1}^{n} (-\Delta_i + \kappa) + \frac{1}{3\tau^3} \sum_{\substack{i,j,k \\ i \neq j \neq k \neq i}}^{n} w(x_i - x_j)\, w(x_j - x_k)\, w(x_k - x_i)\,. \tag{2.66}$$

Note that $H_\tau^{(n)} = \frac{1}{\tau} H^{(n)}$ for $H^{(n)}$ as in (1.10) with $\lambda = \frac{1}{\tau^2}$. The rest of the quantum setting is defined analogously to the cubic case. For self-containedness, we include the definitions below.

We define the *rescaled particle number* as

$$\mathcal{N}_\tau := \int dx\, \varphi_\tau^*(x)\, \varphi_\tau(x)\,. \tag{2.67}$$

The *(untruncated and unnormalised) grand canonical ensemble* is given by

$$P_\tau := \mathrm{e}^{-H_\tau} = \bigoplus_{n=0}^{\infty} \mathrm{e}^{-H_\tau^{(n)}}\,,$$

with $H_\tau^{(n)}$ as in (2.66). For a closed operator $\mathcal{A} : \mathcal{F} \to \mathcal{F}$, the *quantum state* $\rho_\tau \equiv \rho_\tau^f$ is defined as

$$\rho_\tau(\mathcal{A}) := \frac{\operatorname{Tr}_{\mathcal{F}}\left(\mathcal{A} P_\tau f(\mathcal{N}_\tau)\right)}{\operatorname{Tr}_{\mathcal{F}}\left(P_\tau f(\mathcal{N}_\tau)\right)}\,. \tag{2.68}$$

We define the *quantum partition function* and *quantum free partition function* $Z_\tau \equiv Z_\tau^f$ and $Z_{\tau,0}$ respectively as

$$Z_\tau := \operatorname{Tr}\left(P_\tau f(\mathcal{N}_\tau)\right), \qquad Z_{\tau,0} := \operatorname{Tr}\left(\mathrm{e}^{-H_{\tau,0}}\right), \tag{2.69}$$

and the *quantum relative partition function* $\mathcal{Z}_\tau \equiv \mathcal{Z}_\tau^f$ as

$$\mathcal{Z}_\tau := \frac{Z_\tau}{Z_{\tau,0}}\,. \tag{2.70}$$

For $p \in \mathbb{N}^*$ we define the *p-particle quantum correlation function* $\gamma_{\tau,p} \equiv \gamma_{\tau,p}^f$ as operator which acts on $\mathfrak{h}^{(p)}$, with kernel given by

$$\gamma_{\tau,p}(x_1, \dots, x_p, y_1, \dots, y_p) := \rho_\tau(\varphi_\tau^*(y_1) \dots \varphi_\tau^*(y_p) \varphi_\tau(x_1) \dots \varphi_\tau(x_p))\,. \tag{2.71}$$

Note that (2.71) is a quantum analogue of (2.58).

By analogy with (2.59), for a closed linear operator $\xi \in \mathcal{L}(\mathfrak{h}^{(p)})$, we define the lift of ξ to Fock space as

$$\Theta_\tau(\xi) := \int dx_1 \dots dx_p \, dy_1 \dots dy_p \, \xi(x_1, \dots, x_p; y_1, \dots, y_p) \times \varphi_\tau^*(x_1) \dots \varphi_\tau^*(x_p) \varphi_\tau(y_1) \dots \varphi_\tau(y_p) \,. \quad (2.72)$$

Analogously to (2.60), in the quantum setting we have

$$H_\tau = \Theta_\tau(h) + \frac{1}{3}\,\Theta_\tau(W)\,,$$

where W is the 3-body operator with kernel (2.61). We note that, by Assumption 2.2.1, $\Theta_\tau(W)$ is a bounded operator on Fock space. For an operator $\mathcal{A} : \mathcal{F} \to \mathcal{F}$, we define the *quantum time evolution*

$$\Psi_\tau^t \mathcal{A} := \mathrm{e}^{\mathrm{i}t\tau H_\tau} \mathcal{A} \, \mathrm{e}^{-\mathrm{i}t\tau H_\tau} \,. \quad (2.73)$$

2.2.3 Main results

We now state our main results in the quintic case. Theorems 2.2.6 and 2.2.7 are our results in the time-independent case, and Theorems 2.2.11 and 2.2.12 the corresponding time-dependent results. Throughout, we fix a cut-off function $f \in C_0^\infty(\mathbb{R})$ as in Assumption 2.2.4 above.

We first analyse bounded interaction potentials (as in Assumption 2.2.1).

Theorem 2.2.6 (Convergence for bounded interaction potentials)**.** *Let w be as in Assumption 2.2.1. Let $p \in \mathbb{N}^*$ be given. Consider γ_p and $\gamma_{\tau,p}$ defined as in* (2.58) *and* (2.71) *respectively. Then, we have*

$$\lim_{\tau \to \infty} \|\gamma_{\tau,p} - \gamma_p\|_{\mathfrak{S}^1(\mathfrak{h}^{(p)})} = 0 \,. \quad (2.74)$$

Moreover, for z and $\mathcal{Z}_\tau$ defined as in (2.56) *and* (2.70) *respectively, we have*

$$\lim_{\tau \to \infty} \mathcal{Z}_\tau = z \,. \quad (2.75)$$

To obtain a result for w as in Assumption 2.2.2, we use an approximation argument. For a suitable approximation w^ε to w, as defined below, any object with a superscript ε will be the corresponding object defined using w^ε instead of w.

Theorem 2.2.7 (Convergence for $L^{\frac{3}{2}}$ interaction potentials)**.** *Let w be as in Assumption 2.2.2. Suppose that w^ε is a sequence of interaction potentials as in Assumption 2.2.1 converging to w in $L^{\frac{3}{2}}$. With objects defined analogously as for*

Theorem 2.2.6, there exists a sequence $\varepsilon_\tau \to 0$ as $\tau \to \infty$ such that for any $p \in \mathbb{N}^*$, we have

$$\lim_{\tau\to\infty} \|\gamma_{\tau,p}^{\varepsilon_\tau} - \gamma_p\|_{\mathfrak{S}^1(\mathfrak{h}^{(p)})} = 0\,. \tag{2.76}$$

and such that

$$\lim_{\tau\to\infty} \mathcal{Z}_\tau^{\varepsilon_\tau} = z\,. \tag{2.77}$$

Before proceeding with the time-dependent results, let us make a few remarks.

Remark 2.2.8. As in the cubic problem, in the classical setting, our method relies on a good understanding of the nonlocal problem with bounded interaction potential (as in Theorem 2.2.6 above). We are then able to upgrade this to a result for the nonlocal problem with unbounded interaction potential (as in Theorem 2.2.7) above by a suitable approximation argument. Unlike in the cubic Hartree equation, the nonlinearity in (2.45) is not given by a convolution, and hence considering $w \in L^1(\Lambda)$ in Theorem 2.2.7 does not seem to be possible, as the nonlinear term in (2.45) becomes too singular. Note that formally taking $w = -\delta$, (2.45) yields the focusing quintic NLS. In this case, the nonlinearity simplifies due to the presence of delta functions and one can establish a suitable well-posedness theory, as in [8]. Due to the aforementioned lack of a convolution structure, we cannot recover the necessary estimates for $w \in L^1(\Lambda)$ from those given in the analysis of the local problem [8], as was possible in the cubic case. Instead, we work with w as in Assumption 2.2.2, which, in light of Section 4.1 below, we conjecture to be optimal.

Remark 2.2.9. We emphasise that the calculation (2.48) above does not rely on the evenness of w. This is in contrast to the cubic Hartree equation

$$\mathrm{i}\partial_t u + (\Delta - \kappa)u = \int dy\, w(x-y)\,|u(y)|^2\,u(x)\,,$$

which is a Hamiltonian equation of motion associated with Hamiltonian

$$H(u) = \int dx\, \left(|\nabla u(x)|^2 + \kappa|u(x)|^2\right) + \frac{1}{2}\int dx\,dy\,|u(x)|^2\,w(x-y)\,|u(y)|^2\,,$$

and Poisson structure is given by (2.46) if and only if w is even. In fact, for the entire analysis of the classical problem in Section 4.1 below, we can omit the assumption that w is even in Assumption 2.2.2. However, the operator W with kernel (2.61) given above, acts (densely) on $\mathfrak{h}^{(3)}$ when w is even. Moreover, the operator $\Theta_\tau(W)$ is a well-defined operator on Fock space only when w is even. This is all necessary to study the quantum problem. Physically, when considering the n-body

Hamiltonian given in (1.10), it is realistic to assume that w is even (in addition to being bounded) in the sense that the interaction between particle x_i and x_j is the same as the interaction between particle x_j and x_i; see (1.10) above. Note that the approximation in Theorem 2.2.7 above is possible only when w is even. Hence, we will always consider w even in the analysis.

Remark 2.2.10. In Section 3.4.2, we prove that one can obtain an analogue of Theorem 2.1.7 with a cut-off function of the form of $f(x) = e^{-c|x|^2}$. This is not possible in the quintic case because there is no known corresponding analogue of Theorem 2.2.6 for non-negative interaction potentials without a cut-off function.

We now state our time-dependent results. Let us recall the definitions of ρ and ρ_τ in (2.57) and (2.68) respectively. Furthermore, we recall the definitions of $\Psi^t\Theta(\xi)$ and $\Psi^t_\tau\Theta_\tau(\xi)$ in (2.62) and (2.72)–(2.73) respectively. We first state the result for bounded interaction potentials.

Theorem 2.2.11 (Convergence for bounded potentials)**.** *Let w be as in Assumption 2.2.1. Let $m \in \mathbb{N}^*$, $p_1, \ldots, p_m \in \mathbb{N}^*$, $\xi_1 \in \mathcal{L}(\mathfrak{h}^{(p_1)}), \ldots, \xi_m \in \mathcal{L}(\mathfrak{h}^{(p_m)})$, and $t_1, \ldots, t_m \in \mathbb{R}$ be given. Then*

$$\lim_{\tau\to\infty} \rho_\tau(\Psi^{t_1}_\tau\Theta_\tau(\xi_1)\ldots\Psi^{t_m}_\tau\Theta_\tau(\xi_m)) = \rho(\Psi^{t_1}\Theta(\xi_1)\ldots\Psi^{t_m}\Theta(\xi_m))\,.$$

As in the time-independent problem, we can use an approximation result to prove a result for w as in Assumption 2.2.2.

Theorem 2.2.12 (Convergence for $L^{\frac{3}{2}}$ potentials)**.** *Let w be as in Assumption 2.2.2. Let w^ε be defined as in Theorem 2.2.7 above. Then there is a sequence $\varepsilon_\tau \to 0$ as $\tau \to \infty$ such that, for all $m \in \mathbb{N}^*$, $p_1, \ldots, p_m \in \mathbb{N}^*$, $\xi_1 \in \mathcal{L}(\mathfrak{h}^{(p_1)}), \ldots, \xi_m \in \mathcal{L}(\mathfrak{h}^{(p_m)})$, and $t_1, \ldots, t_m \in \mathbb{R}$, we have*

$$\lim_{\tau\to\infty} \rho^{\varepsilon_\tau}_\tau(\Psi^{t_1}_\tau\Theta_\tau(\xi_1)\ldots\Psi^{t_m}_\tau\Theta_\tau(\xi_m)) = \rho(\Psi^{t_1}\Theta(\xi_1)\ldots\Psi^{t_m}\Theta(\xi_m))\,.$$

We make the following remarks about the time-dependent results given in Theorems 2.2.11 and 2.2.12 above.

Remark 2.2.13. Theorems 2.2.11–2.2.12 give the first microscopic derivation of time-dependent correlation functions for a quintic NLS. From the PDE point of view, the study of time-dependent correlation functions is more relevant for quintic nonlinearities than for cubic ones. Namely, in the latter case, one can study the global well-posedness theory in $\mathfrak{h}$ without using Gibbs measures [8]. When studying the quintic problem, local well-posedness is known in $H^s(\mathbb{T})$ only for $s > 0$. Therefore, one needs to use the invariance of the Gibbs measure as the substitute for a conservation law that allows us to obtain (almost sure) global solutions [9].

Remark 2.2.14. By arguing as in [30, Remark 1.4], we can also recover the invariance of the Gibbs measure (2.55) for (2.45) from Theorem 2.2.12.

Remark 2.2.15. We note that Theorems 2.2.11 and 2.2.12 are both generalisations of Theorems 2.2.6 and 2.2.7 respectively. This can be seen by taking $m = 1$ and $t_1 = 0$, and arguing by duality as in Remarks 2.1.10 (5) and 2.1.14 above. Alternatively, see (4.109)–(4.111) below.

2.3 Previously known many body results

The results in Sections 2.1.3 and Section 2.2.3 are called microscopic derivations of Gibbs measures or time-dependent correlations from many-body quantum mechanics. These can be interpreted as the high-density limit where the mass or temperature of the system tends to infinity. One can also consider the parameter $\frac{1}{\tau}$ to be a semiclassical or mean field parameter, for a more detailed explanation of this interpretation, see [29, Section 1.1]. In the defocusing case, with $f = 1$, the first results of this kind were obtained by Lewin, Nam, and Rougerie in [49]. Here they considered the one dimensional problem with a positive translation-invariant interaction, which does not require any Wick ordering. This was based on the Gibbs variational principle and the quantum de Finetti theorem. These methods were used by the same authors to extend their results to one dimensional harmonic traps in [50].

In [29], using a perturbative series expansion of the quantum and classical states and Borel resummation, Fröhlich, Knowles, Schlein, and Sohinger gave an alternative proof of the one-dimensional result obtained in [49]. They also gave a proof for appropriately Wick-ordered Gibbs measures obtained from translation-invariant interaction potentials in two and three dimensions for a suitable modification of the grand canonical ensemble. The results in [29] in two and three dimensions were originally stated for $w \in L^\infty$, and were extended to optimal $w \in L^q$ using the methods of [29] by Sohinger in [80]. The optimal range of these q were originally observed in [11].

The full result for two and three dimensions was later shown simultaneously and independently using different methods by Lewin, Nam, and Rougerie in [53] and Fröhlich, Knowles, Schlein, and Sohinger in [31]. The result for two dimensions in

[53] was previously announced in [51]. The method in [51, 53] is a highly non-trivial extension of the one used in [49]. The proof in [31] is based on a functional integral representation. In [34], the two dimensional result in [31] is proved for the Φ^4_2 theory, which corresponds to taking an interaction potential $w = \delta$ in the cubic case.

The results from [29] were also used to study the time-dependent correlations in one dimension in [30]. Related problems for the lattice were studied in [33, 45, 71], and further details can be found in the expository works [32, 52].

We emphasise that all of the results mentioned here are for the cubic defocusing case, and that ours are the first known results in the focusing case for the two body interaction, and the first known result for the three body interaction. Our methods rely crucially on the cut-off function f, which is a natural assumption when working with focusing potentials, see [9, 20, 47, 54, 65, 68, 87]. One would also expect the results to hold for defocusing three-body interactions without cut-off – we conjecture this is true, but do not address this problem in the thesis.

For works concerning the three-body interaction, we direct the reader to [21, 22, 48, 61–63, 88]. We emphasise that the convention for the three-body interaction in (2.66) differs from the aforementioned works, since we require our effective equation (2.45) to be Hamiltonian; see Lemma 2.2.3 above. The effective evolution equations corresponding to the three-body problems in [21, 88] are the local NLS, and the ones in [22, 48] are nonlocal but have a different form to (2.45). We do not study the former nonlocal models in this thesis.

2.4 Outline of the proof

2.4.1 Cubic problem

We first analyse the time-independent problem for bounded interaction potentials $w \in L^\infty$, so the case of Theorem 2.1.7. The starting point is the perturbative expansion of the $e^{-H_\#}$, similarly to [29, Section 2.2] for the quantum setting and [29, Section 3.2] for the classical setting. Due to the presence of the truncation $f(\mathcal{N}_\#)$, the series that result have infinite radius of convergence; see Propositions 3.2.4 and 3.2.7 below. The analyticity of the expansions means we avoid needing the Borel summation techniques used in [29].

To analyse the remainder term in the quantum setting, we apply the Feynman-Kac formula and use the support properties of the cut-off function from Assumption 2.1.4. This analysis is possible since we do not Wick order the interaction in one dimension; see Lemma 3.2.3. Similarly, the truncation is crucial to the analysis of the remainder term in the classical case; see Lemma 3.2.6.

When proving the convergence of the explicit terms for the obtained series, we use the Helffer-Sjöstrand formula from complex analysis to perform an expansion on the truncation $f(\mathcal{N}_{\#})$. This is the only stage where the technical assumption of the smoothness of the truncation is used. This reduces the analysis to the study of the problem with a shifted chemical potential, but with no truncation. It is important that when we no longer have the control from the truncation, the analysis does not depend on the sign of the interaction. The details of this step is found in Lemma 3.2.8 and Lemma 3.2.10.

The proofs of Theorems 2.1.8 and 2.1.9 are based on an application of Theorem 2.1.7 and a diagonal argument. Here we crucially use [9, Lemma 3.10], which we recall in Lemma 3.1.1 below. When working with L^1 interaction potentials, we apply the local version of this result, see Corollary 3.1.4. For full details, see the proof of Lemma 3.3.1, and in particular (3.62)–(3.63).

To treat the time-dependent case for bounded interaction potentials, we apply a *Schwinger-Dyson expansion* in both the quantum and classical cases, similarly to [30, Sections 3.2-3.3]. Precise statements can be found in Lemmas 3.5.1–3.5.2. This expansion allows us to deduce Theorem 2.1.11 from Theorem 2.1.7. Crucially, since we have the truncation $f(\mathcal{N}_\tau)$, we do not have to consider the large particle regime as in [30, Section 4], which requires the non-negativity assumption of the interaction potential.

Finally, Theorems 2.1.12 and 2.1.13 are deduced from Theorem 2.1.11 by applying an approximation argument. More precisely, we estimate the flow map of the NLS with an interaction potential w with that of an interaction $w^\varepsilon \to w$. For precise statements, see Lemma 3.5.4 when $w \in L^1$ and Lemma 3.5.5 when $w = -\delta$. These results are proved within the framework of $X^{s,b}$ spaces.

2.4.2 Quintic problem

The primary difference in the quintic case is the analysis of the classical equation. We begin by proving the local well-posedness, exsitence of the Gibbs measure, and almost sure global well-posedness of (2.45), see Propositions 4.1.1 and 4.1.2 below for precise statements. The well-posedness results and construction of the Gibbs measure are analogous to the arguments in [8,9]. These results are proved using the multilinear estimates in Lemmas 4.1.3 and 4.1.4. The main difficulty in the analysis compared to [8, 9] is that the nonlinearity in (2.45) no longer has a convolution structure.

On the many body side, the argument is analogous to the cubic case, requiring a perturbative expansion of the quantum and classical states. These are analysed using the same methods, meaning that the cut-off in $f(\mathcal{N}_\tau)$ is again crucial in bounding both the explicit and remainder terms in the quantum and classical expansions. For precise statements, see Section 4.2.2. The same complex analysis argument from the cubic case implies the convergence of the quantum explicit terms to the classical explicit terms, see the proof of Proposition 4.2.13.

To prove the convergence of the power series, we require bounds and convergence of the untruncated terms, defined in (4.113). To prove this, we use a diagrammatic representation of the terms, similar to that of [29, Sections 2 and 4], but adapted to the normal ordered quintic interaction. We then consider interaction potentials in $L^{\frac{3}{2}}$ using a diagonilisation argument similar to the cubic case for unbounded interaction potentials.

For bounded interaction potentials, we treat bounded potentials using a Schwinger-Dyson expansion analogous to the cubic case, but proved for the quintic interaction. Finally, to prove Theorem 2.2.12, we need to prove an approximation result analogous to Lemma 3.5.4. To do this we need to explicitly use the almost sure global well-posedness of (2.45) and a suitable approximation lemma. See Lemma 4.4.4 for precise details.

Chapter 3

Microscopic Derivation of the Gibbs Measure for the Cubic NLS

In Section 3.1 we recall some results we will use in the chapters on the cubic and quintic NLS. Section 3.2 is an analysis of the time-independent case for bounded potentials, and Section 3.3 deals with unbounded potentials in the time-independent case. In Section 3.4, we deal with the cases of different cut-off functions, as mentioned in Remark 2.1.10 (1) and (3). Finally Section 3.5 deals with bounded and unbounded interaction potentials in the time-dependent case.

3.1 Preliminary results and basic estimates

3.1.1 Preliminary results

We recall the notation introduced in Section 1.6. We also note the following auxiliary results, which we will use throughout the thesis and state some basic estimates which will be used throughout the rest of the cubic case and the rest of thesis.

Gibbs measures for the focusing local NLS

When analysing Gibbs measures for the focusing cubic NLS with $w \in L^\infty(\mathbb{T})$, it is straightforward to make rigorous sense of (1.6) due to the presence of the truncation as in Assumption 2.1.4; see Lemma 3.1.6 (1) below.

For unbounded potentials, we will need to make use of the following result of Bourgain, found in [9, Lemma 3.10], whose proof is recalled in Appendix A.

Lemma 3.1.1. *Let* $(\mathbb{C}^{\mathbb{N}}, \mathcal{G}, \mu)$ *be the probability space defined in* (2.11)*. For* $\varphi \equiv \varphi^\omega$ *as in 2.12, the quantity*

$$\mathrm{e}^{\frac{2}{p}\|\varphi\|_{L^p}^p} \chi_{\{\|\varphi\|_{L^2} \leq B\}} \tag{3.1}$$

is in $L^1(d\mu)$ *for* $p \in [4,6)$ *for* $B > 0$ *arbitrary and* $p = 6$ *for* $B > 0$ *sufficiently small. Moreover, in* (3.1)*, we can take the* $\frac{2}{p}$ *in the exponential to be any positive constant.*

Remark 3.1.2. When $p = 6$, the optimal value of B in Lemma 3.1.1 was recently determined in [65, Theorem 1.1 (ii)]. We do not need to use this precise result since we work with $p = 4$ in the cubic case.

Remark 3.1.3. When $p = 6$, the maximum value of B depends on the constant in the exponential. For details, see (A.13) below. We do not use this in the cubic case.

Corollary 3.1.4. *Let* $(\mathbb{C}^{\mathbb{N}}, \mathcal{G}, \mu)$ *be the probability space defined in* (2.11)*, and let* $w \in L^1(\mathbb{T})$*. For* $\varphi \equiv \varphi^\omega$*,*

$$\mathrm{e}^{-\frac{1}{2}\int dx\, dy\, |\varphi(x)|^2\, w(x-y)\, |\varphi(y)|^2} \chi_{\{\|\varphi\|_{L^2} \leq B\}}$$

is in $L^1(d\mu)$ *for* $B > 0$ *arbitrary.*

We note that Corollary 3.1.4 follows from Lemma 3.1.1 with $p = 4$ by the same argument as estimate (3.3) below.

Hölder's inequality for Schatten spaces

We have the following version of Hölder's inequality for Schatten spaces (1.11), found in [76].

Lemma 3.1.5 (Hölder's Inequality)**.** *Given* $p_1, p_2 \in [1, \infty]$ *with* $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$ *and* $\mathcal{A}_j \in \mathfrak{S}^{p_j}(\mathcal{F})$ *we have*

$$\|\mathcal{A}_1 \mathcal{A}_2\|_{\mathfrak{S}^p(\mathcal{F})} \leq \|\mathcal{A}_1\|_{\mathfrak{S}^{p_1}(\mathcal{F})} \|\mathcal{A}_2\|_{\mathfrak{S}^{p_2}(\mathcal{F})}\,.$$

3.1.2 Basic estimates

Let us first note the following bound on the classical interaction.

Lemma 3.1.6. *Suppose that* $\mathcal{W} = \frac{1}{2}\int dx\, dy\, |\varphi(x)|^2\, w(x-y) |\varphi(y)|^2$ *is defined as in* (2.15)*. The following estimates hold.*

1. *For* $w \in L^\infty(\mathbb{T})$

$$|\mathcal{W}| \leq \frac{1}{2}\|w\|_{L^\infty} \|\varphi\|_{L^2}^4. \tag{3.2}$$

2. For $w \in L^1(\mathbb{T})$

$$|\mathcal{W}| \leq \frac{1}{2}\|w\|_{L^1}\|\varphi\|_{L^4}^4. \tag{3.3}$$

Proof. For (1), we note that

$$\begin{aligned}|\mathcal{W}| &= \frac{1}{2}\left|\int dx\,dy\,|\varphi(x)|^2 w(x-y)|\varphi(y)|^2\right| \\ &\leq \frac{1}{2}\|w\|_{L^\infty}\int dx\,dy\,|\varphi(x)|^2|\varphi(y)|^2 = \frac{1}{2}\|w\|_{L^\infty}\|\varphi\|_{L^2}^4.\end{aligned}$$

For (2), we apply Cauchy-Schwarz and Young's inequality to get (3.3). □

For the remainder of this section, we fix $p \in \mathbb{N}^*$. Unless otherwise specified, we consider $\xi \in \mathcal{L}(\mathfrak{h}^{(p)})$. Moreover, $\|\cdot\|$ denotes the operator norm. The following lemma follows from the definition of $\Theta(\xi)$ in (2.14).

Lemma 3.1.7. *We have*

$$|\Theta(\xi)| \leq \|\varphi\|_{\mathfrak{h}}^{2p}\|\xi\|\,.$$

Let us note that with Θ_τ as in (2.28), we have

$$\Theta_\tau(\xi)\big|_{\mathfrak{h}^{(n)}} = \begin{cases}\frac{p!}{\tau^p}\binom{n}{p}P_+\left(\xi \otimes \mathbf{1}^{(n-p)}\right)P_+ & \text{if } n \geq p \\ 0 & \text{otherwise}\,,\end{cases} \tag{3.4}$$

where $\mathbf{1}^{(q)}$ denotes the identity map on $\mathfrak{h}^{(q)}$ and P_+ is the orthogonal projection onto the subspace of symmetric tensors. More details of the above equality can be found in [45, (3.88)]. We also have the quantum analogue of Lemma 3.1.7, which follows from (3.4).

Lemma 3.1.8. *For all* $n \in \mathbb{N}^*$, *we have*

$$\left\|\Theta_\tau(\xi)\big|_{\mathfrak{h}^{(n)}}\right\| \leq \left(\frac{n}{\tau}\right)^p\|\xi\|.$$

3.2 The time-independent problem with bounded interaction potential. Proof of Theorem 2.1.7.

In this section, we study the time-independent problem with bounded interaction potential. In Section 3.2.1, we set up the Duhamel expansion in the quantum setting. For this expansion, bounds on the explicit term are shown in Section 3.2.2 and bounds on the remainder term are shown in Section 3.2.3. The analogous expansion in the classical setting is analysed in Section 3.2.4. In Section 3.2.4, we prove convergence of the explicit terms. The proof of Theorem 2.1.7 is given in Section 3.2.6.

3.2.1 Duhamel expansion

Throughout this section, we take $w \in L^\infty(\mathbb{T})$. Note that with ρ_τ defined as in (2.33), we have

$$\rho_\tau(\Theta_\tau(\xi)) = \frac{\tilde{\rho}_{\tau,1}(\Theta_\tau(\xi))}{\tilde{\rho}_{\tau,1}(\mathrm{I})}, \tag{3.5}$$

where

$$\tilde{\rho}_{\tau,\zeta}(\mathcal{A}) := \frac{1}{Z_{\tau,0}} \mathrm{Tr}\left(\mathcal{A}\mathrm{e}^{-H_{\tau,0}-\zeta\mathcal{W}_\tau} f(\mathcal{N}_\tau)\right), \tag{3.6}$$

and I denotes the identity operator on $\mathcal{F}$. Here, we recall the definition (2.34) of $Z_{\tau,0}$. With notation as above, we define

$$A_\tau^\xi(\zeta) := \tilde{\rho}_{\tau,\zeta}(\Theta_\tau(\xi)) .$$

Performing a Duhamel expansion by up to order $M \in \mathbb{N}$ by iterating the identity $e^{X+\zeta Y} = e^X + \zeta \int_0^1 dt\, e^{(1-t)X} Y e^{t(X+\zeta Y)}$ yields the following result.

Lemma 3.2.1. *For $M \in \mathbb{N}$, we have $A_\tau^\xi(\zeta) = \sum_{m=0}^{M-1} a_{\tau,m}^\xi \zeta^m + R_{\tau,M}^\xi(\zeta)$, where*

$$\begin{aligned} a_{\tau,m}^\xi :=& \frac{(-1)^m}{Z_{\tau,0}} \mathrm{Tr}\bigg(\int_0^1 dt_1 \int_0^{t_1} dt_2 \ldots \int_0^{t_{m-1}} dt_m\, \Theta_\tau(\xi) \mathrm{e}^{-(1-t_1)H_{\tau,0}}\, \mathcal{W}_\tau \\ & \times \mathrm{e}^{-(t_1-t_2)H_{\tau,0}}\, \mathcal{W}_\tau\, \mathrm{e}^{-(t_2-t_3)H_{\tau,0}} \ldots \mathrm{e}^{-(t_{m-1}-t_m)H_{\tau,0}}\, \mathcal{W}_\tau\, \mathrm{e}^{-t_m H_{\tau,0}} f(\mathcal{N}_\tau) \bigg) \end{aligned} \tag{3.7}$$

and

$$\begin{aligned} R_{\tau,M}^\xi(\zeta) := \frac{(-1)^M \zeta^M}{Z_{\tau,0}} \mathrm{Tr}\bigg(& \int_0^1 dt_1 \int_0^{t_1} dt_2 \ldots \int_0^{t_{M-1}} dt_M\, \Theta_\tau(\xi) \mathrm{e}^{-(1-t_1)H_{\tau,0}}\, \mathcal{W}_\tau \\ & \times \mathrm{e}^{-(t_1-t_2)H_{\tau,0}} \ldots \mathcal{W}_\tau\, \mathrm{e}^{-(t_{M-1}-t_M)H_{\tau,0}}\, \mathcal{W}_\tau \\ & \times \mathrm{e}^{-t_M(H_{\tau,0}+\zeta\mathcal{W}_\tau)} f(\mathcal{N}_\tau) \bigg) . \end{aligned}$$

We also define

$$\mathfrak{A} := \{\mathbf{t} \in \mathbb{R}^m : 0 < t_m < t_{m-1} \ldots < t_1 < 1\} . \tag{3.8}$$

3.2.2 Bounds on the explicit terms

Throughout the following proofs, we will use without mention that for any function $g : \mathbb{C} \to \mathbb{C}$, $g(\mathcal{N}_\tau)$ commutes with all operators on $\mathcal{F}$ that commute with $\mathcal{N}_\tau$, which is clear from the definition of $g(\mathcal{N}_\tau)$. Namely, $g(\mathcal{N}_\tau)$ acts on the p^{th} sector of Fock space as multiplication by $g(n/\tau)$. In particular, all of the operators appearing in the integrands of $a^\xi_{\tau,m}$ and $R^\xi_{\tau,M}$ commute with $g(\mathcal{N}_\tau)$.

Lemma 3.2.2. *For $m \in \mathbb{N}$, we have*

$$\left|a^\xi_{\tau,m}\right| \leq \frac{K^p \|\xi\| \left(K^2 \|w\|_{L^\infty}\right)^m}{2^m m!}. \tag{3.9}$$

Proof. Lemma 3.1.5 implies

$$\begin{aligned}\left|a^\xi_{\tau,m}\right| \leq \frac{1}{Z_{\tau,0}} \int_0^1 dt_1 \int_0^{t_1} dt_2 \ldots \int_0^{t_{m-1}} dt_m \left\|\Theta_\tau(\xi) f^{\frac{1}{m+1}}(\mathcal{N}_\tau)\right\|_{\mathfrak{S}^\infty} \\ \times \left\|\mathrm{e}^{-(1-t_1)H_{\tau,0}}\right\|_{\mathfrak{S}^{\frac{1}{1-t_1}}} \left\|\mathcal{W}_\tau f^{\frac{1}{m+1}}(\mathcal{N}_\tau)\right\|_{\mathfrak{S}^\infty} \left\|\mathrm{e}^{-(t_1-t_2)H_{\tau,0}}\right\|_{\mathfrak{S}^{\frac{1}{t_1-t_2}}} \\ \times \ldots \left\|\mathcal{W}_\tau f^{\frac{1}{m+1}}(\mathcal{N}_\tau)\right\|_{\mathfrak{S}^\infty} \left\|\mathrm{e}^{-t_m H_{\tau,0}}\right\|_{\mathfrak{S}^{\frac{1}{t_m}}}.\end{aligned} \tag{3.10}$$

Since $\mathrm{e}^{-sH_{\tau,0}}$ is a positive operator for $s \in [0,1]$, we have $\|\mathrm{e}^{-sH_{\tau,0}}\|_{\mathfrak{S}^{1/s}} = (Z_{\tau,0})^s$. So it follows from (3.10) that

$$\left|a^\xi_{\tau,m}\right| \leq \frac{Z_{\tau,0}}{Z_{\tau,0}} \frac{1}{m!} \left\|\Theta_\tau(\xi) f^{\frac{1}{m+1}}(\mathcal{N}_\tau)\right\|_{\mathfrak{S}^\infty} \left\|\mathcal{W}_\tau f^{\frac{1}{m+1}}(\mathcal{N}_\tau)\right\|^m_{\mathfrak{S}^\infty}. \tag{3.11}$$

From Lemma 3.1.8, for fixed n we have

$$\left\|\Theta_\tau(\xi) f^{\frac{1}{m+1}}(\mathcal{N}_\tau)\big|_{\mathfrak{h}^{(n)}}\right\|_{\mathfrak{S}^\infty} \leq \left(\frac{n}{\tau}\right)^p \left|f^{\frac{1}{m+1}}\left(\frac{n}{\tau}\right)\right| \|\xi\| \leq K^p \|\xi\|, \tag{3.12}$$

where the final inequality follows from Assumption 2.1.4. It follows from (3.12) that, when viewed as an operator on $\mathcal{F}$

$$\left\|\Theta_\tau(\xi) f^{\frac{1}{m+1}}(\mathcal{N}_\tau)\right\|_{\mathfrak{S}^\infty} \leq K^p \|\xi\|. \tag{3.13}$$

To bound $\left\|\mathcal{W}_\tau f^{\frac{1}{m+1}}(\mathcal{N}_\tau)\right\|_{\mathfrak{S}^\infty}$ we note that $\mathcal{W}_\tau$ acts on $\mathfrak{h}^{(n)}$ as multiplication by

$$\frac{1}{\tau^2} \sum_{1 \leq i < j \leq n} w(x_i - x_j). \tag{3.14}$$

In particular, arguing as in (3.13), it follows that

$$\left\| \mathcal{W}_\tau f^{\frac{1}{m+1}}(\mathcal{N}_\tau) \right\|_{\mathfrak{S}^\infty} \leq \frac{1}{2} K^2 \|w\|_{L^\infty}. \tag{3.15}$$

Combining (3.11) with (3.13) and (3.15), we have (3.9). □

3.2.3 Bounds on the remainder term

The following bound holds on the remainder term.

Lemma 3.2.3. *Let $M \in \mathbb{N}$, and $\mathbf{t} \in \mathfrak{A}$ (as in (3.8)) be given. Define*

$$\begin{aligned} \mathcal{R}^\xi_{\tau,M}(\mathbf{t},\zeta) := \Theta_\tau(\xi) \mathrm{e}^{-(1-t_1)H_{\tau,0}} \, \mathcal{W}_\tau \, \mathrm{e}^{-(t_1-t_2)H_{\tau,0}} \ldots \\ \times \mathcal{W}_\tau \, \mathrm{e}^{-(t_{M-1}-t_M)H_{\tau,0}} \, \mathcal{W}_\tau \, \mathrm{e}^{-t_M(H_{\tau,0}+\zeta\mathcal{W}_\tau)} f(\mathcal{N}_\tau) \, . \end{aligned}$$

Then for any $\zeta \in \mathbb{C}$,

$$\frac{1}{Z_{\tau,0}} \left| \mathrm{Tr}\left(\mathcal{R}^\xi_{\tau,M}(\mathbf{t},\zeta) \right) \right| \leq \mathrm{e}^{|\mathrm{Re}(\zeta)| K^2 \|w\|_{L^\infty}} \frac{K^p \|\xi\| \left(K^2 \|w\|_{L^\infty} \right)^M}{2^M}. \tag{3.16}$$

Proof. Define

$$\mathcal{S}(\mathbf{t}) := \Theta_\tau(\xi) \mathrm{e}^{-(1-t_1)H_{\tau,0}} \mathcal{W}_\tau \mathrm{e}^{-(t_1-t_2)H_{\tau,0}} \ldots \mathcal{W}_\tau \, \mathrm{e}^{-(t_{M-1}-t_M)H_{\tau,0}} \mathcal{W}_\tau.$$

Then

$$\mathrm{Tr}\left(\mathcal{R}^\xi_{\tau,M}(\mathbf{t},\zeta) \right) = \sum_{n\geq 0} \mathrm{Tr}\left(\left[\mathcal{S}(\mathbf{t}) f^{\frac{1}{2}}(\mathcal{N}_\tau) \right] \left[\mathrm{e}^{-t_M(H_{\tau,0}+\zeta\mathcal{W}_\tau)} f^{\frac{1}{2}}(\mathcal{N}_\tau) \right] \right)^{(n)}, \tag{3.17}$$

where the trace on the left hand side of (3.17) is taken over Fock space, whereas on the right hand side for each term it is taken over the n^{th} sector of Fock space. For $n \in N$, we have

$$\begin{aligned} &\mathrm{Tr}\left(\left[\mathcal{S}(\mathbf{t}) f^{\frac{1}{2}}(\mathcal{N}_\tau) \right] \left[\mathrm{e}^{-t_M(H_{\tau,0}+\zeta\mathcal{W}_\tau)} f^{\frac{1}{2}}(\mathcal{N}_\tau) \right] \right)^{(n)} \\ &\quad = \int_{\mathbb{T}^n} d\mathbf{x} \int_{\mathbb{T}^n} d\mathbf{y} \left(\mathcal{S}(\mathbf{t}) f^{\frac{1}{2}}(\mathcal{N}_\tau) \right)^{(n)} (\mathbf{y};\mathbf{x}) \left(\mathrm{e}^{-t_M(H_{\tau,0}+\zeta\mathcal{W}_\tau)} f^{\frac{1}{2}}(\mathcal{N}_\tau) \right)^{(n)} (\mathbf{x};\mathbf{y}) \, . \end{aligned}$$

We now rewrite $\left(\mathrm{e}^{-t_M(H_{\tau,0}+\zeta\mathcal{W}_\tau)} f^{\frac{1}{2}}(\mathcal{N}_\tau)\right)^{(n)}(\mathbf{x};\mathbf{y})$ using Proposition 1.7.12.

$$\left(\mathrm{e}^{-t_M(H_{\tau,0}+\zeta\mathcal{W}_\tau)} f^{\frac{1}{2}}(\mathcal{N}_\tau)\right)^{(n)}(\mathbf{x};\mathbf{y}) \\ = \int \mathbb{W}^{t_M}_{\mathbf{x},\mathbf{y}}(d\boldsymbol{\omega}) \mathrm{e}^{-\frac{\kappa n}{\tau}t_M} \mathrm{e}^{-\int_0^{t_M} ds\, \zeta\left(\frac{1}{\tau^2}\sum_{1\le i<j\le n} w_{ij}(\omega(s))\right)} f^{\frac{1}{2}}\left(\frac{n}{\tau}\right),$$

where $\mathbb{W}^t_{\mathbf{x},\mathbf{y}}(d\boldsymbol{\omega}) := \prod_{i=1}^n \mathbb{W}^t_{x_i,y_i}(d\omega_i)$. Here we used that

$$(\mathcal{W}_\tau)^{(n)}(\mathbf{u};\mathbf{v}) = \frac{1}{\tau^2}\sum_{1\le i<j\le n} w(u_i-u_j)\prod_{k=1}^n \delta(u_k-v_k)$$

and defined $w_{ij}(\mathbf{u}) := w(u_i-u_j)$ for $\mathbf{u}=(u_1,\ldots,u_n)\in\mathbb{T}^n$. Then

$$\left|\left(\mathrm{e}^{-t_M(H_{\tau,0}+\zeta\mathcal{W}_\tau)} f^{\frac{1}{2}}(\mathcal{N}_\tau)\right)^{(n)}(\mathbf{x};\mathbf{y})\right| \\ \le \int \mathbb{W}^{t_M}_{\mathbf{x},\mathbf{y}}(d\boldsymbol{\omega}) \mathrm{e}^{-\frac{\kappa n}{\tau}t_M} \left|\mathrm{e}^{-\int_0^{t_M} ds\, \zeta\left(\frac{1}{\tau^2}\sum_{1\le i<j\le n} w_{ij}(\boldsymbol{\omega}(s))\right)} \times f^{\frac{1}{2}}\left(\frac{n}{\tau}\right)\right| \\ \le \sup_{\boldsymbol{\omega}}\left|\mathrm{e}^{-\int_0^{t_M} ds\, \zeta\left(\frac{1}{\tau^2}\sum_{1\le i<j\le n} w_{ij}(\boldsymbol{\omega}(s))\right)} f^{\frac{1}{2}}\left(\frac{n}{\tau}\right)\right| \left(\mathrm{e}^{-t_M H_{\tau,0}}\right)^{(n)}(\mathbf{x};\mathbf{y}), \qquad (3.18)$$

where we have used Proposition 1.7.12 in the second line. We have

$$\sup_{\boldsymbol{\omega}}\left|\mathrm{e}^{-\int_0^{t_M} ds\, \zeta\left(\frac{1}{\tau^2}\sum_{1\le i<j\le n} w_{ij}(\boldsymbol{\omega}(s))\right)} f^{\frac{1}{2}}\left(\frac{n}{\tau}\right)\right| \le \mathrm{e}^{|\mathrm{Re}(\zeta)|t_M\left(\frac{n}{\tau}\right)^2\|w\|_{L^\infty}}\left|f^{\frac{1}{2}}\left(\frac{n}{\tau}\right)\right|. \qquad (3.19)$$

It follows from (3.19) that

$$\sup_{\boldsymbol{\omega}}\left|\mathrm{e}^{-\int_0^{t_M} ds\, \zeta\left(\frac{1}{\tau^2}\sum_{1\le i<j\le n} w_{ij}(\boldsymbol{\omega}(s))\right)} f^{\frac{1}{2}}\left(\frac{n}{\tau}\right)\right| \le \mathrm{e}^{|\mathrm{Re}(\zeta)|K^2\|w\|_{L^\infty}}. \qquad (3.20)$$

Combining (3.18) with (3.20) and the triangle inequality, we have shown

$$\left|\left(\mathrm{e}^{-t_M(H_{\tau,0}+\zeta\mathcal{W}_\tau)} f^{\frac{1}{2}}(\mathcal{N}_\tau)\right)^{(n)}(\mathbf{x};\mathbf{y})\right| \le \mathrm{e}^{|\mathrm{Re}(\zeta)|K^2\|w\|_{L^\infty}}\left(\mathrm{e}^{-t_M H_{\tau,0}}\right)^{(n)}(\mathbf{x};\mathbf{y}). \qquad (3.21)$$

Combining (3.17) with (3.21), it follows that

$$\left|\mathrm{Tr}\left(\mathcal{R}^\xi(\mathbf{t},\zeta)\right)\right| \le \mathrm{e}^{|\mathrm{Re}(\zeta)|K^2\|w\|_{L^\infty}}\mathrm{Tr}\Bigg(\int_0^1 dt_1\int_0^{t_1} dt_2\ldots\int_0^{t_{M-1}} dt_M\,\Theta_\tau(\tilde{\xi}) \\ \times \mathrm{e}^{-(1-t_1)H_{\tau,0}}\widetilde{\mathcal{W}_\tau}\mathrm{e}^{-(t_1-t_2)H_{\tau,0}}\widetilde{\mathcal{W}_\tau}\mathrm{e}^{-(t_2-t_3)H_{\tau,0}}\ldots\widetilde{\mathcal{W}_\tau}\mathrm{e}^{-t_M H_{\tau,0}} f^{\frac{1}{2}}(\mathcal{N}_\tau)\Bigg),$$

where $\tilde{\xi}$ is the operator with kernel $|\xi|$ and

$$\widetilde{\mathcal{W}_\tau} := \frac{1}{2} \int dx\, dy\, \varphi_\tau^*(x) \varphi_\tau^*(y) |w(x-y)| \varphi_\tau(x) \varphi_\tau(y) \,. \tag{3.22}$$

Then (3.16) follows by arguing as in the proof of Lemma 3.2.2. $\square$

Integrating (3.16) in the variables $\mathbf{t} \in \mathfrak{A}$, as defined in (3.8), implies

$$\left| R^\xi_{\tau,M}(\zeta) \right| \leq \mathrm{e}^{|\mathrm{Re}(\zeta)| K^2 \|w\|_{L^\infty}} \frac{K^p \|\xi\| \left(K^2 \|w\|_{L^\infty} \right)^M}{2^M M!} |\zeta|^M . \tag{3.23}$$

We note that this converges to 0 as $M \to \infty$ for any fixed $\zeta \in \mathbb{C}$. Moreover, since the radius of convergence of $a^\xi_{\tau,m}$ is infinite by Lemma 3.2.2, we conclude the following proposition.

Proposition 3.2.4. *The function* $A^\xi_\tau(\zeta) = \sum_{m=0}^\infty a^\xi_{\tau,m} \zeta^m$ *is analytic on* $\mathbb{C}$.

3.2.4 The classical setting

We now analyse the analogous expansion in the classical setting. Let us note that

$$\rho(\Theta(\xi)) = \frac{\tilde{\rho}_1(\Theta(\xi))}{\tilde{\rho}_1(\mathrm{I})} \,, \tag{3.24}$$

where

$$\tilde{\rho}_\zeta(X) := \int d\mu\, X \mathrm{e}^{-\zeta \mathcal{W}} f(\mathcal{N}) \,.$$

Define

$$A^\xi(\zeta) := \tilde{\rho}_\zeta(\Theta(\xi)) \,.$$

Then, for $M \in \mathbb{N}$

$$A^\xi(\zeta) = \sum_{m=0}^{M-1} a^\xi_m \zeta^m + R^\xi_M(\zeta) \,,$$

where

$$a^\xi_m := \frac{(-1)^m}{m!} \int d\mu\, \Theta(\xi) \mathcal{W}^m f(\mathcal{N}) \tag{3.25}$$

$$R^\xi_M(\zeta) = \frac{(-1)^M \zeta^M}{M!} \int d\mu\, \Theta(\xi) \mathcal{W}^M f(\mathcal{N}) \mathrm{e}^{-\tilde{\zeta} \mathcal{W}} \quad \text{for some} \quad \tilde{\zeta} \in [0, \zeta]. \tag{3.26}$$

Lemma 3.2.5. *For each $m \in \mathbb{N}$, we have*

$$\left|a_m^\xi\right| \leq \frac{K^p \|\xi\| \left(K^2 \|w\|_{L^\infty}\right)^m}{2^m m!} \tag{3.27}$$

Proof. We have

$$\left|a_m^\xi\right| \leq \frac{1}{m!} \int d\mu \left|\Theta(\xi) f^{\frac{1}{m+1}}(\mathcal{N})\right| \left|\mathcal{W} f^{\frac{1}{m+1}}(\mathcal{N})\right|^m. \tag{3.28}$$

From Lemma 3.1.7 and Assumption 2.1.4, we have

$$\left|\Theta(\xi) f^{\frac{1}{m+1}}(\mathcal{N})\right| \leq \left\|f^{\frac{1}{m+1}}\right\|_{L^\infty} K^p \|\xi\|. \tag{3.29}$$

Moreover, Lemma 3.1.6 (1) and Assumption 2.1.4 imply

$$\left|\mathcal{W} f^{\frac{1}{m+1}}(\mathcal{N})\right| \leq \frac{1}{2} \|w\|_{L^\infty} \left\|f^{\frac{1}{m+1}}\right\|_{L^\infty} K^2. \tag{3.30}$$

Recalling $\|f\|_{L^\infty} \leq 1$, (3.27) follows from (3.28) combined with (3.29) and (3.30). ☐

Note that Lemma 3.1.6 implies that

$$\left|\mathrm{e}^{-\tilde{\zeta}\mathcal{W}} f^{\frac{1}{M+2}}(\mathcal{N})\right| \leq \mathrm{e}^{\frac{1}{2}|\mathrm{Re}(\zeta)| K^2 \|w\|_{L^\infty}}$$

for $\tilde{\zeta} \in [0, \zeta]$. Applying the same arguments as the proof of Lemma 3.2.5, we have the following lemma.

Lemma 3.2.6. *For any $M \in \mathbb{N}$, we have*

$$\left|R_M^\xi(\zeta)\right| \leq \mathrm{e}^{\frac{1}{2}|\mathrm{Re}(\zeta)| K^2 \|w\|_{L^\infty}} \frac{K^p \|\xi\| \left(K^2 \|w\|_{L^\infty}\right)^M}{M!\, 2^M} |\zeta|^M. \tag{3.31}$$

Like in the quantum case, for each fixed $\zeta \in \mathbb{C}$, $R_M^\xi(\zeta)$ converges to 0 as $M \to \infty$ and a_m^ξ has infinite radius of convergence, so we have the following result.

Proposition 3.2.7. *The function $A^\xi(\zeta) = \sum_{m=0}^\infty a_m^\xi \zeta^m$ is analytic in $\mathbb{C}$.*

3.2.5 Convergence of the explicit terms

When analysing the convergence of the explicit terms, we argue similarly as in [30, Section 3.1] and rewrite $f(\mathcal{N}_\#)$ as an integral of the form

$$f(\mathcal{N}_\#) = \int_{\mathbb{C}} d\zeta \, \frac{\psi(\zeta)}{\mathcal{N}_\# - \zeta}, \tag{3.32}$$

for suitable $\psi \in C_c^\infty(\mathbb{C})$. For the precise setup, see (3.46)–(3.47) below. Using (3.32), we use that

$$\frac{1}{\mathcal{N}_\# - \zeta} = \int_0^\infty d\nu \, \mathrm{e}^{-\nu(\mathcal{N}_\# - \zeta)}, \tag{3.33}$$

for $\operatorname{Re} \zeta < 0$, which leads us to analyse analogues of (3.7) and (3.25) without the truncation $f(\mathcal{N})$ and with chemical potential shifted by $\nu > 0$. More precisely, we note the following boundedness and convergence result. We recall that here we are always considering $w \in L^\infty$.

Lemma 3.2.8. *Fix $\nu > 0$. We recall $\mathcal{C}_p$ given by (1.13) and consider $\xi \in \mathcal{C}_p$. Let*

$$\begin{aligned} b^{\xi,\nu}_{\tau,m} := \frac{(-1)^m}{Z_{\tau,0}} \mathrm{Tr}\bigg(\int_0^1 dt_1 \int_0^{t_1} dt_2 \ldots \int_0^{t_{m-1}} dt_m \, \Theta_\tau(\xi) \mathrm{e}^{-(1-t_1)(H_{\tau,0}+\nu\mathcal{N}_\tau)} \\ \times \mathcal{W}_\tau \mathrm{e}^{-(t_1-t_2)(H_{\tau,0}+\nu\mathcal{N}_\tau)} \mathcal{W}_\tau \mathrm{e}^{-(t_2-t_3)(H_{\tau,0}+\nu\mathcal{N}_\tau)} \ldots \\ \times \mathrm{e}^{-(t_{m-1}-t_m)(H_{\tau,0}+\nu\mathcal{N}_\tau)} \mathcal{W}_\tau \mathrm{e}^{-t_m(H_{\tau,0}+\nu\mathcal{N}_\tau)} \bigg) \\ b^{\xi,\nu}_m := \frac{(-1)^m}{m!} \int d\mu \, \Theta(\xi) \mathcal{W}^m \mathrm{e}^{-\nu\mathcal{N}}. \end{aligned}$$

Then, the following results hold

1. $\left| b^{\xi,\nu}_{\#,m} \right| \leq C(m,p,\nu)$.
2. $b^{\xi,\nu}_{\tau,m} \to b^{\xi,\nu}_m$ *as $\tau \to \infty$ uniformly in $\xi \in \mathcal{C}_p$.*

Proof. Let us first consider the case when $\xi \in \mathfrak{B}_p$. We define

$$h^\nu := h + \nu = \sum_{k \in \mathbb{N}} (\lambda_k + \nu) u_k u_k^* \,.$$

Then the deformed classical state defined by

$$\widetilde{\rho}^\nu_0(X) := \frac{\int d\mu \, X \mathrm{e}^{-\nu\mathcal{N}}}{\int d\mu \, \mathrm{e}^{-\nu\mathcal{N}}} \tag{3.34}$$

satisfies a Wick theorem with Green function given by $G^\nu := \frac{1}{h^\nu}$. This is the same as in Proposition 2.1.2, since all we have done is shift the chemical potential by ν.

Moreover, the deformed quasi-free state defined by

$$\widetilde{\rho}^\nu_{\tau,0}(\mathcal{A}) := \frac{\mathrm{Tr}\left(\mathcal{A} \, \mathrm{e}^{-H_{\tau,0}-\nu\mathcal{N}_\tau}\right)}{\mathrm{Tr}\left(\mathrm{e}^{-H_{\tau,0}-\nu\mathcal{N}_\tau}\right)} \tag{3.35}$$

satisfies a quantum Wick theorem similar to [29, Lemma B.1] (see Lemma 4.2.19) with quantum Green function $G_\tau = \frac{1}{\tau(\mathrm{e}^{h/\tau}-1)}$ replaced by

$$G_\tau^\nu := \frac{1}{\tau(\mathrm{e}^{h^\nu/\tau}-1)}\,.$$

In particular, we have that $\|G_\#^\nu\|_{\mathfrak{S}^2} \leq \|G_\#\|_{\mathfrak{S}^2} < \infty$. Let us define

$$\begin{aligned}\tilde{b}_{\tau,m}^{\xi,\nu} := \frac{(-1)^m}{\mathrm{Tr}\left(\mathrm{e}^{-H_{\tau,0}-\nu\mathcal{N}_\tau}\right)}\mathrm{Tr}\bigg(\int_0^1 dt_1 \int_0^{t_1} dt_2 \ldots \int_0^{t_{m-1}} dt_m\, \Theta_\tau(\xi)\mathrm{e}^{-(1-t_1)(H_{\tau,0}+\nu\mathcal{N}_\tau)} \\ \times \mathcal{W}_\tau \mathrm{e}^{-(t_1-t_2)(H_{\tau,0}+\nu\mathcal{N}_\tau)}\mathcal{W}_\tau \mathrm{e}^{-(t_2-t_3)(H_{\tau,0}+\nu\mathcal{N}_\tau)}\ldots \\ \times \mathrm{e}^{-(t_{m-1}-t_m)(H_{\tau,0}+\nu\mathcal{N}_\tau)}\mathcal{W}_\tau \mathrm{e}^{-t_m(H_{\tau,0}+\nu\mathcal{N}_\tau)}\bigg)\end{aligned}$$

and

$$\tilde{b}_m^{\xi,\nu} := \frac{1}{m!}\frac{(-1)^m}{\int d\mu\, e^{-\nu\mathcal{N}}}\int d\mu\,\Theta(\xi)\mathcal{W}^m e^{-\nu\mathcal{N}}\,.$$

Noting that noting that the arguments in [29, Sections 2.3–2.6] concerning explicit terms do not use any positivity properties of w, we hence obtain that the following properties hold.

(1') $\left|\tilde{b}_{\#,m}^{\xi,\nu}\right| \leq C(m,p,\nu)$.

(2') $\tilde{b}_{\tau,m}^{\xi,\nu} \to \tilde{b}_m^{\xi,\nu}$ as $\tau \to \infty$ uniformly in $\xi \in \mathcal{B}_p$.

More precisely, $(1')$ and $(2')$ correspond to the 1 dimensional versions[1] of [29, Corollary 2.21, Proposition 2.26] proved in [29, Section 4.1], as well as [29, Lemma 3.1].

When $\xi \in \mathfrak{B}_p$, we deduce the claim from (1') and (2') by noting that by [30, Lemma 3.4], we have

$$\lim_{\tau\to\infty}\frac{\mathrm{Tr}\left(\mathcal{A}\,\mathrm{e}^{-H_{\tau,0}-\nu\mathcal{N}_\tau}\right)}{\mathrm{Tr}\left(\mathcal{A}\,\mathrm{e}^{-H_{\tau,0}}\right)} = \int d\mu\,\mathrm{e}^{-\nu\mathcal{N}}\,.$$

It remains to consider the case when $\xi = \mathbf{1}_p$ is the identity operator on $\mathfrak{h}^{(p)}$. We then have

$$\xi(x_1,\ldots,x_p;y_1,\ldots,y_p) = \prod_{j=1}^p \delta(x_j - y_j)\,. \tag{3.36}$$

Since $\widetilde{\rho}_{\tau,0}$ satisfies the quantum Wick theorem, we can argue analogously as in [29, Section 4.2] to get the required bounds and convergence as before. We omit the details. □

[1] Throughout the thesis, when referring to [29, Corollary 2.21, Proposition 2.26], we mean these 1 dimensional versions.

We also need the following result.

Lemma 3.2.9. *Let $\mathcal{A}:\mathcal{F}\to\mathcal{F}$ and $g\in L^\infty(\mathbb{R})$. Then $|\mathrm{Tr}(\mathcal{A}\,g(\mathcal{N}_\tau))|\le\|g\|_{L^\infty}\,\mathrm{Tr}(\hat{\mathcal{A}})$, where $\hat{\mathcal{A}}^{(n)}$ has kernel $\big|\mathcal{A}^{(n)}(x;y)\big|$.*

Proof. For an operator $\mathcal{A}:\mathcal{F}\to\mathcal{F}$, we define $\mathcal{A}^{(n)}:=P^{(n)}\,\mathcal{A}\,P^{(n)}$, where $P^{(n)}$ is the projection of an operator on Fock space to the n^{th} component of Fock space. We also define $\hat{\mathcal{A}}:=\oplus_{n\ge0}\hat{\mathcal{A}}^{(n)}$. We have

$$\begin{aligned}|\mathrm{Tr}(\mathcal{A}\,g(\mathcal{N}_\tau))| &= \left|\sum_{n\ge0}\int_{\mathbb{T}^n} d\mathbf{x}\,\mathcal{A}^{(n)}(\mathbf{x};\mathbf{x})g\left(\frac{n}{\tau}\right)\right| \\ &\le \sup_{n\ge0}\left\{\left|g\left(\frac{n}{\tau}\right)\right|\right\}\sum_{n\ge0}\int_{\mathbb{T}^n} d\mathbf{x}\,|A^{(n)}(\mathbf{x};\mathbf{x})| \\ &\le \|g\|_{L^\infty}\,\mathrm{Tr}(\hat{\mathcal{A}})\,.\end{aligned}$$

□

Lemma 3.2.10. *We recall the definitions* (3.7) *and* (3.25)*. For each $m\in\mathbb{N}$, we have*

$$\lim_{\tau\to\infty} a^\xi_{\tau,m} = a^\xi_m \tag{3.37}$$

uniformly in $\xi\in\mathcal{C}_p$, defined in (1.13)*.*

Proof. For $\zeta\in\mathbb{C}\backslash[0,\infty)$, we define

$$\begin{aligned}\alpha^\xi_{\tau,m}(\zeta) := \frac{1}{Z_{\tau,0}}\mathrm{Tr}\bigg(\int_0^1 dt_1\int_0^{t_1} dt_2\ldots\int_0^{t_{m-1}} dt_m\,\Theta_\tau(\xi)\mathrm{e}^{-(1-t_1)H_{\tau,0}}\mathcal{W}_\tau\mathrm{e}^{-(t_1-t_2)H_{\tau,0}} \\ \times\mathcal{W}_\tau\ldots\mathrm{e}^{-(t_{m-1}-t_m)H_{\tau,0}}\mathcal{W}_\tau\mathrm{e}^{-t_mH_{\tau,0}}\frac{1}{\mathcal{N}_\tau-\zeta}\bigg)\end{aligned}$$

and

$$\alpha^\xi_m(\zeta) := \frac{1}{m!}\int d\mu\,\Theta(\xi)\mathcal{W}^m\frac{1}{\mathcal{N}-\zeta}\,. \tag{3.38}$$

We prove that $\alpha^\xi_{\tau,m}$ and α^ξ_m are analytic in $\zeta\in\mathbb{C}\backslash[0,\infty)$. We first deal with α^ξ_m. Note that

$$\left|\alpha^\xi_m(\zeta)\right| \le \frac{1}{m!}\int d\mu\,|\Theta(\xi)\mathcal{W}^m|\left|\frac{1}{\mathcal{N}-\zeta}\right|.$$

Using Lemma 3.1.7, Lemma 3.1.6 (1), and that $\int d\mu\,\|\varphi\|^{2p}_{\mathfrak{h}}\le C(p)$ by Remark 3.2.11, we have

$$|\alpha^\xi_m(\zeta)| \le \frac{C(m,p)}{\max\{-\mathrm{Re}\,\zeta,|\mathrm{Im}\,\zeta|\}}. \tag{3.39}$$

Arguing similarly to (3.39), it follows that

$$\frac{1}{m!}\int d\mu\,|\Theta(\xi)\mathcal{W}^m|\left|\frac{1}{(\mathcal{N}-\zeta)^2}\right| \leq \frac{C(m,p)}{\max\{-\mathrm{Re}\,\zeta,|\mathrm{Im}\,\zeta|\}^2}\,,$$

so by the dominated convergence theorem, we can differentiate under the integral sign in (3.38) and conclude that α^{ξ}_{m} is analytic in $\mathbb{C}\backslash[0,\infty)$.

To show $\alpha^{\xi}_{\tau,m}$ is analytic in $\mathbb{C}\backslash[0,\infty)$, we first note that $\frac{1}{\mathcal{N}_\tau-\zeta}$ acts as multiplication by $\frac{1}{(n/\tau)-\zeta}$ on the n^{th} sector of Fock space. By using Lemma 3.2.9 we get

$$\begin{aligned}
|\alpha_{\tau,m}(\zeta)| &\leq \frac{1}{Z_{\tau,0}}\mathrm{Tr}\biggl(\biggl[\int_0^1 dt_1\int_0^{t_1}dt_2\ldots\int_0^{t_{m-1}}dt_m\,\Theta_\tau(\xi)\mathrm{e}^{-(1-t_1)H_{\tau,0}}\mathcal{W}_\tau\mathrm{e}^{-(t_1-t_2)H_{\tau,0}}\\
&\qquad\times\mathcal{W}_\tau\ldots\mathrm{e}^{-(t_{m-1}-t_m)H_{\tau,0}}\mathcal{W}_\tau\mathrm{e}^{-t_mH_{\tau,0}}\biggr]^{\wedge}\biggr)\frac{1}{\max\{-\mathrm{Re}\,\zeta,|\mathrm{Im}\,\zeta|\}}\\
&\leq \frac{1}{Z_{\tau,0}}\mathrm{Tr}\biggl(\int_0^1 dt_1\int_0^{t_1}dt_2\ldots\int_0^{t_{m-1}}dt_m\,\Theta_\tau(\tilde{\xi})\mathrm{e}^{-(1-t_1)H_{\tau,0}}\widetilde{\mathcal{W}_\tau}\mathrm{e}^{-(t_1-t_2)H_{\tau,0}}\\
&\qquad\times\widetilde{\mathcal{W}_\tau}\ldots\mathrm{e}^{-(t_{m-1}-t_m)H_{\tau,0}}\widetilde{\mathcal{W}_\tau}\mathrm{e}^{-t_mH_{\tau,0}}\biggr)\frac{1}{\max\{-\mathrm{Re}\,\zeta,|\mathrm{Im}\,\zeta|\}},
\end{aligned}$$

where we recall $\tilde{\xi}$ is the operator with kernel $|\xi|$, and $\widetilde{\mathcal{W}_\tau}$ is as in (3.22). Applying [29, Corollary 2.21], we have

$$|\alpha^{\xi}_{\tau,m}(\zeta)| \leq \frac{C(m,p)}{\max\{-\mathrm{Re}\,\zeta,|\mathrm{Im}\,\zeta|\}}. \tag{3.40}$$

Define

$$\mathfrak{h}^{(\leq p)} := \bigoplus_{n=0}^{p}\mathfrak{h}^{(n)}\,, \tag{3.41}$$

and

$$P^{(\leq p)}:\mathcal{F}\to\mathfrak{h}^{(\leq p)}$$

as the orthogonal projection. Define

$$\begin{aligned}
\alpha_{\tau,m,n}(\zeta) := \frac{1}{Z_{\tau,0}}\,\mathrm{Tr}\biggl(&\int_0^1 dt_1\int_0^{t_1}dt_2\ldots\int_0^{t_{m-1}}dt_m\,P^{(\leq n)}\,\Theta_\tau(\xi)\,\mathrm{e}^{-(1-t_1)H_{\tau,0}}\,\mathcal{W}_\tau\\
&\times\mathrm{e}^{-(t_1-t_2)H_{\tau,0}}\,\mathcal{W}_\tau\ldots\mathrm{e}^{-(t_{m-1}-t_m)H_{\tau,0}}\,\mathcal{W}_\tau\mathrm{e}^{-t_mH_{\tau,0}}\frac{1}{\mathcal{N}_\tau-\zeta}\biggr).
\end{aligned}$$

Since $P^{(\leq n)}$ commutes with $\Theta_\tau(\xi)$, $H_{\tau,0}$, and $\mathcal{W}_\tau$, it follows that $\alpha_{\tau,m,n}$ is analytic in $\mathbb{C}\backslash[0,\infty)$. By construction we have $\lim_{n\to\infty}\alpha_{\tau,m,n}(\zeta)=\alpha_{\tau,m}(\zeta)$ for all $\zeta\in\mathbb{C}\backslash[0,\infty)$

and by the same argument as (3.40), we have

$$|\alpha^{\xi}_{\tau,m,n}(\zeta)| \leq \frac{C(m,p)}{\max\{-\operatorname{Re}\zeta, |\operatorname{Im}\zeta|\}}\,. \tag{3.42}$$

The pointwise convergence, (3.42) and the dominated convergence theorem imply that

$$\lim_{n\to\infty}\int_{\partial T} d\zeta\, \alpha_{\tau,m,n}(\zeta) = \int_{\partial T} d\zeta\, \alpha_{\tau,m}(\zeta)$$

for any triangle T contained in $\mathbb{C}\backslash[0,\infty)$. Morera's theorem implies that $\alpha_{\tau,m}$ is analytic in $\mathbb{C}\backslash[0,\infty)$.

We now prove that $\alpha^{\xi}_{\tau,m}(\zeta) \to \alpha^{\xi}_{m}(\zeta)$ as $\tau\to\infty$ for all $\zeta\in\mathbb{C}\backslash[0,\infty)$. First, for $\operatorname{Re}\zeta<0$, we recall (3.33). Therefore,

$$\begin{aligned}
\left|\alpha^{\xi}_{\tau,m}(\zeta) - \alpha^{\xi}_{m}(\zeta)\right| &= \Bigg|\frac{1}{Z_{\tau,0}}\operatorname{Tr}\Bigg(\int_0^1 dt_1\int_0^{t_1}dt_2\ldots\int_0^{t_{m-1}}dt_m\,\Theta_\tau(\xi)\mathrm{e}^{-(1-t_1)H_{\tau,0}}\mathcal{W}_\tau \\
&\quad\times \mathrm{e}^{-(t_1-t_2)H_{\tau,0}}\mathcal{W}_\tau\ldots\mathrm{e}^{-(t_{m-1}-t_m)H_{\tau,0}}\mathcal{W}_\tau\mathrm{e}^{-t_mH_{\tau,0}}\frac{1}{\mathcal{N}_\tau-\zeta}\Bigg) \\
&\quad - \frac{1}{m!}\int d\mu\,\mathcal{W}^m\frac{1}{\mathcal{N}-\zeta}\Bigg| \\
&\leq \int_0^\infty d\zeta\,\mathrm{e}^{\nu\zeta}\Bigg|\frac{1}{Z_{\tau,0}}\operatorname{Tr}\Bigg(\int_0^1 dt_1\int_0^{t_1}dt_2\ldots\int_0^{t_{m-1}}dt_m\,\Theta_\tau(\xi)\mathrm{e}^{-(1-t_1)(H_{\tau,0}+\nu\mathcal{N}_\tau)} \\
&\quad\times\mathcal{W}_\tau\mathrm{e}^{-(t_1-t_2)(H_{\tau,0}+\nu\mathcal{N}_\tau)}\mathcal{W}_\tau\ldots\mathrm{e}^{-(t_{m-1}-t_m)(H_{\tau,0}+\nu\mathcal{N}_\tau)}\mathcal{W}_\tau \\
&\quad\times\mathrm{e}^{-t_m(H_{\tau,0}+\nu\mathcal{N}_\tau)}\Bigg) - \frac{1}{m!}\int d\mu\,\mathrm{e}^{-\nu\mathcal{N}}\mathcal{W}^m\Bigg|\,,
\end{aligned} \tag{3.43}$$

where we have used part (1) of Lemma 3.2.8 and $\operatorname{Re}\zeta<0$ to apply Fubini's theorem. Lemma 3.2.8, (3.43), and the dominated convergence theorem give

$$\lim_{\tau\to\infty}\alpha^{\xi}_{\tau,m}(\zeta) = \alpha^{\xi}_{m}(\zeta) \tag{3.44}$$

uniformly in $\xi\in\mathcal{C}_p$ for $\operatorname{Re}\zeta<0$.

We define $\beta^{\xi}_{\tau,m} := \alpha^{\xi}_{\tau,m} - \alpha^{\xi}_{m}$. We follow the argument in [30, Proposition 3.3] to prove

$$\lim_{\tau\to\infty}\sup_{\xi\in\mathcal{C}_p}|\beta^{\xi}_{\tau,m}(\zeta)| = 0 \text{ for all } \zeta\in\mathbb{C}\backslash[0,\infty)\,. \tag{3.45}$$

From the analyticity of $\alpha^{\xi}_{\#,m}$ on $\mathbb{C}\backslash[0,\infty)$, (3.39) and (3.40), and (3.44), we know that $\beta^{\xi}_{\tau,m}$ satisfy the following properties.

1. $\beta^{\xi}_{\tau,m}$ is analytic on $\mathbb{C}\backslash[0,\infty)$.

2. $\lim_{\tau\to\infty}\sup_{\xi\in\mathcal{C}_p}|\beta^{\xi}_{\tau,m}(\zeta)| = 0$ for all $\operatorname{Re}\zeta < 0$.

3. $\sup_{\xi\in\mathcal{C}_p}|\beta^{\xi}_{\tau,m}(\zeta)| \leq \frac{C(m,p)}{|\operatorname{Im}\zeta|}$ for all $\zeta \in \mathbb{C}\backslash[0,\infty)$.

Given $\varepsilon > 0$, define

$$\mathcal{D}_\varepsilon := \{\zeta : \operatorname{Im}\zeta > \varepsilon\}$$

and

$$\mathcal{T}_\varepsilon := \{\zeta_0 \in \mathcal{D}_\varepsilon : \lim_{\tau\to\infty}\sup_{\xi\in\mathcal{C}_p}\left|\partial^n_\zeta\beta^{\xi}_{\tau,m}(\zeta_0) = 0\right| \text{ for all } n \in \mathbb{N}\,.$$

So $\mathcal{T}_\varepsilon$ is the set of points in $\mathcal{D}_\varepsilon$ at which all ζ-derivatives of $\beta^{\xi}_{\tau,m}$ converge to 0 as $\tau \to \infty$ uniformly in $\xi \in \mathcal{C}_p$. Using properties (1) - (3) of β, Cauchy's integral formula, and the dominated convergence theorem, we have $\mathcal{D}_\varepsilon \cap \{\zeta : \operatorname{Re}\zeta < 0\} \subset \mathcal{T}_\varepsilon$. In particular, $\mathcal{T}_\varepsilon$ is not empty.

So to prove (3.45) on $\mathcal{D}_\varepsilon$, it suffices to show that $\mathcal{T}_\varepsilon = \mathcal{D}_\varepsilon$. Since $\mathcal{D}_\varepsilon$ is connected, the latter claim follows from showing that $\mathcal{T}_\varepsilon$ is both open and closed in $\mathcal{D}_\varepsilon$. We first show that $\mathcal{T}_\varepsilon$ is open in $\mathcal{D}_\varepsilon$. Given $\zeta_0 \in \mathcal{T}_\varepsilon$, note that $B_{\varepsilon/2}(\zeta_0) \subset \mathcal{D}_{\varepsilon/2}$. So by property (3), $\left|\beta^{\zeta}_{\tau,m}\right| \leq C(\varepsilon)$ on $B_{\varepsilon/2}(\zeta_0)$. Analyticity and Cauchy's integral formula imply that the Taylor series of $\beta^{\xi}_{\tau,m}$ at ζ_0 converges on $B_{\varepsilon/2}(\zeta_0)$. So we can differentiate term by term and use the dominated convergence theorem and $\zeta_0 \in \mathcal{T}_\varepsilon$ to get that $B_\delta(\zeta_0) \subset \mathcal{T}_\varepsilon$ for $\delta \in (0, \varepsilon/2)$ sufficiently small such that $B_\delta(\zeta_0) \subset \mathcal{D}_\varepsilon$. So $\mathcal{T}_\varepsilon$ is open in $\mathcal{D}_\varepsilon$.

To show that $\mathcal{T}_\varepsilon$ is closed in $\mathcal{D}_\varepsilon$, let (ζ_n) be a sequence in $\mathcal{T}_\varepsilon$ which converges to some $\zeta \in \mathcal{D}_\varepsilon$. Since $\zeta \in \mathcal{D}_\varepsilon$ which is open, there is $\varepsilon' \in (0, \varepsilon/2)$ such that $B_{\varepsilon'}(\zeta) \subset \mathcal{D}_\varepsilon$. Since $(\zeta_n) \to \zeta$, for n sufficiently large, $\zeta \in B_{\varepsilon'/2}(\zeta_n)$. Since $B_{\varepsilon'/2}(\zeta_n) \subset B_{\varepsilon'}(\zeta) \subset \mathcal{D}_\varepsilon$, the argument that $\mathcal{T}_\varepsilon$ is open in $\mathcal{D}_\varepsilon$ implies that $B_{\varepsilon'/2}(\zeta_n) \subset \mathcal{T}_\varepsilon$. In particular, $\zeta \in \mathcal{T}_\varepsilon$, so $\mathcal{T}_\varepsilon$ is closed in $\mathcal{D}_\varepsilon$. By symmetry, the same argument shows that (3.45) holds on $\widetilde{\mathcal{D}}_\varepsilon := \{\zeta : \operatorname{Im}\zeta < -\varepsilon\}$. Then (3.45) holds on $\mathbb{C}\backslash[0,\infty)$ by letting $\varepsilon \to 0$ and recalling that (3.45) holds for $\zeta < 0$ by property (2) above.

Applying the Helffer-Sjöstrand formula and arguing as in [30, (3.29)-(3.33)], we can find $\psi \in C^\infty_c(\mathbb{C})$ satisfying

$$|\psi(\zeta)| \leq C|\operatorname{Im}\zeta| \tag{3.46}$$

such that

$$f(\mathcal{N}_\#) = \int_{\mathbb{C}} d\zeta\, \frac{\psi(\zeta)}{\mathcal{N}_\# - \zeta}\,. \tag{3.47}$$

Then (3.39), (3.40), (3.46), and $\psi \in C^\infty_c(\mathbb{C})$ imply that

$$|\alpha^{\xi}_{\#,m}(\zeta)\psi(\zeta)| \leq F(\zeta) \tag{3.48}$$

for some $F \in L^1(\mathbb{C})$. By (3.48), we can use Fubini's theorem to write

$$|a^\xi_{\tau,m} - a^\xi_m| \leq \int_{\mathbb{C}} |\psi(\zeta)| \left|\alpha^\xi_{\tau,m}(\zeta) - \alpha^\xi_m(\zeta)\right| .$$

Using $\beta^\xi_{\tau,m} \to 0$ as $\tau \to \infty$ almost everywhere in $\mathbb{C}$ uniformly in ξ and (3.48), the dominated convergence theorem implies (3.37). □

Remark 3.2.11. In the proof of Lemma 3.2.10, we used that $\int d\mu \, \|\varphi\|^{2p}_{\mathfrak{h}} \leq C(p) < \infty$. To see this, recall (2.12) implies

$$\begin{aligned} \int d\mu \, \|\varphi\|^{2p}_{\mathfrak{h}} &= \mathbb{E}_\omega \left[\left(\sum_{n \in \mathbb{N}} \frac{|\omega_n|^2}{\lambda_n} \right)^p \right] \\ &= \mathbb{E}_\omega \left[\sum_{n_i \in \mathbb{N}} \frac{\omega_{n_1}\overline{\omega_{n_1}} \dots \omega_{n_p}\overline{\omega_{n_p}}}{\lambda_{n_1} \dots \lambda_{n_p}} \right] \\ &\leq C(p) \left(\sum_n \frac{1}{\lambda_n} \right)^p \leq C(p) < \infty \, . \end{aligned}$$

The final line follows from Proposition 2.1.2.

3.2.6 Convergence of correlation functions. Proof of Theorem 2.1.7

Lemma 3.2.12. *$A^\xi_\tau(\zeta) \to A^\xi(\zeta)$ as $\tau \to \infty$ uniformly in $\xi \in \mathcal{C}_p$.*

Proof. Since $A^\xi_\#$ are analytic in $\mathbb{C}$, for all $\zeta \in \mathbb{C}$

$$\sup_{\xi \in \mathfrak{B}_p} \left|A^\xi_\tau(\zeta) - A^\xi(\zeta)\right| \leq \sum_m \sup_\xi \left|a^\xi_{\tau,m} - a^\xi_m\right| |\zeta|^m \to 0$$

as $\tau \to \infty$. Here we have used Lemma 3.2.10, Lemma 3.2.2, and the dominated convergence theorem. We also recall the notation (1.12). □

Recalling (3.5) and (3.24) and taking $\zeta = 1$, we have the following result.

Corollary 3.2.13. *$\rho_\tau(\Theta_\tau(\xi)) \to \rho(\Theta(\xi))$ as $\tau \to \infty$ uniformly in $\xi \in \mathcal{C}_p$.*

Before proceeding to the proof of Theorem 2.1.7, we first need to prove the following technical lemma.

Lemma 3.2.14. *Recalling* (2.22) *and* (2.36)*, we have $\gamma_{\#,p} \geq 0$ in the sense of operators.*

Proof. For $\eta \in \mathfrak{h}^{(p)}$, define the orthogonal projection $\Pi_\eta(\cdot) := \langle \eta, \cdot \rangle \eta$. Let us first note that

$$\langle \eta, \gamma_{\#,p} \eta \rangle_{\mathfrak{h}^{(p)}} = \rho_\#(\Theta_\#(\Pi_\eta)) \,. \tag{3.49}$$

In the quantum setting, we use (2.36) and linearity to compute

$$\begin{aligned} \langle \eta, \gamma_{\tau,p} \eta \rangle_{\mathfrak{h}^{(p)}} &= \int dx_1 \cdots dx_p \, dy_1 \cdots dy_p \, \overline{\eta}(x_1, \dots, x_p) \, \eta(y_1, \dots, y_p) \\ &\qquad \times \rho_\tau \big(\varphi_\tau^*(y_1) \cdots \varphi_\tau^*(y_p) \varphi_\tau(x_1) \cdots \varphi_\tau(x_p) \big) \\ &= \rho_\tau \bigg(\int dx_1 \cdots dx_p \, dy_1 \cdots dy_p \, \overline{\eta}(x_1, \dots, x_p) \, \eta(y_1, \dots, y_p) \\ &\qquad \times \varphi_\tau^*(y_1) \cdots \varphi_\tau^*(y_p) \varphi_\tau(x_1) \cdots \varphi_\tau(x_p) \bigg) \,. \end{aligned} \tag{3.50}$$

By (2.28) and the definition of Π_η we deduce that that the expression in (3.50) equals $\rho_\tau(\Theta_\tau(\Pi_\eta))$, thus showing (3.49) in the quantum setting. Similarly in the classical setting, we use (2.22) and (2.14) to compute

$$\begin{aligned} \langle \eta, \gamma_p \eta \rangle_{\mathfrak{h}^{(p)}} &= \rho \bigg(\int dx_1 \cdots dx_p \, dy_1 \cdots dy_p \, \overline{\eta}(x_1, \dots, x_p) \, \eta(y_1, \dots, y_p) \\ &\qquad \times \overline{\varphi}(y_1) \cdots \overline{\varphi}(y_p) \varphi(x_1) \cdots \varphi(x_p) \bigg) = \rho(\Theta(\Pi_\eta)) \,, \end{aligned}$$

as was claimed.

We now show that the expression on the right-hand side of (3.49) is nonnegative. Let us first show this in the quantum setting. By (2.28), we note that $\Theta_\tau(\Pi_\eta)$ is a positive operator. Furthermore $f(\mathcal{N}_\tau)$ is a positive operator which commutes with $\Theta_\tau(\Pi_\eta)$. In particular, their composition is a positive operator. Recalling (2.32), we know that

$$\mathcal{A} \mapsto \frac{\operatorname{Tr}(\mathcal{A} P_\tau)}{\operatorname{Tr}(P_\tau)}$$

is a quantum state. In particular, when applied to positive operators it is nonnegative, so we obtain that

$$\frac{\operatorname{Tr}(\Theta_\tau(\Pi_\eta) f(\mathcal{N}_\tau) P_\tau)}{\operatorname{Tr}(P_\tau)} \geq 0 \,. \tag{3.51}$$

Since P_τ and $f(\mathcal{N}_\tau)$ commute, by using (3.51), and recalling (2.32) as well as Assumption 2.1.4, it follows that

$$\rho_\tau(\Theta_\tau(\Pi_\eta) f(\mathcal{N}_\tau)) = \frac{\operatorname{Tr}(\Theta_\tau(\Pi_\eta) f(\mathcal{N}_\tau) P_\tau)}{\operatorname{Tr}(P_\tau f(\mathcal{N}_\tau))} \geq 0 \,. \tag{3.52}$$

We deduce the claim in the quantum setting from (3.49) and (3.52).

In the classical setting, we use (2.14) to write

$$\begin{aligned}\rho(\Theta(\Pi_\eta)) = \rho\biggl(\int dx_1 \cdots dx_p\, dy_1 \cdots dy_p \overline{\eta}(x_1), \dots, x_p)\, \eta(y_1, \dots, y_p) \\ \times\, \varphi(x_1) \cdots \varphi(x_p)\, \overline{\varphi}(y_1) \cdots \overline{\varphi}(y_p)\biggr) \\ = \rho\biggl(\biggl|\int dx_1 \cdots dx_p \overline{\eta}(x_1, \dots, x_p)\, \varphi(x_1) \cdots \varphi(x_p)\biggr|^2\biggr) \geq 0\,. \quad (3.53)\end{aligned}$$

For the last inequality in (3.53), we recalled (2.20). We deduce the claim in the classical setting from (3.49) and (3.53). □

Remark 3.2.15. By following the same duality argument as [30, Proposition 3.3 (*ii*)], we can deduce from Lemma 3.2.14 that Corollary 3.2.13 holds for all $\xi \in \mathcal{L}(\mathfrak{h}^{(p)})$.

To prove Theorem 2.1.7, we use the following result, proved in [29, Lemma 4.10].

Lemma 3.2.16. *Let $p \in \mathbb{N}$ be fixed. Suppose that for all $\tau > 0$, $\gamma_\tau \in \mathfrak{S}^1(\mathfrak{h}^{(p)})$ is positive and that $\gamma \in \mathfrak{S}^1(\mathfrak{h}^{(p)})$ is positive. Suppose further that*

$$\lim_{\tau\to\infty} \|\gamma_\tau - \gamma\|_{\mathfrak{S}^2(\mathfrak{h}^{(p)})} = 0, \quad \textit{and} \quad \lim_{\tau\to\infty} \mathrm{Tr}\gamma_\tau = \mathrm{Tr}\gamma. \quad (3.54)$$

Then $\lim_{\tau\to\infty} \|\gamma_\tau - \gamma\|_{\mathfrak{S}^1(\mathfrak{h}^{(p)})} = 0$.

Proof of Theorem 2.1.7. We first prove (2.37). Let $p \in \mathbb{N}^*$ be given. We verify the conditions of Lemma 3.2.16. Using the fact that $\mathfrak{S}^2(\mathfrak{h}^{(p)}) \cong \mathfrak{S}^2(\mathfrak{h}^{(p)})^*$ and recalling that $\mathfrak{B}_p$ is the unit ball of $\mathfrak{S}^2(\mathfrak{h}^{(p)})$, we have

$$\|\gamma_{\tau,p} - \gamma_p\|_{\mathfrak{S}^2(\mathfrak{h}^{(p)})} = \sup_{\xi\in\mathfrak{B}_p} |\mathrm{Tr}\,(\gamma_{\tau,p}\xi - \gamma_p\xi)| = \sup_{\xi\in\mathfrak{B}_p} |\rho_\tau(\Theta_\tau(\xi)) - \rho(\Theta(\xi))| \to 0 \quad (3.55)$$

as $\tau \to \infty$ by Corollary 3.2.13. We also note that $\mathrm{Tr}\gamma_{\#,p} = \rho_\#(\Theta_\#(\mathrm{I}))$. So Corollary 3.2.13 implies

$$\lim_{\tau\to\infty} \mathrm{Tr}\gamma_{\tau,p} = \mathrm{Tr}\gamma_p. \quad (3.56)$$

Combining Lemma 3.2.14, (3.55), and (3.56), Lemma 3.2.16 implies (2.37). The proof of (2.38) is similar. Namely we start from (3.6) with $\mathcal{A} = \mathrm{I}$ and repeat the previous argument (in which we formally set $p = 0$). □

3.3 The time-independent problem with unbounded interaction potentials. Proofs of Theorems 2.1.8 and 2.1.9.

In this section, we analyse the time-independent problem for general w as in Assumption 2.1.1. In particular, we no longer assume that w is bounded, as in Section 3.2. In Section 3.3.1, we consider w satisfying Assumption 2.1.1 (i) and prove Theorem 2.1.8. In Section 3.3.2, we consider w satisfying Assumption 2.1.1 (ii) and prove Theorem 2.1.9. As before, we fix $p \in \mathbb{N}^*$ throughout the section.

3.3.1 L^1 interaction potentials. Proof of Theorem 2.1.8.

We first consider the case where w satisfies Assumption 2.1.1 (i), i.e. when it is taken to be an even and real-valued function in $L^1(\mathbb{T})$. To do this, we approximate w with bounded potentials w^ε, which are even and real-valued. For instance, we can take $w^\varepsilon := w\chi_{\{|w| \leq 1/\varepsilon\}}$. We then use the results of the previous section combined with a diagonal argument.

Let us first note the following result.

Lemma 3.3.1. *Let w be as in Assumption 2.1.1 (i), and suppose $w^\varepsilon \in L^\infty$ is a sequence of even, real-valued interaction potentials satisfying $w^\varepsilon \to w$ in $L^1(\mathbb{T})$ as $\varepsilon \to 0$. Then there exists a sequence (ε_τ) converging to 0 as $\tau \to \infty$ such that for all $p \in \mathbb{N}^*$*

$$\lim_{\tau \to \infty} \rho_\tau^{\varepsilon_\tau} \left(\Theta_\tau(\xi)\right) = \rho\left(\Theta\left(\xi\right)\right) \,, \tag{3.57}$$

uniformly in $\xi \in \mathcal{C}_p$. We recall that $\mathcal{C}_p$ is given by (1.13).

Proof. Using a standard diagonal argument, it suffices to prove that for each fixed $\varepsilon > 0$

$$\lim_{\tau \to \infty} \rho_\tau^{\varepsilon} \left(\Theta_\tau(\xi)\right) \to \rho^\varepsilon\left(\Theta\left(\xi\right)\right) \tag{3.58}$$

uniformly in $\xi \in \mathcal{C}_p$, and

$$\lim_{\varepsilon \to 0} \rho^{\varepsilon} \left(\Theta\left(\xi\right)\right) \to \rho\left(\Theta\left(\xi\right)\right) \tag{3.59}$$

uniformly in $\xi \in \mathcal{C}_p$. The convergence in (3.58) holds by Corollary 3.2.13 because $w^\varepsilon \in L^\infty(\mathbb{T})$. To show (3.59), we first note that by Lemma 3.1.6 (2) and the Sobolev embedding theorem

$$|\mathcal{W}^\varepsilon - \mathcal{W}| \lesssim \|w^\varepsilon - w\|_{L^1} \|\varphi\|_{H^{\frac{1}{2}-}}^4 \,. \tag{3.60}$$

Since $\varphi \in H^{\frac{1}{2}-}$ almost surely, it follows that

$$\lim_{\varepsilon \to 0} \mathcal{W}^\varepsilon = \mathcal{W} \tag{3.61}$$

almost surely. Continuity of the exponential implies that

$$\lim_{\varepsilon \to 0} \mathrm{e}^{-\mathcal{W}^\varepsilon} = \mathrm{e}^{-\mathcal{W}}$$

almost surely. By Lemma 3.1.6 (2), we have

$$|\mathcal{W}| \leq \frac{1}{2} \|w\|_{L^1} \|\varphi\|_{L^4}^4 ,$$

and for ε sufficiently small

$$|\mathcal{W}^\varepsilon| \leq \frac{1}{2} \|w^\varepsilon\|_{L^1} \|\varphi\|_{L^4}^4 \leq \|w\|_{L^1} \|\varphi\|_{L^4}^4 .$$

It follows that

$$\left| \mathrm{e}^{-\mathcal{W}^\varepsilon} - \mathrm{e}^{-\mathcal{W}} \right| \leq 2\mathrm{e}^{\|w\|_{L^1} \|\varphi\|_{L^4}^4} . \tag{3.62}$$

By Lemma 3.1.1 and Assumption 2.1.4, we know that

$$\mathrm{e}^{\|w\|_{L^1} \|\varphi\|_{L^4}^4} f^{\frac{1}{2}}(\mathcal{N}) \in L^1(d\mu) . \tag{3.63}$$

By Lemma 3.1.7, we have that

$$\Theta(\xi) f^{\frac{1}{2}}(\mathcal{N}) \in L^\infty(d\mu) . \tag{3.64}$$

Using (3.62)–(3.64) and the dominated convergence theorem, it follows that

$$\lim_{\varepsilon \to 0} \int d\mu \, |\Theta(\xi)| \, \left| \mathrm{e}^{-\mathcal{W}^\varepsilon} - \mathrm{e}^{-\mathcal{W}} \right| f(\mathcal{N}) = 0 . \tag{3.65}$$

The same argument implies

$$\lim_{\varepsilon \to 0} z^\varepsilon = z . \tag{3.66}$$

Noting that

$$\rho^\varepsilon \left(\Theta \left(\xi \right) \right) - \rho \left(\Theta \left(\xi \right) \right) = \frac{1}{z} \int d\mu \, \Theta(\xi) f(\mathcal{N}) \left(\frac{z}{z^\varepsilon} \mathrm{e}^{-\mathcal{W}^\varepsilon} - \mathrm{e}^{-\mathcal{W}} \right) ,$$

(3.59) follows from (3.65) and (3.66). □

We can now prove Theorem 2.1.8.

Proof of Theorem 2.1.8. We deduce (2.39) from Lemma 3.3.1 by arguing analogously as in the proof of (2.37). The proof of (2.40) is similar to that of (2.39). Instead of (3.58), we use

$$\lim_{\tau\to\infty} \mathcal{Z}_\tau^\varepsilon = z^\varepsilon\,,$$

for fixed $\varepsilon > 0$, which follows from (2.38). Instead of (3.59), we use (3.66). □

3.3.2 The delta function. Proof of Theorem 2.1.9

We now deal with the case $w = -\delta$. Let us first recall the definition (2.42) of w^ε. Let us note that since U is even, it is not necessary to take U to be non-positive, since we can argue as in [30, (5.33)] using $|U|$ (note that in [30], one writes $\widetilde{w}$ for U). In what follows, we again denote objects corresponding to the interaction potential w^ε by using a superscript ε. Again, by following Section 3.2.6, to prove Theorem 2.1.9, it suffices to prove the following proposition.

Lemma 3.3.2. *Let $w := -\delta$, and let w^ε be defined as in* (2.42)*. Then there is a sequence (ε_τ) satisfying ε_τ converging to* 0 *as $\tau \to \infty$ such that*

$$\lim_{\tau\to\infty} \rho_\tau^{\varepsilon_\tau}\left(\Theta_\tau(\xi)\right) = \rho\left(\Theta(\xi)\right)\,, \tag{3.67}$$

uniformly in $\xi \in \mathcal{C}_p$, where $\mathcal{C}_p$ is given by (1.13).

Proof. As in the proof of Lemma 3.3.1, it suffices to prove for fixed ε that

$$\lim_{\tau\to\infty} \rho_\tau^{\varepsilon}\left(\Theta_\tau(\xi)\right) \to \rho^\varepsilon\left(\Theta\left(\xi\right)\right)\,, \tag{3.68}$$

uniformly in $\xi \in \mathcal{C}_p$, and

$$\lim_{\varepsilon\to 0} \rho^\varepsilon\left(\Theta\left(\xi\right)\right) \to \rho\left(\Theta\left(\xi\right)\right) \tag{3.69}$$

uniformly in $\xi \in \mathcal{C}_p$. Since $w^\varepsilon \in L^\infty(\mathbb{T})$, (3.68) follows from Lemma 3.3.1. To prove (3.69), we note that Lemma 3.1.6 (2) now implies

$$\begin{aligned} |\mathcal{W}| &\leq \frac{1}{2}\|U\|_{L^1}\|\varphi\|_{L^4}^4\,,\\ |\mathcal{W}^\varepsilon| &\leq \frac{1}{2}\|U\|_{L^1}\|\varphi\|_{L^4}^4\,. \end{aligned} \tag{3.70}$$

Since $\int dx\, U = -1$ and U is even,

$$\mathcal{W}^\varepsilon - \mathcal{W} = \frac{1}{2}\int dx\, dy\, w^\varepsilon(x-y)\left(|\varphi(x)|^2|\varphi(y)|^2 - |\varphi(x)|^4\right)\,.$$

So

$$|\mathcal{W}^{\varepsilon}-\mathcal{W}| \leq \frac{1}{2}\int dx\,dy\,|w^{\varepsilon}(x-y)||\varphi(x)|^2|\varphi(x)-\varphi(y)|\,(|\varphi(x)|+|\varphi(y)|)\,. \tag{3.71}$$

We can then follow the argument in [30, (5.49) in the proof of Theorem 1.6.] to conclude $\mathcal{W}^{\varepsilon} \to \mathcal{W}$. We omit the details. Arguing as in the proof of Lemma 3.3.1, we obtain (3.69), and thus (3.67). We emphasise that, in order to apply the dominated convergence theorem as in the proof of Lemma 3.3.1, it is important that the upper bound (3.70) is uniform in ε. □

Proof of Theorem 2.1.9. We obtain (2.43) by arguing analogously as for (2.39). Here, instead of Lemma 3.3.1, we use Lemma 3.3.2. The proof of (2.44) is analogous to that of (2.40). □

3.4 Remarks about the cut-off function f

In this section, we expand on Remark 2.1.10 (1) and (3).

3.4.1 Interaction potentials of positive type

For a bounded, real-valued, even interaction potential w of positive type (i.e. $\hat{w} \geq 0$ pointwise almost everywhere), we claim we can apply the methods used in the proof of [29, Theorem 1.8] to get the result of Theorem 2.1.7 for $\rho_{\#}$ defined without a truncation in $\mathcal{N}_{\sharp}$. To do this, we follow the convention from the two and three dimensional cases from [29] and consider a non-normal ordered quantum interaction, namely

$$\mathcal{W}'_{\tau} := \frac{1}{2}\int dx\,dy\,\varphi^{*}_{\tau}(x)\varphi_{\tau}(x)w(x-y)\varphi^{*}_{\tau}(y)\varphi_{\tau}(y)\,, \tag{3.72}$$

which we note is different to the convention adopted in the rest of the cubic case. We also define $H'_{\tau} := H_{\tau,0} + \mathcal{W}'_{\tau}$ in contrast to (2.30). Applying (2.26), we have

$$\mathcal{W}'_{\tau} = \mathcal{W}_{\tau} + \frac{1}{2\tau}w(0)\mathcal{N}_{\tau}^{2}\,,$$

where we recall (2.29) and (2.31). We consider this non-normal ordered interaction since $\mathcal{W}'_{\tau}$ acts on the n^{th} sector of Fock space as multiplication by

$$\frac{1}{2\tau^2}\sum_{i,j=1}^{n} w(x_i - x_j)\,. \tag{3.73}$$

The key difference from (3.14) is (3.73) includes the diagonal terms of the sum. The remark follows from showing that if w is of positive type, (3.73) ≥ 0 almost everywhere, since we can apply Proposition 1.7.12 as in the proof of [29, Proposition 4.5]. We can further reduce this to showing (3.73) ≥ 0 for $w \in C^\infty$ of positive type by taking $w^\varepsilon := w * \varphi^\varepsilon$ for a standard approximation to the identity φ^ε of positive type, since then $w^\varepsilon \to w$ pointwise almost everywhere.

To see this, recall that for $g \in L^2$, Parseval's theorem implies

$$\langle g, w * g \rangle \sim \sum_{k \in \mathbb{Z}} |\hat{g}(k)|^2 \hat{w}(k) \geq 0 \tag{3.74}$$

since w is of positive type. Taking $g^\varepsilon \in C^\infty$ with $g^\varepsilon \to \sum_{j=1}^n \delta(\cdot - x_j)$ weakly with respect to continuous functions, for $w \in C^\infty$ we have, by (3.74)

$$0 \leq \langle g^\varepsilon, w * g^\varepsilon \rangle \to \sum_{i,j=1}^n w(x_i - x_j) \,.$$

Letting $\varepsilon \to 0$ then yields (3.73) ≥ 0 for w smooth of positive type.

3.4.2 General L^∞ interaction potentials

For a general bounded, even, real-valued interaction potential w we show we could have used a Gaussian cut-off rather than a compactly supported one. Notice that since $w \in L^\infty$, there is some c such that $w^c := w + c \geq 0$ pointwise. Throughout this section, for an object $X_\#$, we use $X^c_\#$ to denote $X_\#$ defined using w^c rather than w. Notice that

$$\mathcal{W}^{\prime,c}_\tau = \mathcal{W}'_\tau + \frac{c}{2} \mathcal{N}_\tau^2 \,, \tag{3.75}$$

$$\mathcal{W}^c = \mathcal{W} + \frac{c}{2} \mathcal{N}^2 \,. \tag{3.76}$$

Applying an adapted form of [29, Theorem 1.8] for non-normal ordered interactions, we have

$$\lim_{\tau \to \infty} \frac{\operatorname{Tr}\left(\Theta_\tau(\xi) \mathrm{e}^{-H^{\prime,c}_\tau}\right)}{\operatorname{Tr}\left(\mathrm{e}^{-H^{\prime,c}_\tau}\right)} = \frac{\int d\mu \, \Theta(\xi) \mathrm{e}^{-H^c}}{\int d\mu \, \mathrm{e}^{-H^c}} \,. \tag{3.77}$$

We note that the adapted form of [29, Theorem 1.8] holds by applying the same proof, but using $\sum_{i,j=1}^n w^c(x_i - x_j) \geq 0$ instead of $\sum_{i,j=1, i\neq j}^n w^c(x_i - x_j) \geq 0$ in the proof of [29, Proposition 4.5].

Rewriting (3.77) using (3.75) and (3.76) gives

$$\lim_{\tau\to\infty} \frac{\mathrm{Tr}\left(\Theta_\tau(\xi)\mathrm{e}^{-H'_\tau}\mathrm{e}^{-\frac{c}{2}\mathcal{N}_\tau^2}\right)}{\mathrm{Tr}\left(\mathrm{e}^{-H'_\tau}\mathrm{e}^{-\frac{c}{2}\mathcal{N}_\tau^2}\right)} = \frac{\int d\mu\, \Theta(\xi)\mathrm{e}^{-H}\mathrm{e}^{-\frac{c}{2}\mathcal{N}^2}}{\int d\mu\, \mathrm{e}^{-H}\mathrm{e}^{-\frac{c}{2}\mathcal{N}^2}}\,.$$

3.5 The time-dependent problem

In this section, we consider the time-dependent problem. The analysis for bounded w and the proof of Theorem 2.1.11 are given in Section 3.5.1. The case when w is unbounded is analysed in Section 3.5.2. Here, we prove Theorems 2.1.12 and 2.1.13. Throughout the section, we fix $p \in \mathbb{N}^*$ and $\xi \in \mathcal{L}(\mathfrak{h}^{(p)})$. In particular, we have the following two lemmas.

3.5.1 Bounded interaction potentials. Proof of Theorem 2.1.11.

In order to deal with bounded interaction potentials, we recall the *Schwinger-Dyson expansion* outlined in [30, Sections 3.2 and 3.3].

Lemma 3.5.1. *Given $\mathcal{K} > 0$, $\varepsilon > 0$, and $t \in \mathbb{R}$, there exists $L = L(\mathcal{K}, \varepsilon, t, \|\xi\|, p) \in \mathbb{N}$, a finite sequence $(e^l)_{l=0}^L$, with $\mathrm{e}^l = \mathrm{e}^l(\xi,t) \in \mathcal{L}(\mathfrak{h}^{(p)})$ and $\tau_0 = \tau_0(\mathcal{K}, \varepsilon, t, \|\xi\|) > 0$ such that*

$$\left\| \left(\Psi_\tau^t \Theta_\tau(\xi) - \sum_{l=0}^{L} \Theta_\tau(e^l) \right) \Big|_{\mathfrak{h}^{(\leq \mathcal{K}\tau)}} \right\| < \varepsilon\,,$$

for all $\tau \geq \tau_0$. Here we recall the definition of $\mathfrak{h}^{(\leq p)}$ from (3.41).

In other words, for large τ and restricted numbers of particles, we can approximate the evolution of the lift of an arbitrary operator with finitely many unevolved lifts. We also have the corresponding classical result.

Lemma 3.5.2. *Given $\mathcal{K} > 0$, $\varepsilon > 0$, and $t \in \mathbb{R}$, then there exist $L = L(\mathcal{K}, \varepsilon, t, \|\xi\|, p) \in \mathbb{N}$, $\tau_0 = \tau_0(\mathcal{K}, \varepsilon, t, \|\xi\|) > 0$ both possibly larger than in Lemma 3.5.1, and for the same choice of $\mathrm{e}^l = \mathrm{e}^l(\xi, t)$ as in Lemma 3.5.1, we have*

$$\left| \left(\Psi^t \Theta(\xi) - \sum_{l=0}^{L} \Theta(e^l) \right) \chi_{\{\mathcal{N} \leq \mathcal{K}\}} \right| < \varepsilon\,,$$

for all $\tau \geq \tau_0$.

We note that the proofs of Lemmas 3.5.1 and 3.5.2, respectively [30, Lemmas 3.9 and 3.12], do not use the sign of the interaction potential, so still hold in our case. The proofs of both results also require a compactly supported cut-off function, demonstrating the cut-off function of the form $f(x) = e^{-cx^2}$ discussed in Remark 2.1.10 (3) would not suffice here.

Proof of Theorem 2.1.11. By using Theorem 2.1.7, Lemmas 3.5.1–3.5.2, and following the proof of [30, Proposition 2.1], we obtain Theorem 2.1.11. $\square$

Remark 3.5.3. Recalling the proof of [30, Proposition 2.1], it follows that the convergence in Theorem 2.1.11 is uniform in the set

$$\{w \in L^\infty, m \in \mathbb{N}, t_i \in \mathbb{R}, p_i \in \mathbb{N} : \|w\|_{L^\infty}, |t_i|, p_i, \|\xi^i\|, m \leq M\},$$

for $i \in [1, \ldots, m]$ and for any fixed choice of $M > 0$.

3.5.2 Unbounded interaction potentials. Proofs of Theorems 2.1.12 and 2.1.13

Before proceeding, we need to prove a technical result concerning the flow of the NLS.

Lemma 3.5.4. *Let $w \in L^1(\mathbb{T})$ and $s \geq \frac{3}{8}$ be given, and suppose $\varphi \in H^s$. Consider the Cauchy problem on $\mathbb{T}$ given by*

$$\begin{cases} i\partial_t u + (\Delta - \kappa)u = \left(w * |u|^2\right) u \\ u_0 = \varphi. \end{cases} \tag{3.78}$$

In addition, given $\varepsilon > 0$ and letting $w^\varepsilon \in L^\infty$ be a sequence satisfying $w^\varepsilon \to w$ in L^1, we consider

$$\begin{cases} i\partial_t u^\varepsilon + (\Delta - \kappa)u^\varepsilon = \left(w^\varepsilon * |u^\varepsilon|^2\right) u^\varepsilon \\ u_0^\varepsilon = \varphi. \end{cases} \tag{3.79}$$

Since $s > 3/8 \geq 0$, the flow map defined in (2.5) *is globally well defined. Denote by u and u^ε the solutions of* (3.78) *and* (3.79) *respectively. Then for $T > 0$*

$$\lim_{\varepsilon \to 0} \|u - u^\varepsilon\|_{L^\infty_{[-T,T]}\mathfrak{h}} = 0.$$

In the following, we always take $b = \frac{1}{2} + \nu$, for $\nu > 0$ small.

Proof of Lemma 3.5.4. We recall the details of proof of [30, Proposition 5.1]. Firstly, we can take $\kappa = 0$ by considering $\tilde{u} := \mathrm{e}^{i\kappa t} u$. We construct global mild solutions to (3.78) and (3.79) in the following way.

Let $\zeta, \psi : \mathbb{R} \to \mathbb{R}$ be smooth functions with

$$\zeta(t) = \begin{cases} 1 & \text{if } |t| \leq 1 \\ 0 & \text{if } |t| > 2. \end{cases} \tag{3.80}$$

$$\psi(t) = \begin{cases} 1 & \text{if } |t| \leq 2 \\ 0 & \text{if } |t| > 4. \end{cases} \tag{3.81}$$

We also define $\zeta_\delta(t) := \zeta(t/\delta)$ and $\psi_\delta(t) := \psi(t/\delta)$. We consider

$$(Lv)(\cdot, t) := \zeta_\delta(t) \mathrm{e}^{it\Delta} \varphi_0 - i\zeta_\delta(t) \int_0^t dt' \, \mathrm{e}^{i(t-t')\Delta} \left(w * |v_\delta|^2 \right) v_\delta(t'), \tag{3.82}$$

$$(L^\varepsilon v)(\cdot, t) := \zeta_\delta(t) \mathrm{e}^{it\Delta} \varphi_0 - i\zeta_\delta(t) \int_0^t dt' \, \mathrm{e}^{i(t-t')\Delta} \left(w^\varepsilon * |v_\delta|^2 \right) v_\delta(t'), \tag{3.83}$$

where $v_\delta(x,t) := \psi_\delta(t) v(x,t)$. By proving L and L^ε are both contractions on appropriate function spaces for $\delta > 0$ sufficiently small, we are able to find local mild solutions to (3.78) and (3.79). The arguments used to prove (3.82) and (3.83) are contractions in [30, Proposition 5.1] still hold if we can show that

$$\left\| \left(w * |v_\delta|^2 \right) v_\delta \right\|_{X^{0,b-1}} \lesssim \|w\|_{L^1} \|v_\delta\|^3_{X^{0,b}}. \tag{3.84}$$

To show (3.84), we define $\mathcal{V}_\delta$ as the function satisfying $\widetilde{\mathcal{V}}_\delta = |\tilde{v}_\delta|$. Note that by construction, $\|\mathcal{V}_\delta\|_{X^{0,b}} = \|v_\delta\|_{X^{0,b}}$. Then

$$\begin{aligned} &\left| \left(\left(w * |v_\delta|^2 \right) v_\delta \right)^{\sim} (k, \eta) \right| \\ &\quad \leq \|\hat{w}\|_{\ell^\infty} \int dk_1 \, dk_2 \, dk_3 \, d\eta_1 \, d\eta_2 \, d\eta_3 \, |\tilde{v}_\delta(k_1, \eta_1)| |\tilde{v}_\delta(-k_2, -\eta_2)| |\tilde{v}_\delta(k_3, \eta_3)| \\ &\qquad\qquad \times \delta(k_1 + k_2 + k_3 - k) \delta(\eta_1 + \eta_2 + \eta_3 - \eta) \\ &\qquad\qquad\qquad = \|\hat{w}\|_{\ell^\infty} \left(|\mathcal{V}_\delta|^2 \mathcal{V}_\delta \right)^{\sim} (k, \eta) \leq \|w\|_{L^1} \left(|\mathcal{V}_\delta|^2 \mathcal{V}_\delta \right)^{\sim} (k, \eta). \end{aligned}$$

To prove (3.84), it remains to show

$$\left\| |\mathcal{V}_\delta|^2 \mathcal{V} \right\|_{X^{0,b-1}} \lesssim \|\mathcal{V}_\delta\|^3_{X^{0,b}} = \|v_\delta\|_{X^{0,b}}. \tag{3.85}$$

To show (3.85) we argue as in [79, (2.147)-(2.153)], where similar bounds are proved for the quintic case, and use a duality argument. Choose $c : \mathbb{Z} \times \mathbb{R} \to \mathbb{C}$ such that

$\sum_k \int d\eta\, |c(k,\eta)|^2 = 1$. We consider

$$I := \sum_k \int d\eta\, \left(1 + |\eta + k^2|\right)^{b-1} \left(|\mathcal{V}_\delta|^2 \mathcal{V}_\delta\right)^{\sim}(k,\eta)\, c(k,\eta)\,.$$

We have

$$|I| \le \sum_k \sum_{k_1+k_2+k_3=k} \int_{\eta_1+\eta_2+\eta_3=\eta} d\eta_1\, d\eta_2\, d\eta_3\, d\eta\, \frac{|c(k,\eta)|}{\left(1+|\eta+k^2|\right)^{1-b}} \times |\tilde{v}_\delta(k_1,\eta_1)||\tilde{v}_\delta(-k_2,-\eta_2)||\tilde{v}_\delta(k_3,\eta_3)|.$$

Define

$$F(x,t) := \sum_k \int d\eta\, \frac{|c(k,\eta)|}{\left(1+|\eta+k^2|\right)^{1-b}} \mathrm{e}^{2\pi i k x + 2\pi i t \eta}\,,$$
$$G(x,t) := \sum_k \int d\eta\, |\tilde{v}_\delta(k,\eta)| \mathrm{e}^{2\pi i k x + 2\pi i t \eta}\,.$$

Parseval's identity implies

$$\begin{aligned} I &\lesssim \int\int dx\, dt\, F\overline{G}G\overline{G} = \left|\int\int dx\, dt\, F\overline{G}G\overline{G}\right| \\ &\le \|F\|_{L^4_{t,x}} \|G\|^3_{L^4_{t,x}}\,. \end{aligned} \tag{3.86}$$

Since $b > 3/8$, we have the estimate $\|\phi\|_{L^4_{t,x}} \lesssim \|\phi\|_{X^{0,b}}$ (see [8, Proposition 2.6], [38, Lemma 2.1 (i)], and [83, Proposition 2.13]). So

$$\|F\|_{L^4_{t,x}} \lesssim \|F\|_{X^{0,3/8}} \le \|F\|_{X^{0,b-1}} = \|c\|_{\ell^2_k L^2_\eta} = 1\,. \tag{3.87}$$

Moreover

$$\|G\|_{L^4_{t,x}} \lesssim \|v_\delta\|_{X^{0,3/8}} \lesssim \delta^\theta \|v_\delta\|_{X^{0,b}}\,, \tag{3.88}$$

where $\theta > 0$. Here the final inequality follows from [30, Lemma 5.3 (iv)]. Combining (3.86) with (3.87) and (3.88) yields (3.85).

So, for a time of existence δ that depends only on the L^2 norm of the initial data, we are able to construct local mild solutions, $v_{(n)}$ and $v^\varepsilon_{(n)}$ on $[n\delta, (n+1)\delta]$. We then piece these solutions together to create mild solutions u and u^ε to (3.78) and (3.79) respectively. Using v and v^ε to denote $v_{(0)}$ and $v^\varepsilon_{(0)}$ respectively, we have

$$\begin{aligned}\|u-u^\varepsilon\|_{L^\infty_{[0,\delta]}L^2_x} =&\|v-v^\varepsilon\|_{L^\infty_{[0,\delta]}L^2_x}\\ \leq&\left\|\zeta_\delta(t)\int_0^t dt'\, \mathrm{e}^{i(t-t')\Delta}\left[(w-w^\varepsilon)*|v_\delta(t')|^2\right]v_\delta(t')\right\|_{X^{0,b}}\\ &+\left\|\zeta_\delta(t)\int_0^t dt'\, \mathrm{e}^{i(t-t')\Delta}\left[w^\varepsilon*\left(|v_\delta(t')|^2-|v^\varepsilon_\delta(t')|^2\right)\right]v_\delta(t')\right\|_{X^{0,b}}\\ &+\left\|\zeta_\delta(t)\int_0^t dt'\, \mathrm{e}^{i(t-t')\Delta}\left[w^\varepsilon*|v^\varepsilon_\delta(t')|^2\right]\left(v_\delta(t')-v^\varepsilon_\delta(t')\right)\right\|_{X^{0,b}}. \end{aligned} \tag{3.89}$$

For the first term of (3.89), we have

$$\begin{aligned}&\left\|\zeta_\delta(t)\int_0^t dt'\, \mathrm{e}^{i(t-t')\Delta}\left[(w-w^\varepsilon)*|v_\delta(t')|^2\right]v_\delta(t')\right\|_{X^{0,b}}\\ &\qquad\qquad\leq C\delta^{\frac{1-2b}{2}}\left\|\left[(w-w^\varepsilon)*|v_\delta|^2\right]v_\delta\right\|_{X^{0,b-1}},\end{aligned}$$

where the $\delta^{\frac{1-2b}{2}}$ comes from the estimates for local $X^{s,b}$ spaces proved in [43] and [44]. For a summary of these local $X^{s,b}$ spaces, we direct the reader to [30, Appendix A] and Lemma 1.7.6.

Arguing as in (3.84), we have

$$\left\|\left[(w-w^\varepsilon)*|v_\delta|^2\right]v_\delta\right\|_{X^{0,b-1}}\leq\|w-w^\varepsilon\|_{L^1}\|v_\delta\|^3_{X^{0,b}}\to 0.$$

The bound on the second term in (3.89) follows by the same argument as in the proof of [30, Proposition 5.1], although we note that since $\|w^\varepsilon\|_{L^1}$ is only bounded rather than equal to 1, we may get a larger constant times a positive power of ε, which is not a problem. The third term in (3.89) then follows for the same reasons combined with [30, Proposition 5.1].

Following the remainder of the argument from [30, Proposition 5.1] and noting that there we gain no negative powers of ε, we have

$$\|u-u^\varepsilon\|_{L^\infty_{[0,T]}\mathfrak{h}}\to 0\,.$$

The corresponding negative time estimates follow from an analogous argument. □

We also have the corresponding result for the focusing local NLS.

Lemma 3.5.5. *Let $s\geq\frac{3}{8}$ be given, and suppose $\varphi\in H^s(\mathbb{T})$. Consider the Cauchy* *problem on $\mathbb{T}$ given by*

$$\begin{cases} i\partial_t u + (\Delta - \kappa)u = -|u|^2 u \\ u_0 = \varphi\,. \end{cases} \tag{3.90}$$

In addition, given $\varepsilon > 0$, let w^ε be as in (2.42). *We consider*

$$\begin{cases} i\partial_t u^\varepsilon + (\Delta - \kappa)u^\varepsilon = \left(w^\varepsilon * |u^\varepsilon|^2\right) u^\varepsilon \\ u_0^\varepsilon = \varphi\,. \end{cases} \tag{3.91}$$

Since $s > 3/8 \geq 0$, the flow map defined in (2.5) *is globally well defined. Denote by u and u^ε the solutions of* (3.90) *and* (3.91) *respectively. Then for $T > 0$*

$$\lim_{\varepsilon \to 0} \|u - u^\varepsilon\|_{L^\infty_{[-T,T]}\mathfrak{h}} = 0\,.$$

Proof. We can follow exactly the proof of [30, Proposition 5.1], recalling (2.41), and noting that the function w^ε defined in (2.42) is even and to deduce

$$\left||v_\delta(x)|^2 - (w^\varepsilon * |v^\delta|^2)(x)\right| \leq \int dy\, |w^\varepsilon(x-y)|\, |v_\delta(x) - v_\delta(y)|(|v_\delta(x)| + |v_\delta(y)|)\,,$$

similarly as in [30, (5.27)]. We also have the same point about $\|w^\varepsilon\|_{L^1}$ not necessarily equal to 1 as in the proof of Lemma 3.5.4, which does not affect the argument. □

Before proving Theorem 2.1.12, we recall the following diagonalisation result, proved in [30, Lemma 5.5].

Proposition 3.5.6. *Let $(Z_k)_{k\in\mathbb{N}}$ be an increasing sequence of sets in the sense that $Z_k \subset Z_{k+1}$ and put $Z := \cup_{k\in\mathbb{N}} Z_k$. For $\varepsilon, \tau > 0$, suppose that $g, g^\varepsilon, g^\varepsilon_\tau : Z \to \mathbb{C}$ are functions with the following properties.*

1. *For each fixed $k \in \mathbb{N}$ and $\varepsilon > 0$, $\lim_{\tau\to\infty} g^\varepsilon_\tau(\zeta) = g^\varepsilon(\zeta)$ uniformly in $\zeta \in Z_k$.*
2. *For each fixed $k \in \mathbb{N}$, $\lim_{\varepsilon\to 0} g^\varepsilon(\zeta) = g(\zeta)$ uniformly in $\zeta \in Z_k$.*

Then there is a sequence (ε_τ) such that $\lim_{\tau\to\infty} = 0$ and

$$\lim_{\tau\to\infty} g^{\varepsilon_\tau}_\tau(\zeta) = g(\zeta)$$

for any $\zeta \in Z$.

Proof of Theorem 2.1.12. Throughout this proof we use X^ε or X to denote an object defined using w^ε or w respectively. Define

$$
\begin{aligned}
Z &:= \{(m, t_i, p_i, \xi^i) : m \in \mathbb{N}, |t_i| \in \mathbb{R}, p_i \in \mathbb{N}, \xi^i \in \mathcal{L}(\mathfrak{h}^{(p_i)})\} \\
Z_k &:= \{(m, t_i, p_i, \xi^i) : m \leq k, |t_i| \leq k, p_i \leq k, \|\xi^i\| \leq k\},
\end{aligned}
$$

where $i \in [1, \ldots, m]$. We also define

$$
\begin{aligned}
g^\varepsilon_\#(\zeta) &:= \rho^\varepsilon_\# \left(\Psi^{t_1,\varepsilon}_\#(\Theta_\#(\xi^1)) \ldots \Psi^{t_m,\varepsilon}_\#(\Theta_\#(\xi^m))\right), \\
g(\zeta) &:= \rho\left(\Psi^{t_1}(\Theta(\xi^1)) \ldots \Psi^{t_m}(\Theta(\xi^m))\right).
\end{aligned}
$$

Applying Theorem 2.1.11 and recalling Remark 3.5.3, Proposition 3.5.6 implies it suffices to show that for a fixed $k \in \mathbb{N}$

$$
\lim_{\varepsilon \to 0} \rho^\varepsilon \left(\Psi^{t_1,\varepsilon}\Theta(\xi^1) \ldots \Psi^{t_m,\varepsilon}\Theta(\xi^m)\right) = \rho\left(\Psi^{t_1}\Theta(\xi^1) \ldots \Psi^{t_m}\Theta(\xi^m)\right), \tag{3.92}
$$

uniformly in Z_k. Recalling (3.61), we have

$$
\lim_{\varepsilon \to 0} \mathcal{W}^\varepsilon = \mathcal{W},
$$

almost surely. Using Corollary 3.1.4 and the dominated convergence theorem, we have

$$
\lim_{\varepsilon \to 0} \tilde{\rho}^\varepsilon_1(I) = \tilde{\rho}_1(I). \tag{3.93}
$$

Here we recall

$$
\tilde{\rho}_\zeta(X) := \int X \, \mathrm{e}^{-\zeta \mathcal{W}} d\mu.
$$

So by (3.24) and (3.93), to prove (3.92), it suffices to show

$$
\lim_{\varepsilon \to 0} \tilde{\rho}^\varepsilon_1(\left(\Psi^{t_1,\varepsilon}\Theta(\xi^1) \ldots \Psi^{t_m,\varepsilon}\Theta(\xi^m)\right)) = \tilde{\rho}_1\left(\Psi^{t_1}\Theta(\xi^1) \ldots \Psi^{t_m}\Theta(\xi^m)\right) \tag{3.94}
$$

uniformly in Z_k.

Let S_t and S^ε_t be the flow maps for equations (3.78) and (3.79) respectively. Let $\varphi_0 \in H^{\frac{1}{2}-} \subset \mathfrak{h}$ be the classical free field defined in (2.12). Then for $\xi \in \mathcal{L}(\mathfrak{h}^{(k)})$, we can write

$$
\Psi^{t,\varepsilon}\Theta(\xi) = \left\langle (S^\varepsilon_t \varphi_0)^{\otimes_k}, \xi\, (S^\varepsilon_t \varphi_0)^{\otimes_k} \right\rangle_{\mathfrak{h}^{\otimes_k}}, \quad \Psi^t\Theta(\xi) = \left\langle (S_t \varphi_0)^{\otimes_k}, \xi\, (S_t \varphi_0)^{\otimes_k} \right\rangle_{\mathfrak{h}^{\otimes_k}}.
$$

Lemma 3.5.4 implies $(S^\varepsilon_t)^{\otimes_k} \varphi_0 \to (S_t)^{\otimes_k} \varphi_0$ as $\varepsilon \to 0$. Moreover, $\xi \in \mathcal{L}(\mathfrak{h}^{(k)})$ implies

$$
\lim_{\varepsilon \to 0} \Psi^{\varepsilon,t}\Theta(\xi) = \Psi^t\Theta(\xi),
$$

uniformly in Z_k. Since $\mathcal{W}^\varepsilon \to \mathcal{W}$ as $\varepsilon \to 0$ almost surely, we have

$$\lim_{\varepsilon\to 0} \Psi^{t_1,\varepsilon}\Theta(\xi^1)\dots\Psi^{t_m,\varepsilon}\Theta(\xi^m)\mathrm{e}^{-\mathcal{W}^\varepsilon} = \Psi^{t_1}\Theta(\xi^1)\dots\Psi^{t_m}\Theta(\xi^m)\mathrm{e}^{-\mathcal{W}}, \tag{3.95}$$

almost surely. Using conservation of mass for (3.78) – (3.79) and Lemma 3.1.6 (2), we have

$$\begin{aligned}\left|\Psi^{t_1,\varepsilon}\Theta(\xi^1)\dots\Psi^{t_m,\varepsilon}\Theta(\xi^m)\mathrm{e}^{-\mathcal{W}^\varepsilon} f(\mathcal{N})\right| &\leq \prod_{j=1}^m \|\xi^j\|\|\varphi_0\|_{\mathfrak{h}}^{2p_j}\mathrm{e}^{\frac{1}{2}\|w^\varepsilon\|_{L^1}\|\varphi_0\|_{L^4}^4} f(\mathcal{N}),\\ \left|\Psi^{t_1}\Theta(\xi^1)\dots\Psi^{t_m}\Theta(\xi^m)\mathrm{e}^{-\mathcal{W}} f(\mathcal{N})\right| &\leq \prod_{j=1}^m \|\xi^j\|\|\varphi_0\|_{\mathfrak{h}}^{2p_j}\mathrm{e}^{\frac{1}{2}\|w\|_{L^1}\|\varphi_0\|_{L^4}^4} f(\mathcal{N}).\end{aligned} \tag{3.96}$$

Using Lemma 3.1.1 and Assumption 2.1.4, both of the bounding functions in (3.96) are $L^1(d\mu)$. So (3.94) follows from (3.95) and the dominated convergence theorem. □

Proof of Theorem 2.1.13. We follow the proof of Theorem 2.1.12, with the same definitions of $Z, Z_k, g_\tau^\varepsilon, g^\varepsilon, g$. Applying Theorem 2.1.12 and using Proposition 3.5.6, it suffices to show

$$\lim_{\varepsilon\to 0}\rho^\varepsilon\left(\Psi^{t_1,\varepsilon}\Theta(\xi^1)\dots\Psi^{t_m,\varepsilon}\Theta(\xi^m)\right) = \rho\left(\Psi^{t_1}\Theta(\xi^1)\dots\Psi^{t_m}\Theta(\xi^m)\right),$$

This follows by arguing analogously to the proof of Theorem 2.1.12, recalling that the fact that $\mathcal{W}^\varepsilon \to \mathcal{W}$ almost surely was shown in (3.71). □

Chapter 4

Microscopic Derivation of the Gibbs Measure for the Quintic NLS

4.1 Analysis of the quintic Hartree equation (2.45)

In this section, we consider w as in Assumption 2.2.2 and study the Cauchy problem for the quintic Hartree equation (2.45).

$$\begin{cases} \mathrm{i}\partial_t u + (\Delta - \kappa)u = \int dy\, dz\, w(x-y)\, w(y-z)\, w(z-x)\, |u(y)|^2\, |u(z)|^2\, u(x) \\ u|_{t=0} = u_0 \in H^s(\Lambda)\,. \end{cases} \tag{4.1}$$

4.1.1 Deterministic local well-posedness and invariance of the Gibbs measure

We first prove the following deterministic local well-posedness result, which should be viewed as an analogue of [8, Theorem 1] for the local quintic NLS.

Proposition 4.1.1 (Deterministic local existence for (4.1))**.** *The Cauchy problem* (4.1) *is locally well-posed in* $H^s(\Lambda)$ *for* $s > 0$.

We then prove the following probabilistic result, which should be viewed as an analogue of the result proved in [9] for the focusing local quintic NLS.

Proposition 4.1.2 (Invariance of truncated Gibbs measure and almost sure global existence for (4.1))**.** *The following claims hold.*

(i) Recall the probability space $(\mathbb{C}, \mathcal{G}, \mu)$ *defined in* (2.11), $\varphi \equiv \varphi^\omega$ *defined in* (2.12), *and* $\mathcal{N}$ *defined in* (2.18). *For* $B > 0$ *sufficiently small, we have*

$$\mathrm{e}^{-\frac{1}{3}\int dx\,dy\,dz\,w(x-y)\,w(y-z)\,w(x-z)\,|\varphi(x)|^2|\varphi(y)|^2|\varphi(z)|^2}\chi_{(\mathcal{N}\leq B)} \in L^1(d\mu)\,.$$

In particular, taking $K = B$ *in* (2.54), *we get that the probability measure* $\mathbb{P}^f_{\mathrm{Gibbs}}$ *in* (2.55) *is well-defined.*

(ii) Consider $s \in (0, \frac{1}{2})$. *The measure* $\mathbb{P}^f_{\mathrm{Gibbs}}$ *is invariant under the flow of* (4.1). *Furthermore,* (4.1) *admits global solutions for* $\mathbb{P}^f_{\mathrm{Gibbs}}$*-almost every* $u_0 \in H^s(\Lambda)$.

Before proving Propositions 4.1.1 and 4.1.2, we note several multilinear estimates.

Lemma 4.1.3. *Consider* $w_1, w_2, w_3 \in L^{\frac{3}{2}}(\Lambda)$. *Given* $q \in H^{\frac{1}{3}}(\Lambda)$, *let*

$$\mathcal{N}_1(q) := \int dx\,dy\,dz\,w_1(x-y)\,w_2(y-z)\,w_3(z-x)\,|q(x)|^2\,|q(y)|^2\,|q(z)|^2\,.$$

Then, we have

$$|\mathcal{N}_1(q)| \lesssim \|w_1\|_{L^{\frac{3}{2}}}\,\|w_2\|_{L^{\frac{3}{2}}}\,\|w_3\|_{L^{\frac{3}{2}}}\,\|q\|^6_{H^{\frac{1}{3}}}\,.$$

Proof. By a density argument, we may assume without loss of generality that w_1, w_2, w_3, q are smooth functions, thus making the calculations that follow rigorous. Writing each integrand as a Fourier series, we compute

$$\begin{aligned}\mathcal{N}_1(q) = &\\ \sum_{k_1,\ldots,k_6}\sum_{\zeta_1,\zeta_2,\zeta_3}&\int dx\,dy\,dz\,\widehat{w}_1(\zeta_1)\,\widehat{w}_2(\zeta_2)\,\widehat{w}_3(\zeta_3)\,\widehat{q}\,(k_1)\,\overline{\widehat{q}\,(k_2)}\,\widehat{q}\,(k_3)\,\overline{\widehat{q}\,(k_4)}\,\widehat{q}\,(k_5)\,\overline{\widehat{q}\,(k_6)}\\ &\times \mathrm{e}^{2\pi\mathrm{i}x(k_1-k_2+\zeta_1-\zeta_3)}\,\mathrm{e}^{2\pi\mathrm{i}y(k_3-k_4-\zeta_1+\zeta_2)}\,\mathrm{e}^{2\pi\mathrm{i}z(k_5-k_6-\zeta_2+\zeta_3)} \qquad (4.2)\end{aligned}$$

The summations in (4.2) and in the sequel are taken over $\mathbb{Z}$. By integrating in x, y, z in (4.2) and taking absolute values, we deduce that

$$\begin{aligned}|\mathcal{N}_1(q)| \leq &\\ \sum_{k_1,\ldots,k_6}\sum_{\zeta_1,\zeta_2,\zeta_3}&|\widehat{w}_1(\zeta_1)|\,|\widehat{w}_2(\zeta_2)|\,|\widehat{w}_3(\zeta_3)|\,|\widehat{q}\,(k_1)|\,|\widehat{q}\,(k_2)|\,|\widehat{q}\,(k_3)|\,|\widehat{q}\,(k_4)|\,|\widehat{q}\,(k_5)|\,|\widehat{q}\,(k_6)|\\ &\times \delta(k_1-k_2+\zeta_1-\zeta_3)\,\delta(k_3-k_4-\zeta_1+\zeta_2)\,\delta(k_5-k_6-\zeta_2+\zeta_3)\,. \qquad (4.3)\end{aligned}$$

By the constraints on the summands, we can rewrite the expression on the right-hand side of (4.3) as

$$\sum_{k_1,\ldots,k_6} \left\{ \sum_{\zeta_1} |\widehat{w}_1(\zeta_1)|\, |\widehat{w}_2(\zeta_1 - k_3 + k_4)|\, |\widehat{w}_3(\zeta_1 + k_1 - k_2)| \right\} \times |\widehat{q}\,(k_1)|\, |\widehat{q}\,(k_2)|\, |\widehat{q}\,(k_3)|\, |\widehat{q}\,(k_4)|\, |\widehat{q}\,(k_5)|\, |\widehat{q}\,(k_6)| \times \delta(k_1 - k_2 + k_3 - k_4 + k_5 - k_6)\,, \quad (4.4)$$

which by applying Hölder's inequality in ζ_1 in the curly brackets in (4.4) is

$$\leq \|\widehat{w}_1\|_{L^3}\, \|\widehat{w}_2\|_{L^3}\, \|\widehat{w}_3\|_{L^3} \sum_{k_1,\ldots,k_6} |\widehat{q}\,(k_1)|\, |\widehat{q}\,(k_2)|\, |\widehat{q}\,(k_3)|\, |\widehat{q}\,(k_4)|\, |\widehat{q}\,(k_5)|\, |\widehat{q}\,(k_6)| \times \delta(k_1 - k_2 + k_3 - k_4 + k_5 - k_6)\,. \quad (4.5)$$

By the Hausdorff-Young inequality and Parseval's identity, we have that

$$(4.5) \lesssim \|w_1\|_{L^{\frac{3}{2}}}\, \|w_2\|_{L^{\frac{3}{2}}}\, \|w_3\|_{L^{\frac{3}{2}}} \int dx\, |F(x)|^6\,, \quad (4.6)$$

where F is chosen such that

$$\widehat{F} = |\widehat{q}\,|\,. \quad (4.7)$$

By applying Sobolev embedding, we deduce that

$$(4.6) = \|w_1\|_{L^{\frac{3}{2}}}\, \|w_2\|_{L^{\frac{3}{2}}}\, \|w_3\|_{L^{\frac{3}{2}}}\, \|F\|_{L^6}^6 \lesssim \|w_1\|_{L^{\frac{3}{2}}}\, \|w_2\|_{L^{\frac{3}{2}}}\, \|w_3\|_{L^{\frac{3}{2}}}\, \|F\|_{H^{\frac{1}{3}}}^6 = \|w_1\|_{L^{\frac{3}{2}}}\, \|w_2\|_{L^{\frac{3}{2}}}\, \|w_3\|_{L^{\frac{3}{2}}}\, \|q\|_{H^{\frac{1}{3}}}^6\,. \quad (4.8)$$

For the last equality in (4.8), we used that $\|F\|_{H^{\frac{1}{3}}} = \|q\|_{H^{\frac{1}{3}}}$, which follows by (4.7). □

We recall the definition of the local $X^{s,b}$ spaces from Definition 1.7.5.

Lemma 4.1.4. *Consider $w_1, w_2, w_3 \in L^{\frac{3}{2}}(\Lambda)$. Given $s, \varepsilon > 0$ and $v_j \in X^{s,1/2-\varepsilon}$, for $j = 1, \ldots, 5$, we let*

$$\mathcal{N}_2(v_1, v_2, v_3, v_4, v_5)(x,t) := \int dy\, dz\, w_1(x-y)\, w_2(y-z)\, w_3(z-x)\, v_1(y,t)\, \overline{v_2(y,t)}\, v_3(z,t)\, \overline{v_4(z,t)}\, v_5(x,t)\,. \quad (4.9)$$

For $\varepsilon > 0$ sufficiently small, we have that for all $t_0 \in \mathbb{R}$ and $\delta > 0$ small

$$\|\mathcal{N}_2(v_1, v_2, v_3, v_4, v_5)\|_{X^{s,-\frac{1}{2}+\varepsilon}_{[t_0,t_0+\delta]}} \lesssim_{s,\varepsilon} \delta^{\varepsilon} \|w_1\|_{L^{\frac{3}{2}}} \|w_2\|_{L^{\frac{3}{2}}} \|w_3\|_{L^{\frac{3}{2}}} \prod_{j=1}^{5} \|v_j\|_{X^{s,\frac{1}{2}+\varepsilon}_{[t_0,t_0+\delta]}} . \tag{4.10}$$

Proof. We first prove the following global version of the estimate (4.10).

$$\|\mathcal{N}_2(v_1, v_2, v_3, v_4, v_5)\|_{X^{s,-\frac{1}{2}+\varepsilon}} \lesssim_{s,\varepsilon} \|w_1\|_{L^{\frac{3}{2}}} \|w_2\|_{L^{\frac{3}{2}}} \|w_3\|_{L^{\frac{3}{2}}} \prod_{j=1}^{5} \|v_j\|_{X^{s,\frac{1}{2}-\varepsilon}} . \tag{4.11}$$

As in the proof of Lemma 4.1.3, it suffices to show (4.11) when the w_i, $i = 1, 2, 3$ and v_j, $j = 1, \ldots, 5$ are smooth. Given $k \in \mathbb{Z}$ and $\eta \in \mathbb{R}$, we show that

$$\begin{aligned} \big|\big(\mathcal{N}_2(v_1, v_2, v_3, v_4, v_5)\big)\widetilde{\ }(k, \eta)\big| &\lesssim \|w_1\|_{L^{\frac{3}{2}}} \|w_2\|_{L^{\frac{3}{2}}} \|w_3\|_{L^{\frac{3}{2}}} \\ \times \sum_{k_1,\ldots,k_5} \int d\eta_1 \cdots d\eta_5 \, &\delta(k_1 - k_2 + k_3 - k_4 + k_5 - k) \, \delta(\eta_1 - \eta_2 + \eta_3 - \eta_4 + \eta_5 - \eta) \\ \times |\widetilde{v}_1(k_1, \eta_1)| \, |\widetilde{v}_2(k_2, \eta_2)| \, &|\widetilde{v}_3(k_3, \eta_3)| \, |\widetilde{v}_4(k_4, \eta_4)| \, |\widetilde{v}_5(k_5, \eta_5)| . \end{aligned} \tag{4.12}$$

Let us assume (4.12) for the moment. By Parseval's theorem for the spacetime Fourier transform, we note that the right-hand side of (4.12) can be written as

$$\|w_1\|_{L^{\frac{3}{2}}} \|w_2\|_{L^{\frac{3}{2}}} \|w_3\|_{L^{\frac{3}{2}}} (F_1 F_2 F_3 F_4 F_5)\widetilde{\ }(k, \eta) , \tag{4.13}$$

where for $j = 1, \ldots, 5$, the function F_j is chosen such that

$$\widetilde{F}_j = |\widetilde{v}_j| . \tag{4.14}$$

Using Definition 1.7.5 we note that (4.12)–(4.13) imply that

$$\|\mathcal{N}_2(v_1, v_2, v_3, v_4, v_5)\|_{X^{s,-\frac{1}{2}+\varepsilon}} \lesssim \|w_1\|_{L^{\frac{3}{2}}} \|w_2\|_{L^{\frac{3}{2}}} \|w_3\|_{L^{\frac{3}{2}}} \|F_1 F_2 F_3 F_4 F_5\|_{X^{s,-\frac{1}{2}+\varepsilon}} . \tag{4.15}$$

We now use the known quintic estimate

$$\|F_1 F_2 F_3 F_4 F_5\|_{X^{s,-\frac{1}{2}+\varepsilon}} \lesssim_{s,\varepsilon} \prod_{j=1}^{5} \|F_j\|_{X^{s,\frac{1}{2}-\varepsilon}} , \tag{4.16}$$

for $\varepsilon > 0$ sufficiently small. For a proof, see [26, Proof of (3.56)]. Strictly speaking, the claim [26, (3.56)] is stated for all F_j being equal, but the proof based on the trilinear estimates [26, Lemma 3.29, Corollary 3.30] extends to the general case verbatim. We omit the details. By (4.14) and Definition 1.7.5, we have that for $j = 1, \ldots, 5$

$$\|F_j\|_{X^{s,-\frac{1}{2}+\varepsilon}} = \|v_j\|_{X^{s,-\frac{1}{2}+\varepsilon}}. \tag{4.17}$$

The claim (4.11) then follows from (4.15)–(4.17) (provided that we know (4.12)).

We now show (4.12). By expanding all of the integrands as a Fourier series and arguing analogously as in the proof of Lemma 4.1.3, we compute

$$\begin{aligned}
&\mathcal{N}_2(v_1, v_2, v_3, v_4, v_5)(x,t) = \\
&\quad \sum_{k_1,\ldots,k_5} \sum_{\zeta_1,\zeta_2,\zeta_3} \int dy\, dz \int d\eta_1 \cdots d\eta_5\, \widehat{w}_1(\zeta_1)\, \widehat{w}_2(\zeta_2)\, \widehat{w}_3(\zeta_3)\, \widetilde{v}_1(k_1,\eta_1)\, \overline{\widetilde{v}_2(k_2,\eta_2)} \\
&\qquad \times \widetilde{v}_3(k_3,\eta_3)\, \overline{\widetilde{v}_4(k_4,\eta_4)}\, \widetilde{v}_5(k_5,\eta_5)\, \mathrm{e}^{2\pi \mathrm{i} x(\zeta_1-\zeta_3+k_5)}\, \mathrm{e}^{2\pi \mathrm{i} y(-\zeta_1+\zeta_2+k_1-k_2)} \\
&\qquad\qquad \times \mathrm{e}^{2\pi \mathrm{i} z(-\zeta_2+\zeta_3+k_3-k_4)}\, \mathrm{e}^{2\pi \mathrm{i} t(\eta_1-\eta_2+\eta_3-\eta_4+\eta_5)}\,.
\end{aligned} \tag{4.18}$$

From (4.18), we hence deduce that

$$\begin{aligned}
&\big(\mathcal{N}_2(v_1, v_2, v_3, v_4, v_5)\big)^{\widetilde{}}(k,\eta) = \\
&\quad \sum_{k_1,\ldots,k_5} \sum_{\zeta_1,\zeta_2,\zeta_3} \int d\eta_1 \cdots d\eta_5\, \widehat{w}_1(\zeta_1)\, \widehat{w}_2(\zeta_2)\, \widehat{w}_3(\zeta_3)\, \widetilde{v}_1(k_1,\eta_1)\, \overline{\widetilde{v}_2(k_2,\eta_2)} \\
&\quad \times \widetilde{v}_3(k_3,\eta_3)\, \overline{\widetilde{v}_4(k_4,\eta_4)}\, \widetilde{v}_5(k_5,\eta_5)\, \delta(\zeta_1-\zeta_3+k_5-k)\, \delta(-\zeta_1+\zeta_2+k_1-k_2) \\
&\qquad \times \delta(-\zeta_2+\zeta_3+k_3-k_4)\, \delta(\eta_1-\eta_2+\eta_3-\eta_4+\eta_5-\eta) \\
&\quad = \sum_{k_1,\ldots,k_5} \sum_{\zeta_1} \int d\eta_1 \cdots d\eta_5\, \widehat{w}_1(\zeta_1)\, \widehat{w}_2(\zeta_1-k_1+k_2)\, \widehat{w}_3(\zeta_1+k_5-k) \\
&\qquad \times \widetilde{v}_1(k_1,\eta_1)\, \overline{\widetilde{v}_2(k_2,\eta_2)}\, \widetilde{v}_3(k_3,\eta_3)\, \overline{\widetilde{v}_4(k_4,\eta_4)}\, \widetilde{v}_5(k_5,\eta_5) \\
&\qquad \times \delta(k_1-k_2+k_3-k_4+k_5-k)\, \delta(\eta_1-\eta_2+\eta_3-\eta_4+\eta_5-\eta)\,.
\end{aligned} \tag{4.19}$$

We now deduce (4.12) from (4.19) by arguing analogously as in the proof of Lemma 4.1.3. Hence, we obtain (4.11).

We now prove (4.10) when $t_0 = 0$. The general case follows by a suitable translation in time. We recall (1.21), use (4.11) for $V_j \in X^{s,b}$ such that $V_j|_{\mathbb{T}\times[0,\delta]} = v_j|_{\mathbb{T}\times[0,\delta]}$ for $j = 1, \ldots, 5$, take infima over V_j, and deduce the claim by using the localisation property[1]

$$\|v\|_{X^{s,b_1}_{[0,\delta]}} \lesssim_s \delta^{b_2-b_1} \|v\|_{X^{s,b_2}_{[0,\delta]}}\,, \tag{4.20}$$

for $-\frac{1}{2} < b_1 \leq b_2 < \frac{1}{2}$. For a self-contained proof of (4.20), we refer the reader to [26, Lemma 3.11]; see also [83, Lemma 2.11]. The estimate (4.10) now follows. □

Before the next proof, we recall the properties of the local $X^{s,b}$ spaces from Lemma 1.7.6.

Proof of Proposition 4.1.1. By replacing u with $\mathrm{e}^{\mathrm{i}t\kappa}u$, we can reduce to considering the case[2] when $\kappa = 0$. We consider solutions for non-negative times. The argument for negative times is analogous. Throughout the proof, we consider $w_1 = w_2 = w_3 = w$ in (4.9). Therefore, the nonlinearity in (4.1) is equal to $\mathcal{N}_2(u,u,u,u,u)$. Let us fix $b = \frac{1}{2} + \varepsilon$ for $\varepsilon > 0$ sufficiently small. We consider the map

$$(Lv)(\cdot,t) := \mathrm{e}^{\mathrm{i}t\Delta}\, u_0 - \mathrm{i}\int_0^t dt'\, \mathrm{e}^{\mathrm{i}(t-t')\Delta}\, \mathcal{N}_2(v,v,v,v,v)(t')\,. \tag{4.21}$$

In order to show the local existence of a solution, by the Banach fixed point theorem, it suffices to show that for suitable $\alpha > 0$ and $\delta \sim \|u_0\|_{H^s}^{-\alpha}$ sufficiently small, the map L is a contraction on the ball

$$\mathcal{B} := \Big\{ v \in X^{s,b}_{[0,\delta]}\,, \quad \|v\|_{X^{s,b}_{[0,\delta]}} \leq \mathcal{M}\|u_0\|_{H^s} \Big\} \tag{4.22}$$

in the Banach space $X^{s,b}_{[0,\delta]}$ for suitable $\mathcal{M} > 0$. By construction, each such fixed point u will be a mild solution of (4.1) on the time interval $[0,\delta]$, meaning that for all $t \in [0,\delta]$, we have

$$u(\cdot,t) = \mathrm{e}^{\mathrm{i}t\Delta}\, u_0 - \mathrm{i}\int_0^t dt'\, \mathrm{e}^{\mathrm{i}(t-t')\Delta}\, \mathcal{N}_2(u,u,u,u,u)(t')\,. \tag{4.23}$$

Furthermore, by Lemma 1.7.6 (i), it follows that $\|u\|_{L^\infty_{t\in[0,\delta]}H^s_x} \lesssim_{\mathcal{M},b} \|u_0\|_{H^s}$.

We now show that (4.21) is a contraction on (4.22). By using Lemma 1.7.6 (ii), (iii) and Lemma 4.1.4 with $v_1 = v_2 = v_3 = v_4 = v_5 = v$, we deduce that

$$\|Lv\|_{X^{s,b}_{[0,\delta]}} \leq C_1\|u_0\|_{H^s} + C_2\,\delta^{\varepsilon}\|v\|^5_{X^{s,b}_{[0,\delta]}}\,, \tag{4.24}$$

for suitable constants $C_1, C_2 > 0$. Note that C_1 is the implied constant in Lemma

[1] In fact, this argument shows (4.10) with δ^{ε} replaced by $\delta^{5\varepsilon-}$, but we will not need this estimate in the sequel.

[2] Note that this transformation does not change the H^s norm.

1.7.6 (ii). Similarly, we have

$$\|Lv^{(1)} - Lv^{(2)}\|_{X^{s,b}_{[0,\delta]}} \leq C_3\, \delta^{\varepsilon}(\|v^{(1)}\|^4_{X^{s,b}_{[0,\delta]}} + \|v^{(2)}\|^4_{X^{s,b}_{[0,\delta]}})\|v^{(1)} - v^{(2)}\|_{X^{s,b}_{[0,\delta]}}\,, \tag{4.25}$$

for a suitable constant $C_3 > 0$. In order to deduce (4.25), we used Lemma 1.7.6 (iii), as well as the precise multilinear form of $\mathcal{N}_2$ given by (4.9), and Lemma 4.1.4 with v_j taking values $v^{(1)}, v^{(2)}$, or $v^{(1)} - v^{(2)}$. We have also used the elementary inequality $x^a y^b \leq C(x^{a+b} + y^{a+b})$ for $x, y > 0$ to control the cross terms. From (4.24)–(4.25), it follows that (4.21) is a contraction on (4.22) if we take $\mathcal{M} = 2C_1$ and $\delta \sim \|u_0\|_{H^s}^{-\frac{4}{\varepsilon}}$ sufficiently small. More precisely, we choose $\delta > 0$ such that

$$C_2\, \delta^{\varepsilon} \mathcal{M}^5 \|u_0\|^4_{H^s} \leq C_1\,, \qquad 2C_3\, \delta^{\varepsilon} \mathcal{M}^4 \|u_0\|^4_{H^s} \leq \frac{1}{2}\,. \tag{4.26}$$

The above argument also shows the conditional uniqueness of mild solutions of (4.1) in (4.22). Namely, suppose that $u^{(1)}, u^{(2)} \in \mathcal{B}$ both satisfy (4.23) for $t \in [0, \delta]$. By repeating the earlier arguments, we deduce that

$$\begin{aligned}\|u^{(1)} - u^{(2)}\|_{X^{s,b}_{[0,\delta]}} &\leq C_3\, \delta^{\varepsilon}(\|u^{(1)}\|^4_{X^{s,b}_{[0,\delta]}} + \|u^{(2)}\|^4_{X^{s,b}_{[0,\delta]}})\|u^{(1)} - u^{(2)}\|_{X^{s,b}_{[0,\delta]}} \\ &\leq \frac{1}{2}\, \|u^{(1)} - u^{(2)}\|_{X^{s,b}_{[0,\delta]}}\,. \end{aligned} \tag{4.27}$$

For the second inequality in (4.27), we used the second condition in (4.26). From (4.27), it indeed follows that $u^{(1)} = u^{(2)}$. This concludes the proof. □

Proof of Proposition 4.1.2. We first prove (i). The analysis is similar to that of [9, Lemma 3.10], which we recall in Appendix A.6. We follow the exposition in Appendix A.6 and explain the main differences. By using Proposition 4.1.3 with $w_1 = w_2 = w_3$ and $q = \varphi$, it follows that

$$\left| -\frac{1}{3} \int dx\, dy\, dz\, w(x-y)\, w(y-z)\, w(z-x)\, |\varphi(x)|^2\, |\varphi(y)|^2\, |\varphi(z)|^2 \right| \lesssim \|w\|^3_{L^{\frac{3}{2}}}\, \|\varphi\|^6_{H^{\frac{1}{3}}}\,. \tag{4.28}$$

The estimate (4.28) is the key reduction to the proof of [9, Lemma 3.10]. In particular, given $c_0 > 0$, it suffices to show that for $B > 0$ sufficiently small, depending on c_0, we have

$$\mathrm{e}^{c_0 \|\varphi\|^6_{H^{1/3}}}\, \chi(\mathcal{N} \leq B) \in L^1(d\mu)\,. \tag{4.29}$$

By arguing analogously as for (A.11) (see also [9, (3.11)]), we reduce the proof of

(4.29) to showing that there exists $c > 0$ such that for large enough $\lambda > 0$, we have

$$\mu\left[\left\|\sum_{k\in\mathbb{Z}} \frac{\omega_k}{\sqrt{\lambda_k}}\, \mathrm{e}^{2\pi \mathrm{i} kx}\right\|_{H^{\frac{1}{3}}} > \lambda\,,\ \left(\sum_{k\in\mathbb{Z}} \frac{|\omega_k|^2}{\lambda_k}\right) \leq B\right] \lesssim \exp\bigl(-cM_0^{4/3}\,\lambda^2\bigr)\,, \tag{4.30}$$

where

$$M_0 \sim \left(\frac{\lambda}{B}\right)^3. \tag{4.31}$$

Here, we recall (2.12). In other words, we reduce to the analysis from [9, Lemma 3.10] and Appendix A.6 with $p = 6$, with $\|\cdot\|_{L^6}$ norms replaced by $\|\cdot\|_{H^{\frac{1}{3}}}$ norms[3] Since the norms are now defined in Fourier space, the analysis in fact simplifies. In particular, the analogue of (A.14), which is now given by

$$\left\|\sum_{|k|\sim M} a_k \mathrm{e}^{2\pi \mathrm{i} kx}\right\|_{H^{\frac{1}{3}}} \lesssim M^{\frac{1}{3}} \left\|\sum_{|k|\sim M} a_k \mathrm{e}^{2\pi \mathrm{i} kx}\right\|_{L^2}\,,$$

follows by definition of $\|\cdot\|_{H^{\frac{1}{3}}}$. Here, M is a dyadic integer and $|k| \sim M$ means that $\frac{3M}{4} \leq |k| < \frac{3M}{2}$. Similarly, the analogue of (A.21) which is now given by

$$\left\|\sum_{|k|\sim M} \frac{\omega_k}{\sqrt{\lambda_k}}\, \mathrm{e}^{2\pi \mathrm{i} kx}\right\|_{H^{\frac{1}{3}}} \lesssim \frac{1}{M} \left\|\sum_{|k|\sim M} \omega_k \mathrm{e}^{2\pi \mathrm{i} kx}\right\|_{H^{\frac{1}{3}}}\,,$$

follows by definition of $\|\cdot\|_{H^{\frac{1}{3}}}$.

With M_0 as in (4.31) and

$$\sigma_M \sim M^{-1/6} + (M_0/M)^{1/2}\,, \quad M > M_0\,,$$

as in (A.24), the above modifications allow us to deduce that there is $M > M_0$ a dyadic integer such that

$$\left\|\sum_{|k|\sim M} \omega_k\, \mathrm{e}^{2\pi \mathrm{i} kx}\right\|_{H^{\frac{1}{3}}} > \sigma_M M \lambda\,. \tag{4.32}$$

We now conclude the proof of (4.30) by using (4.32) and a union bound as in (A.25) – (A.30). In order to apply the above union bound argument, one needs to show

[3] These norms are linked by the Sobolev embedding $H^{\frac{1}{3}} \subset L^6$.

that, given a dyadic integer M, and

$$\mathcal{S} \equiv \mathcal{S}_M := \left\{ \sum_{|k|\sim M} a_k \, \mathrm{e}^{2\pi \mathrm{i} kx} \, , a_k \in \mathbb{C} \right\},$$

there exists a set $\Xi \equiv \Xi_M$ contained in the unit sphere of $H^{-\frac{1}{3}}$ satisfying the following properties.

(1) $\max_{\phi\in\Xi} |\langle g, \phi\rangle| \geq \frac{1}{2} \|g\|_{H^{\frac{1}{3}}}$ for all $g \in \mathcal{S}$.

(2) $\|\phi\|_{L^2} \lesssim M^{\frac{1}{3}}$ for all $\phi \in \Xi$.

(3) $\log |\Xi| \lesssim M$.

The property (2) above follows from the frequency localisation of ϕ and the assumption that $\|\phi\|_{H^{-1/3}} = 1$. Properties (1) and (2) follow from the analogue of Lemma A.5.1 with the spaces L^p and $L^{p'}$ replaced with $H^{\frac{1}{3}}$ and $H^{-\frac{1}{3}}$ respectively. The modified claim follows from the same proof. We refer the reader to Appendix A.5 for the full details. Claim (i) now follows.

We now prove claim (ii). Once we have the local well-posedness given by Proposition 4.1.1 above and the tools used in the proof (most notably the multilinear estimate given by Lemma 4.1.4), the argument follows that of [9, Section 4], which corresponds to formally taking $w = \delta$ in (4.1). We outline only the main differences needed to consider the nonlocal problem (4.1). The main idea is to approximate (4.1) by a finite-dimensional system and to prove a suitable approximation result.

Step 1. Introducing the finite-dimensional system.

Given $N \in \mathbb{N}^*$, we denote by P_N the operator

$$P_N g(x) := \sum_{|k|\leq N} \widehat{g}(k) \, \mathrm{e}^{2\pi \mathrm{i} kx} \, ,$$

i.e. the projection onto frequencies $|k| \leq N$. We then compare (4.33) with its finite-dimensional truncation given by the following.

$$\begin{cases} \mathrm{i}\partial_t u^N + (\Delta - \kappa) u^N = \\ P_N \left[\int dy \, dz \, w(x-y) \, w(y-z) \, w(z-x) \, |u^N(y)|^2 \, |u^N(z)|^2 \, u^N(x) \right] \\ u^N|_{t=0} = P_N u_0 \, . \end{cases} \tag{4.33}$$

Let us note that, in (4.33), we are applying $P_N(\cdot)$ in the x variable. We write the solution u^N of the finite-dimensional system (4.33) as

$$u^N(x,t) = \sum_{|k|\leq N} a_k(t)\, \mathrm{e}^{2\pi \mathrm{i} kx}\,. \tag{4.34}$$

In light of (4.34), we identity u^N with $a = (a_k)_{|k|\leq N}$. With this identification, we can write (4.33) as a Hamiltonian system

$$\frac{da_k}{dt} = -\mathrm{i}\,\frac{\partial H_N(a)}{\partial \bar{a}_k}\,, \quad |k| \leq N\,, \tag{4.35}$$

where the Hamiltonian is given by

$$H_N(a) = \sum_{|k|\leq N} (4\pi^2|k|^2+\kappa)|a_k|^2 + \frac{1}{3}\int dx\,dy\,dz\, w(x-y)\, w(y-z)\, w(z-x) \\ \times \Bigg|\sum_{|k|\leq N} a_k\, \mathrm{e}^{2\pi \mathrm{i} kx}\Bigg|^2 \Bigg|\sum_{|k|\leq N} a_k\, \mathrm{e}^{2\pi \mathrm{i} ky}\Bigg|^2 \Bigg|\sum_{|k|\leq N} a_k\, \mathrm{e}^{2\pi \mathrm{i} kz}\Bigg|^2 \tag{4.36}$$

and

$$\frac{\partial}{\partial \bar{a}_k} = \frac{1}{2}\bigg(\frac{\partial}{\partial \mathrm{Re}\, a_k} + \mathrm{i}\,\frac{\partial}{\partial \mathrm{Im}\, a_k}\bigg)\,. \tag{4.37}$$

Let us note that (4.36) and (4.37) differ from the corresponding quantities used in [9]. This is because the convention of the Poisson structure in [9] amounts to taking $\{u(x), \bar{u}(y)\} = 2\mathrm{i}\delta(x-y)$, which differs from (2.46) by a factor of 2. See Remark 4.1.5 below for more details.

Let us show (4.35) in detail. By (4.33) and (4.36), it suffices to show that for $|k| \leq N$, the k-th Fourier coefficient of

$$Q_N(a) := P_N\Bigg[\int dy\,dz\, w(x-y)\, w(y-z)\, w(z-x) \\ \times \Bigg|\sum_{|k_2|\leq N} a_{k_2} \mathrm{e}^{2\pi \mathrm{i} k_2 y}\Bigg|^2 \Bigg|\sum_{|k_3|\leq N} a_{k_3} \mathrm{e}^{2\pi \mathrm{i} k_3 z}\Bigg|^2 \sum_{|k_1|\leq N} a_{k_1} \mathrm{e}^{2\pi \mathrm{i} k_1 x}\Bigg] \tag{4.38}$$

equals $\frac{\partial W_N(a)}{\partial \bar{a}_k}$, where

$$W_N(a) := \frac{1}{3}\int dx\,dy\,dz\, w(x-y)\, w(y-z)\, w(z-x) \\ \times \Bigg|\sum_{|k_1|\leq N} a_{k_1} \mathrm{e}^{2\pi \mathrm{i} k_1 x}\Bigg|^2 \Bigg|\sum_{|k_2|\leq N} a_{k_2} \mathrm{e}^{2\pi \mathrm{i} k_2 y}\Bigg|^2 \Bigg|\sum_{|k_3|\leq N} a_{k_3} \mathrm{e}^{2\pi \mathrm{i} k_3 z}\Bigg|^2\,. \tag{4.39}$$

(The terms corresponding to the kinetic energy are easily seen to be the same and we omit the proof).

Expanding the factors of w in (4.39) Fourier series, using $\frac{\partial}{\partial \bar{a}_k}\bar{a}_k = 1$, $\frac{\partial}{\partial \bar{a}_k} a_k = 0$, we compute for $|k| \leq N$

$$\frac{\partial W_N(a)}{\partial \bar{a}_k} = \sum_{\substack{k_1,\ldots,k_5 \\ |k_j|\leq N}} \sum_{\zeta_1,\zeta_2,\zeta_3} \int dx\, dy\, dz\, \widehat{w}(\zeta_1)\, \widehat{w}(\zeta_2)\, \widehat{w}(\zeta_3)\, a_{k_1}\, a_{k_2}\, \bar{a}_{k_3}\, a_{k_4}\, \bar{a}_{k_5}$$
$$\times\, \mathrm{e}^{2\pi \mathrm{i}(\zeta_1-\zeta_3+k_1-k)x}\, \mathrm{e}^{2\pi \mathrm{i}(-\zeta_1+\zeta_2+k_2-k_3)y}\, \mathrm{e}^{2\pi \mathrm{i}(-\zeta_2+\zeta_3+k_4-k_5)z}\, \chi_{|k|\leq N}$$
$$= \sum_{\substack{k_1,\ldots,k_5 \\ |k_j|\leq N}} \sum_{\zeta_1,\zeta_2,\zeta_3} \widehat{w}(\zeta_1)\, \widehat{w}(\zeta_2)\, \widehat{w}(\zeta_3)\, a_{k_1} a_{k_2} \bar{a}_{k_3} a_{k_4}\, \bar{a}_{k_5}$$
$$\times\, \delta(\zeta_1-\zeta_3+k_1-k)\, \delta(-\zeta_1+\zeta_2+k_2-k_3)\, \delta(-\zeta_2+\zeta_3+k_4-k_5)\, \chi_{|k|\leq N}\,. \quad (4.40)$$

By similar arguments, we can rewrite (4.38) as

$$Q_N(a) = P_N \Bigg[\sum_{\substack{k_1,\ldots,k_5 \\ |k_j|\leq N}} \sum_{\zeta_1,\zeta_2,\zeta_3} \widehat{w}(\zeta_1)\, \widehat{w}(\zeta_2)\, \widehat{w}(\zeta_3)\, a_{k_1} a_{k_2} \bar{a}_{k_3} a_{k_4} \bar{a}_{k_5}$$
$$\times\, \delta(-\zeta_1+\zeta_2+k_2-k_3)\, \delta(-\zeta_2+\zeta_3+k_4-k_5)\, \mathrm{e}^{2\pi \mathrm{i}(\zeta_1-\zeta_3+k_1)x} \Bigg]. \quad (4.41)$$

We hence deduce (4.35) by noting that (4.40) is indeed equal to the k-th Fourier coefficient of (4.41).

By construction, the *truncated Gibbs measure*

$$d\mathbb{P}^f_{\mathrm{Gibbs},N}(a) := \frac{1}{z^f_{\mathrm{Gibbs},N}}\, \mathrm{e}^{-H_N(a)}\, f\Bigg(\sum_{|k|\leq N} |a_k|^2\Bigg) \prod_{|k|\leq N} da_k\,, \quad (4.42)$$

where

$$z^f_{\mathrm{Gibbs},N} := \int \prod_{|k|\leq N} da_k\, \mathrm{e}^{-H_N(a)}\, f\Bigg(\sum_{|k|\leq N} |a_k|^2\Bigg) \quad (4.43)$$

is invariant under the finite-dimensional Hamiltonian flow (4.33). Here, we recall Assumption 2.2.4 and (4.36). The normalisation factor (4.43) is chosen in such that (4.42) is a probability measure. In order to deduce the invariance stated above, we used the fact that the flow (4.33) conserves mass. This is true because w is real-valued by Assumption 2.2.2.

Step 2. Approximation by the finite-dimensional system.

Having defined the finite-dimensional approximation (4.33) of (4.1), we want to compare the flow of the two. We prove an approximation result, which is an analogue of [9, Lemma 2.27] proved for the local quintic NLS. In order to state the claim, we need to introduce some notation. Suppose that $\psi \in C_c^\infty(\mathbb{R})$ is a function such that

$$\psi(\xi) = \begin{cases} 1 & \text{if } |\xi| \leq \frac{1}{2}, \\ 0 & \text{if } |\xi| > 1\,. \end{cases} \tag{4.44}$$

With ψ as in (4.44), we define the following Fourier multiplier operators.

$$(R_N^- g)\hat{}\,(k) := \psi\left(\frac{3k}{N}\right)\widehat{g}(k)\,, \quad R_N^+ := \mathbf{1} - R_N^-\,. \tag{4.45}$$

We note the following result, which corresponds to Bourgain's approximation lemma [9, Lemma 2.27].

Claim (*) : *Let $A, T > 0$ be given. Fix $u_0 \in H^s$ with*

$$\|u_0\|_{H^s} \leq A\,. \tag{4.46}$$

Consider for large N a solution u^N of (4.33) on $[0,T]$ that satisfies

$$\sup_{t\in[0,T]} \|u^N(t)\|_{H^s} \leq \mathcal{M}A\,, \tag{4.47}$$

for some constant $\mathcal{M} > 0$ independent of N. Then the initial value problem (4.1) is well-posed on $[0,T]$ and the following approximation bound holds for all $s_1 \in (0,s)$

$$\sup_{t\in[0,T]} \|u(t) - u^N(t)\|_{H^{s_1}} \leq C(s, s_1, A, T, w, \mathcal{M}) \left(\|R_N^+ w\|_{L^{\frac{3}{2}}} + N^{s_1 - s}\right), \tag{4.48}$$

provided that the expression on the right-hand side of (4.48) is strictly less than 1.

The well-posedness of (4.1) stated above is interpreted in H^{s_1} for $s_1 \in (0,s)$. This claim follows immediately from (4.48) and the local well-posedness in H^{s_1} which we obtain from Proposition 4.1.1. We present the details of the proof of Claim (*) in Section 4.1.2 below.

Let us note that for $w \in L^{\frac{3}{2}}$, we have that

$$\lim_{N\to\infty} \|R_N^+ w\|_{L^{\frac{3}{2}}} = 0\,, \tag{4.49}$$

hence (4.48) is an appropriate bound. In order to prove (4.49), we first recall (4.45) and use Lemma 1.7.2 to deduce that

$$\|R_N^+\|_{L^{\frac{3}{2}} \to L^{\frac{3}{2}}} \lesssim 1\,, \tag{4.50}$$

uniformly in N. Now, we note that (4.49) holds for all $w \in L^{\frac{3}{2}}$ whose Fourier transform is compactly supported. Namely, in this case, we have that $R_N^+ w = 0$ for large enough N. Such w are dense in $L^{\frac{3}{2}}$, so we deduce that (4.49) holds on all of $L^{\frac{3}{2}}$ by using density and (4.50).

Step 3. Conclusion of the proof. Once we have the approximation result from Step 2, the proof follows identically to that given in [9, Sections 3–4]. We refer the reader to [9] for details. □

Remark 4.1.5. In our convention, we write the fields as

$$u(x) = \sum_k (p_k + \mathrm{i} q_k)\, \mathrm{e}^{2\pi \mathrm{i} k x}$$

where $p_k := \mathrm{Re}\, \widehat{u}(k)\,, q_k := \mathrm{Im}\, \widehat{u}(k)$. We work with p_k and q_k as the canonical coordinates. Our convention for the Hamiltonian system is

$$\begin{cases} \dot{p}_k = \frac{1}{2} \frac{\partial H}{\partial q_k} \\ \dot{q}_k = -\frac{1}{2} \frac{\partial H}{\partial p_k}\,, \end{cases} \tag{4.51}$$

and our convention for the Poisson bracket is

$$\{g_1, g_2\} := \frac{1}{2} \sum_k \left(\frac{\partial g_1}{\partial q_k} \frac{\partial g_2}{\partial p_k} - \frac{\partial g_1}{\partial p_k} \frac{\partial g_2}{\partial q_k} \right). \tag{4.52}$$

We note that, with the Poisson bracket as in (4.52), we have

$$\{u(x), \bar{u}(y)\} = \\ \frac{1}{2} \sum_k \left(\mathrm{i} \mathrm{e}^{2\pi \mathrm{i} k x}\, \mathrm{e}^{-2\pi \mathrm{i} k y} - \mathrm{e}^{2\pi \mathrm{i} k x} (-\mathrm{i}) \mathrm{e}^{-2\pi \mathrm{i} k y} \right) = \mathrm{i} \sum_k \mathrm{e}^{2\pi \mathrm{i} k (x-y)} = \mathrm{i} \delta(x-y)\,.$$

Similarly, we have $\{u(x), u(y)\} = \{\bar{u}(x), \bar{u}(y)\} = 0$. Therefore, the Poisson bracket convention in (4.52) coincides with (2.46).

4.1.2 Proof of the approximation lemma from Step 2

In this section, we give the details of the proof of Claim (*) given in Step 2 of the proof of Proposition 4.1.2 (ii). In particular, we prove the estimate (4.48). Before proceeding with the proof, we prove a multilinear estimate.

Lemma 4.1.6. *Let us fix* $b = \frac{1}{2} + \varepsilon$ *for* $\varepsilon > 0$ *small and* $s_1 > 0$. *For* $t_0 \in \mathbb{R}$, $\delta > 0$, *and* $v \in X^{s_1,b}_{[0,\delta]}$, *and* $\mathcal{N}_2(v,v,v,v,v)$ *as in* (4.9) *with* $w_1 = w_2 = w_3 = w$, *we have*

$$\|\mathcal{N}_2(v,v,v,v,v) - P_N \mathcal{N}_2(v,v,v,v,v)\|_{X^{s_1,b-1}_{[t_0,t_0+\delta]}} \\ \lesssim \delta^{\varepsilon} \, \|w\|^2_{L^{\frac{3}{2}}} \, \|v\|^4_{X^{s_1,b}_{[t_0,t_0+\delta]}} \, \Big(\|R_N^+ w\|_{L^{\frac{3}{2}}} \|v\|_{X^{s_1,b}_{[t_0,t_0+\delta]}} + \|w\|_{L^{\frac{3}{2}}} \|R_N^+ v\|_{X^{s_1,b}_{[t_0,t_0+\delta]}} \Big) \, .$$

Here, we recall (4.45).

Proof. We write

$$\mathcal{N}_2(v,v,v,v,v) - P_N \mathcal{N}_2(v,v,v,v,v) = \\ \int dy \, dz \, \Big\{ w(x-y) \, w(x-z) \, v(x,t) - P_N \big[w(x-y) \, w(x-z) \, v(x,t) \big] \Big\} \\ \times w(y-z) \, |v(y,t)|^2 \, |v(z,t)|^2 \, . \quad (4.53)$$

We write

$$w = R_N^+ w + R_N^- w \, , \quad v(\cdot,t) = R_N^+ v(\cdot,t) + R_N^- v(\cdot,t) \quad (4.54)$$

for all the terms appearing in the curly brackets in (4.53). By (4.44)–(4.45), it follows that

$$R_N^- w(x-y) \, R_N^- w(y-z) \, R_N^- v(x,t) - P_N \big[R_N^- w(x-y) \, R_N^- w(y-z) \, R_N^- v(x,t) \big] = 0 \, . \quad (4.55)$$

The claim follows by combining (4.53)–(4.55) and Lemma 4.1.4. □

Proof of Claim (*) from the proof of Proposition 4.1.2 (ii). By arguing analogously as in the proof of Proposition 4.1.1, we can reduce to the case when $\kappa = 0$. Let us note that the local well-posedness argument in the proof of Proposition 4.1.1 carries over immediately to the finite-dimensional approximation (4.33). This is because the operator P_N is a contraction on $X^{s,b}$. In particular, there exists $\delta \sim_A 1$ such that both (4.1) and (4.33) are well-posed on $[0,\delta]$ in H^{s_1} whenever the initial data is bounded in the H^{s_1} norm by $\mathcal{M}A + 1$. By construction, this is possible if we have

$$\delta^{\varepsilon} A^4 \ll 1 \, . \quad (4.56)$$

By recalling (4.33), (4.46), and by using frequency localisation, we have that

$$\|u(0) - u^N(0)\|_{H^{s_1}} \lesssim N^{s_1 - s} \, A \, . \quad (4.57)$$

Define $U := u - u^N$. Then, U solves the following difference equation.

$$\begin{cases} \mathrm{i}\partial_t U + (\Delta - \kappa)U = \mathcal{N}_2(u,u,u,u,u) - P_N \mathcal{N}_2(u^N,u^N,u^N,u^N,u^N), \\ U|_{t=0} = u(0) - u^N(0)\,. \end{cases} \tag{4.58}$$

Arguing analogously as in the proof of Proposition 4.1.1, but now in the context of (4.58) (instead of (4.1)), and using (4.47), it follows that there exists $T_0 \lesssim \delta$ depending only on A such that for all N, we have

$$\sup_{t \in [0,T_0]} \|u(t) - u^N(t)\|_{H^{s_1}} \lesssim \|u(0) - u^N(0)\|_{H^{s_1}}\,. \tag{4.59}$$

We note that the implied constant in (4.59) is independent of N. Let us fix $\sigma_0 \in (0,1)$ arbitrarily small. From (4.57) and (4.59), we have that

$$\sup_{t \in [0,T_0]} \|u(t) - u^N(t)\|_{H^{s_1}} \leq \sigma_0 \tag{4.60}$$

for all N large enough (depending on σ_0, A, s, s_1). For the remainder of the proof, we consider such N.

Combining (4.47) and (4.60), it follows that

$$\|u(T_0)\|_{H^{s_1}} \leq \|u^N(T_0)\|_{H^{s_1}} + \sigma_0 \leq \mathcal{M}A + 1\,. \tag{4.61}$$

We now introduce the following Cauchy problem.

$$\begin{cases} \mathrm{i}\partial_t v^N + (\Delta - \kappa)v^N = \int dy\,dz\, w(x-y)\,w(y-z)\,w(z-x)\,|v^N(y)|^2\,|v^N(z)|^2\,v^N(x) \\ v^N|_{t=T_0} = u^N(T_0)\,. \end{cases} \tag{4.62}$$

Let us compare (4.62) with the flow of (4.1) started at $t = T_0$. Again arguing analogously as in the proof of Proposition 4.1.1, but now in the context of (4.62), it follows that we can take $\delta \sim_A 1$ possibly smaller satisfying (4.56) and obtain that (4.62) has a solution on the time interval $[T_0, T_0 + \delta]$ which satisfies

$$\begin{aligned} \sup_{t \in [T_0, T_0+\delta]} \|u(t) - v^N(t)\|_{H^{s_1}} &\leq \mathcal{K}\|u(T_0) - v^N(T_0)\|_{H^{s_1}} \\ &= \mathcal{K}\|u(T_0) - u^N(T_0)\|_{H^{s_1}} \leq \mathcal{K}\sigma_0\,, \end{aligned} \tag{4.63}$$

for some constant $\mathcal{K} > 0$. In (4.63), we recalled the initial condition in (4.62), as well as (4.60).

Let us now compare $v^N(t)$ and $u^N(t)$ for $t \in [T_0, T_0+\delta]$. For the remainder of the proof, we always set $w_1 = w_2 = w_3 = w$ when working with the quantity $\mathcal{N}_2$ in (4.9). By (4.33), (4.62), and Duhamel's principle, we have that for all $t \in [T_0, T_0+\delta]$

$$v^N(t) - u^N(t) = -\mathrm{i} \int_{T_0}^{t} dt' \, \mathrm{e}^{\mathrm{i}(t-t')\Delta} \, \Gamma(\cdot, t') \,, \tag{4.64}$$

where

$$\Gamma := \mathcal{N}_2(v^N, v^N, v^N, v^N, v^N) - P_N \mathcal{N}_2(u^N, u^N, u^N, u^N, u^N) \,. \tag{4.65}$$

Using (4.64)–(4.65) and Lemma 1.7.6 (iii), we deduce that

$$\begin{aligned} \|v^N - u^N\|_{X^{s_1,b}_{[T_0,T_0+\delta]}} &\lesssim \|\Gamma\|_{X^{s_1,b-1}_{[T_0,T_0+\delta]}} \\ &\leq \big\|\mathcal{N}_2(v^N, v^N, v^N, v^N, v^N) - P_N \mathcal{N}_2(v^N, v^N, v^N, v^N, v^N)\big\|_{X^{s_1,b-1}_{[T_0,T_0+\delta]}} \\ &+ \big\|P_N \mathcal{N}_2(v^N, v^N, v^N, v^N, v^N) - P_N \mathcal{N}_2(u^N, u^N, u^N, u^N, u^N)\big\|_{X^{s_1,b-1}_{[T_0,T_0+\delta]}} \\ &=: I + II \,. \end{aligned} \tag{4.66}$$

We estimate the terms I and II in (4.66) separately.

Let us first estimate I. By Lemma 4.1.6, we have that

$$I \lesssim \delta^{\varepsilon} \, \|w\|_{L^{\frac{3}{2}}}^2 \, \|v^N\|^4_{X^{s_1,b}_{[T_0,T_0+\delta]}} \Big(\|R_N^+ w\|_{L^{\frac{3}{2}}} \|v^N\|_{X^{s_1,b}_{[T_0,T_0+\delta]}} + \|w\|_{L^{\frac{3}{2}}} \|R_N^+ v^N\|_{X^{s_1,b}_{[T_0,T_0+\delta]}} \Big) \,. \tag{4.67}$$

Let us note that

$$\|v^N\|_{X^{s,b}_{[T_0,T_0+\delta]}} \lesssim A \,. \tag{4.68}$$

In order to obtain (4.68), we note that for δ as in (4.56), we can argue as in the proof of Proposition 4.1.1 and obtain well-posedness of (4.62) on the time interval $[T_0, T_0+\delta]$ since $\|v^N(T_0)\|_{H^s} = \|u^N(T_0)\|_{H^s} \leq \mathcal{M}A$ by (4.47). From (4.68), we obtain

$$\|v^N\|_{X^{s_1,b}_{[T_0,T_0+\delta]}} \lesssim A \tag{4.69}$$

and

$$\|R_N^+ v^N\|_{X^{s_1,b}_{[T_0,T_0+\delta]}} \lesssim N^{s_1 - s} \, A \,. \tag{4.70}$$

In order to deduce (4.70) from (4.68), we recall (4.45) and use frequency localisation. Combining (4.56), (4.67), and (4.69)–(4.70), we obtain that

$$I \lesssim \|w\|_{L^{\frac{3}{2}}}^2 \, A \Big(\|R_N^+ w\|_{L^{\frac{3}{2}}} + N^{s_1 - s} \, \|w\|_{L^{\frac{3}{2}}} \Big) \,. \tag{4.71}$$

Let us now estimate II. Since P_N is a contraction on $X^{s_1,b-1}_{[T_0,T_0+\delta]}$, we have that

$$II \leq \big\| \mathcal{N}_2(v^N, v^N, v^N, v^N, v^N) - \mathcal{N}_2(u^N, u^N, u^N, u^N, u^N) \big\|_{X^{s_1,b-1}_{[T_0,T_0+\delta]}} . \tag{4.72}$$

By recalling (4.9), using multilinearity, and using Lemma 4.1.4, it follows that the right-hand side of (4.72) is

$$\lesssim \delta^{\varepsilon} \, \|w\|^3_{L^{\frac{3}{2}}} \, \|u^N - v^N\|_{X^{s_1,b}_{[T_0,T_0+\delta]}} \Big(\|u^N\|^4_{X^{s_1,b}_{[T_0,T_0+\delta]}} + \|v^N\|^4_{X^{s_1,b}_{[T_0,T_0+\delta]}} \Big) . \tag{4.73}$$

By arguing analogously as for (4.69), we have

$$\|u^N\|_{X^{s_1,b}_{[T_0,T_0+\delta]}} \lesssim A . \tag{4.74}$$

Using (4.69), and (4.74) in (4.73), it follows that

$$II \lesssim \delta^{\varepsilon} A^4 \, \|w\|^3_{L^{\frac{3}{2}}} \, \|u^N - v^N\|_{X^{s_1,b}_{[T_0,T_0+\delta]}} . \tag{4.75}$$

We combine (4.66), (4.71), (4.75), and choose $\delta \sim_A 1$ possibly smaller satisfying (4.56) to deduce that

$$\|v^N - u^N\|_{X^{s_1,b}_{[T_0,T_0+\delta]}} \lesssim \|w\|^2_{L^{\frac{3}{2}}} \, A \Big(\|R_N^+ w\|_{L^{\frac{3}{2}}} + N^{s_1 - s} \, \|w\|_{L^{\frac{3}{2}}} \Big) . \tag{4.76}$$

Combining (4.76) and Lemma 1.7.6 (i), it follows that

$$\sup_{t \in [T_0, T_0+\delta]} \|v^N(t) - u^N(t)\|_{H^s} \lesssim \|w\|^2_{L^{\frac{3}{2}}} \, A \Big(\|R_N^+ w\|_{L^{\frac{3}{2}}} + N^{s_1 - s} \, \|w\|_{L^{\frac{3}{2}}} \Big) . \tag{4.77}$$

Using (4.63) and (4.77), it follows that

$$\sup_{t \in [T_0, T_0+\delta]} \|u(t) - u^N(t)\|_{H^s} \leq C_0 \, \|w\|^2_{L^{\frac{3}{2}}} \, A \Big(\|R_N^+ w\|_{L^{\frac{3}{2}}} + N^{s_1 - s} \, \|w\|_{L^{\frac{3}{2}}} \Big) + \mathcal{K} \sigma_0 =: \sigma_1 . \tag{4.78}$$

for some constant C_0 (which depends on s and s_1, but we suppress this dependence here).

We now iterate this construction. Namely, we start from the time interval $[T_0 + (j-1)\delta, T_0 + j\delta]$ on which we have

$$\sup_{t \in [T_0 + (j-1)\delta, T_0 + j\delta]} \|u(t) - u^N(t)\|_{H^s} \leq \sigma_j$$

and use the above arguments to deduce that

$$\sup_{t\in[T_0+j\delta,T_0+(j+1)\delta]} \|u(t)-u^N(t)\|_{H^s} \leq \sigma_{j+1}\,,$$

where

$$\sigma_{j+1} := C_0 \,\|w\|^2_{L^{\frac{3}{2}}}\, A\Big(\|R_N^+ w\|_{L^{\frac{3}{2}}} + N^{s_1-s}\,\|w\|_{L^{\frac{3}{2}}}\Big) + \mathcal{K}\sigma_j\,. \tag{4.79}$$

The iteration step is possible provided that

$$\sigma_j \leq 1\,. \tag{4.80}$$

(Recall (4.60) and (4.61) above).

From (4.79), by recalling (4.49), it follows that (4.80) holds for all $j \leq \lceil T/\delta \rceil$ provided that σ_0 is chosen sufficiently small and provided that N is chosen large enough. Note that

$$[0,T] \subset [0,T_0] \cup \bigcup_{j=1}^{\lceil T/\delta \rceil} [T_0+(j-1)\delta, T_0+j\delta]\,.$$

The claim then follows.

□

4.2 The time-independent problem with bounded interaction potentials. Proof of Theorem 2.2.6

In this section, we consider w as in Assumption 2.2.1 above. Our goal is to prove Theorem 2.2.6. As a preliminary step in the analysis, we expand the quantum and classical states into a power series. As in the cubic case, we note that, due to the presence of the cut-off, the resulting series are analytic in the complex plane. The precise series in the quantum and classical setting are respectively given in (4.83) and (4.88) below. In Section 4.2.1, we state several estimates that will be used in the analysis. In Section 4.2.2, we explicitly compute the expansion for the quantum and classical states mentioned above. In Sections 4.2.3 and 4.2.4, we prove bounds on the explicit and remainder terms of the resulting series. In Section 4.2.5, we comment on how to use this to prove Theorem 2.2.6, provided that we have convergence of the untruncated explicit terms given by Proposition 4.2.13. We prove Proposition 4.2.13 by using graphical methods in Section 4.2.6.

4.2.1 Basic estimates

Throughout this section, we fix $p \in \mathbb{N}^*$ and take $\xi \in \mathcal{L}(\mathfrak{h}^{(p)})$ unless stated otherwise. We have the following estimates on the quantities $\Theta_\tau(\xi)$ and $\Theta(\xi)$ defined in (2.72) and (2.59) respectively. We note that Lemmas 4.2.1 and 4.2.2 are also stated in the cubic case, however we state them again to be self-contained.

Lemma 4.2.1. *For any $n \in \mathbb{N}^*$, we have*

$$\left\| \Theta_\tau(\xi)\big|_{\mathfrak{h}^{(n)}} \right\| \leq \left(\frac{n}{\tau}\right)^p \|\xi\|.$$

Lemma 4.2.2. *We have*

$$|\Theta(\xi)| \leq \|\varphi\|_{\mathfrak{h}}^{2p} \|\xi\|.$$

Lemma 4.2.1 follows from [45, (3.88)], and Lemma 4.2.2 is a consequence of (2.59). We also have the following estimates on the classical interaction, which follows immediately from Hölder's inequality.

Lemma 4.2.3. *Suppose that the classical interaction $\mathcal{W}$ is defined as in* (2.51). *Then for $w \in L^\infty(\Lambda)$, we have*

$$|\mathcal{W}| \leq \frac{1}{3}\|w\|_{L^\infty}^3 \|\varphi\|_{L^2}^6.$$

We also collect some estimates about Schatten space operators. The following result follows from the spectral decomposition of $|\mathcal{A}| = \sqrt{\mathcal{A}^*\mathcal{A}}$.

Lemma 4.2.4. *Let $\mathcal{H}$ be a separable Hilbert space. Suppose $1 \leq p_1 \leq p_2 \leq \infty$ and $\mathcal{A} \in \mathfrak{S}^{p_1}(\mathcal{H}) \cap \mathfrak{S}^{p_2}(\mathcal{H})$. Then*

$$\|\mathcal{A}\|_{\mathfrak{S}^{p_2}(\mathcal{H})} \leq \|\mathcal{A}\|_{\mathfrak{S}^{p_1}(\mathcal{H})}.$$

We also recall Lemma 3.1.5, which is Hölder's inequality for Schatten spaces.

4.2.2 Power series expansions of the classical and quantum states

In this section, we compute the power series for the quantum and classical states. Let us recall (2.68) and (2.72). We note the following identities.

$$\rho_\tau(\Theta_\tau(\xi)) = \frac{\tilde{\rho}_{\tau,1}(\Theta_\tau(\xi))}{\tilde{\rho}_{\tau,1}(\mathbf{1})}, \tag{4.81}$$

where for $z \in \mathbb{C}$, we let

$$\tilde{\rho}_{\tau,z}(\mathcal{A}) := \frac{1}{Z_{\tau,0}} \operatorname{Tr}\left(\mathcal{A} \mathrm{e}^{-H_{\tau,0} - z\mathcal{W}_\tau} f(\mathcal{N}_\tau)\right), \tag{4.82}$$

and recall $\mathbf{1}$ is the identity operator on $\mathcal{F}$. Let us also define

$$F_\tau^\xi(z) := \tilde{\rho}_{\tau,z}(\Theta_\tau(\xi)). \tag{4.83}$$

Let us note the following result.

Lemma 4.2.5. *For $M \in \mathbb{N}^*$, we have $F_\tau^\xi(z) = \sum_{m=0}^{M-1} a_{\tau,m}^\xi z^m + R_{\tau,M}^\xi(z)$. Here,*

$$a_{\tau,m}^\xi := \frac{1}{Z_{\tau,0}} \operatorname{Tr}\bigg((-1)^m \int_0^1 dt_1 \int_0^{t_1} dt_2 \ldots \int_0^{t_{m-1}} dt_m\, \Theta_\tau(\xi) \mathrm{e}^{-(1-t_1)H_{\tau,0}} \mathcal{W}_\tau \\ \times \mathrm{e}^{-(t_1-t_2)H_{\tau,0}} \ldots \mathcal{W}_\tau \mathrm{e}^{-(t_{m-1}-t_m)H_{\tau,0}} \mathcal{W}_\tau \mathrm{e}^{-t_m H_{\tau,0}} f(\mathcal{N}_\tau)\bigg), \tag{4.84}$$

and

$$R_{\tau,M}^\xi(z) := \frac{1}{Z_{\tau,0}} \operatorname{Tr}\bigg((-z)^M \int_0^1 dt_1 \int_0^{t_1} dt_2 \ldots \int_0^{t_{m-1}} dt_M\, \Theta_\tau(\xi) \mathrm{e}^{-(1-t_1)H_{\tau,0}} \mathcal{W}_\tau \\ \times \mathrm{e}^{-(t_1-t_2)H_{\tau,0}} \ldots \mathcal{W}_\tau \mathrm{e}^{-(t_{M-1}-t_M)H_{\tau,0}} \mathcal{W}_\tau \mathrm{e}^{-t_m (H_{\tau,0} + z\mathcal{W}_\tau)} f(\mathcal{N}_\tau)\bigg). \tag{4.85}$$

Proof. By performing a Duhamel expansion up to order M, we obtain the following result. More precisely, we use the identity

$$\mathrm{e}^{X+Y} = \mathrm{e}^X + \int_0^1 dt\, \mathrm{e}^{(1-t)X}\, Y\, \mathrm{e}^{X+tY}$$

M times in (4.82)–(4.83). ☐

As before, given $g : \mathbb{C} \to \mathbb{C}$, and any operator $\mathcal{A} : \mathcal{F} \to \mathcal{F}$, that commutes with $\mathcal{N}_\tau$, we note that $\mathcal{A}$ also commutes with $g(\mathcal{N}_\tau)$. This is because the operator $g(\mathcal{N}_\tau)$ acts on the n^{th} sector of Fock space as multiplication by $\frac{n}{\tau}$. In particular, it follows that that, for every $\alpha > 0$, $f^\alpha(\mathcal{N}_\tau)$ commutes with any operators of the form $\mathcal{W}_\tau$, $\mathrm{e}^{-tH_{\tau,0}}$, $\mathrm{e}^{-t(H_{\tau,0}+z\mathcal{W}_\tau)}$, occurring as factors in the integrands in (4.84)–(4.85) above. We use this fact without further mention in the sequel.

Let us recall (2.57) and (2.59). By analogy with (4.81), we rewrite the classical state as

$$\rho(\Theta(\xi)) = \frac{\tilde{\rho}_1(\Theta(\xi))}{\tilde{\rho}_1(\mathbf{1})}, \tag{4.86}$$

where for a random variable X and $z \in \mathbb{C}$, $\tilde{\rho}_z$ is defined as

$$\tilde{\rho}_z(X) := \int d\mu \, X \mathrm{e}^{-z\mathcal{W}} f(\mathcal{N}) \,. \tag{4.87}$$

For $z \in \mathbb{C}$, we define

$$F^{\xi}(z) := \tilde{\rho}_z(\Theta(\xi)) \,. \tag{4.88}$$

Then we have the following analogue of Lemma 4.2.5.

Lemma 4.2.6. *For $M \in \mathbb{N}$, we have $F^{\xi}(z) = \sum_{m=0}^{M-1} a^{\xi}_m z^m + R^{\xi}_M(z)$. Here*

$$a^{\xi}_m := \frac{(-1)^m}{m!} \int d\mu \, \Theta(\xi) \mathcal{W}^m f(\mathcal{N}) \,, \tag{4.89}$$

and

$$R^{\xi}_M(z) := \frac{(-z)^M}{M!} \int d\mu \, \Theta(\xi) \mathcal{W}^M \mathrm{e}^{-\tilde{z}\mathcal{W}} f(\mathcal{N}) \,, \tag{4.90}$$

for some $\tilde{z} \in [0, z]$.

4.2.3 Analysis of the quantum series (4.83)

In this section we prove that the explicit and remainder terms defined in (4.84) and (4.85) satisfy sufficient bounds for (4.83) to be analytic.

Lemma 4.2.7. *For any $m \in \mathbb{N}$ and $a^{\xi}_{\tau,m}$ defined as in (4.84), we have*

$$|a^{\xi}_{\tau,m}| \leq \frac{(K^3 \|w\|^3_{L^\infty})^m K^p \|\xi\|}{3^m \, m!}. \tag{4.91}$$

Proof. We argue similarly to the proof of Lemma 3.2.2. Lemma 3.1.5 implies

$$\begin{aligned} |a^{\xi}_{\tau,m}| \leq \frac{1}{Z_{\tau,0}} \int_0^1 dt_1 \int_0^{t_1} dt_2 \ldots \int_0^{t_{m-1}} dt_m \left\| \Theta_\tau(\xi) f^{\frac{1}{m+1}}(\mathcal{N}_\tau) \right\|_{\mathfrak{S}^\infty(\mathfrak{h})} \\ \times \left\| \mathcal{W}_\tau f^{\frac{1}{m+1}}(\mathcal{N}_\tau) \right\|^m_{\mathfrak{S}^\infty(\mathfrak{h})} \prod_{j=0}^{m} \left\| \mathrm{e}^{-(t_j - t_{j+1}) H_{\tau,0}} \right\|_{\mathfrak{S}^{\frac{1}{t_j - t_{j+1}}}(\mathfrak{h})} \,, \end{aligned} \tag{4.92}$$

where we take the convention $t_0 := 1$ and $t_{m+1} := 0$. Noting that

$$\|e^{-sH_{\tau,0}}\|_{\mathfrak{S}^{1/s}(\mathfrak{h})} = (Z_{\tau,0})^s \,,$$

since $\mathrm{e}^{-sH_{\tau,0}}$ is a positive operator, it follows from (4.92) that

$$|a^{\xi}_{\tau,m}| \leq \frac{1}{m!} \left\| \Theta_\tau(\xi) f^{\frac{1}{m+1}}(\mathcal{N}_\tau) \right\|_{\mathfrak{S}^\infty(\mathfrak{h})} \left\| \mathcal{W}_\tau f^{\frac{1}{m+1}}(\mathcal{N}_\tau) \right\|^m_{\mathfrak{S}^\infty(\mathfrak{h})} \,. \tag{4.93}$$

Noting that $\mathcal{W}_\tau$ acts on $\mathfrak{h}^{(n)}$ as multiplication by

$$\frac{1}{3\tau^3} \sum_{\substack{i,j,k \\ i\neq j\neq k\neq i}}^{n} w(x_i - x_j) w(x_j - x_k) w(x_k - x_i)\,,$$

and recalling Lemma 4.2.1, we have

$$\left\| \mathcal{W}_\tau f^{\frac{1}{m+1}}(\mathcal{N}_\tau) \right\|_{\mathfrak{S}^\infty(\mathfrak{h}^{(n)})} \leq \frac{1}{3}\left(\frac{n}{\tau}\right)^3 \|w\|_{L^\infty}^3 \left| f\left(\frac{n}{\tau}\right)\right| \leq \frac{1}{3} K^3 \|w\|_{L^\infty}^3\,. \tag{4.94}$$

Here we have also used the support properties of f as in Assumption 2.2.4, namely (2.54), and that $\|f\|_{L^\infty} \leq 1$. Using Lemma 4.2.1 and Assumption 2.2.4, we have

$$\left\| \Theta_\tau(\xi) f^{\frac{1}{m+1}}(\mathcal{N}_\tau) \right\|_{\mathfrak{S}^\infty(\mathfrak{h})} \leq K^p \|\xi\|\,. \tag{4.95}$$

Then (4.91) follows from (4.93), (4.94), and (4.95). □

In order to estimate $R^\xi_{\tau,M}(z)$, we apply the Feynman-Kac formula, which we recall from Proposition 1.7.12.

Lemma 4.2.8. *For any $M \in \mathbb{N}$ and $R^\xi_{\tau,M}(z)$ defined as in (4.85), we have*

$$\left| R^\xi_{\tau,M}(z) \right| \leq \mathrm{e}^{\frac{1}{3}K^3 |\mathrm{Re}(z)| \|w\|_{L^\infty}^3} \frac{(K^3 \|w\|_{L^\infty}^3)^M K^p \|\xi\|}{3^M\, M!} |z|^M\,. \tag{4.96}$$

Proof. By arguing similarly as the proof of Lemma 3.2.3 and (3.23), it suffices to show that for $t \in [0,1]$, we have

$$\left| \left(\mathrm{e}^{-t(H_{\tau,0} + z\mathcal{W}_\tau)} f^{\frac{1}{2}}(\mathcal{N}_\tau) \right)^{(n)} (\mathbf{x};\mathbf{y}) \right| \leq \mathrm{e}^{\frac{1}{3}K^3 |\mathrm{Re(z)}| \|\mathrm{w}\|_{\mathrm{L}^\infty}^3} \left(\mathrm{e}^{-tH_{\tau,0}} \right)^{(n)} (\mathbf{x};\mathbf{y})\,, \tag{4.97}$$

where $\mathcal{A}^{(n)}$ denotes the kernel of $\mathcal{A}$ restricted to the n^{th} sector of Fock space. Noting that

$$(\mathcal{W}_\tau)^{(n)}(\mathbf{x};\mathbf{y}) = \frac{1}{3\tau^3} \sum_{\substack{i,j,k \\ i\neq j\neq k\neq i}}^{n} w(x_i - x_j) w(x_j - x_k) w(x_k - x_i) \prod_{l=1}^{n} \delta(x_l - y_l)\,,$$

we can rewrite $\left(\mathrm{e}^{-t(H_{\tau,0} + z\mathcal{W}_\tau)} f^{\frac{1}{2}}(\mathcal{N}_\tau) \right)^{(n)} (\mathbf{x};\mathbf{y})$ using Proposition 1.7.12 as

$$\int \mathbb{W}^t_{\mathbf{x},\mathbf{y}}(d\widetilde{\omega})\, \mathrm{e}^{-\frac{t\kappa n}{\tau} - \int_0^t ds\, z\left(\frac{1}{3\tau^3}\sum_{i\neq j\neq k\neq i}^{n} w_{i,j}(\widetilde{\omega}(s)) w_{j,k}(\widetilde{\omega}(s)) w_{k,i}(\widetilde{\omega}(s))\right)} f^{\frac{1}{2}}\left(\frac{n}{\tau}\right)\,. \tag{4.98}$$

In (4.98), we have written $\widetilde{\omega} \equiv (\omega_1, \dots, \omega_n)$ and $\mathbb{W}^t_{\mathbf{x},\mathbf{y}}(d\widetilde{\omega}) \equiv \prod_{l=1}^n \mathbb{W}^t_{x_l,y_l}(d\omega_l)$. Furthermore, we have abbreviated $w_{i,j}(\mathbf{x}) \equiv w(x_i - x_j)$ and

$$\sum_{i \neq j \neq k \neq i}^{n} \equiv \sum_{\substack{i,j,k \\ i \neq j \neq k \neq i}}^{n}$$

all of which we use in the sequel. Using the triangle inequality, it follows that (4.98) is in absolute value

$$\leq \int \mathbb{W}^t_{\mathbf{x},\mathbf{y}}(d\widetilde{\omega}) \, \mathrm{e}^{-\frac{t\kappa n}{\tau}} \left| \mathrm{e}^{-\int_0^t ds \, z \left(\frac{1}{3\tau^3} \sum_{i\neq j\neq k\neq i}^n w_{i,j}(\widetilde{\omega}(s)) w_{j,k}(\widetilde{\omega}(s)) w_{k,i}(\widetilde{\omega}(s)) \right)} f^{\frac{1}{2}} \left(\frac{n}{\tau} \right) \right| . \tag{4.99}$$

Using Proposition 1.7.12 once more, we deduce that (4.99) is

$$\leq \sup_{\widetilde{\omega}} \left| \mathrm{e}^{-\int_0^t ds \, z \left(\frac{1}{3\tau^3} \sum_{i\neq j\neq k\neq i}^n w_{i,j}(\widetilde{\omega}(s)) w_{j,k}(\widetilde{\omega}(s)) w_{k,i}(\widetilde{\omega}(s)) \right)} f^{\frac{1}{2}} \left(\frac{n}{\tau} \right) \right| \left(\mathrm{e}^{-tH_{\tau,0}} \right)^{(n)} (\mathbf{x};\mathbf{y}) . \tag{4.100}$$

Arguing as in (4.94), we have

$$\sup_{\widetilde{\omega}} \left| \mathrm{e}^{-\int_0^t ds \, z \left(\frac{1}{3\tau^3} \sum_{i\neq j\neq k\neq i}^n w_{i,j}(\widetilde{\omega}(s)) w_{j,k}(\widetilde{\omega}(s)) w_{k,i}(\widetilde{\omega}(s)) \right)} f^{\frac{1}{2}} \left(\frac{n}{\tau} \right) \right| \leq \mathrm{e}^{\frac{1}{3} K^3 |\mathrm{Re}(z)| \|w\|^3_{L^\infty}} . \tag{4.101}$$

Combining (4.100) and (4.101), we obtain (4.97), thus completing the proof. □

Combining the bounds proved in Lemmas 4.2.7 and 4.2.8 with Taylor's theorem yields the following corollary.

Corollary 4.2.9. *$F^\xi_\tau(z) = \sum_{m=0}^\infty a^\xi_{\tau,m} z^m$ is analytic on the whole of $\mathbb{C}$.*

4.2.4 Analysis of the classical series (4.88)

We now prove that the explicit and remainder terms defined in (4.89) and (4.90) satisfy sufficient bounds for the function defined in (4.88) to be analytic on $\mathbb{C}$.

Lemma 4.2.10. *Let $m \in \mathbb{N}$ and a^ξ_m be defined as in (4.89). Then*

$$\left| a^\xi_m \right| \leq \frac{(K^3 \|w\|^3_{L^\infty})^m K^p \|\xi\|}{3^m \, m!} . \tag{4.102}$$

Proof. Using Lemma 4.2.2 as in the proof of Lemma 3.2.5, it is sufficient to prove that

$$\left| \mathcal{W} f^{\frac{1}{m+1}}(\mathcal{N}) \right| \leq \frac{1}{3} K^3 \|w\|^3_{L^\infty} . \tag{4.103}$$

Using Lemma 4.2.3 and (2.54), we have

$$\left|\mathcal{W} f^{\frac{1}{m+1}}(\mathcal{N})\right| \leq \frac{1}{3} K^3 \|w\|_{L^\infty}^3 \|f^{\frac{1}{m+1}}\|_{L^\infty}.$$

Here we recall that $\mathcal{N} = \|\varphi\|_{\mathfrak{h}}^2$. Noting that $\|f\|_{L^\infty} \leq 1$, (4.103) follows. □

Lemma 4.2.11. *Let $M \in \mathbb{N}$ and $R_m^\xi(z)$ be defined as in* (4.90). *Then*

$$\left|R_M^\xi(z)\right| \leq \mathrm{e}^{\frac{1}{3}K^3|\mathrm{Re}(z)|\|w\|_{L^\infty}^3} \frac{\left(K^3\|w\|_{L^\infty}^3\right)^M K^p \|\xi\|}{M!\,3^M} |z|^M. \tag{4.104}$$

Proof. We note that Lemma 4.2.3 and (2.54) imply that for any $\tilde{z} \in [0, z]$, we have

$$\left|\mathrm{e}^{-z\mathcal{W}} f^{\frac{1}{M+2}}(\mathcal{N})\right| \leq \mathrm{e}^{\frac{1}{3}K^3|\mathrm{Re}(z)|\|w\|_{L^\infty}^3}.$$

Recalling (4.90) and using Lemma 4.2.10, we obtain (4.104). □

Combining Lemmas 4.2.10 and 4.2.11, we have the following corollary.

Corollary 4.2.12. $F^\xi(z) = \sum_{m=0}^\infty a_m^\xi z^m$ *is analytic on the whole of* $\mathbb{C}$.

4.2.5 Proof of Theorem 2.2.6

We note the following result, whose proof we defer to Section 4.2.6 below.

Proposition 4.2.13. *Let $\nu > 0$ be fixed. Let $\mathcal{C}_p$ be as in* (1.13). *Define*

$$\begin{aligned}\alpha_{\tau,m}^{\xi,\nu} := \frac{(-1)^m}{Z_{\tau,0}} \mathrm{Tr}\bigg(\int_0^1 dt_1 \int_0^{t_1} dt_2 \ldots \int_0^{t_{m-1}} dt_m\, &\Theta_\tau(\xi) \mathrm{e}^{-(1-t_1)(H_{\tau,0}+\nu\mathcal{N}_\tau)}\mathcal{W}_\tau \\ &\times \mathrm{e}^{-(t_1-t_2)(H_{\tau,0}+\nu\mathcal{N}_\tau)}\mathcal{W}_\tau \mathrm{e}^{-(t_2-t_3)(H_{\tau,0}+\nu\mathcal{N}_\tau)} \ldots \\ &\times \mathrm{e}^{-(t_{m-1}-t_m)(H_{\tau,0}+\nu\mathcal{N}_\tau)}\mathcal{W}_\tau \mathrm{e}^{-t_M(H_{\tau,0}+\nu\mathcal{N}_\tau)}\bigg),\end{aligned} \tag{4.105}$$

$$\alpha_m^{\xi,\nu} := \frac{(-1)^m}{m!} \int d\mu\, \Theta(\xi) \mathcal{W}^m \mathrm{e}^{-\nu\mathcal{N}}. \tag{4.106}$$

Then, the following results hold uniformly in $\xi \in \mathcal{C}_p$.

(i) $\left|\alpha_{\#,m}^{\xi,\nu}\right| \leq C(m,p,\nu)$.

(ii) $\lim_{\tau\to\infty} \alpha_{\tau,m}^{\xi,\nu} = \alpha_m^{\xi,\nu}$.

Proof of Theorem 2.2.6. By using Proposition 4.2.13 and by arguing analogously as for Lemma 3.2.10, we deduce that for $m \in \mathbb{N}$, we have

$$\lim_{\tau \to \infty} a^{\xi}_{\tau,m} = a^{\xi}_{m}\,, \tag{4.107}$$

uniformly in $\xi \in \mathcal{C}_p$. We then combine (4.107) with Corollary 4.2.9, Corollary 4.2.12, Lemma 4.2.7, Lemma 4.2.10, and the dominated convergence theorem to deduce that for all $z \in \mathbb{C}$, we have

$$\lim_{\tau \to \infty} \sup_{\xi \in \mathcal{C}_p} \big| F^{\xi}_{\tau}(z) - F^{\xi}(z) \big| \leq \lim_{\tau \to \infty} \sum_{m} \sup_{\xi \in \mathcal{C}_p} |a^{\xi}_{\tau,m} - a^{\xi}_{m}| |z|^m = 0\,. \tag{4.108}$$

The claim (2.75) follows from (4.108) by taking $p = 0$ and recalling (2.56), (2.69)–(2.70), (4.82)–(4.83), and (4.87)–(4.88) above. Note that by convention, when $p = 0$, there is no observable ξ in the analysis above. Moreover, we take $\mathcal{A} = \mathbf{1}$ in (4.82) and $X = 1$ in (4.87), respectively.

The claim (2.74) follows from (4.108) by a duality argument. More precisely, by using (2.68) and (2.71)–(2.72), we have that for all $\xi \in \mathcal{L}(\mathfrak{h}^{(p)})$

$$\rho_\tau(\Theta_\tau(\xi)) = \mathrm{Tr}\,(\gamma_{\tau,p}\,\xi)\,. \tag{4.109}$$

Analogously, by using (2.57)–(2.59), we have

$$\rho(\Theta(\xi)) = \mathrm{Tr}\,(\gamma_p\,\xi)\,. \tag{4.110}$$

Recalling (4.81) and (4.86), we see that (4.108) implies

$$\lim_{\tau \to \infty} \sup_{\xi \in \mathcal{C}_p} \big| \rho_\tau(\Theta_\tau(\xi)) - \rho(\Theta(\xi)) \big| = 0\,. \tag{4.111}$$

From (4.109)–(4.110) (and taking suprema over $\xi \in \mathfrak{B}_p$), we immediately deduce the weaker analogue of (2.74) given by

$$\lim_{\tau \to \infty} \|\gamma_{\tau,p} - \gamma_p\|_{\mathfrak{S}^2(\mathfrak{h}^{(p)})} = 0\,. \tag{4.112}$$

We upgrade (4.112) to (2.74) by noting that $\gamma_{\tau,p} \geq 0$ and $\gamma_p \geq 0$ in the sense of operators and that $\lim_{\tau \to \infty} \mathrm{Tr}\gamma_{\tau,p} = \mathrm{Tr}\gamma_p$. The former claim follows from Lemma 3.2.14, whose proof carries over directly to the quintic setting. The latter claim from (4.109)–(4.111) by taking $\xi = \mathbf{1}_p \in \mathcal{C}_p$. For the details of the last step, we refer the reader to [29, Lemma 4.10] and Lemma 3.2.16. □

4.2.6 Graphical analysis of the untruncated explicit terms. Proof of Proposition 4.2.13

In this section, we prove Proposition 4.2.13 stated above. For the proof, we use a graphical argument similar to that used in [29, Sections 2.3-2.6 and 4.1]. The graphs will be different, due to the three-body interaction. The essence of the argument is quite similar. For completeness, we review the proof and refer the reader to [29] for more details and motivation. In what follows, we denote

$$b^{\xi}_{\#,m} := \alpha^{\xi,0}_{\#,m}\,, \tag{4.113}$$

where we recall (4.105)–(4.106). Let us note that it suffices to show Proposition 4.2.13, when $\nu = 0$. The general claim follows from this one (with possibly different constants depending on ν) by replacing κ in (2.6) with $\kappa + \nu$. In Subsection 4.2.6, we show that

$$\left| b^{\xi}_{\tau,m} \right| \leq C(m,p)\,, \tag{4.114}$$

uniformly in $\xi \in \mathfrak{B}_p$. In Subsection 4.2.6, we show that

$$\lim_{\tau \to \infty} b^{\xi}_{\tau,m} = b^{\xi}_{m}\,, \tag{4.115}$$

uniformly in $\xi \in \mathfrak{B}_p$. In Subsection 4.2.6, we show that (4.114)–(4.115) hold for $\xi = \mathbf{1}_p$. The latter requires a slightly modified graphical structure. Putting these steps together, we complete the proof of Proposition 4.2.13.

Proof of (4.114) **uniformly in** $\xi \in \mathfrak{B}_p$

We begin by recalling a number of definitions and results from [29]. Throughout, we abbreviate $\varphi_{\tau,k} := \varphi_\tau(u_k)$, where the u_k are defined as in (2.9) above.

Definition 4.2.14. For $t \in \mathbb{R}$, we define the operator valued distributions $\left(\mathrm{e}^{th/\tau}\varphi_\tau\right)(x)$ and $\left(\mathrm{e}^{th/\tau}\varphi^*_\tau\right)(x)$ as

$$\left(\mathrm{e}^{th/\tau}\varphi_\tau\right)(x) := \sum_{k\in\mathbb{N}} \mathrm{e}^{t\lambda_k/\tau} u_k(x)\varphi_{\tau,k}\,,$$
$$\left(\mathrm{e}^{th/\tau}\varphi^*_\tau\right)(x) := \sum_{k\in\mathbb{N}} \mathrm{e}^{t\lambda_k/\tau} \overline{u}_k(x)\varphi^*_{\tau,k}\,.$$

Here, we recall (2.8).

In what follows, we use the result of [29, Lemma 2.3].

Lemma 4.2.15. *For $t \in \mathbb{R}$ we have,*

$$\mathrm{e}^{tH_{\tau,0}}\varphi_\tau^*(x)\mathrm{e}^{-tH_{\tau,0}} = \left(\mathrm{e}^{th/\tau}\varphi_\tau^*\right)(x), \quad \mathrm{e}^{tH_{\tau,0}}\varphi_\tau(x)\mathrm{e}^{-tH_{\tau,0}} = \left(\mathrm{e}^{-th/\tau}\varphi_\tau\right)(x).$$

Here, we recall the definition (2.64) *of $H_{\tau,0}$.*

The following result follows from Lemma 4.2.15 and the definition (2.63) of $\mathcal{W}_\tau$; see also [29, Corollary 2.4].

Lemma 4.2.16. *For $t \in \mathbb{R}$, we have*

$$\begin{aligned}\mathrm{e}^{tH_{\tau,0}}\,\mathcal{W}_\tau\,\mathrm{e}^{-tH_{\tau,0}} = \frac{1}{3}\int dx\,dy\,dz\,&\left(\mathrm{e}^{th/\tau}\varphi_\tau^*\right)(x)\left(\mathrm{e}^{th/\tau}\varphi_\tau^*\right)(y)\left(\mathrm{e}^{th/\tau}\varphi_\tau^*\right)(z)\\ \times w(x-y)w(y-z)w(z-x)&\left(\mathrm{e}^{-th/\tau}\varphi_\tau\right)(x)\left(\mathrm{e}^{-th/\tau}\varphi_\tau\right)(y)\left(\mathrm{e}^{-th/\tau}\varphi_\tau\right)(z).\end{aligned} \tag{4.116}$$

Given a closed operator $\mathcal{A}$ on $\mathcal{F}$, we define

$$\rho_{\tau,0}(\mathcal{A}) := \frac{\mathrm{Tr}(\mathcal{A}\mathrm{e}^{-H_{\tau,0}})}{\mathrm{Tr}(\mathrm{e}^{-H_{\tau,0}})}. \tag{4.117}$$

Furthermore, let us recall (4.113) and (4.105). Using Lemma 4.2.16 and the cyclicity of the trace, it follows that for all $m, p \in \mathbb{N}$ and $\xi \in \mathfrak{B}_p$, we have

$$b_{\tau,m}^{\xi} = \frac{(-1)^m}{3^m}\int_0^1 dt_1 \int_0^{t_1} dt_2 \ldots \int_0^{t_{m-1}} dt_m\, g_{\tau,m}^{\xi}(\mathbf{t}), \tag{4.118}$$

where $\mathbf{t} := (t_1, \ldots, t_m)$ and

$$\begin{aligned} g_{\tau,m}^{\xi}(\mathbf{t}) := \int dx_1 \ldots dx_{m+p}\, dy_1 \ldots dy_{m+p}\, dz_1 \ldots dz_m \\ \times \left(\prod_{i=1}^m w(x_i - y_i)w(y_i - z_i)w(z_i - x_i)\right)\xi(x_{m+1}, \ldots, x_{m+p}; y_{m+1}, \ldots, y_{m+p}) \\ \times \rho_{\tau,0}\Bigg(\prod_{i=1}^m \Big[\left(\mathrm{e}^{t_i h_\tau/\tau}\varphi_\tau^*\right)(x_i)\left(\mathrm{e}^{t_i h_\tau/\tau}\varphi_\tau^*\right)(y_i)\left(\mathrm{e}^{t_i h_\tau/\tau}\varphi_\tau^*\right)(z_i) \\ \times \left(\mathrm{e}^{-t_i h_\tau/\tau}\varphi_\tau\right)(x_i)\left(\mathrm{e}^{-t_i h_\tau/\tau}\varphi_\tau\right)(y_i)\left(\mathrm{e}^{-t_i h_\tau/\tau}\varphi_\tau\right)(z_i)\Big] \\ \times \prod_{i=1}^p \varphi_\tau^*(x_{m+i}) \prod_{i=1}^p \varphi_\tau(y_{m+i})\Bigg). \end{aligned} \tag{4.119}$$

Here and throughout all of our products are taken in the order of increasing indices. We now fix $m, p \in \mathbb{N}$ and define an abstract vertex set Σ containing $(6m+2p)$

elements as follows.

Definition 4.2.17. Given $m, p \in \mathbb{N}$, we define $\Sigma \equiv \Sigma(m,p)$ to be the set of triples (i, r, δ) with $i \in \{1, \ldots, m+1\}$. If $i \in \{1, \ldots, m\}$, we consider $r \in \{1,2,3\}$ and if $i = m+1$, we take $r \in \{1, \ldots, p\}$. Finally, we take $\delta \in \{\pm 1\}$. We write[4] $\alpha = (i, r, \delta)$ and write the components of α as $i_\alpha, r_\alpha, \delta_\alpha$. We use the lexicographical ordering on Σ to order the vertices, which we denote by $\leq$. If $\alpha \leq \beta$ and $\alpha \neq \beta$, we write $\alpha < \beta$. To each vertex $\alpha = (i, r, \delta) \in \Sigma$, we assign a spatial integration variable x_α. Moreover, to each $i \in \{1, \ldots, m\}$, we assign a time t_i, and take $t_{m+1} := 0$ by convention. Where convenient, we write $x_{i,r,\delta}$ or $t_{i,r,\delta}$ instead of x_α or t_α respectively. Let us write

$$\mathbf{x} := (x_\alpha)_{\alpha \in \Sigma} \in \Lambda^\Sigma, \qquad \mathbf{t} := (t_\alpha)_{\alpha \in \Sigma} \in \mathbb{R}^\Sigma .$$

We only consider $(t_1, \ldots, t_m)$ to be in the support of the integral from (4.118). In other words, we always take $\mathbf{t} \in \mathfrak{V} \equiv \mathfrak{V}(m)$, where

$$\mathfrak{V} := \big\{ \mathbf{t} \in \mathbb{R}^\Sigma : t_{i,r,\delta} = t_i \text{ with } 0 = t_{m+1} < t_m < \cdots < t_1 < 1 \big\} . \tag{4.120}$$

In the discussion that follows, we fix $m, p \in \mathbb{N}$ as well as $\xi \in \mathfrak{B}_p$.

Remark 4.2.18. We interpret the integrand in (4.119) in terms of the set Σ in Definition 4.2.17 as follows. Each occurrence of $\varphi_\tau^*(\cdot)$ or $\varphi_\tau^*(\cdot)$ corresponds to an element of Σ. For $i \in \{1, \ldots, m\}$, i denotes that we are working with the i^{th} interaction, and hence that we are considering a factor of the form $\big(\mathrm{e}^{t_i h_\tau / \tau} \varphi_\tau^*\big)(\cdot)$ or $\big(\mathrm{e}^{-t_i h_\tau / \tau} \varphi_\tau\big)(\cdot)$. When $\delta = +1$, it is the former and when $\delta = -1$, it is the latter. The index $r = 1,2,3$ refers to the integration variable x_i, y_i, z_i respectively. Furthermore, when $i = m+1$, we consider the factor $\varphi_\tau^*(x_{m+r})$ and when $\delta = +1$, and the factor $\varphi_\tau(y_{m+r})$ when $\delta = -1$; see Figure 4.1 for a graphical representation. Let us also note that

$$\alpha < \beta \quad \Rightarrow \quad 0 \leq t_\alpha - t_\beta < 1 . \tag{4.121}$$

Let us recall the *quantum Wick theorem.*

Lemma 4.2.19 (Quantum Wick theorem)**.** *Let $\mathcal{A}_1, \ldots, \mathcal{A}_n$ be operators of the form $\mathcal{A}_i = b(f_i)$ or $\mathcal{A}_i = b^*(f_i)$, for $f_1, \ldots, f_n \in \mathfrak{h}$. Here, we recall* (2.24)–(2.25)*. We then have*

$$\rho_{\tau,0}(\mathcal{A}_1 \cdots \mathcal{A}_n) = \sum_{\Pi \in M(n)} \prod_{(i,j) \in \Pi} \rho_{\tau,0}(\mathcal{A}_i \mathcal{A}_j) ,$$

[4]We emphasise that this is a different object than the $\alpha^{\xi,\nu}_{\#,m}$ in (4.105)–(4.106) above.

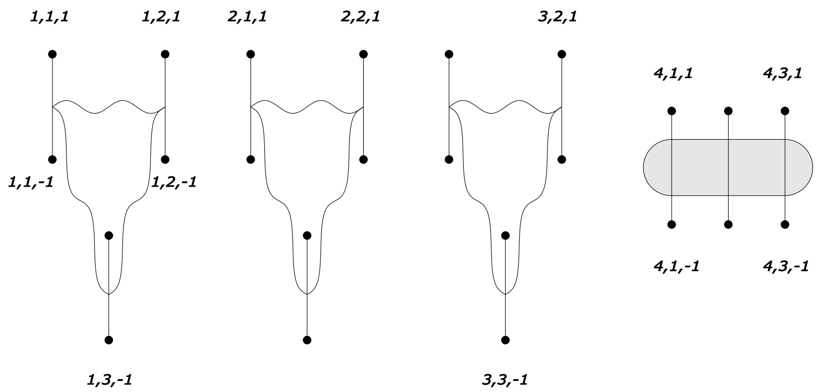

Figure 4.1: An unpaired graph from Definition 4.2.17 with $m = p = 3$. The black dots correspond to factors of $\varphi_\tau(\cdot)$ and $\varphi^*_\tau(\cdot)$. The wavy lines correspond to factors of the interaction potential w.

where, as in Proposition 2.1.2, $M(n)$ *denotes the set of complete pairings of* $\{1, \ldots, n\}$*. The edges of* Π *are now labelled using ordered pairs* (i, j) *with* $i < j$*. Here, we also recall* (4.117)*.*

For a self-contained proof of Lemma 4.2.19, we refer the reader to [29, Lemma B.1]. As in [29, Section 2], we use Lemma 4.2.19 to simplify the expression (4.119). Before proceeding, we define a few objects which we will use in the analysis.

Definition 4.2.20. Given Π a pairing of Σ, i.e. a one-regular graph on Σ, we regard its edges as ordered pairs (α, β) such that $\alpha < \beta$. We then define $\mathfrak{P} \equiv \mathfrak{P}(m, p)$ to be the set of pairings Π of Σ satisfying $\delta_\alpha \delta_\beta = -1$ whenever $(\alpha, \beta) \in \Pi$; see Figure 4.2.

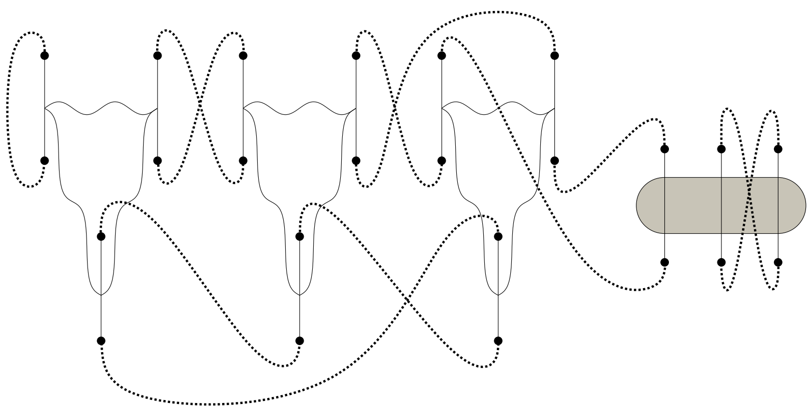

Figure 4.2: A graph with $m = p = 3$ with a valid pairing from Definition 4.2.20.

Definition 4.2.21. For $\alpha \in \Sigma$ and $\mathbf{t} \in \mathfrak{V}$, define

$$\mathcal{B}_\alpha(\mathbf{x}, \mathbf{t}) := \begin{cases} \left(\mathrm{e}^{t_\alpha h/\tau} \varphi_\tau^*\right)(x_\alpha) & \text{if } \delta = 1\,, \\ \left(\mathrm{e}^{-t_\alpha h/\tau} \varphi_\tau\right)(x_\alpha) & \text{if } \delta = -1\,. \end{cases} \tag{4.122}$$

Definition 4.2.22. For $\Pi \in \mathfrak{P}$ and $\mathbf{t} \in \mathfrak{V}$, we define

$$I^\xi_{\tau,\Pi}(\mathbf{t}) := \int_{\Lambda^\Sigma} d\mathbf{x} \prod_{i=1}^m \left(w(x_{i,1,1} - x_{i,2,1}) w(x_{i,2,1} - x_{i,3,1}) w(x_{i,3,1} - x_{i,1,1}) \prod_{r=1}^3 \delta(x_{i,r,1} - x_{i,r,-1}) \right) \times \xi(x_{m+1}, \ldots, x_{m+p}; y_{m+1}, \ldots, y_{m+p}) \prod_{(\alpha,\beta)\in\Pi} \rho_{\tau,0}(\mathcal{B}_\alpha(\mathbf{x},\mathbf{t}) \mathcal{B}_\beta(\mathbf{x},\mathbf{t}))\,. \tag{4.123}$$

We can now state the simplification of (4.119) that we will use in the sequel.

Lemma 4.2.23. *For $\mathbf{t} \in \mathfrak{V}$, we have $g^\xi_{\tau,m}(\mathbf{t}) = \sum_{\Pi \in \mathfrak{P}} I_{\tau,\Pi}(\mathbf{t})$.*

Proof. We follow the proof and use the notation used in [29, Lemma 2.8]. Namely for $\alpha \in \Sigma$, to each x_α, we define a corresponding spectral label $k_\alpha \equiv k_{i,r,\delta}$, and write $\mathbf{k} := (k_\alpha)_{\alpha\in\Sigma}$. Let us recall (2.9). For $l \in \mathbb{N}$, we let

$$u_l^\alpha := \begin{cases} \overline{u}_l \text{ if } \delta = +1\,, \\ u_l \text{ if } \delta = -1\,. \end{cases}$$

From (4.122), it follows that

$$\mathcal{B}_\alpha(\mathbf{x}, \mathbf{t}) = \sum_{k_\alpha \in \mathbb{N}} \mathrm{e}^{\delta_\alpha t_\alpha \lambda_{k_\alpha}/\tau} u^\alpha_{k_\alpha}(x_\alpha) \mathcal{A}_\alpha(\mathbf{k})\,, \tag{4.124}$$

where we define

$$\mathcal{A}_\alpha(\mathbf{k}) := \begin{cases} \varphi^*_{\tau,k_\alpha} \text{ if } \delta = +1\,, \\ \varphi_{\tau,k_\alpha} \text{ if } \delta = -1\,. \end{cases}$$

Using (4.124) in (4.119), we have

$$
\begin{aligned}
&g^{\xi}_{\tau,m}(\mathbf{t}) = \\
&\int_{\Lambda^{\Sigma}} d\mathbf{x} \prod_{i=1}^{m} \left(w(x_{i,1,1} - x_{i,2,1}) w(x_{i,2,1} - x_{i,3,1}) w(x_{i,3,1} - x_{i,1,1}) \prod_{r=1}^{3} \delta(x_{i,r,1} - x_{i,r,-1}) \right) \\
&\qquad \times \xi(x_{m+1,1,1}, \ldots, x_{m+1,p,1}; y_{m+1,1,-1}, \ldots, y_{m+1,p,-1}) \\
&\qquad \times \sum_{\mathbf{k}} \left[\left(\prod_{\alpha \in \Sigma} \mathrm{e}^{\delta_\alpha t_\alpha \lambda_{\tau,k_\alpha}/\tau} u^{\alpha}_{k_\alpha}(x_\alpha) \right) \rho_{\tau,0} \left(\prod_{\alpha \in \Sigma} \mathcal{A}_\alpha(\mathbf{k}) \right) \right] . \quad (4.125)
\end{aligned}
$$

The result follows from applying Lemma 4.2.19 to (4.125) and using (4.124). □

We define the following bounded operators.

$$
\begin{aligned}
S_{\tau,t} &:= \mathrm{e}^{-th/\tau} \text{ if } t \geq 0 \,, \\
G_{\tau,t} &:= \frac{\mathrm{e}^{-th/\tau}}{\tau(\mathrm{e}^{h/\tau} - 1)} \text{ if } t \geq -1 \,.
\end{aligned}
$$

Here we note that $G_{\tau,t}$ is the time-evolved Green function.

Lemma 4.2.24. *For $t \geq 0$, both $G_{\tau,t}$ and $S_{\tau,t}$ have symmetric, non-negative kernels.*

Proof. Since both $S_{\tau,t}$ and $G_{\tau,t}$ are self-adjoint, it follows from Proposition 1.7.12 that their kernels are non-negative and thus symmetric. See [29, Lemma 2.9] for more details. □

We thus have the following result for computing the free quantum states of products of pairs of $\mathcal{B}_\alpha$; see [29, Lemma 2.10].

Lemma 4.2.25. *Suppose $\alpha, \beta \in \Sigma$ with $\alpha < \beta$.*

1. *If $\delta_\alpha = 1$ and $\delta_\beta = -1$ with $t_\alpha - t_\beta < 1$, then*

$$\rho_{\tau,0}\left(\mathcal{B}_\alpha(\mathbf{x},\mathbf{t})\mathcal{B}_\beta(\mathbf{x},\mathbf{t})\right) = G_{\tau,-(t_\alpha - t_\beta)}(x_\alpha; x_\beta) \,.$$

2. *If $\delta_\alpha = -1$ and $\delta_\beta = 1$ with $t_\alpha - t_\beta \geq 0$, then*

$$\rho_{\tau,0}\left(\mathcal{B}_\alpha(\mathbf{x},\mathbf{t})\mathcal{B}_\beta(\mathbf{x},\mathbf{t})\right) = G_{\tau,t_\alpha - t_\beta}(x_\alpha; x_\beta) + \frac{1}{\tau} S_{\tau,t_\alpha - t_\beta}(x_\alpha; x_\beta) \,.$$

3. *For both (1) and (2), by Lemma 4.2.24, we have $\rho_{\tau,0}\left(\mathcal{B}_\alpha(\mathbf{x},\mathbf{t})\mathcal{B}_\beta(\mathbf{x},\mathbf{t})\right) \geq 0$.*

We define a second, coloured graph, where we collapse the left-hand side vertices of the original graph. In other words, we identify $x_{i,r,1}$ and $x_{i,r,-1}$ for $i \leq m$. In order to make this precise, we use first define the following equivalence relation.

Definition 4.2.26. For $\alpha, \beta \in \Sigma$, we write $\alpha \sim \beta$ if and only if $i_\alpha = i_\beta \leq m$ and $r_\alpha = r_\beta$.

Let us now implement Definition 4.2.26 in the graphical structure.

Definition 4.2.27. For each $\Pi \in \mathfrak{P}$, we define the coloured graph $(V_\Pi, E_\Pi, \sigma_\Pi) = (V, E, \sigma)$ as follows. $V := \Sigma/\sim$ is the set of equivalence classes of Σ under $\sim$. For $\alpha \in \Sigma$, let us denote by $[\alpha]$ its equivalence class under $\sim$. For each edge $(\alpha, \beta) \in \Pi$, we obtain an edge $e = \{[\alpha], [\beta]\}$, and we denote by E the set of edges obtained in this way. Finally, we define the colour of an edge $e \in E$ as

$$\sigma(e) := \delta_\beta \,. \tag{4.126}$$

This is well-defined by construction.

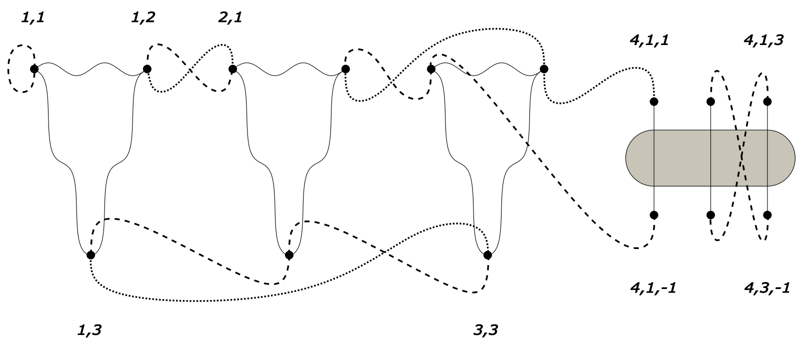

Figure 4.3: An example of the coloured graph from Definition 4.2.27 with the same pairing as Figure 4.2. The wider dotted edges have colour -1, and the finer dotted lines have colour 1.

Remark 4.2.28. We make the following observations about (V, E, σ).

1. The set V inherits a well-defined total order from Σ defined by $[\alpha] \leq [\beta]$ if $\alpha \leq \beta$. We also adopt the same convention as before to write $[\alpha] < [\beta]$ if $\alpha < \beta$.

2. We can write V as the disjoint union $V = V_2 \sqcup V_1$, where

$$V_2 := \{(i, r) \mid i \in \{1, \dots m\}, r \in \{1, 2, 3\}\}, \quad V_1 := \{(m+1, r, \pm 1)\}.$$

Note that then each vertex in V_j has degree j.

3. We write $\operatorname{conn}(E)$ to denote the set of connected components of E. Thus $E = \bigsqcup_{\mathcal{P} \in \operatorname{conn}(E)} \mathcal{P}$. We call $\mathcal{P} \in \operatorname{conn}(E)$ a *path* of E.

We now fix $m, p \in \mathbb{N}$ and $\Pi \in \mathfrak{P}$, and let (V, E, σ) be the associated graph defined in Definition 4.2.27. For each $\mathbf{x} \in \Lambda^{\Sigma}$ and $\mathbf{t} \in \mathbb{R}^{\Sigma}$, we associate integration labels $\mathbf{y} := (y_a)_{a \in V} \in \Lambda^V$ and $\mathbf{s} := (s_a)_{a \in V} \in \mathbb{R}^V$ defined by

$$y_\alpha := x_{[\alpha]}, \qquad s_\alpha := t_{[\alpha]} \tag{4.127}$$

for any $\alpha \in \Sigma$. It follows from Definition 4.2.17 that the definition above does not depend on the choice of vertex α. We note that (4.121) implies

$$a < b \quad \Rightarrow \quad 0 \leq s_a - s_b < 1\,, \tag{4.128}$$

and

$$s_a = s_b \text{ if and only if } i_a = i_b\,, \tag{4.129}$$

where we have used a slight abuse of notation to write $i_\alpha := i_{[\alpha]}$.

Definition 4.2.29. We say that a path $\mathcal{P} \in \operatorname{conn}(E)$ is *closed* if all of its vertices are in V_2. Otherwise we call it *open*. We also denote by $V(\mathcal{P}) := \bigcup_{e \in \mathcal{P}} e$ and $V_i(\mathcal{P}) := V(\mathcal{P}) \cap V_i$.

Definition 4.2.30. For $\mathbf{y} \in \Lambda^V$ and $\mathbf{s} \in \mathbb{R}^V$ satisfying (4.128), and $e = \{a, b\} \in E$, we define $\mathbf{y}_e := (y_a; y_b) \in \Lambda^e$ and the integral kernels

$$\mathcal{J}_{\tau,e}(\mathbf{y}_e, \mathbf{s}) := G_{\tau,\sigma(e)(s_a - s_b)}(\mathbf{y}_e) + \frac{\chi(\sigma(e) = 1)\chi(s_a \neq s_b)}{\tau} S_{\tau, s_a - s_b}(\mathbf{y}_e)\,, \tag{4.130}$$

$$\hat{\mathcal{J}}_{\tau,e}(\mathbf{y}_e, \mathbf{s}) := G_{\tau,\sigma(e)(s_a - s_b)}(\mathbf{y}_e)\,. \tag{4.131}$$

Here, we recall (4.126).

Although both of the operators (4.130)–(4.131) always Hilbert-Schmidt, we never estimate them in the Hilbert-Schmidt norm as we want to prove estimates which are uniform in $\mathbf{s}$. In the sequel, let us write

$$\mathbf{y} = (\mathbf{y}_1, \mathbf{y}_2)\,, \tag{4.132}$$

with $\mathbf{y}_i := (\mathbf{y}_a)_{a \in V_i}$. Slightly abusing notation, let us also write

$$\xi(y_{m+1,1,1}, \ldots, y_{m+1,p,1}; y_{m+1,1,-1}, \ldots, y_{m+1,p,-1}) =: \xi(\mathbf{y}_1)\,.$$

Lemma 4.2.31. *For* $\mathbf{t} \in \mathfrak{V}$ *and* $\mathbf{s}$ *defined as in* (4.127)*, we have*

$$I^{\xi}_{\tau,\Pi}(\mathbf{t}) = \int_{\Lambda^V} d\mathbf{y} \Bigg(\prod_{i=1}^{m} w(y_{i,1} - y_{i,2}) w(y_{i,2} - y_{i,3}) w(y_{i,3} - y_{i,1}) \Bigg) \xi(\mathbf{y}_1) \prod_{e \in E} \mathcal{J}_{\tau,e}(\mathbf{y}_e, \mathbf{s}) . \tag{4.133}$$

Proof. We define a map $T : \Lambda^V \to \Lambda^\Sigma$ by $T\mathbf{y} := (y_{[\alpha]})_{\alpha \in \Sigma}$. Then T is a bijection from Λ^V onto the subset of Λ^Σ defined by

$$\{\mathbf{x} \in \Lambda^\Sigma \mid x_{i,r,1} = x_{i,r,-1} \text{ for all } i \in \{1, \dots, m\}, r \in \{1,2,3\}\} .$$

From Lemma 4.2.25 and Definition 4.2.30, it follows that

$$\rho_{\tau,0} \left(\mathcal{B}_\alpha(T\mathbf{y}, \mathbf{t}) \mathcal{B}_\alpha(T\mathbf{y}, \mathbf{t}) \right) = \mathcal{J}_{\tau,e}(\mathbf{y}_e, \mathbf{s}) , \tag{4.134}$$

where $(\alpha, \beta) \in \Pi$ is chosen such that $[\alpha] = a$ and $[\beta] = b$. By making the change of variables, we obtain $\mathbf{x} = T\mathbf{y}$, (4.133), as was claimed. □

Let us note the following corollary.

Corollary 4.2.32. *For* $\mathbf{t} \in \mathfrak{V}$*, we have*

$$\left| I^{\xi}_{\tau,\Pi}(\mathbf{t}) \right| \leq \|w\|^{3m}_{L^\infty} \int_{\Lambda^V} d\mathbf{y} \, |\xi(\mathbf{y}_1)| \prod_{e \in E} \mathcal{J}_{\tau,e}(\mathbf{y}_e, \mathbf{s}) . \tag{4.135}$$

Proof. This is a consequence of Lemmas 4.2.31 and 4.2.25 (3). □

Lemma 4.2.33. *Suppose that* $\mathcal{P} \in \operatorname{conn}(E)$ *is a closed path as in Definition 4.2.29. Then, we have*

$$\int_{\Lambda^{V(\mathcal{P})}} \prod_{a \in V(\mathcal{P})} dy_a \prod_{e \in \mathcal{P}} \mathcal{J}_{\tau,e}(\mathbf{y}_e, \mathbf{s}) \leq C^{|V(\mathcal{P})|} . \tag{4.136}$$

Moreover

$$\left| \int_{\Lambda^{V(\mathcal{P})}} \prod_{a \in V(\mathcal{P})} dy_a \left[\prod_{e \in \mathcal{P}} \mathcal{J}_{\tau,e}(\mathbf{y}_e, \mathbf{s}) - \prod_{e \in \mathcal{P}} \hat{\mathcal{J}}_{\tau,e}(\mathbf{y}_e, \mathbf{s}) \right] \right| \to 0 \text{ as } \tau \to \infty . \tag{4.137}$$

We note that h defined as in (2.6) satisfies $h > 0$ and $h^{-1} \in \mathfrak{S}^1(\mathfrak{h}) \subset \mathfrak{S}^2(\mathfrak{h})$.

Proof of Lemma 4.2.33. Since $\mathcal{P}$ is closed, we need to consider two cases depending on the length of $\mathcal{P}$. If $|\mathcal{P}| = 1$, the path is a loop. Then the left-hand side of (4.136) equals

$$\int_\Lambda dy_a\, G_\tau(y_a;y_a) = \|G_\tau\|_{\mathfrak{S}^1(\mathfrak{h})} = \mathrm{Tr}(G_\tau) = \sum_{k\geq 0} \frac{1}{\tau(e^{\lambda_k/\tau}-1)}$$
$$\leq \sum_{k\geq 0}\frac{1}{\lambda_k} = \mathrm{Tr}(h^{-1}) = \|G\|_{\mathfrak{S}^1(\mathfrak{h})} < \infty\,. \quad (4.138)$$

Here we have used the positivity of G_τ as an operator as well as (2.8).

We henceforth need to consider the case when $|\mathcal{P}| \geq 2$. Here, we argue as in [29, Lemma 2.17]. Let $|\mathcal{P}| = q$, and write $\mathcal{P} = \{e_1,\ldots,e_q\}$, where e_j and e_{j+1} are incident for $j=1,\ldots,q$. Throughout the proof we take the index j to be modulo q. We denote by a_j the unique vertex in $e_{j-1}\cap e_j$. Without loss of generality, we take $a_1 < a_2$.

An induction argument shows that the colour of e_j is determined by the colour of e_1. Namely, recalling (4.126), we have

$$\sigma(e_j) = \begin{cases} \sigma(e_1) & \text{if } a_j < a_{j+1}, \\ -\sigma(e_1) & \text{if } a_j > a_{j+1}. \end{cases} \quad (4.139)$$

For $j=1,\ldots,q$, we define $a_{j,-} := \min\{a_j,a_{j+1}\}$ and $a_{j,+} := \max\{a_j,a_{j+1}\}$. From (4.139), we have

$$\sigma(e_j)(s_{a_{j,-}} - s_{a_{j,+}}) = \sigma(e_1)(s_{a_j} - s_{a_{j+1}})\,. \quad (4.140)$$

Moreover, it is clear from (4.128) and (4.129) that $0 \leq \sigma(e_1)(s_{a_j}-s_{a_{j+1}}) < 1$. Thus, $G_{\tau,\sigma(e_1)(s_{a_j}-s_{a_{j+1}})}$ and $S_{\tau,\sigma(e_1)(s_{a_j}-s_{a_{j+1}})}$ are both well-defined. Substituting (4.140) into (4.130), we have

$$\mathcal{J}_{\tau,e_j}(\mathbf{y}_{e_j};\mathbf{s}) = G_{\tau,\sigma(e_1)(s_{a_j}-s_{a_{j+1}})}(y_{a_j};y_{a_{j+1}})$$
$$+ \frac{\chi(\sigma(e_j)=1)\chi(s_{a_j}\neq s_{a_{j+1}})}{\tau} S_{\tau,\sigma(e_1)(s_{a_j}-s_{a_{j+1}})}(y_{a_j};y_{a_{j+1}})\,. \quad (4.141)$$

Here we use that the kernels of $G_{\tau,t}$ and $S_{\tau,t}$ are symmetric. Rewriting the left-hand side of (4.136) using (4.141), we get

$$\mathrm{Tr}\left[\prod_{j=1}^{q}\left(G_{\tau,\sigma(e_1)(s_{a_j}-s_{a_{j+1}})} + \frac{\chi(\sigma(e_j)=1)\chi(s_{a_j}\neq s_{a_{j+1}})}{\tau} S_{\tau,\sigma(e_1)(s_{a_j}-s_{a_{j+1}})}\right)\right]. \quad (4.142)$$

By definition, all of these operators commute, and hence the order of the above product does not matter. We define

$$J_{\mathcal{P}} := \big\{ j \in \{1,\dots,q\} : \mathcal{J}_{\tau,e_j} \neq \hat{\mathcal{J}}_{\tau,e_j} \big\} \,.$$

We note that $J_{\mathcal{P}} \neq \big\{1,\dots,q\big\}$. Namely, we recall (4.130)–(4.131) and note that by construction, the smallest vertex in $\mathcal{P}$ (with respect to $\leq$) is incident to an edge e with $\sigma(e) = -1$. Therefore, we can rewrite (4.142) as

$$\begin{aligned}
&\sum_{I \subset J_{\mathcal{P}}} \mathrm{Tr}\left[\left(\prod_{j \in \{1,\dots,q\}\setminus I} G_{\tau,\sigma(e_1)(s_{a_j}-s_{a_{j+1}})}\right)\left(\prod_{j\in I} \frac{1}{\tau} S_{\tau,\sigma(e_1)(s_{a_j}-s_{a_{j+1}})}\right)\right] \\
&\quad = \sum_{I \subset J_{\mathcal{P}}} \mathrm{Tr}\left[\left(\prod_{j \in \{1,\dots,q\}\setminus I} G_{\tau,0}\right)\left(\prod_{j \in \{1,\dots,q\}\setminus I} \frac{1}{\tau} S_{\tau,0}\right)\right] = \sum_{J \subset J_{\mathcal{P}}} \frac{1}{\tau^{|I|}} \mathrm{Tr}(G_\tau^{q-|I|}) \,.
\end{aligned} \tag{4.143}$$

The first equality holds since $\sum_{j=1}^q \big(s_{a_j} - s_{a_{j+1}}\big) = s_{a_1} - s_{a_{q+1}} = 0$ because $s_{a_1} = s_{a_{q+1}}$. By Lemma 4.2.4, we note that for $|I| \leq q-2$,

$$\mathrm{Tr}(G_\tau^{q-|I|}) = \|G_\tau\|^{q-|I|}_{\mathfrak{S}^{q-|I|}(\mathfrak{h})} \leq \|G_\tau\|^{q-|I|}_{\mathfrak{S}^2(\mathfrak{h})} \,. \tag{4.144}$$

Using (4.144) we have that (4.143) is

$$\begin{aligned}
&\leq \sum_{\substack{I \subset J_{\mathcal{P}} \\ |I| \leq q-2}} \frac{1}{\tau^{|I|}} \|G_\tau\|^{q-|I|}_{\mathfrak{S}^2(\mathfrak{h})} + \frac{q}{\tau^{q-1}} \|G_\tau\|_{\mathfrak{S}^1(\mathfrak{h})} \\
&\leq C^{|V(\mathcal{P})|} \left(1 + \|G_\tau\|_{\mathfrak{S}^2(\mathfrak{h})} + \frac{1}{\tau}\|G_\tau\|_{\mathfrak{S}^1(\mathfrak{h})}\right)^{|V(\mathcal{P})|} \,.
\end{aligned} \tag{4.145}$$

In the first inequality above, we used $q \geq 2$. We now deduce (4.136) from (4.145), by using

$$\|G_\tau\|_{\mathfrak{S}^2(\mathfrak{h})} \leq \|G_\tau\|_{\mathfrak{S}^1(\mathfrak{h})} \leq C \,, \tag{4.146}$$

for some $C > 0$ independent of τ, which follows from (4.138), and Lemma 4.2.4.

To obtain (4.137), we split into the same two cases. If $|\mathcal{P}| = 1$, then $\mathcal{P} = \{e\}$, so the path is a loop, so $s_a = s_a$. So $\mathcal{J}_e = \hat{\mathcal{J}}_e$, and there is nothing to prove. If $|\mathcal{P}| \geq 2$, we apply the same argument as used in the proof of (4.136). The only difference is that we now sum over *non-empty* subsets I of $J_{\mathcal{P}}$ in (4.143). This results in the an extra power of $\frac{1}{\tau}$, and one less power of $\big(1 + \|G_\tau\|_{\mathfrak{S}^2(\mathfrak{h})} + \frac{1}{\tau}\|G_\tau\|_{\mathfrak{S}^1(\mathfrak{h})}\big)$ in (4.145). We hence deduce (4.137). $\square$

Lemma 4.2.34. *Suppose that $\mathcal{P} \in \mathrm{conn}(E)$ is an open path with endpoints $b_1, b_2 \in V_2(\mathcal{P})$. Then, we have*

$$\left\| \int_{\Lambda^{V_2(\mathcal{P})}} \prod_{a \in V_2(\mathcal{P})} dy_a \prod_{e \in \mathcal{P}} \mathcal{J}_{\tau,e}(\mathbf{y}_e, \mathbf{s}) \right\|_{L^2_{y_{b_1}, y_{b_2}}} \leq C^{|V_2(\mathcal{P})|} . \tag{4.147}$$

Moreover

$$\left\| \int_{\Lambda^{V(\mathcal{P})}} \prod_{a \in V(\mathcal{P})} dy_a \left[\prod_{e \in \mathcal{P}} \mathcal{J}_{\tau,e}(\mathbf{y}_e; \mathbf{s}) - \prod_{e \in \mathcal{P}} \hat{\mathcal{J}}_{\tau,e}(\mathbf{y}_e; \mathbf{s}) \right] \right\|_{L^2_{y_{b_1}, y_{b_2}}} \to 0 \text{ as } \tau \to \infty . \tag{4.148}$$

Proof. We argue similarly as for [29, Lemma 2.18]. We first prove (4.147). Let b_1, b_2 be as in the statement of the lemma. Without loss of generality, suppose $b_1 < b_2$, so $\delta_{b_1} = 1$ and $\delta_{b_2} = -1$. Let $q := |V_2(\mathcal{P})|$. If $q = 0$, then $\mathcal{J}_{\tau,e}(\mathbf{y}; \mathbf{s}) = G_\tau(y_{b_1}, y_{b_2})$, since $\sigma(e) = -1$. Hence, (4.147) follows from (4.146).

Suppose that $q \geq 1$. Write $\mathcal{P} := \{e_1, \ldots, e_{q+1}\}$, where $b_1 \in e_1$, $b_2 \in e_{q+1}$ and a_j is the unique vertex in $e_j \cap e_{j+1}$. An induction argument shows that the colour of e_j is determined by the colour of e_1. Namely, we have

$$\sigma(e_j) = \begin{cases} -1 & \text{if } a_{j-1} < a_j, \\ 1 & \text{if } a_j < a_{j-1} . \end{cases} \tag{4.149}$$

Define $a_{j,-} := \min\{a_{j-1}, a_j\}$ and $a_{j,+} := \max\{a_{j-1}, a_j\}$. Then (4.149) implies

$$\sigma(e_j)(s_{a_{j,-}} - s_{a_{j,+}}) = s_{a_j} - s_{a_{j-1}} . \tag{4.150}$$

As in the proof of Lemma 4.2.33, we use (4.128) and (4.129) to deduce that $0 \leq \sigma(e_j)(s_{a_{j,-}} - s_{a_{j,+}}) < 1$. Substituting (4.149) into (4.2.30), we have

$$\mathcal{J}_{\tau,e}(\mathbf{y}_{e_j}, \mathbf{s}) = G_{\tau, s_{a_j} - s_{a_{j-1}}}(y_{a_{j-1}}; y_{a_j}) + \frac{\chi(\sigma(e_j) = 1)\chi(s_{a_j} \neq s_{a_{j-1}})}{\tau} S_{\tau, s_{a_j} - s_{a_{j-1}}}(y_{a_{j-1}}; y_{a_j}). \tag{4.151}$$

Here, we have used the symmetry of the kernels of $G_{\tau,t}$ and $S_{\tau,t}$. Using (4.151), we have

$$\int_{\Lambda^{V_2(\mathcal{P})}} \prod_{a\in V_2(\mathcal{P})} dy_a \prod_{e\in\mathcal{P}} \mathcal{J}_{\tau,e}(\mathbf{y},\mathbf{s})$$
$$= \left[\prod_{j=1}^{q+1}\left(G_{\tau,s_{a_j}-s_{a_{j-1}}}\right) + \frac{\chi(\sigma(e_j)=1)\chi(s_{a_j}\neq s_{a_{j-1}})}{\tau} S_{\tau,s_{a_j}-s_{a_{j-1}}}\right](y_{b_1};y_{b_2})\,. \tag{4.152}$$

Define $J_{\mathcal{P}} := \{j \in \{1,\ldots,q+1\} : \mathcal{J}_{\tau,e_j} \neq \hat{\mathcal{J}}_{\tau,e_j}\}$. Since $\sigma(e_1)=1$ and $s_{b_1}\neq s_{a_1}$ (because $q\geq 1$), we have that $1\in J_{\mathcal{P}}$. Moreover, $\sigma(e_{q+1})=-1$, so $q+1\notin J_{\mathcal{P}}$, so $1\leq |J_{\mathcal{P}}|\leq q$. We rewrite (4.152) as

$$\sum_{I\subset J_{\mathcal{P}}}\left[\left(\prod_{j\in\{1,\ldots,q\}\setminus I} G_{\tau,s_{a_j}-s_{a_{j-1}}}\right)\left(\prod_{j\in I}\frac{1}{\tau}S_{\tau,s_{a_j}-s_{a_{j-1}}}\right)\right](y_{b_1};y_{b_2})$$
$$= \sum_{I\subset J_{\mathcal{P}}}\left[\left(\prod_{j\in\{1,\ldots,q\}\setminus I} G_{\tau,0}\right)\left(\prod_{j\in I}\frac{1}{\tau}S_{\tau,0}\right)\right](y_{b_1};y_{b_2})$$
$$= \sum_{I\subset J_{\mathcal{P}}}\frac{1}{\tau^{|I|}}\left(G_\tau^{q+1-|I|}\right)(y_{b_1};y_{b_2})\,. \tag{4.153}$$

In the first inequality, we uses $\sum_{j=1}^{q+1}(s_{a_j}-s_{a_{j-1}}) = s_{a_{q+1}}-s_{a_0}=0$, which is true since $s_{a_{q+1}}=s_{a_0}=0$.

For $k\geq 2$, applying the Cauchy-Schwarz inequality to the operator kernels implies that

$$G_\tau^k(y_{b_1};y_{b_2})$$
$$:= \int_{\Lambda^{k-1}}\prod_{j=1}^{k-1} dx_j\, G_\tau(y_{b_1};x_1)G_\tau(x_1;x_2)\ldots G_\tau(x_{k-2};x_{k-1})G_\tau(x_{k-1};y_{b_2})$$
$$\leq \|G_\tau\|_{\mathfrak{S}^2(\mathfrak{h})}^{k-2}\|G_\tau(y_{b_1},\cdot)\|_{\mathfrak{h}}\|G_\tau(y_{b_2},\cdot)\|_{\mathfrak{h}}\,. \tag{4.154}$$

Applying (4.154), we deduce that (4.153) is

$$\leq \sum_{\substack{I\subset J_{\mathcal{P}}\\ |I|\leq q-1}}\frac{1}{\tau^{|I|}}\|G_\tau\|_{\mathfrak{S}^2(\mathfrak{h})}^{q+1-|I|}\|G_\tau(y_{b_1},\cdot)\|_{\mathfrak{h}}\|G_\tau(y_{b_2},\cdot)\|_{\mathfrak{h}} + \frac{q+1}{\tau^q}G_\tau(y_{b_1};y_{b_2})$$
$$\leq C^{|V_2(\mathcal{P})|}\left(1+\|G_\tau\|_{\mathfrak{S}^2(\mathfrak{h})}\right)^{|V_2(\mathcal{P})|}\left(\|G_\tau(y_{b_1},\cdot)\|_{\mathfrak{h}}\|G_\tau(y_{b_2},\cdot)\|_{\mathfrak{h}} + G_\tau(y_{b_1};y_{b_2})\right)\,. \tag{4.155}$$

Then (4.147) follows from (4.155) and (4.146).

To prove (4.148), we note that if $q = 0$, we have $\mathcal{J}_{\tau,e} = \hat{\mathcal{J}}_{\tau,e}$, since $\delta_{b_2} = -1$. In this case, (4.148) automatically holds. If $q \geq 1$, we argue as for (4.147), except we sum over *non-empty* subsets of $J_{\mathcal{P}}$ in (4.153). This results in an extra factor of $\frac{1}{\tau}$ and one less power of $(1 + \|G_\tau\|)_{\mathfrak{S}^2(\mathfrak{h})}$ in (4.155). (4.148) then follows from (4.146). □

We can now bound the quantity (4.123).

Lemma 4.2.35. *For* $\Pi \in \mathfrak{P}$ *and* $\mathbf{t} \in \mathfrak{V}$, *we have*

$$\left| I^{\xi}_{\tau,\Pi}(\mathbf{t}) \right| \leq C^{m+p} \|w\|^{3m}_{L^\infty} \,. \tag{4.156}$$

Proof. We argue as in [29, Lemma 2.19]. We use the splitting (4.132) and Corollary 4.2.32 to rewrite (4.135) as

$$\left| I^{\xi}_{\tau,\Pi}(\mathbf{t}) \right| \leq \|w\|^{3m}_{L^\infty} \int_{\Lambda^{V_1}} d\mathbf{y}_1 \, |\xi(\mathbf{y}_1)| \int_{\Lambda^{V_2}} d\mathbf{y}_2 \prod_{e \in E} \mathcal{J}_{\tau,e}(\mathbf{y}_e, \mathbf{s}) \,. \tag{4.157}$$

Let us introduce the partition $\mathrm{Conn}(E) = \mathrm{Conn}_c(E) \sqcup \mathrm{Conn}_o(E)$. In other words, we partition $\mathrm{Conn}(E)$ into the closed connected paths $\mathrm{Conn}_c(E)$ and the open connected paths $\mathrm{Conn}_o(E)$. Then, we have

$$\begin{aligned}\int_{\Lambda^{V_2}} d\mathbf{y}_2 \prod_{e \in E} \mathcal{J}_{\tau,e}(\mathbf{y}_e, \mathbf{s}) = & \prod_{\mathcal{P} \in \mathrm{Conn}_c(E)} \left(\int_{\Lambda^{V(\mathcal{P})}} \prod_{a \in V(\mathcal{P})} dy_a \prod_{e \in \mathcal{P}} \mathcal{J}_{\tau,e}(\mathbf{y}_e, \mathbf{s}) \right) \\ & \times \prod_{\mathcal{P} \in \mathrm{Conn}_o(E)} \left(\int_{\Lambda^{V_2(\mathcal{P})}} \prod_{a \in V_2(\mathcal{P})} dy_a \prod_{e \in \mathcal{P}} \mathcal{J}_{\tau,e}(\mathbf{y}_e, \mathbf{s}) \right). \end{aligned} \tag{4.158}$$

Substituting (4.158) into (4.157) and using (4.136), we have

$$\begin{aligned} & |I^{\xi}_{\tau,\Pi}(\mathbf{t})| \\ & \leq C^m \|w\|^{3m}_{L^\infty} \int_{\Lambda^{V_1}} d\mathbf{y}_1 \, |\xi(\mathbf{y}_1)| \prod_{\mathcal{P} \in \mathrm{Conn}_o(E)} \left(\int_{\Lambda^{V_2(\mathcal{P})}} \prod_{a \in V_2(\mathcal{P})} dy_a \prod_{e \in \mathcal{P}} \mathcal{J}_{\tau,e}(\mathbf{y}_e, \mathbf{s}) \right). \end{aligned} \tag{4.159}$$

We note that (4.156) follows from (4.159) by applying the Cauchy-Schwarz inequality in the $\mathbf{y}_1$ variables, followed by (4.147). We also use that $|\mathrm{Conn}_o(E)| = p$ to get the factor of C^p on the right-hand side of (4.156). □

Using Lemma 4.2.35, we can now bound the quantity $g^{\xi}_{\tau,m}(\mathbf{t})$ defined in (4.119).

Lemma 4.2.36. *For* $\mathbf{t} \in \mathfrak{V}$*, we have*

$$\left| g^{\xi}_{\tau,m}(\mathbf{t}) \right| \leq C^{m+p} \|w\|^{3m}_{L^\infty} (3m+p)!\,.$$

Proof. The claim follows from Lemma 4.2.23, (4.156), and the observation that $|\mathfrak{P}| \leq (3m+p)!$. ☐

We can now bound the quantity $b^{\xi}_{\tau,m}$ given by (4.113) and (4.105) above and obtain (4.114) uniformly in $\xi \in \mathfrak{B}_p$.

Corollary 4.2.37. *Uniformly in* $\xi \in \mathfrak{B}_p$*, we have*

$$|b^{\xi}_{\tau,m}| \leq (Cp)^p C^m (m!)^2 \|w\|^{3m}_{L^\infty} =: C(m,p)\,. \tag{4.160}$$

Proof. The claim follows from (4.118) and Lemma 4.2.36, after integrating over the simplex (4.120) (which gives a factor of $\frac{1}{m!}$) and using Stirling's formula. ☐

Proof of (4.115) **uniformly in** $\xi \in \mathfrak{B}_p$

Let us make the following definition.

Definition 4.2.38. For $e = \{a,b\} \in E$, we define

$$\mathcal{J}_e(\mathbf{y}_e) := G(y_a; y_b)\,.$$

Proposition 1.7.12 implies the following lemma.

Lemma 4.2.39. *The kernel of* G *is non-negative and symmetric.*

Definition 4.2.40. For $\Pi \in \mathfrak{P}$, we define

$$I^{\xi}_{\Pi} := \int_{\Lambda^V} d\mathbf{y} \left(\prod_{i=1}^{m} w(y_{i,1}-y_{i,2}) w(y_{i,2}-y_{i,3}) w(y_{i,3}-y_{i,1}) \right) \xi(\mathbf{y}_1) \prod_{e \in E} \mathcal{J}_e(\mathbf{y}_e)\,. \tag{4.161}$$

Lemma 4.2.41. *For each* $\Pi \in \mathfrak{P}$ *and* $\mathbf{t} \in \mathfrak{V}$*, we have*

$$I^{\xi}_{\tau,m}(\mathbf{t}) \to I^{\xi}_{m} \text{ uniformly in } \xi \in \mathfrak{B}_p \text{ as } \tau \to \infty\,. \tag{4.162}$$

Proof. The proof is similar to the proof of [29, Lemma 2.25]. For $\mathbf{t} \in \mathfrak{V}$, we define

$$\hat{I}^{\xi}_{\tau,\Pi}(\mathbf{t}) := \int_{\Lambda^V} d\mathbf{y} \left(\prod_{i=1}^{n} w(y_{i,1} - y_{i,2}) w(y_{i,2} - y_{i,3}) w(y_{i,3} - y_{i,1}) \right) \xi(\mathbf{y}_1) \prod_{e \in E} \hat{\mathcal{J}}_{\tau,e}(\mathbf{y}_e) \,. \quad (4.163)$$

Let first show that

$$\hat{I}^{\xi}_{\tau,\Pi}(\mathbf{t}) \to I^{\xi}_{\Pi} \text{ uniformly in } \xi \in \mathfrak{B}_p \text{ as } \tau \to \infty \,. \quad (4.164)$$

Namely, from (4.161) and (4.163), we have

$$\hat{I}^{\xi}_{\tau,\Pi}(\mathbf{t}) - I^{\xi}_{\Pi} = \int_{\Lambda^V} d\mathbf{y} \left(\prod_{i=1}^{m} w(y_{i,1} - y_{i,2}) w(y_{i,2} - y_{i,3}) w(y_{i,3} - y_{i,1}) \right) \xi(\mathbf{y}_1) \times \left[\prod_{e \in E} \hat{\mathcal{J}}_{\tau,e}(\mathbf{y}_e) - \prod_{e \in E} \mathcal{J}_{e}(\mathbf{y}_e) \right] . \quad (4.165)$$

By telescoping, we can write

$$\prod_{e \in E} \hat{\mathcal{J}}_{\tau,e}(\mathbf{y}_e, \mathbf{s}) - \prod_{e \in E} \mathcal{J}_{e}(\mathbf{y}_e) = \sum_{e_0 \in E} \left[\prod_{\substack{e \in E \\ e < e_0}} \hat{\mathcal{J}}_{\tau,e}(\mathbf{y}_e, \mathbf{s}) \left(\hat{\mathcal{J}}_{\tau,e_0}(\mathbf{y}_{e_0}, \mathbf{s}) - \mathcal{J}_{e_0}(\mathbf{y}_{e_0}) \right) \prod_{\substack{e \in E \\ e > e_0}} \mathcal{J}_{e}(\mathbf{y}_e) \right] , \quad (4.166)$$

where we order the elements of E arbitrarily. Substituting (4.166) into (4.165), we have

$$\left| \hat{I}^{\xi}_{\tau,\Pi}(\mathbf{t}) - I^{\xi}_{\Pi} \right| \leq \sum_{e_0 \in E} \|w\|^{3m}_{L^\infty} \int_{\Lambda^V} d\mathbf{y} \, |\xi(\mathbf{y}_1)| \times \left[\prod_{\substack{e \in E \\ e < e_0}} \hat{\mathcal{J}}_{\tau,e}(\mathbf{y}_e, \mathbf{s}) \left| \hat{\mathcal{J}}_{\tau,e_0}(\mathbf{y}_{e_0}, \mathbf{s}) - \mathcal{J}_{e_0}(\mathbf{y}_{e_0}) \right| \prod_{\substack{e \in E \\ e > e_0}} \mathcal{J}_{e}(\mathbf{y}_e) \right] . \quad (4.167)$$

Here, we have used that $\mathcal{J}_{\tau,e}(\mathbf{y}_e; \mathbf{s})$ and $\mathcal{J}_e(\mathbf{y}_e) \geq 0$ by Lemmas 4.2.24 and 4.2.39 (recalling Definitions 4.2.30 and 4.2.38).

Let us denote by $\sigma^{\xi}_{\tau,e_0}(\mathbf{t})$ the summand in (4.167) corresponding to e_0. Since (4.167) is a finite sum, in order to obtain (4.2.41), it suffices to show that for each

$e_0 \in E$, we have

$$\sigma^{\xi}_{\tau,e_0}(\mathbf{t}) \to 0 \text{ uniformly in } \xi \in \mathfrak{B}_p \text{ as } \tau \to \infty\,. \tag{4.168}$$

We fix $e_0 \in E$. Let us define an integral kernel associated with an edge $e \in E$ by

$$\tilde{\mathcal{J}}_{\tau,e}(\mathbf{y}_e,\mathbf{s}) := \begin{cases} \hat{\mathcal{J}}_{\tau,e}(\mathbf{y}_e,\mathbf{s}) & \text{if } e < e_0\,, \\ \Big|\hat{\mathcal{J}}_{\tau,e}(\mathbf{y}_e,\mathbf{s}) - \mathcal{J}_e(\mathbf{y}_e)\Big| & \text{if } e = e_0\,, \\ \mathcal{J}_e(\mathbf{y}_e) & \text{if } e > e_0\,. \end{cases} \tag{4.169}$$

We have the following estimates for $\tilde{\mathcal{J}}_{\tau,e}$.

1. If $e = \{a,a\}$ (i.e. a loop in the graph) and $e \neq e_0$, then

$$\|\tilde{\mathcal{J}}_{\tau,e}(\cdot,\mathbf{s})\|_{\mathfrak{S}^1(\mathfrak{h})} \leq \|G_\tau\|_{\mathfrak{S}^1(\mathfrak{h})} + \|G\|_{\mathfrak{S}^1(\mathfrak{h})} \leq C\,, \tag{4.170}$$

   which holds by (4.138).

2. If $e = \{a,a\}$ and $e = e_0$, then

$$\begin{aligned} \|\tilde{\mathcal{J}}_{\tau,e}(\cdot,\mathbf{s})\|_{\mathfrak{S}^1(\mathfrak{h})} &= \int dy\, \Big|\hat{\mathcal{J}}_{\tau,e_0}(\mathbf{y}_{e_0},\mathbf{s}) - \mathcal{J}_{e_0}(\mathbf{y}_{e_0})\Big| \\ &= \int dy\, \left|\frac{1}{\tau(\mathrm{e}^{h/\tau}-1)}(y;y) - \frac{1}{h}(y;y)\right| = \int dy \sum_{k\in\mathbb{N}} \left(\frac{1}{\lambda_k} - \frac{1}{\tau(e^{\lambda_k/\tau}-1)}\right) \\ &= \|G_\tau - G\|_{\mathfrak{S}^1(\mathfrak{h})} \to 0\,, \end{aligned} \tag{4.171}$$

   as $\tau \to \infty$ by spectral decomposition and the dominated convergence theorem. The third equality above follows by comparing Taylor series.

3. If $e = \{a,b\}$ with $a < b$ and $e < e_0$, then $\tilde{\mathcal{J}}_{\tau,e}(\mathbf{y}_e,\mathbf{s}) = G_{\tau,\sigma(e)(s_a-s_b)}(y_a;y_b)$. Let us note that

$$\lim_{\tau\to\infty}(1+t)\left\|\frac{\mathrm{e}^{-th/\tau}}{\tau(\mathrm{e}^{h/\tau}-1)} - h^{-1}\right\|_{\mathfrak{S}^2(\mathfrak{h})} \to 0\,, \tag{4.172}$$

   uniformly in $t \in (-1,1)$. The claim (4.172) follows by a spectral argument; see [29, Lemma C.2] for the proof of a more general claim. Then (4.172) implies

$$\|\tilde{\mathcal{J}}_{\tau,e}(\cdot,\mathbf{s})\|_{\mathfrak{S}^2(\mathfrak{h})} \leq C_{\mathbf{s}}\,, \tag{4.173}$$

   where the constant depends on $\mathbf{s}$.

4. If $e = \{a, b\}$ with $a < b$ and $e = e_0$. Then

$$\tilde{\mathcal{J}}_{\tau,e}(\mathbf{y}_e, \mathbf{s}) = \left(G_{\tau,\sigma(e)(s_a - s_b)} - G\right)(y_a; y_b)\,. \tag{4.174}$$

Therefore, (4.174)–(4.172) imply that

$$\|\tilde{\mathcal{J}}_{\tau,e}(\cdot, \mathbf{s})\|_{\mathfrak{S}^2(\mathfrak{h})} \to 0 \text{ as } \tau \to \infty\,. \tag{4.175}$$

5. If $e = \{a, b\}$ with $a < b$ and $e > e_0$. Then $\tilde{\mathcal{J}}_{\tau,e}(\mathbf{y}_e, \mathbf{s}) = G(y_a; y_b)$. Then

$$\|\tilde{\mathcal{J}}_{\tau,e}(\cdot, \mathbf{s})\|_{\mathfrak{S}^2(\mathfrak{h})} = \|G\|_{\mathfrak{S}^2(\mathfrak{h})} \leq C\,. \tag{4.176}$$

Applying the same decomposition from the proof of Lemma 4.2.35, we have

$$\sigma^{\xi}_{\tau,e_0}(\mathbf{t}) = \|w\|^{3m}_{L^\infty} \int_{\Lambda^{V_1}} d\mathbf{y}_1 \prod_{\mathcal{P} \in \mathrm{Conn}_c(E)} \left(\int_{\Lambda^{V(\mathcal{P})}} \prod_{a \in V(\mathcal{P})} dy_a \prod_{e \in \mathcal{P}} \tilde{\mathcal{J}}_{\tau,e}(\mathbf{y}_e, \mathbf{s}) \right)$$
$$\times \prod_{\mathcal{P} \in \mathrm{Conn}_o(E)} \left(\int_{\Lambda^{V_2(\mathcal{P})}} \prod_{a \in V_2(\mathcal{P})} dy_a \prod_{e \in \mathcal{P}} \tilde{\mathcal{J}}_{\tau,e}(\mathbf{y}_e, \mathbf{s}) \right). \tag{4.177}$$

For $\mathcal{P} \in \mathrm{Conn}_c(E)$, apply the Cauchy-Schwarz inequality in y_a for $a \in V(\mathcal{P})$ to deduce that

$$\int_{\Lambda^{V(\mathcal{P})}} \prod_{a \in V(\mathcal{P})} dy_a \prod_{e \in \mathcal{P}} \tilde{\mathcal{J}}_{\tau,e}(\mathbf{y}_e, \mathbf{s}) \leq \prod_{e \in \mathcal{P}} \|\tilde{\mathcal{J}}_{\tau,e}(\cdot, \mathbf{s})\|_{\mathfrak{S}^2(\mathfrak{h})}\,. \tag{4.178}$$

For $\mathcal{P} \in \mathrm{Conn}_o(E)$, we recall the notation from the proof of Lemma 4.2.34. Namely

$$\mathcal{P} = \{e_1, \dots, e_{q+1}\}, \quad V_1(\mathcal{P}) = \{b_1, b_2\}, \qquad V_2(\mathcal{P}) = \{a_1, \dots, a_q\}.$$

We apply the Cauchy-Schwarz inequality in the y_a for $a \in V_2(\mathcal{P})$ to obtain

$$\int_{\Lambda^{V_2(\mathcal{P})}} \prod_{a \in V_2(\mathcal{P})} dy_a \prod_{e \in \mathcal{P}} \tilde{\mathcal{J}}_{\tau,e}(\mathbf{y}_e, \mathbf{s}) \leq$$
$$\prod_{j=2}^{q} \|\tilde{\mathcal{J}}_{\tau,e_j}(\cdot, \mathbf{s})\|_{\mathfrak{S}^2(\mathfrak{h})} \|\tilde{\mathcal{J}}_{\tau,e_1}(y_{b_1}, \cdot)\|_{\mathfrak{h}} \|\tilde{\mathcal{J}}_{\tau,e_{q+1}}(y_{b_2}, \cdot)\|_{\mathfrak{h}}\,. \tag{4.179}$$

Substituting (4.178) and (4.179) into (4.177) and applying the Cauchy-Schwarz inequality in the $\mathbf{y}_1$ variables yields

$$\sigma^{\xi}_{\tau,e_0}(\mathbf{t}) \leq \|w\|^{3m}_{L^\infty} \|\xi\|_{\mathfrak{S}^2(\mathfrak{h})} \prod_{e \in E} \|\tilde{\mathcal{J}}_{\tau,e}(\cdot,\mathbf{s})\|_{\mathfrak{S}^2(\mathfrak{h})} \,. \tag{4.180}$$

Recalling (4.170)–(4.176) and Lemma 4.2.4, we obtain (4.168) from (4.180). Hence, (4.164) follows.

Let us also note that

$$I^{\xi}_{\tau,\Pi}(\mathbf{t}) \to \hat{I}^{\xi}_{\tau,\Pi}(\mathbf{t}) \text{ uniformly in } \xi \in \mathfrak{B}_p \text{ as } \tau \to \infty \,. \tag{4.181}$$

To obtain (4.181), we use a telescoping argument analogous to (4.166) above, perform a decomposition into open and closed paths as in the proof of Lemma 4.2.35, and use Lemmas 4.2.33 and 4.2.34. We omit the details, see [29, (2.62)]. We obtain the claim of Lemma 4.2.41 from (4.164) and (4.181). □

For $m \in \mathbb{N}$, let us define

$$b^{\xi}_{\infty,m} := \frac{(-1)^m}{m!\,3^m} \sum_{\Pi \in \mathfrak{P}} I^{\xi}_{\Pi} \,. \tag{4.182}$$

We now conclude the claimed convergence result.

Lemma 4.2.42. *For $b^{\xi}_{\tau,m}, b^{\xi}_m$ defined as in* (4.113), *we have*

$$b^{\xi}_{\tau,m} \to b^{\xi}_m \textit{ as } \tau \to \infty \textit{ uniformly in } \xi \in \mathfrak{B}_p \,. \tag{4.183}$$

Proof. By Lemma 4.2.23 and Proposition 4.2.35, we have that $g^{\xi}_{\tau,m}(\mathbf{t})$ is bounded uniformly in $\tau > 0$, $\xi \in \mathfrak{B}_p$, and $\mathbf{t} \in \mathfrak{V}$. Moreover, Lemmas 4.2.23 and 4.2.41 imply that for any $\mathbf{t} \in \mathfrak{V}$

$$g^{\xi}_{\tau,m}(\mathbf{t}) \to \sum_{\Pi \in \mathfrak{P}} I^{\xi}_{\Pi} \text{ as } \tau \to \infty \text{ uniformly in } \xi \in \mathfrak{B}_p \,. \tag{4.184}$$

Recalling (4.118) and applying (4.182), as well as (4.184), combined with the dominated convergence theorem, we obtain that $b^{\xi}_{\tau,m} \to b^{\xi}_{\infty,m}$ as $\tau \to \infty$ uniformly in $\xi \in \mathfrak{B}_p$. The claim of the lemma follows by noting that, by Wick's theorem (Proposition 2.1.2), we have $b^{\xi}_{\infty,m} = b^{\xi}_m$. □

Proof of (4.114)–(4.115) **for** $\xi = \mathbf{1}_p$

To conclude the proof of Proposition 4.2.13, it remains to prove (4.114)–(4.115) for $\xi = \mathbf{1}_p$. We hence consider the operator with kernel

$$\xi(x_1, \ldots, x_p; y_1, \ldots, y_p) := \prod_{j=1}^{p} \delta(x_j - y_j)\,. \tag{4.185}$$

To do this, we need to introduce a slightly modified version of the graphs defined in Definition 4.2.17 above. The modification is analogous to that used in [29, Section 4.2].

Definition 4.2.43. Let $m, p \in \mathbb{N}$ be given. We consider the same abstract vertex set $\Sigma \equiv \Sigma(m,p)$ with $6m + 2p$ elements and set of matchings of $\mathfrak{P} \equiv \mathfrak{P}(m,p)$ as in Definition 4.2.17. For each $\Pi \in \mathfrak{P}$, we consider a coloured multigraph $(\tilde{V}, \tilde{E}, \tilde{\sigma}) \equiv (\tilde{V}_\Pi, \tilde{E}_\Pi, \tilde{\sigma}_\Pi)$, with $\tilde{\sigma} : \tilde{E} \to \{\pm 1\}$, defined as follows.

1. We say $\alpha \sim \beta$ if and only if $i_\alpha = i_\beta$ and $r_\alpha = r_\beta$. We define the set $\tilde{V} := \{[\alpha] : \alpha \in \Sigma\}$ and write $\tilde{V} = \tilde{V}_1 \sqcup \tilde{V}_2$, where

   $$\tilde{V}_1 := \{(m+1, r) : r \in \{1, \ldots, p\}\}$$

   and

   $$\tilde{V}_2 := \{(i, r) : i \in \{1, \ldots, m\}, r \in \{1, 2, 3\}\}\,.$$

   The set $\tilde{V}$ inherits an order from the lexicographical ordering (given in Definition 4.2.17). Namely $[\alpha] \leq [\beta]$ if $\alpha \leq \beta$.

2. For $\Pi \in \mathfrak{P}$, we note that (α, β) induces an edge $e := \{[\alpha], [\beta]\}$ in $\tilde{E}$. Let us $\tilde{\sigma}(e) := \delta_\beta$. This is well-defined by construction.

3. Let $\operatorname{conn}(\tilde{E})$ denote the set of connected components of $\tilde{E}$. Note we can write $\tilde{E} = \bigsqcup_{\mathcal{P} \in \operatorname{conn}(\tilde{E})} \mathcal{P}$. We call the connected components $\mathcal{P}$ of $\tilde{E}$ paths.

Remark 4.2.44. In a slight abuse of notation, we denote the equivalence relation in Definition 4.2.27 and in Definition 4.2.43 by $\sim$. From context, it will always be clear to which equivalence relation we are referring. The same holds for the order $\leq$ induced by lexicographical order on Σ.

Remark 4.2.45. The difference between the graph structure in Definition 4.2.43 and the one in Definition 4.2.27 is that, in the former, we identify the nodes corresponding to the observable ξ with kernel (4.185); see Figure 4.4.

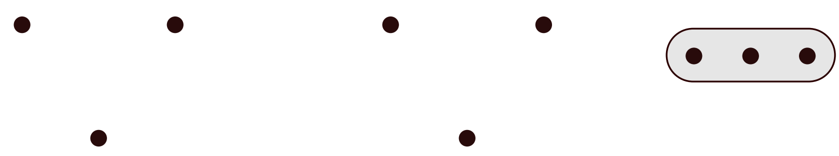

Figure 4.4: An unpaired graph with $m = 2$, $p = 3$ corresponding to Definition 4.2.43.

By construction, every vertex in $\tilde{V}$ has degree 2. Hence all paths $\mathcal{P} \in \mathrm{conn}(\tilde{E})$ are closed. For fixed $\Pi \in \mathfrak{P}$ and $\mathbf{t} \in \mathfrak{V}$, we define

$$I^{\xi}_{\tau,\Pi}(\mathbf{t}) := \int_{\Lambda^{\tilde{V}}} \prod_{a \in \tilde{V}} dy_a \left(\prod_{i=1}^{m} w(y_{i,1} - y_{i,2})\, w(y_{i,2} - y_{i,3})\, w(y_{i,3} - y_{i,1}) \right) \prod_{e \in \tilde{E}} \mathcal{J}_{\tau,e}(\mathbf{y}_e; \mathbf{s}) \,. \tag{4.186}$$

As in Definition 4.2.17, we consider the spatial labels $\mathbf{y} = (y_a)_{a \in \tilde{V}}$ and time labels $\mathbf{s} = (s_a)_{a \in \tilde{V}}$, except that we now adopt $\tilde{V}$ as the vertex set and let the equivalence relation be the one from Definition 4.2.43 above. Moreover, we adopt the convention that $\mathbf{y}_i := (y_a)_{a \in \tilde{V}_i}$. Given $\mathcal{P} \in \mathrm{conn}(\tilde{E})$, we denote the set of vertices of $\mathcal{P}$ by $\tilde{V}(\mathcal{P})$, and write $\tilde{V}_i(\mathcal{P}) := \tilde{V}(\mathcal{P}) \cap \tilde{V}_i$ for $i = 1, 2$, analogously as in Definition 4.2.29. Let us also define $\mathcal{J}_{\tau,e}$ analogously to (4.130), replacing (V, E) with $(\tilde{V}, \tilde{E})$.

Lemma 4.2.46. *Suppose that $\mathcal{P} \in \mathrm{conn}(\tilde{E})$. Then*

$$\int_{\Lambda^{\tilde{V}(\mathcal{P})}} \prod_{a \in \tilde{V}(\mathcal{P})} dy_a \prod_{e \in \mathcal{P}} \mathcal{J}_{\tau,e}(\mathbf{y}_e, \mathbf{s}) \leq C^{|\tilde{V}(\mathcal{P})|} \,. \tag{4.187}$$

Proof. We follow the approach of the proof given in [29, Lemma 4.9]. We have two cases, depending on whether $\tilde{V}(\mathcal{P}) \subset \tilde{V}_2$ or $\tilde{V}_1(\mathcal{P}) \neq \emptyset$. In the first case, all the vertices lies in $\tilde{V}_2$, so we can argue as in the proof of Lemma 4.2.33.

Let us henceforth assume that $\tilde{V}_1(\mathcal{P}) \neq \emptyset$. If $|\tilde{V}(\mathcal{P})| = 1$, then $\mathcal{P}$ is a loop, and the left-hand side of (4.187) is $\|G_\tau\|_{\mathfrak{S}^1}$. Therefore, we can argue as in the proof of Lemma 4.2.33 to get the required bound.

Let us now suppose that $|\tilde{V}(\mathcal{P})| > 1$. Since $\mathcal{P}$ is a closed path, there exist $b_1 \dots, b_k \in \tilde{V}_1(\mathcal{P})$ such that $\mathcal{P} = \sqcup_{j=1}^k \mathcal{P}_j$, where for each $j = 1, \dots, k$, $\mathcal{P}_j = \{e_1^j, \dots, e_{q_j}^j\}$, with $b_j \in e_1^j$, $b_{j+1} \in e_{q_j}^j$, and $e_k^j \cap e_{k+1}^j \in \tilde{V}_2$ for $k \in \{1, \dots, q_{j-1}\}$. Since $\mathcal{P}$ is closed, we set $b_{k+1} := b_1$. Let us note that if b_j and b_{j+1} are connected by a path of length one, then we have $q_j = 1$. Therefore, the left-hand side of (4.187) can be written as

$$\int_{\Lambda^k} dy_{b_1} \dots dy_{b_1} \prod_{j=1}^k \left[\int_{\Lambda^{\tilde{V}_2(\mathcal{P}_j)}} \prod_{a \in \tilde{V}_2(\mathcal{P}_j)} dy_a \prod_{e \in \mathcal{P}_j} \mathcal{J}_{\tau,e}(\mathbf{y}_e, \mathbf{s}) \right] . \tag{4.188}$$

Arguing as in the proof of Lemma 4.2.34 (in particular as in the proof of (4.155)), we have that the j^{th} factor in (4.188) is less than or equal to

$$C^{|\tilde{V}_2(\mathcal{P}_j)|} \left(1 + \|G_\tau\|_{\mathfrak{S}^2(\mathfrak{h})}\right)^{|\tilde{V}_2(\mathcal{P}_j)|} \left(\|G_\tau(y_{b_j}, \cdot)\|_{\mathfrak{h}} \|G_\tau(y_{b_{j+1}}, \cdot)\|_{\mathfrak{h}} + G_\tau(y_{b_j}; y_{b_{j+1}})\right) . \tag{4.189}$$

(4.187) follows from applying the Cauchy-Schwarz inequality in each of the $y_{b_1}, \dots, y_{b_k}$ variables, and using (4.146). □

Let us note that Lemma 4.2.46 implies (4.114) for $\xi = \mathbf{1}_p$.

For fixed $\Pi \in \mathfrak{P}$, we define

$$I_\Pi^\xi := \int_{\Lambda^{\tilde{V}}} d\mathbf{y} \left(\prod_{i=1}^m w(y_{i,1} - y_{i,2}) w(y_{i,2} - y_{i,3}) w(y_{i,3} - y_{i,1}) \right) \prod_{e \in \tilde{E}} \mathcal{J}_e(\mathbf{y}_e) . \tag{4.190}$$

Let $b_{\infty,m}^\xi$ be defined as in (4.182), where I_Π^ξ is now given by (4.190) instead of by (4.161). The same telescoping argument used in the proof of Lemma 4.2.41 (adapted to the framework of the proof of Lemma 4.2.46) implies that (4.115) holds for $\xi = \mathbf{1}_p$. We omit the details. This completes the proof of Proposition 4.2.13.

4.3 The time-independent problem with unbounded interaction potentials. Proof of Theorem 2.2.7

In this section, we consider w as in Assumption 2.2.2 above. Our goal is to prove Theorem 2.2.7 and thus complete the analysis of the time-independent problem outlined in Section 2.2.3. Let us first note the following claim, which follows from (4.109)–(4.110) and Theorem 2.2.6 by duality.

Lemma 4.3.1. *Suppose that $w \in L^\infty$ is even and real-valued (as in Assumption 2.2.1). Then, for all $p \in \mathbb{N}^*$, we have $\rho_\tau(\Theta_\tau(\xi)) \to \rho(\Theta(\xi))$ as $\tau \to \infty$ uniformly in $\xi \in \mathcal{C}_p$. Here, we recall the definition* (1.13) *of $\mathcal{C}_p$.*

We now prove the following result for w as in Assumption 2.2.2 (the result in Lemma 4.3.1 will be applied for bounded approximations of the original interaction potential).

Lemma 4.3.2. *Suppose that w is as in Assumption 2.2.2 and that $w^\varepsilon \in L^\infty$ is a sequence of interaction potentials as in Assumption 2.2.1 satisfying $w^\varepsilon \to w$ in $L^{3/2}$ as $\varepsilon \to 0$. Then, there is a sequence (ε_τ) tending to 0 as $\tau \to \infty$ such that for any $p \in \mathbb{N}^*$*

$$\lim_{\tau \to \infty} \rho_\tau^{\varepsilon_\tau}(\Theta_\tau(\xi)) = \rho(\Theta(\xi)) \tag{4.191}$$

uniformly in $\xi \in \mathcal{C}_p$ with $\mathcal{C}_p$ defined as in (1.13).

Proof. By a diagonal argument, it suffices to prove that for fixed $\varepsilon > 0$, we have

$$\lim_{\tau \to \infty} \rho_\tau^{\varepsilon}(\Theta_\tau(\xi)) = \rho^\varepsilon(\Theta(\xi)) \tag{4.192}$$

uniformly in $\xi \in \mathcal{C}_p$, and that

$$\lim_{\varepsilon \to 0} \rho^\varepsilon(\Theta(\xi)) = \rho(\Theta(\xi))\,, \tag{4.193}$$

uniformly in $\xi \in \mathcal{C}_p$. The convergence (4.192) follows by noting that w^ε satisfies Assumption 2.2.1 for each fixed $\varepsilon > 0$ and by using Lemma 4.3.1.

Let us now prove the convergence (4.193). We recall (2.51) (and the corresponding analogue for interaction w^ε). Let us first show that

$$\lim_{\varepsilon \to 0} \mathcal{W}^\varepsilon = \mathcal{W}\,, \quad \mu\text{-almost surely}\,. \tag{4.194}$$

We have

$$3\,|\mathcal{W} - \mathcal{W}^\varepsilon| \leq \int dx\,dy\,dz\,|\varphi(x)|^2 |\varphi(y)|^2 |\varphi(z)|^2 \\ \times |w(x-y)w(y-z)w(z-x) - w^\varepsilon(x-y)w^\varepsilon(y-z)w^\varepsilon(z-x)|\,. \tag{4.195}$$

Using the triangle inequality, we can rewrite the term inside the absolute value on the right-hand side of (4.195) as

$$\begin{aligned}
&\leq |w(x-y)w(y-z)w(z-x) - w^{\varepsilon}(x-y)w(y-z)w(z-x)| \\
&+ |w^{\varepsilon}(x-y)w(y-z)w(z-x) - w^{\varepsilon}(x-y)w^{\varepsilon}(y-z)w(z-x)| \\
&+ |w^{\varepsilon}(x-y)w^{\varepsilon}(y-z)w(z-x) - w^{\varepsilon}(x-y)w^{\varepsilon}(y-z)w^{\varepsilon}(z-x)| \,. \qquad (4.196)
\end{aligned}$$

To bound (4.195), we apply Lemma 4.1.3 separately to each of the terms obtained (4.196) to deduce that

$$|\mathcal{W} - \mathcal{W}^{\varepsilon}| \lesssim \|w - w^{\varepsilon}\|_{L^{\frac{3}{2}}} \left(\|w\|_{L^{\frac{3}{2}}}^2 + \|w^{\varepsilon}\|_{L^{\frac{3}{2}}}^2\right) \|\varphi\|_{H^{\frac{1}{3}}}^6 \,, \qquad (4.197)$$

We hence obtain (4.194) from (4.197), the Sobolev embedding theorem, and the fact that $w^{\varepsilon} \to w$ in $L^{3/2}$.

By using (4.28) and the fact that $w^{\varepsilon} \to w$ in $L^{3/2}$ as $\varepsilon \to 0$, it follows that there exists $c_0 > 0$ (depending on w) such that for $\varepsilon > 0$ sufficiently small, we have

$$\left|\mathrm{e}^{-\mathcal{W}^{\varepsilon}} - \mathrm{e}^{-\mathcal{W}}\right| \leq 2\mathrm{e}^{c_0 \|\varphi\|_{H^{1/3}}^6} \,. \qquad (4.198)$$

By (4.29) and Assumption 2.2.4 (with K sufficiently small), we know

$$\mathrm{e}^{c_0 \|\varphi\|_{H^{1/3}}^6} f^{\frac{1}{2}}(\mathcal{N}) \in L^1(d\mu) \,. \qquad (4.199)$$

By Lemma 4.2.2 and Assumption 2.2.4, we have

$$\Theta(\xi)\, f^{\frac{1}{2}}(\mathcal{N}) \in L^{\infty}(d\mu) \,. \qquad (4.200)$$

Combining (4.194) and (4.198)–(4.200), it follows that

$$\lim_{\varepsilon \to 0} \int d\mu \left|\mathrm{e}^{-\mathcal{W}^{\varepsilon}} - \mathrm{e}^{-\mathcal{W}}\right| |\Theta(\xi)|\, f(\mathcal{N}) = 0 \,. \qquad (4.201)$$

By analogous arguments, we obtain

$$\lim_{\varepsilon \to 0} z^{\varepsilon} = z \,. \qquad (4.202)$$

We write

$$\rho^{\varepsilon}(\Theta(\xi)) - \rho(\Theta(\xi)) = \frac{1}{z} \int d\mu \left(\frac{z}{z^{\varepsilon}}\, \mathrm{e}^{-\mathcal{W}^{\varepsilon}} - \mathrm{e}^{-\mathcal{W}}\right) \Theta(\xi)\, f(\mathcal{N}) \,, \qquad (4.203)$$

and deduce the claim of the lemma from (4.201)–(4.203). For the last step, we also used (4.199) and $\mathrm{e}^{-\mathcal{W}^\varepsilon} \leq \mathrm{e}^{c_0 \|\varphi\|^6_{H^{1/3}}}$, which is proved as (4.198) above. □

We now have all the tools necessary to prove Theorem 2.2.7.

Proof of Theorem 2.2.7. We deduce (2.76) (with the same subsequence (ε_τ)) from Lemma 4.3.2 by a duality argument based on (4.109)–(4.110) as in the proof of Theorem 2.2.6.

The proof of (2.77) is similar. Let us first note that by (2.75), we have that for fixed $\varepsilon > 0$,

$$\lim_{\tau \to \infty} \mathcal{Z}^\varepsilon_\tau = z^\varepsilon \,. \tag{4.204}$$

The proof of (2.77) proceeds analogously as the proof of (2.76). The only difference is that instead of (4.192) and (4.193), we use (4.204) and (4.202) respectively. □

4.4 The time-dependent problem. Proofs of Theorems 2.2.11 and 2.2.12

4.4.1 Bounded interaction potentials. Proof of Theorem 2.2.11

Let us consider w as in Assumption 2.2.1. To prove Theorem 2.2.11, we note the following analogues of the Schwinger-Dyson expansion results from [30, Sections 3.2 and 3.3].

Lemma 4.4.1 (Quantum Schwinger-Dyson Expansion). *Let $\xi \in \mathcal{L}(\mathfrak{h}^{(p)})$, $\mathcal{K} > 0$, $\varepsilon > 0$, and $t \in \mathbb{R}$ be given. Then there exists $L(\mathcal{K}, \varepsilon, t, \|\xi\|, p) \in \mathbb{N}$, $(e^l)_{l=0}^L$, where $e^l = e^l(x_i, t) \in \mathcal{L}(\mathfrak{h}^{(l)})$, and $\tau_0 = \tau_0(\mathcal{K}, \varepsilon, t, \|\xi\|) > 0$ such that*

$$\left\| \left(\Psi^t_\tau \Theta_\tau(\xi) - \sum_{l=0}^L \Theta_\tau(e^l) \right) \Big|_{\mathfrak{h}^{(\leq \mathcal{K}\tau)}} \right\| < \varepsilon$$

for all $\tau > \tau_0$.

Proof. We use the same proof and notation as [30, Lemma 3.9]. We begin by noting that all of the calculations up to [30, (3.44)] hold since they do not use the explicit form of $\mathcal{W}$. Instead of [29, 3.46], the same argument yields that the norm of $A^t_{\tau,\infty}(\xi)$ is bounded by

$$\frac{|t|^j}{j!}(p+j)^j 2^j \left(\frac{n}{\tau}\right)^{p+j} \|w\|^3_{L^\infty} \|\xi\| \leq \mathrm{e}^p \mathcal{K}^p \left(2\mathrm{e}\mathcal{K}\|w\|^3_{L^\infty}|t|\right)^j \|\xi\|,$$

which differs only from [30, (3.47)] in that the $\|w\|_{L^\infty}$ is cubed. An analogous argument to [30, Lemma 3.9] yields the result. We refer the reader to [30, Lemma 3.9] for the precise definitions and arguments. □

Lemma 4.4.2 (Classical Schwinger-Dyson Expansion)**.** *Let $\xi \in \mathcal{L}(\mathfrak{h}^{(p)})$, $\mathcal{K} > 0$, $\varepsilon > 0$, and $t \in \mathbb{R}$ be given. Then for $L(\mathcal{K}, \varepsilon, t, \|\xi\|, p) \in \mathbb{N}$ and $\tau_0(\mathcal{K}, \varepsilon, t, \|\xi\|) > 0$ chosen possibly larger than in Lemma 4.4.1 and the same choice of $e^l \in \mathcal{L}(\mathfrak{h}^{(l)})$ as in Lemma 4.4.1 we have*

$$\left| \left(\Psi^t \Theta(\xi) - \sum_{l=0}^{L} \Theta(e^l) \right) \chi_{\{\mathcal{N} \leq \mathcal{K}\}} \right| \leq \varepsilon$$

for all $\tau \geq \tau_0$.

Proof. The proof is analogous to the proof of Lemma 4.4.1. □

Proof of Theorem 2.2.11. By the proof of [30, Proposition 2.1], we note that the time-dependent claim follows from the corresponding time-independent claim, provided that we have the results of Lemmas 4.4.1 and 4.4.2 above. The time-independent claim was shown in Theorem 2.2.6. The claim now follows. □

Remark 4.4.3. The analogous reduction of the time-dependent result to the time-dependent result was also used in the proof of Theorem 2.1.11.

4.4.2 Unbounded interaction potentials. Proof of Theorem 2.2.12

In this section, we prove Theorem 2.2.12. Throughout the section, we consider w as in Assumption 2.2.2. Before proceeding with the proof of Theorem 2.2.12, we note an approximation result concerning the approximation of the flow of (2.45).

Let $s \in (0, \frac{1}{2})$ be given. We denote by $\mathcal{G} \subset H^s(\Lambda)$ the set constructed in Proposition 4.1.2 (ii) above. We know $\mathbb{P}^f_{\mathrm{Gibbs}}(\mathcal{G}) = 1$ and initial data in $\mathcal{G}$ gives rise to global solutions of (4.1). Here, we recall (2.55).

Lemma 4.4.4. *Let $w^\varepsilon \in L^\infty(\Lambda)$ be a sequence such that*

$$\lim_{\varepsilon \to 0} \|w^\varepsilon - w\|_{L^{3/2}} = 0\,. \tag{4.205}$$

Furthermore, fix $s \in (0, \frac{1}{2})$, $T > 0$, and consider $\psi \in \mathcal{G}$, with $\mathcal{G} \in H^s(\Lambda)$ as above.

(i) For $\varepsilon > 0$, the following Cauchy problem is well-posed on $[-T, T]$.

$$\begin{cases} \mathrm{i}\partial_t u^\varepsilon + (\Delta - \kappa)u^\varepsilon = \int dy\, dz\, w^\varepsilon(x-y)w^\varepsilon(y-z)w^\varepsilon(z-x)|u^\varepsilon(y)|^2|u^\varepsilon(z)|^2 u^\varepsilon(x) \\ u_0^\varepsilon = \psi\,. \end{cases} \tag{4.206}$$

(ii) We have

$$\lim_{\varepsilon \to 0} \|u^\varepsilon - u\|_{L^\infty_{[-T,T]} H^s} = 0\,, \tag{4.207}$$

where

$$\begin{cases} \mathrm{i}\partial_t u + (\Delta - \kappa)u = \int dy\, dz\, w(x-y)w(y-z)w(z-x)|u(y)|^2|u(z)|^2 u(x) \\ u_0 = \psi\,. \end{cases} \tag{4.208}$$

Proof. As in the proof of Proposition 4.1.1, it suffices to consider the case $\kappa = 0$. By symmetry, it suffices to consider only positive times. Using [9, Lemma 4.4] and [9, (4.10)], we have a more quantitative description of the set $\mathcal{G}$ in Proposition 4.1.2 (ii) above[5]. Namely, given $\nu > 0$, we can write

$$\mathcal{G} = \bigcup_{\eta, A > 0} \mathcal{K}_{\eta, A}\,, \tag{4.209}$$

where

$$\mathcal{K}_{\eta,A} := \left\{ \psi \in H^s(\Lambda) : \|\psi\|_{L^2}^2 \leq K\,, \quad \|u(t)\|_{H^s} \leq A \log\left(\frac{1+|t|}{\eta}\right)^{s+\nu} \right\}. \tag{4.210}$$

In (4.210), the constant K is as in Assumption 2.2.4 and u is the solution of (4.208) with initial data ψ.

By (4.209), we deduce that the claim follows if we show that it holds for $\psi \in \mathcal{K}_{\eta,A}$ with $\eta, A > 0$ fixed. Throughout the proof, we consider ε sufficiently small such that

$$\|w^\varepsilon\|_{L^{3/2}} \leq 2\|w\|_{L^{3/2}}\,. \tag{4.211}$$

For fixed η, A, we let

$$\mathcal{A} := A \log\left(\frac{1+|T|}{\eta}\right)^{s+\nu}. \tag{4.212}$$

[5] Once we have the setup of Section 4.1, we can directly apply these arguments from [9].

We fix $\theta > 0$ small and consider

$$b = \frac{1}{2} + \theta \,. \tag{4.213}$$

With parameters as above, we consider $\delta > 0$ such that

$$\delta^{\theta} \, \|w\|_{L^{3/2}}^{3} \, (\mathcal{A} + 1)^4 \ll 1 \,, \tag{4.214}$$

where the smallness of the right-hand side of (4.214) will be determined later (depending on the other parameters above). For the remainder of the proof, we let

$$n = \lfloor T/\delta \rfloor \,. \tag{4.215}$$

Let us now show that the following properties hold for $k = 0, 1, \ldots, n-1$.

(1) For $\varepsilon > 0$ small, we have $u^{\varepsilon} \in L^{\infty}_{[0,k\delta]} H^s_x$ and

$$\lim_{\varepsilon \to 0} \|u^{\varepsilon}(k\delta) - u(k\delta)\|_{H^s} = 0 \,. \tag{4.216}$$

(2) There exists $\varepsilon_k > 0$ such that for all $\varepsilon \in (0, \varepsilon_k)$, and for a constant $C > 0$ independent of k, we have $u^{\varepsilon} \in X^{s,b}_{[k\delta,(k+1)\delta]}$ and

$$\begin{aligned} \|u^{\varepsilon} - u\|_{X^{s,b}_{[k\delta,(k+1)\delta]}} &\leq C \|u^{\varepsilon}(k\delta) - u(k\delta)\|_{H^s} \\ &\quad + C\delta^{\theta} \, \|w - w^{\varepsilon}\|_{L^{3/2}} \, \|w\|_{L^{3/2}}^{2} \, (\mathcal{A} + 1)^5 \,. \end{aligned} \tag{4.217}$$

We show (4.216)–(4.217) by induction on k.

Base We consider $k = 0$. Note that (4.216) is automatically satisfied since $u^{\varepsilon}(0) = u(0) = \psi$. Using (4.211)–(4.214) and arguing analogously as in the proof of Proposition 4.1.1, we deduce that

$$u^{\varepsilon} \in X^{s,b}_{[0,\delta]} \,, \qquad \|u^{\varepsilon}\|_{X^{s,b}_{[0,\delta]}} \lesssim \mathcal{A} \,. \tag{4.218}$$

Let us note that we also have

$$u \in X^{s,b}_{[0,\delta]} \,, \qquad \|u\|_{X^{s,b}_{[0,\delta]}} \lesssim \mathcal{A} \,, \tag{4.219}$$

by the same argument.

We use Duhamel's principle for the difference equation solved by $u^\varepsilon - u$ and write for $t \in [0,\delta]$

$$
\begin{aligned}
&u^\varepsilon(t) - u(t) = \\
&-\mathrm{i}\int_0^t dt'\,\mathrm{e}^{\mathrm{i}(t-t')\Delta}\bigg(\int dy\,dz\,w^\varepsilon(x-y)\,w^\varepsilon(y-z)\,w^\varepsilon(z-x)|u^\varepsilon(y)|^2|u^\varepsilon(z)|^2u^\varepsilon(x) \\
&\qquad\qquad - \int dy\,dz\,w(x-y)\,w(y-z)\,w(z-x)\,|u(y)|^2\,|u(z)|^2\,u(x)\bigg) \qquad (4.220)
\end{aligned}
$$

By using telescoping, Lemma 1.7.6 (iii), Lemma 4.1.4, and (4.211), we deduce from (4.220) that

$$
\begin{aligned}
\|u^\varepsilon - u\|_{X^{s,b}_{[0,\delta]}} &\leq C\delta^\theta\,\|w^\varepsilon - w\|_{L^{3/2}}\,\|w\|^2_{L^{3/2}}\left(\|u^\varepsilon\|^5_{X^{s,b}_{[0,\delta]}} + \|u\|^5_{X^{s,b}_{[0,\delta]}}\right) \\
&\quad + C\delta^\theta\,\|w\|^3_{L^{3/2}}\,\|u^\varepsilon - u\|_{X^{s,b}_{[0,\delta]}}\left(\|u^\varepsilon\|^4_{X^{s,b}_{[0,\delta]}} + \|u\|^4_{X^{s,b}_{[0,\delta]}}\right). \qquad (4.221)
\end{aligned}
$$

Using (4.218)–(4.219), followed by (4.214) (with sufficiently small right-hand side), we obtain from (4.221) that

$$
\|u^\varepsilon - u\|_{X^{s,b}_{[0,\delta]}} \leq C\delta^\theta\,\|w^\varepsilon - w\|_{L^{3/2}}\,\|w\|^2_{L^{3/2}}\,\mathcal{A}^5\,, \qquad (4.222)
$$

with a different choice of C. We obtain (4.217) from (4.222).

Inductive Step Suppose that (4.216)–(4.217) hold for some $0 \leq k \leq n-2$. Let us observe that by Lemma 1.7.6 (i) and (4.217) for k, we have that for $\varepsilon \in (0,\varepsilon_k)$

$$
\begin{aligned}
\|u^\varepsilon((k+1)\delta) - u((k+1)\delta)\|_{H^s} &\lesssim \|u^\varepsilon - u\|_{X^{s,b}_{[k\delta,(k+1)\delta]}} \\
&\leq C\|u^\varepsilon(k\delta) - u(k\delta)\|_{H^s} + C\delta^\theta\|w^\varepsilon - w\|_{L^{3/2}}\,\|w\|^2_{L^{3/2}}\,(\mathcal{A}+1)^5\,. \qquad (4.223)
\end{aligned}
$$

We deduce

$$
\lim_{\varepsilon\to 0}\|u^\varepsilon((k+1)\delta) - u((k+1)\delta)\|_{H^s} = 0 \qquad (4.224)
$$

from (4.223), combined with (4.216) for k and (4.205). This shows that (4.216) holds for $k+1$.

By (4.224), it follows that there exists $\varepsilon_{k+1} \in (0,\varepsilon_k)$ small enough such that for all $\varepsilon \in (0,\varepsilon_{k+1})$, we have

$$
\|u^\varepsilon((k+1)\delta) - u((k+1)\delta)\|_{H^s} \leq 1\,. \qquad (4.225)
$$

From (4.225), we deduce that for all $\varepsilon \in (0, \varepsilon_{k+1})$, we have

$$\|u^\varepsilon((k+1)\delta)\|_{H^s} \leq \|u((k+1)\delta)\|_{H^s} + 1 \leq \mathcal{A} + 1 \,. \tag{4.226}$$

For (4.226), we recalled 4.210 and (4.212) above. Using (4.226), recalling (4.211), (4.213)–(4.214) and arguing analogously as in the proof of Proposition 4.1.1, we deduce that

$$u^\varepsilon \in X^{s,b}_{[(k+1)\delta,(k+2)\delta]} \,, \qquad \|u^\varepsilon\|_{X^{s,b}_{[(k+1)\delta,(k+2)\delta]}} \lesssim \mathcal{A} + 1 \,. \tag{4.227}$$

As in (4.219) (with the same argument), we have

$$u \in X^{s,b}_{[(k+1)\delta,(k+2)\delta]} \,, \qquad \|u\|_{X^{s,b}_{[(k+1)\delta,(k+2)\delta]}} \lesssim \mathcal{A} \,. \tag{4.228}$$

Similarly as for (4.220), we use Duhamel's principle to write for $t \in [(k+1)\delta, (k+2)\delta]$

$$\begin{aligned} u^\varepsilon(t) - u(t) &= \mathrm{e}^{\mathrm{i}(t-(k+1)\delta)\Delta}\Big(u^\varepsilon((k+1)\delta) - u((k+1)\delta)\Big) \\ &- \mathrm{i} \int_{(k+1)\delta}^t dt' \, \mathrm{e}^{\mathrm{i}(t-t')\Delta}\bigg(\int dy\, dz\, w^\varepsilon(x-y)\, w^\varepsilon(y-z)\, w^\varepsilon(z-x) |u^\varepsilon(y)|^2 |u^\varepsilon(z)|^2 u^\varepsilon(x) \\ &\qquad - \int dy\, dz\, w(x-y)\, w(y-z)\, w(z-x)\, |u(y)|^2\, |u(z)|^2\, u(x)\bigg) \,. \end{aligned} \tag{4.229}$$

Using Lemma 1.7.6 (ii) to estimate the first term on the right-hand side of (4.229), and using telescoping, Lemma 1.7.6 (iii), Lemma 4.1.4, and (4.211) (as for (4.221)) to estimate the second term, we deduce that

$$\begin{aligned} \|u^\varepsilon - u\|_{X^{s,b}_{[(k+1)\delta,(k+2)\delta]}} &\leq C \|u^\varepsilon((k+1)\delta) - u((k+1)\delta)\|_{H^s} \\ &+ C\delta^\theta \, \|w^\varepsilon - w\|_{L^{3/2}} \, \|w\|^2_{L^{3/2}} \left(\|u^\varepsilon\|^5_{X^{s,b}_{[(k+1)\delta,(k+2)\delta]}} + \|u\|^5_{X^{s,b}_{[(k+1)\delta,(k+2)\delta]}}\right) \\ &+ C\delta^\theta \, \|w\|^3_{L^{3/2}} \, \|u^\varepsilon - u\|_{X^{s,b}_{[(k+1)\delta,(k+2)\delta]}} \left(\|u^\varepsilon\|^4_{X^{s,b}_{[(k+1)\delta,(k+2)\delta]}} + \|u\|^4_{X^{s,b}_{[(k+1)\delta,(k+2)\delta]}}\right) . \end{aligned} \tag{4.230}$$

We now use (4.227)–(4.228) and (4.214) (for sufficiently small right-hand side) to deduce that

$$\begin{aligned} \|u^\varepsilon - u\|_{X^{s,b}_{[(k+1)\delta,(k+2)\delta]}} &\leq C \|u^\varepsilon((k+1)\delta) - u((k+1)\delta\|_{H^s} \\ &+ C\delta^\theta \, \|w^\varepsilon - w\|_{L^{3/2}} \, \|w\|^2_{L^{3/2}} \, (\mathcal{A} + 1)^5 \,, \end{aligned} \tag{4.231}$$

for a different choice of C. From (4.231), we conclude the induction.

Using (4.216)–(4.217), Lemma 1.7.6 (i), and (4.205), it follows that

$$\lim_{\varepsilon \to 0} \|u^\varepsilon - u\|_{L^\infty_{[0,n\delta]} H^s_x} = 0\,, \tag{4.232}$$

for n as in (4.215). By using $\|u\|_{L^\infty_{[n\delta,T]} H^s_x} \leq \mathcal{A}$, and by repeating the above argument, we also obtain

$$\lim_{\varepsilon \to 0} \|u^\varepsilon - u\|_{L^\infty_{[n\delta,T]} H^s_x} = 0\,. \tag{4.233}$$

The lemma follows from (4.232)–(4.233). □

Remark 4.4.5. The proof of Lemma 4.4.4 above is more involved than that of the analogous cubic results [30, Proposition 5.1] and Lemma 3.5.4. It requires full use of the set $\mathcal{G}$ of initial data leading to global solutions of (2.45).

Let us recall the following form of the diagonal argument [30, Lemma 5.5].

Lemma 4.4.6 (Diagonal argument)**.** *Let (Γ_k) be a sequence of sets with $\Gamma_k \subset \Gamma_{k+1}$ and let $\Gamma := \cup_k \Gamma_k$. Given $\varepsilon, \tau > 0$, let $g, g^\varepsilon, g^\varepsilon_\tau : \Gamma \to \mathbb{C}$ be functions satisfying the following properties.*

(i) For $k \in \mathbb{N}$ and $\varepsilon > 0$ fixed, we have $\lim_{\tau \to \infty} g^\varepsilon_\tau(\zeta) = g^\varepsilon(\zeta)$, uniformly in $\zeta \in \Gamma_k$.

(ii) For fixed $k \in \mathbb{N}$, we have $\lim_{\varepsilon \to 0} g^\varepsilon(\zeta) = g(\zeta)$, uniformly in $\zeta \in \Gamma_k$.

Then, there exists a sequence (ε_τ) converging to zero as $\tau \to \infty$ such that

$$\lim_{\tau \to \infty} g^{\varepsilon_\tau}_\tau(\zeta) = g(\zeta)\,.$$

We now have all of the necessary tools to prove Theorem 2.2.12.

Proof of Theorem 2.2.12. Once we have Lemma 4.4.4 at our disposal, the proof is quite similar to that of Theorem 2.1.12. We just outline the main differences and refer the reader to Section 3.5.2 for more details. We adopt the convention that the superscript ε denotes an object defined using the interaction potential w^ε. Using Theorem 2.2.11, Lemma 4.4.6, and arguing analogously as in the proof of Theorem 2.1.12, the claim follows if we show that

$$\lim_{\varepsilon \to 0} \tilde{\rho}^\varepsilon_1 \left(\Psi^{t_1,\varepsilon}\Theta(\xi^1) \cdots \Psi^{t_m,\varepsilon}\Theta(\xi^m)\right) = \tilde{\rho}_1 \left(\Psi^{t_1}\Theta(\xi^1) \cdots \Psi^{t_m}\Theta(\xi^m)\right)\,, \tag{4.234}$$

uniformly in Γ_k

$$\Gamma_k := \left\{(m, t_i, p_i, \xi^i) : \, m \leq k, \, |t_i| \leq k, \, p_i \in \mathbb{N}^* \, , p_i \leq k, \, \|\xi^i\| \leq k \, , 1 \leq i \leq m\right\} . \tag{4.235}$$

In (4.235), we take $\xi^i \in \mathcal{L}(\mathfrak{h}^{(p_i)})$. We also recall (2.62), (4.87), and define the quantities with superscript ε accordingly.

Let φ denote the classical free field given by (2.12). Recall we have that for $s \in (0, \frac{1}{2})$, $\varphi \in H^s$ μ-almost surely. Let us recall the definition (2.59) of $\Theta(\xi)$ and the definition (2.50) of the flow map $S_t(\cdot)$ of (2.45). By suitably defining the quantities with superscript ε, we have for $\xi \in \mathcal{L}(\mathfrak{h}^{(k)})$

$$\Psi^{t,\varepsilon}\Theta(\xi) = \left\langle (S_t^\varepsilon \varphi)^{\otimes_k} \, , \xi \, (S_t^\varepsilon \varphi)^{\otimes_k} \right\rangle_{\mathfrak{h}^{\otimes_k}} \, , \quad \Psi^t\Theta(\xi) = \left\langle (S_t\varphi)^{\otimes_k} \, , \xi \, (S_t\varphi)^{\otimes_k} \right\rangle_{\mathfrak{h}^{\otimes_k}} \, . \tag{4.236}$$

When $\xi \in \mathcal{L}(\mathfrak{h}^{(k)})$, Lemma 4.4.4 and (4.236) imply that

$$\lim_{\varepsilon \to 0} \Psi^{t,\varepsilon}\Theta(\xi) = \Psi^t\Theta(\xi) \, , \tag{4.237}$$

μ-almost surely.

Recalling (4.194), and using (4.237), it follows that

$$\lim_{\varepsilon \to 0} \Psi^{t_1,\varepsilon}\Theta(\xi^1) \cdots \Psi^{t_m,\varepsilon}\Theta(\xi^m) \, \mathrm{e}^{-\mathcal{W}^\varepsilon} = \Psi^{t_1}\Theta(\xi^1) \cdots \Psi^{t_m}\Theta(\xi^m) \, \mathrm{e}^{-\mathcal{W}} \, , \tag{4.238}$$

μ-almost surely.

Using Lemma 4.1.3, Lemma 4.2.2, conservation of mass for (4.206) and (4.208), as well as (4.205), we have the following bounds for $\varepsilon > 0$ sufficiently small.

$$\begin{aligned} \left|\Psi^{t_1,\varepsilon}\Theta(\xi^1) \cdots \Psi^{t_m,\varepsilon}\Theta(\xi^m) \mathrm{e}^{-\mathcal{W}^\varepsilon} f(\mathcal{N})\right| &\leq \left(\prod_{j=1}^m \|\xi^j\| \|\varphi\|_{\mathfrak{h}}^{2p_j}\right) \mathrm{e}^{c\|w\|_{L^{3/2}}^3 \|\varphi\|_{H^{1/3}}^3} f(\mathcal{N}) \\ \left|\Psi^{t_1}\Theta(\xi^1) \ldots \Psi^{t_m}\Theta(\xi^m) e^{-\mathcal{W}} f(\mathcal{N})\right| &\leq \left(\prod_{j=1}^m \|\xi^j\| \|\varphi\|_{\mathfrak{h}}^{2p_j}\right) \mathrm{e}^{c\|w\|_{L^{3/2}}^3 \|\varphi\|_{H^{1/3}}^3} f(\mathcal{N}) \, . \end{aligned} \tag{4.239}$$

The claim now follows from (4.238)–(4.239), by using Proposition 4.1.2 (i), Assumption 2.2.4 (with K sufficiently small) and the dominated convergence theorem.

□

Appendix A

Outline of the Construction of the Gibbs Measure for the Cubic NLS

In this section we recall the construction of the Gibbs measure for the cubic NLS in one dimension in more detail, following [9]. We recall from (1.5) that for non-negative interaction potential w, we heuristically define

$$d\mathbb{P}_{\text{Gibbs}} = \frac{1}{z_{\text{Gibbs}}} \mathrm{e}^{-H(u)} du, \tag{A.1}$$

where $z_{\text{Gibbs}} \equiv z$ is a normalisation constant, H is the Hamiltonian and du is the formally defined infinite dimensional Lebesgue measure on the space of fields. We begin with some formal arguments to establish why we expect the measure to be invariant, and to establish the support of the measure. To rigorously construct the Gibbs measure, we need the following.

1. A way of realising the infinite Lebesgue measure as a rigorously defined infinite dimensional measure times a weight function.
2. To show that the weight function mentioned above is an L^1 function with respect to the infinite dimensional measure, allowing us to define z.
3. To establish the invariance of the measure.

To give a brief outline of how we establish 1. – 3., we will write the Gibbs measure as a weighted Wiener measure. In the case of a defocusing potential, we can use the Sobolev embedding theorem to show that z is well defined, and we use a Galerkin like approximation to establish the invariance of the measure. We also include the arguments for a focusing potential, which requires us to truncate the Gibbs measure defined in (A.1).

A.1 Formal arguments

In this section, we consider the defocusing cubic NLS given by

$$\begin{cases} i\partial_t u + (\Delta - \kappa)u = |u|^2 u \\ u(x,0) = u_0(x), \end{cases} \tag{A.2}$$

which has associated Hamiltonian

$$H(u) = \int dx\, |\nabla u(x)|^2 + \kappa |u(x)|^2 + \frac{1}{2}\int dx\, |u(x)|^4.$$

We can write (A.2) as an infinite dimensional Hamiltonian system. Namely

$$u_t = -i\frac{\partial H}{\partial \bar{u}}, \tag{A.3}$$

Thus, drawing analogy with the finite dimensional system, Liouville's theorem means we would formally expect the measure $d\mathbb{P}_{\text{Gibbs}}$ to be invariant under the flow of (A.2).

We rewrite

$$d\mathbb{P}_{\text{Gibbs}} = \mathrm{e}^{-\int dx\, |u|^4} d\nu,$$

where $d\nu$ is the unormalised Wiener measure given by

$$d\nu := \mathrm{e}^{-\int dx\, |\nabla u|^2} du.$$

We define the normalised Wiener measure by

$$d\mu := \frac{1}{Z_{\text{Wiener}}} \mathrm{e}^{-\int dx\, |\nabla u|^2} du,$$

where Z_{Wiener} is the normalisation constant associated with the Wiener measure. We note that in one dimension, the Wiener measure is equivalent to Brownian motion, and in higher dimensions it is a Gaussian Free Field (GFF). For a review of the GFF, we direct the reader to [74]. We wish to understand, heuristically, what functions lie in the support of the Wiener measure. To do this, we write $a_k := \hat{u}(k)$, and use Plancherel's theorem to write

$$\int_{\mathbb{T}^d} |\nabla u|^2 \, dx = c \sum_{k \in \mathbb{Z}^d} |k|^2 |a_k|^2.$$

Putting $\kappa = 0$, we have

$$\begin{aligned} d\mu &= \frac{\exp\left(-c \sum_{k\in\mathbb{Z}^d} |k|^2 |a_k|^2\right) \prod_{k\in\mathbb{Z}^d} da_k}{\int \exp\left(-c \sum_{k\in\mathbb{Z}^d} |k|^2 |a_k|^2\right) \prod_{k\in\mathbb{Z}^d} da_k} \\ &= \prod_{k\in\mathbb{Z}^d} \frac{\exp\left(-c|k|^2 |a_k|^2\right) da_k}{\int \exp\left(-c|k|^2|a_k|^2\right) da_k}. \end{aligned}$$

So we get a Gaussian distribution for

$$|k| a_k = |k| \hat{u}(k).$$

Thus we can think of each Fourier coefficient as a random variable, $\hat{u}(k) = g_k(\omega)/|k|$. So each φ in the support of $d\rho$ has the random Fourier series

$$\varphi(x) = \varphi^\omega(x) = \frac{1}{2\pi} \sum_{k\in\mathbb{Z}^d} \frac{\omega_k}{|k|} \mathrm{e}^{2\pi i \langle k, x\rangle}, \tag{A.4}$$

where (ω_k) are suitable i.i.d. complex Gaussian random variables. To get around the problem at the zero mode in (A.4), we take $\kappa > 0$. Repeating the computation for a non-zero value of κ, we find that a typical element in the support of μ is given by the random Fourier series

$$\varphi(x) = \varphi^\omega(x) = \sum_{k\in\mathbb{Z}^d} \frac{\omega_k}{\sqrt{\lambda_k}} \mathrm{e}^{2\pi i \langle k, x\rangle}, \tag{A.5}$$

where we recall $\lambda_k := 4\pi^2 |k|^2 + \kappa$ are the eigenvalues of the one-body Hamiltonian $h = -\Delta + \kappa$. Thus we recover (2.12). By Wick's theorem, we calculate that

$$\mathbb{E}_\mu \left[\|\varphi\|^2_{H^s(\mathbb{T}^d)} \right] = \sum_{k\in\mathbb{Z}^d} \frac{\mathbb{E}[|\omega_k|^2]}{\lambda_k} (1+|k|^2)^s \sim \sum_{k\in\mathbb{Z}^d} |k|^{2s-2}.$$

Comparing with the harmonic series, we find that this is summable if and only if $s < 1 - \frac{d}{2}$. So it follows that $\varphi \in H^s(\mathbb{T}^d)$ almost surely for $s < 1 - d/2$. So $d\mu$ is a probability measure on $H^s(\mathbb{T}^d)$ for any such s, and $d\mu(H^s(\mathbb{T}^d)) = 0$ for any $s \geq 1 - d/2$. For more details, we direct the reader to the exposition in [49, Section 3.1] and the classical texts [46, 77].

We now fix $d = 1$ in (A.2) and continue to consider the defocusing case. Then the Sobolev embedding theorem implies that $\frac{2}{p}\int |u|^p \, dx$ is finite almost surely for $u = \varphi^\omega$. Therefore, $\mathrm{e}^{-\frac{2}{p}\int |u|^p \, dx}$ is in $L^1(d\omega)$. Similarly, for a pointwise non-negative $V \in L^1(\mathbb{T})$, the Sobolev embedding theorem and Young's inequality imply that $\mathrm{e}^{-\frac{1}{2}\int \left(V * |u|^2\right)|u|^2 \, dx}$ is in $L^1(d\omega)$. So, if we can make our arguments rigorous, then using the Radon-Nikodym theorem, we will be able to define the Gibbs measure for the defocusing NLS and defocusing Hartree equation using the Wiener measure. By normalising the Gibbs measure, we can show it is an invariant probability measure on $H^s(\mathbb{T})$ for $s < 1/2$. The focusing case and corresponding Hartree equation require more work to show the density functions are in L^1, and are considered later in the appendix.

A.2 Rigorous arguments

We henceforth fix $d = 1$. We want to make our arguments in the previous section rigorous. To do this, we will consider the truncated NLS given by

$$\begin{cases} iv_t^N + \Delta v^N = \pm P_N \left(|v^N|^{p-2} v^N\right) \\ v_0^N = P_N \varphi. \end{cases} \tag{A.6}$$

We note that for $\varphi \in H^s(\mathbb{T})$, $P_N\varphi \in H^\infty(\mathbb{T})$, so (A.6) is locally well-posed for $p \in [4, 6]$. We then define the Wiener and Gibbs measures associated with (A.6) as truncations of what we had previously, so

$$d\nu_N := \prod_{|k| \leq N} \exp\left(-c|k|^2|a_k|^2\right) da_k,$$

$$d\mu_N := \prod_{|k| \leq N} \frac{\exp\left(-c|k|^2|a_k|^2\right) \, da_k}{\int \exp\left(-c|k|^2|a_k|^2\right) \, da_k}$$

$$d\mathbb{P}_{\mathrm{Gibbs},N} := \frac{\exp\left(\pm\frac{1}{p}\int_{\mathbb{T}} |P_N(u)|^p \, dx\right) d\nu_N}{\int \exp\left(\pm\frac{1}{p}\int_{\mathbb{T}} |P_N(u)|^p \, dx\right) d\nu_N}.$$

We note that all of the previously formal arguments are rigorous in the case of $d\mu_N$, since we are working in a finite dimensional space ($\mathbb{C}^{2N+1}$). In particular, $d\mu_N$ is invariant under the flow of (A.6).

For U an open subset of $H^{\frac{1}{2}-}(\mathbb{T})$, we define the Wiener and Gibbs measures of (A.2) by:

$$
\begin{aligned}
d\rho(U) &:= \lim_{N\to\infty} d\rho_N(U \cap E_N),\\
d\mu(U) &:= \lim_{N\to\infty} d\mu_N(U \cap E_N),
\end{aligned}
\tag{A.7}
$$

where $E_N := \mathrm{Span}_{\mathbb{C}}\left\{\mathrm{e}^{2\pi ikx} \,:\, |k| \leq N\right\}$. It was shown by Zhidkov in [89] definition (A.7) of the Gibbs measure is equivalent to the original invariant measure shown to exist by Lebowitz, Rose, and Speer in [47]. We want to show that this measure is invariant under the flow of the nonlinear Schrödinger equation (A.2). A key ingredient for this is the approximation result given by Lemma A.2.1, stated in the next section, which relates the solutions of (A.6) to the solutions to (A.2).

Approximation lemma

Lemma A.2.1. *Let $s > 0$ and suppose that $\varphi \in H^s(\mathbb{T})$, $\|\varphi\|_{H^s} \leq A$, and that $N \in \mathbb{N}$. Suppose that the solution of*

$$
\begin{cases}
iv_t^N + \Delta v = \pm P_N(|v^N|^2 v^N)\\
v_0^N(x) = P_N\varphi(x)
\end{cases}
$$

satisfies $\|v^N(t)\|_s \leq A$ for all $t \in [0,T]$. Then for u satisfying

$$
\begin{cases}
iu_t + \Delta u = \pm|u|^2 u\\
u_0(x) = \varphi(x),
\end{cases}
$$

for $t \in [0,T]$, if $s_1 \in (0,s)$, we have the approximation

$$
\|u(t) - v^N(t)\|_{H^{s_1}} \leq \exp\left(C(A+1)^{C_1}T\right) N^{s_1-s}
$$

provided that the quantity on the right hand side of (A.2.1) *remains less than* 1.

Lemma (A.2.1) is proved in [9, Lemma 2.27] and is similar to the proof of the approximation result for the quintic case proved Section 4.1.2. We do not include the details of the remainder of the construction of the measure for the defocusing case, which can be found in [9].

A.3 Focusing case

We now focus on the case of a non-positive nonlinearity. In particular, we recall the proof of Lemma 3.1.1, which was proven in [9, Lemma 3.10]. For the convenience of the reader, we present the full details of the proof in a self-contained way. For an alternative summary, see also [65, Section 2]. Before proceeding with the proof, we recall in Section A.4 several auxiliary results concerning Fourier multipliers in the periodic setting and concentration inequalities. In Section A.5, we recall the notion of a *norming set*, which we use to prove duality results in L^p spaces. The proof of Lemma 3.1.1 is given in Section A.6.

A.4 Auxiliary results

We recall the definition of a sub-gaussian random variable.

Definition A.4.1. Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space. We say a random variable X is sub-gaussian if there exist constants $C, v > 0$ such that for all $t > 0$ we have

$$\mathbb{P}(|X| > t) \leq C\mathrm{e}^{-vt^2}.$$

We will use the following inequality about sub-gaussian random variables. For a proof, see [86, Proposition 5.10].

Lemma A.4.2 (Hoeffding's Inequality)**.** *Suppose that $X_1, \ldots, X_N$ are all independent, centred sub-gaussian random variables. Let $Q := \max_i \|X_i\|_{\psi_2}$ for*

$$\|X\|_{\psi_2} := \sup_{p \geq 1} p^{-1/2} (\mathbb{E}|X|^p)^{1/p}$$

and let $a \in \mathbb{R}^N$. Then, for any $t > 0$, we have

$$\mathbb{P}\left[\left|\sum_{i=1}^{N} a_i X_i\right| > t\right] \lesssim \exp\left(-\frac{ct^2}{Q^2\|a\|_{\ell^2}^2}\right).$$

We will also need the Riesz-Thorin interpolation theorem, which is proved in [81, V].

Theorem A.4.3 (Riesz-Thorin theorem)**.** *Let (X, μ) and (Y, ν) be σ-finite measure spaces. Suppose that $p_0, p_1, q_0, q_1 \in [1, \infty]$ and suppose that $T : L^{p_j}(X, \mu) \to L^{q_j}(Y, \nu)$ is a bounded linear operator with norm K_j for $j = 0, 1$. Then $T : L^{p_\theta}(X, \mu) \to L^{q_\theta}(Y, \nu)$ is a bounded linear map with norm $K_\theta \leq K_0^{1-\theta} K_1^{\theta}$ for all*

$\theta \in [0,1]$. *Here*

$$\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1},$$
$$\frac{1}{q_\theta} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$

A.5 Norming sets

To prove Lemma 3.1.1, we need the following result about duality in L^p spaces. We emphasise that this is a known result, but whose proof we could not find in the literature, so we write out the proof for the convenience of the reader.

Lemma A.5.1. *Suppose that $\mathcal{M} \subset \mathbb{Z}$ has cardinality m, and let*

$$S := \mathrm{Span}_{\mathbb{C}} \left\{ \mathrm{e}^{2\pi i k x} : k \in \mathcal{M} \right\}.$$

Then there is some subset Ξ of the unit sphere of $L^{p'}$ satisfying the following properties.

1. $\max_{\varphi \in \Xi} |\langle g, \varphi \rangle| \geq \frac{1}{2} \|g\|_{L^p}$ *for all $g \in S$.*
2. $\log |\Xi| \leq Cm$ *for some universal constant $C > 0$.*

Remark. This result can be extended to finite dimensional subsets of normed vector spaces, but we do not need the result in full generality.

A.5.1 Norming sets and ε-nets

Before proceeding, we introduce several notions in Banach spaces.

Definition A.5.2. Let X be a Banach space, $Y \subset X$ a linear subspace, and $\theta \in (0,1]$. We denote by X^* the (continuous) dual space of X. We say that a set $F \subset X^*$ is *θ-norming over Y* if

$$\sup_{g \in F \setminus \{0\}} \frac{g(y)}{\|g\|} \geq \theta \|y\|$$

for all $y \in Y$.

Definition A.5.3. Let X be a Banach space. Given $x \in X$ and $\varepsilon > 0$, we write $B_\varepsilon(x) = \{y \in X : \|x - y\| < \varepsilon\}$ for the ball in X of radius ε around x. Let $Y \subset X$ be a subset of X. Given $\varepsilon > 0$, we call $N_\varepsilon \subset Y$ an *ε-net of Y* if $Y \subset \bigcup_{x \in N_\varepsilon} B_\varepsilon(x)$.

We write $S_X := \{x \in X : \|x\| = 1\}$ for the unit sphere of X.

We want to relate norming sets to ε-nets. To do this, we take inspiration from the following result, the proof of which comes from [42, Section 17.2.4, Theorem 1].

Lemma A.5.4. *Suppose X is a Banach space, $Y \subset X$ is a linear subspace, and $G \subset S_{X^*}$ a set that is 1-norming over Y. Let $\varepsilon \in (0,1)$, and suppose that N_ε is an ε-net on the unit sphere of Y. For each element $x \in N_\varepsilon$, fix a functional $g_x \in G$ such that $g_x(x) > 1-\varepsilon$ (which we can do since $N_\varepsilon \subset S_Y$ and G is 1-norming over Y). Then the set $F = \{g_x\}_{x\in N_\varepsilon}$ is θ-norming over S_Y for $\theta = 1-2\varepsilon$.*

Proof. Let $y \in S_Y$. By definition, there is some $x_y \in N_\varepsilon$ satisfying $\|y - x_y\| < \varepsilon$. Then, by definition of F, linearity, the definition of x_y, and $G \subset S_{X^*}$, we have

$$\begin{aligned}\sup_{g\in F} |g(y)| = \sup_{x\in N_\varepsilon} |g_x(y)| \geq |g_{x_y}(y)| &= |g_{x_y}(x_y) - g_{x_y}(y - x_y)| \\ &> 1 - \varepsilon - \|y - x_y\| > 1 - 2\varepsilon = \theta\,.\end{aligned}$$

□

A.5.2 Conclusion of the proof of Lemma A.5.1

We begin by bounding the size of an ε-net of $\mathbb{C}^m$. For $\mathcal{M} \subset \mathbb{Z}$ with $|\mathcal{M}| = m$, we consider the following norm on $\mathbb{C}^m$

$$|||(a)_{k\in\mathcal{M}}||| := \left\| \sum_{k\in\mathcal{M}} a_k \mathrm{e}^{2\pi i k x} \right\|_{L^{p'}} .$$

We define $\Sigma := \{(a_k)_{k\in\mathcal{M}} : |||a||| = 1\}$. Notice that since $\mathbb{C}^m$ is finite dimensional, the unit ball with respect to any norm is compact. Let N_ε be a maximal subset of Σ satisfying the property

$$x, y \in N_\varepsilon, x \neq y \implies |||x - y||| > \varepsilon\,. \tag{A.8}$$

In other words, any subset of Σ strictly containing N_ε fails to have property (A.8). Such a set exists and is finite by the compactness of Σ. Any such set must be an ε-net of Σ by maximality. We have the following bound, whose proof is an adaptation of [86, Lemma 5.2].

Lemma A.5.5. *For $N_\varepsilon \subset \Sigma$ maximal satisfying* (A.8)*, we have*

$$|N_\varepsilon| \leq \left(1 + \frac{2}{\varepsilon}\right)^m =: C_\varepsilon^m\,.$$

Proof. The result follows from a volume bound. Since N_ε is ε-separated, it follows that $\{B_{\varepsilon/2}(x)\}_{x\in N_\varepsilon}$ are pairwise disjoint. Moreover, since $x \in \Sigma$, it follows from the triangle inequality that all such balls lie inside the ball of radius $1+\varepsilon/2$ centred at the origin. So

$$\mathrm{vol}[B_{\varepsilon/2}(x)] \cdot |N_\varepsilon| \leq \mathrm{vol}[B_{1+\varepsilon/2}(0)]. \tag{A.9}$$

We also have the following identity

$$\begin{aligned}\mathrm{vol}\big[cB_1(0)\big] &= \mathrm{vol}\left[\left\{(ca_k)_{k\in\mathcal{M}} : \Big\|\sum_{k\in\mathcal{M}} a_k \mathrm{e}^{2\pi ikx}\Big\|_{L^{p'}} \leq 1\right\}\right] \\ &= c^m\,\mathrm{vol}[B_1(0)]\,.\end{aligned}$$

Combining this with (A.9) (and using translation invariance), we have

$$|N_\varepsilon| \leq \left(\frac{1+\varepsilon/2}{\varepsilon/2}\right)^m = \left(1+\frac{2}{\varepsilon}\right)^m.$$

□

We are now able to prove Lemma A.5.1.

Proof of Lemma A.5.1. We let $\Xi := N_{1/4} \subset \Sigma$ be obtained by setting $\varepsilon = \frac{1}{4}$ in the construction above. Let $g = \sum_{k\in\mathcal{M}} a_k \mathrm{e}^{2\pi ikx}$. By duality, there is some $\psi \in \Sigma$ with $|\langle g,\psi\rangle| \geq \frac{3}{4}\|g\|_{L^p}$. Moreover, since $N_{1/4}$ is a $\frac{1}{4}$ net of Σ, we can find $\varphi \in N_{1/4} \equiv \Xi$ with $\|\psi-\varphi\|_{L^{p'}} \leq \frac{1}{4}$. Hence, it follows that $|\langle g,\psi-\varphi\rangle| \leq \frac{1}{2}\|g\|_{L^p}$. Therefore, we obtain

$$\frac{1}{2}\|g\|_{L^p} \leq |\langle g,\psi\rangle| - |\langle g,\psi-\varphi\rangle| \leq |\langle g,\psi\rangle - \langle g,\psi-\varphi\rangle| = |\langle g,\varphi\rangle|\,, \tag{A.10}$$

where in the second step above, we used the reverse triangle inequality. The result follows from (A.10) and Lemma A.5.5. □

A.6 Proof of Lemma 3.1.1

We now prove Lemma 3.1.1, which was originally proved in [9, Lemma 3.10]. For the convenience of the reader, we present the full details of the proof in a self-contained way. Throughout, $(\mathbb{C}^{\mathbb{N}},\mathcal{G},\mu)$ is the probability space defined in (2.11) above.

Proof of Lemma 3.1.1. We show the following bound for large λ.

$$\mu\left[\left\|\sum_{k\in\mathbb{Z}}\frac{\omega_k}{\sqrt{\lambda_k}}\mathrm{e}^{2\pi ikx}\right\|_{L^p}>\lambda,\left(\sum_{k\in\mathbb{Z}}\frac{|\omega_k|^2}{\lambda_k}\right)^{1/2}\le B\right]\lesssim\exp(-cM_0^{1+2/p}\lambda^2),\quad\text{(A.11)}$$

where

$$M_0\sim\left(\frac{\lambda}{B}\right)^{\frac{1}{1/2-1/p}}.\quad\text{(A.12)}$$

Let us assume (A.11) and we show that it implies the claim. We write

$$F:=\left\|\sum_{k\in\mathbb{Z}}\frac{\omega_k}{\sqrt{\lambda_k}}\mathrm{e}^{2\pi ikx}\right\|_{L^p}^p\chi_{\left(\sum_{k\in\mathbb{Z}}\frac{|\omega_k|^2}{\lambda_k}\right)^{1/2}\le B}$$
$$G:=\mathrm{e}^{\frac{2}{p}\|\sum_{k\in\mathbb{Z}}\frac{\omega_k}{\sqrt{\lambda_k}}\mathrm{e}^{2\pi ikx}\|_{L^p}^p}\chi_{\left(\sum_{k\in\mathbb{Z}}\frac{|\omega_k|^2}{\lambda_k}\right)^{1/2}\le B}.$$

Then

$$\begin{aligned}\|G\|_{L^1}&=\int_{y>0}dy\,y\,\mu\left(|G|>y\right)\\&\le\int_{y>1}dy\,y\,\mu\left(|G|>y\right)+1,\end{aligned}$$

where the inequality follows because μ is a probability measure. Now defining $y:=\exp\left(\frac{2}{p}\lambda^p\right)$, we have

$$\begin{aligned}\|G\|_{L^1}&\le\int_{\lambda>0}d\lambda\,2\lambda^{p-1}\mathrm{e}^{\frac{2}{p}\lambda^p}\mu\left(|F|>\lambda\right)+1\\&\lesssim\int_{\lambda>0}d\lambda\,\exp\left(\frac{2}{p}\lambda^p-cB^{\frac{-2p+4}{p-2}}\lambda^{\frac{4p}{p-2}}\right)\lambda^{p-1}+1.\end{aligned}\quad\text{(A.13)}$$

Since $p<\frac{4p}{p-2}$ for $p\in[4,6)$ for $\|G\|_{L^1}$ to be finite, B can be arbitrary. We have $p=\frac{4p}{p-2}$ for $p=6$, so in this case we have to take B sufficiently small.

We now prove (A.11). Throughout, M is a dyadic integer and $|k|\sim M$ means $\frac{3M}{4}\le|k|<\frac{3M}{2}$. We make use of the following inequality.

$$\left\|\sum_{|k|\sim M}a_k\mathrm{e}^{2\pi ikx}\right\|_{L^p}\lesssim M^{1/2-1/p}\left\|\sum_{|k|\sim M}a_k\mathrm{e}^{2\pi ikx}\right\|_{L^2}.\quad\text{(A.14)}$$

For $p=2$, (A.14) is trivial, and for $p=\infty$, it follows from Cauchy-Schwarz and Plancherel's theorem. We then use the Riesz-Thorin interpolation theorem to deduce

(A.14) for all $p \in (2, \infty)$.

With M_0 as in (A.12), we consider a sequence $(\sigma_M)_{M>M_0}$ of positive numbers with

$$\sum_{M>M_0} \sigma_M = \delta \,, \tag{A.15}$$

with $\delta > 0$ sufficiently small to be determined later.

Consider $\omega \in \Omega$ such that

$$\Big\| \sum_{k\in\mathbb{Z}} \frac{\omega_k}{\sqrt{\lambda_k}} \mathrm{e}^{2\pi i k x} \Big\|_{L^p} > \lambda, \quad \Big(\sum_{k\in\mathbb{Z}} \frac{|\omega_k|^2}{\lambda_k} \Big) \leq B \,. \tag{A.16}$$

With ω as in (A.16), we show that there is some $M > M_0$ such that

$$\Big\| \sum_{|k|\sim M} \omega_k \mathrm{e}^{2\pi i k x} \Big\|_{L^p} > \sigma_M M \lambda \,. \tag{A.17}$$

We argue by contradiction. First, we note that for ω as in (A.16), we have

$$\sum_{M\leq M_0} \Big\| \sum_{|k|\sim M} \frac{\omega_k}{\sqrt{\lambda_k}} \mathrm{e}^{2\pi i k x} \Big\|_{L^p} \lesssim M_0^{1/2-1/p} \Big(\sum_{k\in\mathbb{Z}} \frac{|\omega_k|^2}{\lambda_k} \Big)^{1/2} \lesssim \lambda \,. \tag{A.18}$$

We used (A.14), Plancherel's theorem, and summed a geometric sequence for the first inequality in (A.18). For the second inequality in (A.18), we used the L^2 bound in (A.16), and (A.12). By taking the implied constant in (A.12) to be sufficiently small, let us note that the proof of (A.18) implies

$$\sum_{M\leq M_0} \Big\| \sum_{|k|\sim M} \frac{\omega_k}{\sqrt{\lambda_k}} \mathrm{e}^{2\pi i k x} \Big\|_{L^p} \leq \frac{\lambda}{2}. \tag{A.19}$$

We henceforth work with such a small implied constant in (A.12).

Suppose that (A.17) did not hold for any $M > M_0$. Then it would follow that, for an appropriate choice of δ in (A.15), we would have

$$\sum_{M>M_0} \Big\| \sum_{|k|\sim M} \frac{\omega_k}{\sqrt{\lambda_k}} \mathrm{e}^{2\pi i k x} \Big\|_{L^p} \leq \frac{\lambda}{2} \,. \tag{A.20}$$

We note that (A.20) combined with (A.18) would give us a contradiction with the first inequality in (A.16). Let us explain how we have obtained (A.20). First we note

$$\left\| \sum_{|k|\sim M} \frac{\omega_k}{\sqrt{\lambda_k}} \mathrm{e}^{2\pi i k x} \right\|_{L^p} \leq \frac{C}{M} \left\| \sum_{|k|\sim M} \omega_k \mathrm{e}^{2\pi i k x} \right\|_{L^p}, \tag{A.21}$$

which we justify as follows. Let $\Phi : \mathbb{R} \to \mathbb{C}$ be a smooth, compactly supported function which is equal to 1 on $1/2 \leq |\xi| \leq 2$ and zero for $|\xi| \leq 1/4$ and $|\xi| \geq 4$. Consider the function

$$\Psi_M(\xi) := \frac{1}{\sqrt{\frac{4\pi^2|\xi|^2}{M^2} + \frac{\kappa}{M^2}}} \Phi\left(\frac{\xi}{M}\right). \tag{A.22}$$

Since

$$\tilde{\Phi}(\xi) := \frac{1}{\sqrt{4\pi^2|\xi|^2 + (\kappa/M^2)}} \Phi(\xi)$$

has bounded derivatives of all order (with bound depending on κ), the same holds for $\Psi_M = \tilde{\Phi}(\xi/M)$ given by (A.22) above. Hence, the Mikhlin multiplier theorem (on $\mathbb{R}$) implies that the map T_M defined by $(T_M f)\hat{}(\xi) := \Psi_M(\xi)\hat{f}(\xi)$ is bounded as a map on $L^p(\mathbb{R})$. Applying the support properties of Φ and using Lemma 1.7.2, we obtain

$$\begin{aligned} M \left\| \sum_{|k|\sim M} \frac{\omega_k}{\sqrt{\lambda_k}} \mathrm{e}^{2\pi i k x} \right\|_{L^p} &= M \left\| \sum_{|k|\sim M} \Phi_M(k) \frac{\omega_k}{\sqrt{\lambda_k}} \mathrm{e}^{2\pi i k x} \right\|_{L^p} \\ &= \left\| \sum_{|k|\sim M} \Psi_M(k) \omega_k \mathrm{e}^{2\pi i k x} \right\|_{L^p} \\ &\lesssim \left\| \sum_{|k|\sim M} \omega_k \mathrm{e}^{2\pi i k x} \right\|_{L^p}. \end{aligned}$$

Here we use the fact that the Fourier coefficients are supported on $|k| \sim M$, for which $\Phi_M(k) = 1$. We hence deduce (A.21). By summing in $M > M_0$ and applying (A.15) with δ sufficiently small, we obtain (A.20). Therefore (A.17) holds for some $M > M_0$.

To estimate the contribution for each dyadic M, we consider the subspace S of L^p given by $\mathrm{Span}_{\mathbb{C}}\{e^{2\pi i k x} : |k| \sim M\}$. We want to construct a $\frac{1}{2}$-norming set, Ξ, contained in the unit sphere of $L^{p'}$ with the following properties.

1. $\max_{\varphi \in \Xi} |\langle g, \varphi \rangle| \geq \frac{1}{2} \|g\|_{L^p}$ for all $g \in S$.

2. $\|\varphi\|_{L^2} \lesssim M^{1/2-1/p}$ for any $\varphi \in \Xi$.

3. $\log|\Xi| \lesssim M$.

To find this set, we apply Lemma A.5.1 and take the orthogonal projection of Ξ onto S. We obtain the first and third properties from Lemma A.5.1, and the second follows from Plancherel's theorem, Hölder's inequality, and the Hausdorff-Young inequality (applied to p'). Namely, for $\varphi \in \Xi$, we have

$$\begin{aligned}\|\varphi\|_{L^2} = \|\hat{\varphi}\|_{\ell^2} &\lesssim M^{1/2-1/p}\|\hat{\varphi}\|_{\ell^p} \\ &\lesssim M^{1/2-1/p}\|\varphi\|_{L^{p'}} = M^{1/2-1/p}\,.\end{aligned}$$

Having constructed the set, we now estimate the norm. We choose $M > M_0$ satisfying (A.17). Then

$$\begin{aligned}\sigma_M M\lambda < \left\| \sum_{|k|\sim M} \omega_k e^{2\pi ikx} \right\|_{L^p} &\leq 2\max_{\varphi\in\Xi} \left| \sum_{|k|\sim M} \omega_k \hat{\varphi}(k) \right| \\ &= 2\max_{\varphi\in\Xi} \left| \sum_{|k|\sim M} \omega_k \frac{\hat{\varphi}(k)}{\|\varphi\|_{L^2}} \right| \|\varphi\|_{L^2} \\ &\lesssim 2M^{1/2-1/p} \max_{\varphi\in\Xi} \left| \sum_{|k|\sim M} \omega_k \frac{\hat{\varphi}(k)}{\|\varphi\|_{L^2}} \right| ,\end{aligned}$$

where the first line uses property (1) of Ξ and the final inequality follows from property (2) of Ξ. So

$$\sigma_M M^{1/2+1/p}\lambda \lesssim \max_{\varphi\in\Xi} \left| \sum_{|k|\sim M} \omega_k \frac{\hat{\varphi}(k)}{\|\varphi\|_{L^2}} \right| . \tag{A.23}$$

Let us take $(\sigma_M)_{M>M_0}$ satisfying (A.15) to be of the form

$$\sigma_M \sim M^{-1/p} + (M_0/M)^{1/2}\,, \tag{A.24}$$

for a suitable choice of implied constant. For $M > M_0$, let X_M denote the event

(A.23). Then

$$\begin{aligned}
&\mathbb{P}_\omega\left[\left\|\sum_{k\in\mathbb{Z}}\frac{\omega_k}{\sqrt{\lambda_k}}e^{2\pi ikx}\right\|_{L^p}>\lambda, \left(\sum_{k\in\mathbb{Z}}\frac{|\omega_k|^2}{\lambda_k}\right)^{1/2}\leq B\right] \\
&\leq \mathbb{P}_\omega\left(\cup_{M>M_0}X_M\right) && \text{(A.25)}\\
&\leq \sum_{M>M_0}\sum_{\varphi\in\Xi}\mathbb{P}_\omega\left[\left|\sum_{|k|\sim M}\omega_k\frac{\hat{\varphi}(k)}{\|\varphi\|_{L^2}}\right|\gtrsim \sigma_M M^{1/2+1/p}\lambda\right] && \text{(A.26)}\\
&\lesssim \sum_{M>M_0}\sum_{\varphi\in\Xi}\exp\left(-cM^{1+2/p}\sigma_M^2\lambda^2\right) && \text{(A.27)}\\
&\lesssim \sum_{M>M_0}\exp\left(CM-cM^{1+2/p}\sigma_M^2\lambda^2\right) && \text{(A.28)}\\
&= \sum_{M>M_0}\exp\left(CM-c\left(M+M_0M^{2/p}+2M_0^{1/2}M^{1/2+1/p}\right)\lambda^2\right) && \text{(A.29)}\\
&\lesssim \sum_{M>M_0}\exp\left(-cM_0M^{2/p}\lambda^2\right) && \text{(A.30)}\\
&\lesssim \exp\left(-cM_0^{1+2/p}\lambda^2\right).
\end{aligned}$$

Here, (A.25) follows from (A.23). (A.26) follows from a union bound. (A.27) comes from applying Lemma A.4.2 with $X_i = \omega_i$ and $a_i = \hat{\varphi}(i)/\|\varphi\|_{L^2}$ (so that $Q \sim 1$ and $\|a\|_{\ell^2} \leq 1$ by Plancherel's theorem), and for (A.28), we use property (3) of Ξ. (A.29) comes from (A.24). We obtain (A.30) from the fact that λ is large and noticing that the second term will give a factor less than one. The final inequality follows from the fact we have a geometric series with common ratio equal to $1 - \zeta_{M_0}$, with $\zeta_{M_0} > 0$. So we have shown (A.11), which completes the proof. □

A.7 Higher dimensional constructions

In higher dimensions, the random Fourier series in (2.12) is less regular. In particular, in two spatial dimensions, a typical element in the support of the measure is contained in H^{0-}, meaning that the L^2 norm is almost surely infinite. To get around this issue, one has to (Wick) renormalise the interaction, in essence subtracting a diverging quantity to end up with an interaction which is non-infinite. This makes proving that the weight function is in L^1 significantly more difficult. It also means that there is no longer a deterministic global well-posedness theory for the underlying equation. One instead needs to prove probabilistic local well-posed results, showing that the equation is locally well-posed for typical element in the support of the Gibbs measure.

In two dimensions and for a defocusing potential, this is the content of [10]. It was also shown in [18] that Wick ordering and a suitable truncation does not produce a well-defined measure in the case of focusing local NLS, although Bourgain showed in [11] that given suitable assumption on the interaction potential, one can construct the measure for the Hartree equation.

In two dimensions and p an even integer, the construction of the measure for the suitably renormalised equation with nonlinearity $|u|^p u$ also began with the works of Nelson [58] (as well as [37, 75]). A summary of the construction of the Gibbs measure in this case can be found in [64].

In three dimensions, the measure has been constructed for the Hartree equation for certain interaction potentials w_β which act like $|x|^{-(d-\beta)}$. This was done for $\beta > 2$ by Bourgain in [9] and by Deng-Nahmod-Yue for $\beta > 1 - \varepsilon$ in [23]. The construction of the measure in the case of general $\beta > 0$ and the local nonlinearity remains open in three dimensions.

In the cubic case, in dimensions greater than three, it is expected that the measure will be a Gaussian measure for any normalisation of the potential energy. This was proved for $d \geq 5$ independently by Aizenman and Fröhlich in [1] and [28] respectively. It was also proved for $d = 4$ in the case of the real-valued measure by Aizenman and Duminil-Copin in [2]. The same result is expected to hold for $d = 4$ in the case of the complex-valued measure.

For a summary of the construction results of Gibbs measures for non-Schrödinger equations, we point the reader for example to the summary in [16, Section 1.1].

Appendix B

Creation and annihilation operators

B.1 Basic computations

In this section, we record some basic computations. First we compute the commutation relations. We recall that for $f \in L^2$ and $\Psi = (\psi^{(0)}, \psi^{(1)} \ldots) \in \mathcal{F}$, we define

$$(b(f)\Psi)^{(n)}(x_1, \ldots, x_n) := \sqrt{n+1} \int dx \, \overline{f(x)} \psi^{(n+1)}(x, x_1, \ldots, x_n),$$
$$(b^*(f)\Psi)^{(n)}(x_1, \ldots, x_n) := \frac{1}{\sqrt{n}} \sum_{i=1}^{n} f(x_i) \psi^{(n-1)}(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n).$$

To simplify notation, we denote by $\mathbf{x} := x_1, \ldots, x_n$ and $\mathbf{x}_{\not i} := x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n$. We show that

$$[b(f), b^*(g)] = \langle f, g \rangle_{L^2} \mathbf{1}_{\mathcal{F}}. \tag{B.1}$$

We have

$$\begin{aligned}(b(f)b^*(g)\Psi)^{(n)}(\mathbf{x}) &= \sqrt{n+1} \int dx \, \overline{f(x)} \, (b^*(g)\Psi)^{(n+1)} (x, \mathbf{x}) \\ &= \frac{\sqrt{n+1}}{\sqrt{n+1}} \int dx \left(\overline{f(x)} \sum_{i=1}^{n} g(x_i) \psi^{(n)}(x, \mathbf{x}_{\not i}) + \overline{f(x)} g(x) \psi^{(n)}(\mathbf{x}) \right).\end{aligned}$$

Similarly

$$(b^*(f)b(g)\Psi)^{(n)}(\mathbf{x}) := \frac{1}{\sqrt{n}}\sum_{i=1}^{n} g(x_i)\,(b(g)\Psi)^{(n-1)}\,(\mathbf{x}_{\not{i}})$$
$$= \frac{\sqrt{n}}{\sqrt{n}}\sum_{i=1}^{n} g(x_i)\int dx\,\overline{f(x)}\psi^{(n)}(x,\mathbf{x}_{\not{i}}).$$

(B.1) follows from the definition of the commutator.

We also show that

$$\left(\int dx\, b^*(x)b(x)\right)\Big|_{\mathfrak{h}^{(n)}} = n\mathbf{1}_{\mathfrak{h}^{(n)}},$$

where we recall that $b(x) = b(\delta_x)$ and similarly $b^*(x) = b^*(\delta_x)$. Then

$$\left[\left(\int dx\, b^*(x)b(x)\right)\Psi\right]^{(n)}(\mathbf{x}) = \frac{1}{\sqrt{n}}\int dx\,\sum_{i=1}^{n}\delta(x-x_i)(b^*(x)\Psi)^{(n-1)}(\mathbf{x}_{\not{i}})$$
$$= \frac{\sqrt{n}}{\sqrt{n}}\int dx\,\sum_{i=1}^{n}\delta(x-x_i)\int dy\,\delta(x-y)\psi^{(n)}(y,\mathbf{x}_{\not{i}})$$
$$= \sum_{i=1}^{n}\int dx\,\delta(y-x_i)\psi^{(n)}(y,\mathbf{x}_{\not{i}}) = n\psi^{(n)}(\mathbf{x}),$$

where the final line uses the fact bosonic wavefunctions are symmetric.

More details on creation and annihilation operators, both in the bosonic and fermionic case can be found for example in [13].

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