Convex hulls of Lévy processes in space time

by

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Thesis

Submitted to the University of Warwick

for the degree of

Doctor of Philosophy in Statistics (Research)

Department of Statistics

August 2023

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Acknowledgements

I want to express my sincere thanks to my supervisor Aleksandar Mijatović for his guidance during my PhD, and his help in making me a better researcher. Also, thanks to Jorge Ignacio González Cázares for the many wonderful hours spend together doing research. A special thanks to my wife Kamilla, my brother Frederik, my mum Helle and my dad Jens, as well as the rest of my family, for their continued support, motivation and cheering throughout my PhD.

Declaration

This thesis is submitted to the University of Warwick in support of my application for the degree of Doctor of Philosophy in Statistics (Research). It has been composed by myself and has not been submitted in any previous application for any degree. Parts of this thesis has been published or submitted by the author and was developed in collaboration with Jorge Ignacio González Cázares and Aleksandar Mijatović: [8, 10, 11, 12, 13].

Abstract

In this thesis we apply a stick-breaking representation of the convex minorant and concave majorant of a one-dimensional Lévy process to show multiple probabilistic and geometric properties for the convex hull of a Lévy process. We show a central limit theorem for the fluctuations of the length of the concave majorant of a Lévy process when there is a finite second moment and consider the asymptotic dependence with the extrema of the process itself. The limit fluctuations of the length is also considered in the case where the Lévy process is in the domain of attraction of an α-stable law. In the rest of the thesis we study smoothness properties of the convex hull. Indeed, we characterise a class of Lévy processes whose graph has a continuously differentiable convex hull. Moreover, we also study how smooth the convex hull can be, by studying the growth rate of the convex minorant whenever the right derivative of the convex minorant increases continuously. Lastly, we characterise the Hölder continuity of the convex hull of a one-dimensional Lévy process.

Chapter 1

Introduction

Through the past decades, the topic of Lévy processes has constituted an important field within probability theory. In recent years there has been a surge of new advances and ideas within the field of Lévy processes, especially within the theory of convex hulls of Lévy processes, i.e. convex minorants and concave majorants of Lévy processes. Convex hulls of random walks and stochastic processes, including Cauchy processes, Brownian motion and Lévy processes, have been of interest for many decades, see e.g. [1, 4, 21, 30, 38, 40, 44, 49, 54, 56, 64, 66] and references therein. The topic of this thesis is convex hulls of one-dimensional Lévy processes, and the thesis will contain many new probabilistic results relating to geometric properties and smoothness properties for the convex hull. We will explore multiple areas within the topic of convex hulls, such as a central limit theorem, fluctuation theory, smoothness and connections to known results from other areas of probability theory.

To control the convex hull, it is enough to study the convex minorant and the concave majorant, which make up the convex hull. The convex minorant and the concave majorant are piecewise linear functions, with possibly infinitely many pieces of linearity, and controlling these functions is delicate and can be quite difficult. The surge of new results within the theory of convex hulls is, in part, due to the strong results from [38, 64], wherein the authors characterise the law of the piecewise linear faces of the convex minorant (and concave majorant) of a Lévy path. Having such a characterisation of the law of the faces, makes it possible to prove many probabilistic results of the convex hull. This characterisation will, throughout this thesis, be a crucial tool to prove our main results.

In Chapter 2 and Appendix A, we will go through the general notation, the main definitions and state the most important results used throughout this thesis. We do this to make the thesis almost surely self-contained.

Within the theory of convex hulls, the area of geometric properties such as length, diameter and volume are of specific interest, especially studied for random walks and isotropic stable processes (see references in §3.1.2). In Chapter 3 we study the asymptotic behaviour of the fluctuations of the length of the concave majorant of a Lévy process as the time horizon tends to infinity, in terms of a central limit theorem whenever there exists a finite second moment for the Lévy process. The scale of the fluctuations of the length and other statistics, as well as their asymptotic dependence, vary significantly with the tail behaviour of the Lévy measure. Moreover, in the case of finite second moment and zero mean, we describe the asymptotic dependence between the fluctuations of the length of the concave majorant and a triplet of processes that only depend on the Lévy process itself, i.e. the supremum, final value and time of the supremum of the Lévy process. Additionally, we study the cases where the Lévy process is in the domain of attraction of an α-stable law for $\alpha \in (0, 2] \setminus \{1\}$, which has a different dependence structure than in the case of finite second moment.

Another important topic, studied in the case of planar Brownian motion and Cauchy process (see references in §4.1), is the smoothness properties of the boundary of a convex hull of a Lévy process. In Chapter 4 we characterise, in terms of their transition laws, the class of one-dimensional Lévy processes whose graph has a continuously differentiable (planar) convex hull. We show that this phenomenon is exhibited by a broad class of infinite variation Lévy processes and depends subtly on the behaviour of the Lévy measure at zero. We introduce a class of strongly eroded Lévy processes, whose Dini derivatives vanish at every local minimum of the trajectory for all perturbations with a linear drift and prove that these are precisely the processes with smooth convex hulls. We study how the smoothness of the convex hull can break and construct examples exhibiting a variety of smooth/non-smooth behaviours. In the finite variation case, we characterise the points of smoothness of the convex hull in terms of the Lévy measure. We study these properties of smoothness both for finite and infinite time horizon. Finally, we conjecture that an infinite variation Lévy process is either strongly eroded or abrupt, a claim implied by Vigon's point-hitting conjecture. Studying the smoothness of a convex hull has many connections to many other classical areas of probability. Indeed, through the work of Vigon, there are strong connections to areas such as hitting points, potential theory, local time and regularity of 0.

The study of smoothness is continued in Chapter 5, wherein we use the knowledge from the previous chapter to study and quantify the smoothness of the boundary of the convex hull in terms of the growth rate of the right derivative of the convex minorant. Since the convex minorant is piecewise linear, its right derivative may increase continuously either at a vertex time of finite slope or at time 0 where the slope is $-\infty$. While the convex hull depends on the entire path, we show that the local fluctuations of the derivative C' depend only on the fine structure of the small jumps of the Lévy process and are the same for all time horizons. When points of smoothness exist, i.e. when the right derivative of the convex minorant increases continuously, we study the behaviour of the convex hull at these points by finding upper and lower functions for the right derivative, meaning that we study the modulus of continuity. The main process of interest, turns out to be the vertex time process which is the right inverse of the right derivative of the convex minorant. This process has independent increments but not necessarily stationary increments. Therefore, it is crucial to extend multiple known fluctuation results for subordinators to the time-inhomogeneous case, which is not easily done since the Laplace exponent is bivariate. Moreover, we also study what implications these properties of the convex hull have for the path of the Lévy process. We find that under certain conditions, we can use the fluctuation of the right derivative of the convex hull to describe the fluctuations of the Lévy process, and especially we find novel results in terms of the local growth of the post-minimum process and corresponding Lévy meander.

In Chapter 6, our study of convex hulls is finalised, by characterising the Hölder continuity of the convex minorant of most Lévy processes. Indeed, Hölder continuity of random functions is a classical area, well studied for Brownian motion and fractional Brownian motion. The methods in the chapter are based on a novel connection between the path properties of the Lévy process at zero and the boundedness of the set of r-slopes of the convex minorant.

Chapter 2

Notation & important results

§2.1 Notation

In this section we introduce important notation that is used uniformly throughout the thesis. We start by introducing some limit behaviour notation, and to do so we let $a \in [0, \infty]$. Given two positive functions f and g, we say $f(x) = \mathcal{O}(g(x))$ as $x \to a$ if $\limsup_{x \to a} f(x)/g(x) < \infty$. Similarly, we write $f(x) \approx g(x)$ as $x \to a$ if $f(x) = \mathcal{O}(g(x))$ and $g_2(x) = \mathcal{O}(g_1(x))$ as $x \to a$. For the two functions f and g, we write $f(x) = \mathrm{o}(g(x))$ as $x \to a$ if $\lim_{x \to a} f(x)/g(x) = 0$. The notation $f(x) \sim g(x)$ as $x \to a$ is used if $f(x)/g(x) \to 1$ as $x \to a$. Note, in the case where $a = 0$, that we take $x \downarrow 0$ and that a is most often taken to be 0 or ∞.

Denote the positive (resp. negative) part of x by $x^+ := \max\{x, 0\}$ (resp. $x^- := \max\{-x, 0\}$). Denote $\mathbb{R}_+ := [0, \infty)$ and $\mathbb{R}_- := (-\infty, 0]$. Throughout the thesis, δ_z will denote the Dirac delta measure of the point z, i.e. $\delta_z(A) = \mathbb{1}_A(z)$ for a measurable A. For any stochastic process $(X_t)_{t \geq 0}$, we denote by X_{t-} is the left limit of the trajectory at time t, i.e. $X_{t-} := \lim_{s \uparrow t} X_s$, with $X_{0-} := X_0$.

§2.2 Lévy processes

A one-dimensional *Lévy process* $X = (X_t)_{t \geq 0}$ is a stochastic process with $X_0 = 0$ a.s., it has stationary and independent increments and its paths are càdlàg[1] (see [70, Def. 1.6, Ch. 1]). Most one-dimensional Lévy processes applied in the literature includes Brownian motion, Cauchy process, Poisson process and compound Poisson process. This thesis will assume basic knowledge of one-dimensional Lévy processes, and for a thorough background on Lévy processes, we refer to the monograph [70].

[1] A function $f : [0, \infty) \to \mathbb{R}$ is càdlàg if it is right-continuous and has left limits.

In this section and Appendix A, we introduce the most important results on Lévy processes, that are used the most throughout the thesis.

The characteristic function of X, given by $\varphi_{X_t}(\theta) := \mathbb{E}[e^{i\theta X_t}]$ for $\theta \in \mathbb{R}$, is often used to characterise the Lévy process. Let ψ be the Lévy–Khintchine exponent [70, Thm 8.1 & Def. 8.2] of the Lévy process X, defined, for $\theta \in \mathbb{R}$ and $t > 0$, as

$$\psi(\theta) := t^{-1} \log(\varphi_{X_t}(\theta)) = -\frac{1}{2}\sigma^2\theta^2 + i\theta\gamma + \int_{\mathbb{R}} \left(e^{i\theta x} - 1 - i\theta x \mathbb{1}_{(-1,1)}(x)\right)\nu(\mathrm{d}x), \quad (2.1)$$

where $\sigma^2 \geq 0$ is called the *Gaussian coefficient*, $\gamma \in \mathbb{R}$ and ν is a measure on $\mathbb{R}$ called the *Lévy measure* which satisfies $\int_{\mathbb{R}}(x^2 \wedge 1)\nu(\mathrm{d}x) < \infty$. The triplet (σ^2, γ, ν), which characterises the Lévy process X, is called the *generating triplet*, and is given w.r.t. the cut-off function $x \mapsto \mathbb{1}_{(-1,1)}(x)$.

Remark 2.1 ([70, p. 39]). Let X be a one-dimensional Lévy process with generating triplet (σ^2, γ, ν).

(a) If $\int_{(-1,1)} |x|\nu(\mathrm{d}x) < \infty$, we can re-write the Lévy-Khintchine exponent ψ as

$$\psi(\theta) = -\frac{1}{2}\sigma^2\theta^2 + i\theta\gamma_0 + \int_{\mathbb{R}} \left(e^{i\theta x} - 1\right)\nu(\mathrm{d}x), \qquad \text{for } \theta \in \mathbb{R},$$

where $\gamma_0 = \gamma - \int_{(-1,1)} x\nu(\mathrm{d}x) \in \mathbb{R}$, and γ_0 is called the *drift* of X.

(b) If $\int_{\mathbb{R}\setminus(-1,1)} |x|\nu(\mathrm{d}x) < \infty$, then the Lévy-Khintchine exponent can be expressed as

$$\psi(\theta) = -\frac{1}{2}\sigma^2\theta^2 + i\theta\gamma_1 + \int_{\mathbb{R}} \left(e^{i\theta x} - 1 - i\theta x\right)\nu(\mathrm{d}x), \qquad \text{for } \theta \in \mathbb{R},$$

where $\gamma_1 = \gamma + \int_{\mathbb{R}\setminus(-1,1)} x\nu(\mathrm{d}x) = \mathbb{E}[X_1]$ is the *center* of X (see also Example A.1).

◊

The class of diffuse Lévy processes will be important throughout the thesis. We therefore state Doeblin's diffuseness lemma, which characterise when a Lévy process is diffuse in terms of the generating triplet.

Lemma 2.2 ([43, Lem. 15.22]). *A Lévy process with generating triplet (σ^2, γ, ν) is a diffuse Lévy process if and only if $\sigma^2 \neq 0$ or $\nu(\mathbb{R}) = \infty$.*

For all $\varepsilon > 0$, we define the following functions of the generating triplet (σ^2, γ, ν) of X, used throughout the thesis:

$$\overline{\sigma}^2(\varepsilon) := \int_{(-\varepsilon,\varepsilon)} x^2\nu(\mathrm{d}x), \quad \overline{\gamma}(\varepsilon) := \int_{(-1,1)\setminus(-\varepsilon,\varepsilon)} x\nu(\mathrm{d}x), \quad \overline{\nu}(\varepsilon) := \nu(\mathbb{R} \setminus (-\varepsilon, \varepsilon)), \quad (2.2)$$

with $\overline{\nu}^+(\varepsilon) := \nu([\varepsilon, \infty))$ and $\overline{\nu}^-(\varepsilon) := \nu((-\infty, -\varepsilon])$.

In the following example, we will consider the Laplace exponent of the specific class of Lévy processes where the paths are increasing.

Example 2.1 (Subordination of a Lévy process). We say that a process $Y = (Y_t)_{t\geq 0}$ is a *subordinator* if it is a Lévy process with increasing paths. Let Y be a driftless subordinator, then from [20, Sec. 1.2, p. 7] and [70, Thm 30.1], we know that the Fourier-Laplace exponent $\phi(u) = \log \mathbb{E}[e^{uY_1}]$ for $u \in \mathbb{C}$ where $\Re u \leq 0$, of Y, has the form $u^{-1}\phi(u) = d + \int_{(0,\infty)} e^{-yu}\,\overline{\nu}_Y^{+}(y)\mathrm{d}y$, where d is the drift coefficient and ν_Y is the Lévy measure of the subordinator Y.

Let X be a Lévy process on $\mathbb{R}$. The process $Z = (Z_t)_{t\geq 0} = (X_{Y_t})_{t\geq 0}$ is then called the subordination of X by the subordinator Y, and by [70, Thm 30.1], the Lévy measure ν_Z of $(Z_t)_{t\geq 0}$ is given by $\nu_Z(\mathrm{d}x) = \int_0^1 \mathbb{P}(X_t \in \mathrm{d}x)\nu_Y(\mathrm{d}t)$. △

Knowing the activity of the Lévy measure ν of a Lévy process X is very important when working with the fine structures of Lévy processes. The following indices will therefore be used frequently throughout the thesis. The *Blumenthal–Getoor index* of X (see [26]), denoted $\beta_+ \in [0,2]$, is defined by

$$\beta_+ \coloneqq \inf\{q \in [0,2] \,:\, I_q < \infty\}, \quad \text{where } I_q \coloneqq \int_{(-1,1)} |x|^q \nu(\mathrm{d}x), \quad q > 0. \tag{2.3}$$

The *lower-activity index*, denoted $\beta_- \in [0,2]$ (inspired by Pruitt [65]), is given by

$$\beta_- \coloneqq \inf\left\{p > 0 : \liminf_{u\downarrow 0} u^{p-2}\overline{\sigma}^2(u) = 0\right\}. \tag{2.4}$$

It is easy to see that the inequalities $0 \leq \beta_- \leq \beta_+ \leq 2$ hold. We note that β_- (resp. β_+) presents a lower (resp. upper) bound on the activity of the Lévy measure ν at zero. Thus, in general, we may have $\beta_- < \beta_+$.

We will throughout this thesis often split the class of all Lévy processes in three classes; processes of (I) finite activity, (II) infinite activity and finite variation and (III) infinite variation. To be of type (I) means that the process X is a compound Poisson process with drift, i.e. $\sigma^2 = 0$ and $\nu(\mathbb{R}) < \infty$ in terms of the characteristic triplet. Type (II) consists of the Lévy processes where the paths are of finite variation[2] but is not compound Poisson, i.e. $\sigma^2 = 0$ and $\int_{(-1,1)} |x|\nu(\mathrm{d}x) < \infty$. Lastly, type (III) then consists of the Lévy processes, where the paths are *not* of finite variation, i.e. $\sigma^2 > 0$ or $\int_{(-1,1)} |x|\nu(\mathrm{d}x) = \infty$. Often we will exclude the case of compound Poisson processes, since some results are trivial in this case, and thus only work with processes of type (II) or (III).

In the following theorem, we state the Lévy–Itô decomposition, which states that a Lévy process X can be decomposed into a pure-jump process called the *jump part* and an independent continuous process called the *continuous part*. The theorem is stated in higher generality for additive processes. An additive process is a Lévy

[2] A function $f : [0,\infty) \to \mathbb{R}$ is said to be of finite variation if, for any interval $[a,b]$ where $0 \leq a < b < \infty$, it holds that $\sup_{n\in\mathbb{N}} \sup_{a=x_0<\cdots<x_n=b} \sum_{i=1}^n |f(x_i) - f(x_{i-1})| < \infty$, where the inner supremum is taken over all partitions $a = x_0 < \cdots < x_n = b$ of $[a,b]$.

processes where we drop the requirement of stationary increments.

Theorem 2.3 ([70, Thm 19.2]). *Let $(X_t)_{t\geq 0}$ be an additive process on $\mathbb{R}$ defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with system of generating triplets $(A_t, \gamma(t), \nu_t)$, and define the measure $\widetilde{\nu}$ on $H := (0,\infty)\times\mathbb{R}$ by $\widetilde{\nu}((0,t]\times B) = \nu_t(B)$ for $B \in \mathcal{B}(\mathbb{R})$. Let $\Omega_0 \in \mathcal{F}$ with $\mathbb{P}(\Omega_0) = 1$ such that, for all $\omega \in \Omega_0$, $X_t(\omega)$ is càdlàg. We define, for all $B \in \mathcal{B}(H)$,*

$$J(B,\omega) := \begin{cases} \#\{s \in (0,\infty) : (s, X_s(\omega) - X_{s-}(\omega)) \in B\}, & \text{for } \omega \in \Omega_0, \\ 0, & \text{for } \omega \in \Omega_0. \end{cases}$$

Then the following statements hold.

(i) *$\{J(B) : B \in \mathcal{B}(H)\}$ is a Poisson random measure on H with intensity $\widetilde{\nu}$.*

(ii) *There is a $\Omega_1 \in \mathcal{F}$ with $\mathbb{P}(\Omega_1) = 1$, such that, for any $\omega \in \Omega_1$,*

$$\begin{aligned} X_t^1(\omega) = \lim_{\varepsilon\downarrow 0} \int_{(0,t]\times((-1,-\varepsilon)\cup(\varepsilon,1))} & x(J(\mathrm{d}(s,x),\omega) - \widetilde{\nu}(\mathrm{d}(s,x))) \\ &+ \int_{(0,t]\times(\mathbb{R}\setminus(-1,1))} xJ(\mathrm{d}(s,x),\omega), \end{aligned}$$

is defined for all $t \in [0,\infty)$ and the convergence is uniform in t on any bounded interval. The process $(X_t^1)_{t\geq 0}$ is then an additive process on $\mathbb{R}$ with system of generating triplets $(0, 0, \nu_t)$.

(iii) *Define $X_t^2(\omega) = X_t(\omega) - X_t^1(\omega)$, for $\omega \in \Omega_1$. There exist a $\Omega_2 \in \mathcal{F}$ with $\mathbb{P}(\Omega_2) = 1$, such that, for all $\omega \in \Omega_2$, $X_2(t)$ is continuous in t. The process $(X_t^2)_{t\geq 0}$ is an additive process on $\mathbb{R}$, and has system of generating triplet given by $(A_t, \gamma(t), 0)$.*

(iv) *The processes $(X_t^1)_{t\geq 0}$ and $(X_t^2)_{t\geq 0}$ are independent.*

§2.2.1 Technical results

Throughout this subsection we will state some of the most important technical results on Lévy processes used throughout the thesis. Some of the more classical results are stated in Appendix §1.3.

We start with the following technical lemma, extended in [33, Thm 1.1], which shows a small-time ergodic property for the Lévy process X and unbounded moment functions $x \mapsto |x|^p$ for $p \geq 2$.

Lemma 2.4 ([6, Lem. 3.1]). *Let X be a one-dimensional Lévy process with Lévy measure ν. Suppose for some $p \geq 2$, that $\mathbb{E}[|X_1|^p] < \infty$ and $\mathbb{E}[X_1] = 0$, then*

$$\lim_{n\to\infty} n\mathbb{E}[|X_{1/n}|^p] = \int_{\mathbb{R}} |x|^p \nu(\mathrm{d}x).$$

We continue stating results related to moments, by introducing some bounds in the ensuing lemma, that depends on the Blumenthal–Getoor index β_+. Recall from (2.3), that $I_p < \infty$ for all $p > \beta_+$, and if $I_{\beta_+} = \infty$ then $\beta_+ < 2$, implying that we can find some $\delta \in (0, 2 - \beta_+)$ such that $\beta_+ + \delta < 1$ when $\beta_+ < 1$, and we define in $\widetilde{\beta}_+ := \beta_+ + \delta \mathbb{1}_{\{I_{\beta_+} = \infty\}} \in [\beta_+, 2]$, and note that $I_{\widetilde{\beta}_+} < \infty$.

Lemma 2.5 ([39, Lem. 1]). *Let X be a one-dimensional Lévy process with Lévy measure ν. The measure ν then satisfies, for all $\kappa \in (0, 1]$, that $\overline{\nu}(\kappa) \leq \kappa^{-\widetilde{\beta}_+} I_{\widetilde{\beta}_+} + \overline{\nu}(1)$ and $\overline{\sigma}^2(\kappa) \leq \kappa^{2-\widetilde{\beta}_+} I_{\widetilde{\beta}_+}$. Moreover, $\int_{(-1,-\kappa]\cup[\kappa,1)} |x|^p \nu(\mathrm{d}x) \leq \kappa^{-(\widetilde{\beta}_+-p)^+} I_{\widetilde{\beta}_+}$ for $p \in \mathbb{R}$ and $\int_{(-\kappa,\kappa)} |x|^p \nu(\mathrm{d}x) \leq \kappa^{p-\widetilde{\beta}_+} I_{\widetilde{\beta}_+}$ for $p \geq \widetilde{\beta}_+$.*

For a Lévy process X on $[0, T]$, we denote by $\overline{X}_t := \sup_{0 \leq s \leq t} X_s$ for $t \in [0, T]$, the running supremum of X.

Lemma 2.6 ([39, Lem. 2]). *Consider the Lévy process X on a fixed time horizon $[0, T]$, for some $T > 0$. Then, for all $t \in [0, T]$ and $p > 0$, the assumption $I_{p,+} := \int_{[1,\infty)} x^p \nu(\mathrm{d}x) < \infty$ implies that*

$$\mathbb{E}\left[\overline{X}_t^p\right] \leq C_{p,1} t^{p/\widetilde{\beta}_+} + C_{p,2} t^{p/2} + C_{p,3} t^p + C_{p,4} t^{\min\{1, p/\widetilde{\beta}_+\}}, \tag{2.5}$$

for positive finite constants $\{C_{p,i}\}_{i=1}^4$. Moreover, if $I_{1,+} < \infty$, then

$$\mathbb{E}\left[\overline{X}_t\right] \leq \widetilde{C}_1 \sqrt{t} + \begin{cases} \widetilde{C}_2 t + \widetilde{C}_3 \sqrt{t}, & \widetilde{\beta}_+ = 2, \\ \widetilde{C}_4 \sqrt{t} + \widetilde{C}_5 t^{1/\widetilde{\beta}_+}, & \widetilde{\beta}_+ \in (1, 2), \\ \widetilde{C}_6 t, & \widetilde{\beta}_+ \leq 1, \end{cases}$$

for positive finite constants $\{\widetilde{C}_i\}_{i=1}^6$.

Note that the constants $\{C_{p,i}\}_{i=1}^4$ and $\{\widetilde{C}_i\}_{i=1}^6$ are given in full explicit form in [39, Lem. 2] and that they might depend on T. However, since the explicit form is not necessary for the applications within this thesis, and the explicit forms are rather lengthy, this has been omitted.

§2.2.2 Fluctuation theory

Let $\lambda = \lambda_\theta \sim \mathrm{Exp}(\theta)$, i.e. λ is an exponentially distributed random variable with parameter $\theta > 0$. We define the time on the time-horizon $[0, \lambda]$ at which the supremum of X_t is attained by $\tau_\lambda = \sup\{t < \lambda : X_t = \overline{X}_t\}$. Define the hitting time $T_x := \inf\{t > 0 : X_t = x\}$ for any $x \in \mathbb{R}$. We say that 0 is regular (resp. irregular) for X, if $\mathbb{P}(T_0 = 0) = 1$ (resp. $\mathbb{P}(T_0 = 0) = 0$).

Lemma 2.7 ([18, Lem. 6, Sec. VI.2]). *Let $\theta > 0$ and X be a Lévy process. Then the following statements hold.*

(i) *If* 0 *is irregular for the process* $(\overline{X}_t - X_t)_{t\geq 0}$*, then the processes* $(X_t)_{0\leq t\leq \tau_\lambda}$ *and* $(X_{\tau_\lambda+t} - X_{\tau_\lambda})_{0\leq t<\lambda-\tau_\lambda}$ *are independent.*

(ii) *If* 0 *is regular for the process* $(\overline{X}_t - X_t)_{t\geq 0}$*, then the processes* $(X_t)_{0\leq t\leq \tau_\lambda}$ *and* $(X_{\tau_\lambda+t} - X_{\tau_\lambda-})_{0\leq t<\lambda-\tau_\lambda}$ *are independent.*

One can take $\theta \downarrow 0$ (in which case $\lambda \to \infty$) in Lemma 2.7, and thus, if $\tau_\infty < \infty$ a.s., we see that the post-supremum process and the pre-supremum processes on $[0,\infty)$ are also independent, i.e. if $\tau_\infty < \infty$ a.s. and 0 is regular for $(\overline{X}_t - X_t)_{t\geq 0}$, then $(X_t)_{0\leq t\leq \tau_\infty}$ and $(X_{\tau_\infty+t} - X_{\tau_\infty-})_{t\geq 0}$ are independent. Note that $\tau_\infty < \infty$ a.s. is equivalent to the "all-time" supremum being finite a.s., i.e. $\overline{X}_\infty \coloneqq \sup_{t\geq 0} X_t < \infty$ a.s. This can be done by checking if the Laplace transform of $\overline{X}_\infty$ is not identically 0 for some $u > 0$, and hence all $u > 0$. The Laplace exponent of $\overline{X}_\infty$ is given by (see [38, Thm 2.7]):

$$\mathbb{E}[e^{-u\overline{X}_\infty}] = \exp\left(-\int_0^\infty \int_{(0,\infty)} (1 - e^{-ux})\mathbb{P}(X_t \in \mathrm{d}x)\frac{\mathrm{d}t}{t}\right).$$

Similarly to the paragraph above, we say that 0 is regular for the half-line $(0,\infty)$ (resp. $(-\infty,0)$) for X if X visits $(0,\infty)$ (resp. $(-\infty,0)$) a.s. immediately after time 0, i.e. $\mathbb{P}(\bigcap_{t>0}\bigcup_{s\leq t}\{X_s > 0\}) = 1$ (resp. $\mathbb{P}(\bigcap_{t>0}\bigcup_{s\leq t}\{X_s < 0\}) = 1$). We can now state the useful Rogozin's criterion, which gives integral criteria in terms of the transition probabilities of X for when 0 is regular for the halfline $(0,\infty)$ (see also [18, Prop. 11, Sec. VI.3]).

Theorem 2.8 ([38, Thm 2.6])**.** *Let* X *be a Lévy process. Then, the starting point* 0 *of* X *is regular for* $(0,\infty)$ *if and only if* $\int_0^1 \mathbb{P}(X_t > 0)t^{-1}\mathrm{d}t < \infty$*.*

This result is extremely important, however, it is often not tractable to have an integral criterion in terms of the transition probabilities. Thus, we state the following theorem, which holds for finite variation Lévy processes, where the integral criteria is stated in terms of the characteristics of the Lévy process. Define $I^-(x) \coloneqq \int_0^x \overline{\nu}^-(y)\mathrm{d}y$ for any $x \geq 0$, and note that I^- is well defined by the definition of ν.

Theorem 2.9 ([19, Thm 1])**.** *Let* X *be a Lévy process of finite variation and zero-drift. Then,* 0 *is regular for* $(0,\infty)$ *if and only if* $\int_{0+} \overline{\nu}^+(x)\mathrm{d}(x/I^-(x)) < \infty$*.*

Let $(R_x)_{x\geq 0}$ be the first passage time process of the Lévy process X, defined as $R_x = \inf\{t > 0 : X_t > x\}$ for all $x \geq 0$. In the following theorem, we give conditions in terms of the generating triplet for whether $R_0 = 0$ a.s. or $R_0 > 0$ a.s.

Theorem 2.10 ([70, Thm 47.5])**.** *Let* X *be a Lévy process with generating triplet* (σ^2,γ,ν)*, and let* γ_0 *given as in Remark 2.1.*

(a) *Assume that* $\sigma^2 = 0$ *and* $\nu(\mathbb{R}) < \infty$*, then:* (a-i) *if* $\gamma_0 > 0$ *then* $R_0 = 0$ *a.s.;* (a-ii) *if* $\gamma_0 \leq 0$ *then* $R_0 > 0$ *a.s.*

(b) *Assume that* $\sigma^2 = 0$*,* $\nu(\mathbb{R}) = \infty$ *and* $\int_{(-1,1)} |x|\nu(\mathrm{d}x) < \infty$*, then:*

(b-i) *If* $\gamma_0 > 0$ *then* $R_0 = 0$ *a.s.*

(b-ii) *If* $\gamma_0 < 0$ *then* $R_0 > 0$ *a.s.*

(b-iii) *If* $\gamma_0 = 0$ *and* $\nu((-\infty, 0)) < \infty$*, then* $R_0 = 0$ *a.s.*

(b-iv) *If* $\gamma_0 = 0$ *and* $\nu((0, \infty)) < \infty$*, then* $R_0 > 0$ *a.s.*

(b-v) *If* $\gamma_0 = 0$*,* $\nu((-\infty, 0)) = \infty$ *and* $\nu((0, \infty)) = \infty$*, then both* $R_0 > 0$ *a.s. and* $R_0 = 0$ *a.s. are possible.*

(c) *Assume that* $\sigma^2 \neq 0$ *or* $\int_{(-1,1)} |x|\nu(\mathrm{d}x) = \infty$*, then* $R_0 = 0$ *a.s.*

Consider a Lévy process X on $[0, T]$. We define the post-minimum process $X^{\to} = (X_t^{\to})_{t\in[0,T-\tau_0]}$ given by $X_t^{\to} := X_{t+\tau_0} - \inf_{0\leq t\leq T} X_t$, where τ_0 is the time that X_t attains its minimum on $[0, T]$. As in the paragraph preceding [76, Thm 2], we define a Lévy meander of length T as the weak limit, as $\varepsilon \downarrow 0$, of the Lévy process X conditioned to stay above $-\varepsilon$ on $[0, T]$ under $\mathbb{P}$ (the law of the Lévy process).

Theorem 2.11 ([76, Thm 2])**.** *Assume that* 0 *is regular for both half-lines* $(-\infty, 0)$ *and* $(0, \infty)$ *and, for any* $t > 0$*,* $\int_{\mathbb{R}} |\mathbb{E}[e^{iuX_t}]|\mathrm{d}u < \infty$*. Then, the law of* $X^{\to}$ *is the same as the law of the Lévy meander of length* $T - \tau_0$ *of the Lévy process* X*.*

§2.3 α-stable processes & stable domain of attraction

Throughout the thesis we will often work with *α-stable processes*, since they have many important applications, explicit characteristics and nice scaling properties. We say that a process $Z = (Z_t)_{t\geq 0}$ is an α-stable process, if it is a one-dimensional Lévy process such that $Z_t \stackrel{d}{=} t^{1/\alpha} Z_1$ for all $t > 0$ (see also §1.3.4 or [70, Ch. 3]). Examples of α-stabel processes include Brownian motion ($\alpha = 2$) and Cauchy processes ($\alpha = 1$).

If Z is an α-stable process with $\alpha \in (0, 2)$, then Z has generating triplet $(0, \gamma, \nu)$, where $\nu(\mathrm{d}x) = |x|^{-1-\alpha}(c_+ \mathbb{1}_{(0,\infty)}(x) + c_- \mathbb{1}_{(-\infty,0)}(x))\mathrm{d}x$ for some $c_+, c_- \geq 0$ such that $c_+ + c_- > 0$ by [70, Thm 14.3(ii)]. Note, when $\alpha < 2$, that an α-stable process has no Gaussian component. By studying the function $x \mapsto |x|^{-1-\alpha}$ from the closed form of ν, we can see that an α-stable processes mainly has big jumps if α is close to 0, whereas if α is close to 2, then Z mainly has small jumps. In the case where Z is α-stable the activity indices β_- and β_+ will agree, and be equal to α, see also §1.3.4.

Another important class of processes closely related to α-stable processes, used frequently throughout this thesis, are the processes X that are in the *domain of attraction of an α-stable law*, see [42, §4] for a full characterisation of this class. We say that a Lévy process X is in the domain of attraction of an α-stable law for some $\alpha \in (0, 2]$, if

$$X_t/g(t) \xrightarrow{d} Z_1, \quad \text{as } t \to \infty, \text{ for a positive function } g(t) = t^{1/\alpha}l(t), \tag{2.6}$$

where l is a slowly varying function at infinity (i.e. $l(cx)/l(x) \to 1$ as $x \to \infty$ for all $c > 0$) and $(Z_t)_{t\geq 0}$ is an α-stable process (see also [42, Eq. (8)]). The function g is in this context called the *scaling function*. We note that (2.6) is equivalent to

$$(X_{st}/g(t))_{s\in[0,1]} \xrightarrow{d} (Z_s)_{s\in[0,1]}, \quad \text{as } t \to \infty, \tag{2.7}$$

in the Skorokhod space $\mathcal{D}[0, 1]$ equipped with the J_1-topology [23, Ch. 3], with g as in (2.6). Note that $\mathcal{D}[0, 1]$ is the space of functions on $[0, 1]$ that are right-continuous with left-limits. If the slowly varying function l converges to a positive finite constant, and (2.6) holds, then we say that X is in the *domain of normal attraction* of an α-stable law. When l does not converge to a finite positive constant but (2.6) still holds (e.g. l converges to 0, ∞ or fluctuates), then we say that X is in the *domain of non-normal attraction* of an α-stable law.

Similarly to the setting above, we say that X is in the small-time domain of attraction of an α-stable law, if $X_t/g(t) \xrightarrow{d} Z_1$ as $t \downarrow 0$, for some positive function $g(t) = t^{1/\alpha}l(t)$, where l is slowly varying at 0. This can also be characterised through the characteristics of the process, as explained by the succeeding theorem.

Theorem 2.12 ([42, Thm 2])**.** *Consider the Lévy process X on $\mathbb{R}$ with generating tripet (σ^2, γ, ν) and the α-stable process Z with parameters $c_+, c_- > 0$ and $\widehat{\gamma}$ (see definition of Z above).*

(i) *X is attracted to the non-zero linear drift $(t\widehat{\gamma})_{t\geq 0}$ if and only if $\sigma = 0$, $(\gamma - \overline{\gamma}(x))/\widehat{\gamma}$ is eventually positive, $x\overline{\nu}(x)/(\gamma - \overline{\gamma}(x)) \to 0$ as $x \downarrow 0$ and $g(t)$ is chosen such that $g(t)/(\gamma - \overline{\gamma}(g(t))) \sim t/\widehat{\gamma}$ as $t \downarrow 0$.*

(ii) *X is in the small-time domain of attraction of an α-stable process Z if and only if the following hold:*
   (a) *when X is of finite variation, then $\sigma^2 = 0$ and $\gamma_0 = 0$,*
   (b) *the functions $\varepsilon \mapsto \overline{\nu}^{\pm}(\varepsilon)$ are regularly varying at 0 with index $-\alpha$ if $c_{\pm} > 0$, and $\overline{\nu}^{+}(\varepsilon)/\overline{\nu}^{-}(\varepsilon) \to c_+/c_-$ as $\varepsilon \downarrow 0$.*
   (c) *when $\alpha = 1$ it is also required that $(\varepsilon\overline{\nu}^{+}(\varepsilon))^{-1}(\gamma - \overline{\gamma}(\varepsilon)) \to \widehat{\gamma}/c_+$ as $\varepsilon \downarrow 0$, and $g(t)$ is chosen to satisfy $\overline{\nu}^{\pm}(g(t)) \sim t^{-1}c_{\pm}/\alpha$ if $c_{\pm} > 0$.*

(iii) *X is attracted to a Brownian motion with variance $\widehat{\sigma}^2$ if and only if $x^2\overline{\nu}(x)/(\sigma^2 + \overline{\sigma}^2(x)) \to 0$ as $x \downarrow 0$ and $g(t)$ is chosen to satisfy $t/\widehat{\sigma}^2 \sim g(t)^2/(\sigma^2 + \overline{\sigma}^2(g(t)))$ as $t \downarrow 0$.*

In the following example, we consider a non-strictly 1-stable process that is attracted to a linear drift.

Example 2.2 ([42, Ex. 4.2.2]). Assume that X is a 1-stable process with Lévy measure where $c_+ \neq c_-$. Directly from the definition of the Lévy measure ν of X, we see that $\overline{\nu}(x) = (c_+ + c_-)/x$ and that $\gamma - \overline{\gamma}(x) = \gamma + (c_+ - c_-)\log(x)$. Hence, we see that the conditions of Theorem 2.12(i) are fulfilled for any $\widehat{\gamma}$ having the same sign as $(c_- - c_+)$. Thus, a non-strictly 1-stable process is attracted to a non-zero linear drift process, and the scaling function $g(t)$ must satisfy $-g(t)/\log(g(t)) \sim t(c_- - c_+)/\widehat{\gamma}$ as $t \downarrow 0$. △

In the following two lemmas, we assume that X is in the small-time domain of attraction of an α-stable process Z with Lévy measure ν_Z, i.e. assume that (2.6) holds, and let $X_t^{(n)} = b_n X_{t/n}$ where $b_n = 1/g(1/n)$, with corresponding generating triplet $(\sigma^2_{(n)}, \gamma_{(n)}, \nu_{(n)})$.

Lemma 2.13 ([25, Lem. 4.8]). *Assume that X is in the small time domain of attraction of an α-stable law. Then $\gamma_{(n)}, \sigma^2_{(n)}$ and $\int_{(-1,1)} x^2 \nu_{(n)}(\mathrm{d}x)$ have finite limits as $n \to \infty$. Moreover, for any $p < \alpha$ such that $\int_{[1,\infty)} x^p \nu_{(n)}(\mathrm{d}x) < \infty$, we have that $\int_{[1,\infty)} x^p \nu_{(n)}(\mathrm{d}x) \to \int_{[1,\infty)} x^p \nu_Z(\mathrm{d}x) < \infty$ as $n \to \infty$.*

Lemma 2.14 ([25, Lem. 4.9]). *Assume that $\max\{\gamma_{(n)}, 0\}$, $\sigma^2_{(n)}$, $\int_{(-1,1)} x^2 \nu_{(n)}(\mathrm{d}x)$ and $\int_{[1,\infty)} x^p \nu_{(n)}(\mathrm{d}x)$ are bounded. Then $\mathbb{E}[(\sup_{t\in[0,1]} X_t^{(n)})^p]$ is bounded.*

We will, in the ensuing lemma, consider the discrete time version of domain of attraction of an α-stable law. We consider a sequence $(X_n)_{n\in\mathbb{N}}$ of iid random variables with the same distribution function F. If we can find normalising constants A_n and B_n, such that the distribution F_n of $(X_1 + \cdots + X_n - A_n)/B_n$ converges weakly to a distribution function G of an α-stable random variable, then we say that F is in the domain of attraction of G with exponent α.

Lemma 2.15 ([41, Lem. 5.2.2]). *If F is in the domain of attraction of an α-stable law G, then for all $\delta < \alpha$, the moments $\int_{\mathbb{R}} |x|^\delta F_n(\mathrm{d}x)$ are uniformly bounded in n.*

§2.4 Convex hull & the stick-breaking process

In this section we introduce the main objects of this entire thesis, the *convex hull*, *concave majorant* and the *convex minorant* of a one-dimensional Lévy process X. We also define the closely related *stick-breaking process*, and how this can be used to describe the law of the faces of the convex minorant and concave majorant.

Definition 2.16. *Let X be a one-dimensional Lévy process, and fix a time interval $[0, T]$ for some positive time horizon $T > 0$. The convex minorant $C_T^{\smile}(t)$ (resp.*

concave majorant $C_T^{\frown}(t)$) of a path of X is the largest (resp. smallest) function that is point-wise smaller (resp. larger) than the path of X, i.e. $C_T^{\smile}(t) \leq X_t$ (resp. $C_T^{\frown}(t) \geq X_t$) for all $t \in [0,T]$.

The boundary of the convex hull of the graph of a Lévy process X over $[0,T]$ is a union of the graphs of the convex minorant and the concave majorant.

In chapters 4, 5 & 6, when considering only the convex minorant for a fixed time horizon $T > 0$, we will, for convenience, drop the superscript and subscript, and denote by $C = (C(t))_{t\in[0,T]}$ the convex minorant on the fixed time interval $[0,T]$. In the case where we are considering the convex minorant of X on $[0,\infty)$, we use the notation $C_\infty = (C_\infty(t))_{t\in[0,\infty)}$. Note that the convex minorant and concave majorant are piecewise linear functions for any Lévy process X, with countably, but possibly infinitely, many pieces of linearity, as seen in Theorem 2.18. To describe the law of these pieces of linearity, we need the process $(\ell_n)_{n\in\mathbb{N}}$, called a *uniform stick breaking* process.

Definition 2.17. *For any $T > 0$, a uniform stick-breaking process $(\ell_n)_{n\in\mathbb{N}}$ on $[0,T]$ is defined recursively by an iid-*U$(0,1)$ *sequence $(U_n)_{n\in\mathbb{N}}$ as follows: $L_0 \coloneqq T$, $\ell_n \coloneqq U_n L_{n-1}$ and $L_n \coloneqq L_{n-1} - \ell_n$ for $n \in \mathbb{N}$. The process $(L_n)_{n\in\mathbb{N}\cup\{0\}}$ will be referred to as the stick-remainders.*

Note that if $(\ell_n)_{n\in\mathbb{N}}$ is a uniform stick-breaking process on $[0,T]$, then $(a\ell_n)_{n\in\mathbb{N}}$ is a stick-breaking process on $[0,aT]$ for any $a > 0$. There is a very close relationship between the convex minorant (or the concave majorant) and the stick-breaking process, as seen in the following theorem.

Theorem 2.18 ([38, Thm 3.1])**.** *Let X be a one-dimensional Lévy process and $(\ell_n)_{n\in\mathbb{N}}$ be a uniform stick-breaking process on $[0,T]$ independent of X for a fixed $T > 0$. Then, the convex minorant $C(t)$ of X on $[0,T]$ has the same law (in the space of continuous functions on $[0,T]$) as the piecewise linear convex function on $[0,T]$, given by the formula*

$$t \mapsto \sum_{n=1}^{\infty} \xi_n \min\{\max\{t-a_n,0\}/\ell_n, 1\}, \quad \textit{where} \quad \xi_n \coloneqq X_{L_{n-1}} - X_{L_n} \quad \textit{and}$$
$$a_n \coloneqq \sum_{k=1}^{\infty} \ell_k \cdot \mathbb{1}_{\{\xi_k/\ell_k<\xi_n/\ell_n\}} + \sum_{k=1}^{n-1} \ell_k \cdot \mathbb{1}_{\{\xi_k/\ell_k=\xi_n/\ell_n\}}, \quad \textit{for } n \in \mathbb{N}. \tag{2.8}$$

In particular, the face of the piecewise linear function with length ℓ_n has vertical height ξ_n.

Note that a similar result exists in the case where we consider the concave majorant. In that case, the indicator function in the first part of a_n from (2.8) would have the opposite inequality, since we would sort by decreasing slope instead (in the case of the convex minorant the faces are in order of increasing slope). Moreover, note that Theorem 2.18, generalises to *all* Lévy processes the characterisation of the law of the convex minorant (and the concave majorant) established in [64] for diffuse Lévy processes. This extension is very important for the results in this thesis, since it allows us to study the convex minorant (and the concave majorant) of all Lévy processes, including Poisson processes with drift.

Corollary 2.19 ([38, Cor. 3.2]). *Let $\theta \in [0,\infty)$ and $\lambda_\theta \sim \mathrm{Exp}(\theta)$ (with $\lambda_0 = \infty$) be independent of the Lévy process X. When $\theta = 0$ we assume that $l := \liminf_{t\to\infty} X_t/t > -\infty$. Define the σ-finite measure $\mu_\theta(\mathrm{d}t, \mathrm{d}x)$ on $(0,\infty)\times\mathbb{R}$:*

$$\mu_\theta(\mathrm{d}t,\mathrm{d}x) := \begin{cases} t^{-1}e^{-\theta t}\mathbb{P}(X_t \in \mathrm{d}x)\mathrm{d}t, & \theta > 0, \\ \mathbb{1}_{\{x/t<l\}}t^{-1}\mathbb{P}(X_t \in \mathrm{d}x)\mathrm{d}t, & \theta = 0. \end{cases}$$

Let $\Xi_\theta = \sum_{n\in\mathbb{N}} \delta_{(\ell_n^{(\theta)},\xi_n^{(\theta)})}$ be a Poisson point process with mean measure μ_θ. Then the convex minorant $C_{\lambda_\theta} = (C_{\lambda_\theta}(t))_{t\in[0,\lambda_\theta]}$ of X has the same law as the piecewise linear function given in (2.8). In particular, the face of the piecewise linear function with horizontal length $\ell_n^{(\theta)}$ has vertical height $\xi_n^{(\theta)}$, and when $\theta = 0$, the corresponding slope $\xi_n^{(\theta)}/\ell_n^{(\theta)}$ lies on the interval $(-\infty, l)$.

Remark 2.20 ([38, Sec. 2.2 & App. A]). Let $\theta \in (0,\infty)$ and $\lambda_\theta \sim \mathrm{Exp}(\theta)$ be independent of the Lévy process X. Let $(\ell_n^{(\theta)})_{n\in\mathbb{N}}$ be a stick-breaking process on $[0,\lambda_\theta]$. Then the random measure $\sum_{n=1}^\infty \delta_{\ell_n^{(\theta)}}$ on $(0,\infty)$ is a Poisson point process with mean measure satisfying $\mathbb{E}[\sum_{n=1}^\infty \delta_{\ell_n^{(\theta)}}(A)] = \int_A t^{-1}e^{-\theta t}\mathrm{d}t$ for any measurable set A. ◊

In the following remark, we see the relation between the values of l and the finiteness of the convex minorant C_∞.

Remark 2.21 ([64, Cor. 3]). Note that the value $l = \liminf_{t\to\infty} X_t/t$ is a.s. constant, and $l \in (-\infty,\infty]$ if and only if C_∞ is a.s. finite. ◊

For a Lévy process that is not a compound Poisson process, we let ρ_t denote the a.s. unique time at which the minimum of X is attained on $[0,t]$, and in the ensuing theorem we see a property for the distribution of ρ.

Theorem 2.22 ([64, Thm 2]). *Let X be a Lévy process (but not compound Poisson process) for which 0 is regular for both half-lines $(-\infty,0)$ and $(0,\infty)$. Then the distribution of ρ_1 is equivalent to the Lebesgue measure on $[0,1]$.*

In the last part of this section we will discuss the *vertex time process* of the convex minorant C_{λ_θ} with exponential time horizon. For some $\theta \in [0,\infty)$, let $\lambda_\theta \sim \mathrm{Exp}(\theta)$ (with $\lambda_0 = \infty$) be independent of the Lévy process X. Define the right derivative of the convex minorant $C_{\lambda_\theta} = (C_{\lambda_\theta}(t))_{t\in[0,\lambda_\theta]}$ by $C'_{\lambda_\theta}(t) := \lim_{\varepsilon\downarrow 0}(C_{\lambda_\theta}(t+\varepsilon) - C_{\lambda_\theta}(t))/h$, which exists for all $t \in [0, \lambda_\theta)$. The vertex time process $\widehat{\tau} = (\widehat{\tau}_s)_{s\in\mathbb{R}}$ with an exponential time horizon (see [38, Sec. 2.3]), is defined by $\widehat{\tau}_s := \inf\{t \in (0, \lambda_\theta) : C'_{\lambda_\theta}(t) > s\} \wedge \lambda_\theta$, and is the right-inverse of the non-decreasing process C'_{λ_θ}.

Theorem 2.23 ([38, Thm 2.9])**.** *Let $\theta \in [0,\infty)$ and let l be as in Corollary 2.19. Then, $\widehat{\tau}$ has independent but non-stationary increments and its Laplace transform is given by*

$$\mathbb{E}[e^{-w\widehat{\tau}_s}] = \exp\left(-\int_0^\infty (1-e^{-wt})e^{-\theta t}\mathbb{P}(X_t \le st)\frac{\mathrm{d}t}{t}\right),$$

for all $w \ge 0$ and either $s \in \mathbb{R}$ (if $\theta > 0$) and or $s \in (-\infty, l)$ (if $\theta = 0$).

§2.4.1 Cauchy case

In this section, we will state some of the results known about the derivative of the convex minorant C' in the case where X is a Cauchy process (see also Example 5.2). Throughout the rest of this section, we assume $X = (X_t)_{t\in[0,1]}$ to be a standard one-dimensional Cauchy process and let $C = (C(t))_{t\in[0,1]}$ be its convex minorant, with right-derivative C'.

Theorem 2.24 ([21, Thm 2])**.** *The process $(C'(t))_{t\in(0,1)}$ is continuous and has the same law as $-\cot(\pi L(t\gamma_1))$ for $t \in (0,1)$, where $\gamma = (\gamma_t)_{t\ge 0}$ is a standard gamma process and, for $x \ge 0$, $L(x) = \inf\{s \ge 0 : \gamma_s > x\}$ is the inverse process of γ.*

Corollary 2.25 ([21, Cor. 3])**.** *With probability one, it holds that*

$$\liminf_{s\downarrow 0}\frac{|C'(s)|\log\log\log(1/s)}{\log(1/s)} = \liminf_{s\downarrow 0}\frac{|C'(1-s)|\log\log\log(1/s)}{\log(1/s)} = \frac{1}{\pi}.$$

Moreover, if $f : [0,\infty) \to [0,\infty)$ is an increasing function so that $t \mapsto f(t)/t$ is decreasing, then both $\limsup_{s\downarrow 0}|C'(s)|f(s)$ and $\limsup_{s\downarrow 0}|C'(1-s)|f(s)$ equals 0 or ∞ according as $x \mapsto f(x)/x$ is integrable at 0 or not.

Let $\tau_s := \inf\{t \in (0,1) : C'(t) > s\} \wedge 1$ be the right inverse of C'. As explained in the paragraph ensuing the proof of [21, Cor. 3], we can also study the behaviour of C' on $(0,1)$, and show, using a variation of the arguments used in the proof [21, Cor. 3], that

$$\limsup_{s\downarrow 0}\frac{(C'(s+\tau_x)-x)\log(1/s)}{\log\log\log(1/s)} = \pi(1+x^2) \qquad \text{a.s.} \tag{2.9}$$

§2.5 Vigon theorems & related results

Vigon proved many important results, and in this section we introduce some of these results, as well as some related results. We start by defining the notion of abruptness, which is important when characterising smoothness, was introduced by Vigon in his PhD thesis [78, Def. 12.1.1] (see also [79, Def. 1.1]).

Definition 2.26 ([79, Def. 1.1]). *Let X be a Lévy process of infinite variation, and set $M := \{t \in [0,\infty) : \exists \varepsilon > 0 \, \forall s \in (t-\varepsilon, t+\varepsilon) : X_s \leq X_t \text{ or } X_s \leq X_{t-}\}$ to be the set of local maxima of X. Then X is said to be abrupt if the following Dini derivatives are infinite at every local minimum t of the path of X, i.e., for all $t \in M$,*

$$\limsup_{\varepsilon \uparrow 0} \frac{X_{t+\varepsilon} - X_{t-}}{\varepsilon} = -\infty \qquad \text{and} \qquad \liminf_{\varepsilon \downarrow 0} \frac{X_{t+\varepsilon} - X_t}{\varepsilon} = \infty.$$

The main result of [79] gives an integral criteria, in terms of the transition probabilities of X, for when an infinite variation Lévy process is abrupt.

Theorem 2.27 ([79, Thm 1.3]). *A Lévy process X of infinite variation is abrupt if and only if $\int_0^1 \mathbb{P}(X_t/t \in [a,b]) t^{-1} \mathrm{d}t < \infty$ for all $a, b \in \mathbb{R}$ where $a < b$.*

The notion of an *eroded* process was introduced in [80, Def. 1.2] (see also [78, App. D, p. 10]).

Definition 2.28 ([80, Def. 1.2]). *An infinite variation Lévy process X is* eroded *if the following Dini derivatives equal zero at every local minimum t of the path of X, i.e. for all $t \in M$,*

$$\limsup_{\varepsilon \uparrow 0} \frac{X_{t+\varepsilon} - X_{t-}}{\varepsilon} = 0 \qquad \text{and} \qquad \liminf_{\varepsilon \downarrow 0} \frac{X_{t+\varepsilon} - X_t}{\varepsilon} = 0.$$

In a similar fashion to Definition 2.28, we define a Lévy process X to be *strongly eroded* if $(X_t - rt)_{t \geq 0}$ is eroded for every $r \in \mathbb{R}$. In the ensuing theorem, we see a characterisation of when X is eroded in terms of the transition probabilities. This theorem can also be derived using [79, Prop. 3.6].

Theorem 2.29 ([80, Thm 1.4]). *An infinite variation Lévy process X is eroded if and only if the measure S, given by $S(\mathrm{d}x) = \int_0^1 \mathbb{P}(-X_t/t) \in \mathrm{d}x) t^{-1} \mathrm{d}t$, gives infinite mass to all neighbourhoods of 0.*

Another important class of Lévy processes, is the class of processes that *creeps*. Recall that the first passage time of X is defined as $R_x = \inf\{t > 0 : X_t > x\}$.

Definition 2.30 ([77, Def. 1.1]). *The Lévy process X is said to creep upwards if, for all $x > 0$, $\mathbb{P}(X_{R_x} = x) > 0$. We say that X creeps downwards if $-X$ creeps upwards.*

Example 2.3 ([79, Ex. 1.5]). The property of creeping is closely related to abruptness. Indeed, if a Lévy process X creeps upwards or downwards, then X is abrupt. △

In the following theorem, we see a characterisation of when X creeps upwards given in terms of the characteristics of X, which is more tractable than criteria given in terms of the transition probabilities.

Theorem 2.31 ([77, Thm Kaa]). *Let X be an infinite variation Lévy process on $\mathbb{R}$ with Lévy measure ν and no Gaussian component. Then, X creeps upwards, if and only if*

$$\int_0^1 \frac{x}{\int_{-x}^0 \overline{\overline{\pi}}_1(u)\mathrm{d}u} \overline{\pi}(x)\mathrm{d}x < \infty,$$

where $\overline{\pi}(x) = \int_{\mathbb{R}}(\mathbb{1}_{\{0<x\le s\}} + \mathbb{1}_{\{s\le x<0\}})\nu(\mathrm{d}s)$ for all $x \in \mathbb{R}$ and $\overline{\overline{\pi}}_1(x) = \int_{-1}^x \overline{\pi}(u)\mathrm{d}u$ for all $x \in [-1, 0)$.

Remark 2.32 ([55, Eq. (1.7)]). Let X be a Lévy process with $\sigma^2 = 0$ and Lévy measure ν. Consider the following asumptions:
(a) $\int_{(0,1)} x\nu(\mathrm{d}x) < \infty$ and $\int_{(-1,0)} |x|\nu(\mathrm{d}x) = \infty$.
(b) $\int_{(-1,0)} |x|\nu(\mathrm{d}x) < \infty$ and $\int_{(0,1)} x\nu(\mathrm{d}x) = \infty$,
Assuming (a) (resp. (b)) implies that X will creep upwards (resp. downwards). ◊

§2.5.1 Integrability of $\mathfrak{s}_p(r)$

Throughout the remainder of the section we assume the Lévy process X to have infinite activity. Recall that ψ is the characteristic exponent of X, satisfying $\psi(u) = \log \mathbb{E}[\exp(iuX_1)]$ for $u \in \mathbb{R}$. Define for any $p > 0$ and $r \in \mathbb{R}$,

$$\mathfrak{s}_p(r) := \frac{1}{2\pi}\int_{\mathbb{R}} \Re \frac{1}{p + iur - \psi(u)}\mathrm{d}u.$$

The relation $\mathfrak{s}_p(r) \in (0, \infty]$ holds since, by $\Re\psi(u) \le 0$ and (4.5), the integrand in the definition of $\mathfrak{s}_p(r)$ is positive for all $u \in \mathbb{R}$. Define for any $q > p > 0$ the measures μ_p and $\mu_{p,q}$ given by

$$\mu_p(A) := \int_0^\infty \mathbb{P}(X_t/t \in A)e^{-pt}\frac{\mathrm{d}t}{t}, \quad \mu_{p,q}(A) := \int_0^\infty \mathbb{P}(X_t/t \in A)(e^{-pt} - e^{-qt})\frac{\mathrm{d}t}{t},$$

for any measurable $A \subset \mathbb{R}$. We note here that both measures are diffuse since the law of X_t/t is diffuse by Lemma 2.2. Moreover, $\mu_{p,q}(\mathbb{R}) < \infty$ for any finite $q > p > 0$ since $t \mapsto (e^{-pt} - e^{-qt})/t$ is integrable on $(0, \infty)$, while clearly $\mu_p(\mathbb{R}) = \infty$ for any Lévy process X. In fact, X is strongly eroded if and only if $\mu_p(I) = \infty$ for all bounded intervals I in $\mathbb{R}$ (see Theorem 4.2 below).

Remark 2.33. For any $q > 0$ and $z, w \in \mathbb{C}$ with $\Re z, \Re w \ge 0$ and $q \ge |z - w|$ we have $\Re(1/(q+z)) \le 8\Re(1/(q+w))$. Indeed, the inequality is equivalent to $(q+\Re z)|q+w|^2 \le 2(q+\Re w)\cdot(2|q+z|)^2$, which follows from $|q+w| = |q+z+(w-z)| \le |q+z|+q \le 2|q+z|$ and $q + \Re z = q + \Re w + \Re(z - w) \le 2(q + \Re w)$. Thus $(1/8)\mathfrak{s}_q(r) \le \mathfrak{s}_{p+q}(r) \le 8\mathfrak{s}_q(r)$

for any $q \geq p > 0$ and $r \in \mathbb{R}$, implying that the finiteness and local integrability of $\mathfrak{s}_p$ do not depend on $p \in (0, \infty)$. $\diamondsuit$

For Lévy processes with bounded jumps, Theorem 2.34 below was established in [80, Thm 1.5]. We extend this result to all infinite activity Lévy processes. Our proof follows the same strategy as the one in [80] but is shorter and has the advantage of being almost completely elementary, requiring only basic facts about Fourier inversion and Brownian motion. The key step in [80], relying heavily on the fluctuation theory of Lévy processes, is replaced by a simple Gaussian perturbation of the Lévy process. Moreover, almost no potential theory is used in our proof. More specifically, we apply Theorem A.27 only once to show that $\lim_{q\to\infty} \mathfrak{s}_q(r) = 0$ whenever $\mathfrak{s}_p(r) < \infty$ for some $p > 0$.

Theorem 2.34. *Suppose X has infinite activity. Then for any $p \in (0, \infty)$ and $-\infty \leq a < b \leq \infty$, we have*

$$\int_a^b \mathfrak{s}_p(r)\mathrm{d}r = \mu_p((a, b)). \tag{2.10}$$

The equivalence (4.3) below is immediate from Theorem 2.34. The proof of Proposition 2.35 below is elementary, requiring no knowledge of potential theory for Lévy processes. Theorem 2.34 follows easily from Proposition 2.35 and Theorem A.27 as we will see below. Moreover, fluctuation identities are not used in the proof of Theorem 2.34, which is what one would expect since identity (2.10) involves only the marginal laws of X.

Proposition 2.35. *Suppose X has infinite activity. For $p > 0$ and $a, b \in \mathbb{R}$ with $a < b$ we have:*

(a) *if $\int_a^b \mathfrak{s}_p(r)\mathrm{d}r < \infty$, then for any $q \in [p, \infty)$ we have*

$$\int_a^b (\mathfrak{s}_p(r) - \mathfrak{s}_q(r))\mathrm{d}r = \mu_{p,q}((a, b)); \tag{2.11}$$

(b) *if $\mu_p((a, b)) < \infty$, then $\int_{a+\varepsilon}^{b-\varepsilon} \mathfrak{s}_p(r)\mathrm{d}r < \infty$ for every $\varepsilon \in (0, (b - a)/2)$.*

Note that (2.11), applied to every open subinterval of (a, b), implies $\mathfrak{s}_p \geq \mathfrak{s}_q$ a.e. on the interval (a, b) for any $q \in [p, \infty)$. We now show that Proposition 2.35 implies Theorem 2.34.

Proof of Theorem 2.34. First assume $a, b \in \mathbb{R}$ and $\int_a^b \mathfrak{s}_p(r)\mathrm{d}r < \infty$. Then $\mathfrak{s}_p$ is finite a.e. on (a, b). Recall that the q-capacity c_r^q of the set $\{0\}$ (see Definition A.25) for the process $(X_t - rt)_{t\geq 0}$ satisfies $(4\mathfrak{s}_q(r))^{-1} \leq c_r^q \leq \mathfrak{s}_q(r)^{-1}$ for any $q > 0$ by Remark A.26. If $\mathfrak{s}_p(r) < \infty$, then, by Theorem A.27 (see also Remark A.30), we have $c_r^q \to \infty$ and hence $\mathfrak{s}_q(r) \to 0$ as $q \to \infty$. Since, by (2.11) we have $\mathfrak{s}_q \leq \mathfrak{s}_p$ a.e.

on the interval (a,b) for any $q \in [p,\infty)$, the monotone convergence theorem (along a countable sub-sequence) implies $\int_a^b (\mathfrak{s}_p(r) - \mathfrak{s}_q(r))\mathrm{d}r \uparrow \int_a^b \mathfrak{s}_p(r)\mathrm{d}r$ as $q \to \infty$. Again, by monotone convergence, we have $\mu_{p,q}((a,b)) \uparrow \mu_p((a,b))$ as $q \to \infty$, implying the identity $\mu_p((a,b)) = \int_a^b \mathfrak{s}_p(r)\mathrm{d}r$ by (2.11).

Next we show that, for $a,b \in \mathbb{R}$, $\int_a^b \mathfrak{s}_p(r)\mathrm{d}r = \infty$ implies $\mu_p((a,b)) = \infty$. Suppose $\mu_p((a,b)) < \infty$. Then $\int_{a+1/n}^{b-1/n} \mathfrak{s}_p(r)\mathrm{d}r < \infty$ for all sufficiently large $n \in \mathbb{N}$ by Proposition 2.35(b). Hence, (2.10) holds over every interval $(a+1/n, b-1/n)$. Since $\mathfrak{s}_p \geq 0$, taking $n \to \infty$ and applying the monotone convergence theorem gives (2.10) over the interval (a,b), completing the proof for $a,b \in \mathbb{R}$.

Take any real sequences $a_n \downarrow a$ and $b_n \uparrow b$ as $n \to \infty$. Since (2.10) holds over the intervals (a_n, b_n), the monotone convergence theorem implies (2.10) over the possibly infinite interval (a,b). □

The following result can be deduced from the results in [70, Sec. 42] on the potential theory of Lévy processes. Since Lemma 2.36 is key in the proof of Proposition 2.35, we include an elementary short proof for completeness.

Lemma 2.36. *Suppose the Lévy process X is of infinite activity and $\mathfrak{s}_p(r) < \infty$ for some $p > 0$, $r \in \mathbb{R}$. Then, for any $\varepsilon > 0$ we have*

$$\frac{1}{2\pi}\int_{\mathbb{R}} \frac{\sin(u\varepsilon)}{u\varepsilon}\Re\frac{1}{p + iur - \psi(u)}\mathrm{d}u = \frac{1}{2\varepsilon}\int_0^\infty \mathbb{P}(X_t - rt \in (-\varepsilon,\varepsilon))e^{-pt}\mathrm{d}t. \tag{2.12}$$

In particular, the following limit holds

$$\mathfrak{s}_p(r) = \lim_{\varepsilon \downarrow 0}\frac{1}{2\varepsilon}\int_0^\infty \mathbb{P}(X_t - rt \in (-\varepsilon,\varepsilon))e^{-pt}\mathrm{d}t, \qquad p > 0. \tag{2.13}$$

Note from (2.13) that $p \mapsto \mathfrak{s}_p(r)$ is a non-increasing function for each $r \in \mathbb{R}$.

Proof of Lemma 2.36. Since $(X_t - rt)_{t\geq 0}$ has infinite activity, we may assume without loss of generality that $r = 0$. Recall the measure $U^p(\mathrm{d}x) := \int_0^\infty \mathbb{P}(X_t \in \mathrm{d}x)e^{-pt}\mathrm{d}t$ on $\mathbb{R}$ from Definition A.24. Note that U^p is diffuse by Lemma 2.2 and, by Fubini's theorem, the Fourier transform of U^p equals $\int_{\mathbb{R}} e^{iux}U^p(\mathrm{d}x) = \int_0^\infty e^{-(p-\psi(u))t}\mathrm{d}t = 1/(p - \psi(u))$. Fourier inversion formula from Theorem A.5 and Fubini's theorem yield

$$\begin{aligned} U^p((-\varepsilon,\varepsilon)) &= \lim_{c\to\infty}\frac{1}{2\pi}\int_{-c}^{c}\Re\left[\frac{e^{iu\varepsilon} - e^{-iu\varepsilon}}{iu}\frac{1}{p - \psi(u)}\right]\mathrm{d}u \\ &= \lim_{c\to\infty}\frac{1}{\pi}\int_{-c}^{c}\frac{\sin(u\varepsilon)}{u}\Re\frac{1}{p - \psi(u)}\mathrm{d}u. \end{aligned}$$

Since $0 < \Re(1/(p - \psi(u)))^2 \leq \Re(1/(p - \psi(u)))/p$ for all u (as in (4.5)), and since $\mathfrak{s}_p(0) < \infty$, the function $u \mapsto \Re(1/(p - \psi(u)))$ is square-integrable. Since

$u \mapsto \sin(u\varepsilon)/u$ is also square-integrable, their product is integrable by Cauchy–Schwarz. Thus, $U^p((-\varepsilon,\varepsilon)) = \pi^{-1}\int_{\mathbb{R}}(\sin(u\varepsilon)/u)\Re(1/(p-\psi(u)))\mathrm{d}u$ for any $\varepsilon > 0$, implying (2.12). Since $|\sin(x)/x| \le 1$ and the map $u \mapsto \Re(1/(p-\psi(u))) \ge 0$ is integrable, taking $\varepsilon \downarrow 0$ in (2.12) gives (2.13) by the dominated convergence theorem. □

Proof of Proposition 2.35. (a). Since $\mathfrak{s}_p$ is integrable on (a,b), it is finite a.e. on (a,b). By Remark 2.33, for each $r \in (a,b)$ with $\mathfrak{s}_p(r) < \infty$ we have $\mathfrak{s}_q(r) < \infty$ for all $q > 0$. Hence $\mathfrak{s}_q(r) \ge (2\varepsilon)^{-1}\int_0^\infty \mathbb{P}(X_t - rt \in (-\varepsilon,\varepsilon))e^{-qt}\mathrm{d}t \to \mathfrak{s}_q(r)$ as $\varepsilon \downarrow 0$ by (2.12) in Lemma 2.36 since $|\sin(x)/x| \le 1$ for $x \in \mathbb{R}$. Thus, the dominated convergence theorem and Fubini's theorem give

$$\begin{aligned}\int_a^b \mathfrak{s}_q(r)\mathrm{d}r &= \int_a^b \lim_{\varepsilon\downarrow 0}\frac{1}{2\varepsilon}\int_0^\infty \mathbb{P}(X_t - rt \in (-\varepsilon,\varepsilon))e^{-qt}\mathrm{d}t\mathrm{d}r\\ &= \lim_{\varepsilon\downarrow 0}\int_0^\infty \frac{1}{2\varepsilon}\mathbb{E}\bigg[\int_a^b \mathbb{1}_{((X_t-\varepsilon)/t,(X_t+\varepsilon)/t)}(r)\mathrm{d}r\bigg]e^{-qt}\mathrm{d}t.\end{aligned}$$

The random variable $\frac{1}{2}(t/\varepsilon)\int_a^b \mathbb{1}_{((X_t-\varepsilon)/t,(X_t+\varepsilon)/t)}(r)\mathrm{d}r$ is bounded by 1 and converges to $\mathbb{1}_{(a,b)}(X_t/t) + \frac{1}{2}\mathbb{1}_{\{a,b\}}(X_t/t)$ as $\varepsilon \downarrow 0$, which equals $\mathbb{1}_{(a,b)}(X_t/t)$ a.s. since X has infinite activity. Since the function $t \mapsto (e^{-pt} - e^{-qt})/t$ is integrable on $(0,\infty)$ for any $q > p$, the dominated convergence theorem implies

$$\begin{aligned}\int_a^b (\mathfrak{s}_p(r) - \mathfrak{s}_q(r))\mathrm{d}r &= \lim_{\varepsilon\downarrow 0}\int_0^\infty \mathbb{E}\bigg[\frac{t}{2\varepsilon}\int_a^b \mathbb{1}_{((X_t-\varepsilon)/t,(X_t+\varepsilon)/t)}(r)\mathrm{d}r\bigg](e^{-pt} - e^{-qt})\frac{\mathrm{d}t}{t}\\ &= \int_0^\infty \mathbb{E}\big[\mathbb{1}_{(a,b)}(X_t/t)\big](e^{-pt} - e^{-qt})\frac{\mathrm{d}t}{t} = \int_0^\infty \mathbb{P}(X_t/t \in (a,b))(e^{-pt} - e^{-qt})\frac{\mathrm{d}t}{t},\end{aligned}$$

establishing (2.11).

(b). Assume $\mu_p((a,b)) < \infty$ and that there exists some $\varepsilon > 0$ such that $\int_{a+\varepsilon}^{b-\varepsilon}\mathfrak{s}_p(r)\mathrm{d}r = \infty$. We now show that these assumptions lead to a contradiction. Let Brownian motion B be independent of X. The characteristic exponent of $X + \varsigma B$ equals $u \mapsto \psi(u) - \varsigma^2u^2/2$ for any $\varsigma > 0$. For any $p > 0$ let

$$\mathfrak{s}_p^\varsigma(r) := \frac{1}{2\pi}\int_{\mathbb{R}}\Re\frac{1}{p + iur - \psi(u) + \varsigma^2u^2/2}\mathrm{d}u, \quad \text{and note}$$

$$0 < \int_a^b \mathfrak{s}_p^\varsigma(r)\mathrm{d}r \le \frac{b-a}{2\pi}\int_{\mathbb{R}}\frac{\mathrm{d}u}{p + \varsigma^2u^2/2} < \infty.$$

Thus (2.11) holds for the process $X + \varsigma B$, the interval $(a+\varepsilon, b-\varepsilon)$ and any $q > p$. Since, by the monotone convergence theorem, the upper bound in the last display tends to zero as $q \to \infty$, the monotone convergence theorem applied to the right-hand side of (2.11) yields

$$\begin{aligned}\int_{a+\varepsilon}^{b-\varepsilon}\frac{1}{2\pi}\int_{\mathbb{R}}\Re\frac{1}{p+iur-\psi(u)+\varsigma^2u^2/2}\mathrm{d}u\mathrm{d}r\\ =\int_0^\infty \mathbb{P}((X_t+\varsigma B_t)/t\in(a+\varepsilon,b-\varepsilon))e^{-pt}\frac{\mathrm{d}t}{t}.\end{aligned}\tag{2.14}$$

Since we assumed that $\int_{a+\varepsilon}^{b-\varepsilon}\mathfrak{s}_p(r)\mathrm{d}r=\infty$, then for every $M>0$ there exist some $K>0$ such that the inequality $(2\pi)^{-1}\int_{a+\varepsilon}^{b-\varepsilon}\int_{-K}^{K}\Re(1/(p+iur-\psi(u)))\mathrm{d}u\mathrm{d}r\geq 2M$ holds. The bound $\Re(1/(p+iur-\psi(u)+\varsigma^2u^2/2))\leq 1/p$ and the dominated convergence theorem (applied as $\varsigma\downarrow 0$) give

$$\frac{1}{2\pi}\int_{a+\varepsilon}^{b-\varepsilon}\int_{-K}^{K}\Re\frac{1}{p+iur-\psi(u)+\varsigma^2u^2/2}\mathrm{d}u\mathrm{d}r\geq M,$$

for all sufficiently small $\varsigma>0$. Since $M>0$ is arbitrary, this implies that the integral on the right side of (2.14) diverges as $\varsigma\downarrow 0$.

To complete the proof, we show that the assumption $\mu_p((a,b))<\infty$ implies that the integral on the right side of (2.14) is bounded as $\varsigma\downarrow 0$. We will first bound the integral on $[\varsigma^2,\infty)$. Note that

$$\mathbb{P}((X_t+\varsigma B_t)/t\in(a+\varepsilon,b-\varepsilon))\leq\mathbb{P}(|\varsigma B_t/t|\geq\varepsilon)+\mathbb{P}(X_t/t\in(a,b)).$$

By assumption, the integral $\int_{\varsigma^2}^\infty\mathbb{P}(X_t/t\in(a,b))e^{-pt}t^{-1}\mathrm{d}t$ is finite and converges to $\mu_p((a,b))<\infty$ as $\varsigma\downarrow 0$. The elementary bound $\mathbb{P}(|B_1|\geq x)\leq e^{-x^2/2}/(\sqrt{2\pi}x)$ implies that,

$$\int_{\varsigma^2}^\infty\mathbb{P}(|\varsigma B_t/t|\geq\varepsilon)e^{-pt}\frac{\mathrm{d}t}{t}=\int_1^\infty\mathbb{P}(|B_1|\geq\varepsilon\sqrt{t})e^{-p\varsigma^2t}\frac{\mathrm{d}t}{t}\leq\int_1^\infty\frac{e^{-\varepsilon^2t/2}}{\varepsilon\sqrt{2\pi}}\frac{\mathrm{d}t}{t^{3/2}}<\infty.$$

It remains to bound the integral on the right side of (2.14) over the interval $(0,\varsigma^2)$ as $\varsigma\downarrow 0$. To do this, note that

$$\sup_{x\in\mathbb{R}}\mathbb{P}(\varsigma B_t/t+x\in(c,d))=\mathbb{P}\big(|B_1|\leq(d-c)\sqrt{t}/(2\varsigma)\big)\leq(d-c)\sqrt{t}/\big(\varsigma\sqrt{2\pi}\big),$$

for any $c<d$. Thus, elementary inequalities yield

$$\begin{aligned}&\int_0^{\varsigma^2}\mathbb{P}((X_t+\varsigma B_t)/t\in(a+\varepsilon,b-\varepsilon))\frac{\mathrm{d}t}{t}\\ &=\int_0^{\varsigma^2}\int_{\mathbb{R}}\mathbb{P}(\varsigma B_t/t+x\in(a+\varepsilon,b-\varepsilon))\mathbb{P}(X_t/t\in\mathrm{d}x)\frac{\mathrm{d}t}{t}\\ &\leq\int_0^{\varsigma^2}\frac{(b-a-2\varepsilon)\sqrt{t}}{\varsigma\sqrt{2\pi}}\frac{\mathrm{d}t}{t}=\int_0^1\frac{(b-a-2\varepsilon)}{\sqrt{2\pi}}\frac{\mathrm{d}t}{\sqrt{t}}<\infty.\end{aligned}$$

Hence, the right side of (2.14) is bounded as $\varsigma\downarrow 0$, completing the proof. □

§2.5.2 Characterisation of infinite variation

The following lemma is proved in [78, Prop. 1.5.3]. The basic idea for its proof is already present in [27], see the first display on page 34 of [27]. As this lemma is very important for the examples in Chapter 4, we give a proof below.

Lemma 2.37 ([78, Prop. 1.5.3])**.** *Let ψ be the characteristic exponent of a Lévy process X. Then the following equivalence holds: $\int_1^\infty u^{-2}|\Re\psi(u)|\mathrm{d}u = \infty$ if and only if X has paths of infinite variation.*

Proof. If the Gaussian component $\sigma^2 > 0$, the integral in the lemma is infinite and X is of infinite variation. We thus assume $\sigma^2 = 0$. Since the compound Poisson process composed of the jumps of X of magnitude at least 1 has a bounded characteristic function, we may assume that the Lévy measure ν of X is supported in the interval $(-1, 1)$. Recall that $\overline{\nu}(x) = \nu((-1, 1) \setminus (-x, x))$. Define $\widetilde{\nu}(x) := \int_x^1 \overline{\nu}(y)\mathrm{d}y$ for $x \in [0, 1)$ and 0 otherwise. By Fubini's theorem we get $\widetilde{\nu}(0) = \int_{(-1,1)} |x|\nu(\mathrm{d}x)$. Moreover, for any twice differentiable function $f : [0, 1) \to [0, \infty)$ with $f(0) = f'(0+) = 0$, Fubini's theorem implies

$$\int_{(-1,1)} f(|x|)\nu(\mathrm{d}x) = \int_{(-1,1)} \int_0^{|x|} f'(y)\mathrm{d}y\nu(\mathrm{d}x)$$
$$= \int_0^1 f'(y)\overline{\nu}(y)\mathrm{d}y = \int_0^1 f''(z)\widetilde{\nu}(z)\mathrm{d}z.$$

The choice $f(x) = x^2$ yields $2\int_0^1 \widetilde{\nu}(x)\mathrm{d}x = \int_{(-1,1)} x^2\nu(\mathrm{d}x)$, implying that $\widetilde{\nu}$ is integrable. Similarly, the choice $f(x) = (1 - \cos(ux))/u^2$ gives $u^{-2}|\Re\psi(u)| = \int_{(-1,1)} u^{-2}(1 - \cos(ux))\nu(\mathrm{d}x) = \int_0^1 \widetilde{\nu}(z)\cos(uz)\mathrm{d}z$. Fix $\lambda \in (0, \infty)$, integrate the last identity on $(0, \lambda)$ and apply Fubini's theorem again to obtain

$$\int_0^\lambda \frac{|\Re\psi(u)|}{u^2}\mathrm{d}u = \int_0^1 \frac{\sin(\lambda x)}{x}\widetilde{\nu}(x)\mathrm{d}x.$$

Note that the integrand is integrable since $|\sin(\lambda x)/x| \le \lambda$ for all $x \in \mathbb{R}$ and $\widetilde{\nu}$ is integrable. Recall that $\widetilde{\nu}(x) = 0$ for $x \in \mathbb{R} \setminus [0, 1)$ and $\widetilde{\nu}$ is of bounded variation (since $\widetilde{\nu} \ge 0$ is non-increasing with $\widetilde{\nu}(0) < \infty$). Hence, Fourier's single-integral formula in Theorem A.6 gives

$$\int_{(0,\infty)} u^{-2}|\Re\psi(u)|\mathrm{d}u = \lim_{\lambda\to\infty} \int_{\mathbb{R}} x^{-1}\sin(\lambda x)\widetilde{\nu}(x)\mathrm{d}x$$
$$= (\pi/2)\widetilde{\nu}(0) = (\pi/2)\int_{(-1,1)} |x|\nu(\mathrm{d}x).$$

This quantity is infinite if and only if X is of infinite variation, completing the proof. □

Chapter 3

Asymptotic shape of the concave majorant of a Lévy process

§3.1 Introduction and main results

The main objective of this chapter is to understand the asymptotic shape of the concave majorant of a Lévy process as the time horizon tends to infinity (see Figure 3.1).

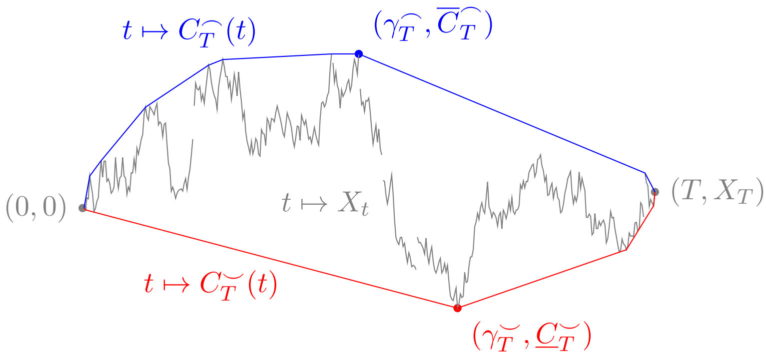

Figure 3.1: A sample path of a Lévy process X on the interval $[0, T]$, the graphs of the concave majorant $C_T^{\frown}$ and the convex minorant $C_T^{\smile}$ and the time and space position of their respective supremum $(\gamma_T^{\frown}, \overline{C}_T^{\frown})$ and infimum $(\gamma_T^{\smile}, \underline{C}_T^{\smile})$.

Let $X = (X_t)_{t \geq 0}$ be a one-dimensional Lévy process and fix a time interval $[0, T]$ for some positive time horizon $T > 0$. As in Definition 2.16, let $(C_T^{\frown}(t))_{t \in [0,T]}$ (resp. $(C_T^{\smile}(t))_{t \in [0,T]}$) be the concave majorant (resp. the convex minorant) of a path of a Lévy process X on the interval $[0, T]$. Let $\Upsilon_T^{\frown}$ (resp. $\Upsilon_T^{\smile}$) denote the length of the graph of the concave (resp. convex) function $t \mapsto C_T^{\frown}(t)$ (resp. $t \mapsto C_T^{\smile}(t)$) over the interval $[0, T]$. The following inequalities are immediate from Figure 3.2 on page 28 below:

$$1 \le \Upsilon_T^{\frown}/T \le \Big(T + 2\overline{C}_T^{\frown} - C_T^{\frown}(T)\Big)/T, \qquad \text{where } \overline{C}_T^{\frown} := \sup_{t\in[0,T]} C_T^{\frown}(t). \tag{3.1}$$

If $\mathbb{E}|X_1|^{1+\epsilon} < \infty$ for some $\epsilon > 0$ and $\mathbb{E}X_1 = 0$, the bounds in (3.1) and Proposition A.45 imply that $\Upsilon_T^{\frown}/T \to 1$ a.s. as $T \to \infty$ (note $\overline{C}_T^{\frown} = \sup_{t\in[0,T]} X_t$ and $C_T^{\frown}(T) = X_T$). Our main aim is to identify the precise asymptotic behaviour and the dependence of the shape parameters $\Upsilon_T^{\frown}$, supremum $\overline{C}_T^{\frown}$, time of supremum $\gamma_T^{\frown}$ and final position $C_T^{\frown}(T)$ of the concave majorant $C_T^{\frown}$. More precisely, we seek to identify the correct asymptotic mean, analyse the fluctuations of the length $\Upsilon_T^{\frown}$ around its asymptotic mean and study their dependence on other shape parameters. If the second moment is infinite, we study analogous questions for X in the domain of attraction of a stable process.

Our main result describes the asymptotic dependence between the fluctuations of the length of the concave majorant, its supremum, final position and the time the supremum is attained, for Lévy processes that have zero mean and finite variance (see Theorem 3.1 below). We also describe this dependence in the case the process is in the domain of attraction of a stable law with stability parameter $\alpha \in (0,2] \setminus \{1\}$ (see Theorems 3.4, 3.6 and 3.7 for $\alpha \in (1,2)$ with zero mean, $\alpha \in (1,2]$ with nonzero mean and $\alpha \in (0,1)$, respectively). As we shall see, the dependence has very different structure in each of these cases, with Theorem 3.1 being the most subtle. In particular, for example, the dependence between the fluctuations of the length of the concave majorant and the other statistics weakens with increasing α. For a short overview of the results in this chapter see YouTube [9].

Before stating our results, recall that the concave majorant of a path of a Lévy process X is a piecewise linear function with countably many faces (see Theorem 2.18). Each face is given by a horizontal length $l > 0$ and a vertical height $h \in \mathbb{R}$, thus having the slope h/l. Note that all the faces with slope equal to a given real value $s \in \mathbb{R}$ must lie next to each other in the graph of the concave majorant and can be concatenated into a *maximal* face with slope s. Let H_T equal the number of maximal faces with horizontal length l at least 1.

Theorem 3.1. *Let $X = (X_t)_{t\ge 0}$ be a Lévy process with Lévy measure ν. Assume that the Lévy process has zero mean $\mathbb{E}[X_1] = 0$ and finite positive variance $\sigma := \sqrt{\mathbb{E}[X_1^2]} \in (0,\infty)$. For $T > 0$ define $\Theta(T) := \frac{1}{2}\int_{\mathbb{R}} x^2 \log^+(\min\{T, x^2\})\nu(\mathrm{d}x)$. Then the following weak limit holds as $T \to \infty$:*

$$\left(\frac{\Upsilon_T^{\frown} - T - (\sigma^2/2)H_T + \Theta(T)}{\sqrt{\log T}}, \frac{H_T - \log T}{\sqrt{\log T}}, \frac{\overline{C}_T^{\frown}}{\sqrt{T}}, \frac{C_T^{\frown}(T)}{\sqrt{T}}, \frac{\gamma_T^{\frown}}{T}\right) \xrightarrow{d} \left(\tfrac{\sigma^2}{\sqrt{2}} Z_1, Z_2, \sigma\overline{B}_1, \sigma B_1, \rho\right), \tag{3.2}$$

where the standard Brownian motion $B = (B_t)_{t\ge 0}$ is independent of the normal random vector (Z_1, Z_2) with zero mean, satisfying $\mathbb{E}Z_1^2 = \mathbb{E}Z_2^2 = 1$ and $\mathbb{E}[Z_1Z_2] = 0$, $\overline{B}_1 := \sup_{t\in[0,1]} B_t$ and $\rho \in [0, 1]$ is the a.s. unique time such that $B_\rho = \overline{B}_1$.

The weak limit in (3.2) shows that the asymptotic centering of the length $\Upsilon_T^{\frown}$ of the concave majorant $C_T^{\frown}$ is stochastic. Moreover, the fluctuations around the centering are asymptotically independent of the centering itself and the randomness in the centering is a function of the horizontal lengths of the faces of $C_T^{\frown}$ only. A linear transformation of the vector in (3.2) yields a deterministic centering of $\Upsilon_T^{\frown}$ at the cost of increasing the asymptotic variance. Put differently, the variance of the centering contributes $\sigma^4/4$ (recall $\sigma^2 = \mathbb{E}[X_1^2]$) to the total asymptotic variance of the length $\Upsilon_T^{\frown}$.

Corollary 3.2. *Under the assumptions of Theorem 3.1, we have*

$$\frac{1}{\sqrt{\log T}}\left(\Upsilon_T^{\frown} - T - \frac{\sigma^2}{2}\log T + \Theta(T)\right) \xrightarrow{d} \frac{\sqrt{3}}{2}\sigma^2 Z, \qquad \text{as } T \to \infty, \tag{3.3}$$

where $\Theta(T) = \frac{1}{2}\int_{\mathbb{R}} x^2 \log^+(\min\{T, x^2\})\nu(\mathrm{d}x) = \mathrm{o}(\log T)$ and Z is a standard normal variable. Moreover, if $\int_{\mathbb{R}} x^2 \log^+(|x|)^{1/2}\nu(\mathrm{d}x) < \infty$, then $\Theta(T) = \mathrm{o}(\sqrt{\log T})$, and thus

$$\frac{1}{\sqrt{\log T}}\left(\Upsilon_T^{\frown} - T - \frac{\sigma^2}{2}\log T\right) \xrightarrow{d} \frac{\sqrt{3}}{2}\sigma^2 Z, \qquad \text{as } T \to \infty. \tag{3.4}$$

Further remarks about Theorem 3.1 and Corollary 3.2 are in order.

Remark 3.3. **(i)** The limit in (3.2) reveals that the fluctuations of the asymptotic length of the concave majorant $C_T^{\frown}$ are independent of its asymptotic supremum, time of supremum and final position. In the case only the first moment of X_1 is finite, the dependence of these shape statistics persists in the limit (see Theorem 3.4 below), while if even the first moment of X_1 is infinite, the length $\Upsilon_T^{\frown}$ becomes a deterministic function of the asymptotic supremum and final position (see Theorem 3.7 below).
(ii) Corollary 3.2 is stated for the deterministic centering of the length only. However, the same linear transform yields a quintuple limit analogous to (3.2). Put differently, as $T \to \infty$, we have

$$\left(\frac{\Upsilon_T^{\frown} - T - (\sigma^2/2)\log T + \Theta(T)}{\sqrt{\log T}}, \frac{H_T - \log T}{\sqrt{\log T}}, \frac{\overline{C}_T^{\frown}}{\sqrt{T}}, \frac{C_T^{\frown}(T)}{\sqrt{T}}, \frac{\gamma_T^{\frown}}{T}\right) \xrightarrow{d} \left(\frac{\sigma^2}{\sqrt{2}}Z_1 + \frac{\sigma^2}{2}Z_2, Z_2, \sigma\overline{B}_1, \sigma B_1, \rho\right).$$

The dependence structure of the length $\Upsilon_T^{\frown}$ and H_T is intractable for any finite $T > 0$, but, as shown by this limit, is asymptotically rather simple.
(iii) There exist Lévy processes for which (3.3) holds and (3.4) does not. Indeed, by

Fubini's theorem, the integral in the asymptotic mean satisfies

$$2\Theta(T) = \int_{\mathbb{R}} x^2 \log^+(\min\{T, x^2\})\nu(\mathrm{d}x) = \int_1^T \frac{1}{t} \int_{\mathbb{R}\setminus(-\sqrt{t},\sqrt{t})} x^2\nu(\mathrm{d}x)\mathrm{d}t,$$

(iv) Note that in the weak limit of Theorem 3.1 neither X nor its concave majorant $C_T^{\frown}$ are scaled before the length $\Upsilon_T^{\frown}$ is calculated. Since X is in the domain of attraction of Brownian motion, one could scale space by $1/\sqrt{T}$ and time by $1/T$ and *then* compute the length of the graph of the resulting concave majorant. This length would, by continuity, converge to the length of the concave majorant of a Brownian motion on $[0, 1]$. For the original length $\Upsilon_T^{\frown}$, this approach only yields $\Upsilon_T^{\frown}/T \overset{d}{\to} 1$.
(v) To the best of our knowledge, Theorem 3.1 had been established neither for Brownian motion nor compound Poisson processes. Moreover, the marginal convergence in Corollary 3.2 does not follow easily from the random walk case, recently analysed in [4], since, for instance, the law of the length of the convex minorant is not invariant under stochastic time-changes, see Figure 3.3 below.
(vi) Consider the counting measure $h_T(A)$, where A is a Borel subset in $\mathbb{R}$, recording the number of maximal faces with horizontal lengths in A. In (3.2) we considered the variable $H_T = h_T([1, \infty))$, but the same weak limit holds for any $h_T([a, \infty))$ with $a \in (0, \infty)$. Moreover, for any bounded set A, the mean measure $\mathbb{E}[h_T(A)]$ equals $\int_A t^{-1}\mathrm{d}t$ for all $T > 0$ satisfying $A \subset [0, T]$ by Lemma 3.9 below. ♢

Recall from (2.6), that a Lévy process X is in the domain of attraction of an α-stable law for some $\alpha \in (0, 2]$ if $X_T/a_T \overset{d}{\to} S_\alpha(1)$, as $T \to \infty$, with scaling function $a_T = T^{1/\alpha}l(T)$ (positive function) and $(S_\alpha(t))_{t\geq 0}$ denote an α-stable process (see §2.3). We note that if X is as in Theorem 3.1 ($\mathbb{E}[X_1] = 0$ and $\sigma = \sqrt{\mathbb{E}[X_1^2]} < \infty$), the standard CLT implies that X satisfies (2.6) with $\alpha = 2$ and scaling function $a_T = \sqrt{T}$. Results analogous to Theorem 3.1 for Lévy process in the domain of attraction of an α-stable law will now be presented: the case $\alpha \in (1, 2)$ with $\mathbb{E}[X_1] = 0$ (resp. $\alpha \in (1, 2]$ with $\mathbb{E}[X_1] \neq 0$; $\alpha \in (0, 1)$) is considered in Theorem 3.4 (resp. Theorem 3.6; Theorem 3.7). The case $\alpha = 2$ with $\mathbb{E}[X_1^2] = \infty$ and $\mathbb{E}[X_1] = 0$ as well as the case $\alpha = 1$ are not considered. To state these theorems, recall the definition of a uniform stick-breaking process $(\ell_n)_{n\in\mathbb{N}}$ on $[0, 1]$ from Definition 2.17.

Theorem 3.4. *Assume X is in the domain of attraction of an $\alpha-$stable law with $\alpha \in (1, 2)$ and $\mathbb{E}[X_1] = 0$. Then, as $T \to \infty$, we have*

$$\begin{aligned}&\left(\frac{T}{a_T^2}\left(\Upsilon_T^{\frown} - T\right), \frac{\overline{C}_T^{\frown}}{a_T}, \frac{C_T^{\frown}(T)}{a_T}, \frac{\gamma_T^{\frown}}{T}\right)\\ &\overset{d}{\to} \left(\frac{1}{2}\sum_{n=1}^{\infty} \ell_n^{2/\alpha-1}\big(S_\alpha^{(n)}\big)^2, \sum_{n=1}^{\infty} \ell_n^{1/\alpha}\big(S_\alpha^{(n)}\big)^+, \sum_{n=1}^{\infty} \ell_n^{1/\alpha}S_\alpha^{(n)}, \sum_{n=1}^{\infty} \ell_n \mathbb{1}_{\{S_\alpha^{(n)}>0\}}\right),\end{aligned} \tag{3.5}$$

where $(\ell_n)_{n\in\mathbb{N}}$ *is a uniform stick-breaking process that is independent of the sequence* $(S_\alpha^{(n)})_{n\in\mathbb{N}}$ *of independent copies of* $S_\alpha(1)$*.*

Under the assumptions of Theorem 3.4, the Lévy process X has infinite variance. By (3.5), the fluctuations of $\Upsilon_T^{\frown}$ about its centering function are typically of order $T^{2/\alpha-1}$, compared with the fluctuations of order $\sqrt{\log T}$ in the finite variance case (see Theorem 3.1 above). The last three coordinates of the limit law in (3.5) have the same law as $(\sup_{t\in[0,1]} S_\alpha(t), S_\alpha(1), \gamma^{\alpha\frown})$, where $\gamma^{\alpha\frown}$ is the time at which the supremum of $S_\alpha(t)$ over $t \in [0,1]$ is attained. We do not know of an interpretation of the law of the first coordinate as a simple functional of the path of the stable process S_α. In particular, it is *not* equal to the law of the length of the concave majorant of S_α on $[0,1]$. However, the tail decay of this coordinate can be characterised using the fact that the law of the series $\sum_{n=1}^\infty \ell_n^{2/\alpha-1}(S_\alpha^{(n)})^2$ satisfies a stochastic perpetuity equation.

Proposition 3.5. *The following asymptotic equivalence holds*

$$\lim_{x\to\infty} \frac{\mathbb{P}\big(\frac{1}{2}\sum_{n=1}^\infty \ell_n^{2/\alpha-1}(S_\alpha^{(n)})^2 > x\big)}{\mathbb{P}((S_\alpha^{(1)})^2 > x)} = \lim_{x\to\infty} \frac{\mathbb{P}\big(\frac{1}{2}\sum_{n=1}^\infty \ell_n^{2/\alpha-1}(S_\alpha^{(n)})^2 > x\big)}{(c_+ + c_-)x^{-\alpha/2}} = \frac{2^{1-\alpha/2}}{2-\alpha},$$

for the constants $c_+, c_- \ge 0$ *defined by* $c_\pm := \lim_{x\to\infty} \mathbb{P}(\pm S_\alpha^{(1)} > \sqrt{x})/x^{-\alpha/2}$*, which satisfy* $c_+ + c_- > 0$*.*

Note that in Theorems 3.1 and 3.4, we have assumed that X has a finite first moment and $\mathbb{E}[X_1] = 0$. If the mean is not zero, the behaviour in these cases is described by the following result. In this description, it is important to distinguish between the cases of positive and negative mean.

Theorem 3.6. *Assume* $\mu := \mathbb{E}X_1 \neq 0$ *and that* X *is in the domain of attraction of an* α*-stable law with* $\alpha \in (1,2]$*.*
(a) Suppose $\mu > 0$*, then, as* $T \to \infty$*, we have*

$$\left(\frac{\Upsilon_T^{\frown} - \sqrt{1+\mu^2}T}{a_T}, \frac{\overline{C}_T^{\frown} - \mu T}{a_T}, \frac{C_T^{\frown}(T) - \mu T}{a_T}\right) \xrightarrow{d} S_\alpha(1)\left(\frac{\mu}{\sqrt{1+\mu^2}}, 1, 1\right).$$

(b) Suppose $\mu < 0$ *and let* $(\overline{X}_\infty, \gamma_\infty^{\frown})$ *be the a.s. finite limit of the supremum and its time* $(\overline{C}_T^{\frown}, \gamma_T^{\frown})$ *as* $T \to \infty$*. Then, as* $T \to \infty$*, we have*

$$\left(\frac{\Upsilon_T^{\frown} - \sqrt{1+\mu^2}T}{a_T}, \overline{C}_T^{\frown}, \frac{C_T^{\frown}(T) - \mu T}{a_T}, \gamma_T^{\frown}\right) \xrightarrow{d} \left(\frac{\mu}{\sqrt{1+\mu^2}}S_\alpha(1), \overline{X}_\infty, S_\alpha(1), \gamma_\infty^{\frown}\right),$$

where $S_\alpha(1)$ *and* $(\overline{X}_\infty, \gamma_\infty^{\frown})$ *are independent.*

Note that the centering function of $\Upsilon_T^{\frown}$ in Theorem 3.6 equals the length of the graph of the linear function $t \mapsto \mu t$ on $[0,T]$. Moreover, the order of the fluctuations of $\Upsilon_T^{\frown}$ in this case is different than that in Theorems 3.1 and 3.4. Asymptotically, $\Upsilon_T^{\frown}$ and $C_T^{\frown}(T)$ are positively correlated when $\mu > 0$ and negatively correlated when $\mu < 0$.

When X is in the domain of attraction of an α-stable law with $\alpha \in (0,1)$, the tails of X are very heavy. The large jumps of X make its concave majorant thin and tall, implying that the length $\Upsilon_T^{\frown}$ will be well approximated by the extremes of X. Define $\underline{C}_T^{\smile} := \inf_{t\in[0,T]} C_T^{\smile}(t)$ and let $\gamma_T^{\smile}$ be the time at which the infimum is attained (see Figure 3.1). Denote $\overline{S}_\alpha(1) := \sup_{t\in[0,1]} S_\alpha(t)$, $\underline{S}_\alpha(1) := \inf_{t\in[0,1]} S_\alpha(t)$ and let $\gamma^{\alpha\frown}$ (resp. $\gamma^{\alpha\smile}$) be the time at which $(S_\alpha(t))_{t\in[0,1]}$ attains its supremum (resp. infimum).

Theorem 3.7. *Let X be in the domain of attraction of the α-stable law $S_\alpha(1)$ for $\alpha \in (0,1)$. Define*

$$\Lambda_T^1 := \left(\frac{\Upsilon_T^{\frown}}{a_T}, \frac{\overline{C}_T^{\frown}}{a_T}, \frac{C_T^{\frown}(T)}{a_T}, \frac{\gamma_T^{\frown}}{T}\right), \quad \Lambda^1 := \left(2\overline{S}_\alpha(1) - S_\alpha(1), \overline{S}_\alpha(1), S_\alpha(1), \gamma^{\alpha\frown}\right),$$
$$\Lambda_T^2 := \left(\frac{\Upsilon_T^{\smile}}{a_T}, \frac{\underline{C}_T^{\smile}}{a_T}, \frac{C_T^{\smile}(T)}{a_T}, \frac{\gamma_T^{\smile}}{T}\right), \quad \Lambda^2 := \left(S_\alpha(1) - 2\underline{S}_\alpha(1), \underline{S}_\alpha(1), S_\alpha(1), \gamma^{\alpha\smile}\right).$$

Then the following joint convergence holds: $\left(\Lambda_T^1, \Lambda_T^2\right) \xrightarrow{d} \left(\Lambda^1, \Lambda^2\right)$ as $T \to \infty$.

The Lévy process X in Theorem 3.7 has a thin and tall concave majorant, so the asymptotic centering by T, present in Theorems 3.1 and 3.4, is no longer required. Moreover, note that in Theorems 3.1 and 3.4 the fluctuations of $\Upsilon_T^{\frown}$ about this centering were significantly smaller than T, which is no longer the case here. The proof of Theorem 3.7 in §3.3.2 below is based on and approximation of $C_T^{\frown}$ by simpler geometric figures such as the ones in Figure 3.2.

The concave majorant lies between two natural geometric figures. Under the concave majorant lies the 'hut' $C_T^{\wedge}$, defined as the linear path connecting the vertices: $(0,0)$, $(\gamma_T^{\frown}, \overline{X}_T)$ and (T, X_T), where $\gamma_T^{\frown} = \arg\inf\{t > 0 : X_t \vee X_{t-} = \overline{X}_T\}$ is the time X attains its supremum on $[0,T]$. Over the concave majorant lies the 'box-top' $C_T^{\sqcap}$, defined as the linear path connecting the vertices: $(0,0)$, $(0,\overline{X}_T)$, $(T,\overline{X}_T)$ and (T, X_T).

Suppose that the lengths of the hut $C_T^{\wedge}$ and the box-top $C_T^{\sqcap}$ are $\Upsilon_T^{\wedge}$ and $\Upsilon_T^{\sqcap}$, respectively. It is clear from the triangle inequality that $\Upsilon_T^{\wedge} \le \Upsilon_T^{\frown} \le \Upsilon_T^{\sqcap}$. These lengths do not generally all have the same asymptotic behaviour. The next result provides a short comparison in the cases $\alpha \in (1,2]$ with $\mathbb{E}[X_1] = 0$ and $\alpha \in (0,1)$.

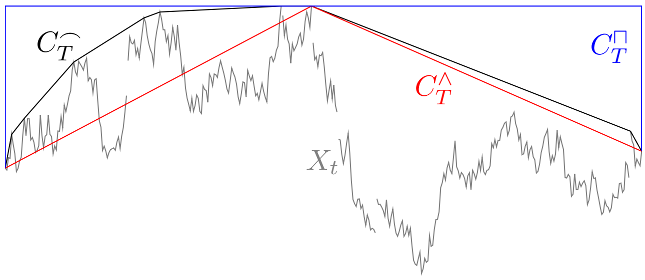

Figure 3.2: The figure shows a sample of the path of X, the concave majorant $C_T^{\frown}$, the hut $C_T^{\wedge}$ and the box-top $C_T^{\sqcap}$.

Proposition 3.8. *Define $\Upsilon_T^{\wedge}$ and $\Upsilon_T^{\sqcap}$ as before then the following statements hold as $T \to \infty$.*

(a) *Suppose $\mathbb{E}[X_1] = 0$ and $\sigma^2 = \mathbb{E}[X_1^2] < \infty$, then*

$$\left(\Upsilon_T^{\wedge} - T, \frac{1}{\sqrt{\log T}}\left(\Upsilon_T^{\frown} - T - \frac{\sigma^2}{2}\log T + \Theta(T)\right), \frac{1}{\sqrt{T}}(\Upsilon_T^{\sqcap} - T)\right) \xrightarrow{d} \left(\frac{\sigma^2}{2}\left(\frac{\overline{B}_1^2}{\rho} + \frac{(\overline{B}_1 - B_1)^2}{1-\rho}\right), \frac{\sqrt{3}}{2}\sigma^2 Z, \sigma(2\overline{B}_1 - B_1)\right),$$

where Z is a standard normal variable independent of the standard Brownian motion $B = (B_t)_{t\ge0}$, $\overline{B}_1 = \sup_{t\in[0,1]} B_t$ and $\rho \in [0,1]$ is the a.s. unique time such that $B_\rho = \overline{B}_1$.

(b) *Suppose the limit in* (2.6) *holds for some $\alpha \in (1,2)$, scaling function a_T and $\mathbb{E}[X_1] = 0$, then*

$$\left(\frac{T}{a_T^2}(\Upsilon_T^{\wedge} - T), \frac{T}{a_T^2}(\Upsilon_T^{\frown} - T), \frac{1}{a_T}(\Upsilon_T^{\sqcap} - T)\right) \xrightarrow{d}$$
$$\frac{1}{2}\left(\left(\sum_{n=1}^{\infty} \ell_n^{1/\alpha}(S_\alpha^{(n)})^+\right)^2 + \left(\sum_{n=1}^{\infty} \ell_n^{1/\alpha}(S_\alpha^{(n)})^-\right)^2, \sum_{n=1}^{\infty} \ell_n^{2/\alpha-1}(S_\alpha^{(n)})^2, 2\sum_{n=1}^{\infty} \ell_n^{1/\alpha}|S_\alpha^{(n)}|\right),$$

where $(\ell_n)_{n\in\mathbb{N}}$ is a uniform stick-breaking process that is independent of the sequence $(S_\alpha^{(n)})_{n\in\mathbb{N}}$ of independent copies of $S_\alpha(1)$.

(c) *Suppose the limit in* (2.6) *holds for some $\alpha \in (0,1)$ and scaling function a_T, then*

$$\left(\frac{\Upsilon_T^{\wedge}}{a_T}, \frac{\Upsilon_T^{\frown}}{a_T}, \frac{\Upsilon_T^{\sqcap}}{a_T}\right) \xrightarrow{d} (2\overline{S}_\alpha(1) - S_\alpha(1))(1,1,1).$$

Under the assumptions of either Theorem 3.1 or Theorem 3.4, the centering functions of $\Upsilon_T^{\wedge}$, $\Upsilon_T^{\frown}$ and $\Upsilon_T^{\sqcap}$ in Proposition 3.8 are of the form $T + o(T)$ as $T \to \infty$. However, even though the three statistics are closely related, the order of their fluctuations, measured via the scaling functions, exhibits a wide variety of behaviours, see Table 3.1 below. In the case $\alpha \in (0,1)$, the centering functions are all zero and the corresponding scaling functions coincide with the scale of the process.

Setting	Scaling of X_T	$\Upsilon_T^{\wedge}$	$\Upsilon_T^{\frown}$	$\Upsilon_T^{\sqcap}$
Theorem 3.1 ($\mathbb{E}[X_1^2] < \infty$)	$a_T = \sqrt{T}$	1	$\sqrt{\log T}$	$\sqrt{T}$
Theorem 3.4 ($1 < \alpha < 2$)	$a_T = T^{1/\alpha} l(T)$	$T^{2/\alpha-1} l(T)^2$	$T^{2/\alpha-1} l(T)^2$	$T^{1/\alpha} l(T)$
Theorem 3.7 ($0 < \alpha < 1$)	$a_T = T^{1/\alpha} l(T)$	$T^{1/\alpha} l(T)$	$T^{1/\alpha} l(T)$	$T^{1/\alpha} l(T)$

Table 3.1: The table shows the scaling functions (after centering) in the weak limits of the lengths $\Upsilon_T^{\wedge}$, $\Upsilon_T^{\frown}$ and $\Upsilon_T^{\sqcap}$ under the assumptions of the corresponding theorems with a_T as in (2.6).

Recall that $\Upsilon_T^{\wedge} \le \Upsilon_T^{\frown} \le \Upsilon_T^{\sqcap}$. Interestingly, for X with finite variance, by Proposition 3.8(a) the fluctuations of $\Upsilon_T^{\frown}$ are asymptotically independent of those of $\Upsilon_T^{\wedge}$ and $\Upsilon_T^{\sqcap}$, while the fluctuations of the sandwiching lengths $\Upsilon_T^{\wedge}$ and $\Upsilon_T^{\sqcap}$ exhibit a strong asymptotic dependence, both being deterministic functions of the vector $(B_1, \overline{B}_1, \rho)$. Proposition 3.8(b)&(c) states that the dependence of the fluctuations of all three statistics persists in the limit when $\alpha < 2$.

§3.1.1 Overview of the proofs

Our starting point is Theorem 2.18, which implies the following crucial identity for any Lévy process and time horizon $T > 0$:

$$\begin{aligned}&(\Upsilon_T^{\frown}, H_T', C_T^{\frown}(T), \overline{C}_T^{\frown}, \gamma_T^{\frown})\\ &\quad \stackrel{d}{=} \sum_{n=1}^{\infty} \Big(\sqrt{(T\ell_n)^2 + \xi_n^2}, \mathbb{1}_{\{T\ell_n \ge 1\}}, \xi_n, \xi_n^+, T\ell_n \mathbb{1}_{\{\xi_n > 0\}} \Big),\end{aligned} \tag{3.6}$$

where $\xi_n := X_{TL_{n-1}} - X_{TL_n}$, H_T' is a random variable such that $|H_T - H_T'|$ is bounded in L^1 as $T \to \infty$ (see Lemma 3.14 below for details) and ℓ is a uniform stick-breaking process independent of X with stick-remainders $(L_n)_{n\in\mathbb{N}\cup\{0\}}$. This identity is essential in all that follows as it reduces the claims in Theorems 3.1, 3.4 and 3.6 to limit statements for the sum in (3.6), which is given in terms of the increments of the Lévy process over independent stick-breaking lengths. Establishing those limits as time horizon $T \to \infty$ turns out to be a delicate task.

In the case of finite variance and zero mean, the proof of Theorem 3.1 requires splitting the weak limits into three asymptotically independent weak limits. The faces of $C_T^{\frown}$ of length smaller than 1 do not contribute to the fluctuations of $(\Upsilon_T^{\frown}, H_T')$. However, all faces of $C_T^{\frown}$ of moderate size contribute in aggregate to its fluctuations, with any finite set of faces the of moderate size not surviving in the limiting fluctuations. In contrast, *only the largest few* faces of $C_T^{\frown}$ influence the scaling limit of the vector $(C_T^{\frown}(T), \overline{C}_T^{\frown}, \gamma_T^{\frown})$, making its limit independent of the limiting fluctuations of $(\Upsilon_T^{\frown}, H_T')$. Moreover, the CLT for $(\Upsilon_T^{\frown}, H_T')$ consists of two asymptotically independent weak limits. The first captures the fluctuations *due to* the stick-breaking process while the second describes the fluctuations *conditional on* a manifestation of the stick-breaking process. The remaining work in the proof of Theorem 3.1 is mostly concerned with establishing weak limits, conditional on the stick-breaking process, and crucially depends on Theorem 3.20. Theorem 3.20 requires different methods to those used in this section. Consequently, Theorem 3.20 is stated and proved in §3.4 below.

In the case of finite first moment and infinite variance, the proofs of Theorems 3.4 and 3.6 split the sum in (3.6) into two sums according to whether the faces are shorter or longer than one. However, unlike in the finite-variance zero-mean case, here this is just a technical step: in the proof of Theorems 3.4 all the faces of the concave majorant survive in the limit, contributing both to the fluctuations of its length as well as the remaining statistics of $C_T^{\frown}$. It follows from the proof of Theorem 3.6 that only the vertical heights of the faces of $C_T^{\frown}$ in aggregate contribute to the fluctuations of its length, which are determined by the asymptotic behaviour of its final point $C_T^{\frown}(T) = X_T$ as $T \to \infty$.

In the infinite first moment case, Theorem 3.7 follows by a sandwiching argument involving the weak limits for the lengths $\Upsilon_T^{\wedge}$ and $\Upsilon_T^{\sqcap}$ as in Proposition 3.8 above. As in the proof of Theorem 3.6, only the heights of the faces of $C_T^{\frown}$ in aggregate contribute to this limit.

§3.1.2 Connections with the literature

Convex hulls of stochastic processes are of longstanding interest, see e.g. [1] and the references therein. Of particular interest are the geometric properties of convex hulls such as the length, area and diameter, see [4, 54, 66, 81, 82] for random walks and [44] for isotropic stable process. Concave majorants of one-dimensional Lévy processes are also of interest in physics. In the monograph [57, Ch. XI], for example, the problem of whether a quantum particle stays within the light cone is analysed using concave majorants of one-dimensional Lévy processes.

If the Lévy process is in the domain of attraction of a stable law, one can pose two types of questions about the limiting behaviour of its convex hulls. A limit of a geometric quantity (e.g. perimeter) of the convex hull of the original process may be considered *or* the limit of the convex hull of the scaled process may be analysed. Since taking the convex hull of the graph of a function is a continuous mapping[1],

[1]This mapping takes càdlàg functions equipped with the Skorokhod topology to compact sets

in the latter case it is natural to expect that the limit will be given in terms of the corresponding geometric quantity of the convex hull of the stable limit, which is what happens in [56, Sec. 5]. In this chapter, we consider the former type of question for the length of the concave majorant. It is clear from Theorems 3.1 and 3.4 above that in this case the asymptotic mean and the scale of the fluctuations around them are of different order than those of the process. Moreover, the limit is not given in terms of the corresponding quantity for the stable process. Differently put, we analyse the statistics describing the geometry of the convex hull of the original process as the time horizon tends to infinity without scaling the process first and then considering the limiting behaviour of such statistics.

The object of study in [56] is the convex hull of the scaled multi-dimensional Lévy process attracted to an isotropic α-stable process. This confers upon the convex hull a spatial homogeneity not enjoyed by the concave majorant, which is a one dimensional object in space-time that behaves very differently in space and time coordinates. A further difference with problem considered in [56] is that our aim is to understand the fluctuations around the asymptotic centering rather than obtaining the limit, which in our case is straightforward, see (3.1) above.

A related question about the fluctuations of the length of the convex minorant of a random walk, as the time horizon tends to infinity, was studied in the recent paper [4]. CLT-type results for the length of the convex minorant of a random walk ware established in [4] under hypothesis analogous to ours (i.e. the increments either have finite variance and zero mean or are in the domain of attraction of an α-stable law for $\alpha \in (0,2) \setminus \{1\}$). The joint limits for the shape statistics in the random walk case are not discussed in [4]. Moreover, we stress that the fluctuations of the length of the concave majorant in our Theorems 3.1, 3.4 and 3.6 cannot be deduced easily from the results of [4] even in the case of a compound Poisson process since the random time-change connecting it with a random walk distorts the concave majorant, see Figure 3.3 below.

As mentioned in §3.1.1 above, a crucial structure used to establish our main results is the characterisation of the law of the concave majorant for all Lévy processes given Theorem 2.18, proved in the recent article [38].

Finally we note that in [70, Sec. 28], Sato explores the long time behaviour of a Lévy process and its supremum of the process. Since the concave majorant on $[0,T]$ always coincides with the process at times T and $\gamma_{\widehat{T}}$, our results may be viewed as an extension of those in [70, Sec. 28].

The remainder of the chapter is organised as follows. Theorem 3.1 is proved

in $\mathbb{R}^2$ equipped with the Hausdorff distance.

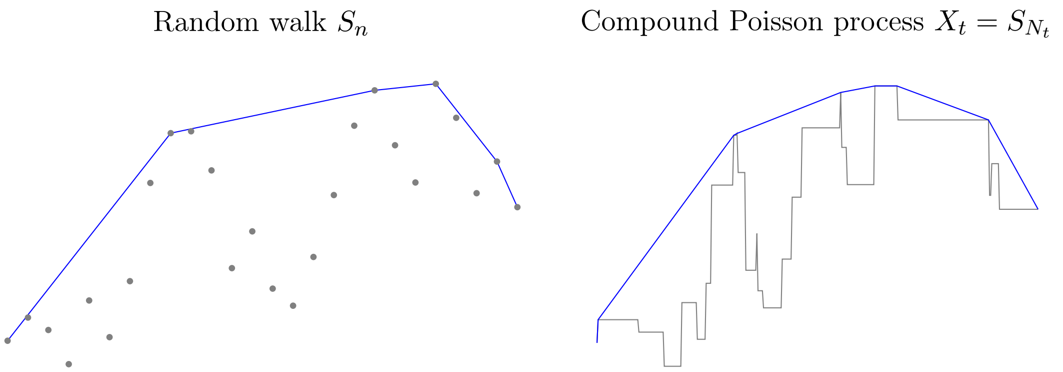

Figure 3.3: The figure shows a sample of the path of a random walk S_n (left) and that of the compound Poisson process $X_t = S_{N_t}$ (right), where N_t is a Poisson process independent of S_n. Note that both processes visit the same states and in the same order, but the random time-change induced by N_t distorts the shape of the concave majorant, since the two concave majorants have a different number of faces.

in §3.2. Theorems 3.4, 3.6 and 3.7 as well as the two propositions in the introduction are proved in §3.3. A Gaussian approximation theorem (Theorem 3.20), is proved in §3.4.

§3.2 Proof of Theorem 3.1

Recall that $\xi_n = X_{TL_{n-1}} - X_{TL_n}$ and denote $t_n := T\ell_n$ for $n \in \mathbb{N}$, where $\ell = (\ell_n)_{n\in\mathbb{N}}$ is a uniform stick-breaking process on $[0,1]$, independent of the Lévy process X, and $L = (L_n)_{n\in\mathbb{N}\cup\{0\}}$ is its stick-remainder process. Recall that $(t_n)_{n\in\mathbb{N}}$ is a uniform stick-breaking process on $[0,T]$. Define the following set of indices $\mathfrak{I}_T := \{n \in \mathbb{N} : t_n \geq 1\}$.

The strategy for the proof of Theorem 3.1 is the following. We will show that the cardinality of $\mathfrak{I}_T$ is by Theorem 2.18 closely related to the random variable H_T appearing in Theorem 3.1 (see Lemma 3.14 below for more details). Setting

$$\varpi_T := \left(0, \frac{|\mathfrak{I}_T| - \log T}{\sqrt{\log T}}, \frac{\sum_{n=1}^{\infty} \xi_n^+}{\sqrt{T}}, \frac{\sum_{n=1}^{\infty} \xi_n}{\sqrt{T}}, \frac{\sum_{n=1}^{\infty} t_n \mathbb{1}_{\{\xi_n>0\}}}{T}\right),$$

and using the aforementioned close relationship and Theorem 2.18, we will find that Theorem 3.1 is equivalent to the vector

$$\left(\frac{\sum_{n=1}^{\infty}(\sqrt{\xi_n^2 + t_n^2} - t_n) - \frac{\sigma^2}{2}|\mathfrak{I}_T| + \Theta(T)}{\sqrt{\log T}}, 0, 0, 0, 0\right) + \varpi_T, \tag{3.7}$$

converging weakly to $\zeta := (\sigma^2 Z_1/\sqrt{2}, Z_2, \sigma\overline{B}_1, \sigma B_1, \rho)$ as $T \to \infty$. We next apply certain moment estimates for X and limit results for the stick-breaking process ℓ to show that the quintuple in (3.7) converges weakly to ζ if and only if the following weak limit holds as $T \to \infty$:

$$\left(\frac{\sum_{n\in\mathfrak{I}_T}(\xi_n^2/t_n - \sigma_{t_n}^2)}{2\sqrt{\log T}}, 0, 0, 0, 0\right) + \varpi_T \xrightarrow{d} \zeta, \tag{3.8}$$

where $\sigma_t^2 := \sigma^2 - \int_{\mathbb{R}\setminus(-\kappa\sqrt{t},\kappa\sqrt{t})} x^2\nu(\mathrm{d}x)$ for any $t \geq 1$ and some $\kappa \geq 1$ such that $\sigma_1 > 0$ (see Proposition 3.16 below for details). Note that the second coordinate in the quintuple in (3.8) is a deterministic function of the stick-breaking process ℓ and denote the remaining quadruple by ζ'_T. In order to establish (3.8), we condition ζ'_T on ℓ and prove that its weak limit under the conditional law is $(\sigma^2 Z_1/\sqrt{2}, \sigma\overline{B}_1, \sigma B_1, \rho)$. Since this limit law does not depend on ℓ, applying Proposition 3.11 below will complete the proof of Theorem 3.1.

The steps described in this strategy require a variety of technical results. The details of the proof of Theorem 3.1 are given after the technical results have been established (see page 43 below).

§3.2.1 Limit properties of the stick-breaking process

The proof of Theorem 3.1 requires a detailed analysis of certain asymptotic properties of the stick-breaking process. We start with a compensation formula for the point process based on a stick-breaking process, analogous to Campbell's formula for Poisson point processes.

Lemma 3.9. *Define the point process $\Xi_T := \sum_{n\in\mathbb{N}} \delta_{t_n}$, where δ_x is the Dirac measure at x. Then for any measurable function $f : [0, T] \to \mathbb{R}_+$ the following identities hold (with all quantities possibly equal to $+\infty$):*

$$\mathbb{E}\left[\int_{\mathbb{R}_+} f(x)\Xi_T(\mathrm{d}x)\right] = \mathbb{E}\left[\sum_{n\in\mathbb{N}} f(t_n)\right] = \int_0^T \frac{f(t)}{t}\mathrm{d}t. \tag{3.9}$$

The point process Ξ_T converges weakly as $T \to \infty$ to a Poisson point process Ξ_∞ on $(0, \infty)$ with intensity $t \mapsto t^{-1}$. Moreover, there exists a coupling of point processes $\overline{\Xi}_\infty$ and $\overline{\Xi}_T$ for all $T > 0$ such that: $\overline{\Xi}_T \overset{d}{=} \Xi_T$ and $\overline{\Xi}_\infty \overset{d}{=} \Xi_\infty$, $\overline{\Xi}_T \to \overline{\Xi}_\infty$ a.s. in the vague topology and for every compact set $A \subset (0, \infty)$, we have $\overline{\Xi}_T|_A = \overline{\Xi}_\infty|_A$ for all sufficiently large T.

The distributional convergence in Lemma 3.9 holds in the vague topology of locally finite measures on $(0, \infty)$, i.e. $\int f(x)\overline{\Xi}_T(\mathrm{d}x) \to \int f(x)\overline{\Xi}_\infty(\mathrm{d}x)$ a.s. as $T \to \infty$ for any continuous function f on $(0, \infty)$ that vanishes at 0 and ∞ (see also [43, Ch. 16, p. 316]).

Proof. Note that $-\log \ell_n$ is gamma distributed with density $t \mapsto t^{n-1}e^{-t}/(n-1)!$.

Thus, Fubini's theorem implies (3.9):

$$\mathbb{E}\left[\sum_{n\in\mathbb{N}} f(t_n)\right] = \sum_{n\in\mathbb{N}} \int_0^\infty \frac{f(Te^{-t})t^{n-1}}{(n-1)!} e^{-t}\mathrm{d}t = \int_0^\infty f(Te^{-t})\mathrm{d}t = \int_0^T \frac{f(t)}{t}\mathrm{d}t.$$

To prove $\Xi_T \xrightarrow{d} \Xi_\infty$ as $T \to \infty$, it suffices to provide a coupling $(\overline{\Xi}_T, \overline{\Xi}_\infty)$ with $\overline{\Xi}_T \overset{d}{=} \Xi_T$ and $\overline{\Xi}_\infty \overset{d}{=} \Xi_\infty$ such that $\overline{\Xi}_T \to \overline{\Xi}_\infty$ a.s. as $T \to \infty$. To that end, let Y be a subordinator with infinite mean $\mathbb{E}[Y_1] = \infty$ and the convex minorant C_∞ on $\mathbb{R}_+$. By Corollary 2.19, for any enumeration of the horizontal lengths $(l_n)_{n\in\mathbb{N}}$ and vertical heights $(h_n)_{n\in\mathbb{N}}$ of the faces of C_∞, the point process $\widetilde{\Xi}_\infty := \sum_{n\in\mathbb{N}} \delta_{(l_n,h_n)}$ on $(0,\infty)^2$ is Poisson with mean measure $t^{-1}\mathbb{P}(Y_t \in \mathrm{d}x)\mathrm{d}t$, $(t,x) \in (0,\infty)^2$. Similarly, let $\widetilde{\Xi}_T$ be the point process of lengths and heights of the convex minorant C_T of Y on $[0,T]$.

For any $s > 0$ define the set $A_s := \{(t,x) \in (0,\infty)^2 : x/t < s\}$ and let T_s be the last time the right derivative of C_∞ was smaller than s, which is a.s. finite by Remark 2.21. It follows that $C_T = C_\infty$ on $[0,T_s]$ for any $T > T_s$, implying that $\widetilde{\Xi}_T$ and $\widetilde{\Xi}_\infty$ agree on A_s for any $T > T_s$. Since $\bigcup_{s>0} A_s = (0,\infty)^2$ and any compact set in $(0,\infty)^2$ is contained in some A_s, we have

$$\int_{(0,\infty)^2} f(y)\widetilde{\Xi}_T(\mathrm{d}y) = \int_{(0,\infty)^2} f(y)\widetilde{\Xi}_\infty(\mathrm{d}y), \qquad T > T_s,$$

for any compactly supported continuous function $f : (0,\infty)^2 \to \mathbb{R}_+$. Since $T_s < \infty$ for all $s > 0$, we therefore have $\widetilde{\Xi}_T \to \widetilde{\Xi}_\infty$ a.s. in the vague topology. Moreover, this implies that the projections $\overline{\Xi}_T := \widetilde{\Xi}_T(\cdot \times \mathbb{R}_+) \overset{d}{=} \Xi_T$ converge to $\overline{\Xi}_\infty := \widetilde{\Xi}_\infty(\cdot \times \mathbb{R}_+) \overset{d}{=} \Xi_\infty$ a.s. in the vague topology. □

Recall that $\mathfrak{I}_T = \{n \in \mathbb{N} : t_n \geq 1\}$ is the finite set of indices of sticks in $[0,T]$ of length greater than one and denote by $\mathfrak{I}_T^{\mathsf{c}} := \mathbb{N} \setminus \mathfrak{I}_T$ its infinite complement.

Corollary 3.10. *(a) Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ be a measurable function and $T \geq 1$. Then the following equalities hold:*

$$\mathbb{E} \sum_{n\in\mathfrak{I}_T} f(t_n) = \int_1^T \frac{f(t)}{t}\mathrm{d}t \qquad \text{and} \qquad \mathbb{E} \sum_{n\in\mathfrak{I}_T^{\mathsf{c}}} f(t_n) = \int_0^1 \frac{f(t)}{t}\mathrm{d}t. \tag{3.10}$$

In particular, the first expectation in (3.10) always has a (possibly infinite) limit as $T \to \infty$ and for any $q > 0$ we have $\lim_{T\to\infty} \mathbb{E}\sum_{n\in\mathfrak{I}_T} t_n^{-q} = 1/q$.

(b) For any bounded and measurable function $f : [1,\infty) \to \mathbb{R}$ with $\lim_{t\to\infty} f(t) = 0$ we have $\mathbb{E}\sum_{n\in\mathfrak{I}_T} f(t_n) = \mathrm{o}(\log T)$, implying that $(\log T)^{-1} \sum_{n\in\mathfrak{I}_T} f(t_n) \xrightarrow{L^1} 0$.

Proof. (a) Note that $f(t_n)\mathbb{1}_{\{n\in\mathfrak{I}_T\}} = h(t_n)$ where $h(t) = f(t)\mathbb{1}_{\{T>1\}}$, so (3.10) follows from (3.9). The formulae for the power functions then follow easily.

(b) Let $T > 1$ and note that

$$\frac{1}{\log T}\mathbb{E}\sum_{n\in\mathfrak{I}_T} f(t_n) = \int_1^T \frac{f(t)}{t\log T}\mathrm{d}t = \mathbb{E}[f(Z_T)],$$

where Z_T has the density $t \mapsto (t\log T)^{-1}$, $t \in [1,T]$. Since $Z_T \xrightarrow{\mathbb{P}} \infty$, we have $f(Z_T) \xrightarrow{\mathbb{P}} 0$ and since the variables $|f(Z_T)|$ are bounded by $\sup_{t\in[1,\infty)}|f(t)|$, the dominated convergence theorem implies that $\mathbb{E}[f(Z_T)] \to 0$. □

We now prove the following CLT for the cardinality of the set $\mathfrak{I}_T$ defined above.

Proposition 3.11. *The cardinality $|\mathfrak{I}_T|$ of the set $\mathfrak{I}_T$ satisfies the limits*

$$|\mathfrak{I}_T|/\log T \xrightarrow{L^1} 1 \qquad \text{and} \qquad (|\mathfrak{I}_T| - \log T)/\sqrt{\log T} \xrightarrow{d} N(0,1) \qquad \text{as } T \to \infty.$$

Moreover, for any T we have $\mathfrak{I}_T \subset \{1,\dots,\tau(T)+1\}$ and $\mathbb{E}[\tau(T)] = \mathbb{E}[|\mathfrak{I}_T|] = \log^+(T)$, where we define $\tau(T) \coloneqq |\{n \in \mathbb{N} : L_n \ge 1/T\}|$.

Proof. Recall by definition of the stick-remainder that $L_n = \prod_{i=1}^n(1-U_i)$ for an iid sequence $(U_i)_{i\in\mathbb{N}}$ of uniform random variables on the unit interval. Thus $S_n \coloneqq -\log L_n$ is a random walk with exponential increments of unit mean or, equivalently, the jump times of a Poisson process with unit intensity. Thus, the definition of $\tau(T)$ implies that, for $T > 1$, $\tau(T)$ follows the marginal distribution of the Poisson process with unit intensity at time $\log T$. Put differently, $\tau(T)$ is Poisson distributed with mean $\log T$. In particular, we have $(\tau(T) - \log T)/\sqrt{\log T} \xrightarrow{d} N(0,1)$ as $T \to \infty$.

Recall that $\ell_m = L_n\prod_{i=n+1}^m U_i < L_n$ for all $m > n$. Since $L_{\tau(T)+1} < 1/T$ we get $\ell_m < 1/T$ for all $m > \tau(T)+1$ and thus $\mathfrak{I}_T \subset \{1,\dots,\tau(T)+1\}$ and $\tau(T) + 1 - |\mathfrak{I}_T| \ge 0$. Corollary 3.10(a) gives $\mathbb{E}[\tau(T)+1-|\mathfrak{I}_T|] = 1$ and thus $\mathbb{E}[|\tau(T) - |\mathfrak{I}_T||] \le 2$ for all $T > 0$, implying $(\tau(T) - |\mathfrak{I}_T|)/\sqrt{\log T} \xrightarrow{L^1} 0$. Hence, the CLT for $\tau(T)$ yields the CLT for $|\mathfrak{I}_T|$. Since the random variables $\tau(T)/\log T$, $T \ge 2$, are uniformly integrable, we have $\tau(T)/\log T \xrightarrow{L^1} 1$ and thus

$$|\mathfrak{I}_T|/\log T = (|\mathfrak{I}_T| - \tau(T))/\log T + \tau(T)/\log T \xrightarrow{L^1} 1.$$ □

Remark 3.12. The law $|\mathfrak{I}_T|$ is much more complicated than that of $\tau(T)$, which follows a Poisson distribution with mean $\log T$ (for $T > 1$). The reason for this lies in the fact that $\tau(T)$ is a stopping time in a correct filtration, while $|\mathfrak{I}_T|$ is not, making its moments hard to control. In Proposition 3.11 we circumvent this problem by approximating $|\mathfrak{I}_T|$ with $\tau(T)$. We note that, even though the expectation $\mathbb{E}[|\tau(T) - |\mathfrak{I}_T||] \le 2$ is bounded for all $T > 0$, the difference $|\tau(T) - |\mathfrak{I}_T||$ takes arbitrarily large values with positive probability. ◊

The following L^1 limit holds.

Proposition 3.13. *Let $f : [1,\infty) \to \mathbb{R}_+$ be measurable and non-increasing with $\lim_{t\to\infty} f(t) = 0$. Then, as $T \to \infty$,*

$$\frac{1}{\sqrt{\log T}}\Big(\sum_{n\in\mathfrak{I}_T} f(t_n) - \mathbb{E}\sum_{n\in\mathfrak{I}_T} f(t_n)\Big) = \frac{1}{\sqrt{\log T}}\Big(\sum_{n\in\mathfrak{I}_T} f(t_n) - \int_1^T \frac{f(t)}{t}\mathrm{d}t\Big) \xrightarrow{L^1} 0.$$

Proof. Define for every T the random variables

$$A_T := \sum_{n\in\mathfrak{I}_T} f(t_n) - \sum_{n=1}^{\tau(T)} f(TL_n), \qquad \text{and} \qquad B_T := \sum_{n=1}^{\tau(T)} f(TL_n) - \int_1^T \frac{f(t)}{t}\mathrm{d}t.$$

Note that it suffices to show that $\mathbb{E}|A_T|$ is bounded for $T > 1$ and $B_T/\sqrt{\log T} \xrightarrow{L^1} 0$.

By Lemma 3.9 and the equality in law $t_n \overset{d}{=} TL_n$, we have

$$\mathbb{E}[A_T] = \sum_{n\in\mathbb{N}} \mathbb{E}[f(t_n)\mathbb{1}_{\{t_n\geq 1\}}] - \sum_{n\in\mathbb{N}} \mathbb{E}[f(TL_n)\mathbb{1}_{\{TL_n\geq 1\}}] = 0. \tag{3.11}$$

Since the function f is non-increasing and $t_n \leq TL_{n-1}$ for all $n \in \mathbb{N}$, we have $C_T := \sum_{n\in\mathfrak{I}_T}(f(t_n) - f(TL_{n-1})) \leq 0$. Similarly, as f is non-increasing, by Proposition 3.11

$$\begin{aligned}|C_T - A_T| &= \Big|\sum_{n=2}^{\tau(T)+1} f(TL_{n-1}) - \sum_{n\in\mathfrak{I}_T} f(TL_{n-1})\Big| \\ &= \Big| - f(T) + \sum_{n\in\{1,\dots,\tau(T)+1\}\setminus\mathfrak{I}_T} f(TL_{n-1})\Big| \leq f(1)|\tau(T) + 2 - |\mathfrak{I}_T||.\end{aligned}$$

Thus (3.11) and Proposition 3.11 yield $\mathbb{E}[|C_T|] = -\mathbb{E}[C_T] = \mathbb{E}[A_T - C_T] \leq 2f(1)$, implying that $\mathbb{E}|A_T|$ is bounded by $4f(1)$ for all $T > 1$.

It remains to show that $B_T/\sqrt{\log T} \xrightarrow{L^1} 0$. Let $S_n := -\log L_n$ and note that $\Xi_T := \sum_{i=1}^{\tau(T)} \delta_{S_i}$ is a random measure with atoms at the jump times on the interval $[0, \log T]$ of a Poisson process with unit intensity. Thus Ξ_T is a Poisson point process on $[0, \log T]$ with the Lebesgue measure as its mean measure. By the reflection and translation invariance of the Lebesgue measure, the mapping theorem for Poisson point processes (Theorem A.47) gives $\Xi_T \overset{d}{=} \sum_{i=1}^{\tau(T)} \delta_{\log T - S_i}$, implying

$$D_T := \sum_{n=1}^{\tau(T)} f(e^{S_n}) \overset{d}{=} \sum_{n=1}^{\tau(T)} f(e^{\log T - S_n}) = \sum_{n=1}^{\tau(T)} f(TL_n) = B_T + \int_1^T \frac{f(t)}{t}\mathrm{d}t.$$

Campbell's formula (Theorem A.48) yields

$$\mathbb{E}[D_T] = \int_0^{\log T} f(e^x)\mathrm{d}x = \int_1^T \frac{f(t)}{t}\mathrm{d}t, \quad \mathrm{Var}[D_T] = \int_0^{\log T} f(e^x)^2\mathrm{d}x = \int_1^T \frac{f(t)^2}{t}\mathrm{d}t.$$

Thus, it suffices to show that $\mathbb{E}[B_T^2]/\log T = \mathrm{Var}[D_T]/\log T \to 0$ as $T \to \infty$. Consider the distribution functions $g_T(t) = \log t/\log T$ for $t \in [1, T]$ and define $Z_T := g_T^{-1}(U) = T^U$ for all $T > 1$ and some fixed uniform random variable U on $(0, 1)$. Then $Z_T \to \infty$ a.s. and hence $f(Z_T)^2 \to 0$ a.s. as $T \to \infty$. By the dominated convergence theorem, $\mathrm{Var}[D_T]/\log T = \mathbb{E}[f(Z_T)^2] \to 0$ as $T \to \infty$. $\square$

§3.2.2 A conditional limit theorem and the proof of Theorem 3.1

Recall from the first paragraph of §3.2 that $(t_n)_{n\in\mathbb{N}}$ denotes a uniform stick-breaking process on $[0,T]$, independent of X, and that $\mathfrak{I}_T$ denotes the set $\{n \in \mathbb{N} : t_n \geq 1\}$. Each horizontal length t_n has an associated slope given by ξ_n/t_n, where $\xi_n = X_{TL_{n-1}} - X_{TL_n}$ is the corresponding vertical height. Aggregate all the horizontal lengths with a common slope in the sequence $(t_n)_{n\in\mathbb{N}}$ into a maximal horizontal length corresponding to that slope. Consider the set $\mathfrak{F}_T$ of the maximal horizontal lengths with size at least 1. Note that, by Theorem 2.18, $|\mathfrak{F}_T| \stackrel{d}{=} H_T$, where H_T is the number of all horizontal lengths greater or equal to 1 of the maximal faces of the concave majorant $t \mapsto C_T^{\frown}(t)$. The analysis of the set $\mathfrak{F}_T$ is based on the properties of the $\mathfrak{I}_T$ established in §3.2.1 above. This strategy is feasible because the difference of sets $\mathfrak{F}_T$ and $\{t_n : n \in \mathfrak{I}_T\}$ is bounded in L^1 in the following sense.

Lemma 3.14. *For any bounded function $f : [1,\infty) \to \mathbb{R}$, the following holds*

$$\sup_{T>0} \mathbb{E}\left| \sum_{t\in\mathfrak{F}_T} f(t) - \sum_{n\in\mathfrak{I}_T} f(t_n) \right| < \infty.$$

Proof. Suppose X is not compound Poisson with drift. Then, by Doeblin's diffuseness lemma (Lemma 2.2) and Theorem 2.18, no two slopes in the sequence $(\xi_n/t_n)_{n\in\mathbb{N}}$ coincide, implying the identity $\mathfrak{F}_T = \{t_n : n \in \mathfrak{I}_T\}$ a.s. The claim then follows since both random sums are equal a.s.

Suppose X is compound Poisson with drift γ (see Remark 2.1 for the definition of the drift of a Lévy processes of finite variation). Consider two horizontal lengths t_n and t_m such that the corresponding slopes ξ_n/t_n and ξ_m/t_m are equal with positive probability. Since the pair (t_n, t_m) has a density $f_{n,m} : (0,T)\times(0,T) \to (0,\infty)$, the result in Proposition A.12 implies

$$\mathbb{P}\left(\frac{\xi_n}{t_n} = \frac{\xi_m}{t_m}\right) = \int_{(0,T)^2} \mathbb{P}\left(\frac{X_s}{s} = \frac{X'_u}{u}\right) f_{n,m}(s,u)\mathrm{d}s\mathrm{d}u = \mathbb{P}\left(\frac{\xi_n}{t_n} = \gamma = \frac{\xi_m}{t_m}\right),$$

where $X' \stackrel{d}{=} X$ is a Lévy process independent of X. Thus all slopes ξ_n/t_n different from γ are also different from each other with probability one and therefore the corresponding faces are already maximal. Hence the set equality $\{t_n : n \in \mathfrak{I}_T\} \setminus \mathfrak{F}_T = \{t_n : n \in \mathfrak{I}_T,\, \xi_n = \gamma t_n\}$ holds a.s.

To complete the proof, it is sufficient to show that the number of faces with length at least 1 and slope $\xi_n/t_n = \gamma$ is bounded in L^1. By Corollary 3.10(a), we have

$$\begin{aligned}\mathbb{E}|\{n \in \mathfrak{I}_T : \xi_n/t_n = \gamma\}| &= \mathbb{E} \sum_{n\in\mathfrak{I}_T} \mathbb{P}(X_{t_n} = \gamma t_n | t_n)\\ &= \int_1^T \frac{\mathbb{P}(X_t = \gamma t)}{t}\mathrm{d}t \xrightarrow[T\to\infty]{} \int_1^\infty \frac{\mathbb{P}(X_t = \gamma t)}{t}\mathrm{d}t,\end{aligned}$$

where the limit is finite by Lemma A.44. □

Remark 3.15. The proof of Lemma 3.14 implies that the only maximal face of the concave majorant $C_T^{\frown}$ of a compound Poisson process X with drift γ that corresponds to more than one face in the representation in Theorem 2.18 is the face whose slope equals γ. All the other faces in this representation are finite in number and have slopes different from each other. ◊

The following result, a conditional CLT given ℓ, is the final ingredient for the proof of Theorem 3.1.

Proposition 3.16. *Suppose* $\mathbb{E}[X_1] = 0$ *and* $\sigma := \sqrt{\mathbb{E}[X_1^2]} \in (0, \infty)$. *If* $\nu \neq 0$, *choose* $\kappa \geq 1$ *such that* $\nu((-\kappa, \kappa)) \in (0, \infty]$ *and otherwise set* $\kappa := 1$ *and recall that* $\sigma_t^2 = \sigma^2 - \int_{\mathbb{R}\setminus(-\kappa\sqrt{t},\kappa\sqrt{t})} x^2\nu(\mathrm{d}x)$ *for* $t > 0$. *Then we have the following limit in probability as* $T \to \infty$*:*

$$\Sigma_T - \frac{\sum_{n=1}^\infty(\sqrt{\xi_n^2 + t_n^2} - t_n) - \frac{1}{2}(\sigma^2|\mathfrak{I}_T| - \int_{\mathbb{R}} x^2 \log^+(\min\{T, x^2\})\nu(\mathrm{d}x))}{\sqrt{\log T}} \xrightarrow{\mathbb{P}} 0,$$
$$\text{where} \qquad \Sigma_T := \frac{1}{2\sqrt{\log T}} \sum_{n\in\mathfrak{I}_T} \left(\frac{\xi_n^2}{t_n} - \sigma_{t_n}^2\right). \tag{3.12}$$

Proof. Define for every $T > 1$, the random variables

$$\Sigma_T^{(1)} := \frac{1}{\sqrt{\log T}} \sum_{n\in\mathfrak{I}_T^c} (\sqrt{t_n^2 + \xi_n^2} - t_n),\ \Sigma_T^{(2)} := \frac{1}{\sqrt{\log T}} \sum_{n\in\mathfrak{I}_T} \left(\sqrt{t_n^2 + \xi_n^2} - t_n - \frac{\xi_n^2}{2t_n}\right),$$
$$\text{and} \qquad \Sigma_T^{(3)} := \frac{1}{2\sqrt{\log T}}\left(\sum_{n\in\mathfrak{I}_T} (\sigma^2 - \sigma_{t_n}^2) - \int_{\mathbb{R}} x^2 \log^+(\min\{T, x^2\})\nu(\mathrm{d}x)\right),$$

and note that, since $\mathbb{N} = \mathfrak{I}_T^c \cup \mathfrak{I}_T$, (3.12) states that $\Sigma_T^{(3)} - \Sigma_T^{(1)} - \Sigma_T^{(2)} \xrightarrow{\mathbb{P}} 0$ as $T \to \infty$. It is therefore sufficient to prove that the expectations $\mathbb{E}[|\Sigma_T^{(1)}|]$, $\mathbb{E}[|\Sigma_T^{(2)}|^{1/2}]$ and $\mathbb{E}[|\Sigma_T^{(3)}|]$ all tend to 0 as $T \to \infty$.

Since $\sqrt{t_n^2 + \xi_n^2} - t_n \leq |\xi_n|$ and $\mathbb{E}[|\xi_n||\ell] \leq \mathbb{E}[\xi_n^2|\ell]^{1/2} = \sigma\sqrt{t_n}$, by Corollary 3.10(a),

$$\mathbb{E}[|\Sigma_T^{(1)}|] \leq \frac{1}{\sqrt{\log T}}\mathbb{E} \sum_{n\in\mathfrak{I}_T^c} \mathbb{E}[|\xi_n||\ell] \leq \frac{1}{\sqrt{\log T}}\mathbb{E} \sum_{n\in\mathfrak{I}_T^c} \sigma\sqrt{t_n} \xrightarrow[T\to\infty]{} 0.$$

Taylor's theorem for the function $x \mapsto \sqrt{1+x^2}$ around $x=0$ applied to $\sqrt{1+\xi_n^2/t_n^2}$ yields

$$|\Sigma_T^{(2)}| = \frac{1}{\sqrt{\log T}}\bigg|\sum_{n\in\mathfrak{I}_T}\frac{\xi_n^4}{8t_n^3}\cdot\theta(|\xi_n|/t_n)\bigg| \le \frac{1}{\sqrt{\log T}}\sum_{n\in\mathfrak{I}_T}\frac{\xi_n^4}{8t_n^3},$$

where $\theta:[0,\infty)\to[0,1]$ is a bounded function. Recall that $\mathbb{E}[X_t^2]=\mathrm{Var}(X_t)=\sigma^2 t$ for all $t\ge 0$. Since $x\mapsto\sqrt{x}$ is concave and starts at 0, we have

$$\begin{aligned}\mathbb{E}\bigg[\bigg|\frac{1}{\sqrt{\log T}}\sum_{n\in\mathfrak{I}_T}\frac{\xi_n^4}{t_n^3}\bigg|^{1/2}\bigg] &\le \frac{\mathbb{E}\sum_{n\in\mathfrak{I}_T}t_n^{-3/2}\xi_n^2}{\log^{1/4}T} = \frac{\mathbb{E}\sum_{n\in\mathfrak{I}_T}\mathbb{E}[t_n^{-3/2}\xi_n^2|\ell]}{\log^{1/4}T}\\ &= \sigma^2\frac{\mathbb{E}\sum_{n\in\mathfrak{I}_T}t_n^{-1/2}}{\log^{1/4}T} = 2\sigma^2\frac{1-T^{-1/2}}{\log^{1/4}T}\xrightarrow[T\to\infty]{} 0,\end{aligned}$$

where the last equality follows from Corollary 3.10(a). This implies $\mathbb{E}[|\Sigma_T^{(2)}|^{1/2}]\to 0$ as $T\to\infty$.

It remains to prove that $\mathbb{E}[|\Sigma_T^{(3)}|]\to 0$ as $T\to\infty$. Applying Corollary 3.10(a) and Fubini's theorem, for any $T>1$ we obtain

$$\begin{aligned}\mathbb{E}\sum_{n\in\mathfrak{I}_T}(\sigma^2-\sigma_{t_n}^2) &= \int_1^T\frac{1}{t}\int_{\mathbb{R}\setminus(-\kappa\sqrt{t},\kappa\sqrt{t})}x^2\nu(\mathrm{d}x)\mathrm{d}t\\ &= \int_{\mathbb{R}\setminus(-\kappa,\kappa)}\int_1^{T\wedge(x^2/\kappa^2)}\frac{\mathrm{d}t}{t}x^2\nu(\mathrm{d}x) = \int_{\mathbb{R}}x^2\log^+(\min\{T,x^2/\kappa^2\})\nu(\mathrm{d}x).\end{aligned}$$

Moreover, since $\kappa\ge 1$, we have

$$\begin{aligned}0 &\le \int_{\mathbb{R}}x^2\log^+(\min\{T,x^2\})\nu(\mathrm{d}x) - \mathbb{E}\sum_{n\in\mathfrak{I}_T}(\sigma^2-\sigma_{t_n}^2)\\ &= \int_{\mathbb{R}}(\log(\kappa^2)\mathbb{1}_{\{|x|<\sqrt{T}\}} + \log(T\kappa^2/x^2)\mathbb{1}_{\{\sqrt{T}\le|x|<\kappa\sqrt{T}\}})x^2\nu(\mathrm{d}x)\\ &\le \log(\kappa^2)\int_{\mathbb{R}}x^2\nu(\mathrm{d}x)<\infty.\end{aligned}$$

Thus, Proposition 3.13 implies that, as $T\to\infty$,

$$\begin{aligned}\Sigma_T^{(3)} &= \frac{1}{2\sqrt{\log T}}\bigg(\sum_{n\in\mathfrak{I}_T}(\sigma^2-\sigma_{t_n}^2) - \mathbb{E}\sum_{n\in\mathfrak{I}_T}(\sigma^2-\sigma_{t_n}^2)\bigg)\\ &\quad + \frac{1}{2\sqrt{\log T}}\bigg(\mathbb{E}\sum_{n\in\mathfrak{I}_T}(\sigma^2-\sigma_{t_n}^2) - \int_{\mathbb{R}}x^2\log^+(\min\{T,x^2\})\nu(\mathrm{d}x)\bigg)\xrightarrow{L^1} 0. \qquad \square\end{aligned}$$

The conditional limit result is a key ingredient for the proof of Theorem 3.1 is the following conditional limit result.

Proposition 3.17. *Let Σ_T be as in* (3.12) *in Proposition 3.16. Then the following conditional limit holds: for any $x\in\mathbb{R}$,*

$$\mathbb{P}(\Sigma_T\le x|\ell)\xrightarrow{L^1}\Phi(\sqrt{2}x/\sigma^2), \qquad \textit{as } T\to\infty, \tag{3.13}$$

where Φ *is the distribution function of a standard normal random variable.*

The limit law in (3.13) is $N(0,\sigma^4/2)$ and the convergence in L^1 is equivalent to the convergence in probability since the random variables $\mathbb{P}(\Sigma_T \le x|\ell)$ are bounded. In particular, (3.13) implies the weak convergence $\mathbb{P}(\Sigma_T \le x) \to \Phi(\sqrt{2}x/\sigma^2)$ for all $x \in \mathbb{R}$. The proof of Proposition 3.17 requires certain limit results for stick-breaking processes from §3.2.1 and Theorem 3.20.

Proof of Proposition 3.17. The proof consists of three steps.

Step 1. Let $Z \sim N(0,1)$ be independent of the stick-breaking process ℓ. Fix $r > 0$ and $\gamma \ge 0$, let

$g_T(t) := (\log T)^{-\gamma/2}|\mathbb{E}[|X_t^2/t|^\gamma \mathbb{1}\{X_t^2/t \le r\sqrt{\log T}\} - |\sigma_t^2 Z^2|^\gamma \mathbb{1}\{\sigma_t^2 Z^2 \le r\sqrt{\log T}\}]|,$

for $t > 0$, where we recall that $\sigma_t^2 = \sigma^2 - \int_{\mathbb{R}\setminus(-\kappa\sqrt{t},\kappa\sqrt{t})} x^2\nu(\mathrm{d}x)$. In this step we establish the following limit:

$$\sum_{n\in\mathfrak{I}_T} g_T(t_n) \xrightarrow{L^1} 0, \quad \text{as } T \to \infty. \tag{3.14}$$

The integration-by-parts formula implies that for any non-negative random variable ζ and constant $a \in (0,\infty)$ we have

$$a^{-\gamma}\mathbb{E}[\zeta^\gamma \mathbb{1}_{\{\zeta\le a\}}] = \mathbb{P}(\zeta \le a) - \gamma\int_0^1 x^{\gamma-1}\mathbb{P}(\zeta \le ax)\mathrm{d}x.$$

Applying the identity in the previous display twice yields

$$\begin{aligned} 0 \le g_T(t) \le r^\gamma K_T(t) \le 2r^\gamma K(t), \quad \text{where} \qquad (3.15)\\ K_T(t) := |\mathbb{P}(X_t^2/t \le r\sqrt{\log T}) - \mathbb{P}(\sigma_t^2 Z^2 \le r\sqrt{\log T})| \\ + \gamma\int_0^1 x^{\gamma-1}|\mathbb{P}(X_t^2/t \le xr\sqrt{\log T}) - \mathbb{P}(\sigma_t Z \le xr\sqrt{\log T})|\mathrm{d}x \end{aligned}$$

and $K(t) := \sup_{x\in\mathbb{R}}|\mathbb{P}(X_t/\sqrt{t} \le x) - \mathbb{P}(\sigma_t Z \le x)|$. Since the normal distribution has a bounded density, the weak limits $X_t/\sqrt{t} \xrightarrow{d} N(0,\sigma^2)$ and $\sigma_t Z \xrightarrow{d} N(0,\sigma^2)$ as $t \to \infty$ hold in the Kolmogorov distance by Theorem A.8, implying $\lim_{t\to\infty} K(t) = 0$. Moreover, by the dominated convergence theorem, we have $\lim_{T\to\infty} K_T(t) = 0$ and thus $\lim_{T\to\infty} g_T(t) = 0$ for all $t > 0$.

Let $\overline{\Xi}_T$ and $\overline{\Xi}_\infty$ be the coupled point processes described in Lemma 3.9 and recall that $\overline{\Xi}_T \to \overline{\Xi}_\infty$ in the vague topology and, for any $N > 1$, we have $\overline{\Xi}_\infty([1,N]) < \infty$ and $\overline{\Xi}_T|_{[1,N]} = \overline{\Xi}_\infty|_{[1,N]}$ for all sufficiently large T. By the definition of vague topology, we have $\int_{[1,\infty)} K(x)\overline{\Xi}_T(\mathrm{d}x) \to \int_{[1,\infty)} K(x)\overline{\Xi}_\infty(\mathrm{d}x)$ a.s. Since $g_T(t) \to 0$ as

$T \to \infty$ for every atom t of $\overline{\Xi}_\infty|_{[1,N]}$, we have

$$\begin{aligned}
&\limsup_{T\to\infty} \int_{[1,\infty)} g_T(x)\overline{\Xi}_T(\mathrm{d}x) \\
&\le \limsup_{T\to\infty} \int_{[1,N]} g_T(x)\overline{\Xi}_T(\mathrm{d}x) + \limsup_{T\to\infty} \int_{(N,\infty)} 2r^\gamma K(x)\overline{\Xi}_T(\mathrm{d}x) \\
&= \limsup_{T\to\infty} \int_{[1,N]} g_T(x)\overline{\Xi}_\infty(\mathrm{d}x) + \int_{(N,\infty)} 2r^\gamma K(x)\overline{\Xi}_\infty(\mathrm{d}x) = 2r^\gamma \int_{(N,\infty)} K(x)\overline{\Xi}_\infty(\mathrm{d}x).
\end{aligned}$$

By Theorem 3.20 we have $\mathbb{E}\int_{[1,\infty)} K(x)\overline{\Xi}_\infty(\mathrm{d}x) = \int_1^\infty t^{-1}K(t)\mathrm{d}t < \infty$. Therefore, the display above and Fatou's lemma imply, as $N \to \infty$,

$$\begin{aligned}
\limsup_{T\to\infty} \mathbb{E} \sum_{n\in\mathfrak{I}_T} g_T(t_n) &\le \mathbb{E} \limsup_{T\to\infty} \int_{[1,\infty)} g_T(x)\overline{\Xi}_T(\mathrm{d}x) \\
&\le 2r^\gamma \mathbb{E} \int_{(N,\infty)} K(x)\overline{\Xi}_\infty(\mathrm{d}x) = 2r^\gamma \int_N^\infty \frac{K(x)}{x}\mathrm{d}x \to 0,
\end{aligned}$$

thus proving (3.14).

Step 2. Denote $S_{n,T} := \xi_n^2/(2t_n\sqrt{\log T})$ for all $n \in \mathbb{N}$ and $T > 1$. Assume that the following limits in probability hold as $T \to \infty$:

$$\sum_{n\in\mathfrak{I}_T} \mathbb{P}_\ell(S_{n,T} \ge \epsilon) \xrightarrow{\mathbb{P}} 0, \qquad \text{for every } \epsilon > 0, \tag{3.16}$$

$$\sum_{n\in\mathfrak{I}_T} \mathrm{Var}_\ell\left(S_{n,T}\mathbb{1}_{\{S_{n,T}\le r\}}\right) \xrightarrow{\mathbb{P}} \frac{\sigma^4}{2}, \qquad \text{for some } r > 0, \tag{3.17}$$

$$\sum_{n\in\mathfrak{I}_T} \left(\mathbb{E}_\ell\left[S_{n,T}\mathbb{1}_{\{S_{n,T}\le r'\}}\right] - \frac{\sigma_{t_n}^2}{2\sqrt{\log T}}\right) \xrightarrow{\mathbb{P}} 0, \qquad \text{for some } r' > 0, \tag{3.18}$$

where we denote $\mathbb{P}_\ell(\cdot) = \mathbb{P}(\cdot|\ell)$, $\mathbb{E}_\ell[\cdot] = \mathbb{E}[\cdot|\ell]$ and $\mathrm{Var}_\ell(\cdot) := \mathrm{Var}(\cdot|\ell)$. We now prove that (3.16)–(3.18) imply the L^1 limit in (3.13).

Since the random variables in (3.13) are bounded, it suffices to prove the limit in probability. Fix a sequence $(T_k)_{k\in\mathbb{N}}$ such that $T_k \to \infty$. By a diagonal argument, there exists a subsequence, again denoted $(T_k)_{k\in\mathbb{N}}$ for ease of notation, such that the limit in (3.16) holds for all positive rational ϵ as $T_k \to \infty$ almost surely. Thus, the limit in (3.16) holds for *all* $\epsilon > 0$ as $T_k \to \infty$ a.s. Moreover, we may assume that the limits in (3.17)–(3.18) hold a.s. as $T_k \to \infty$. Recall that, given the stick-breaking process ℓ, the variables $\{S_{n,T_k} : n \in \mathfrak{I}_{T_k}\}$ are independent, making $(\{S_{n,T_k} : n \in \mathfrak{I}_{T_k}\})_{k\in\mathbb{N}}$ a triangular array of row-wise independent random variables. Applying the CLT for triangular arrays in Theorem A.9, we deduce that (3.13) holds a.s. as $T_k \to \infty$.

Step 3. In this step we prove (3.16)–(3.18). Recall that $Z \sim N(0,1)$ is independent of ℓ. By (3.14) with $\gamma = 0$ and $r = \epsilon$, Markov's inequality and Proposition 3.11, we have

$$\lim_{T\to\infty}\mathbb{E}\sum_{n\in\mathfrak{I}_T}\mathbb{P}_\ell\left(\frac{\xi_n^2/t_n}{\sqrt{\log T}}>\epsilon\right)=\lim_{T\to\infty}\mathbb{E}\sum_{n\in\mathfrak{I}_T}\mathbb{P}_\ell\left(\frac{\sigma_{t_n}^2Z^2}{\sqrt{\log T}}>\epsilon\right)$$
$$\leq\lim_{T\to\infty}\mathbb{E}\sum_{n\in\mathfrak{I}_T}\frac{\sigma_{t_n}^6\mathbb{E}[Z^6]}{(\epsilon\sqrt{\log T})^3}\leq\lim_{T\to\infty}\frac{15\sigma^6\mathbb{E}|\mathfrak{I}_T|}{\epsilon^3(\log T)^{3/2}}=0,$$

for all $\epsilon>0$, implying (3.16) (recall that $S_{n,T}=\xi_n^2/(2t_n\sqrt{\log T})$).

To prove the limit in (3.17), first note that $|a^2-b^2|\leq(a+b)|a-b|$ for $a,b\geq 0$, implying

$$\left|\mathbb{E}_\ell\left[\tfrac{1}{2}t_n^{-1}\xi_n^2\mathbb{1}_{\{\xi_n^2\leq 2t_n\epsilon\sqrt{\log T}\}}\right]^2-\mathbb{E}_\ell\left[\tfrac{1}{2}\sigma_{t_n}^2Z^2\mathbb{1}_{\{\sigma_{t_n}^2Z^2\leq 2\epsilon\sqrt{\log T}\}}\right]^2\right|$$
$$\leq 2\epsilon\sqrt{\log T}\left|\mathbb{E}_\ell\left[\tfrac{1}{2}t_n^{-1}\xi_n^2\mathbb{1}_{\{\xi_n^2\leq 2t_n\epsilon\sqrt{\log T}\}}\right]-\mathbb{E}_\ell\left[\tfrac{1}{2}\sigma_{t_n}^2Z^2\mathbb{1}_{\{\sigma_{t_n}^2Z^2\leq 2\epsilon\sqrt{\log T}\}}\right]\right|.$$

Thus, by applying (3.14) with $\gamma=1$ and $\gamma=2$ and $r=2\epsilon$, we find (all limits are taken in L^1):

$$\lim_{T\to\infty}\frac{\sum_{n\in\mathfrak{I}_T}\mathrm{Var}_\ell\big(\tfrac{1}{2}t_n^{-1}\xi_n^2\mathbb{1}_{\{\xi_n^2\leq 2t_n\epsilon\sqrt{\log T}\}}\big)}{\log T}$$
$$=\lim_{T\to\infty}\frac{\sum_{n\in\mathfrak{I}_T}\mathrm{Var}_\ell\big(\tfrac{1}{2}\sigma_{t_n}^2Z^2\mathbb{1}_{\{\sigma_{t_n}^2Z^2\leq 2\epsilon\sqrt{\log T}\}}\big)}{\log T}=\lim_{T\to\infty}\frac{\sum_{n\in\mathfrak{I}_T}\mathrm{Var}_\ell\big(\tfrac{1}{2}\sigma_{t_n}^2Z^2\big)}{\log T}$$
$$=\lim_{T\to\infty}\frac{1}{2\log T}\left(\sigma^4|\mathfrak{I}_T|+\sum_{n\in\mathfrak{I}_T}(\sigma_{t_n}^4-\sigma^4)\right)=\frac{\sigma^4}{2},$$

where the first equality in the second line follows from the fact that $\mathrm{Var}(Z^2)=2$ and the last equality in the same line follows from Proposition 3.11 and Corollary 3.10(b) applied to the bounded function $t\mapsto\sigma_t^4-\sigma^4$ with zero limit as $t\to\infty$. This establishes (3.17) since $S_{n,T}=\xi_n^2/(2t_n\sqrt{\log T})$.

It remains to prove (3.18). Markov's inequality, the equality $\mathbb{E}[Z^2]=1$ and Proposition 3.11 imply

$$\frac{1}{\sqrt{\log T}}\sum_{n\in\mathfrak{I}_T}\left|\mathbb{E}_\ell\left[\tfrac{1}{2}\sigma_{t_n}^2Z^2\mathbb{1}_{\{\sigma_{t_n}^2Z^2\leq 2\epsilon\sqrt{\log T}\}}\right]-\frac{\sigma_{t_n}^2}{2}\right|$$
$$=\frac{1}{\sqrt{\log T}}\sum_{n\in\mathfrak{I}_T}\mathbb{E}_\ell\left[\tfrac{1}{2}\sigma_{t_n}^2Z^2\mathbb{1}_{\{\sigma_{t_n}^2Z^2>2\epsilon\sqrt{\log T}\}}\right]\leq\frac{1}{\sqrt{\log T}}\sum_{n\in\mathfrak{I}_T}\frac{\mathbb{E}_\ell[\sigma_{t_n}^8Z^8]}{(2\epsilon\sqrt{\log T})^3}$$
$$=\frac{1}{8\epsilon^3\log^2 T}\sum_{n\in\mathfrak{I}_T}\sigma_{t_n}^8\mathbb{E}[Z^8]\leq\frac{105\sigma^8}{8\epsilon^3\log^2 T}|\mathfrak{I}_T|\xrightarrow{L^1}0.$$

The display above and (3.14) with $\gamma=1$ and $r=2\epsilon$ imply (3.18), completing the proof. □

Proof of Theorem 3.1. The proof of Theorem 3.1 consists of several steps.

Step 1. In this step we show that (3.2) follows from the limits in (3.20) below.

By Theorem 2.18, Lemma 3.14 and Proposition 3.16, the weak limit in (3.2) of Theorem 3.1 is equivalent to the following limit as $T \to \infty$:

$$\zeta_T := \left(\frac{\sum_{n\in\mathfrak{I}_T}(\xi_n^2/t_n - \sigma_{t_n}^2)}{2\sqrt{\log T}}, \frac{|\mathfrak{I}_T| - \log T}{\sqrt{\log T}}, \frac{\sum_{n=1}^\infty \xi_n^+}{\sqrt{T}}, \frac{\sum_{n=1}^\infty \xi_n}{\sqrt{T}}, \frac{\sum_{n=1}^\infty t_n \mathbb{1}_{\{\xi_n>0\}}}{T} \right)$$
$$\xrightarrow{d} \zeta = (\sigma^2 Z_1/\sqrt{2}, Z_2, \sigma\overline{B}_1, \sigma B_1, \rho), \tag{3.19}$$

where the standard Brownian motion B, the stick-breaking process ℓ and the standard normal variables Z_1 and Z_2 are all independent.

Define $\eta_n := \xi_n/\sqrt{t_n}$ for $n \in \mathbb{N}$ and note that

$$\zeta_T = \left(\frac{\sum_{n\in\mathfrak{I}_T}(\eta_n^2 - \sigma_{t_n}^2)}{2\sqrt{\log T}}, \frac{|\mathfrak{I}_T| - \log T}{\sqrt{\log T}}, \sum_{n=1}^\infty \ell_n^{1/2}\eta_n^+, \sum_{n=1}^\infty \ell_n^{1/2}\eta_n, \sum_{n=1}^\infty \ell_n \mathbb{1}_{\{\eta_n>0\}} \right).$$

Let $W_1, W_2 \ldots$ be an iid sequence of standard normal random variables independent of ℓ, Z_1 and Z_2. For $k \in \mathbb{N}$ and $T > 1$ define the random variables

$$\chi_{k,T} := \left(\frac{\sum_{n=k}^\infty(\eta_n^2 - \sigma_{t_n}^2)\mathbb{1}_{\{t_n\geq 1\}}}{2\sqrt{\log T}}, \frac{\sum_{n=k}^\infty \mathbb{1}_{\{t_n\geq 1\}} - \log T}{\sqrt{\log T}}, 0, 0, 0 \right)$$
$$+ \left(0, 0, \sum_{n=1}^{k-1} \ell_n^{1/2}\eta_n^+, \sum_{n=1}^{k-1} \ell_n^{1/2}\eta_n, \sum_{n=1}^{k-1} \ell_n \mathbb{1}_{\{\eta_n>0\}} \right), \quad \text{and}$$
$$\chi_k := \left(\frac{\sigma^2}{\sqrt{2}} Z_1, Z_2, \sum_{n=1}^{k-1} \ell_n^{1/2}\sigma W_n^+, \sum_{n=1}^{k-1} \ell_n^{1/2}\sigma W_n, \sum_{n=1}^{k-1} \ell_n \mathbb{1}_{\{\sigma W_n>0\}} \right).$$

By Theorem A.7, (3.19) will follow if we prove that the following limits hold:

$$\text{(a) } \chi_{k,T} \xrightarrow[T\to\infty]{d} \chi_k, \text{ (b) } \chi_k \xrightarrow[k\to\infty]{d} \zeta, \text{ (c) } \lim_{k\to\infty}\limsup_{T\to\infty} \mathbb{P}(\|\chi_{k,T} - \zeta_T\| > \epsilon) = 0, \forall\epsilon > 0, \tag{3.20}$$

where $\|x\| = \sum_{i=1}^d |x_i|$ denotes the ℓ^1-norm in $\mathbb{R}^d$, $d \geq 1$.

Step 2. In this step we establish (3.20a). Define $\ell^{(k)} := (\ell_1, \ldots, \ell_{k-1})$. To prove (3.20a), it suffices to show that $\mathbb{E}[\phi(\chi_{k,T})|\ell^{(k)}] \to \mathbb{E}[\phi(\chi_k)|\ell^{(k)}]$ a.s. as $T \to \infty$ for any continuous and bounded function $\phi : \mathbb{R}^5 \to \mathbb{R}$. With this in mind, denote by $\mathbb{P}^{(k)}$ the conditional probability measure $\mathbb{P}$ given $\ell^{(k)}$.

Under $\mathbb{P}^{(k)}$, the process $(\ell_k, \ell_{k+1}, \ldots)$ is a uniform stick-breaking process on $[0, L_{k-1}]$ independent of $(\eta_n)_{n<k}$. Thus the first two coordinates of $\chi_{k,T}$ are independent under $\mathbb{P}^{(k)}$ of the last three coordinates. Moreover, since $X_t/\sqrt{t} \xrightarrow{d} \sigma Z_1$ as $t \to \infty$, then, under $\mathbb{P}^{(k)}$, we have $(\eta_1, \ldots, \eta_{k-1}) = (\xi_1/\sqrt{t_1}, \ldots, \xi_{k-1}/\sqrt{t_{k-1}}) \xrightarrow{d}$ $(\sigma W_1, \ldots, \sigma W_{k-1})$ as $T \to \infty$ (recall that $t_n = T\ell_n$). Thus, to prove (3.20a), it suffices to show that the first two coordinates of $\chi_{k,T}$ converge weakly to the first two coordinates of χ_k under $\mathbb{P}^{(k)}$.

Recall that, under $\mathbb{P}^{(k)}$, the process $(\ell_k, \ell_{k+1}, \ldots)$ is a uniform stick-breaking process on $[0, L_{k-1}]$ and $\sum_{n=k}^{\infty} t_n = TL_{k-1}$. Thus, Proposition 3.11 implies that

$$\frac{\sum_{n=k}^{\infty} \mathbb{1}_{\{t_n \geq 1\}} - \log(TL_{k-1})}{\sqrt{\log(TL_{k-1})}} \xrightarrow{d} Z_2, \quad \text{as } T \to \infty \text{ under } \mathbb{P}^{(k)}.$$

Since $\log(TL_{k-1}) = \log T + \log L_{k-1}$, where L_{k-1} is deterministic under $\mathbb{P}^{(k)}$, then

$$M_T := \frac{\sum_{n=k}^{\infty} \mathbb{1}_{\{t_n \geq 1\}} - \log T}{\sqrt{\log T}} \xrightarrow{d} Z_2, \quad \text{as } T \to \infty \text{ under } \mathbb{P}^{(k)}.$$

Moreover, since $\mathbb{P}^{(k)}(\cdot|\ell) = \mathbb{P}(\cdot|\ell)$, Proposition 3.17 implies that $\mathbb{P}^{(k)}(\Sigma_T \leq x|\ell) \xrightarrow{L^1} \mathbb{P}(\sigma^2 Z_1/\sqrt{2} \leq x)$ for all $x \in \mathbb{R}$ as $T \to \infty$, where Σ_T is as in (3.12). Denote by $\mathbb{E}^{(k)}$ the expectation under $\mathbb{P}^{(k)}$. Thus, taking limits in the following identity

$$\begin{aligned}\mathbb{E}^{(k)}[\mathbb{1}_{\{M_T \leq y\}} \mathbb{P}^{(k)}(\Sigma_T \leq x|\ell)] &= \mathbb{P}^{(k)}(M_T \leq y)\mathbb{P}^{(k)}(\sigma^2 Z_1/2 \leq x) \\ &\quad + \mathbb{E}^{(k)}[\mathbb{1}_{\{M_T \leq y\}}(\mathbb{P}^{(k)}(\Sigma_T \leq x|\ell) - \mathbb{P}^{(k)}(\sigma^2 Z_1/\sqrt{2} \leq x))],\end{aligned}$$

implies $\mathbb{P}^{(k)}(M_T \leq y, \Sigma_T \leq x) \to \mathbb{P}^{(k)}(Z_2 \leq y)\mathbb{P}^{(k)}(\sigma^2 Z_1/\sqrt{2} \leq x)$ as $T \to \infty$. To see that the first two coordinates of $\chi_{k,T}$ converge weakly to those of χ_k under $\mathbb{P}^{(k)}$, note that $\mathbb{E}^{(k)} \sum_{n=1}^{k-1} |\eta_n^2 - \sigma_{t_n}^2|/\sqrt{\log T} \leq 2(k-1)\sigma^2/\sqrt{\log T} \to 0$ as $T \to \infty$.

Step 3. In this step we establish (3.20b)–(3.20c). To prove (3.20b), it suffices to show the convergence for the last three coordinates. Note that

$$\sum_{n=1}^{k-1} \big(\sqrt{\ell_n}\sigma W_n^+, \sqrt{\ell_n}\sigma W_n, \ell_n \mathbb{1}_{\{\sigma W_n > 0\}}\big) \xrightarrow[k\to\infty]{a.s.} \sum_{n=1}^{\infty} (\sqrt{\ell_n}\sigma W_n^+, \sqrt{\ell_n}\sigma W_n, \ell_n \mathbb{1}_{\{\sigma W_n > 0\}}),$$

where the limit has the same law as $(\sigma\overline{B}_1, \sigma B_1, \rho)$ by the scaling property of Brownian motion and (3.6) applied to σB, implying (2.14b).

If we prove $\lim_{m\to\infty} \limsup_{T\to\infty} \mathbb{E}\|\chi_{k,T} - \zeta_T\| = 0$, (3.20c) will follow by Markov's inequality. Moreover, the previous limit is a consequence of the following limits

$$\limsup_{T\to\infty} \frac{\mathbb{E}\sum_{n=1}^{k-1} |\eta_n^2 - \sigma_{t_n}^2| \mathbb{1}_{\{t_n \geq 1\}}}{2\sqrt{\log T}} = 0, \qquad \limsup_{T\to\infty} \frac{\mathbb{E}\sum_{n=1}^{k-1} \mathbb{1}_{\{t_n \geq 1\}}}{2\sqrt{\log T}} = 0,$$

$$\lim_{k\to\infty} \limsup_{T\to\infty} \mathbb{E}\sum_{n=k}^{\infty} \sqrt{\ell_n}|\eta_n| = 0, \qquad \lim_{k\to\infty} \mathbb{E}\sum_{n=k}^{\infty} \ell_n = 0.$$

The first two limits in the display obviously hold. The fourth limit holds since $\sum_{n=k}^{\infty} \ell_k = L_{k-1}$ and $\mathbb{E}L_{k-1} = 2^{1-k}$. Finally, the third limit in the display above follows from the bounds

$$\mathbb{E}\sum_{n=k}^{\infty} \sqrt{\ell_n}|\eta_n| \leq \sum_{n=k}^{\infty} \mathbb{E}[\sqrt{\ell_n}\mathbb{E}_\ell[\eta_n^2]^{1/2}] = \sigma\sum_{n=k}^{\infty} \mathbb{E}\sqrt{\ell_n} = \sigma\sum_{n=k}^{\infty} (2/3)^n = 3\sigma(2/3)^k,$$

implying (3.20c) and completing the proof. □

Proof of Corollary 3.2. By Theorem 3.1, it suffices to prove the claims on the integral

$\int_{\mathbb{R}} x^2 \log^+(\min\{T, x^2\})\nu(\mathrm{d}x)$. Since $x^2 \log^+(\min\{T, x^2\})/\log T$ tends to 0 pointwise on x as $T \to \infty$ and is upper bounded by the ν-integrable function $x \mapsto x^2$, the dominated convergence theorem implies that the integral is $\mathrm{o}(\log T)$. Similarly, the integral is $\mathrm{o}(\sqrt{\log T})$ if $x \mapsto x^2(\log^+|x|)^{1/2}$ is ν-integrable. □

§3.3 Stable domain of attraction

This section is dedicated to proving Theorems 3.4, 3.6 and 3.7, stated in §3.1. Assume the limit in (2.6) holds for some $\alpha \in (0, 2] \setminus \{1\}$, and recall that this is equivalent to (2.7), i.e. $(X_{tT}/a_T)_{t\in[0,1]} \overset{d}{\to} (S_\alpha(t))_{t\in[0,1]}$, as $T \to \infty$, in the Skorokhod space $\mathcal{D}[0, 1]$ equipped with the J_1-topology, with the scaling function a_T is as in (2.6). Since $a_T \to \infty$ as $T \to \infty$, we assume without loss of generality that $a_T > 1$ is locally bounded for all $T \geq 1$. The following lemma provides a key step in the proofs of Theorems 3.4 and 3.6.

Lemma 3.18. *Suppose a Lévy process X satisfies* (2.7) *for some $\alpha \in (0, 2]$. Then, for every $p \in [0, \alpha)$, there exists a constant $K_p \in (0,\infty)$ such that $\mathbb{E}[|X_t/a_t|^p] \leq K_p$ for all $t \geq 1$.*

Proof. By the the concavity of $x \mapsto x^p$ (when $p \in [0, 1]$) and Jensen's inequality (when $p \in (1, \alpha)$), we have $(a + b)^p \leq 2^{(p-1)^+}(a^p + b^p)$ for any $a, b \geq 0$. Thus, $\mathbb{E}[|X_t|^p] \leq 2^{(p-1)^+}(\mathbb{E}[|X_{\lfloor t\rfloor}|^p] + \mathbb{E}[|X_{t-\lfloor t\rfloor}|^p])$ for all $t \geq 1$, where $\lfloor t\rfloor \coloneqq \sup\{m \in \mathbb{N} : m \leq t\}$. By Lemma 2.15, $\mathbb{E}[|X_n/a_n|^p]$ is bounded for all $n \in \mathbb{N}$. By the regular variation of $a_t \geq 1$, we have

$$1 \leq \liminf_{t\to\infty} \frac{a_t}{a_{\lfloor t\rfloor}} \leq \limsup_{t\to\infty} \frac{a_t}{a_{\lfloor t\rfloor}} \leq \limsup_{t\to\infty} \frac{a_t}{a_{ct}} = c^{-1/\alpha},$$

for any $c \in (0, 1)$, implying $a_t/a_{\lfloor t\rfloor} \to 1$ as $t \to \infty$. Thus, it suffices to show that $\mathbb{E}[|X_s|^p]$ is bounded for $s \in [0, 1]$. This bound follows from Lemma 2.6 and the inequality $\mathbb{E}[|X_s|^p] \leq \mathbb{E}[\overline{X}_s^p] + \mathbb{E}[|\underline{X}_s|^p]$ implied by $|X_s|^p \leq \max\{\overline{X}_s^p, |\underline{X}_s|^p\}$. □

Remark 3.19. An explicit upper bound in Lemma 3.18 can be obtained in terms of the characteristics of X and the regularly varying function a_t by using methods analogous to the ones in the proof of Lemma 2.6 (see the proof of [39, Lem. 2]). Since the explicit value of the upper bound C_p is not important in our context, we only provide the short proof above. ♢

§3.3.1 The case of finite mean

Proof of Theorem 3.4. Recall $\mathbb{P}_\ell(\cdot) = \mathbb{P}(\cdot|\ell)$ and $\mathbb{E}_\ell[\cdot] = \mathbb{E}[\cdot|\ell]$, where ℓ is the stick-breaking process on $[0, 1]$, and $t_n = T\ell_n$. Denote $\eta_n \coloneqq \xi_n/a_{t_n}$ and $\varrho_n \coloneqq a_{t_n}/a_T$ for

$n \in \mathbb{N}$ and note that $\sqrt{t_n^2 + \xi_n^2} - t_n = \xi_n^2/(t_n + \sqrt{t_n^2 + \xi_n^2})$. Thus, by (3.6), we have

$$\left(\frac{\Upsilon_{\widehat{T}} - T}{a_T^2/T}, \frac{\overline{C}_{\widehat{T}}}{a_T}, \frac{C_{\widehat{T}}(T)}{a_T}, \frac{\gamma_{\widehat{T}}}{T}\right)$$
$$\overset{d}{=} \sum_{n=1}^{\infty} \left(\frac{\varrho_n^2 \eta_n^2}{\ell_n + \sqrt{\ell_n^2 + \varrho_n^2 \eta_n^2 a_T^2/T^2}}, \varrho_n \eta_n^+, \varrho_n \eta_n, \ell_n \mathbb{1}_{\{\varrho_n \eta_n > 0\}}\right).$$

By Theorem A.7, (3.5) will follow if we prove the following limits: for any $k \in \mathbb{N}$,

$$\sum_{n=1}^{k-1} \left(\frac{\varrho_n^2 \eta_n^2}{\ell_n + \sqrt{\ell_n^2 + \varrho_n^2 \eta_n^2 a_T^2/T^2}}, \varrho_n \eta_n^+, \varrho_n \eta_n, \ell_n \mathbb{1}_{\{\varrho_n \eta_n > 0\}}\right)$$
$$\xrightarrow[T \to \infty]{d} \sum_{n=1}^{k-1} \left(\frac{1}{2} \ell_n^{2/\alpha - 1} \big(S_\alpha^{(n)}\big)^2, \ell_n^{1/\alpha} \big(S_\alpha^{(n)}\big)^+, \ell_n^{1/\alpha} S_\alpha^{(n)}, \ell_n \mathbb{1}_{\{S_\alpha^{(n)} > 0\}}\right), \tag{3.21}$$

and, for all $\epsilon > 0$,

$$\lim_{k \to \infty} \limsup_{T \to \infty} \mathbb{P}\left(\sum_{n=k}^{\infty} \big\| \big(R_n, \varrho_n \eta_n^+, \varrho_n \eta_n, \ell_n \mathbb{1}_{\{\varrho_n \eta_n > 0\}}\big) \big\| > \epsilon\right) = 0,$$
$$\text{where} \quad R_n := \frac{\varrho_n^2 \eta_n^2}{\ell_n + \sqrt{\ell_n^2 + \varrho_n^2 \eta_n^2 a_T^2/T^2}}, \tag{3.22}$$

and $\|x\| = \sum_{i=1}^d |x_i|$ denotes the ℓ^1-norm in $\mathbb{R}^d$, $d \geq 1$.

To prove (3.21), it suffices to show that the weak convergence holds conditional on ℓ. By assumption, we have $X_t/a_t \xrightarrow{d} S_\alpha^{(1)}$, $a_{ct}/a_t \to c^{1/\alpha}$ and $a_t/t \to 0$ as $t \to \infty$. Thus, given ℓ, the random variables $\eta_1, \ldots, \eta_k$ are independent and we have the following convergences as $T \to \infty$: $(\eta_1, \ldots, \eta_k) \xrightarrow{d} (S_\alpha^{(1)}, \ldots, S_\alpha^{(k)})$, $(\varrho_1, \ldots, \varrho_k) \to (\ell_1^{1/\alpha}, \ldots, \ell_k^{1/\alpha})$ and $a_T/T \to 0$. The continuous mapping theorem then yields the weak convergence in (3.21) conditional on ℓ.

Next we prove (3.22). To prove this, we note that $\sum_{n=k}^{\infty} \ell_k = L_{k-1}$ and $\mathbb{P}(L_{k-1} > \epsilon) \leq \epsilon^{-1} \mathbb{E} L_{k-1} \to 0$ as $k \to \infty$, so it suffices to show that, for all $\epsilon > 0$, the following limits hold as $k \to \infty$:

$$\limsup_{T \to \infty} \mathbb{P}\left(\sum_{n=k}^{\infty} R_n > \epsilon\right) \to 0, \quad \limsup_{T \to \infty} \mathbb{P}\left(\sum_{n=k}^{\infty} \varrho_n |\eta_n| > \epsilon\right) \to 0. \tag{3.23}$$

We will prove both limits via Markov's inequality $\mathbb{P}(|\zeta| > \epsilon) \leq \epsilon^{-p} \mathbb{E}[|\zeta|^p]$ for $p > 0$, and bounding the first moment by splitting the summation over the sets $\mathfrak{I}_T$ and $\mathfrak{I}_T^{\mathsf{c}}$ (recall that $\mathfrak{I}_T = \{n \in \mathbb{N} : t_n \geq 1\}$). First note that $R_n \leq |\xi_n|(T/a_T^2)$ and $\rho_n |\eta_n| = |\xi_n|/a_T$, where $a_T \to \infty$ and $a_T^2/T \to \infty$ as $T \to \infty$. There exists a constant

K such $\mathbb{E}[|X_t|] \le K\sqrt{t}$ for all $t \le 1$ (see, e.g. Lemma 2.6), so Corollary 3.10(a) yields

$$\limsup_{T\to\infty} \mathbb{E} \sum_{n\in\mathfrak{I}_T^c, n\ge k} \varrho_n \mathbb{E}_\ell |\eta_n| \le \limsup_{T\to\infty} \frac{K}{a_T} \mathbb{E} \sum_{n\in\mathfrak{I}_T^c} \ell_n^{1/2} = \limsup_{T\to\infty} \frac{2K}{a_T} = 0, \quad \text{and}$$

$$\limsup_{T\to\infty} \mathbb{E} \sum_{n\in\mathfrak{I}_T^c, n\ge k} R_n \le \limsup_{T\to\infty} \frac{KT}{a_T^2} \mathbb{E} \sum_{n\in\mathfrak{I}_T^c} \ell_n^{1/2} = \limsup_{T\to\infty} \frac{2KT}{a_T^2} = 0.$$

It remains to consider the summation sets $\mathfrak{I}_T \cap \{k, k+1, \ldots\}$. By Lemma 3.18, for any $p \in (0, \alpha)$, we have $\mathbb{E}_\ell[|\eta_n|^p] \le K_p$ for some $K_p > 0$. Since $t \mapsto a_t$ is regularly varying at infinity with index $1/\alpha$, Potter's bound (Theorem A.53) implies that for all $q \in (0, 1/\alpha)$ there exists a constant $K_q' > 0$ such that $a_s/a_t \le K_q'(s/t)^q$ for all $t > s \ge 1$. Thus, the second limit in (3.23) follows from the limit

$$\limsup_{T\to\infty} \mathbb{E} \sum_{n\in\mathfrak{I}_T, n\ge k} \varrho_n \mathbb{E}_\ell |\eta_n| \le K_1 K_{1/2}' \sum_{n=k}^{\infty} \mathbb{E}[\ell_n^{1/2}] = 3K_1 K_{1/2}' (2/3)^{k-1} \xrightarrow[k\to\infty]{} 0.$$

Fix any $p \in (0, \alpha/2)$ and $q \in (1/2, 1/\alpha)$ and note that $R_n \le \varrho_n^2 \eta_n^2/\ell_n$. By Markov's inequality and the subadditivity of $x \mapsto x^p$, the first limit in (3.23) follows from

$$\begin{aligned}\limsup_{T\to\infty} \mathbb{E} \sum_{n\in\mathfrak{I}_T, n\ge k} R_n^p &\le K_{2p}(K_q')^{2p} \sum_{n=k}^{\infty} \mathbb{E}[\ell_n^{p(2q-1)}] \\ &= \frac{K_{2p}(K_q')^{2p}(1+p(2q-1))^{1-k}}{p(2q-1)} \xrightarrow[k\to\infty]{} 0. \quad \square\end{aligned}$$

Proof of Proposition 3.5. Note that $Q \coloneqq \frac{1}{2}\sum_{n=1}^{\infty} \ell_n^{2/\alpha-1}(S_\alpha^{(n)})^2$ satisfies

$$\begin{aligned}2Q &= \ell_1^{2/\alpha-1}(S_\alpha^{(1)})^2 + \sum_{i=2}^{\infty} \ell_n^{2/\alpha-1}(S_\alpha^{(n)})^2 \\ &= \ell_1^{2/\alpha-1}(S_\alpha^{(1)})^2 + L_1^{2/\alpha-1} \sum_{i=2}^{\infty} \left(\frac{\ell_n}{L_1}\right)^{2/\alpha-1} (S_\alpha^{(n)})^2.\end{aligned}$$

Let $A \coloneqq L_1^{2/\alpha-1}$, $B \coloneqq \frac{1}{2}\ell_1^{2/\alpha-1}(S_\alpha^{(1)})^2$ and $Q' \coloneqq \frac{1}{2}\sum_{i=2}^{\infty}(\ell_n/L_1)^{2/\alpha-1}(S_\alpha^{(n)})^2$ and note that $Q = AQ' + B$. Since $(\ell_n/L_1)_{n\ge 2}$ is a stick-breaking process on $[0,1]$ independent of L_1 and $S_\alpha^{(1)}$, we conclude that $Q' \stackrel{d}{=} Q$ is independent of (A, B).

By Theorem A.59 it follows that $\mathbb{P}(Q > x) \sim (1 - \mathbb{E}[A^{\alpha/2}])^{-1}\mathbb{P}(B > x)$, as $x \to \infty$. Furthermore, by Lemma A.60, we have

$$\mathbb{P}(B > x) \sim \mathbb{E}\big[\big(\tfrac{1}{2}\ell_1^{2/\alpha-1}\big)^{\alpha/2}\big]\mathbb{P}((S_\alpha^{(1)})^2 > x), \qquad \text{as } x \to \infty.$$

Recall that $L_1 = 1 - \ell_1 \sim U(0,1)$. Similarly, we have that $\ell_1 \sim U(0,1)$. Thus, it follows that

$$\big(1-\mathbb{E}[A^{\alpha/2}]\big)^{-1}\mathbb{E}\big[\big(\tfrac{1}{2}\ell_1^{2/\alpha-1}\big)^{\alpha/2}\big] = 2^{-\alpha/2}\big(1-\mathbb{E}\big[V_1^{1-\alpha/2}\big]\big)^{-1}\mathbb{E}\big[V_1^{1-\alpha/2}\big]$$
$$= 2^{-\alpha/2}\Big(1-\frac{2}{4-\alpha}\Big)^{-1}\frac{2}{4-\alpha} = \frac{2^{1-\alpha/2}}{2-\alpha}.$$

Thus we have $\mathbb{P}(Q > x) \sim 2^{1-\alpha/2}\mathbb{P}((S_\alpha^{(1)})^2 > x)/(2-\alpha)$, as $x \to \infty$. The last asymptotic equivalence in Proposition 3.5 follows from the identity $\mathbb{P}((S_\alpha^{(1)})^2 > x) = \mathbb{P}(S_\alpha^{(1)} > \sqrt{x}) + \mathbb{P}(-S_\alpha^{(1)} > \sqrt{x})$. □

Proof of Theorem 3.6. (a) Assume $\mu > 0$. We assume without loss of generality that $t \mapsto a_t$ is continuous and $a_t \geq 1$ for all $t > 0$. Define

$$Z_T := \Big(\frac{\mu}{\sqrt{1+\mu^2}}, 1, 1\Big)\frac{X_T - \mu T}{a_T}, \quad Z_T' := \frac{1}{a_T}\big(\Upsilon_{\widehat{T}} - \sqrt{1+\mu^2}T, X_T - \mu T, \overline{X}_T - \mu T\big).$$

Since $Z_T \xrightarrow{d} (\mu/\sqrt{1+\mu^2}, 1, 1)S_\alpha(1)$ as $T \to \infty$, it suffices to show that $\|Z_T - Z_T'\| \xrightarrow{\mathbb{P}} 0$ as $T \to \infty$. Define

$$\Delta_T := \Upsilon_{\widehat{T}} - \sqrt{1+\mu^2}T - \frac{\mu}{\sqrt{1+\mu^2}}(X_T - \mu T), \qquad T > 0.$$

Note that $|\underline{X}_T/a_T| \xrightarrow{\mathbb{P}} 0$ as $T \to \infty$ since the positive drift $\mu > 0$ implies that $-\underline{X}_T \to -\underline{X}_\infty < \infty$ a.s. as $T \to \infty$. Since $\|Z_T' - Z_T\| = a_T^{-1}\|(\Delta_T, 0, (\overline{X}_T - X_T)/a_T)\|$, and $\overline{X}_T - X_T \overset{d}{=} -\underline{X}_T$, part (a) will follow if we show that $\Delta_T/a_T \xrightarrow{\mathbb{P}} 0$ as $T \to \infty$.

By (3.6), we have $(\Upsilon_{\widehat{T}} - T, X_T - \mu T) \overset{d}{=} \sum_{n=1}^\infty(\sqrt{t_n^2 + \xi_n^2} - t_n, \widetilde{\xi}_n)$, where we define $\widetilde{\xi}_n := \xi_n - \mu t_n$. Thus we have $\Delta_T \overset{d}{=} \sum_{n\in\mathbb{N}}\zeta_n$, where

$$\zeta_n := \sqrt{t_n^2 + \xi_n^2} - \sqrt{1+\mu^2}t_n - \frac{\mu}{\sqrt{1+\mu^2}}\widetilde{\xi}_{t_n}$$
$$= \sqrt{1+\mu^2}t_n\bigg(\bigg(1 + \frac{\widetilde{\xi}_n^2 + 2\mu t_n\widetilde{\xi}_n}{t_n^2(1+\mu^2)}\bigg)^{1/2} - 1 - \frac{\mu}{1+\mu^2}\frac{\widetilde{\xi}_n}{t_n}\bigg).$$

To prove that $\Delta_T/a_T \xrightarrow{\mathbb{P}} 0$, we again split the summation set with $\mathfrak{I}_T$ and $\mathfrak{I}_T^\complement$. Define: $\Delta_T^{(1)} := \sum_{n\in\mathfrak{I}_T}\zeta_n$ and $\Delta_T^{(2)} := \sum_{n\in\mathfrak{I}_T^\complement}\zeta_n$ and note that $\Delta_T \overset{d}{=} \Delta_T^{(1)} + \Delta_T^{(2)}$.

Fix some $p \in (0, \alpha/2)$ and use the inequality $\sqrt{1+z} \leq 1 + z/2$ for $z \geq -1$ and the subadditivity of $x \mapsto x^p$ to obtain

$$\mathbb{E}[|\Delta_T^{(1)}/a_T|^p] \leq \mathbb{E}\bigg[\bigg|\sum_{n\in\mathfrak{I}_T}\frac{\widetilde{\xi}_n^2}{2a_T\sqrt{1+\mu^2}t_n}\bigg|^p\bigg] \leq \mathbb{E}\sum_{n\in\mathfrak{I}_T}\frac{|\widetilde{\xi}_n|^{2p}}{a_T^p t_n^p}.$$

Recall that $(X_t - \mu t)/a_t \xrightarrow{d} S_\alpha(1)$ as $t \to \infty$. Thus, by Lemma 3.18, there exists a constant $K_{2p} > 0$ such that $\mathbb{E}[|X_t - \mu t|^{2p}] \leq K_{2p}a_t^{2p}$ for all $t \geq 1$. Therefore $\mathbb{E}_\ell[|\widetilde{\xi}_n|^{2p}] \leq K_{2p}a_{t_n}^{2p}$ for $n \in \mathfrak{I}_T$.

Suppose $\alpha \in (1, 2)$. Pick $q \in (1/2, 1/\alpha)$ and apply Potter's bound (Theorem A.53) to obtain $a_t/a_T \leq K_q'(t/T)^q$ for all $T > t \geq 1$ and some $K_q' > 0$.

Thus, Corollary 3.10(a) yields

$$\begin{aligned}\mathbb{E}[|\Delta_T^{(1)}/a_T|^p] &\le K_{2p}\mathbb{E}\sum_{n\in\mathfrak{I}_T}\frac{a_{t_n}^{2p}}{a_T^p t_n^p} = K_{2p}\Big(\frac{a_T}{T}\Big)^p\mathbb{E}\sum_{n\in\mathfrak{I}_T}\ell_n^{-p}\Big(\frac{a_{t_n}}{a_T}\Big)^{2p}\\ &\le K_{2p}(K_q')^{2p}\Big(\frac{a_T}{T}\Big)^p\mathbb{E}\sum_{n=1}^{\infty}\ell_n^{p(2q-1)} = \frac{K_{2p}(K_q')^{2p}}{p(2q-1)}\Big(\frac{a_T}{T}\Big)^p,\end{aligned}$$

which tends to 0 as $T\to\infty$, implying $\Delta_T^{(1)}/a_T \xrightarrow{\mathbb{P}} 0$.

Suppose $\alpha=2$. We may assume $a_t=\sqrt{t}l(t)$ for a locally bounded and slowly varying function l. Thus, by Proposition A.54, $\widetilde{l}(T) := \int_1^T t^{-1}l(t)^{2p}\mathrm{d}t$ is also slowly varying and Corollary 3.10(a) yields

$$\mathbb{E}[|\Delta_T^{(1)}/a_T|^p] \le K_{2p}\mathbb{E}\sum_{n\in\mathfrak{I}_T}\frac{a_{t_n}^{2p}}{a_T^p t_n^p} = K_{2p}\mathbb{E}\sum_{n\in\mathfrak{I}_T}\frac{l(t_n)^{2p}}{a_T^p} = K_{2p}\frac{\widetilde{l}(T)}{a_T^p}\xrightarrow[T\to\infty]{} 0.$$

It remains to show that $\Delta_T^{(2)}/a_T \xrightarrow{\mathbb{P}} 0$ as $T\to\infty$. The inequality $\sqrt{1+x+y}\ge 1+y/2$ for $x\ge y^2/4$ and $x+y\ge -1$ shows that $\Delta_T^{(2)}\ge 0$ a.s. By the subadditivity of $x\mapsto\sqrt{x}$, we obtain

$$\begin{aligned}\frac{1}{a_T}\mathbb{E}[|\Delta_T^{(2)}|] &\le \frac{\sqrt{1+\mu^2}}{a_T}\mathbb{E}\sum_{n\in\mathfrak{I}_T^c}t_n\bigg|\frac{\widetilde{\xi}_n}{t_n\sqrt{1+\mu^2}}+\frac{(2|\mu||\widetilde{\xi}_n|)^{1/2}}{\sqrt{t_n(1+\mu^2)}}-\frac{\mu}{1+\mu^2}\frac{\widetilde{\xi}_n}{t_n}\bigg|\\ &\le \frac{1}{a_T}\mathbb{E}\sum_{n\in\mathfrak{I}_T^c}\bigg(\Big(1-\frac{\mu}{\sqrt{1+\mu^2}}\Big)|\widetilde{\xi}_n|+\sqrt{2|\mu|t_n|\widetilde{\xi}_n|}\bigg).\end{aligned}$$

By (2.5) and Jensen's inequality, there exists a constant $K>0$ such that $\mathbb{E}[|X_t-\mu t|]\le K\sqrt{t}$ for all $t\le 1$. Thus, Corollary 3.10(a) yields $\Delta_T^{(2)}/a_T \xrightarrow{L^1} 0$ as $T\to\infty$, completing the proof of part (a).

(b) Note that $\overline{X}_T\to\overline{X}_\infty<\infty$ a.s. and $\gamma_T^{\frown}\to\gamma_\infty^{\frown}<\infty$ a.s. as $T\to\infty$. Next, we split the length of the concave majorant in two at the time of the supremum, so the total length $\Upsilon_T^{\frown}$ up to time T is equal to the sum of the length $\Delta_T^{(1)}$ up to time $\gamma_T^{\frown}$ and the length $\Delta_T^{(2)}$ from $\gamma_T^{\frown}$ to T. It follows that $\Delta_T^{(1)}\to\Upsilon_{\gamma_\infty^{\frown}}^{\frown}$ a.s. as $T\to\infty$, implying $\Delta_T^{(1)}/a_T\to 0$ a.s. Thus, it suffices to consider $\Delta_T^{(2)}$ for the weak limit of $\Upsilon_T^{\frown}$. Since the post-supremum process is independent of the pre-supremum process by Lemma 2.7, as in part (a) we conclude that, as $T\to\infty$,

$$\bigg(\frac{\Delta_T^{(2)}-(T-\gamma_T^{\frown})}{a_T},\frac{(C_T^{\frown}(T)-\overline{X}_T)-\mu(T-\gamma_\infty^{\frown})}{a_T}\bigg)\bigg|(\overline{X}_\infty,\gamma_\infty^{\frown}) \xrightarrow{d} \Big(\tfrac{\mu}{\sqrt{1+\mu^2}},1\Big)S_\alpha(1).$$

Note here that the limit law does not depend on $(\overline{X}_\infty,\gamma_\infty^{\frown})$, so the limit is independent of $(\overline{X}_\infty,\gamma_\infty^{\frown})$. Since we also have $|\overline{X}_\infty-\overline{X}_T|\to 0$ and $|\gamma_\infty^{\frown}-\gamma_T^{\frown}|\to 0$ a.s. as $T\to\infty$, the result follows. □

§3.3.2 Sandwiching the concave majorant

When the tails of X are sufficiently heavy for it not to have the first moment, the asymptotic behaviour of the boundary of its convex hull is straightforward.

Proof of Theorem 3.7. The supremum, infimum and the times at which they are attained are functionals that are continuous a.s. in J_1-topology with respect to the law of an α-stable process, since the times at which the extrema are attained are a.s. unique (see Lemma A.4 and Theorem 2.22). Thus, by the continuous mapping theorem, it suffices to prove $|\Upsilon_T^{\frown} - (2\overline{C}_T^{\frown} - C_T^{\frown}(T))|/a_T \to 0$ and $|\Upsilon_T^{\smile} - (C_T^{\smile}(T) - 2\underline{C}_T^{\smile})|/a_T \to 0$ a.s. as $T \to \infty$. Recall $X_T = C_T^{\frown}(T) \le \overline{X}_T = \overline{C}_T^{\frown}$ and $\gamma_T^{\frown} \in [0, T]$. Hence, by Figure 3.2, the following inequalities hold:

$$
\begin{aligned}
2\overline{X}_T - X_T &\le ((\gamma_T^{\frown})^2 + (\overline{X}_T)^2)^{1/2} + ((T - \gamma_T^{\frown})^2 + (\overline{X}_T - X_T)^2)^{1/2} \\
&\le \Upsilon_T^{\frown} \le 2\overline{X}_T - X_T + T.
\end{aligned}
$$

Since $\alpha \in (0,1)$ we have $\lim_{T\to\infty} T/a_T = 0$, implying $|\Upsilon_T^{\frown} - (2\overline{C}_T^{\frown} - C_T^{\frown}(T))|/a_T \to 0$ a.s. as $T \to \infty$. The proof of the second limit is analogous. $\square$

Proof of Proposition 3.8. (a)&(b) In part (a), define $a_T := \sqrt{T}$ for all $T > 0$. Note that $\Upsilon_T^{\sqcap} - T = 2\overline{X}_T - X_T$ and

$$
\Upsilon_T^{\wedge} - T = \left(\sqrt{(\gamma_T^{\frown})^2 + \overline{X}_T^2} - \gamma_T^{\frown}\right) + \left(\sqrt{(T - \gamma_T^{\frown})^2 + (\overline{X}_T - X_T)^2} - (T - \gamma_T^{\frown})\right).
$$

We will show that

$$
\frac{T}{a_T^2}\left|\Upsilon_T^{\wedge} - T - \frac{\overline{X}_T^2}{2\gamma_T^{\frown}} - \frac{(\overline{X}_T - X_T)^2}{2(T - \gamma_T^{\frown})}\right| \xrightarrow{\mathbb{P}} 0, \qquad \text{as } T \to \infty. \tag{3.24}
$$

The conclusions of parts (a) & (b) will then follow from (3.24), an application of the continuous mapping theorem and Theorems 3.1 & 3.4, respectively.

To prove (3.24), by symmetry, it suffices to establish, as $T \to \infty$, the limit $Ta_T^{-2}|((\gamma_T^{\frown})^2 + \overline{X}_T^2)^{1/2} - \gamma_T^{\frown} - \overline{X}_T^2/(2\gamma_T^{\frown})| \xrightarrow{\mathbb{P}} 0$. Taylor's theorem yields $\sqrt{1+x^2} = 1 + x^2/2 + x^4\theta(|x|)/8$, where $\theta : [0,\infty) \to [0,1]$ is a bounded function. Thus, the limit in probability is implied by the limit $Ta_T^{-2}\overline{X}_T^4/(\gamma_T^{\frown})^3 \xrightarrow{\mathbb{P}} 0$ as $T \to \infty$, which is itself a direct consequence of the fact that $a_T/T \to 0$, the continuous mapping theorem and the weak limits $\gamma_T^{\frown}/T \xrightarrow{d} \gamma^{\alpha\frown}$ and $\overline{X}_T/a_T \xrightarrow{d} \overline{S}_\alpha(1)$ as $T \to \infty$.

(c) The proof follows as in the proof of Theorem 3.7, using the triangle inequality to obtain

$$
2\overline{X}_T - X_T \le \Upsilon_T^{\wedge} \le \Upsilon_T^{\sqcap} = T + 2\overline{X}_T - X_T,
$$

and then using the fact that $T/a_T \to 0$ as $T \to \infty$. $\square$

§3.4 A Gaussian approximation theorem for Lévy processes

The classical central limit theorem (CLT), applied to a one-dimensional Lévy processes $X = (X_t)_{t\geq 0}$ with zero mean and finite variance, states that $\mathbb{P}(X_t/\sqrt{t} \leq x) \to \Phi(x/\sigma)$ as $t \to \infty$ for all $x \in \mathbb{R}$, where σ^2 is the variance of X_1 and Φ is the distribution function of a standard normal random variable Z. Since the law Φ has a bounded density, this weak convergence is well-known to imply the convergence in the Kolmogorov distance $\sup_{x\in\mathbb{R}} |\mathbb{P}(X_t/\sqrt{t} \leq x) - \Phi(x/\sigma)| \to 0$ as $t \to \infty$, see Theorem A.8. It is natural to inquire about the rate of this convergence without additional assumptions on the Lévy process. In this section we answer this question, thus extending to the continuous-time setting the classical random walk result by Friedman, Katz and Koopmans [35].

Theorem 3.20. *Let $X = (X_t)_{t\geq 0}$ be a Lévy process satisfying $\sigma := \mathbb{E}[X_1^2]^{1/2} \in (0,\infty)$, $\mathbb{E}X_1 = 0$ and $X_0 = 0$ a.s. If the Lévy measure ν of X is nontrivial, choose $\kappa \geq 1$ such that $0 < \nu((-\kappa,\kappa)) \leq \infty$ and otherwise set $\kappa := 1$. Defining $\sigma_t^2 := \sigma^2 - \int_{\mathbb{R}\setminus(-\kappa\sqrt{t},\kappa\sqrt{t})} x^2\nu(\mathrm{d}x)$ for $t > 0$, we have*

$$\int_1^\infty \sup_{x\in\mathbb{R}} \left|\mathbb{P}\big(X_t/\sqrt{t} \leq x\big) - \Phi\big(x/\sigma_t\big)\right| \frac{\mathrm{d}t}{t} < \infty. \tag{3.25}$$

Heuristically, (3.25) states that the Kolmogorov distance between the laws of $X_t/\sqrt{t}$ and $\sigma_t Z$ decays faster than say $1/\log(t)$ as $t \to \infty$. The parameter $\kappa \geq 1$ is chosen to ensure $\sigma_t > 0$, with its precise value not being important for (3.25). We stress that σ_t in (3.25) *cannot*, in general, be replaced by σ. Such a replacement would require strictly more than second moment. This result is proved in [8, Thm 1.2], but has been omitted here since it is outside of the scope of this thesis. The main idea behind the proof of Theorem 3.20 is to apply Berry-Esseen bounds to a Lévy process possessing all moments, obtained by removing from a path of X the finitely many jumps with magnitude greater than $\kappa\sqrt{t}$ during the time interval $[0,t]$.

§3.4.1 Proof of Theorem 3.20

Let (Σ^2, β, ν) be the generating triplet of the Lévy process $X = (X_t)_{t\geq 0}$ corresponding to the cutoff function $x \mapsto \mathbb{1}_{(-1,1)}(x)$, where $\Sigma^2 \geq 0$ and $\beta \in \mathbb{R}$ (see §2.2). All generating triplets in the following proof are with respect to the cutoff function $x \mapsto \mathbb{1}_{(-1,1)}(x)$.

Proof of Theorem 3.20. For any $t \geq 1$, let $\widetilde{Y}^{(t)} = (\widetilde{Y}_s^{(t)})_{s\geq 0}$ be the compound Poisson process consisting of jumps of X with magnitude at least $\kappa\sqrt{t}$ and define $Y^{(t)} =$

$(Y^{(t)}_s)_{s\geq 0}$ as $Y^{(t)}_s := X_s - \widetilde{Y}^{(t)}_s$. Then, by Theorem 2.3, $Y^{(t)}$ is a Lévy process with generating triplet $(\Sigma^2, \nu|_{(-\kappa\sqrt{t},\kappa\sqrt{t})}, \beta)$ whose jumps are of magnitude smaller than $\kappa\sqrt{t}$. Since the support of the Lévy measure of $Y^{(t)}$ is compact, by Theorem A.11, $Y^{(t)}_t$ has moments of all orders. Thus, we may define the real number $\mu_t := \mathbb{E}Y^{(t)}_t$. Moreover, the constant $\kappa \geq 1$, chosen in the statement of Theorem 3.20, ensures

$$0 < \sigma_t^2 = \Sigma^2 + \int_{(-\kappa\sqrt{t},\kappa\sqrt{t})} x^2\nu(\mathrm{d}x) = \mathrm{Var}(Y^{(t)}_t)/t < \infty \quad \text{for all } t \geq 1.$$

The first equality in the last display follows from the identity $\sigma^2 = \Sigma^2 + \int_{\mathbb{R}} x^2\nu(\mathrm{d}x)$, which holds by Example A.1 applied to X. The same argument applied to the Lévy process $Y^{(t)}$ yields the second equality in the display.

Define the function

$$K(t) := \sup_{x\in\mathbb{R}} |\mathbb{P}(X_t/\sqrt{t} \leq x) - \Phi(x/\sigma_t)| \quad \text{for all } t > 0.$$

Let J_t denote the event on which X only has jumps of magnitude smaller than $\kappa\sqrt{t}$ during the time interval $[0, t]$. Note that on the event J_t we have $X_t = Y^{(t)}_t$, implying, for all $x \in \mathbb{R}$ and $t \geq 1$, the inequality

$$|\mathbb{P}(X_t \leq x) - \mathbb{P}(Y^{(t)}_t \leq x)| \leq \mathbb{E}|\mathbb{1}_{\{X_t\leq x\}} - \mathbb{1}_{\{Y^{(t)}_t\leq x\}}| \leq \mathbb{E}[\mathbb{1}_{J_t^{\mathsf{c}}}] = \mathbb{P}(J_t^{\mathsf{c}}).$$

By adding and subtracting the probability $\mathbb{P}(Y^{(t)}_t/\sqrt{t} \leq x)$ in the definition of $K(t)$, for all $t \geq 1$ we obtain the inequality

$$K(t) \leq A(t) + \mathbb{P}(J_t^{\mathsf{c}}), \quad \text{where} \quad A(t) := \sup_{x\in\mathbb{R}} \big|\mathbb{P}\big(Y^{(t)}_t/\sqrt{t} \leq x\big) - \Phi\big(x/\sigma_t\big)\big|. \tag{3.26}$$

The triangle inequality implies

$$A(t) = \sup_{x\in\mathbb{R}} \big|\mathbb{P}\big((Y^{(t)}_t - \mu_t)/\sqrt{t} \leq x\big) - \Phi\big((x + \mu_t/\sqrt{t})/\sigma_t\big)\big| \leq B(t) + D(t),$$

where for any $t \geq 1$ we define

$$B(t) := \sup_{x\in\mathbb{R}} \big|\mathbb{P}\big((Y^{(t)}_t - \mu_t)/\sqrt{t} \leq x\big) - \Phi\big(x/\sigma_t\big)\big| \quad \text{and}$$
$$D(t) := \sup_{x\in\mathbb{R}} \big|\Phi\big(x/\sigma_t\big) - \Phi\big((x + \mu_t/\sqrt{t})/\sigma_t\big)\big|.$$

To complete the proof, it suffices to show that the following integrals are finite:

$$\text{(a)} \int_1^\infty \mathbb{P}(J_t^{\mathsf{c}})\frac{\mathrm{d}t}{t} < \infty, \qquad \text{(b)} \int_1^\infty B(t)\frac{\mathrm{d}t}{t} < \infty, \qquad \text{(c)} \int_1^\infty D(t)\frac{\mathrm{d}t}{t} < \infty.$$

The integrals in (a)–(c) exist since the integrands are non-negative. It remains to prove they are finite. Fubini's theorem yields

$$I := \int_{\mathbb{R}} x^2\nu(\mathrm{d}x) = \int_0^\infty 2x\overline{\nu}(x)\mathrm{d}x = \int_0^\infty \overline{\nu}(\sqrt{x})\mathrm{d}x < \infty, \tag{3.27}$$

where ν is the Lévy measure of X and we recall that $\overline{\nu}(x) = \nu(\mathbb{R}\setminus(-x, x))$ for $x > 0$.

(a) Since $\widetilde{Y}^{(t)} = X - Y^{(t)}$ is a compound Poisson process with intensity $\overline{\nu}(\kappa\sqrt{t})$, the first jump of $\widetilde{Y}^{(t)}$ is exponentially distributed with mean $1/\overline{\nu}(\kappa\sqrt{t})$. As the event J_t can be defined by the first jump of $\widetilde{Y}^{(t)}$ being greater than t, it has probability $\mathbb{P}(J_t) = e^{-t\overline{\nu}(\kappa\sqrt{t})}$. Thus, we have

$$\mathbb{P}(J_t^{\mathsf{c}}) = 1 - e^{-t\overline{\nu}(\kappa\sqrt{t})} \le t\overline{\nu}(\kappa\sqrt{t}), \quad \text{for } t > 0,$$

implying the bound $\int_1^\infty t^{-1}\mathbb{P}(J_t^{\mathsf{c}})\mathrm{d}t \le \int_1^\infty \overline{\nu}(\kappa\sqrt{t})\mathrm{d}t \le I/\kappa^2$.

(b) For any $t \ge 1$, $Y_t^{(t)}$ is nontrivial and infinitely divisible with a finite third moment. More precisely, $Y_t^{(t)} = \sum_{k=1}^n Z_k$, where the variables $Z_k := Y_{tk/n}^{(t)} - Y_{t(k-1)/n}^{(t)} \overset{d}{=} Y_{t/n}^{(t)}$ are independent. The Berry-Esseen inequality for independent random variables yields a constant $c > 0$ such that it for all $n \in \mathbb{N}$ holds that

$$\begin{aligned} B(t) &= \sup_{y\in\mathbb{R}} \left| \mathbb{P}\left(\frac{\sum_{k=1}^n Z_k - \mathbb{E}\sum_{k=1}^n Z_k}{\mathrm{Var}\big(\sum_{k=1}^n Z_k\big)^{1/2}} \le y \right) - \Phi(y) \right| \\ &\le \frac{cn\mathbb{E}\big[|Y_{t/n}^{(t)} - \mathbb{E}Y_{t/n}^{(t)}|^3\big]}{\big(n\mathrm{Var}(Y_{t/n}^{(t)})\big)^{3/2}} \le \frac{4cn\big(\mathbb{E}\big[|Y_{t/n}^{(t)}|^3\big] + \big|\mathbb{E}\big[Y_{t/n}^{(t)}\big]\big|^3\big)}{\big(n\mathrm{Var}\big(Y_{t/n}^{(t)}\big)\big)^{3/2}}. \end{aligned}$$

The second inequality in the display above follows from the inequality $|(a+b)/2|^p \le (|a|^p + |b|^p)/2$ for any $a, b \in \mathbb{R}$ and $p \ge 1$ (which holds by convexity), applied with $a = Y_{t/n}^{(t)}$, $b = -\mathbb{E}Y_{t/n}^{(t)}$ and $p = 3$. Since, by Lemma 2.4, the limit $\lim_{n\to\infty} n\mathbb{E}\big[\big|Y_{t/n}^{(t)}\big|^3\big] = t\int_{(-\kappa\sqrt{t},\kappa\sqrt{t})} |x|^3\nu(\mathrm{d}x)$ holds, the equalities $\mathbb{E}\big[Y_{t/n}^{(t)}\big] = \mathbb{E}\big[Y_1^{(t)}\big]t/n$ and $\mathrm{Var}\big(Y_{t/n}^{(t)}\big) = \mathrm{Var}\big(Y_1^{(t)}\big)t/n$ imply

$$\begin{aligned} B(t) &\le \lim_{n\to\infty} \frac{4cn(\mathbb{E}\big[|Y_{t/n}^{(t)}|^3\big] + \big|\mathbb{E}\big[Y_{t/n}^{(t)}\big]\big|^3)}{\big(n\mathrm{Var}\big(Y_{t/n}^{(t)}\big)\big)^{3/2}} \\ &= \frac{4c(\lim_{n\to\infty} n\mathbb{E}\big[|Y_{t/n}^{(t)}|^3\big] + \lim_{n\to\infty}\big|\mathbb{E}\big[Y_1^{(t)}\big]\big|^3 t^3/n^2)}{\big(\mathrm{Var}\big(Y_1^{(t)}\big)t\big)^{3/2}} \\ &= \frac{4ct\int_{(-\kappa\sqrt{t},\kappa\sqrt{t})} |x|^3\nu(\mathrm{d}x)}{\big(\Sigma^2 t + t\int_{(-\kappa\sqrt{t},\kappa\sqrt{t})} x^2\nu(\mathrm{d}x)\big)^{3/2}} \le \frac{4c}{\sqrt{t}\sigma_1^3}\int_{(-\kappa\sqrt{t},\kappa\sqrt{t})} |x|^3\nu(\mathrm{d}x) \end{aligned}$$

for any $t \ge 1$ (recall that $t \mapsto \sigma_t^2$, defined in Theorem 3.20, is non-decreasing). We thus obtain

$$\begin{aligned} \int_1^\infty B(t)\frac{\mathrm{d}t}{t} &\le \frac{4c}{\sigma_1^3}\int_1^\infty t^{-3/2}\int_{(-\kappa\sqrt{t},\kappa\sqrt{t})} |x|^3\nu(\mathrm{d}x)\mathrm{d}t \\ &\le \frac{12c}{\sigma_1^3}\int_1^\infty t^{-3/2}\int_0^{\kappa\sqrt{t}} x^2\overline{\nu}(x)\mathrm{d}x\mathrm{d}t, \end{aligned} \tag{3.28}$$

where the second inequality follows from the identity $\int_{(-w,w)} |x|^3\nu(\mathrm{d}x) = -w^3\overline{\nu}(w) + 3\int_0^w x^2\overline{\nu}(x)\mathrm{d}x$ for all $w > 0$. The limit $0 \le y^2\overline{\nu}(y) \le \int_{\mathbb{R}\setminus(-y,y)} x^2\nu(\mathrm{d}x) \to 0$ as $y \to \infty$ implies $\int_0^{\kappa\sqrt{T}} x^2\overline{\nu}(x)\mathrm{d}x/\sqrt{T} \to 0$ as $T \to \infty$. Thus, the bound in (3.28) and integration-by-parts imply that the integral in (b) is finite:

$$\begin{aligned}
&\int_1^\infty t^{-3/2}\int_0^{\kappa\sqrt{t}} x^2\overline{\nu}(x)\mathrm{d}x\mathrm{d}t\\
&\quad=\Big[-2t^{-1/2}\int_0^{\kappa\sqrt{t}} x^2\overline{\nu}(x)\mathrm{d}x\Big]_1^\infty+\int_1^\infty 2t^{-1/2}\cdot\kappa^2 t\overline{\nu}(\kappa\sqrt{t})\cdot\frac{\kappa}{2\sqrt{t}}\mathrm{d}t\\
&\quad=2\int_0^\kappa x^2\overline{\nu}(x)\mathrm{d}x+\kappa^3\int_1^\infty\overline{\nu}(\kappa\sqrt{t})\mathrm{d}t=2\int_0^\kappa x^2\overline{\nu}(x)\mathrm{d}x+\kappa\int_{\kappa^2}^\infty\overline{\nu}(\sqrt{y})\mathrm{d}y\\
&\quad\le 2\int_0^\kappa x^2\overline{\nu}(x)\mathrm{d}x+\kappa I,
\end{aligned}$$

where the final inequality follows from (3.27).

(c) Since the distribution Φ is unimodal and symmetric, the mean-value theorem implies that $D(t)$ satisfies

$$D(t)=\big|\Phi\big(\mu_t/(2\sqrt{t}\sigma_t)\big)-\Phi(-\mu_t/(2\sqrt{t}\sigma_t))\big|=e^{-c^2/2}|\mu_t|/(\sqrt{2\pi t}\sigma_t)\le|\mu_t|/(\sigma_1\sqrt{t})$$

for $t\ge 1$ and some $c\in(-|\mu_t|/(2\sqrt{t}\sigma_t),|\mu_t|/(2\sqrt{t}\sigma_t))$ (recall that $t\mapsto\sigma_t^2$ is non-decreasing). Since $\sigma_1>0$, it suffices to prove that $\int_1^\infty|\mu_t|t^{-3/2}\mathrm{d}t<\infty$. By $0=\mathbb{E}[X_t]=\beta t+t\int_{\mathbb{R}\setminus(-1,1)}x\nu(\mathrm{d}x)$, we have $\mu_t=\mathbb{E}[Y_t^{(t)}]=-t\int_{\mathbb{R}\setminus(-\kappa\sqrt{t},\kappa\sqrt{t})}x\nu(\mathrm{d}x)$. Hence $|\mu_t|\le t\int_{\mathbb{R}\setminus(-\sqrt{t},\sqrt{t})}|x|\nu(\mathrm{d}x)$ for all $t\ge1$ since $\kappa\ge1$. Apply Fubini's theorem to obtain

$$\int_1^\infty\frac{|\mu_t|}{t^{3/2}}\mathrm{d}t\le\int_1^\infty\frac{1}{\sqrt{t}}\int_{\mathbb{R}\setminus(-\sqrt{t},\sqrt{t})}|x|\nu(\mathrm{d}x)\mathrm{d}t\le 2\int_{\mathbb{R}\setminus(-1,1)}x^2\nu(\mathrm{d}x)<\infty.$$

This implies that the integral in (c) is finite, completing the proof of Theorem 3.20.

□

Chapter 4

When is the convex hull of a Lévy path smooth?

§4.1 Introduction and main results

The boundary of the convex hull of the range of a planar Brownian motion consists of piecewise linear segments but is well-known to be smooth (i.e. continuously differentiable) everywhere [30]. The convex hull of a graph of a path of a standard Cauchy process also possesses a smooth boundary almost surely [21], a fact not easily discerned from the simulation in Figure 4.1 below (but *cf.* discussion following Theorem 4.2 below). Since the law of the graph of the standard Cauchy process scales linearly in time, it is natural to ask whether smoothness of the hull occurs at all for Lévy process without a linear scaling property (note that the range of a planar Brownian motion also possesses a temporal scaling property). In this chapter we characterise (in terms of transition laws) what turns out to be a rich and interesting class of Lévy processes whose graphs have smooth convex hulls almost surely. We study its properties by analysing how the smoothness of the hull of a graph may fail for a general Lévy process (see YouTube [14] for a short presentation on the results).

Recall that the boundary of the convex hull of the graph of a Lévy process X over a finite interval $[0, T]$ is a union of the graphs of the convex minorant and the concave majorant defined in Definition 2.16 (see Figure 4.1). Recall that the minorant and majorant are piecewise linear functions for any Lévy process X. The set of slopes $\mathcal{S} \subset \mathbb{R}$ of the convex minorant, which has the same law as the set of slopes of the concave majorant (see e.g. Theorem 2.18), plays a key role in the question of smoothness of the boundary of the hull. Unless otherwise stated, X is assumed to be of infinite activity (i.e. X is *not* compound Poisson with drift,

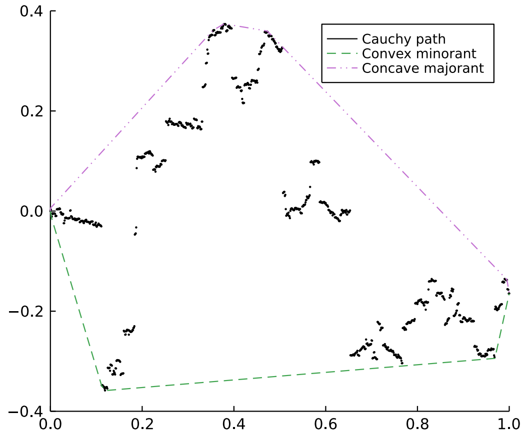

Figure 4.1: The path of a Cauchy process and its convex hull.

which would make $\mathcal{S}$ clearly a finite set and thus smoothness infeasible), making X_t diffuse (i.e. the law of X_t has no atoms) for all $t > 0$ (see Lemma 2.2) and thus the cardinality of the set of slopes $\mathcal{S}$ is infinite by Theorem 2.18. The following zero-one law characterises the local finiteness of $\mathcal{S}$ in terms of the increments of X. The characterisation holds for all Lévy processes X with diffuse increments.

Theorem 4.1. *For any measurable set $I \subseteq \mathbb{R}$, the set $\mathcal{S} \cap I$ is either finite a.s. or infinite a.s. Moreover, the cardinality $|\mathcal{S} \cap I|$ of the intersection $\mathcal{S} \cap I$ is infinite almost surely if and only if*

$$\int_0^1 \mathbb{P}(X_t/t \in I)\frac{\mathrm{d}t}{t} = \infty. \tag{4.1}$$

Theorem 4.1 follows from a novel zero-one law for stick-breaking processes in Theorem 4.18 and Corollary 4.19, established in §4.3 below, and the characterisation of the law of convex minorant of a Lévy process (see e.g. Theorem 2.18). Since the set of slopes of the concave majorant of the path of X has the law of $\mathcal{S}$, Theorem 4.1 can be used to establish the following characterisation of the smoothness of the convex hull.

Theorem 4.2. *The boundary of the convex hull of the graph $t \mapsto (t, X_t)$, $t \in [0, T]$, of a path of any Lévy process X is continuously differentiable (as a closed curve in $\mathbb{R}^2$) a.s. if and only if (4.1) holds for all bounded intervals I in $\mathbb{R}$. Moreover, this is equivalent to the set $\mathcal{S}$ being dense in $\mathbb{R}$ a.s.*

It is clear that if the set of slopes $\mathcal{S}$ is not dense in $\mathbb{R}$, the convex hull cannot possess a smooth boundary. Indeed, a gap in $\mathcal{S}$ (i.e. an open interval contained in the complement $\mathbb{R} \setminus \mathcal{S}$) results in the jump of the derivative of the convex minorant and concave majorant (see §4.4.2 below for the proof of Theorem 4.2). Intuitively, as suggested by the simulation in Figure 4.1, $\mathcal{S}$ is dense if every contact point of X with the boundary of the hull is both preceded and followed by infinitely many contact points between the path and the boundary. More generally, Theorem 4.1 (applied to intervals $I = (a, b)$ with rational $a < b$) implies that, interestingly, the set $\mathcal{L}(\mathcal{S})$ of the accumulation points (see §4.5 for definition) of the random set $\mathcal{S}$ is almost surely constant for any Lévy process X. Theorem 4.2 thus states that the convex hull of the path of X has a smooth boundary if and only if $\mathcal{L}(\mathcal{S}) = \mathbb{R}$ a.s. Note that the criterion in Theorem 4.1 depends neither on the time horizon T nor (by the Lévy-Itô decomposition of X) on the behaviour of the Lévy measure of X on the complement of any neighbourhood of zero, even though the set of slopes $\mathcal{S}$ does depend on both. In particular, if the paths of X have finite variation, then both sides of the equivalence in Theorem 4.2 fail, see §4.1.1.1 below for details.

It was conjectured (without proof) in [3, Rem. 3.4.4] that if the paths of X have infinite variation (or, equivalently, if the integral over any neighbourhood of zero of the distance from zero against the Lévy measure of X is infinite), then $\mathcal{S}$ has finitely many points on every interval (a, b) if and only if (4.1) fails for all $a < b$. This is implied by Theorem 4.1 above and is furthermore equivalent to $\mathcal{L}(\mathcal{S}) = \emptyset$ a.s. Moreover, as we will see below (Proposition 4.3), since X is not compound Poisson with drift, $\mathcal{L}(\mathcal{S}) = \emptyset$ a.s. in fact implies that X must be of infinite variation.

Recall from Definition 2.26, that an infinite variation process X is *abrupt* if, $\limsup_{\varepsilon\uparrow 0}(X_{t+\varepsilon} - X_{t-})/\varepsilon = -\infty$ and $\liminf_{\varepsilon\downarrow 0}(X_{t+\varepsilon} - X_t)/\varepsilon = \infty$ at every local minimum t of the path of X. The notion of abruptness captures Lévy processes that approach and leave very rapidly each local minimum of their trajectory. Interestingly, Theorem 2.27 states that an infinite variation process X is abrupt if and only if condition (4.1) fails for all intervals in $\mathbb{R}$. Since the minimum is a contact point between the path of X and its convex minorant, abrupt Lévy processes are unlikely to have smooth convex minorants. In fact, the criteria in Theorem 4.2 and Theorem 2.27 imply that an infinite variation Lévy process X is abrupt if and only if $\mathcal{L}(\mathcal{S}) = \emptyset$ a.s. Recall from Definition 2.28, that an infinite variation Lévy process X is *eroded* if $\limsup_{\varepsilon\uparrow 0}(X_{t+\varepsilon} - X_{t-})/\varepsilon = 0$ and $\liminf_{\varepsilon\downarrow 0}(X_{t+\varepsilon} - X_t)/\varepsilon = 0$. Eroded Lévy processes approach and leave their local minima very slowly and are good candidates to possess a smooth convex minorant. However, eroded processes appear not to satisfy the condition in Theorem 4.2 that the limit set $\mathcal{L}(\mathcal{S})$ equals $\mathbb{R}$ a.s. Indeed, it follows from Theorem 2.29 and Theorem 4.1 that an infinite variation Lévy process is eroded if and only if $0 \in \mathcal{L}(\mathcal{S})$ a.s. is approached continuously by the slopes of the minorant from *both* sides, i.e. $0 \in \mathcal{L}^-(\mathcal{S}) \cap \mathcal{L}^+(\mathcal{S})$ a.s. (see §4.5 below for the definition of $\mathcal{L}^{\pm}(\mathcal{S})$).

It is clear that, if X is abrupt, then $(X_t - rt)_{t\geq 0}$ is also abrupt for any $r \in \mathbb{R}$. However, since an infinite variation Lévy process X is eroded if and only if $0 \in \mathcal{L}^-(\mathcal{S}) \cap \mathcal{L}^+(\mathcal{S})$ a.s. and the left- and right-accumulation points of the slopes of the process $(X_t - rt)_{t\geq 0}$ equal $\mathcal{L}^-(\mathcal{S})$ and $\mathcal{L}^+(\mathcal{S})$ shifted by the drift $-r$, respectively, this invariance may fail for an eroded process. Recall that a Lévy process X is *strongly eroded* if $(X_t - rt)_{t\geq 0}$ is eroded for every $r \in \mathbb{R}$, which is equivalent to $\mathcal{L}^-(\mathcal{S}) \cap \mathcal{L}^+(\mathcal{S}) = \mathbb{R}$ a.s. Since the interior of $\mathcal{L}(\mathcal{S})$ is contained in $\mathcal{L}^-(\mathcal{S}) \cap \mathcal{L}^+(\mathcal{S}) \subseteq \mathcal{L}(\mathcal{S})$ by definition, the process X is strongly eroded if and only if $\mathcal{L}(\mathcal{S}) = \mathbb{R}$ a.s. or, equivalently, if the boundary of its convex hull is smooth. Since the respective criteria on the law of X in Theorem 4.2 and Theorem 2.27 are *not* complementary, an interesting question, closely related to Vigon's point-hitting conjecture discussed below (see Conjecture 4.10), is which (if any) infinite variation processes satisfy (4.1) for some bounded intervals I but not for others. In particular, are there any eroded processes that are not strongly eroded? See §4.1.1 below for further discussion of these questions.

The class of strongly eroded Lévy processes, defined in terms of the transition probabilities by the criterion in Theorem 4.2, has a rich structure. For example, it contains families of processes with symmetric and asymmetric Lévy measures, including a standard Cauchy process but excluding all non-standard Cauchy (i.e. weakly 1-stable) processes with asymmetric Lévy measures, see §4.2 below. Moreover, a strongly eroded process has no Gaussian component (by Proposition 4.6) and, since it satisfies $\mathcal{L}(\mathcal{S}) = \mathbb{R}$ a.s., has paths of infinite variation (by Proposition 4.3). Its Blumenthal–Getoor index is thus greater or equal to one while the related index β_-, defined in (2.4), is less or equal to one (see Proposition 4.6). More generally, for any strongly eroded Lévy process X, the Lévy process $X + Y$ is strongly eroded for any Lévy process Y of finite variation (see Proposition 4.5 below). In contrast, if Y and X are both strongly eroded and independent of each other, the Lévy process $X + Y$ need not (but, of course, could) be strongly eroded, see Example 4.6 below. The properties of strongly eroded Lévy processes will be discussed in more detail in the remainder of §4.1 and in §4.2.

It is natural to attempt to construct the non-random set of limit slopes $\mathcal{L}(\mathcal{S})$ directly from the characteristics of an arbitrary Lévy process X. It turns out that, if X is of finite variation, $\mathcal{L}(\mathcal{S})$ is a singleton given by the natural drift of the process, see Table 4.1 below for an overview of our results. In the infinite variation case, we characterise $\mathcal{L}(\mathcal{S})$ up to Conjecture 4.9 stated below, which is implied by Vigon's point-hitting conjecture [80, Conj. 1.6] (see the discussion of Conjectures 4.9 and 4.10

below). More precisely, if Vigon's conjecture were true, our sufficient condition for X to be strongly eroded (i.e. $\mathcal{L}(\mathcal{S}) = \mathbb{R}$ a.s.), given in terms of the characteristic exponent of X, would also be necessary, and its complement would imply abruptness (i.e. $\mathcal{L}(\mathcal{S}) = \emptyset$ a.s.). Moreover, via Orey's process in Example 4.7 below, if Conjecture 4.10 were true, there would exist a strongly eroded Lévy processes whose path variation is arbitrarily close to two. This is in contrast to all known examples of strongly eroded Lévy processes, which turn out to have Blumenthal–Getoor index equal to one (see §4.2 below). However, if Vigon's conjecture is not true, there would exist an infinite variation Lévy process with slopes of the convex minorant accumulating at some deterministic values but not at others. Differently put, in this case the non-random set $\mathcal{L}(\mathcal{S})$ would be a proper closed subset of $\mathbb{R}$, implying that kinks in the boundary of the hull would constitute a proper subset of the contact points between the boundary and the closure (in $\mathbb{R}^2$) of the graph of the path. As it is not easy to imagine a boundary of the hull of the path being smooth in some regions but not in others, our results could perhaps be viewed as further evidence for Vigon's point-hitting conjecture.

§4.1.1 Where and how does the continuous differentiability of the boundary fail?

This change of perspective sheds light on where the smoothness features of the boundary discussed above come from. Before stating our results in detail, we give an overview in Table 4.1. Let $C : [0, T] \to \mathbb{R}$ denote the piecewise linear convex minorant of the path of X on $[0, T]$, see e.g. Theorem 2.18. Its right-derivative $C' : (0, T) \to \mathbb{R}$ is a non-decreasing piecewise constant function with image $\{C'(x)|0 < x < T\} \supset \mathcal{S}$. Moreover, we have $\{C'(x)|0 < x < T\} \subset \mathcal{L}^+(\mathcal{S}) \cup \mathcal{S} \subset \mathcal{L}(\mathcal{S}) \cup \mathcal{S}$, see Table 4.2 in §4.5 below for details. Differently put, for every $r \in \mathcal{S}$ there exists a maximal open interval $I_r \subset (0, T)$, satisfying $C'(I_r) = \{r\}$. Note that, in general, C' may but need not be discontinuous at a boundary point of I_r. However, as an increasing right-continuous process, its path is completely determined by the set $\mathcal{S}$ of values on the dense set $\bigcup_{r\in\mathcal{S}} I_r$ and its discontinuities are in a one-to-one relationship with the gaps of $\mathcal{S}$.

The second derivative (as a distribution) is given by a positive Radon measure $\mathrm{d}C'$ on $(0, T)$. Since the set of slopes of the concave majorant and the convex minorant have the same law, the second derivative of the concave majorant has the same law as the (negative) Radon measure $-\mathrm{d}C'$. Thus, the derivative of the boundary of the convex hull over the open interval $(0, T)$ is discontinuous at a point if and only if the point is an atom of the measure $\mathrm{d}C'$. Over the set $\{0, T\} \subset [0, T]$, the discontinuity of the boundary occurs if and only if the derivative C' is either bounded below or above.

Lévy process X	Derivative C' and the limt set $\mathcal{L}(\mathcal{S})$	Measure $\mathrm{d}C'$
Finite variation	C' bounded below *and* above; C' discontinuous on boundary ∂I_r, $\forall r \in \mathcal{S}$; $\mathcal{L}(\mathcal{S}) = \{\gamma_0\}$, where $\gamma_0 = \lim_{t\downarrow 0} X_t/t$ a.s., and $\gamma_0 \notin \mathcal{S}$	atomic; atoms accumulate from left/right or from both sides at a unique (random) accumulation point in $[0,T]$
Infinite variation & $\mathfrak{s}_1$ locally integrable	C' discontinuous on boundary ∂I_r, $\forall r \in \mathcal{S}$; $-\lim_{t\downarrow 0} C'(t) = \lim_{t\uparrow T} C'(t) = \infty$; $\mathcal{L}(\mathcal{S}) = \emptyset$	atomic; atoms accumulate only at 0 and T
Infinite variation & $\mathfrak{s}_1(r) = \infty$, $\forall r \in \mathbb{R}$	C' is continuous on $(0,T)$; $-\lim_{t\downarrow 0} C'(t) = \lim_{t\uparrow T} C'(t) = \infty$; $\mathcal{L}(\mathcal{S}) = \mathbb{R}$	singular continuous

Table 4.1: Summary of the regularity results of the convex minorant $C : [0,T] \to \mathbb{R}$ of any Lévy process X of infinite activity (i.e. not compound Poisson with drift) over time horizon T. The function $\mathfrak{s}_1$, defined in (4.2) in terms of the generating triplet of X, is either locally integrable or everywhere infinite under Conjecture 4.10. By Theorem 4.8 below, this conjecture is known to hold for most Lévy processes.

Let ψ be the Lévy-Khintchine exponent of the Lévy process X (see (2.1)). Note that $\Re(1/(1+iur-\psi(u))) > 0$ for all $r, u \in \mathbb{R}$ since $\Re\psi(u) \le 0$, where $\Re z$ is the real part of a complex $z \in \mathbb{C}$ and $i^2 = -1$. Hence $0 < \mathfrak{s}_1(r) \le \infty$ for any $r \in \mathbb{R}$, where

$$\mathfrak{s}_1(r) := \frac{1}{2\pi}\int_{\mathbb{R}} \Re\frac{1}{1+iur-\psi(u)}\mathrm{d}u, \qquad r \in \mathbb{R}. \tag{4.2}$$

The identity in Theorem 2.34 below (first established in [80] for Lévy processes with bounded jumps) yields the following equivalence for any real $a < b$:

$$\int_a^b \mathfrak{s}_1(r)\mathrm{d}r < \infty \qquad \text{if and only if} \qquad \int_0^1 \mathbb{P}(X_t/t \in (a,b))\frac{\mathrm{d}t}{t} < \infty. \tag{4.3}$$

By definition, $\mathfrak{s}_1 \in L^1_{\mathrm{loc}}(r)$ if and only if $\mathfrak{s}_1$ is Lebesgue integrable on a neighbourhood of $r \in \mathbb{R}$. Thus $r \in \mathcal{L}(\mathcal{S})$ is (by (4.3) and Theorem 4.1) equivalent to the condition $\mathfrak{s}_1 \notin L^1_{\mathrm{loc}}(r)$, involving only the characteristic exponent of X and not the law of X_t for $t > 0$. For example, the presence of a non-trivial Brownian component implies that $\mathcal{L}(\mathcal{S}) = \emptyset$ and, equivalently, that $\mathcal{S}$ is locally finite a.s., see Proposition 4.6 below.

The sum of the lengths of the open maximal intervals $\{I_s : s \in \mathcal{S}\}$ of constancy of C' equals the time horizon T. As suggested by Table 4.1, the random Radon measure $\mathrm{d}C'$, supported on the complement of $\bigcup_{s\in\mathcal{S}} I_s$, must therefore be singular a.s. with respect to the Lebesgue measure. Thus the Lebesgue decomposition of $\mathrm{d}C'$ is in general a sum of an atomic and a purely singular continuous components. Moreover, if Conjecture 4.10 holds, only one of these summands is non-trivial for any Lévy process.

The convex minorant of a compound Poisson process with drift has only finitely many faces, making its derivative necessarily discontinuous at all boundary points of its maximal intervals of constancy. If X has infinite activity but finite variation, then C' is bounded on $[0, T]$ and discontinuous at every point in $\bigcup_{s\in\mathcal{S}} \partial I_s$, but is possibly continuous at the single (random) accumulation point of this set (see Proposition 4.3 below and discussion thereafter for details). If X has too much jump activity or a Brownian component, then C' is discontinuous at every point in $\bigcup_{s\in\mathcal{S}} \partial I_s$ and infinite on $\{0, T\}$ (see Proposition 4.6 below and the discussion thereafter for details). Hence, for C to be continuously differentiable on the open interval $(0, T)$ (i.e., for X to be strongly eroded), the process X must have sufficient jump activity to be of infinite variation, but not too much, as its index β_- (see (2.4)) must be bounded above by one. These features are discussed in more detail in the following subsections (see Table 4.2 in §4.5 for all possible behaviours of the right-derivative of a piecewise linear convex function).

§4.1.1.1 Finite variation

Throughout this subsection we assume X has finite variation but infinite activity. Let γ_0 denote the natural drift of X defined in terms of the characteristics in (4.17) below. Since, by Lemma 2.2, it follows that $\mathbb{P}(X_t = \gamma_0 t) = 0$ for all $t > 0$, the integrals

$$I_- := \int_0^1 \mathbb{P}(X_t < t\gamma_0)\frac{\mathrm{d}t}{t} \qquad \text{and} \qquad I_+ := \int_0^1 \mathbb{P}(X_t > t\gamma_0)\frac{\mathrm{d}t}{t} \tag{4.4}$$

satisfy $I_- + I_+ = \infty$, implying that at least one is infinite. Moreover, the integrals $I_\pm$ are given in terms of the law of a pure-jump Lévy process $(X_t - \gamma_0 t)_{t\geq 0}$, uniquely determined by the Lévy measure of X. Let $\mathcal{L}^+(\mathcal{S})$ (resp. $\mathcal{L}^-(\mathcal{S})$) be the set of right-accumulation (resp. left-accumulation) points of $\mathcal{S}$ (see §4.5 for definition). Equivalence (4.3) and Theorem 4.1 imply that, for any $s \in \mathbb{R}$, $\mathbb{P}(s \in \mathcal{L}^+(\mathcal{S})) \in \{0, 1\}$ (resp. $\mathbb{P}(s \in \mathcal{L}^-(\mathcal{S})) \in \{0, 1\}$) and that it equals 1 if and only if $\mathfrak{s}_1 \notin L^1_{\mathrm{loc}}(s+)$ (resp. $\mathfrak{s}_1 \notin L^1_{\mathrm{loc}}(s-)$), where a function f is in $L^1_{\mathrm{loc}}(s+)$ (resp. $L^1_{\mathrm{loc}}(s-)$) if and only if $f \cdot \mathbb{1}_{(s,\infty)} \in L^1_{\mathrm{loc}}(s)$ (resp. $f \cdot \mathbb{1}_{(-\infty,s)} \in L^1_{\mathrm{loc}}(s)$). In particular, if the a.s. constant set $\mathcal{L}(\mathcal{S})$ is countable, then limit sets $\mathcal{L}^+(\mathcal{S})$ and $\mathcal{L}^-(\mathcal{S})$ (which are subsets of $\mathcal{L}(\mathcal{S})$) are also constant a.s.

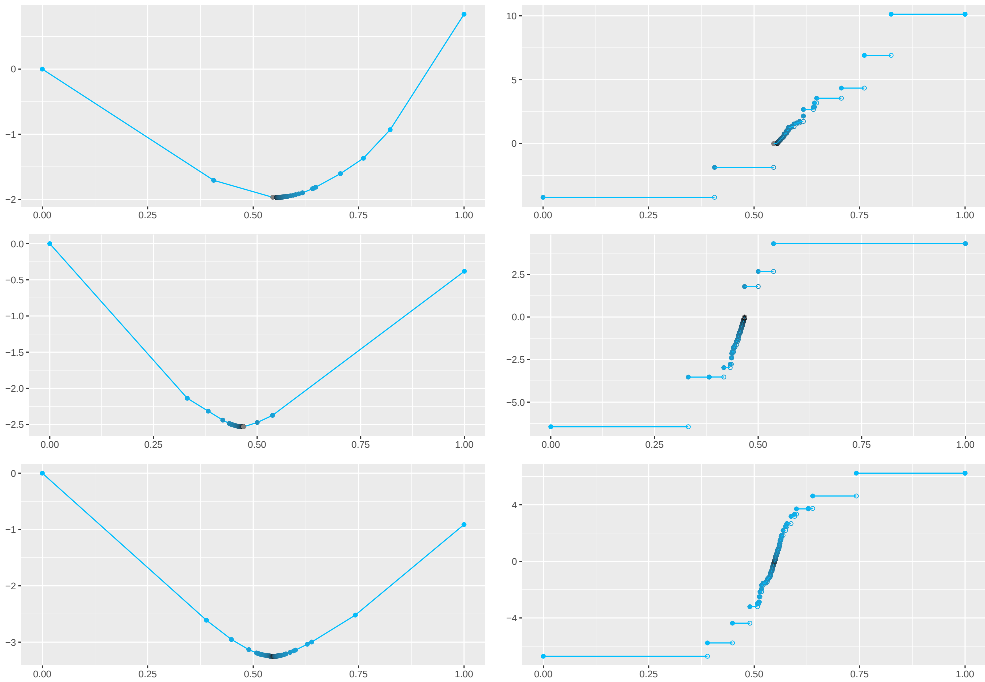

Figure 4.2: Behaviour of C and C' for a finite variation infinite activity process X. The left panels graph the piecewise linear convex function C (circles mark contact points with the path of X). The panels on the right graph the corresponding right-continuous derivative C' (jump-size equals the difference of consecutive slopes). The top (resp. middle) panels correspond to right-accumulation (resp. left-accumulation) in Prop. 4.3(b) (resp. Prop. 4.3(c)). Note that in both of these cases, C' is only right-continuous at the accumulation point of jump-times. The bottom panels depict two-sided accumulation in Prop. 4.3(a), making C' continuous at the unique accumulation point of the jump-times.

Proposition 4.3. *Let X have infinite activity and finite variation. Then the derivative C' (and thus the set of slopes $\mathcal{S}$) is bounded from below and above. C' is discontinuous on $\bigcup_{r\in\mathcal{S}}\partial I_r$, where I_r is the maximal interval of constancy of C' corresponding to slope r, and the limit set of slopes $\mathcal{L}(\mathcal{S})$ is a singleton $\{\gamma_0\}$ (the natural drift γ_0 of X is defined in (4.17)). Time $v\in[0,T]$ at which the process $(X_t-\gamma_0 t)_{t\geq 0}$ attains its minimum in $[0,T]$ is a.s. unique. If $v>0$, denote the left limit of C' at v by $C'(v-):=\lim_{t\uparrow v}C'(t)$. Then we have:*

- *(a) if $I_+=I_-=\infty$, then $v\in(0,T)$, $\mathcal{L}^-(\mathcal{S})=\mathcal{L}^+(\mathcal{S})=\{\gamma_0\}$ and $C'(v-)=C'(v)=\gamma_0$ a.s.;*
- *(b) if $I_-<\infty$, then $v\in[0,T)$ with $\mathbb{P}(v=0)>0$, $\mathcal{L}^-(\mathcal{S})=\emptyset$, $\mathcal{L}^+(\mathcal{S})=\{\gamma_0\}$, $C'(v)=\gamma_0$ a.s. and, on the event $\{v>0\}$, we have $C'(v-)<\gamma_0$ a.s.;*
- *(c) if $I_+<\infty$, then $v\in(0,T]$ with $\mathbb{P}(v=T)>0$, $\mathcal{L}^+(\mathcal{S})=\emptyset$, $\mathcal{L}^-(\mathcal{S})=\{\gamma_0\}$, $C'(v-)=\gamma_0$ a.s. and, on the event $\{v<T\}$, we have $C'(v)>\gamma_0$ a.s.*

By Rogozin's criterion (see Theorem 2.8), the integral conditions in Proposition 4.3(a) are equivalent to 0 being regular for both half-lines for the process $(X_t - \gamma_0 t)_{t\geq 0}$. In particular, for the two conditions to hold concurrently it is necessary (but not sufficient) for X to exhibit infinitely many positive and negative jumps in any finite time interval (see Theorem 2.10) and sufficient (but not necessary) for X to be spectrally symmetric. The other cases are also possible since $I_- < \infty$ (resp. $I_+ < \infty$) if X is the difference of two stable subordinators and the positive (resp. negative) jumps have a larger stability index. The following corollary is a simple consequence of Proposition 4.3, equivalence (4.3) and Theorem 2.9. It characterises which case in Proposition 4.3 occurs in terms of either the Lévy measure ν or the characteristic exponent ψ (via $\mathfrak{s}_1$ defined in (4.2)) of the process X.

Corollary 4.4. *Let X have infinite activity but finite variation with natural drift γ_0. Then we have*

$$I_+ = \infty \iff \mathfrak{s}_1 \notin L^1_{\mathrm{loc}}(\gamma_0+) \iff \int_{(-1,1)} \frac{\max\{x,0\}\nu(\mathrm{d}x)}{\int_0^{\max\{x,0\}} \nu((y,\infty))\mathrm{d}y} = \infty;$$

$$I_- = \infty \iff \mathfrak{s}_1 \notin L^1_{\mathrm{loc}}(\gamma_0-) \iff \int_{(-1,1)} \frac{\max\{-x,0\}\nu(\mathrm{d}x)}{\int_0^{\max\{-x,0\}} \nu((-\infty,-y))\mathrm{d}y} = \infty.$$

Proposition 4.3 and Corollary 4.4 give a complete description (in terms of the characteristics of any infinite activity, finite variation Lévy process X) of how the continuity of the derivative C' fails, see Figure 4.2 for all possible behaviours. The proof of Proposition 4.3 is based on the criterion in Theorem 4.1 and the crucial fact that, as $t \to 0$, the quotient X_t/t a.s. stops visiting closed intervals that do not contain the natural drift γ_0 (since $\lim_{t\to 0} X_t/t = \gamma_0$ a.s.), see §4.4.1 below for details. The smoothness of the minorant in the infinite variation case is more intricate precisely because in that case we do not have a good understanding in general of how frequently the quotient X_t/t visits intervals in $\mathbb{R}$ as $t \to 0$.

§4.1.1.2 Infinite variation

The set of slopes $\mathcal{S}$ is unbounded on both sides for any X of infinite variation, i.e. $\sup\mathcal{S} = -\inf\mathcal{S} = \infty$, and hence $-\lim_{t\downarrow 0} C'(t) = \lim_{t\uparrow T} C'(t) = \infty$ a.s. (cf. Figure 4.3). Indeed, by Theorem A.38, asserting that X_t/t takes arbitrarily large positive and negative values at arbitrarily small times t for any X of infinite variation, it is impossible for the convex minorant to start at 0 or end at T with a finite slope: $-\infty = \liminf_{t\downarrow 0} X_t/t \geq \inf\mathcal{S}$, and, by time reversal, $\infty = \limsup_{t\downarrow 0}(X_T - X_{T-t})/t \leq \sup\mathcal{S}$. Thus the boundary of the convex hull of the path of X is differentiable over the set $\{0, T\}$.[1] However, the smoothness of the boundary of the convex hull of the graph of $t \mapsto X_t$ over the open interval $(0, T)$ is a much more delicate matter. We now state some results to elucidate this structure.

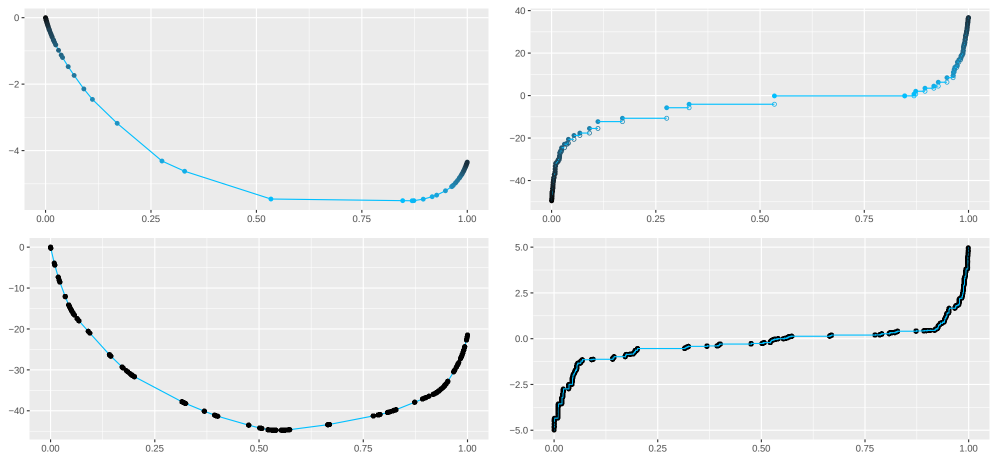

Figure 4.3: The left pictures show the graph of a piecewise linear convex function C with circle marks on $\bigcup_{r \in \mathcal{S}} \partial I_r$. The right pictures show the graph of the derivative C' with visible points of increase marked with a circle. In the top two pictures, there are no points of continuity as C' tends to $\pm\infty$ near times 0 and 1. In the bottom two pictures, the function C is continuously differentiable with a singular continuous derivative C'.

A strongly eroded Lévy process, perturbed by a finite variation process, is still strongly eroded. In fact, for any Lévy process, such a perturbation shifts the set $\mathcal{L}(\mathcal{S})$ by the natural drift, defined in (4.17), of the finite variation process (given a set $A \subset \mathbb{R}$, define $A + b := \{a + b : a \in A\}$ for any $b \in \mathbb{R}$ with convention $\emptyset + b := \emptyset$).

Proposition 4.5. *Suppose the Lévy process X is of the form $X = Y + Z$ for (possibly dependent) Lévy processes Y and Z. Let $\mathcal{S}_X$ and $\mathcal{S}_Z$ be the sets of slopes of the faces of the convex minorants of X and Z, respectively. If Y is of finite variation (possibly finite activity) with natural drift b defined in* (4.17)*, then $\mathcal{L}(\mathcal{S}_X) = \mathcal{L}(\mathcal{S}_Z) + b$.*

The proof of Proposition 4.5 relies on the a.s. limit $b = \lim_{t \downarrow 0} Y_t/t$ and the stick-breaking representation of the convex minorant in Theorem 2.18. The main idea is that if Z_t/t frequently visits any neighborhood of a point $x \in \mathbb{R}$ as $t \downarrow 0$, then X_t/t visits the neighborhoods of $x + b$ just as frequently. Crucially, when the visits of Z_t/t to the neighborhoods of x occur, Y_t/t must necessarily be close to b (since the limit $Y_{t_n}/t_n \to b$ holds along any random sequence of times $t_n \downarrow 0$).

[1] More precisely, the boundary of the convex hull of the graph of $t \mapsto X_t$, as a closed curve in $\mathbb{R}^2$, contains points $(0, 0), (T, 0) \in \mathbb{R}^2$ and possesses local parametrisations $\varphi_0, \varphi_T : (-\varepsilon, \varepsilon) \to \mathbb{R}^2$, for some $\varepsilon > 0$, such that $\varphi_0(0) = (0, 0) \in \mathbb{R}^2$, $\varphi_T(0) = (T, 0) \in \mathbb{R}^2$ and the derivatives $\varphi_0'(0)$ and $\varphi_T'(0)$ exist.

Proposition 4.5 and its proof may suggest that, if Y_t/t and Z_t/t were to visit all open intervals as $t \downarrow 0$ with such frequency that their respective sets of slopes $\mathcal{S}_Y$ and $\mathcal{S}_Z$ are dense, then the same should be true for X_t/t. This intuition, however, turns out to be false, as Example 4.6 below illustrates. Intuitively, the reason for this is that the frequent visits of Y_t/t and Z_t/t to such neighborhoods may be sufficiently rare so that they do not occur simultaneously with sufficient frequency even when Y and Z are independent.

Too much jump activity breaks smoothness. Denote by $\sigma \geq 0$ the volatility of the Brownian component of X and recall the function $\overline{\sigma}^2(u) = \int_{(-u,u)} x^2 \nu(\mathrm{d}x)$ for $u > 0$ (see (2.2)). Moreover, from (2.4) & (2.3) recall the definitions of the lower-activity index β_- and the Blumenthal–Getoor index β_+, that $0 \leq \beta_- \leq \beta_+ \leq 2$, and that we in general may have $\beta_- < \beta_+$. However, both indices agree if the tails of ν are regularly varying at 0 (e.g. if X_t is in the domain of attraction of an α-stable law as $t \downarrow 0$).

Proposition 4.6. *If $\int_1^\infty (1 + u^2(\sigma^2 + \overline{\sigma}^2(1/u)))^{-1} \mathrm{d}u < \infty$, then $\mathcal{L}(\mathcal{S}) = \emptyset$ a.s. and hence X is abrupt. In particular, this is the case if $\sigma^2 > 0$ or $\beta_- > 1$.*

Proposition 4.6 shows that a strongly eroded process necessarily satisfies $\beta_- \leq 1 \leq \beta_+$. This is natural since, in some sense, the running supremum of the process X fluctuates between the functions $t \mapsto t^{1/\beta_+}$ and $t \mapsto t^{1/\beta_-}$ as $t \downarrow 0$ (see Proposition A.39) and the visits of X_t/t to compact intervals determine whether X is strongly eroded. In other words, it is natural that strongly eroded processes require the linear map $t \mapsto t$ to lie between the functions $t \mapsto t^{1/\beta_+}$ and $t \mapsto t^{1/\beta_-}$ as $t \downarrow 0$, which is equivalent to $\beta_- \leq 1 \leq \beta_+$. We remark that, despite the necessary condition on the indices β_- and β_+ allowing a strict inequality, all our examples of strongly eroded processes lie in the boundary case $\beta_- = 1 = \beta_+$. However, as explained in Example 4.7 below, Conjecture 4.10 implies that a strict inequality is feasible for certain strongly eroded Lévy processes.

Too much asymmetry breaks smoothness. Recall from Definition 2.30, that a Lévy process is said to creep *upwards* (resp. *downwards*) if $\mathbb{P}(T_{(x,\infty)} < \infty, X_{T_{(x,\infty)}} = x) > 0$ (resp. $\mathbb{P}(T_{(-\infty,x)} < \infty, X_{T_{(-\infty,x)}} = x) > 0$) for some $x > 0$ (resp. $x < 0$), where $T_A := \inf\{t > 0 : X_t \in A\}$ denotes the first hitting time of set $A \subset \mathbb{R}$ (with convention $\inf \emptyset = \infty$), i.e., if the process crosses levels continuously with positive probability. Processes that creep (upward or downward), all of which are abrupt by Example 2.3, tend to have Lévy measures that are asymmetric on a neighborhood of 0 (see Theorem 2.31 for a characterisation of such processes in terms of ν). For instance, if $\sigma = 0$ and ν is of infinite variation but $\int_{(0,1)} x\nu(\mathrm{d}x) < \infty$ (resp.

$\int_{(-1,0)} |x|\nu(\mathrm{d}x) < \infty$), then X creeps upwards (resp. downwards), see Remark 2.32.

These facts suggest that asymmetry tends to produce abrupt processes. This heuristic is also suggested by the following:

$$\Re\frac{1}{1+iur-\psi(u)} = \frac{1-\Re\psi(u)}{|1+iur-\psi(u)|^2} \le \frac{1}{1-\Re\psi(u))} \le 1, \tag{4.5}$$

where we note that the characteristic exponent of the symmetrisation $\widehat{X}$ of X (a process with the same law as $X - Y$, where Y is independent of X but shares its law) equals $2\Re\psi$. In particular, under Vigon's point-hitting conjecture (see Conjecture 4.10 below), this inequality and (4.3) yield the following implications: (I) if $\widehat{X}$ is abrupt then X is abrupt and (II) if X is strongly eroded then $\widehat{X}$ is strongly eroded. We complement these observations with the following result, further supporting this heuristic.

Proposition 4.7. *Let X be an infinite variation Lévy process and suppose there exist constants $c > 1$ and $x_0 \in (0, \infty]$ such that $\nu([x, y)) \ge c\nu((-y, -x])$ for all $0 < x < y < x_0$. Then X is abrupt.*

We stress that the process in Proposition 4.7 is abrupt but need *not* creep. The assumption in Proposition 4.7 requires, on a neighborhood of 0, the restriction of ν on the negative half-line to be absolutely continuous with respect to its restriction on the positive half-line with Radon–Nikodym derivative $\varphi(x) = \nu(\mathrm{d}(-x))/\nu(\mathrm{d}x)$ bounded above by $1/c < 1$ (equivalently, $\limsup_{x\downarrow 0} \varphi(x) < 1$). This condition is nearly optimal, since there exist strongly eroded processes with positive asymmetry $1 - \varphi(x)$ that vanishes (arbitrarily) slowly $1 - \varphi(x) \downarrow 0$ as $x \downarrow 0$, see Example 4.10 below.

The domination of the positive jumps in the assumption of Proposition 4.7 is not essential. If the negative jumps dominate, i.e. $c\nu([x, y)) \le \nu((-y, -x])$ for all $0 < x < y < x_0$, then Proposition 4.7 implies that $-X$ is abrupt. Thus, by Theorem 2.27, condition (4.1) with $-X$ fails for all intervals in $\mathbb{R}$. Therefore (4.1) also fails with X for all intervals in $\mathbb{R}$, making it abrupt. Finally, we note that by using Propositions 4.7 and 4.5 jointly, we obtain a simple recipe to construct abrupt processes with $\beta_- \le 1 \le \beta_+$.

Sufficient conditions for X to be strongly eroded (or abrupt). The following theorem, implied by Theorem 4.1 above and the results in [79], shows that most Lévy processes of infinite variation are either strongly eroded or abrupt (cf. paragraph following Theorem 4.8). Moreover, Theorem 4.8 offers simple conditions to ascertain whether X is strongly eroded or abrupt. The criteria are (mostly) in terms of the behaviour at infinity of the Lévy–Khintchine exponent ψ of X. More precisely, it is connected to the ratio $\psi(u)/u$ for large $|u|$. This ratio appears naturally since the characteristic exponent of X_t/t is given by $u \mapsto t\psi(u/t)$, whose behaviour for small t is described by the behaviour of $\psi(u)/u$ for large $|u|$.

Let $\Im z$ and $|z|$ denote the imaginary part and the modulus of the complex number $z \in \mathbb{C}$. Recall from (2.1), that $\psi(u) = -u^2\sigma^2/2 + iu\gamma + \int_{\mathbb{R}}(e^{iux} - 1 - iux\mathbb{1}_{(-1,1)}(x))\nu(\mathrm{d}x)$, for $u \in \mathbb{R}$, where $\Re\psi$ (resp. $\Im\psi$) is an even (resp. odd) function on $\mathbb{R}$, making $|\psi|$ an even function on $\mathbb{R}$.

Theorem 4.8. *Let X be a Lévy process of infinite variation with $e^{\psi(u)} = \mathbb{E}e^{iuX_1}$ for $u \in \mathbb{R}$. Then the following hold.*

(i) If $\limsup_{|u|\to\infty} |\psi(u)/u| < \infty$, then X is strongly eroded.

(ii) If $\lim_{|u|\to\infty} |\psi(u)/u| = \infty$, then X is either abrupt or strongly eroded.

(ii-a) *Assume that $\lim_{|u|\to\infty} |\Re\psi(u)/u| = \infty$, then X is strongly eroded if and only if $\int_1^\infty \Re(1/(1-\psi(u)))\mathrm{d}u = \infty$,*

(ii-b) Assume that $0 < \liminf_{|u|\to\infty} |\Re\psi(u)/u| \le \limsup_{|u|\to\infty} |\Re\psi(u)/u| < \infty$ and $\lim_{|u|\to\infty} |\Im\psi(u)/u| = \infty$, then X is strongly eroded if and only if $\int_1^\infty u(1 + |\Im\psi(u)|^2)^{-1}\mathrm{d}u = \infty$,

(ii-c) Assume $\lim_{|u|\to\infty} |\Re\psi(u)/u| = 0$ and $\lim_{|u|\to\infty} |\Im\psi(u)/u| = \infty$, then X is strongly eroded if and only if we have $\int_1^\infty (1-\Re\psi(u))(1+|\Im\psi(u)|^2)^{-1}\mathrm{d}u = \infty$.

In fact, the proof of Theorem 4.8 shows that, under the assumptions of Theorem 4.8, $\mathfrak{s}_1$ is either locally bounded (making X abrupt) or everywhere infinite (making X strongly eroded). Cases (i) and (ii) in Theorem 4.8 are in some sense generic, but they do not exhaust the class of infinite variation Lévy processes, see Examples 4.5 and 4.7 below. In fact, Example 4.5 defines a strongly eroded Lévy process, outside of the scope of Theorem 4.8, with the characteristic exponent that fluctuates between linear and superlinear behaviour as $u \to \infty$. However, the class of processes in the union of case (i) and (ii) is closed under addition of independent summands. Similarly, cases (ii-a), (ii-b) and (ii-c) are not exhaustive within (ii). However, for neither case (i) nor case (ii) to hold, it is necessary that $|\psi(u)|$ fluctuate between (sub-)linear and superlinear functions of $|u|$, suggesting that most Lévy processes satisfy the assumptions of Theorem 4.8. Furthermore, any infinite variation process satisfies $\int_1^\infty u^{-2}|\Re\psi(u)|\mathrm{d}u = \infty$ (by Lemma 2.37, see also [78, Prop. 1.5.3]), which further restricts the Lévy–Khintchine exponent of any process outside of the domain of Theorem 4.8. Note that this class is not empty, see Examples 4.6, but for those specific examples we can nevertheless determine whether they are strongly eroded or abrupt.

A conjectural dichotomy. Our results may be viewed as further evidence for Vigon's point-hitting conjecture (see Conjecture 4.10 below), whose origins go back to Vigon's PhD thesis [78, p. 10] in 2002 and which implies the following dichotomy for infinite variation Lévy processes:

Conjecture 4.9. Any infinite variation Lévy process is either abrupt or strongly eroded.

In order to understand the relationship between Conjecture 4.9 and Vigon's point-hitting conjecture, recall first that the process X hits points if for some $x \in \mathbb{R}$, the hitting time $T_x \coloneqq \inf\{t > 0 : X_t = x\}$ is finite with positive probability. If X has infinite variation, Theorem A.29(b) implies that $\mathbb{P}(T_x < \infty) > 0$ for some $x \in \mathbb{R}$ if and only if $\mathbb{P}(T_x < \infty) > 0$ for all $x \in \mathbb{R}$.

Conjecture 4.10 ([80, Conject. 1.6]). Let X be an infinite variation process and for any $r \in \mathbb{R}$ define the Lévy process $X^{(r)} = (X_t - rt)_{t \geq 0}$. Then the following statements are equivalent.

(i) There exists some $r \in \mathbb{R}$ such that the process $X^{(r)}$ hits points.
(ii) For all $r \in \mathbb{R}$ the process $X^{(r)}$ hits points.
(iii) The process X is abrupt.

By Theorem A.29(a), $\mathfrak{s}_1(r) < \infty$ is equivalent to $X^{(r)}$ hitting points (recall the definition of $\mathfrak{s}_1$ in (4.2)). Moreover, by the equivalence in (4.3) and Theorem 2.27, X is abrupt if and only if $\mathfrak{s}_1$ is locally integrable on $\mathbb{R}$. Conjecture 4.10 thus says that the following three statements are equivalent for any infinite variation Lévy process X: (i) $\mathfrak{s}_1(r) < \infty$ for some $r \in \mathbb{R}$, (ii) $\mathfrak{s}_1(r) < \infty$ for all $r \in \mathbb{R}$ and (iii) the function $\mathfrak{s}_1$ is locally integrable on $\mathbb{R}$. In particular, under Conjecture 4.10, the function $\mathfrak{s}_1$ is either everywhere infinite or locally integrable, thus implying Conjecture 4.9 by Theorem 4.1, equivalence (4.3) and Theorem 2.27.

The finiteness of $\mathfrak{s}_1(r)$, hinging entirely on the integrability at infinity of the positive bounded function $u \mapsto \Re(1/(1 + iur - \psi(u)))$, becomes a focal point under Conjecture 4.10. For instance, Conjecture 4.10 holds if X satisfies the assumptions of either Proposition 4.7 or Theorem 4.8 (the below proofs of these results establish that $\mathfrak{s}_1$ is locally finite on $\mathbb{R}$). The condition $\mathfrak{s}_1(0) < \infty$ is equivalent to a number of probabilistic statements about the infinite variation process X:

- the potential measure of X is absolutely continuous with a bounded density (by Theorem A.28(a)),
- the point 0 is regular for itself for the process X, i.e. $\mathbb{P}(T_0 = 0) = 1$ (by Theorem A.29),
- the process X possesses a local time field (by Theorem A.22).

In principle, any of these properties may hold for X but not for some $X^{(r)}$. Conjecture 4.10, which asserts that this is not the case, can thus be equivalently stated by substituting "hitting of points" with any of the three properties in the bullet-point list.

The structure of Conjecture 4.10, in terms of varying drifts, is natural as a number of properties of infinite variation processes are known to be invariant under addition of a deterministic drift and, more generally, under a perturbation by an independent finite variation process Y. For instance, as Vigon shows in Theorem 2.31, if the infinite variation process X creeps in either direction, then $X+Y$ also creeps in the same direction. Despite Vigon's extensive body of work in the area over the years (see [77, 78, 79, 80]), to the best of our knowledge Conjecture 4.10 remains unsolved. In Conjecture 4.9 we offer a weaker conjectural dichotomy and prove that, if it holds for X then it holds for $X+Y$, where Y is any finite variation process independent of X, see Proposition 4.5. As Conjecture 4.9 remains unsolved in spite of our efforts, in conclusion we only remark that it implies the existence of a strongly eroded Lévy process with high activity of small jumps, i.e. Blumenthal–Getoor index arbitrarily close to two (see Example 4.7 below).

§4.1.1.3 Infinite time horizon

Given any Lévy process X (possibly compound Poisson with drift), define the quantity $l := \liminf_{t\to\infty} X_t/t \in [-\infty,\infty]$. The convex minorant C_∞ of X on the time interval $[0,\infty)$ is finite a.s. if and only if $l \in (-\infty,\infty]$ by Remark 2.21 (recall from Kolmogorov's zero-one law shows that the limit l is a.s. constant); otherwise, C_∞ equals $-\infty$ on $(0,\infty)$. By Corollary 2.19, when $l \in (-\infty,\infty]$, C_∞ is also piecewise linear and the slopes of the faces of C_∞ lie on the interval $(-\infty,l)$. Whenever the expectation $\mathbb{E}X_1$ is well defined, i.e., if $\min\{\mathbb{E}X_1^+,\mathbb{E}X_1^-\}<\infty$, Theorems A.13 & A.14 imply that $l=\mathbb{E}X_1=\lim_{t\to\infty}X_t/t$ a.s. Otherwise, we have $\mathbb{E}X_1^+=\mathbb{E}X_1^-=\infty$ and the following theorem (see also [32]) characterises l.

Theorem 4.11 ([31, Thm 15])**.** *If $\mathbb{E}X_1^+=\mathbb{E}X_1^-=\infty$, then $l\in\{-\infty,\infty\}$ and*

$$l=\infty \qquad \textit{if and only if} \qquad \int_{(-\infty,-1)} \frac{|x|}{\int_0^{|x|}\nu([\max\{1,y\},\infty))\mathrm{d}y}\nu(\mathrm{d}x)<\infty. \tag{4.6}$$

Hence, the convex minorant and concave majorant of X are both finite a.s. if and only if $\mathbb{E}|X_1|<\infty$, and in that case $l=\mathbb{E}X_1=\lim_{t\to\infty}X_t/t$ a.s.

Proposition 4.12. *Suppose $l\in(-\infty,\infty]$, then we have $\lim_{t\to\infty}C_\infty'(t)=l$ a.s.*

Proposition 4.12 implies that the set of slopes $\mathcal{S}_\infty$ of the convex minorant C_∞ satisfies $l\in\mathcal{L}^-(\mathcal{S}_\infty)$ and $\mathcal{S}_\infty\subseteq(-\infty,l)$ a.s. whenever $l\in\mathbb{R}$. This means that C_∞ becomes nearly parallel to the line $t\mapsto lt$ as $t\to\infty$; however, this does not entail any additional continuity for C'_∞ (other than during its intervals of constancy) as it does not occur at a finite time. In particular, Proposition 4.12 shows that, if $l \in (-\infty, \infty]$, then $\mathcal{S}_\infty$ is an infinite set even for compound Poisson processes.

Proposition 4.13. *Suppose $l \in (-\infty, \infty]$. Let $\mathcal{S}$ be the set of slopes of the convex minorant C of an arbitrary Lévy process X (possibly compound Poisson with drift) on the time interval $[0, 1]$. Then we have*

$$\mathcal{L}(\mathcal{S}_\infty) = \{l\} \cup (\mathcal{L}(\mathcal{S}) \cap (-\infty, l)) \ \ a.s. \tag{4.7}$$

Moreover, for any $s \in (-\infty, l)$, we have $\mathbb{P}(s \in \mathcal{L}^\pm(\mathcal{S}_\infty)) = \mathbb{P}(s \in \mathcal{L}^\pm(\mathcal{S})) \in \{0, 1\}$.

As a consequence of Propositions 4.12 & 4.13, the limit set $\mathcal{L}(\mathcal{S}_\infty)$ is constant a.s. The results in §4.1.1.1 & §4.1.1.2 together with Propositions 4.12 & 4.13 yield the following.

Corollary 4.14. *Suppose X has finite variation and $l \in (-\infty, \infty]$. Let $I_{r,\infty}$ be the maximal open interval of constancy of C'_∞ corresponding to slope r. Then C'_∞ is discontinuous on $\bigcup_{r\in\mathcal{S}_\infty} \partial I_{r,\infty}$, lower bounded and $\lim_{t\to\infty} C'_\infty(t) = l$ a.s. Moreover, the following statements hold:*
If X has finite activity, then

(i) *C_∞ has infinitely many faces with $\mathcal{L}(\mathcal{S}_\infty) = \emptyset$ when $l = \infty$ and otherwise $\mathcal{L}^-(\mathcal{S}_\infty) = \{l\}$ and $\mathcal{L}^+(\mathcal{S}_\infty) = \emptyset$.*

If X has infinite activity, then:

(ii) *If $l \in (\gamma_0, \infty]$, then the process $t \mapsto X_t - \gamma_0 t$ attains its infimum on $[0, \infty)$ at a unique time v and $\mathcal{L}(\mathcal{S}_\infty) = \{\gamma_0, l\}$ if $l < \infty$ and otherwise $\mathcal{L}(\mathcal{S}_\infty) = \{\gamma_0\}$. Moreover,*
   - *(ii-a) if $I_+ = I_- = \infty$, then $v \in (0, \infty)$ and $C'_\infty(v-) = C'_\infty(v) = \gamma_0 \in \mathcal{L}^-(\mathcal{S}_\infty) \cap \mathcal{L}^+(\mathcal{S}_\infty)$ a.s.,*
   - *(ii-b) if $I_- < \infty$, then $v \in [0, \infty)$, $\mathbb{P}(v = 0) \in (0, 1]$, $C'_\infty(v) = \gamma_0 \notin \mathcal{L}^-(\mathcal{S}_\infty)$ and, on the event $\{v \neq 0\}$, we have $C'_\infty(v-) < \gamma_0$ a.s.,*
   - *(ii-c) if $I_+ < \infty$, then $v \in (0, \infty)$, $C'(v-) = \gamma_0 \notin \mathcal{L}^+(\mathcal{S}_\infty)$ and we have $C'(v) > \gamma_0$ a.s.*

(iii) *If $l \in (-\infty, \gamma_0]$, then $\mathcal{L}^-(\mathcal{S}_\infty) = \{l\}$ and $\mathcal{L}^+(\mathcal{S}_\infty) = \emptyset$.*

Corollary 4.15. *Assume that X has infinite variation and $l \in (-\infty, \infty]$, then $\inf \mathcal{S}_\infty = -\infty$ a.s. Moreover, C'_∞ is continuous if and only if $\int_0^1 t^{-1}\mathbb{P}(X_t/t \in (a, b))\mathrm{d}t = \infty$ for all $a < b < l$.*

Again, under Conjecture 4.9, the Lebesgue–Stieltjes measure $\mathrm{d}C'_\infty$ is either purely atomic or purely singular continuous.

§4.1.2 Related literature

The smoothness of the convex hull of planar Brownian motion goes back to Paul Lévy [51]. In [30], the authors establish lower bounds on the modulus of continuity of the derivative of the boundary of the convex hull (see [30] and the references therein). These results all concern the spatial convex hull of Brownian motion while we consider the time-space convex hull of a real-valued Lévy process X, i.e. the convex hull of $t \mapsto (t, X_t)$. However, in our context it is also natural to enquire about the modulus of continuity of the convex minorant of X, a topic that is addressed in Chapter 5.

In [21], Bertoin describes the law of the convex minorant of Cauchy process in terms of a gamma process, establishing the continuity of its derivative. The result relies on an explicit description of the right-continuous inverse of the slope process of Cauchy process (see [38, 40] and [57, Ch. XI] for similar characterisations for other Lévy processes). Our approach is instead based on the stick-breaking representation of the convex minorant of Lévy process first established in [64] (see also [1, 38, 63]). This is an important stepping stone for our results in §4.3 below.

The abruptness of a Lévy process X is closely connected via (4.1) to the properties of the contact set between X and its α-Lipschitz minorant (the largest Lipschitz function with derivative equal to $\pm\alpha$ a.e.) or between X and its convex minorant. This connection also has a geometric interpretation. By [2, Thm 3.8], the subordinator associated to the contact set between the process X and its α-Lipschitz minorant has infinite activity if and only if (4.1) holds for the interval $I = [-\alpha, \alpha]$. By Theorem 4.1 this subordinator has infinite activity if and only if C has infinitely many faces whose slope lies on $[-\alpha, \alpha]$. When this occurs, the Lévy process remains close to the α-Lipschitz minorant after touching it [2, Rem. 4.4]. We also observe this behaviour at every contact point between the Lévy process and its convex minorant when the latter is continuously differentiable. In contrast, an abrupt Lévy process must leave its convex minorant sharply after every contact point in the same way it leaves its running supremum [77] (see also [80]). This strengthens the contrasting behaviour between abrupt processes and strongly eroded processes, cf. the conjectural dichotomy in §4.1.1.2 above.

§4.1.3 Organisation of the chapter.

The remainder of this chapter is organised as follows. In §4.2 we illustrate the breath of the class of strongly eroded Lévy processes. In particular, we show that even within the class of Lévy processes with regularly varying Lévy measure at zero, a wide variety of behaviours is possible. In §4.3 we introduce and establish a zero-one law for the stick-breaking process (see Theorem 4.18 below), which implies Theorem 4.1. Theorem 4.2 and all other results of §4.1 are established in §4.4. In §4.6 we describe informally, in terms of the path behaviour of the process X as it leaves 0, what appears to be the main stumbling block for establishing the dichotomy in Conjecture 4.9 and, more strongly, Vigon's point-hitting conjecture (see Conjecture 4.10 above). The analytical behaviour of an arbitrary piecewise linear convex function and its right-derivative is described in §4.5.

§4.2 Is an infinite variation Lévy process strongly eroded? Examples and counterexamples

The class of strongly eroded Lévy processes has a delicate structure, depending crucially on the fine behaviour of the Lévy measure at zero. In this section we present evidence for the following principles for constructing strongly eroded Lévy processes, as well as study their limitations. Heuristically, the boundary of the convex hull of the path of an infinite variation Lévy process X becomes smoother as any of the following occur:

(I) the jump activity decreases (cf. Example 4.9);

(II) the small jumps become more symmetric (cf. Examples 4.9 and 4.10);

(III) at zero, the Lévy measure "approaches" that of a Cauchy process (cf. Example 4.1).

However, as we shall see from the examples below, the following features are also demonstrated: (I) a decrease of the straightforward measure of the jump activity, such as the Blumenthal–Getoor index, appears not to be sufficient for X to become strongly eroded, cf. Example 4.7; (II) there exist both asymmetric strongly eroded processes with one of the tails of the Lévy measure at zero dominating (i.e. $|\Im\psi(u)|/\Re\psi(u) \to \infty$ as $|u| \to \infty$, where ψ is the characteristic exponent of X), cf. Example 4.10, and symmetric abrupt processes; (III) there exist abrupt processes attracted to Cauchy process, cf. Example 4.2. We further show that the classes of abrupt processes (i.e., with $\mathcal{L}(\mathcal{S}) = \emptyset$) and strongly eroded processes (i.e., with $\mathcal{L}(\mathcal{S}) = \mathbb{R}$) are *not* closed under addition of independent summands. Moreover, the sum of a strongly eroded and an independent abrupt process may be either abrupt or strongly eroded, cf. Examples 4.6 and 4.10. In addition, a subordinated abrupt process of infinite variation may be either abrupt or strongly eroded, while a subordinated strongly eroded process may (but need not) be strongly eroded, cf. Example 4.11.

§4.2.1 Near-Cauchy processes

We begin with processes in the domain of attraction of a Cauchy process. Already in this class, we will see how easily a minor change in the jump activity of a process turn a strongly eroded process into an abrupt one. In particular, it is clear that information that does not capture the asymmetry of ψ (such as the indices β_+ and β_- defined in (2.3) & (2.4)) will have limitations in determining whether X is strongly eroded or abrupt.

Example 4.1 (Domain of normal attraction to Cauchy process). Suppose that X_t/t converges weakly as $t \downarrow 0$ to a Cauchy random variable S with density proportional to $x \mapsto 1/(\lambda_1^2 + (x - \lambda_2)^2)$ on $\mathbb{R}$ for some $\lambda_1 > 0$ and $\lambda_2 \in \mathbb{R}$. Then X is strongly eroded. Indeed, $\lim_{t\downarrow 0} \mathbb{P}(X_t/t \in (a,b)) = \mathbb{P}(S \in (a,b)) > 0$ for any $a < b$ by assumption, so Theorem 4.1 immediately gives the claim. Such an assumption is satisfied if, for instance, $\overline{\nu}^+(x)x \to c$, $\overline{\nu}^-(x)x \to c$ and $\int_{(-1,-x]\cup[x,1)} y\nu(\mathrm{d}y) \to c'$ as $x \downarrow 0$ for some $c > 0$ and $c' \in \mathbb{R}$ by Theorem 2.12. △

Example 4.2 (Domain of non-normal attraction to Cauchy process). Assume the characteristic exponent of X is given by $\psi(u) = iu\gamma + \int_{\mathbb{R}}(e^{iux} - 1 - iux\mathbb{1}_{(-1,1)}(x))\nu(\mathrm{d}x)$ for $u \in \mathbb{R}$ where ν is symmetric with $\overline{\nu}^+(x) = x^{-1}\rho(x)$, $x > 0$, for a slowly varying function ρ at 0 with $\lim_{x\downarrow 0} \rho(x) = \infty$. By symmetry, we have $\int_{\{x\le|y|<1\}} y\nu(\mathrm{d}y) = 0$ implying that $(\gamma - \int_{\{x\le|y|<1\}} y\nu(\mathrm{d}y))/(x\overline{\nu}^+(x)) = \gamma/\rho(x) \to 0$ as $x \downarrow 0$. Thus, Theorem 2.12 imply that $X_t/h(t)$ converges weakly as $t \downarrow 0$ to a Cauchy random variable S for an appropriate function h (given in terms of the de Bruijn inverse of ρ) with non-constant ratio $h(t)/t$ that is slowly varying at 0. However, X may be abrupt or strongly eroded. In fact, $\Im\psi(u) = \gamma u$ and $\Re\psi(u)$ is bounded between multiples of $|u|\rho(1/|u|)$ as $|u| \to \infty$ by Lemma 4.17 below. Hence (4.3) and Theorem 4.1 show that X is abrupt (resp. strongly eroded) if $\int_0^1 (x\rho(x))^{-1}\mathrm{d}x$ is finite (resp. infinite). Intuitively, as shown by the following examples, a sufficiently large ρ may make X sufficiently different from a Cauchy process, resulting in an abrupt process X. Pick $\rho(x) \coloneqq \log(1/x)^2\mathbb{1}_{(0,1/2)}(x)$, then X has infinite variation (since $\int_0^1 x^{-1}\rho(x)\mathrm{d}x = \infty$) and is abrupt since $\int_0^1 (x\rho(x))^{-1}\mathrm{d}x = 1/\log(2) < \infty$. If instead $\rho(x) \coloneqq \log(1/x)\mathbb{1}_{(0,1/2)}(x)$, then X is strongly eroded as $\int_0^1 (x\rho(x))^{-1}\mathrm{d}x = \infty$. △

Next we consider 1-semi-stable and weakly 1-stable processes, both of which have relatively simple characteristic exponents.

Example 4.3 (Weakly stable processes). Let X be a (possibly weakly) 1-stable process, i.e., with Lévy measure $\nu(\mathrm{d}x) = |x|^{-2}(c_+\mathbb{1}_{(0,\infty)}(x) + c_-\mathbb{1}_{(-\infty,0)}(x))\mathrm{d}x$ for some $c_\pm \geq 0$ with $c_+ + c_- > 0$. If $c_+ = c_-$, then X is Cauchy (strictly 1-stable), has infinite variation, $\psi(u) = -\theta|u|$ for some $\theta > 0$ and X is strongly eroded by Theorem 4.8(i). If $c_+ \neq c_-$, then X is weakly 1-stable, has infinite variation, is not attracted to a Cauchy process as $t \downarrow 0$ (see Example 2.2), $\psi(u) = -\theta|u|(1 + i\beta\mathrm{sgn}(u)\log|u|)$ for some $\theta > 0$, $\beta \in \mathbb{R} \setminus \{0\}$ (where $\mathrm{sgn}(x) \coloneqq \mathbb{1}_{(0,\infty)}(x) - \mathbb{1}_{(-\infty,0)}(x)$) and X is abrupt by Proposition 4.7. Since the symmetrisation of a weakly 1-stable process is Cauchy (strictly 1-stable), the class of abrupt Lévy processes is not closed under addition even within the class of weakly stable processes. △

Example 4.4 (Semi-stable processes). Let X be a 1-semi-stable process (see Definition A.31). Then, by Theorem A.32 and Proposition A.33, X has infinite variation and there exists a positive constant $b > 1$ such that the Lévy measure ν is uniquely defined as a periodic extension of its restriction to $(-b, b) \setminus (-1, 1)$. Moreover, by Theorem A.36, if X is strictly 1-semi-stable (i.e. if $\int_{(-b,b)\setminus(-1,1)} x\nu(\mathrm{d}x) = 0$), then $\mathfrak{s}_1(r) = \infty$ for all $r \in \mathbb{R}$, in which case X is strongly eroded by Theorem 4.1 and (4.3). Otherwise (i.e., if X is not strictly 1-semi-stable), then $\mathfrak{s}_1$ is locally bounded by Theorem A.36, making X abrupt. In both cases, the tails of the Lévy measure of X need not be regularly varying at 0. In particular, this gives examples of strongly eroded processes with possibly asymmetric Lévy measures that are not regularly varying at 0. However, all these examples have $\beta_- = \beta_+ = 1$. △

§4.2.2 Oscillating characteristic exponent

The fact that abrupt processes are not closed under addition is obvious since any strongly eroded process is the sum of two spectrally one-sided processes, both of which creep and are thus abrupt. In contrast, proving that strongly eroded processes are not closed under addition requires us to look at processes with oscillating characteristic function in the sense that $|\psi(u)/u|$ has a finite lower limit and an infinite upper limit as $|u| \to \infty$.

Example 4.5 (Strongly eroded with mild oscillation). Define $\rho : (0, e^{-e}) \to (0, \infty)$ given by $\rho(1/x) \coloneqq (1 + \sin(\log\log x)^2 \cdot \log x \cdot (\log\log x)^2)\mathbb{1}_{(e^e,\infty)}(x)$. We claim that ρ is slowly varying at 0. By Karamata's representation theorem (see Theorem A.50), it suffices to show that $h(x) \coloneqq \log\rho(e^{-x})$ (i.e. $\rho(x) = e^{h(\log(1/x))}$) satisfies $h'(x) \to 0$ as $x \to \infty$. This is easy to see in our case since we have $h(x) = \log(1 + \sin(\log x)^2 \cdot x \cdot (\log x)^2)$, establishing the slow variation of ρ.

Let X be symmetric with $\nu(\mathrm{d}x) = x^{-2}\rho(|x|)\mathbb{1}_{(0,e^{-e})}(|x|)\mathrm{d}x$ (note that ν is a Lévy measure since $\rho \in L^1_{\mathrm{loc}}(0)$ as $|\rho(x)| = \mathcal{O}(|x|^{-\varepsilon})$ for any $\varepsilon > 0$ by Potter's bound (Theorem A.53)) and $\sigma = 0$, implying that $\Im\psi(u) = 0$ and the even function $-\Re\psi(u) = |\Re\psi(u)|$ is bounded between multiples of $|u|\rho(1/|u|)$ as $|u| \to \infty$ (see Lemma 4.17 below). Thus, for some $c > 0$ and all sufficiently large $|u|$, (4.5) gives

$$\begin{aligned}\Re\frac{1}{1+iur-\psi(u)} &= \frac{1-\Re\psi(u)}{u^2r^2+(1-\Re\psi(u))^2}\\ &\geq \frac{c}{|u|(1+\sin(\log\log|u|)^2\cdot\log|u|\cdot(\log\log|u|)^2)}.\end{aligned}$$

We claim that X is strongly eroded. To see this, note that $\sin(k\pi+u)^2 \leq u^2$ for any $k \in \mathbb{N}$ and $u \in \mathbb{R}$, implying that $\sin(\log\log u)^2 \cdot (\log\log u)^2 \leq 1$ for $u \in [\exp(e^{k\pi-1/(k\pi)}), \exp(e^{k\pi})]$. Hence, we have

$$\begin{aligned}\int_{e^e}^{\infty}\frac{\mathrm{d}u}{u(1+\sin(\log\log u)^2\cdot\log u\cdot(\log\log u)^2)} &\geq \sum_{k=1}^{\infty}\int_{\exp(e^{k\pi-1/(k\pi)})}^{\exp(e^{k\pi})}\frac{\mathrm{d}u}{2u\log u}\\ &= \sum_{k=1}^{\infty}\frac{1}{2k\pi} = \infty,\end{aligned}$$

proving that X is strongly eroded. We note here that Theorem 4.8 is inapplicable as $\liminf_{u\to\infty}|\psi(u)/u| < \infty$ and $\limsup_{u\to\infty}|\psi(u)/u| = \infty$. △

Example 4.6 (Eroded processes are not closed under addition). Define the function $\varrho : (0, e^{-e}) \to (0, \infty)$ given by $\varrho(1/x) = (1+\cos(\log\log x)^2\cdot\log x\cdot(\log\log x)^2)\mathbb{1}_{(e^e,\infty)}(x)$. A similar argument to the one made in Example 4.5 above shows ϱ is slowly varying at 0 and yields another strongly eroded process. However, the sum X of such a process and the one from Example 4.5 above is symmetric with Lévy measure $\nu([x,\infty)) = x^{-1}(2+\log(1/x)(\log\log(1/x))^2)\mathbb{1}_{(0,e^{-e})}(x)$ for all $x > 0$. Lemma 4.17 then implies that for some $c > 0$ and all sufficiently large $|u|$, (4.5) gives

$$\Re\frac{1}{1+iur-\psi(u)} \leq \frac{1}{1-\Re\psi(u)} \leq \frac{c}{|u|\cdot\log|u|\cdot(\log\log|u|)^2}.$$

Since $\int_{e^e}^{\infty}(u\cdot\log u\cdot(\log\log u)^2)^{-1}\mathrm{d}u = 1 < \infty$, the process X is abrupt by Theorem 4.1 and (4.3). △

The fluctuations present in the previous examples are tame enough for us to determine decisively that $\mathfrak{s}_1$ is identically infinite and hence not locally integrable. In the following example we find a symmetric process for which we may show that $\mathfrak{s}_1(0) = \infty$ but for which, as a consequence of the heavy oscillations of its characteristic function, it is incredibly hard to find whether $\mathfrak{s}_1(r)$ is finite or not for *any* given $r \neq 0$. The oscillations of the characteristic exponent in particular satisfy both $\liminf_{|u|\to\infty}|\psi(u)| = 0$ (hence $\beta_- = 0$) and $\limsup_{|u|\to\infty}|\psi(u)|/|u|^\alpha > 0$ for a constant $\alpha \in (1,2)$ (which in fact agrees with β_+) that may be taken arbitrarily close to 2. In particular, this symmetric process is a prime candidate for one of the two interesting possibilities: (I) a counter-example to Conjecture 4.10, as $\mathfrak{s}_1(0) = \infty$

but $\mathfrak{s}_1(r)$ is possibly finite for some $r \neq 0$ (which may possibly result in a non-eroded, non-abrupt process) or (II) a strongly eroded process with path variation $\beta_+ = \alpha \in (1,2)$ arbitrarily close to that of a Brownian motion. When the process is asymmetric, however, it is abrupt.

Example 4.7 (Can an eroded Lévy process have path variation close to that of Brownian motion?). We recall the definition of Orey's process [58]. Fix any $\alpha \in (1,2)$, $c_\pm \geq 0$ with $c_+ + c_- > 0$ and integer $\eta > 2/(2-\alpha)$. Set $\sigma = 0$, $\gamma = 0$ and $\nu = \sum_{n\in\mathbb{N}} a_n^{-\alpha}(c_+\delta_{a_n} + c_-\delta_{-a_n})$ for $a_n = 2^{-\eta^n}$. Then we have $\int_{(-1,1)} |x|\nu(\mathrm{d}x) = (c_+ + c_-)\sum_{n\in\mathbb{N}} a_n^{1-\alpha} = \infty$, making the associated Lévy process of infinite variation. Since $\cos(2k\pi) = 1$ for every integer k and $1 - \cos(x) \leq x^2$ for $x \in [0,1]$,

$$\begin{aligned}\frac{-\Re\psi(2\pi/a_n)}{(c_+ + c_-)} &= \sum_{k=1}^{\infty} a_k^{-\alpha}(1 - \cos(2\pi a_k/a_n)) = \sum_{k=1}^{\infty} 2^{\alpha\eta^k}\left(1 - \cos\left(2\pi 2^{\eta^n - \eta^k}\right)\right) \\ &\leq 4\pi^2 \sum_{k=n+1}^{\infty} 2^{2\eta^n - (2-\alpha)\eta^k} = 4\pi^2 \sum_{k=1}^{\infty} 2^{-\eta^n((2-\alpha)\eta^k - 2)} \xrightarrow[n\to\infty]{} 0,\end{aligned}$$

where the limit follows by the monotone convergence theorem and the inequality $\eta > 2/(2-\alpha)$. Thus, we conclude that $\liminf_{u\to\infty} |\Re\psi(u)| = 0$. Similarly, we have $-a_n^\alpha \Re\psi(\pi/a_n) = (c_+ + c_-)a_n^\alpha \sum_{k=1}^\infty a_k^{-\alpha}(1 - \cos(\pi a_k/a_n)) \geq 2(c_+ + c_-)$, implying $\limsup_{u\to\infty} |\psi(u)|/|u|^\alpha \geq 2(c_+ + c_-)$.

Suppose $c_+ = c_-$. Then $\Im\psi = 0$ and, since $u \mapsto 1/(1-\psi(u))$ is the characteristic function of X_{e_1}, where e_1 is a unit-mean exponential time independent of X, and $\limsup_{|u|\to\infty} 1/(1-\psi(u)) = 1 > 0$, the Riemann–Lebesgue lemma implies that X_{e_1} is singular continuous. In particular, $u \mapsto 1/(1-\psi(u))$ is not integrable on $\mathbb{R}$, giving $\mathfrak{s}_1(0) = \infty$. If instead $c_+ \neq c_-$, then X is abrupt by Proposition 4.7.

Furthermore, we note that $\beta_- = 0$ and $\beta_+ = \alpha$, with the strong oscillation of ψ resulting in a large gap between these indices. To see that $\beta_+ = \alpha$, note that $\int_{(-1,1)} |x|^p \nu(\mathrm{d}x) = 2\sum_{n=1}^\infty 2^{(\alpha-p)\eta^n}$, where the sum diverges for all $p \leq \alpha$ and converges for all $p > \alpha$. Since $\overline{\sigma}^2(u) = 2\sum_{n\in\mathbb{N},\, a_n<u} a_n^{2-\eta}$, the lower limit of $u^{-2}\overline{\sigma}^2(u)$ is attained along the sequence a_n and

$$\liminf_{n\to\infty} a_n^{-2}\overline{\sigma}^2(a_n) = \liminf_{n\to\infty} 2\sum_{k=n+1}^{\infty} 2^{2\eta^n - (2-\alpha)\eta^k} = \liminf_{n\to\infty} 2\sum_{k=1}^{\infty} 2^{-\eta^n((2-\alpha)\eta^k - 2)} = 0,$$

by the monotone convergence theorem. △

§4.2.3 Lévy measure with regularly varying tails

We begin with some estimates on the characteristic exponent $\psi(u)$ for $u \in \mathbb{R}$ (see (2.1)) in terms of commonly used functions of the Lévy measure for the proofs of Proposition 4.6 and Lemma 4.17 below. Recall, from (2.2), the functions $\overline{\sigma}^2(x)$,

$\overline{\nu}(x) = \nu(\mathbb{R} \setminus (-x, x))$ and $\overline{\gamma}(x) = \int_{(-1,1)\setminus(-x,x)} y\nu(\mathrm{d}y)$, for $x > 0$.

Lemma 4.16. (a) *For any* $u \neq 0$, $\frac{1}{3}u^2\overline{\sigma}^2(|u|^{-1}) \leq -\Re\psi(u) - \frac{1}{2}u^2\sigma^2 \leq 2\overline{\nu}(2|u|^{-1}) + \frac{1}{2}u^2\overline{\sigma}^2(2|u|^{-1})$.
(b) *For any* $|u| > 1$, *we have* $|\Im\psi(u) + (\overline{\gamma}(|u|^{-1}) - \gamma)u| \leq \frac{1}{3}u^2\overline{\sigma}^2(|u|^{-1}) + \overline{\nu}(|u|^{-1})$.

Proof. (a) Note that $\frac{1}{3}x^2\mathbb{1}_{\{|x|<1\}} \leq 1 - \cos(x) \leq \frac{1}{2}\min\{x^2, 4\}$ for all $x \in \mathbb{R}$. Integrating then gives

$$\frac{1}{3}\int_{\mathbb{R}} (ux)^2\mathbb{1}_{\{|ux|<1\}}\nu(\mathrm{d}x) \leq \int_{\mathbb{R}} (1 - \cos(ux))\nu(\mathrm{d}x) \leq \frac{1}{2}\int_{\mathbb{R}} \min\{(ux)^2, 4\}\nu(\mathrm{d}x),$$

implying the inequality in (a).
(b) First note that

$$\int_{\mathbb{R}} (\sin(ux) - ux\mathbb{1}_{\{|x|<1\}})\nu(\mathrm{d}x) = -u\overline{\gamma}(|u|^{-1}) + \int_{\mathbb{R}} (\sin(ux) - ux\mathbb{1}_{\{|ux|<1\}})\nu(\mathrm{d}x).$$

Hence, integrating $|\sin(ux) - ux\mathbb{1}_{\{|ux|<1\}}| \leq \frac{1}{3}\mathbb{1}_{\{|ux|<1\}}u^2x^2 + \mathbb{1}_{\{|ux|\geq 1\}}$ gives the result. □

Assume throughout the remainder of this section that, for some $\alpha \in [0, 2]$, the functions

$$\wp_-(x) := x^\alpha\nu((-\infty, -x]) \quad \text{and} \quad \wp_+(x) := x^\alpha\nu([x, \infty)) \quad \text{for } x \in (0, 1), \tag{4.8}$$

are slowly varying at 0 (see definition in §1.5). The infinite variation of X requires either $\alpha \geq 1$ or $\sigma^2 > 0$. However, if either $\alpha > 1$ or $\sigma^2 > 0$, then X is abrupt by Proposition 4.6. Thus, without loss of generality we assume $\alpha = 1$ and $\sigma = 0$ throughout the remainder of this section. Moreover, since we may modify arbitrarily the Lévy measure of X away from 0 without changing $\mathcal{L}(\mathcal{S})$ (by Proposition 4.5), we may assume that ν is supported on $(-1, 1)$.

The following result controls the real and imaginary parts of the characteristic exponent ψ. This is important in determining whether X is abrupt or strongly eroded because they feature in the integrand in the definition of $\mathfrak{s}_1(r)$. The proof of Lemma 4.17 is based on Karamata's theorem and the elementary estimates from Lemma 4.16. Define $\wp(x) := \wp_+(x) + \wp_-(x)$ and $\widetilde{\wp}_\pm(x) := \int_x^1 t^{-1}\wp_\pm(t)\mathrm{d}t$ for $x \in (0, 1)$. Note that the infinite variation of X is equivalent to $\int_0^1 x^{-1}\wp(x)\mathrm{d}x = \infty$, which is further equivalent to $\lim_{x\downarrow 0}(\widetilde{\wp}_+(x) + \widetilde{\wp}_-(x)) = \infty$. Moreover, we see that the functions $\widetilde{\wp}_\pm$ are slowly varying at 0 with $\lim_{x\downarrow 0}\widetilde{\wp}_\pm(x)/\wp_\pm(x) = \infty$ by Proposition A.54.

Lemma 4.17. *Suppose* $\sigma = 0$ *and the Lévy measure* ν *is supported on* $(-1, 1)$ *and satisfies* (4.8) *for* $\alpha = 1$. *Then* $\overline{\gamma}$ *in* (2.2) *satisfies* $\overline{\gamma}(x) = \wp_+(x) + \widetilde{\wp}_+(x) - (\wp_-(x) +$

$\widetilde{\wp}_-(x))$ *for all* $x \in (0,1)$ *and, as* $|u| \to \infty$,

$$\Re\psi(u) \approx |u|\wp\big(|u|^{-1}\big) \quad \textit{and} \quad \Im\psi(u) = \big(\gamma - \overline{\gamma}\big(|u|^{-1}\big)\big)u + \mathcal{O}\big(|u|\wp\big(|u|^{-1}\big)\big). \tag{4.9}$$

Proof. The formula for $\overline{\gamma}$ follows by applying Fubini's theorem. Similarly, Fubini's theorem yields

$$\overline{\sigma}^2(x) = \int_{(-x,x)} y^2\nu(\mathrm{d}y) = \int_0^x 2y(\overline{\nu}(y) - \overline{\nu}(x))\mathrm{d}y = \int_0^x 2y\overline{\nu}(y)\mathrm{d}y - x^2\overline{\nu}(x).$$

Note that $\overline{\nu}(x) = x^{-1}\wp(x)$ for all $x \in (0,1)$. Karamata's theorem (see Theorem A.55) then shows that $\int_0^x 2y\overline{\nu}(y)\mathrm{d}y \sim 2x\wp(x)$ while $x^2\overline{\nu}(x) = x\wp(x)$, implying that $\overline{\sigma}^2(x) \sim x\wp(x)$ as $x \downarrow 0$. Then the estimates in (4.9) follow from Lemma 4.16. □

Lemma 4.17 provides sufficient control over the characteristic exponent of X in two regimes: if ν is near-symmetric (defined by the condition $\overline{\gamma}(x) = \mathcal{O}(1)$ as $x \downarrow 0$, see definition in (2.2)), or if ν is skewed (defined by $\lim_{x\downarrow 0} x^{-1}\overline{\gamma}(x)/\overline{\nu}(x) \in \{-\infty, \infty\}$, which is equivalent to $\lim_{x\downarrow 0}(\widetilde{\wp}_+(x) - \widetilde{\wp}_-(x))/\wp(x) \in \{-\infty, \infty\}$ by Lemma 4.17), motivating the two ensuing subsections. In the remainder of this section we freely apply equivalence (4.3) and Theorems 4.2 and 4.8.

§4.2.3.1 Near-symmetric Lévy measure

Suppose $\overline{\gamma}$ in (2.2) satisfies $\overline{\gamma}(x) = \mathcal{O}(1)$ as $x \downarrow 0$ (e.g., ν symmetric). By Lemma 4.17, $|\Im\psi(u)| = \mathcal{O}(|u|(1 \vee \wp(|u|^{-1}))) = \mathcal{O}(|u| \vee \Re\psi(u))$ as $|u| \to \infty$, where $x \vee y :=\max\{x, y\}$. Thus, the integrand in the definition of $\mathfrak{s}_1(r)$ is asymptotically sandwiched between multiples of $\wp(1/|u|)/(u(1 + \wp(1/|u|)^2))$ as $|u| \to \infty$ for all $r \neq \gamma$.

Example 4.8 (Near-symmetric with high activity). Suppose $\liminf_{x\downarrow 0} \wp(x) = \infty$. Then $\liminf_{|u|\to\infty} |\psi(u)/u| = \infty$ by Lemma 4.17, so Theorem 4.8(ii) shows that X is either eroded or abrupt. Moreover, by Lemma 4.17, we have as $|u| \to \infty$,

$$\Re\frac{1}{1 + iur - \psi(u)} = \frac{1 - \Re\psi(u)}{(1 - \Re\psi(u))^2 + (ur - \Im\psi(u))^2} \approx \frac{1}{\Re\psi(u)} \approx \frac{1}{|u|\wp(|u|^{-1})}.$$

Thus, X is strongly eroded if and only if $x \mapsto 1/(x\wp(x))$ is not integrable at 0. △

Example 4.9 (Near-symmetric with low activity). Suppose $\limsup_{x\downarrow 0} \wp(x) < \infty$. Then Lemma 4.17 implies that $\limsup_{|u|\to\infty} |\psi(u)/u| < \infty$ so Theorem 4.8(i) shows that X is strongly eroded. △

§4.2.3.2 Skewed Lévy measure

Suppose $\lim_{x\downarrow 0} x^{-1}\overline{\gamma}(x)/\overline{\nu}(x) \in \{-\infty, \infty\}$ (which is equivalent to $\lim_{x\downarrow 0}(\widetilde{\wp}_+(x) - \widetilde{\wp}_-(x))/\wp(x) \in \{-\infty, \infty\}$), where $\overline{\gamma}$ is defined as in (2.2). This is the case if, for instance, we have $\liminf_{x\downarrow 0} \wp_+(x)/\wp_-(x) > 1$ or $\limsup_{x\downarrow 0} \wp_+(x)/\wp_-(x) < 1$

by Proposition A.54. Moreover, either of these inequalities essentially imply that X is abrupt. More precisely, if the maps $x \mapsto \nu([x,1)) = x^{-1}\wp_+(x)$ and $x \mapsto \nu((-1,-x]) = x^{-1}\wp_-(x)$ are eventually differentiable with a monotone derivative as $x \downarrow 0$, then the monotone density theorem (see Theorem A.57) shows that the respective derivatives are asymptotically equivalent to $-x^{-2}\wp_+(x)$ and $-x^{-2}\wp_-(x)$. Hence, the Radon-Nikodym derivative $\nu(\mathrm{d}x)/\nu(\mathrm{d}(-x))$ is asymptotically equivalent to $\wp_+(x)/\wp_-(x)$ as $x \downarrow 0$. Thus, either of the limits $\liminf_{x\downarrow 0} \wp_+(x)/\wp_-(x) > 1$ or $\limsup_{x\downarrow 0} \wp_+(x)/\wp_-(x) < 1$ imply that X is abrupt by Proposition 4.7.

The following example shows that the condition in Proposition 4.7 is close to being sharp. It constructs strongly eroded processes whose asymmetry, quantified by the quotient $(\wp_+(x)-\wp_-(x))/\wp_-(x) > 0$, converges (arbitrarily) slowly to 0 as $x \downarrow 0$. Clearly, the roles of $\wp_+$ and $\wp_-$ could be reversed without affecting these conclusions. Moreover, since Cauchy process is strongly eroded but spectrally one-sided infinite variation processes are abrupt, the following example also shows that the sum of an abrupt process and an independent strongly eroded process may result in an abrupt or a strongly eroded process.

Example 4.10 (Low asymmetry). Suppose that $\wp_+(x) = (p(x)+q(x))\mathbb{1}_{(0,\varepsilon)}(x)$ and $\wp_-(x) = p(x)\mathbb{1}_{(0,\varepsilon)}(x)$ for positive slowly varying functions p and q defined on $(0,\varepsilon)$. Define recursively $\log^{(1)}(x) = \log x$ and $\log^{(n+1)}(x) = \log(\log^{(n)}(x))$ for $n \in \mathbb{N}$. Fix $n \in \mathbb{N} \cup \{0\}$, define the functions $p(x) = 1/\prod_{k=1}^{n} \log^{(k)}(1/x)$ and $q(x) = p(x)/\log^{(n+1)}(1/x)$ (where an empty product equals 1 by convention) and choose ε sufficiently small to ensure p and q are both positive and the maps $x \mapsto x^{-1}\wp_\pm(x)$ are monotone on $(0,\varepsilon)$. Then Lemma 4.17 gives $|\Re\psi(u)| \approx |u|p(1/|u|)$ and $|\Im\psi(u)| \approx |u|\log^{(n+2)}(|u|)$ as $|u| \to \infty$. Since $|u|^{-1}p(1/|u|)/(\log^{(n+2)}(|u|))^2$ is not integrable at infinity, then $\mathfrak{s}_q(r) = \infty$ for all $r \in \mathbb{R}$, making X strongly eroded with slowly varying asymmetry $\wp_+(x)/\wp_-(x) - 1 = q(x)/p(x) = 1/\log^{(n+1)}(1/x) \to 0$ as $x \downarrow 0$ and, furthermore, with $|\Im\psi(u)/\Re\psi(u)| \approx \log^{(n+2)}(|u|)/p(1/|u|) \to \infty$ as $|u| \to \infty$.

A similar analysis shows that the choice $q(x) = p(x)/\log^{(n)}(1/x)$ instead leads to an abrupt process. We point out that, in either case, p cannot be much smaller since the infinite variation of X requires the function $u^{-2}|\Re\psi(u)| \approx |u|^{-1}p(1/|u|)$ to be non-integrable at infinity by Lemma 2.37. △

§4.2.4 Subordination

Let X be an infinite variation Lévy process and Y be an independent driftless subordinator with Fourier–Laplace exponent $\phi(u) := \log \mathbb{E}[\exp(uY_1)]$ for any $u \in \mathbb{C}$ with $\Re u \leq 0$. Then, for any $c \geq 0$, the subordinated process $Z = (X_{ct+Y_t})_{t\geq 0}$ is Lévy with characteristic exponent given by $u \mapsto \phi(\psi(u)) + c\psi(u)$. The following example shows that subordinating an abrupt processes can result in either an abrupt or a strongly eroded process Z and that subordinated strongly eroded processes can still be strongly eroded.

Example 4.11 (Subordinating abrupt and strongly eroded processes). (a) Suppose X is a Brownian motion, Y is an α-stable subordinator (with $\alpha \in (0,1)$) and $c = 0$. Then Z is a symmetric 2α-stable process, making it abrupt for $\alpha > 1/2$, strongly eroded for $\alpha = 1/2$ and of finite variation for $\alpha < 1/2$.

(b) Suppose $c > 0$ and $(X_{Y_t})_{t\ge0}$ is of finite variation. Then $Z = (X_{ct+Y_t})_{t\ge0}$ can be decomposed as the sum of two independent processes, one with the law of $(X_{ct})_{t\ge0}$ and the other with the law of $(X_{Y_t})_{t\ge0}$. Thus, Proposition 4.5 implies $\mathcal{L}(\mathcal{S}_X) = c\mathcal{L}(\mathcal{S}_Z)$, where $\mathcal{S}_X$ and $\mathcal{S}_Z$ are the set of slopes of the convex minorants of X and Z, respectively, with convention $cA := \{ca : a \in A\}$ and $c\emptyset := \emptyset$.

(c) Suppose $\lim_{|u|\to\infty} |\psi(u)/u| = \infty$ and $c > 0$ (implying that Z is of infinite variation). For any $R \in (0,c)$ there exists some $K > 0$ such that for all $|z| > K$, we have $|\phi(z)| \le R|z|$ by Example 2.1. Thus, Z satisfies the conditions of Theorem 4.8(ii), making the process either strongly eroded or abrupt.

(d) Suppose $\limsup_{|u|\to\infty} |\psi(u)/u| < \infty$ and Z is of infinite variation. Then, for some $R \ge 0$ there exists $K > 0$ such that we for all $|z| > K$ have $|\phi(z)| \le R|z|$ by Example 2.1 implying that Z satisfies the assumptions of Theorem 4.8(i), making it strongly eroded. In particular, this is the case if X is symmetric Cauchy of unit scale (i.e. the law of X_1 has parameters $(\lambda_1, \lambda_2) = (1,0)$ as in Example 4.1 above), $c = 0$ and Y has Lévy measure $\nu_Y(\mathrm{d}t) = t^{-2}(\log(1/t))^{-2}\mathbb{1}_{(0,1)}(t)\mathrm{d}t$. Indeed, it suffices to verify that Z has infinite variation. By Example 2.1 the Lévy measure ν_Z of Z is given by the formula $\nu_Z(\mathrm{d}x) = \int_0^1 \mathbb{P}(X_t \in \mathrm{d}x)\nu_Y(\mathrm{d}t)$, thus Fubini's theorem yields

$$\begin{aligned}
\int_{(-1,1)} |x|\nu_Z(\mathrm{d}x) &= \int_{(-1,1)} |x| \int_0^1 \mathbb{P}(X_t \in \mathrm{d}x)\frac{\mathrm{d}t}{t^2(\log(1/t))^2} \\
&= \int_0^1 \int_0^1 \frac{x}{t^2+x^2}\mathrm{d}x \frac{2}{\pi t(\log(1/t))^2}\mathrm{d}t = \int_0^1 \frac{\log(1+t^{-2})}{\pi t(\log(1/t))^2}\mathrm{d}t \\
&\ge \int_0^1 \frac{\mathrm{d}t}{\pi t\log(1/t)} = \infty.
\end{aligned}$$

△

§4.3 Zero-one law for stick breaking and the slopes of the minorant

For $T > 0$, let $(\ell_n)_{n\in\mathbb{N}}$ be a uniform stick-breaking process on $[0,T]$, defined in Definition 2.17. Let $(V_n)_{n\in\mathbb{N}}$ be an i.i.d. sequence of $\mathrm{U}(0,1)$ random variables, independent of the stick-breaking process $(\ell_n)_{n\in\mathbb{N}}$. Recall that a measurable function f on $\mathbb{R}$ is in $L^1_{\mathrm{loc}}(0+)$ if for some $\varepsilon > 0$ it satisfies $\int_0^\varepsilon |f(t)|\mathrm{d}t < \infty$. The following zero-one law, which does not involve the Lévy process X, is key for the analysis of the regularity of the convex hull of X.

Theorem 4.18. *Let $\varphi : \mathbb{R}_+ \times [0,1] \to \mathbb{R}_+$ be measurable and bounded. Define $\Sigma_T := \sum_{n=1}^\infty \varphi(\ell_n, V_n)$ and the function $\phi : t \mapsto \int_0^1 \varphi(t,u)\mathrm{d}u$. Then Σ_T is either finite a.s. or infinite a.s., characterised by*

$$\Sigma_T < \infty \ \text{a.s.} \qquad \Longleftrightarrow \qquad t \mapsto t^{-1}\phi(t) \in L^1_{\mathrm{loc}}(0+). \tag{4.10}$$

Moreover, the mean of Σ_T is given by $\mathbb{E}\Sigma_T = \int_0^T t^{-1}\phi(t)\mathrm{d}t$.

Note that, by (4.10) in Theorem 4.18, Σ_T is either finite a.s. for all $T > 0$ or infinite a.s. for all $T > 0$. Furthermore, the proof of Theorem 4.18 implies that $\Sigma_T < \infty$ a.s. if and only if $t \mapsto t^{-1}\Phi(t) \in L^1_{\mathrm{loc}}(0+)$, where $\Phi(t) := t^{-1}\int_0^t \phi(s)\mathrm{d}s$.

Proof. Proving that Σ_T is either finite a.s. or infinite a.s., according to (4.10), requires three steps. First, we show that the events $\{\Sigma_T = \infty\}$ and $\{\widetilde{\Sigma}_T = \infty\}$ agree a.s., where $\widetilde{\Sigma}_T := \sum_{n=1}^\infty \Phi(L_n)$. Second, we use the Poisson process embedded in the stick remainders $(L_n)_{n\in\mathbb{N}}$ to establish that $\mathbb{P}(\widetilde{\Sigma}_T = \infty) = 1$ if $\int_0^1 t^{-1}\Phi(t)\mathrm{d}t = \infty$ and otherwise $\mathbb{P}(\widetilde{\Sigma}_T = \infty) = 0$. Third, we use the Poisson point process, given by the stick-breaking process on an independent exponential time horizon, to establish (4.10).

Define the filtration $(\mathcal{F}_n)_{n\in\mathbb{N}\cup\{0\}}$ by $\mathcal{F}_0 := \{\emptyset, \Omega\}$ and $\mathcal{F}_n := \sigma((U_k, V_k); k \le n)$ for $n \ge 1$. Note that the conditional distribution of ℓ_n, given $\mathcal{F}_{n-1}$, is uniform on the interval $(0, L_{n-1})$, implying

$$\mathbb{E}[\varphi(\ell_n, V_n)|\mathcal{F}_{n-1}] = \mathbb{E}[\phi(\ell_n)|\mathcal{F}_{n-1}] = L_{n-1}^{-1}\int_0^{L_{n-1}} \phi(s)\mathrm{d}s = \Phi(L_{n-1}), \quad n \in \mathbb{N}.$$

Hence, the process $(M_n)_{n\in\mathbb{N}\cup\{0\}}$, given by $M_0 := 0$ and $M_n := \sum_{k=1}^n(\varphi(\ell_k, V_k) - \Phi(L_{k-1}))$ for $n \in \mathbb{N}$, is a $(\mathcal{F}_n)$-martingale with bounded increments (recall that φ is bounded). By Proposition A.10, the event $A := \{M_n \text{ converges to a finite limit}\}$ satisfies the following equality

$$A = \left\{\sup_{n\in\mathbb{N}} M_n < \infty\right\} = \left\{\inf_{n\in\mathbb{N}} M_n > -\infty\right\} \quad \text{a.s.} \tag{4.11}$$

On A we have $\Sigma_T = \widetilde{\Sigma}_T + \lim_{n\to\infty} M_n$, implying that $\Sigma_T = \infty$ if and only if $\widetilde{\Sigma}_T = \infty$. On the complement of A, by (4.11), we must have $\sup_{n\in\mathbb{N}} M_n = -\inf_{n\in\mathbb{N}} M_n = \infty$, implying $\Sigma_T = \widetilde{\Sigma}_T = \infty$. Thus, the events $\{\Sigma_T = \infty\}$ and $\{\widetilde{\Sigma}_T = \infty\}$ agree a.s.

Note that $(-\log(U_n))_{n\ge 1}$ are iid exponential random variables with unit mean. Hence, the process $(N(t))_{t\ge 0}$, given by $N(t) \coloneqq \sum_{n=1}^{\infty} \mathbb{1}_{\{-\log(T^{-1}L_n)\le t\}}$, is a standard Poisson process. Denote by $(N(\mathrm{d}x); x \in (0,\infty))$ the corresponding Poisson point process on $(0,\infty)$. Since $\widetilde{\Sigma}_T = \int_{(0,\infty)} \Phi(Te^{-x})N(\mathrm{d}x)$, Campbell's formula (Theorem A.48) yields the Laplace transform of $\widetilde{\Sigma}_T$:

$$\log \mathbb{E}[\exp(-q\widetilde{\Sigma}_T)] = -\int_0^\infty \big(1 - e^{-q\Phi(Te^{-x})}\big)\mathrm{d}x = -\int_0^T \big(1 - e^{-q\Phi(t)}\big)\frac{\mathrm{d}t}{t}, \quad q \ge 0.$$

Since φ is bounded, there exists $q_0 > 0$ such that $0 \le \Phi(t) \le 1/q_0$ for all $t > 0$. The inequalities $x/2 \le 1 - e^{-x} \le x$, valid for $x \in [0,1]$, imply the following for all $q \in (0, q_0]$:

$$(q/2)\int_0^T \Phi(t)\frac{\mathrm{d}t}{t} \le \int_0^T \big(1 - e^{-q\Phi(t)}\big)\frac{\mathrm{d}t}{t} \le q\int_0^T \Phi(t)\frac{\mathrm{d}t}{t}. \tag{4.12}$$

The monotone convergence theorem implies

$$\begin{aligned}\mathbb{P}(\widetilde{\Sigma}_T < \infty) = \lim_{q\downarrow 0} \mathbb{E}\big[e^{-q\widetilde{\Sigma}_T}\big] &= \lim_{q\downarrow 0} \exp\Big(-\int_0^T \big(1 - e^{-q\Phi(t)}\big)\frac{\mathrm{d}t}{t}\Big)\\ &= \begin{cases} 1, & t \mapsto t^{-1}\Phi(t) \in L^1_{\mathrm{loc}}(0+),\\ 0, & t \mapsto t^{-1}\Phi(t) \notin L^1_{\mathrm{loc}}(0+).\end{cases}\end{aligned}$$

Since $\{\Sigma_T = \infty\} = \{\widetilde{\Sigma}_T = \infty\}$ a.s., the first claim in the theorem follows.

In order to prove the equivalence in (4.10), note first that whether the function $t \mapsto t^{-1}\Phi(t)$ is in $L^1_{\mathrm{loc}}(0+)$ does not depend on T. Hence the random variable $\widetilde{\Sigma}_T$ (and thus Σ_T) is either finite a.s. for all $T > 0$ or infinite a.s. for all $T > 0$. Let E be an exponential random variable with unit mean, independent of (ℓ, V) where $\ell = (\ell_n)_{n\in\mathbb{N}}$ and $V = (V_n)_{n\in\mathbb{N}}$. Thus $\{\Sigma_T = \infty\} = \{\Sigma = \infty\}$ almost surely, where $\Sigma \coloneqq \sum_{n\in\mathbb{N}} \varphi(\ell_n E/T, V_n)$. It is hence sufficient to prove the equivalence between $\mathbb{P}(\Sigma < \infty) = 1$ and $t \mapsto t^{-1}\phi(t) \in L^1_{\mathrm{loc}}(0+)$.

Since $(\ell_n E/T)_{n\in\mathbb{N}}$ is a stick-breaking process on the random interval $[0, E]$, Remark 2.20 and the marking theorem (see Theorem A.49) imply that the process $\Xi = \sum_{n=1}^{\infty} \delta_{(\ell_n E/T, V_n)}$ is a Poisson point process with mean measure

$$\mu(\mathrm{d}t, \mathrm{d}u) \coloneqq t^{-1}e^{-t}\mathrm{d}t\mathrm{d}u, \quad (t, x) \in \mathbb{R}_+ \times [0,1].$$

Moreover, as $\Sigma = \int_{\mathbb{R}_+\times[0,1]} \varphi(t,u)\Xi(\mathrm{d}t, \mathrm{d}u)$, Campbell's formula (Theorem A.48) implies

$$
\begin{aligned}
-\log \mathbb{E}\big[e^{-q\Sigma}\big] &= \int_{\mathbb{R}_+\times[0,1]} \big(1-e^{-q\varphi(t,u)}\big)\mu(\mathrm{d}t,\mathrm{d}u) \\
&= \int_{\mathbb{R}_+\times[0,1]} \big(1-e^{-q\varphi(t,u)}\big)e^{-t}\frac{\mathrm{d}t}{t}\mathrm{d}u.
\end{aligned}
\tag{4.13}
$$

There exists $q_1 > 0$ such that for all $q \in (0, q_1]$ we have $0 \le \varphi(t,u) \le 1/q$ for all $(t,u) \in \mathbb{R}_+ \times [0,1]$. The elementary inequalities that implied (4.12) yield the following for all $q \in (0, q_1]$:

$$
\frac{q}{2}\int_0^\infty \phi(t)e^{-qt}\frac{\mathrm{d}t}{t} \le \int_{\mathbb{R}_+\times[0,1]} \big(1-e^{-q\varphi(t,u)}\big)e^{-qt}\frac{\mathrm{d}t}{t}\mathrm{d}u \le q\int_0^\infty \phi(t)e^{-qt}\frac{\mathrm{d}t}{t}. \tag{4.14}
$$

The monotone convergence theorem yields $\mathbb{P}(\Sigma < \infty) = \lim_{q\downarrow 0}\mathbb{E}[e^{-q\Sigma}]$. Since $\int_0^\infty t^{-1}\phi(t)e^{-t}\mathrm{d}t < \infty$ if and only if $t \mapsto t^{-1}\phi(t) \in L^1_{\mathrm{loc}}(0+)$, (4.13)-(4.14) imply that $\mathbb{P}(\Sigma < \infty) = 1$ if and only if $t \mapsto t^{-1}\phi(t) \in L^1_{\mathrm{loc}}(0+)$, establishing (4.10).

Recall that $\sum_{n=1}^\infty \mathbb{E}[f(\ell_n)] = \int_0^T t^{-1}f(t)\mathrm{d}t$ for any measurable $f : [0,T] \to \mathbb{R}_+$, see e.g. (3.9). Since the stick-breaking process $(\ell_n)_{n\in\mathbb{N}}$ and the iid sequence $(V_n)_{n\in\mathbb{N}}$ are independent, the following holds $\mathbb{E}\Sigma_T = \sum_{n=1}^\infty \mathbb{E}\left[\varphi(\ell_n, V_n)\right] = \sum_{n=1}^\infty \mathbb{E}\left[\phi(\ell_n)\right] = \int_0^T \phi(t)t^{-1}\mathrm{d}t$. □

Recall that X is a Lévy process, assumed not to be compound Poisson with drift. Let $F(t,x) := \mathbb{P}(X_t \le x)$ for all $t \ge 0$ and $x \in \mathbb{R}$. Let $u \mapsto F^{-1}(t,u) := \inf\{u \in \mathbb{R} : F(t,u) \ge x\}$ be the right-inverse of $x \mapsto F(t,x)$ for every $t > 0$ and note that $X_t \overset{d}{=} F^{-1}(t,U)$ for any $U \sim \mathrm{U}(0,1)$. The convex minorant C of the path of X on the interval $[0,T]$ is piecewise linear. Take *any* enumeration of the maximal faces of C and let g_n and d_n be the left and right endpoints, respectively, of the n-th interval of linearity. Then the set of length-height pairs $\{(d_n - g_n, C(d_n) - C(g_n)) : n \in \mathbb{N}\}$ (note that the n-th face of C has length $d_n - g_n$ and height $C(d_n) - C(g_n)$) of the maximal faces satisfies the following identity in law:

$$
\{(d_n - g_n, C(d_n) - C(g_n)) \,:\, n \in \mathbb{N}\} \overset{d}{=} \{(\ell_n, F^{-1}(\ell_n, V_n)) \,:\, n \in \mathbb{N}\}, \tag{4.15}
$$

see Theorem 2.18 (see also [64, Thm 1]). We stress that the specific enumeration of the length-height pairs on the left-hand side of (4.15) is not important for our purposes because the smoothness of C is determined by the limit set of the quotients of these pairs. The identity in law in (4.15) and Theorem 4.18 yield the following result.

Corollary 4.19. *Pick a measurable function $f : (0,\infty) \times \mathbb{R} \to \mathbb{R}$ and a measurable* *set* $I \subseteq \mathbb{R}$*. Then we have that* $\mathbb{P}(\{f(d_n - g_n, C(d_n) - C(g_n))) \in I\}\ \ i.o.) \in \{0,1\}$ *and*

$$\mathbb{P}(\{f(d_n - g_n, C(d_n) - C(g_n))) \in I\}\ \ i.o.) = 1 \iff \int_0^1 \mathbb{P}(f(t, X_t) \in I)\frac{\mathrm{d}t}{t} = \infty. \tag{4.16}$$

Note that Corollary 4.19 (with $f(t,x) = x/t$) directly implies Theorem 4.1 above.

Proof. Define the function $\varphi : (s,u) \mapsto \mathbb{1}_{\{f(s,F^{-1}(s,u)) \in I\}}$. For any $U \sim \mathrm{U}(0,1)$ we have

$$\phi(s) = \int_0^1 \varphi(s,u)\mathrm{d}u = \mathbb{P}(f(s, F^{-1}(s,U)) \in I) = \mathbb{P}(f(s, X_s) \in I) \quad \text{for all } s > 0.$$

An application of Theorem 4.18 and (4.15) imply the corollary. □

Remark 4.20. **(i)** The equality in law in (4.15) follows from the representation theorem for convex minorants of Lévy processes in Theorem 2.18 because X is assumed not to be compound Poisson with drift. Under this assumption, the law of X_t (for any $t > 0$) is diffuse (i.e. non-atomic) by Lemma 2.2, implying that no two linear segments in the piecewise linear convex function defined in Theorem 2.18 have the same slope. Thus all these linear segments are maximal faces. The identity in law in (4.15) is essentially the content of [64, Thm 1]. Since the proof of [64, Thm 1] is highly non-trivial and moreover relies on deep results in the fluctuation theory of Lévy processes, we chose the route above based on Theorem 2.18, whose proof is short and elementary, requiring only the definition of a Lévy process.
(ii) The limit points of the countable random set $\mathcal{S}_f := \{f(d_n - g_n, C(d_n) - C(g_n)) : n \in \mathbb{N}\}$ can be determined via Corollary 4.19. Indeed, the set $\mathcal{L}(\mathcal{S}_f)$ is determined by the following countable family of events $\{|\mathcal{S}_f \cap (a,b)| = \infty\} = \{f(d_n - g_n, C(d_n) - C(g_n)) \in (a,b)\ i.o.\}$, where $a < b$ range over the rational numbers. Indeed, $x \in \mathcal{L}(\mathcal{S}_f)$ if and only if $|\mathcal{S}_f \cap (a,b)| = \infty$ for all rational a, b with $a < x < b$. By Corollary 4.19, the indicator of any such event is almost surely constant, making the limit set $\mathcal{L}(\mathcal{S}_f)$ also almost surely constant. Thus, $\mathcal{L}(\mathcal{S}_f)$ is stochastically independent of $\mathcal{S}_f$ itself (recall that any a.s. constant random element is stochastically independent of all other random variables) and is not affected under conditioning on an event of positive probability such as X not having jumps larger than some $\varepsilon > 0$ on $[0,T]$. In particular, we may modify the Lévy measure of X by adding or removing a finite amount of mass anywhere on $\mathbb{R} \setminus \{0\}$ without altering $\mathcal{L}(\mathcal{S}_f)$. These facts will be used throughout the chapter.
(iii) It can be easily shown that, when the time horizon T is an exponential variable independent of X with mean $1/p$, $|\mathcal{S} \cap I|$ is a Poisson random variable with mean

$\int_0^\infty \mathbb{P}(X_t/t \in I)e^{-pt}t^{-1}\mathrm{d}t$. ♢

§4.4 Continuous differentiability of the boundary of the convex hull – proofs

This section is dedicated to proving the results stated in §4.1. Let ψ be the Lévy–Khintchine exponent of the Lévy process X, defined as in (2.1), and let (σ^2, γ, ν) be the generating triplet of X corresponding to the cutoff function $x \mapsto \mathbb{1}_{(-1,1)}(x)$.

§4.4.1 Finite variation – proofs

Recall that a Lévy process X has paths of finite variation if and only if $\sigma^2 = 0$ and $\int_{(-1,1)} |x|\nu(\mathrm{d}x) < \infty$, and recall from Remark 2.1 that the natural drift $\gamma_0 \in \mathbb{R}$ of X is defined by

$$\gamma_0 \coloneqq \gamma - \int_{(-1,1)} x\nu(\mathrm{d}x). \tag{4.17}$$

Proof of Proposition 4.3. Since X has finite variation, Theorem A.37 yields that $\lim_{t\downarrow 0} X_t/t = \gamma_0$ a.s. Recall that the positive (resp. negative) half-line is regular for a process Z with Z_0 if and only if $\inf\{t > 0 : Z_t > 0\} = 0$ a.s. (resp. $\inf\{t > 0 : Z_t < 0\} = 0$ a.s.). Hence, the positive (resp. negative) half-line is not regular for the process $(X_t - ct)_{t\geq 0}$ if $c > \gamma_0$ (resp. $c < \gamma_0$). Rogozin's criterion (see Theorem 2.8) then yields $\int_0^1 t^{-1}\mathbb{P}(X_t - (\gamma_0 + \varepsilon)t > 0)\mathrm{d}t < \infty$ and $\int_0^1 t^{-1}\mathbb{P}(X_t - (\gamma_0 - \varepsilon)t < 0)\mathrm{d}t < \infty$ for all $\varepsilon > 0$. By Theorem 4.1 we get that, for every $\varepsilon > 0$, the set $\mathcal{S} \cap (\mathbb{R} \setminus (\gamma_0 - \varepsilon, \gamma_0 + \varepsilon))$ is finite a.s. In particular, $\mathcal{S}$ is bounded and for any $r \in \mathbb{R} \setminus \{\gamma_0\}$, $\mathbb{P}(r \in \mathcal{L}(\mathcal{S})) = 0$ a.s. Moreover, since X is of infinite activity, by Theorem 2.18 and Lemma 2.2, the cardinality of the set of slopes $\mathcal{S}$ is infinite implying that it has an accumulation point, which can only be γ_0, i.e. $\gamma_0 \in \mathcal{L}(\mathcal{S})$ a.s. In particular, the set of slopes $\mathcal{S}$ consists of isolated points with a single accumulation point $\mathcal{L}(\mathcal{S}) = \{\gamma_0\}$ satisfying $\gamma_0 \notin \mathcal{S}$ a.s. (in fact, since X is diffuse, for any $x \in \mathbb{R}$ we have $\mathbb{P}(x \notin \mathcal{S}) = 1$ by (4.15)). Hence C', whose image is contained in the closure of $\mathcal{S}$, is bounded and discontinuous on $\bigcup_{r\in\mathcal{S}} \partial I_r$ (recall that $I_r = (C')^{-1}(\{r\})$).

Without loss of generality we may assume that X has right-continuous paths with left limits. In particular, recall that $X_{t-} = \lim_{s\uparrow t} X_s$ if $t > 0$ and $X_{0-} = X_0$ otherwise. Let v be the last time in $[0, T]$ the process $(X_t - \gamma_0 t)_{t\geq 0}$ attains its minimum, i.e. v is the greatest time in $[0, T]$ satisfying $\min\{X_v, X_{v-}\} - \gamma_0 v = \inf_{t\in[0,T]}(X_t - \gamma_0 t)$. Since $t \mapsto C(t) - \gamma_0 t$ is the convex minorant on $[0, T]$ of $t \mapsto X_t - \gamma_0 t$, if the latter function attained its minimum at two or more times with positive probability, the former function, which is piecewise linear and convex, would have a face of slope zero with positive probability. Since the increments of X are diffuse by Lemma 2.2, this contradicts the formula for the slopes in Theorem 2.18. Moreover, v is the a.s. unique time at which the convex function $t \mapsto C(t) - \gamma_0 t$ on $[0, T]$ attains its minimum.

The probability $\mathbb{P}(v = 0) = \mathbb{P}(\inf_{t \in [0,T]}(X_t - \gamma_0 t) = 0)$ (resp. $\mathbb{P}(v = T) = \mathbb{P}(\inf_{t \in [0,T]}(\gamma_0 t + X_{(T-t)-} - X_T) = 0) = \mathbb{P}(0 = \inf_{t \in [0,T]}(\gamma_0 t - X_t)))$ is positive if zero is not regular for the half-line $(-\infty, 0)$ for $(X_t - \gamma_0 t)_{t \geq 0}$ (resp. $(-X_t + \gamma_0 t)_{t \geq 0}$), which is by Rogozin's criterion (see Theorem 2.8) equivalent to $I_- < \infty$ (resp. $I_+ < \infty$). In particular, $v \in (0, T)$ a.s. is equivalent to $I_+ = I_- = \infty$. We proved above that γ_0 is the only limit point of $\mathcal{S}$ a.s. Thus, by definition of $\mathcal{L}^-(\mathcal{S})$ (resp. $\mathcal{L}^+(\mathcal{S})$) in the paragraph containing (4.4) above, γ_0 is a left (resp. right) limit point of $\mathcal{S}$ if and only if the set $\mathcal{S} \cap (-\infty, \gamma_0)$ (resp. $\mathcal{S} \cap (\gamma_0, \infty)$) has infinitely many elements a.s., which is by Theorem 4.1 equivalent to $I_- = \infty$ (resp. $I_+ = \infty$). Thus $I_\pm < \infty$ implies $\mathcal{L}^\pm(\mathcal{S}) = \emptyset$ and $\mathcal{L}^\mp(\mathcal{S}) = \{\gamma_0\}$, where $\mp = -(\pm)$. Furthermore, if γ_0 is in $\mathcal{L}^+(\mathcal{S})$ (resp. $\mathcal{L}^-(\mathcal{S})$), then $C'(v) = \inf\{s > \gamma_0 : s \in \mathcal{S}\} = \gamma_0$ (resp. $C'(v-) = \sup\{s < \gamma_0 : s \in \mathcal{S}\} = \gamma_0$), where the infimum (resp. supremum) is necessarily taken over a non-empty set. If $\mathcal{L}^+(\mathcal{S}) = \emptyset$ (resp. $\mathcal{L}^-(\mathcal{S}) = \emptyset$), on the event $v \in (0, T)$ we have $C'(v) = \inf\{s > \gamma_0 : s \in \mathcal{S}\} > \gamma_0$ (resp. $C'(v-) = \sup\{s < \gamma_0 : s \in \mathcal{S}\} < \gamma_0$), where the infimum (resp. supremum) is necessarily taken over a finite non-empty set. This concludes the proof of the proposition. □

Proof of Corollary 4.4. The equivalence $I_\pm = \infty \iff \mathfrak{s}_1 \notin L^1_{\text{loc}}(\gamma_0 \pm)$ follows from the equivalence in (4.3). We now prove that $I_+ = \infty$ is equivalent to the integral condition in the corollary (the equivalence involving $I_- = \infty$ follows from this one by considering $-X$). Define the function $\varpi(x) := x / \int_0^x \nu((-\infty, -y))\mathrm{d}y$, $x > 0$. We consider two cases:

(I) $\nu((-\infty, 0)) < \infty$: then both integrals are infinite. Indeed, since $X_t > t\gamma_0$ whenever X does not have a negative jump on $[0, t]$, we have $\mathbb{P}(X_t > t\gamma_0) \geq \exp(-t\nu((-\infty, 0)))$ and hence $I_+ = \infty$ by definition. The function ϖ is bounded below by the positive constant $1/\nu((-\infty, 0)) \in (0, \infty]$, implying $\int_{(0,1)} \varpi(x)\nu(\mathrm{d}x) = \infty$ as the infinite activity of X requires $\nu((0, 1)) = \infty$.

(II) $\nu((-\infty, 0)) = \infty$: then $\varpi(x)$ is finite for $x > 0$ and $I_+ = \infty$ is equivalent to $\int_{(0,1)} \nu((x, \infty))\mathrm{d}\varpi(x) = \infty$ by Theorem 2.9, where the Radon measure $\mathrm{d}\varpi(x)$ is well defined since $1/\varpi(x) = x^{-1} \int_0^x \nu((-\infty, -y))\mathrm{d}y$ is a non-increasing function as it is the average over the interval $(0, x)$ of the non-increasing function $y \mapsto \nu((-\infty, -y))$. The function ϖ is continuous on $(0, \infty)$ and, since $1/\varpi(x) \geq \nu((-\infty, -x)) \to \infty$ as

$x \downarrow 0$, we have $\lim_{x\downarrow 0} \varpi(x) = 0$. By Fubini's theorem,

$$\int_{(0,1)} \nu((x,\infty))\mathrm{d}\varpi(x) = \int_{(0,1)} \int_{(x,\infty)} \nu(\mathrm{d}y)\mathrm{d}\varpi(x) = \int_{(0,\infty)} \int_{(0,1\wedge y)} \mathrm{d}\varpi(x)\nu(\mathrm{d}y)$$
$$= \int_{(0,\infty)} \varpi(1 \wedge y)\nu(\mathrm{d}y) = \nu([1,\infty))\varpi(1) + \int_{(0,1)} \varpi(y)\nu(\mathrm{d}y).$$

Thus $\int_{(0,1)} \varpi(y)\nu(\mathrm{d}y) = \infty$ is equivalent to $\int_{(0,1)} \nu((x,\infty))\mathrm{d}\varpi(x) = \infty$ and hence to $I_+ = \infty$ by Theorem 2.9. □

§4.4.2 Infinite variation – proofs

Proof of Theorem 4.2. First note that the smoothness of the boundary of the convex hull of X requires X to have infinite variation by Proposition 4.3. Similarly, if X is of finite variation, then (4.1) fails for any compact interval I with $\gamma \notin I$. Thus, both conditions in Theorem 4.2 require X to have infinite variation, which we assume in the remainder of this proof.

Since X is of infinite variation, the set of slopes $\mathcal{S}$ is unbounded below and above by Rogozin's theorem as explained in the first paragraph of §4.1.1.2. This makes the boundary of the convex hull of X smooth at times 0 and T. Indeed, let $C^{\smile}$ (resp. $C^{\frown}$) be the convex minorant (resp. concave majorant) of X over the interval $[0,T]$. For a sufficiently small $\varepsilon > 0$, we may locally parametrise the curve $(t, C^{\smile}(t))$; $t \in [0,\varepsilon]$ (resp. $(t, C^{\frown}(t))$; $t \in [0,\varepsilon]$) as the curve $(\varsigma_{\smile}(u), u)$; $u \in [C^{\smile}(\varepsilon), 0]$ (resp. $(\varsigma_{\frown}(u), u)$; $u \in [0, C^{\frown}(\varepsilon)]$), using a local inverse $\varsigma_{\smile}$ (resp. $\varsigma_{\frown}$) of $C^{\smile}$ (resp. $C^{\frown}$). The function $\varsigma_{\smile}$ is continuous, so for any $u \in (C^{\smile}(\varepsilon), 0]$ and all sufficiently small $h > 0$ we have

$$\frac{\varsigma_{\smile}(u) - \varsigma_{\smile}(u-h)}{h} = \frac{h'}{u - C^{\smile}(\varsigma_{\smile}(u) - h')} = \frac{h'}{C^{\smile}(\varsigma_{\smile}(u)) - C^{\smile}(\varsigma_{\smile}(u) - h')},$$

where $h' := \varsigma_{\smile}(u) - \varsigma_{\smile}(u-h) \uparrow 0$ as $h \downarrow 0$. Thus, we have $\varsigma_{\smile}'(u) = 1/(C^{\smile})'(\varsigma_{\smile}(u))$ where $\varsigma_{\smile}'$ is the left-derivative of $\varsigma_{\smile}$ and $(C^{\smile})'$ is the right-derivative of $C^{\smile}$. Similarly, the right-derivatives of $\varsigma_{\frown}$ and $C^{\frown}$ satisfy $\varsigma_{\frown}'(u) = 1/(C^{\frown})'(\varsigma_{\frown}(u))$ and hence, the concatenation $\varsigma(u) = \varsigma_{\smile}(u)\mathbb{1}_{(C^{\smile}(\varepsilon),0)}(u) + \varsigma_{\frown}(u)\mathbb{1}_{(0,C^{\frown}(\varepsilon))}(u)$ can be used to locally parametrise the boundary of the convex hull of X around $(0,0)$ as the curve given by $(\varsigma(u), u)$; $u \in [C^{\smile}(\varepsilon), C^{\frown}(\varepsilon)]$. Moreover, ς is differentiable at 0 with $\varsigma'(0) = 0$ since $\lim_{t\downarrow 0} 1/|(C^{\smile})'(t)| = \lim_{t\downarrow 0} 1/|(C^{\frown})'(t)| = 0$ a.s., implying the smoothness of the boundary of the convex hull of X at time 0. By time reversal, the boundary of the convex hull of X is also smooth at time T.

It remains to prove that the convex minorant C of X is continuously differentiable if and only if the condition (4.1) holds for all bounded intervals I. Recall that the right-derivative C' is right-continuous by definition, and thus, its image equals $\mathcal{L}^+(\mathcal{S}) \cup \mathcal{S}$ (see Table 4.2 for all possible behaviours of the right-derivative of a piecewise linear convex function).

Suppose the boundary of the convex hull of X is smooth a.s., making C' continuous a.s. By the intermediate value theorem, since C' is unbounded from below and above, its image $\mathcal{L}^+(\mathcal{S}) \cup \mathcal{S}$ must equal $\mathbb{R}$. Since $\mathcal{S}$ is countable, $\mathcal{L}^+(\mathcal{S}) \cup \mathcal{S} = \mathbb{R}$ a.s. implies $\mathcal{L}^+(\mathcal{S}) = \mathbb{R}$ a.s. Since $\mathcal{L}^+(\mathcal{S}) \subset \mathcal{L}(\mathcal{S})$, we have $\mathcal{L}(\mathcal{S}) = \mathbb{R}$, implying that $\mathcal{S}$ is dense in $\mathbb{R}$ a.s. and thus condition (4.1) holds for all bounded intervals I.

Now assume (4.1) holds for all bounded intervals I. Note that $\mathcal{L}^+(\mathcal{S}) \cap \mathcal{L}^-(\mathcal{S})$ contains the interior of $\mathcal{L}(\mathcal{S})$, so the condition $\mathcal{L}(\mathcal{S}) = \mathbb{R}$ implies $\mathcal{L}^+(\mathcal{S}) = \mathbb{R}$. Since C' is right-continuous and non-decreasing with image $\mathcal{L}^+(\mathcal{S}) \cup \mathcal{S} = \mathbb{R}$, it must be continuous, completing the proof. □

Proof of Proposition 4.5. Recall that Y and Z are possibly dependent Lévy processes, $X = Y + Z$ and Y is of finite variation with natural drift b. Let $(\ell_n)_{n\in\mathbb{N}}$ be an independent uniform stick-breaking process on $[0, T]$ as defined in Definition 2.17. For $n \geq 1$ define $\zeta_n^X := X_{L_n} - X_{L_{n+1}}$, $\zeta_n^Y := Y_{L_n} - Y_{L_{n+1}}$ and $\zeta_n^Z := Z_{L_n} - Z_{L_{n+1}}$. By Theorem 2.18, the convex minorant of X (resp. Y; Z) has the same law as the unique piecewise linear convex function with faces $((\ell_n, \zeta_n^X) : n \in \mathbb{N})$ (resp. $((\ell_n, \zeta_n^Y) : n \in \mathbb{N})$; $((\ell_n, \zeta_n^Z) : n \in \mathbb{N})$). In particular, the sets of slopes $\mathcal{S}_X$, $\mathcal{S}_Y$ and $\mathcal{S}_Z$ (with a.s. constant limit sets) have the same law as the sets $\{\zeta_n^X/\ell_n : n \in \mathbb{N}\}$, $\{\zeta_n^Y/\ell_n : n \in \mathbb{N}\}$ and $\{\zeta_n^Z/\ell_n : n \in \mathbb{N}\}$, respectively, and hence their respective limit sets are a.s. constant and equal to $\mathcal{L}(\mathcal{S}_X)$, $\mathcal{L}(\mathcal{S}_Y)$ and $\mathcal{L}(\mathcal{S}_Z)$, respectively. These limit sets must be constant a.s. by Theorem 4.1. In particular, $\mathcal{L}(\{\zeta_n^Y/\ell_n : n \in \mathbb{N}\}) = \{b\}$ a.s. by Proposition 4.3. The result now follows from the fact that, for any deterministic sequences $(y_n)_{n\in\mathbb{N}}$ and $(z_n)_{n\in\mathbb{N}}$ with $\lim_{n\to\infty} y_n = b$, we have $\mathcal{L}(\{y_n + z_n : n \in \mathbb{N}\}) = \mathcal{L}(\{z_n : n \in \mathbb{N}\}) + b$. □

Proof of Proposition 4.6. Our assumption implies that $\mathfrak{s}_1$ is finite and uniformly bounded. Indeed, by Lemma 4.16(a) and (4.5), we obtain

$$\begin{aligned}2\pi\mathfrak{s}_1(r) = \int_{\mathbb{R}} \Re \frac{1}{1 + iru - \psi(u)} \mathrm{d}u &\leq \int_{\mathbb{R}} \frac{1}{1 - \Re\psi(u)} \mathrm{d}u \\ &\leq \int_{\mathbb{R}} \frac{3}{1 + u^2(\sigma^2 + \overline{\sigma}^2(1/|u|))} \mathrm{d}u < \infty.\end{aligned}$$

Thus, Theorem 4.1 and (4.3) imply $\mathcal{L}(\mathcal{S}) = \emptyset$.

It remains to show that the assumption holds if $\sigma^2 > 0$ or $\beta_- > 1$. If $\sigma^2 > 0$ (resp. $\beta_- > 1$) fix some $\alpha \in (1, 2)$ (resp. $\alpha \in (1, \beta_-)$) and note that, by the definition of β_-, there exists some $K > 0$ such that $u^2(\sigma^2 + \overline{\sigma}^2(|u|^{-1})) \geq K|u|^\alpha$ for all $u \in \mathbb{R} \setminus (-1,1)$. Hence, we have

$$\int_1^\infty \frac{\mathrm{d}u}{1+u^2(\sigma^2+\overline{\sigma}^2(1/u))} \le \int_1^\infty \frac{\mathrm{d}u}{1+Ku^\alpha} < \infty. \qquad \square$$

Proof of Proposition 4.7. We may assume without loss of generality that $x_0 = 1$, $\nu(\mathbb{R}\setminus(-1,1)) = 0$ and $\gamma = 0$, due to Proposition 4.5. If $\sigma^2 > 0$ then X is abrupt by Proposition 4.6 and if $\sigma^2 = 0$ and either $\int_{(-1,0)} |x|\nu(\mathrm{d}x) < \infty$ or $\int_{(0,1)} x\nu(\mathrm{d}x) < \infty$, then X creeps and is thus abrupt by Remark 2.32. Thus, it suffices to assume that $\sigma^2 = 0$ and $\int_{(-1,0)} |x|\nu(\mathrm{d}x) = \int_{(0,1)} x\nu(\mathrm{d}x) = \infty$. Decompose $X = Y + Z$ where the Lévy processes Y and Z are independent of each other and have generating triplets $(0,0,\nu|_{(0,1)})$ and $(0,0,\nu|_{(-1,0)})$, respectively. Let ψ_Y and ψ_Z be the characteristic exponents of Y and Z, respectively. Note that $\psi = \psi_Y + \psi_Z$ and recall that the functions $\Re\psi$, $\Re\psi_Y$ and $\Re\psi_Z$ are even while $\Im\psi$, $\Im\psi_Y$ and $\Im\psi_Z$ are odd. The idea is to bound the function $\mathfrak{s}_1$ of X uniformly over compact sets by the corresponding function of Y (note that Y is of infinite variation and creeps, making it abrupt by Example 2.3).

The assumption implies that $\int_{(0,1)} f(x)\nu(\mathrm{d}x) \ge c\int_{(-1,0)} f(-x)\nu(\mathrm{d}x) \ge 0$ for any measurable function $f : (0,1) \to [0,\infty)$. Thus, the following inequalities hold for all $u > 0$:

$$|\Re\psi_Y(u)| \le |\Re\psi(u)| \le (1+1/c)|\Re\psi_Y(u)|, \;\text{ and }\; \Im\psi(-u) \ge (1-1/c)\Im\psi_Y(-u) > 0. \tag{4.18}$$

Fubini's theorem and the infinite variation of Y imply that $\int_{(0,1)} \nu([x,1))\mathrm{d}x = \int_{(0,1)} y\nu(\mathrm{d}y) = \infty$. Moreover, for any $u > 0$, Fubini's theorem yields

$$\begin{aligned}\frac{\Im\psi_Y(-u)}{u} &= \frac{1}{u}\int_{(0,1)} (ux - \sin(ux))\nu(\mathrm{d}x) \ge \frac{1}{u}\int_{(1/u,1)} (ux-1)\nu(\mathrm{d}x)\\ &= \int_{(1/u,1)}\int_{(1/u,x]} \mathrm{d}y\,\nu(\mathrm{d}x) = \int_{(1/u,1)} \nu([y,1))\mathrm{d}y \xrightarrow[u\to\infty]{} \infty.\end{aligned}$$

Fix any $R > 0$ and let $M > 0$ satisfy $\frac{1}{2}(1-1/c)|\Im\psi_Y(u)| \ge |u|R$ for all $|u| \ge M$ (recall that $\Im\psi_Y$ is an odd function). Then, by (4.18), for all $|u| \ge M$ and $|r| \le R$, we have

$$|ur - \Im\psi(u)| \ge |\Im\psi(u)| - |ur| \ge (1-1/c)|\Im\psi_Y(u)| - |ur| \ge \tfrac{1}{2}(1-1/c)|\Im\psi_Y(u)|.$$

Since by the first inequality in (4.18) we have $|1-\Re\psi(u)| = 1-\Re\psi(u) \ge 1-\Re\psi_Y(u) \ge \frac{1}{2}(1-1/c)|1-\Re\psi_Y(u)|$, the inequality in the last display yields $|1+iur-\psi(u)| \ge \frac{1}{2}(1-1/c)|1-\psi_Y(u)|$ for all $|u| \ge M$ and $|r| \le R$. Recall that $\Re\left(1/(1+iru-\psi(u))\right) \le 1$

for all $u, r \in \mathbb{R}$. Applying (4.18) and (4.5) then gives, for all $u \in \mathbb{R}$ and $r \in [-R, R]$,

$$\begin{aligned}\Re \frac{1}{1+iru-\psi(u)} &= \frac{1-\Re\psi(u)}{|1+iur-\psi(u)|^2} \\ &\le \mathbb{1}_{\{|u|<M\}} + \mathbb{1}_{\{|u|\ge M\}} \frac{1+1/c}{(1-1/c)^2/4} \cdot \frac{1-\Re\psi_Y(u)}{|1-\psi_Y(u)|^2}.\end{aligned}$$

Since all the jumps of Y are positive, the infinite variation Lévy process Y creeps and hence hits points. By Theorem A.29(a), this is equivalent to $\int_{\mathbb{R}}(1-\Re\psi_Y(u))|1-\psi_Y(u)|^{-2}\mathrm{d}u = \int_{\mathbb{R}} \Re\left(1/(1-\psi_Y(u))\right) \mathrm{d}u < \infty$, implying that the left-hand side of the last display is integrable over $u \in \mathbb{R}$ for every $r \in [-R, R]$. Thus, the function $r \mapsto \mathfrak{s}_1(r)$ is finite, uniformly bounded and hence integrable on $[-R, R]$. Since $R > 0$ was arbitrary, X is abrupt by (4.3) and Theorem 4.1. □

Proof of Theorem 4.8. For the proofs of Parts (i) and (ii), we adapt the arguments given in [80].

Part (i). Assume that there exists some $k \in (0, \infty)$ such that $(1 + |\psi(u)|)/u \le k$ for all $u \ge 1$. Recall that $\Re\psi(u) \le 0$, and note from (4.5) that

$$\begin{aligned}\int_1^\infty \Re\frac{1}{1+iur-\psi(u)}du &= \int_1^\infty \frac{1-\Re\psi(u)}{|1+iur-\psi(u)|^2}\mathrm{d}u \\ &\ge \int_1^\infty \frac{|\Re\psi(u)|}{(1+|iur|+|\psi(u)|)^2}\mathrm{d}u \ge \frac{1}{(k+|r|)^2}\int_1^\infty \frac{|\Re\psi(u)|}{u^2}\mathrm{d}u.\end{aligned}$$

Since X has infinite variation the right hand side is always infinite by Lemma 2.37. Hence $\mathfrak{s}_1(r) = \infty$ for all r, implying the claim.

Part (ii). Suppose that $\liminf_{u\to\infty} |\psi(u)/u| = \infty$. It suffices to show that, if $\mathfrak{s}_1(r_0) < \infty$ for some $r_0 \in \mathbb{R}$, then $\sup_{r\in[r_0-R,r_0+R]} \mathfrak{s}_1(r) < \infty$ for any $R > 0$. Indeed, this would imply that either $\mathfrak{s}_1(r) = \infty$ for all r (making X strongly eroded) or $\mathfrak{s}_1$ is bounded uniformly on compact sets (making X abrupt). Suppose $\mathfrak{s}_1(r_0) < \infty$ for some r_0 and fix $R > 0$. By assumption, there exists some $M \ge 1$ such that $|\psi(u)/u| \ge 3(1 + |r_0|) + 2R$ for all $|u| \ge M$. Thus, for $|u| \ge M$ and $|r - r_0| \le R$, we have the inequalities $|\psi(u)| \ge 3|1 + ir_0u| + 2R|u| \ge 2|1 + iru| + |1 + ir_0u|$, and hence

$$|1 + ir_0u - \psi(u)| \le |\psi(u)| + |1 + ir_0u| \le 2(|\psi(u)| - |1 + iur|) \le 2|1 + iur - \psi(u)|.$$

Thus, for any $r \in [r_0 - R, r_0 + R]$, we have

$$\int_0^\infty \Re\frac{1}{1+iru-\psi(u)}du \le \int_0^M \Re\frac{1}{1+iru-\psi(u)}\mathrm{d}u + 2\int_M^\infty \Re\frac{1}{1+ir_0u-\psi(u)}\mathrm{d}u,$$

implying $\mathfrak{s}_1(r) \le M/\pi + 2\mathfrak{s}_1(r_0)$ since $\Re(1/(1 + r - \psi(u))) \le 1$ for all $r, u \in \mathbb{R}$ by (4.5).

It remains to prove parts (ii-a)–(ii-c). By Part (ii), $\mathfrak{s}_1(r)$ is either everywhere finite and locally integrable, or $\mathfrak{s}_1(r) = \infty$ for all r. Thus, in the remainder of the proof it suffices to check if $\mathfrak{s}_1(0) < \infty$. Since $\Re(1/(1-\psi(u))) \le 1$ is locally integrable,

$\Im\psi$ is an odd function and $\Re\psi$ is an even function, the finiteness of $\mathfrak{s}_1(0)$ depends only on that of the following integral:

$$\int_1^\infty \frac{1-\Re\psi(u)}{(1-\Re\psi(u))^2+\Im\psi(u)^2}\mathrm{d}u. \tag{4.19}$$

Part (ii-a). Assume that $\lim_{|u|\to\infty}|\Re\psi(u)/u|=\infty$. The integral in (4.19) equals $\int_1^\infty \Re(1/(1-\psi(u)))\mathrm{d}u$, giving (ii-a).

Part (ii-b). Assume now that the upper and lower limits of $|\Re\psi(u)/u|$ as $|u|\to\infty$ lie in $(0,\infty)$ and that $\lim_{|u|\to\infty}|\Im\psi(u)/u|=\infty$. In this case the denominator of the integrand in (4.19) is asymptotically equivalent to $|\Im\psi(u)|^2$ and the numerator of (4.19) is asymptotically sandwiched between multiples of $|u|$. Hence the integral in (4.19) is infinite if and only if $\int_1^\infty u(1+|\Im\psi(u)|^2)^{-1}\mathrm{d}u=\infty$.

Part (ii-c). Assume now $\lim_{|u|\to\infty}|\Re\psi(u)/u|=0$ and $\lim_{|u|\to\infty}|\Im\psi(u)/u|=\infty$. In this case the denominator of the integrand in (4.19) is asymptotically equivalent to $|\Im\psi(u)|^2$. Hence the integral in (4.19) is infinite if and only if $\int_1^\infty(1-\Re\psi(u))(1+|\Im\psi(u)|^2)^{-1}\mathrm{d}u=\infty$. □

§4.4.3 Infinite time horizon – proofs

Proof of Proposition 4.12. Let $\Xi=\sum_{n\in\mathbb{N}}\delta_{(\ell_n,\xi_n)}$ be a Poisson point process with mean measure given by $\mu(\mathrm{d}t,\mathrm{d}x)=\mathbb{1}_{\{x/t<l\}}t^{-1}\mathbb{P}(X_t\in\mathrm{d}x)\mathrm{d}t$. By Corollary 2.19 and the convexity of C_∞, the result will follow if we show that $\Xi(\{(t,x):c\le x/t<l,\,t\ge 1\})=\infty$ a.s. for any $c<l$. Since Ξ is Poisson, it suffices to show that its mean is infinite. To that end, we will prove that $\mu(\{(t,x):x/t<c,\,t\ge1\})<\infty$ and $\mu(\{(t,x):x/t<l,\,t\ge1\})=\infty$.

Fix any $c<l$ and define the Lévy process $X^{(c)}=(X_t^{(c)})_{t\ge0}:=(X_t-ct)_{t\ge0}$. Since $\lim_{t\to\infty}X_t^{(c)}=\infty$ a.s. (by definition of l), Theorem A.46 yields

$$\mu(\{(t,x):x/t<c,\,t\ge1\})=\int_1^\infty\mathbb{P}\big(X_t^{(c)}<0\big)\frac{\mathrm{d}t}{t}<\infty.$$

It remains to establish that $\mu(\{(t,x):x/t<l,\,t\ge1\})=\infty$. If we assume $l=\infty$ then $\mu(\{(t,x):x/t<l,\,t\ge1\})=\int_1^\infty t^{-1}\mathrm{d}t=\infty$. Assume instead that $l<\infty$ and let $X^{(l)}$ be as before with $c=l$. In this case $\mathbb{E}X_1^{(l)}=0$, making $X^{(l)}$ recurrent by Remark A.15, so the event $\{\liminf_{t\to\infty}X_t^{(l)}=\infty\}$ has probability 0. Hence, Theorem A.46 yields

$$\mu(\{(t,x):x/t<l,\,t\ge1\})=\int_1^\infty\mathbb{P}\big(X_t^{(l)}<0\big)\frac{\mathrm{d}t}{t}=\infty.$$ □

Proof of Proposition 4.13. By Theorem 4.1, with probability 1, for all rational $T>0$, the set of slopes of the convex minorant of X on the interval $[0,T]$ have $\mathcal{L}(\mathcal{S})$ as their limit set, where $\mathcal{S}$ is the set of slopes of the convex minorant of X on the interval $[0,1]$. Without loss of generality, in this proof, we restrict the underlying probability space to this event.

Fix any $c \in (-\infty, l)$. By Proposition 4.12, the random time $\tau \coloneqq \inf\{t > 0 : C'_\infty(t) > c\}$ is finite a.s. Let $T = \lfloor \tau \rfloor + 1$ be the smallest integer larger than τ and let $\widetilde{C}$ be the convex minorant of X on the time interval $[0, T]$. Observe that $X_t \geq C_\infty(t \wedge \tau) + c(t \vee \tau - \tau)$ and $X_t \geq \widetilde{C}(t \wedge \tau) + c(t \vee \tau - \tau)$ for all $t \geq 0$, where $x \wedge y \coloneqq \min\{x, y\}$. Note that the term $c(t \vee \tau - \tau)$ vanishes for $t \in [0, \tau]$ and that the functions $t \mapsto C_\infty(t \wedge \tau) + c(t \vee \tau - \tau)$ and $t \mapsto \widetilde{C}(t \wedge \tau) + c(t \vee \tau - \tau)$ are convex. The maximality of the convex minorant $\widetilde{C}$ (resp. C_∞) implies that $\widetilde{C} \geq C_\infty$ (resp. $C_\infty \geq \widetilde{C}$) on $[0, \tau]$. Thus, $\widetilde{C} = C_\infty$ on $[0, \tau]$ and all the faces of $\widetilde{C}$ and C_∞ with slope smaller than c occur during the time interval $[0, \tau]$.

Let $\widetilde{\mathcal{S}}$ be the set of slopes of $\widetilde{C}$ and note that $\mathcal{L}(\widetilde{\mathcal{S}}) = \mathcal{L}(\mathcal{S})$, because the random time horizon τ is rational. Thus, $\mathcal{L}(\mathcal{S}_\infty) \cap (-\infty, c) = \mathcal{L}(\widetilde{\mathcal{S}}) \cap (-\infty, c) = \mathcal{L}(\mathcal{S}) \cap (-\infty, c)$ a.s. Moreover, by Theorem 4.1, we have $\mathbb{P}(s \in \mathcal{L}^\pm(\mathcal{S})) = \mathbb{P}(s \in \mathcal{L}^\pm(\widetilde{\mathcal{S}})) \in \{0, 1\}$ for any $s \in \mathbb{R}$, which is further equal to $\mathbb{P}(s \in \mathcal{L}^\pm(\mathcal{S}_\infty))$ if $s \in (-\infty, c)$. By taking $c \uparrow l$ along a countable sequence, (4.7) follows. $\square$

§4.5 Vertices, slopes and derivatives of piecewise linear convex functions

A point $x \in \mathbb{R}$ is an *accumulation* (or *limit*) *point* of a set $\mathcal{A} \subset \mathbb{R}$ if every neighborhood of x in $\mathbb{R}$ intersects $\mathcal{A} \setminus \{x\}$. Denote by $\mathcal{L}(\mathcal{A})$ the set of all accumulation points in $\mathbb{R}$ of the set $\mathcal{A}$. A point $x \in \mathbb{R}$ is a *right-accumulation* (or *right-limit*) *point* of $\mathcal{A}$ if every neighborhood of x in $\mathbb{R}$ intersects $\mathcal{A} \cap (x, \infty)$. Denote by $\mathcal{L}^+(\mathcal{A})$ the set of all right-accumulation points in $\mathbb{R}$ of the set $\mathcal{A}$. A set of *left-accumulation* (or *left-limit*) *points* of $\mathcal{A}$, denoted by $\mathcal{L}^-(\mathcal{A})$, is defined analogously. Note that $\mathcal{L}(\mathcal{A}) = \mathcal{L}^+(\mathcal{A}) \cup \mathcal{L}^-(\mathcal{A})$ with the intersection $\mathcal{L}^+(\mathcal{A}) \cap \mathcal{L}^-(\mathcal{A})$ consisting of points in $\mathbb{R}$ that are limits of a strictly decreasing and a strictly increasing sequence of elements in $\mathcal{A}$. Moreover, the closure $\overline{\mathcal{A}}$ of $\mathcal{A}$ in $\mathbb{R}$ equals $\mathcal{A} \cup \mathcal{L}(\mathcal{A})$. Throughout the chapter, the sets $\mathcal{L}(\mathcal{A})$ and $\mathcal{L}^\pm(\mathcal{A})$ are called the *limit sets* of $\mathcal{A}$.

Let $C : [0, T] \to \mathbb{R}$ be a piecewise linear convex function on a bounded interval. Let $\mathcal{S}$ be the set of the slopes of the linear segments of C and $\{I_r : r \in \mathcal{S}\}$ the family of maximal open intervals of constancy of the right-continuous derivative C' of C. Denote by $V \coloneqq \bigcup_{r \in \mathcal{S}} \partial I_r$ a subset of $[0, T]$ consisting of all the boundary points of the intervals of constancy of C'. Both $\mathcal{S}$ and V are countable sets. Table 4.2 describes all possible behaviours of the derivative C'.

Times in $[0, T]$	Slopes $\mathcal{S}$	Derivative C'
$v \in [0, T] \setminus \overline{V}$	$C'(v) \in \mathcal{S}$	C' is constant on a neighbourhood of v
$v \in V \setminus \mathcal{L}(V)$	$C'(v) \in \mathcal{S} \setminus \mathcal{L}(\mathcal{S})$	C' is equal to a constant on $(v - \varepsilon, v)$ and a different constant on $[v, v + \varepsilon)$ for some $\varepsilon > 0$
$v \in \mathcal{L}^+(V) \setminus \mathcal{L}^-(V)$ (thus $v \in V \cap [0, T)$)	$C'(v) \in \mathcal{L}^+(\mathcal{S}) \cap \mathcal{S}$	C' is continuous at v; $C'(v) < C'(v + \delta)$ for all $\delta \in (0, T - v)$; C' is constant on $(v - \varepsilon, v]$ for some $\varepsilon > 0$
	$C'(v) \in \mathcal{L}^+(\mathcal{S}) \setminus \mathcal{S}$	$C'(v) < C'(v + \delta)$ for all $\delta \in (0, T - v)$; if $v > 0$, $C'(v) > C'(v-)$ and C' constant on $(v - \varepsilon, v)$ for some $\varepsilon > 0$;
$v \in \mathcal{L}^-(V) \setminus \mathcal{L}^+(V)$ (thus $v \in V \cap (0, T]$)	$C'(v-) \in \mathcal{L}^-(\mathcal{S}) \cap \mathcal{S}$	C' is continuous at v, $C'(v - \varepsilon) < C'(v-)$ for any $\varepsilon > 0$ and C' is constant on $[v, v + \varepsilon)$ for some $\varepsilon > 0$
	$C'(v-) \in \mathcal{L}^-(\mathcal{S}) \setminus \mathcal{S}$	$C'(v - \varepsilon) < C'(v-)$ for any $\varepsilon > 0$ and, if $v \neq T$, $C'(v) > C'(v-)$ with C' constant on $[v, v + \varepsilon)$ for some $\varepsilon > 0$
$v \in \mathcal{L}^-(V) \cap \mathcal{L}^+(V)$ (thus $v \notin V$)	$C'(v) \in \mathcal{L}^+(\mathcal{S}) \setminus \mathcal{L}^-(\mathcal{S})$ (and $C'(v) \notin \mathcal{S}$)	C' is discontinuous at v with $C'(v-) < C'(v) < C'(v + \varepsilon)$ for any $\varepsilon > 0$
	$C'(v) \in \mathcal{L}^-(\mathcal{S}) \cap \mathcal{L}^+(\mathcal{S})$ (and $C'(v) \notin \mathcal{S}$)	C' is continuous at v with $C'(v - \varepsilon) < C'(v) < C'(v + \varepsilon)$ for any $\varepsilon > 0$

Table 4.2: The behaviours of C' and of the sets of times V and slopes $\mathcal{S}$ of an arbitrary piecewise linear function $C : [0, T] \to \mathbb{R}$. The table exhausts all possibilities. Recall that C' is non-decreasing and right-continuous on $(0, T)$, defined by its limits on $\{0, T\}$, $C'(0) \coloneqq \lim_{t \downarrow 0} C'(t) \in [-\infty, \infty)$ and $C'(T) \coloneqq \lim_{t \uparrow T} C'(t) \in (-\infty, \infty]$, and has left-limits $C'(v-) \coloneqq \lim_{t \uparrow v} C'(t)$ for all $v \in (0, T]$.

§4.6 Concluding remarks

The probabilistic arguments used in the proofs of Theorem 4.1 (see §4.3) and Proposition 4.5 (see §4.4.2) strongly suggest that "frequent" visits of the process $(X_t/t)_{t\in(0,1]}$ to bounded intervals as $t \downarrow 0$ play a major role in X being strongly eroded. The *time spent* during such visits, and not the number of visits, appears to be the key quantity for the following reasons.

(I) The integral $\int_0^1 \mathbb{P}(X_t/t \in I)t^{-1}\mathrm{d}t$ in Theorem 4.1, which needs to be infinite if X is to be strongly eroded, is equal to the mean of the (weighted) occupation measure $\mathcal{T}(I) := \int_0^1 \mathbb{1}_I(X_t/t)t^{-1}\mathrm{d}t$ of the interval I corresponding to the process $(X_t/t)_{t\in(0,1]}$.

(II) For any abrupt process X, the process $(X_t/t)_{t\in(0,\varepsilon]}$ visits every bounded interval infinitely many times for every $\varepsilon > 0$. Indeed, since $\mathfrak{s}_1$ is locally integrable, $\mathfrak{s}_1(r)$ is finite for a.e. $r \in \mathbb{R}$. Moreover, if $\mathfrak{s}_1(r) < \infty$, then 0 is regular for itself for the process $(X_t - rt)_{t\geq 0}$ and hence $(X_t/t)_{t\in(0,\varepsilon]}$ visits r infinitely often for every $\varepsilon > 0$. These visits, however, are brief since X is abrupt and thus $\mathbb{E}\mathcal{T}(I) < \infty$ for all bounded intervals I.

In the finite variation case, our ability to obtain a complete picture of how and where smoothness of the derivative C' fails is due to the fact that, for every open interval I, the process $(X_t/t)_{t\in(0,\varepsilon]}$ spends all of the (resp. no) time in I for all sufficiently small $\varepsilon > 0$ if the limit $\lim_{t\downarrow 0} X_t/t$ lies inside (resp. outside) of I. In order to establish Conjectures 4.9 and 4.10, we would need a better understanding (in the infinite variation case) of how much time the process $(X_t/t)_{t\in(0,1]}$ spends on any bounded interval. Such a result would allow us to apply Theorem 4.18 above to obtain the conjectured dichotomy. However, a result of this type appears to be delicate because the jumps of $(X_t/t)_{t\in(0,\varepsilon]}$ visit all bounded intervals infinitely many times for all $\varepsilon > 0$ whenever the positive and negative jumps of X both have infinite variation. (Recall that if $\int_{(-1,0)} |x|\nu(\mathrm{d}x) < \infty$ or $\int_{(0,1)} x\nu(\mathrm{d}x) < \infty$, then the process X creeps and is therefore abrupt.) Indeed, let $\Delta_t := X_t - X_{t-}$ denote the jump of X at time $t > 0$ and let $\Xi = \sum_{\Delta_t\neq 0} \delta_{(t,\Delta_t)}$ be the Poisson random measure on the set $(0,\infty) \times (\mathbb{R} \setminus \{0\})$ of the jumps of X with mean measure $\mathrm{Leb} \otimes \nu$. For any $\varepsilon > 0$, the Poisson variable $\mathscr{T}(I) := \sum_{t\in(0,\varepsilon]} \mathbb{1}_I(\Delta_t/t) = \int_{(0,\varepsilon]\times(\mathbb{R}\setminus\{0\})} \mathbb{1}_I(x/t)\Xi(\mathrm{d}t, \mathrm{d}x)$ is infinite a.s. for any interval $I = [a, b) \subset (0,\infty)$ since its mean is infinite: by Campbell's formula,

$$\mathbb{E}\mathscr{T}([a, b)) = \int_{(0,\infty)} \int_0^{\varepsilon} \mathbb{1}_{[a,b)}(x/t)\mathrm{d}t\nu(\mathrm{d}x) \geq \left(\tfrac{1}{a} - \tfrac{1}{b}\right) \int_{(0,a\varepsilon]} x\nu(\mathrm{d}x) = \infty.$$

Theorem 4.1 can be rephrased as follows: for a given interval I, we have $|\mathcal{S}\cap I| = \infty$ a.s. if and only if $\mathbb{E}\mathcal{T}(I) = \infty$ where we recall $\mathcal{T}(I) = \int_0^1 \mathbb{1}_I(X_t/t)t^{-1}\mathrm{d}t$. In light of Theorem 4.18, it is natural to speculate that something stronger is true, namely, $\mathbb{E}\mathcal{T}(I) = \infty$ if and only if $\mathcal{T}(I) = \infty$ a.s. We make the final observation that this occupation measure equals the total time X_t/t spends in I under an exponential change of variable: $\mathcal{T}(I) = \int_0^\infty \mathbb{1}_I(X_{e^{-u}}/e^{-u})\mathrm{d}u$. This emphasis on the time spent by X_t/t over exponentially small times is in line with the geometric decay of the length of the sticks in the stick-breaking representation for the convex minorant C, the main tool in proving Theorem 4.1.

Chapter 5

How smooth can a convex hull of a Lévy path be?

§5.1 Introduction

The class of Lévy processes with paths whose graphs have convex hulls in the plane with smooth boundary almost surely was characterised in Chapter 4. In fact, as explained in Chapter 4, to understand whether the boundary is smooth at a point with tangent of a given slope, it suffices to analyse whether the right-derivative $C' = (C'(t))_{t\in(0,T)}$ of the convex minorant $C = (C(t))_{t\in[0,T]}$ of a Lévy process $X = (X_t)_{t\in[0,T]}$ is continuous as it attains that slope (recall C from Definition 2.16). The main objective of this chapter is to quantify the smoothness of the boundary of the convex hull of X by quantifying the modulus of continuity of C' via its lower and upper functions. In the case of times 0 and T, we quantify the degree of smoothness of the boundary of the convex hull by analysing the rate at which $|C'(t)| \to \infty$ as t approaches either 0 or T (see YouTube [14] for a short presentation of the results).

It is known that C is a piecewise linear convex function (see e.g. Theorem 2.18 or [64]) and the image of the right-derivative C' over the open intervals of linearity of C is a countable random set $\mathcal{S}$ with a.s. deterministic limit points that do not depend on the time horizon T, see Theorem 4.1. These limit points of $\mathcal{S}$ determine the continuity of C' on $(0, T)$ outside of the open intervals of constancy of C', see §4.5. Indeed, the *vertex time process* $\tau = (\tau_s)_{s\in\mathbb{R}}$, given by $\tau_s := \inf\{t \in (0, T) : C'(t) > s\} \wedge T$ (where $a \wedge b := \min\{a, b\}$ and $\inf \emptyset := \infty$), is the right-inverse of the non-decreasing process C'. The process τ finds the times in $[0, T]$ of the vertices of the convex minorant C (see §2.4 or [38, Sec. 2.3]), so the only possible discontinuities of C' lie in the range of τ. Clearly, it suffices to analyse only the times τ_s for which

C' is non-constant on the interval $[\tau_s, \tau_s + \varepsilon)$ for every $\varepsilon > 0$ (otherwise, τ_s is the time of a vertex isolated from the right). At such a time, the continuity of C' can be described in terms of a limit set of $\mathcal{S}$. In this chapter we analyse the quality of the right-continuity of C' at such points. By time reversal, analogous results apply for the left-continuity of $t \mapsto C'(t)$ on $(0, T)$ (i.e., as $t \uparrow \tau_s$ for $s \in \mathbb{R}$) and for the explosion of $C'(t)$ as $t \uparrow T$. Throughout this chapter, the variable $s \in \mathbb{R}$ will be reserved for *slope*, indexing the vertex time process τ.

§5.1.1 Contributions

We describe the small-time fluctuations of the derivative of the boundary of the convex hull of X at its points of smoothness. This requires studying the local growth of C' in two regimes: at *finite slope* (FS) s in the deterministic set $\mathcal{L}^* \subset \mathbb{R}$ of points s that are a.s. in the set $\mathcal{L}^+(\mathcal{S})$ of right-limit points[1] of the set of slopes $\mathcal{S}$ and at *infinite slope* (IS) for Lévy processes of infinite variation, see Figure 5.1 below. In terms of times, regime (FS) with $s \in \mathcal{L}^*$ analyses how C' leaves the slope s at vertex time τ_s in $[0, T)$ and regime (IS) analyses how C' enters from $-\infty$ at time $0 = \lim_{u \downarrow -\infty} \tau_u$. At all other times $t \in (0, T) \setminus \{\tau_s : s \in \mathcal{L}^*\}$, the derivative C' is a.s. constant on $[t, t + \varepsilon)$ for some sufficiently small $\varepsilon > 0$. In particular, in what follows we exclude all Lévy processes that are compound Poisson with drift, since C' only takes finitely many values in that case.

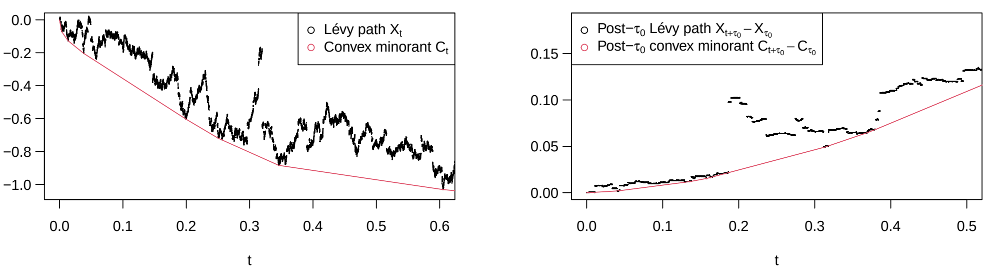

Figure 5.1: The picture on the left shows the path of an α-stable Lévy process X with $\alpha \in (1, 2)$ and its convex minorant C starting at time 0. The picture on the right shows the post-minimum process $(X_{t+\tau_0} - X_{\tau_0})_{t \in [0, T-\tau_0]}$ of an α-stable process with $\alpha \in (0, 1)$ and its corresponding convex minorant $(C(t + \tau_0) - C(\tau_0))_{t \in [0, T-\tau_0]}$. Note that, in the case $\alpha \in (0, 1)$, the derivative C' is continuous only at τ_0, i.e. at $t = 0$ in the graph, and at no other contact point between the path and its convex minorant.

[1] A point x is a right-limit point of $A \subset \mathbb{R}$, denoted $x \in \mathcal{L}^+(A)$ if $A \cap (x, x + \varepsilon) \neq \emptyset$ for all $\varepsilon > 0$ (see also §4.5).

Regime (FS): C' immediately after τ_s. Given a slope $s \in \mathbb{R}$, we have $s \notin \mathcal{S}$ a.s. by Theorem 2.18 since the law of X is diffuse. By Theorem 4.1, $s \in \mathcal{L}^*$ if and only if, with probability 1, the derivative C' attains level s at a unique time $\tau_s \in (0, T)$ (i.e. $C'(\tau_s) = s$) and is not constant on every interval $[\tau_s, \tau_s + \varepsilon)$, $\varepsilon > 0$, a.s. Moreover, $s \in \mathcal{L}^*$ if and only if $\int_0^1 \mathbb{P}(X_t/t \in (s, s+\varepsilon))t^{-1}\mathrm{d}t = \infty$ for all $\varepsilon > 0$. The regime (FS) includes an infinite variation process X if it is strongly eroded (implying $\mathcal{L}^* = \mathbb{R}$) or, more generally, if $(X_t - st)_{t \geq 0}$ is eroded (implying $s \in \mathcal{L}^*$), see Chapter 4. Moreover, regime (FS) includes a finite variation process X at slope $s \in \mathcal{L}^*$ if and only if the natural drift $\gamma_0 = \lim_{t \downarrow 0} X_t/t$ equals s and $\int_0^1 \mathbb{P}(X_t > \gamma_0 t)t^{-1}\mathrm{d}t = \infty$ or, equivalently, if the positive half-line is regular for $(X_t - \gamma_0 t)_{t \geq 0}$ (see Corollary 4.4 for a characterisation in terms of the Lévy measure of X or its characteristic exponent).

Our results in regime (FS) are summarised as follows. For any process with $s \in \mathcal{L}^*$, Theorem 5.2 establishes general sufficient conditions identifying when the limit $\liminf_{t \downarrow 0}(C'(t + \tau_s) - s)/f(t)$ is either 0 a.s. or ∞ a.s. In particular, we show that $\liminf_{t \downarrow 0}(C'(t + \tau_s) - s)/f(t)$ cannot take a positive finite value if X has jumps of both signs and is an α-stable with $\alpha \in (0, 1]$ (recall that, if $\alpha > 1$, then $\mathcal{L}^* = \emptyset$ by Proposition 4.6).

For processes X in the small-time domain of attraction of an α-stable process with $\alpha \in (0, 1)$ (see §5.2.2 below for definition), Theorem 5.7 finds a parametric family of functions f that essentially determine the upper fluctuations of $C'(t + \tau_s) - s$ up to sublogarithmic factors. In particular, Theorem 5.7 determines when $\limsup_{t \downarrow 0}(C'(t + \tau_s) - s)/f(t)$ equals 0 a.s. or ∞ a.s., essentially characterising the right-modulus of continuity[2] of C' at τ_s. The family of functions f is given in terms of the regularly varying normalising function of X.

Regime (IS): C' immediately after 0. The boundary of the convex hull of X is smooth at the origin if and only if $\lim_{t \downarrow 0} C'(t) = -\infty$ a.s., which is equivalent to X being of infinite variation (see Proposition 4.5 & §4.1.1.2). If X has finite variation, then C' is bounded (see Proposition 4.3). In this case, C' has positive probability of being non-constant on the interval $[0, \varepsilon)$ for every $\varepsilon > 0$ if and only if the negative half-line is not regular. Moreover, if this event occurs, then $C'(t)$ approaches the natural drift γ_0 as $t \downarrow 0$ by Proposition 4.3(b) and the local behaviour of C' at 0 would be described by the results of regime (FS). Thus, in regime (IS) we only consider Lévy processes of infinite variation.

Results in regime (IS) are summarised as follows. For any infinite variation pro-

[2]We say that a non-decreasing function $\varphi : [0, \infty) \to [0, \infty)$ is a right-modulus of continuity of a right-continuous function g at $x \in \mathbb{R}$ if $\limsup_{y \downarrow x} |g(y) - g(x)|/\varphi(y - x) < \infty$.

cess X, Theorem 5.9 establishes general sufficient conditions for $\limsup_{t\downarrow 0}|C'(t)|f(t)$ to equal either 0 a.s. or ∞ a.s. In particular, we show that $\limsup_{t\downarrow 0}|C'(t)|f(t)$ cannot take a positive finite value if X is α-stable with $\alpha\in[1,2)$ and has (at least some) negative jumps.

If the Lévy process lies in the domain of attraction of an α-stable process, with $\alpha\in(1,2]$, Theorem 5.13 finds a parametric family of functions f that essentially determine the lower fluctuations of C' up to sublogarithmic functions. The function f is given in terms of the regularly varying normalising function of X. Again, these results describe the right-modulus of continuity of the derivative of the boundary of the convex hull of X (as a closed curve in $\mathbb{R}^2$) at the origin. In this case, for a sufficiently small $\varepsilon>0$, we may locally parametrise the curve $((t,C(t));t\in[0,\varepsilon])$, as $((\varsigma(t),t);t\in[C(\varepsilon),0])$, using a local inverse $\varsigma(t)$ of $C(t)$ with left-derivative $\varsigma'(t)=1/C'(\varsigma(t))$ that vanishes at 0 (since $\lim_{t\downarrow 0}1/|C'(t)|=0$ a.s.). Thus, the left-modulus of continuity of ς at 0 is described by the upper and lower limits of $(|C'(t)|f(t))^{-1}$ as $t\downarrow 0$, the main focus of our results in this regime.

Consequences for the path of a Lévy process and its meander. In §5.2.5 we present some implications the results in this chapter have for the path of X. We find that, under certain conditions, the local fluctuations of X can be described in terms of those of C', yielding novel results for the local growth of the post-minimum process of X and the corresponding Lévy meander (see Lemma 5.15 and Corollaries 5.16 and 5.17 below).

§5.1.2 Strategy and ideas behind the proofs

An overview of the proofs of our results is as follows. First we show that, under our assumptions, the local properties of C' do not depend on the time horizon T. This reduces the problem to the case where the time horizon T is independent of X and exponentially distributed. When $T\sim\text{Exp}(\lambda)$, we denote the convex minorant by $(\widehat{C}(t))_{t\in[0,T]}$, with the corresponding right-derivative denoted by $\widehat{C}'$. Second, we translate the problem of studying the local behaviour of $\widehat{C}'$ to the problem of studying the local behaviour of its inverse: the vertex time process $\widehat{\tau}$. Third, we exploit the fact that, since the time horizon T is an independent exponential random variable with mean $1/\lambda$, the vertex time process $\widehat{\tau}$ is a time-inhomogeneous non-decreasing additive process (i.e., a process with independent but non-stationary increments) and its Laplace exponent is given by (see Theorem 2.23)

$$\mathbb{E}[e^{-w\widehat{\tau}_u}]=e^{-\Phi_u(w)},\quad \Phi_u(w):=\int_0^\infty(1-e^{-wt})e^{-\lambda t}\mathbb{P}(X_t\le ut)\frac{\mathrm{d}t}{t},\tag{5.1}$$

for all $w \geq 0$ and $u \in \mathbb{R}$. These three observations reduce the problem to the analysis of the fluctuations of the additive process $\widehat{\tau}$.

The local properties of $\widehat{C}'$ are entirely driven by the small jumps of X. However, different facets of the small-jump activity of X dominate in each regime, resulting in related but distinct results and criteria. Indeed, regime (FS) corresponds to the short-term behaviour of $\widehat{\tau}_{s+u} - \widehat{\tau}_s$ as $u \downarrow 0$ while regime (IS) corresponds to the long-term behaviour of $\widehat{\tau}_u$ as $u \to -\infty$ (note that, when X is of infinite variation, $\widehat{\tau}_u > 0$ for $u \in \mathbb{R}$ and $\lim_{u\to-\infty} \widehat{\tau}_u = 0$ a.s.). This bears out in a difference in the behaviour of the Laplace exponent Φ of $\widehat{\tau}$ at either bounded or unbounded slopes and leads to an interesting diagonal connection in behaviour that we now explain.

Our main tool is the novel description of the upper and lower fluctuations of a non-decreasing time-inhomogeneous additive process Y started at $Y_0 = 0$, in terms of its time-dependent Lévy measure and Laplace exponent. In our applications, the process Y is given by $(\widehat{\tau}_{u+s} - \widehat{\tau}_s)_{u\geq 0}$ in regime (FS) and $(\widehat{\tau}_{-1/u})_{u\geq 0}$ (with conventions $-1/0 = -\infty$ and $\widehat{\tau}_{-\infty} = 0$) in regime (IS). Then our main technical tools, Theorems 5.22 & 5.24 of §5.3 below, describing the upper and lower fluctuations of Y, also serve to describe the lower and upper fluctuations, respectively, of the right-inverse L of Y. Since, in regime (FS), we have $\widehat{C}'(t + \widehat{\tau}_s) - s = L_t$ but, in regime (IS), we have $\widehat{C}'(t) = -1/L_t$, the lower (resp. upper) fluctuations of $\widehat{C}'$ in regime (FS) will have a similar structure to the upper (resp. lower) fluctuations of $\widehat{C}'$ in regime (IS). This diagonal connection is *a priori* surprising as the processes considered by either regime need not have a clear connection to each other. Indeed, regime (FS) considers most finite variation processes and only some infinite variation processes while regime (IS) considers exclusively infinite variation processes. This diagonal connection is reminiscent of the duality between stable process with stability index $\alpha \in (1, 2]$ and a corresponding stable process with stability index $1/\alpha \in [1/2, 1)$ arising in the famous time-space inversion first observed by Zolotarev for the marginals and later studied by Fourati [34] for the ascending ladder process (see also [45] for further extensions of this duality).

The lower and upper fluctuations of the corresponding process Y require varying degrees of control on its Laplace exponent Φ in (5.1). The assumptions of Theorem 5.22 require tight two-sided estimates of Φ, not needed in Theorem 5.24. When applying Theorem 5.22, we are compelled to assume X lies in the domain of attraction of an α-stable process. In regime (FS) this assumption yields sharp estimates on the density of X_t as $t \downarrow 0$, which in turn allows us to control the term $\mathbb{P}(0 < X_t - st \leq ut)$ for small $t > 0$ in the Laplace exponent $\Phi_{s+u} - \Phi_s$ of $\widehat{\tau}_{u+s} - \widehat{\tau}_s$ as $u \downarrow 0$, cf. (5.1) above. The growth rate of the density of X_t as $t \downarrow 0$ is controlled is by *lower* estimates on the small-jump activity of X given in Lemma 5.33 below, a refinement of the results in [62] for processes attracted to a stable process. In regime (IS) we require control over the negative tail probabilities $\mathbb{P}(X_t \le ut)$ for small $t > 0$ appearing in the Laplace exponent Φ_u of $\widehat{\tau}_u$ as $u \to -\infty$, cf. (5.1). The behaviour of these tails are controlled by *upper* estimates of the small-jump activity of X, which are generally easier to obtain. In this case, moment bounds for the small-jump component of the Lévy process and the convergence in Kolmogorov distance implied by the attraction to the stable law, give sufficient control over these tail probabilities.

§5.1.3 Connections with the literature

In [21], Bertoin finds the law of the convex minorant of Cauchy process on $[0, 1]$ and finds the exact asymptotic behaviour (in the form of a law of interated logarithm with a positive finite limit) for the derivative C' at times 0, 1 and any τ_s, $s \in \mathbb{R}$. The methods in [21] are specific to Cauchy process with its linear scaling property, making the approach hard to generalise. In fact, the results in [21] are a direct consequence of the fact that the vertex time process $\widehat{\tau}$ has a Laplace transform Φ in (5.1) that factorises as $\Phi_u(w) = \mathbb{P}(X_1 \le u)\Phi_\infty(w)$, making $\widehat{\tau}$ a gamma subordinator under the deterministic time-change $u \mapsto \mathbb{P}(X_1 \le u)$, cf. Example 5.2 below.

Paul Lévy showed that the boundary of the convex hull of a planar Brownian motion has no corners at any point, see [51], motivating [30] to characterise the modulus of continuity of the derivative of that boundary. Given the characterisation of the smoothness of the convex hull of a Lévy path in Chapter 4, the results in this chapter are likewise motivated by the study of the modulus of continuity of the derivative of the boundary in this context.

The literature on the growth rate of the path of a Lévy process X is vast, particularly for subordinators, see e.g. [20, 36, 37, 47, 71, 72, 83]. The authors in [36, 37] study the growth rate of a subordinator at 0 and ∞. In [36] (see also [20, Prop 4.4]) Fristedt fully characterises the upper fluctuations of a subordinator in terms of its Lévy measure, a result we generalise in Theorem 5.24 to processes that need not have stationary increments. In [20, Thm 4.1] (see also [37, Thm 1]), a function essentially characterising the exact lower fluctuations of a subordinator is constructed in terms of its Laplace exponent. These methods are not easily generalised to the time-inhomogeneous case since the Laplace exponent is now bivariate and there is neither a one-parameter lower function to propose nor a clear extension to the proofs.

In [68], Sato establishes results for time-inhomogeneous non-decreasing additive processes similar to our result in §5.3. The assumptions in [68] are given in terms of the transition probabilities of the additive process, which are generally intractable, particularly for the processes $(\widehat{\tau}_{-1/u})_{u>0}$ and $(\widehat{\tau}_{u+s} - \widehat{\tau}_s)_{u\geq 0}$, considered here. Our results are also easier to apply in other situations as well, for example, to fractional Poisson processes (see definition in [16]).

The upper fluctuations of a Lévy process at zero have been the topic of numerous studies, see [17, 72] for the one-sided problem and [47, 71, 83] for the two-sided problem. Similar questions have been considered for more general time-homogeneous Markov processes [29, 50]. The time-homogeneity again plays an important role in these results. The lower fluctuations of a stochastic process is only qualitatively different from the upper fluctuations if the process is positive. This is the reason why this problem has mostly only been addressed for subordinators (see the references above) and for the running supremum of a Lévy process, see e.g. [7]. We stress that the results in this chapter, while related in spirit to this literature, are fundamentally different in two ways. First, we study the *derivative* of the convex minorant of a Lévy path on $[0, T]$, which (unlike e.g. the running supremum) cannot be constructed locally from the restriction of the path of the Lévy process to any short interval. Second, the convex minorant and its derivative are neither Markovian nor time-homogeneous. In fact, the only result in our context prior to our work is in the Cauchy case [21], where the derivative of the convex minorant is an explicit gamma process under a deterministic time-change, cf. Example 5.2 below.

§5.1.4 Organisation of the chapter

In §5.2 we present the main results of this chapter, and the section is split in four, according to regimes (FS) and (IS) and whether the upper or lower fluctuations of C' are being described. The implications of the results in §5.2 for the Lévy process and meander are covered in §5.2.5. In §5.3, technical results for general time-inhomogeneous non-decreasing additive processes are established. In §5.4 we recall from [38] the definition and law of the vertex time process τ and provide the proofs of the results stated in §5.2. The chapter is concluded with §5.6.

§5.2 Growth rate of the derivative of the convex minorant

Let $X = (X_t)_{t\geq 0}$ be an infinite activity Lévy process, and let $C = (C(t))_{t\in[0,T]}$ be the convex minorant of X on $[0, T]$ for some $T > 0$. In this section we analyse the growth rate of the right derivative of C, denoted by $C' = (C'(t))_{t\in(0,T)}$, near time 0 and at the vertex time $\tau_s = \inf\{t > 0 : C'(t) > s\} \wedge T$ of the slope $s \in \mathbb{R}$ (i.e., the first time C' attains slope s). More specifically, we give sufficient conditions to identify the values of the possibly infinite limits (for appropriate increasing functions f with $f(0) = 0$): $\limsup_{t\downarrow 0}(C'(t+\tau_s) - s)/f(t)$ & $\liminf_{t\downarrow 0}(C'(t+\tau_s) - s)/f(t)$ in the finite slope (FS) regime and $\limsup_{t\downarrow 0}|C'(t)|f(t)$ & $\liminf_{t\downarrow 0}|C'(t)|f(t)$ in the infinite slope (IS) regime. The values of these limits are constants in $[0,\infty]$ a.s. by Corollary 5.32 below. We note that these limits are invariant under certain modifications of the law of X, which we describe in the following remark.

Remark 5.1.

(a) Let $\mathbb{P}$ be the probability measure on the space where X is defined. If the following limits $\limsup_{t\downarrow 0}|C'(t)|f(t)$, $\liminf_{t\downarrow 0}|C'(t)|f(t)$, $\limsup_{t\downarrow 0}(C'(t+\tau_s) - s)/f(t)$ and $\liminf_{t\downarrow 0}(C'(t+\tau_s)-s)/f(t)$ are $\mathbb{P}$-a.s. constant, then they are also $\mathbb{P}'$-a.s. constant with the same value for any probability measure $\mathbb{P}'$ absolutely continuous with respect to $\mathbb{P}$. In particular, we may modify the Lévy measure of X on the complement of any neighborhood of 0 without affecting these limits (see e.g. Theorems A.16 & A.17).

(b) We may add a drift process to X without affecting the limits at 0 since such a drift would only shift $|C'(t)|$ by a constant value and $f(t) \to 0$ as $t \downarrow 0$. Similarly, for the limits of $(C'(t+\tau_s) - s)/f(t)$ as $t \downarrow 0$, it suffices to analyse the post-minimum process (i.e., the vertex time τ_0) of the process $(X_t - st)_{t\geq 0}$. For ease of reference, our results are stated for a general slope s. ◊

§5.2.1 Regime (FS): lower functions at time τ_s

The following theorem describes the lower fluctuations of $C'(t+\tau_s) - s$ as $t \downarrow 0$. Recall that $\mathcal{L}^*$ is the deterministic set of points that are a.s. right-limit points of the set of slopes $\mathcal{S}$.

Theorem 5.2. *Let $s \in \mathcal{L}^*$ and f be continuous and increasing, satisfying $f(t) \leq 1 = f(1)$ for $t \in (0,1]$ and $f(0) = 0 = \lim_{c\downarrow 0}\limsup_{t\downarrow 0} f(ct)/f(t)$. Let $c > 0$ and consider the following conditions:*

$$\int_0^1 \mathbb{P}(0 < (X_t - st)/t \leq f(t/c))\frac{\mathrm{d}t}{t} < \infty, \tag{5.2}$$

$$\int_0^1 \mathbb{E}\left[\frac{t}{f^{-1}((X_t - st)/t)^2}\mathbb{1}_{\{f(t/2)<(X_t-st)/t\leq 1\}}\right]\mathrm{d}t < \infty, \tag{5.3}$$

$$2^n\int_0^{2^{-n}} \mathbb{P}(f(t/2) < (X_t - st)/t \leq f(2^{-n}))\mathrm{d}t \to 0, \quad \text{as } n \to \infty. \tag{5.4}$$

Then the following statements hold.

(i) *If* (5.2)–(5.4) *hold for $c = 1$, then $\liminf_{t\downarrow 0}(C'(t+\tau_s) - s)/f(t) = \infty$ a.s.*

(ii) If (5.2) *fails for every* $c > 0$, *then* $\liminf_{t\downarrow 0}(C'(t+\tau_s)-s)/f(t) = 0$ *a.s.*

(iii) *If* $\liminf_{t\downarrow 0}(C'(t+\tau_s)-s)/f(t) > 1$ *a.s., then* (5.2) *holds for any* $c > 1$.

Some remarks are in order.

Remark 5.3.

(a) Any continuous regularly varying function f of index $r > 0$ satisfies the assumption in the theorem: $\lim_{c\downarrow 0}\lim_{t\downarrow 0} f(ct)/f(t) = \lim_{c\downarrow 0} c^r = 0$. Moreover, the assumption $f(t) \le 1 = f(1)$ for $t \in (0,1]$ is not necessary but makes conditions (5.2)–(5.4) take a simpler form.

(b) The proof of Theorem 5.2 is based on the analysis of the upper fluctuations of τ at slope s. Condition (5.2) ensures $(\tau_{u+s}-\tau_s)_{u\ge 0}$ jumps finitely many times over the boundary $u \mapsto f^{-1}(u)$, condition (5.4) makes the small-jump component of $(\tau_{u+s}-\tau_s)_{u\ge 0}$ (i.e. the sum of the jumps at times $v \in [s, u+s]$ of size at most $f^{-1}(v)$) have a mean that tends to 0 as $u \downarrow 0$ and condition (5.3) controls the deviations of $(\tau_{u+s}-\tau_s)_{u\ge 0}$ away from its mean.

(c) Eq. (5.4) holds if $\int_0^1 \mathbb{P}(f(2^{-n}t/2) < (X_{2^{-n}t} - s2^{-n}t)/(2^{-n}t) \le f(2^{-n}))\mathrm{d}t \to 0$ as $n \to \infty$, which, by the dominated convergence theorem, holds if we have the limit $\mathbb{P}(f(u/2) < (X_u - su)/u \le f(u/t)) \to 0$ as $u \downarrow 0$ for a.e. $t \in (0,1)$.

(d) Condition (5.3) in Theorem 5.2 requires access to the inverse f^{-1} of the function f. In the special case when the function f is concave, this assumption can be replaced with an assumption given in terms of f (cf. Proposition 5.26 and Corollary 5.28). However, it is important to consider non-concave functions f, see Corollary 5.4 below. ♢

§5.2.1.1 Simple sufficient conditions for the assumptions of Theorem 5.2

Let f be as in Theorem 5.2. By Theorem 5.24(c) below (with measure $\Pi(\mathrm{d}x,\mathrm{d}t) = \mathbb{P}((X_t - st)/t \in \mathrm{d}x)t^{-1}\mathrm{d}t$), the following condition implies (5.3)–(5.4):

$$\int_0^1 \mathbb{E}\left[\frac{1}{f^{-1}((X_t-st)/t)}\mathbb{1}_{\{f(t/2)<(X_t-st)/t\le 1\}}\right]\mathrm{d}t < \infty. \tag{5.5}$$

If estimates on the density of X_t are available (e.g., via assumptions on the generating triplet of X), (5.5) can be simplified further, see Corollary 5.4 below.

Throughout the chapter, we denote by (σ^2, γ, ν) the generating triplet of X (corresponding to the cutoff function $x \mapsto \mathbb{1}_{(-1,1)}(x)$, see §2.2). For $\varepsilon > 0$, we recall $\overline{\gamma}(\varepsilon) = \int_{(-1,1)\setminus(-\varepsilon,\varepsilon)} x\nu(\mathrm{d}x)$ and $\overline{\sigma}^2(\varepsilon) = \overline{\sigma}^2_+(\varepsilon) + \overline{\sigma}^2_-(\varepsilon)$ from (2.2), where $\overline{\sigma}^2_+(\varepsilon) := \int_{(0,\varepsilon)} x^2\nu(\mathrm{d}x)$ and $\overline{\sigma}^2_-(\varepsilon) := \int_{(-\varepsilon,0)} x^2\nu(\mathrm{d}x)$. Recall that, in regime (FS), we have $\sigma^2 = 0$ (see Proposition 4.6).

Corollary 5.4. *Fix* $\beta \in (0,1]$ *and let* $s \in \mathcal{L}^*$ *and* f *be as in Theorem 5.2.*

(a) *If* $\liminf_{\varepsilon\downarrow 0}\varepsilon^{\beta-2}(\overline{\sigma}^2(\varepsilon)+\sigma^2)>0$, *$f$ is differentiable with positive derivative* $f'>0$ *and the integrals* $\int_0^1\int_{t/2}^1(f'(y)/y)t^{1-1/\beta}\mathrm{d}y\mathrm{d}t$ *and* $\int_0^1 t^{-1/\beta}f(t)\mathrm{d}t$ *are finite, then* $\liminf_{t\downarrow 0}(C'(t+\tau_s)-s)/f(t)=\infty$ *a.s.*

(b) *Assume* $\int_0^1((t^{-1/\beta}f(t))\wedge t^{-1})\mathrm{d}t=\infty$ *and either of the following hold:*

  (i) $(\overline{\sigma}^2(\varepsilon)+\sigma^2)\approx\varepsilon$ *and* $|\overline{\gamma}(\varepsilon)|=\mathcal{O}(1)$ *as* $\varepsilon\downarrow 0$,

  (ii) $\beta\in(0,1)$ *and* $\overline{\sigma}^2_\pm(\varepsilon)\approx\varepsilon^{2-\beta}$ *as* $\varepsilon\downarrow 0$ *for both signs of* $\pm$,

then $\liminf_{t\downarrow 0}(C'(t+\tau_s)-s)/f(t)=0$ *a.s.*

We stress that the sufficient conditions in Corollary 5.4 are all in terms of the characteristics of the Lévy process X and the function f.

Remark 5.5.

(a) The assumptions in Corollary 5.4 are satisfied by most processes in the class $\mathcal{Z}_{\alpha,\rho}$ of Lévy processes in the small-time domain of attraction of an α-stable distribution, see §5.2.2 below (cf. (2.6)). Thus, the assumptions of part (a) in Corollary 5.4 hold for any $X\in\mathcal{Z}_{\alpha,\rho}$ and $\beta<\alpha$ (by Karamata's theorem in Theorem A.55, we can take $\beta=\alpha$ if the normalising function g of X satisfies $\liminf_{t\downarrow 0}t^{-1/\alpha}g(t)>0$). Moreover, the assumptions of cases (b-i) and (b-ii) hold for processes in the domain of normal attraction (i.e. if the normalising function equals $g(t)=t^{1/\alpha}$ for all $t>0$) with $\rho\in(0,1)$ and $\beta=\alpha\in(0,1]$, see Theorem 2.12. In particular, these assumptions are satisfied by stable processes with $\alpha\in(0,1]$ and $\rho\in(0,1)$.

(b) Both integrals in part (a) of Corollary 5.4 are finite or infinite simultaneously whenever f' is regularly varying at 0 with nonzero index by Karamata's theorem (see Theorem A.55). Thus, in that case, under the conditions of either (b-i) or (b-ii), the limit $\liminf_{t\downarrow 0}(C'(t+\tau_s)-s)/f(t)$ equals 0 or ∞ according to whether $\int_0^1 t^{-1/\beta}f(t)\mathrm{d}t$ is infinite or finite, respectively.

(c) The case $\beta>1$ is not considered in Corollary 5.4(a) and (b-ii) since in this case we would have $\mathcal{L}^*=\emptyset$ by Proposition 4.6. ◊

Proof of Corollary 5.4. Assume without loss of generality that $s=0\in\mathcal{L}^*$ (equivalently, we consider the process $(X_t-st)_{t\ge 0}$ for $s\in\mathcal{L}^*$).

(a) Our assumptions and Theorem A.19 show that the density $x\mapsto p_X(t,x)$ of X_t exists for $t>0$ and moreover $\sup_{x\in\mathbb{R}}p_X(t,x)\le Ct^{-1/\beta}$ for some $C>0$ and all $t\in(0,1]$. Thus, (5.5) is implied by

$$\int_0^1\int_{tf(t/2)}^t\frac{1}{f^{-1}(x/t)}t^{-1/\beta}\mathrm{d}x\mathrm{d}t=\int_0^1\int_{t/2}^1\frac{f'(y)}{y}t^{1-1/\beta}\mathrm{d}y\mathrm{d}t<\infty,\tag{5.6}$$

where we have used the change of variable $x=tf(y)$. Similarly, the bound on the density of X_t shows that condition (5.2) holds if $\int_0^1 t^{-1/\beta}f(t)\mathrm{d}t<\infty$. Thus, the result follows from Theorem 5.2.

(b) In either case (i) or (ii), our assumptions and Theorem A.20 show that $Ct^{-1/\beta} \le p_X(t,x)$ for some $C > 0$ and all $|x| \le t^{1/\beta}$. Thus $\mathbb{P}(0 < X_t \le tf(t/c)) \ge ((tf(t/c)) \wedge t^{1/\beta})Ct^{-1/\beta}$, implying that (5.2) fails for some $c > 0$ whenever we have $\int_0^1 ((t^{-1/\beta} f(t/c)) \wedge t^{-1})\mathrm{d}t = \infty$. A simple change of variables shows that this integral is either finite for all $c > 0$ or infinite for all $c > 0$. The result then follows from Theorem 5.2(ii). □

The following is another simple corollary of Theorem 5.2. This result can also be established using similar arguments to those used to prove Corollary 2.25

Corollary 5.6. *Let X be a Cauchy process, f be as in Theorem 5.2 and pick $s \in \mathbb{R}$. Then $\liminf_{t\downarrow 0}(C'(t+\tau_s) - s)/f(t)$ equals 0 (resp. ∞) a.s. if $\int_0^1 t^{-1} f(t)\mathrm{d}t$ is infinite (resp. finite).*

Proof. Assume without loss of generality that $s = 0$. Then the law of X_t/t does not depend on $t > 0$ and hence the integral in (5.5) equals

$$\int_0^1 \mathbb{E}\left[\frac{\mathbb{1}_{\{t/2<f^{-1}(X_1)\le 1\}}}{f^{-1}(X_1)}\right]\mathrm{d}t = \mathbb{E}\left[\int_0^1 \frac{\mathbb{1}_{\{t/2<f^{-1}(X_1)\le 1\}}}{f^{-1}(X_1)}\mathrm{d}t\right] \le 2\mathbb{P}(X_1 \in (0,1]) < \infty.$$

Moreover, condition (5.2) simplifies to $\int_0^1 \mathbb{P}(0 < X_1 \le f(t/c))t^{-1}\mathrm{d}t < \infty$, which is equivalent to the integral $\int_0^1 t^{-1} f(t/c)\mathrm{d}t$ being finite since X_1 has a bounded density that is bounded away from zero on $[0,1]$. The change of variables $t' = t/c$ shows that this integral is either finite for all $c > 0$ or infinite for all $c > 0$. Thus, Theorem 5.2 gives the result. □

§5.2.2 Regime (FS): upper functions at time τ_s

The upper fluctuations of $C'(t + \tau_s) - s$ are harder to describe than the lower fluctuations studied in §5.2.1 above. The main reason for this is that in Theorem 5.7 below the lim sup of C' at a vertex time τ_s can be expressed in terms of the lim inf of the vertex time process τ, which requires strong two-sided control on the Laplace exponent $\Phi_{u+s}(w) - \Phi_s(w)$, defined in (5.1), of the variable $\tau_{u+s} - \tau_s$ as $w \to \infty$ and $u \downarrow 0$. (In the proof of Theorem 5.2, lim sup of the vertex time process τ is needed, which is easier to control.) In turn, by (5.1), this requires sharp two-sided estimates on the probability $\mathbb{P}(0 < X_t - st \le ut)$ as a function of (u,t) for small $u, t > 0$. In particular, it is important to have strong control on the density of X_t for small $t > 0$ on the "pizza slice" $\{(t,x) : s < x/t \le u + s\}$ as $u \downarrow 0$. We establish these estimates for the processes in the domain of attraction of an α-stable process, leading to Theorem 5.7 below.

We denote by $\mathcal{Z}_{\alpha,\rho}$ the class of Lévy processes in the small-time domain of attraction of an α-stable process with positivity parameter $\rho \in [0,1]$ (see (2.6)). In the case $\alpha < 1$, relevant in the regime (FS) at slope s equal to the natural drift γ_0, for each Lévy process $X \in \mathcal{Z}_{\alpha,\rho}$ there exists a normalising function g that is regularly varying at 0 with index $1/\alpha$ and an α-stable process $(Z_u)_{u\in[0,T]}$ with $\rho = \mathbb{P}(Z_1 > 0) \in [0,1]$ such that the weak convergence $((X_{ut} - \gamma_0 ut)/g(t))_{u\in[0,T]} \xrightarrow{d} (Z_u)_{u\in[0,T]}$ holds as $t \downarrow 0$. Given $X \in \mathcal{Z}_{\alpha,\rho}$ with normalising function g, we define $G(t) \coloneqq t/g(t)$ for $t \in (0,\infty)$.

Theorem 5.7. *Suppose $X \in \mathcal{Z}_{\alpha,\rho}$ for some $\alpha \in (0,1)$ and $\rho \in (0,1]$. Define $f : (0,1) \to (0,\infty)$ through $f(t) \coloneqq 1/G(t\log^p(1/t))$, $t \in (0,1)$, for some $p \in \mathbb{R}$. Then the following hold for $s = \gamma_0$:*

(i) if $p > 1/\rho$, then $\limsup_{t\downarrow 0}(C'(t+\tau_s) - s)/f(t) = 0$ a.s.,

(ii) if $p < 1/\rho$, then $\limsup_{t\downarrow 0}(C'(t+\tau_s) - s)/f(t) = \infty$ a.s.

The class $\mathcal{Z}_{\alpha,\rho}$ is quite large and the assumption $X \in \mathcal{Z}_{\alpha,\rho}$ is essentially reduced to the Lévy measure of X being regularly varying at 0, see Theorem 2.12. In particular, α agrees with the Blumenthal–Getoor index β_+ defined in (2.3). Moreover, for $\alpha < 1$ and $\rho \in (0,1]$, the assumption $X \in \mathcal{Z}_{\alpha,\rho}$ implies that X is of finite variation with $\mathbb{P}(X_t - \gamma_0 t > 0) \to \rho$ as $t \downarrow 0$, implying $\mathcal{L}^* = \{\gamma_0\}$ by Proposition 4.3 and Corollary 4.4.

Note that the function f in Theorem 5.7 is regularly varying at 0 with index $1/\alpha - 1$. The appearance of the positivity parameter ρ, a nontrivial function of the Lévy measure of X, in Theorem 5.7 suggests that the upper fluctuations of C' at time τ_s (for $s = \gamma_0$) are more delicate than its lower fluctuations described in Theorem 5.13. Indeed, if $X \in \mathcal{Z}_{\alpha,\rho}$ is in the domain of normal attraction (i.e. $g(t) = t^{1/\alpha}$) and $\rho \in (0,1)$, then the fluctuations of C' at vertex time τ_s, characterised by Corollary 5.4(a) & (b-ii) (with $\beta = \alpha$) and Remark 5.5(a), do not involve parameter ρ. In particular, by Theorem 5.7 and Corollary 5.4(b-ii), we have $\liminf_{t\downarrow 0}(C'(t+\tau_s) - s)/f(t) = 0$ and $\limsup_{t\downarrow 0}(C'(t+\tau_s) - s)/f(t) = \infty$ a.s. for $f(t) = t^{1/\alpha-1}\log^q(1/t)$ and any $q \in [-1, (1/\alpha - 1)/\rho)$, demonstrating the gap between the lower and upper fluctuations of C' at vertex time τ_s.

Remark 5.8.

(a) The case where X is attracted to Cauchy process with $\alpha = 1$ is expected to hold for the functions f in Theorem 5.7. For such $X \in \mathcal{Z}_{1,\rho}$, a multitude of cases arise including X having (i) less activity (e.g., X is of finite variation), (ii) similar amount of activity (i.e., X is in the domain of normal attraction) or (iii) more activity than Cauchy process (see, e.g. Examples 4.1–4.2. In terms of the normalising function g of X, these cases correspond to the limit $\lim_{t\downarrow 0} t^{-1/\alpha} g(t)$ being equal to: (i) zero, (ii) a finite and positive constant or (iii) infinity. (Recall that in cases (ii) and (iii) X is strongly eroded with $\mathcal{L}^* = \mathbb{R}$, see Examples 4.1–4.2, and in case (i) X may be strongly eroded, by Theorem 4.8, or of finite variation with $\mathcal{L}^* = \{\gamma_0\}$ by Proposition 4.3 and the fact that $\lim_{t\downarrow 0} \mathbb{P}(X_t > 0) = \rho \in (0,1)$.) However, we stress that our methodology can be used to obtain a description of the lower fluctuations of C' at τ_s in cases (i), (ii) and (iii). This would require an application of Theorem 5.22 along with two-sided estimates of the Laplace exponent Φ of the vertex time process in (5.1), generalising Lemma 5.34 to the case $\alpha = 1$. In the interest of brevity we do not give the details of this extension.

(b) The boundary case $p = 1/\rho$ can be analysed along similar lines. In fact, our methods can be used to get increasingly sharper results, determining the value of $\limsup_{t\downarrow 0}(C'(t+\tau_s) - s)/f(t)$ for functions f containing powers of iterated logarithms, when stronger control over the densities of the marginals of X is available. Such refinements are possible when X is a stable process cf. §5.6. In particular, we may prove the following law of iterated logarithm given in (2.9) for a Cauchy process X with density $x \mapsto p_X(t,x)$ at time $t > 0$: for any $s \in \mathbb{R}$ and the function $f(t) = (\log\log\log(1/t))/\log(1/t)$, we have $\limsup_{t\downarrow 0}(C'(t+\tau_s) - s)/f(t) = 1/p_X(1,s)$ a.s. ◊

§5.2.3 Regime (IS): upper functions at time 0

Throughout this subsection we assume X has infinite variation, which is equivalent to $\liminf_{t\downarrow 0} C'(t) = -\infty$ a.s. by §4.1.1.2. The following theorem describes the upper fluctuations of $C'(t)$ as $t \downarrow 0$.

Theorem 5.9. *Let f be continuous and increasing with $\lim_{c\downarrow 0}\limsup_{t\downarrow 0} f(ct)/f(t) = 0$, $f(0) = 0$ and $f(t) \le 1 = f(1)$ for $t \in (0,1]$. Let $c > 0$, denote $F(t) \coloneqq t/f(t)$ for $t > 0$ and consider the conditions*

$$\int_0^1 \mathbb{P}(X_t \le -cF(t))\frac{\mathrm{d}t}{t} < \infty, \tag{5.7}$$

$$\int_0^1 \mathbb{E}[(X_t/F(t))^2 \mathbb{1}_{\{-2F(t)<X_t\le -t\}}]\frac{\mathrm{d}t}{t} < \infty, \tag{5.8}$$

$$2^n \int_0^{2^{-n}} \mathbb{P}(-t/f(2^{-n}) \ge X_t > -2F(t/2))\mathrm{d}t \to 0, \qquad \text{as } n \to \infty. \tag{5.9}$$

Then the following statements hold.

(i) *If* (5.7)–(5.9) *hold for $c = 1$ and f is concave, then* $\limsup_{t\downarrow 0} |C'(t)|f(t) = 0$ *a.s.*

(ii) If (5.7) *fails for all* $c > 0$*, then* $\limsup_{t\downarrow 0} |C'(t)|f(t) = \infty$ *a.s.*

(iii) If $\limsup_{t\downarrow 0} |C'(t)|f(t) < 1$ *a.s., then* (5.7) *holds for any* $c > 1$*.*

Some remarks are in order.

Remark 5.10.

(a) Any continuous regularly varying function f of index $r > 0$ satisfies the assumption in the theorem, see Remark 5.3(a) above.

(b) The proof of Theorem 5.9 is based on the analysis of the upper fluctuations of the vertex time $\tau_{-1/u}$ as $u \downarrow 0$. The interpretation and purpose of conditions (5.7)–(5.9) are analogous to those of conditions (5.2)–(5.4), respectively, see Remark 5.3(b) above.

(c) Note that (5.9) holds if $\int_0^1 \mathbb{P}(-2F(2^{-n}t/2) < X_{2^{-n}t} \le -tF(2^{-n}))\mathrm{d}t \to 0$ as $n \to \infty$, which, by the dominated convergence theorem, is the case if we have the limit $\mathbb{P}(-2F(u/2) < X_u \le -tF(u/t)) \to 0$ as $u \downarrow 0$ for a.e. $t \in (0, 1)$.

(d) The assumed concavity of f in part (ii) can be dropped by modifying assumption (5.8) into a condition involving the inverse of f (cf. Corollary 5.28 and Proposition 5.26). We do not make this explicit in the statement of Theorem 5.9 because the functions of interest in this context are typically concave. ♢

§5.2.3.1 Simple sufficient conditions for the assumptions of Theorem 5.9

The tail probabilities of X_t appearing in the assumptions of Theorem 5.9 are not analytically available in general. In this subsection we present sufficient conditions, in terms of the generating triplet (σ^2, γ, ν) of X, implying the assumptions in (5.7)–(5.9) of Theorem 5.9. Let f and F be as in Theorem 5.9 and note that $F(t) \in (0, 1]$ since f is concave with $f(1) = 1$. The inequalities in Lemma 5.36 (with $p = 2$, $\varepsilon = F(t) \in (0, 1]$ and $K = cF(t)$), applied to $\mathbb{P}(|X_t| \ge cF(t))$ and $\mathbb{E}[\min\{X_t^2, 4F(t)^2\}] \ge \mathbb{E}[X_t^2 \mathbb{1}_{\{|X_t|\le 2F(t)\}}]$, show that the condition

$$\int_0^1 \big[F(t)^{-2}\big((\gamma - \overline{\gamma}(F(t)))^2 t + \overline{\sigma}^2(F(t)) + \sigma^2\big) + \overline{\nu}(F(t))\big]\mathrm{d}t < \infty, \tag{5.10}$$

implies (5.7)–(5.8). Similarly, by Remark 5.10(c) and Lemma 5.36, the following condition implies (5.9):

$$\big[F(t)^{-2}\big((\gamma - \overline{\gamma}(F(t)))^2 t + \overline{\sigma}^2(F(t)) + \sigma^2\big) + \overline{\nu}(F(t))\big]t \to 0, \qquad \text{as } t \downarrow 0. \tag{5.11}$$

These simplifications lead to the following corollary.

Corollary 5.11. *Suppose* $\overline{\nu}(\varepsilon) + \varepsilon^{-2}(\overline{\sigma}^2(\varepsilon) + \sigma^2) + \varepsilon^{-1}|\gamma - \overline{\gamma}(\varepsilon)| = \mathcal{O}(\varepsilon^{-\beta})$ *as* $\varepsilon \downarrow 0$ *for some* $\beta \in [1, 2]$ *and, as before, let* $F(t) = t/f(t)$*. If we have* $F(t)^{-\beta}t \to 0$ *as* $t \to 0$ *and* $\int_0^1 F(t)^{-\beta}\mathrm{d}t < \infty$*, then* $\limsup_{t\downarrow 0} |C'(t)|f(t) = 0$ *a.s.*

Proof. By virtue of Theorem 5.9(i), it suffices to verify (5.10) and (5.11). By assumption, we have $[F(t)^{-2}(\overline{\sigma}^2(F(t))+\sigma^2)+\overline{\nu}(F(t))]t = \mathcal{O}(F(t)^{-\beta}t)$ and $F(t)^{-2}(\gamma-\overline{\gamma}(F(t)))^2t^2 = \mathcal{O}((F(t)^{-\beta}t)^2)$, which tend to 0 as $t \downarrow 0$, implying (5.11). Condition (5.10) follows similarly, completing the proof. □

Recall the definition of the Blumenthal–Getoor index $\beta_+ \in [0, 2]$ from (2.3). Note that, in our setting, X has infinite variation and hence $\beta_+ \geq 1$. Since $I_\beta < \infty$ for any $\beta > \beta_+$, Lemma 2.5 shows that β satisfies the assumptions of Corollary 5.11. Hence $\limsup_{t\downarrow 0} |C'(t)|t^p = 0$ a.s. for any $p > 1 - 1/\beta_+ \in [0, 1/2]$ by Corollary 5.11.

Stronger results are possible when stronger conditions are imposed on the law of X. For instance, for stable processes we have the following consequence of Theorem 5.9.

Corollary 5.12. *Let X be an α-stable process with $\alpha \in [1, 2)$. Then the following statements hold.*

(a) *If $t \mapsto t^{-1/\alpha}F(t)$ is bounded as $t \downarrow 0$, then $\limsup_{t\downarrow 0} |C'(t)|f(t) = \infty$ a.s.*

(b) *If $t^{-1/\alpha}F(t) \to \infty$ as $t \downarrow 0$ and X is not spectrally positive, then the limit $\limsup_{t\downarrow 0} |C'(t)|f(t)$ is equal to ∞ (resp. 0) a.s. if the integral $\int_0^1 F(t)^{-\alpha}\mathrm{d}t$ is infinite (resp. finite).*

Proof. The scaling property of X gives $\mathbb{P}(X_t \leq -cF(t)) = \mathbb{P}(X_1 \leq -ct^{-1/\alpha}F(t))$ for any $c, t > 0$. If $t \mapsto t^{-1/\alpha}F(t)$ is bounded, then $\liminf_{t\downarrow 0} \mathbb{P}(X_t \leq -cF(t)) > 0$ making (5.7) fail for all $c > 0$. In that case, we have $\limsup_{t\downarrow 0} |C'(t)|f(t) = \infty$ a.s. by Theorem 5.9(ii), proving part (a).

To prove part (b), suppose X is not spectrally positive and let $t^{-1/\alpha}F(t) \to \infty$ as $t \downarrow 0$. Then $x^\alpha \mathbb{P}(X_1 \leq -x)$ converges to a positive constant as $x \to \infty$, implying the following equivalence: $\int_0^1 t^{-1}\mathbb{P}(X_t \leq -ct^{-1/\alpha}F(t))\mathrm{d}t < \infty$ if and only if $\int_0^1 F(t)^{-\alpha}\mathrm{d}t < \infty$, where we note that the last integral does not depend of $c > 0$. If $\int_0^1 F(t)^{-\alpha}\mathrm{d}t < \infty$, then (5.10)–(5.11) hold and Theorem 5.9(i) gives $\limsup_{t\downarrow 0} |C'(t)|f(t) = 0$ a.s. If instead $\int_0^1 F(t)^{-\alpha}\mathrm{d}t = \infty$, then $\int_0^1 t^{-1}\mathbb{P}(X_t \leq -ct^{-1/\alpha}F(t))\mathrm{d}t = \infty$ for all $c > 0$, so Theorem 5.9(ii) implies $\limsup_{t\downarrow 0} |C'(t)|f(t) = \infty$ a.s., completing the proof. □

For a Cauchy process (i.e. $\alpha = 1$), Corollary 5.12 contains the dichotomy in Corollary 2.25 for the upper functions of C' at time 0. We note here that results analogous to Corollary 5.12 can be derived for a spectrally positive stable process X (and for Brownian motion), using the exponential (instead of polynomial) decay of the probability $\mathbb{P}(X_1 \leq x)$ in x as $x \to -\infty$. The exponential decay follows by Markov's inequality, since this implies that $\mathbb{P}(X_1 \leq x) = \mathbb{P}(e^{-cX_1} \geq e^{-cx}) \leq$

$\mathbb{E}[e^{-cX_1}]e^{cx}$ for any $c > 0$, where $\mathbb{E}[e^{-cX_1}] < \infty$ since X is spectrally positive. The exact asymptotic behaviour of $\mathbb{P}(X_1 \le x)$ can be found in [75, Thm 4.7.1], but is not necessary here, and is mentioned for the sake of completeness.

§5.2.4 Regime (IS): lower functions at time 0

As explained before, obtaining fine conditions for the lower fluctuations of C' is more delicate than in the case of upper fluctuations of C' at 0. The main reason is that the proof of Theorem 5.13 requires strong control on the Laplace exponent $\Phi_u(w)$ of τ_u, defined in (5.1), as $w \to \infty$ and $u \to -\infty$. This in turn requires sharp two-sided estimates on the negative tail probability $\mathbb{P}(X_t \le ut)$ as a function of (u, t) as $u \to -\infty$ and $t \downarrow 0$ jointly.

Due to the necessity of such strong control, in the following result we assume $X \in \mathcal{Z}_{\alpha,\rho}$ for some $\alpha > 1$. In other words, we assume there exist some normalising function g that is regularly varying at 0 with index $1/\alpha$ and an α-stable process $(Z_s)_{s\in[0,T]}$ with $\rho = \mathbb{P}(Z_1 > 0) \in (0, 1)$ such that $(X_{ut}/g(t))_{u\in[0,T]} \overset{d}{\to} (Z_u)_{u\in[0,T]}$ as $t \downarrow 0$. Recall that $G(t) = t/g(t)$ for $t > 0$.

Theorem 5.13. *Let $X \in \mathcal{Z}_{\alpha,\rho}$ for some $\alpha \in (1, 2]$ (and hence $\rho \in (0, 1)$). Let $f : (0, 1) \to (0, \infty)$ be given by $f(t) := G(t \log^p(1/t))$, for some $p \in \mathbb{R}$ and all $t \in (0, 1)$. Then the following statements hold:*

(i) if $p > 1/(1 - \rho)$, then $\liminf_{t\downarrow 0} |C'(t)|f(t) = \infty$ a.s.,

(ii) if $p < 1/(1 - \rho)$, then $\liminf_{t\downarrow 0} |C'(t)|f(t) = 0$ a.s.

Remark 5.14.

(a) The assumption $X \in \mathcal{Z}_{\alpha,\rho}$ for some $\alpha > 1$ implies that X is of infinite variation. Note that the function f in Theorem 5.13 is regularly varying at 0 with index $1 - 1/\alpha$. The 'negativity' parameter $1 - \rho = \lim_{t\downarrow 0} \mathbb{P}(X_t < 0) \in (0, 1)$ is a nontrivial function of the Lévy measure of X. The fact that $1 - \rho$ features as a boundary point in the power of the logarithmic term in Theorem 5.13 indicates that the lower fluctuations of C' at time 0 depends in a subtle way on the characteristics of X. Such dependence is, for instance, not present for the upper fluctuations of C' at time 0 when X is α-stable, see Corollary 5.12 above. Indeed, for an α-stable process X, Theorem 5.13 and Corollary 5.12(b) show that $\liminf_{t\downarrow 0} |C'(t)|f(t) = 0$ and $\limsup_{t\downarrow 0} |C'(t)|f(t) = \infty$ a.s. for $f(t) = t^{1-1/\alpha} \log^q(1/t)$ and any $q \in [-1/\alpha, (1 - 1/\alpha)/(1 - \rho))$, demonstrating the gap between the lower and upper fluctuations of C' at time 0.

(b) The case where X is attracted to Cauchy process with $\alpha = 1$ is expected to hold for the functions f in Theorem 5.13. As explained in Remark 5.8(a) above, many cases arise, with even some abrupt processes being attracted to Cauchy process (see Example 4.2). We again stress that, in this case, our methodology can be used to obtain a description of the upper fluctuations of C' at time 0 via Theorem 5.24 and two-sided estimates, analogous to Lemma 5.35, of the Laplace exponent Φ in (5.1) of the vertex time process. In the interest of brevity, we omit the details of such extensions.

(c) As with Theorem 5.7 above (see Remark 5.8(b)), the boundary case $p = 1/(1-\rho)$ in Theorem 5.13 can be analysed along similar lines. In fact, our methods can be used to get increasingly sharper results for the lower fluctuations of C' at time 0 when stronger control over the negative tail probabilities for the marginals X is available. Such improvements are possible, for instance, when X is α-stable. We decided to leave such results for future work in the interest of brevity. For completeness, however, we mention that the following law of iterated logarithm from Corollary 2.25 can also be proved using our methods (see Example 5.2 below): $\liminf_{t\downarrow 0} |C'(t)| f(t) = p_X(1,0)$ a.s., where $x \mapsto p_X(t,x)$ is the density of X_t. $\diamondsuit$

§5.2.5 Upper and lower function of the Lévy path at vertex times

In this section we establish consequences for the lower (resp. upper) fluctuations of the Lévy path at vertex time τ_s (resp. time 0) in terms of those of C'. Recall $X_{t-} = \lim_{u\uparrow t} X_u$ for $t > 0$ (and $X_{0-} = X_0$) and define $m_s := \min\{X_{\tau_s}, X_{\tau_s -}\}$ for $s \in \mathcal{L}^*$.

Lemma 5.15. *Suppose $s \in \mathcal{L}^*$. Let the function $f : [0,\infty) \to [0,\infty)$ be continuous and increasing and define the function $\tilde{f}(t) := \int_0^t f(u)\mathrm{d}u$, $t \geq 0$. Then the following statements hold for any $M > 0$.*

(i) If $\liminf_{t\downarrow 0}(C'(t+\tau_s) - s)/f(t) > M$ a .s. then $\liminf_{t\downarrow 0}(X_{t+\tau_s} - m_s - st)/\tilde{f}(t) \geq M$ a.s.

(ii) If $\limsup_{t\downarrow 0}(C'(t+\tau_s) - s)/f(t) < M$ a.s. then $\liminf_{t\downarrow 0}(X_{t+\tau_s} - m_s - st)/\tilde{f}(t) \leq M$ a.s.

The proof of Lemma 5.15 is pathwise. The lemma yields the following implications

(i) $\liminf_{t\downarrow 0}(C'(t+\tau_s) - s)/f(t) = \infty \implies \liminf_{t\downarrow 0}(X_{t+\tau_s} - m_s - st)/\tilde{f}(t) = \infty$,

(ii) $\limsup_{t\downarrow 0}(C'(t+\tau_s) - s)/f(t) = 0 \implies \liminf_{t\downarrow 0}(X_{t+\tau_s} - m_s - st)/\tilde{f}(t) = 0$.

The upper fluctuations of X at vertex time τ_s cannot be controlled via the fluctuations of C' since the process may have large excursions away from its convex minorant between contact points. Moreover, the limits $\liminf_{t\downarrow 0}(C'(t+\tau_s)-s)/f(t) = 0$ or

$\limsup_{t\downarrow 0}(C'(t+\tau_s)-s)/f(t)=\infty$, do not provide sufficient information to ascertain the value of the lower limit $\liminf_{t\downarrow 0}(X_{t+\tau_s}-m_s-st)/\tilde{f}(t)$, since this limit may not be attained along the contact points between the path and its convex minorant.

Theorems 5.2 and 5.7 give sufficient conditions, in terms of the law of X, for the assumptions in Lemma 5.15 to hold. This leads to the following corollaries.

Corollary 5.16. *Let $s\in\mathcal{L}^*$ and let f be a continuous and increasing function with $f(0)=0=\lim_{c\downarrow 0}\limsup_{t\downarrow 0}f(ct)/f(t)$, $f(1)=1$ and $f(t)\le 1$ for $t\in(0,1]$. If conditions (5.2)–(5.4) hold for $c=1$, then $\liminf_{t\downarrow 0}(X_{t+\tau_s}-m_s-st)/\tilde{f}(t)=\infty$ a.s. where we denote $\widetilde{f}(t):=\int_0^t f(u)\mathrm{d}u$.*

Denote by $\varpi(t):=t^{-1/\alpha}g(t)$ the slowly varying (at 0) component of the normalising function g of a process in the class $\mathcal{Z}_{\alpha,\rho}$. Recall that $G(t)=t/g(t)$ for $t>0$.

Corollary 5.17. *Let $X\in\mathcal{Z}_{\alpha,\rho}$ for some $\alpha\in(0,1)$ and $\rho\in(0,1]$. Given $p\in\mathbb{R}$, denote $\tilde{f}(t):=\int_0^t G(u\log^p(u^{-1}))^{-1}\mathrm{d}u$ for $t>0$. Then the following statements hold for $s=\gamma_0$.*

- *(i) If $p>1/\rho$, then $\liminf_{t\downarrow 0}(X_{t+\tau_s}-m_s-st)/\tilde{f}(t)=0$ a.s.*
- *(ii) If $\alpha\in(1/2,1)$, $p<-\alpha/(1-\alpha)$ and $(\varpi(c/t)/\varpi(1/t)-1)\log\log(1/t)\to 0$ as $t\downarrow 0$ for some $c\in(0,1)$, then $\liminf_{t\downarrow 0}(X_{t+\tau_s}-m_s-st)/\tilde{f}(t)=\infty$ a.s.*
- (iii) *If $\alpha\in(0,1/2]$, then $\liminf_{t\downarrow 0}(X_{t+\tau_s}-m_s-st)/t^q=\infty$ a.s. for any $q>1/\alpha\ge 2$.*

Remark 5.18.

- (a) The function $\tilde{f}$ is regularly varying at 0 with index $1/\alpha$. This makes conditions in Corollary 5.17 nearly optimal in the following sense: the polynomial rate in all three cases is either $1/\alpha$ (cases (i) and (ii) in Corollary 5.17) or arbitrarily close to it (case (iii) in Corollary 5.17). If $\alpha>1/2$, then the gap is in the power of the logarithm in the definition of $\tilde{f}$.
- (b) When the natural drift $\gamma_0=0$, Corollary 5.17 describes the lower fluctuations (at time 0) of the post-minimum process $X^{\rightarrow}=(X^{\rightarrow}_t)_{t\in[0,T-\tau_0]}$ given by $X^{\rightarrow}_t:=X_{t+\tau_0}-m_0$ (note that $m_0=\inf_{t\in[0,T]}X_t$). The closest result in this vein is [79, Prop. 3.6] where Vigon shows that, for any infinite variation Lévy process X and $r>0$, we have $\liminf_{t\downarrow 0}X^{\rightarrow}_t/t\ge r$ a.s. if and only if $\int_0^1\mathbb{P}(X_t/t\in[0,r])t^{-1}\mathrm{d}t<\infty$. Our result considers non-linear functions and a large class of finite variation processes.
- (c) By Theorem 2.11, the assumption $X\in\mathcal{Z}_{\alpha,\rho}$ and $\gamma_0=0$ implies that the post-minimum process, conditionally given τ_0, is a Lévy meander. Hence, Corol-

lary 5.17 also describes the lower functions of the meanders of Lévy processes in $\mathcal{Z}_{\alpha,\rho}$. A similar remark applies to the results in Corollary 5.16. $\diamondsuit$

When X has infinite variation, the process X and C touch each other infinitely often on any neighborhood of 0 (see Chapter 4), leading to the following connection in small time between the paths of X and its convex minorant C.

Lemma 5.19. *Let $f : [0,\infty) \to [0,\infty)$ be continuous and increasing with $f(0) = 0$ and finite $\tilde{f}(t) := \int_0^t f(u)^{-1}\mathrm{d}u$, $t \geq 0$. Then the following statements hold for any $M > 0$.*

(i) If $\limsup_{t\downarrow 0} |C'(t)|f(t) < M$ a.s., then $\limsup_{t\downarrow 0}(-X_t)/\tilde{f}(t) \leq M$ a.s.

(ii) If $\liminf_{t\downarrow 0} |C'(t)|f(t) > M$ a.s., then $\limsup_{t\downarrow 0}(-X_t)/\tilde{f}(t) \geq M$ a.s.

Theorem 5.9 and the corollaries thereafter give sufficient explicit conditions for the assumption in Lemma 5.19(i) to hold. Similarly, Theorem 5.13 gives a fine class of functions f satisfying the assumption in Lemma 5.19(ii) for a large class of processes. Such conclusions on the fluctuations of the Lévy path of X would not be new as the fluctuations of X at 0 are already known, see [29, 71, 72]. In particular, the upper functions of X and $-X$ at time 0 were completely characterised in [72] in terms of the generating triplet of X. Let us comment on some two-way implications of our results, the literature and Lemma 5.19.

Remark 5.20.

(a) By Lemma 6.7 ([47, Fundamental lemma]), the assumption in Theorem 5.9(ii) implies $\limsup_{t\downarrow 0} |X_t|/F(t) = \infty$ a.s. where we recall that $F(t) = t/f(t)$. Similarly, by Lemma 6.7, if $\limsup_{t\downarrow 0} |X_t|/F(t) = \infty$ a.s. then the assumption in Theorem 5.9(ii) must hold for either X or $-X$, which, by time reversal, implies that at least one of the limits $\limsup_{t\downarrow 0} |C'(t)|f(t)$ or $\limsup_{t\downarrow 0} |C'(T-t)|f(t)$ is infinite a.s. This conclusion is similar to that of Lemma 5.19, the main difference being the use of either $\tilde{f}$ or F. Note however, that if f is regularly varying with index different from 1, then Theorem A.55 implies $\lim_{t\downarrow 0} \tilde{f}(t)/F(t) \in (0,\infty)$.

(b) The contrapositive statements of Lemma 5.19 give information on C' in terms of $-X$. Indeed, if we have $\limsup_{t\downarrow 0}(-X_t)/\tilde{f}(t) > 0$, then $\limsup_{t\downarrow 0} |C'(t)|f(t) > 0$. Similarly, if $\limsup_{t\downarrow 0}(-X_t)/\tilde{f}(t) < \infty$, then $\liminf_{t\downarrow 0} |C'(t)|f(t) < \infty$. $\diamondsuit$

The connections between the fluctuations of X and those of C' at time 0 are intricate. Although the one-sided fluctuations of X at 0 were essentially characterised in Theorem A.40, its combination with Lemma 5.19 is not sufficiently strong to obtain conditions for any of the following statements: $\limsup_{t\downarrow 0} |C'(t)|f(t) = 0$, $\limsup_{t\downarrow 0} |C'(t)|f(t) > 0$, $\liminf_{t\downarrow 0} |C'(t)|f(t) < \infty$ or $\liminf_{t\downarrow 0} |C'(t)|f(t) = \infty$ a.s.

§5.3 Small-time fluctuations of non-decreasing additive processes

Consider a pure-jump right-continuous non-decreasing additive (i.e. with independent and possibly non-stationary increments) process $Y = (Y_t)_{t\geq 0}$ with $Y_0 = 0$ a.s. and its mean jump measure $\Pi(\mathrm{d}t, \mathrm{d}x)$ for $(t,x) \in [0,\infty) \times (0,\infty)$, see Theorem 2.3. Then, by Campbell's formula (Theorem A.48), its Laplace transform satisfies

$$\mathbb{E}\big[e^{-uY_t}\big] = e^{-\Psi_t(u)}, \quad \text{where} \quad \Psi_t(u) := \int_{(0,\infty)} (1-e^{-ux})\Pi((0,t],\mathrm{d}x), \quad \text{for } u \geq 0. \tag{5.12}$$

Let $L_t := \inf\{u > 0 : Y_u > t\}$ for $t \geq 0$ (with convention $\inf\emptyset = \infty$) denote the right-continuous inverse of Y. Our main objective in this section is to describe the upper and lower fluctuations of L, extending known results, such as the following theorem, for the case where Y has stationary increments (making Y a subordinator) in which case $\Pi(\mathrm{d}t, \mathrm{d}x) = \Pi((0,1],\mathrm{d}x)\mathrm{d}t$ for all $(t,x) \in [0,\infty)\times(0,\infty)$.

Theorem 5.21 ([20, Thm 4.1])**.** *Assume that Y is a subordinator. Then there exist a finite and positive constant c_Ψ, such that*

$$\limsup_{t\downarrow 0} \frac{L_t \Psi_1(t^{-1}\log\log\Psi_1(t^{-1}))}{\log\log\Psi_1(t^{-1})} = c_\Psi \qquad a.s.$$

§5.3.1 Upper functions of L

The following theorem is the main result of this subsection.

Theorem 5.22. *Let $f : (0,1) \to (0,\infty)$ be increasing with $\lim_{t\downarrow 0} f(t) = 0$ and $\phi : (0,\infty) \to (0,\infty)$ be decreasing with $\lim_{u\to\infty}\phi(u) = 0$. Let the positive sequence $(\theta_n)_{n\in\mathbb{N}}$ satisfy $\lim_{n\to\infty}\theta_n = \infty$ and define the associated sequence $(t_n)_{n\in\mathbb{N}}$ via $t_n := \phi(\theta_n)$ for $n \in \mathbb{N}$.*

(a) *If we have $\sum_{n=1}^\infty \exp(\theta_n t_n - \Psi_{f(t_n)}(\theta_n)) < \infty$ then the following inequality holds a.s. $\limsup_{t\downarrow 0} L_t/f(t) \leq \limsup_{n\to\infty} f(t_n)/f(t_{n+1})$.*

(b) *If we have $\lim_{u\to\infty}\phi(u)u = \infty$, $\sum_{n=1}^\infty[\exp(-\Psi_{f(t_n)}(\theta_n)) - \exp(-\theta_n t_n)] = \infty$ and $\sum_{n=1}^\infty \Psi_{f(t_{n+1})}(\theta_n) < \infty$, then we a.s. have $\limsup_{t\downarrow 0} L_t/f(t) \geq 1$.*

Remark 5.23.

(a) Theorem 5.22 plays a key role in the proofs of Theorems 5.7 and 5.13. Before applying Theorem 5.22, one needs to find appropriate choices of the free infinite-dimensional parameters h and $(\theta_n)_{n\in\mathbb{N}}$. This makes the application of Theorem 5.22 hard in general and is why, in Theorems 5.7 and 5.13, we are required to assume that X lies in the domain of attraction of an α-stable process.

(b) If Y has stationary increments (making Y a subordinator), the proof of Theorem 5.21 follows from Theorem 5.22 by using the specific form of f, and finding a sequences $(\theta_n)_{n\in\mathbb{N}}$ (the specific choices of $(\theta_n)_{n\in\mathbb{N}}$ are omitted here, but can be found in [20, Lem. 4.2 & 4.3]) satisfying the assumptions of Theorem 5.22. In this case, the function f is given in terms of the single-parameter Laplace exponent Ψ_1 as seen in Theorem 5.21. $\Diamond$

Proof of Theorem 5.22. (a) We have that $\{L_{t_n} > f(t_n)\} = \{t_n \geq Y_{f(t_n)}\}$ for $n \in \mathbb{N}$, since L is the right-inverse of Y. Using Chernoff's bound (Markov's inequality), we obtain

$$\mathbb{P}\big(t_n \geq Y_{f(t_n)}\big) \leq e^{\theta_n t_n}\mathbb{E}\big[\exp\big(-\theta_n Y_{f(t_n)}\big)\big] = \exp(\theta_n t_n - \Psi_{f(t_n)}(\theta_n)), \quad \text{for all } n \geq 1.$$

The assumption $\sum_{n=1}^{\infty}\exp(\theta_n t_n - \Psi_{f(t_n)}(\theta_n)) < \infty$ thus implies $\sum_{n=1}^{\infty}\mathbb{P}(L_{t_n} > f(t_n)) < \infty$. Hence, the Borel–Cantelli lemma yields $\limsup_{n\to\infty} L_{t_n}/f(t_n) \leq 1$ a.s. Since L is non-decreasing and $(t_n)_{n\in\mathbb{N}}$ is decreasing monotonically to zero, we can conclude (a), since it a.s. holds that

$$\begin{aligned}\limsup_{t\downarrow 0}\frac{L_t}{f(t)} &\leq \limsup_{n\to\infty}\sup_{t\in[t_{n+1},t_n]}\frac{L_{t_n}}{f(t)} \leq \limsup_{n\to\infty}\frac{L_{t_n}}{f(t_n)}\limsup_{n\to\infty}\frac{f(t_n)}{f(t_{n+1})}\\ &\leq \limsup_{n\to\infty}\frac{f(t_n)}{f(t_{n+1})}.\end{aligned}$$

(b) It suffices to establish the following limits: $\limsup_{n\to\infty} Y_{f(t_{n+1})}/t_n \leq \delta$ a.s. for any $\delta > 0$ and $\liminf_{n\to\infty}(Y_{f(t_n)} - Y_{f(t_{n+1})})/t_n \leq 1$ a.s. Indeed, by taking $\delta \downarrow 0$ along a countable sequence, the first limit gives $\limsup_{n\to\infty} Y_{f(t_{n+1})}/t_n = 0$ a.s. and hence $\liminf_{n\to\infty} Y_{f(t_n)}/t_n \leq 1$ a.s. For any $t > 0$ with $Y_{f(t)} \leq t$ we have $L_t > f(t)$. Since the former holds for arbitrarily small values of $t > 0$ a.s., we obtain $\limsup_{t\downarrow 0} L_t/f(t) \geq 1$ a.s.

We use the Borel–Cantelli lemmas to prove $\liminf_{n\to\infty}(Y_{f(t_n)} - Y_{f(t_{n+1})})/t_n \leq 1$ a.s. and $\limsup_{n\to\infty} Y_{f(t_{n+1})}/t_n \leq \delta$ a.s. for any $\delta > 0$. Applying Markov's inequality, we obtain the upper bound $\mathbb{P}(Y_t > s) \leq (1 - e^{-\theta s})^{-1}\mathbb{E}[1 - e^{-\theta Y_t}]$ for all $t, s, \theta > 0$, implying

$$\mathbb{P}\big(Y_{f(t_n)} \leq t_n\big) \geq \frac{\exp(-\Psi_{f(t_n)}(\theta_n)) - \exp(-\theta_n t_n)}{1 - \exp(-\theta_n t_n)}, \qquad \text{for all } n \geq 1.$$

Since $\theta_n t_n = \theta_n \phi(\theta_n) \to \infty$ as $n \to \infty$, the denominator of the lower bound in the display above tends to 1 as $n \to \infty$, and hence the assumption $\sum_{n=1}^{\infty}[\exp(-\Psi_{f(t_n)}(\theta_n)) - \exp(-\theta_n t_n)] = \infty$ implies $\sum_{n=1}^{\infty}\mathbb{P}(Y_{f(t_n)} < t_n) = \infty$. Since Y has non-negative independent increments and

$$\sum_{n=1}^{\infty}\mathbb{P}(Y_{f(t_n)} - Y_{f(t_{n+1})} < t_n) \geq \sum_{n=1}^{\infty}\mathbb{P}(Y_{f(t_n)} < t_n) = \infty,$$

the second Borel–Cantelli lemma yields $\liminf_{n\to\infty}(Y_{f(t_n)} - Y_{f(t_{n+1})})/t_n \le 1$ a.s.

To prove the second limit, use Markov's inequality and the elementary bound $1 - e^{-x} \le x$ to get

$$\begin{aligned}\mathbb{P}\big(Y_{f(t_{n+1})} > \delta t_n\big) &\le \frac{\mathbb{E}[1-\exp(-\theta_n Y_{f(t_{n+1})})]}{1-\exp(-\delta\theta_n t_n)}\\ &= \frac{1-\exp(-\Psi_{f(t_{n+1})}(\theta_n))}{1-\exp(-\delta\theta_n t_n)} \le \frac{\Psi_{f(t_{n+1})}(\theta_n)}{1-\exp(-\delta\theta_n t_n)},\end{aligned}$$

for all $n \in \mathbb{N}$. Again, the denominator tends to 1 as $n \to \infty$ and the assumption $\sum_{n=1}^{\infty} \Psi_{f(t_{n+1})}(\theta_n) < \infty$ implies $\sum_{n=1}^{\infty} \mathbb{P}(Y_{f(t_{n+1})} > \delta t_n) < \infty$. The Borel–Cantelli lemma implies $\limsup_{n\to\infty} Y_{f(t_{n+1})}/t_n \le \delta$ a.s. and completes the proof. □

§5.3.2 Lower functions of L

To describe the lower fluctuations of L, it suffices to describe the upper fluctuations of Y. The following result extends known results for subordinators (see, e.g. [36, Thm 1] i.e. Remark 5.30). Given a continuous increasing function h with $h(0) = 0$ and $h(1) = 1$, consider the following statements, used in the following result to describe the upper fluctuations of Y:

$$\limsup_{t\downarrow 0} Y_t/h(t) = 0, \quad \text{a.s.}, \tag{5.13}$$

$$\limsup_{t\downarrow 0} Y_t/h(t) < 1, \quad \text{a.s.}, \tag{5.14}$$

$$\Pi(\{(t,x) : t \in (0,1],\, x \ge h(t)\}) < \infty, \tag{5.15}$$

$$\int_{(0,1]\times(0,1)} \frac{x^2}{h(t)^2} \mathbb{1}_{\{2h(t)>x\}} \Pi(\mathrm{d}t, \mathrm{d}x) < \infty, \tag{5.16}$$

$$2^n \int_{(0,h^{-1}(2^{-n})]\times(0,2^{-n})} x \mathbb{1}_{\{2h(t)>x\}} \Pi(\mathrm{d}t, \mathrm{d}x) \downarrow 0, \quad \text{as } n\to\infty, \quad \text{and} \tag{5.17}$$

$$\int_{(0,1]\times(0,1)} \frac{x}{h(t)} \mathbb{1}_{\{2h(t)>x\}} \Pi(\mathrm{d}t, \mathrm{d}x) < \infty. \tag{5.18}$$

Theorem 5.24. *Let h be continuous and increasing with $h(0) = 0$ and $h(1) = 1$. Then the following implications hold:*

(a) (5.13) $\Longrightarrow$ (5.14) $\Longrightarrow$ (5.15),
(b) (5.15)–(5.17) $\Longrightarrow$ (5.13),
(c) (5.18) $\Longrightarrow$ (5.16)–(5.17).

Remark 5.25. If h is as in Theorem 5.24 and $\Pi(\{(t,x) : t \in (0,1],\, x \ge ch(t)\}) = \infty$ for all $c > 0$, then we have that $\limsup_{\downarrow 0} Y_t/h(t) = \infty$ a.s., which follows from the negation of Theorem 5.24(a). ◊

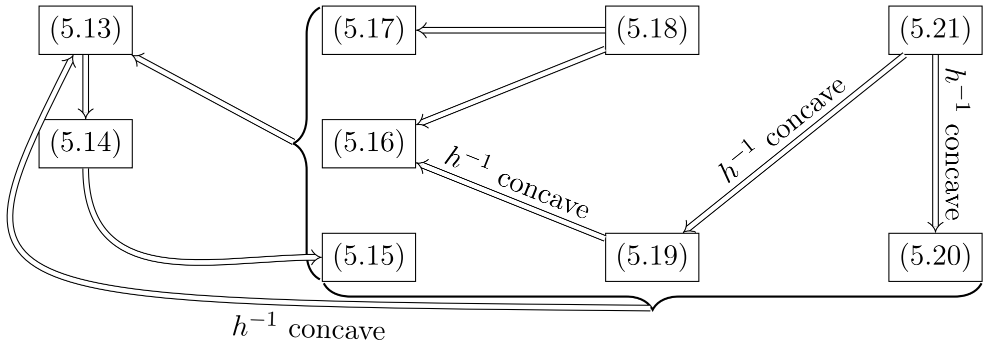

Figure 5.2: A graphical representation of the implications in Theorem 5.24 and Proposition 5.26.

In the description of the lower fluctuations of L, we are typically given the function h^{-1} directly instead of h. In those cases, the conditions in Theorem 5.24 may be hard to verify directly (see e.g. the proof of Theorem 5.9(i)). To alleviate this issue, we introduce alternative conditions describing the upper fluctuations of Y in terms of the function h^{-1}. However, this requires the additional assumption that h^{-1} is concave, see Proposition 5.26 below. Consider the following conditions on h^{-1}:

$$\int_{(0,1]\times(0,1)} \frac{h^{-1}(x)^2}{t^2} \mathbb{1}_{\{2t\ge h^{-1}(x)\}} \Pi(\mathrm{d}t,\mathrm{d}x) < \infty, \tag{5.19}$$

$$2^n \int_{(0,2^{-n}]\times(0,h(2^{-n}))} h^{-1}(x) \mathbb{1}_{\{2t\ge h^{-1}(x)\}} \Pi(\mathrm{d}t,\mathrm{d}x) \downarrow 0, \quad \text{as } n\to\infty, \quad \text{and} \tag{5.20}$$

$$\int_{(0,1]\times(0,1)} \frac{h^{-1}(x)}{t} \mathbb{1}_{\{2t\ge h^{-1}(x)\}} \Pi(\mathrm{d}t,\mathrm{d}x) < \infty. \tag{5.21}$$

Proposition 5.26. *Let h be convex and increasing with $h(0) = 0$ and $h(1) = 1$. Then the following implications hold:*

(a) (5.19) $\Longrightarrow$ (5.16),
(b) (5.21) $\Longrightarrow$ (5.19)–(5.20),
(c) (5.15) *&* (5.19)–(5.20) $\Longrightarrow$ (5.13).

The relation between the assumptions of Theorem 5.24 and Proposition 5.26 (concerning h and h^{-1}) is described in Figure 5.2. The following elementary result explains how the upper fluctuations of Y (described by Theorem 5.24) are related to the lower fluctuations of L.

Lemma 5.27. *Let h be a continuous increasing function with $h(0) = 0$ and denote by h^{-1} its inverse. Then the following implications hold for any $c > 0$:*

(a) $\liminf_{t\downarrow 0} L_t/h^{-1}(t/c) > 1 \implies \limsup_{t\downarrow 0} Y_t/h(t) \le c$,
(b) $\limsup_{t\downarrow 0} Y_t/h(t) < c \implies \liminf_{t\downarrow 0} L_t/h^{-1}(t/c) \ge 1$.

Proof. The result follows from the implications $L_u > t \implies u \ge Y_t \implies L_u \ge t$ for any $t, u > 0$. Indeed, if $\liminf_{u\downarrow 0} L_u/h^{-1}(u/c) > 1$ then $L_u > h^{-1}(u/c)$ for all sufficiently small $u > 0$ implying that $Y_t \le ch(t)$ for all sufficiently small $t > 0$ and hence $\limsup_{t\downarrow 0} Y_t/h(t) \le c$. This establishes part (a). Part (b) follows along similar lines. □

A combination of Lemma 5.27, Theorem 5.24, Proposition 5.26 and Remark 5.25 yield the following corollary.

Corollary 5.28. *Let h be a continuous and increasing function with $h(0) = 0$ and $h(1) = 1$ such that $\lim_{c\downarrow 0} \limsup_{t\downarrow 0} h^{-1}(ct)/h^{-1}(t) = 0$. Then the following results hold:*

(i) If $\liminf_{t\downarrow 0} L_t/h^{-1}(t/c) > 1$ a.s. for some $c \in (0, 1)$ then (5.15) *holds.*
(ii) Suppose (5.15)–(5.17) *hold, then $\liminf_{t\downarrow 0} L_t/h^{-1}(t) = \infty$ a.s.*
(ii') Suppose h is convex and conditions (5.15) *and* (5.19)–(5.20) *hold, then we a.s. have $\liminf_{t\downarrow 0} L_t/h^{-1}(t) = \infty$.*
(iii) If $\Pi(\{(t, x) : t \in (0, 1], x \ge ch(t)\}) = \infty$ for all $c > 0$ then we a.s. have $\liminf_{t\downarrow 0} L_t/h^{-1}(t) = 0$.

To prove Theorem 5.24 we require the following lemma. For all $t \ge 0$ denote by $\Delta_t := Y_t - Y_{t-}$ the jump of Y at time t, so that $Y_t = \sum_{u\le t} \Delta_u$ since Y is a pure-jump additive process. We also let N denote the Poisson jump measure of Y, given by $N(A) := |\{t : (t, \Delta_t) \in A\}|$ for $A \subset [0, \infty) \times (0, \infty)$ and note that its mean measure is $\Pi(\mathrm{d}t, \mathrm{d}x)$.

Lemma 5.29. *Let h be continuous and increasing with $h(0) = 0$ and $h(1) = 1$. Assume* (5.15)–(5.17) *hold, then $\limsup_{t\downarrow 0} Y_t/h(t) = \limsup_{t\downarrow 0} Y_{h^{-1}(t)}/t = 0$ a.s.*

Proof. For all $n \in \mathbb{N}$, we let $B_n := [2^{-n}, \infty)$ and set $C_n := h^{-1}((2^{-n-1}, 2^{-n}]) \times B_n$. Then we have

$$\sum_{n\in\mathbb{N}} \mathbb{P}(N(C_n) \ge 1) = \sum_{n\in\mathbb{N}} \left(1 - e^{-\Pi(C_n)}\right) \le \sum_{n\in\mathbb{N}} \Pi(C_n),$$

by the definition of N and the inequality $1 - e^{-x} \le x$. Note that $\sum_{n\in\mathbb{N}} \Pi(C_n) < \infty$ by (5.15), since

$$\sum_{n\in\mathbb{N}} \Pi(C_n) \le \Pi(\{(t, x) : t \in [0, 1], x \ge h(t)\}) < \infty.$$

By the Borel–Cantelli lemma, there exists some $n_0 \in \mathbb{N}$ with $N(h^{-1}((2^{-n-1}, 2^{-n}]) \times B_n) = 0$ a.s. for all $n \ge n_0$. By the mapping theorem, the random measure

$N_h(A\times B) := N(h^{-1}(A)\times B)$ for any measurable $A, B \subset [0,\infty)$, is a Poisson random measure with mean measure $\Pi_h(A \times B) := \Pi(h^{-1}(A), B)$. Note that $Y_{h^{-1}(t)} = \int_{(0,h^{-1}(t)]\times(0,\infty)} xN(\mathrm{d}u, \mathrm{d}x) = \int_{(0,t]\times(0,\infty)} xN_h(\mathrm{d}u, \mathrm{d}x)$ for $t \geq 0$ and, for any $n \geq n_0$ and $t \in (2^{-n-1}, 2^{-n}]$, we have $|Y_{h^{-1}(t)}/t| \leq \zeta_n := 2^{n+1}\sum_{m=n}^{\infty}\xi_m$, where

$$\xi_m := \int_{(2^{-m-1},2^{-m}]\times(0,2^{-m})} xN_h(\mathrm{d}u, \mathrm{d}x), \qquad m \in \mathbb{N}.$$

To complete the proof, it suffices to show that $\zeta_n \downarrow 0$ a.s. as $n \to \infty$. Fubini's theorem yields

$$\begin{aligned}
2^{-n-1}\mathbb{E}[\zeta_n] &= \sum_{m=n}^{\infty} \int_{(2^{-m-1},2^{-m}]\times(0,2^{-m})} x\Pi_h(\mathrm{d}u, \mathrm{d}x) \\
&= \int_{(0,2^{-n}]\times(0,2^{-n})} x \sum_{m=n}^{\infty} \mathbb{1}_{\{x<2^{-m}\}}\mathbb{1}_{\{u\leq 2^{-m}<2u\}}\Pi_h(\mathrm{d}u, \mathrm{d}x) \\
&\leq \int_{(0,2^{-n}]\times(0,2^{-n})} x\mathbb{1}_{\{2u>x\}}\Pi_h(\mathrm{d}u, \mathrm{d}x) \\
&= \int_{(0,h^{-1}(2^{-n})]\times(0,2^{-n})} x\mathbb{1}_{\{2h(v)>x\}}\Pi(\mathrm{d}v, \mathrm{d}x).
\end{aligned}$$

By assumption (5.17), we deduce that $\mathbb{E}[\zeta_n] \downarrow 0$ as $n \to \infty$. Similarly, note that

$$\mathrm{Var}(\zeta_n) = 4^{n+1}\sum_{m=n}^{\infty}\int_{(2^{-m-1},2^{-m}]\times(0,2^{-m})} x^2\Pi_h(\mathrm{d}u, \mathrm{d}x),$$

and hence, by Fubini's theorem and assumption (5.16), we have

$$\begin{aligned}
\sum_{n=1}^{\infty}\mathrm{Var}(\zeta_n) &= \sum_{m=1}^{\infty}\sum_{n=1}^{m} 4^{n+1}\int_{(2^{-m-1},2^{-m}]\times(0,2^{-m})} x^2\Pi_h(\mathrm{d}t, \mathrm{d}x) \\
&\leq \sum_{m=1}^{\infty} 4^{m+2}\int_{(2^{-m-1},2^{-m}]\times(0,2^{-m})} x^2\Pi_h(\mathrm{d}u, \mathrm{d}x) \\
&= \int_{(0,1]\times(0,1)} x^2 \sum_{m=1}^{\infty} 4^{m+2}\mathbb{1}_{\{x<2^{-m}\}}\mathbb{1}_{\{u<2^{-m}<2u\}}\Pi_h(\mathrm{d}u, \mathrm{d}x) \\
&\leq 4^2\int_{(0,1]\times(0,1)} \frac{x^2}{u^2}\mathbb{1}_{\{2u>x\}}\Pi_h(\mathrm{d}u, \mathrm{d}x) \\
&= 4^2\int_{(0,h^{-1}(1)]\times(0,1)} \frac{x^2}{h(v)^2}\mathbb{1}_{\{2h(v)>x\}}\Pi(\mathrm{d}v, \mathrm{d}x) < \infty.
\end{aligned}$$

Thus, the sum $\sum_{n=1}^{\infty}(\zeta_n - \mathbb{E}[\zeta_n])^2$ has finite mean equal to $\sum_{n=1}^{\infty}\mathrm{Var}(\zeta_n) < \infty$ and is thus finite a.s. Hence, the summands must tend to 0 a.s. and, since $\mathbb{E}[\zeta_n] \to 0$, we deduce that $\zeta_n \downarrow 0$ a.s. as $n \to \infty$. □

Proof of Theorem 5.24. It is obvious that (5.13) implies (5.14). If (5.14) holds, then $Y_t < h(t)$ for all sufficiently small t. Thus, the path bound $Y_t \geq \Delta_t$ implies

$\mathbb{P}(N(\{(t,x) : t \in [0,1],\, x > h(t)\}) < \infty) = 1$ and hence (5.15). By Lemma 5.29, conditions (5.15)–(5.17) imply (5.13), so it remains to show that (5.18) implies (5.16) and (5.17).

It is easy to see that (5.18) implies (5.16). Moreover, if (5.18) holds, then

$$2^n \int_{(0,h^{-1}(2^{-n})]\times(0,2^{-n})} x \mathbb{1}_{\{2h(t)>x\}} \Pi(\mathrm{d}t, \mathrm{d}x) \\ \le \int_{(0,h^{-1}(1)]\times(0,1)} \frac{x}{h(t)} \mathbb{1}_{\{2h(t)>x\}} \mathbb{1}_{(0,h^{-1}(2^{-n})]\times(0,2^{-n})}(t,x) \Pi(\mathrm{d}t, \mathrm{d}x),$$

where the upper bound is finite for all $n \in \mathbb{N}$ and tends to 0 as $n \to \infty$ by the monotone convergence theorem, implying (5.17). □

Proof of Proposition 5.26. Since h^{-1} is concave with $h^{-1}(0) = 0$, then $x \mapsto h^{-1}(x)/x$ is decreasing, so the condition $h(t) > x/2$ implies $(x/2)/h(t) \le h^{-1}(x/2)/t \le h^{-1}(x)/t$. The inequality $h^{-1}(x)/x \le h^{-1}(x/2)/(x/2)$ gives $\{(t,x) : 2h(t) > x\} \subset \{(t,x) : 2t > h^{-1}(x)\}$, proving the first claim: (5.19) implies (5.16).

Since h^{-1} is concave with $h^{-1}(0) = 0$, it is subadditive, implying

$$\zeta_t \coloneqq \sum_{u \le t} h^{-1}(\Delta_u) \ge h^{-1}(Y_t).$$

Since $\limsup_{t\downarrow 0} \zeta_t/t \le c$ implies $\limsup_{t\downarrow 0} Y_t/h(ct) \le 1$ for $c > 0$ and h is a convex function, it suffices to show that $\limsup_{t\downarrow 0} \zeta_t/t = 0$ a.s. Note that ζ is an additive process with jump measure $\Pi(\mathrm{d}t, h(\mathrm{d}x))$. Applying Theorem 5.24 to ζ with the identity function yields the result, completing the proof. □

Remark 5.30. We now show that, when the increments of Y are stationary (making Y a subordinator), Theorem 5.24 gives a complete characterisation of the upper functions of Y, recovering [36, Thm 1] (see also [20, Prop. 4.4]). This is done in two steps.

Suppose h is convex and Y has stationary increments with mean jump measure $\Pi(\mathrm{d}t, \mathrm{d}x) = \Pi((0,1], \mathrm{d}x)\mathrm{d}t$. Then h^{-1} is concave and the non-decreasing additive process $\widetilde{Y}_t \coloneqq \sum_{s\le t} h^{-1}(\Delta_s) \ge h^{-1}(Y_t)$ has mean jump measure $\Pi(\mathrm{d}t, h(\mathrm{d}x))$, making it a subordinator. Theorem 5.24 applied to $\widetilde{Y}$ and the identity function makes all conditions (5.15)–(5.17) equivalent to $\int_{(0,1)} h^{-1}(x)\Pi((0,1], \mathrm{d}x) < \infty$ and therefore, by Theorem 5.24, also equivalent to the condition $\limsup_{t\downarrow 0} \widetilde{Y}_t/t = 0$ a.s.

Note that condition (5.15) for $\widetilde{Y}$ and the identity function coincides with condition (5.15) for Y and h. This equivalence, together with the fact that the limit $\limsup_{t\downarrow 0} \widetilde{Y}_t/t = 0$ implies $\limsup_{t\downarrow 0} Y_t/h(t) = 0$, shows that both limits are either 0 a.s. or positive a.s. jointly. Thus, $\limsup_{t\downarrow 0} Y_t/h(t) = 0$ a.s. if and only if $\int_{(0,1)} h^{-1}(x)\Pi((0,1], \mathrm{d}x) < \infty$ and, if the latter condition fails, then Remark 5.25

implies that $\limsup_{t\downarrow 0} Y_t/h(t) = \infty$ a.s. This is precisely the criterion given in [36, Thm 1] (see also [20, Prop. 4.4]). $\diamondsuit$

Remark 5.30 shows that condition (5.15) perfectly describes the upper fluctuations of Y when Y has stationary increments, making conditions (5.16) & (5.17) appear superfluous. These conditions are, however, not superfluous since (5.15) by itself cannot fully characterise the upper fluctuations of Y, as the following example shows.

Example 5.1. Let $\Pi(\mathrm{d}t, \mathrm{d}x) = \sum_{n\in\mathbb{N}} n^{-1}2^n \delta_{(2^{-n},2^{-n}/n)}(\mathrm{d}t, \mathrm{d}x)$, where δ_x denotes the Dirac measure at x, and consider the corresponding additive process Y (whose existence is ensured by Theorem 2.3). Equation [59, Eq. (6)] says that $\mathbb{P}(\xi \geq \mu) \geq 1/5$ for every Poisson random variable ξ with mean $\mu \geq 2$, implying that $\sum_{n\in\mathbb{N}} \mathbb{P}(N(\{(2^{-n}, 2^{-n}/n)\}) \geq 2^n/n) = \infty$. The second Borel–Cantelli lemma then shows that $\Delta_{2^{-n}} \geq 1/n^2$ i.o. Thus, $Y_{2^{-n}}/2^{-n} \geq 2^n\Delta_{2^{-n}} \geq 2^n/n^2$ i.o., implying $\limsup_{t\downarrow 0} Y_t/t = \infty$ a.s. even when condition (5.15) holds. In fact, $\Pi(\{(t, x) : t \in (0, 1], x \geq ct\}) < \infty$ for all $c > 0$. $\triangle$

§5.4 The vertex time process and the proofs of the results in §5.2

We first recall basic facts about the vertex time process $\tau = (\tau_s)_{s\in\mathbb{R}}$. Fix a deterministic time horizon $T > 0$, let C be the convex minorant of X on $[0, T]$ with right-derivative C' and recall the definition $\tau_s = \inf\{t > 0 : C'(t) > s\}$ for any slope $s \in \mathbb{R}$. By the convexity of C, the right-derivative C' is non-decreasing and right-continuous, making τ a non-decreasing right-continuous process with $\lim_{s\to-\infty} \tau_s = 0$ and $\lim_{s\to\infty} \tau_s = T$. Intuitively put, the process τ finds the times in $[0, T]$ at which the slopes increase as we advance through the graph of the convex minorant $t \mapsto C(t)$ chronologically. We remark that the vertex time process can be constructed directly from X without any reference to the convex minorant C, as follows (cf. [57, Thm 11.1.2]): for each slope $s \in \mathbb{R}$ and time epoch $t \geq 0$, define $X^{(s)}_t := X_t - st$, $\underline{X}^{(s)}_t := \inf_{u\in[0,t]} X^{(s)}_u$ and note $\tau_s = \sup\big\{t \in [0, T] : X^{(s)}_{t-} \wedge X^{(s)}_t = \underline{X}^{(s)}_T\big\}$, where $X^{(s)}_{u-} := \lim_{v\uparrow u} X^{(s)}_{u-}$ for $u > 0$ and $X^{(s)}_{0-} := X^{(s)}_0 = 0$. Put differently, subtracting a constant drift s from the Lévy process X "rotates" the convex hull so that the vertex time τ_s becomes the time the minimum of $X^{(s)}$ during the time interval $[0, T]$ is attained.

§5.4.1 The vertex time process over exponential times

Fix any $\lambda > 0$ and let E be an independent exponential random variable with unit mean. Let $\widehat{C} := (\widehat{C}(t))_{t\in[0,E/\lambda]}$ be the convex minorant of X over the exponential time-horizon $[0, E/\lambda]$ and denote by $\widehat{\tau}$ the right-continuous inverse of $\widehat{C}'$, i.e. $\widehat{\tau}_s := \inf\{u \in [0, E/\lambda] : \widehat{C}'(u) > s\}$ for $s \in \mathbb{R}$. Hence, in the remainder of the chapter, the processes with (resp. without) a 'hat' will refer to the processes whose definition is based on the path of X on $[0, E/\lambda]$ (resp. $[0, T]$), where E is an exponential random variable with unit mean independent of X and $T > 0$ is fixed and deterministic.

It is more convenient to consider the vertex time processes over an independent exponential time horizon rather than the fixed time horizon T, as this does not affect the small-time behaviour of the process (see Corollary 5.32 below), while making its law more tractable. Moreover, as we will see, to analyse the fluctuations of $\widehat{C}'$ over short intervals, it suffices to study those of $\widehat{\tau}$. By Theorem 2.23, the process $\widehat{\tau}$ has independent but non-stationary increments and its Laplace exponent is given by

$$\mathbb{E}[e^{-u\widehat{\tau}_s}] = e^{-\Phi_s(u)}, \quad \text{where} \quad \Phi_s(u) := \int_0^\infty (1 - e^{-ut})e^{-\lambda t}\mathbb{P}(X_t \le st)\frac{\mathrm{d}t}{t}, \quad (5.22)$$

for all $u \ge 0$ and $s \in \mathbb{R}$. The following lemma states that, after a vertex time, the convex minorants C and $\widehat{C}$ must agree for a positive amount of time, see Figure 5.3 for a pictorial description.

Lemma 5.31. *For any $s \in \mathcal{L}^*$, on the event $\{\tau_s < E/\lambda \le T\}$, we have $\tau_s = \widehat{\tau}_s$ and the convex minorants C and $\widehat{C}$ agree on and interval $[0, \tau_s + m]$ for a random $m > 0$. If X is of infinite variation, the functions C and $\widehat{C}$ agree on an interval $[0, m]$ for a random variable m satsifying $0 < m \le \min\{T, E/\lambda\}$ a.s.*

Since the Lévy process X and the exponential time E are independent, we have $\mathbb{P}(\tau_s < E/\lambda \le T) > 0$.

Proof. The proof follows directly from the definition of the convex minorant of f as the greatest convex function dominated by the path of f over the corresponding interval. Let f be a measurable function on $[0, t]$ with piecewise linear convex minorant $M^{(t)}$. Then, for any vertex time $v \in (0, t)$ of $M^{(t)}$ and any $u \in (v, t]$, the convex minorant $M^{(u)}$ of f on $[0, u]$ equals $M^{(t)}$ over the interval $[0, v]$. The result then follows since the condition $s \in \mathcal{L}^*$ (resp. X has infinite variation) implies that there are infinitely many vertex times immediately after τ_s (resp. 0). □

The following result shows that local properties of C agree with those of $\widehat{C}$. Multiple extensions are possible, but we opt for the following version as it is simple and sufficient for our purpose.

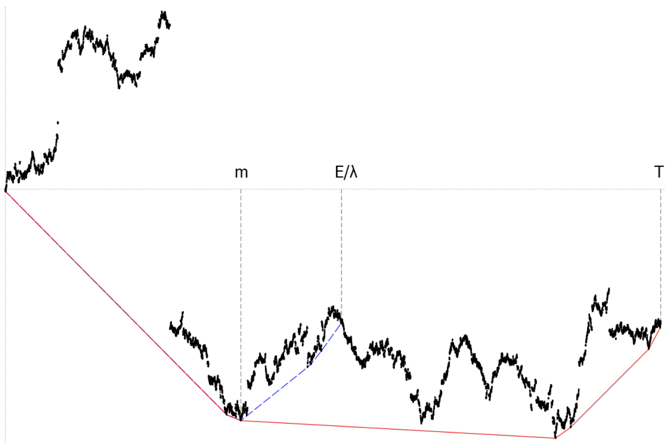

Figure 5.3: The picture shows a path of X (black) and its convex minorants C (red) on $[0, T]$ and $\widehat{C}$ (blue) on $[0, E/\lambda]$. Both convex minorants agree until time m, after which they may behave very differently.

Corollary 5.32. *Fix any measurable function* $f : (0, \infty) \to (0, \infty)$*.*
(a) *If* $s \in \mathcal{L}^*$*, then the following limits are a.s. constants on* $[0, \infty]$*:*

$$\limsup_{t\downarrow 0} \frac{C'(t+\tau_s)-s}{f(t)} = \limsup_{t\downarrow 0} \frac{\widehat{C}'(t+\widehat{\tau}_s)-s}{f(t)} \qquad \textit{and}$$

$$\liminf_{t\downarrow 0} \frac{C'(t+\tau_s)-s}{f(t)} = \liminf_{t\downarrow 0} \frac{\widehat{C}'(t+\widehat{\tau}_s)-s}{f(t)}.$$

(b) *If* X *is of infinite variation, then the following limits are a.s. constants on* $[0, \infty]$*:*

$$\limsup_{t\downarrow 0} C'(t)/f(t) = \limsup_{t\downarrow 0} \widehat{C}'(t)/f(t) \quad \textit{and} \quad \liminf_{t\downarrow 0} C'(t)/f(t) = \liminf_{t\downarrow 0} \widehat{C}'(t)/f(t).$$

Proof. We will prove part (a) for lim inf, with the remaining proofs being analogous. First note that the assumption $s \in \mathcal{L}^*$ implies that $(\tau_{u+s} - \tau_s)_{u\geq 0}$ and the additive processes $(\widehat{\tau}_{u+s} - \widehat{\tau}_s)_{u\geq 0}$ have infinite activity as $u \downarrow 0$ a.s. Thus, applying Blumenthal's 0–1 law (see Corollary A.1) to $(\widehat{\tau}_{u+s} - \widehat{\tau}_s)_{u\geq 0}$ (and using the fact that $\widehat{C}'(\widehat{\tau}_s) = s$ a.s.), implies that $\liminf_{t\downarrow 0}(\widehat{C}'(t + \widehat{\tau}_s) - s)/f(t)$ is a.s. equal to some constant μ in $[0, \infty]$. Moreover, by the independence of the increments of $\widehat{\tau}_s$, this limit holds even when conditioning on the value of $\widehat{\tau}_s$. Recall further that $\widehat{\tau}_s = \tau_s$ on the event $\{\tau_s < E/\lambda \leq T\}$ by Lemma 5.31. By Lemma 5.31 and the independence of E and X, we a.s. have

$$0<\mathbb{P}\bigg(\tau_s<\frac{E}{\lambda}\le T\,\bigg|\,\tau_s\bigg)=\mathbb{P}\bigg(\liminf_{t\downarrow 0}\frac{\widehat{C}'(t+\widehat{\tau}_s)-s}{f(t)}=\mu,\,\tau_s<\frac{E}{\lambda}\le T\,\bigg|\,\tau_s\bigg)$$
$$=\mathbb{P}\bigg(\liminf_{t\downarrow 0}\frac{C'(t+\tau_s)-s}{f(t)}=\mu,\,\tau_s<\frac{E}{\lambda}\le T\,\bigg|\,\tau_s\bigg)$$
$$=\mathbb{P}\bigg(\liminf_{t\downarrow 0}\frac{C'(t+\tau_s)-s}{f(t)}=\mu\,\bigg|\,\tau_s\bigg)\mathbb{P}\bigg(\tau_s<\frac{E}{\lambda}\le T\,\bigg|\,\tau_s\bigg),$$

implying that $\liminf_{t\downarrow 0}(C'(t+\tau_s)-s)/f(t)=\mu$ a.s. □

By virtue of Corollary 5.32 it suffices to prove all the results in §5.2 for $\widehat{C}$ instead of C. This allows us to use the independent increment structure of the right inverse $\widehat{\tau}$ of the right-derivative $\widehat{C}'$.

Example 5.2 (Cauchy process). If X is a Cauchy process, then the Laplace exponent of $\widehat{\tau}_u$ factorises $\Phi_u(w)=\mathbb{P}(X_1\le u)\int_0^\infty(1-e^{-wt})e^{-\lambda t}t^{-1}\mathrm{d}t$ for any $u\in\mathbb{R}$ and $w\ge 0$. This implies that $\widehat{\tau}$ has the same law as a gamma subordinator time-changed by the distribution function $u\mapsto\mathbb{P}(X_1\le u)=\frac{1}{2}+\frac{1}{\pi}\arctan(cu+\mu)$ for some $c>0$ and $\mu=\tan(\pi(\frac{1}{2}-\rho))$. This result can be used as an alternative to Theorem 2.24, in conjunction with classical results on the fluctuations of a gamma process (see, e.g. [20, Ch. 4]), to establish Corollary 2.25 and all the other results in [21]. △

The proofs of the results in §5.2 are based on the results of §5.3: we will construct a non-decreasing additive process $Y=(Y_t)_{t\ge 0}$, started at 0, in terms of $\widehat{\tau}$ and apply the results in §5.3 to Y and its inverse $L=(L_u)_{u\ge 0}$. These proofs are given in the following subsections.

§5.4.2 Upper and lower functions at time τ_s - proofs

Let $s\in\mathcal{L}^*$. Fix any $\lambda>0$ and let $Y_u:=\widehat{\tau}_{u+s}-\widehat{\tau}_s$, $u\ge 0$. Then the right-inverse $L_u:=\inf\{t>0: Y_t>u\}$ of Y equals $L_u=\widehat{C}'(u+\widehat{\tau}_s)-s$ for $u\ge 0$. Note that Y has independent increments and (5.22) implies

$$\Psi_u(w):=-\log\mathbb{E}[e^{-wY_u}]=\int_0^\infty(1-e^{-wt})\Pi((0,u],\mathrm{d}t),\quad\text{for all } w,u\ge 0,\qquad(5.23)$$

where $\Pi(\mathrm{d}u,\mathrm{d}t)=e^{-\lambda t}\mathbb{P}((X_t-st)/t\in\mathrm{d}u)t^{-1}\mathrm{d}t$ is the mean jump measure of Y.

Proof of Theorem 5.2. Since $s\in\mathcal{L}^*$, all three parts of the result follow from a direct application of Proposition 5.26 and Corollary 5.28 to the definitions of Y and L above. □

To prove Theorem 5.7, we require the following two lemmas. The first lemma establishes some general regularity for the densities of X_t as a function of t and the second lemma provides a strong asymptotic control on the function $\Psi_s(u)$ as $s \downarrow 0$ and $u \to \infty$. Recall that, when X is of finite variation, $\gamma_0 = \lim_{t\downarrow 0} X_t/t$ denotes the natural drift of X.

Lemma 5.33. *Let $X \in \mathcal{Z}_{\alpha,\rho}$ for some $\alpha \in (0,1)$ and $\rho \in (0,1]$ and denote by g its normalising function.*
(a) *Define $Q_t := (X_t - \gamma_0 t)/g(t)$, then Q_t has an infinitely differentiable density p_t such that p_t and each of its derivatives $p_t^{(k)}$ are uniformly bounded: for any $k \in \mathbb{N}\cup\{0\}$ it holds that $\sup_{t\in(0,1]}\sup_{x\in\mathbb{R}} |p_t^{(k)}(x)| < \infty$.*
(b) *Define $\widetilde{Q}_t := X_t/\sqrt{t}$, then $\widetilde{Q}_t$ has an infinitely differentiable density $\tilde{p}_t$ such that $\tilde{p}_t$ and each of its derivatives $\tilde{p}_t^{(k)}$ are uniformly bounded: for any $k \in \mathbb{N}\cup\{0\}$ it holds that $\sup_{t\in[1,\infty)}\sup_{x\in\mathbb{R}} |\tilde{p}_t^{(k)}(x)| < \infty$.*

Proof. Part (a). We assume without loss of generality that $g(t) \le 1$ for $t \in (0,1]$, and note that Q_t is infinitely divisible. Denote by ν_{Q_t} the Lévy measure of Q_t, and note for $A \subset \mathbb{R}$ that $\nu_{Q_t}(A) = t\nu(g(t)A)$ and

$$\overline{\sigma}^2_{Q_t}(u) := \int_{(-u,u)} x^2 \nu_{Q_t}(\mathrm{d}x) = \frac{t}{g(t)^2}\int_{(-ug(t),ug(t))} x^2\nu(\mathrm{d}x) = \frac{t}{g(t)^2}\overline{\sigma}^2(ug(t)),$$

for $t \in (0,1]$ and $u \in \mathbb{R}\setminus\{0\}$. The regular variation of $\overline{\nu}$ (see Theorem 2.12), Fubini's theorem and Theorem A.55(ii) imply that, as $u \downarrow 0$,

$$\begin{aligned}\overline{\sigma}^2(u) &= -\int_0^u x^2\overline{\nu}(\mathrm{d}x) = -\int_0^u 2\int_0^x z\mathrm{d}z\overline{\nu}(\mathrm{d}x) = -\int_0^u\int_z^u 2z\overline{\nu}(\mathrm{d}x)\mathrm{d}z \\ &= \int_0^u 2z(\overline{\nu}(z) - \overline{\nu}(u))\mathrm{d}z = \int_0^u 2z\overline{\nu}(z)\mathrm{d}z - u^2\overline{\nu}(u) \sim \frac{\alpha}{2-\alpha}u^2\overline{\nu}(u).\end{aligned}$$

Since $X \in \mathcal{Z}_{\alpha,\rho}$, Theorem 2.12 implies that $g^{-1}(u)u^{-2}\overline{\sigma}^2(u) \to c_0$ for some $c_0 > 0$ as $u \downarrow 0$. Thus,

$$0 < \inf_{z\in(0,1]} \frac{g^{-1}(z)}{z^2}\overline{\sigma}^2(z) \le \inf_{u,t\in(0,1]} \frac{g^{-1}(ug(t))}{u^2 g(t)^2}\overline{\sigma}^2(ug(t)).$$

Since g is regluarly varying with index $1/\alpha$, we suppose that $g(t) = t^{1/\alpha}\varpi(t)$ for a slowly varying function ϖ. Thus, Potter's bound (Theorem A.53) imply that, for some constant $c > 1$ and all $t, u \in (0,1]$, we have $\varpi(t)/\varpi(tu^\beta) \le cu^{-\beta\delta}$ for $\delta = 1/\beta - 1/\alpha > 0$. Hence, we obtain $ug(t) \le cg(tu^\beta)$ and moreover $g^{-1}(ug(t)) \le c^\beta tu^\beta$ for all $t \in (0,1]$ and $u \in (0,1/c]$. Multiplying the rightmost term on the display above (before taking infimum) by $tu^\beta/g^{-1}(ug(t))$ gives

$$\inf_{t\in(0,1]}\inf_{u\in(0,1/c]} u^{\beta-2}\overline{\sigma}^2_{Q_t}(u) = \inf_{t\in(0,1]}\inf_{u\in(0,1/c]} \frac{tu^\beta}{u^2g(t)^2}\overline{\sigma}^2(ug(t)) > 0. \tag{5.24}$$

Hence, Lemma A.18 gives the desired result.

Part (b). As before, we see that $\overline{\sigma}^2_{\widetilde{Q}_t}(u) = \overline{\sigma}^2(u\sqrt{t})$. Hence, the left side of (5.24) gives

$$\inf_{t\in[1,\infty)}\inf_{u\in(0,1]} u^{\beta-2}\overline{\sigma}^2_{\widetilde{Q}_t}(u) = \inf_{u\in(0,1]} u^{\beta-2}\overline{\sigma}^2(u) > 0,$$

for any $\beta \in (0,\alpha)$. Thus, Lemma A.18 gives the desired result. □

Lemma 5.34. *Let $X \in \mathcal{Z}_{\alpha,\rho}$ for some $\alpha \in (0,1)$ and $\rho \in (0,1]$, denote by g its normalising function and define $G(t) = t/g(t)$ for $t > 0$. The following statements hold for any sequences $(u_n)_{n\in\mathbb{N}} \subset (0,\infty)$ and $(s_n)_{n\in\mathbb{N}} \subset (0,\infty)$ such that $u_n \to \infty$ and $s_n \downarrow 0$ as $n \to \infty$:*

(i) if $u_nG^{-1}(s_n^{-1}) \to \infty$, then $\Psi_{s_n}(u_n) \sim \rho\log(u_nG^{-1}(s_n^{-1}))$,

(ii) if $u_nG^{-1}(s_n^{-1}) \to 0$, then $\Psi_{s_n}(u_n) = \mathcal{O}([u_nG^{-1}(s_n^{-1})]^q + s_n)$ for any $q \in (0,1]$ with $q < 1/\alpha - 1$.

Proof. Part (i). Define $Q_t \coloneqq (X_t - \gamma_0 t)/g(t)$ and note that

$$\Psi_{s_n}(u_n) = \int_0^\infty (1-e^{-tu_n})e^{-\lambda t}\mathbb{P}\big(0 < Q_t \le s_nG(t)\big)\frac{\mathrm{d}t}{t}, \quad \text{for all } n \in \mathbb{N}.$$

Fix $\delta \in (0,\rho/3)$, let $\kappa_n \coloneqq G^{-1}(\delta/s_n)$ and note that $\kappa_n \downarrow 0$ as $n \to \infty$. We will now split the integral in the previous display at κ_n and 1 and find the asymptotic behaviour of each of the resulting integrals.

The integral on $[1,\infty)$ is bounded as $n \to \infty$:

$$\int_1^\infty (1-e^{-tu_n})e^{-\lambda t}\mathbb{P}\big(0 < Q_t \le s_nG(t)\big)\frac{\mathrm{d}t}{t} \le \int_1^\infty e^{-\lambda t}\frac{\mathrm{d}t}{t} < \infty.$$

Next, we consider the integral on $[\kappa_n, 1)$. By Lemma 5.33(a), there exists a uniform upper bound $C > 0$ on the densities of Q_t, $t \in (0,1]$. An application of Theorem A.55(i) gives, as $n \to \infty$,

$$\begin{aligned}\int_{\kappa_n}^1 (1-e^{-u_nt})e^{-\lambda t}\mathbb{P}\big(0 < Q_t \le s_nG(t)\big)\frac{\mathrm{d}t}{t} &\le C\int_{\kappa_n}^1 s_nG(t)\frac{\mathrm{d}t}{t} \\ &\sim \frac{\alpha C}{1-\alpha}s_nG(\kappa_n) = \frac{\delta\alpha C}{1-\alpha},\end{aligned}$$

which is finite. Since we will prove that $\Psi_{s_n}(u_n) \to \infty$ as $n \to \infty$, the asymptotic behaviour of $\Psi_{s_n}(u_n)$ will be driven by asymptotic behaviour of the integral on $(0,\kappa_n)$:

$$J_n^0 \coloneqq \int_0^1 (1-e^{-u_n\kappa_nt})e^{-\lambda\kappa_nt}\mathbb{P}\big(0 < Q_{\kappa_nt} \le s_nG(\kappa_nt)\big)\frac{\mathrm{d}t}{t}. \tag{5.25}$$

We will show that, asymptotically as $n \to \infty$, we may replace the probability in the integrand with the probability $\mathbb{P}(0 < Z < \delta t^{1-1/\alpha})$ in terms of the limiting α-stable random variable Z. Since Z has a bounded density (see, e.g. Remark A.35 or [75, Ch. 4]), the weak convergence $Q_t \xrightarrow{d} Z$ as $t \downarrow 0$ implies that the distributions functions converge in Kolmogorov distance by Theorem A.8. Thus, since $\kappa_n \to 0$ as

$n \to \infty$, there exists some $N_\delta \in \mathbb{N}$ such that

$$\sup_{n \geq N_\delta} \sup_{t \in (0,\kappa_n]} \sup_{x \in \mathbb{R}} |\mathbb{P}(0 < Q_t \leq x) - \mathbb{P}(0 < Z \leq x)| < \delta,$$

where $\delta \in (0, \rho/3)$ is as before, arbitrary but fixed. In particular, the following inequality holds $\sup_{n \geq N_\delta} \sup_{t \in (0,\kappa_n]} |\mathbb{P}(0 < Q_t \leq s_n G(t)) - \mathbb{P}(0 < Z \leq s_n G(t))| < \delta$. For any $N \geq N_\delta$ the triangle inequality yields

$$\begin{aligned} B_{\delta,N} &:= \sup_{n \geq N} \sup_{t \in (0,1]} |\mathbb{P}(0 < Z < \delta t^{1-1/\alpha}) - \mathbb{P}(0 < Q_{t\kappa_n} \leq s_n G(t\kappa_n))| \\ &\leq \delta + \sup_{n \geq N} \sup_{t \in (0,1]} |\mathbb{P}(0 < Z < \delta t^{1-1/\alpha}) - \mathbb{P}(0 < Z \leq s_n G(t\kappa_n))| \\ &\leq \delta + \sup_{n \geq N} \sup_{t \in (0,1]} \mathbb{P}(m_{t,n} < Z < M_{t,n}), \end{aligned}$$

where $m_{t,n} := \min\{s_n G(t\kappa_n), \delta t^{1-1/\alpha}\}$ and $M_{t,n} := \max\{s_n G(t\kappa_n), \delta t^{1-1/\alpha}\}$. We aim to show that $B_{\delta,N'_\delta} < 2\delta$ for some $N'_\delta \in \mathbb{N}$.

By Remark A.35 (see also [75, Ch. 4]), there exists $K > 0$ such that the stable density of Z is bounded by the function $x \mapsto Kx^{-\alpha-1}$ for all $x > 0$. Thus, since $M_{t,n} - m_{t,n} = |\delta t^{1-1/\alpha} - s_n G(t\kappa_n)|$, we have

$$\begin{aligned} \mathbb{P}(m_{t,n} < Z < M_{t,n}) &\leq K m_{t,n}^{-\alpha-1} |\delta t^{1-1/\alpha} - s_n G(t\kappa_n)| \\ &\leq K((\delta t^{1-1/\alpha})^{-\alpha-1} + (s_n G(t\kappa_n))^{-\alpha-1}) |\delta t^{1-1/\alpha} - s_n G(t\kappa_n)|. \end{aligned} \tag{5.26}$$

To show that this converges uniformly in $t \in (0, 1]$, we consider both summands. First, we have

$$(\delta t^{1-1/\alpha})^{-\alpha-1} |\delta t^{1-1/\alpha} - s_n G(t\kappa_n)| = \delta^{-\alpha} \left| t^{1-\alpha} - \frac{(t\kappa_n)^{(1-\alpha^2)/\alpha} G(t\kappa_n)}{\kappa_n^{(1-\alpha^2)/\alpha} G(\kappa_n)} \right|,$$

which tends to 0 as $n \to \infty$ uniformly in $t \in (0, 1]$ by Theorem A.51 since $t \mapsto t^{(1-\alpha^2)/\alpha} G(t)$ is regularly varying at 0 with index $1 - \alpha > 0$ (recall that g is regularly varying at 0 with index $1/\alpha$ and $G(t) = t/g(t)$). Similarly, since $s_n = \delta/G(\kappa_n)$, we have

$$(s_n G(t\kappa_n))^{-\alpha-1} |\delta t^{1-1/\alpha} - s_n G(t\kappa_n)| = \delta^{-\alpha} \left| \frac{(t\kappa_n)^{1-1/\alpha} G(t\kappa_n)^{-\alpha-1}}{\kappa_n^{1-1/\alpha} G(\kappa_n)^{-\alpha-1}} - \frac{G(t\kappa_n)^{-\alpha}}{G(\kappa_n)^{-\alpha}} \right|.$$

Since both terms in the last line converge to $\delta^\alpha t^{1-\alpha}$ as $n \to \infty$ uniformly in $t \in (0, 1]$ by Theorem A.51, the difference tends to 0 uniformly too. Hence, the right side of (5.26) converges to 0 as $n \to \infty$ uniformly in $t \in (0, 1]$. Thus, for a sufficiently large N'_δ, we have

$$\sup_{n \geq N'_\delta} \sup_{t \in (0,1]} |\mathbb{P}(0 < Z < \delta t^{1-1/\alpha}) - \mathbb{P}(0 < Q_{t\kappa_n} \leq s_n G(t\kappa_n))| = B_{\delta,N'_\delta} < 2\delta. \tag{5.27}$$

We now analyse a lower bound on the integral J_n^0 in (5.25). By (5.27), for all

$n \geq N'_\delta$, we have

$$J_n^0 \geq \int_0^1 (1 - e^{-u_n\kappa_n t}) e^{-\lambda\kappa_n t} \big(\mathbb{P}\big(0 < Z \leq \delta t^{1-1/\alpha}\big) - 2\delta\big) \frac{\mathrm{d}t}{t}.$$

Recall that $\kappa_n = G^{-1}(\delta/s_n)$, define $\xi_n := G^{-1}(1/s_n)$ and note from the regular variation of G^{-1} that $\kappa_n/\xi_n \to \delta^{\alpha/(\alpha-1)}$ as $n \to \infty$, implying $\log(u_n\kappa_n) \sim \log(u_n\xi_n)$ as $n \to \infty$ since $u_n\xi_n \to \infty$. We split the integral from the display above at $\log(u_n\kappa_n)^{-1}$ and note, as $n \to \infty$, that

$$\begin{aligned}
&\int_{\log(u_n\kappa_n)^{-1}}^1 (1 - e^{-u_n\kappa_n t}) e^{-\lambda\kappa_n t} \big(\mathbb{P}\big(0 < Z \leq \delta t^{1-1/\alpha}\big) + 2\delta\big) \frac{\mathrm{d}t}{t} \\
&\leq \big(1 + 2\delta\big) \int_{\log(u_n\kappa_n)^{-1}}^1 \frac{\mathrm{d}t}{t} = \big(1 + 2\delta\big) \log(\log(u_n\kappa_n)) \sim \big(1 + 2\delta\big) \log(\log(u_n\xi_n)).
\end{aligned}$$

For the integral over $(0, \log(u_n\kappa_n)^{-1})$, note, for all sufficiently large $n \in \mathbb{N}$, that

$$\mathbb{P}(0 < Z \leq \delta t^{1-1/\alpha}) \geq \mathbb{P}(0 < Z \leq \delta \log(u_n\kappa_n)^{1/\alpha-1}) > \rho - \delta, \quad t \in (0, \log(u_n\kappa_n)^{-1}),$$

since $u_n\kappa_n \to \infty$. Thus, as $n \to \infty$, we have

$$\begin{aligned}
&\int_0^{\log(u_n\kappa_n)^{-1}} (1 - e^{-u_n\kappa_n t}) e^{-\lambda\kappa_n t} \big(\mathbb{P}\big(0 < Z \leq \delta t^{1-1/\alpha}\big) - 2\delta\big) \frac{\mathrm{d}t}{t} \\
&\geq \big(\rho - 3\delta\big) e^{-\lambda\kappa_n/\log(u_n\kappa_n)} \int_0^{\log(u_n\kappa_n)^{-1}} (1 - e^{-u_n\kappa_n t}) \frac{\mathrm{d}t}{t} \sim \big(\rho - 3\delta\big) \log(u_n\xi_n),
\end{aligned}$$

where the asymptotic equivalence follows from the fact that $u_n\kappa_n/\log(u_n\kappa_n) \to \infty$ as $n \to \infty$ and $\int_0^1 (1 - e^{-xt}) t^{-1} \mathrm{d}t \sim \log x$ as $x \to \infty$. (In fact, we have $\int_0^1 (1 - e^{-xt}) t^{-1} \mathrm{d}t = \log x + \Gamma(0, x) + \gamma$ for $x > 0$ where $\Gamma(0, x) = \int_x^\infty t^{-1} e^{-t} \mathrm{d}t$ is the upper incomplete gamma function and γ is the Euler–Mascheroni constant.) This shows that $\liminf_{n\to\infty} J_n^0/\log(u_n\xi_n) \geq \rho - 3\delta > 0$ since $\delta \in (0, \rho/3)$.

Similarly, (5.27) implies that for all $n \geq N'_\delta$, we have

$$\begin{aligned}
J_n^0 &\leq \int_0^1 (1 - e^{-u_n\kappa_n t}) e^{-\lambda\kappa_n t} \big(\mathbb{P}\big(0 < Z \leq \delta t^{1-1/\alpha}\big) + 2\delta\big) \frac{\mathrm{d}t}{t} \\
&\leq (\rho + 2\delta) \int_0^1 (1 - e^{-u_n\kappa_n t}) \frac{\mathrm{d}t}{t} \sim (\rho + 2\delta) \log(u_n\xi_n), \quad \text{as } n \to \infty,
\end{aligned}$$

implying $\limsup_{n\to\infty} J_n^0/\log(u_n\xi_n) \leq \rho + 2\delta$. Altogether, we deduce that

$$\rho - 3\delta \leq \liminf_{n\to\infty} \Psi_{s_n}(u_n)/\log(u_n\xi_n) \leq \limsup_{n\to\infty} \Psi_{s_n}(u_n)/\log(u_n\xi_n) \leq \rho + 2\delta.$$

Since $\delta \in (0, \rho/3)$ is arbitrary and the sequence $\Psi_{s_n}(u_n)/\log(u_n\xi_n)$ does not depend on δ, we may take $\delta \downarrow 0$ to obtain Part (i).

Part (ii). We will bound each of the terms in $\Psi_{s_n}(u_n) = J_n^1 + J_n^2 + J_n^3$, where

$\xi_n = G^{-1}(1/s_n)$ and

$$J_n^1 \coloneqq \int_0^{\xi_n} (1 - e^{-u_n t}) e^{-\lambda t} \mathbb{P}(0 < Q_t \le s_n G(t)) \frac{\mathrm{d}t}{t},$$
$$J_n^2 \coloneqq \int_{\xi_n}^1 (1 - e^{-u_n t}) e^{-\lambda t} \mathbb{P}(0 < Q_t \le s_n G(t)) \frac{\mathrm{d}t}{t} \quad \text{and}$$
$$J_n^3 \coloneqq \int_1^{\infty} (1 - e^{-u_n t}) e^{-\lambda t} \mathbb{P}(0 < X_t - \gamma_0 t \le s_n t) \frac{\mathrm{d}t}{t}.$$

Recall that our assumption in part (ii) states that $u_n \xi_n \to 0$ as $n \to \infty$. Using the elementary inequality $1 - e^{-x} \le x$ for $x \ge 0$, we obtain $J_n^1 = \mathcal{O}(u_n \xi_n)$ as $n \to \infty$. Next we bound J_n^3. Lemma 5.33(b) shows the existence of a uniform upper bound $\widetilde{C} > 0$ on the densities of $X_t/\sqrt{t}$. Thus, $\mathbb{P}(0 < X_t - \gamma_0 t \le s_n t) = \mathbb{P}(\gamma_0 \sqrt{t} < X_t/\sqrt{t} \le (\gamma_0 + s_n)\sqrt{t}) \le \widetilde{C} s_n \sqrt{t}$ and hence

$$J_n^3 \le \widetilde{C} s_n \int_1^{\infty} t^{-1/2} e^{-\lambda t} \mathrm{d}t = \mathcal{O}(s_n), \qquad \text{as } n \to \infty.$$

It remains to bound J_n^2. Let $q \in (0, 1]$ with $q < 1/\alpha - 1$ and $C > 0$ be a uniform bound on the densities of Q_t (whose existence is guaranteed by Lemma 5.33(a)). The elementary bound $1 - e^{-x} \le x^q$ for $x \ge 0$ for $q \in (0, 1]$ and Theorem A.55(i) yield

$$J_n^2 \le C u_n^q s_n \int_{\xi_n}^1 t^q G(t) \frac{\mathrm{d}t}{t} \sim \frac{C}{1/\alpha - q - 1} u_n^q s_n G(\xi_n) \xi_n^q = \mathcal{O}(u_n^q \xi_n^q), \qquad \text{as } n \to \infty. \ \square$$

Proof of Theorem 5.7. Throughout this proof we let $\phi(u) \coloneqq \gamma u^{-1} (\log \log u)^r$, for some $\gamma > 0$, $r \in \mathbb{R}$.

Part (i). Since p is arbitrary on $(1/\rho, \infty)$ and $f(t) = 1/G(t \log^p(1/t))$, it suffices to show that $\limsup_{t \downarrow 0} (\widehat{C}'(t + \widehat{\tau}_s) - s)/f(t) = \limsup_{t \downarrow 0} L_t/f(t) < \infty$ a.s. (Recall that $L_t = C'(t + \widehat{\tau}_s) - s$ and $\Psi_u(w) = -\log \mathbb{E}[e^{-w Y_u}]$ for all $u, w \ge 0$.) By Theorem 5.22(a), it suffices to find a positive sequence $(\theta_n)_{n \in \mathbb{N}}$ with $\lim_{n \to \infty} \theta_n = \infty$ such that $\sum_{n=1}^{\infty} \exp(\theta_n t_n - \Psi_{f(t_n)}(\theta_n)) < \infty$ and $\limsup_{n \to \infty} f(t_n)/f(t_{n+1}) < \infty$ where $t_n \coloneqq \phi(\theta_n)$.

Let $\theta_n \coloneqq e^n$ and $r = 0$. Since the function f is regularly varying at 0, it follows that $\limsup_{n \to \infty} f(t_n)/f(t_{n+1}) = \lim_{n \to \infty} f(t_n)/f(t_{n+1}) = e^{1 - 1/\alpha}$. Thus, it suffices to prove that the series above is finite. Since $t_n = \phi(\theta_n)$, it follows that $t_n \theta_n = \gamma$. Note from the definition of f that, as $u \to \infty$,

$$u G^{-1}(f(\phi(u))^{-1}) = u h(u) (\log(\phi(u)^{-1}))^p = \gamma (\log(\gamma^{-1} u))^p \sim \gamma (\log u)^p \to \infty. \tag{5.28}$$

Since $\theta_n G^{-1}(f(t_n)^{-1}) \sim \gamma (\log \theta_n)^p \to \infty$ as $n \to \infty$ by (5.28), Lemma 5.34(i) implies that $\Psi_{f(t_n)}(\theta_n) \sim \rho \log(\theta_n G^{-1}(f(t_n)^{-1}))$ as $n \to \infty$.

Fix some $\varepsilon > 0$ with $(1 - \varepsilon)\rho p > 1$. Note that $\Psi_{f(t_n)}(\theta_n) \ge (1 - \varepsilon)\rho p \log \log \theta_n$

for all sufficiently large n. It suffices to show that the following sum is finite: $\sum_{n=1}^{\infty} \exp\big(\gamma - (1-\varepsilon)\rho p \log\log\theta_n\big)$. Since $(1-\varepsilon)\rho p > 1$, this sum is bounded by a multiple of the series $\sum_{n=1}^{\infty} n^{-(1-\varepsilon)\rho p} < \infty$.

Part (ii). Since p is arbitrary in $(0, 1/\rho)$, it suffices to show $\limsup_{t\downarrow 0} L_t/f(t) \geq 1$ a.s. By Theorem 5.22(b), it is therefore sufficient to find a positive sequence $(\theta_n)_{n\in\mathbb{N}}$ satisfying $\lim_{n\to\infty}\theta_n = \infty$, $\sum_{n=1}^{\infty}(\exp(-\Psi_{f(t_n)}(\theta_n)) - \exp(-\theta_n t_n)) = \infty$ and $\sum_{n=1}^{\infty}\Psi_{f(t_{n+1})}(\theta_n) < \infty$.

Let $r = \gamma = 1$, choose $\sigma > 1$ and $\varepsilon > 0$ to satisfy $\sigma(1+\varepsilon)\rho p < 1$ and set $\theta_n := e^{n^\sigma}$ for $n \in \mathbb{N}$. We start by showing that the second sum in the paragraph above is finite. Since $\sigma > 1$, (5.28) yields

$$\theta_n G^{-1}(f(t_{n+1})^{-1}) \sim \frac{\theta_n}{\theta_{n+1}}(\log\theta_{n+1})^p \log\log\theta_{n+1} \downarrow 0, \quad \text{as } n\to\infty. \tag{5.29}$$

Hence, Lemma 5.34(ii) with $q \in (0,1]$ and $q < 1/\alpha - 1$ and (5.29) imply

$$\Psi_{f(t_{n+1})}(\theta_n) = \mathcal{O}\big([\theta_n G^{-1}(f(t_{n+1})^{-1})]^q + f(t_{n+1})\big), \quad \text{as } n\to\infty.$$

By (5.29), it is enough to show that

$$\sum_{n=1}^{\infty}\left(\frac{\theta_n}{\theta_{n+1}}(\log\theta_{n+1})^p\log\log\theta_{n+1}\right)^q < \infty, \qquad \text{and} \qquad \sum_{n=1}^{\infty} f(t_{n+1}) < \infty.$$

Newton's generalised binomial theorem implies that $\theta_n/\theta_{n+1} = \exp(n^\sigma - (n+1)^\sigma) \leq \exp(-\sigma n^{\sigma-1}/2)$ for all sufficiently large n. Since $\log\theta_{n+1} \sim n^\sigma$, we conclude that the first series in the previous display is indeed finite. The second series is also finite since $f\circ h$ is regularly varying at infinity with index $(\alpha-1)/\alpha < 0$ (recall that $t_{n+1} = \phi(\theta_{n+1})$).

Next we prove that $\sum_{n=1}^{\infty}(\exp(-\Psi_{f(t_n)}(\theta_n)) - \exp(-\theta_n t_n)) = \infty$. Note that we have $\exp(-\theta_n t_n) = \exp(-\log\log\theta_n) = n^{-\sigma}$, which is summable. Applying Lemma 5.34(i) and (5.28), we see that $\Psi_{f(t_n)}(\theta_n) \sim \rho\log(\theta_n G^{-1}(f(t_n)^{-1}))$ as $n\to\infty$. As in Part (i), it is easy to see that for every $\varepsilon > 0$, the inequality $\Psi_{f(t_n)}(\theta_n) \leq (1+\varepsilon)\rho p\log\log\theta_n$ holds for all sufficiently large n. Thus $\exp(-\Psi_{f(t_n)}(\theta_n)) \geq n^{-\sigma(1+\varepsilon)\rho p}$ is not summable (since $\sigma(1+\varepsilon)\rho p < 1$): $\sum_{n=1}^{\infty}\exp(-\Psi_{f(t_n)}(\theta_n)) = \infty$, completing the proof. □

§5.4.3 Upper and lower functions at time 0 - proofs

Fix any $\lambda > 0$. Let $Y_s := \widehat{\tau}_{-1/s}$ for $s \in (0,\infty)$ and note that the mean jump measure of Y_s is given by

$$\Pi(\mathrm{d}s, \mathrm{d}t) := t^{-1}e^{-\lambda t}\mathbb{P}(-t/X_t \in \mathrm{d}s)\mathrm{d}t,$$

implying $\Pi((0,s],\mathrm{d}t) = t^{-1}e^{-\lambda t}\mathbb{P}(X_t \le -t/s)\mathrm{d}t$. Since $\widehat{C}'$ is the right-inverse of $\widehat{\tau}$, we have the identity $\widehat{C}'(t) = -1/L_t$ where $L_t := \inf\{s > 0 : Y_s > t\}$. Thus, $\limsup_{t\downarrow 0} |\widehat{C}'(t)|f(t)$ equals 0 (resp. ∞) if and only if $\liminf_{t\downarrow 0} L_t/f(t)$ equals ∞ (resp. 0). Corollary 5.28 and Proposition 5.26 above are the ingredients in the proof of Theorem 5.9.

Proof of Theorem 5.9. Since the conditions in Theorem 5.24 only involve integrating the mean measure Π of Y near the origin, we may ignore the factor $e^{-\lambda t}$ in the definition of the mean measure Π above. After substituting $\Pi(\mathrm{d}u,\mathrm{d}t) = t^{-1}\mathbb{P}(-t/X_t \in \mathrm{d}u)\mathrm{d}t$ in conditions (5.15) and (5.19)–(5.20), we obtain the conditions in (5.7)–(5.9). Thus, Corollary 5.28 and the identity $\widehat{C}(t) = -1/L_t$ yield the claims in Theorem 5.9. □

The following technical lemma which establishes the asymptotic behaviour of the characteristic exponent Φ defined in (5.22). This result plays an important role in the proof of Theorem 5.13. We will assume that $X \in \mathcal{Z}_{\alpha,\rho}$. For simplicity, by virtue of (A.4) and Theorem A.52, we assume without loss of generality that: $g(t) = 1$ for $t \ge 1$, g is continuous and decreasing on $(0,1]$ and the function $G(t) = t/g(t)$ is continuous and increasing on $(0,\infty)$. Hence, the inverse G^{-1} of G is also continuous and increasing.

Lemma 5.35. *Let $X \in \mathcal{Z}_{\alpha,\rho}$ for some $\alpha \in (1,2]$ and $\rho \in (0,1)$ and assume $\mathbb{E}[X_1^2] < \infty$ and $\mathbb{E}[X_1] = 0$. The following statements hold for any sequences $(u_n)_{n\in\mathbb{N}} \subset (0,\infty)$ and $(s_n)_{n\in\mathbb{N}} \subset \mathbb{R}_-$ such that $u_n \to \infty$ and $s_n \to -\infty$ as $n \to \infty$:*

(i) if $u_nG^{-1}(|s_n|^{-1}) \to \infty$, then $\Phi_{s_n}(u_n) \sim (1-\rho)\log(u_nG^{-1}(|s_n|^{-1}))$,

(ii) if $u_nG^{-1}(|s_n|^{-1}) \downarrow 0$, then $\Phi_{s_n}(u_n) = \mathcal{O}([u_nG^{-1}(|s_n|^{-1})]^{(\alpha-1)/2} + |s_n|^{-2})$.

Proof. Part (i). Denote $Q_t := X_t/g(t)$ and note that, for all $n \in \mathbb{N}$,

$$\Phi_{s_n}(u_n) = \int_0^\infty (1-e^{-tu_n})e^{-\lambda t}\mathbb{P}\big(Q_t \le s_nG(t)\big)\frac{\mathrm{d}t}{t}.$$

For every $\delta > 0$ let $\kappa_n := G^{-1}(\delta/|s_n|)$ and note that $\kappa_n \downarrow 0$ as $n \to \infty$. The integral in the previous display is split at κ_n and we control the two resulting integrals.

We start with the integral on $[\kappa_n,\infty)$. For any $q \in (0,\alpha)$ we claim that $K := \sup_{t\ge 0}\mathbb{E}[|Q_t|^q] < \infty$. Indeed, since $\mathbb{E}[X_t^2] < \infty$, $t^{-1/2}g(t)Q_t$ converges weakly to a normal random variable as $t \to \infty$. Applying Lemma 3.18 gives $\sup_{t\ge 1}\mathbb{E}[|Q_t|^q]t^{-q/2}g(t)^q < \infty$, and hence $\sup_{t\ge 1}\mathbb{E}[|Q_t|^q] < \infty$ since $t^{-1/2}g(t)$ is bounded from below for $t \ge 1$. Similarly, $\sup_{t\le 1}\mathbb{E}[|Q_t|^q] < \infty$ by Lemmas 2.13 & 2.14, implying that $K < \infty$. Markov's inequality then yields

$$K \ge \sup_{n\in\mathbb{N}}\sup_{t\ge\kappa_n} |s_n|^qG(t)^q\mathbb{P}(Q_t \le s_nG(t)). \tag{5.30}$$

Let $q' \coloneqq q(1-1/\alpha) > 0$ and note that $G(t)^{-q}$ is regularly varying at 0 with index $-q'$. By (5.30) we have $\mathbb{P}(Q_t \le s_n G(t)) \le K|s_n|^{-q} G(t)^{-q}$ for all $t \ge \kappa_n$ and $n \in \mathbb{N}$. Hence, Karamata's theorem (see Theorem A.55) gives, as $n \to \infty$, that

$$\int_{\kappa_n}^{\infty} (1-e^{-u_n t}) e^{-\lambda t} \mathbb{P}\big(Q_t \le s_n G(t)\big) \frac{\mathrm{d}t}{t} \le K \int_{\kappa_n}^{\infty} |s_n|^{-q} e^{-\lambda t} G(t)^{-q} \frac{\mathrm{d}t}{t}$$
$$\sim \frac{K}{q'} |s_n|^{-q} G(\kappa_n)^{-q} = \frac{K}{q'\delta^q} < \infty.$$

Thus, the integral $\int_{\kappa_n}^{\infty} (1-e^{-u_n t}) e^{-\lambda t} \mathbb{P}\big(Q_t \le s_n G(t)\big) t^{-1} \mathrm{d}t$ is bounded as $n \to \infty$.

It remains to establish the asymptotic growth of the corresponding integral on $(0, \kappa_n)$. Since the limiting α-stable random variable Z has a bounded density (see, e.g. Remark A.35 or [75, Ch. 4]), the weak convergence of $Q_t \xrightarrow{d} Z$ as $t \downarrow 0$ extends to convergence in Kolmogorov distance by Theorem A.8. Thus, there exists some $N_\delta \in \mathbb{N}$ such that

$$\sup_{n \ge N_\delta} \sup_{t \in [0,\kappa_n]} |\mathbb{P}(Q_t \le s_n G(t)) - \mathbb{P}(Z \le s_n G(t))| < \delta.$$

Since $G(\kappa_n) = \delta/|s_n|$ and $\mathbb{P}(Z \le 0) = 1-\rho$, the triangle inequality yields

$$B_\delta \coloneqq \sup_{n \ge N_\delta} \sup_{t \in [0,\kappa_n]} |1-\rho-\mathbb{P}(Q_t \le s_n G(t))| \le |1-\rho-\mathbb{P}(Z \le -\delta)| + \delta.$$

which tends to 0 as $\delta \downarrow 0$.

Define $\xi_n \coloneqq G^{-1}(1/|s_n|)$ for and note from the regular variation of G^{-1} that $\kappa_n/\xi_n \to \delta^{\alpha/(\alpha-1)}$ as $n \to \infty$, implying $\log(u_n\kappa_n) \sim \log(u_n\xi_n)$ as $n \to \infty$ since $u_n\xi_n \to \infty$. As in the proof of Lemma 5.34 above, we have $\int_0^1 (1-e^{-xt}) t^{-1} \mathrm{d}t \sim \log x$ as $x \to \infty$. Since $u_n\xi_n \to \infty$ and $\xi_n \downarrow 0$ as $n \to \infty$, we have

$$\int_0^{\kappa_n} (1-e^{-u_n t}) e^{-\lambda t} \mathbb{P}\big(Q_t \le s_n G(t)\big) \frac{\mathrm{d}t}{t} \le (1-\rho+B_\delta) \int_0^{\kappa_n} (1-e^{-u_n t}) e^{-\lambda t} \frac{\mathrm{d}t}{t}$$
$$\sim (1-\rho+B_\delta) \log(u_n\xi_n), \quad \text{as } n \to \infty.$$

This implies that $\limsup_{n\to\infty} \Phi_{s_n}(u_n)/\log(u_n\xi_n) \le 1-\rho+B_\delta$. A similar argument can be used to obtain $\liminf_{n\to\infty} \Phi_{s_n}(u_n)/\log(u_n\xi_n) \ge 1-\rho-B_\delta$. Since $\delta > 0$ is arbitrary and $B_\delta \downarrow 0$ as $\delta \downarrow 0$, we deduce that $\Phi_{s_n}(u_n) \sim (1-\rho)\log(u_n\xi_n)$ as $n \to \infty$.

Part (ii). We will bound each of the terms in $\Phi_{s_n}(u_n) = J_n^1 + J_n^2 + J_n^3$, where $\xi_n = G^{-1}(1/|s_n|)$ and

$$J_n^1 \coloneqq \int_0^{\xi_n} (1-e^{-u_n t}) e^{-\lambda t} \mathbb{P}(X_t \le s_n t) \frac{\mathrm{d}t}{t}, \; J_n^2 \coloneqq \int_1^{\infty} (1-e^{-u_n t}) e^{-\lambda t} \mathbb{P}(X_t \le s_n t) \frac{\mathrm{d}t}{t},$$
$$\text{and} \qquad J_n^3 \coloneqq \int_{\xi_n}^{1} (1-e^{-u_n t}) e^{-\lambda t} \mathbb{P}(Q_t \le s_n G(t)) \frac{\mathrm{d}t}{t}.$$

The elementary inequality $1-e^{-x} \le x$ for $x \ge 0$ implies that the integrand of J_n^1 is bounded by u_n. Hence, we have $J_n^1 = \mathcal{O}(u_n\xi_n) = \mathcal{O}((u_n\xi_n)^{(\alpha-1)/2})$ as $n \to \infty$.

To bound J_n^2, we use Markov's inequality as follows: since $\mathbb{E}[X_t^2] = \mathbb{E}[X_1^2]t$ for all $t > 0$, we have $\mathbb{P}(X_t \le s_n t) \le \mathbb{E}[X_1^2]t/(|s_n|^2 t^2) = \mathbb{E}[X_1^2]|s_n|^{-2}t^{-1}$, for all $n \in \mathbb{N}$, $t > 0$. Thus, we get

$$J_n^2 \le \frac{\mathbb{E}[X_1^2]}{|s_n|^2} \int_1^\infty \frac{\mathrm{d}t}{t^2} = \frac{\mathbb{E}[X_1^2]}{|s_n|^2} = \mathcal{O}(|s_n|^{-2}), \qquad \text{as } n \to \infty.$$

It remains to bound J_n^3. Let $r := (\alpha - 1)/2$, pick any $q \in (\alpha/2, \alpha)$ and recall from Part (i) that $K = \sup_{t \ge 0} \mathbb{E}[|Q_t|^q] < \infty$. Note that $q' = q(1 - 1/\alpha) > r$, so Karamata's theorem (see Theorem A.55), the inequality in (5.30) and the elementary bound $1 - e^{-x} \le x^r$ for $x \ge 0$ yield

$$J_n^3 \le K u_n^r \int_{\xi_n}^1 t^r |s_n|^{-q} G(t)^{-q} \frac{\mathrm{d}t}{t} \sim \frac{K u_n^r}{q' - r} \xi_n^r |s_n|^{-q} G(\xi_n)^{-q} = \frac{K}{q' - r}(u_n \xi_n)^r,$$

as $n \to \infty$. We conclude that $J_n^3 = \mathcal{O}((u_n \xi_n)^r)$ as $n \to \infty$, completing the proof. □

Proof of Theorem 5.13. Throughout this proof we let $\phi(u) := \gamma u^{-1}(\log\log u)^r$, for some $\gamma > 0$ and $r \in \mathbb{R}$. By Remark 5.1 we may and do assume without loss of generality that $(X_t)_{t \ge 0}$ has a finite second moment and zero mean.

Part (i). Since p is arbitrary on the interval $(1/(1-\rho), \infty)$, it suffices to show that $\liminf_{t \downarrow 0} |\widehat{C}'(t)| f(t) > 0$ a.s. where $f(t) = G(t \log^p(1/t))$. Since $\widehat{C}'(t) = -1/L_t$, this is equivalent to $\limsup_{t \downarrow 0} L_t / f(t) < \infty$ a.s. Recall that $\Psi_u(w) = \log \mathbb{E}[e^{-wY_u}] = \log \mathbb{E}[e^{-w\widehat{\tau}_{-1/u}}] = \Phi_{-1/u}(w)$ for all $u > 0$ and $w \ge 0$. By virtue of Theorem 5.22(a), it suffices to show that $\sum_{n=1}^\infty \exp(\theta_n t_n - \Psi_{f(t_n)}(\theta_n)) < \infty$ and $\limsup_{n \to \infty} f(t_n)/f(t_{n+1}) < \infty$ for $t_n := \phi(\theta_n)$ and a positive sequence $(\theta_n)_{n \in \mathbb{N}}$ with $\lim_{n \to \infty} \theta_n = \infty$.

Let $\theta_n := e^n$ and $r = 0$. Since f is a regularly variation function at 0, it holds that $\limsup_{n \to \infty} f(t_n)/f(t_{n+1}) = \lim_{n \to \infty} f(t_n)/f(t_{n+1}) = e^{1 - 1/\alpha}$. Thus, it suffices to prove that the series is finite. Since $t_n = \phi(\theta_n)$, it follows that $t_n \theta_n = \gamma$. Note from the definition of f that, as $u \to \infty$,

$$uG^{-1}(f(\phi(u))) = u\phi(u)(\log(\phi(u)^{-1}))^p = \gamma(\log(\gamma^{-1}u))^p \sim \gamma(\log u)^p \to \infty. \quad (5.31)$$

By Lemma 5.35(i) we have $\Psi_{f(t_n)}(\theta_n) = \Phi_{-1/f(t_n)}(\theta_n) \sim (1 - \rho)\log(\theta_n G^{-1}(f(t_n)))$ as $n \to \infty$, since $\theta_n G^{-1}(f(t_n)) \sim \gamma(\log \theta_n)^p \to \infty$ as $n \to \infty$ by (5.31).

Fix $\varepsilon > 0$ with $(1-\varepsilon)(1-\rho)p > 1$. Note that $\Psi_{f(t_n)}(\theta_n) \ge (1-\varepsilon)(1-\rho)p \log\log\theta_n$ for all sufficiently large n. It is enough to show that the following sum is finite: $\sum_{n=1}^\infty \exp\left(\gamma - (1-\varepsilon)(1-\rho)p\log\log\theta_n\right)$. Since $(1-\varepsilon)(1-\rho)p > 1$, this sum is bounded by a multiple of $\sum_{n=1}^\infty n^{-(1-\varepsilon)(1-\rho)p} < \infty$.

Part (ii). As before, since p is arbitrary in $(0, 1/(1-\rho))$, it suffices to show that we have $\liminf_{t \downarrow 0} |\widehat{C}'(t)| f(t) < \infty$ a.s. By Theorem 5.22(b), it suffices to show that there exists some $r > 0$ and a positive sequence $(\theta_n)_{n \in \mathbb{N}}$ satisfying $\lim_{n \to \infty} \theta_n = \infty$, such that $\sum_{n=1}^{\infty}(\exp(-\Psi_{f(t_n)}(\theta_n)) - \exp(-\theta_n t_n)) = \infty$ and $\sum_{n=1}^{\infty} \Psi_{f(t_{n+1})}(\theta_n) < \infty$.

Let $\gamma = r = 1$, choose $\sigma > 1$ and $\varepsilon > 0$ satisfying $\sigma(1+\varepsilon)p(1-\rho) < 1$ (recall $p(1-\rho) < 1$) and set $\theta_n := e^{n^\sigma}$. We start by showing that the second sum is finite. Since $\sigma > 1$, (5.31) yields

$$\theta_n G^{-1}(f(t_{n+1})) \sim \frac{\theta_n}{\theta_{n+1}}(\log\theta_{n+1})^p \downarrow 0, \quad \text{as } n \to \infty. \tag{5.32}$$

Hence, the time-change $\widehat{C}'(t) = -1/L_t$, Lemma 5.35(ii) and (5.32) imply

$$\Psi_{f(t_{n+1})}(\theta_n) = \Phi_{-1/f(t_{n+1})}(\theta_n) = \mathcal{O}\big([\theta_n G^{-1}(f(t_{n+1}))]^{(\alpha-1)/2} + f(t_{n+1})^2\big),$$

as $n \to \infty$. By (5.32), it is enough to show that

$$\sum_{n=1}^{\infty}\left(\frac{\theta_n}{\theta_{n+1}}(\log\theta_{n+1})^p \log\log\theta_{n+1}\right)^{(\alpha-1)/2} < \infty, \qquad \text{and} \qquad \sum_{n=1}^{\infty} f(t_{n+1})^2 < \infty.$$

Newton's generalised binomial theorem implies that $\theta_n/\theta_{n+1} = \exp(n^\sigma - (n+1)^\sigma) \le \exp(-\sigma n^{\sigma-1}/2)$ for all sufficiently large n. Since $\log\theta_{n+1} \sim n^\sigma$, we conclude that the first series in the previous display is indeed finite. The second series is also finite since $f \circ h$ is regularly varying at infinity with index $-(\alpha-1)/\alpha$ (recall that $t_{n+1} = \phi(\theta_{n+1})$).

Next we prove that $\sum_{n=1}^{\infty}(\exp(-\Psi_{f(t_n)}(\theta_n)) - \exp(-\theta_n t_n)) = \infty$. First observe that the terms $\exp(-\theta_n t_n) = \exp(-\log\log\theta_n) = n^{-\sigma}$ are summable. Applying Lemma 5.35(i) and (5.31), we obtain $\Psi_{f(t_n)}(\theta_n) \sim (1-\rho)\log(\theta_n G^{-1}(f(t_n)))$ as $n \to \infty$. As in Part (i), $\Psi_{f(t_n)}(\theta_n) \le (1+\varepsilon)p(1-\rho)\log\log\theta_n$ for all sufficiently large n. Thus $\exp(-\Psi_{f(t_n)}(\theta_n)) \ge n^{-\sigma(1+\varepsilon)p(1-\rho)}$ and, since $\sigma(1+\varepsilon)p(1-\rho) < 1$, we deduce that $\sum_{n=1}^{\infty}\exp(-\Psi_{f(t_n)}(\theta_n)) = \infty$, completing the proof. □

§5.4.4 Proofs of §5.2.5

In this subsection we prove the results stated in §5.2.5.

Proofs of Lemmas 5.15 and 5.19. We first prove Lemma 5.15. Let $s \in \mathcal{L}^*$ and let the function $f : [0,\infty) \to [0,\infty)$ be continuous and increasing with $f(0) = 0$ and define the function $\tilde{f}(t) := \int_0^t f(u)\mathrm{d}u$, $t \ge 0$. Note that $m_s = X_{\tau_s} \wedge X_{\tau_s-}$ equals $C(\tau_s)$ since τ_s is a contact point between $t \mapsto X_t \wedge X_{t-}$ and its convex minorant C.

Part (i). By assumption, for any $M > 0$ there exists $\delta > 0$ such that $C'(t+\tau_s) - s \ge Mf(t)$ for $t \in (0,\delta)$. Since $\int_0^t (C'(u+\tau_s) - s)\mathrm{d}u = C(t+\tau_s) - m_s - st$ it follows that $C(t+\tau_s) - m_s - st \ge M\tilde{f}(t)$ for all $t \in [0,\delta)$. Note that the path of X stays above its convex minorant, implying $C(t+\tau_s) - m_s - st \le X_{t+\tau_s} - m_s - st$. Thus, $X_{t+\tau_s} - m_s - st \ge M\tilde{f}(t)$ for all $t \in [0,\delta)$, implying that $\liminf_{t\downarrow 0}(X_{t+\tau_s} - m_s - st)/\tilde{f}(t) \ge M$.

Part (ii). Assume $\tilde{f}$ is convex on a neighborhood of 0, and that $\limsup_{t\downarrow 0}(C'(t+\tau_s)-s)/f(t)=0$. Then, for all $M>0$ there exists some $\delta>0$ such that $C'(t+\tau_s)-s\le Mf(t)$ for all $t\in[0,\delta)$. Integrating this inequality gives $C(t+\tau_s)-m_s-st\le M\tilde{f}(t)$ for all $t\in[0,\delta)$. Since $s\in\mathcal{L}^*$, there exists a decreasing sequence of slopes $s_n\downarrow s$ such that $t_n=\tau_{s_n}-\tau_s\downarrow 0$ and $X_{t_n+\tau_s}\wedge X_{t_n+\tau_s-}=C(t_n+\tau_s)$ for all $n\in\mathbb{N}$. Thus, either $X_{t_n+\tau_s}-m_s-st_n\le M\tilde{f}(t_n)$ i.o. or $X_{t_n+\tau_s-}-m_s-st_n\le M\tilde{f}(t_n)$ i.o. Since $\tilde{f}$ is continuous, we deduce that $\liminf_{t\downarrow 0}(X_{t+\tau_s}-m_s-st)/\tilde{f}(t)\le M$.

The proof of Lemma 5.19 follows along similar lines with $\tilde{f}(t)=\int_0^t f(u)^{-1}\mathrm{d}u$, $t>0$, the slope $s=-\infty$ and $m_{-\infty}=X_0=0$. □

Proof of Corollary 5.17. Part (i) follows from Theorem 5.7 and Lemma 5.15(ii).

Part (ii). Assume $\alpha\in(1/2,1)$. By Theorem 5.2 and Lemma 5.15(i) it suffices to prove that (5.2)–(5.4) hold for $c=1$. As described in §5.2.1.1, condition (5.6) implies (5.3)–(5.4). By Lemma 5.33, the density of $(X_t-st)/g(t)$ is uniformly bounded in $t>0$. Hence, the following condition implies (5.6):

$$\int_0^1\int_{f(t/2)}^1\frac{1}{f^{-1}(x)}\mathrm{d}x\frac{t}{g(t)}\mathrm{d}t<\infty. \tag{5.33}$$

Similarly, (5.2) holds with $c=1$ if $\int_0^1(f(t)/g(t))\mathrm{d}t<\infty$. Thus, it remains to show that (5.33) holds and $\int_0^1(f(t)/g(t))\mathrm{d}t<\infty$.

We first establish (5.33). Let $a=\alpha/(1-\alpha)$, where $f(t)\coloneqq 1/G(t(\log t^{-1})^p)=t^{1/a}\widetilde{\varpi}(t)$ with slowly varying function $\widetilde{\varpi}$ given by $\widetilde{\varpi}(t)=\log^{p/a}(1/t)\varpi(t\log^p(1/t))$. Thus, by Theorem A.56, the inverse f^{-1} of f admits the representation $f^{-1}(t)=t^a\widehat{\varpi}(t)$ for some slowly varying function $\widehat{\varpi}(t)$. This slowly varying function satisfies

$$t=f^{-1}(f(t))=f(t)^a\widehat{\varpi}(f(t))\implies\widehat{\varpi}(f(t))\sim t/f(t)^a\sim 1/\widetilde{\varpi}(t)^a,\quad\text{as } t\downarrow 0. \tag{5.34}$$

Since $a>1$, the function f^{-1} is not integrable at 0. Thus, by Karamata's theorem (see Theorem A.55) and (5.34), the inner integral in (5.33) satisfies

$$\int_{f(t/2)}^1\frac{1}{f^{-1}(x)}\mathrm{d}x\sim\frac{1}{a-1}f(t/2)^{1-a}\widehat{\varpi}(f(t))^{-1}\sim\frac{2^{(a-1)/a}}{a-1}f(t)^{1-a}\widetilde{\varpi}(t)^a,\quad\text{as } t\downarrow 0.$$

Since $t/g(t)=t^{-1/a}/\varpi(t)$ for $t>0$, condition (5.33) holds if and only if the following integral is finite

$$\int_0^1 f(t)^{1-a}\frac{\widetilde{\varpi}(t)^a}{\varpi(t)}t^{-1/a}\mathrm{d}t=\int_0^1\log^{p/a}(1/t)\frac{\varpi(t\log^p(1/t))}{\varpi(t)}\frac{\mathrm{d}t}{t}.$$

The integrand is asymptotically equivalent to $\log^{p/a}(1/t)t^{-1}$, since we have that $\varpi(t\log^p(1/t))/\varpi(t)\to 1$ as $t\downarrow 0$ uniformly on $[0,1]$ by Theorem A.58 and our assumption on ϖ. Thus, the condition $p<-a$ makes the integral in display finite, proving condition (5.33).

To prove that $\int_0^1 (f(t)/g(t))\mathrm{d}t < \infty$, take any $\delta > 0$ with $p(1/a - \delta) < -1$ (recall $p/a < -1$ by assumption) and apply Potter's bound, i.e. Theorem A.53(iii), with δ to obtain, for some constant $K > 0$,

$$\int_0^1 \frac{f(t)}{g(t)}\mathrm{d}t = \int_0^1 \frac{g(t\log^p(1/t))}{g(t)\log^p(1/t)}\frac{\mathrm{d}t}{t} \leq K\int_0^1 \log^{p(1/a-\delta)}(1/t)\frac{\mathrm{d}t}{t} < \infty.$$

Part (iii). The result follows from Corollary 5.4 and Lemma 5.15(i). □

§5.5 Elementary estimates

Recall that (σ^2, γ, ν) is the generating triplet of X and the definition of the functions $\overline{\gamma}$, $\overline{\sigma}^2$ and $\overline{\nu}$ in (2.2).

Lemma 5.36. *For any $p \in (0, 2]$, $t, K > 0$ and $\varepsilon \in (0, 1]$, the following bounds hold*

$$\mathbb{E}[(|X_t| \wedge K)^p] \leq \big((\gamma - \overline{\gamma}(\varepsilon))^2 t^2 + (\overline{\sigma}^2(\varepsilon) + \sigma^2)t\big)^{p/2} + K^p\overline{\nu}(\varepsilon)t,$$
$$\mathbb{P}(|X_t| \geq K) \leq \big((\gamma - \overline{\gamma}(\varepsilon))^2 t^2 + (\overline{\sigma}^2(\varepsilon) + \sigma^2)t\big)/K^2 + \overline{\nu}(\varepsilon)t.$$

Proof. Let $X_t = (\gamma - \overline{\gamma}(\varepsilon))t + J_t + M_t$ be the Lévy-Itô decomposition of X at level ε, where J is compound Poisson containing all of the jumps of X with magnitude at least ε and M_t is a martingale with jumps of size smaller than ε. Fix $t > 0$ and define the event A of not observing any jump of J on the time interval $[0, t]$. Clearly $1 - \mathbb{P}(A) = 1 - e^{-\overline{\nu}(\varepsilon)t} \leq \overline{\nu}(\varepsilon)t$. Consider the elementary inequality $|X_t|^p \wedge K^p \leq |(\gamma - \overline{\gamma}(\varepsilon))t + M_t|^p \mathbb{1}_A + K^p \mathbb{1}_{A^c}$. Taking expectations and applying Jensen's inequality (with the concave function $x \mapsto x^{p/2}$ on $(0, \infty)$), we obtain the bound

$$\mathbb{E}\big(|X_t|^p \wedge K^p\big) \leq \big((\gamma - \overline{\gamma}(\varepsilon))^2 t^2 + \mathbb{E}\big[M_t^2\big]\big)^{p/2} + K^p(1 - \mathbb{P}(A)),$$

because $\mathbb{E}M_t = 0$. The first inequality readily follows. The second inequality follows from the first one: using Markov's inequality we get

$$\mathbb{P}(|X_t| \geq K) = \mathbb{P}(|X_t| \wedge K \geq K) \leq \mathbb{E}(X_t^2 \wedge K^2)/K^2.$$

Thus, the second result follows from the first with $p = 2$. □

§5.6 Concluding remarks

The points on the boundary of the convex hull of a Lévy path where the slope increases continuously were characterised (in terms of the law of the process) in Chapter 4. In this chapter we address the question of the rate of increase for the derivative of the boundary at these points in terms of lower and upper functions, both when the tangent has finite slope and when it is vertical (i.e. of infinite slope). Our results cover a large class of Lévy processes, presenting a comprehensive picture of this behaviour. Our aim was not to provide the best possible result in each case and indeed many extensions and refinements are possible. Below we list a few that arose while discussing our results in §5.2 as well as other natural questions.

- Find an explicit description of the lower (resp. upper) fluctuations in the finite (resp. infinite) slope regime for Lévy processes in the domain of attraction of an α-stable process in terms of the normalising function (cf. Corollaries 5.4 and 5.12). In the finite slope regime, this appears to require a refinement of Theorem A.20 for processes in this class.
- In Theorems 5.7 and 5.13 we find the correct power of the logarithmic factor, in terms of the positivity parameter ρ, in the definition of the function f for processes in the domain of attraction of an α-stable process. It is natural to ask what powers of iterated logarithm arise and how the boundary value is linked to the characteristics of the Lévy process. This question might be tractable for α-stable processes since power series and other formulae exist for their transition densities [75, Sec. 4], allowing higher order control of the Laplace transform Φ in Lemmas 5.34 and 5.35.
- Find the analogue of Theorems 5.7 and 5.13 for processes attracted to Cauchy process (see Remarks 5.8(a) and 5.14(b) for details).
- Find Lévy processes for which there exists a deterministic function f such that any of the following limits is positive and finite: $\limsup_{t\downarrow 0}(C'(t+\tau_s)-s)/f(t)$, $\liminf_{t\downarrow 0}(C'(t+\tau_s)-s)/f(t)$, $\limsup_{t\downarrow 0}|C'(t)|f(t)$ or $\liminf_{t\downarrow 0}|C'(t)|f(t)$. By Corollaries 5.4 and 5.12, such a function does not exist for $\liminf_{t\downarrow 0}(C'(t+\tau_s)-s)/f(t)$ or $\limsup_{t\downarrow 0}|C'(t)|f(t)$ within the class of regularly varying functions and α-stable processes with jumps of both signs.

Chapter 6

Hölder continuity of the convex minorant of a Lévy process

§6.1 Introduction and main results

The Hölder continuity[1] of continuous random functions is a classical topic, analysed extensively for Brownian motion and related processes, see e.g. [73] for fractional Brownian motion. Typically, such results make use of Kolmogorov's extension theorem (see Theorem A.2). The convex minorant of a Lévy process is a continuous random function, which may but need not be smooth as described in Chapters 4 & 5, motivating the question of its Hölder continuity. However, as the increments of the convex minorant (and their moments) are not tractable and its local behaviour varies greatly with the characteristics of the Lévy process (as seen in Chapter 4), Kolmogorov's extension theorem is not the right tool. In this chapter we establish sufficient and necessary conditions for the Hölder continuity the convex minorant of a Lévy process, using a generalisation of the 0–1 law in Theorem 4.18, the characterisation of small time behaviour of the Lévy path in Theorem A.41 and an elementary lemma by Khinchine (see Lemma 6.7 below). We prove for example that, in the absence of a Brownian component, the critical Hölder exponent is the reciprocal of the Blumenthal–Getoor index for most infinite variation Lévy processes (complete results are given in Table 6.1 below). A short YouTube [15] video describes the results and the structure of our proofs.

Let $C = (C_t)_{t\in[0,T]}$ be the convex minorant of the one-dimensional Lévy process X (see Definition 2.16). By Proposition 4.3, C is Lipschitz (i.e., 1-Hölder) continuous

[1] Given $r \in (0, 1]$ the function $f : [0, T] \to \mathbb{R}$ is r-Hölder continuous by definition if its Hölder constant $\sup_{0\le x<y\le T} |f(y) - f(x)|/(y - x)^r$ is finite.

if and only if X is of finite variation. In what follows we assume that X is of infinite variation. Then C is not Lipschitz on $[0,T]$ but, by convexity, is Lipschitz on every interval $[\varepsilon, T-\varepsilon]$, $\varepsilon > 0$, with Lipschitz constant $\max\{|C'(\varepsilon)|, C'(T-\varepsilon)\}$ given in terms of the right-derivative C' of C. Note that the rate at which $\max\{|C'(\varepsilon)|, C'(T-\varepsilon)\}$ tends to infinity as $\varepsilon \downarrow 0$, analysed in Chapter 5, is insufficient to characterise the r-Hölder continuity of C on $[0,T]$ for $r \in (0,1)$, since $C'(\varepsilon)$ may fluctuate between functions that are not asymptotically equivalent as $\varepsilon \downarrow 0$, see Remark 5.14(a) (analogous behaviour is observed for $C'(T-\varepsilon)$).

Let σ and ν be the Gaussian coefficient and Lévy measure of X, respectively. Recall the definition of the Blumenthal–Getoor index β_+ from (2.3). Note that $\beta_+ \in [1,2]$ since X is of infinite variation. Finally, when $\beta_+ \in (1,2]$, define

$$J_{\beta_+} := \int_0^1 \mathbb{E}\big[\min\{|X_t|/t^{1/\beta_+}, 1\}^{\beta_+/(\beta_+-1)}\big]\frac{\mathrm{d}t}{t} \in (0,\infty].$$

Our results are summarised in Table 6.1 below.

<table>
<tr><th colspan="3">Infinite variation Lévy process X</th><th>r</th><th>Is C r-Hölder continuous?</th></tr>
<tr><td colspan="3" rowspan="2">$\sigma^2 > 0$</td><td>$0 < r < 1/2$</td><td>Yes</td></tr>
<tr><td>$1/2 \le r < 1$</td><td>No</td></tr>
<tr><td rowspan="5">$\sigma^2 = 0$</td><td colspan="2">$\beta_+ = 1$</td><td>$0 < r < 1$</td><td>Yes</td></tr>
<tr><td rowspan="4">$\beta_+ \in (1,2]$</td><td rowspan="2">$\int_{(-1,1)} |x|^{\beta_+}\nu(\mathrm{d}x) = \infty$</td><td>$0 < r < 1/\beta_+$</td><td>Yes</td></tr>
<tr><td>$1/\beta_+ \le r < 1$</td><td>No</td></tr>
<tr><td rowspan="2">$J_{\beta_+} < \infty$</td><td>$0 < r \le 1/\beta_+$</td><td>Yes</td></tr>
<tr><td>$1/\beta_+ < r < 1$</td><td>No</td></tr>
</table>

Table 6.1: Critical level $r \in (0,1)$ for r-Hölder continuity of the convex minorant is $1/2$ in the presence of a Brownian component and $1/\beta_+$ in its absence if the Blumenthal–Getoor index β_+ is greater than one.

Our results present an almost complete picture in the sense that only a small portion of Lévy processes are not covered in Table 6.1 above. Indeed, the conditions $\int_{(-1,1)} |x|^{\beta_+}\nu(\mathrm{d}x) = \infty$ and $J_{\beta_+} < \infty$ are mutually exclusive (by Theorem 6.2 and (6.1)) but not necessarily complementary. However, as discussed below, Proposition 6.3 suggests that there are very few Lévy processes with Blumenthal–Getoor index $\beta_+ < 2$ satisfying $\int_{(-1,1)} |x|^{\beta_+}\nu(\mathrm{d}x) < \infty = J_{\beta_+}$. In contrast, when X has no Brownian component and $\beta_+ = 2$, there is a class of Lévy processes where our methods are inconclusive, see Proposition 6.4 below for details.

§6.1.1 r-Hölder continuity and sets of r-slopes

The convex minorant C is piecewise linear with countably many maximal intervals of linearity (see, e.g. Theorem 2.18). Denote the corresponding sequences of horizontal lengths and vertical heights by $(\ell_n)_{n\in\mathbb{N}}$ and $(\xi_n)_{n\in\mathbb{N}}$, respectively. Thus, over the n-th interval of linearity (where C has slope ξ_n/ℓ_n), C is clearly r-Hölder with Hölder constant $|\xi_n|/\ell_n^r$. Our main objective is to characterise the Lévy processes with convex minorants that are r-Hölder continuous for $r \in (0,1)$. It turns out that, for a large class of Lévy processes, the a.s. finiteness of $k_r := \sup_{n\in\mathbb{N}} |\xi_n|/\ell_n^r$ implies that C is r-Hölder a.s. It is important to note that neither 0 nor T are the endpoints of an interval of linearity of C since X is of infinite variation (see §4.1.1.2), implying that, even though C is "locally r-Hölder" on $(0,T)$ (i.e. $k_r < \infty$), it may fail to be r-Hölder on $[0,T]$.

For any $r \in (0,1)$, define the set of r-slopes by $\mathcal{S}_r := \{\xi_n/\ell_n^r : n \in \mathbb{N}\}$, which is either a.s. bounded ($k_r < \infty$) or a.s. unbounded ($k_r = \infty$) by Corollary 4.19. By Lemma 6.5 below, we have:

$$k_r = \sup_{s\in\mathcal{S}_r} |s| \le \sup_{0\le u<t\le T} \frac{|C(t)-C(u)|}{(t-u)^r} \le \left(\sum_{s\in\mathcal{S}_r} |s|^{1/(1-r)}\right)^{1-r} =: K_r \quad \text{a.s.} \tag{6.1}$$

Note that the upper bound K_r on the r-Hölder constant in (6.1) is in fact the L^p-norm of C' for $p = 1/(1-r)$. The utility of (6.1) lies in the fact that it controls the Hölder continuity of convex minorant C, since C is r-Hölder if $K_r < \infty$ and it is *not* r-Hölder if $k_r = \infty$. Our main results, Theorems 6.1 and 6.2 below, show that, for most Lévy processes, k_r and K_r are simultaneously finite or infinite, i.e., $\mathbb{P}(\{K_r = \infty\} \cap \{k_r < \infty\}) = 0$, yielding Table 6.1. Since, by Proposition 6.6 below, for any Lévy process X and any $r \in (0,1)$, we have $\mathbb{P}(k_r = \infty) \in \{0,1\}$ and $\mathbb{P}(K_r = \infty) \in \{0,1\}$, the main function of Theorems 6.1 and 6.2 is thus to rule out the possibility of having $k_r < \infty = K_r$ a.s.

Theorem 6.1. *Let X be a Lévy process of infinite variation. If $\sigma^2 > 0$, then $k_r = \infty$ for $1/2 \le r < 1$ and $K_r < \infty$ for $0 < r < 1/2$. If $\sigma^2 = 0$, then $k_r = \infty$ for $1/\beta_+ < r < 1$ and $K_r < \infty$ for $0 < r < 1/\beta_+$.*

By the inequalities in (6.1), Theorem 6.1 characterises Hölder continuity of C when either $\beta_+ = 1$ or $\sigma^2 > 0$, implying the rows one, two and three in Table 6.1. Moreover, Theorem 6.1 reveals that the critical level of the Hölder exponent is $r = 1/\beta_+$ with $\beta_+ \in (1,2]$, considered next.

Theorem 6.2. *Let X be a Lévy process of infinite variation and suppose that $\sigma^2 = 0$ and $\beta_+ \in (1,2]$. The equivalences hold: (i) $\int_{(-1,1)} |x|^{\beta_+}\nu(\mathrm{d}x) = \infty \iff k_{1/\beta_+} = \infty$ a.s.; (ii) $J_{\beta_+} < \infty \iff K_{1/\beta_+} < \infty$ a.s.*

Implicit in Theorem 6.2 is the fact that $J_{\beta_+} < \infty$ implies $\int_{(-1,1)} |x|^{\beta_+}\nu(\mathrm{d}x) < \infty$. Checking the finiteness of the integral J_{β_+} in Theorem 6.2(ii) may appear hard as it is given in (§6.1) in terms of the truncated moments of the marginals of X. For $\beta_+ \in (1,2)$, we now give sufficient conditions for $J_{\beta_+} < \infty$ in terms of the Lévy measure ν. Recall the functions $\overline{\nu}$ and $\overline{\gamma}$ in (2.2). Note that, by Fubini's theorem, we have $\int_{(-1,1)} |x|^p\nu(\mathrm{d}x) = \int_0^1 \overline{\nu}(x^{1/p})\mathrm{d}x - \overline{\nu}(1)$ for any $p > 0$. In particular, the condition $\int_{(-1,1)} |x|^{\beta_+}\nu(\mathrm{d}x) < \infty$ is equivalent to $\int_0^1 \overline{\nu}(x^{1/\beta_+})\mathrm{d}x < \infty$.

Proposition 6.3. *Suppose $\sigma^2 = 0$, $\beta_+ \in (1,2)$ and $\int_0^1 \overline{\nu}(x^{1/\beta_+})\mathrm{d}x < \infty$. Consider the conditions:*

$$\text{(i)}\ \limsup_{x\downarrow 0} x\overline{\nu}\big(x^{1/\beta_+}\big) < \infty, \quad \text{(ii)}\ \int_0^1 x\overline{\nu}\big(x^{1/\beta_+}\big)^2\mathrm{d}x < \infty,$$

$$\text{(iii)}\ \int_0^1 x^{1-2/\beta_+}\overline{\gamma}\big(x^{1/\beta_+}\big)^2\mathrm{d}x < \infty, \quad \text{(iv)}\ \int_0^1 \mathbb{E}\big[\min\{|X_t|/t^{1/\beta_+},1\}^2\big]\frac{\mathrm{d}t}{t} < \infty,$$

$$\text{(v)}\ J_{\beta_+} = \int_0^1 \mathbb{E}\big[\min\{|X_t|/t^{1/\beta_+},1\}^{\beta_+/(\beta_+-1)}\big]\frac{\mathrm{d}t}{t} < \infty.$$

Then the following implications hold: (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii) $\Longrightarrow$ (iv) $\Longrightarrow$ (v).

By Proposition 6.3, for most processes with $\beta_+ \in (1,2)$, Theorem 6.2 characterises the $(1/\beta_+)$-Hölder continuity of C. Indeed, for Theorem 6.2 not to imply $K_{1/\beta_+} < \infty$ a.s., by Proposition 6.3, a Lévy measures ν would have to satisfy $\int_0^1 \overline{\nu}(x^{1/\beta_+})\mathrm{d}x < \infty = \int_0^1 x\overline{\nu}\big(x^{1/\beta_+}\big)^2\mathrm{d}x$. Put differently, the function $x \mapsto x\overline{\nu}(x^{1/\beta_+})$ would have to be integrable but not square integrable with respect to the measure $x^{-1}\mathrm{d}x$ on $(0,1)$. While such a ν could be constructed, it does not arise frequently in applications. Moreover, it is not clear whether a Lévy measure ν, satisfying $\int_0^1 \overline{\nu}(x^{1/\beta_+})\mathrm{d}x < \infty = J_{\beta_+}$, exists.

If X has no Brownian component (i.e. $\sigma^2 = 0$) but satisfies $\beta_+ = 2$, it is possible to have $k_{1/2} < \infty$ and $K_{1/2} = \infty$ a.s., rendering (6.1) insufficient to ascertain whether C is $\frac{1}{2}$-Hölder continuous. Indeed, the phenomenon $k_{1/2} < \infty = K_{1/2}$ occurs whenever the a.s. constant value $\lambda := \limsup_{t\downarrow 0} |X_t|/\sqrt{t} \in [0,\infty]$ (Theorem A.42 expresses λ in terms of ν) lies in $(0,\infty)$. In fact, we have the following.

Proposition 6.4. *Suppose $\sigma^2 = 0$, $\beta_+ = 2$ and set $\lambda := \limsup_{t\downarrow 0} |X_t|/\sqrt{t} \in [0,\infty]$. Then* (i) *$\lambda = \infty$ implies $k_{1/2} = \infty$ a.s.,* (ii) *$\lambda \in (0,\infty)$ implies $k_{1/2} < \infty = K_{1/2}$ a.s.,* (iii) *$\lambda = 0$ implies $k_{1/2} < \infty$ a.s.*

We suspect that $K_{1/2} < \infty$ a.s. in the case $\lambda = 0$ of Proposition 6.4. We were unable to establish this because the lower bound on the $(1/2)$-slopes in the proof of Proposition 6.4 is zero for $\lambda = 0$.

§6.1.2 Strategy for the proofs and connections with the literature

For any $r \in (0,1)$, we give sufficient as well as necessary conditions for the convex minorant C to be r-Hölder continuous in terms of the set of r-slopes $\mathcal{S}_r := \{\xi_n/\ell_n^r : n \in \mathbb{N}\}$ (see (6.1) and Lemma 6.5). In Proposition 6.6 we generalise the 0–1 law in Theorem 4.18 and use it to characterise the finiteness of K_r in terms of the truncated moments of the marginals of X. Through Khintchine's characterisation of the two-sided upper functions of X [47], given in Lemma 6.7 below, and the 0–1 law in Corollary 4.19, we find that $k_r < \infty$ a.s. if and only if $\limsup_{t\downarrow 0} |X_t|/t^r < \infty$ a.s., see Corollary 6.9 below. The final ingredient in the proofs of Theorems 6.1 and 6.2 are the characterisations of the limit $\limsup_{t\downarrow 0} |X_t|/t^r$ given in [70, Sec. 47] and [17], respectively (see §1.3.5). In §6.4 we discuss a possible extension of Theorem 6.1 and its connection to the characterisation in [83] of the limits $\limsup_{t\downarrow 0} |X_t|/h(t)$ for a non-decreasing h.

§6.2 Proofs of results from §6.1.1

We begin with an elementary deterministic lemma that implies the inequalities in (6.1).

Lemma 6.5. *Let f be an absolutely continuous, piece-wise linear function with infinitely many faces, defined on the interval $[a,b]$. Given any enumeration of the maximal intervals of linearity of f, let $(l_n)_{n\in\mathbb{N}}$ and $(h_n)_{n\in\mathbb{N}}$ be the corresponding sequences of horizontal lengths and vertical heights, respectively, of those line segments. Then for any $r \in (0,1)$ we have*

$$\sup_{n\in\mathbb{N}} |h_n| l_n^{-r} \le \sup_{a\le u<t\le b} \frac{|f(t)-f(u)|}{(t-u)^r} \le \bigg(\sum_{n\in\mathbb{N}} (|h_n| l_n^{-r})^{1/(1-r)}\bigg)^{1-r}.$$

Proof. Fix $r \in (0,1)$ and let $p = 1/(1-r) > 1$. Let (g_n, d_n), $n \in \mathbb{N}$, be the maximal intervals of linearity of f where the slope of f over (g_n, d_n) equals h_n/l_n and $\sum_{n\in\mathbb{N}} (d_n - g_n) = b - a$. The lower bound is obvious since it is attained by restricting the supremum to the values $(u,t) = (g_n, d_n)$. To establish the upper bound, first note that f' exists on the set $\bigcup_{n\in\mathbb{N}} (g_n, d_n)$ of measure $b-a$ and

$$\int_a^b |f'(t)|^p \mathrm{d}t = \sum_{n\in\mathbb{N}} \int_{g_n}^{d_n} |f'(t)|^p \mathrm{d}t = \sum_{n\in\mathbb{N}} \frac{|h_n|^p}{l_n^p} \int_{g_n}^{d_n} \mathrm{d}t = \sum_{n\in\mathbb{N}} \frac{|h_n|^p}{l_n^{p-1}}.$$

By Hölder's inequality with p and $q = p/(p-1) = 1/r > 1$, it follows

$$|f(t) - f(u)| \le \int_u^t |f'(x)| \mathrm{d}x = \int_a^b \mathbb{1}_{[u,t]}(x) |f'(x)| \mathrm{d}x \le (t-u)^{1/q} \bigg(\int_a^b |f'(x)|^p \mathrm{d}x\bigg)^{1/p}.$$

Thus, we have

$$\sup_{a\le u<t\le b}\frac{|f(t)-f(u)|}{(t-u)^r}\le\bigg(\int_a^b|f'(x)|^p\mathrm{d}x\bigg)^{1/p}$$
$$=\bigg(\sum_{n\in\mathbb{N}}\frac{|h_n|^p}{l_n^{p-1}}\bigg)^{1/p}=\bigg(\sum_{n\in\mathbb{N}}(|h_n|l_n^{-r})^{1/(1-r)}\bigg)^{1-r}.\quad\square$$

The proofs of Theorems 6.1 & 6.2 hinge on two key tools. First is the 0–1 law in Proposition 6.6, generalising Theorem 4.18 to unbounded functionals of the faces of C, and second is Khintchine's characterisation of the upper functions of $|X|$ at zero given in Lemma 6.7 below. Recall that since X is of infinite activity, its convex minorant C is a piecewise linear function whose *maximal* intervals of linearity have corresponding sequences of horizontal lengths $(\ell_n)_{n\in\mathbb{N}}$ and vertical heights $(\xi_n)_{n\in\mathbb{N}}$ given by the formulae in Theorem 2.18.

Proposition 6.6. *Let $\phi:\mathbb{R}\times(0,\infty)\to[0,\infty)$ be measurable. Then the sum $\sum_{n\in\mathbb{N}}\phi(\xi_n,\ell_n)$ is either a.s. finite or a.s. infinite. Moreover, we have*

$$\sum_{n\in\mathbb{N}}\phi(\xi_n,\ell_n)<\infty\quad a.s.\qquad\iff\qquad\int_0^1\mathbb{E}[\min\{\phi(X_t,t),1\}]\frac{\mathrm{d}t}{t}<\infty.\tag{6.2}$$

Proof. Note that $\sum_{n\in\mathbb{N}}a_n<\infty$ if and only if $\sum_{n\in\mathbb{N}}\min\{a_n,1\}<\infty$ for any sequence $(a_n)_{n\in\mathbb{N}}$ in $[0,\infty)$. Thus, it follows that $\sum_{n\in\mathbb{N}}\phi(\xi_n,\ell_n)<\infty$ a.s. if and only if $\sum_{n\in\mathbb{N}}\min\{\phi(\xi_n,\ell_n),1\}<\infty$ a.s. and the equivalence in (6.2) follows from Theorem 2.18 and the 0–1 law in Theorem 4.18 applied to the bounded function $(t,x)\mapsto\min\{\phi(x,t),1\}$. $\square$

The following characterisation due to Khintchine [47] is central in relating the upper fluctuations of $|X|$ and the faces of C. Recall that, for any positive measurable function $h:(0,\infty)\to(0,\infty)$, $\limsup_{t\downarrow 0}|X_t|/h(t)$ is a.s. a constant on $[0,\infty]$ by Blumenthal's 0–1 law Corollary A.1 (see also [70, Prop. 40.4]).

Lemma 6.7 (Khintchine)**.** *Suppose X is not compound Poisson with drift. Let $h:(0,\infty)\to(0,\infty)$ be measurable and increasing at 0 and fix $R>0$. Then the following statements hold.*

(i) If $\int_0^1\mathbb{P}(|X_t|/h(t)>R/4)t^{-1}\mathrm{d}t<\infty$, then $\limsup_{t\downarrow 0}|X_t|/h(t)\le R$ a.s.

(ii) If $\int_0^1\mathbb{P}(|X_t|/h(t)>8R)t^{-1}\mathrm{d}t=\infty$, then $\limsup_{t\downarrow 0}|X_t|/h(t)\ge R$ a.s.

Remark 6.8. For completeness, we give a short elementary proof of Lemma 6.7 in §6.3 below. It is based on Khintchine's proof of a closely related result in [47, Fundamental lemma]. It is not essential for the results in this thesis, but it is natural to enquire whether Lemma 6.7 holds with the constants $R/4$ and $8R$ in the integral conditions substituted by R. $\Diamond$

Corollary 6.9. *Suppose X is not compound Poisson with drift. Let $h:(0,\infty)\to(0,\infty)$ be measurable and increasing at 0. Define the set of h-slopes $\mathcal{S}_h := \{\xi_n/h(\ell_n) : n\in\mathbb{N}\}$ and set $k_h := \sup_{s\in\mathcal{S}_h}|s|$. Then $\mathbb{P}(k_h=\infty)\in\{0,1\}$. Moreover, $k_h<\infty$ a.s. if and only if $\limsup_{t\downarrow 0}|X_t|/h(t)<\infty$ a.s.*

Proof. Suppose there exists $R\in(0,\infty)$ such that $\int_0^1 \mathbb{P}(|X_t|/h(t)>R)t^{-1}\mathrm{d}t<\infty$. Then Corollary 4.19 (applied to $f(t,x)=|x|/h(t)$) implies that $\mathcal{S}_h\cap(\mathbb{R}\setminus[-R,R])$ is a.s. a finite set and hence $k_h<\infty$ a.s. Similarly, since $\limsup_{t\downarrow 0}|X_t|/h(t)$ is a.s. constant, Lemma 6.7(i) implies $\limsup_{t\downarrow 0}|X_t|/h(t)\le 4R$.

Next assume that for all $R\in(0,\infty)$ we have $\int_0^1 \mathbb{P}(|X_t|/h(t)>R)t^{-1}\mathrm{d}t=\infty$. Then Corollary 4.19 (applied to $f(t,x)=|x|/h(t)$) implies that $\mathcal{S}_h\cap(\mathbb{R}\setminus[-R,R])$ is a.s. an infinite set for any $R>0$. Hence $k_h\ge R$ a.s. for any $R>0$, implying that $k_h=\infty$ a.s. Similarly, Lemma 6.7(ii) implies $\limsup_{t\downarrow 0}|X_t|/h(t)\ge R/8$ a.s. for any $R>0$ and hence $\limsup_{t\downarrow 0}|X_t|/h(t)=\infty$.

Since $\int_0^1 \mathbb{P}(|X_t|/h(t)>R)t^{-1}\mathrm{d}t$ is either finite for some R or infinite for all R, it follows that $\mathbb{P}(k_h=\infty)$ is either 0 or 1, respectively. Moreover, the former (resp. latter) case implies that $\limsup_{t\downarrow 0}|X_t|/h(t)$ is finite (resp. infinite) a.s. $\square$

Proof of Theorem 6.1. First note that, for any $p>0$ the sum $\sum_{n\in\mathbb{N}}\ell_n^p$ is finite a.s. (with mean T^p/p) by Theorem 4.18. Pick any $r'>r$ and note that $|\xi_n|/\ell_n^r\le k_{r'}\ell_n^{r'-r}$ for every $n\in\mathbb{N}$, implying

$$K_r^{1/(1-r)}=\sum_{s\in\mathcal{S}_r}|s|^{1/(1-r)}=\sum_{n\in\mathbb{N}}(|\xi_n|/\ell_n^r)^{1/(1-r)}\le k_{r'}^{1/(1-r)}\sum_{n\in\mathbb{N}}\ell_n^{(r'-r)/(1-r)}. \quad (6.3)$$

In particular, $K_r<\infty$ whenever $k_{r'}<\infty$ for some $r'>r$.

Assume first that $\sigma^2>0$, then $\limsup_{t\downarrow 0}|X_t|/\sqrt{t\log\log(1/t)}=\sqrt{2}|\sigma|>0$ by Proposition A.43. Thus, the limit $\limsup_{t\downarrow 0}|X_t|/t^r$ equals 0 (resp. ∞) a.s. for $r\in(0,1/2)$ (resp. $r\in[1/2,1)$). Then, by Corollary 6.9, we have $k_r=\infty$ for all $r\in[1/2,1)$ and $k_{(r+1/2)/2}<\infty$ for $r\in(0,1/2)$ since $(r+1/2)/2<1/2$. In the latter case, $r<(r+1/2)/2$ and hence $K_r<\infty$ by (6.3).

Next assume $\sigma^2=0$. By Proposition A.39, the limit $\limsup_{t\downarrow 0}|X_t|/t^r$ equals 0 (resp. ∞) a.s. for $r\in(0,1/\beta_+)$ (resp. $r\in(1/\beta_+,1)$) where β_+ is the Blumenthal–Getoor index defined in (2.3). As before, by Corollary 6.9, we have $k_r=\infty$ for all $r\in(1/\beta_+,1)$ and $k_{(r+1/\beta_+)/2}<\infty$ for $r\in(0,1/\beta_+)$ since $(r+1/\beta_+)/2<1/\beta_+$. In the latter case, $r<(r+1/\beta_+)/2$ and hence $K_r<\infty$ by (6.3). $\square$

Recall, by Fubini's theorem, $\int_{(-1,1)}|x|^p\nu(\mathrm{d}x)=\int_0^1\overline{\nu}(t^{1/p})\mathrm{d}t-\overline{\nu}(1)$ for any $p>0$.

Proof of Theorem 6.2. Let $r = 1/\beta_+ \in [1/2, 1)$. By Theorem A.41, $\int_0^1 \overline{\nu}(t^r)\mathrm{d}t$ is finite (resp. infinite) if and only if $\limsup_{t\downarrow 0} |X_t|/t^r$ is finite (resp. infinite) a.s. Thus, by Corollary 6.9, $\int_0^1 \overline{\nu}(t^r)\mathrm{d}t = \infty$ if and only if $k_r = \infty$. By Proposition 6.6 (with $\phi(x,t) = (|x|/t^r)^{1/(1-r)}$): $J_{\beta_+} = \int_0^1 \mathbb{E}[\min\{|X_t|/t^r, 1\}^{1/(1-r)}]t^{-1}\mathrm{d}t$ is finite if and only if $K_r^{1/(1-r)} = \sum_{n\in\mathbb{N}} |\xi_n|^{1/(1-r)}/\ell_n^{r/(1-r)}$ is finite a.s., completing the proof. $\square$

Proof of Proposition 6.3. Let $r = 1/\beta_+$ and note that $\int_0^1 \overline{\nu}(t^r)\mathrm{d}t < \infty$. It is thus clear that (i) $\limsup_{t\downarrow 0} t\overline{\nu}(t^r) < \infty$ implies (ii) $\int_0^1 t\overline{\nu}(t^r)^2\mathrm{d}t < \infty$. Since $1/(1-r) = \beta_+/(\beta_+ - 1) > 2$ and $\min\{|x|, 1\}^p \le \min\{|x|, 1\}^q$ for $p \ge q$, (iv) implies (v). It remains to show that (ii) $\implies$ (iii) $\implies$ (iv).

Let us show that (ii) implies (iii) $\int_0^1 t^{1-2r}\overline{\gamma}(t^r)^2\mathrm{d}t < \infty$. Denote $\overline{\nu}_1(x) := \overline{\nu}(x) - \overline{\nu}(1)$ for $x \in (0,1]$ and recall $\overline{\gamma}(u) = \int_{(-1,1)\setminus(-u,u)} x\nu(\mathrm{d}x)$ for $u \in (0,1]$. By Fubini's theorem, we have

$$|\overline{\gamma}(u)| \le \int_{(-1,1)} \mathbb{1}_{\{u\le|x|<1\}} \int_0^{|x|} \mathrm{d}y\nu(\mathrm{d}x) = \int_0^1 \overline{\nu}_1(\max\{y,u\})\mathrm{d}y$$
$$= u\overline{\nu}_1(u) + \int_u^1 \overline{\nu}_1(y)\mathrm{d}y.$$

Hence, the elementary inequality $(a+b)^2 \le 2(a^2+b^2)$ yields

$$\frac{1}{2}t^{1-2r}\overline{\gamma}(t^r)^2 \le t\overline{\nu}(t^r)^2 + t^{1-2r}\bigg(\int_{t^r}^1 \overline{\nu}(y)\mathrm{d}y\bigg)^2, \quad t \in (0,1].$$

Since $\int_0^1 t\overline{\nu}(t^r)^2\mathrm{d}t < \infty$ by assumption (ii), to establish (iii) we need only show that the integral $\int_0^1 t^{1-2r}(\int_{t^r}^1 \overline{\nu}(y)\mathrm{d}y)^2\mathrm{d}t$ is finite. Since $\min\{a,b\}^2 \le ab$ and $r = 1/\beta_+ < 1$, Fubini's theorem yields

$$\begin{aligned}
(2-2r)&\int_0^1 t^{1-2r}\bigg(\int_{t^r}^1 \overline{\nu}(y)\mathrm{d}y\bigg)^2\mathrm{d}t \\
&= (2-2r)\int_0^1\int_0^1 \bigg(\int_0^{\min\{x,y\}^{1/r}} t^{1-2r}\mathrm{d}t\bigg)\overline{\nu}(x)\overline{\nu}(y)\mathrm{d}x\mathrm{d}y \\
&= \int_0^1\int_0^1 \min\{x,y\}^{2/r-2}\overline{\nu}(x)\overline{\nu}(y)\mathrm{d}x\mathrm{d}y \\
&\le \bigg(\int_0^1 x^{1/r-1}\overline{\nu}(x)\mathrm{d}x\bigg)^2 = \bigg(r\int_0^1 \overline{\nu}(t^r)\mathrm{d}t\bigg)^2 < \infty.
\end{aligned}$$

It remains to show that the condition (iii) $\int_0^1 t^{1-2r}\overline{\gamma}(t^r)^2\mathrm{d}t < \infty$ implies the following (iv) $\int_0^1 \mathbb{E}[\min\{|X_t|/t^r, 1\}^2]t^{-1}\mathrm{d}t < \infty$. Let γ be the drift parameter of X corresponding to the cutoff function $x \mapsto \mathbb{1}_{(-1,1)}(x)$ (see §2.2) and recall the function $u \mapsto \overline{\sigma}^2(u)$ from (2.2). Apply Lemma 5.36 (with $\varepsilon = K = t^r$) to obtain

$$\int_0^1 \mathbb{E}\big[\min\{|X_t|, t^r\}^2\big]\frac{\mathrm{d}t}{t^{1+2r}} \le \int_0^1 \big[t^2(\gamma - \overline{\gamma}(t^r))^2 + t\overline{\sigma}^2(t^r) + t^{2r+1}\overline{\nu}(t^r)\big]\frac{\mathrm{d}t}{t^{1+2r}}.$$

Since the integrals $\int_0^1 \gamma^2 t^{1-2r}\mathrm{d}t = \gamma^2/(2-2r)$, $\int_0^1 t^{1-2r}\overline{\gamma}(t^r)^2\mathrm{d}t$ and $\int_0^1 \overline{\nu}(t^r)\mathrm{d}t$ are all finite (recall $r = 1/\beta_+ < 1$), it remains to show that $\int_0^1 t^{-2r}\overline{\sigma}^2(t^r)\mathrm{d}t < \infty$. By Fubini's theorem, we obtain

$$\overline{\sigma}^2(x) = \int_{(-x,x)} \left(\int_0^{|u|} 2y\mathrm{d}y\right)\nu(\mathrm{d}u) = 2\int_0^x y(\overline{\nu}(y) - \overline{\nu}(x))\mathrm{d}y \le 2\int_0^x y\overline{\nu}(y)\mathrm{d}y,$$

for $x \in (0,1]$. Thus, again by Fubini's theorem (recall that $r = 1/\beta_+ \in (1/2,1)$),

$$\int_0^1 t^{-2r}\overline{\sigma}^2(t^r)\mathrm{d}t \le 2\int_0^1\int_0^{t^r} t^{-2r}y\overline{\nu}(y)\mathrm{d}y\mathrm{d}t = 2\int_0^1 \frac{y^{1/r-2}-1}{2r-1}y\overline{\nu}(y)\mathrm{d}y.$$

Since, $\int_0^1 y^{1/r-1}\overline{\nu}(y)\mathrm{d}y = r\int_0^1 \overline{\nu}(t^r)\mathrm{d}t < \infty$, the integral on the right side of the display above is finite. □

Proof of Proposition 6.4. Define $\lambda := \limsup_{t\downarrow 0}|X_t|/\sqrt{t} \in [0,\infty]$. If $\lambda = \infty$, then $k_{1/2} = \infty$ by Corollary 6.9. If $\lambda \in [0,\infty)$ then $k_{1/2} < \infty$ by Corollary 6.9. Finally, assume $\lambda \in (0,\infty)$. Then $\int_0^1 t^{-1}\mathbb{P}(|X_t|/\sqrt{t} > R)\mathrm{d}t = \infty$ for $R < \lambda/4$ by Lemma 6.7(i). Thus, for any $\varepsilon \in (0,\lambda/4)$, $\mathcal{S}_{1/2}$ has infinitely many points with magnitude on the interval $[\varepsilon,\infty)$ by Corollary 4.19, implying $K_r = \infty$. □

§6.3 Proof of Lemma 6.7

We present a short proof of Lemma 6.7, based on the proof of [47, Fundamental lemma].

Proof of Lemma 6.7. Fix $0 < s < t$ and $0 < y < x$, then $\{|X_t| \ge x\} \subset \{|X_s| \ge y\} \cup \{|X_t - X_s| \ge x - y\}$. Since $X_t - X_s \overset{d}{=} X_{t-s}$, this yields

$$\mathbb{P}(|X_t| \ge x) \le \mathbb{P}(|X_s| \ge y) + \mathbb{P}(|X_{t-s}| \ge x - y). \tag{6.4}$$

In particular, the choice $s = t/2$ and $y = x/2$ gives $\mathbb{P}(|X_t| \ge x) \le 2\mathbb{P}(|X_{t/2}| \ge x/2)$. Without loss of generality, we assume throughout h is non-decreasing on $(0,1]$.

Part (i). It suffices to show that, given $R > 0$, the condition $\int_0^1 \mathbb{P}(|X_t| > Rh(t))t^{-1}\mathrm{d}t < \infty$ implies $\limsup_{t\downarrow 0}|X_t|/h(t) \le 4R$ a.s. The proof is split in three steps.

Step 1. We first show that $\mathbb{P}(|X_t| > 2Rh(t)) \to 0$ as $t \downarrow 0$ and, in particular, there exists some $\varepsilon > 0$ such that $\mathbb{P}(|X_t| > 2Rh(t)) < 1/2$ for all $t \in (0,\varepsilon)$. Since h is non-decreasing, (6.4) implies

$$\mathbb{P}(|X_t| > 2Rh(t)) \le \mathbb{P}(|X_s| > Rh(s)) + \mathbb{P}(|X_{t-s}| > Rh(t-s)).$$

Integrating the previous inequality over $[t/2, t]$ w.r.t. the measure $s^{-1}\mathrm{d}s$, yields

$$\begin{aligned}\mathbb{P}(|X_t| > 2Rh(t))\log 2 &\le \int_{t/2}^{t} \mathbb{P}(|X_s| > Rh(s))\frac{\mathrm{d}s}{s} + \int_{t/2}^{t} \mathbb{P}(|X_{t-s}| > Rh(t-s))\frac{\mathrm{d}s}{s}\\ &\le \int_{t/2}^{t} \mathbb{P}(|X_s| > Rh(s))\frac{\mathrm{d}s}{s} + \int_{t/2}^{t} \mathbb{P}(|X_{t-s}| > Rh(t-s))\frac{\mathrm{d}s}{t-s}\\ &= \int_{0}^{t} \mathbb{P}(|X_s| > Rh(s))\frac{\mathrm{d}s}{s} < \infty.\end{aligned}$$

The integral $\int_0^t \mathbb{P}(|X_s| > Rh(s))s^{-1}\mathrm{d}s$ is finite and vanishes as $t \downarrow 0$, implying the following limit $\mathbb{P}(|X_t| > 2Rh(t)) \to 0$ as $t \downarrow 0$.

Step 2. Define $\overline{X}_t := \sup_{s\in[0,t]} X_s$ for $t \ge 0$. We will show that $\mathbb{P}(\overline{X}_t > 4Rh(t)) \le 2\mathbb{P}(X_t > 2Rh(t))$ for $t \in (0, \varepsilon)$ where ε is as in Step 1. Fix $n \in \mathbb{N}$, set $t_k := tk/n$ for $k \in \{1, \dots, n\}$ and define the events

$$A_k := \{X_{t_i} \le 4Rh(t) \text{ for all } i \in \{1, \dots, k-1\}\} \cap \{X_{t_k} > 4Rh(t)\}, \quad k \in \{1, \dots, n\}.$$

Since the increments of X are independent and stationary, we have

$$\begin{aligned}\mathbb{P}(X_t > 2Rh(t)|A_k) &\ge \mathbb{P}(X_t - X_{t_k} > -2Rh(t)|A_k) = \mathbb{P}(X_t - X_{t_k} > -2Rh(t))\\ &\ge \mathbb{P}(|X_t - X_{t_k}| < 2Rh(t)) = \mathbb{P}(|X_{t-t_k}| < 2Rh(t))\\ &\ge \mathbb{P}(|X_{t-t_k}| < 2Rh(t - t_k)).\end{aligned}$$

By step 1, for all $t \in (0, \varepsilon)$ we have $t - t_k < \varepsilon$ and hence $\mathbb{P}(X_t > 2Rh(t)|A_k) > 1/2$ for all $k \in \{1, \dots, n\}$.

Define $M_t^{(n)} := \max_{1\le k\le n} X_{t_k}$, then $\{M_t^{(n)} > 4Rh(t)\} = \bigcup_{k=1}^n A_k$. Since the sets A_k are disjoint, for any $t \in (0, \varepsilon)$ we have

$$\begin{aligned}\mathbb{P}\big(M_t^{(n)} > 4Rh(t)\big) = \sum_{k=1}^{n} \mathbb{P}(A_k) &\le 2\sum_{k=1}^{n} \mathbb{P}(A_k)\mathbb{P}(X_t > 2Rh(t)|A_k)\\ &\le 2\mathbb{P}(X_t > 2Rh(t)).\end{aligned}$$

Since X is right-continuous with limits from the left, $M_t^{(2^n)} \uparrow \overline{X}_t$ a.s. as $n \to \infty$. Hence, the monotone convergence theorem yields that $\mathbb{P}(\overline{X}_t > 4Rh(t)) = \lim_{n\to\infty} \mathbb{P}(M_t^{(2^n)} > 4Rh(t)) \le 2\mathbb{P}(X_t > 2Rh(t))$.

Step 3. Define the probability $p_n := \mathbb{P}(\sup_{t\in[2^{-n},2^{1-n}]}(X_t/h(t)) > 4R)$ for $n \in \mathbb{N}$ and let n_ε be the smallest positive integer larger than $1 + \log(1/\varepsilon)/\log 2$, where ε is as in Step 1. Since h is non-decreasing, Step 2 and (6.4) imply that for all $n \ge n_\varepsilon$

and $t \in [2^{-n}, 2^{1-n}]$ we have

$$
\begin{aligned}
p_n &\le \mathbb{P}\Big(\sup_{t \in [2^{-n}, 2^{1-n}]} X_t > 4Rh(2^{-n}) \Big) = \mathbb{P}\big(\overline{X}_{2^{-n}} > 4Rh(2^{-n})\big) \\
&\le \mathbb{P}\big(\overline{X}_t > 4Rh(2^{-n})\big) \le 2\mathbb{P}(X_t > 2Rh(2^{-n})) \le 2\mathbb{P}(X_t > 2Rh(t/2)) \\
&\le 4\mathbb{P}(X_{t/2} > Rh(t/2)).
\end{aligned}
$$

Integrating the previous inequality over $t \in [2^{-n}, 2^{1-n}]$ and summing over $n \ge n_\varepsilon$ gives

$$
\begin{aligned}
\sum_{n=n_\varepsilon}^{\infty} p_n \frac{\log 2}{4} &= \sum_{n=n_\varepsilon}^{\infty} \int_{2^{-n}}^{2^{1-n}} \frac{p_n}{4} \frac{\mathrm{d}t}{t} \le \sum_{n=1}^{\infty} \int_{2^{-n}}^{2^{1-n}} \mathbb{P}(X_{t/2} > Rh(t/2)) \frac{\mathrm{d}t}{t} \\
&= \int_0^2 \mathbb{P}(X_t > Rh(t)) \frac{\mathrm{d}t}{t} < \infty.
\end{aligned}
$$

The Borel–Cantelli lemma implies $\sup_{t \in [2^{-m-1}, 2^{-m}]}(X_t/h(t)) \le 4R$ for all but finitely many n, implying $\limsup_{t \to 0} X_t/h(t) \le 4R$ a.s. By symmetry, it follows that $\limsup_{t \to 0}(-X_t)/h(t) \le 4R$ a.s., proving part (i).

Part (ii). It suffices to show that, given $R > 0$, the condition $\int_0^1 \mathbb{P}(|X_t|/h(t) > 8R) t^{-1} \mathrm{d}t = \infty$ implies $\limsup_{t \downarrow 0} |X_t|/h(t) \ge R$ a.s. The proof is split in three steps.

Step 1. Define $M(t) \coloneqq \sup_{s \in (0,t]}(|X_s|/h(s))$. We will show that

$$
B_n \coloneqq \Big\{ \sup_{t \in [2^{-n-1}, 2^{-n}]} |X_t - X_{2^{-n-1}}| > 2Rh(2^{-n}),\ M(2^{-n-1}) \le R \Big\} \subset \{M(2^{-n}) > R\}. \tag{6.5}
$$

To see (6.5) note that, on the event B_n there exists some $t \in [2^{-n-1}, 2^{-n}]$ satisfying

$$
\begin{aligned}
M(2^{-n}) \ge |X_t| \ge |X_t - X_{2^{-n-1}}| - |X_{2^{-n-1}}| &> 2Rh(2^{-n}) - Rh(2^{-n-1}) \\
&\ge Rh(2^{-n}) \ge Rh(t).
\end{aligned}
$$

Step 2. We claim $\sum_{n \in \mathbb{N}} q_n = \infty$, where $q_n \coloneqq \mathbb{P}(\sup_{t \in [0, 2^{-n-1}]} |X_t| > 2Rh(2^{-n}))$. For $t \le 2^{-n-1}$, apply (6.4) twice to get $4q_n \ge 4\mathbb{P}(|X_t| > 2Rh(2^{-n})) \ge \mathbb{P}(|X_{4t}| > 8Rh(2^{-n}))$. Hence, for any $t \in [2^{-n-2}, 2^{-n-1}]$, we have $4q_n \ge \mathbb{P}(|X_{4t}| > 8Rh(4t))$. Integrating the previous inequality on $[2^{-n-2}, 2^{-n-1}]$ with respect to $t^{-1}\mathrm{d}t$ yields

$$
(4 \log 2) q_n \ge \int_{2^{-n-2}}^{2^{-n-1}} \mathbb{P}(|X_{4t}| > 8Rh(4t)) \frac{\mathrm{d}t}{t} = \int_{2^{-n}}^{2^{-n+1}} \mathbb{P}(|X_t| > 8Rh(t)) \frac{\mathrm{d}t}{t},
$$

for all $n \in \mathbb{N}$. Thus, $\int_0^1 \mathbb{P}(|X_t| > 8Rh(t)) t^{-1} \mathrm{d}t = \infty$ implies $\sum_{n \in \mathbb{N}} q_n = \infty$.

Step 3. Define $r_n \coloneqq \mathbb{P}(M(2^{-n}) > R)$ for $n \in \mathbb{N} \cup \{0\}$. By Step 1, the event $B_n \subset \{M(2^{-n-1}) \le R\}$ in (6.5) satisfies

$$
q_n(1 - r_{n+1}) = \mathbb{P}(B_n) \le \mathbb{P}(M(2^{-n-1}) \le R,\ M(2^{-n}) > R) = r_n - r_{n+1}.
$$

This further implies that, for any $k \ge 0$ and $n \in \mathbb{N}$,

$$0 \le 1 - r_n \le (1 - r_{n+1})(1 - q_n) \le (1 - r_{n+k+1}) \prod_{i=n}^{n+k} (1 - q_i).$$

Since $\sum_{n\in\mathbb{N}} q_n = \infty$, it follows that $\prod_{i=n}^{n+k}(1 - q_i) \to 0$ as $k \to \infty$ (see, e.g. Lemma A.3), implying $r_n = 1$ for all $n \in \mathbb{N} \cup \{0\}$. Thus, $\limsup_{t\downarrow 0} |X_t|/h(t) = \lim_{t\downarrow 0} M(t) \ge R$ a.s. □

§6.4 Concluding remarks

It is natural to consider the question of whether the convex minorant C is h-Hölder continuous, i.e., if $\sup_{0\le u<t\le T} |C(t)-C(u)|/h(t-u) < \infty$, for an appropriate general concave increasing function $h : (0, \infty) \to (0, \infty)$. In this context, it is also easy to see that

$$\sup_{0\le u<t\le T} \frac{|C(t) - C(u)|}{h(t - u)} \ge k_h = \sup_{n\in\mathbb{N}} \frac{|\xi_n|}{h(\ell_n)} = \sup_{s\in\mathcal{S}_h} |s|,$$

where the finiteness of k_h can be completely characterised via Corollary 6.9 and Theorem 6.10 in terms of the Lévy measure ν (see Corollary 6.11 in §6.4.1 below for details).

It is not however immediately clear how to construct a tractable upper bound, say K_h, satisfying $K_h < \infty$ whenever $k_h < \infty$. Indeed, a crucial step in proving Lemma 6.5 (and hence (6.1)) is the application of Hölder's inequality to establish that the r-Hölder constant of C is bounded by the L^p-norm of the derivative C' for $p = 1/(1-r)$. This step is not easily extendable to a general concave function h since there is no sufficiently sharp extension of Hölder's inequality (see, e.g. [52, 53]). Thus, it appears that a generalisation of our results beyond the case where h is a power function would require analysing the integral $\int_u^t |C'(v)|\mathrm{d}v$ for all $0 \le u < t \le T$ by other means. For instance, the results in Chapter 5 obtain upper and lower functions for $|C'|$ at 0 and T, yielding upper and lower bounds on $\int_u^t |C'(v)|\mathrm{d}v$ for $u < t$ close to either 0 or T. Note however, that there may exist a large gap between the upper and lower functions of C', see Remark 5.14(a), showing that this question is nontrivial.

§6.4.1 When is k_h finite?

Recall the definition of the functions $\overline{\nu}$, $\overline{\gamma}$ and $\overline{\sigma}^2$ in (2.2) and let γ be the drift parameter of X (for the cutoff function $x \mapsto \mathbb{1}_{(-1,1)}(x)$). We start by considering the main theorem from [83].

Theorem 6.10 ([83, Thm (a)]). *Let $(X_t)_{t\geq 0}$ be a Lévy process with generating triplet (σ^2, γ, ν), and assume that $\sigma^2 = 0$. Then there exists some function $\beta(t)$ that is a positive non-decreasing function defined on $[0, \infty]$ with $\beta(0) = 0$ and $\beta(\infty) = \infty$ such that $\limsup_{t\downarrow 0} |X_t|/\beta(t) \in (0, \infty)$ a.s. if and only if*

$$\liminf_{x\downarrow 0} \overline{\nu}(x)/(\overline{\nu}(x) + x^{-2}\overline{\sigma}^2(x) + x^{-1}|\gamma - \overline{\gamma}(x)|) = 0. \tag{6.6}$$

If (6.6) fails, and β is a function as above, then $\limsup_{t\downarrow 0} |X_t|/\beta(t) = 0$ or ∞ a.s. according as $\int_0^1 \overline{\nu}(\beta(t))\mathrm{d}t$ converges or diverges.

The ensuing corollary is a direct consequence of Corollary 6.9 and Theorem 6.10.

Corollary 6.11. *Suppose X is not compound Poisson with drift. Then, for any function h increasing at 0 with $h(0) = 0$, the variable $k_h < \infty$ a.s. (resp. $k_h = \infty$ a.s.) if and only if $\limsup_{t\downarrow 0} |X_t|/h(t)$ is a.s. finite (resp. infinite). Moreover, the following statements hold.*

(i) *If $\sigma^2 > 0$, then $k_h < \infty$ a.s. if and only if $\liminf_{t\downarrow 0} h(t)/\sqrt{t \log\log(1/t)} > 0$ a.s.*

(ii) *If $\sigma^2 = 0$ and $\limsup_{x\downarrow 0}(x^{-2}\overline{\sigma}^2(x)+x^{-1}|\gamma-\overline{\gamma}(x)|)/\overline{\nu}(x) < \infty$, then the random variable $k_h < \infty$ a.s. if and only if $\int_0^1 \overline{\nu}(h(t))\mathrm{d}t < \infty$.*

(iii) *Suppose $\sigma^2 = 0$ and $\limsup_{x\downarrow 0}(x^{-2}\overline{\sigma}^2(x) + x^{-1}|\gamma - \overline{\gamma}(x)|)/\overline{\nu}(x) = \infty$. Then there exists a non-decreasing function h^* such that $\limsup_{t\downarrow 0} |X_t|/h^*(t) \in (0, \infty)$ a.s. (h^* constructed in the paragraph below). Moreover, the following implications hold*

$$\limsup_{t\downarrow 0} h^*(t)/h(t) < \infty \quad \Longrightarrow \quad k_h < \infty \ a.s.$$
$$\liminf_{t\downarrow 0} h^*(t)/h(t) = \infty \quad \Longrightarrow \quad k_h = \infty \ a.s.$$

Wee and Kim proved in Theorem 6.10 that $\limsup_{t\downarrow 0} |X_t|/h^*(t) \in (0, \infty)$ a.s. for a non-decreasing function h^* if and only if $\sigma^2 = 0$ and $\liminf_{x\downarrow 0} \overline{\nu}(x)/(\overline{\nu}(x) + x^{-2}\overline{\sigma}^2(x) + x^{-1}|\gamma - \overline{\gamma}(x)|) = 0$. In the following two cases, which are exhaustive by Lemma 6.12 (stated below), we describe a construction of h^*, implicitly given in the proof of [83, Thm 3.4].

(a) Suppose that $\liminf_{x\downarrow 0}(\overline{\nu}(x) + x^{-1}|\gamma - \overline{\gamma}(x)|)/(x^{-2}\overline{\sigma}^2(x)) = 0$. Choose a sequence $u_n \downarrow 0$, such that $u_{n+1}^{-2}\overline{\sigma}^2(u_{n+1}) > 2u_n^{-2}\overline{\sigma}^2(u_n)$ for all $n \in \mathbb{N}$ and $\sum_{n\in\mathbb{N}} \log(n)(\overline{\nu}(u_n) + u_n^{-1}|\gamma - \overline{\gamma}(u_n)|)/(u_n^{-2}\overline{\sigma}^2(u_n)) < \infty$. For all $n \in \mathbb{N}$, let $t_n = \log(n)/(u_n^{-2}\overline{\sigma}^2(u_n))$, and define $h^*(t) := u_n \log(n)$ for $t_{n+1} < t \leq t_n$ and $n \in \mathbb{N}$.

(b) Suppose $\liminf_{x\downarrow 0}(\overline{\nu}(x) + x^{-2}\overline{\sigma}^2(x))/(x^{-1}|\gamma - \overline{\gamma}(x)|) = 0$. Choose a sequence $u_n \downarrow 0$, such that $u_{n+1}^{-1}|\gamma - \overline{\gamma}(u_{n+1})| \geq 2u_n^{-1}|\gamma - \overline{\gamma}(u_n)|$ for $n \in \mathbb{N}$ and $\sum_{n\in\mathbb{N}}(\overline{\nu}(u_n) + u_n^{-2}\overline{\sigma}^2(u_n))/(u_n^{-1}|\gamma - \overline{\gamma}(u_n)|) < \infty$. Let $t_n = 1/(u_n^{-1}|\gamma - \overline{\gamma}(u_n)|)$ and define $h^*(t) := u_n$ for $t_{n+1} < t \leq t_n$ and $n \in \mathbb{N}$.

Lemma 6.12 ([83, Lem. 3.3])**.** $\liminf_{x\downarrow 0} \overline{\nu}(x)/(\overline{\nu}(x) + x^{-2}\overline{\sigma}^2(x) + x^{-1}|\gamma - \overline{\gamma}(x)|) = 0$ *hold if and only if at least one of the following conditions hold:* $\liminf_{x\downarrow 0}(\overline{\nu}(x) + x^{-1}|\gamma - \overline{\gamma}(x)|)/(x^{-2}\overline{\sigma}^2(x)) = 0$ *or* $\liminf_{x\downarrow 0}(\overline{\nu}(x) + x^{-2}\overline{\sigma}^2(x))/(x^{-1}|\gamma - \overline{\gamma}(x)|) = 0$.

Appendix A

Probabilistic & analytical results

Throughout this appendix, we will state the results applied in this thesis, that are not already stated in Chapter 2. We start by introducing some classical results from [43].

In the ensuing theorem we introduce Blumenthal's 0–1 law, that is a 0–1 law for canonical Feller processes [43, Ch. 19] (see [70, Prop. 40.4] for the special case of Lévy processes). Let S be a locally compact, separable space, and let $C_0 = C_0(S)$ denote the class of continuous functions $f : S \to \mathbb{R}$ where $f(x) \to 0$ as $x \to \infty$. We say that T is a positive contraction operator, if $0 \leq f \leq 1$ implies that $0 \leq Tf \leq 1$. A semi-group of positive contraction operators T_t on C_0 is called a Feller semi-group, if it has the additional properties: $T_t C_0 \subset C_0$ for any $t \geq 0$ and $T_f f(x) \to f(x)$ as $t \to 0$, for any $f \in C_0$ and $x \in S$. Let $(\mathcal{F}_t)$ be the right continuous filtration generated by X and define the shift-operator θ_t by $(\theta_t \omega)_s = \omega_{s+t}$ for $s, t \geq 0$ and $\omega \in \Omega$. We say that X has distribution $\mathbb{P}_\rho$, where ρ is the initial distribution of X. If $\rho = \delta_x$ we write $\mathbb{P}_x$. The process X with associated distribution $\mathbb{P}_\rho$, filtration $(\mathcal{F}_t)$ and shift operators θ_t is called the canonical Feller process with semi-group (T_t).

Corollary A.1 ([43, Cor. 19.18]). *For any canonical Feller process, it holds that* $\mathbb{P}_x(A) = 1$ *or* 0, *for* $x \in S$ *and* $A \in \mathcal{F}_0 = \bigcap_{s>0} \mathcal{F}_t$.

In the following we state Kolmogorov's extension theorem, which is a classical probabilistic result used often in the analysis of Hölder continuity of stochastic processes such as Brownian motion and fractional Brownian motion.

Theorem A.2 ([43, Thm 3.23]). *Let* X *be a process on* $\mathbb{R}$ *with values in a complete metric space* (S, ρ), *and assume for some* $a, b > 0$, *that* $\mathbb{E}[\rho(X_s, X_t)^a] \leq K|s-t|^{1+b}$, *for all* $s, t \in \mathbb{R}$, *for some constant* $K < \infty$. *Then* X *has a continuous version, and this version is a.s. locally Hölder continuous with exponent* c, *for any* $c \in (0, b/a)$.

Lemma A.3 ([43, Lem. 5.8]). *Consider a null array of constants $q_{nj} \geq 0$, and fix $c \in [0, \infty]$. Then $\Pi_j(1 - q_{nj}) \to e^{-c}$ as $n \to \infty$ if and only if $\sum_j q_{nj} \to c$ as $n \to \infty$*

Recall that the Skorokhod space $\mathcal{D}[0,1]$, is the the space of functions on $[0,1]$ that are right-continuous with left-limits.

Lemma A.4 ([43, Lem. 14.12]). *Consider on $\mathcal{D}[0,1]$ the functional $f(x) = \inf\{t \in [0,1] : x_t \vee x_{t-} = \sup_{s\in[0,1]} x_s\}$. Then, f is continuous at x if and only if $x_t \vee x_{t-}$ has a unique maximum.*

§1.1 Fourier formulas

In this section we state some general Fourier formulas. We start with Fourier inversion formula.

Theorem A.5 ([22, Thm 26.2]). *If a probability measure μ has characteristic function φ, and if $\mu(\{a\}) = \mu(\{b\}) = 0$, then*

$$\mu((a,b]) = \lim_{T\to\infty} \frac{1}{2\pi} \int_{-T}^{T} \frac{e^{-ita} - e^{-itb}}{it} \varphi(t) \mathrm{d}t.$$

The ensuing theorem is called the Fourier's single-integral formula.

Theorem A.6 ([74, Sec. 1.14, Thm 12, p. 25]). *The formula*

$$\frac{1}{2}(f(x+0) + f(x-0)) = \lim_{\lambda\to\infty} \frac{1}{\pi} \int_{\infty}^{\infty} f(t) \frac{\sin(\lambda(x-t))}{x-t} \mathrm{d}t$$

holds if both of the following holds:

(i) *$f(x)/x$ is of bounded variation in (a, ∞) and $(-\infty, -a)$ for some $a > 0$, and $f(x)/x$ tends to 0 as $x \to \infty$,*

(ii) *$f(x)$ is of bounded variation in an interval including x.*

§1.2 Convergence results

The following result is stated in the setting of random variables on $\mathbb{R}$, but can be proven for general random elements on a metric space S.

Theorem A.7 ([23, Thm 3.2]). *Suppose that (ξ_{un}, ξ_n) are random variables on $\mathbb{R}\times\mathbb{R}$. If $\xi_{un} \xrightarrow[u\to\infty]{d} \zeta_n \xrightarrow[n\to\infty]{d} \xi$ and $\lim_{u\to\infty} \limsup_{n\to\infty} \mathbb{P}(|\xi_{un} - \xi_n| \geq \epsilon) = 0$ for all $\epsilon > 0$, then $\xi_n \xrightarrow{d} \xi$ as $n \to \infty$.*

For two random variables ξ, ζ on $\mathbb{R}$, we define the Kolmogorov distance d_K and the Lévy metric as d_L as

$$d_K(\xi, \zeta) := \sup_{x \in \mathbb{R}} |\mathbb{P}(\xi \le x) - \mathbb{P}(\zeta \le x)|, \tag{A.1}$$

$$d_L(\xi, \zeta) := \inf\{h > 0 : \mathbb{P}(\xi \le x - h) - h \le \mathbb{P}(\zeta \le x) \le \mathbb{P}(\xi \le x + h) + h, \, \forall x \in \mathbb{R}\}. \tag{A.2}$$

Theorem A.8 ([61, 1.8.31 & 1.8.32, p. 43])**.** (a) *Let ξ, ζ be two random variables on $\mathbb{R}$. If ζ has an absolutely continuous distribution function $F(x) = \mathbb{P}(\zeta \le x)$, then*

$$d_K(\xi, \zeta) \le \left(1 + \sup_{x \in \mathbb{R}} F'(x)\right) d_L(\xi, \zeta).$$

(b) *Let $(\xi_n)_{n \in \mathbb{N}}$ be a sequence of random variables on $\mathbb{R}$ and ξ be a random variable on $\mathbb{R}$. Then, $\xi_n \xrightarrow{d} \xi$ as $n \to \infty$ if and only if $d_L(\xi_n, \xi) = 0$ as $n \to \infty$. Moreover, by part* (a) *above, $\xi_n \xrightarrow{d} \xi$ as $n \to \infty$ implies that $d_K(\xi_n, \xi) = 0$ as $n \to \infty$, if ξ has an absolutely continuous distribution function.*

Denote by $(\xi_{nk})_{k=1,\dots,k_n, \, n \in \mathbb{N}}$ the sequence of random variables

$$\xi_{11}, \dots, \xi_{1k_1}, \quad \xi_{21}, \dots, \xi_{2,k_2}, \quad \dots \quad \xi_{n1}, \dots, \xi_{nk_n}, \quad \dots,$$

where the random variables are independent within each series $\xi_{m1}, \dots, \xi_{mk_m}$ for all $m \in \mathbb{N}$, and such that $k_n \to \infty$ as $n \to \infty$. Such a sequence is called a triangular array of row-wise independent random variables, and the following theorem is a CLT for such a sequence.

Theorem A.9 ([60, Thm 18, Chap. IV, §4])**.** *Let $(\xi_{nk})_{k=1,\dots,k_n, \, n \in \mathbb{N}}$ triangular array of row-wise independent random variables, and let $F_{nk}(x)$ denote the distribution function of ξ_{nk}. Then it holds that $\max_{1 \le k \le k_n} \mathbb{P}(|\xi_{nk}| \ge \epsilon) \to 0$ as $n \to \infty$ for every fixed $\epsilon > 0$, and there will exist a sequence $(b_n)_{n \in \mathbb{N}}$ such that the distribution of the sums $\sum_{k=1}^{k_n} \xi_{nk} - b_n$ converges in distribution to a standard normal distribution, if and only if the following conditions are fulfilled:*

$$\sum_{k=1}^{k_n} \mathbb{P}(|\xi_{nk}| \ge \epsilon) \to 0, \qquad \text{as } n \to \infty \text{ for every } \epsilon > 0, \text{ and}$$

$$\sum_{k=1}^{k_n} \left(\int_{|x|<\tau} x^2 \mathrm{d}F_{nk}(x) - \left(\int_{|x|<\tau} x \mathrm{d}F_{nk}(x) \right)^2 \right) \to 1, \qquad \text{as } n \to \infty \text{ for some } \tau > 0.$$

As stated in [43, p.119], given a filtration $\mathcal{F} = (\mathcal{F}_n)_{n \in \mathbb{N}}$, we say that a sequence $M = (M_n)_{n \in \mathbb{N}}$ is a martingale wrt. $\mathcal{F}$ if $\mathbb{E}[M_n | \mathcal{F}_{n-1}] = M_{n-1}$ a.s. for all n. Note that $\Delta M_n = M_n - M_{n-1}$ for all $n \in \mathbb{N}$.

Proposition A.10 ([43, Prop. 7.19])**.** *Let M be a martingale on $\mathbb{Z}_+$ with $\Delta M_n \le c$ a.s. for some constant $c < \infty$. Then, $\{M_n \text{ converges}\} = \{\sup_{n \in \mathbb{N}} M_n < \infty\}$ a.s.*

§1.3 Classical resluts for Lévy processes

We say that a function g is submultiplicative, if it is non-negative and there exists some constant $a > 0$, such that $g(x + y) \leq ag(x)g(y)$ for all $x, y \in \mathbb{R}$.

Theorem A.11 ([70, Thm 25.3])**.** *Let g be a submultiplicative, locally bounded and measurable function on $\mathbb{R}$. Moreover, let X be a Lévy process on $\mathbb{R}$ with Lévy measure ν. Then, $\mathbb{E}[g(X_t)] < \infty$ for all $t > 0$ if and only if $\int_{\mathbb{R}\setminus(-1,1)} g(x)\nu(\mathrm{d}x) < \infty$.*

Example A.1 ([70, Example 25.12]). Let X be a Lévy process with generating triplet (σ^2, γ, ν). Then, $\mathbb{E}[X_t] < \infty$ for all $t > 0$ if and only if $\int_{\mathbb{R}\setminus(-1,1)} |x|\nu(\mathrm{d}x) < \infty$. In this case

$$\mathbb{E}[X_t] = t\left(\int_{\mathbb{R}\setminus(-1,1)} x\nu(\mathrm{d}x) + \gamma\right) = t\gamma_1,$$

with γ_1 defined as in Remark 2.1. Moreover, $E[X_t^2] < \infty$ for all $t > 0$ if and only if $\int_{\mathbb{R}\setminus(-1,1)} x^2\nu(\mathrm{d}x) < \infty$, in which case

$$\mathbb{E}[(X_t - \mathbb{E}[X_t])^2] = t\left(\sigma^2 + \int_{\mathbb{R}} x^2\nu(\mathrm{d}x)\right).$$

△

For a discrete measure ρ, we denote by $\mathcal{C}_\rho \coloneqq \{x \in \mathbb{R} : \rho(\{x\}) > 0\}$ the *carrier* of ρ. If a random variable X on $\mathbb{R}$ has discrete distribution $\mathbb{P}_X$, we denote by $\mathcal{C}_X$ the carrier of $\mathbb{P}_X$.

Proposition A.12 ([70, Prop. 27.6])**.** *Assume that the Lévy process X has Gaussian coefficient $\sigma^2 = 0$, Lévy measure ν which satisfies $\nu(\mathbb{R}) < \infty$ and drift γ_0. Then $\mathcal{C}_{X_t} = (\{0\} \cup \bigcup_{n=1}^\infty \{x_1 + \cdots + x_n : x_1, \ldots, x_n \in C_\nu\}) + \gamma_0 t$ for all $t > 0$.*

In the following, we state the weak law of large numbers.

Theorem A.13 ([70, Thm 36.4])**.** *Let $(S_n)_{n\in\mathbb{N}}$ be a random walk on $\mathbb{R}$ and let $\gamma \in \mathbb{R}$. Then, $n^{-1}S_n \to \gamma$ in probability as $n \to \infty$ if and only if $\lim_{r\to\infty} r\mathbb{P}(|S_1| > r) = 0$ and $\lim_{r\to\infty} \mathbb{E}[S_1 \mathbb{1}_{\{|S_1|\leq r\}}] = \gamma$.*

Strong law of large numbers is likewise given in the ensuing theorem.

Theorem A.14 ([70, Thm 36.5])**.** *Let X be a Lévy process on $\mathbb{R}$. If $\mathbb{E}[|X_1|] < \infty$ and $\mathbb{E}[X_1] = \gamma$, then $\lim_{t\to\infty} t^{-1}X_t = \gamma$ a.s. and $\lim_{t\to\infty} \mathbb{E}[|t^{-1}X_t - \gamma|] = 0$. If $\mathbb{E}[|X_1|] = \infty$ then $\limsup_{t\to\infty} t^{-1}X_t = \infty$ a.s.*

Recall that a Lévy process X on $\mathbb{R}$ is called recurrent (resp. transient) if $\liminf_{t\to\infty} |X_t| = 0$ (resp. $\lim_{t\to\infty} |X_t| = \infty$) a.s.

Remark A.15 ([70, Rem. 37.9]). If $\mathbb{E}[X_1^+] < \infty$ or $\mathbb{E}[X_1^-] < \infty$, then a necessary and sufficient condition for X to be recurrent, is that $\mathbb{E}[X_1] = 0$. ♢

§1.3.1 Densities and density transformations of Lévy processes

As in [70, Sec. 33], we will now consider density transformation of Lévy processes. Let $\mathcal{D} := \mathcal{D}([0,\infty),\mathbb{R})$ be the space of mappings ξ from $[0,\infty)$ to $\mathbb{R}$ of right-continuous mappings with left limits. We use the local notation that $\xi(t) = X_t(\xi)$. Define $\mathcal{F}_{\mathcal{D}}$ (resp. $\mathcal{F}_t$) as the smallest σ-algebra that makes X_t, $t \in [0,\infty)$ (resp. X_s, $s \in [0,t)$) measurable. The Lévy processes in the next two theorems, are a Lévy process $(X_t)_{t\geq 0}$ with probability measure $\mathbb{P}$ on $(\mathcal{D},\mathcal{F}_{\mathcal{D}})$, denoted $((X_t)_{t\geq 0},\mathbb{P})$ where we specify the probability measure. We say that two measures ρ_1,ρ_2 on a common measurable space $(M,\mathcal{F}_M)$ are mutually absolutely continuous, stated as $\rho_1 \approx \rho_2$, if $\{B \in \mathcal{F}_M : \rho_1(B) = 0\}$ and $\{B \in \mathcal{F}_M : \rho_2(B) = 0\}$ are identical. Recall that the Radon-Nikoým derivative of ρ_2 wrt. ρ_1 is denoted by $\frac{\mathrm{d}\rho_2}{\mathrm{d}\rho_1}$.

Theorem A.16 ([70, Thm 33.1])**.** *Let $((X_t)_{t\geq 0},\mathbb{P})$ and $((X_t)_{t\geq 0},\widehat{\mathbb{P}})$ be Lévy processes on $\mathbb{R}$ with generating triplets (σ^2,γ,ν) and $(\widehat{\sigma}^2,\widehat{\gamma},\widehat{\nu})$ respectively. Then the following are equivalent:*

(i) $\mathbb{P}|_{\mathcal{F}_t} \approx \widehat{\mathbb{P}}|_{\mathcal{F}_t}$ *for all $t \in (0,\infty)$.*

(ii) *The generating triplets satisfy that $\sigma^2 = \widehat{\sigma}^2$, $\nu \approx \widehat{\nu}$, with the function $\varphi(x)$, defined as $e^{\varphi(x)} = \frac{\mathrm{d}\widehat{\nu}}{\mathrm{d}\nu}$, satisfying $\int_{\mathbb{R}}(e^{\varphi(x)/2}-1)^2\nu(\mathrm{d}x) < \infty$ and*

$$\widehat{\gamma} - \gamma - \int_{(-1,1)} x(\widehat{\nu}(\mathrm{d}x) - \nu(\mathrm{d}x)) \begin{cases} \in \mathbb{R}, & \text{if } \sigma^2 > 0, \\ = 0, & \text{if } \sigma^2 = 0. \end{cases}$$

Theorem A.17 ([70, Thm 33.2])**.** *Let $((X_t)_{t\geq 0},\mathbb{P})$ and $((X_t)_{t\geq 0},\widehat{\mathbb{P}})$ be Lévy processes on $\mathbb{R}$ with generating triplets (σ^2,γ,ν) and $(\widehat{\sigma}^2,\widehat{\gamma},\widehat{\nu})$ respectively. Suppose that the equivalent conditions from Theorem A.16 are satisfied. Chose some $\eta \in \mathbb{R}$, such that $\widehat{\gamma} - \gamma - \int_{(-1,1)} x(\widehat{\nu}(\mathrm{d}x) - \nu(\mathrm{d}x)) = \sigma^2\eta$. We can now define, $\mathbb{P}$-a.s.,*

$$\begin{aligned} U_t &= \eta(X_t - X_t^{\nu}) - \frac{1}{2}t\eta^2\sigma^2 - t\gamma\eta \\ &\quad + \lim_{\varepsilon\downarrow 0}\left(\sum_{(s,X_s-X_{s-})\in(0,t]\times\{|x|\geq\varepsilon\}} \varphi(X_s - X_{s-}) - t\int_{\{|x|\geq\varepsilon\}}(e^{\varphi(x)}-1)\nu(\mathrm{d}x)\right), \end{aligned}$$

where φ is the function defined in Theorem A.16(ii) and $((X_t - X_t^{\nu})_{t\geq 0},\mathbb{P})$ is the continuous part of $((X_t)_{t\geq 0},\mathbb{P})$. The convergence of the right hand side of the equation in the display above is uniform in t on any bounded interval, $\mathbb{P}$-a.s. The process $((U_t)_{t\geq 0},\mathbb{P})$ is a Lévy process on $\mathbb{R}$ with generating triplet $(\sigma_U^2,\gamma_U,\nu_U)$ given by $\sigma_U^2 = \eta^2\sigma^2$, $\nu_U = \nu\varphi^{-1}$ and $\gamma_U = -\frac{1}{2}\eta^2\sigma^2 - \int_{\mathbb{R}}(e^y - 1 - y\mathbb{1}_{(-1,1)}(y))(\nu\varphi^{-1})(\mathrm{d}y)$.

As defined in [70, Def. 7.1] and the preceding paragraph, we say that a measure μ on $\mathbb{R}$ is infinitely divisible if, for any $n \in \mathbb{N}$ there exist a probability measure μ_n on $\mathbb{R}$ such that $\mu = \mu_n * \cdots * \mu_n = \mu_n^{n*}$. Note that μ^{n*} denotes the n-fold convolution of μ with itself. Moreover, assuming that two processes with the same law are considered to be the same, then the collection of all infinitely divisible distributions has a one-to-one correspondence with the collection of all Lévy processes (see [70, Thm 7.10]).

Next, we consider the following technical results, describing when the density of an infinitely divisible law is smooth, and when the density is uniformly bounded. Note, that we denote the k'th derivative with respect to x of a differentiable function f, by $f^{(k)}(x)$, i.e. $f^{(k)}(x) = \frac{\mathrm{d}^k}{\mathrm{d}x^k} f(x)$.

Lemma A.18 ([62, Lem. 2.3]). *Consider a family of infinitely divisible laws Q_i with generating triplets $(\sigma_i^2, \gamma_i, \nu_i)$, and suppose that $c\varepsilon^{2-\beta} \leq \int_{(-\varepsilon,\varepsilon)} x^2 \nu_i(\mathrm{d}x) + \sigma^2$ for any $\varepsilon \in [0,1]$ and for some $c > 0$ and $\beta \in (0,2]$. Then Q_i has a smooth density p_i, and $p_i(x)$ (as well as all its derivatives $p_i^{(k)}(x)$) are uniformly bounded in (i,x).*

In the following results we will consider, when it exists, the density $x \mapsto p_t(x)$ of the Lévy process X_t for $t > 0$.

Theorem A.19 ([62, Thm 3.1]). *Let X be a Lévy process with Lévy measure ν. Suppose that $\liminf_{\varepsilon\downarrow 0} \varepsilon^{\beta-2}(\overline{\sigma}^2(\varepsilon) + \sigma^2) > 0$ for some $\beta \in (0,2]$. Then X_t has an infinitely differentiable density satisfying $\sup_{x\in\mathbb{R}} p_t(x) = \mathcal{O}(t^{-1/\beta})$ as $t \downarrow 0$, and more generally $\sup_{x\in\mathbb{R}} |p_t^{(k)}(x)| = \mathcal{O}(t^{-(k+1)/\beta})$ as $t \downarrow 0$.*

Theorem A.20 ([62, Thm 4.3]). *Let X be a Lévy process on $\mathbb{R}$ with generating triplet (σ^2, γ, ν). Assume that $c\varepsilon^{\beta-2} \leq \overline{\sigma}^2(\varepsilon) + \sigma^2 \leq C\varepsilon^{\beta-2}$, for any $\varepsilon \in [0,1]$ and for some $\beta \in (0,2]$. Suppose that one of the following is true.*

(i) $\beta > 1$;

(ii) $\beta = 1$ *and* $\limsup_{\varepsilon\downarrow 0} \left| \int_{(-1,-\varepsilon]\cup[\varepsilon,1)} x\nu(\mathrm{d}x) \right| < \infty$;

(iii) $\beta < 1$ *(making $\sigma^2 = 0$), both $\overline{\sigma}_+^2(\varepsilon) = \int_{(0,\varepsilon)} x^2\nu(\mathrm{d}x)$ and $\overline{\sigma}_-^2(\varepsilon) = \int_{(-\varepsilon,0)} x^2\nu(\mathrm{d}x)$ satisfy $c\varepsilon^{\beta-2} \leq \overline{\sigma}_\pm^2(\varepsilon) \leq C\varepsilon^{\beta-2}$, and $\gamma - \int_{(-1,1)} x\nu(\mathrm{d}x) = 0$.*

Then, for any $\rho > 0$, we have that $|x| \leq \rho t^{1/\beta}$ implies that $ct^{-1/\beta} \leq p_t(x) \leq Ct^{-1/\beta}$ for small enough t, so in particular $p_t(0)$ is of order $t^{-1/\beta}$.

§1.3.2 Local times of a Lévy process

In this section of the appendix, we will introduce the terminology of local times of a Lévy process.

Definition A.21 ([18, Defn. (Occupation measure), Sec. V]). *For all $t > 0$, the occupation measure on the time interval $[0,t]$ of the Lévy process X, is the measure μ_t, given for any measurable function $f : \mathbb{R} \to [0,\infty)$ by $\int_{\mathbb{R}} f(x)\mu_t(\mathrm{d}x) = \int_0^t f(X_s)\mathrm{d}s$.*

Theorem A.22 ([18, Thm 1, Sec. V]). *Let X be a Lévy process on $\mathbb{R}$, and $\mathfrak{s}_q(r)$ as in §2.5.1. Then, $\mathfrak{s}_q(0) < \infty$ if and only if for every $t \geq 0$, μ_t is absolutely continuous w.r.t.* Leb*, with density in $L^2(\mathrm{d}x \otimes \mathrm{d}\mathbb{P})$. Moreover, if $\mathfrak{s}_q(0) = \infty$ then μ_t is singular for every $t > 0$.*

Under the assumption that Theorem A.22 holds, we can consider the following particular version of the density of the occupation measure, called the *local time.*

Definition A.23 ([18, Defn. (Local times), Sec. V]). *We denote by $L(x,t)$ the local time of X at level x and time t, and define $L(x,t) \coloneqq \limsup_{\varepsilon\downarrow 0} \frac{1}{2\varepsilon} \int_0^t \mathbb{1}_{\{|X_s - x| < \varepsilon\}} \mathrm{d}s$, for every $t \geq 0$ and $x \in \mathbb{R}$.*

Since ε can be restricted to the rational numbers $\mathbb{Q}$, we can consider $(L(x,t))_{x\in\mathbb{R}}$ as an $(\mathcal{F}_t)$-measurable version of μ_t. We say that X has has a local time field, if μ_t is absolutely continuous.

§1.3.3 Potential theory and q-capacity

In this section, we introduce some potential theory, and define the q-capacity and related necessary results, which are closely related to the behaviour of $\mathfrak{s}_q(r)$ defined in §2.5.1. To understand and define the bascis of potential theory, we start by introducing the q-potential measure V^q [70, Def. 30.9], the q-potential operator U^q [70, Def. 41.2], an assumption called the *(ACP)* assumption [70, Def. 41.11] and the q-co-excessive function u^q [70, Thm 41.16] in the following definition.

Definition A.24. *Let X be a Lévy process on $\mathbb{R}$. Then we define the q-potential measure V^q, as*

$$V^q(B) = \mathbb{E}\left[\int_0^\infty e^{-qt}\mathbb{1}_B(X_t)\mathrm{d}t\right], \qquad \text{for } B \in \mathcal{B}(\mathbb{R}),\ q \geq 0.$$

We say that the absolute continuity of potential measure (ACP) assumption is satisfied, if V^q is absolutely continuous for all $q \geq 0$. The q-potential kernel $U^q(x,B)$ is defined as

$$U^q(x,B) = \int_0^\infty e^{-qt}\mathbb{P}(X_t + x \in B)\mathrm{d}t, \qquad \text{for } B \in \mathcal{B}(\mathbb{R}),\ q \geq 0,\ x \in \mathbb{R}.$$

The q-potential operator U^q is then defined as $U^q f(x) = \int_{\mathbb{R}} f(y) U^q(x, \mathrm{d}y)$. Under the assumption of (ACP), there is a unique q-co-excessive function u^q, such that

$$U^q f(x) = \int_{\mathbb{R}} u^q(y-x) f(y)\mathrm{d}y, \qquad \text{for } q > 0,$$

and any non-negative universally measurable function f.

Definition A.25 ([70, Defs. 42.6 & 43.1]). *The q-capacity of the set $\{0\}$ for the process $(X_t)_{t\geq 0}$ is defined to be*

$$c^q := q \int_{\mathbb{R}} \mathbb{E}\big[e^{-qT_x} \mathbb{1}_{\{X_{T_x}=0\}}\big] \mathrm{d}x, \qquad \text{for all } q > 0,$$

where $T_x = \inf\{t > 0 : X_t = x\}$.

Remark A.26 ([70, Eqs. (42.32) & (42.34)]). The q-capacity c_r^q of the set $\{0\}$ for the process $(X_t - rt)_{t\geq 0}$, satisfies that

$$\frac{1}{4\mathfrak{s}_q(r)} \leq c_r^q \leq \frac{1}{\mathfrak{s}_q(r)}, \qquad \text{for any } q > 0. \qquad \Diamond$$

For all $q > 0$, we define the h^q function for the process $(X_t)_{t\geq 0}$, by $h^q(x) := \mathbb{E}[e^{-qT_x}]$ and $h^0(x) := \mathbb{P}(T_x < \infty)$, where T_x is defined as Definition A.25.

Theorem A.27 ([67, Thms 1.5 & 2.6]). *If h^q is continuous, then $c^q \uparrow \infty$ as $q \to \infty$.*

The following theorem allows us to translate the statement from Theorem A.27, into a statement on the q-co-excessive function u^q of X. We say that the set $\{x\}$ for $x \in \mathbb{R}$ is essentially polar, if and only if $\mathbb{P}(T_{x-y} = \infty) = 1$ for a.e. y.

Theorem A.28 ([70, Thms 43.3 & 43.5]). *Let $q > 0$, and let the h^q and u^q functions be for X.*

(a) *A one-point set is not essentially polar if and only if (ACP) is satisfied, u^q is bounded and $c^q u^q(x) = h^q(x)$, for all $x \in \mathbb{R}$. Furthermore, this is equvalent to $V^q(\mathrm{d}x)$ having a bounded density.*

(b) *The set $\{0\}$ is not essentially polar and 0 is regular for itself if and only if (ACP) holds and u^q is bounded, continuous and positive on $\mathbb{R}$.*

Note that some one-point set is essentially polar if and only if any one-point set is essentially polar. As a consequence of Theorem A.28, we see that if u^q is continuous, then so is h^q.

Theorem A.29 ([27, Thms 7 & 8]). *Let X be a Lévy process with generating triplet (σ^2, γ, ν). Let $\mathcal{P} = \{x \in \mathbb{R} : \mathbb{P}(X_t = x \text{ for at least one } t > 0) > 0\}$. Then the following statements hold.*

(a) *A necessary and sufficient condition for $\mathrm{Leb}(\mathcal{P}) > 0$ is that $\mathfrak{s}_q(0) < \infty$.*

(b) *Consider the following cases:*

  (b-i) *If $\sigma^2 > 0$, then we have that $\mathcal{P} = \mathbb{R}$, and 0 is regular for itself.*

  (b-ii) *If $\sigma^2 = 0$ and $\int_{\mathbb{R}}(|x| \wedge 1)\nu(\mathrm{d}x) = \infty$, then $\mathcal{P} = \emptyset$ or $\mathcal{P} = \mathbb{R}$ (use the integral criteria from part (a) to see which) and if $\mathcal{P} = \mathbb{R}$, then 0 is regular for itself.*

(b-ii) *Assume that* $\sigma^2 = 0$ *and* $\int_{\mathbb{R}}(|x| \wedge 1)\nu(\mathrm{d}x) < \infty$*, and denote by* γ_0 *the drift of* X *from Remark 2.1. If* $\gamma_0 = 0$ *then* $\mathcal{P} = \emptyset$*; if* $\gamma_0 > 0$ *and* ν *is carried by* $\mathbb{R}_+$ *then* $\mathcal{P} = \mathbb{R}_+$ *and* 0 *is not regular for itself; if* $\gamma_0 > 0$ *and* ν *gives positive mass to* $\mathbb{R}_-$ *then* $\mathcal{P} = \mathbb{R}$ *and* 0 *is not regular for itself.*

We say that ν is carried by $(0, \infty)$ if $\nu((-\infty, 0)) = 0$. For an alternative reference for a similar result, see [46, Thms 1 & 2]. If $\mathrm{Leb}(\mathcal{P}) > 0$, we say that $(X_t)_{t\geq 0}$ visits points, and note that if $\mathrm{Leb}(\mathcal{P}) > 0$ then we know that $\mathbb{P}(T_x < \infty) > 0$.

Remark A.30. The application throughout the thesis will be on the process $X^{(r)} = (X_t - rt)_{t\geq 0}$, and we can now see from Theorem A.27 that $\mathfrak{s}_q(r) < \infty$ implies that the q-capacity of $\{0\}$ for $X^{(r)}$, satisfies $c_r^q \uparrow \infty$ as $q \to \infty$ for all $r \in \mathbb{R}$.

Indeed, to see this, we note that, $\mathfrak{s}_q(r) < \infty$ implies that 0 is regular for itself for $X^{(r)}$, this holds since $X^{(r)}$ hits points by Theorem A.29. Since 0 is regular for itself, it follows by Theorem A.28, that u^q and hence h_r^q is continuous, and hence Theorem A.27 implies that $c_r^q \uparrow \infty$ as $q \to \infty$ for all $r \in \mathbb{R}$. ♢

§1.3.4 Stable and semi-stable processes

We will in the section define semi-stable processes, as well as add some details for α-stable processes as introduced in §2.3.

Definition A.31 ([70, Defs. 13.1 & 13.2])**.** *Let* X *be a Lévy process on* $\mathbb{R}$*, and denote by* μ *the distribution of* X_t *at* $t = 1$*. We say that* X *is semi-stable if for some* $a > 0$ *with* $a \neq 1$ *there exists* $b > 0$ *and* $c \in \mathbb{R}$ *such that* $\widehat{\mu}(z)^a = \widehat{\mu}(bz)e^{icz}$*, where* $\widehat{\mu}$ *is the characteristic function of* μ*. Moreover,* X *is said to be strictly semi-stable, if for some* $a > 0$ *with* $a \neq 1$ *there is a* $b > 0$ *such that* $\widehat{\mu}(z)^a = \widehat{\mu}(bz)$*. If* $b = a^{1/\alpha}$ *in the first case (resp. second case), for some* $\alpha \in (0, 2]$*, we say that* X *is* α*-semi-stable (resp. strictly* α*-semi-stable).*

Let $S_n(b) = \{x \in \mathbb{R} : b^n < |x| \leq b^{n+1}\}$ for $n \in \mathbb{Z}$, and note that $S_n(b) = b^n S_0(b)$. If μ is the distribution of a random variable X, then we denote by $T_r\mu$ the distribution of rX. Moreover, the restriction of a measure ρ to a Borel set B, is denoted by $\rho|_B$. Recall that a distribution is trivial if it is a δ-distribution. In the following we see equivalent definitions of being α-semi-stable and α-stable.

Theorem A.32 ([70, Thm 14.3])**.** *Let* μ *be infinitely divisible and non-trivial with generating triplet* (σ^2, γ, ν) *and let* $\alpha \in (0, 2)$*.*

(i) Let $b > 1$*. Then the following are equivalent:*

(a) μ *is* α*-semi-stable with* b *as span.*

(b) $\sigma^2 = 0$ *and, for each integer* n *the measure* ν *on* $S_n(b)$ *is determined by the measure* ν *on* $S_0(b)$ *by* $\nu|_{S_n(b)} = b^{-n\alpha} T_{b^n}(\nu|_{S_0(b)})$.

(ii) *The following are equivalent:*

(a) μ *is* α*-stable;*

(b) $\sigma^2 = 0$ *and there is a finite measure* λ *on* $\{-1, 1\}$ *such that* $\nu(B) = \int_{\{-1,1\}} \int_0^\infty \mathbb{1}_B(rx) \frac{\mathrm{d}r}{r^{1+\alpha}} \lambda(\mathrm{d}x)$*, for* $B \in \mathcal{B}(\mathbb{R})$.

Proposition A.33 ([70, Prop. 14.5])**.** *Let* μ *be non-trivial and* α*-semi-stable on* $\mathbb{R}$ *with* $\alpha \in (0, 2)$ *and Lévy measure* ν*. Then,* $\int_{(-1,1)} |x|\nu(\mathrm{d}x) < \infty$ *if and only if* $\alpha < 1$*. Moreover,* $\int_{\mathbb{R}\setminus(-1,1)} |x|\nu(\mathrm{d}x) < \infty$ *if and only if* $\alpha > 1$*. The total mass of* ν *is always infinite.*

The characteristic function of an α-stable law has a specific closed form, which is introduced in the ensuing theorem.

Theorem A.34 ([70, Prop. 14.15])**.** *Assume that* $\alpha \in (0, 2) \setminus \{1\}$*. If* μ *is non-trivial and* α*-stable, then*

$$\widehat{\mu}(z) = \exp\left(-c|z|^\alpha \left(1 - i\beta \operatorname{sgn}(z) \tan\left(\pi\alpha/2\right)\right) + i\tau z\right),$$

with $c > 0$*,* $\beta \in [-1, 1]$ *and* $\tau \in \mathbb{R}$.

A non-trivial process X with ditribution μ as in Theorem A.34, is called a stable process with parameters (α, β, τ, c), where $\tau = \gamma_0$ if $\alpha \in (0, 1)$ and $\tau = \gamma_1$ if $\alpha \in (1, 2)$. In the case where we consider an α-stabel process, the asymptotic behaviour of its density is known, which is explained in the following remark.

Remark A.35 ([70, Rem. 14.18]). If μ is non-trivial and α-stable, then it has a continuous density. Let $(X_t)_{t\geq 0}$ be a stable process with parameters (α, β, τ, c) and $\alpha \in (0, 2) \setminus \{1\}$, and let $X_t^0 = X_t - t\tau$. By $p_t(x)$ (resp. $p_t^0(x)$) we denote the continuous density of X_t (resp. X_t^0) for $t > 0$. Then

$$p_t(x) = t^{-1/\alpha} p_1(t^{-1/\alpha} x + (1 - t^{(\alpha-1)/\alpha})\tau) = t^{-1/\alpha} p_1^0(t^{-1/\alpha}(x - \tau t)),$$

where $p_1^0(x) \sim \frac{1}{\pi}\Gamma(1+\alpha)\sin(\pi\rho\alpha)x^{-\alpha-1}$, if $\beta \neq -1$ as $x \to \infty$. We can also see that the density $p_1(x)$ is bounded, since it is continuous and bounded at infinity, since its asymptotic behaviour at ∞ is bounded. ◊

The full characterisation of the densities can be found in the full remark [70, Rem. 14.18] and the ensuing paragraphs.

Recall that a one-point set $\{x\}$ is polar for the process X, if $\mathbb{P}(T_{x-y} = \infty)$ for all $y \in \mathbb{R}$, where T_x is as in Definition A.25.

Theorem A.36 ([69, Thm 7.4]). *If X is strictly 1-semi-stable, then a one-point set is polar. If X is 1-semi-stable and not strictly 1-semi-stable, then a one-point set is non-polar.*

§1.3.5 Small-time results

Throughout this section, we introduce some important small-time fluctuation results for Lévy processes.

Theorem A.37 ([70, Thm 43.20]). *Let X be a Lévy process with generating triplet $(0, \gamma, \nu)$, where $\nu(\mathbb{R}) < \infty$ or $\nu(\mathbb{R}) = \infty$ with $\int_{(-1,1)} |x|\nu(\mathrm{d}x) < \infty$ with drift γ_0. Then $\mathbb{P}(\lim_{t\downarrow 0} t^{-1}X_t = \gamma_0) = 1$.*

Next, we introduce Rogozin's theorem (see also [5, Thm 1]).

Theorem A.38 ([70, Thm 47.1]). *Let X be a Lévy process with generating triplet (σ^2, γ, ν), where $\sigma^2 \neq 0$ or $\int_{(-1,1)} |x|\nu(\mathrm{d}x) = \infty$. Then*

$$\mathbb{P}(\limsup_{t\downarrow 0} t^{-1}X_t = \infty \quad \text{and} \quad \liminf_{t\downarrow 0} t^{-1}X_t = -\infty) = 1.$$

For the following proposition, we consider a Lévy process X with generating triplet $(0, \gamma, \nu)$. Let $h(r) := \overline{\nu}(r) + r^{-2}\overline{\sigma}^2(r) + r^{-1}|\gamma - \overline{\gamma}(r)|$ for $r < 1$, and define the small-time indices:

$$\beta_L = \inf\{\eta > 0 : \limsup_{r\downarrow 0} r^{\eta}h(r) = 0\}, \quad \delta_L = \inf\{\eta > 0 : \liminf_{r\downarrow 0} r^{\eta}h(r) = 0\}.$$

From [70, p. 362], we know that, in the special case where X is α-stable, $\alpha = \beta_L = \delta_L$. Recall the index β_+ from (2.3) and γ_0 from Remark 2.1, and note moreover that $\beta_L = \beta_+$, except when $\int_{(-1,1)} |x|\nu(\mathrm{d}x) < \infty$ and $\gamma_0 \neq 0$. Moreover, the index β_- from (2.4) coincides with δ_L.

Proposition A.39 ([70, Prop. 47.24]). *Let X be a Lévy process on $\mathbb{R}$ with generating triplet $(0, \gamma, \nu)$. Let $\eta > 0$, then*

$$\limsup_{t\downarrow 0} \frac{\sup_{0\le s\le t} |X_t|}{t^{1/\eta}} = \begin{cases} 0 \text{ a.s.}, & \eta > \beta_L, \\ \infty \text{ a.s.}, & \eta < \beta_L. \end{cases}$$

$$\liminf_{t\downarrow 0} \frac{\sup_{0\le s\le t} |X_t|}{t^{1/\eta}} = \begin{cases} 0 \text{ a.s.}, & \eta > \delta_L, \\ \infty \text{ a.s.}, & \eta < \delta_L. \end{cases}$$

In the following theorem, we will state the full characterisation of the upper fluctuations of a Lévy process X with no Gaussian component. For the following theorem, we let b be an increasing positive function, with $b(0) = 0$, $b(1) = 1$ and the following properties: $b(t)t^{\epsilon-1} \uparrow \infty$ as $t \downarrow 0$ for some $\epsilon > 0$ and, for some $\alpha > 1/2$,

$b(t)t^{-\alpha} \downarrow 0$ as $t \downarrow 0$. Let $b^{\leftarrow}$ be the right inverse of b and let $w(t) = v^{\leftarrow}(t)$ where $v(t) = b^{\leftarrow}(t)t^{-1}$. Moreover, define $W_-(x) = 2\int_0^x y\overline{\nu}^-(y)\mathrm{d}y + \int_x^1 \overline{\nu}^-(y)\mathrm{d}y$ and $\lambda_J = \inf\{\lambda > 0 : J(\lambda) < \infty\}$, where

$$J(\lambda) = \int_0^1 \exp\left(-w\left(\lambda \frac{\rho}{W_-(\rho)}\right)\rho^{-1}\right)\frac{\mathrm{d}\rho}{\rho}.$$

For the ensuing theorem, we define $\nu^{\pm}$ as $\nu^+((x,\infty))) = \nu((x,\infty))$ and $\nu^-((x,\infty)) = \nu((-\infty,-x))$ for $x > 0$.

Theorem A.40 ([72, Thm 3.1])**.** *Let X be an infinite variation Lévy process with no Gaussian component, i.e. $\sigma^2 = 0$. Let b be a function as described in the paragraph above. Then $\limsup_{t\downarrow 0} X_t/b(t) = \infty$ a.s. if and only if*

(i) $\int_0^1 b^{\leftarrow}(t)\nu^+(\mathrm{d}t) = \infty$, *or*

(ii) $\int_0^1 b^{\leftarrow}(t)\nu^+(\mathrm{d}t) < \int_0^1 b^{\leftarrow}(t)\nu^-(\mathrm{d}t) = \infty$ *and* $\lambda_J = \infty$.

If both (i) *and* (ii) *fail, then $\limsup_{t\downarrow 0} X_t/b(t) = 0$ a.s. if and only if*

(iii) $\int_{-1}^1 b^{\leftarrow}(t)\nu(\mathrm{d}t) < \infty$, *or*

(iv) $\int_0^1 b^{\leftarrow}(t)\nu^+(\mathrm{d}t) < \int_0^1 b^{\leftarrow}(t)\nu^-(\mathrm{d}t) = \infty$ *and* $\lambda_J = 0$.

Alternatively, suppose that $\int_0^1 b^{\leftarrow}(t)\nu^+(\mathrm{d}t) < \int_0^1 b^{\leftarrow}(t)\nu^-(\mathrm{d}t) = \infty$ and $\lambda_J \in (0,\infty)$. Then $\limsup_{t\downarrow 0} X_t/b(t) \in (0,\infty)$ a.s.

Theorem A.41 ([17, Thm 2.1])**.** *Let X be a Lévy process (not a compound Poisson process) on $\mathbb{R}$ with generating triplet (σ^2,γ,ν), where $\sigma^2 = 0$, and take $\kappa > 1/2$. Moreover, if X is of finite variation, we assume that its drift is* 0.

(i) *If $\int_0^1 \overline{\nu}(x^\kappa)\mathrm{d}x < \infty$ then $\lim_{t\downarrow 0} X_t/t^\kappa = 0$ a.s.*

(ii) *Conversely, if $\int_0^1 \overline{\nu}(x^\kappa)\mathrm{d}x < \infty$ fails, then $\limsup_{t\downarrow 0} |X_t - a(t)|/t^\kappa = \infty$ a.s. for any non-stochastic function $a : [0,\infty) \to \mathbb{R}$.*

Theorem A.42 ([17, Thm 2.2])**.** *Let X be a Lévy process (not a compound Poisson process) on $\mathbb{R}$ with generating triplet (σ^2,γ,ν), where $\sigma^2 = 0$, and put*

$$I(\lambda) = \int_0^1 \exp\left(-\frac{\lambda^2}{2\overline{\sigma}^2(x)}\right)\frac{\mathrm{d}x}{x}, \qquad \text{and} \qquad \lambda_J^* := \inf\{\lambda > 0 : I(\lambda) < \infty\} \in [0,\infty].$$

Then,

$$-\liminf_{t\downarrow 0} \frac{X_t}{\sqrt{t}} = \limsup_{t\downarrow 0} \frac{X_t}{\sqrt{t}} = \limsup_{t\downarrow 0} \frac{|X_t|}{\sqrt{t}} = \lambda_I^* \qquad \text{a.s.}$$

Proposition A.43 ([70, Prop. 47.11])**.** *Let X be a Lévy process with $\sigma^2 \geq 0$, then* $\limsup_{t\downarrow 0} |X_t|/\sqrt{t \log\log(1/t)} = \sqrt{2}|\sigma|$.

§1.3.6 Long-time results

Lemma A.44 ([70, Lem. 48.3])**.** *Let $(X_t)_{t\geq 0}$ be a non-zero Lévy process on $\mathbb{R}$. Then, for any finite interval K, we have that $\mathbb{P}(X_t \in K) = \mathcal{O}(t^{-1/2})$, as $t \to \infty$.*

Proposition A.45 ([70, Prop. 48.10]). *Let $(X_t)_{t\geq 0}$ be a non-zero Lévy process on $\mathbb{R}$, where $\mathbb{E}[|X_1|^{1+\epsilon}] < \infty$ for some $\epsilon > 0$ and $\mathbb{E}[X_1] = 0$. Let $\eta > 0$ and define $\overline{\beta} := \sup\{p \in [0,2] : \int_{\mathbb{R}\setminus(-1,1)} |x|^p \nu(\mathrm{d}x) < \infty\}$, then*

$$\limsup_{t\to\infty} \frac{\sup_{0\leq s\leq t}|X_t|}{t^{1/\eta}} = \begin{cases} 0 \ a.s., & \eta < \overline{\beta}, \\ \infty \ a.s., & \eta > \overline{\beta}. \end{cases}$$

In the following theorem, we introduce Rogozin's criterion, which gives criteria in terms of the transition probabilities for when a Lévy process drifts to ∞, $-\infty$ or oscillates.

Theorem A.46 ([70, Thm 48.1]). *Let $(X_t)_{t\geq 0}$ be a Lévy process on $\mathbb{R}$. Then X is drifting to ∞ if and only if $\int_1^\infty t^{-1}\mathbb{P}(X_t > 0)\mathrm{d}t < \infty$; drifting to $-\infty$ if and only if $\int_1^\infty t^{-1}\mathbb{P}(X_t < 0)\mathrm{d}t < \infty$; and oscillating if and only if $\int_1^\infty t^{-1}\mathbb{P}(X_t > 0)\mathrm{d}t = \int_1^\infty t^{-1}\mathbb{P}(X_t < 0)\mathrm{d}t = \infty$.*

§1.4 Poisson processes

In the following theorem we state the mapping theorem for Poisson processes.

Theorem A.47 ([48, Sec. 2.3]). *Let Π be a Poisson process on S with σ-finite mean measure μ, and let $f : S \to T$ be a measurable function such that the induced measure $\mu^* = \mu^*(B) = \mu(f^{-1}(B))$ (B is a measurable subset of T) has no atoms. Then $f(\Pi)$ is a Poisson process on T having the induced measure μ^* as its mean measure.*

Next, Campbell's formula from [48] (see also [43, Lem. 12.2]) is stated.

Theorem A.48 ([48, Campbell's Theorem, p. 28]). *Let Π be a Poisson process on S with mean measure μ, and let $f : S \to \mathbb{R}$ be measurable. Then the sum $\Sigma := \sum_{N\in\Pi} f(N)$ is absolutely convergent with probability if and only if*

$$\int_S \min\{|f(x)|, 1\}\mu(\mathrm{d}x) < \infty.$$

If this condition holds, then $\mathbb{E}[e^{\theta\Sigma}] = \exp\left\{\int_S (e^{\theta f(x)} - 1)\mu(\mathrm{d}x)\right\}$, for any $\theta \in \mathbb{C}$ for which the integral on the right converges, and in particular whenever θ is purely imaginary. Moreover,

$$\mathbb{E}[\Sigma] = \int_S f(x)\mu(\mathrm{d}x), \tag{A.3}$$

in the sense that the expectation exists if and only if the integral converges and they are equal. If (A.3) converges, then $Var(\Sigma) = \int_S x^2\mu(\mathrm{d}x)$, which can be finite or infinite.

In the following, we present the marking theorem for Poisson processes, see [48, Sec. 5.2] for the theorem and the ensuing discussion. Let Π be a Poisson process on S with mean measure μ. Assume that we with each point X of the random set Π associate some random variable m_X (called the mark of X) taking values in a space M. Note that the distribution of m_X may depend on X but not the other points of Π, and that m_X for different X are independent. We denote by X^* the pair (X, m_X) in $S \times M$, and note that the totality of such points form a random countable subset $\Pi^* = \{(X, m_X) : X \in \Pi\}$ of $S \times M$. The following marking theorem then tells us that Π^* is a Poisson process on the product space $S \times M$.

Theorem A.49 ([48, Marking Thm]). *Let Π be a Poisson process on S with mean measure μ and probability distribution $p(x, \cdot)$ on M depending on $x \in S$ such that $p(\cdot, B)$ is a measurable function on S for $B \subseteq M$. The random subset Π^* is a Poisson process on $S \times M$ with mean measure μ^*, given as*

$$\mu^*(C) = \iint_{(x,m)\in C} \mu(\mathrm{d}x)p(x, \mathrm{d}m).$$

§1.5 Asymptotic theory

We say that a function l is slowly varying at 0 (resp. ∞), if $l(cx)/l(x) \to 1$ as $x \downarrow 0$ (resp. $x \to \infty$) for all $c > 0$. Note that slowly varying functions may be wildly oscillating, i.e. $\liminf_{x\to\infty} l(x) = 0$ but $\limsup_{x\to\infty} l(x) = \infty$. Karamata's representation theorem gives us an exact representation of slowly varying functions.

Theorem A.50 ([24, Thm 1.3.1]). *A function l is slowly varying at infinity if and only if it can be written in the form*

$$l(x) = c(x) \exp\left(\int_a^x \varepsilon(u)u^{-1}\mathrm{d}u\right), \qquad \text{for } x \geq a,$$

for some $a > 0$, where $x \mapsto c(x)$ is a measurable function and $c(x) \to c \in (0, \infty)$, $\varepsilon(x) \to 0$ as $x \to \infty$.

In a similar fashion to above, we say that a positive function f is regularly varying at 0 (resp. ∞) with index α, if $f(\lambda x)/f(x) \to \lambda^\alpha$ as $x \downarrow 0$ (resp. $x \to \infty$) (see [24, Sec. 1.4.2]). From [24, Eq. (1.5.1)], we know that if f is regularly varying at ∞ with index α, then

$$f(x) = x^\alpha c(x) \exp\left(\int_a^x \varepsilon(u)u^{-1}\mathrm{d}u\right), \qquad \text{for } x \geq a, \tag{A.4}$$

for some $a > 0$, where $c(x) \to c \in (0, \infty)$ and $\varepsilon(x) \to 0$ as $x \to \infty$.

Theorem A.51 ([24, Thm 1.5.2]). *Assume that f is regularly varying at ∞ with index α, then (in the case $\alpha > 0$, assuming f is bounded on each interval $(0, K]$), $f(\lambda x)/f(x) \to \lambda^\alpha$ as $x \to \infty$ uniformly in λ*

(i) *on each $[a, b]$, with $0 < a \le b < \infty$ if $\alpha = 0$;*

(ii) *on each $(0, b]$, with $0 < b < \infty$ if $\alpha > 0$;*

(iii) *on each $[a, \infty)$, with $0 < a < \infty$ if $\alpha < 0$.*

Theorem A.52 ([24, Thm 1.5.4]). *A positive, measurable function l is slowly varying at ∞ if and only if, for every $\alpha > 0$, there exists a non-decreasing function ϕ and non-increasing function ψ with $x^\alpha l(x) \sim \phi(x)$ and $x^{-\alpha} l(x) \sim \psi(x)$ as $x \to \infty$.*

In the following theorem we state the results known as Potter's bound.

Theorem A.53 ([24, Thm 1.5.6]). (i) *If l is a slowly varying function at ∞, then for any chosen constants $A > 1$ and $\delta > 0$ there exists some $K = K(A, \delta)$, such that*

$$l(y)/l(x) \le A \max\{(y/x)^\delta, (y/x)^{-\delta}\}, \qquad \text{for all } x, t \ge K.$$

(ii) *If further, l is bounded away from 0 and ∞ on every compact subset of $[0, \infty)$, then for any $\delta > 0$, there exists a $A' = A'(\delta) > 1$ such that*

$$l(y)/l(x) \le A' \max\{(y/x)^\delta, (y/x)^{-\delta}\}, \qquad \text{for all } x, y > 0.$$

(iii) *If f is a regularly varying function at ∞ with index α, then for any chosen constants $A > 1$ and $\delta > 0$ there exists some $K = K(A, \delta)$, such that*

$$f(y)/f(x) \le A \max\{(y/x)^{\alpha+\delta}, (y/x)^{\alpha-\delta}\}, \qquad \text{for all } x, y \ge K.$$

Proposition A.54 ([24, Prop. 1.5.9a]). *Let l be slowly varying at ∞, and choose K such that $l \in L^1_{\mathrm{loc}}([K, \infty))$. Then $\int_K^x l(t) t^{-1} \mathrm{d}t$ is slowly varying at ∞ and $\int_K^x l(t) t^{-1} \mathrm{d}t / l(x) \to \infty$ as $x \to \infty$.*

Note that a similar result holds in the case where l is slowly varying at 0. We now state Karamata's theorem.

Theorem A.55 ([24, Thm 1.5.11]). *Let f be a regularly varying function at ∞ with index ρ, and assume that f is locally bounded in $[K, \infty)$. Then,*

(i) *for any $\kappa \ge -(\rho + 1)$,*

$$\frac{x^{\kappa+1} f(x)}{\int_K^x t^\kappa f(t) \mathrm{d}t} \to \kappa + \rho + 1, \qquad \text{as } x \to \infty;$$

(ii) *for any $\kappa < -(\rho + 1)$ (and for $\kappa = -(\rho + 1)$ if $t \mapsto t^{-(\rho+1)} f(t) \mathrm{d}t$ is integrable at ∞)*

$$\frac{x^{\kappa+1} f(x)}{\int_x^\infty t^\kappa f(t) \mathrm{d}t} \to -(\kappa + \rho + 1), \qquad \text{as } x \to \infty.$$

Theorem A.56 ([24, Thm 1.5.12]). *If f is regularly varying at ∞ (resp. 0) with index α, then there exists a function g that is regularly varying at ∞ (resp. 0) with index $1/\alpha$, so that $f(g(x)) \sim g(f(x)) \sim x$ as $x \to \infty$ (resp. $x \downarrow 0$).*

Let U be absolutely continuous with density u. In the following theorem, we will describe the asymptotic behaviour of u if we know the behaviour of U, this is called the monotone density theorem.

Theorem A.57 ([24, Thm 1.7.2]). *Let $U(x) = \int_0^x u(y)\mathrm{d}y$. If $U(x) \sim cx^\rho l(x)$ as $x \to \infty$, where $c \in \mathbb{R}$, $\rho \in \mathbb{R}$, l is slowly varying at ∞ and if u is ultimately monotone, then $u(x) \sim c\rho x^{\rho-1} l(x)$, as $x \to \infty$.*

Theorem A.58 ([24, Thm 2.3.1(i)]). *If l is slowly varying at ∞, and l satisfies $(l(\lambda x)/l(x) - 1)\log(f(x)) \to 0$ as $x \to \infty$ for some $\lambda > 1$ and $f(x) > 0$, then for $\gamma > 0$, if $x^\gamma f(x)$ is eventually non-decreasing, $l(xf(x)^\delta)/l(x) \to 1$ as $x \to \infty$ uniformly in $\delta \in [0, \Delta]$ for $0 < \Delta < 1/\gamma$.*

§1.5.1 Regularly varying random variables

We say that a random variable ξ is *regularly varying with index* $\alpha \geq 0$, if there exists $p, q \geq 0$ with $p+q = 1$ and a slowly varying function l, such that $\mathbb{P}(\xi > x) \sim px^{-\alpha}l(x)$ and $\mathbb{P}(\xi < -x) \sim qx^{-\alpha}l(x)$ as $x \to \infty$.

Theorem A.59 ([28, Thm 2.4.3]). *Assume for the random variable A, that $A \geq 0$ a.s. and $\mathbb{P}(A = 0) < 1$. Moreover, assume that $\mathbb{E}[A^\alpha] < 1$ for some $\alpha > 0$ and $\mathbb{E}[A^{\alpha+\delta}] < \infty$ for some $\delta > 0$. Then the following holds:*

(i) *Assume that ξ solves the equation $\xi \stackrel{d}{=} A\xi + B$, where ξ is independent of the pair of random variables (A, B). If the random variable ξ is regularly varying with index $\alpha > 0$, then B is regularly varying with index α.*

(ii) *Conversely, if B is regularly varying with index $\alpha > 0$, then there exists a solution to the equation $\xi \stackrel{d}{=} A\xi + B$, where ξ is independent of the random variables (A, B). If $\lim_{x\to\infty} \mathbb{P}(\pm B > x)/\mathbb{P}(|B| > x) = c_\pm$ for some positive constants c_- and c_+, then $\mathbb{P}(\pm\xi > x) \sim (1 - \mathbb{E}[A^\alpha])^{-1}\mathbb{P}(\pm B > x)$, as $x \to \infty$.*

Breiman's Lemma is stated in the following lemma.

Lemma A.60 ([28, Lem. B.5.1]). *Assume that ξ and ζ are independent and non-negative random variables, where ξ is regularly varying with index $\alpha > 0$, and one of the following conditions holds*

(i) $\mathbb{E}[\zeta^{\alpha+\epsilon}] < \infty$ *for some* $\epsilon > 0$.

(ii) $\mathbb{P}(\xi > x) \sim c_0 x^{-\alpha}$ *as* $x \to \infty$ *for some* $c_0 > 0$ *and* $\mathbb{E}[\zeta^\alpha] < \infty$.

Then $\mathbb{P}(\xi\zeta > x) \sim \mathbb{E}[\zeta^\alpha]\mathbb{P}(\xi > x)$, *as* $x \to \infty$.

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