Key varieties for surfaces of general type

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Contents

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Declaration

Chapters 1, 2 are of an expository nature, and chapter 3 is a motivating example in the style of the preprint [R]. Aside from this I declare that, to the best of my knowledge, the material contained in this thesis is original work of the author except where otherwise indicated.

Chapter 1

Introduction

The study of canonical models of surfaces of general type is a subject which has been of interest for many years, since the time of Enriques. The major question is: given particular values of p_g and K^2 can one construct the moduli space of regular surfaces with these invariants? In particular, we want to study surfaces with $p_g = 0$ and $K^2 = 1$. The first example of such a surface was due to L. Godeaux [G], constructed as the quotient of a quintic surface in $\mathbb{P}^3$ by a free $\mathbb{Z}/5$ group action. Surfaces with these invariants are called (numerical) Godeaux surfaces.

It is known that Godeaux surfaces have cyclic algebraic fundamental group π_1^{alg} of order ≤ 5. Thus if X is a Godeaux surface with nontrivial π_1^{alg} then we construct X as a free quotient of the finite Galois étale covering associated to the fundamental group. This is essentially the method used in [G]. The Godeaux method was expanded by M. Reid in [R1] to give explicit constructions for the canonical models of surfaces with $p_g = 0$ and $K^2 = 1$ as $\mathbb{Z}/5$, $\mathbb{Z}/4$ and $\mathbb{Z}/3$ quotients. In particular the moduli spaces of these surfaces are all 8-dimensional and irreducible.

Suppose X is a Godeaux surface with cyclic torsion group $\mathrm{Tors}(X) \subset \mathrm{Pic}(X)$ generated by σ. Then the canonical ring of the étale covering $Y \to X$

$$R(Y, K_Y) = \bigoplus_{n \geq 0} H^0(Y, nK_Y)$$

splits into eigenspaces under the action induced by $\mathrm{Tors}(X)$. In particular, we can study $R(X, K_X)$ as a subring of $R(Y, K_Y)$ by considering the latter as a bigraded ring:

$$R(Y, K_Y) = \bigoplus_{n \geq 0, \tau \in \mathrm{Tors}(X)} H^0(Y, nK_X + \tau).$$

The novel feature of [R1] is the systematic study of this bigraded canonical ring and its application to constructing the finite Galois étale coverings $Y \to$

X when $\text{Tors}(X) = \mathbb{Z}/5, \mathbb{Z}/4, \mathbb{Z}/3$. Later, Barlow constructed some examples of Godeaux surfaces with $\text{Tors}(X) = \mathbb{Z}/2$ and 0 using similar techniques but with more obtuse covering surfaces and group actions [B1, B2]. However, the problem of constructing all such surfaces remains unsolved.

In this thesis we study the étale coverings $Y \to X$ of Godeaux surfaces X with $\text{Tors}\, X = \mathbb{Z}/2$. These coverings are surfaces of general type with $p_g = 1$ and $K^2 = 2$, which were studied by Catanese and Debarre in [CD] following Enriques. We give an alternative description of the canonical model of such surfaces, using the concept of key varieties. Roughly speaking a key variety is a large "simple" variety containing lots of interesting and complicated varieties, usually as simple linear sections inside it. We hope to be able to describe our key variety as a homogeneous space or something similar. There follows an overview of this thesis, chapter by chapter.

Chapter 2 gives a brief introduction to Hilbert series methods for polarised varieties. There is nothing new here, but the calculations described underpin much of the work on key varieties which the following chapters build upon.

Chapter 3 is a substantial example of a calculation in the method of unprojection. Here the objective is to calculate the equations of a polarised K3 surface of high codimension as an automatic calculation in linear algebra and Gröbner bases. In particular I aim to demonstrate that we are close to being able to think of such calculations as a black box which automatically calculates the equations of a surface in high codimension from its projection.

Chapter 4 contains the main result of the thesis. We construct a key variety for surfaces Y of general type with $p_g = 1$ and $K^2 = 2$ in the case where the curve section of $|K_Y|$ is not hyperelliptic. This is intimately related with symmetric determinantal quartic hypersurfaces, although the detailed connections remain somewhat elusive.

Chapter 5 deals with the hyperelliptic degeneration of the construction of the preceeding chapter. We begin by discussing the structure of graded rings over hyperelliptic curves, before giving a structure theorem for hyperelliptic K3 surfaces. Finally we construct a hyperelliptic key variety using unprojection.

Chapter 6 gives a construction of Godeaux surfaces using the hyperelliptic version of our key variety. The surface is given by taking a codimension 4 complete intersection of type $(1, 1, 1, 2)$ in the key variety of chapter 5, then dividing by a fixed point free $\mathbb{Z}/2$-action.

Chapter 2

Preliminaries

Here we collect together various easy properties of graded rings and their associated Hilbert series. Most of these are well known and proofs are sketched. However, many of the calculations described here are a valuable tool in establishing the expected shape taken by the embedding of a projective variety into some ambient space. Indeed, many of the results of this thesis, even when proved independently, are underpinned by such Hilbert series considerations, as is the graded ring database [GRDB].

2.1 Graded rings and Hilbert series

Let (V, A) be a pair consisting of an algebraic variety V and a choice of ample divisor A. We call A the polarisation of V. We study the graded ring

$$R(V, A) = \bigoplus_{n \geq 0} H^0(V, nA),$$

and aim to describe such rings explicitly in terms of generators and relations. If $R(V, A)$ is generated in degree 1 so that A is very ample, then writing down generators and relations corresponds to embedding V in some projective space. More generally $R(V, A)$ is generated in higher degrees and so V lives in some weighted projective space.

The graded summands of R are the vector spaces

$$H^0(V, nA) = \{f \in k(V) | \operatorname{div} f + nA \geq 0\},$$

and there is an obvious multiplication map

$$H^0(V, mA) \times H^0(V, nA) \to H^0(V, (m + n)A)$$

which induces the grading on $R(V, A)$.

The first step on the way to writing down generators for the ring $R(V, A)$ is to calculate the dimension of each graded summand. For this we use the Riemann–Roch theorem. In applications, it is better to water down the full strength theorem for the particular case in question.

2.1.1 Example: Curve of genus 3

The Riemann–Roch theorem for curves says that if we have a nonsingular curve C of genus g together with a polarisation A, then

$$h^0(C, nA) - h^1(C, nA) = 1 - g + \deg nA.$$

Let C be a curve of genus 3 which is not hyperelliptic and choose $A = K_C$. Then we can tabulate the data from the Riemann–Roch theorem as follows:

n	0	1	2	3	4	...
$h^0(C, nK_C)$	1	3	6	10	14	...

Now choosing generators for $R(C, K_C)$ is easy in this case. We must have 3 generators in degree 1, which we call x_1, x_2, x_3. Then there are 6 quadratic monomials in the x_i, which we can assume are linearly independent in the vector space $H^0(C, 2K_C)$. This is precisely because we assumed that C is not hyperelliptic. In the hyperelliptic case, the 6 quadratic monomials will not be linearly independent. Similarly $H^0(C, 3K_C)$ is generated by cubic monomials in the x_i. However, in degree 4, we have 15 monomials but $h^0(C, 4K_C) = 14$, so there must be a linear relation between the monomials of degree 4. One can check that there are no new relations in higher degrees, and the presentation of $R(C, K_C)$ in terms of generators and relations is $k[x_1, x_2, x_3]/(f_4)$. Hence C is a plane quartic, which is well known.

2.1.2 An easy Hilbert series calculation

In fact one can take this analysis a step further, by introducing the Hilbert series. The Hilbert series is a generating function which records the dimension of each summand of $R(V, A)$, and is defined as:

$$P_{V,A}(t) = \sum_{n=0}^{\infty} h^0(V, nA)t^n.$$

So for our curve of genus 3,

$$P_{C,K_C}(t) = 1 + 3t + 6t^2 + 10t^3 + 14t^4 + \dots.$$

The following observation is the key to understanding presentations of graded rings $R(V, A)$ using Hilbert series.

Proposition 2.1.1 *Let $C_4 \subset \mathbb{P}^2$ be a plane quartic curve, and let H be the hyperplane class in $\mathbb{P}^2$, so that $K_C = H|_C$. Then the Hilbert series $P_{C,K_C}(t)$ can be expressed as the rational function*

$$P_C(t) = \frac{1-t^4}{(1-t)^3}.$$

Proof Let us start from the elementary observation that the Hilbert series of $(\mathbb{P}^2, H)$ is equal to the following rational function:

$$\begin{aligned} P_{\mathbb{P}^2}(t) &= 1 + 3t + 6t^2 + 10t^3 + 15t^4 + \cdots + \binom{n+2}{2} t^n + \dots \\ &= \frac{1}{(1-t)^3}. \end{aligned}$$

To explain the numerator of $P_C(t)$, observe that

$$\begin{aligned} P_{\mathbb{P}^2}(t) - P_C(t) &= t^4 + 3t^5 + 6t^6 + 10t^7 + \dots \\ &= t^4 P_{\mathbb{P}^2}(t). \end{aligned}$$

This expresses the fact that $h^0(\mathbb{P}^2, 4H)$ and $h^0(C, 4K_C)$ differ by one because of the quartic relation f_4, whereas $h^0(\mathbb{P}^2, 5H)$ and $h^0(C, 5K_C)$ differ by 3 because of the three combinations $x_i f_4$, etc. It is clear that we can use this fact to write $P_C(t)$ as the rational function displayed above.

Remark 2.1.2 (1) The form of the rational function suggests that the ring $R(C, K_C)$ is generated by 3 elements in degree 1 with one degree 4 relation between those generators. More complicated situations exhibit analogous properties as we shall see later.

(2) Write $S = R(\mathbb{P}^2, H)$ and consider the short exact sequence

$$0 \leftarrow S/(f_4) \leftarrow S \leftarrow (f_4) \leftarrow 0,$$

which is equivalent to

$$0 \leftarrow R(C, K_C) \leftarrow S \leftarrow S(-4) \leftarrow 0.$$

The denominator $(1-t)^3$ suggests that $R(C, K_C)$ is a module over the polynomial ring S and the numerator $1 - t^4$ tells us what to expect the free resolution of $R(C, K_C)$ as an S-module to be. In fact, we could have proved the proposition by starting from the above short exact sequence.

2.2 K3 surfaces and Fano 3-folds

A Fano 3-fold is a 3-dimensional normal projective variety V with $-K_V$ ample. We consider the class of Fano 3-folds with $\mathbb{Q}$-factorial terminal singularities. Furthermore we make the harmless assumption that V has a basket consisting only of cyclic quotient singularities of type $\frac{1}{r}(1, a, r-a)$. This does not affect the Hilbert series calculations we use. The obvious line of attack is to study the anticanonical ring $R(V, -K_V)$, and for this we need to use the orbifold Riemann–Roch formula for Fano 3-folds from [YPG], Chapter III. In this instance, the Riemann–Roch formula is best viewed as a formula for the Hilbert series of $R(V, -K_V)$:

$$P_{V,-K_V}(t) = \frac{1+t}{(1-t)^2} + \frac{t(1+t)}{(1-t)^4}\frac{(-K_V)^3}{2} - \sum_{\mathcal{B}} \frac{1}{(1-t)(1-t^r)} \sum_{i=1}^{r-1} \frac{\overline{bi}(r-\overline{bi})t^i}{2r},$$

where $\mathcal{B} = \{\frac{1}{r}(1, a, r-a)\}$ is the basket of cyclic quotient singularities on V, b is the inverse of a modulo r, and $\overline{j}$ means take the residue of j modulo r. The formula is derived in [ABR] using the orbifold Riemann–Roch formula of [YPG].

2.2.1 Example: A codimension 2 Fano 3-fold

Suppose we take the initial data $(-K_V)^3 = \frac{9}{2}$ and $\mathcal{B} = \{\frac{1}{2}(1,1,1)\}$. The Hilbert series for V is calculated using the above formula to obtain:

$$\begin{aligned} P_{V,-K_V}(t) &= \frac{1+t}{(1-t)^2} + \frac{t(1+t)}{(1-t)^4}\frac{9}{4} - \frac{t}{4(1-t)(1-t^2)} \\ &= 1 + 5t + 16t^2 + 38t^3 + 76t^4 + 134t^5 + 217t^6 + \dots. \end{aligned}$$

Playing the game familiar to us from example (2.1.1), we see that since $h^0(V, -K_V) = 5$, we need 5 generators in degree 1, which we call $x_1, \dots, x_5$. However, this time these are not sufficient to generate the entire ring. There are only 15 quadratic monomials in x_i, and so we need one further generator y in degree 2. Moreover there are now 40 monomials in degree 3 but only 38 of these can be linearly independent. This forces 2 relations in degree 3 between our generators. So the Hilbert series suggests that our Fano 3-fold is a codimension 2 complete intersection

$$V_{3,3} \subset \mathbb{P}(1^5, 2)$$

with one $\frac{1}{2}(1,1,1)$ orbifold point. Indeed, one checks that for general choices of f_3, g_3 defining V, this really is a Fano 3-fold of degree $\frac{9}{2}$ with the requisite orbifold point at $(0,0,0,0,0,1)$.

Moreover there is an expression for the Hilbert series $P_{V,-K_V}$ as a rational function:

$$P_{V,-K_V}(t) = \frac{(1-t^3)(1-t^3)}{(1-t)^5(1-t^2)} = \frac{1-2t^3+t^6}{(1-t)^5(1-t^2)}.$$

Once again the Hilbert series denominator reflects the ambient space, since the anticanonical model of V lives in $\mathbb{P}(1^5, 2)$ and the numerator $(1-t^3)(1-t^3)$ tells us that V is a complete intersection of two cubics. The multiplied out numerator calculates the free resolution of $R(V, -K_V)$ as a module over $S = k[x_1, \dots, x_5, y]$:

$$0 \leftarrow R(V, -K_V) \leftarrow S \leftarrow S(-3) \oplus S(-3) \leftarrow S(-6) \leftarrow 0.$$

Remark 2.2.1 (1) Suppose we start from the surface

$$X_8 \subset \mathbb{P}(1,2,3,4),$$

which has Hilbert series

$$P_X(t) = \frac{1-t^8}{(1-t)(1-t^2)(1-t^3)(1-t^4)}.$$

Then by the same argument as in the above example, the curve C obtained as $X_8 \cap Y_6$ where Y_6 is a general sextic has Hilbert series

$$P_C(t) = \frac{(1-t^6)(1-t^8)}{(1-t)(1-t^2)(1-t^3)(1-t^4)} = (1-t^6)P_X(t).$$

In particular, the Hilbert series of a hyperplane section is calculated simply by multiplying by $1-t$. For more on hyperplane sections, see (2.2.3).

(2) We recall the plane quartic curve of example (2.1.1). Suppose that the curve is hyperelliptic now, then there is a degree 2 relation between the generators. How is this degeneration realised by the Hilbert series? We multiply top and bottom by $1-t^2$ to obtain

$$P_C(t) = \frac{(1-t^2)(1-t^4)}{(1-t)^3(1-t^2)},$$

so that $P_C(t)$ remains unchanged, but now C is the complete intersection

$$C_{2,4} \subset \mathbb{P}(1^3, 2),$$

where the relation in degree 2 does not eliminate the new generator in degree 2. This is called "masking," where there are generators and relations of the same degree which can not be seen by the Hilbert series. See (2.2.5) for more on masking.

2.2.2 Elephants and K3 surfaces

If V is a Fano 3-fold, then any surface $S \in |-K_V|$ which is not too singular has trivial canonical class by adjunction. If S exists at all, then it is given by the hyperplane section $(x = 0)$ for some $x \in H^0(V, -K_V)$. In good cases, S has at worst Du Val singularities, and then S is a K3 surface. Our knowledge of Hilbert series allows us to derive the Riemann–Roch formula for K3 surfaces with Du Val singularities from the formula for Fano 3-folds or vice versa, see [ABR]. Indeed, if we assume S exists and is a K3 surface, then its polarisation by $A := -K_V|_S$ has Hilbert series $(1-t)P_{V,-K_V}(t)$:

$$P_{S,A}(t) = \frac{1+t}{1-t} + \frac{t(1+t)}{(1-t)^3}\frac{A^2}{2} - \sum_{\mathcal{B}} \frac{1}{(1-t^r)} \sum_{i=1}^{r-1} \frac{\overline{bi}(r-\overline{bi})t^i}{2r}.$$

Further, if we choose some $y \in |2A|$, the hyperplane section $(y = 0) \cap S$ is a subcanonical curve C. Thus we have the famous tower

$$C \subset S \subset V.$$

Conversely, suppose we start from the curve C polarised by an ample divisor A such that $2A = K_C$. The problem is how to construct S and subsequently V from C. One solution is to study the underlying graded rings in the picture above. We start from $R(C, A)$ and we want to construct $R(S, A)$, $R(V, -K_V)$. We will use the following principle:

2.2.3 The hyperplane section principle

Let $R = \bigoplus R_i$ be a graded ring, and choose $x_0 \in R_i$. Write $\bar{R} = R/(x_0)$ and suppose $\bar{R}$ is generated by $y_1, \dots, y_n$, where the y_i are homogeneous. Then

(1) R is generated by $x_0, y_1, \dots, y_n$. In other words if $\bar{R} = k[y_1, \dots, y_n]/\bar{I}$ then $R = k[x_0, y_1, \dots, y_n]/I$ for some ideal I of R;

(2) $\bar{I} = I/(x_0)$, so that if $\bar{I} = (f_1, \dots, f_m)$ then there are f'_i in R such that

$$I = (f_1 + x_0 f'_1, \dots, f_m + x_0 f'_m);$$

(3) Similar statements for syzygies and higher syzygies.

There are some obvious hypotheses on R here. The hyperplane section principle will only work if R is Cohen–Macaulay and x_0 should be a regular element. Some discussion of the hyperplane section principle can be found in [Aq]. Essentially we use the principle to guide our calculations, and to check that we get sensible outcomes.

Viewed from the wrong way up, the hyperplane section principle tells us what we know about the shape of the graded ring R given $\bar{R}$. We know the degrees of its generators, relations and syzygies, and we know that these should reduce modulo x_0 to those of $\bar{R}$.

2.2.4 More complicated examples

Extrapolating from the hypersurface and complete intersection cases, we can use rational expressions for Hilbert series to suggest properties of more complicated rings. Suppose we start from the pair (V, A) with

$$\operatorname{Proj} R(V, A) \subset \mathbb{P}(a_0, \ldots, a_n),$$

and the free resolution of $R(V, A)$ is

$$0 \leftarrow R(V, A) \leftarrow S \leftarrow \bigoplus_{i=1}^{m} S(-b_i) \leftarrow \bigoplus_{i=1}^{n} S(-c_i) \cdots \leftarrow S(-d) \leftarrow 0,$$

where as usual S is the weighted polynomial ring $k[x_0, \ldots, x_n]$. Then the Hilbert series $P_{V,A}(t)$ is the rational function with numerator

$$1 - \sum_{i=1}^{m} t^{b_i} + \sum_{i=1}^{n} t^{c_i} - \cdots \pm t^d,$$

and denominator

$$\prod_{i=0}^{n} (1 - t^{a_i}).$$

This simple observation sometimes allows us to guess a possible presentation for $R(V, A)$ from its Hilbert series without any prior knowledge.

2.2.5 Gorenstein symmetry

If the graded ring $R(V, A)$ is Gorenstein, then the numerator of the Hilbert series will have palindromic coefficients. For example

$$1 - 3t^2 + 3t^4 - t^6 \quad \text{but not} \quad 1 - 3t^2 + 2t^3.$$

Indeed, given a Gorenstein graded ring $R(V, A)$, suppose the corresponding structure sheaf $\mathcal{O}_V$ has locally free resolution

$$0 \leftarrow \mathcal{O}_V \leftarrow \mathcal{L}_0 \leftarrow \mathcal{L}_1 \leftarrow \cdots \leftarrow \mathcal{L}_n \leftarrow 0.$$

Then since $R(V, A)$ is Gorenstein, $\mathcal{L}_n \cong \mathcal{O}_{\mathbb{P}}(-k)$ for some integer k and by Serre duality $\mathcal{L}_i \cong \mathcal{L}_{n-i}^{\vee} \otimes \mathcal{O}_{\mathbb{P}}(-k)$. Thus each locally free sheaf has the same rank as its palindromic counterpart. This phenomenon is called Gorenstein symmetry. In practice the rings appearing in this thesis are Gorenstein.

Warnings! The Hilbert series is a *coarser* invariant than the free resolution. This fact can manifest itself in many ways, most commonly:

1. We do not know for certain that the variety in question has the presentation suggested by its Hilbert series. We still have to exhibit an example with the requisite presentation. Furthermore the simplest expression as a rational function is not necessarily the correct one. The hyperelliptic degeneration described above is one instance where we would have been lead astray by using the simplest rational expression.

2. Another more subtle problem is "masking", when there are generators and relations, or relations and syzygies of the same degree. These cancel one another in the Hilbert series numerator but they are still present. For example, to the untrained eye the numerator

$$1 - 3t^2 - 2t^3 + t^3 + 3t^4 - t^6 + \dots$$

   might appear to have 1 relation in degree 3 and 3 syzygies in degree 4, whereas actually there are 2 relations in degree 3 with a syzygy in degree 3 between the 3 quadrics.

2.2.6 Unprojection and Hilbert series

For the definition of unprojection and further details, see section (3.2). For now I want to describe the numerical effect of a Gorenstein projection on the Hilbert series of a Fano 3-fold. Suppose we have a Fano 3-fold whose anticanonical model Y has an orbifold point $P = \frac{1}{r}(1, a, r-a)$ in its basket. The Gorenstein projection from P will result in a new Fano 3-fold X, whose recalculated anticanonical model has the following numerics:

- The basket of X is the same as the basket of Y with P replaced by two orbifold points $\frac{1}{a}(1, r-a, -r)$ and $\frac{1}{r-a}(1, a, -a)$. In some cases, these points will be smooth and are omitted. For example, if we project from $\frac{1}{3}(1, 1, 2)$ on Y then X will only have a $\frac{1}{2}(1, 1, 1)$ point, since the other orbifold point is trivial.

- The anticanonical degree is recalculated according to:

$$(-K_X)^3 = (-K_Y)^3 - \frac{1}{ar(r-a)}.$$

Thus the Hilbert series of X and Y are related by

$$P_Y(t) - P_X(t) = \frac{1}{(1-t)(1-t^a)(1-t^{r-a})(1-t^r)},$$

as demonstrated in [CPR], [ABR]. Indeed, these facts can be deduced immediately from the description of Gorenstein projection using the Kawamata $(1, a, r-a)$-weighted blowups of [CPR]. Thus we can calculate the Hilbert series of the image of a Gorenstein projection to predict the result without having to calculate the projection itself. The numerics are implemented more fully in [GRDB].

Chapter 3

Large codimension K3 surfaces

This is a computer calculation in the style of [R] which works out a fairly representative and substantial example of a type IV unprojection calculation. The one I write out in detail is just a construction of a codimension 4 K3 surface, but with some masking. However, there are lots of other similar examples which I have calculated, with jumps from codimension 1, 2 up to codimension $3, \dots, 8$ and beyond. Other interesting examples are the extreme K3 surfaces from [BR] and there are any number of possible examples in the K3 database of [GRDB].

One of the problems with relying exclusively on computer algebra is that for the more complicated unprojections it not feasible to calculate with anything other than surfaces and 3-folds. Examples of dimension larger than 3 are in general too large for the computer to handle effectively.

Another problem with using computer algebra is that we lose some of the structure of the equations for our variety. For many reasons it would be useful to find out if the equations fit into some format like the 4×4 Pfaffians of a 6×6 extrasymmetric matrix, or a "quasi"-Pfaffian format (see [Ki]). This would give an intrinsic description of the variety, independent of the unprojection construction and of the limitations of the computer.

3.1 The example

From the graded ring database [GRDB], consider the K3 surface Y polarised by A so that it is a codimension 4 subvariety

$$Y \subset \mathbb{P}(2^3, 3^3, 4)$$

with basket $8 \times \frac{1}{2}(1,1)$, $\frac{1}{3}(1,2)$ and $A^2 = \frac{2}{3}$. The Gorenstein projection from any one of the $\frac{1}{2}$ points has image X where

$$\mathbb{P}^1 \subset X_{14} \subset \mathbb{P}(2,2,3,7),$$

with basket $7 \times \frac{1}{2}(1,1)$, $\frac{1}{3}(1,2)$. The above copy of $\mathbb{P}^1$ is just the image of the $\frac{1}{2}$ point under the projection. Also note that we have somehow picked up a new variable of weight 7, even though Y did not have any generators of degree greater than 4. This is the first indication that something slightly nonstandard is happening, perhaps inside a larger ring with some redundant generators.

3.2 Gorenstein projection

The surface Y polarised by A is given by the following graded ring construction:

$$Y = \operatorname{Proj} R(Y, A) = \operatorname{Proj} \bigoplus_{n \geq 0} H^0(Y, nA).$$

Choosing appropriate generators x_i, y_i, z of $R(Y, A)$ of degrees 2, 3, 4 respectively, we see that Y lives in $\mathbb{P}(2^3, 3^3, 4)$ with one of the $\frac{1}{2}$ points at $P_{x_3} := (0, 0, 1, 0, 0, 0, 0)$.

The Gorenstein projection from a $\frac{1}{2}$ point P of Y is analogous to an operation in the Mori minimal model program. It is calculated by first performing a $(1,1)$-weighted Kawamata blowup of P to get $E \subset \widetilde{Y} \to Y$ where $\widetilde{Y}$ is the blown up surface and $E \cong \mathbb{P}^1$ the exceptional divisor. Then project away from E on $\widetilde{Y}$ by taking $X = \operatorname{Proj} R(\widetilde{Y}, B)$, where $B = \sigma^* A - \frac{1}{2}E$.[1] Then X is a hypersurface containing a copy of $\mathbb{P}^1$

$$\mathbb{P}^1 \subset X_{14} \subset \mathbb{P}(2, 2, 3, 7),$$

with basket $7 \times \frac{1}{2}(1,1)$, $\frac{1}{3}(1,2)$ as predicted by the database.

The variables x_3, y_2 and y_3 are also eliminated by the projection from P_{x_3} since they "do not vanish enough at the singular point P" and so are not in the ring $R(\widetilde{Y}, B)$. For example,

$$x_3 \in H^0(\widetilde{Y}, 2A) \not\subseteq H^0(\widetilde{Y}, 2A - E).$$

Another way of saying this is that y_2 and y_3 are eliminated because they are the local coordinates for Y near the singular point P_{x_3}, and x_3 is eliminated because we blow up at the coordinate point P_{x_3}. The mysterious thing is that we also lose the generator z in degree 4 because $z \notin H^0(\widetilde{Y}, 4A - 2E)$ and gain a new one in degree 7. More on this later, see (3.5).

[1] The projection in question should be thought of as an "elephant section" of the corresponding projection of Fano 3-folds. In the Fano case we have $A = -K_Y$, $B = -K_X = \sigma^*(-K_Y) - \frac{1}{2}E$ so that X is the anticanonical model of the Kawamata blowup of P in Y.

Up to now, everything is just a calculus of Hilbert series as in the [GRDB]. The weighted blowup will remove a $\frac{1}{2}$ point from the basket of Y and $B^2 = A^2 - \frac{1}{2}$. So we anticipate that the image of the projection is $X_{14} \subset \mathbb{P}(2,2,3,7)$ as a calculation in Hilbert series. The problem we address is how to prove the existence of such an example rather than just examining the numerology.

3.3 Extended example

We demonstrate how to construct the reverse map to this projection, which is called Gorenstein unprojection. We first need to set up an appropriate image $X \subset \mathbb{P}(2,2,3,7)$, containing a copy of $\mathbb{P}^1$. Choose coordinates u, v on $\mathbb{P}^1$ and x_1, x_2, y, z on $\mathbb{P} := \mathbb{P}(2,2,3,7)$. To write down an embedding $\varphi\colon \mathbb{P}^1 \to \mathbb{P}$ is the same as having $H^0(\mathbb{P}^1, \varphi^*\mathcal{O}_{\mathbb{P}}(n))$ span $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(n))$ for all $n \gg 0$. Counting dimensions of vector spaces, it is impossible to fulfil this condition until $n = 12$. Conversely, thereafter it will be virtually impossible not to span if φ is general! Thus if we choose random combinations of monomials in the appropriate degrees to define the map φ, we will get an embedding. Indeed, one could even analyse all possible monomial combinations to try and find an optimal or "general" solution. To keep things simple, we make the choice

$$\varphi\colon (u,v) \mapsto (u^2, v^2, u^2v + uv^2 + v^3, u^7 + v^7).$$

The equations of the image of φ with small enough degree to be relevant are

$$\begin{aligned} f_{12}&\colon x_1^4x_2^2 + 2x_1^3x_2^3 + 3x_1^2x_2^4 - \dots \\ g_{14}&\colon x_1^6x_2 - x_1^4y^2 - x_1^3x_2^4 + 3x_1^3x_2y^2 + \dots \\ h_{14}&\colon x_1^7 - x_1^3x_2y^2 + 3x_1^2x_2^5 - 5x_1^2x_2^2y^2 + \dots. \end{aligned}$$

There are various ways to calculate the equations defining the image of φ. This is just linear algebra, although it is best done on a computer. One can write down the coefficient matrix of monomials in $H^0(\mathbb{P}^1, \varphi^*\mathcal{O}(n))$ in terms of $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(n))$ and calculate its kernel to get a relation in degree n. Other methods include Gröbner style elimination. We know how many equations to expect and in which degree by dimension counting as described above.

We choose an appropriate homogeneous combination of f, g and h in degree 14, and check that this defines a quasismooth hypersurface. For our calculations we take $(x_1 + x_2)f + g + h$. This gives us our variety X and we have the following picture:

$$\mathbb{P}^1 \xrightarrow{\simeq} D \subset X_{14} \subset \mathbb{P}(2,2,3,7).$$

3.4 Gorenstein unprojection

The unprojection of $D \subset X$ is obtained by writing down functions on X with poles along D and then adjoining them to the coordinate ring of X. We take the standard short exact sequence

$$0 \to \mathcal{I}_D \to \mathcal{O}_X \to \mathcal{O}_D \to 0$$

and apply the derived functor of $\mathcal{H}om_{\mathcal{O}_X}(-, \omega_X)$ to get the short exact sequence of unprojection

$$0 \to \omega_X \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{I}_D, \omega_X) \to \omega_D \to 0. \tag{3.1}$$

Here we used Grothendieck–Serre duality to calculate ω_D plus the fact that D is a codimension 1 effective divisor in X, so that $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_D, \omega_X) = 0$. Now, since X is Gorenstein, $\omega_X \cong \mathcal{O}_X$ as an $\mathcal{O}_X$-module. Thus elements of $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{I}_D, \omega_X)$ can be viewed as functions on X with poles along D, because they are $\mathcal{O}_X$-homomorphisms from $\mathcal{I}_D = \mathcal{O}_X(-D)$ to $\mathcal{O}_X$. It remains to calculate the generators s_i of $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{I}_D, \omega_X)$ and the relations over $\mathcal{O}_X$ holding between them. The s_i are called unprojection variables and the unprojection of $D \subset X$ is the ring $\mathcal{O}_X[s_i]$. For further details on the theory behind unprojection, see [PR] and [Ki].

In the simplest cases, $D \subset X$ is projectively Gorenstein and so ω_D has only one generator s as an $\mathcal{O}_X$-module. This means that we only need to find relations expressing s as a rational function in order to write down $\mathcal{O}_X[s]$. In our examples D does not have such nice properties and we must make use of the normalisation $\widetilde{D} \to D$. There are several unprojection variables and finding relations between them is quite complicated.

3.5 The calculation

There are at least two ways to calculate the unprojection variables and relations between them. We will use complexes to calculate ω_X, ω_D and then lift their generators to $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{I}_D, \omega_X)$ using the short exact sequence (3.1). See below for a discussion of other methods.

Clearly the locally free resolution of $\mathcal{O}_X$ as an $\mathcal{O}_{\mathbb{P}}$-module will calculate ω_X for us as the last nonzero entry in the complex. However, calculating ω_D is not as easy, since $\mathcal{O}_D$ is not Cohen–Macaulay by construction. Instead we make use of the projective normalisation $\widetilde{D} \to D$ where the map is our familiar $\varphi\colon \mathbb{P}^1 \to D$. As an $\mathcal{O}_X$-module, $\mathcal{O}_{\widetilde{D}}$ is generated by $e_0 = 1$, $e_1 = u$, $e_2 = v$, $e_3 = uv$. Note that $\omega_D \cong \varphi_* \omega_{\widetilde{D}} \cong \varphi_* \mathcal{O}_{\widetilde{D}}$ because φ is an isomorphism in codimension 1, so we can calculate generators of ω_D by considering $\mathcal{O}_{\widetilde{D}}$. The locally free resolutions of $\mathcal{O}_X$ and $\mathcal{O}_{\widetilde{D}}$ fit into the commutative diagram

$$\begin{array}{ccccccccc}
\mathcal{O}_X & \longleftarrow & \mathcal{O} & \xleftarrow{F_{14}} & \mathcal{O}(-14) & \longleftarrow & 0 & & \\
\downarrow & & \cap & & \downarrow & & \downarrow & & \\
\mathcal{O}_{\widetilde{D}} & \longleftarrow & \mathcal{L}_0 & \xleftarrow{M_1} & \mathcal{L}_1 & \xleftarrow{M_2} & \mathcal{L}_2 & \longleftarrow & 0
\end{array}$$

where $\mathcal{O} = \mathcal{O}_{\mathbb{P}}$, $\mathcal{L}_0 = 4\mathcal{O}$, $\mathcal{L}_1 = 8\mathcal{O}$, $\mathcal{L}_2 = 4\mathcal{O}$, and the maps between them are given by the block matrices

$$M_1 = \begin{pmatrix} Y & Z \end{pmatrix}, \quad M_2 = \begin{pmatrix} -Z \\ Y \end{pmatrix}$$

where

$$Y = \begin{pmatrix} y & -x_1x_2 & -x_1x_2 - x_2^2 & 0 \\ -x_2 & y & 0 & -x_1x_2 - x_2^2 \\ -x_1 - x_2 & 0 & y & -x_1x_2 \\ 0 & -x_1 - x_2 & -x_2 & y \end{pmatrix},$$

$$Z = \begin{pmatrix} z & -x_1^4 & -x_2^4 & 0 \\ -x_1^3 & z & 0 & -x_2^4 \\ -x_2^3 & 0 & z & -x_1^4 \\ 0 & -x_2^3 & -x_1^3 & z \end{pmatrix}.$$

Note that the matrix Y (respectively Z) encodes the multiplication table of y (resp. z) on the $\mathcal{O}_X$-module $\mathcal{O}_{\widetilde{D}}$. In particular

$$\begin{pmatrix} e_0, & e_1, & e_2, & e_3 \end{pmatrix} Y = \begin{pmatrix} e_0, & e_1, & e_2, & e_3 \end{pmatrix} Z = 0.$$

Now, the locally free sheaf $\mathcal{H}om(\mathcal{L}_2, \omega_{\mathbb{P}})$ is a presentation of $\omega_{\widetilde{D}}$, which is equal to ω_D because the affine cone on D is normal in codimension 1. Thus we use (3.1) to lift its generators $\bar{s}_0, \dots, \bar{s}_3$ to $s_0, \dots, s_3$ generating $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{I}_D, \omega_X)$. Recall that the unprojection ring is $\mathcal{O}_Y = \mathcal{O}_X[s_i]$, and to write down Y explicitly we have to find the relations holding between the s_i. The remaining generator of $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{I}_D, \omega_X)$ is not an unprojection variable, it is the trivial inclusion map corresponding to $\omega_X \to \mathcal{H}om$ in (3.1). Since the diagram of complexes commutes, we deduce that there are 8 linear relations between the s_i given by

$$M_2 \begin{pmatrix} s_3 \\ s_2 \\ s_1 \\ s_0 \end{pmatrix} = D_2, \tag{3.2}$$

where D_2 is the down arrow $\mathcal{O}(-14) \to \mathcal{L}_1$. For further details on the theory of unprojection and a proof of the linear relations, see [PR].

The relations (3.2) hold between the generators of $\mathcal{O}_Y$ and as such are some of the equations defining the K3 surface Y. However, we expect there to be relations which are quadratic in the s_i as well. Some justification for this is that the generators $e_0, \ldots, e_3$ of $\mathcal{L}_0$ are Serre dual to $\bar{s}_0, \ldots, \bar{s}_3$ generating $\mathcal{L}_2$, (see section 2.2.5). Thus the obvious monomial relations

$$\begin{array}{ll} e_1^2 = x_1 e_0^2 & e_1 e_3 = x_1 e_0 e_2 \\ e_1 e_2 = e_0 e_3 & e_2 e_3 = x_2 e_0 e_1 \\ e_2^2 = x_2 e_0^2 & e_3^2 = x_1 x_2 e_0^2 \end{array}$$

present in $\mathcal{L}_0$ should remain after lifting to $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{I}_D, \omega_X)$, but with a (Serre) twist.

There are various ways to calculate these quadratic relations in general. One can write the syzygies yoking the linear relations (3.2) in some Pfaffian format to try and work out the quadratic relations by hand. Another, more direct approach to calculating the unprojection ring is to ask for $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{I}_D, \omega_X)$. Computer algebra packages are capable of this calculation pretty quickly, which essentially yields rational expressions for the unprojection variables. It is then possible to recover the linear and quadratic relations between the s_i using these rational expressions. Finally, a less attractive (but also less labour intensive) alternative is to use computer algebra Gröbner basis methods to work out a primary decomposition of the ideal generated by the linear relations. The largest component of this decomposition is the ideal of the surface that we are looking for. This last method is what we use here. In the present example, the 8 linear relations (3.2) reduce to just 4:

$$\begin{array}{l} z + y s_3 - x_2^2 s_1 - 5x_2^2 y - x_1 x_2 s_2 - x_1 x_2 s_1 - 3x_1 x_2 y \\ y s_2 - y^2 - x_2 s_3 - x_2^2 s_0 + 3x_2^3 - x_1 x_2 s_0 + x_1 x_2^2 + x_1^2 x_2 + x_1^3 \\ y s_1 - x_2 s_3 + 6x_2^3 - x_1 s_3 - x_1 x_2 s_0 + 5x_1 x_2^2 + 3x_1^2 x_2 \\ y s_0 - x_2 s_2 - x_2 s_1 + 3x_2 y - x_1 s_2, \end{array}$$

with the other 4 linear relations absorbed by the 6 quadratic relations

$$\begin{array}{l} s_3^2 - x_1 x_2 s_0^2 - 3x_2 y^2 - 10x_2^2 s_3 + x_2^3 s_0 + 27x_2^4 - 6x_1 x_2 s_3 - 5x_1 x_2^2 s_0 + 24x_1 x_2^3 \\ \quad + 8x_1^2 x_2^2 + x_1^3 s_0 + 3x_1^3 x_2 \\ s_2 s_3 - x_2 s_0 s_1 - y s_3 - 5x_2^2 s_2 - 2x_2^2 s_1 + 2x_2^2 y - 3x_1 x_2 s_2 + 3x_1 x_2 y + x_1^2 s_1 \\ s_1 s_3 - x_1 s_0 s_2 + x_2^2 s_2 - 5x_2^2 s_1 - 4x_2^2 y - 2x_1 x_2 s_2 - 2x_1 x_2 s_1 - 3x_1 x_2 y + x_1^2 s_2 \\ s_2^2 - x_2 s_0^2 - y^2 - 2x_2 s_3 - 6x_2^2 s_0 - x_2^3 - x_1 x_2 s_0 + 4x_1 x_2^2 + x_1^2 s_0 + 4x_1^2 x_2 + x_1^3 \\ s_1 s_2 - s_0 s_3 - 3x_2 s_3 + 5x_2^2 s_0 + 19x_2^3 + 2x_1 x_2 s_0 + 9x_1 x_2^2 \\ s_1^2 - x_1 s_0^2 + x_2^2 s_0 - x_2^3 - 5x_1 x_2 s_0 - 13x_1 x_2^2 + x_1^2 s_0. \end{array}$$

Reviewing the equations of Y, we see that the variable z of weight 7 is eliminated by the first linear equation, so that we have $Y \subset \mathbb{P}(2^3, 3^3, 4)$ defined by 9 equations with 16 syzygies. As we alluded to earlier, the projection is actually happening in some bigger ambient space corresponding to a bigger ring with redundant generators. We take a weight 7 hyperplane section $z = \dots$ defined by the first linear equation of $Y' \subset \mathbb{P}(2^3, 3^3, 4, 7)$ to obtain Y. This explains the occurrence of the 7 in the weights for X. This is a common phenomenon in higher codimensions called "masking". One can also check directly by computer algebra that Y is quasismooth and has the requisite orbifold points.

3.6 Analysis of our result

An interesting question is whether we can fit the equations of Y into some kind of recognised format. This example is a codimension 4 ring so perhaps the equations are 4×4 Pfaffians of a 6×6 skewsymmetric matrix, rolling factors format or have something to do with Tom and Jerry. This is a difficult problem especially in harder examples than codimension 4 (see unpublished work of [BR]). We would expect this example to be related to a deformation of one of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, $\mathbb{P}^2 \times \mathbb{P}^2$ or $\mathbb{P}^1 \times \mathbb{P}^3$.

There are various automorphisms involved in choosing the unprojection variables s_i. We can use these to try and fit the equations into a nicer format, and some progress can be made along these lines. However, the results are mixed, and it is not clear how to make this kind of calculation into a general principle.

Moreover since projection is a birational map this example must be the elephant section of one half of a Sarkisov link. See [BZ] for examples of graded rings for Sarkisov links.

There is another unprojection construction for Y, projecting from the $\frac{1}{3}(1,2)$ point to yield

$$\mathbb{P}(1,2) \xrightarrow{\simeq} D \subset X_{6,6} \subset \mathbb{P}(2,2,2,3,3),$$

which might be a worthwhile pursuit in the future.

Chapter 4

Symmetric determinantal quartics

This chapter is concerned with the symmetric determinantal format for quartics and its extensions. The first section is a brief overview of properties of symmetric determinantal varieties from various perspectives. The main result of the chapter is theorem (4.2.1), which gives a construction for a particular stratum of extensions of symmetric determinantal quartics with remarkable properties.

4.1 Symmetric determinantal varieties

In this paragraph I collect together various descriptions and facts about symmetric determinantal varieties and ineffective theta characteristics. Each viewpoint has its advantages, and it is useful to be able to switch between them. For simplicity I describe the quartic surface case, but much of what follows is true more generally (see remark 4.1.1).

1. Take a symmetric 4×4 matrix M with general linear entries in the variables $y_1, \dots, y_4$. The projective variety T defined by $\det M = 0$ is a quartic K3 surface in $\mathbb{P}^3$ with 10 nodes. Alternatively one can consider a model of T in weighted projective space $\mathbb{P}(2^4, 3^4)$ defined by the equations

$$\begin{pmatrix} z_1, & z_2, & z_3, & z_4 \end{pmatrix} M = 0, \quad \bigwedge_{i,j}^{3} M = z_i z_j, \tag{4.1}$$

where now the 10 nodes become $\frac{1}{2}(1,1)$ points of the weighted ambient space. The notation $\bigwedge_{i,j}^3 M$ means $(-1)^{i+j} \det M_{ij}$, the (i,j)th cofactor of M. One should think of this as taking the "square-root" of the hypersurface model of T. To generate confusion I refer to the hypersurface as $T_4 \subset \mathbb{P}^3$ and the alternative model as $T \subset \mathbb{P}(2^4, 3^4)$.

2. A coherent sheaf $\mathcal{A}$ on $\mathbb{P}^3$ is called projectively Cohen–Macaulay if $\mathcal{A}$ is locally Cohen–Macaulay and $H^i(\mathbb{P}^3, \mathcal{A}(j)) = 0$ for all integers j and for $1 \le i \le \dim T - 1$, where $T \subset \mathbb{P}^3$ is the support of $\mathcal{A}$. These conditions are equivalent to $\Gamma_* \mathcal{A}$ being a Cohen–Macaulay graded $k[\mathbb{P}^3]$-module. Thus if $\mathcal{A}$ is supported on a hypersurface, then using the free resolution of $\Gamma_* \mathcal{A}$ we get a locally free resolution of $\mathcal{A}$

$$0 \leftarrow \mathcal{A} \leftarrow \bigoplus_{i=1}^{n} \mathcal{O}_{\mathbb{P}^3}(-d_i) \xleftarrow{M} \bigoplus_{i=1}^{n} \mathcal{O}_{\mathbb{P}^3}(-e_i) \leftarrow 0,$$

where the determinant of the matrix M defines T set theoretically, and consequently $\sum_{i=1}^{n}(e_i - d_i)$ is the degree of T. If in addition we require

$$H^0(\mathcal{A}(-1)) = H^2(\mathcal{A}(-2)) = 0,$$

then all the $d_i = 0$, $e_i = 1$ so that M has linear entries and T has degree n. Finally, M will be symmetric if $\mathcal{A}(-1)^{[2]} = \mathcal{O}_T(1)$.

Conversely if $T_4 \subset \mathbb{P}^3$ is defined by the determinant of a 4×4 symmetric matrix then $\mathcal{A} := \operatorname{coker} M$ is a projectively Cohen–Macaulay sheaf on T_4 with $\mathcal{A}(-1)^{[2]} = \mathcal{O}_T(1)$. Writing $\mathcal{O}_T(A) = \mathcal{A}(-1)$ and twisting by -1 the short exact sequence becomes

$$0 \leftarrow \mathcal{O}_T(A) \xleftarrow{(z_i)} 4\mathcal{O}_{\mathbb{P}^3}(-1) \xleftarrow{M} 4\mathcal{O}_{\mathbb{P}^3}(-2) \xleftarrow{(z_i)^t} 0.$$

We define

$$T = \operatorname{Proj} R(T_4, \mathcal{O}_T(A)) \subset \mathbb{P}(2^4, 3^4),$$

which is the variety defined by equations (4.1). The first part of equations (4.1) follows immediately because the sequence is exact, while the second part is determined by what happens when the matrix M drops rank from 3 to 2. See [Be, Cat] for further details on projectively Cohen–Macaulay sheaves.

3. Let $\mathbb{P}^9 = \mathbb{P}H^0(\mathbb{P}^3, \mathcal{O}(2))$ be the space of quadrics in $\mathbb{P}^3$, or if you prefer, the space of symmetric 4×4 matrices up to scalar multiplication. There is a natural stratification of this space by rank:

$$\mathbb{P}^9 \supset V_4^8 \supset V_{10}^6 \supset V_8^3.$$

For example, V_4^8 is a hypersurface of degree 4 in $\mathbb{P}^9$, which corresponds to quadrics in $\mathbb{P}^3$ of rank ≤ 3, or equivalently 4×4 symmetric matrices whose determinant vanishes. Similarly V_{10}^6 (respectively V_8^3) is the locus of quadrics of rank ≤ 2 (resp. ≤ 1).

Now take a web $\mathcal{M}$ of quadrics in $\mathbb{P}^3$, i.e. $\mathcal{M}$ is a linear system of projective dimension 3 inside $\mathbb{P}H^0(\mathbb{P}^3, \mathcal{O}(2))$. Choose coordinates $y_1, \dots, y_4$ for $\mathcal{M}$, and define

$$T_4 = \mathcal{M} \cap V_4^8.$$

Then T_4 is the locus of quadrics of rank ≤ 3 in $\mathcal{M}$, and it is defined by the vanishing of the determinant of a 4×4 symmetric matrix with linear entries in $y_1, \dots, y_4$. In general if $\mathcal{M}$ is base point free, this is an irreducible quartic hypersurface in $\mathbb{P}^3$.

The singularities of T_4 are given by $T_4 \cap V_{10}^6$, the locus of quadrics of rank ≤ 2 in $\mathcal{M}$. There are 10 isolated points in this locus, corresponding to 10 nodes on T_4. Of course $T_4 \cap V_{10}^6$ is defined algebraically by the vanishing of the 3×3 minors of M. These minors generate the linear system of contact cubics to the quartic hypersurface T_4. See [Tj] for details of this and [Cay] for a classical proof.

Remark 4.1.1 Most of the above holds in general for symmetric determinantal varieties. One easy specialisation of the quartic surface case is when the matrix in question has linear entries in just 3 variables instead of 4. Then

$$D_4 \colon (\det M = 0) \subset \mathbb{P}^2$$

is a nonhyperelliptic plane curve of genus 3. Then D is endowed with an *ineffective theta characteristic* i.e. a divisor class A such that $2A = K_D$ and $h^0(A) = 0$. Geometrically D has 28 bitangent lines, and the ineffective theta is a sum of bitangents $\beta_1 + \beta_2 - \beta_3$, of which there are 36 choices. Furthermore $\mathcal{O}_D(A)(1)$ is a projectively Cohen–Macaulay sheaf, and we recover many of the properties listed above immediately. In particular, we get a short exact sequence

$$0 \leftarrow \mathcal{O}_D(A) \xleftarrow{(z_i)} 4\mathcal{O}_{\mathbb{P}^2}(-1) \xleftarrow{M} 4\mathcal{O}_{\mathbb{P}^2}(-2) \xleftarrow{(z_i)^t} 0$$

and the equations of $D := \operatorname{Proj} R(D_4, A)$ are analogous to (4.1) with the matrix M having entries in just 3 variables.

4.1.1 A projection construction for T

Let $T_4 \subset \mathbb{P}^3$ be a hypersurface with 10 nodes. Choose a node and project away from it onto the complementary plane $\mathbb{P}^2$. Explicitly, we can choose coordinates so that the equation of T_4 is

$$\alpha_2(y_1,y_2,y_3)y_4^2+\beta_3(y_1,y_2,y_3)y_4+\gamma_4(y_1,y_2,y_3)=0,$$

with a node at $P=(0,0,0,1)$. Then linear projection onto the plane with coordinates y_1, y_2, y_3 gives a double covering of the plane branched in the sextic curve $\beta^2-4\alpha\gamma$. The image of P under the projection is the conic $\alpha=0$, which touches the branch curve doubly in each of 6 points. We say that the conic is totally tangent to the sextic.

If we further assume that T is a symmetric determinantal hypersurface, then the branch curve breaks up into two distinct cubics. These two cubics intersect one another transversally to give 9 nodes, and the additional node from the centre of projection makes 10 nodes on T_4.

The same map can also be viewed as a calculation in Gorenstein projection. See section (3.2) for a discussion of Gorenstein unprojection. Start with the K3 surface $T\subset\mathbb{P}(2^4,3^4)$ which has $10\times\frac{1}{2}$ orbifold points. Let A denote the polarising divisor for this model of T, choose one of the $\frac{1}{2}$ points and call it P. Then write $\sigma\colon\widetilde{T}\to T$ for the $(1,1)$-weighted Kawamata blowup of P. The exceptional locus $E\cong\mathbb{P}^1\subset\widetilde{T}$ is the centre for our projection, and the projection map is determined by the linear system $\sigma^*A-\frac{1}{2}E$ on $\widetilde{T}$. The image of this projection is $T'_{6,6}\subset\mathbb{P}(2^3,3^2)$.

The surface $T'_{6,6}\subset\mathbb{P}(2^3,3^2)$ is a double cover of $\mathbb{P}(2,2,2)$ branched in the two cubics defined by the relations of weight 6. The image of the exceptional curve E is embedded as a conic which is totally tangent to the branch sextic.

A further way to calculate this projection is via explicit commutative algebra. Fairly generally we can assume the matrix M is of the form

$$M=\begin{pmatrix} b & y_4 & B & 0\\ & a & 0 & A\\ \text{sym} & & y_1 & y_2\\ & & & y_3\end{pmatrix},$$

where a, b are general linear forms in $y_1,\dots,y_3$ and $A=y_1+\alpha_1y_2+\alpha_2y_3$, $B=\beta_1y_1+\beta_2y_2+y_3$. One can check that the K3 surface determined by this matrix has a $\frac{1}{2}$ point at $(0,0,0,1)$, with local coordinates near the singularity given by the variables z_3, z_4. Thus if we project away from this point we expect to eliminate y_4, z_3, z_4. Calculating cofactors $(1,1)$, $(2,2)$ of M we obtain equations:

$$\begin{aligned} z_1^2&=F_6(y_1,y_2,y_3)=a(y_1y_3-y_2^2)-y_1A^2,\\ z_2^2&=G_6(y_1,y_2,y_3)=b(y_1y_3-y_2^2)-y_3B^2.\end{aligned}\tag{4.2}$$

These are the only equations remaining from (4.1) that do not involve y_4, z_3, z_4. In particular the cofactor $-M_{12}$ for z_1z_2 involves y_4 and so does not survive the projection. Further, the product F_6G_6 is the defining equation $\beta^2 - 4\alpha\gamma$ of the totally tangent sextic. The equations (4.2) define the image of our projection map

$$T'_{6,6} \subset \mathbb{P}(2,2,2,3,3).$$

Remark 4.1.2 The truncated graded ring $R(T')^{[2]}$ which is the even part of $R(T')$ no longer contains z_1, z_2 as generators because they have odd degree. However, we win the new generator z_1z_2 since we observed above that there is no equation involving this product in $R(T')$. Thus the truncation defines the familiar double cover

$$T_6 \subset \mathbb{P}(1,1,1,3)$$

defined by $u^2 = FG$ where $u = z_1z_2$ and we have divided degrees by 2.

Conversely, given $T_6 \subset \mathbb{P}(1,1,1,3)$ defined by $z^2 = F_3G_3(y_1,y_2,y_3)$, the above argument shows there is a divisor class A on T_6 with $\mathcal{O}_{T_6}(2A) = \mathcal{O}(1)$.

4.1.2 Description of T as an almost homogeneous space

Let $V = \mathbb{C}^4$ be a vector space of dimension 4, then there is a natural $G := \mathrm{GL}(4,\mathbb{C})$ group action on V by matrix multiplication. There is an induced action of G on the vector spaces $S^2(V)$ and $\bigwedge^3 V$. We define the almost homogeneous space X to be the closure of the G-orbit of the vector

$$(N,P) \in S^2(V) \oplus \bigwedge^3 V$$

where

$$N = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 0 \end{pmatrix}, \quad P = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

The K3 surface T is the intersection of X with a 4-dimensional subspace $\mathcal{M} \subset S^2(V)$ and naturally lives in weighted projective space $\mathbb{P}(2^4,3^4)$ with equations (4.1).

4.2 Extending determinantal formats

In this section we treat extensions of symmetric determinantal quartic surfaces which were discussed in some detail above. Let D be the model of a nonhyperelliptic curve of genus 3 determined by one of its ineffective theta characteristics in $\mathbb{P}(2^3,3^4)$. As we know, the structure of $R(D,A)$ is completely determined by a symmetric 4×4 matrix with linear entries in 3 variables y_1,y_2,y_3 of weight 2. If we add another variable y_0 of weight 2 into the matrix preserving the linearity and symmetry, we get the graded ring of a K3 surface $T \subset \mathbb{P}(2^4, 3^4)$ with $10 \times \frac{1}{2}$ points. The curve D is contained in T as a hyperplane section of weight 2, hence we have an easy illustration of the hyperplane section principle. The graded ring of the hyperplane section D is related to that of T by $R(T, A) = R(D, A)/(y_0)$. A priori we know that both T and A are always symmetric determinantal varieties, so this is the only way to extend D to a K3 surface with $10 \times \frac{1}{2}$ points.

Furthermore, T is a K3 surface and so is naturally the elephant hyperplane section of a Fano 3-fold $W^3 \subset \mathbb{P}(1, 2^4, 3^4)$ with $10 \times \frac{1}{2}$ points. In other words, there is an element $x_0 \in H^0(W^3, -K_W)$ such that

$$T = (x_0 = 0) \cap W^3 \subset \mathbb{P}(1, 2^4, 3^4),$$

or in terms of graded rings

$$R(T, A) = R(W^3, -K_W)/(x_0).$$

In fact, this process can be iterated and we can continue incorporating more variables x_1, x_2, x_3 of degree 1 into the ring. In this way we obtain a tower of inclusions

$$D \subset T \subset W^3 \subset W^4 \subset W^5 \subset W^6 \subset \mathbb{P}(1^4, 2^4, 3^4),$$

where each W^n is a Fano n-fold of Fano index $n - 2$. Of course it is not immediately clear how to perform the extension procedure; it is certainly not as simple as generalising the symmetric matrix to have entries involving the x_i. Amazingly, after having built our tower as far as the Fano 6-fold, we discover that there is a one to one correspondence between the moduli of the K3 surface T and the moduli of the 6-fold W^6.

Theorem 4.2.1 *For each quasismooth symmetric determinantal K3 surface $T \subset \mathbb{P}(2^4, 3^4)$ with $10 \times \frac{1}{2}$ orbifold points there is a unique extension to a quasismooth Fano 6-fold $W \subset \mathbb{P}(1^4, 2^4, 3^4)$ with $10 \times \frac{1}{2}$ points and such that*

$$T = W \cap H_1 \cap H_2 \cap H_3 \cap H_4,$$

where the H_i are hyperplanes of the projective space $\mathbb{P}(1^4, 2^4, 3^4)$.

Jan Stevens first observed this phenomenon in 1993 when calculating the deformation–extension theory for the special case of the Klein quartic curve, which has maximal symmetry group of order 168. This extra symmetry restricts the deformation extension space enough to make the computation viable. Using quite different methods, we are able to give a proof of the theorem for any symmetric determinantal K3 surface.

In our proof we will use the Gorenstein projection construction for the K3 surface T, which is not as symmetric as the determinantal representation but is very beautiful in its own way. The projection is described in section (4.1): we project from one of the $\frac{1}{2}$ points on T to get the surface

$$\mathbb{P}^1 \xrightarrow{\varphi} T'_{6,6} \subset \mathbb{P}(2,2,2,3,3)$$

with $9 \times \frac{1}{2}$ points. The surface T' is a double cover of $\mathbb{P}^2$ branched in a sextic curve which breaks into two cubics. The image of the map φ is a conic in the plane $\mathbb{P}(2,2,2)$ which touches both branch cubics at exactly 3 points each. Hence constructing a K3 surface $T \subset \mathbb{P}(2^4, 3^4)$ with $10 \times \frac{1}{2}$ points is equivalent to exhibiting a suitable projected surface $T' \subset \mathbb{P}(2^3, 3^2)$ along with a map φ embedding $\mathbb{P}^1$ inside T' with appropriate tangency.

Write y_i, z_i for the coordinates on $\mathbb{P}(2,2,2,3,3)$ of weight 2, 3 respectively. After coordinate changes, for general T' the embedding of $\mathbb{P}^1$ is

$$\varphi \colon \mathbb{P}^1 \to \mathbb{P}(2,2,2,3,3)$$
$$(u,v) \mapsto (u^2, uv, v^2, u^3 + \alpha_1 u^2 v + \alpha_2 uv^2, \beta_1 u^2 v + \beta_2 uv^2 + v^3). \tag{4.3}$$

We have assumed that u is a factor of $\varphi^*(z_1)$ and likewise v divides $\varphi^*(z_2)$. Moreover we assume that $\varphi^*(z_1)$ and $\varphi^*(z_2)$ have no common factor.

Since $S^3(u^2, uv, v^2)$ generates $S^6(u,v)$ we see that the image of φ is given by the equations

$$C_1 \colon z_1^2 = y_1(y_1 + \alpha_1 y_2 + \alpha_2 y_3)^2, \tag{4.4}$$
$$C_2 \colon z_1 z_2 = y_2(y_1 + \alpha_1 y_2 + \alpha_2 y_3)(\beta_1 y_1 + \beta_2 y_2 + y_3), \tag{4.5}$$
$$C_3 \colon z_2^2 = y_3(\beta_1 y_1 + \beta_2 y_2 + y_3)^2, \tag{4.6}$$
$$Q \colon y_1 y_3 = y_2^2. \tag{4.7}$$

Note that the choice of representation for the first three equations is only unique modulo the conic Q of equation (4.7); for example I could have written $z_2^2 = \beta_1^2 y_1^2 y_3 + 2\beta_1\beta_2 y_2^3 + (\beta_2^2 + 2\beta_1\beta_3) y_2^2 y_3 + \beta_3^2 y_3^3$ instead.

The projected surface T' is given by taking two combinations

$$\begin{aligned} &C_1 + l_1(y_1, y_2, y_3) Q \\ &C_3 + l_3(y_1, y_2, y_3) Q, \end{aligned} \tag{4.8}$$

where l_i are linear. There are 9 moduli for this construction: 3 from the parameters α_i, β_i and a further 3 for each of the linear forms l_1, l_3. As an illustration, we could choose

$$z_1^2 = y_1(y_1 + \alpha_1 y_2 + \alpha_2 y_3)^2 + (y_2 + 2y_3)(y_1 y_3 - y_2^2)$$
$$z_2^2 = y_3(\beta_1 y_1 + \beta_2 y_2 + y_3)^2 + y_1(y_1 y_3 - y_2^2),$$

which corresponds to the symmetric matrix

$$M = \begin{pmatrix} y_1 & y_4 & \beta_1 y_1 + \beta_2 y_2 + y_3 & 0 \\ & y_2 + 2y_3 & 0 & y_1 + \alpha_1 y_2 + \alpha_2 y_3 \\ \text{sym} & & y_1 & y_2 \\ & & & y_3 \end{pmatrix}.$$

Remark 4.2.2 We have made a trade off here between simplifying the equations of T' and simplifying the map φ. Denote the branch cubics by B_1, B_2 and the conic by Q. Then the restrictions $B_i|_Q$ generate a pencil of cubics on $Q \cong \mathbb{P}^1$. We have chosen $\varphi^*(z_i) := B_i|_Q$, which means that the equations of T' take the simpler form $z_i^2 = f_i(y_1, y_2, y_3)$. We could have reduced the number of terms involved in the definition of φ by choosing $\varphi^*(z_i)$ to be generators for the pencil of the form $u^3 + \alpha u^2 v$ and $\beta uv^2 + v^3$. However, were we to do this, the price we pay is that we are only able to assume the equations for T' are of the form $(\lambda_i z_1 + \mu_i z_2)^2 = f_i(y_1, y_2, y_3)$.

Proof of theorem The key point is that there is an analogous Gorenstein projection of the Fano 6-fold W, which has image $\mathbb{P}^5 \subset W'_{6,6} \subset \mathbb{P}(1^4, 2^3, 3^2)$. If we can write down the extension of T' to W', then this is as good as extending T to W itself. Of course we have reduced to a much easier problem because we can work explicitly with T' and W' as they are codimension 2 Gorenstein varieties, which are well understood.

We define φ as in (4.3) and write $\varphi_0 \colon \mathbb{P}^1 \to \mathbb{P}(2, 2, 2)$ for the standard parametrisation of the conic in $\mathbb{P}(2, 2, 2)$:

$$\varphi_0^*(y_1) = u^2, \quad \varphi_0^*(y_2) = uv, \quad \varphi_0^*(y_3) = v^2.$$

If we write u, v, a, b, c, d for the coordinates of $\mathbb{P}^5$ then up to automorphisms of $\mathbb{P}^5$ and $\mathbb{P}(1^4, 2^3)$, the general extension of φ_0 to $\Phi_0 \colon \mathbb{P}^5 \to \mathbb{P}(1^4, 2^3)$ is

$$\Phi_0^*(a) = a, \quad \Phi_0^*(b) = b, \quad \Phi_0^*(c) = c, \quad \Phi_0^*(d) = d,$$

$$\begin{aligned} \Phi_0^*(y_1) &= u^2 \qquad\qquad -dv\ + bd - c^2, \\ \Phi_0^*(y_2) &= uv + bu + cv\ - ad + bc, \\ \Phi_0^*(y_3) &= v^2 - au \qquad\qquad + ac - b^2. \end{aligned} \tag{4.9}$$

The curious extra terms $bd - c^2$, $-ad + bc$, $ac - b^2$ are added with hindsight. They are the 2×2 cofactors of the matrix

$$\begin{pmatrix} a & b & c \\ b & c & d \end{pmatrix}$$

and are harmless, but ensure that the matrix (4.10) below is more beautiful.

We prove that there is a unique map $\Phi\colon \mathbb{P}^5 \to \mathbb{P}(1^4, 2^3, 3^2)$ extending $T'_{6,6}$ to $W'_{6,6}$ and lifting Φ_0 so that the following diagram commutes:

$$\begin{array}{ccc} & & \mathbb{P}(1^4, 2^3, 3^2) = \operatorname{Proj} S \\ & \overset{\Phi}{\nearrow} & \Big\downarrow \pi \\ \mathbb{P}^5 & \xrightarrow{\Phi_0} & \mathbb{P}(1^4, 2^3) = \operatorname{Proj} R \end{array}$$

Write M, R, S for the coordinate rings of $\mathbb{P}^5$, $\mathbb{P}(1^4, 2^3)$ and $\mathbb{P}(1^4, 2^3, 3^2)$ respectively. Then M is a graded R-module via Φ_0^* generated by 1, u, v (see equation (4.9)) with presentation

$$0 \leftarrow M \xleftarrow{(1,u,v)} R \oplus 2R(-1) \xleftarrow{A} 2R(-3) \oplus R(-4)$$

where A is the matrix

$$\begin{pmatrix} by_1 + cy_2 + dy_3 & ay_1 + by_2 + cy_3 & A_{13} \\ -y_2 & y_3 & ay_1 + by_2 + cy_3 \\ y_1 & -y_2 & by_1 + cy_2 + dy_3 \end{pmatrix} \tag{4.10}$$

and the outsized entry is

$$A_{13} = y_1y_3 - y_2^2 + b^2y_1 + (2bc - ad)y_2 + c^2y_3.$$

Moreover, M is also a graded module over S via Φ^*, with the same generators and of course more relations. Finally, S is a module over R which is not finite. We will not insist on writing φ^*, Φ_0^*, Φ^* when it is clear that we are dealing with the module structure.

Since Φ is a lift of Φ_0 and φ, we can assume the general forms for $\Phi^*(z_i)$ are

$$\begin{aligned} \Phi^*(z_1) &= u^3 + \alpha_1u^2v + \alpha_2uv^2 + s_1u^2 + s_2uv + s_3v^2 + s_4u + s_5v, \\ \Phi^*(z_2) &= \beta_1u^2v + \beta_2uv^2 + v^3 + t_1u^2 + t_2uv + t_3v^2 + t_4u + t_5v \end{aligned}$$

where $s_i(a, b, c, d)$, $t_i(a, b, c, d)$ are homogeneous polynomials of degree 1 or 2 as appropriate. Now using the R-module structure of M, we can write

$$\begin{aligned} \Phi^*(z_1) &= (f + s_4)u + s_5v, \\ \Phi^*(z_2) &= t_4u + (g + t_5)v \end{aligned} \tag{4.11}$$

where

$$f = y_1 + \alpha_1y_2 + \alpha_2y_3, \quad g = \beta_1y_1 + \beta_2y_2 + y_3.$$

Here we use coordinate changes such as $z_1 \mapsto z_1 + s_1 y_1$ so that z_1, z_2 absorb the values of s_i, t_i for $i = 1, 2, 3$. We are required to find suitable values of s_4, s_5, t_4, t_5 so that the kernel of Φ^* contains equations extending (4.4), (4.6) and (4.7). Constructing the extension Φ of φ amounts to the following algebraic result:

Theorem 4.2.3 *(I) The kernel of $\Phi^*\colon S \to M$ contains equations extending (4.4), (4.6) of the form*

$$z_1^2 - y_1 f^2 \in R + Rz_1 + Rz_2,$$
$$z_2^2 - y_3 g^2 \in R + Rz_1 + Rz_2$$

if and only if $s_4 = s_5 = t_4 = t_5 = 0$.

(II) Given part (I), the equations are

$$\begin{aligned} z_1^2 - y_1 f^2 = (c^2 - bd) f^2 - ((1-\alpha_2\beta_1)L_1 + (\alpha_2\beta_2 - \alpha_1)L_2)\, df \\ + ((1-\alpha_2\beta_1)y_2 - (\alpha_2\beta_2 - \alpha_1)y_3)\, dz_1 + \alpha_2 df\, z_2 \end{aligned} \tag{4.12}$$

$$\begin{aligned} z_2^2 - y_3 g^2 = (b^2 - ac) g^2 - ((\beta_1\alpha_1 - \beta_2)L_1 + (1-\beta_1\alpha_2)L_2)\, ag \\ + \beta_1 ag z_1 + (-(\beta_1\alpha_1 - \beta_2)y_1 + (1-\beta_1\alpha_2)y_2)\, az_2, \end{aligned} \tag{4.13}$$

where $L_1 = by_1 + cy_2 + dy_3$, $L_2 = ay_1 + by_2 + cy_3$.

Corollary 4.2.4 *The kernel of Φ^* contains the following equation extending (4.5)*

$$z_1 z_2 - fgy_2 = fg(ad - bc) - bgz_1 - cfz_2,$$

and equations extending multiples of (4.7), of the form

$$y_i A_{13} \in R + Rz_1 + Rz_2$$

for $i = 1, 2, 3$.

Remark 4.2.5 Part (I) of the theorem uniquely determines Φ up to automorphism. Moreover, the coordinate changes used do not alter the original setup

$$\varphi\colon \mathbb{P}^1 \hookrightarrow T'_{6,6} \subset \mathbb{P}(2,2,2,3,3),$$

so Φ is completely determined by φ.

As an aside, observe that since we expect the image of Φ not to be Cohen–Macaulay, our strategy of using the hyperplane section principle from (2.2.3) goes awry. The equation $y_1 y_3 - y_2^2$ does not extend directly, and we need three separate equations replacing it in the kernel of Φ^*. The image of $\Phi_0\colon \mathbb{P}^5 \to \mathbb{P}(1^4, 2^3)$ is defined by the vanishing of the determinant of the matrix A from (4.10).

Proof The "if" part is a straightforward verification that when $s_4 = s_5 = t_4 = t_5 = 0$, equations (4.12), (4.13) are in the kernel of Φ^* by direct substitution. The remainder of the proof is the "only if" part.

The ring $k[u,v]$ is a graded module over $k[y_1,y_2,y_3]$ via φ_0^*, so referring to equation (4.3), we can write $\varphi^*(z_i)$ as:

$$\varphi^*(z_1) = (y_1 + \alpha_1 y_2 + \alpha_2 y_3)u$$
$$\varphi^*(z_2) = (\beta_1 y_1 + \beta_2 y_2 + y_3)v.$$

If we square these two expressions and use the module structure to render residual terms u^2, v^2 as y_1, y_3 we obtain the two equations (4.4), (4.6). Moreover we can write down the equation for $z_1 z_2$ by rendering uv as y_2.

We attempt the same elimination calculation with Φ^*. Observe that by definition of Φ^*, we can write u^2, uv, v^2 as

$$\begin{aligned} u^2 &= \Phi^*(y_1 - bd + c^2) + dv \\ uv &= \Phi^*(y_2 + ad - bc) - bu - cv \\ v^2 &= \Phi^*(y_3 - ac + b^2) + au. \end{aligned}$$

Thus by squaring $\Phi^*(z_i)$ defined in (4.11) and rendering u^2, uv, v^2 as above, we arrive at

$$\begin{aligned} \Phi^*\left(z_1^2 - \widetilde{f}^2(y_1 - bd + c^2) - 2\widetilde{f}s_5(y_2 + ad - bc) - s_5^2(y_3 - ac + b^2)\right) &\equiv 0 \\ \Phi^*\left(z_2^2 - t_4^2(y_1 - bd + c^2) - 2\widetilde{g}t_4(y_2 + ad - bc) - \widetilde{g}^2(y_3 - ac + b^2)\right) &\equiv 0 \end{aligned}$$

modulo $(a,b,c,d)M$, where $\widetilde{f} = f + s_4$ and $\widetilde{g} = g + t_5$. The residual parts to these congruences are

$$\begin{aligned} K&\colon (f+s_4)^2 dv - 2(f+s_4)s_5(bu+cv) + s_5^2 au \\ L&\colon t_4^2 dv - 2(g+t_5)t_4(bu+cv) + (g+t_5)^2 au, \end{aligned} \tag{4.14}$$

which are homogeneous expressions of degree 6 in $(a,b,c,d)M$. We prove that for the unique values $s_4 = s_5 = t_4 = t_5 = 0$ the two residual terms K, L are contained in the submodule

$$R + Rz_1 + Rz_2 \subset M = R + Ru + Rv.$$

This is necessary and sufficient to obtain equations for z_i^2 of the required form in the kernel of Φ^*.

By referring to the definition of $\Phi^*(z_i)$ from (4.11), we see that the submodule $R + Rz_1 + Rz_2$ is the image of the composite map

$$M \xleftarrow{(1,u,v)} R \oplus 2R(-1) \xleftarrow{B} R \oplus 4R(-3) \oplus R(-4)$$

where B is the matrix

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & by_1+cy_2+dy_3 & ay_1+by_2+cy_3 & A_{13} \\ 0 & f+s_4 & t_4 & -y_2 & y_3 & ay_1+by_2+cy_3 \\ 0 & s_5 & g+t_5 & y_1 & -y_2 & by_1+cy_2+dy_3 \end{array}\right)$$

Note that the first 3 columns of B represent the submodule generators 1, z_1, z_2 respectively while the last 3 columns are the matrix A of (4.10), which maps to 0 under the composite.

We must write K, L of (4.14) as expressions in the image of this composite map. We stratify the problem according to degree in y_1, y_2, y_3, so that

$$\begin{aligned} K &= K^{(2)} + K^{(\leq 1)}, \\ L &= L^{(2)} + L^{(\leq 1)} \end{aligned}$$

where $K^{(2)} = df^2v$, $L^{(2)} = ag^2u$ are the terms of K, L which are degree 2 in y_i. First work in degree 2 so that we can assume that the matrix B does not involve s_i, t_i. I demonstrate how to calculate the preimage of df^2v under B, as ag^2u is exactly similar.

We have to find some $\eta := \eta^{(1)} + \eta^{(0)}$ in $R \oplus 4R(-3) \oplus R(-4)$ such that

$$df^2v = \begin{pmatrix} 1, & u, & v \end{pmatrix} B\eta^{(1)},$$

where $\eta^{(i)}$ has degree i in y_1, y_2, y_3. We can do this explicitly: the first column of B can be used to eliminate any terms in the first row, so the important part of B is the submatrix

$$B' = \begin{pmatrix} f & 0 & -y_2 & y_3 \\ 0 & g & y_1 & -y_2 \end{pmatrix}.$$

Since the bottom row of B' only involves y_3 as part of g, we must write

$$\begin{aligned} f &= y_1 + \alpha_1 y_2 + \alpha_2 y_3 \\ &= y_1 + \alpha_1 y_2 + \alpha_2(g - \beta_1 y_1 - \beta_2 y_2) \end{aligned}$$

or as an expression in the bottom row of B',

$$f = \begin{pmatrix} 0, & g, & y_1, & -y_2 \end{pmatrix} \begin{pmatrix} * \\ \alpha_2 \\ (1-\alpha_2\beta_1) \\ (-\alpha_1+\alpha_2\beta_2) \end{pmatrix}.$$

We are still free to use the first column of B' to remove spurious terms from the middle row by adjusting the starred entry to solve

$$0 = \begin{pmatrix} f, & 0, & -y_2, & y_3 \end{pmatrix} \begin{pmatrix} * \\ \alpha_2 \\ (1-\alpha_2\beta_1) \\ (-\alpha_1+\alpha_2\beta_2) \end{pmatrix}.$$

This is where we use the extra multiple of f to avoid having to divide by f, so we must have

$$\eta_2^{(1)} = \frac{1}{f}(\eta_4^{(1)} y_2 - \eta_5^{(1)} y_3) \qquad \eta_4^{(1)} = (1-\alpha_2\beta_1)df$$
$$\eta_3^{(1)} = \alpha_2 df \qquad \eta_5^{(1)} = (-\alpha_1+\alpha_2\beta_2)df,$$

where $\eta_2^{(1)}$ is the starred entry whose value is completely determined by the rest of $\eta^{(1)}$. Finally, referring back to the large matrix B and in the same manner as for B', we use the first column to remove any accidental terms from the top row so that the remaining entries of the vector $\eta^{(1)}$ are

$$\eta_1^{(1)} = -(by_1+cy_2+dy_3)\eta_4^{(1)} - (ay_1+by_2+cy_3)\eta_5^{(1)}$$
$$\eta_6^{(1)} = 0.$$

An exactly similar argument proves that

$$ag^2u = \begin{pmatrix} 1, & u, & v \end{pmatrix} B\xi^{(1)}$$

where $\xi^{(1)}$ is the vector

$$\xi_1^{(1)} = -(by_1+cy_2+dy_3)\xi_4^{(1)} - (ay_1+by_2+cy_3)\xi_5^{(1)}$$
$$\xi_2^{(1)} = \beta_1 ag$$
$$\xi_3^{(1)} = \frac{1}{g}(-\xi_4^{(1)} y_1 + \xi_5^{(1)} y_2)$$
$$\xi_4^{(1)} = (\beta_1\alpha_1 - \beta_2)ag$$
$$\xi_5^{(1)} = (1-\beta_1\alpha_2)ag$$
$$\xi_6^{(1)} = 0.$$

Now we reinstate s_i, t_i to the matrix B and use the degree 1 solutions $\eta^{(1)}$, $\xi^{(1)}$ to compute the full vectors η, ξ. The easiest way to do this is via a direct computation. Evaluate the remaining residual terms

$$K' := K - \begin{pmatrix} 1, & u, & v \end{pmatrix} B\eta^{(1)}$$
$$L' := L - \begin{pmatrix} 1, & u, & v \end{pmatrix} B\xi^{(1)}$$

to obtain two expressions in M of degree 6 and involving u, v in degrees ≤ 3. In particular all terms involve some s_i or t_i by construction, and the terms of degree 3 in u, v have coefficients which must be linear in s_i, t_i. We attempt to write K', L' as expressions in $R + Rz_1 + Rz_2$, first using z_1, z_2 to remove terms involving u^3, v^3 respectively. Then the coefficients of u^2v and uv^2 in K', L' must vanish because we have no other expressions in $R + Rz_1 + Rz_2$ that are of degree 3 in u, v. We write these 4 coefficients as simultaneous linear equations in the s_i, t_i:

$$C \begin{pmatrix} s_4 \\ s_5 \\ t_4 \\ t_5 \end{pmatrix} = 0 \tag{4.15}$$

where C is the coefficient matrix

$$\begin{pmatrix} \delta_1 d & -2\delta_1 c + \beta_1\delta_2 d & 0 & -\alpha_2\delta_1 d \\ \delta_2 d & -2\delta_2 c + (\beta_2\delta_2 - \delta_1)d & 0 & -\alpha_2\delta_2 d \\ -\beta_1\delta_3 a & 0 & (\alpha_1\delta_3 - \delta_1)a - 2\delta_3 b & \delta_3 a \\ -\beta_1\delta_1 a & 0 & \alpha_2\delta_3 a - 2\delta_1 b & \delta_1 a \end{pmatrix}$$

and $\delta_1 = 1 - \alpha_2\beta_1$, $\delta_2 = \alpha_1 - \alpha_2\beta_2$, $\delta_3 = \beta_2 - \alpha_1\beta_1$. The δ_i are the 3 cross ratios of the 6 points of tangency on the conic and they appear in equations (4.12), (4.13). The matrix C can be calculated by hand or using the computer routine below. Assume Δ, δ_1 are nonzero[1], where $\Delta = \delta_1^2 - \delta_2\delta_3$ is the determinant of the resultant matrix of f, g displayed as (4.16) below. Then row operations on C imply that $s_4 = s_5 = t_4 = t_5 = 0$. Hence $K' = L' = 0$ and so we have proved that $\eta = \eta^{(1)}$ and $\xi = \xi^{(1)}$.

Here is the computer code to calculate C:

```
Q:=Rationals();
K<al1,al2,be1,be2>:=FunctionField(Q,4);
S<a,b,c,d,u,v,s4,s5,t4,t5>:=PolynomialRing(K,10);

y1 := u^2 - d*v + b*d - c^2;
y2 := u*v + b*u + c*v - a*d + b*c;
y3 := v^2 - a*u + a*c - b^2;
f := y1 + al1*y2 + al2*y3;
g := be1*y1 + be2*y2 + y3;
z1 := u*(f+s4) + v*s5;
z2 := u*t4 + v*(g+t5);

K := (f+s4)^2*d*v-2*(f+s4)*s5*(b*u+c*v)+s5^2*a*u;
L := t4^2*d*v-2*(g+t5)*t4*(b*u+c*v)+(g+t5)^2*a*u;
```

[1] If $\delta_1 = 0$ the solution is still $s_i = t_i = 0$ but there is an interesting anomaly. See section (4.2.1).

```
Kdash := K + (1-al2*be1)*(b*y1+c*y2+d*y3)*d*f
           + (-al1+al2*be2)*(a*y1+b*y2+c*y3)*d*f
           - ((1-al2*be1)*y2-(-al1+al2*be2)*y3)*d*z1
           - al2*f*d*z2;

Ldash := L + (1-be1*al2)*(a*y1+b*y2+c*y3)*a*g
           + (be1*al1-be2)*(b*y1+c*y2+d*y3)*a*g
           - be1*a*g*z1
           - ((1-be1*al2)*a*y2-(be1*al1-be2)*a*y1)*z2;

Kdash:=Kdash-(-2*b*s5 - al2*d*t4)*z1
            -(-2*al2*c*s5 + 2*al2*d*s4
               + (-al1 + al2*be2)*d*s5 - al2^2*d*t5)*z2;
Ldash:=Ldash-(-be1^2*a*s4 + (al1*be1 - be2)*a*t4
               + 2*be1*a*t5 - 2*be1*b*t4)*z1
            -(-be1*a*s5 - 2*c*t4)*z2;

D := [Coefficient(Coefficient(Kdash,u,3-i),v,i):i in [1..2]] cat
     [Coefficient(Coefficient(Ldash,u,3-i),v,i):i in [1..2]];
List:=[Coefficient(f,m,1):m in [s4,s5,t4,t5],f in D];
C:=Matrix(S,4,List);

Determinant(C);
```

The full form of equation $z_1^2 - y_1 f^2 \in R + Rz_1 + Rz_2$ is obtained by writing out the vector η inside $R + Rz_1 + Rz_2$ in terms of the generators 1, z_1, z_2:

$$z_1^2 = f^2(y_1 - bd + c^2) + \eta_1 + \eta_2 z_1 + \eta_3 z_2.$$

Likewise using ξ, the equation for z_2^2 is

$$z_2^2 = g^2(y_3 - ac + b^2) + \xi_1 + \xi_2 z_1 + \xi_3 z_2.$$

Written out in full, these are equations (4.12), (4.13) stated in the theorem. This concludes the proof of theorem (4.2.3) and its corollary is proved in section (4.3).

Given the existence of equations extending (4.4–4.7), we can prove the main theorem (4.2.1): define the unique Fano 6-fold

$$\mathbb{P}^5 \xrightarrow{\Phi} W'_{6,6} \subset \mathbb{P}(1^4, 2^3, 3^2)$$

extending $\mathbb{P}^1 \xrightarrow{\varphi} T'_{6,6} \subset \mathbb{P}(2,2,2,3,3)$ by taking the combination of equations constructed in theorem (4.2.3) and its corollary which correspond to the choice (4.8) made in the definition of $T'_{6,6}$.

4.2.1 General position of tangency points

First, if $\Delta = 0$ then $\varphi^*(z_i)$ have a shared root, which implies one of the tangency points P is common to both branch curves. Thus P is a $\frac{1}{2}$ point of $T'_{6,6} \subset \mathbb{P}(2,2,2,3,3)$. However, the two branch curves will not intersect transversally at P by construction and so this contradicts the hypothesis that T is quasismooth.

Now to fill in the gap I left in the proof that $s_i = t_i = 0$, suppose $\delta_1 = 0$ so that $\alpha_2 = \beta_1^{-1}$. Then if $\delta_2 = 0$ or $\delta_3 = 0$ this implies $\Delta = 0$ which was discounted above. Hence we assume that $\delta_2\delta_3 \neq 0$ and studying the first and last rows of C we see that this forces $s_5 = t_4 = 0$. However, the remaining two rows of C reduce to $s_4 = \alpha_2 t_5$, which no longer has a unique solution!

As a result we get an extension of φ to

$$\widetilde{\Phi}\colon \mathbb{P}^5 \to \mathbb{P}(1^4, 2^4, 3^2)$$

where the extra coordinate of weight 2 is s_4 (or t_5). Further, the kernel of $\widetilde{\Phi}^*$ contains equations

$$z_1^2 - y_1 f^2 \in R + Rs_4 + Rz_1 + Rz_2$$
$$z_2^2 - y_3 g^2 \in R + Rs_4 + Rz_1 + Rz_2,$$

but the analogue of corollary (4.2.4) does not hold unless we insist that $s_4 \equiv 0$ so that we recover our original hypothesis.

Thus for those particular configurations of degenerate branch curves on $T'_{6,6} \subset \mathbb{P}(2,2,2,3,3)$ with $\delta_1 = 0$, there is an extension to some Fano 7-fold

$$V'_{6,6} \subset \mathbb{P}(1^4, 2^4, 3^2).$$

This does not invalidate the main theorem (4.2.1), since we are looking for Fano 6-folds $W'_{6,6} \subset \mathbb{P}(1^4, 2^3, 3^2)$. However, this is a curious extra stratum of extensions of the K3 surface which demands further investigation.

Some further remarks

1. It is possible to write down all the equations for W explicitly by using unprojection. However, this is not very illuminating unless we have a format or structure for them. I have attempted to persuade the equations into some kind of extension of the symmetric determinantal format, but so far they have resisted.

2. We can choose the map $\Phi\colon \mathbb{P}^5 \to \mathbb{P}(1^4, 2^3, 3^2)$ to be $\mathrm{GL}(2, \mathbb{C})$ invariant. This may be of some use in finding a possible description of W inside some homogeneous space, although I have not investigated this fully.

4.3 Proof of corollary (4.2.4)

To prove the corollary we must calculate the equations extending (4.5) and multiples of (4.7). First note that

$$z_1 z_2 - fg(y_2 + ad - bc) = -fg(bu + cv),$$

so if we can write $fg(bu + cv)$ as an expression in the module $R + Rz_1 + Rz_2$ then we are done. We must find some ν in $R \oplus 4R(-3) \oplus R(-4)$ such that

$$fg(bu + cv) = \begin{pmatrix} 1, & u, & v \end{pmatrix} B\nu.$$

Indeed, we can choose the vector ν such that $\nu_2 = bg$, $\nu_3 = cf$ and the other $\nu_i = 0$. Thus the equation extending (4.5) is

$$z_1 z_2 = fg(y_2 + ad - bc) - bgz_1 - cfz_2.$$

The equations extending (4.7) are more complicated. First note from the definition of the matrix A of (4.10) that

$$A_{13} + L_2 u + L_1 v = 0,$$

where $L_1 = by_1 + cy_2 + dy_3$, $L_2 = ay_1 + by_2 + cy_3$. Thus to write down an equation for $y_i A_{13}$ in the kernel of Φ^* we seek some ν in $R \oplus 4R(-3) \oplus R(-4)$ such that

$$y_i L_2 u + y_i L_1 v = \begin{pmatrix} 1, & u, & v \end{pmatrix} B\nu.$$

Since we used the last column of B to calculate the residual part of $y_i A_{13}$, to avoid trivial solutions we only use the first 5 columns of B. As previously, the important part is the submatrix

$$B' = \begin{pmatrix} f & 0 & -y_2 & y_3 \\ 0 & g & y_1 & -y_2 \end{pmatrix}.$$

Let us calculate the equation for $y_1 A_{13}$. We construct the preimages of $y_1 L_2 u$ and $y_1 L_1 v$ under B separately and then sum these two expressions to get the preimage of the residual part. The idea is to try to write down two separate expressions for $y_1 y_i$ in terms of $y_i f$ and in terms of $y_i g$. With this in mind, consider the resultant matrix

$$T = \left(\begin{array}{ccc|ccc} 1 & \alpha_1 & \alpha_2 & & & \\ & 1 & \alpha_1 & \alpha_2 & & \\ & & 1 & \alpha_1 & \alpha_2 & \\ \hline & \beta_1 & \beta_2 & 1 & & \\ & & \beta_1 & \beta_2 & 1 & \\ & & & \beta_1 & \beta_2 & 1 \end{array}\right). \qquad (4.16)$$

The matrix T and its inverse have block form

$$T = \begin{pmatrix} V_1 & V_2 \\ W_1 & W_2 \end{pmatrix}, \quad T^{-1} = \begin{pmatrix} v_1 & v_2 \\ w_1 & w_2 \end{pmatrix},$$

so that in particular,

$$v_1 V_1 + v_2 W_1 = I_3 \qquad v_1 V_2 + v_2 W_2 = 0 \tag{4.17}$$
$$w_1 V_1 + w_2 W_1 = 0 \qquad w_1 V_2 + w_2 W_2 = I_3 \tag{4.18}$$

The reason for writing T in block form is that

$$\begin{pmatrix} V_1 & V_2 \end{pmatrix} \begin{pmatrix} y_1^2 \\ y_1 y_2 \\ y_1 y_3 \\ y_2 y_3 \\ y_3^2 \\ * \end{pmatrix} = \begin{pmatrix} y_1 f \\ y_2 f - \alpha_1 (y_2^2 - y_1 y_3) \\ y_3 f \end{pmatrix},$$

where here and elsewhere a star means that entry is irrelevant because it is multiplied by zero. Now if we try to "invert" this matrix equation we get an expression for $y_1 y_i$ in terms of $y_i f$ after a small correction. Multiplying both sides by block v_1 and using identities (4.17) we get

$$\begin{pmatrix} y_1^2 \\ y_1 y_2 \\ y_1 y_3 \end{pmatrix} = v_1 \begin{pmatrix} y_1 f \\ y_2 f - \alpha_1 (y_2^2 - y_1 y_3) \\ y_3 f \end{pmatrix} + v_2 W_1 \begin{pmatrix} * \\ y_1 y_2 \\ y_1 y_3 \end{pmatrix} + v_2 W_2 \begin{pmatrix} y_2 y_3 \\ y_3^2 \\ * \end{pmatrix}.$$

A similar treatment multiplying the bottom half of T by v_2 leads to the matrix equation

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = v_2 \begin{pmatrix} y_1 g \\ y_2 g - \beta_2 (y_2^2 - y_1 y_3) \\ y_3 g \end{pmatrix} - v_2 W_1 \begin{pmatrix} * \\ y_1^2 \\ y_1 y_2 \end{pmatrix} - v_2 W_2 \begin{pmatrix} y_1 y_3 \\ y_2 y_3 \\ * \end{pmatrix}.$$

Now we can write these two equations in terms of the columns of B' by collecting the terms together appropriately to obtain

$$\begin{aligned} y_1 Y &= v_1 Y f && +(v_2 X_4 + Z_4)(-y_2) && +(v_2 X_5 + Z_5) y_3 \\ 0 &= \quad v_2 Y g && +(v_2 X_4 + Z_4) y_1 && +(v_2 X_5 + Z_5)(-y_2), \end{aligned}$$

where

$$X_4 = \begin{pmatrix} -\beta_1 y_1 - y_3 \\ -\beta_2 y_3 \\ -\beta_1 y_3 \end{pmatrix}, \quad X_5 = \begin{pmatrix} \beta_2 y_1 \\ \beta_1 y_1 + y_3 \\ \beta_2 y_3 \end{pmatrix}, \quad Y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix},$$

$$Z_4 = v_1 \begin{pmatrix} 0 \\ \alpha_1 y_2 \\ 0 \end{pmatrix} + v_2 \begin{pmatrix} 0 \\ \beta_2 y_3 \\ 0 \end{pmatrix}, \quad Z_5 = v_1 \begin{pmatrix} 0 \\ \alpha_1 y_1 \\ 0 \end{pmatrix} + v_2 \begin{pmatrix} 0 \\ \beta_2 y_2 \\ 0 \end{pmatrix}.$$

The matrices X_4, X_5 express the terms multiplying W_1, W_2 above in terms of the columns of B'. Similarly Z_4 and Z_5 express the correction terms involving $y_2^2 - y_1 y_3$. Thus multiplying on the left by the matrix $\Lambda_2 := \begin{pmatrix} a, & b, & c \end{pmatrix}$ we get

$$y_1 L_2 u = \begin{pmatrix} 1, & u, & v \end{pmatrix} B\nu,$$

where

$$\nu_2 = \Lambda_2 v_1 Y, \quad \nu_3 = \Lambda_2 v_2 Y, \quad \nu_4 = \Lambda_2(v_2 X_4 + Z_4), \quad \nu_5 = \Lambda_2(v_2 X_5 + Z_5)$$

and

$$\nu_1 = -\nu_4 L_1 - \nu_5 L_2$$

is chosen to cancel extra terms arising from the first row of B.

We perform a similar calculation to get an expression for $y_1 L_1 v$ in the image of B. However, this time it is necessary to alter T. Let σ be the cyclic permutation $(3, 4, 5, 6, 1, 2)$ of order 3 acting on the columns of T, and let σ^{-1} act on the rows of T^{-1}. I write $\sigma(T)$ and $\sigma^{-1}(T^{-1})$ in block form as

$$\sigma(T) = \begin{pmatrix} \widehat{V}_1 & \widehat{V}_2 \\ \widehat{W}_1 & \widehat{W}_2 \end{pmatrix}, \quad \sigma^{-1}(T^{-1}) = \begin{pmatrix} \widehat{v}_1 & \widehat{v}_2 \\ \widehat{w}_1 & \widehat{w}_2 \end{pmatrix}.$$

Then

$$\begin{pmatrix} \widehat{W}_1 & \widehat{W}_2 \end{pmatrix} \begin{pmatrix} y_2 y_3 \\ y_3^2 \\ * \\ y_1^2 \\ y_1 y_2 \\ y_1 y_3 \end{pmatrix} = \begin{pmatrix} y_1 g \\ y_2 g - \beta_2(y_2^2 - y_1 y_3) \\ y_3 g \end{pmatrix},$$

so that multiplying by $\widehat{w}_2$ and using permuted versions of identities (4.18) we obtain

$$y_1 Y = \widehat{w}_2 \begin{pmatrix} y_1 g \\ y_2 g - \beta_2(y_2^2 - y_1 y_3) \\ y_3 g \end{pmatrix} + \widehat{w}_1 \widehat{V}_1 \begin{pmatrix} y_2 y_3 \\ * \\ * \end{pmatrix} + \widehat{w}_1 \widehat{V}_2 \begin{pmatrix} y_1^2 \\ y_1 y_2 \\ y_1 y_3 \end{pmatrix}.$$

A similar equation arises when we multiply the top half of $\sigma(T)$ by $\widehat{w}_1$:

$$0 = \widehat{w}_1 \begin{pmatrix} y_1 f \\ y_2 f - \alpha_1(y_2^2 - y_1 y_3) \\ y_3 f \end{pmatrix} - \widehat{w}_1 \widehat{V}_1 \begin{pmatrix} y_3^2 \\ * \\ * \end{pmatrix} - \widehat{w}_1 \widehat{V}_2 \begin{pmatrix} y_1 y_2 \\ y_1 y_3 \\ y_2 y_3 \end{pmatrix}.$$

Then separate out these two equations as expressions in the columns of B

$$\begin{array}{rll} 0 & = \widehat{w}_1 Y f & +(\widehat{w}_1 \widehat{X}_4 + Z_4)(-y_2) \ +(\widehat{w}_1 \widehat{X}_5 + Z_5) y_3 \\ y_1 Y & = \qquad \widehat{w}_2 Y g & +(\widehat{w}_1 \widehat{X}_4 + Z_4) y_1 \quad\ \ +(\widehat{w}_1 \widehat{X}_5 + Z_5)(-y_2), \end{array}$$

where

$$\widehat{X}_4 = \begin{pmatrix} \alpha_1 y_1 \\ y_1 + \alpha_2 y_3 \\ \alpha_1 y_3 \end{pmatrix}, \quad \widehat{X}_5 = \begin{pmatrix} -\alpha_2 y_1 \\ -\alpha_1 y_1 \\ -y_1 - \alpha_2 y_3 \end{pmatrix}$$

and Z_4, Z_5 are as above. We multiply on the left by $\Lambda_1 := \begin{pmatrix} b, & c, & d \end{pmatrix}$ to obtain an expression for $y_1 L_1 v$ in the image of B. The preimage $\widehat{\nu}$ is the vector

$$\widehat{\nu}_1 = -\widehat{\nu}_4 L_1 - \widehat{\nu}_5 L_2, \quad \widehat{\nu}_2 = \Lambda_1 \widehat{w}_1 Y, \quad \widehat{\nu}_3 = \Lambda_1 \widehat{w}_2 Y,$$

$$\widehat{\nu}_4 = \Lambda_1 (\widehat{w}_1 \widehat{X}_4 + Z_4), \quad \widehat{\nu}_5 = \Lambda_1 (\widehat{w}_1 \widehat{X}_5 + Z_5),$$

Hence

$$y_1 (L_2 u + L_1 v) = \begin{pmatrix} 1, & u, & v \end{pmatrix} B(\nu + \widehat{\nu})$$

is the residual part to $y_1 A_{13}$ and so we can write out an equation in $R + Rz_1 + Rz_2$:

$$y_1 A_{13} + (\nu_1 + \widehat{\nu}_1) + (\nu_2 + \widehat{\nu}_2) z_1 + (\nu_3 + \widehat{\nu}_3) z_2 = 0.$$

The calculation of $y_2 A_{13}$, $y_3 A_{13}$ requires further cyclic permutations of the columns of T. I do not write out the details here but it follows the same pattern as the calculations above.

Chapter 5

The hyperelliptic case

The following chapter discusses the hyperelliptic degeneration of the quartic determinantal K3 surface and its extensions. First I discuss various easy results on graded rings for hyperelliptic curves. These results are then generalised to K3 surfaces. Finally, theorem (5.3.1) about extensions of hyperelliptic varieties is a complementary result to theorem (4.2.1)

5.1 Graded rings over hyperelliptic curves

For curves of genus $g \geq 2$ the canonical linear system $|K_D|$ is base point free and so defines a morphism $\varphi\colon D \to \mathbb{P}^{g-1}$. There are two possibilities:

- φ is an embedding of D;
- φ is a double cover of the rational normal curve of degree $g-1$.

In the latter case φ fits into the commutative diagram

$$\begin{array}{ccc} D & \xrightarrow{\quad\varphi\quad} & \mathbb{P}^{g-1} \\ & {\scriptstyle\pi}\searrow \qquad \nearrow{\scriptstyle v_{g-1}} & \\ & \mathbb{P}^1 & \end{array}$$

where π is a double cover of $\mathbb{P}^1$ and v_{g-1} is the Veronese embedding in degree $g-1$. We call such a curve hyperelliptic. The double cover π of $\mathbb{P}^1$ determines and is determined by a free linear system on D of dimension 1 and degree 2, which is called the g^1_2. Since the diagram commutes it is clear that $K_D \sim (g-1)g^1_2$. By the Riemann–Hurwitz formula there are $2g+2$ ramification (or Weierstrass) points $P_1, \ldots, P_{2g+2}$ on D and we denote the corresponding branch points on $\mathbb{P}^1$ by $Q_1, \ldots, Q_{2g+2}$. There is a natural "hyperelliptic" involution h on D which swaps the two sheets of the double covering and π is the quotient map of this involution. We will see shortly how to write down the graded ring $R(D, A)$ for any divisor class A which is invariant under h. First we make some remarks:

(1) It is easy to see that $2P_i \sim g_2^1$ for any Weierstrass point P_i.

(2) Choose generators s_1, s_2 of $H^0(D, g_2^1)$. These are coordinates on $\mathbb{P}^1$, and there is an equation $F_{2g+2}(s_1, s_2)$ defining the branch locus $Q_1 + \cdots + Q_{2g+2}$ on $\mathbb{P}^1$. The double covering is then defined by the equation $w^2 = F_{2g+2}(s_1, s_2)$. Hence for a hyperelliptic curve,

$$(g+1)g_2^1 \sim P_1 + \cdots + P_{2g+2},$$

or more generally,

$$P_1 + \cdots + P_a + (2g+2-a)g_2^1 \sim P_{a+1} + \cdots + P_{2g+2} + (g+1)g_2^1.$$

Indeed, (1) is a direct consequence of Riemann–Roch and the fact that π is ramified at P_i. For (2) consider the rational function w/s^{g+1}, where s is the local equation for a Weierstrass point P.

Now let us write $B_1 = P_1 + \cdots + P_a$, $B_2 = P_{a+1} + \cdots + P_{2g+2}$ then since the B_i are effective Cartier divisors on D we can choose constant sections as follows:

$$u \colon \mathcal{O}_D \to \mathcal{O}_D(B_1),$$
$$v \colon \mathcal{O}_D \to \mathcal{O}_D(B_2).$$

Then u^2, uv, v^2 are sections of ag_2^1, $(g+1)g_2^1$, $(2g+2-a)g_2^1$ respectively, so we have two relations $u^2 = f(s_1, s_2)$, $v^2 = g(s_1, s_2)$ and the identity $w = uv$, where w is the variable defining the double cover. Here $f(s_1, s_2)$ is a homogeneous function of degree a on $\mathbb{P}^1$ with zeros at $Q_1, \ldots, Q_a$, similarly $g(s_1, s_2)$. Clearly the only divisor classes that are invariant under h are of the form

$$A \sim P_1 + \cdots + P_a + bg_2^1 \sim P_{a+1} + \cdots + P_{2g+2} + (a+b-g-1)g_2^1.$$

We assert that for such a divisor A, the graded ring

$$R(D, A) = \bigoplus_{n \geq 0} H^0(D, nA)$$

is generated by monomials in s_1, s_2, u, v along with relations derived in a trivial way from the ones mentioned above.

Proposition 5.1.1 (Reid, [Aq]) *Let D be a hyperelliptic curve of genus g with Weierstrass points $P_1, \dots, P_{2g+2}$, and write $\pi\colon D \to \mathbb{P}^1$ for the natural quotient by the hyperelliptic involution. Then*

(1) $\pi_*\mathcal{O}_D = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-g-1)$;

(2) $\pi_*\mathcal{O}_D(g^1_2) = \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(-g)$;

(3) $\pi_*\mathcal{O}_D(P_1 + \cdots + P_a) = \mathcal{O}_{\mathbb{P}^1}u \oplus \mathcal{O}_{\mathbb{P}^1}(a-g-1)v$;

where in each case the first summand is invariant under the involution and the second is antiinvariant.

Remark 5.1.2 Note that in case (1) the direct image sheaf is a sheaf of $\mathcal{O}_{\mathbb{P}^1}$-algebras, where the multiplication

$$\mathcal{O}_{\mathbb{P}^1}(-g-1) \otimes \mathcal{O}_{\mathbb{P}^1}(-g-1) \to \mathcal{O}_{\mathbb{P}^1}$$

is defined via $w^2 = F(s_1, s_2)$.

Proof I will give a sketch proof of part (1): Considered locally, $\pi_*\mathcal{O}_D$ is generated by 1 and w as an $\mathcal{O}_{\mathbb{P}^1}$-module, where w is a local equation for the branch locus. Thus a local section of $\pi_*\mathcal{O}_D$ can be written as $\psi = \psi_1 + w\psi_2$, where ψ must be homogeneous of degree 0. Thus ψ_2 is a local section of $\mathcal{O}_{\mathbb{P}^1}(-g-1)$, since its numerator must have degree $g+1$ less than its denominator, and we are done. The other parts are similar.

5.1.1 Example: A hyperelliptic curve of genus 3

A hyperelliptic curve of genus 3 will branch in 8 points over $\mathbb{P}^1$. Consider the ineffective theta characteristic

$$A \sim P_1 + \cdots + P_4 - g^1_2 \sim P_5 + \cdots + P_8 - g^1_2.$$

One can check very quickly that $2A \sim K_D$, and using the proposition we obtain the following generators for $R(D, A)$:

n	$H^0(D, \mathcal{O}_D(nA)$	$H^0(\mathbb{P}^1, \pi_*\mathcal{O}_D(nA))$
0	1	1
1	ϕ	ϕ
2	$y_1,\ y_2,\ y_3$	$s_1^2,\ s_1s_2,\ s_2^2$
3	$z_1,\ z_2,\ z_3,\ z_4$	$s_1u,\ s_2u,\ s_1v,\ s_2v$
4	t	uv

The relations between these are either of the trivial monomial kind, or derived from

$$u^2 = f_4(s_1, s_2), \ v^2 = g_4(s_1, s_2).$$

For example, it is clear that $z_1^2 = s_1^2 u^2 = y_1 f(y_1, y_2, y_3)$, where $f(y_1, y_2, y_3)$ is a rendering of $f_4(s_1, s_2)$ in the quadratic monomials s_1^2, $s_1 s_2$, s_2^2. In fact we can present the equations as

$$\operatorname{rank}\left(\begin{array}{cc|cc} y_1 & y_2 & z_1 & z_3 \\ y_2 & y_3 & z_2 & z_4 \\ \hline z_1 & z_2 & f_2 & t \\ z_3 & z_4 & t & g_2 \end{array}\right) \leq 1,$$

where f_2 and g_2 are quadrics in y_1, y_2, y_3. The curve is then

$$\operatorname{Proj} R(D, A) \subset \mathbb{P}(2^3, 3^4, 4),$$

and the double cover to $\mathbb{P}^1$ can be clearly seen as the conic defined by the first 2×2 minor of the matrix.

Alternatively D is a codimension 2 complete intersection inside a weighted homogeneous variety as follows: let X be the second Veronese embedding of $\mathbb{P}^3$ with coordinates s_1, s_2, u, v and take the affine cone $\mathcal{C}X \subset \mathbb{A}^{10}$ over X. Aside from the obvious $\mathbb{C}^\times$-action on $\mathcal{C}X$ there are many other possibilities, and we choose a weighted $\mathbb{C}^\times$-action with weights $(1, 1, 2, 2)$. Then the quotient $Y = \mathcal{C}X /\!\!/_1 \mathbb{C}^\times$ of $\mathcal{C}X$ is contained in $\mathbb{P}(2^3, 3^4, 4^3)$ and is defined by the equations

$$\operatorname{rank}\left(\begin{array}{cc|cc} y_1 & y_2 & z_1 & z_3 \\ y_2 & y_3 & z_2 & z_4 \\ \hline z_1 & z_2 & x_1 & t \\ z_3 & z_4 & t & x_2 \end{array}\right) \leq 1.$$

The hyperelliptic curve D is simply the codimension 2 complete intersection $x_1 = f_2$, $x_2 = g_2$ inside Y. We will see in the next section that there is a similar description of a K3 surface extending this example.

5.2 Graded rings over hyperelliptic K3 surfaces

A surface T is called a K3 surface if $K_T \sim 0$ and $H^1(T, \mathcal{O}_T) = 0$. These two conditions restrict the behaviour of linear systems on K3 surfaces tremendously, see [SD] for details. If L is a complete linear system on a K3 surface T with $L^2 > 0$ and no fixed components then L is basepoint free.

Further, $|L|$ contains an irreducible curve D of arithmetic genus g, where $g = h^0(T, \mathcal{O}_T(L)) - 1$. Then L determines one of the following:

- a birational morphism onto a surface of degree $2g - 2$ in $\mathbb{P}^g$;
- a 2-to-1 map onto a surface F of degree $g - 1$ in $\mathbb{P}^g$.

In the latter situation T is called hyperelliptic and it is easy to see that the branch locus of the double covering is some divisor in $|-2K_F|$.

Del Pezzo classified surfaces of degree $g - 1$ in $\mathbb{P}^g$ not contained in a hyperplane as the following two cases:

1. the Veronese surface $V \subset \mathbb{P}^5$;
2. a rational scroll $\mathbb{F}(a, b)$ with $a, b \geq 0$.

Since these two possibilities for the base F have very simple explicit descriptions, we can analyse graded rings over hyperelliptic K3 surfaces relative to the base F in the same way as we did for hyperelliptic curves relative to $\mathbb{P}^1$ in section (5.1).

5.2.1 Example $F = \mathbb{P}^1 \times \mathbb{P}^1$

Suppose that $g = 3$ and T is a double cover of the rank 4 quadric surface $F = \mathbb{F}(0, 0) \subset \mathbb{P}^3$, branched in a curve of bidegree $(4, 4)$. We further assume that the branch curve C splits into two components $C_1 + C_2$ of bidegree $(3, 1)$ and $(1, 3)$ respectively, so that they intersect one another transversally in 10 nodes.

As before there is a hyperelliptic involution $h \colon T \to T$ exchanging the two sheets of the double cover, and we write $\pi \colon T \to Q$ for the quotient map. Let H_1, H_2 be the generators of $\operatorname{Pic} Q$, then we omit π^* to write $\pi^* H_i = H_i$ on T and $\pi^* C_i = 2D_i$.

Write s_1, s_2 for the generators of $H^0(T, \mathcal{O}_T(H_1))$, similarly t_1, t_2 for $H^0(T, \mathcal{O}_T(H_2))$. Then there is an equation $F_{4,4}(s_1, s_2, t_1, t_2)$ defining the branch curve C on Q and C breaks into two curves C_1, C_2. The double cover T is given by $w^2 = F$, and we have $2D_1 \sim 3H_1 + H_2$ and $2D_2 \sim H_1 + 3H_2$. Considering the rational function $w/(t_1^2 s_1^2)$ on T we find

$$2(H_1 + H_2) \sim D_1 + D_2.$$

We would like to write down graded rings

$$R(T, A) = \bigoplus_{n \geq 0} H^0(T, nA)$$

where A is a divisor class which is invariant under the hyperelliptic involution. It is clear that any such A can be written in the form

$$A \sim D_1 + n_1 H_1 + n_2 H_2 \sim D_2 + (n_1 + 1) H_1 + (n_2 - 1) H_2.$$

The following proposition allows us to describe $R(T, A)$ relative to $R(Q, \pi_* A)$.

Proposition 5.2.1 *Let T be a hyperelliptic K3 surface double covering of the rank 4 quadric $Q \subset \mathbb{P}^3$, with ramification properties as described above. Let $u\colon \mathcal{O}_T \to \mathcal{O}_T(D_1)$ and $v\colon \mathcal{O}_T \to \mathcal{O}_T(D_2)$ be local equations for the components D_i of the ramification curve. Clearly we have $u^2 = f_{3,1}(s_i, t_i)$, $uv = w$ and $v^2 = g_{1,3}(s_i, t_i)$, where $F = fg$. Moreover,*

(1) $\pi_* \mathcal{O}_T = \mathcal{O}_Q \oplus \mathcal{O}_Q(-2, -2)$;

(2) $\pi_* \mathcal{O}_T(H_1) = \mathcal{O}_Q(1, 0) \oplus \mathcal{O}_Q(-1, -2)$;

(3) $\pi_* \mathcal{O}_T(D_1) = \mathcal{O}_Q u \oplus \mathcal{O}_Q(1, -1) v$;

with similar results for H_2, D_2 respectively.

Remark 5.2.2 Once again we note the $\mathcal{O}_Q$-algebra structure on $\pi_* \mathcal{O}_T$. The multiplication map

$$\mathcal{O}_Q(-2, -2) \otimes \mathcal{O}_Q(-2, -2) \to \mathcal{O}_Q$$

is defined via the equation $w^2 = F_{4,4}(s_1, s_2, t_1, t_2)$.

Proof We will prove a more general version of this proposition later. See proposition (5.2.4).

Write $A \sim D_1 - H_1 \sim D_2 - H_2$, then we can describe the ring $R(T, A)$ in much the same way as we did for the hyperelliptic curve. By the above proposition, it is clear that the generators for $R(T, A)$ are:

n	$H^0(T, \mathcal{O}_T(nA))$	$H^0(Q, \pi_* \mathcal{O}_T(nA))$
0	1	1
1	0	0
2	$y_1,\ y_2,\ y_3,\ y_4$	$s_1 t_1,\ s_2 t_1,\ s_1 t_2,\ s_2 t_2$
3	$z_1,\ z_2,\ z_3,\ z_4$	$t_1 u,\ t_2 u,\ s_1 v,\ s_2 v$
4	t	$uv = w$

The relations are again mostly trivial monomial relations, together with those derived from $u^2 = f_{3,1}$ and $v^2 = g_{1,3}$. Some are slightly more difficult to write down than others, but it is more or less obvious how to proceed. For example,

$$z_1 t = t_1 u^2 v = t_1 v f_{3,1} = s_1 v q_{2,2} + s_2 v q'_{2,2} = z_3 q(y_i) + z_4 q'(y_i),$$

where q and q' are suitable quadrics in $y_1, \dots, y_4$. The trick here is to make $f_{3,1}$ bihomogeneous by incorporating the factor t_1 into f and simultaneously taking out the excess in s_1, s_2. Of course in general f will not be divisible by s_1 or by s_2 and so we have a choice of ways to break up f into quadrics. However, this choice is arbitrary, as any discrepancy is accounted for by considering the minors of (5.1). We can present all the equations of T as follows

$$\operatorname{rank}\left(\begin{array}{cc|c} y_1 & y_2 & z_1 \\ y_3 & y_4 & z_2 \\ \hline z_3 & z_4 & t \end{array}\right) \le 1, \tag{5.1}$$

$$\begin{aligned} z_1^2 &= t_1^2 f_{3,1} & \qquad z_3^2 &= s_1^2 g_{1,3} \\ z_1 z_2 &= t_1 t_2 f_{3,1} & z_3 z_4 &= s_1 s_2 g_{1,3} \\ z_2^2 &= t_2^2 f_{3,1} & z_4^2 &= s_2^2 g_{1,3} \\ z_1 t &= q_1 z_3 + q_1' z_4 & z_3 t &= q_3 z_1 + q_3' z_2 \\ z_2 t &= q_2 z_3 + q_2' z_4 & z_4 t &= q_4 z_1 + q_4' z_2 \end{aligned}$$

$$t^2 = F(y_i),$$

where for example $z_1^2 = t_1^2 f_{3,1}$ means we render the bihomogeneous expression $t_1^2 f_{3,1}$ in the variables $y_1, \dots, y_4$.

The equations of T as written out above have many structural properties, which are described succinctly by the "rolling factors" format. Rolling factors is a term first coined by Duncan Dicks in his PhD thesis [D] as a way of describing the equations of divisors inside scrolls and similar situations. The key observation is the following: consider the first column of equations $z_1^2 = t_1^2 f_{3,1} \dots$ displayed above. The first of these is a linear combination in the first row of the matrix (5.1): $a_1 z_1 + a_2 y_1 + a_3 y_2$. To obtain the second equation from the first, we take the same combination but this time write it out using the second row of the matrix: $a_1 z_2 + a_2 y_3 + a_3 y_4$. This is called "rolling the equations". As we travel down the first column of equations, we roll down the matrix, but as we travel down the second column, we roll across the matrix from left to right. The final relation $t^2 = F(y_i)$ is the end product from either column, which is why it appears centred.

There is an obvious reason why rolling factors works: the ratio between two equations in the same column is fixed, and equal to the ratio between the rows in the matrix due to the rank condition. However, the point is that rolling factors gives a method for writing down the equations for such a variety automatically.

Remark 5.2.3 My favourite way to think about T is as a codimension 2 complete intersection in the weighted homogeneous space we now describe. Consider the following $(\mathbb{C}^\times)^2$-action on $\mathbb{C}^2 \times \mathbb{C}^2 \times \mathbb{C}^2$:

$$\begin{aligned} \lambda &\colon (s_1, s_2, t_1, t_2, u, v) \mapsto (\lambda^2 s_1, \lambda^2 s_2, t_1, t_2, \lambda^3 u, \lambda v) \\ \mu &\colon (s_1, s_2, t_1, t_2, u, v) \mapsto (s_1, s_2, \mu^2 t_1, \mu^2 t_2, \mu u, \mu^3 v). \end{aligned}$$

The quotient $X = (\mathbb{C}^2 \times \mathbb{C}^2 \times \mathbb{C}^2) /\!\!/_{(1,1)} (\mathbb{C}^\times)^2$ is embedded in $\mathbb{P}(2^4, 3^4, 4)$ by the determinantal equations (5.1). The surface T is the complete intersection $u^2 = f_{3,1} \in (6, 2)$, $v^2 = g_{1,3} \in (2, 6)$ in X. Note that taking the hyperplane section $(y_2 = y_3) \cap T$, we retrieve the hyperelliptic curve D of section (5.1).

5.2.2 A structure theory for hyperelliptic K3 surfaces

Rational surface scrolls $\mathbb{F}(a, b)$ have a nice explicit description in terms of $(\mathbb{C}^\times)^2$-quotients of $\mathbb{C}^2 \times \mathbb{C}^2$. Consider the $(\mathbb{C}^\times)^2$-action on $\mathbb{C}^2 \times \mathbb{C}^2$ with coordinates s_1, s_2, t_1, t_2 defined by

$$\begin{aligned} \lambda &\colon (s_1, s_2, t_1, t_2) \mapsto (\lambda s_1, \lambda s_2, \lambda^{-a} t_1, \lambda^{-b} t_2) \\ \mu &\colon (s_1, s_2, t_1, t_2) \mapsto (s_1, s_2, \mu t_1, \mu t_2). \end{aligned}$$

The scroll $\mathbb{F} = \mathbb{F}(a, b)$ is the quotient of $\mathbb{C}^2 \times \mathbb{C}^2$ by this $(\mathbb{C}^\times)^2$-action. The projection map $\pi\colon \mathbb{F}(a, b) \to \mathbb{P}^1$ is obtained by forgetting the coordinates t_1, t_2 and π endows $\mathbb{F}$ with a fibre bundle structure over $\mathbb{P}^1$. The group $\operatorname{Pic} \mathbb{F}$ has a very nice description derived from the $(\mathbb{C}^\times)^2$-action. We say that an effective divisor is of type (α, β) if it is the zero locus of a polynomial on $\mathbb{F}$ of bidegree (α, β) under the grading induced by the group action. Then $\operatorname{Pic} \mathbb{F}$ is $\mathbb{Z}^2$ generated by $L \in (1, 0)$ and $M \in (0, 1)$. For example, one could write $L\colon (s_1 = 0)$ and $M\colon (s_1^a t_1 = 0)$. There are other choices of basis, but for simplicity we fix this one. In this basis the canonical divisor class of $\mathbb{F}$ is

$$K_{\mathbb{F}} \sim (-2 + a + b)L - 2M.$$

If T is a hyperelliptic K3 surface which is a double cover of $\mathbb{F}(a, b)$ then the branch locus of this double cover is a divisor in $|-2K_{\mathbb{F}}|$. Thus we have an explicit description for the branch curve on the scroll $\mathbb{F}(a, b)$. Hence it is possible to describe any graded ring over a hyperelliptic K3 surface as a $(\mathbb{C}^\times)^2$-invariant subring. Indeed, suppose that $\pi\colon T \to \mathbb{F}$ is a hyperelliptic K3 surface and for simplicity we assume the branch curve $B \in |-2K_{\mathbb{F}}|$ is irreducible. Then write $L = \pi^* L$, $M = \pi^* M$ and $2C = \pi^* B$, so that $C \sim (2-a-b)L+2M$. We have the following proposition, which is a generalisation of proposition (5.2.1):

Proposition 5.2.4 *Suppose T is a hyperelliptic K3 surface double covering the rational scroll $\mathbb{F} = \mathbb{F}(a,b)$ via the map $\pi\colon T \to \mathbb{F}$ as described above. Let $w\colon \mathcal{O}_T \to \mathcal{O}_T(C)$ be the local equation for the ramification curve C. Then,*

(1) $\pi_*\mathcal{O}_T = \mathcal{O}_{\mathbb{F}} \oplus \mathcal{O}_{\mathbb{F}}((-2+a+b)L - 2M)$;

(2) $\pi_*\mathcal{O}_T(L) = \mathcal{O}_{\mathbb{F}}(L) \oplus \mathcal{O}_{\mathbb{F}}((-1+a+b)L - 2M)$;

(3) $\pi_*\mathcal{O}_T(M) = \mathcal{O}_{\mathbb{F}}(M) \oplus \mathcal{O}_{\mathbb{F}}((-2+a+b)L - M)$ *and*

(4) $\pi_*\mathcal{O}_T(C) = \mathcal{O}_{\mathbb{F}}w \oplus \mathcal{O}_{\mathbb{F}}((2-a-b)L + 2M)$,

with appropriate $\mathcal{O}_{\mathbb{F}}$-algebra structure on $\pi_*\mathcal{O}_T$.

Proof Part (1): If we consider $\pi_*\mathcal{O}_T$ locally then clearly its generators are 1 and w as an $\mathcal{O}_{\mathbb{F}}$-module. So any local section ψ of $\pi_*\mathcal{O}_T$ can be written as $\psi = \psi_1 + w\psi_2$. For the degrees to match, the numerator of ψ_2 will need to be of bidegree $((-2+a+b), -2)$ less than the denominator. The other parts are similar.

5.2.3 Example of a genus 4 hyperelliptic K3 surface

Suppose T is a K3 surface with hyperelliptic complete linear system of genus 4. Then T is a double cover of the degree 3 scroll $\mathbb{F}(1,2) \subset \mathbb{P}^4$ and we suppose that the branch curve B is irreducible. The scroll $\mathbb{F}(1,2)$ is embedded in $\mathbb{P}^4$ via the linear system $|M|$ and it is not difficult to write down its equations

$$\operatorname{rank}\begin{pmatrix} x_1 & y_1 & y_2 \\ x_2 & y_2 & y_3 \end{pmatrix} \leq 1,$$

where $x_i = s_i t_1$ and $y_{i+j-1} = s_i s_j t_2$. We write $F_{-2,4}(s_i, t_i)$ for the equation of the curve $B \in |-2L+4M|$. Now polarise T by M, and calculate the graded ring $R(T,M)$ relative to $R(\mathbb{F}, \pi_* M)$. As before we call the ramification curve C, and observe that since $C \sim 2M - L$, we can also write $2M \sim C + L$. The following table is calculated using proposition (5.2.4), and we only list the new generators in each degree:

n	$H^0(T, \mathcal{O}_T(nM)$	$H^0(\mathbb{F}, \pi_*\mathcal{O}_T(M))$
0	1	1
1	s_1t_1, s_2t_1	x_1, x_2
	$s_1^2t_2, s_1s_2t_2, s_2^2t_2$	y_1, y_2, y_3
2	s_1w, s_2w	z_1, z_2

Notice that in degree 2 we had to use $2M \sim C + L$ to find the new generators. Once more we can derive the equations of T from trivial monomial relations between the generators and the double cover relation $w^2 = F_{-2,4}(s_i, t_i)$

$$\operatorname{rank}\begin{pmatrix} x_1 & y_1 & y_2 & z_1 \\ x_2 & y_2 & y_3 & z_2 \end{pmatrix} \leq 1,$$

$$\begin{aligned} z_1^2 &= s_1^2 F_{-2,4}(s_i, t_i) \\ z_1z_2 &= s_1s_2 F_{-2,4}(s_i, t_i) \\ z_2^2 &= s_2^2 F_{-2,4}(s_i, t_i) \end{aligned}$$

where for example, $s_1^2F_{-2,4}$ is rendered in terms of variables x_i, y_i. Hence T is codimension 4 surface in $\mathbb{P}(1^5, 2^2)$. Again we see the rolling factors format come to the fore, except this time the branch curve is irreducible so we only roll down the matrix. This is commonplace in situations where we are describing divisors inside scrolls. The surface T can also be written as a $(\mathbb{C}^\times)^2$-quotient of $\mathbb{C}^2 \times \mathbb{C}^2 \times \mathbb{C}$ where the action is

$$\begin{aligned} \lambda \colon (s_1, s_2, t_1, t_2, w) &\mapsto (\lambda s_1, \lambda s_2, \lambda^{-1}t_1, \lambda^{-2}t_2, \lambda^{-1}w) \\ \mu \colon (s_1, s_2, t_1, t_2, w) &\mapsto (s_1, s_2, \mu t_1, \mu t_2, \mu^2 w), \end{aligned}$$

and T is the hypersurface $w^2 = F_{-2,4}(s_i, t_i)$ inside $(\mathbb{C}^2 \times \mathbb{C}^2 \times \mathbb{C}) /\!\!/_{(0,1)} (\mathbb{C}^\times)^2$.

If the branch curve breaks up into two or more components there are similar results, and there is also a treatment for the case where T is a double cover of the Veronese surface. It is more or less obvious how to construct a $(\mathbb{C}^\times)^2$-quotient construction of T in each example.

Unfortunately, it seems to be difficult to generalise this description of hyperelliptic varieties to higher dimensions since we are dependent on an explicit description of the base F, which becomes more complicated for 3-folds.

5.3 Extending hyperelliptic graded rings

The calculations of example (5.1.1) give us the graded ring of a hyperelliptic curve D of genus 3 in $\mathbb{P}(2^3, 3^4, 4)$. Similarly, by the results of example (5.2.1), we know how to write down the graded ring of a hyperelliptic K3 surface $T \subset \mathbb{P}(2^4, 3^4, 4)$ with $10 \times \frac{1}{2}$ orbifold points and whose hyperplane section of weight 2 yields the curve D. Writing A for the polarising divisor on T, we have

$$A|_D = P_1 + \cdots + P_4 - g_2^1,$$

where the P_i are Weierstrass points on D. We want to make an analogous extension construction to that of section (4.2). Once again, T is the general elephant of a Fano 3-fold. There are extensions of T up to a Fano 6-fold W^6, with each successive extension containing T as an appropriate number of hyperplane sections. So we have the tower

$$D \subset T \subset W^3 \subset W^4 \subset W^5 \subset W^6 \subset \mathbb{P}(1^4, 2^4, 3^4, 4).$$

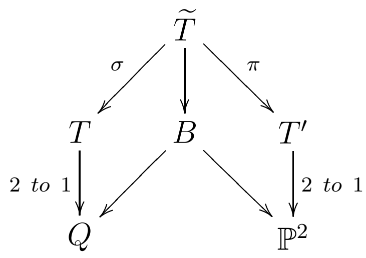
Of course, this is a degeneration of the symmetric determinantal case, so one would expect an analogue of theorem (4.2.1). Indeed, we have

Theorem 5.3.1 *Each quasismooth hyperelliptic K3 surface* $T \subset \mathbb{P}(2^4, 3^4, 4)$ *with* $10 \times \frac{1}{2}$ *orbifold points has a unique extension to a quasismooth Fano 6-fold* $W \subset \mathbb{P}(1^4, 2^4, 3^4, 4)$ *with* $10 \times \frac{1}{2}$ *orbifold points and such that*

$$T = W \cap H_1 \cap H_2 \cap H_3 \cap H_4,$$

where the H_i *are hyperplanes of the projective space* $\mathbb{P}(1^4, 2^4, 3^4, 4)$.

As in the symmetric determinantal case, the most convenient way to extend the K3 surface T is by considering the Gorenstein projection from one of the $\frac{1}{2}$ orbifold points. Let $Q \subset \mathbb{P}^3$ be the quadric of rank 4 so that T is a double cover of Q branched in a curve C of bidegree $(4, 4)$. Since we want T to have 10 orbifold $\frac{1}{2}$ points, C breaks into two curves C_1 and C_2 of bidegree $(3, 1)$ and $(1, 3)$ respectively, which intersect transversally so that C has 10 nodes, each corresponding to a $\frac{1}{2}$ point on T. Choose one of these $\frac{1}{2}$ points and call it P. The Gorenstein projection away from P in T to $T'_{6,6} \subset \mathbb{P}(2^3, 3^2)$ is the diagram



where $\sigma\colon \widetilde{T} \to T$ is the $(1,1)$-weighted Kawamata blowup of P in T,

$$T' = \operatorname{Proj} \bigoplus_{n \geq 0} H^0 \left(\widetilde{T}, \mathcal{O}_{\widetilde{T}}(\sigma^*(A) - \tfrac{1}{2}E) \right)$$

and $E \cong \mathbb{P}^1$ is the exceptional divisor of the Kawamata blowup.

Since we are dealing with hyperelliptic varieties, there is also a projection map over the base, which is a well known classical map between del Pezzo surfaces: blow up a point in Q to obtain the del Pezzo surface B. Then contract the two (-1)-curves on B arising from the rulings of Q to get $\mathbb{P}^2$. Now, the centre of projection P upstairs was chosen to be the point above one of the nodes of C. Thus the two components C_1, C_2 of C are projected to nodal plane cubics, and the centre of projection is mapped to the line L through these two nodes.

Returning to the double cover, we conclude that the image T' of the Gorenstein projection is a double cover of $\mathbb{P}^2$ branched over two nodal cubics. The image of the centre of projection P is a rational curve of arithmetic genus 2 double covering the line L away from the two nodes and branched over the residual intersection with C.

5.3.1 Hyperelliptic projection in commutative algebra

By analogy with section (4.1), the projection to T' can be expressed explicitly as an operation in commutative algebra. Assume the centre of projection is a $\frac{1}{2}$ point at the coordinate point P_{y_4}. Then using the notation of example (5.2.1), write down the matrix relations

$$\operatorname{rank} \begin{pmatrix} y_2 & f & z_1 \\ g & y_4 & z_3 \\ z_2 & z_4 & t \end{pmatrix} \leq 1, \tag{5.2}$$

where I reserve the right to choose f, g later. These equations are a subset of those for $T \subset \mathbb{P}(2^4, 3^4, 4)$ after a trivial change of coordinates. The remaining equations for T are completely determined by

$$\begin{aligned} z_1^2 &= L_1 y_2^2 + L_2 y_2 f + L_3 f^2 \\ z_2^2 &= M_1 y_2^2 + M_2 y_2 g + M_3 g^2 \end{aligned}$$

where a priori L_i, M_i are linear in $y_1, \ldots, y_4$. Indeed, the equations we have written down so far are sufficient to determine the two components of the branch curve, so we can fill in the remaining equations using the rolling factors format or by other means. Since T has a $\frac{1}{2}$ point at P_{y_4}, the last equation for T can be written as

$$t^2 = a_2(y_1, y_3)y_4^2 + b_3(y_1, y_2, y_3)y_4 + c_4(y_1, y_2, y_3).$$

Now the tangent cone to P must factorise because the branch curve C splits into two components, so we can choose coordinates

$$f = y_1 + \alpha y_3, \quad g = \beta y_1 + y_3$$

so that $a = y_1y_3$. This in turn forces $L_3 = y_1$, $M_3 = y_3$ so that modulo the minors of matrix (5.2), the equations involving z_1^2 and z_2^2 take the form

$$\begin{aligned} z_1^2 &= L_1(y_1, y_2, y_3)y_2^2 + l_4y_2fg + y_1f^2 \\ z_2^2 &= M_1(y_1, y_2, y_3)y_2^2 + m_4y_2fg + y_3g^2, \end{aligned} \tag{5.3}$$

where L_1, M_1 do not involve y_4 and l_4, m_4 are scalars.

We are finally in a position to describe the projection centred at P_{y_4} in terms of the explicit equations. The local coordinates near P are z_3, z_4 so we expect the projection to eliminate these variables along with y_4 from the ring $R(T, A)$. In fact when we calculate the projected subring we also lose t and so we are left with equations (5.3) defining a complete intersection

$$T'_{6,6} \subset \mathbb{P}(2, 2, 2, 3, 3).$$

To reverse the projection we must set up

$$\mathbb{P}^1 \xrightarrow{\varphi} T'_{6,6} \subset \mathbb{P}(2, 2, 2, 3, 3),$$

where φ maps to a curve of arithmetic genus 2 double covering a line in the plane $\mathbb{P}(2, 2, 2)$. If we can give a construction of φ and $T'_{6,6}$, then via Gorenstein unprojection, that is equivalent to constructing T itself.

We assume that φ is a double cover of the line $(y_2 = 0) \subset \mathbb{P}(2, 2, 2)$ branched over the points $\varphi(1, 0)$ and $\varphi(0, 1)$. Then for general T' the map φ is

$$\begin{aligned} \varphi \colon \mathbb{P}^1 &\to \mathbb{P}(2, 2, 2, 3, 3) \\ (u, v) &\mapsto \left(u^2, 0, v^2, u(u^2 + \alpha v^2), v(\beta u^2 + v^2)\right). \end{aligned} \tag{5.4}$$

Rendering $\varphi^*(z_i^2)$ in terms of y_1, y_3 we see that the image of φ is defined by the three equations:

$$C_1 \colon z_1^2 = y_1(y_1 + \alpha y_3)^2 \tag{5.5}$$

$$C_2 \colon z_2^2 = y_3(\beta y_1 + y_3)^2 \tag{5.6}$$

$$y_2 = 0, \tag{5.7}$$

and one can check that the two cubics have nodes at $(-\alpha, 0, 1)$ and $(1, 0, -\beta)$ in $\mathbb{P}(2,2,2)$ respectively. We assume that $\varphi^*(z_1)$ and $\varphi^*(z_2)$ have no common factor so that these nodes are distinct.

To define $T' \subset \mathbb{P}(2,2,2,3,3)$ we must choose two appropriate combinations of weight 6 in equations (5.5–5.7). Note that if we want the branch curves to be nondegenerate then we should ensure that both equations for T' involve y_2 nontrivially. Moreover, after incorporating y_2 into the equations we should check that there are still two bona fide nodes at $(-\alpha, 0, 1)$ and $(1, 0, -\beta)$. So, calculating the tangent cone to each curve at these points forces the equations of T' to take the form

$$\begin{aligned} &C_1 + l_1 Q_1 + l_2 Q_2 + l_3 Q_3 + l_4 Q_4 \\ &C_2 + m_1 Q_1 + m_2 Q_2 + m_3 Q_3 + m_4 Q_4 \end{aligned} \tag{5.8}$$

where α, β, l_i, m_i are scalar parameters and

$$Q_1 := (y_1 + \alpha y_3) y_2^2, \quad Q_2 := y_2^3, \quad Q_3 := (\beta y_1 + y_3) y_2^2,$$

$$Q_4 := (y_1 + \alpha y_3)(\beta y_1 + y_3) y_2.$$

Proof of theorem The proof follows a similar approach to that of theorem (4.2.1) and it is informative to compare the two at each stage. We explicitly extend the projected image $T'_{6,6} \subset \mathbb{P}(2,2,2,3,3)$ to a Fano 6-fold $W'_{6,6} \subset \mathbb{P}(1^4, 2^3, 3^2)$ containing the image of $\mathbb{P}^5$ under some map Φ. Define $\varphi \colon \mathbb{P}^1 \to \mathbb{P}(2,2,2,3,3)$ as in (5.4) and write $\varphi_0 \colon \mathbb{P}^1 \to \mathbb{P}(2,2,2)$ for the map

$$\varphi_0^*(y_1) = u^2, \quad \varphi_0^*(y_2) = 0, \quad \varphi_0^*(y_3) = v^2.$$

Then writing u, v, a, b, c, d for the coordinates on $\mathbb{P}^5$, up to automorphisms of $\mathbb{P}^5$ and $\mathbb{P}(1^4, 2^3)$ the general extension of φ_0 to $\Phi_0 \colon \mathbb{P}^5 \to \mathbb{P}(1^4, 2^3)$ is

$$\Phi_0^*(a) = a, \quad \Phi_0^*(b) = b, \quad \Phi_0^*(c) = c, \quad \Phi_0^*(d) = d,$$

$$\begin{aligned} \Phi_0^*(y_1) &= u^2 \qquad\quad +2av, \\ \Phi_0^*(y_2) &= 0 + \quad bu \; + \; cv, \\ \Phi_0^*(y_3) &= v^2 + \, 2du \end{aligned} \tag{5.9}$$

We prove that there is a unique map $\Phi \colon \mathbb{P}^5 \to \mathbb{P}(1^4, 2^4, 3^2)$ which is a lift of Φ_0 and which extends $T'_{6,6}$ to $W'_{6,6}$.

We use the same notation as theorem (4.2.1), writing M, R, S for the coordinate rings of $\mathbb{P}^5$, $\mathbb{P}(1^4, 2^3)$ and $\mathbb{P}(1^4, 2^3, 3^2)$ respectively. By equation (5.9), the map Φ_0^* induces a graded R-module structure on M with generators

1, u, v and uv. Similarly Φ^* makes M into a graded S-module with the same generators. The presentation of M as a module over R is

$$0 \leftarrow M \xleftarrow{(1,u,v,uv)} R \oplus 2R(-1) \oplus R(-2) \xleftarrow{A} R(-2) \oplus 2R(-3) \oplus R(-4)$$

where A is the matrix

$$\begin{pmatrix} -y_2 & by_1 & cy_3 & -2cdy_1 + 4ady_2 - 2aby_3 \\ b & -y_2 & -2cd & cy_3 \\ c & -2ab & -y_2 & by_1 \\ 0 & c & b & -y_2 \end{pmatrix}. \tag{5.10}$$

Since Φ is a lift of φ we assume that the general forms of $\Phi^*(z_i)$ are

$$\begin{aligned} \Phi^*(z_1) &= u^3 + \alpha uv^2 + s_1u^2 + s_2uv + s_3v^2 + s_4u + s_5v \\ \Phi^*(z_2) &= \beta u^2v + v^3 + t_1u^2 + t_2uv + t_3v^2 + t_4u + t_5v \end{aligned}$$

where the $s_i(a, b, c, d)$, $t_i(a, b, c, d)$ are homogeneous polynomials of degree 1 or 2 as appropriate. Then using the R-module structure of M we can write

$$\begin{aligned} \Phi^*(z_1) &= u(f + s_4) + s_2uv + s_5v \\ \Phi^*(z_2) &= v(g + t_5) + t_2uv + t_4u \end{aligned} \tag{5.11}$$

where

$$f = y_1 + \alpha y_3, \quad g = \beta y_1 + y_3.$$

We have used coordinate changes $z_1 \mapsto z_1 + s_1y_1$ and similar to absorb the values of s_1, s_3, t_1, t_3 into z_1, z_2. The following theorem shows that there are unique values of s_i, t_i for $i = 2, 4, 5$ for which there are equations extending (5.5), (5.6). Moreover for these unique s_i, t_i, it follows that $Q_1, \ldots, Q_4$ extend.

Theorem 5.3.2 *(I) The kernel of $\Phi^* \colon S \to M$ contains equations extending (5.5), (5.6) of the form*

$$\begin{aligned} z_1^2 - y_1f^2 &\in R + Rz_1 + Rz_2, \\ z_2^2 - y_3g^2 &\in R + Rz_1 + Rz_2 \end{aligned}$$

if and only if

$$\begin{aligned} s_2 &= (1 - \alpha\beta)a, \quad s_4 = \beta a^2 + \alpha^2 d^2, \quad s_5 = \alpha(\alpha\beta - 1)ad, \\ t_2 &= (1 - \alpha\beta)d, \quad t_4 = \beta(\alpha\beta - 1)ad, \quad t_5 = \beta^2a^2 + \alpha d^2. \end{aligned}$$

(II) Given part (I), the equations are

$$\begin{aligned} z_1^2 - y_1(f+s_4)^2 &= -4(f+s_4)s_2ay_3 - 4s_2s_5dy_1 + s_2^2y_1y_3 + s_5^2y_3 \\ &\quad + 2(1-\alpha\beta)a^2(3dz_1 - az_2) - 2\alpha a(f+s_4)z_2 \end{aligned} \tag{5.12}$$

$$\begin{aligned} z_2^2 - y_3(g+t_5)^2 &= -4(g+t_5)t_2dy_1 - 4t_2t_4ay_3 + t_2^2y_1y_3 + t_4^2y_1 \\ &\quad + 2(1-\alpha\beta)d^2(3az_2 - dz_1) - 2\beta d(g+t_5)z_1 \end{aligned} \tag{5.13}$$

Corollary 5.3.3 *The kernel of* Φ^* *also contains equations extending* Q_i *for* $i = 1, \ldots, 4$ *of the form*

$$fy_2^2,\ y_2^3,\ gy_2^2,\ fgy_2 \in R + Rz_1 + Rz_2$$

respectively.

Proof The "if" part of the theorem is proved by evaluating equations (5.12), (5.13) under Φ^* with s_i, t_i taking the values stated in the theorem. The remainder of the proof is for the "only if" part.

Using the graded module structure of $k[u,v]$ over $k[y_1,y_2,y_3]$ via φ_0^* we write

$$\begin{aligned} \varphi^*(z_1) &= (y_1 + \alpha y_3)u \\ \varphi^*(z_2) &= (\beta y_1 + y_3)v. \end{aligned}$$

Then squaring either of these expressions and rendering u^2, v^2 as y_1, y_3 gives equations (5.5), (5.6) immediately. We attempt to do the same rendering calculation for the extended map Φ^*, using

$$\begin{aligned} u^2 &= \Phi^*(y_1) - 2av \\ v^2 &= \Phi^*(y_3) - 2du. \end{aligned}$$

We can eliminate all terms involving u^2 or v^2 from $\Phi^*(z_i^2)$ to obtain

$$\Phi^*\left(z_1^2 - y_1(f+s_4)^2 + 4(f+s_4)s_2ay_3 + 4s_2s_5dy_1 - s_2^2y_1y_3 - s_5^2y_3\right) \equiv 0$$
$$\Phi^*\left(z_2^2 - y_3(g+t_5)^2 + 4(g+t_5)t_2dy_1 + 4t_2t_4ay_3 - t_2^2y_1y_3 - t_4^2y_1\right) \equiv 0$$

modulo $(a,b,c,d)M$. The residual parts to these congruences are

$$\begin{aligned} K &= K_u u + K_v v + K_{uv} uv, \\ L &= L_u u + L_v v + L_{uv} uv \end{aligned}$$

respectively, where

$$\begin{aligned} K_u &= 8(f+s_4)s_2ad - 2s_5^2d - 2s_2^2dy_1 + 2s_2s_5y_3 \\ K_v &= -2(f+s_4)^2a + 8s_2s_5ad + 2(f+s_4)s_2y_1 - 2s_2^2ay_3 \\ K_{uv} &= 2(f+s_4)s_5 + 4s_2^2ad \end{aligned} \tag{5.14}$$

and

$$\begin{aligned} L_u &= -2(g+t_5)^2 d + 8t_2 t_4 ad + 2(g+t_5)t_2 y_3 - 2t_2^2 dy_1 \\ L_v &= 8(g+t_5)t_2 ad - 2t_4^2 a - 2t_2^2 ay_3 + 2t_2 t_4 y_1 \\ L_{uv} &= 2(g+t_5)t_4 + 4t_2^2 ad. \end{aligned} \tag{5.15}$$

Now K, L are homogeneous expressions of degree 6 in $(a,b,c,d)M$, and we prove that if they are to be contained in the submodule $R + Rz_1 + Rz_2 \subset M$ then s_i, t_i must take the values stated in the theorem. From the definition of $\Phi^*(z_i)$ in (5.11), the submodule $R + Rz_1 + Rz_2$ is the image of the composite

$$M \xleftarrow{(1,u,v,uv)} R \oplus 2R(-1) \oplus R(-2) \xleftarrow{B} R \oplus 2R(-3) \oplus R(-2) \oplus 2R(-3) \oplus R(-4)$$

where B is the matrix

$$\left(\begin{array}{ccc|cccc} 1 & 0 & 0 & -y_2 & by_1 & cy_3 & -2cdy_1 + 4ady_2 - 2aby_3 \\ 0 & f+s_4 & t_4 & b & -y_2 & -2cd & cy_3 \\ 0 & s_5 & g+t_5 & c & -2ab & -y_2 & by_1 \\ 0 & s_2 & t_2 & 0 & c & b & -y_2 \end{array}\right).$$

The first 3 columns of B are the generators 1, z_1, z_2 and the last 4 columns are the matrix A from (5.10), which is mapped to 0 under the composite.

We seek vectors $\xi, \eta \in R \oplus 2R(-3) \oplus R(-2) \oplus 2R(-3) \oplus R(-4)$ such that

$$\begin{aligned} K &= \begin{pmatrix} 1, & u, & v, & uv \end{pmatrix} B\xi, \\ L &= \begin{pmatrix} 1, & u, & v, & uv \end{pmatrix} B\eta. \end{aligned} \tag{5.16}$$

In order to solve for ξ, η and consequently fix the values of s_i, t_i we stratify K, L according to degree in y_1, y_2, y_3. In other words, write

$$\begin{aligned} K &= K^{(0)} + K^{(1)} + K^{(2)} \\ L &= L^{(0)} + L^{(1)} + L^{(2)} \end{aligned}$$

where $K^{(i)}$, $L^{(i)}$ have degree i in y_1, y_2, y_3 and similarly we write

$$\begin{aligned} \xi &= \xi^{(0)} + \xi^{(1)} \\ \eta &= \eta^{(0)} + \eta^{(1)}. \end{aligned}$$

We begin with $K^{(2)}$, which is calculated from (5.14) as

$$K^{(2)} = 2f(y_1 s_2 - fa)v.$$

We must find $\xi^{(1)}$ such that

$$K^{(2)} = \begin{pmatrix}1, & u, & v, & uv\end{pmatrix} B\xi^{(1)} + \text{ lower order terms.} \tag{5.17}$$

Comparing coefficients of y_1^2 and y_3^2, the only solution is

$$\xi_3^{(1)} = \frac{2}{\beta}(s_2 - a)y_1 - 2\alpha^2 a y_3,$$

with the other $\xi_i^{(1)} = 0$. Then the coefficient of $y_1 y_3$ in (5.17) dictates that

$$s_2 = (1 - \alpha\beta)a$$

and therefore $\xi_3^{(1)} = -2\alpha a f$. An exactly similar calculation with $L^{(2)}$ and $\eta_2^{(1)}$ yields

$$t_2 = (1 - \alpha\beta)d$$

and $\eta_2^{(1)} = -2\beta d g$.

Proceeding to the calculation for $K^{(1)}$, we must solve

$$K^{(1)} - \xi_3^{(1)}(t_4 u + t_5 v + t_2 uv) = \begin{pmatrix}1, & u, & v, & uv\end{pmatrix} B\xi^{(0)} + \text{ lower order terms} \tag{5.18}$$

where the term involving $\xi_3^{(1)}$ is necessary to account for the lower order terms from equation (5.17). Now examining the coefficient of uv in (5.18), we obtain

$$2f(s_5 + \alpha a t_2) = s_2\xi_2^{(0)} + t_2\xi_3^{(0)}.$$

However, $\xi^{(0)}$ has degree 0 in y_i by construction, so the left hand side must be identically 0. Hence

$$s_5 = -\alpha a t_2$$

and by considering the coefficient of uv in $L^{(1)}$ we find

$$t_4 = -\beta d s_2.$$

Comparing coefficients of u and v in equation (5.18) we obtain

$$\begin{aligned} 6(1-\alpha\beta)a^2 df &= (f + s_4)\xi_2^{(0)} + t_4\xi_3^{(0)} + \text{ lower order terms} \\ 2a(-s_4(f + \alpha g) + \alpha f t_5 - s_2^2 y_3) &= s_5\xi_2^{(0)} + (g + t_5)\xi_3^{(0)} + \text{ lower order terms.} \end{aligned}$$

Since $\xi^{(0)}$ has degree 0 in y_i we must have $\xi_2^{(0)} = 6(1 - \alpha\beta)a^2 d$. Moreover the coefficient of v must be divisible by g, which is equivalent to

$$\alpha t_5 - s_4 = -\beta(1 - \alpha\beta)a^2. \tag{5.19}$$

By considering the coefficients of u, v in $L^{(1)}$ in the same way we get $\eta_3^{(0)} = 6(1-\alpha\beta)ad^2$ and a further restriction on s_4, t_5:

$$t_5 - \beta s_4 = \alpha(1-\alpha\beta)d^2. \tag{5.20}$$

Solving equations (5.19), (5.20) simultaneously forces

$$\begin{aligned} s_4 &= \beta a^2 + \alpha^2 d^2 \\ t_5 &= \beta^2 a^2 + \alpha d^2, \end{aligned}$$

which in turn means that

$$\begin{aligned} \xi_3^{(0)} &= -2(1-\alpha\beta)a^3 - 2\alpha a s_4 \\ \eta_2^{(0)} &= -2(1-\alpha\beta)d^3 - 2\beta d t_5. \end{aligned}$$

We can finally write out ξ and η in full

$$\begin{aligned} \xi_2 &= 6(1-\alpha\beta)a^2 d & \eta_2 &= -2\beta d(g+t_5) - 2(1-\alpha\beta)d^3 \\ \xi_3 &= -2\alpha a(f+s_4) - 2(1-\alpha\beta)a^3 & \eta_3 &= 6(1-\alpha\beta)ad^2, \end{aligned}$$

where the other $\xi_i = \eta_i = 0$. It is necessary to check that ξ and η actually solve equations (5.16) when all the lower order terms are replaced, which can be verified directly.

The extended equations (5.12), (5.13) are obtained by writing out the vectors ξ, η in terms of the generators of $R + Rz_1 + Rz_2$

$$\begin{aligned} z_1^2 - y_1(f+s_4)^2 &= -4(f+s_4)s_2 a y_3 - 4 s_2 s_5 d y_1 + s_2^2 y_1 y_3 + s_5^2 y_3 \\ &\quad + \xi_2 z_1 + \xi_3 z_2 \\ z_2^2 - y_3(g+t_5)^2 &= -4(g+t_5)t_2 d y_1 - 4 t_2 t_4 a y_3 + t_2^2 y_1 y_3 + t_4^2 y_1 \\ &\quad + \eta_2 z_1 + \eta_3 z_2. \end{aligned}$$

This concludes the proof of theorem (5.3.2).

Proof of corollary First observe that the fourth column of B is equivalent to $y_2 = bu + cv$. Thus the extension of Q_1 is calculated by expressing $f y_2(bu + cv)$ in terms of the other columns of B. We have to find ν such that

$$(f+s_4)y_2(bu+cv) = \begin{pmatrix} 1, & u, & v, & uv \end{pmatrix} B\nu.$$

The solution to this linear algebra problem is

$$\begin{aligned} \nu_2 &= 2by_2 + 2(\beta ab - cd)c & \nu_3 &= 2(\alpha b^2 + c^2)a \\ \nu_4 &= -\beta a s_2 y_2 - 2ac(g+t_5) + 2(cd - \beta ab)s_5 & \nu_5 &= b(f+s_4) - \beta a b s_2 \\ \nu_6 &= -c(f+s_4) - \beta a c s_2 + 2bs_5 & \nu_7 &= 2bs_2, \end{aligned}$$

where $\nu_1 = y_2\nu_4 - by_1\nu_5 - cy_3\nu_6 - (-2cdy_1 + 4ady_2 - 2aby_3)\nu_7$ uses the first column of B to remove any excess terms. Thus the equation extending Q_1 is

$$\widetilde{Q}_1\colon (f + s_4)y_2^2 = \nu_1 + \nu_2 z_1 + \nu_3 z_2.$$

Similar calculations give the equations extending Q_2, Q_3, Q_4 for which I list the corresponding vectors below. The equation extending Q_2 is

$$\widetilde{Q}_2\colon y_2^3 = \nu_1 + \nu_2 z_1 + \nu_3 z_2,$$

where

$$\begin{aligned}
\nu_1 &= y_2\nu_4 - by_1\nu_5 - cy_3\nu_6 - (-2cdy_1 + 4ady_2 - 2aby_3)\nu_7\\
\nu_2 &= \frac{2}{\alpha\beta - 1}(b^2 + \beta c^2)b\\
\nu_3 &= \frac{2}{\alpha\beta - 1}(\alpha b^2 + c^2)c\\
\nu_4 &= \frac{2}{1 - \alpha\beta}\left(b^2(f + s_4) + c^2(g + t_5)\right) + (\beta ac + \alpha bd)y_2 + 2(2 - \alpha\beta)abcd\\
\nu_5 &= -by_2 + 2c^2d + (\beta ac + \alpha bd)b\\
\nu_6 &= -cy_2 + 2ab^2 + (\beta ac + \alpha bd)c\\
\nu_7 &= -2bc.
\end{aligned}$$

The equation extending Q_3 is

$$\widetilde{Q}_3\colon (g + t_5)y_2^2 = \nu_1 + \nu_2 z_1 + \nu_3 z_2,$$

where

$$\nu_1 = y_2\nu_4 - by_1\nu_5 - cy_3\nu_6 - (-2cdy_1 + 4ady_2 - 2aby_3)\nu_7$$

$$\begin{aligned}
\nu_2 &= 2(b^2 + \beta c^2)d & \nu_3 &= 2cy_2 - 2(ab - \alpha cd)b\\
\nu_4 &= -\alpha dt_2y_2 - 2bd(f + s_4) + 2(ab - \alpha cd)t_4 & \nu_5 &= -b(g + t_5) - \alpha bdt_2 + 2ct_4\\
\nu_6 &= c(g + t_5) - \alpha cdt_2 & \nu_7 &= 2ct_2.
\end{aligned}$$

Finally equation Q_4 is extended by

$$\widetilde{Q}_4\colon (f + s_4)(g + t_5)y_2 = \nu_1 + \nu_2 z_1 + \nu_3 z_2$$

where

$$\nu_1 = y_2\nu_4 - by_1\nu_5 - cy_3\nu_6 - (-2cdy_1 + 4ady_2 - 2aby_3)\nu_7$$

$$\nu_2 = b(g + t_5) + ct_4 - t_2 y_2 \qquad \nu_3 = c(f + s_4) + bs_5 - s_2 y_2$$
$$\nu_4 = -s_5 t_4 \qquad \nu_5 = -t_2(f + s_4) - s_2 t_4$$
$$\nu_6 = -s_2(g + t_5) - s_5 t_2 \qquad \nu_7 = -2s_2 t_2.$$

This completes the proof of the corollary.

Given theorem (5.3.2) and its corollary, we can prove that there is a unique hyperelliptic Fano 6-fold $W'_{6,6} \subset \mathbb{P}(1^4, 2^3, 3^2)$ extending any given projected hyperelliptic K3 surface $T'_{6,6}$. Simply take the combination of equations (5.12), (5.13) and $\widetilde{Q}_i$ corresponding to the choice (5.8) made in the definition of $T'_{6,6}$. This proves the main theorem (5.3.1).

Remark 5.3.4 Reviewing our calculations concerning the equations of the curve D and the K3 surface T, we might hope to find a format for the equations of W. The curve D is a codimension 2 complete intersection inside a weighted quotient of the affine cone over the degree 2 Veronese embedding of $\mathbb{P}^3$. Similarly T is a codimension 2 complete intersection inside a weighted $\mathbb{P}^2 \times \mathbb{P}^2$. The equations for W must be a common generalisation of these formats although I have not yet found such a format.

Chapter 6

Godeaux surfaces

This final chapter contains a general construction of a surface of general type with $p_g = 0$, $K^2 = 1$ and torsion $\mathbb{Z}/2$. This builds on the results of theorems (4.2.1) and (5.3.1), using the constructions proved there as so called "key varieties" containing surfaces of general type as linear sections.

6.1 Surfaces of general type

A nonsingular projective surface S is called general type if K_S is nef and $K_S^2 > 0$. In classical terminology, general type means the surface has Kodaira dimension 2. This is a very large class of surfaces; for example, almost every surface complete intersection in projective space is of general type. However, unlike the classification of curves, genus alone is not sufficient to give a nice description of the moduli space of surfaces of general type, we must use degree as well. More precisely, we subclassify according to the genus $p_g = h^0(S, K_S)$ and canonical degree K^2. For simplicity we often assume that S is regular. That is, the irregularity $q := h^1(S, \mathcal{O}_S)$ is zero.

The canonical graded ring of a surface S is defined to be

$$R(S, K_S) = \bigoplus_{n \geq 0} H^0(S, nK_S).$$

When S is a surface of general type, Mumford showed that $R(S, K_S)$ is finitely generated, and we can define the canonical model X of S by

$$X = \operatorname{Proj} R(S, K_S).$$

The canonical model X is birational to S, since K_S is nef and big, and has finitely many rational double points. Indeed, for some $m \gg 0$ (usually 3 and at most 5) the map induced by the linear system $|mK_S|$ is an isomorphism outside of the support of those curves C on S with $K_S C = 0$ (see [Bom] for details). The study of canonical models of surfaces of general type has been of interest for many years. Except for some special values of p_g and K^2, their descriptions are very difficult to obtain.

6.1.1 Example: Horikawa surfaces

The Noether inequality states that for a minimal surface of general type,

$$K^2 \geq 2p_g - 4.$$

The points where $K^2 = 2p_g - 4$ make up the Noether–Horikawa line. In the 1970s Horikawa studied the moduli of surfaces on and near the Noether line (see for example [Hor]), and showed that the situation becomes more and more complicated the further we move away from the line. For example, for most choices[1] of p_g and K^2 on the line $K^2 = 2p_g - 4$, all surfaces with these invariants are deformations of one another; the moduli space is connected. However, if we move to the line $K^2 = 2p_g - 2$, the moduli space can have several components of different dimensions and is not necessarily connected.

6.1.2 Example: Godeaux surfaces

A Godeaux surface is a minimal regular surface of general type with $p_g = 0$ and $K^2 = 1$. In fact, a surface with $p_g = 0$ will always be regular (see [Bom]). The Godeaux surfaces are one of the initial cases of a surface of general type and as such are important and should be relatively simple to construct. After the classification of surfaces was completed in 1914, geometers were interested in criteria for rationality and examples of non-rational surfaces with $p_g = q = 0$. This led to Enriques' discovery of a sextic surface in $\mathbb{P}^3$ passing doubly through a tetrahedron. The Enriques surface has invariants $p_g = q = 0$ but is not rational by construction. Later, Godeaux discovered the following example of a surface of general type with $p_g = q = 0$:

Take a quintic hypersurface $Y_5 \subset \mathbb{P}^3$ which is invariant under the $\mathbb{Z}/5$ group action $x_i \mapsto \varepsilon^i x_i$ where ε is a primitive 5th root of unity. For example, the Fermat quintic

$$x^5 + y^5 + z^5 + t^5 = 0.$$

If the group action is fixed point free, then the quotient X is a surface of general type with invariants $p_g = 0$ and $K^2 = 1$.

[1] If $8 \mid K^2$ then the moduli space is disconnected.

6.1.3 Algebraic fundamental group

Another invariant used to further subclassify surfaces of general type is the algebraic fundamental group which is defined by

$$\pi_1^{alg}(X) = \varprojlim \mathrm{Gal}(Y/X),$$

where the inverse limit is taken over all Galois finite étale covers $Y \to X$, (see [SGA1] for details).

In applications it is easier to work with the torsion subgroup $\mathrm{Tors}\, X \subset \mathrm{Pic}\, X$. Suppose $\mathrm{Tors}\, X$ is cyclic of order n and generated by $\mathcal{O}_X(\sigma)$. Then

$$Y := \mathrm{Proj} \bigoplus_{m,n \geq 0} H^0(X, mK_X + n\sigma)$$

is a finite cyclic Galois étale cover of X of degree n. Thus $\mathrm{Tors}\, X$ is a quotient of $\pi_1^{alg}(X)$. The strategy we follow is to use $\mathrm{Tors}\, X$ to construct a particular étale cover of X and then try to prove that $\mathrm{Tors}\, X = \pi_1^{alg}(X)$. Write $f\colon Y \to X$ for the cover determined by $\mathrm{Tors}\, X$. Then if X is a Godeaux surface, Y is a surface of general type with $K_Y^2 = n$ and $p_g(Y) = n - 1 + q(Y)$. Indeed, since f is étale, $T_Y = f^*T_X$, so that $K_Y = f^*K_X$ and $c_i(Y) = f^*c_i(X)$. Hence we can calculate K_Y^2 and $\chi(\mathcal{O}_Y)$ in terms of K_X^2 and $\chi(\mathcal{O}_X)$.

This is precisely the strategy used by Godeaux: a quintic surface Y in $\mathbb{P}^3$ is the canonical model of a regular surface of general type with $p_g = 4$ and $K^2 = 5$. If Y has a fixed point free $\mathbb{Z}/5$-action then the quotient map $Y \to X$ is a $\mathbb{Z}/5$-Galois étale cover of a surface X with invariants $p_g = 0$, $K^2 = 1$ and $\mathrm{Tors}\, X = \mathbb{Z}/5$.

If X is a Godeaux surface then it is known that $\pi_1^{alg}(X)$ is cyclic of order ≤ 5 (see [Miy]). The cases $\pi_1^{alg}(X) = \mathbb{Z}/5$, $\mathbb{Z}/4$, $\mathbb{Z}/3$ were studied by Reid in [R1] and he showed that in each case the moduli space is 8-dimensional, irreducible and unirational. There are examples ([B1, B2]) with $\pi_1^{alg}(X) = \mathbb{Z}/2$ and 0 but no classification.

6.2 "Plan of action"

From now on we assume that X is the canonical model of a surface of general type with $p_g = 0$, $K^2 = 1$ and $\mathrm{Tors}\, X = \mathbb{Z}/2$. We intend to give a construction of the étale double cover Y of X complete with its fixed point free $\mathbb{Z}/2$-action. The surface Y will have invariants $p_g = 1$ and $K_Y^2 = 2$. In general we can not expect all surfaces Y with these invariants to be a double cover of some X, so we rely on an explicit description of Y to determine the appropriate subfamily. Catanese and Debarre gave an explicit description of surfaces with

$p_g = 1$ and $K^2 = 2$ in [CD] but were unable to determine precisely the family of surfaces Y having a free $\mathbb{Z}/2$-action.

If Y is the double cover of a Godeaux surface then each summand of the canonical ring $R(Y, K_Y)$ splits into eigenspaces:

$$H^0(Y, nK_Y) = H^0(X, nK_X) \oplus H^0(X, nK_X + \sigma).$$

The first summand is invariant under the $\mathbb{Z}/2$-action and the second is anti-invariant. Hence we get a bigrading on the canonical ring $R(Y, K_Y)$ by $\mathbb{Z} \oplus \mathbb{Z}/2$. It is a simple exercise in Hilbert series and the Riemann–Roch theorem for surfaces to determine the dimension of each eigenspace and hence to figure out generators and relations for the canonical ring of Y with its involution.

6.2.1 Hilbert series basics

The Riemann–Roch theorem for surfaces reduces in the case of surfaces of general type to

$$h^0(X, nK_X) = \begin{cases} 1 & n = 0 \\ p_g & n = 1 \\ 1 + p_g + \frac{n(n-1)}{2} K_X^2 & n \geq 2. \end{cases}$$

Then the Hilbert series of a surface of general type is

$$P_X(t) = \sum_{n=0}^{\infty} h^0(X, nK_X) t^n = 1 + p_g t + (1 + p_g + K_X^2) t^2 + \dots.$$

For a Godeaux surface X this series can be written as the rational function

$$P_X(t) = \frac{1 - 6t^6 - 12t^7 - 18t^8 - 4t^9 + \dots}{(1-t^2)^2(1-t^3)^4(1-t^4)^4(1-t^5)^3}$$

which suggests that the canonical model is given (at best) by

$$\operatorname{Proj} \bigoplus_{n \geq 0} H^0(X, nK_X) \subset \mathbb{P}(2^2, 3^4, 4^4, 5^3),$$

where the surface is defined by 40 equations in codimension 10! Similar Hilbert series considerations show that the canonical model of the étale double cover Y is at least

$$\operatorname{Proj} \bigoplus_{n \geq 0} H^0(Y, nK_Y) \subset \mathbb{P}(1, 2^3, 3^4)$$

with 14 relations. This still a codimension 5 variety, but bearing in mind the relationship between complexity of graded rings and codimension, this is a vast improvement over codimension 10.

6.2.2 The Godeaux curve

A standard line of attack when constructing surfaces of general type is to first consider the restricted algebra $R(D, K_Y|_D)$, where D is a nonsingular irreducible curve in the linear system $|K_Y|$. Then using the hyperplane section principle, we can try to deduce the structure of the canonical ring of Y from that of the restricted algebra. By an easy application of the adjunction formula, the genus $g(D) = 3$, and $2K_Y|_D = K_D$. Furthermore, D is not just any old curve of genus 3, as the following lemma from [CD] shows:

Lemma 6.2.1 *If Y is an unramified double cover of a Godeaux surface X then the curve section D in $|K_Y|$ must be hyperelliptic.*

Proof The original proof of this lemma used the Riemann–Roch theorem and monomial counting. We reproduce this proof and then give a second proof using Hilbert series afterwards. Write down generators for the ring $R(Y, K_Y)$ and separate them according to their eigenspace under the $\mathbb{Z}/2$-action. We can calculate the dimension of each eigenspace using the Riemann–Roch theorem, so choosing generators for each graded summand and observing the rules of multiplication, we get the following table:

n	$H^0(Y, nK_X)$	$H^0(Y, nK_X + \sigma)$
0	k	ϕ
1	0	x
2	x^2, y_1	y_2, y_3
3	xy_2, xy_3, z_1, z_2	x^3, xy_1, z_3, z_4
4	$x^4, x^2y_1, y_1^2, y_2^2,$	$x^2y_2, x^2y_3, xz_1, xz_2,$
	$y_2y_3, y_3^2, xz_3, xz_4$	y_1y_2, y_1y_3

Now $h^0(Y, 4K_X) = h^0(Y, 4K_X + \sigma) = 7$, so it is clear from the table that there are too many generators for $H^0(4K_X)$ and not enough for $H^0(4K_X + \sigma)$. Thus we must have a relation between the generators of $H^0(4K_X)$ and in turn, an extra generator t for $H^0(4K_X + \sigma)$. Hence the canonical curve D is hyperelliptic, double covering the conic defined by the relation in $H^0(4K_X)$.

The alternative proof is an argument in bigraded Hilbert series. We define

$$P_Y(t, e) = \sum_{m \geq 0, n \in \mathbb{Z}/2} h^0(Y, mK_X + n\sigma)t^m e^n$$

where e keeps track of the eigenspace, so accordingly $e^2 = 1$. Bearing in mind the eigenspace decomposition of $H^0(Y, nK_Y)$ from section (6.2), we see that

$$P_Y(t, e) = P_X(t) + (P_Y(t) - P_X(t))\, e.$$

When we express $P_Y(t,e)$ as a rational function, we must keep track of the eigenspace of each new generator as well as its degree:

$$P_Y(t,e) = \frac{1 + (e-1)t^4 + (-2e-2)t^5 + (-4e-6)t^6 + (7e+8)t^8 + \dots}{(1-et)(1-t^2)(1-et^2)^2(1-t^3)^2(1-et^3)^2}.$$

Notice that the first nontrivial coefficient in the numerator is $e - 1$ which is not negative, so we need a further new generator of degree 4 in the negative eigenspace. In other words, we must divide by $(1-et^4)$ so that the numerator becomes

$$1 - t^4 + (-2e-2)t^5 + (-4e-6)t^6 + \dots.$$

Now the extra $-t^4$ term in the numerator indicates that there is a relation in degree 4 not involving the new generator and the denominator suggests $D \subset \mathbb{P}(2^3, 3^4, 4)$. Hence the curve D is hyperelliptic.

This calculation in Hilbert series is exactly the same as the monomial counting proof. However, we gain additional information, because the numerator of the bigraded Hilbert series tells us about the eigenspace of some of the relations and syzygies of $R(Y, K_Y)$.

6.3 Surfaces with $p_g = 1$, $K^2 = 2$

Consider the famous tower

$$D \subset T \subset W$$

where D is a curve, T a K3 surface, W a Fano n-fold and each inclusion is a complete intersection of one or more weighted hyperplane sections. In particular I have in mind the symmetric determinantal tower constructed in section (4.2) and its hyperelliptic degeneration of section (5.3), so that W is a Fano 6-fold. Each of these Fano 6-folds W is an example of a *key variety*. Without attempting to give a formal definition, a key variety is a "large" variety which contains many interesting varieties as transverse intersections with various (weighted) hyperplane sections. A good illustration of the philosophy of key varieties is the work of Mukai on canonical curves, K3 surfaces and Fano 3-folds [Muk]. Mukai showed that nonsingular prime Fano 3-folds occur naturally as intersections of hyperplanes with certain homogeneous spaces.

Unfortunately I do not have an analogue of Mukai's result for surfaces with $p_g = 1$, $K^2 = 2$, in that until now I have not discovered any universal key variety for these surfaces. However, I am able to use family of Fano 6-folds W as a substitute for a universal key variety. Recall the Fano 6-fold $W \subset \mathbb{P}(1^4, 2^4, 3^4)$ constructed in theorem (4.2.1) and its hyperelliptic counterpart $W \subset \mathbb{P}(1^4, 2^4, 3^4, 4)$ of theorem (5.3.1).

Theorem 6.3.1 *(I) There is a* 16 *parameter family of surfaces* Y *of general type with* $p_g = 1$, $K^2 = 2$ *and no torsion, each of which is a complete intersection of type* $(1, 1, 1, 2)$ *in a Fano* 6*-fold* $W \subset \mathbb{P}(1^4, 2^4, 3^4)$ *with* $10 \times \frac{1}{2}$ *points.*

(II) There is a 15 *parameter family of hyperelliptic surfaces* Y *with* $p_g = 1$, $K^2 = 2$ *and no torsion, each of which is a complete intersection of type* $(1, 1, 1, 2)$ *in a hyperelliptic Fano* 6*-fold* $W \subset \mathbb{P}(1^4, 2^4, 3^4, 4)$ *with* $10 \times \frac{1}{2}$ *points.*

Proof I prove part (I) since the proof of part (II) is identical. Then to obtain Y from W take 3 transverse hyperplane sections of weight 1 and one hyperplane section of weight 2, avoiding the isolated orbifold $\frac{1}{2}$ points. Since W is quasismooth, by the adjunction formula the surface Y has $\omega_Y = \mathcal{O}_Y(1)$ and is smooth. Furthermore it is clear from the construction of Y that

$$p_g(Y) = h^0(Y, K_Y) = h^0(W, -K_W) - 3 = 1.$$

Consider Y as a quadric section of a Fano 3-fold W^3. Then the standard short exact sequence

$$0 \to \mathcal{O}_{W^3}(-2) \to \mathcal{O}_{W^3} \to \mathcal{O}_Y \to 0,$$

implies that $H^1(\mathcal{O}_Y) = 0$ by Kodaira vanishing, so Y is regular. Finally the Riemann–Roch formula gives $K_Y^2 = 2$.

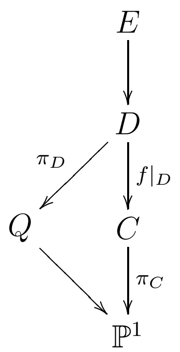
Theorem (4.2.1) says that the family of Fano 6-folds depends on the same number of moduli as the family of K3 surface sections, which is 9. Furthermore, naively counting the number of choices for linear and quadric sections of W suggests that we have a $9 + 3 + 4 = 16$ parameter family of surfaces Y. This is the expected dimension of the moduli space of surfaces with $p_g = 1$, $K^2 = 2$, which suggests that we have constructed the general surface (see [CD] for further justification).

The remainder of this chapter aims to construct the étale double cover Y of the Godeaux surface X with torsion $\mathbb{Z}/2$. The theorem shows that we can use the hyperelliptic tower $D \subset T \subset W$ to construct the hyperelliptic surface Y so the remaining obstacle is to find the appropriate subfamily of W for which the surface $Y \subset W$ has a Godeaux involution.

6.4 A picture in curves

Recall that a hyperelliptic curve of genus g is a double cover of $\mathbb{P}^1$ branched in $2g + 2$ Weierstrass points. There is a hyperelliptic involution h on such a curve which exchanges the two sheets of this double covering and fixes the Weierstrass points. Of course the quotient by h gives $\mathbb{P}^1$ marked with the $2g+2$ fixed points. Now let $f\colon Y \to X$ be the étale double cover of a Godeaux surface X and suppose D is a nonsingular curve in $|K_Y|$, similarly C in $|K_X+\sigma|$. By lemma (6.2.1), D is a hyperelliptic curve and C has genus 2 so is automatically hyperelliptic too. Let $\pi_D\colon D \to Q \cong \mathbb{P}^1$ denote the quotient map of the hyperelliptic involution on D, similarly $\pi_C\colon C \to \mathbb{P}^1$. Since D is an unramified double cover of C via $f|_D\colon D \to C$, we note that Q is naturally a double cover of $\operatorname{Im}\pi_C = \mathbb{P}^1$.

We get the following picture:



There is a fixed point free involution on the curve D induced by the unramified double cover $f|_D$, which is called the Godeaux involution. We use the same notation to denote the involution $\sigma\colon D \to D$ and the torsion element $\sigma \in \operatorname{Pic} X$. It is clear that the Weierstrass points of D must be invariant under σ, so there is a natural division of these eight points into two sets $\{P_1,\dots,P_4\}$ and $\{P_5,\dots,P_8\}$, which are interchanged by σ.

Now consider the linear system $A_D := K_Y|_D$ on D which is determined by the surface Y. The divisor class A_D is of degree 2, ineffective and invariant under both σ and the hyperelliptic involution, so a priori the only possibilities are

$$\begin{aligned} A_D &\sim P_1+P_2+P_3+P_4-g^1_2 \sim P_5+P_6+P_7+P_8-g^1_2, \\ A_D &\sim P_1+P_3+P_5+P_7-g^1_2 \sim P_2+P_4+P_6+P_8-g^1_2. \end{aligned} \tag{6.1}$$

The difference between these is that the former is only σ-invariant as a divisor class, whereas the latter is an σ-invariant divisor.

Observe that we have already constructed the determinantal graded ring $R(D,A_D)$ in example (5.1.1). Furthermore by the adjunction formula, it is clear that $2g^1_2 \sim 2A_D \sim K_D$. However, these two divisor classes A_D, g^1_2 are distinct, because the g^1_2 is effective whereas A_D is ineffective. Thus we have a 2-torsion class

$$\tau := A_D - g^1_2$$

on D, which corresponds to a genus 5 unramified double cover E of D, where

$$E = \operatorname{Proj} R(D, A_D, \tau) = \operatorname{Proj} \bigoplus_{m \geq 0, n \in \mathbb{Z}/2} H^0(D, mA_D + n\tau).$$

Now we have sufficient information to construct the bigraded ring $R(D, A_D, \tau)$ over the covering curve E. Indeed, using the notation of example (5.1.1) write t_1, t_2 for the sections of the g_2^1 and

$$u\colon \mathcal{O}_D \to \mathcal{O}_D(P_1 + \cdots + P_4), \quad v\colon \mathcal{O}_D \to \mathcal{O}_D(P_5 + \cdots + P_8).$$

We can very quickly write down generators and relations for $R(D, A_D, \tau)$:

n	$H^0(D, nA_D)$	$H^0(D, nA_D + \tau)$
0	k	ϕ
1	ϕ	t_1, t_2
2	t_1^2, t_1t_2, t_2^2	u, v
3	$\dots$	$\dots$

Thus E is a complete intersection

$$E_{4,4} \subset \mathbb{P}(1, 1, 2, 2),$$

defined by equations $u^2 = f_4(t_1, t_2)$ and $v^2 = g_4(t_1, t_2)$. The polynomials f and g are functions on $\mathbb{P}^1$ whose vanishing determines the splitting of the Weierstrass points of D into two sets of four.

Of course E comes bundled at no extra cost with the fixed point free involution $\tau\colon E \to E$ associated to the torsion τ of D. We recover the restricted algebra $R(D, K_Y|_D)$ of (5.1.1) by taking the τ-invariant subring of $R(D, A_D, \tau)$:

$$R(D, A_D) = R(D, A_D, \tau)^{\langle\tau\rangle}.$$

For future reference, we write out the action of τ on E using the eigenspace table above

$$t_1 \mapsto -t_1, \quad t_2 \mapsto -t_2, \quad u \mapsto -u, \quad v \mapsto -v.$$

Now, I claim the covering curve E completely determines the Godeaux involution σ on D. First observe that D is a quotient of E, and that this covering curve only exists because D is the curve section of $|K_Y|$. Thus σ lifts to the curve E and should be compatible with the involution τ on E, so that $\sigma^2 = 1$ or τ on E.

Proposition 6.4.1 *The action of σ on E is given by*

$$t_1 \mapsto it_1, \quad t_2 \mapsto -it_2, \quad u \mapsto iv, \quad v \mapsto iu,$$

so that $\sigma^2 = \tau$ and the group $\langle\sigma, \tau\rangle$ is a copy of $\mathbb{Z}/4$. Moreover, the polarising divisor of D is

$$A_D \sim P_1 + P_2 + P_3 + P_4 - g_2^1 \sim P_5 + P_6 + P_7 + P_8 - g_2^1.$$

Proof The table and Hilbert series of section (6.2.2) give the eigenspace decomposition of σ on D, which we must abide by. In particular, $R(D, A_D)$ should have only one invariant generator in degree 2, and the generator in degree 4 should be antiinvariant. This forces $\sigma^2 = \tau$, so that the group $\langle \sigma, \tau \rangle$ acting on E is $\mathbb{Z}/4$ rather than $\mathbb{Z}/2 \oplus \mathbb{Z}/2$.

Now, there are two possibilities for σ depending on the representation of A_D chosen from equation (6.1). These are

$$t_1 \mapsto it_1, \quad t_2 \mapsto -it_2, \quad u \mapsto iv, \quad v \mapsto iu$$
$$t_1 \mapsto it_1, \quad t_2 \mapsto -it_2, \quad u \mapsto iu, \quad v \mapsto iv$$

respectively. I claim that the second choice can not possibly be the Godeaux involution because it has fixed points on D. Indeed, in this second case the ring $R(D, A_D)$ and the action on it by σ are represented by the following display

$$\operatorname{rank}\left(\begin{array}{cc|cc} y_1 & y_2 & z_1 & z_3 \\ y_2 & y_3 & z_2 & z_4 \\ \hline z_1 & z_2 & f_2 & t \\ z_3 & z_4 & t & g_2 \end{array}\right) \leq 1 \quad \mapsto \quad \operatorname{rank}\left(\begin{array}{cc|cc} -y_1 & y_2 & -z_1 & -z_3 \\ y_2 & -y_3 & z_2 & z_4 \\ \hline -z_1 & z_2 & -f_2 & -t \\ -z_3 & z_4 & -t & -g_2 \end{array}\right) \leq 1,$$

where

$$f_2 = \alpha_1 y_1 y_2 + \alpha_2 y_2 y_3,$$
$$g_2 = \alpha_3 y_1 y_2 + \alpha_4 y_2 y_3.$$

However, then σ has two fixed points at the coordinate points P_{y_1} and P_{y_3}, which is a contradiction.

Therefore the only possibility is

$$A_D \sim P_1 + P_2 + P_3 + P_4 - g_2^1 \sim P_5 + P_6 + P_7 + P_8 - g_2^1,$$

with corresponding action on $R(D, A_D)$ given by

$$\operatorname{rank}\left(\begin{array}{cc|cc} y_1 & y_2 & z_1 & z_3 \\ y_2 & y_3 & z_2 & z_4 \\ \hline z_1 & z_2 & f_2 & t \\ z_3 & z_4 & t & g_2 \end{array}\right) \leq 1 \quad \mapsto \quad \operatorname{rank}\left(\begin{array}{cc|cc} -y_1 & y_2 & -z_3 & -z_1 \\ y_2 & -y_3 & z_4 & z_2 \\ \hline -z_3 & z_4 & -g_2 & -t \\ -z_1 & z_2 & -t & -f_2 \end{array}\right) \leq 1,$$

where

$$f_2 = \alpha_1 y_1^2 + \alpha_2 y_1 y_2 + \alpha_3 y_1 y_3 + \alpha_4 y_2^2 + \alpha_5 y_2 y_3 + \alpha_6 y_3^2$$
$$g_2 = -\alpha_1 y_1^2 + \alpha_2 y_1 y_2 - \alpha_3 y_1 y_3 - \alpha_4 y_2^2 + \alpha_5 y_2 y_3 - \alpha_6 y_3^2$$

and the involution will have no fixed points as long as α_1 and α_6 are not zero. This proves the proposition.

6.5 Involution on the K3 surface

Moving one step up the tower, there is a hyperelliptic K3 surface T containing D as a quadric section. In fact T is the surface $T \subset \mathbb{P}(2^4, 3^4, 4)$ with $10 \times \frac{1}{2}$ orbifold points which appeared in the graded ring calculation of example (5.2.1). With the notation used there, T is a double cover of the rank 4 quadric surface $Q \subset \mathbb{P}^3$ branched in the curve $C_1 + C_2$ where $C_1 \in |3H_1 + H_2|$ and $C_2 \in |H_1 + 3H_2|$. Let D_1, D_2 be the ramification curves on T, then

$$A_T \sim D_1 - H_1 \sim D_2 - H_2.$$

We recover the hyperelliptic curve D by taking a quadric section of T and $A_T|_D = A_D$, where A_D is defined by equation (6.1).

The whole argument becomes quite transparent when viewed in terms of commutative algebra. The graded ring $R(T, A_T)$ is described explicitly in example (5.2.1) and on eliminating one of the generators in degree 2, we obtain $R(D, A_D)$. Now if D is the unramified double covering of a Godeaux curve C with its involution $\sigma\colon D \to D$ from proposition (6.4.1), then:

Proposition 6.5.1 *There is at least one K3 surface T containing the curve D such that the involution σ on D has a unique lift to T. Moreover, such a lift $\sigma\colon T \to T$ has 4 fixed points which are $\frac{1}{2}$ points of T. We call σ the Godeaux involution on T.*

Remark 6.5.2 This is surprising because we are looking for a fixed point free involution on the covering surface Y, so it would be reasonable to expect that the involution on the K3 surface is free.

Proof *Step (1) Determining the character of σ.* First choose coordinates on T so that $D = T \cap (y_4 = 0)$, where y_4 must be semiinvariant under any putative involution. Then the determinantal part of the equations for T take the general form

$$\operatorname{rank} \begin{pmatrix} y_1 + \alpha y_4 & y_2 + \beta y_4 & z_1 \\ y_2 + \gamma y_4 & y_3 + \delta y_4 & z_2 \\ z_3 & z_4 & t \end{pmatrix} \leq 1,$$

where $\alpha, \beta, \gamma, \delta$ are scalars. Now, if σ lifts to T then our choice of coordinates means that the action of σ on T is predetermined by $\sigma|_D$ excepting the new variable y_4. Since $Q \subset \mathbb{P}^3$ is a quadric of rank 4, the determinantal equations force $\alpha = \delta$ and $\beta = -\gamma$. In particular this means y_4 is antiinvariant and the signature of σ on Q is $(1,3)$. Now recalibrate the coordinate system so that the determinantal equations for T are

$$\operatorname{rank}\left(\begin{array}{cc|c} y_1 & y_2 & z_1 \\ y_3 & y_4 & z_2 \\ \hline z_3 & z_4 & t \end{array}\right) \leq 1,$$

where

$$\sigma(y_2) = y_3, \quad \sigma(y_3) = y_2$$

and D is obtained by taking the quadric section $y_2 = y_3$.

Step (2) Fixed points of σ. First observe that the involution on T swaps the two branch curves of the double cover. Thus using the description of T as a $(\mathbb{C}^\times)^2$-quotient from example (5.2.1), we can write σ as

$$\begin{array}{ccc} s_1 \mapsto -t_1 & t_1 \mapsto s_1 & u \mapsto -v \\ s_2 \mapsto t_2 & t_2 \mapsto -s_2 & v \mapsto u, \end{array}$$

where the notation is described in example (5.2.1). In more explicit terms,

$$\operatorname{rank}\left(\begin{array}{cc|c} y_1 & y_2 & z_1 \\ y_3 & y_4 & z_2 \\ \hline z_3 & z_4 & t \end{array}\right) \leq 1 \quad \mapsto \quad \operatorname{rank}\left(\begin{array}{cc|c} -y_1 & y_3 & -z_3 \\ y_2 & -y_4 & z_4 \\ \hline -z_1 & z_2 & -t \end{array}\right) \leq 1$$

$$\begin{array}{rcl} z_1^2 = t_1^2 f_{3,1} & \leftrightarrow & z_3^2 = s_1^2 g_{1,3} \\ z_1 z_2 = t_1 t_2 f_{3,1} & \leftrightarrow & -(z_3 z_4 = s_1 s_2 g_{1,3}) \\ z_2^2 = t_2^2 f_{3,1} & \leftrightarrow & z_4^2 = s_2^2 g_{1,3} \\ z_1 t = q_1 z_3 + q_1' z_4 & \leftrightarrow & z_3 t = q_3 z_1 + q_3' z_2 \\ z_2 t = q_2 z_3 + q_2' z_4 & \leftrightarrow & -(z_4 t = q_4 z_1 + q_4' z_2) \\ & t^2 = F(y_i) \circlearrowleft & \end{array}$$

where

$$\begin{aligned} f_{3,1} = & \alpha_1 s_1^3 t_1 + \alpha_2 s_1^2 s_2 t_1 + \alpha_3 s_1 s_2^2 t_1 + \alpha_4 s_2^3 t_1 \\ & + \beta_1 s_1^3 t_2 + \beta_2 s_1^2 s_2 t_2 + \beta_3 s_1 s_2^2 t_2 + \beta_4 s_2^3 t_2, \\ g_{1,3} = & -\alpha_1 s_1 t_1^3 + \alpha_2 s_1 t_1^2 t_2 - \alpha_3 s_1 t_1 t_2^2 + \alpha_4 s_1 t_2^3 \\ & + \beta_1 s_2 t_1^3 - \beta_2 s_2 t_1^2 t_2 + \beta_3 s_2 t_1 t_2^2 - \beta_4 s_2 t_2^3, \end{aligned}$$

to ensure that the branch curves are interchanged by σ. Note that there is more than one choice of $f_{3,1}$, $g_{1,3}$ for which $T \cap (y_2 = y_3) = D$, so I can not claim T is unique in the statement of the proposition.

For a point to be fixed under σ on T one of two things must happen:

$$y_1 = y_4 = y_2 - y_3 = 0 \text{ or } y_2 + y_3 = 0.$$

The only case we need to worry about is when $y_2 + y_3 = 0$ since the other case reduces to the curve D, on which σ is fixed point free by hypothesis. We can assume $y_2 = 1$, $y_3 = -1$ and kill the weighted $\mathbb{C}^\times$-action with $i \in \mathbb{C}^\times$. Then t must be zero since the $\mathbb{C}^\times$-action by i does not affect t. This in turn implies that all the z_i are zero using the determinantal relations. Thus for a general choice of branch curve there are 4 fixed points on T which are $(\lambda, 1, -1, -1/\lambda, 0, 0, 0, 0, 0)$, where λ is a root of the quartic equation

$$\alpha_1\lambda^4 + (\alpha_2 - \beta_1)\lambda^3 + (\alpha_3 - \beta_2)\lambda^2 + (\alpha_4 - \beta_3)\lambda - \beta_4,$$

and these are 4 of the ten orbifold $\frac{1}{2}$ points on T, which proves the proposition.

6.6 G-equivariant unprojection

We would like to extend the involution on T to the Fano 6-fold W constructed in theorem (5.3.1). Even though we only have a construction of W via unprojection it is still possible to extend the involution using G-equivariant unprojection.

Suppose we start from the unprojection data

$$E \subset X \subset \mathbb{P},$$

where E is the divisor to be unprojected, X is the projected variety, and $\mathbb{P}$ is a suitable ambient space. Further, assume that X and consequently E has an action on it by some finite group G, which is compatible with the inclusion $E \subset X$. In other words E is an invariant subvariety under the group action. Then the standard short exact sequence of unprojection from section (3.4) is G-equivariant:

$$G \curvearrowright \left[0 \to \omega_X \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{I}_E, \omega_X) \to \omega_E \to 0\right].$$

Now, unprojection is essentially calculated by choosing generators $s_0, \dots, s_n$ of $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{I}_E, \omega_X)$ and then adjoining them to $\mathcal{O}_X$ as new generators. Hence if we choose G-invariant generators s_i, the G-action will still be present on the unprojected variety Y and the centre of projection will be fixed under the group action.

6.6.1 Involution on the Fano 6-fold

We begin by constructing the K3 surface T with an involution by using $\mathbb{Z}/2$-unprojection. Recall from proposition (6.5.1) that if T has a Godeaux involution σ, then there are 4 fixed points, each of which is a $\frac{1}{2}$ point, so choose one of the fixed points P and project from it. The image

$$\mathbb{P}^1 \xrightarrow{\varphi} T'_{6,6} \subset \mathbb{P}(2,2,2,3,3)$$

of this projection is a double cover of $\mathbb{P}^2$ branched in 2 nodal cubics, while the image of P in T' is a double cover of the line through the two nodes. The induced involution on T' swaps the two branch cubics and leaves the image of P invariant. Examining the projection $T \to T'$ as described in section (5.3.1), the determinantal equations for T are

$$\operatorname{rank} \begin{pmatrix} y_2 & f & z_1 \\ g & y_4 & z_3 \\ z_2 & z_4 & t \end{pmatrix} \leq 1,$$

where $f = y_1 + \alpha y_3$, $g = \alpha y_1 + y_3$ because the two branch curves are interchanged by σ. Proposition (6.5.1) fixes the involution on T as

$$\begin{gathered} f \mapsto g, \quad y_2 \mapsto -y_2, \quad g \mapsto f, \quad z_1 \mapsto -z_2, \quad z_2 \mapsto -z_1, \\ y_4 \mapsto -y_4, \quad z_3 \mapsto z_4, \quad z_4 \mapsto z_3, \quad t \mapsto -t \end{gathered} \tag{6.2}$$

and note that this implies $\sigma(y_1) = y_3$, $\sigma(y_3) = y_1$. Hence referring to section (5.3.1) the equations of $T'_{6,6} \subset \mathbb{P}(2,2,2,3,3)$ must be of the form

$$\begin{aligned} z_1^2 &= y_1 f^2 + y_2^2(l_1 f + l_2 y_2 + l_3 g) + l_4 y_2 f g \\ z_2^2 &= y_3 g^2 + y_2^2(l_3 f - l_2 y_2 + l_1 g) - l_4 y_2 f g, \end{aligned} \tag{6.3}$$

where l_i are scalars. The remaining equations of T can be calculated from those of T' using unprojection.

There are 3 isolated fixed points on T' when $z_1 = z_2 = y_1 + y_3 = 0$, which correspond to 3 of the $9 \times \frac{1}{2}$ points as expected. Further, T' has 2 fixed points on the unprojection divisor which arise from the fact that the centre of projection P was itself a fixed point. Indeed, suppose we have a local orbifold chart for a neighbourhood of the $\frac{1}{2}$ point P in T. This is just the quotient of $\mathbb{C}^2$ by $\mathbb{Z}/2$ acting by -1 on both coordinates. Then writing u, v for the coordinates of $\mathbb{C}^2$, σ lifts to the chart as

$$u \mapsto -iv, \quad v \mapsto -iu$$

by equation (6.2). The Kawamata blowup at P introduces the ratio $(u : v)$ as a copy of $\mathbb{P}^1$ and so the induced action of σ on the unprojection divisor inside T' has two fixed points at $\varphi(1,1)$ and $\varphi(-1,1)$.

It is important to note that the unprojection construction for T relies on the choice of $\frac{1}{2}$ point P. As such we can no longer assume there is a canonical choice of curve $D \subset T$ defined by setting $f = g$ as we did in the proof of proposition (6.5.1). The choice of covering curve D is made by taking any antiinvariant quadric section of T which avoids the $10 \times \frac{1}{2}$ points. In particular the quadric $f = g$ contains the point P and so is not valid.

Now I claim that the involution on T can be extended to the Fano 6-fold W at the top of the tower.

Proposition 6.6.1 *Suppose $T \subset \mathbb{P}(2^4, 3^4, 4)$ is a K3 surface with $10 \times \frac{1}{2}$ points and $\sigma \colon T \to T$ is a Godeaux involution lifted from some quadric section $D \subset T$. Then there is a lift of σ to the unique Fano 6-fold $W \subset \mathbb{P}(1^4, 2^4, 3^4, 4)$ extending T which was constructed in theorem (5.3.1). Moreover the fixed locus of the involution $\sigma \colon W \to W$ consists of 4 isolated $\frac{1}{2}$ points.*

Proof Project from one of the fixed $\frac{1}{2}$ points on T to get

$$\varphi \colon \mathbb{P}^1 \to T'_{6,6} \subset \mathbb{P}(2^3, 3^2).$$

Following the extension procedure outlined in the proof of theorem (5.3.1), the extended map

$$\Phi \colon \mathbb{P}^5 \to W'_{6,6} \subset \mathbb{P}(1^4, 2^3, 3^2)$$

must be

$$\Phi \colon (a, b, c, d, u, v) \mapsto (a, b, c, d, u^2 + 2av, bu + cv, v^2 + 2du, f_1, f_2),$$

where

$$f_1 = u\left(f + \alpha(a^2 + \alpha d^2)\right) + (1 - \alpha^2)auv + \alpha(\alpha^2 - 1)adv,$$
$$f_2 = v\left(g + \alpha(\alpha a^2 + d^2)\right) + (1 - \alpha^2)duv + \alpha(\alpha^2 - 1)adu.$$

To make Φ compatible with the lift of $\sigma \colon T \to T$ defined by equation (6.2), the action on $\mathbb{P}^5$ must be

$$u \mapsto -v, \quad v \mapsto -u, \quad a \mapsto -d, \quad b \mapsto c, \quad c \mapsto b, \quad d \mapsto -a.$$

Thus Φ is σ-equivariant, so the equations defining the image of Φ are invariant and consequently $W' \subset \mathbb{P}(1^4, 2^3, 3^2)$ can be chosen to be invariant. Alternatively, a direct calculation following the proof of theorem (5.3.1) demonstrates explicitly that the equations of the image of Φ are invariant. Hence by G-equivariant unprojection, the involution lifts to the 6-fold W.

Now outside the unprojection divisor, there are just 3 isolated points on W' that are fixed under σ. These are the same $\frac{1}{2}$ points that were fixed under $\sigma|_{T'}$. On the unprojection divisor itself there are two copies of $\mathbb{P}^2 \subset \mathbb{P}^5$ whose image under Φ are fixed by σ. These are defined by

$$\mathbb{P}^5 \cap (u = v, a = d, b = -c), \quad \mathbb{P}^5 \cap (u = -v, a = -d, b = c),$$

and they are the analogue of the two fixed points on $\varphi(\mathbb{P}^1) \subset T'$. These nonisolated fixed loci are contracted to the centre of projection P on W, so that σ fixes just 4 isolated $\frac{1}{2}$ points there. This proves the proposition.

6.7 Godeaux surfaces with torsion $\mathbb{Z}/2$

Given a hyperelliptic tower $D \subset T \subset W$ where W is the unique Fano 6-fold extending the K3 surface T, suppose the curve D is a double cover of a Godeaux curve C. Now, suppose further that the tower is constructed so that the Godeaux involution σ on D lifts to T and subsequently W as described in propositions (6.5.1), (6.6.1). Write A for the hyperplane class on W so that $\mathcal{O}_W(A) := \mathcal{O}_W(1)$ and $-K_W = 4A$. Then σ induces a $\mathbb{Z} \oplus \mathbb{Z}/2$-bigrading on the ring $R(W, A)$ according to eigenspace:

n	$H^0(W, nA)^+$	$H^0(W, nA)^-$
1	$a - d, b + c$	$a + d, b - c$
2	$y_1 + y_3$	$y_1 - y_3, y_2, y_4$
3	$z_1 - z_2, z_3 + z_4$	$z_1 + z_2, z_3 - z_4$
4		t

Now by theorem (6.3.1), we can construct a surface Y of general type with $p_g = 1$, $K^2 = 2$ as a complete intersection inside W as long as Y avoids the $\frac{1}{2}$ points of W, which is an open condition. Referring to the above table and the eigenspace decomposition on Y given in section (6.2.2), if we take

$$Y = (1^+, 1^+, 1^-, 2^-) \cap W$$

then $\sigma|_Y$ will be the fixed point free Godeaux involution on Y. Hence we have:

Theorem 6.7.1 *There is an 8 parameter family of Godeaux surfaces with $\mathbb{Z}/2$-torsion.*

The parameter count is a matter of calculating the moduli of W using theorem (5.3.1), section (6.6.1) and then counting the number of free parameters involved in choosing the complete intersection $(1^+, 1^+, 1^-, 2^-)$.

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