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QUALITATIVELY DIFFERENT APPROACHES TO SIMPLE ARITHMETIC

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Ph.D Thesis in Mathematics Education

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Tables are presented in plain type and figures given in italics.

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Eddie Gray, April 1993

DECLARATION

I declare that the material within this thesis has not been presented in any other thesis. Chapters 4, 5 ,6, 7, and 8 contains material which has formed the basis of several papers:

Gray, E.M.: 1991(b), 'An analysis of diverging approaches to simple arithmetic: preference and its consequences', *Educational Studies in Mathematics*, 22, 551-574.

Gray, E.M.: 1993, 'Count-on: The Parting of the Ways for Simple Arithmetic', *Proceedings of PME XVII*, Vol. I pp. 204-211. Tsukuba (Japan).

Gray, E.M. & Tall, D. O.: 1991, 'Duality, ambiguity & flexibility in successful mathematical thinking', *Proceedings of PME XV*, Vol. II pp. 72-79. Assisi (Italy).

Gray, E.M. & Tall, D. O: (in press), 'Duality, ambiguity and flexibility: A proceptual view of mathematical thinking', *Journal for Research in Mathematics Education.*

I acknowledge David Tall's guidance and assistance which was seminal to the publication of Gray & Tall (1991).

SUMMARY

This study explores the qualitative difference in performance between those who are more successful and those who are less successful in simple arithmetic.

In the event that children are unable to retrieve a basic number combination the study identifies that there is a spectrum of performance between children who mainly use procedures, such as count-all in addition and take-away in subtraction, to those who handle simple arithmetic in a much more flexible way.

Two independent studies are described The first contrasts the performances of children in simple arithmetic. It considers teacher selected pupils of different ability from within each year group from 7+ to 12+. It takes a series of snapshots of different groups of children and considers their responses to a series of simple number combinations. This first experiment shows qualitatively different thinking in which the less successful children are seen to focus more on the use of procedures and in the development of competence in utilising them. The more successful appear to have developed a flexible mode of thinking which is not only capable of stimulating their selection of more efficient procedures but, the procedures they select are then used in an efficient and competent way.

However, the use of procedures amongst the more successful is seen to be only one of two alternative approaches that they use. The other approach involves the flexible use of mathematical objects, numbers, that are derived from encapsulated processes. The below-average children demonstrate little evidence of the flexible use of encapsulated processes.

It is the ability of the more able children to demonstrate flexibility through the use of efficient procedures and/or the use of encapsulated processes that stimulates the development of the theory of procepts. This theory utilises the duality which is ambiguously inherent in arithmetical symbolism to establish a framework from which we may identify the notion of proceptual thinking.

The second study considers the development of a group of children over a period of nearly a year. This study relates to aspects of the numerical component of the standardised tests in mathematics which form part of the National Curriculum. It provides the data which gives support to the theory and provides evidence to confirm the snap shots taken of children at the age of 7+ and 8+. It indicates that children who possess procedural competence may achieve the same level of attainment as those who display proceptual flexibility at one level of difficulty but they may not possess the appropriate mental tools to cope with the next.

The evidence of the study supports the hypothesis that there is a qualitative difference in children's arithmetical thinking.

CHAPTER 1

DEVELOPMENT OF THE STUDY

1.1 THE RESEARCH RATIONALE

Mathematics has been notorious over the centuries for the fact that so many of the population fail to understand what a small minority regard as being almost trivially simple. Within the context of school mathematics in Britain, the overwhelming impression gained by those involved in the CSMS research (Hart, 1981), which shows that for some understanding improves only slightly as the child gets older, "is that mathematics is a *very difficult* subject for most children" (p. 209). In the context of arithmetic this impression would seem to be substantiated by the results of the APU studies (D. E. S., 1980 –1989) which indicate that the lower 10% to 35% of eleven-year-old children were unable to perform many of the arithmetic tasks that pupils were regarded as capable of doing by the time they moved to secondary school. Indeed, the serious difficulties that many children have with coping with elementary arithmetical operations were reported as early as 1935 by Brownell.

It is the observation of such difficulties that provides the rationale for this study. Earlier observations (Gray, 1988b) had indicated that there appeared to be a link between the way in which children responded to basic number combinations and the difficulties they were experiencing with addition and subtraction algorithms . For example, children who relied extensively on counting to obtain the number combinations to twenty frequently tended to compute their two and three digit addition and subtraction problems from left to right. Focusing particularly on the basic number combinations (Gray, 1988a) provided a notion of the differences in behaviour in children's attempts to obtain solutions to these combinations if they could not retrieve them from memory.

Consider Andrew who was eight. He placed unifix blocks one by one on his fingers to represent the amounts to be added when resolving the solutions to addition

combinations to ten. He then recounted the combined set by removing the unifix blocks, one at a time, from each finger. Neil, aged 9, used his fingers for combinations to ten and integrated the use of his fingers with counters when finding a solution to 20 – 5. To obtain her solutions, Mary, 9, counted imaginary lines in her head. In contrast, Andrew, 7, knew all of the addition and subtraction combinations to twenty whilst Jonathan, who was also 7, knew many of them but others were obtained through the use of counting or the flexible use of other known combinations. To obtain the solution to 17 – 6, for example, he counted down six from seventeen. When attempting 20–5 he knew that "fifteen add five is twenty so twenty take away five is fifteen".

Such differences, perceptible amongst children of different ages, obviously indicate that some know basic number combinations whilst others do not. But, when they don't know solutions why, for example, should a nine-year-old be demonstrating an approach which contrasts so sharply with that of a child two years younger when both are attempting 20 – 5? What is it that suggests to Neil that he should count, that the problem should be interpreted as the integrated use of fingers and counters to represent twenty and then counting off five from this twenty and recounting the remainder, whilst Jonathan sees the problem quite differently? This study suggests that the answer lies in the fact that the children are doing qualitatively different arithmetic.

Gray (1991) revealed the divergence that may be observed when children solve simple number combinations when they do not know the solution. This led to initial conclusions about the preferences children had when dealing with unknown combinations. What is interesting was that the preferences indicated by the children show that some wished to remain at a procedural level, which, in terms of information processing made things very difficult for them, whilst others operated at a conceptual level which was very flexible. It fact what the children were revealing, without articulating, was a divergence based upon their procedural and conceptual thinking (Hiebert & Lefevre, 1986). Through this divergence it appeared as if the less able, who

relied on procedures, were "making things more difficult for themselves and as a consequence become less able"(Gray, 1991, p. 570) In contrast, "condensing the long sequences (of procedures) appears to be almost intuitive to the above-average child"(ibid.)

Several different studies (e.g. Dubinsky, 1991; Sfard, 1989) suggest that such differences may occur because some children go through a process of 'procedural encapsulation' whilst others do not. This study proposes that there is a more supple underlying theory that relates not just to conceptual differences but to the versatility that may be achieved through the flexible use of process and concept. Such differences suggest that the arithmetic of the successful is conceived in such a way as to be, for them, relatively simple, whilst the less successful are doing a different kind of arithmetic which is often intolerably hard.

1.2 THE THEORETICAL FRAMEWORK

Piaget (1971) believed that children gave us the best chance of studying, amongst other things, the development of mathematical knowledge and physical knowledge. He drew the distinction between the two by indicating that mathematical knowledge is abstract whilst physical knowledge, "knowledge based on experience in general – is concrete" (p.16). His belief was that the abstraction that is mathematical knowledge stemmed not from abstraction from the objects themselves but from the abstraction of the action carried out upon the objects.

Underpinning the Piagetian notions of the cognitive development in mathematics are the means through which a coordinated series of actions, for example, counting, became objects of thought, for example, the number '5'. Using objects as referents is qualitatively different to using representations of numbers as if they were objects. Piaget referred to the notion of "reflective abstraction" (Piaget, 1973) to indicate the cognitive shift from action to object. The notion of reflective abstraction may be linked

to other notions that attempt to account for actions as processes becoming objects of thought, or concepts; it has been variously termed "encapsulation" (Dubinsky, 1991), "entification" (Kaput, 1982), and "reification" (Sfard, 1989).

Within this study the action through which a process is carried out – the counting procedure – and the means whereby the counting process is encapsulated, as first a number concept and then as the concept of sum, provides a framework from which a theoretical appraisal of the link between static conceptual entity and dynamic process is made. The analysis grows out of Piaget's work as exemplified by Sfard (1989, 1991) and Dubinsky (1991) and it is placed within the field of information processing as exemplified Klahr & Wallace (1976), Greeno (1983) and Kaput (1982).

1.3 SIMPLE ARITHMETIC AS AN AREA OF STUDY

Abstraction from the use of physical objects forms the background for the conceived cognitive development of simple arithmetic (e.g. Piaget, 1965, 1973; Steffe *et al*, 1981, Kamii, 1985). An analysis of the literature into the development of number concepts amongst young children indicates that counting plays a sophisticated and central role in the procedural encapsulation of number (e.g. Steffe *et al*, 1982; Wagner & Walters; 1982, Fuson & Hall, 1983; Gelman & Meck 1986a); the sequence of number words become part of a procedure to point at successive elements; each number word is uttered in turn until the last word is identified as the number of elements in the collection; an object is produced through a counting process.

Children's simple arithmetic is an area of research that has received considerable attention in recent years. Not only has this attention provided a wealth of information about children's cognitive development, but it has also provided a wealth of information about the different approaches they use to produce the solutions to basic number combinations within both contextual and context free situations (e. g. Carpenter & Moser, 1982; Gray, 1991) and within information processing paradigms (e.g. Groen

& Parkman, 1972; Ashcraft, 1982). Thus, children's solution strategies are well documented and relatively well understood and yet, children still find it difficult to learn number combinations.

1.4 FRAMEWORK FOR A THEORY

Within the context of simple arithmetic, the notion of procedural encapsulation is somewhat complicated because as well as carrying out a procedure to establish conceptual understanding of number some children learn specific facts by rote which can muddy the theoretical waters, in the sense that known facts may be either rote or meaningful (in the sense of Ausubel, 1968). The distinction between a rote learned fact and a fact that is known, and, then used to establish one that cannot be retrieved from memory, is a distinction that may be identified in the flexibility a child may exhibit to produce number triples. For example, $5 + 3 = 8$ and $3 + 5 = 8$ which in turn incorporate the subtraction combinations $8 - 3 = 5$. The object 8, may be flexibly decomposed and recomposed $5 + 3 = 4 + 4 = 8$.

The theoretical notion of *procept* (Chapter 6) aids the description of the flexibility attached to the phenomena of decomposition and recomposition which is inherent within encapsulated mathematical objects. Thinking that projects the ability to use this flexibility through the use of a procedure, an encapsulated procedure or the flexible compression and decompression of encapsulated procedures, *proceptual thinking*, provides a framework for discussing qualitatively different thinking.

Within the United Kingdom the imposition of a National Curriculum (1989) is aimed at "raising standards" of performance in all subjects, including mathematics. The requirements of this curriculum distinguish between the skills or procedures that an individual needs to have acquired in order that they can *do* things, and the concepts or basic facts which they are expected to know on which they operate with their skills. Such a requirement in itself suggests a fundamental dichotomy between procedures and

concepts, between things to *do* and things to *know*. It is conjectured that it is through their applications of doing or knowing that children display qualitatively different forms of mathematical behaviour.

1.5 STATEMENT OF HYPOTHESIS

In this study I do not pretend to 'prove' the theory of procepts, but I do offer specific hypotheses and evidence which are consistent with the theory.

From the theoretical construct of procept it is hypothesised that there is a qualitative difference between children which is:

(i) on the one hand manifest as a spectrum of performance in the operations on numbers as a procedure that relates to counting and

(ii) on the other, the flexible manipulation of procepts.

Identifying these distinctions leads to an analysis of the attempts of two samples of children to solve basic addition and subtraction combinations. From the evidence 'qualitative difference' is considered against the following operational hypotheses which are consistent with the theory that is developed :

1. Count-all does not lead to an encapsulated process therefore it is very unlikely that children who use count-all will also use a derived fact.

2 Children who remain procedural and use count-on for addition are more likely to use count-back for subtraction.

3. Where children are using count-on for addition it will be seen that:

   (i) Procedurally oriented children will generally use a count-back procedure for subtraction

(ii) The proceptually oriented children will use count-back or count-up in a flexible, and more efficient manner.

4. Children who are procedurally oriented are likely to show evidence of improving procedural competence.

5. Less successful children who demonstrate procedural competence may not demonstrate this competence through procedural flexibility.

6. Children who think proceptually will demonstrate the flexible use of symbol as process and object. More specifically they will do it through the use of derived facts.

7. More successful children will provide evidence of the more efficient use of procedures than do the less able.

8. Procedurally oriented children will take longer than children who display proceptual flexibility to acquire basic number facts.

9. Even when they know proportionately the same number of facts as the younger more successful able children, older less successful children will not use them to derive new number facts to the same extent.

The implications behind several of the operational hypotheses indicated above is that although procedural competence improves with the age of the child, and this in turn may increase procedural efficiency, some children are in fact improving in a technique which in the longer term may not provide them with the competence to cope with more subtle situations. Because children are faster at procedures it does not imply that they are conceptually better at them. On the other hand, children who display procedural efficiency coupled with procedural flexibility may be at the interface of procedural encapsulation.

It is hypothesised that children's qualitatively different thinking in simple arithmetic, as evidenced through the notions of proceptual and procedural thinking, may be explained through mathematical symbolism which inherently projects process/object ambiguity. To support the thesis two experiments are used. Through individual interviews, the first considers groups of children of different abilities over a six year span to reveal

evidence of the diverging approaches to simple arithmetic. The second establishes that the divergence revealed through the snap shots of the first experiment provide an indication of the behaviour of a group of seven and eight year old children with a ten month time gap between interviews. The conclusions not only support the view that there are qualitative differences in children's arithmetical thinking, but they indicate that children who possess procedural competence may achieve the same levels of attainment as those who think in more flexible ways, However, the conceptual differences are such that the quality of their arithmetical thinking may differ to such an extent that the long term prognosis points to very different outcomes.

It is not the intention of the study to speculate on relationships that are too complex for application within the classroom. It is the intention to provide a tool which may go some way forward in helping our understanding of why some children are successful in simple arithmetic and others are not.

1.6 STRUCTURE OF THE STUDY

Two chapters are presented which review the research literature. Chapter 2 considers theories that relate to cognitive development whilst Chapter 3 looks particularly at the cognitive development of simple arithmetic.

Chapter 2 acknowledges the existence of a wide range of influences on children's cognitive development. It considers this development from three perspectives; the organisation of mathematical knowledge, the formation of mathematical knowledge and mental knowledge structures and the representations which lead to mathematical behaviour. Piagetian notions of cognitive development emphasise the need for actions to become seen as objects of thought. The process/object interface is placed within the context of current theories on procedural encapsulation and the role that symbolism may play in ambiguously representing a process that may become an object of thought.

Chapter 3 extends the discussion in Chapter 2 to the cognitive development of simple arithmetic. It presents a review of the literature into the development of children's understanding of number and the way in which this understanding is used in number fact acquisition. Underlying this development is the notion of process encapsulated as object; the process of counting encapsulated into first the concept of number and secondly the concept of sum. Within the discussion different approaches that children may use to resolve number combinations are considered.

Chapter 4 and Chapter 5 consider the method and analysis of the results of the first experiment. This was designed to examine whether or not a divergence in arithmetical thinking may be determined through the procedural and conceptual approaches that children employ to solve basic number combinations. It is clear from the analysis presented that two issues clouded the distinction. Firstly those who may be regarded as conceptual thinkers also used procedures, whilst all children used known facts to a greater or lesser extent.

The more subtle nature of the dichotomy that exists between those who, in contemporary terms, would be defined as procedural thinkers and those who would be defined as conceptual thinkers is discussed within Chapter 5. Here, the "duality ambiguity and flexibility" (Gray & Tall, 1991, 1993a,) which may arise from, and is inherent within, mathematical symbolism is placed within a perspective which leads to the notion the cognitive construct of *procept* (Gray & Tall, 1991). This notion is then used to discuss the theoretical development of simple arithmetic from a proceptual perspective.

Because the first study is seen as a snapshot of different children, the second study, reported within Chapter 7, considers the arithmetical thinking of children of different arithmetical attainment. The ability of the children was identified through their response to numerical aspects of the 1992 Key Stage 1 Standard Attainment Tasks in Mathematics. This second study provides data which supports the theory. The analysis presented within the theoretical framework of procepts concludes that there is

qualitatively different thinking between children who are identified by ability through their level of arithmetical attainment.

Chapter 8 reflects upon the conclusions of this thesis within the realms of simple arithmetic and suggests a broader development of the theory of procepts in other areas of mathematics which opens up avenues for future research action.

Comment

Several areas of the study have been the subject of published papers or of papers currently in press.

Chapter 5 draws fairly extensively on an article within *Educational Studies in Mathematics* (Gray, 1991), whilst Chapters 5 and 6 formed the basis for an article to be published in the *Journal of Research in Mathematics Education* (Gray & Tall, in press).The pivotal role that count-on plays in the development of thinking in simple arithmetic forms the focus of a paper be published within the proceedings of *PME XV11* (Gray, 1993).

CHAPTER 2

MATHEMATICAL THINKING: A REVIEW OF THE LITERATURE

2.1 INTRODUCTION

Following their evaluation of the instructional principle of structure-oriented mathematics, Resnick & Ford (1981) address an interesting issue:

> "When one teaches a part of a larger mathematical structure, even in a mathematically "honest" way, is the partial structure not qualitatively different from the way it is later to be understood as a component of the whole mathematical edifice. In other words, can we be sure that such an approach is not creating double work for the child?".
>
> (Resnick & Ford, 1981, p124)

Resnick & Ford place the notion of structure in mathematics within a context which views mathematical knowledge as a body of knowledge which is internally organised and interrelated. For them, an important relationship that needs a sharper focus is that which exists between formal mathematical structures and the intuitive psychological structures that enable children to use and acquire mathematical knowledge efficiently and flexibly. Behr *et al* (1992) conjecture that intuitive knowledge stems from the possession of an organised set of associations, propositions and relations and is similar to the qualitative knowledge that an individual has about a situation. They describe qualitative knowledge as "...knowledge that "belongs" to the individual, is constructed from real experience and provides for considerable flexibility in thought" (p.321).

Whilst acknowledging the existence of a wide range of influences on a child's cognitive development, this chapter largely focuses on mathematics learning and thinking. Theoretical notions of cognitive development stem from a Piagetian standpoint. The perspective then moves towards information processing paradigms from which contemporary views of mathematical thinking are considered. Historically there have

been various ways of describing and distinguishing between different but complementary forms of mathematical thought; Piaget (1970) used the notions of "figurative and operative", Skemp (1978) introduced the terms "instrumental and relational" and more recently Hiebert & Lefevre (1986) consolidated the contemporary views of mathematical thinking into notions they described as "procedural and conceptual". Since pairs of these notions help to describe a quality of thinking that may be ascribed to mathematical actions they are also used within this study underscore the intuitive psychological structures possessed by children within the first experiment. Drawing upon the work of, for example, Dubinsky (1991) (who suggests that sophisticated mathematical structures can be found in the thinking of young children) and Sfard (1989, 1991), the later part of the chapter considers the cognitive shift implied by the encapsulation of process as object and the role mental representations may play in holding mathematical information.

2.2 INFLUENCES ON MATHEMATICAL BEHAVIOUR

Many issues have roles of different importance to play in the development of maturity in mathematical thinking. Gruszczyk-Kolczynska & Semadeni (1988) indicate the importance of mutual connections that should be made between levels of operational thinking, manual skill, perception and emotional maturity in the development of young children's understanding of number. They see operational thinking as the ability to handle symbolism and manual skill as the ability to handle, for example, the counting process. Such concerns could, of course, be firmly placed within the behaviourist tradition. Within such a framework behaviour is specified and analyse it into components but von Glasersfeld (1988) believes that such a tradition has no room for what is ordinarily called 'understanding'. Since Gruszczyk–Kolczynska & Semadeni's wider considerations are to be seen within the context of the child's level of understanding it would be an injustice to suggest that are purely behaviourists. However, by identifying some of the components that play a part in mathematical development they give some notion of the wider context within which mathematical

behaviour may be viewed. Amongst those influences that affect the way in which the young child develops mathematically Gruszczyk-Kolczynska & Semadeni specify the home, interpersonal relationships made by the child with adults outside school and the perspectives within school. However, the child's social context is only one context – there are cultural contexts too.

Smith, Stanley & Shores (1971) consider the relationship between the culture of the society and its context within a school curriculum. The emphasis arising from such a relationship may well be on a cultural core and its values and its sentiments, together with, for example, the "knowledge, skills and understanding" (D.E.S., 1989) associated with them. Nickson (1992) points out the danger of assuming that there is only one culture associated with the mathematics classroom. For example Cobb (1987) indicated that it was possible to relate children's mathematical behaviour to social interaction patterns that typified classroom life during arithmetic instruction whilst teacher's hidden perspectives (e.g., Bassam, 1962; Begle, 1979; Desforges & Cockburn, 1987) and those of pupils (e.g. Biggs, 1967; Desforges & Cockburn, 1987; Jaworski, 1989) also need consideration.

Cobb (1988) draws the conclusion that students' mathematical and social cognition's are interdependent; the process of accounting for students' mathematical activity involves co-ordinating analyses of their mathematical and social cognition's. He clarifies this by extending the notion of "context" to include, for example, "an understanding of the institutionalised, taken for granted activities and practices that give rise to observed patterns and norms" (p. 9).

2.3 THE ORGANISATION OF KNOWLEDGE

That there was qualitatively different thinking in the kinds of learning engaged in by individuals was a conclusion drawn by Katona (1967). To him learning did not consist merely of memorising a set of associations or a procedure but it could also mean the reorganisation of information so as to form a structure that had the power to explain

other similarly structured problems. An important conclusion that may be drawn from his work is that people who remember the principle for generating knowledge have less to remember than those who do not.

The influence of gestalt thinking is clear in Katona's explanations: knowledge that is organised into a structured whole is retained as part of that whole. Though each part may be remembered because of its place within the whole, the whole is greater than the sum of its parts. Gestaltists believe that such an organisation of learning may lead to more efficient remembering partly because it decreases the number of separate pieces of information that must be retained. However, from a cognitive point of view, the notion of gestalts appears to be linked to notions of mathematical structure in such a way that they do not provide us with insight into the ways in which children acquire knowledge and mentally represent it in the way that notions of concept and schema do.

At its simplest, Skemp (1971) sees a concept as an idea, whilst Vergnaud (1988) identified a concept is a triplet of three sets:

(i) the set of situations that make the concept meaningful in a variety of aspects,

(ii) the set of operation invariants (properties, relationships, objects, theorems in action) that are progressively grasped by students in a hierarchical fashion,

(iii) the set of linguistic and non linguistic symbols that represent those invariants and are used to point at them, to communicate and discuss about them, and therefore to represent situations and procedures.

(Vergnaud, 1988, p. 45)

The notion of concept may provide us with some useful insights into mathematical behaviour which to Vergnaud (1988) is a behaviour that relies on some mathematical ideas. He considers that different kinds of mathematical behaviours and different levels of such behaviours are tied to understanding of mathematical concepts and that such understanding is only implicit through the child's behaviour.

The distinction between the idea and its name is a theme developed by Skemp. He believes that naming, which is identified as the sound, sign or symbol associated with the concept, can play a useful, indeed sometimes an essential part, in the formation of new concepts and its communication. Skemp indicates that hearing the same name in connection with different experiences helps us to collect ideas together in our minds and "helps us abstract their intrinsic similarities" (p. 23). However, just as Vergnaud draws a note of caution to indicate that it does not mean that a child is fully aware of the relationship between the ideas and the way they behave just because a concept exists and a child uses it, Skemp draws a note of caution within the context of naming a concept:

> The criterion for having a concept is not that of being able to say its name, but that of behaving in a way indicative of classifying new data according to the similarities which go to form this concept.
>
> (Skemp, 1971, p. 27).

The interrelationship of concepts Skemp called a "schema" (p.37). As with the notion of gestalt, the idea of a schema extends beyond the separate properties of its individual concepts. A schema functions as the integrator of existing knowledge, a tool for future learning and an enabler for "relational" understanding and, as such, it may be considered as a major instrument in adaptability. However, Skemp indicated that whilst a schema may be the most effective organiser of existing knowledge, its very strength may be the source of its potential downfall; a strong tendency may emerge towards the self perpetuation of existing schema. It may then be necessary to change the structure of the schema. Skemp signifies that this may be difficult and, if it fails, the new experience can no longer be successfully interpreted. Adaptive behaviour may break down.

> The central importance of the schema as a tool of learning means that inappropriate early schema's will make the assimilation of later ideas much more difficult, perhaps impossible.
>
> (Skemp, 1971, p. 48)

2.4 THE ACQUISITION OF MATHEMATICAL KNOWLEDGE

2.4.1. The Piagetian View

The observation of children's mathematical actions form a central component of this study. It is from such actions that it is believed that qualitative differences in thinking may be inferred. In this sense the study follows a Piagetian approach although the outcome is a much coarser grained analysis than that given by Piaget.

The formation of mathematical knowledge and the possession of knowledge identified by mathematical behaviour was one aspect of the work of Piaget. He worked within the realm of genetic–epistemology which "deals with both the formation and meaning of knowledge" (Piaget, 1971, p.12). The concern of genetic–epistemology was to resolve the question "By what means does the human mind go from state of less sufficient knowledge to a state of higher knowledge?" (ibid.). Piaget believed that learning as well as performing mathematics was a matter of active thinking and of operating on the environment. It was not a matter of passively noting or even of memorising what is presented.

The basic processes that underpinned the ability to think mathematically Piaget termed the 'logico-mathematical' experiences which he considered gathered information, "not from the physical properties of particular objects, but from the actual actions (or more precisely their coordinations) carried out by the child on the objects" (Piaget, 1973, p.80). Thus, activity with objects was seen as indispensable to the comprehension of arithmetical relations. Whilst the importance of logico-mathematical experiences were noted, Piaget also felt that these were strongly linked to 'physical experience' which "consists of acting on objects to discovery the properties of the objects themselves" (Piaget, 1973, p.80).

The observation and co-ordination of actions are central to the Piagetian view of the development of mathematical thinking. Indeed, he thought that all humans would

develop certain structures of thinking as long as they maintained a normal interaction with both the social and physical environment. Structure was defined as "a totality; that is it is system governed by laws that apply to the system itself and not only to one or another element in the system" (Piaget, 1970, p.22). The term was used as a means of describing the organisation of experience by an active learner; the system of whole numbers was identified as an example. However, many different mathematical structures could be discovered in this "number" system, for example, the additive group with its rules for associativity and commutativity, whilst number itself was seen as part of a larger system which includes fractions. Piaget saw the notion of number resting on the combination of two primitive structures, class inclusion, in the sense that two is included in three and three is included in four, and relationships of order, in the sense that elements in a set are counted one after the other. Thus any structure could be seen as part of a larger structure. Resnick & Ford (1981) indicate that building structures in the Piagetian sense appears to involve constructing relationships such that change in any part of the system effects the whole system. They identify Piaget's notion of "scheme" as a "small scale version of structure" (p.182); "whatever is repeatable and generisable in an action" (Piaget, 1971, P42). Piaget, then, sees the notion of scheme in terms of organised mental and physical patterns of behaviour. Each scheme is itself a co-ordination of a number of sub schemes; it is the relationship of class inclusion – the sub schemes included within the total scheme – which gives rise to concepts. Counting may be seen as an example. When called upon to count a child quantifies the collection numerically only when all of the objects are put into a single relationship, synthesising (reciprocal assimilation) the scheme of ordering and the scheme of class inclusion. Through such action the child eventually develops the concept of number. Vergnaud's (1988) sense of concept would appear to arise from the Piagetian view whereas Skemp's concept to schema relationship appears to be the reverse of Piaget's scheme to concept. However, the important departure of these two from Piaget is that, in the mathematical context, they not only see concept in terms of

naming but also in terms of symbolic representation: "a concept is not a concept until it has a name and one or more symbolic representations" (Vergnaud, 1987, p. 52).

One of Piaget's concerns was the means through which the co-ordination of actions, for him the roots of mathematical structures, became mental operations and how these operations became structures. An operation is seen as a special kind of mental action—it could be reversed. He identified it as "an action that can be internalised" (Piaget, 1971, p.21), spoke of "actions that are destined to become interiorized as operation" (Beth and Piaget, 1966, p.251) and indicated that an operation can be carried out in thought as well as executed materially; "actions or operations become thematised objects of thought or assimilation" (Piaget, 1985, p. 49). "Interiorised actions" are the mental or intellectual operations that stem from actions. However, he believed that once this interiorisation, with the co-ordination it supposes, is sufficient for deductive thought then logico-mathematical experience in the form of material actions is no longer necessary and interiorised deduction is sufficient. Further, whilst coordinations of actions and logico-mathematical experiences may interiorise themselves they also give rise to logico-mathematical abstraction which he termed "reflective abstraction" (Piaget, 1973, p.81). Such a form of abstraction reflects both the process through which action is projected to thought or mental representation, and the sense of reorganisation of mental activity which reconstructs at a higher level everything drawn from the coordinations of actions.

Reflective abstraction is seen as essentially the self referential use of existing structures to construct new ones by observing ones thoughts and abstracting from them. It involves the construction of relationships between and amongst objects but such relationships may not have an existence in external reality; it only exists in the minds of those that can create it between objects. Kamii (1985) suggests that 'constructive abstraction' might be a more appropriate term than reflective abstraction since this would indicate that this form of abstraction is a construction of the mind rather than something that exists in objects.

Underscoring Piaget's theory of intelligence and mental development is the belief that children's sophistication in thinking increases as they become older; children do not just acquire more knowledge but they also develop more complex cognitive structures. Evidence of qualitatively different cognitive structures was drawn from the interpretations of observations on children engaged in mathematical tasks. These structures were not only seen as evidence of different understandings which led to the solutions of the tasks but they were also believed to develop in sequence encompassing defined stages.

From a Piagetian standpoint we note that the fundamental assumption is that new knowledge is in part constructed by the learner through the use of "active methods" and these "require that every new truth to be learned be rediscovered or at least reconstructed by the student" (Piaget, 1976, p.15). von Glasersfeld (1988) indicates that Piaget took cognitive construction for granted. Such a view is a basic assumption of cognitive learning psychology and in the field of early number development can be seen in the work of, for example, Groen & Resnick (1977), Steffe *et al* (1982), Kamii (1985).

The importance of the Piagetian perspective for this study rests in the type of thinking that children acquire from actions that may be identified as 'logico-mathematical experiences'. The real issue is whether all children who display competence with experiences which are almost 'partial structures', for example, count-on, actually undergo the process of reflective abstraction to internalise the action, or whether for some competence with the experience is an end within itself.

2.4.2 The Constructivist Perspective

Piaget's work established the basis for the constructivist perspective and Vergnaud (1987) described him as the "most systematic theorist of constructivism" (p. 43). Jaworski (1988) defined constructivism as "an abstract philosophical stance about

knowledge and its relation to the world and to people's attempts, through their experiences, to try and rationalise the world" (p.292). The philosophy has, she believes, important consequences for mathematics education although some commentators "share a wary and cautious approach to it, especially the radical variety" (Wheeler, 1987 p.63) and indicate that the "virtue some constructivists need most is that of humility" (Kilpatrick, 1987, p.22).

von Glasersfeld, who I believe is currently one of the most formidable intellectual advocates of constructivism, indicates that the task of education is

> a task of inferring first of all models of the students conceptual constructs and then generating hypotheses as to how students could be given the opportunity to modify their structures so that they lead to mathematical actions that might be considered compatible with the instructors expectations and goals.
>
> (von Glasersfeld, 1988, p.6).

The constructivist approach to education "is predominantly interested in the students conceptual structures and operations and focuses on behavioural manifestations only insofar as they serve the teacher or experimenter to infer the student's understanding" (ibid., p. 7). It is this notion that has guided Steffe *et al* as they take on a mainly Piagetian conception of knowledge to present the epistemological position of constructivism as a key to the way in which concepts are organised. Constructivists believe that children are fundamentally constructivist rather than analytic, and, like Piaget, they place an emphasis on the need for reflective abstraction.

The one implication of Piaget's work, and that of the constructivists, is that the knowledge and beliefs that students bring to a given learning situation can influence the meanings they construct in that situation; a child's mathematical knowledge is viewed as co-ordinated schemes of action that are functioning reliably and effectively. Questions central to the thesis of this study evolve around the meaning that may be placed on 'reliably and effectively'. More particularly, if a co-ordinated scheme of actions is functioning in such a way, is it a limiting action or is it an action that

promotes reflective abstraction? Is the observed behaviour stemming from such action an indicator of the inflexible use of a procedure or of flexible thinking? As an additional point, a constructivist perspective of children's cognitive growth in counting (Steffe et al, 1981) will be considered in Chapter 3.

2.5 STRUCTURING THE USE OF KNOWLEDGE

2.5.1 Remembering

Byers and Erlwanger (1985) indicate that one of the outcomes of learning is remembered knowledge. Although Piaget and the constructivists are interested in the formation of conceptual structures they are less interested in questions which provide some notion of how knowledge is stored in the mind and cued from memory; they have not explored the nature of conceptual representations. However a central tenet they hold is that memory is a constructive or reconstructive process. Complex information is structured to impose some meaning on it and this implies modification of the information that can be remembered. Ausubel (1968) did not accept the full implications of Piaget's stage theory but he accepted his notions of assimilation and accommodation. For him meaningful learning was a process through which new knowledge was absorbed by connecting it to some existing relevant aspect of the individual's knowledge structure. If there were no relevant concepts already in the mind to which the new knowledge could be linked it would have to be learned by rote and it would have to be stored in an arbitrary and disconnected manner.

One advantage of the inclination to create connections between knew and existing knowledge is that well connected knowledge is better remembered (Baddeley, 1976; Bruner, 1960). Memory, if viewed as a reconstructive process, involves the same cognitive activity as understanding. Indeed, this may well be the rationale for Skemp's (1978) distinction between instrumental and relational knowledge. An entire network of knowledge is less likely to deteriorate than an isolated piece of knowledge and retrieval

of information is enhanced if it is connected to a larger network; there are more routes of recall. The possessor of a conceptual structure established through learning relational mathematics "can (in principle) produce an unlimited number of plans for getting from any starting point...to any finishing point" (Skemp, 1978, p. 25).

2.5.2 Information Processing

The structure of knowledge in the mind and the mechanisms by which that knowledge is manipulated, transformed and generated is the focus of attention of information processing methodologies. Through such processes, knowledge structures can be hypothesised for particular learners and the learner's performance on mathematical tasks used to verify the content and organisation of mathematical knowledge. The resulting network structure models account for how people know things without having to list every separate item of knowledge.

Davis (1983) sees the ability to match input data with some kind of knowledge representation structure stored in memory as fundamental to the ability to make links between two areas of mathematics. He has postulated four steps which offer a typical information processing explanation of the process. After cues are used to trigger from memory some specific knowledge representation structure, a search through the specific present input enables certain specific information to enter into the "slots" or "variables" that exist within that structure. Following checks to evaluate the suitability of this process, and a cycling back if necessary, then the result is used for the next stage in the information processing. Within the context of this study this view of information processing can relate to the notion of the strategies children may use to obtain solutions to addition and subtraction combinations; the cues are taken from either written or verbal symbolism and related to the underlying knowledge structure possessed by the child and then these used for an appropriate solution strategy.

Information processing paradigms are in essence seeking to provide models of the relationship between working memory, where actual actions are carried out and where operations are performed on information, short term memory, where information is stored to be used in a relatively short period of time, and, long term memory, where everything a person knows is stored. Short term memory plays a crucial part in the actions carried out by individuals. Resnick & Ford (1981) believe that only through being processed in working memory can information from the sensory part of the system enter a person's long term memory; only when information is called out of the person's long term memory can stored information be used in the course of thinking. The most important aspect of short term memory is its limited capacity. Miller (1956) indicated that despite the huge amounts of information humans can remember in general, they can only keep and operate on about seven "chunks" of information in short term memory. It is possible to retain information in short term memory through "rehearsal" (Resnick & Ford, p. 31), but this does not increase its basic capacity. They suggest that "automacity" and "chunking" may be ways of extending working memory's processing capacity.

There have been strong suggestions that automacity, or automatic response to basic number combinations, can prevent competition for space between them and higher level problem solving processes. Indeed, the importance of automacity had been indicated by Thorndike (1922) who saw the basis of learning as the internalisation of facts best accomplished by means of didactic instruction and drill. Part of the means of achieving this end was to ensure that children leant number combinations as "the association between two digits and a response" (Baroody & Ginsburg, 1986, p. 100). Such a belief led to such information processing models as that of Cambell & Graham (1985) whose autonomous fact retrieval system, labelled the "arithmecon", saw each fact stored as an independent item of knowledge each distinct and bearing no reference to any other.

Any view based on autonomous systems is in contrast to those which hold that well structured knowledge is the extent to which concepts are associated with each other in rich and orderly ways. In an integrated knowledge structure certain concepts are central in that they are associated with a large number of other concepts. Larkin (1977) pursued the notion that internal integration effects temporal patterns. Like Miller, he too used the idea of 'chunk' to identify the information grouped around an organising concept. Simon (1980) pointed out that a "chunk is any perceptual configuration (visual, auditory, or what not) that is familiar and recognisable. For those of us who know the English language, spoken and printed words are chunks....."(p. 83). The notion of chunk provides meaning to the aggregation of tiny "meaningless" bits of information – "small pieces of input data are suddenly linked up with an important memorised data representation structure" (Davis, 1983, p. 264). To avoid loss of information during working memory processes, Harel & Kaput (1991) indicate how large units of information must be chunked into single units. Such units they see as analogous to the notion of conceptual entity, "a cognitive object for which the mental system has procedures that can take that object as an argument, as an input" (Greeno, 1983, p. 227).

The important focus that information processing presents for this study is its central idea that humans are information processors who construct symbolic representations of the world in their minds. This view sees thinking about and acting in the world as mental operations on such representations and then taking actions externally that correspond to the results of the mind's internal workings. A fundamental issue for this study is the nature of the knowledge and meaning that may be behind symbolic representations and the interpretation that is placed on children's mathematical thinking as a result of their interaction with symbolism.

2.6 THE DUALITY OF MATHEMATICAL KNOWLEDGE

Piaget believed that human knowledge is essentially active. He made a distinction between two aspects of thinking; the figurative aspect and the operative aspect. In the field of cognition he described the figurative functions as perception, imitation and mental imagery. Operative aspects of thought includes the actions which transform objects or states and it also includes intellectual operations. Piaget indicated that the essential aspect of thought was its operative and not its figurative aspect.

There has been a long standing distinction between two types of knowledge, characterised by Ryle (1949) respectively, as "knowing that" and "knowing how". Skemp (1978) placed these two types of knowledge in the context of notions of "relational understanding – 'knowing both what do and why'", and "instrumental understanding – 'rules without reason'"(Skemp, 1978, p. 20). Although the differences between relational and instrumental understanding are clearly specified, the relationship is not clear cut. Hiebert & Lefevre (1986), using the notions of "procedural knowledge" and "conceptual knowledge", brought together a set of contemporary studies exploring the connections between them.

2.6.1 Process and Procedure

Before examining in some depth the notions of these two types of knowledge it is important for the context of this study to distinguish between the use of the terms "process" and "procedure". The term "process" will be used in a general sense, as in the "process of addition", the "process of multiplication", the "process of solving an equation"; to mean the cognitive representation of a mathematical operation. It need not be a process that is currently being carried out in thought, for instance we may speak of the process of addition without actually performing it. Nor is there any implication that the process must be carried out in a unique manner (for instance, as we shall see, the process of addition may be carried out by counting, by subitising, by deduction from

known facts or by some other method). Flexibility in carrying out a process will play a fundamental role in the theory that is to be developed within Chapter 6. The term "procedure" is used in the sense of Davis (1983, p.257); it is a specific algorithm for implementing a process. Thus "count-on" is identified as a procedure used to carry out the process of addition and as such it may be spontaneously constructed and "invented" by children (Baroody & Ginsburg, 1986), "personalised" (Gray, 1991), or taught (Fuson & Fuson, 1992).

2.6.2 The Distinction Between Procedural and Conceptual Knowledge

The focus of procedural knowledge is doing through applying. Hiebert & Lefevre indicate that procedural knowledge has two components: the formal language or symbol system of mathematics and the algorithms or rules for completing the mathematical task. The latter, which are usually hierarchically arranged, prescribe instructions on how to complete tasks which operate upon objects. The objects themselves can be distinguished as the written symbols of mathematics or objects that are non-symbolic in that they are concrete or mental images.

Almost in contrast, conceptual knowledge makes use of the underlying relationships which exist within, or between, the objects themselves. Hiebert and Lefevre describe conceptual knowledge as knowledge that can be though of as:

> a connected web ... a network in which the linking relationships are as prominent as the discrete pieces of information ... a unit of conceptual knowledge cannot be an isolated piece of information; by definition it is part of conceptual knowledge only if the holder recognises its relationship to other pieces of information.
>
> (Hiebert and Lefevre 1986, pp. 3-4)

Conceptual knowledge growth within mathematics may take on two forms:

i. In one form the understanding is based upon building relationships between existing bits of knowledge.

ii. A second form is the creation of relationships between existing knowledge and new knowledge that is entering the system: new knowledge is connected to existing knowledge.

Amongst young pre-school children who are learning to count, Gelman & Meck (1986) see the relationship between procedural and conceptual understanding as being both intricate and dynamic. Carpenter (1986) emphasises the difficulties in sharpening the relationship between procedural and conceptual thinking by indicating that a persistent problem is the ability to measure conceptual knowledge directly,

> it is often inferred through the observation of particular procedures for which it presumably is a prerequisite but there is strong evidence that major advances in solving addition and subtraction problems are characterised both by more sophisticated procedures and more elaborated conceptual knowledge and that both need to be taken into account to understand children's problem solving processes and to plan for instruction.
>
> (Carpenter, 1986, p. 121).

Although Baroody & Ginsburg indicate that

> 'procedural and conceptual knowledge are integral aspects of the learning, representation, and efficient production of the basic number facts'
>
> (Baroody & Ginsburg, 1986, p.91)

there is an element of caution in Carpenters' discussion; learning procedures does not ensure that related conceptual knowledge has been acquired. We can often apply procedures mechanically without thinking about related conceptual knowledge and, whilst there is nothing wrong with automated efficient procedures, there is a need for the limits of the procedures to be recognised.

2.7 THE ENCAPSULATION OF PROCESS AS OBJECT

Sinclair & Sinclair (1986) focus on the theme of *action* (process) becoming the *object* of thinking through a brief discussion of the foundation of mathematical reasoning. They remain cognisant of the distinction between procedural and conceptual knowledge

even though they believe this is inappropriate within the pre-school child's environment. The problem is that focusing on separate aspects of mathematical knowledge tends to reinforce the belief that the two kinds of knowledge are not only distinct but discrete. Nothing could be further from the truth. We have already seen the strong relationship that Piaget saw between actions and thematised objects of thought. Procedures form a basic part of arithmetical development and yet thinking based on procedures is likely to be very different to the flexible thinking based on conceptual knowledge.

The important issue is the cognitive shift from mathematical processes into manipulable mental objects:

> ... the whole of mathematics may therefore be thought of in terms of the construction of structures,... mathematical entities move from one level to another; an operation on such 'entities' becomes in its turn an object of the theory, and this process is repeated until we reach structures that are alternately structuring or being structured by 'stronger' structures.
>
> (Piaget 1972, p. 70)

Within the field of arithmetic the connection between 'count meanings' and 'cardinal meanings' is an example of this shift. We have already seen that, in Piagetian terms, behaviour which illustrates the ability to quantify an amount involves the synthesis of ordering and class inclusion. This co-ordination of sub–schemes underscores the counting process which generates and exploits one to one correspondence and eventually leads to number invariance, i.e. the "threeness" of three. Full conceptual understanding implies that the relationships inherent in all of the different components that form 3 are also available (1 and 1 and 1; 2 and 1; 1 and 2; one less than 4 etc.). However, shifting the focus from action to the value of the set, presents us all with uncertainty – we know it *is* done but we do not know *how* it is done. At one and the same time the process is the object; a mathematical process that is turned into a concept and which can be quickly turned back to a process provides a considerable degree of flexibility. We can see therefore see the notion of a process being "chunked" into the

object. Greeno (1983) distinguishes between cognitive objects from attributes, operations and relations which attach to or act upon the objects. He suggests that to qualify as an object such a construct must be permanently available in the individuals mental representation. Thus we may see how the process of counting five leads to the object "five" which can qualify as an object when it is permanently available for future manipulation. The notion of object means that we can now refer to the mathematical entity as if it was a real thing; we can recognise it at a glance and manipulate it as a whole. In contrast, interpreting a notion as a process implies regarding it as a potential rather than an actual entity. It comes into existence through a series of actions. Such a distinction will have an important bearing on the theory developed later within the thesis.

The problem of maintaining the distinction between process and concept in this context is that it may place an inappropriate emphasis on the nature of the mental object. Dubinsky (1991), for example, speaks of the process being *encapsulated* as an object and of the object being *de-encapsulated* into a process, thus maintaining two separate terms: process and object with transformations between them. The uncertainties linked to encapsulation are noted above, but once this is achieved the individual now has both process and object available, so de-encapsulation is trivial. In an analogous way, Greeno defines a "conceptual entity" as a cognitive object which can be manipulated as the input to a mental procedure. The cognitive process of forming a (static) conceptual entity from a (dynamic) process has variously been called "entification" (Kaput, 1982) and "reification" (Sfard, 1989, 1991). These terms shall be used interchangeably in the remainder of this study although the word "encapsulation" is favoured.

Sfard (1991) draws the conclusion that abstract notions such as number or function can be conceived of structurally, as objects, or operationally, as processes and she claims that the operational conception is the first step in new mathematical notions. Sfard's notions of "structural" arises from definitions that treat mathematical concepts as if they referred to some abstract objects whilst her notion of operational refers to processes,

algorithms and actions rather than objects. There are of course links with the Piagetian notions of structure and operation. This is particularly true of her notion of structural although her notion of operational does not on the surface appear to carry the same underlying notion as that of Piaget "an action that can be internalised". She uses a form of stage theory to supply an explanation of the mental changes: interiorisation, condensation and reification. Although she indicates that she uses the notion of "interiorization" in much the same sense as Piaget, she actually sees it as a stage when a "person gets acquainted with processes which will eventually give rise to a new concept" (p. 18). In one sense she expands upon the Piagetian notion since she implies that interiorization is the stage when a person develops the skill to perform mental operations efficiently. Condensation is identified as the stage of "squeezing" lengthy sequences of operations into more manageable units. It continues to refer to the operational component of mathematics and it appears to be analogous to chunking.

Both interiorization and condensation are seen to be lengthy sequences of gradual quantitative rather than qualitative changes; it is reification that provides the quantum leap from operational to structural thinking.

The importance of Sfard's work for this study is her emphasis on the ontological gap between her two forms of thinking. Structural thinking is the ability to see a mathematical entity as an object reified from operational thought.

2.8 MENTAL REPRESENTATIONS

To investigate the notions of qualitatively different thinking must lead to discussion of the means whereby the different forms of thinking may be triggered. A partial answer to this question may be resolved by considering how episodes are represented within the child's mind. Piaget and the constructivists have suggested that mathematical development involves the successive restructuring of knowledge which results from children's interactions with their environment. Tall & Vinner (1981) made a distinction

between the individual's way of thinking of a concept and its formal definition. Thus they distinguish between mathematics as a mental activity and mathematics as a formal system. They used the term *concept image* to describe the total cognitive structure that is associated with a concept. They concur with Piaget and the constructivists in that they see the concept image successively changing in response to new stimuli.

The total cognitive structure perceived by Tall & Vinner included all of the mental pictures and associated properties and processes that were associated with the concept. Such mental pictures were the focus of attention of Bruner (1968). His concern was the way in which children conserve past experience in a model and he suggested three ways in which in which this was accomplished; enactive representations, iconic representation and symbolic representation.

Enactive representation is seen as a mode of representing past events through appropriate motor response and it "is based, it seems, upon a learning of responses and forms of habituation" (Bruner, 1968, p. 11). Resnick & Ford (1981) believe that this mode may well be what we are seeing in children who use finger tapping to support addition strategies; counting, for these children, may still be represented as a motor act.

Iconic representation takes us a step away from the concrete and the physical to the realm of mental imagery. Bruner suggests that iconic representation is what happens when a child pictures an operation or manipulation as a way of not only remembering the act but also of recreating it mentally when necessary. Such a stage, and indeed formulation of a sequence of modes of representation amount to a stage theory of the development of intellect, may well provide some indication of the way knowledge is stored during Sfard's stage of interiorisation and may well be the way that, in a Piagetian sense actions are initially interiorised. Bruner's third stage, that of symbolic representation would appear to be the point where concept formation, in the sense defined by Skemp and Vergnaud, actually takes place. To Bruner representations in words or language are the hallmarks of symbolic representation, to Skemp and

Vergnaud symbolic representation play an essential part in concept formation and mathematical communication. But like many others, Bruner had no explanation for the transition from action to object represented by symbol.

> To put the matter very briefly, it would seem as if some sort of image formation or schema formation–whatever we should call the device that renders a sequence of action simultaneous–comes rather automatically as an accompaniment of response stabilisation's. But how the nervous system converts a sequence of responses into an image or schema is simply not understood.
>
> (Bruner, J. S., 1968, p.14)

We have seen that information processing theorists believe that the knowledge structures possessed by humans are symbolic representations of the world. The implication of work such as Piaget's, Skemp, Vergnaud, Bruner's and Sfard's is that these symbolic representations may change from simply representing an action or process to representing an idea or object. But they can take on double meanings; they may be both the process and the object. Sfard (1989) reminds us that the ability to conceive mathematical notions as processes and objects at the same time, although ostensibly incompatible, is in fact complementary. Yet she asks "How can anything be a process and an object at the same time?" It is suggested later within this study (Chapter 6) that the answer to this question is seen in the way that mathematicians use symbols.

2.9 THE ROLE OF SYMBOLS

The manner in which symbols are used plays a pivotal role in the discussion of the relationship between process and concept. In the Piagetian sense a symbol is a signifier that bears a figurative resemblance to the thing represented. As such it can be invented by the child and does not need to be taught. This is not a view that is taken within this study. For this purpose a symbol is regarded as a conventional signifier, something which is perceived by the senses but bears no similarity to the thing represented. It can be written or spoken so that it can be seen or heard; the word "five" and the numeral

"5" are symbols which require social transmission. The spoken symbol "five" can be heard whilst the written symbol "5" can be seen. They both convey the same message only the form is different.

Barnes (1976) indicates that language is the fundamental instrument the learner can use to bring order to the environment, but many mathematical language patterns are complex and do not come easily to the child (Haylock, 1991). Austin & Howson (1979) see the role of language as a means of supplying verbal symbols which can represent concepts and be used as stimuli for the internalised manipulation of these concepts. However, though concept and language are inextricably linked, concept formation depends on linguistic development (Vygotsky, 1962). Pimm (1987) accepts these general principles but provides some indications of the confusion that may arise if the focus of attention is on the language rather than on the meanings behind the language. To Skemp (1986) the meaning of the symbol whether spoken or written is the idea it expresses: "without an idea attached a symbol is empty, meaningless" (p. 65).

Written symbols can be thought of as intellectual tools which serve the public function of recording what is already known in order to share and communicate it, and the personal function that involves organising and manipulating ideas. To facilitate the latter symbols are seen as not only revealing their referents but objects in their own right. Mathematical symbols are an efficient means of storing and conveying information not least because they allow the compression of a lot of information into a small space (Pimm (1987) and elevate mathematical activity to a new plane (Resnick & Ford (1981). Ernest (1987) notes that the economy of symbolism facilitates and encourages chunking, whilst Harel & Kaput (1991) indicate that through the use of symbols "complex ideas or mental ideas can be chunked and thus represented by physical notations which, in turn can be reflected up or manipulated to generate new ideas" (p. 88). The paradox is that symbolism is both the reason for the power of

mathematics and the reason for its complexity for many of those trying to learn it (Cockcroft, 1982).

2.10 SUMMARY

Within the chapter we have traced the notion of cognitive development from physical action on objects to interiorised actions represented by symbolism. In contemporary terms this cognitive shift has been theorised as the encapsulation of a process giving rise to mathematical objects that are available for manipulation as if they were real things. Objects formed at one level of difficulty may be utilised within the procedural aspects of the next until that process is encapsulated as a new object. Each successive encapsulation may increase the complexity of the object. Symbols have a crucial role to play in being representations of objects as encapsulated processes.

Within this study we consider cognitive shifts as they relate to simple arithmetic. We look at both the flexibility and the inflexibility that may arise from the interpretation of mathematical processes. We see the consequences of using an un–encapsulated procedure and reflect upon the advantages that may be gained from the flexible use of process and object. It is such a distinction that is hypothesised to be a source of qualitatively different thinking in simple arithmetic.

CHAPTER 3

FROM COUNTING TO NUMBER FACT RETENTION: A REVIEW OF THE LITERATURE

3.1 INTRODUCTION

Action interiorised as object is central to the discussion of the development of early number skills. It is the action of counting interiorised as "sum" or "difference" that is the focus of this chapter.

Such has been the attention given to the development of numerical concepts and skills in young children that the importance of meaningful counting as a basis for arithmetical development would now appear to be beyond question. Researchers have been able to provide some insight into the development of early counting skills (e.g. Steffe *et al*, 1982; Fuson *et al*, 1982; Gelman & Meck 1983(a), 1983(b), Gelman & Gallistel, 1986), the development of children's ability to solve arithmetical word problems (e.g. Briars & Larkin, 1984; Riley, Greeno, & Heller, 1983), the strategies used to solve elementary arithmetical word problems (e.g. Carpenter *et al* 1981, 1982; Hiebert *et al*, 1982), the results of instruction in the development of higher order strategies (e.g. Secada *et al*, 1983; Fuson, 1986; Thornton, 1990), and the order in which children acquire number concepts (Denvir & Brown, 1986a, 1986b).

The purpose of this chapter is to review some of the literature relevant to the development of counting skills and place this within a framework which considers the development of competence in the addition and subtraction of basic number combinations. We consider the subskills and principles that are coordinated into the process of counting. The encapsulation of the process of counting into the concept of number forms the next stage of the chapter and this forms a basis for the ensuing discussion on the strategies that children use to solve simple number combinations. These enable the focus to be redirected at procedural and conceptual approaches to

simple arithmetic when the methodology for the first investigation is discussed within Chapter 4.

3.2 THE DEVELOPMENT OF COUNTING SKILLS

Within the mathematical context a major use of counting is to determine the number within a given set; knowing how to count is fundamental to the acquisition of early arithmetic skills. Herscovics & Bergeron (1983) see understanding based on counting procedures as only one stage in the process of constructing a conceptual schema for number. Fuson & Hall (1983) indicate that number concepts and meaningful counting interact in powerful and sophisticated ways: through their application of more efficient counting procedures children gradually discover or construct numerical concepts. This belief supplies implicit support for the Piagetian view of action becoming interiorised as object and, although the counting procedures do not involve formal mathematics, efficient and meaningful (in the sense of Ausubel) counting forms the basis for early arithmetical development.

3.2.1 Counting Principles

It has been inferred by Gelman & Meck (1983a, 1983b) and Gelman & Gallistel (1986) that specific principles govern and define counting. As developmental psychologists they were committed to the empirical investigation of the counting skills possessed by young children. Within their analysis they suggest five principles which are either wholly or partly possible candidates for the initial competence that children bring to the task of acquiring skill in counting:

(a) one-one correspondence – each item counted is assigned a unique name;

(b) the stable order principle – the names are drawn from a stable list;

(c) the cardinal principle – the last name used in the count has a special status;

(d) the item indifference (or irrelevance) principle – there are no restrictions on the collection of items that can be counted

(e) the order-indifference principle – the order in which the items are named is irrelevant.

Within these principles we see a series of sub schemes that would comprise the Piagetian notion of scheme. Gelman & Meck believe that knowledge of these principles provides an indication of the constraints actions must reflect if they are to yield acceptable counting behaviour but it is hypothesised that the principles are available to children *before* they produce such behaviour. It is not suggested that knowledge of the principles guarantees correct performance nor is it indicated that such knowledge may guide successful performance: "the first three principles deal with rules of procedure or how to count; the fourth with the definition of countables, or what to count. The final principle involves a composite of features of the other four principles" (Gelman & Gallistell, 1986, p. 77).

Although the "how to count principles" form a scheme in the Piagetian sense because they reveal characteristic properties that Piaget attributes to schemes, Gelman & Gallistell claim that their conclusions cast doubt on aspects of Piaget's theory, not least because they believe his focus was more on what children couldn't do rather than what they could do. Their study shows that children at Piaget's pre-operational stage are much more competent at number than he allows for.

Klahr & Wallace, (1973), working within an information processing paradigm, present a contrasting view. They suggest that initially children memorise, without understanding, various counting behaviours and only eventually induce principles or components of the counting principles. The implication is that counting ability develops in a hierarchical fashion and gradually, through practice, becomes automatic.

Counting may be the dominant, but not the only, way that children may quantify. Klahr & Wallace (1976) argue that subitising – immediate recognition of a quantity – develops earlier and is a prerequisite to the development of counting. The conjecture is that

subitising is constructed by the child as the result of experience with small collections of objects. Klahr, Langley & Neches (1987) see this process as completely different from quantification by counting which in information processing terms "requires conscious management of attention for effective application of the technology of counting in ones" (p. 385). Counting, they argue, "remains essentially a ritual without semantic basis until the realisation that it performs the same function as subitising confers the status of an indicator of quantity" (ibid.).

Gelman & Gallistel challenge this view. They suggest counting precedes subitising and believe that "*perceptual chunking*" (p. 70) arises from practice at counting. Indeed, they indicate that the notion of subitising should be redefined; it should not be thought of as a developmental mechanism that *precedes* counting but rather as a way of grouping elements to *enhance* counting. (There is the possibility that both views are correct. Current research by the writer indicates that subitising of very small numbers is a characteristic possessed by some children who cannot count at the point of entry to school. It will also be seen later that older children use a "subitised" display of fingers to support a very efficient procedure when dealing with addition and subtraction combinations to ten. Such an approach, which involved no actual counting, involved the visual display of numbers to ten in a concrete form represented by the appropriate number of fingers).

3.2.2 The Number Word Sequence and Counting

Although counting is the method used by all cultures to differentiate and label quantities not easily or accurately differentiated by perceptual means, central to the ability to count is the acquisition of the number word sequence but this is not spontaneous. Memory plays an important part in this process; children have to have the ability to memorise and recite the number word sequence of their own culture. Within the English speaking world this is particularly true of the number sequence one to twelve but 'after

THIRTEEN a preliminary composition procedure may set in and after TWENTY a general one' (Steffe et al, 1982, p.22).

Fuson & Hall (1983) indicate that the number word sequence seems to be an unstructured list for young children until the decade structure is evident. Distinct advantages that may be gained if the number system, as spoken in English, contained links between number words and concrete objects were highlighted by Hughes (1986). Kamii (1985) believes it is unfortunate that in English the counting (spoken) words from eleven through to nineteen correspond so poorly to the Arabic written system from the standpoint of the child who, at the same time as learning to write numerals, is also learning phonics for reading and spelling for writing.

Schaeffer *et al* (1974) indicate that the most basic skill in their hypothesised hierarchy of counting skills, which attempts to account for number conservation development, is the learning of number words. Secondary to this skill is the skill of enumeration, the co-ordination of the number words with each object in the set that is counted, to create one-to-one correspondence. Only after repeated exposure and possible adult intervention, does the child learn that the last number in the set represents the number of items in that set – the action of counting leads to the naming of the value of the set. The implications behind this model are Piagetian but we see emphasised here the importance of intervention to enhance reflective abstraction to promote encapsulation of the process.

Fuson *et al* (1982) verified a hierarchy for the development of several abilities associated with the memorisation of the number word sequence. In doing so they were following Gagné's (1962) approach of analysing skills into a set of subskills. The hierarchy is generated by considering the target task, i.e. the ability to count, and then considering what is to be known or performed during counting. Underlying the Fuson hierarchy, and indeed every hierarchy, there is an implicit theory of the processes that are engaged in to perform the tasks that appear in the various levels of the hierarchy.

Using rational task analysis, Fuson *et al* identified two phases: an initial acquisition phase in which the conventional number words were learnt as a meaningless sequence, and an elaboration phase during which the number sequence is decomposed into separate words for arithmetical and relational purposes. Although ten abilities are indicated within the elaboration phase, these abilities may be identified within two groups : those directly related to memorisation of the number word sequence and those that involve counting. They indicate, for example, the ways in which children acquire advanced sequence skills that involve 'counting-on' for addition and 'counting-up' or 'counting-down from' and 'to' specified number words.

The six abilities within the first group, which commences with the ability to recite starting at one, are learned gradually and are exemplified by the ability to count-on and count-back within a given range of numbers. Those in the second group of four abilities involve keeping track of the amount counted on or counted back. These latter procedures would involve double counting, which, as well as being frequently explicit, may also be implicit in the action. Steffe *et al* (1982) consider such abilities as evidence of abstraction of the number concept; they indicate that there is a profound difference between counting and double counting.

3.2.3 The Counting Act

Steffe et al (1982) and Herscovics & Bergeron (1983) stress the importance of the need to distinguish between the ability to recite the number word sequence and the act of counting, a procedure based on one-to-one correspondence. In the view of the former

> "in order to qualify as [an] act of counting, the vocal or sub-vocal production of each individual number word of the sequence must be accompanied by at least a momentary awareness of the individual unit to which the number word can be related"
>
> (Steffe et al, 1982, p 24).

Units are defined as the objects that the child creates to aid the counting process. Herscovics & Bergeron claim that any interpretation which treats the Fuson hierarchy

as appropriate to counting must be tentative since, although it has been verified for the number word sequence, it has not been verified for counting procedures.

For counting to take place Steffe *et al* indicate that children initially require perceptual material or things which are countable items. Their 'counting type' model, influenced to a considerable extent by Piaget's genetic epistemology, documents five increasingly sophisticated types of units that children create when they count. Decreasing dependence on *perceptual* material will allow children to eventually count *figural* representations of perceptual material; the counting process continues in the absence of the actual items. A further step away from the use of perceptual items is that an increasing awareness of the *motor* acts, such as pointing, nodding and grasping, that accompany the counting process can be taken as further substitute units for perceptual items. Counting with motor acts is seen as an important step in the development of *abstract* numerical concepts. Dependence on the first three forms of unit is further reduced by the realisation that the utterance of a segment of the number word, the *verbal* unit, can be taken as a substitute for countable items that could have been co-ordinated with the uttered number word. The belief is that the concept of unit becomes wholly abstract when the child no longer needs any material in order to create countable items, and that as a corollary of that development, 'the notion of any number word can be taken as a substitute for a series of counting acts involving the co-ordination of units of some kind with number words' (Steffe *et al*, 1982, p.43). An essential point that arises from the work of Steffe *et al* is that, following Piaget, it uses both actual and re-represented sensory motor activity to characterise children's concepts. Cobb (1987) indicates that children at the lower four levels in counting i.e. perceptual, figural, motor and verbal, give meaning to number words and numerals by actually performing or re-presenting activity.

Fuson & Hall (1983) indicate how the sequence of number words become a representational tool that is used for solving the operations of addition, subtraction, multiplication and division in cardinal contexts. The skills identified by Fuson *et al*

(1982), when applied in cardinal problem contexts, have become the solution procedures that have been termed 'counting-on' for addition and 'counting-back' or 'counting-back to' for subtraction.

3.3 COUNTING PROCEDURES

The 'counting types' model proposed by Steffe *et al* attempts to account for a qualitative change in children's arithmetical thinking. It may be seen as a development path in which the child first constructs number as an arithmetical object and then gradually develops the ability to establish relationships between numbers. However, only the fifth of the five qualitatively distinct types of number word meaning, that of abstract counting, corresponds to a set with a numerical value so that there is no actual performing or representing activity. Arising from this Cobb (1987) suggests that "children who have constructed number as an abstract, arithmetical object can initially construct only enactive relationships between numbers" (p. 172).

Riley, Greeno & Heller (1983), working within an information processing paradigm, promote a schema based model of early number development which allows for qualitative change by adding conceptual abilities. The three computer based models which they propose advance in conceptual development from a first level which is limited to the establishment of a new set by either adding or removing objects from a collection, through level two which, through the use of internal representations, can indicate a relationship between the sets that are established, to level 3 which explicitly represents the part-whole relationship.

Both the models of both Steffe *et al* and Riley *et al*, which derive their basis from children's solutions to word problems, specify a sequence of conceptual levels that culminates in the numerical part-whole concept. Resnick (1983) believes that the ability to interpret problems in terms of part and whole relationships is probably the major conceptual achievement of the early school years. She indicates that if it is to function

as a tool in problem solving, "the part-whole knowledge structure must be tied to procedures for constructing or evaluating quantities" (p. 114) and that such a schema may play a role in the subsequent development of number knowledge.

With the application of the part-whole schema it becomes possible for children to think about numbers as compositions of other numbers; such a schema provides an interpretation of number which is quite close to Piaget's definition of the operational number schema; "...all the relations in question form an operational system such that the whole, which has become invariant, is the result of composition through addition of the parts, and that the relationship between the parts are univocally determined by combined additions and subtraction" (Piaget, 1965, p.190). Thus, any quantity, the whole, can be partitioned into parts provided the sum of the parts equals the whole; by implication the parts are included in the whole.

Although the part-whole analysis provides a unifying framework for connecting numbers of different additive and subtractive situations, Carpenter & Moser (1984) "question whether such a specific model [as that of Riley *et al*] can capture the variability of children's performance"(p. 199); it does not correspond to the way most young children think about word problems.

3.3.1 Addition and Counting

Carpenter & Moser (1982) show that, prior to formal instruction in addition and subtraction, children make use of physical objects or fingers to represent quantities to be operated upon. The work of Fuson (1982), Resnick (1983) and Siegler & Shrager (1984) supports this view. In the same way that, for example, Fuson (1982) and Steffe *et al* (1982) indicate that there is a change in the pattern of counting, there are also indications that there are various counting strategies available to children to solve problems.

Whilst the primary objective of Carpenter *et al* (1981) was to consider children's success at solving addition and subtraction problems prior to formal instruction, a secondary one was to characterise the strategies that the children used to solve such problems. It was their belief that different strategies implied different conceptions of addition and subtraction. Within this study it will be seen that although this belief may go a long way towards explaining different conceptions of understanding in addition and subtraction, it may not go the whole way. Considering the integrated use of strategies may provide a clearer picture.

The counting strategies that they identify "counting all", "counting on from smaller number" and "counting on from larger number" (Carpenter, Hiebert & Moser, 1981. p. 32) were established from an analysis of what children said they were doing. They closely resembled the three basic counting models proposed earlier by Groen & Parkman(1972) who used number sentences to test a hypothesised series of models for the addition process using mental counting.

Assuming the addition of $m + n$, the three models specified different duration for task performance:

- Model A: Starting a mental count at zero and incrementing m times then n times.
- Model B: Setting the mental counter to m and incrementing n times.
- Model C: Setting the mental counter to whichever is the larger of m and n and incrementing the smaller. Since the minimum of counting is required for this model it has been termed MIN.

Although identifying two classes of counting-on, Carpenter & Moser's later discussion of the 'counting-on from larger' strategy (Carpenter & Moser, 1982, p.15) indicates that this is the MIN strategy of Groen & Parkman. Carpenter *et al* did not identify a strategy analogous to MIN using counting-all. Neither did they make the distinction in the additional paper (Hiebert *et al*, 1982). However, they indicate that there is some

evidence that young children carrying out the count-all procedure do not always respect addend order and in fact do count out the largest first. They imply that these children have a more advanced conception of addition than do those children who always respect addend order.

Groen & Resnick (1977) indicate that the addition algorithm analogous to the count-all strategy is one of the accepted mathematical definitions for addition and that it is relatively easy to demonstrate and teach to young children. Their belief that young children can spontaneously move from count-all to count-on has been supported by other research (e.g. Carpenter et al, 1981; Houlihan & Ginsburg, 1981). Herscovics & Bergeron (1983) suggest that combining two sets of objects and counting them from one – count-all – may be considered as an initial procedure whereas taking the first set as a starting point for enumeration – count-on – is a procedure so much more efficient that it can be viewed as the beginning of abstraction. The theoretical perspective developed within this study (Chapter 6) would support this view but it indicates that a count-on procedure may have qualitatively different outcomes.

Focusing on the transition from count-all to count-on in addition, Secada *et al* (1983) worked with children whose ages ranged from six years three months to seven years six months. They concentrated on the simplest type of count-on which occurs when objects are presented for the second addend. They identified three sub-skills which aided the procedural growth; the ability to produce the correct sequence of counting words starting from an arbitrary point, the ability to see the cardinal/ordinal qualities of a number, and the ability of the child to shift from seeing objects within two separate sets to seeing one set of all of the objects taken together. None of these subskills are required in count-all.

The general impression gleaned from the literature is that count-on is regarded as a product of children's invention (e.g. Ginsburg, 1977; Baroody & Ginsburg, 1986; Sinclair & Sinclair, 1986). Such invention is seen as an integral component of what has

come to be regarded as children's use of informal counting methods. Baroody & Ginsburg regard such informal arithmetic as an important base from which to learn more formal arithmetic. Not only do they subscribe to two classifications of the counting-on procedure when considering counting as a mental activity, but they also draw together views that enhance the distinction between 'counting-all from the first addend' (CAF) and 'counting-all starting with the larger term' (CAL) (Baroody & Ginsburg, 1986, p.81). Such a distinction makes us meet the problem of commutativity head-on. In contrast to those who regard the invention of an addition procedure that disregards addend order as evidence and understanding of commutativity (e.g. Groen & Resnick, 1977), Baroody and Ginsburg indicate that the use of largest first may be more an indicator of a child's attempts to relieve cognitive stress: it reduces the number of steps within the keeping track process and reduces the length of time required to carry out a procedure.

3.3.2 Subtraction and Counting

Woods, Resnick & Groen (1975) confirmed hypothesised models for subtraction procedures which were identified as decrementing (count-down), incrementing (count-up) and a model which indicated "choice", either incrementing or decrementing depending upon which is the quicker.

Carpenter *et al* (1981) point out that it is more difficult to identify general solution strategies for subtraction word problems because there are more problems and more distinct strategies that can be used for each problem. However, Carpenter & Moser (1984) identified four strategies that are analogous to the strategies identified for addition: direct modelling strategies were "separating from", "adding on" and "matching" whilst the counting strategies were identified as "counting down from", and "counting up from given" (p. 182).

Implicit within the strategy classifications for addition and subtraction that have been summarised above, is the implication that the use of counting algorithms can reveal something about the child's understanding of counting beyond the mere application of a procedure to solve the problems. Young children who use count-all or count-on as appropriate strategies to deal with addition will very frequently see subtraction as the inverse of these approaches. Consequently the physical aspect of separating or the count down strategy would be seen as the most appropriate. Indeed, Steffe *et al* (1982) see that the crucial preconditions for a child to relate addition and subtraction are:

(i) having constituted addition as count-on with numerical extension (a sequence of forward counting acts) and subtraction as counting-back;

and

(ii) having constructed reversibility of numerical extension and declension (backward counting with double count).

An implication of this view is that unless such preconditions are met children are unable to identify the benefits of the part-whole relationship. As is later seen within the study (Chapters 5 and 7) children who flexibly use either count-back or count-up are illustrating procedural flexibility which seems to accompany flexible thinking skills.

There is frequently within the literature a body of opinion which seems to specify that counting procedures are age related. Ilg & Ames (1951), for example, indicate that whilst five-year-old children used a strategy to be later described as count-all, six-year-old children used count-on from the largest. The inverse processes were used for subtraction, 'separating' by five-year-old children and 'counting down from' by six-year-old children. Woods, Resnick & Groen come to the conclusion that second grade children in America (children aged approximately 7+) solved simple subtraction problems by methods that could be described as 'counting-back from' and 'counting-up to' whereas children of the fourth grade (9+) used the "choice" model.

Of course methods that involve counting, although indicating the complementary nature of addition and subtraction do not necessarily reinforce the nature of the links between the two. As the difference between the numbers increases, the difficulties of not only keeping track of the amount counted in the counting-back process but of also reciting the numbers backwards also increases (Baroody, 1984; Carpenter & Moser, 1984). Fuson (1986) and Fuson & Willis (1988) have shown how, in an attempt to alleviate these problems, six-year-old children can learn to solve subtraction problems using counting up as the single counting strategy for subtraction.

The cumulative research on the use of counting to resolve addition and subtraction combinations presents us with refined classifications which help us identify the nature of the actions which children are undertaking and the units that are used to support these actions and represent the numbers involved. Fingers are identified as very common perceptual units used to support counting. Hughes (1986) presents some interesting case studies of children using fingers whilst Steffe *et al* indicate how fingers may function as countable items, perceptual units, or as a check in the double counting process. Indeed such was the anticipated dominance of their use that one portion of the item bank included in their study indicated that children should not use fingers. Brownell (1935) had indicated that a substantial number of number combinations were obtained through the use of fingers. Indeed, he had earlier specified their use as evidence of a learned procedure to execute the calculation of a number combination (Brownell, 1928). Whilst Baroody (1987) has shown how children may use fingers in single digit addition, Fuson (1986) and Fuson & Secada (1986) taught children how to use their fingers to support counting procedures in both addition and subtraction.

In the absence of physical objects, less sophisticated counting procedures, for example, count-all and count-back, do not appear to be readily transferable as problem size increases; 'it seems that children must learn to count-on before they can solve addition problems with sums greater than 10 when physical objects are not available' (Carpenter & Moser, 1982, p.21). Conclusions within this study would support this view but they

also indicate that procedural competence in count-on at one level does not guarantee procedural competence at the next highest level.

3.4 FACT RETRIEVAL

It is recognised that children gradually replace their slower procedural (count-based) approaches with efficient fact retrieval processes. Amongst their sample of first grade children Hiebert *et al* (1982) identified two further solution strategies that were common to both addition and subtraction:

1. Known Fact

   The response of the child was based on the recall of a particular fact.

2. Derived Fact

   The solution that was generated by the child was based on a related number fact or a property of the number system. For example, to solve particular addition and subtraction problems the child may use one of the following approaches:

   (i) 4 + 3 = , "4 + 4 = 8, 4 is one more than 3 so 4 + 3 = 7".

   (ii) 8 – 3 = , "two 4's are 8, 4 is one more than 3 so 8 – 3 = 5".

   (iii) 6 + 8 = , "6 + 4 = 10, 10 + 4 = 14".

   (iv) 14 – 6 = , "14 – 4 = 10, 10 = 5 + 5, 5 – 2 = 3, 3 + 5 = 8".

3.4.1 Derived Facts

As a result of their longitudinal study with children, those in grades one through three (children aged 6+ through to 8+), Carpenter & Moser (1984) indicate that children learn some number facts before others and, before they have mastered addition tables, they make use of a small set of memorised facts to derive solutions for some addition and subtraction problems. The results of the current study would indicate that even when they know some facts some children do not do this, and, the difference between them and those that do, is taken as evidence of qualitatively different thinking about simple

arithmetic. The results of Groen & Parkman and of Woods *et al* suggest that doubles e.g. 4 + 4, 8 – 4, fall into the category of "easier remembered" combinations.

Strategies analogous to derived-fact strategies have been described in a number of studies. Thornton (1978) used the term "thinking strategies" whilst Houlihan & Ginsburg (1981) placed the label "indirect memory" on strategies now considered to fall under the classification derived facts. Resnick (1983) notes that such "shortcut procedures provide evidence that children understand the compositional structure of numbers and are able to partition and recombine quantities with some flexibility" (p. 122).

For the purpose of this study, such strategies will not be interpreted as procedural since the notions of "thinking" and "flexibility" do not seem to sit easily with the notion of procedure considered earlier. Such qualities would appear to provide evidence of the use of conceptual knowledge and relational thinking. The notion of procedure can imply instrumental thinking. It is conjectured that the use of derived facts provides evidence of qualitatively different thinking in simple arithmetic; their use provides some children with the flexibility required to achieve success over a range of basic number combinations. Children who continually need to resort to procedures may have some success but this can be limited since their procedure may not be easily generalisable. The "invented strategies" considered by Baroody, Ginsburg & Waxman (1983) may also include the notion of derived facts but some care has to be taken with their use of the term "invented". Houlihan & Ginsburg (1981) indicate that the invented strategy is the result of children assimilating what is taught into what they already know. Such a strategy can be count based or non count based; that the distinction between count based and non-count based strategies is made quite clear is very important for this study. Carpenter *et al* (1981) used the term "heuristic strategies" to describe strategies used to generate solutions from a small set of known basic facts.

Steffe *et al* (1982) indicate that some factual knowledge seems crucial for the development of thinking strategies. The use of derived facts not only indicates a more sophisticated use of informal methods but has been seen by some researchers (e.g. Steinberg, 1985; Thornton, 1987) as a strong influence on positive fact acquisition. Thornton (1990) suggests that time explicitly spent on teaching such an approach is more effective than drill in facilitating the learning, retention and transfer of basic number combinations. Interestingly, when reporting the results of the assessment component of their main study into the learning of number concepts by low attainers Denvir & Brown (1986a) indicated that their sample of "low attainers" made no use of derived facts, an observation generally supported by the conclusions within this study.

3.4.2 Knowing Combinations

A further interesting point emerging from the Brown & Denvir pilot study was the observation that the "most notable failure of the teaching was the attempt to teach addition number facts for some small numbers" (1986b, p.154).

Learning and using the number facts is the focus of attention of many who teach young children, and, particularly at the present time, also a focus of those who set the criteria against which children may be judged ((DES, 1989, 1991). A central issue within this study is to consider children's use of counting procedures and their interplay between the use of fact retrieval methods that may be identified as 'known facts' and fact retrieval methods based on the reconstruction of facts that are known to establish new known facts – derived facts. Before looking with a little more detail at this relationship we first consider how counting, seen as a partial structure for obtaining solutions to the number combinations may in fact become internalised as a known combination.

Stemming from such seminal research as that of Thorndike (1922), a dominant view saw the basis of learning as the internalisation of facts best accomplished by means of didactic instruction and drill – repeated exposure and practice led to memorised

individual facts. No importance was placed upon invented approaches that may be discerned either through procedures or through the use of related facts. Classroom implementation of the most extreme forms of these beliefs, practised prior to and practised for a decade and a half after, the second world war, followed the course of attempting to provide a systematic training in arithmetic whilst at the same time providing direct attention to speed and accuracy (Gray, 1991a). Practice was seen as the link between initially establishing and then cementing the relationship between problem and answer (Ashcraft, 1985; Siegler & Shrager, 1984). The importance of procedural methods in setting the context for the learning to take place was not discounted but the act of memorising facts stemmed initially from the practice such methods afforded; the relationship was established between the problem and the solution, and then was reinforced through further practice to increase the efficiency of the retrieval method. Thorndike (1922) had argued that the frequency of practice was not sufficient to account totally for number fact learning and neither should number bonds be formed independently. Brownell (1935) indicated that although drill was clearly effective in increasing the speed of recall of facts and in improving fluency in a skill, it made no improvement in the understanding of relationships, as shown by the ability to work out forgotten facts from remembered ones; in other words simply knowing a combination did not mean that it may be used to derive an unknown one. Such views can be placed in the learning context set by Skemp (1976): one consequence of drill could well be the enhanced instrumental responses without relational competence.

How solutions to basic number combinations are stored in memory is a contentious issue. Much of the research on fact retrieval has involved a chronometric approach in which reaction times for different number combinations have been used to infer the nature of the processing involved and thus, by implication, provide some notion of what is remembered (e.g. Groen & Parkman, 1972; Ashcraft, 1982; Siegler & Shrager; 1984).

A number fact may be seen as a relationship between concepts (in the sense of Skemp) and an important issue is how such relationships may be established. In a Piagetian sense we are looking for evidence that an action has become internalised; for information processors such evidence may be incorporated into a processing model. Siegler & Shrager (1984), for example, believe that each problem is associated with a particular solution. The link between a procedure and automatic fact retrieval comes through the child's inability to make the appropriate response to a stimulus through the initial attempts to use fact retrieval. They suggest that in the first instance, any problem that is presented to the child i.e. 2 + 6, is associated with a counting string and in most cases, in a first attempt at a solution, the child simply advances to the number in the count string after the second addend in the string, in this case '7'. This response produces a mental trace which forms a bond between the problem and the (incorrect) response. Within this model the child then applies a corrective mechanism, a procedural approach. As the child continues to use such a procedure to compute the sum correctly each time it is faced, the bond between the problem and the correct sum is gradually strengthened. Eventually the correct solution is produced so frequently that the association between the problem and the solution becomes very strong. Byers and Erlwanger (1985) believe that such an information processing approach reinforces the argument of Bruner (1973) that the old distinction between rote learning and understanding is a 'pseudo issue'.

Ashcraft (1982) claims that older children and adults store simple arithmetic facts in memory and then retrieve them from memory as they are needed. To explain such retrieval he proposed a 'network retrieval model' of mental arithmetic performance which contained two major long term memory components generating performance; the 'declarative' component and the 'procedural' component: the essence of his theory is that knowledge of arithmetic involves

knowledge of arithmetic in stored facts and a body of procedural knowledge about arithmetic

(Ashcraft, 1982, p.232)

Ashcraft's declarative component indicates how facts are stored and retrieved. Basic addition and multiplication facts are stored in network representations with each learned fact represented as a 'node' in the network structure. Nodes vary in respect to accessibility; more difficult facts are less accessible during the memory search and consequently take longer to retrieve. The procedural component indicates essential knowledge about arithmetic, such as algorithms, heuristics, rules, informal procedures etc. Ashcraft predicted that, in general, the use of procedural approaches to generate number facts would be slower than fact retrieval methods, a view supported by the analysis of Siegler & Shrager (1984) and confirmed by Siegler & Jenkins (1989).

Ashcraft reports his research in a purely descriptive way and draws the conclusion from his results that it is not 'farfetched that some of us remember our number facts' (Ashcraft, 1985, p.99): to do otherwise "may overwhelm the resources of memory when more complex computational problems are handled". In Ashcraft's view, even a simple and basic phenomena such as incidental learning would predict eventual memory for most, if not all, of the simple addition facts. His research indicates that by the third grade (children of about eight years of age), children's reaction times to the stimulus of simple addition problems resembles that of adults and if facts were not recalled the children relied on procedures such as counting.

The assumption that when children cease overt counting they have switched to automacity is thrown into doubt through the work of, for example, Groen & Parkman; the non-recall procedure MIN may account for alternative mental procedures.

Baroody (1985) and Baroody & Ginsburg (1986) present an alternative view to that of Ashcraft and Siegler & Shrager. They argue that, although combinations may be obtained through the specific numerical association, many can be accurately and

automatically produced from stored rules, procedures, or principles. Such a view believes that mastery of the basic number fact combinations is achieved through a system of interrelated experiences; the mental representation of number combinations involves relations as well as facts.

The development of 'meaning' implicitly underscores this alternative model. Research interest in counting partially carries with it the implied belief that children will attain mastery in, and understanding of, basic number combinations after a period in which they obtain solutions through the use of procedures considered to be less well advanced than automatic responses. Baroody (1985) believes that such experience provides underlying meaning to habitual production of the number combinations through what he terms reconstructive and reproductive[1] processes. These would appear to work together to generate number combinations; they are parallel operations – each underlying the production of different families. Thus the addition of zero or the addition to zero may be a rule generated from experience (although he is not explicit as to what this experience is or how it may differ from experience in other number combinations). So too may be the addition to, or of, one be a rule. More complex problems i.e. 7 + 9 may be generated through counting on whilst other combinations may arise from the relationship between triples such as 3,4,7 where 3 + 4 = 7 and 7 – 4 = 3. Thus we see Baroody presenting a theory which involves the integrated use of strategies. The evidence of this thesis is that the group of integrated strategies that Baroody identifies as the reconstructive process is not available to all children.

When we view the Ashcraft and Baroody models perhaps the real issue of the difference between them is a matter of degree – "when do the procedural methods of one become the reconstructive methods of the other?" The answer may point to what is

[1]Baroody uses the term "reconstructive" to describe the way that many basic number combinations can be accurately and automatically produced from stored rules, procedures, or principles. 'Reproductive' refers to the 'automatic recall of specific numerical associations' (Baroody, 1985, pp. 83,84). Reconstructive, in the sense that it is used by Baroody, would seem to be a general classification which implies the inclusion of strategies which involve counting procedural and strategies based on the use of derived facts. Baroody's basic argument suggests that these strategies, which had previously been used in learning the number combinations, may continue to operate in adults in a more automatic way.

in effect a qualitatively different approach to mathematics. Ashcraft's focus is mainly on the reproduction from memory of factual knowledge; the use of a procedure is evoked only through the failure to recall from memory; procedures are supplementary to retrieval. Baroody & Ginsburg consider a balance between the use of remembered knowledge and 'reconstructive' processes which make use of known facts, relations and procedures linked through rules or principles; the use of procedures is an integral part of the retrieval process. Proponents of the former frequently attach caution to the extensive belief in models which emphasise the strengths of the latter (e.g. Ashcraft, 1985): reliance on rules and procedures may easily overwhelm the resources of working memory.

3.5 NOTION OF MATHEMATICAL ABILITY

At the start of Chapter 2 a question drawn from Resnick & Ford focuses attention on the possible dichotomy that may exist between the qualitative difference that may develop from the understanding of a partial structure and the way in which that partial structure is to be understood within the context of a hoped for final outcome. In the context of this study a partial structure may be seen as that developed through the use of counting procedures and a final outcome that which evolves around knowledge and using that knowledge as a result of the ability to identify relationships. Katona's conclusion that people engage in qualitatively different learning must be seen as a signal that learning is reflected in qualitatively different thinking. Denvir & Brown draw our attention to the potential differences when summarising the points that emerge from their pilot study:

> It appeared that whilst abler children may perceive relationships which are not made explicit, the low attainer may need to engage in both practical activities and discussion which explicitly draw attention to such relationships.
>
> (Denvir & Brown, 1986(b), p.156)

The problem is that even when relationships are made explicit some children prefer to use their partial structure, a counting procedure, rather than run the risk of adopting approaches built upon relationships whilst others "may see attempts to make relationships explicit as a new set of rote procedural rules" (Steinberg, 1985, p.348).

The notion of ability to deal with particular aspects of the mathematics is an important one for this study. The counting types presented by Steffe *et al* is an analysis which would enable individual differences to be determined by the ability of the individual to abstract from particular perceptual properties. Duncker (1965) believes such ability is essential for discovering the general in the concrete fact and he regards this ability as one of the elements in mathematical thinking. Krutetskii (1976), investigated mathematical ability as the ability for a creative mastery of the school mathematics course and drew the conclusion that mathematical ability can take shape at a very early age, "...and for the most part, in the form of computational abilities – abilities to operate with numbers" (p. 222). He recognises that computational abilities are not true mathematical abilities but on the basis of computational abilities real mathematical abilities may be formed.

For Krutetskii some of the components of mathematical ability were seen to be:

> an ability to generalise mathematical material (an ability to discover the general in what is externally different or isolated; a flexibility of mental processes (an ability to switch rapidly from one train of thought to another); a striving to find the easiest, clearest and most economical ways to solve problems; and ability to chiefly remember generalised relations, reasoning schemes, and methods of solving type problems; curtailment of the reasoning process, a shortening of its individual links;
>
> (Krutetskii, 1975, p. 222–223)

Within his analysis of the age dynamics of the structure of mathematical abilities, Krutetskii frequently refers to the ability of children within a framework which corresponds, in general, to the concept of the structure of mathematical ability seen above. Children's mathematical ability is freely described as "less capable or less able",

"average or capable," and "more capable or most capable" (p. 332-333). He indicates that whilst mathematically able pupils demonstrate flexibility in their mental processes, mathematically less able pupils have difficulty switching from one mental operation to another and are usually bound to "conventional trains of thought" (p.338). He adds that "for such children it is hard to even switch from a harder to an easier method if the first is habitual and familiar and the second new and unfamiliar. One method of solution is an obstacle to another". Such a comment would no doubt be supported by Steinberg (1985) and Thornton (1990) both of whom found that less able children who were confident counters were reluctant to change their strategy to solve basic number combinations. Haylock (1987) provides an indication of the use of stereotyped or fixed ways of solving mathematical problems even if such ways are inappropriate.

Whilst Krutetskii is primarily interested in the mathematically able, Haylock (1991) is concerned with low attainers whose mathematical attainment is such that they are below the 20th percentile. The outcome of his analysis of possible factors associated with low attainers suggests that such children may be identified as such by the age of eight and that they often remain in this category throughout the upper primary years. The mathematical difficulties of such children may well carry on into the secondary school. Hart (1981) presents evidence to show that 10% of the children examined on their understanding of four basic number operations at the end of the first year in secondary school "could at best cope with addition and subtraction" (p.44). She indicates that of the few examples presented within the CSMS test papers that required computation 'there was no apparent increase in facility commensurate with age' (p.211). Indeed, there is the suggestion that if a child could not do subtraction of a three digit number from a four digit number with decomposition "when he entered the secondary school he was unlikely ever to be able to do it" (ibid.). We are left to wonder whether or not this is a condemnation of the primary school or the secondary school or whether such difficulties are reflections of a factor which reflects Krutetskii's notions of mathematical ability.

Ruthven (1987) expresses concern at any notions that may lead to "ability stereotyping" since the perceptions and expectations of pupils so stereotyped may lead to inappropriate patterns of differential treatment within the classroom. He indicates that such perceptions may be manifest, for those with below-average, perhaps even average, levels of attainment, in an emphasis on repetitive activities, on instrumental learning and on computation and the setting of a limited range of stereotyped goals. As such these may enhance the successful use of informal methods which mitigates against the child being confronted with more general, formal and powerful mathematical concepts and techniques. Within the broader context of school mathematics informal methods, those not teacher taught, may be successful within familiar contexts (Booth, 1981), may not necessarily be idiosyncratic (Hart, 1982) but they may not be generalisable.

3.6 CHAPTER CONCLUSION

Within Chapter 1 specific reasons were given for choosing children's arithmetic as an area within which to examine qualitative difference in thinking. Within this chapter we have considered both the development of counting and examined the outcome of well documented research into children's solution strategies when they solve basic addition and subtraction combinations. Although there are suggestions that children's conceptual knowledge of the relations between quantities is related to the acquisition of more efficient counting procedures (e. g. Riley, Greeno & Heller, 1983), an implication behind some of the cognitive models developed for children's approaches to simple arithmetic appears to indicate that children of a given age consistently use a single addition strategy. Groen & Parkman's (1972) MIN model is a good example of such a model. A variety of other findings tend to support the view that young children constantly add in the way postulated by the model. These complement the theoretical construct of the MIN model in that the smaller addend has consistently been the best predictor of six and seven year old children solution times (Ashcraft, 1982; Haye et al,

1986). Siegler & Jenkins (1989) indicate that one discordant note arises, chronometric analysis does not exactly fit what children said they were doing (e.g. Carpenter and Moser, 1982; Fuson, 1982). Of course the conflict between conclusions drawn from chronometric data and those drawn from verbal reports may be resolved if the children's verbal reports are accurate. In an attempt to resolve this issue Siegler & Jenkins conclude that "children used the strategies that they reported using and they employed them on those trials where they said they had" (p.25).

Groen & Parkman and Ashcraft, in common with, for example, Siegler & Shrager and Carpenter & Moser, all indicate how children's solution strategies for resolution of basic number combinations change over time. The latter together with other researchers e.g. Baroody (1984), Carpenter & Moser (1982), Fuson (1982), all suggest that children use a variety of approaches when dealing with arithmetic word problems. The outcome may well be that several children perform successfully at the same task but this does not mean that they share the same underlying knowledge (e.g. Dean, Chabaud & Bridges, 1981). The issues for this study are whether strategy changes can be attributable in equal measure to different groups of children, whether there are some common factors that may be identified in the variety of strategies used by children within different groups and whether different forms of knowledge can be attributed to the children on the basis of their ability to solve basic number combinations – are there qualitatively different approaches to simple arithmetic?

CHAPTER 4
RESEARCH METHOD

4.1 INTRODUCTION

Within Chapter 2 we saw that several different studies hypothesised that there are two forms of mathematical thinking which Hiebert & Lefevre refer to as procedural and conceptual thinking. Other studies come forward with the hypothesis that there is a process of procedural encapsulation such that process or actions become encapsulated as objects of thought. Central to this study is the notion that some children remain within a procedural dimension which lacks the flexibility of being able to use both process and object with ambiguity.

This leads to the hypothesis for this thesis:

> there is a qualitative difference in children's arithmetical thinking as identified by their mathematical behaviour which
>
> (i) on the one hand is manifest as a spectrum of performance in the operations on numbers as a procedure that relates to counting and
>
> (ii) on the other is manifest in the flexible manipulation of process and concepts.

Two different experiments are considered to support this thesis. The aim of the first series of interviews was to obtain an indication of the qualitative differences that may be observed between children of different ages and different levels of success with the mathematics normally expected of them. Such differences were to be considered through an integrated analysis of the behaviour observed amongst the different groups of children to provide some notion of how this may change over a period of time. The aim of the second series of interviews was to replicate the first and to provide a platform for the theoretical analysis which arises from the first. The second also considers more completely the consequences of the use of procedural methods

compared with those that involved flexible within the context of defined levels of mathematical achievement.

Thus the first series of interviews considers through a series of snapshots of children's mathematical behaviour the divergence of thinking that occurs between those that demonstrate procedural thinking and those who think conceptually.

4.2 METHOD

It was the intention that the analysis within the current study built upon many of the earlier studies reported within Chapter 3. In particular the counting strategies described by Carpenter & Moser (1982), that is count-all and count-on are regarded by them (Carpenter & Moser, 1983) and others (e.g. Steffe *et al*, 1981) as characteristic of distinct stages of development in number understanding. Consequently it should be possible to infer some notion of a child's understanding by observing children's integrated use of such strategies

To obtain a notion of the strategies used by children a form of 'structured' and 'open interviewing' techniques, (Cohen & Manion, 1985, p.309) were used. The interviews had a structured component in that both samples were asked to provide answers to sets of predetermined structured questions. However, where appropriate, subjects were asked to reflect upon what had been done to resolve each question such that the integration of the structured and open interview components may best described using the term *revised clinical interview.*

The clinical interview technique was originally used by Piaget to achieve three aims central to the study of cognitive development; a description of intellectual activity, the specification of the nature and organisation of cognitive processes and an evaluation of the child's level of cognitive competence. Designed as an unstructured and open ended method intended to give the child his "natural inclination", "the clinical examination is dependent on direct observation.....the practitioner lets himself be led and takes

account of the whole of the mental context"(Piaget ,1929, p8). Interviews of this form provided the protocols, formal accounts of interactions between the subject and the interviewer, which formed the backdrop for Piaget to develop his stage theory and also for Steffe *et al* to establish their theory of 'counting types'.

Children's words and behaviour received heavy emphasis within the work of Steffe *et al.* Talking aloud, in a proportion defined by the subject, in addition to the observation of elements of behaviour became central to protocols established as the result of children's attempts at a particular task. Although subjects were not explicitly asked to verbalise, verbalisation, non-verbal behaviour, and responses to clinical interviewing were integrated components of the intention of Steffe *et al* to provide instantations of counting types and the development of their theoretical constructs. Concentration on the totality of the action provided the protocols from which both analysis and theory resulted.

4.2.1 Research Limitations

The resolution of the central issues within the study make the assumption that external representations of mathematics provides an indication of the nature of the internal representations. Although the study draws upon the work of, for example, Greeno (1988) and Kaput, (1988, 1991) who indicate that the nature of external mathematical representations influences the nature of internal representations, it makes the contrary assumption that the way in which the child makes an external representation reveals something of how the child has represented similar information internally. Although it is not made explicit, it is conjectured that such an assumption underlies much of the recent work which has provided us with greater insight into children's development of the number concept (see for example, Carpenter *et al* 1982, 1983, 1984; Fuson, 1988; Fuson *et al*, 1982; Hughes, 1986).

The external representations of mathematics may be identified through interaction with abstract symbolism, physical manipulation or verbal description but, though it may provide an indication of the way in which information is stored internally, no precise claims can be made about the nature of internal representations (Kaput, 1991). Although revised clinical interviewing is the mode of data collection within the study and such interviewing may lead a child to make a verbal statement which complements external representation, both external representation and verbal statement may require interpretation by the interviewer. It is recognised however, that the interpretation of the outcome of a child's responses to numerical sentences given verbally may be problematic (e.g. Hughes, 1986, p.39).

There is no presumption that children who work with counters represent all quantities internally as mental images of counters. What is presumed is that children who interact with counters or fingers or some other apparatus to represent the quantities and actions through which a relationship between the quantities is identified, represent the quantities and actions differently for themselves than do children who interact only with the symbols. Kaput not only provides a notion of the distinctions that are to be central to this study – the world of mental operations and the world of physical operations – but he also provides the note of caution within which the framework of the later analysis must be viewed; mental operations are always hypothetical whereas physical operations are usually observable.

The analysis of children's observable behaviour is placed within a context that recognises that such behaviour is selected and organised by the researcher. Thus the theoretical constructs placed on the analysis are not entirely free of the observers expectations and theories, however objective these may appear to be at the time of observation. Although this study will attempt to establish a new theoretical construct to explain observations, such observations become the data which is established from classifications which in turn may also have been subject to expectations and theories of their proponents. However, is believed that qualitative differences in thinking cannot be

automatically inferred from discontinuity in quantitative measurement if such measurement comes from a complex hierarchy of classification. Indeed, the results of the second experiment confirm this. It is partially for this reason that the study attempts to view the differences in children's behaviour in a qualitative way based upon their integrated use of strategies. Consequently, analysis of children's behaviour does not concern itself with the finer gradations of performance in simple arithmetic that have been the focus of studies by, for example, Steffe *et al* (1982) on counting, Siegler & Shrager (1984), addition, and Baroody (1984) on subtraction. Thus, it is not the intention of the study to speculate on fine grained behaviours that are too complex for application within the classroom but rather to consider a tool which may, through the integrated application of these behaviours, provide a little insight into why some children may be successful and others not.

Within this study protocols within the first sample were established through the analysis of a comprehensive series of field notes and, in some instances taped interviews.

Thus attempts at description will be constrained by the aspects of data collection and the levels of reporting subsequently presented. Since the study is concerned with an integrated rather than a fine grained analysis of children's behaviour, the levels of recording, to be considered later, will reflect this. A further possible source of error may be that the nature of the interaction could cause children to employ different approaches during interview to those that may have been used if the interviewer was not present. The request to the child that (s)he reflect upon what has been done may alter the nature of the response. The use of revised clinical interviewing is aimed at minimising this since, at least for those children who use physical manipulation and/or unsolicited verbal utterances during the attempt to obtain a solution, verbal explanation, in response to open questioning, should serve to reinforce the interviewers interpretation of the action.

A major concern within the study was to present problems which, theoretically at least, minimised the level of personal interpretation of the requirement of the problem; a concern was that each child understood the question in the way intended. The aim was to consider children's responses to combinations that could be seen as generalised arithmetic. Although much of the research considered in the previous chapter has its focus of attention on the arithmetical word problem, the guiding light within the current study was the numerical generalisations that children may bring to the development of computational algorithms. Contextual based arithmetic has considerable importance in the development of arithmetical skills but the application of algorithms has as its base the knowledge of decontextualised basic number combinations.

4.3 THE SAMPLE

4.3.1 The Schools

A fuller description of the second series of interviews will be provided at the start of Chapter 7. Here we will confine ourselves to the first series.

Two schools, considered to be fairly representative of English schools, were approached to take part in the investigation. At least one teacher from each school had participated within INSET courses at the University of Warwick.

Both schools were combined schools, that is both schools taught children aged five to twelve.

School A: A one form entry combined school in a small town and in a semi rural location. The children feeding into the school were from backgrounds representing a cross section of the social structure.

The school was a well integrated school with explicit aims and objectives to direct teaching and learning. A designated teacher co-ordinated

mathematics activities and this teacher had attended numerous INSET courses organised by the local authority and by the University of Warwick. Teaching approaches in individual classes did vary somewhat; very active and very passive learning was noted. At the instigation of the mathematics co-ordinator, class teachers ensured that there was a continual "tick-over" of previously learned facts and skills. The Nuffield mathematics scheme was in use within the school and this scheme was used with a considerable degree of flexibility. New topics were introduced through class teaching and although various teaching styles were in evidence i.e. individual learning, group work and class teaching, organisational aspects of children's learning were balanced to suit the learning situation. The general ability of the children was regarded as average to above average.

School B: A two form entry school operating on a split site within a medium sized town. The children were drawn from an urban catchment area.

The teachers in school B were less homogeneous than those in school A. Although there was a designated mathematics co-ordinator this teacher was far less in evidence than the one in school A. There was no explicit statement of aims and objectives in mathematics and, moreover, school B lacked the flexibility of school A. It appeared to be far more traditional, and, through a more formal approach, was skills oriented. The lack of homogeneity amongst the teaching staff emphasised the superordinate nature of each teacher within their own classroom. The more traditional view of the mathematics curriculum was emphasised in the rigid adherence to the commercial scheme in use – Scottish Primary Mathematics. There was very little evidence of class teaching in mathematics; children tended to work through the commercial scheme. There was however some intensive small group teaching although this

took up a very small proportion of the time allocated to mathematics. The general ability of the children was considered to be average.

4.3.2 The Children

The children were chosen in such a way that in the class teacher's opinion they would be representative of those who may be considered "above average", "average" and "below average" in mathematical attainment. The notion of "below average" is used in the sense indicated by Ginsburg & Allardice (1984) in that the children were not performing at the normal level in classroom mathematics. Such children may equally be described as 'low attainers' (Denvir & Brown, 1986a, 1986b; Haylock, 1991).

Class teachers were asked to consider their classes as being divided equally into ability groups reflecting three levels of mathematical achievement; those who were performing at what may be considered a normal level in arithmetic, those who had less success than the normal and those who had more success than the normal. Since the interviews were carried out during the summer term each teacher had a period of two terms to gain intimate knowledge of the mathematical achievement of the children in their class. Following this initial grouping the teacher's were then asked to select two children from each group who they would consider to be "representative" of the group. In this way a total of 72 children representing the chronological ages 7+ to 12+ were identified. Apart from the 12+ group, which through movement and sickness eventually contained only nine children, equally spread over the three arithmetical ability levels, each of the other five age groups contained twelve children equally divided over the three teacher identified ability levels: "below-average", "average", and "above-average" in their performance in arithmetic. In the analysis of the study I shall refer to the year groups by age, so that, for instance, 9+ refers to children who would be nine during the school year. The children were interviewed over a two month period starting six months after the beginning of the year, so at the time of interview a child designated as 9+ would be in the range 8 years 6 months to 9 years 8 months.

Although the focus of attention was mainly on the solutions used by children who had moved beyond the point of pedagogic input in developing knowledge of number combinations, a group of children who were still working at this stage, the 7+ children, were included. The class teachers felt that the 8+ and 9+ age groups knew sufficient number combinations to move on to the development of computational skills in addition and subtraction with and without exchange, whilst those above these ages were felt to be at least reasonably competent with such problems.

4.4 INTERVIEW COMPONENTS

4.4.1 Questionnaire Design

Five issues were uppermost in selecting appropriate number combinations and arithmetical number problems which would reflect the approaches that the children used to obtain solutions:

- The number combinations should be representative of the addition and subtraction combinations that a child would normally be expected to remember by the time they were entering the middle primary school years i.e. ages 8 to 9:

    i. Addition combinations for pairs of numbers where the sum of the addends was ten or less

    ii. Addition combinations for pairs of numbers where the sum of the addends was between ten and twenty.

    iii. Subtraction combination for numbers which had a subtrahend of ten or less.

    iv. Subtraction combinations which had a subtrahend of between 10 and twenty.

- They should include a sample of a range of number combinations where the difference between the pair of numbers to be either added or subtracted varied in size.
- They should include samples of number combinations where the solution may reasonably be expected to be obtained by other known knowledge, for example, doubles and halves.
- Both the addition and subtraction problems within the range 10 to 20 should include a sample of problems which make use of basic number combinations previously solved within the range of problems dealing with combinations to ten.
- A small sample of the subtraction problems should include some that make use of previously solved addition facts.
- The problems should be representative of problems commonly found in school mathematics texts used by average ability seven and eight year old children.

With these considerations in mind, the number combinations were selected partially through frequency of occurrence within school textbooks (Fletcher, 1972; Hollands, 1983; Albany, 1985) and partially to satisfy the parameters presented above, the range of problems presented to the children were classified into the following categories:

Category A: Addition and subtraction combinations to ten. The numerical problems within this stage included:

- the addition and subtraction of zero, and the addition and subtraction of one.
- addition and subtraction involving doubles i.e. 4 + 4, 6 – 3,
- addition and subtraction involving two evens i.e. 6 + 2, 8 – 2; odd and even i.e. 5 + 2, 9 – 4, and two odds i.e. 3 + 5, 7 - 5.
- addition and subtraction of a pair of numbers with a difference of one i.e. 4 + 5, 9 – 8.

Category B: Addition and subtraction combinations within the range ten to twenty which included:

- a sample of addition problems involving teens where the units to be added included pairs that were considered in Stage 1 i.e. 12+ 0, 13 + 5, 3 + 16.
- the subtraction problems also included a sample which involved the use of Stage 1 subtraction combinations i.e. 15 – 4, 16 – 3 and "new" or extended subtraction combinations to twenty i.e. 12 – 8, 18 – 9, 15 – 9.

Category C: problems included:

- the addition of single digit numbers the sum of which was between ten and twenty i.e. 9 + 8 and 4 + 7, and 8 + 6. These were considered as "new" or extended addition combinations.
- simple subtraction problems which involved double digit subtraction without exchange i.e. 16 – 10, 17 – 13 and 19 – 17, and one additional problem involved exchange i.e. 20 – 8.

The full range of combinations are given in Appendix A (A1 and A2)

4.4.2 Item Presentation

After an initial series of visits to the schools to establish mutual familiarity with children in each class, each child was interviewed separately on at least two occasions with a week in between each interview. At the start of the first interview each child was told that the interviewer would present several problems and the child would be asked to find an answer to each problem using the method (s)he thought was best.

During each of the two interviews the children were presented with between 18 and 20 addition and subtraction numerical problems in two stages of difficulty. At the first interview the child was presented with the Category A combinations, consisting of number combinations to ten, and at the second interview the Category B and Category

C combinations which involved number combinations to twenty and a sample of arithmetical combinations which included 'teens' numbers. A third interview was given to those children who had required a substantial amount of time for interviews one and two. Each interview lasted approximately half an hour. Responses were recorded through the use of field notes.

In an attempt to eliminate variables which may have provided a cue to the solution procedure of the problems each problem was presented orally, and on paper. The oral presentation of the problems involved the attempt to use language in such a way that it did not present a clue to the use of a particular strategy. At the start of the first block of questions the exchange with the child would take the form of the following:

> "Would you like to do some addition (subtraction) sums for me?
>
> Do you like doing addition sums?
>
> I think you will get all of the ones you are going to do correct. They are all like this one". (At this point the sum 3 + 2 = was written for the child).
>
> "Try doing this sum best way you can".
>
> (After the problem was completed the child and the interviewer discussed the way that the child had completed the problem. The child was also asked why a particular method was chosen.)
>
> "I am now going to give you some more sums that are like this one. Do each one the best way that you can."

Further combinations were then presented. A similar introduction was given as each category of problems changed.

As each addition problem was presented symbolically on paper, for example, when presenting the combination 4 + 5 = the interviewer stated "four add five". In the case of as each problem was presented the interviewer stated used the word 'subtract' so that 6 – 3 = was written at the same time as the words "six subtract three" were spoken. The use of the word "subtract" was intended to eliminate any procedural cues that a

child would receive had "take–away" or "difference" been used. The solution strategy that each child used was recorded. If this was not completely clear, the child was asked to describe how the answer had been obtained.

After the first problem the children were reminded that they were to use the method that they thought was best. When children changed strategy they were asked to try to give a reason for the change. The problems within each stage were presented separately until the child had completed a section. If a child was either unable to give an explanation, or began to experience considerable difficulty, as measured by three incorrect solutions or by the length of time involved, the interview was terminated. Structured apparatus i.e. counters, unifix blocks, and colour factor rods, was available. Children were told that, if they wished, they could use it. However, the usual practice in both schools was pen and paper so these too were available.

4.5 STRATEGY CLASSIFICATION

Carpenter *et al* (1981,1982) had established a classification for a range of solution strategies used by first grade American children in solving addition and subtraction verbal problems. It is their classification that is taken as the basis for identifying strategies used by the children within the current experiment.

We have seen earlier in the chapter that qualitative difference in the children's thinking is to be identified through the use of a procedure or the use of fact retrieval through derived facts. It is of course fact retrieval that can cloud the issues of qualitative difference. Children may either remember a combination or use a remembered combination to establish one that cannot be remembered. Here we face the issue of known-fact or derived fact. Indeed what may appear to be a known fact may be an almost instantaneous derived fact; what may be a derived fact which may provide evidence of the use of a "thinking strategy" may be in essence a procedure, for example when adding 5 + 4 a child may indicate that "I always do five and five for this and then

take away one". Such a response may reasonably be seen as a procedure rather than a thinking strategy. The problems of interpretation have already been indicated and with this in mind, in general terms, the strategies that children use were classified according to the Carpenter & Moser's terminology. Previous interviewing (Gray, 1988) had provided indications of procedures that could be included within each of the integrated groups.

4.5.1 Addition

So that we may describe the strategies that were used to solve the addition problems, let **m + a = p** where **m** is the first of a pair of numbers given to the child and **a** is the second or the number to be added (additament) and **p** is the sum of the two.

Procedural Methods: (following Davis' (1983, p. 257) notion of procedure)

Count–all

Carpenter et al (1981) indicate that this strategy could be carried out making use of physical objects i.e. cubes, fingers, or, it could be carried out mentally. In the former case it involved the representation of both sets to be added and then the union of the two sets was recounted. Thus the strategy commences with the construction of the sets **m** and **a** and then the counting sequence begins at 1 and ends with **p**.

The following examples provide an indication of strategies classified as count-all within the current study:

Interviewer: "Three add five".

(i) James took three counters, one at a time, from the tray. As he placed each counter in from of him to form a line he uttered "*one..two...three*". Using his right index finger he recounted the three pointing to each counter in term and uttering "one... two...three". In the same way he took five

counters, again one at a time and placed them in a line slightly to the right of the line of three. As he placed each counter in the line he uttered "one...two...three...four...five". He recounted the five, starting from the left. Starting from the left hand counter in the set of three and pointing with his right index finger he indicated each counter in turn and counted "one....two....three...four...five...six ....seven...eight" He looked at the interviewer and said : "Eight".

(ii) Using her right index finger Felicity pointed to each of the three middle fingers of her right hand and uttered "one...two....three". Maintaining the display of these three fingers, whilst bending the small finger and the thumb of the left hand she then displayed the five fingers of her right hand. Starting with the thumb of her right hand she touched her forehead with each finger in turn whilst uttering "four...five...six...seven... eight". She looked at the interviewer and said "Eight".

Any equivalent strategy, even though it initially involved the construction of the sets **m** and **a** but does not involve the recounting of them both, was also to be considered as count-all.

Interviewer: "Five add nothing".

Sean displayed five on the fingers of his left hand and nodded at each finger from the left in turn with the accompanying verbal utterances one to five. He then looked at his right hand and closed his fingers. He looked from left to right and then back to the left.

"Five add nothing.....that's five."

Carpenter *et al* indicate that in "mental count-all", the sequence begins at one and then ends with the number representing the total of the two quantities. There was no evidence of the use of this strategy without the use of some external referent i.e. a subitised display of fingers, and then the motor act of head nodding which momentary focused on each finger in turn. Count-all, supported by a "subitised" display of fingers with and without verbal

utterances, was also identified. An example of a motor act (head nodding) being used as the referent (Steffe et al, 1982) was seen when one child added 2 + 1.

Count-on

Two variants were considered as count-on:

a) The sequence of counting starts at **m** or **m+1**, involves **a** increments and ends with **p**. This strategy made use of similar units to count-all. The sequence starts with the first given number or the successor of that number.

Interviewer: "Five add 2".

(i) Subject uttered "five", and extended her right thumb and right index finger at the same time uttering "six...seven".

(ii) Subject uttered "six" and extended his right forefinger to utter "seven".

b) The sequence of counting starts at **a** or **a+1** and involves **m** or **m - 1** increments and ends with **p.** Again similar units are involved but the counting sequence starts at the largest given number.

Interviewer: "Three add five"

(i) Subject uttered "five" and then extended right thumb and two adjacent fingers in sequence at the same time uttering, "six, seven eight".

(ii) Subject uttered "six" and extended right thumb and index finger uttering, "seven, eight."

Although frequently evident through the use of fingers and the accompaniment of verbal utterances, there were many instances where the identification of the count on from first strategy as distinct from count on from largest was identified through almost imperceptible head nodding or the slight flexing of fingers but no verbal utterances. Many of these instances relied on verbal evidence of support from the child. Examples of such procedures are:

(i) Interviewer: "Three add seven".

Subject looked at her left hand and the interviewer noted the faintest pressure exerted on fingers of the left hand which was placed on the desk. Pressure sequence moved from left to right on all five fingers and then back to little finger and the next one. No verbal signals.

Jane: "Ten".

Interviewer "Tell me how you did that one?"

Jane: "I started from three and counted up seven in my head–its ten"

Interviewer "Did you use your fingers?"

Jane: "No, I just looked at them."

(ii) Interviewer: "Three add seven".

Enzo stared at the desk – no external evidence of counting

Enzo "Ten."

Interviewer "Tell me how you did that one?"

Enzo "You have seven and you need three more to make ten".

Interviewer "How did you do seven add three then?"

Enzo "'Cos, if you count up that makes ten doesn't it.....first I started at seven.....and then I went eight, nine, ten...I just did it in my head"

As with count-all there was evidence of the subitised display of fingers used as counting referents to support count-on.

Retrieval Methods

A distinction was made between those methods which did and did not make use of a counting procedure. The analysis of methods which did not make use of counting will later be identified under the broad classification of retrieval. The identification of such methods generally relied extensively on children's explanations of their approach.

Derived Fact: The solution to a particular problem is generated by the use of:

i. a related number fact, i.e. "3 + 5 is 8, because 4 x 2 = 8 (3 + 1 = 4 and 5 – 1 = 4)", or

ii. a property of the number system, i.e. "8 + 6 = 14, because, 6 + 4 = 10 and 10 + 4 = 14".

Known Fact: The strategy is identified through retrieval of the particular fact.

4.5.2 Subtraction

The possible strategies available to solve subtraction problems of the form **m – s = d**, where **m** is the minuend, **s** is the subtrahend and **d** is the outcome of the subtraction process, presents a more complex picture. Reference to the literature helped to resolve some of the issues.

Procedural Methods

First there is a need to categorise the subtractive process that may be considered analogous to the addition 'count-all' strategy. Children who operated subtraction in unary form by counting out the minuend and then, from within the set formed, counted out the value of the subtrahend and recounting the remainder, were considered to be using a strategy the direct inverse to the addition strategy of count-all. To make a distinction between the subtractive and additive procedures it was decided to refer to the subtractive strategy as 'take-away' (Siegler, 1987).

Take-away

The strategy commences with the modelling of the set **m. s** objects are then counted and removed from **m.** The remaining objects are then counted from 1 to **d.** This process is analogous to the count-on process in addition. As with count-on it may be carried out with counters, bricks or fingers. Its most frequent application within the current study was with fingers.

The following examples provide an indication of the use of take-away:

Interviewer: "Eight subtract two".

(i) Christopher looked at the closed fist of his left hand and then the closed fist of his right. Both hands were extended before him.

Extending the fingers of his left hand, one at a time starting from the thumb he counted to five and continued to eight using successive fingers on the right hand starting at the little finger. As each finger was extended he uttered the appropriate number name. At each count he nodded towards the appropriate finger. With eight fingers extended he asked, "How many do I take away?". Interviewer replied "Eight".

Starting from the thumb of the left hand Christopher counted "one...two" at the same time bending down the thumb and the index finger. This action caused him some motor difficulty. As he bent down the index finger the next adjacent finger also had a tendency to bend. Christopher counted the remaining extended fingers, nodding towards and bending down each one as it was counted. Finally he gave the answer "six".

(ii) As the eight was spoken by the interviewer, Simon quickly extended five fingers on the left hand and three on the right. There was no evidence of counting. Two on the left hand were bent inwards when the two was spoken by the interviewer. The remaining six extended fingers were then counted starting at one.

(iii) As the eight was spoken by the interviewer, Jean put up eight fingers. As the interviewer said "*two*" Jean bent down two fingers and said "*six*" without counting.

Interviewer: "Six subtract three".

Rebecca extended both hands before her with the fists closed. She started the counting process for six at the thumb of her left hand. She extended six fingers, including the little finger of the right hand, one at a time synchronised with each counting word. Starting from the little finger of the right hand she counted three moving right to left to include the first

two fingers of the left hand. At each count she bent down the counted finger. She looked at the remaining extended fingers on the left hand and said "Three".

Count-back. There are two variants of count-back:

a) A backward counting sequence which starts at **m** and involved **s** decrements and ends with **d.**

The following example illustrate this count back procedure:

"Seven subtract five".

Peter uttered "seven" then extended his left thumb and uttered "six". He then extended the remaining fingers of the left hand one at a time, uttering "five...four...three...two". Two was given as the solution.

b) The backward counting sequence begins at **m - 1** involves **s** decrements and ends at **d.**

Example:

Shelly extended the little finger of the right hand and uttered "six". She continued to extend the fingers of the left hand in sequence uttering the words, "five...four...three...two". The solution was given as two.

Count-back to

The sequence of backward counting starts at **m** or **m – 1** and involves as many decrements as necessary to reach **s.** The number of decrements, which may or may not need recounting, equals **d.**

Where this procedure was used with external referents it was analogous to the count-back procedure but it was the number of extended fingers that provided the solution.

Interviewer: "Seven subtract five"

Carl uttered "seven....six.....five". As he uttered six he extended his right thumb, at five he extended his right index finger. He looked at the two fingers and gave the solution "two".

Count-up

The sequence of counting starts at **s** and continues forward as many increments as is necessary to reach **m**. The number of increments, which may or may not need recounting from 1, equals **d.**

Some mental procedures involving the count-up or count-back procedure were difficult to discern. Such procedures involved explanation by the child, which at times the child was not able to completely describe. Final identification of the procedure was subject to interpretation by the interviewer.

Some examples of interpretation would seem to be important here.

Interviewer: "Nine subtract five".

(i) Peter: "I had nine in my head then went, nine....eight..seven..six ..five and that leaves me with four.".....Count back to:

(ii) Jane: (The only visible action was very slight flexing of the fingers of the left hand.) "Four".

Int.: "Tell me how you did that one."

Jane: "I thinked it in my brain"

Int.: "Can you tell me how?"

Jane: "I counted it."

Int.: "How did you count it?"

Jane: "I started with nine and I.....what was it again?"

Int.: "Nine subtract five"

Jane "Oh yes....I went eight....seven...si...six....like that..."

Int.: "Did you use your fingers?"

Jane.: "No!"

The conclusion was that the procedure had been count back.

Interviewer: "Seven subtract five".

David: (No external evidence of use of counting) "Two"

Int: "Can you tell me how you did that one?"

David: "Five is five so I added on two.....I went six, seven."

The conclusion was that this was count–up.

The general distinctions between count-up and count-back were made at the point of interview. Count-up occurred infrequently but, whilst it does not form part of the descriptive analysis, where appropriate it is referred to in later sections i.e. sections 5.4.2. and 7.5.2. Equivalent strategies, even though they initially involved the construction of the sets **m** and **a** but did not involve the recounting of them both, were also considered to be in the count-up/count back category.

Retrieval Strategies

Derived Fact

The solution to a particular problem is generated by the use of

(i) a related number fact, i.e. "9 take away 5 is 4, because 5 + 4 = 9", or

(ii) a property of the number system, i.e. "15 – 9 = 6 because 15 – 10 = 5 but ten is one more than 9 so 6".

Known Fact

The strategy is based on the recall of that particular fact.

Final recording of the strategies used by the children for the purposes of this study is given within the appendices.

The interviews were carried out during the summer term of 1988.

CHAPTER 5

ANALYSIS OF RESULTS

5.1 INTRODUCTION

This chapter presents an analysis of the results of the first series of interviews.

Two central sections form the focus of the qualitative analysis contained within this chapter: the use of retrieval methods (Section 5.2) and the use of procedural methods (Section 5.4) by each of the age and ability groups which formed the sample. Subsections detailing the different application of known facts by groups of children (5.2.1) and providing some indication of changes in the use of known facts (5.2.2) are established within the discussion on retrieval methods. Section (5.3.3) deals with the use of derived facts.

At the end of the discussion on the use of each strategy there is a brief summary which highlights the essential differences between the three ability groups. A Chapter Summary is presented in section 5.5.

Discussion within the analysis is based on tables of raw scores presented within Appendices 3 to 8.

5.2 USE OF RETRIEVAL METHODS

5.2.1 The Known Fact Strategy

In a preliminary analysis of the results, the overall percentage of the solutions that were established through the use of a known fact by each age and ability group are considered.

1. Number Combinations To Ten: Category A

Figure 5.1 shows the mean-percentage of solutions to the Category A combinations that were obtained through the use of the known-fact strategy by each age and ability group. The range[1] of percentage facts indicates the highest and lowest percentage use of known facts by children within each group.

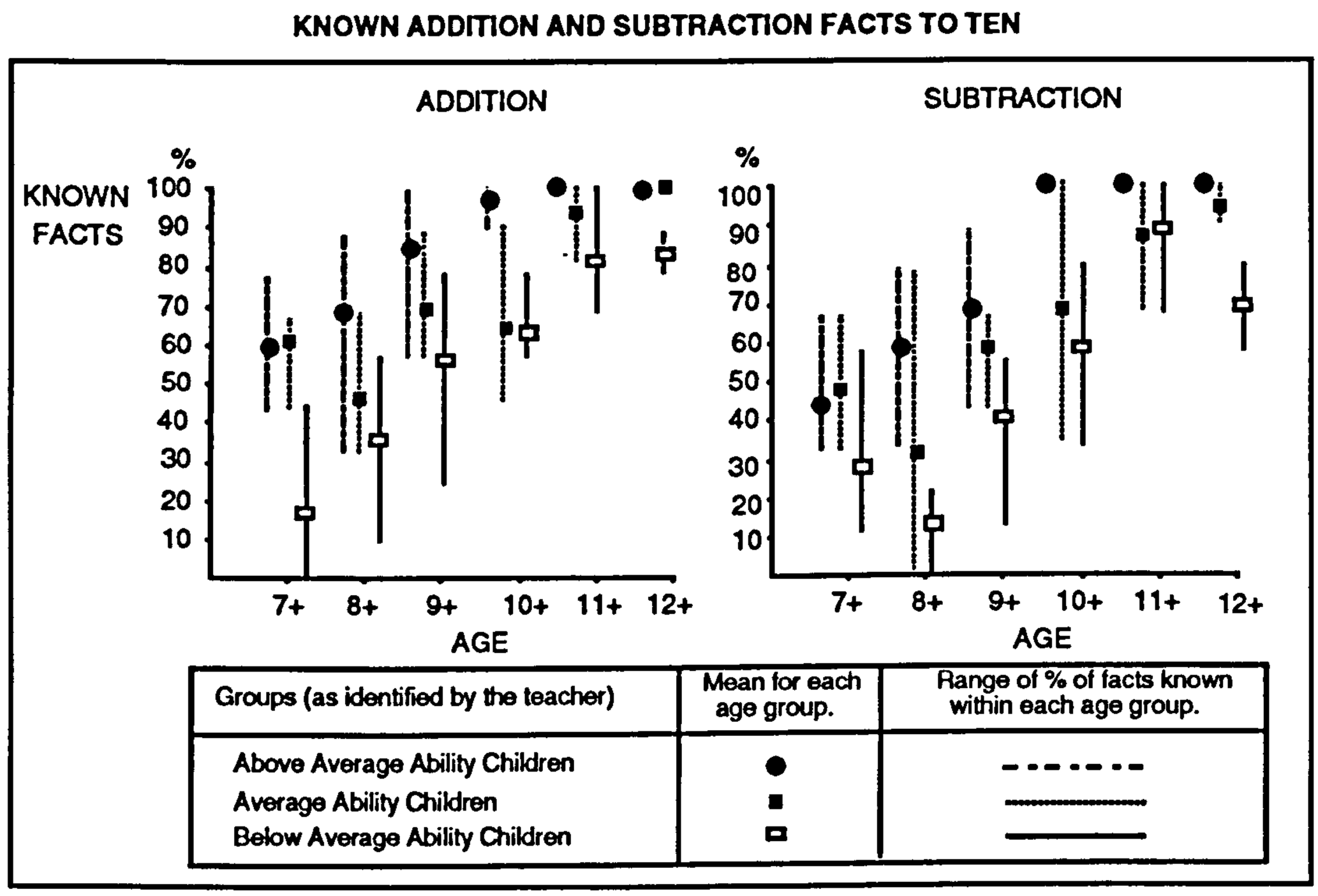


Figure 5.1 Addition and subtraction combinations to ten: the use of known facts

Although the general evidence presented by figure 5.1 would seem to indicate that children acquire knowledge of basic number combinations over a period of years, which would support the notions indicated within earlier research, for example, Groen & Parkman (1972), Ashcraft (1982), other issues which are worth noting are:

- the solutions to all of the complete range of addition and subtraction combinations were only known by the 'above-average' eleven and twelve-year-old children.

[1] The 'range' indicated in Figure 5.1, indicates the range between the high and low percentage use of known facts by children with each group. In fact, its representation on the figure does no more that place the mean in the context of the two most extreme scores within each distribution.

- of the children identified as achieving the normal standard within the class, those identified as 'average', only twelve-year-olds knew every addition combination to ten.
- no complete group of children identified as performing below the normal within the class, the 'below-average' group, knew all of the addition combinations or the subtraction combinations.

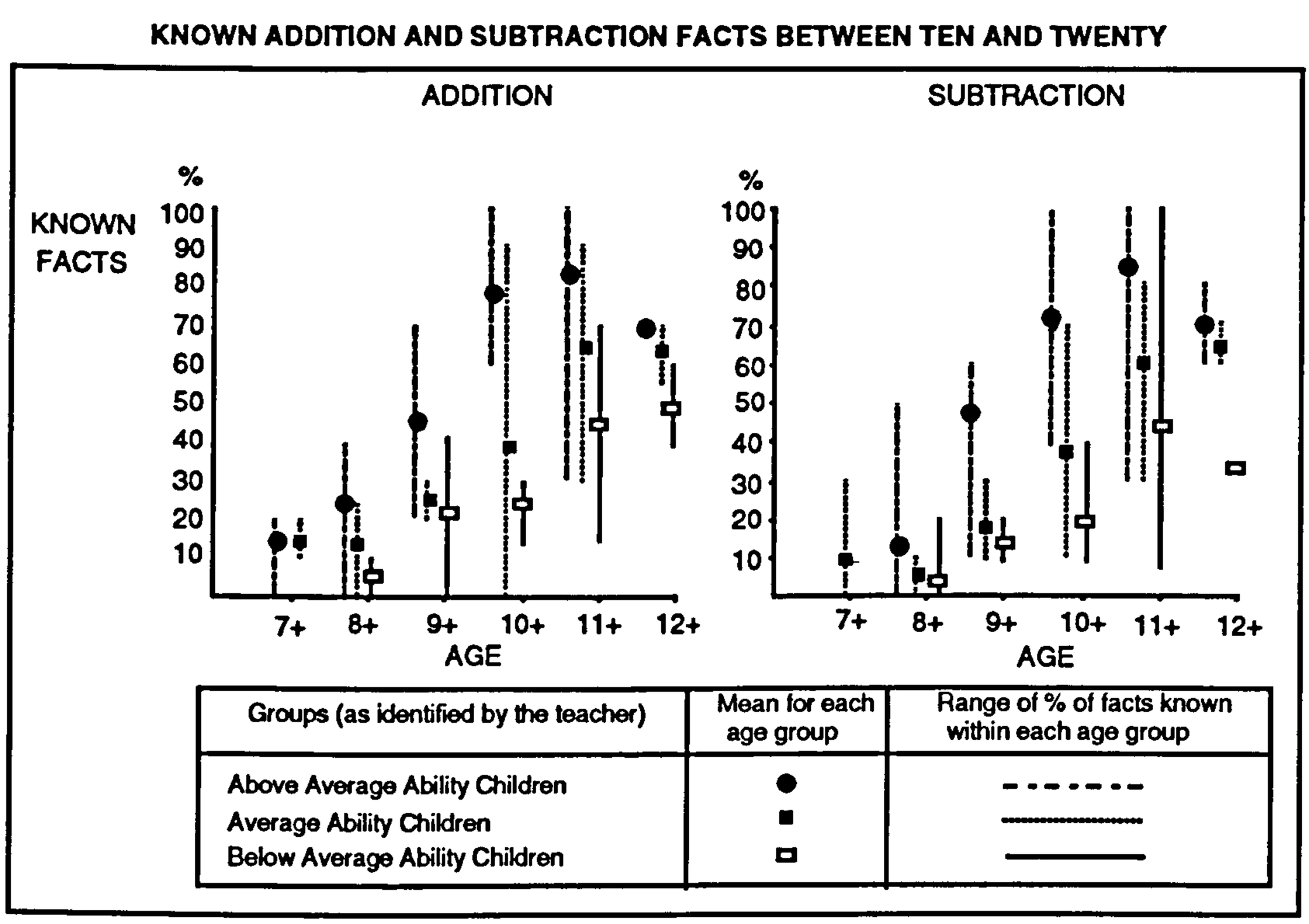
A further feature emerges which is worth a comment. Differences between the children identified as 'average' and 'above-average' were not consistent. Whilst the mean-percentage of solutions obtained through known facts appears to increase steadily as children grow older, the 'average' ability eight and ten-tear-old children do not appear to fit the general trend. The 'average-ability' eight-year-old children used approximately 15% fewer known facts than their seven-year-old counterparts. This may be no more than a feature of selection but it could be a reaction to the current emphasis within the schools. These children were being taught the addition and subtraction algorithms with exchange. One wonders if the complexity of the broader issues of 'exchange' may be responsible for the apparent decrease in the use of known facts.

Additionally, presenting the data in this way, masks the qualitative difference in the responses which are used if children do not know a combination. For example the range for the 'above-average' eight-year-olds extends over 60% points for addition whilst that for the 'average' group extends over 33% percentage points. What the data does not show is that the children represented by the lowest points of the range have back-up strategies that are qualitatively different – the child within the 'above-average' group used all derived facts, the one within the 'average' group used counting. But more of that later.

2. Number Combinations to Twenty: Categories B and C

The age and ability "attainment gap" identified in figure 5.1 becomes more apparent when the addition and subtraction combinations to twenty are considered (Figure 5.2).

This displays the equivalent mean percentages for the number combinations to twenty. The parameters established for figure 5.2 are similar to those established for figure 5.1 Children identified as 'below-average' amongst the seven-year-olds did not attempt any of the subtraction combinations to twenty.



N=4 for each ability group apart from the 12+ average and below average ability children where N=3
The seven year old below average ability group did not attempt the Stage 2 combinations.

Figure 5.2 Addition and subtraction combinations between ten and twenty; the use of known facts

We note within table 5.1 that over a period of time the percentage of facts known by the 'below-average' group converges towards the other two groups. A similar trend is apparent from table 5.2 for addition. For a few years the difference between the percentage of subtraction facts known by two groups seems to diverge. Once again the notion of using only one strategy to provide an indication of differences masks the qualitative difference in the use of back-up strategies. In every instant the back-up strategy used by the eight-year-old 'above-average' children proved to be a derived fact. In only two instances was this the case amongst children of the same age within the other two groups.

Although once again it may be reasoned that children need time to acquire the ability to recall the number combinations to twenty and that once again differences between those children identified as 'above-average' and 'average' are not consistent other issues arise that are worthy of note:

- No one complete group of children solved either every addition combination or every subtraction combination through the use of known facts.
- There was an age-related increase in the mean-percentage use of known facts up to and including the age of eleven.

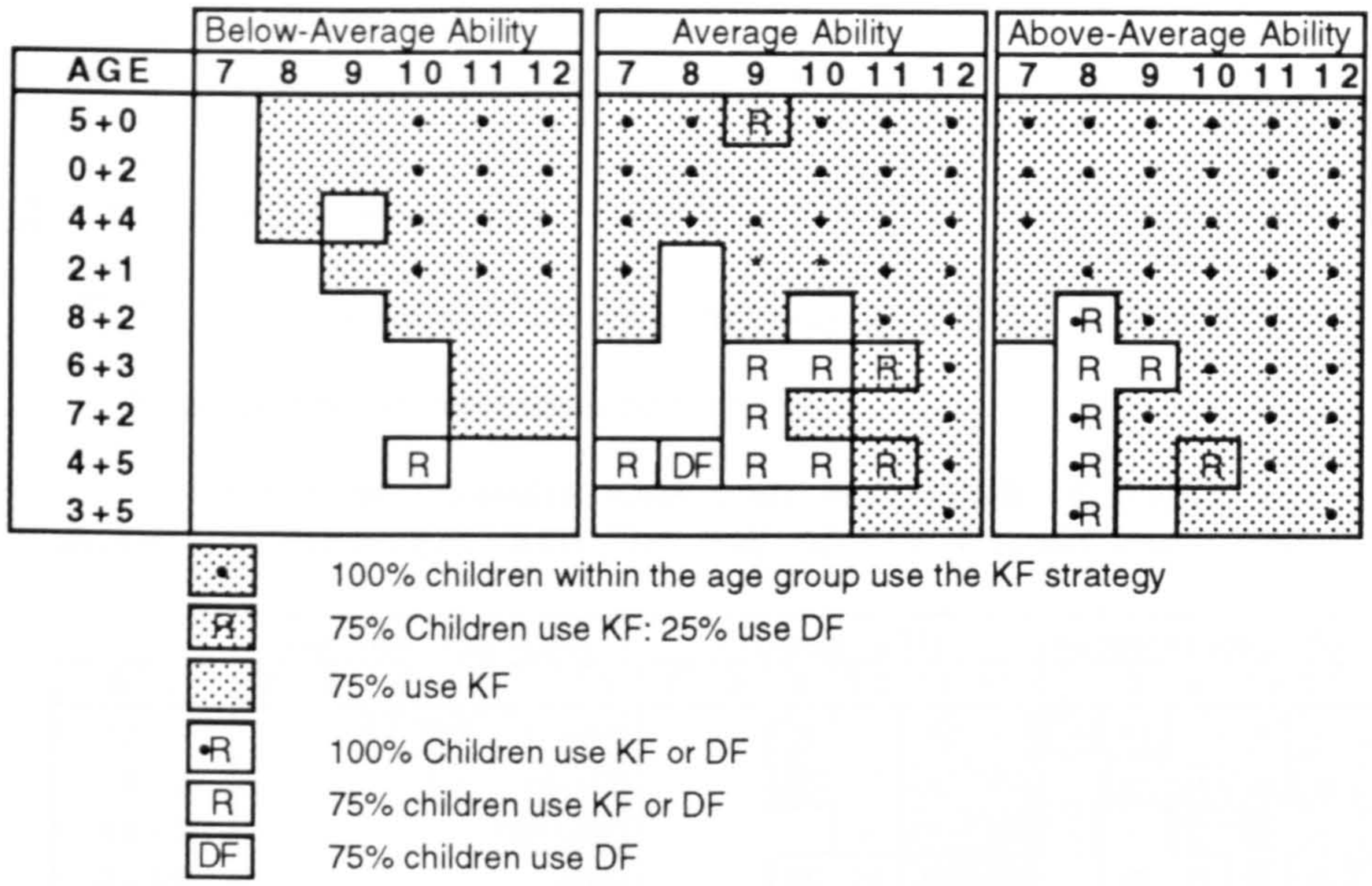
The picture that emerges from this presentation of the use known-facts by the children is that generally the number of solutions obtained through the use of known facts increases as children get older. However, it seems that 'below-average' children do not make the same extensive use of the strategy as the 'above-average' and the 'average' ability children. Although the proportion of addition combinations to ten that are known is almost equal across the three ability groups at the age of 11+, this is not be the case for the combinations to twenty; the 'below-average' ability children do not appear to reach the same facility level as the other two groups in their knowledge of subtraction facts during their period in the combined school.

5.2.2. Growing Competence with Known Combinations:

5.2.2.1. Number Combinations to Ten

Table 5.3 shows the extent to which the different age and ability groups used the known fact strategy to obtain solutions to the separate number combinations to ten. The known fact (KF) strategy is placed within a context which also considers the extent to which derived facts (DF) and/or, where appropriate, the combined use of KF and DF identified as retrieval methods (R), were used.

ADDITION COMBINATIONS TO TEN
INDIVIDUAL PROBLEMS AND THE USE OF THE KNOWN FACT STRATEGY



SUBTRACTION COMBINATIONS TO TEN
INDIVIDUAL PROBLEMS AND THE USE OF THE KNOWN FACT STRATEGY

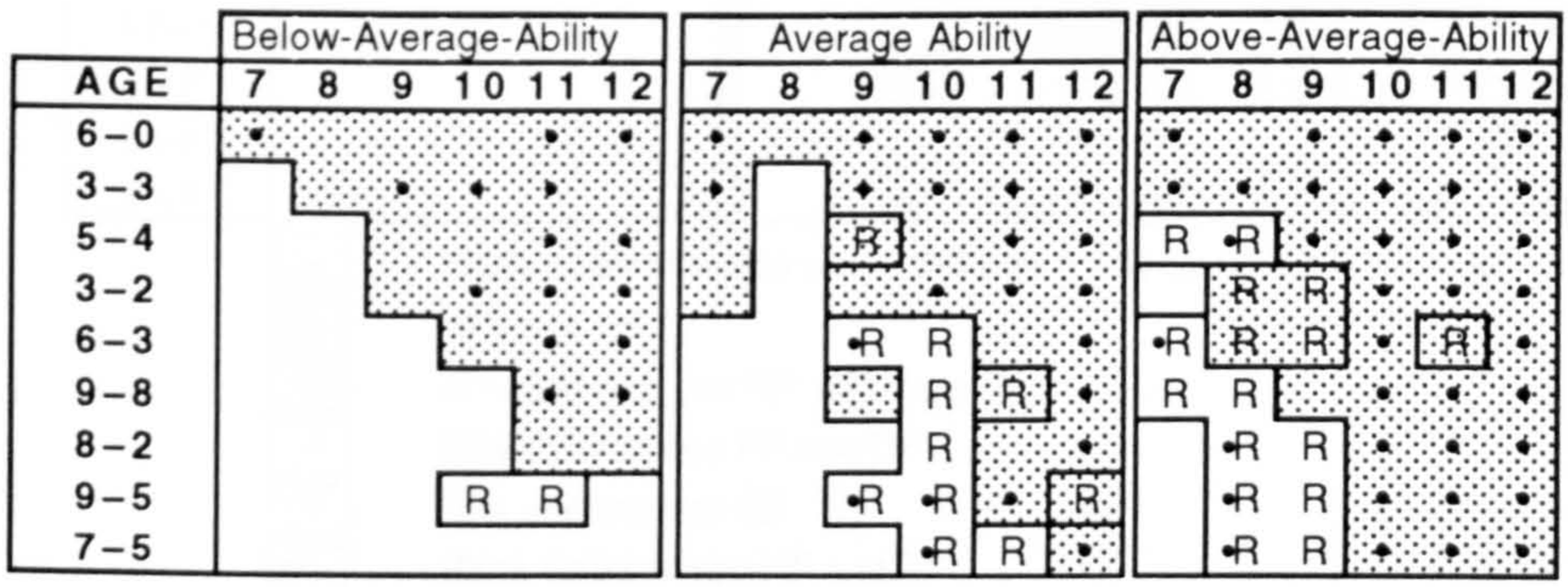


Table 5.3 Addition and subtraction combinations to ten: Individual problems and the use of the known fact strategy.

What is quickly apparent from Table 5.3 is the extent to which groups use known facts and, more interestingly, the extent to which each group appears to use derived facts prior to the complete use of known facts.

- The use of derived facts appears to support the known fact strategy amongst the 'average' and the 'above-average' groups of children.
- Some combinations seem to encourage the use of derived facts amongst the 'average' ability children, for example, 4 + 5 and 9 – 5. These are both solved with the use of known doubles which, for example, Groen &

Parkman (1972) and Woods *et al* (1975) have shown, are learned before most other combinations.

- The 'below-average' children show little evidence that they use derived facts.

5.2.2.2. Number Combinations to Twenty

Table 5.4 is established in the same way as Table 5.3. It indicates the responses obtained for the combinations to twenty.

ADDITION COMBINATIONS AND PROBLEMS TO TWENTY
INDIVIDUAL PROBLEMS AND THE USE OF THE KNOWN FACT STRATEGY

	Below-Average Ability						Average Ability						Above-Average Ability					
AGE	7	8	9	10	11	12	7	8	9	10	11	12	7	8	9	10	11	12
12+1				•	•	•					•	•		•R		•	•	•
10+2				•		•				R	•	•			•	•	R	•
18+2					•	•				R	•	•		R		•	•	•
3+16						•			DF	DF	R	•		•R	R	•	R	•
14+4									•R	•R	R	•		•R	R	•	•	•
15+4									•R	•	R	•		•R	•R	•	R	•
13+5											R	•		DF			R	•
4+7*									DF		•R	•DF		•R		•R		•DF
8+6*										R	•R	•DF			R	R	R	•DF
9+8*						DF			•DF	•R	•DF	•DF		R		•R	R	•DF

•	100% children within within the age group use the KF strategy
	75% use KF
R	75% Children use KF: 25% use DF
R	75% children use KF and DF
DF	75% children use DF
•R	100% Children use KF and DF
•DF	100% children use DF
9+8*	Indicates a Category C number combination

SUBTRACTION COMBINATIONS AND PROBLEMS TO TWENTY
INDIVIDUAL PROBLEMS AND THE USE OF THE KNOWN FACT STRATEGY

	Below-Average Ability						Average Ability						Above-Average Ability					
AGE	7	8	9	10	11	12	7	8	9	10	11	12	7	8	9	10	11	12
13−2					•				R	•R	•	•		R	•R	•	R	•
15−4									DF	DF	•	•		•R	•R	R	R	•
16−3						R			R	R	•	•		R	R	R	R	•
18−9*									DF	•R	•R	•R		•DF	R	R	R	•R
15−9*									DF	•R	•DF	•R		•R	R		R	•R
12−8*						DF				R	•R	•DF		DF	R	R		•DF
16−10						•				•	•	•				•	•	•
17−13											R	•		R		R	•	R
20−8						DF			•DF	R	•R	•DF		•DF	R	R	R	•
19−17											R	•		DF		•R	•	•R

Table 5.4 Addition and subtraction combinations and problems to twenty: Individual problems and the use of the known fact strategy.

As the combinations become less known we note two main trends amongst children aged nine and older.

(i) The general decline in the use of known facts is supported by the use of derived facts amongst the 'above-average' and the 'average' children.

(ii) The overall decline in the general use of known facts by the 'below-average' appears to be supported by the use of procedural methods as indicated by the none marked cells.

In particular we note that:

- No 'below-average' child used a known fact to obtain a solution to any of the category C addition problems. A derived fact was used to solve 9+8 by 15% of the children, mostly from the older age groups. There was slightly more extensive evidence of the use of known facts and derived facts when dealing with category C subtraction combination but this was only amongst the twelve year olds.
- Amongst the 'average' groups only one ten-year-old child used a known fact to establish the solutions to all of the category C addition problems. A combination of known facts and derived facts was used extensively by this and other ten to twelve-year-old children to obtain solutions to both addition and subtraction category C number combinations.
- Only amongst the eleven-year-old 'above-average' ability children was it observed that 75% of the age group obtain the solutions to the category C combinations through the use of a known fact. Once again there was a limited use of known facts to solve Category C subtraction combinations although derived solutions were extensively used by children aged eight to twelve.

One conclusion that may be drawn from the results displayed within Table 5.4 is that the number relationships appear to be more firmly established by younger 'above-

average' children than they are amongst the equivalent aged 'average' children. The evidence for this stems from the greater use these children made of derived facts.

The evidence obtained from tables 5.3 and 5.4 would seem to indicate that amongst the 'below-average' children the use of known facts is complemented with the use of a procedure. In contrast the more extensive use of derived facts, prior to the use of known facts amongst the 'average' and the 'above-average' groups of children, would appear to act as a cushion to the acquisition of known facts at one level whilst their use may also become a viable alternative strategy at another. In one sense the evidence that derived facts are used prior to the use of known facts would superficially lend support to notions that we should teach "thinking strategies" (Thornton, 1987, 1990; Steinberg, 1985) as a stepping stone to the acquisition of known facts. The essence of the use of a thinking strategies is to use what is known. The 'below-average' children do not appear to do this easily, whilst the 'above-average' and to a slightly lesser extent the 'average' seem to achieve their success from doing it.

5.3 USING DIVERSE STRATEGIES

The relationship between the use of procedures and the use of thinking strategies is brought into sharper focus when we consider in greater depth the integration of back-up strategies used by the children if they fail to recall a combination. The integration of the alternatives with the ability to recall some combinations automatically will provide a sense of the complexity of the issues that evolve from the children's attempts to solve the number combinations within the two stages.

5.3.1 An Overview of Integrated Strategy Use

The network presented in Figure 5.5 is a composite picture of the strategies used by all of the children to solve the addition and subtraction problems based on combinations to ten. The percentages, obtained by considering the numbers of children who illustrated

the use of a particular range of strategies (or one strategy), are rounded to the nearest whole and enable comparisons to be made between the proportions of children using a range of strategies.

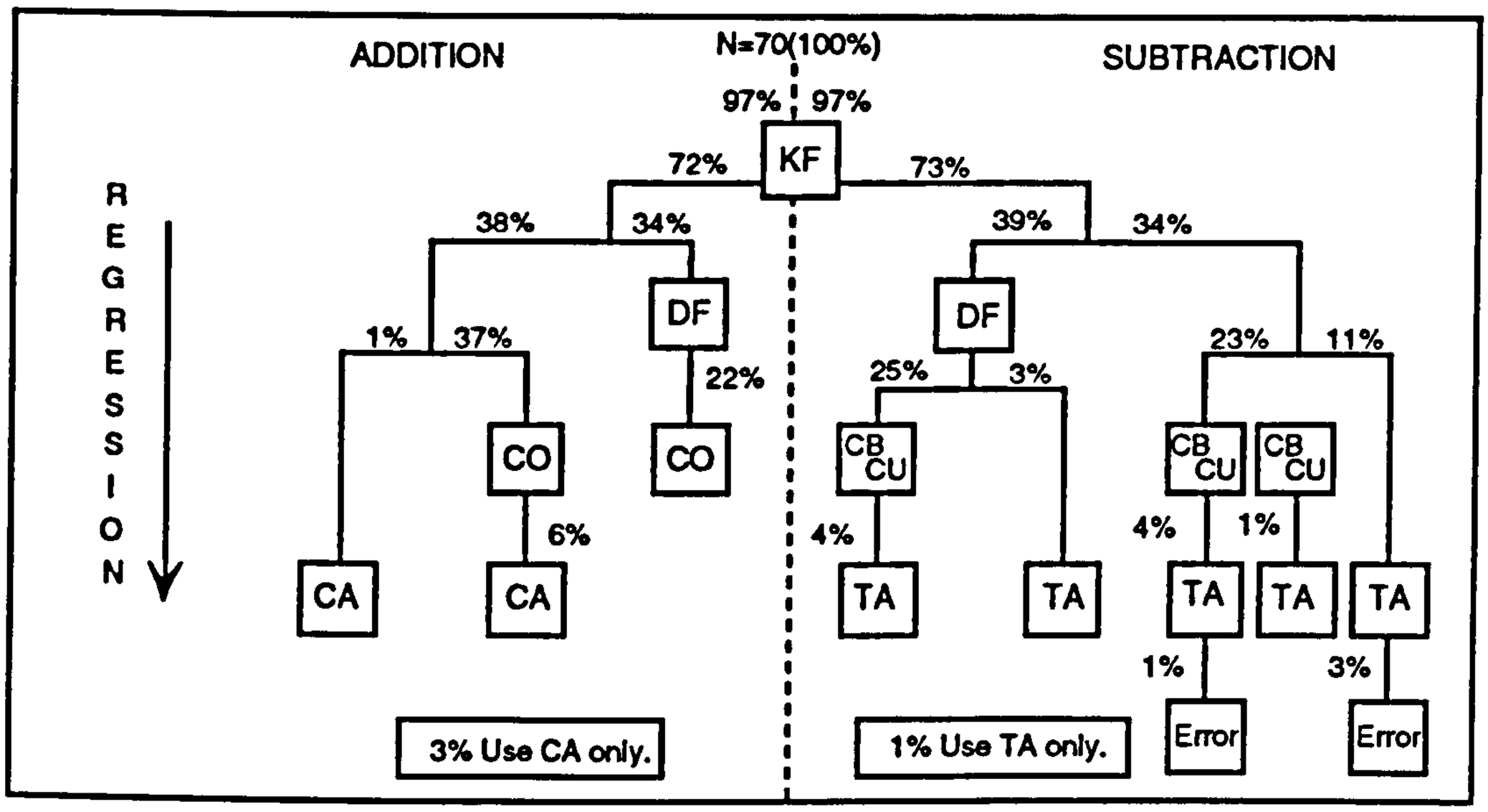


Figure 5.5 Addition and subtraction combinations to ten: composite strategy use.

Through following particular routes from the point of entry (the top centre of the diagram) we are able to trace the extent to which various combinations of strategies were used. A link may well have been formed between the derived fact use (DF) in addition and subtraction, but at this point it is easier to represent routes if these are considered to be discrete although, in practice, knowledge of complementary addition facts was used to solve subtraction combinations.

We can see from figure 5.5 that 97% of the children use known facts (KF) for the addition combinations to ten but 72% supplement their use with alternative strategies. The comparable figures for subtraction are 97% and 73%. From these figures it can be established that 25% of the children knew all of the presented addition facts whilst 24% knew the subtraction facts. 3% of the children use count-all only and 1% (one child) used take-away only. One other child used only procedural methods for subtraction.

It is striking that an almost equal proportion of children used a similar range of alternatives for subtraction. The balance between the use of retrieval methods and procedural methods is almost equal for addition and subtraction. The notion of regression is used to indicate the move from the use of a retrieval method, such as known fact or derived fact, to the use of a more naive strategy, such as, for example, count-on and then count-all (Gray, 1991a); these more naive strategies are slower to operate. The greater need for children to 'regress' to a take-away for subtraction compares strikingly with those who regress similarly to count-all for addition. It is conjectured that a reason for this may be identified through the application of the two methods. Count-on for addition is a natural process. Count-back is its natural reverse but not the easiest reverse, and therefore, its difficulties are likely to force weaker children to the easier, if less sophisticated, strategy of take-away. (Note that there was so little evidence of count-up that it has not been placed within the figures. It was generally only used by 'above-average' children as we shall see later.)

Apart from this one major difference the percentages that included procedural methods as the alternative strategies to known facts were otherwise almost equal; 60% for addition and 62% for subtraction. The balance, by implication, used only derived facts together with known facts; 12% addition and 11% for subtraction indicating that 88% of the sample did not use derived facts for addition and 89% did not use them for subtraction.

The varied use of strategies combinations increases in complexity when a similar analysis is considered for the combinations to twenty as seen in Figure 5.6.

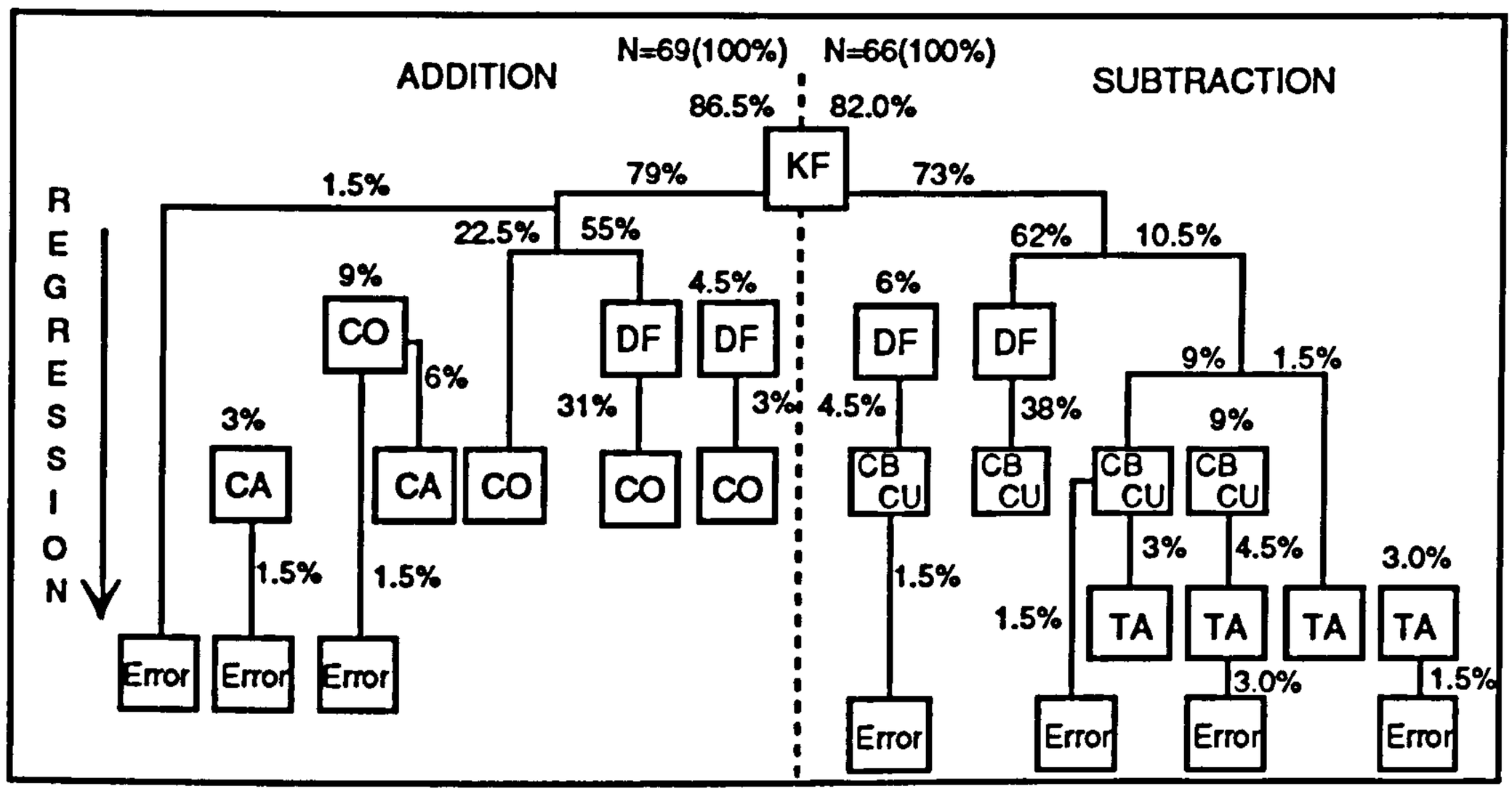


Figure 5.6 Addition and subtraction combinations to twenty: composite strategy use.

The following conclusions can be drawn from figure 5.6 when it is compared to figure 5.5:

(i) the overall picture increases in complexity since there was a considerable decline in the percentage of children who knew all of the facts.

(ii) there is a considerable increase in the percentage of children who use derived facts. Once again it is important to note that though no relationship is established between the use of, for example, an addition fact used to resolve a subtraction combination, this does not imply that there were no links between addition and subtraction in the use of derived facts but as previously noted it was generally restricted to the 'above-average' children.

(iii) there was an increase in the percentage of children who only used procedural methods.

(iv) there was greater evidence of errors

(v) the seven-year-old, 'below-average' children were not asked to attempt these combination, hence the change in totals from figure 5.5.

It is worth noting that apart from one eleven-year-old 'below-average' ability child who solved each of the subtraction combinations to twenty through the use of known facts,

other instances of the complete use of this strategy was confined to children identified as 'above-average'.

5.3.2 Integrated Use of Strategies: Age and Ability Considerations.

Composite pictures such as those presented within Figures 5.3 and 5.4 give an indication of what is happening across the spectrum of the age range but they do not help us to easily identify the contrast in strategies used by children of different abilities and ages. Clearer pictures are obtained if the integrated use of diverse strategies are related to the age and the teacher identified level of the children's arithmetical ability.

Figures 5.7 and 5.7 are area graphs which illustrate the cumulative percentage of the strategies used by the children to obtain solutions to the combinations. The graphs indicate *trends* between age groups and reflect proportions of individual strategies that the children of the three ability ranges, and the six age groups, used. Each group identified within the graphs, apart from the 'below-average' children and the 'average' children aged 12 +, contains a sample of four. The exceptions contain a sample of three.

Several features may be determined from Figure 5.7 that over-ride the obvious limitations of the sample sizes and the nature of the samples, which no doubt may account for some of the irregularities that are apparent. Although the trends indicate an increase in the use of known facts and a decline in the use of procedural methods, it is the interface between these two, the use of derived facts, that is of particular interest. For the 'above-average' and to a slightly lesser extent the 'average' groups the decline in procedural approaches is not accompanied by an immediate corresponding increase in known facts. The use of derived facts would appear to be a transitional stage which cushions the children against the need to use procedures. Such a cushion is hardly apparent amongst the 'below-average' groups. During the time when 'below-average' ability children are effecting the change between count-all and count-on, 'above-

average' children, in particular, appear to be resolving the issue of combinations that are not immediately known by using the fact that are known.

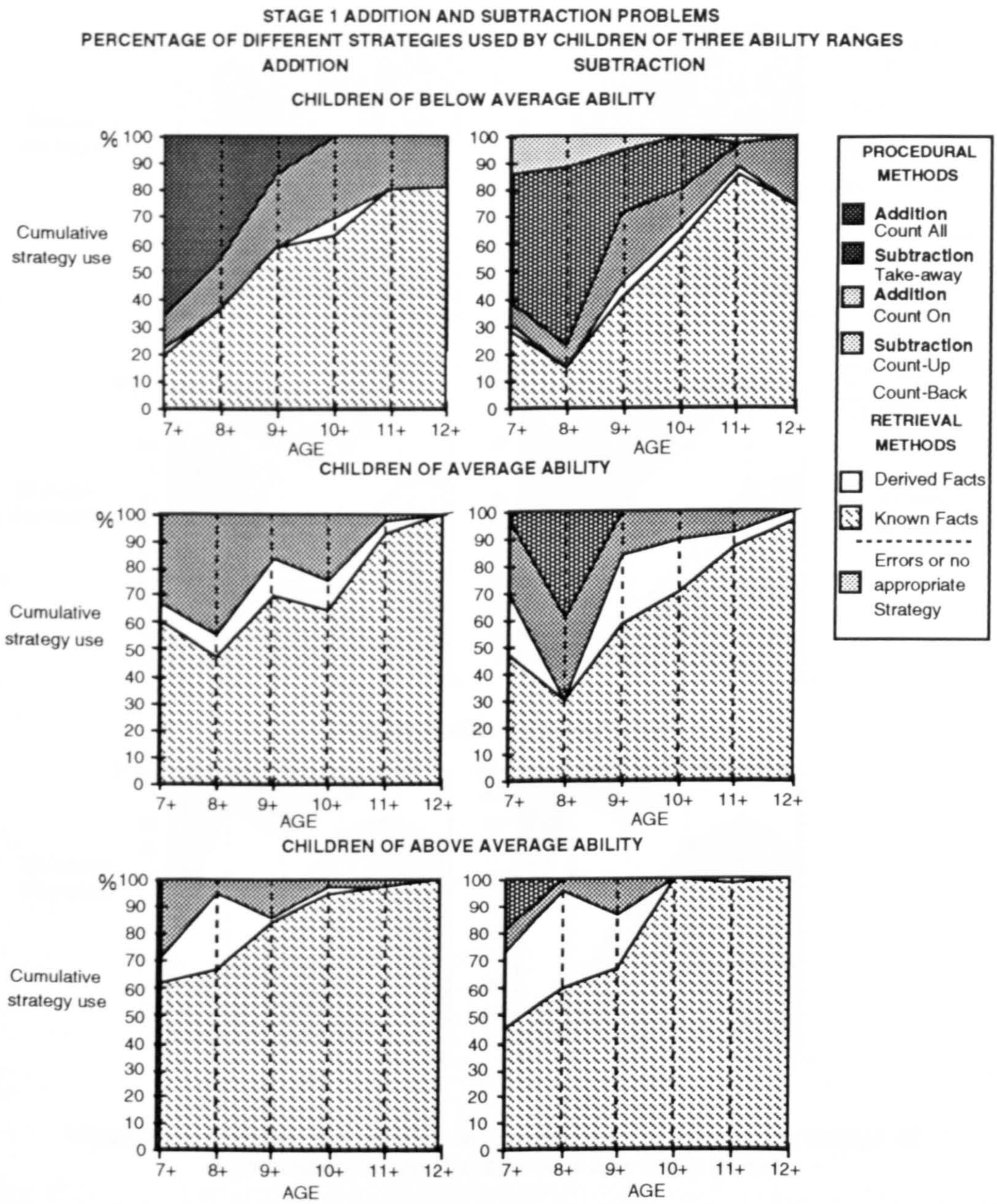


Figure 5.7 Addition and subtraction combinations to ten: Percentage of different strategies used by children of three ability groups.

Such distinctions are enhanced when we consider the equivalent combinations for the number combinations to twenty (Figure 5.8).

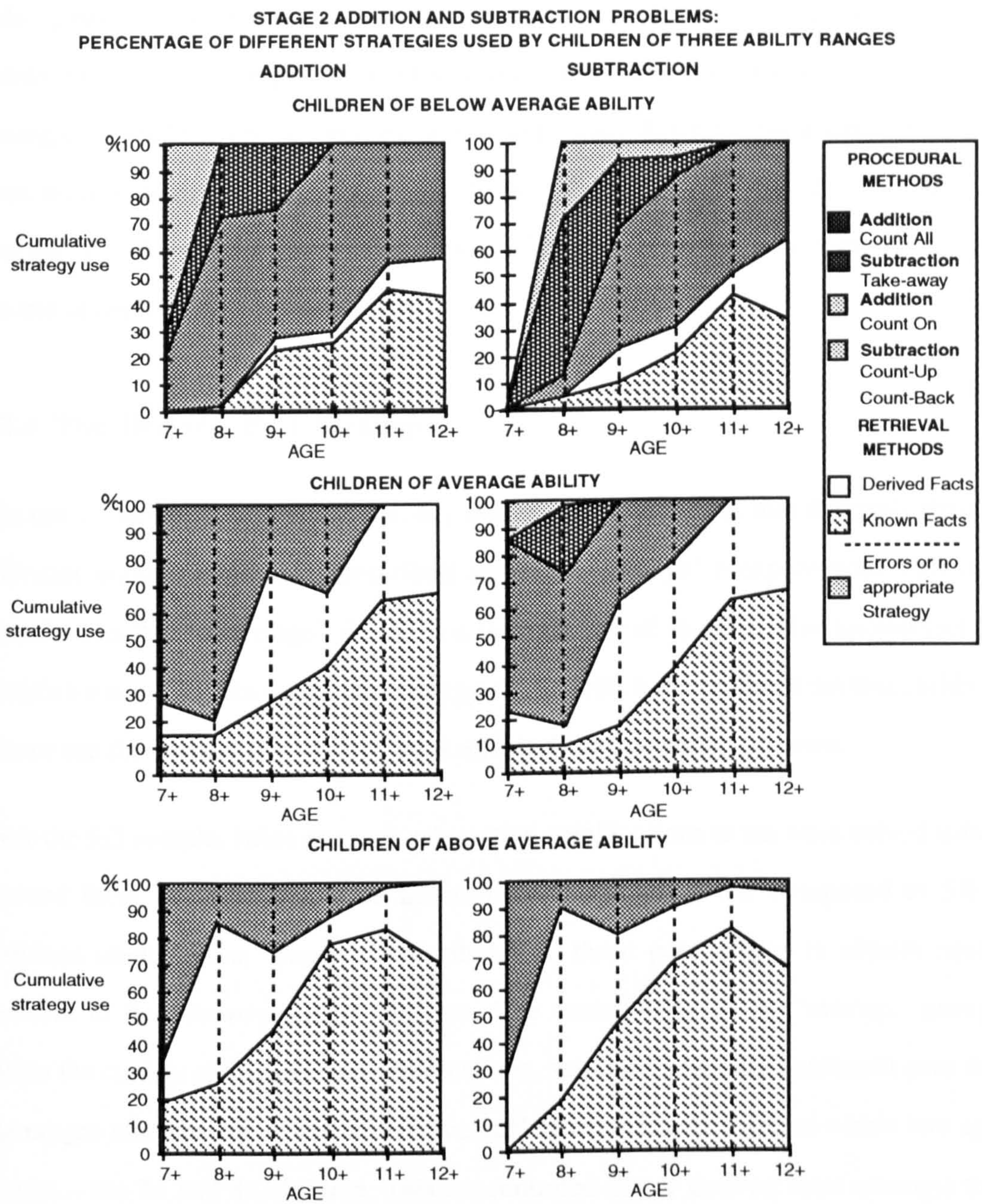


Figure 5.8 Addition and subtraction combinations to twenty: Percentage of different strategies used by children of three ability groups.

The extensive use of procedural methods by the 'below-average' group is clearly identified and so too is the "cushion" formed through the use of derived facts by the other two groups.

The extent to which different groups make use of derived facts is striking. Even when a particular age group of 'below-average' children, for example, the ten and eleven-year-

olds, appear to respond to a similar percentage of the addition combinations through known facts as do the eight and nine-year-old 'above-average' children, the back-up strategies when facts are not recalled appear to be quite different. Such differences are even more striking when strategies used by 8+ 'above-average' and the 10+ 'below-average' to solve the subtraction combinations are compared. We now turn to consider the use of derived facts in more detail.

5.3.3 The Derived Fact Strategy

The use of derived facts may be one key that provides an insight into the qualitatively different ways that children described as 'above-average' compare with children described as 'below-average'. Based as it is on the use of an alternative known fact to establish a new fact, the use of this strategy carries with it the implications that children cannot use the derived fact strategy until such time as *something* is known.

Over the full sample, twice as many subtraction combinations to ten were solved using derived facts strategy than were addition combinations (10% compared to 5%). Children identified as 'average' contributed to these percentages in almost equal measure to the 'above-average' children. The contribution of the 'average' group, within the context of the addition combinations, remained generally consistent over the age ranges whereas that of the 'above-average' group was concentrated within two age groups – the 7+ and 8+ children. This concentrated use of derived facts amongst the 'above-average' children is also noticeable when the subtraction combinations are considered, but, there are inconsistencies between different age groups of the 'average' children. The complete use of derived facts by the 'below-average' children represented slightly more than one twentieth of the total for addition and about one tenth of the total for subtraction.

5.3.3.1 Derived Facts and Number Combinations to Ten

The pattern of the contributions of each age group differed considerably.

a) Children identified as 'above-average':

- The most extensive use of derived facts was noticed amongst the seven and eight-year-old children, seven of the eight children making use of the strategy at least once to solve either an addition or subtraction problem.
- The use of known doubles dominated the derivatives used to obtain solutions:

  a) 5 x 2 ("five two's') or 5 + 5 ("five and five") and then 1 subtracted, were the most frequent strategies used to solve 5 + 4 through a derived fact. 5 x 2 was the usual known fact used to derive a solution to 9 – 5.

  b) Similarly 3 + 3 = 6 enabled children to solve 6 – 3 but knowledge of multiplication facts was used in some instances to derive the solution to 6+3, : 3 x 2 = 6, 3 x 3 = 9.

  c) A known double helped children solve 3 + 5: most frequently from 4 + 4 ("four and four") or 4 x 2 ("two fours").
- Related addition facts were used to derive solutions to subtraction problems such as 7 – 5, 9 – 8, 8 – 2, 3 – 2 and 5 – 4. The seven and eight-year-old children made most use of related addition facts, (all of the older children, apart from two nine-year-olds, one of whom used count-on, knew the solutions to the presented subtraction problems).
- The use of known pairs of numbers that make ten helped children to derive solutions to 7 + 2 and 8 + 2.

b) Children identified as 'average'

Apart from evidence that 4 + 5 was solved through the use of a derived fact by 39% of the 'average' children spread over the age ranges seven to eleven, other evidence of the use of the strategy for the addition problems by this group was extremely fragmentary.

- Only one child, a ten year old, solved more than one addition problem using a derived fact.
- On the whole, evidence of the use of derived facts for addition was restricted to the solutions for 4 + 5 and 6 + 3, the strategies used being similar to those used by the 'above-average' ability group of children.
- Two children obtained the solution to 7 + 2 through knowledge of a pair that made ten whilst one used 4 + 4 to obtain the solution to 5 + 3.
- In contrast to the evidence noted amongst the 'above-average' children, no 'average' eight-year-olds derived the solution to any subtraction combination.
- The nine-year-olds demonstrated the highest incidence of use of derived facts. It was the complementary combination to 4 + 5, 9 – 5, that reflected greatest use: 35% of the full sample of solved this combination by deriving it; 50% using the related addition fact, 4 + 5; 50% using knowledge of combinations to ten. Three children, two seven-year-olds and one nine-year-old, used a derived fact to obtain the solution to both of these problems.
- The ability to use the related addition fact to solve 7 – 5 and 9 – 8 was noticed amongst the nine and ten year olds.

The trend amongst the 'average' children was to use the derived fact strategy later and over a longer period of time than the 'above-average' children but, perhaps, there was a narrower range of problems solved through the use of the strategy although the evidence is not conclusive.

c) Children identified as 'below-average'

Only one eight year old solved 4 +5, through what must by now be recognised as a standard solution strategy, 5 + 5 – 1, although the solution to 9 – 5 was obtained through the use of a derived fact by four children across the ages nine to eleven. The only other subtraction problem solved by the use of a derived fact was 6 – 3, the solution derived through the use of the known addition fact, but only one child actually did this.

5.3.3.2. An Interim Summary of the Use of Derived Facts

From the evidence obtained through the analysis of the use of derived facts to establish a solution to either an addition or subtraction combination to ten, the differences between the three ability groups can be summarised as follows:

- The 'above-average' children make use of derived facts at a young age and their use suggests the strategy is a stepping stone to establishing *new* known combinations.
- The 'average' children use derived facts at a later age than do the 'above-average' children. The evidence suggest that they require a firmer base of known facts from which to use derived facts but then they are also used as a stepping stone to establishing new known facts.
- Their is very limited use of derived facts amongst the 'below-average' children even when they have a fairly extensive repertoire of known facts.

5.3.3.3 Derived Facts and Number Combinations to Twenty

There is a considerable increase in the use of derived facts to solve the combinations to twenty compared to their use to solve combinations to ten. This increase is marked not only by an increase within each separate ability group but by changes within each group that relate to both age and problem type as can be seen in Table 5.9.

Table 5.9 indicates the percentage of each complete ability group that made use of a derived fact to solve each addition and subtraction combination to twenty. The

problems are arranged high to low in an order identified by the percentage of the full sample of children (*N*=70) who solved a particular problem through the derived fact strategy.

ADDITION

		n	Ability Group %	Full Sample % N=70
9+8	Below av.	23	22	44
	Average	23	57	
	Above av.	24	54	
4+7	Below av.	23	9	31
	Average	23	43	
	Above av.	24	42	
8+6	Below av.	23	22	28
	Average	23	43	
	Above av.	24	38	
14+4	Below av.	23	17	24
	Average	23	35	
	Above av.	24	21	
15+4	Below av.	23	0	24
	Average	23	35	
	Above av.	24	38	
3+16	Below av.	23	0	18
	Average	23	26	
	Above av.	24	29	
13+5	Below av.	23	0	10
	Average	23	9	
	Above av.	24	21	
18+2	Below av.	23	0	6
	Average	23	9	
	Above av.	24	8	
10+2	Below av.	23	0	4
	Average	23	4	
	Above av.	24	8	
12+1	Below av.	23	0	3
	Average	23	0	
	Above av.	24	8	

SUBTRACTION

		n	Ability Group %	Full Sample % N=66
18–9	Below av.	19	21	39
	Average	23	52	
	Above av.	24	42	
15–9	Below av.	19	21	39
	Average	23	52	
	Above av.	24	46	
12–8	Below av.	19	26	38
	Average	23	35	
	Above av.	24	50	
20–8	Below av.	19	21	33
	Average	23	48	
	Above av.	24	29	
16–3	Below av.	19	11	26
	Average	23	30	
	Above av.	24	33	
15–4	Below av.	19	0	17
	Average	23	17	
	Above av.	24	33	
17–13	Below av.	19	5	17
	Average	23	22	
	Above av.	24	25	
19–17	Below av.	19	0	15
	Average	23	17	
	Above av.	24	29	
13 – 2	Below av.	19	0	12
	Average	23	13	
	Above av.	24	17	
16–10	Below av.	19	11	11
	Average	23	4	
	Above av.	24	17	

'n' indicates the number in each ability group.
N indicates the total in the full sample.

Table 5.9 Percentage of combinations to twenty problems solved by the derived fact strategy: ability group and full sample considerations.

There are several points to note from Table 5.9 :

- Addition combinations from category C, that is, 9 + 8, 4 + 7 and 8 + 6 were solved more extensively by derived facts than those combinations that contain units previously used in the category A number combinations – those of category B.

- Subtraction combinations where a single digit number is to be subtracted from a teen number where the unit is smaller than the number to be subtracted, that is, 18 – 9, 15 – 9, 12 – 9 and 20 – 8 were solved more extensively through the use of derived facts.
- Each combination to twenty was solved through the use of a derived fact by at least one 'above-average' child.
- 12 – 8 and 20 – 8 were solved extensively through the use of derived facts by the twelve-year-old 'below-average' children.

The general trends within each ability group and the differences in the use of derived facts between each ability groups of children require closer examination.

a) Children identified as 'above-average'

(i) Addition Combinations:

1. Category B Combinations, for example, 14 + 4, 3 + 16 etc.

   The focus of attention here is on the 'above-average' children younger that ten – older children obtained the solutions to these combinations through the use of known facts. The younger children used their knowledge of 4 + 4 and 3 + 6 and 3 + 5 etc. to compute the units component of 14 + 4 and 3 + 16 and 13 + 5 and then add ten to each solution.

2. Category C Combinations, for example, 9 + 8, 4 + 7, 8 + 6.

   - Doubles were increasingly and extensively used amongst the children to obtain solutions to 9 + 8 and 8 + 6.
   - Eight year old children, in addition to using the double nine, demonstrated the use of number pair to make ten e.g. when solving 4 + 7 they used (7 + 3) + 1 but also evidence of (6 + 4) + 1. This was a trend continued spasmodically amongst the older children but it dominated the solution strategy used by the twelve-year-olds.

- Although every twelve-year-old child knew all of the solutions to the addition number combinations to ten all them used derived facts for the category C combinations. The strategies used were based on known doubles or number pairs that make ten. (This level of use of derived facts by the twelve-year-old children was in contrast to that displayed by the eleven-year-olds, only one of whom failed to use the known fact strategy).

(ii) Subtraction Combinations.

- Combinations such as 12 – 8, 18 – 9 and 15 – 9 were solved extensively by the two youngest age groups and the two oldest age groups through the use of a derived fact.
- Combinations such as 17 – 13 and 19 – 17, either using a known addition fact, for example, "7 – 3 = 4 because 3 + 4 = 7", and then dealing with the tens, or, used knowledge of tens, for example, "20 – 8 is 12 because 10 and 10 is 20, 8 from 10 is 2 add 10 is 12".

Overall there was extensive use of derived facts by the 'above-average' children when solving combinations to twenty . Their use dominated as a back-up strategy amongst all of the age groups.

b) Children identified as 'average'

(i) Addition Combinations:

1. Category B Combinations, for example, 14 + 4, 3 + 16 etc.
   - Solutions derived by nine and ten-year-olds were known by eleven and twelve-year-olds.
   - The ways in which solutions were derived were similar to the methods used by the 'above-average' children.
2. Category C Combinations, for example, 9 + 8, 4 + 7, 8 + 6.
   - From the age of nine, 'average' children made considerably more use of derived facts to solve these problems than did the "above-average"

children: derived facts were used in 73% of the instances by average ability children, in 48% by the 'above-average' ability children.

(ii) Subtraction Combinations.

The general pattern in responses reflects those given by the 'above-average' children but it also follows the trends set when the children dealt with addition:

- Although there are very few examples of derived solutions by children below the age of nine there is widespread use of the strategy from that age onwards.
- Children above nine use the strategy more substantially for subtraction than for addition, more–so even than 'above-average' children of the same ages.

It is worth noting that the 'average' children of ten and older know solutions to fewer combinations than their more able counterparts – 52% compared to 74%.

Every 'average' child above the age of nine derived at least one solution to the subtraction combinations. 37% of the solutions obtained by children nine and older were through derived facts. The 'above-average' children within the same age bracket derived 20% of the possible solutions. The difference in the figures can partly be explained by extensive differences noted in responses to particular problems.

- 81% of 'average' children derived the solution to 15 – 9, by far the most common method being 10 – 9, 5 + 1. In contrast 33% of the "above-average" children derived the solution (50% displayed the use of a known fact).
- 63% of the 'average' group derived the solution to 18 – 9, all from the known double; 66% of the 'above-average' group knew the solution, 33% derived it from the known double. This problem may present particular difficulty in interpretation. It could be that we may be seeing evidence of an

instantaneous derived fact which the 'above-average' children do so quickly that when requested to reflect upon their method they are unable to distinguish a derived fact strategy from a known fact.

Apart from the solution to 16–10, which was known by every child above nine in both ability groups, solutions to 17 – 13 and 19 – 17 were as likely to be counted by the 'average' children who did not use the known fact strategy as they were to be derived.

c) Children identified as 'below-average'.

In general, deriving facts amongst 'below-average' children was limited to individuals or to specific problems. In only three separate instances did children derive solutions to any problems that had involved combinations previously met within Category A. In each case these illustrated the use of the known double to solve 4 + 4 and then the addition of ten to provide the solution for 14 + 4.

There were no derived solutions to any of the addition problems involving single digits facts by any 'below-average child younger than eleven. (No child from a 'below-average' group solved any of these problems making use of known facts). One eleven-year-old derived the solutions to all such combinations but in general it was the solution to 9 + 8, from "two nines", that was most frequently derived. However, Stuart (aged 11+) responded to 8 + 6 by saying "I know 8 and 2 is ten, but I have a lot of trouble taking 2 from 6. Now 8 is 4 and 4; 6 and 4 is 10; and another 4 is 14." We may feel we should congratulate Stuart for the breadth of arithmetical manipulation that he displays but the truth of the matter is that his particular approach indicates not so much what he knows as what he does not know. He knows number combinations that make ten but in this context has difficulty with 6 – 2. Such a method may place a severe burden of "inventiveness" upon him which in the long term prove too great a burden to bear.

There was slightly more widespread use of derived facts to solve the subtraction combinations to 20 but rather than efficient contracted approaches usually seen within the other groups, the 'below-average' children frequently used derived facts in a complex way. There was evidence that fingers were used as a visual support, an action not identified within the other two groups. Solutions resolved in this way were categorised as "derived fact" because there was no visible evidence of actual counting and verbal explanations satisfied the interviewer that although fingers were needed to support thinking, the explanation indicated that the child used a thinking strategy.

For instance, Karen (11+), showed a subtle understanding of number relationships (Figure 5.10).

	DISPLAY		EXPLANATION TO CALCULATE 15 –9	
	Left Hand	Right Hand	Child's Explanation	Interviewer's Comment
STAGE 1			*Fifteen is ten and five. Forget the ten.*	Five fingers shown on left hand, (Other ten presumably held in mind.) Right hand closed.
STAGE 2				The child displays the nine to be taken away as a five and four.
STAGE 3				The left hand is closed, to cancel the displayed five with the previously displayed five.
STAGE 4			*Four from one of the fives making ten leaves one.*	The remaining four is taken from one of the fives held in the mind.
STAGE 5			*One and the other five from the ten make six.*	Remaining five in mind now displayed, giving a total of 5 and 1, which is 6.

Figure 5.10: Subtracting nine from fifteen by an inventive route

Karen made considerable use of her fingers. To perform the calculation 15 – 9, she held out five fingers on her left hand and closed it completely; she then held up four fingers on her right hand closed them and opened the right thumb, then

redisplayed the five fingers of her left hand at the same time and responded "six." The whole procedure took about three seconds.

Other 'below-average' children who attempted to derive facts often had to do this based on a limited number of known facts that might not furnish the most efficient way to perform the calculation.

Michael (aged 12+), faced with "16 – 3", said "ten from sixteen leaves six, three from ten leaves seven, three and seven makes ten and another three is thirteen." Michael appears to seek familiar number bonds to solve the problem. He sees 16 as 6 and 10, but takes the three from the 10 rather than from the 6 and ends up having to do the additional sum "six and seven".

Amongst 'below-average' children then, derived facts were used in a limited if occasionally very complex way. The breadth of combinations resolved through the use of derived facts did not match the range resolved through their use by the other two groups.

5.3.4 A Summary of the Use of Derived Facts

Overall the use of derived facts by the children in the sample can be summarised as follows:

(i) Derived facts were more extensively used for the number combinations to 20 than for number combinations to 10.

(ii) There was more evidence of the use of derived facts for subtraction than for addition.

(iii) 'Above-average' children tended to make extensive use of derived facts at a younger age than children from the other two groups

(iv) Evidence of more than a very limited use of derived facts by the 'below-average' children was only apparent amongst the twelve year olds.

Some important issues are noted that begin to suggest qualitative differences in the mathematical behaviour of the 'above-average' children compared to the 'below-average' children:

- Amongst the 'above-average' children there is evidence of more extensive use of derived facts as an alternative method of resolving solutions to number combinations if they do not recall the solution through a known facts.
- 'Above-average' children use derived facts efficiently.
- The use of derived facts as an alternative to known facts by the 'above-average' children may:
  a) help younger children to establish new known facts as seen in the trends apparent when the deal with number combinations to 10.
  b) release older children from the need to remember all of the combinations as seen in the trends apparent when they deal with number combinations to 20.

  'Average' ability children appear to follow the same trends as the above average ability children but at a later stage.
- Apart from some common exceptions that appear to be problem and age related, children from the 'below-average' group did not generally use derived facts as a viable alternative to the use of known facts. Some examples of their use display complexity rather than efficiency.

The 'above-average' and the 'average' ability children use derived facts more and with greater efficiency than do the 'below-average' children.

The 'below–average' children, even as they grow older, seem to rely extensively on procedural methods. It is these methods that are now considered.

5.4 THE USE OF PROCEDURAL METHODS

5.4.1 Number combinations to Ten

Tables 5.11 and 5.12 show the percentage of each individual number combination solved through the use of procedural methods by each of the three groups of children. The order in which the problems is presented was established from the overall use of known facts. Thus 5 + 0 was most frequently solved using a known fact whilst 3 + 5 least frequently.

	N	5+0	0+2	4+4	2+1	8+2	6+3	7+2	4+5	3+5
'Below-Average' Group	23	13	17	21	26	48	52	56	69	83
'Average' Group	23	4	4	0	9	30	48	43	9	52
'Above-Average' Group	24	0	0	0	13	8	13	21	13	25
Full Sample	70	6	7	7	16	29	38	40	30	53

The percentage use of procedural methods to solve addition combinations to 10

Table 5.11

	N	6−0	3−3	5−4	3−2	6−3	9−8	8−2	9−5	7−5
'Below-Average' Group	23	13	13	30	35	52	61	65	52	73
'Average' Group	23	4	9	18	22	22	30	47	26	47
'Above-Average' Group	24	0	0	4	8	0	12	12	21	12
Full Sample	70	6	7	17	22	25	34	41	33	44

The percentage use of procedural methods to solve subtraction combinations to 10

Table 5.12

47% of the solutions to the additions combinations to ten were obtained through the use of procedural methods by the 'below-average' group of children. The overall use of count-all and count-on was almost equal amongst the children although the younger children tended to use the former and older children the latter. In contrast amongst the 'average' and 'above-average' there was no evidence of count-all. Count-on was extensively used amongst the younger 'average' children, 7 + 2, 4 + 5, 3 + 5 and 8 + 2 evoking the greatest proportion of procedural responses.

When dealing with the subtraction combinations to ten a similar pattern emerged amongst the 'below-average' children. Take-away, the complementary strategy to count-all, was extensively used amongst the younger children groups. It remained in

evidence amongst the ten year old children but it was gradually replaced by the direct complementary strategy to count-on – count-back. Thus the procedural growth of the 'below-average' children was from take-away to count-back. There was very little evidence of count-up.

Although take-away was used extensively by the 8+ 'average' children the general trend amongst this whole group was to use of count-back. There was some evidence of children using only count–up.

Apart from the seven year olds, the use of procedures did not figure prominently amongst the 'above-average' children, but count-up was the dominant form of procedure used.

5.4.2. Number Combinations to Twenty.

Tables 5.11 and 5.12 indicate the extent of the use of procedural methods to obtain solutions to the addition and subtraction combinations to twenty. The bracketed figures in table 5.11 indicate the percentage of solutions attempted through the use of procedural methods. In some cases these methods led to errors. Non-bracketed figures show the success rate through procedures.

		Category A Combinations							Category B combinations		
	N	10+2	12+1	14+4	3+16	18+2	15+4	13+5	9+8	8+6	4+7
'Below-Average' Group	23	31(34)	34(36)	48(58)	61(68)	39(45)	79(90)	78(90)	83(90)	61(74)	78(90)
'Average' Group	23	26	26	26	39	39	22	57	35	48	48
'Above-Average' Group	24	8	8	8	17	13	13	33	25	38	33
Full Sample		22	23	27	29	30	38	56	48	49	53

The percentage use of procedural methods to solve addition combinations to twenty

Table 5.13

		Category A Combinations						Category B combinations			
	N	13–2	15–4	16–3	18–9	15–9	12–8	16–10	17–13	20–8	19–17
'Below-Average' Group	19	42	63	58	63	68	68	36	84	68	95
'Average' Group	23	30	43	25	26	39	47	17	53	34	58
'Above-Average' Group	24	21	13	21	17	21	25	8	29	17	29
Full Sample	70	31	40	35	35	43	47	20	55	40	61

The percentage use of procedural methods to solve subtraction combinations to twenty

Table 5.14

It is apparent from Tables 5.13 and 5.14 that, compared to their use in the Category A combinations, the percentage use of procedural methods increased when children solved the addition and subtraction combinations to twenty. What is of particular interest is that this increase was not reflected by an increase in count-all and take-away but to a large extent was identified as an increase in count-on and its complementary subtraction procedure, count-back.

Amongst the 'below-average' children procedural approaches were used in equal measure to resolve both addition and subtraction combinations: 61% for addition and 63% for subtraction. Take-away was used more extensively for subtraction than count-all was used for addition. Neither of these procedures was used by children ten and older.

The most frequent use of take-away was seen amongst the eight- year-olds, but both it and count-back when used by these children caused errors. Indeed even ten-year-olds had difficulty with count-back which was used as the dominant procedure amongst all 'below-average' children above the age of nine. Samples of errors include:

Michael (10+), identified as "below–average" chose to write 18 – 9 in the standard vertical layout as

$$\begin{array}{r} 18 \\ \underline{9} \end{array} -$$

and, as is usual in the decomposition process, put a 'little one' by the eight.

"This is the easy way of working it out. I can't take nine from eight but if I put a little one it makes it easier because now its nine from eighteen."

He failed to realise that this is the same combination he had started with, and, after a considerable time trying to cope with this problem, he resorted to his more usual procedure for subtraction by placing eighteen marks from left to right on his paper, then starting from the left and counting from one to nine as he crossed out nine marks. He

recounted the remaining marks from left to right to complete the correct solution by "take away."

Jay (9+) solved the problem 5 – 4 by casually displaying five fingers on the edge of the desk and counted back, "five, ... four, three, two, one." At each count apart from the five he put slight pressure on each finger in sequence. The solution was provided by last count in the sequence. When attempting 15 – 4 he wanted to use a similar method but had a problem, declaring "I'm too old for counters!" but neither did he want to be seen using his fingers because "My class don't use counters or fingers." He felt he should operate in the same way as other children in his class (most of whom appeared to recall the basic facts from memory) yet he did not know the solutions and knew that he required a counting support. His use of fingers for obtaining solution for number combinations was almost always covert. When dealing with combinations to twenty he combined a casual display of ten splayed fingers on the edge of his desk with an imagined repetition of his fingers just off the desk. He spent a considerable time obtaining individual solutions and had a tendency to be very cautious in giving responses. He used his imaginary fingers to attempt to find a solution to 15 – 9 by counting back. Eventually he became confused and couldn't complete the problem.

There was no evidence of the use of count-all amongst 'average' children but take–away was the sole strategy used by one eight year old to solve the subtraction combinations.

Evidence of the flexibility to use count-back, count-up to or count-up was only obtained from one seven-year-old, who displayed the use of both count-up and count-back; the solutions for 17–13 and 19–17 being achieved through count-up. This flexibility was also demonstrated by a ten year old. Thus amongst the 'average' children we see some evidence of "choice".

None of the 'above-average' children used either count-all or take-away. Once again count-up was the most frequently used procedure for subtraction. The evidence of children using both count-back and count-up came from the seven year olds.

5.4.3 Summary of the Use of Procedural Methods

There were only a very few instances where children carried out their procedures without the external evidence of the use of a referent, whether it be fingers, head nodding or pointing. The 'above average' group of children were the general exceptions to this pattern. Much of their counting was abstract counting though at times it was accompanied by inaudible verbal utterances.

Overall the use of procedural methods by the children in the sample can be summarised as follows:

(i) Procedural methods were more extensively used for the addition and subtraction combinations to twenty than for the addition and subtraction combinations to ten.

(ii) There was no strong evidence to indicate that procedural methods were used more for subtraction than for addition.

(iii) The general evidence shows that where 'below-average' children used count-on for addition they used 'count-back for subtraction.

Some of the issues that arise from the analysis of the use of procedural methods would seem to provide additional support for the view that suggests that 'below–average' children' are doing a qualitatively different form of arithmetic than their apparently more able peers.

- 'Above-average' children make less, and, more sophisticated use of procedural methods than do the less-able.

- 'Average' ability children appear to follow the same trends as the above average ability children but over a wider age span.
- 'Below-average' ability children make extensive use of procedural methods and, apart from both age and problem related exceptions noted for some subtraction combinations to twenty, appear to use procedural methods as the dominant alternative to the use of known facts. Where children used count-all or take-away there was no evidence of the use of derived facts.

5.5 CHAPTER SUMMARY

Any assumption that children of a particular age "know the number facts" must be treated with extreme caution. However, the failure to retrieve the solution to a number combination from memory generally triggers the application of an back-up strategy which is usually successfully applied so that a correct response to a problem can be made. The analysis of the approaches used by the children extends the notion of the strategies outlined by Carpenter et al (1982) to numerical problems, indicates that these strategies change over time (e. g. Carpenter & Moser, 1984; Groen and Parkman, 1972, Woods *et al* 1975), confirms, to some extent the notion of an interaction between both knowledge and procedures (e.g. Ashcraft, 1982, Riley, Greeno & Heller, 1983) and thus provides some indication of the link between the use of procedure and automatic fact retrieval (Siegler & Shrager, 1984). It is in the interaction between knowledge and procedure that we see the qualitative differences between the 'above-average' and the 'below-average'.

- 'Below-average' children display the use of known facts with a counting procedure. The 'above-average' display the use of known facts, and derived facts and, more generally among the younger, children, a procedure.

- The procedures used by the 'below-average' children evoked the use of external referents. They may be defined as perceptual counters (Steffe *et al*, 1981)

- The procedures used by the 'above-average' children did not generally evoke the use of referents. They may be described as 'motor' counters or 'abstract' counters (Steffe *et al*, 1981)

- 'Below-average' children generally used count-back to resolve subtraction combinations they did not know. It many cases this caused difficulty and for the most part it was the hardest subtraction procedure the children may have used.

- 'Above-average' children only showed evidence of "choice" when in the 7+ group. The general trend was to use count-up.

- Even when they knew proportionally the same number of known facts as a younger 'above-average' group, older 'below-average' children did not use derived facts.

It would appear from the evidence of this sample that the less able do not simply learn the same techniques more slowly. They develop *different* techniques. The more able who use procedures, for example, display not only procedural competence at all ages, but also procedural efficiency. The less able appear to take some considerable time to develop procedural competence and then operate within the security of that competence.

It is concluded that the more able demonstrate procedural efficiency within a conceptual framework whilst the less able develop procedural competency within a procedural framework.

From the evidence it would be concluded that there are qualitative differences in the way that the 'above-average' and the 'below-average' children think about simple arithmetic.

CHAPTER 6

DUALITY, AMBIGUITY AND FLEXIBILITY: A THEORETICAL PERSPECTIVE

6.1 INTRODUCTION

Within Chapter 5 we see the strategies that are integrated by different groups of children to solve basic combinations in addition and subtraction. The evidence suggests that there is a dichotomy between the procedural approaches used by the 'below-average' children and the more flexible strategies used within a conceptual framework by the 'above-average' children. Although the dichotomy is easily seen amongst the younger children, the evidence suggests that eventually the difficulties the 'below-average children' experience with their procedures are overcome, and, their procedural approaches become efficient enough to enable them to retrieve the solution to most number combinations. The evidence of diverging approaches to simple arithmetic remains however.

In the study described in Chapter 5, verbal descriptions of the strategies used by the sample of 70 children to solve the complete range of 19 addition combinations and 19 subtraction combinations indicated an important difference in the children's perceptions of mathematics. Such a difference, also noted by Cobb (1991), may be described in the following terms:

- children who relied extensively on procedural methods explained the mathematics they used in terms of a procedure. It seemed that for them mathematics was an activity to be carried out in accordance with a sequence of actions. The numbers used did not symbolise anything but themselves.

- even though many of the 'above-average' children were unable to articulate their use of derived facts there was an implication behind what was explained and what was observed; these children saw their mathematics as actions on numbers

as arithmetical objects. The numbers had a quality which was both concrete and available for manipulation; they could symbolise other relationships.

Within this chapter the focus is placed on such differences seen from the theoretical perspectives that arise from an interpretation of symbolism. The early part of the chapter illustrates how, in mathematics, symbols take on a double meaning and represent both process and object. To draw together the procedural and conceptual possibilities that are represented by symbolism the theoretical the notion of *procept* is introduced. The distinctions between those who demonstrate thinking based upon the flexible use of processes and concepts and those who think procedurally leads to the notions of *proceptual* and *procedural* thinking, which is followed by the formulation of a theoretical analysis of the growth of proceptual thinking in simple arithmetic. This suggests that the difference between proceptual and procedural thinking leads to a divergence termed the *proceptual divide*.

6.2 THE AMBIGUITY OF MATHEMATICAL SYMBOLISM

The notion that there are two forms of thinking in mathematics, procedural thinking and conceptual thinking, tends to enforce the notion of dichotomy and yet almost by definition the latter subsumes the former and certainly within the field of arithmetic it is notions of procedure that leads us into notions of concept. This of course is the very essence of Piagetian belief of action interiorised as object. It is the representation of this 'interiorised action' which is the focus of interest.

Dubinsky (1991) provides a sense of the transformation which may be attached to the action becoming internalised through the use of the term 'encapsulation' which has frequented these pages. By implication, it is this notion which enables us abstract the similarities from different experiences and provide a name for these abstractions. In the mathematical sense such abstractions are not concepts until they are named and have symbolic representations (Skemp, 1971; Vergnaud, 1987). Thus we may see the notion of the experiences or actions encapsulated within a symbol which represents a concept.

As we have seen we may regard a symbol as something which is perceived by the senses. It can be written or spoken so that it can be seen or heard. What is important about the physical representation for the current theoretical development is the way in which it is interpreted by different individuals or by the same individual at different times. In particular the interest is in the way in which a symbol can be conceived as representing a process or an object a conception which lead to Sfard's rhetorical question "How can anything be object and process at the same time?" (Sfard, 1989)

Gray and Tall (1991) suggest that the answer to the question lies in the working practices of professional mathematicians and all those who are successful in mathematics.

> They employ the simple device of using the same notation to represent both a process and the product of that process. It is through using the notation to represent either process or product, whichever is convenient at the time, that the mathematician manages to encompass both – neatly side-stepping the problem. We believe that this ambiguity is the root of successful mathematical thinking. It enables the process of mathematics to be tamed into a state of subjection.
>
> (Gray & Tall, 1991, p. 2)

In practice then, for the successful mathematician, since the same notation can be used to represent the process or the product of that process, there is hardly a variation between the two.

Such ambiguous use of symbols for process and concept pervades the whole of mathematics:

- The symbol 5+4 represents both the process of adding through *counting all* or *counting on* and the concept of *sum* (5+4 is 9),
- The symbol 4 x 3 stands for the process of repeated addition "four multiplied by three" which must be carried out to produce the product of four and three which is the number 12.
- The symbol 3/4 stands for both the process of division and the concept of fraction,

- The symbol +4 stands for both the process of "add four" or shift four units along the number line, and the concept of the positive number +4,

- The symbol –7 stands for both the process of "subtract seven", or shift seven units in the opposite direction along the number line, and the concept of the negative number –7,

- The algebraic symbol $3x+2$ stands both for the process "add three times x and two" and for the product of that process, the expression "$3x+2$",

- The trigonometric ratio sine = $\frac{\text{opposite}}{\text{hypotenuse}}$ represents both the process for calculating the sine of an angle and its value,

- The function notation $f(x)=x^2-3$ simultaneously tells both how to calculate the value of the function for a *particular* value of x and encapsulates the complete concept of the function for a *general* value of x,

- An "infinite" decimal representation $\pi=3.14159\ldots$ is both a process of approximating π by calculating ever more decimal places and the specific numerical limit of that process,

- The notation $\lim_{x\to a} f(x)$ represents both the process of *tending to a limit* and the concept of the *value of the limit*,

Within the context of the analysis given in Chapter 5 it is conjectured that the 'above-average' ability children, instead of having to cope consciously with the duality of concept and process, think ambiguously about the symbolism for product and process. They simplify "*the cognitive complexity of process–concept duality by the notational convenience of process–product ambiguity*" (Gray & Tall, in press).

It is conjectured that the children who formed sections of the sample considered within Chapter 5 provide examples of the qualitatively different thinking that stems from the ability or inability to interpret symbolism in such an ambiguous way. For example James (8+) gave an almost instantaneous response to the sum 6 + 3. When asked how he did it he replied "Well its easy isn't it. Three threes are nine aren't they?" Faced with

3 + 2 such a child might see 3 as "one more than 2", he might know the double 2+2=4 and hence derive the fact that 3 + 2 is "one more", namely 5. Such ability provides a powerful feedback loop using known facts to derive new known facts and develops great flexibility. The addition of 3 + 2 may be seen equivalently as 2 + 3 = 5, 5 – 2 = 3, allowing subtraction to be seen as directly related to addition giving a fluent and easy way to develop subtraction facts.

It is conjectured that the seeming inability of the below-average children to recognise the object/process ambiguity contained within mathematics symbolism encourages them to use a procedure as a fall back to knowing a combination. Thus 5 is the process of counting five ones or it is "5": it does not signal 3 + 2, 6 – 1 etc.

6.3 THE NOTION OF PROCEPT

Within Chapter 2 we saw how the contemporary view of Piaget's notions of "actions" and "actions as thematised objects of thought" is expressed by the contemporary notions of "procedural" and "conceptual". We have also seen within Chapter 2 how the transformation between the two forms of knowledge has been variously described as 'encapsulation', 'reification' and 'entification'. The need to see the two forms of knowledge bridged through such processes as encapsulation etc. implies that there is a permanent dichotomy between notions of process and concept. Yet, we have just seen how a mathematical symbol can represent both at one and the same time.

In an attempt to consider children's behaviour in an integrated way it has been felt necessary to introduce new language for a theoretical construct which draws together the procedural and conceptual possibilities that are represented by symbolism. A mathematical symbol cannot be described as a concept since a concept is an idea held within the mind of an individual. We have already noted how symbolism allows the compression of a considerable amount of information in a small space. It is the means through which complex ideas, which include any procedural aspects which led to the

formulation of the symbolism, can be chunked and represented and then manipulated as an object. Thus symbolism may evoke either conceptual considerations or procedural considerations. The actions of the children reported within Chapter 5 indicates that they used one or other or both of these considerations. Whichever they used was triggered by their interpretation of the qualities inherent within symbolism – it either evoked an automatic response (which, as has already been indicated can cloud the theoretical issues), indicated something to do, for example, count, or signalled a relationship which was used to resolve a problem. Since a mathematical symbol can be seen as a carrier of two notions and an individual interprets symbolism to use one or both of these notions, it is felt essential to furnish the cognitive combination of process and concept with its own terminology.

As a first step towards this terminology, and in an attempt to:

(i) see it in a way which reflects the cognitive reality, and

(ii) avoid the complexities of a two phase definition which becomes more complex.

(Gray & Tall, in press) present the notion of an *elementary procept* . This is identified as the amalgam of three components; the *process* which produces a mathematical *object* and the *symbol* which is used to represent either the process or object.

This *preliminary definition* allows the symbolism to evoke either process or concept, so that a symbol such as 5 can evoke either the process of counting five or the concept of '5'; 2 + 3 can be seen to evoke either the process of addition of the two numbers or the concept of sum

But not only is a single symbol viewed in a flexible way but the same object can be represented symbolically in different ways. These are often seen, not only as different processes to give the same object, but as different names for the same object. In order

to reflect this growing flexibility of the notation and the versatility of thinking processes that encompass such relationships, the definition is refined so that:

a *procept* consists of the collection of 'elementary procepts' which have the same object.

The development of a two stage definition is an attempt to avoid the complexities of a definition that, in the first instance, identifies "procept", and then take this definition a stage further to identify, for example, a "complex procept". In particular the two phase definition as given encompasses the growing compressibility of knowledge characteristic of successful mathematicians.

Thus we can talk about the *procept* 6. It includes the process of counting 6, and a collection of other representations such as 3 + 3, 4 + 2, 2 + 4, 2 x 3, 8 – 2, etc. All of these symbols may be considered to represent the same object, yet indicate the flexible way in which "six" may be decomposed and recomposed using different processes.

A procept, then, is a special kind of concept: one in which appropriate symbolism is used to evoke either a process or the object produced by that process. It is an extremely powerful kind of concept, for *it embodies a procedure to be able to compute the product and it is a mental object that can be manipulated at a higher level.* The procept 2 + 3 represents both the process of calculating the sum of 2 add 3 (perhaps by counting), and also the expression which can be manipulated as part of a larger expression such as (2 + 3) - (2 + 1).

It is to be hoped that the elementary procept grows in richness as the individual's knowledge grows, so that the procept "5" includes not only the result of the process 4+1, but also 3+2. As *processes* 4+1 and 3+2 are quite different: the first might count-on one from four, the second counts on two from three. However, as *outputs*, they are the same thing. For these to be seen as representing the same underlying procept

requires a degree of flexible thinking; a procept can be decomposed in different ways, be reorganised and transformed into an equivalent symbolism which represents a different process but the same product. A procept is envisaged as being *plastic* - something flexible that can be re-moulded and reconstructed at will. In such a way the various different forms combine to give a rich conceptual structure in which the symbol 5 expresses all of these links, the conceptual ones and the procedural ones, the processes and the product of those processes.

The use of the notion of procept can to produce a theoretical synthesis of the development of arithmetical concepts.

6.4 PROCEDURAL AND PROCEPTUAL THINKING

The fundamentally different ways of thinking exhibited by children performing arithmetic usually represented by the terms procedural and conceptual, may be described more incisively as *procedural* and *proceptual.* Proceptual thinking *includes* the use of procedures. However, it also includes the flexible facility to view symbolism either as a trigger for carrying out a procedure or as the representation of a mental object which may be decomposed, recomposed and manipulated at a higher level. This ambiguous use of symbolism is at the root of powerful mathematical thinking to overcome the limited capacity of short-term memory. It enables a symbol to be maintained in short-term memory in a compact form for mental manipulation or to trigger a sequence of actions in time to carry out a mathematical process. It includes both concepts to *know* and processes to *do.*

The need for flexibility in arithmetic is a regular feature in the literature. For instance, Steffe, Richards & von Glaserfeld (1981) and Fuson, Richards & Briars (1982) suggest that the use of the sequence of number words for the solution of addition and subtraction problems leads to the understanding that addition and subtraction are inverse operations, and this contributes to the flexibility of solving addition and

subtraction problems. However, proceptual flexibility gives new insight. The existence of flexible proceptual knowledge means not only that the number 5 can be seen as 3 + 2 or 2 + 3 but that if 3 and something makes 5, then the "something" must be 2. When thinking *proceptually*, addition and subtraction are so closely linked that subtraction is simply a flexible reorganisation of addition facts.

6.5 THE GROWTH OF PROCEPTUAL THINKING IN ARITHMETIC

The procedural and conceptual approaches that children use to form the sum of two or more amounts introduced through word problems have already been documented (for example, Fuson, 1982; Carpenter et al, 1981, 1982) and identified within Chapter 4. Translating some or all of these approaches into a framework for cognitive development has been the focus of these and other studies (Herscovics & Bergeron, 1983; Secada, Fuson and Hall, 1983; Gray, 1991; Fuson & Fuson, 1992).

Although even finer gradations of these categories have been proposed (e.g.. Steffe *et al*, 1982), and these can be helpful in distinguishing the development of children's thinking processes, the underlying theme within this study has been to view the approaches children use to solve basic arithmetic combinations in an integrated manner so that the distinctions between procedural and conceptual thinking may remain to the fore. Now the notion of procept is used to conceptualise the cognitive development in an integrated manner, referring only to the growing facility for compression of ideas from procedures of counting to the procept of number. However it should be recognised that this is a coarse analysis that builds upon the fine grained analyses of, for example, Steffe *et al* (1982) and Carpenter *et al* (1981), but it perhaps goes some way towards explaining the divergent thinking identified within the last chapter.

By both utilising the summary given by Carpenter, Moser & Hiebert (1981) and resorting to analysis of the evidence given within Chapter 5 it is specified that the procedure of "count-all" consists of three separate sub-procedures: count the first set,

count the second set, then combine the sets as a single set and count all the objects (figure 6.1).

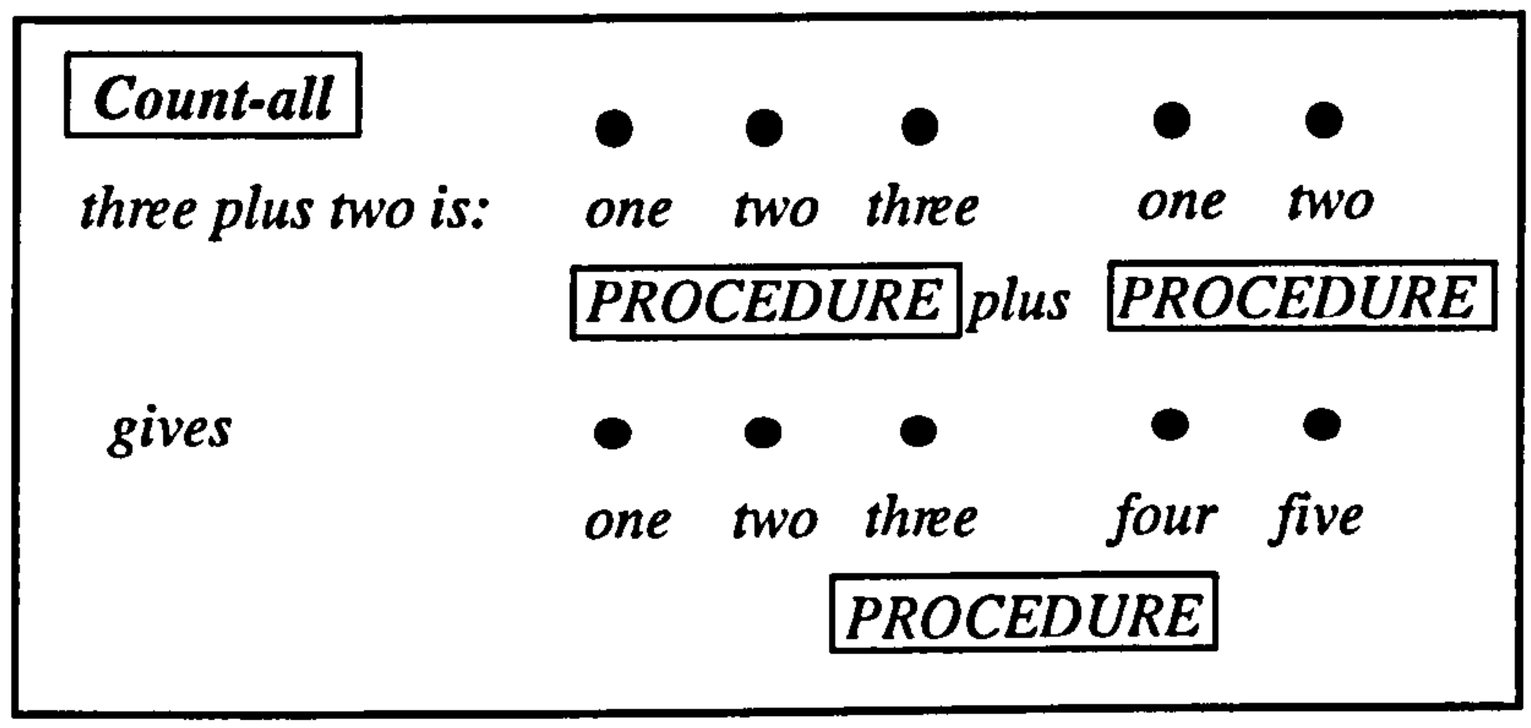


Figure 6.1: count-all as a combination of procedures

It is conjectured that the most salient memory that the child has of this process is the final object counted. This represents the value of the set which is the union of two sets formed from the two sub procedures which involved counting two and counting three. The total of this set, five, is the last point of reference for the child. Since such a procedure occurs in time, it is hypothesised that any proceptual relationship between the input (3 plus 2) and the output (5) is likely to obscured by the lengthy counting routine used to obtain the solution. The nature of such a procedure can mitigate against the encapsulation of 3 + 2 = 5 as a known fact. It is suggested then, that count-all is a procedure which extends the counting process and is unlikely to lead directly to an encapsulated procept. Within Chapter 5 we have already noted that there was no evidence of the use of a derived fact by a child who used count-all or take-away.

The count-on procedure is a more sophisticated strategy than count-all (see, for example, Secada, Fuson & Hall, 1983; Carpenter 1986; Baroody & Ginsburg, 1986; Gray 1991). The notion of elementary procept helps the analysis of cognitive development that may arise from implementing the procedure. To one number, the second is added through a count-on procedure. (We have seen how it is actually a sophisticated double-counting procedure where 3 + 2 involves saying "four, five" whilst simultaneously keeping track that "two" extra numbers are being counted (e.g.

Steffe *et al*, 1981). Count-on is therefore seen as an elementary procept plus procedure, one number is incremented in ones to form a successive series of elementary procepts through a counting procedure (figure 6.2).

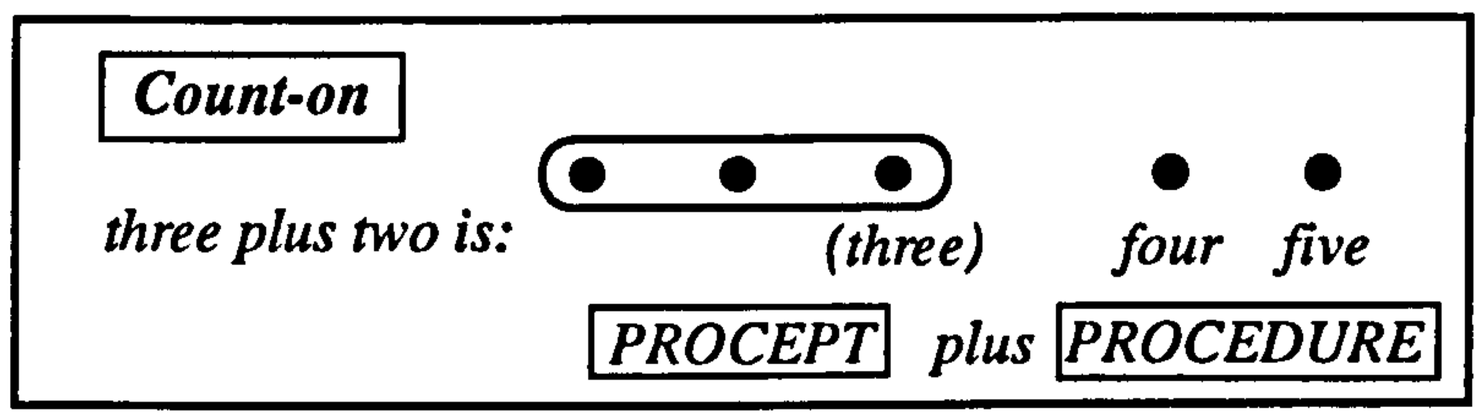


Figure 6.2: counting-on as procept plus procedure

It is suggested that "count-on" as a procedure can have two qualitatively different outcomes, as a (counting) procedure of addition or as the procept of sum.

(i) *Count-on as procedure* is essentially a compression of count-all into a shorter procedure. It remains a procedure that takes place in time so that the child is able to compute the result without necessarily linking input and output in a form that will be remembered as a new fact. It is conjectured that some children – often those with a limited array of known facts – may become so efficient in counting, that they use it as a universal method that does not involve them in the risk of attempting to use a limited number of known facts (see also Steinberg, 1985).

(ii) *Count-on leading to procept* produces a result that is seen both as a counting procedure and a number concept. The notation 3 + 2 is seen to represent both the process of addition and the product of that process, the sum.

When input numbers and their sum can be held in the mind simultaneously then the result is a meaningful known fact which may be envisioned as a flexible combination of procept and procept to give a procept (figure 6.3).

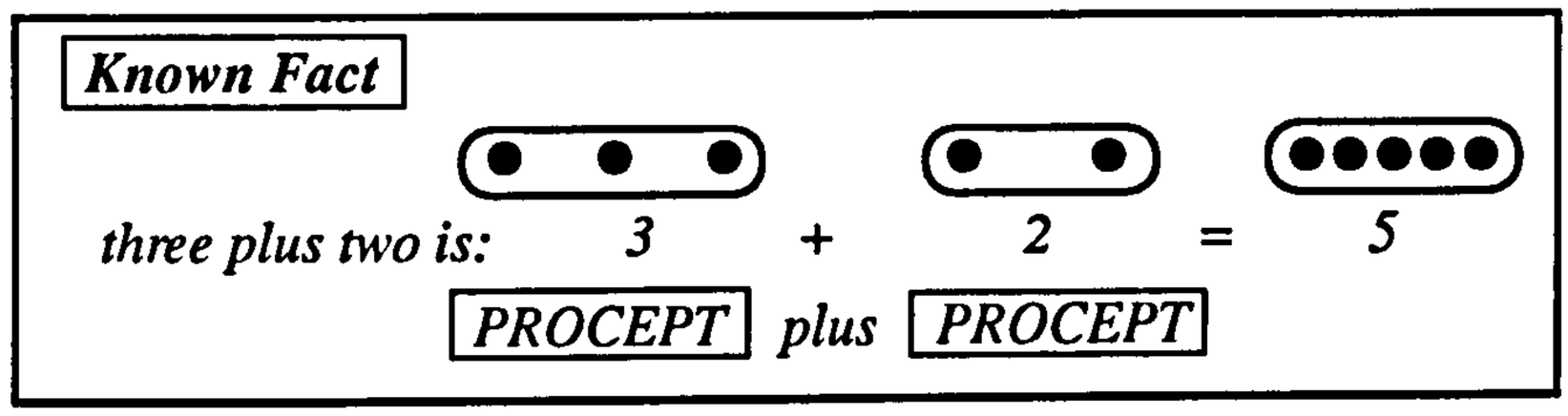


Figure 6.3: (Meaningful) known fact as procept plus procept

It is important here to distinguish between a *meaningful* "known fact" generated by this flexible form of thinking and a fact that is remembered by rote. In any isolated incident such a distinction may be hard to make. The difference becomes more apparent when such facts are decomposed and recomposed to give "derived facts." As we have seen merely "knowing facts" does not necessarily lead to "deriving facts". But the language used by children who do derive facts shows that they freely decompose and recompose the component parts in a proceptual way. For instance, faced with "four and five," one may know that "four and four makes eight," and respond that it is "one more," which is "nine." Some facts such as those for, say, "3 + 16 is 19" based on "3 + 6 is 9," can be so fast as to be virtually instantaneous. On occasion it may be difficult to distinguish between a "known fact" and a quickly constructed "derived fact."

When thinking *procedurally*, addition as "count-on" is considered to have subtraction as inverse through "count-back" or "count-up." Within the context of this study we have already seen that children identified as 'below-average' often favour "count-back" as the natural reverse process even though its cognitive complexity is enormous. Such a procedure, especially when carried out by less successful children, is highly prone to error. Because the proceptual thinker has a simpler task than the procedural thinker, the likely divergence between success and failure is widened.

6.6 A PROCEPTUAL DIVIDE IN SIMPLE ARITHMETIC

The snapshots presented within Chapter 5 present a picture of how children respond to basic number combinations if they cannot immediately retrieve them from memory. As

snapshots, it is conjectured that they provide an indication of what may be happening over a period of time.

Figures 6.4 and 6.5 illustrate the ways strategies used by the more able and less able in the study diverge.

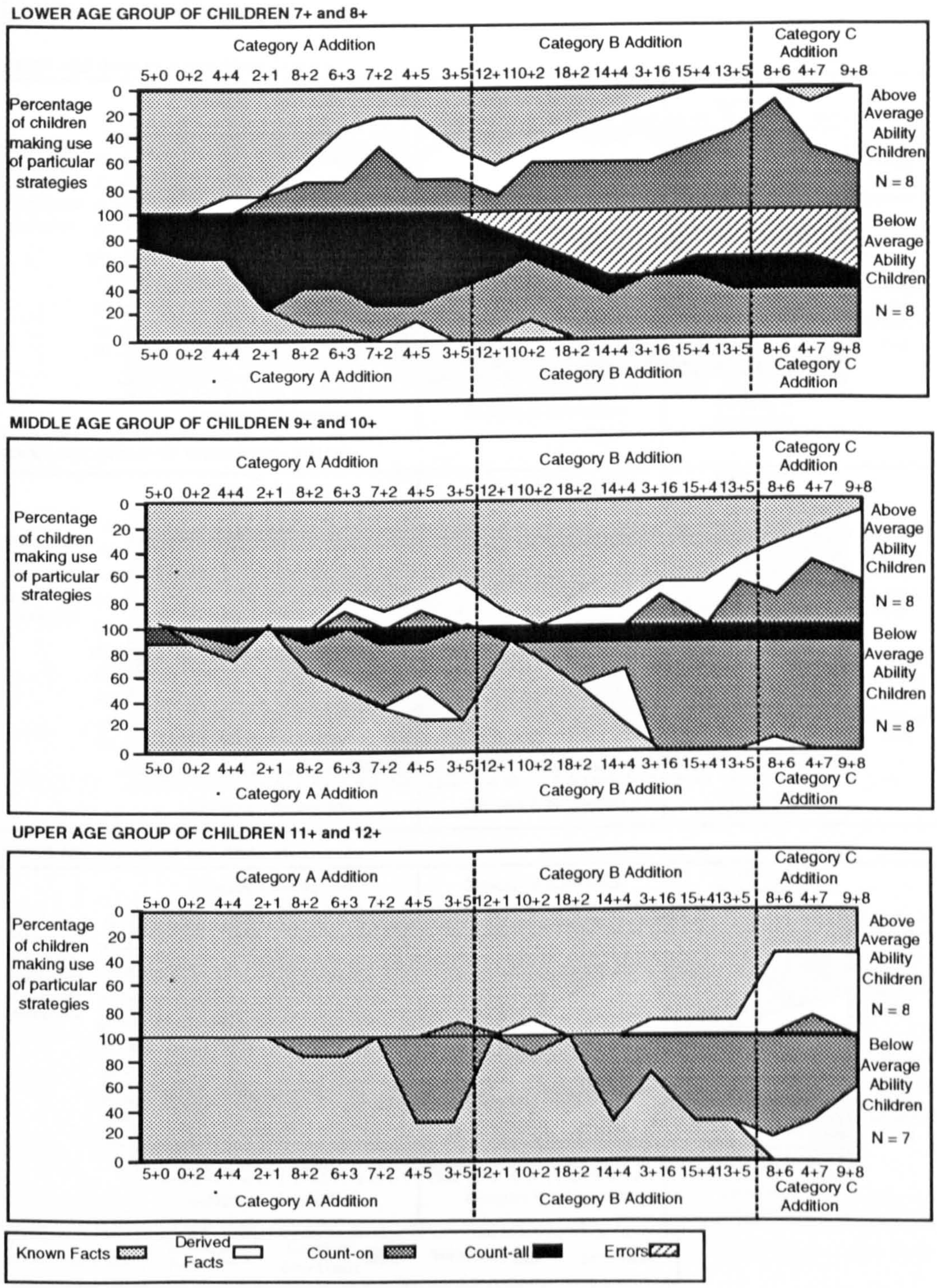


Figure 6.4: Diverging approaches to basic addition combinations: Age and ability comparisons.

They pair age groups together, two years at a time to give more viable group sizes. Combinations are arranged in order of difficulty, established by considering the overall percentage of children within the sample who responded to individual combinations through the use of known facts. The categories of combinations are as those previously identified within Chapter 4.

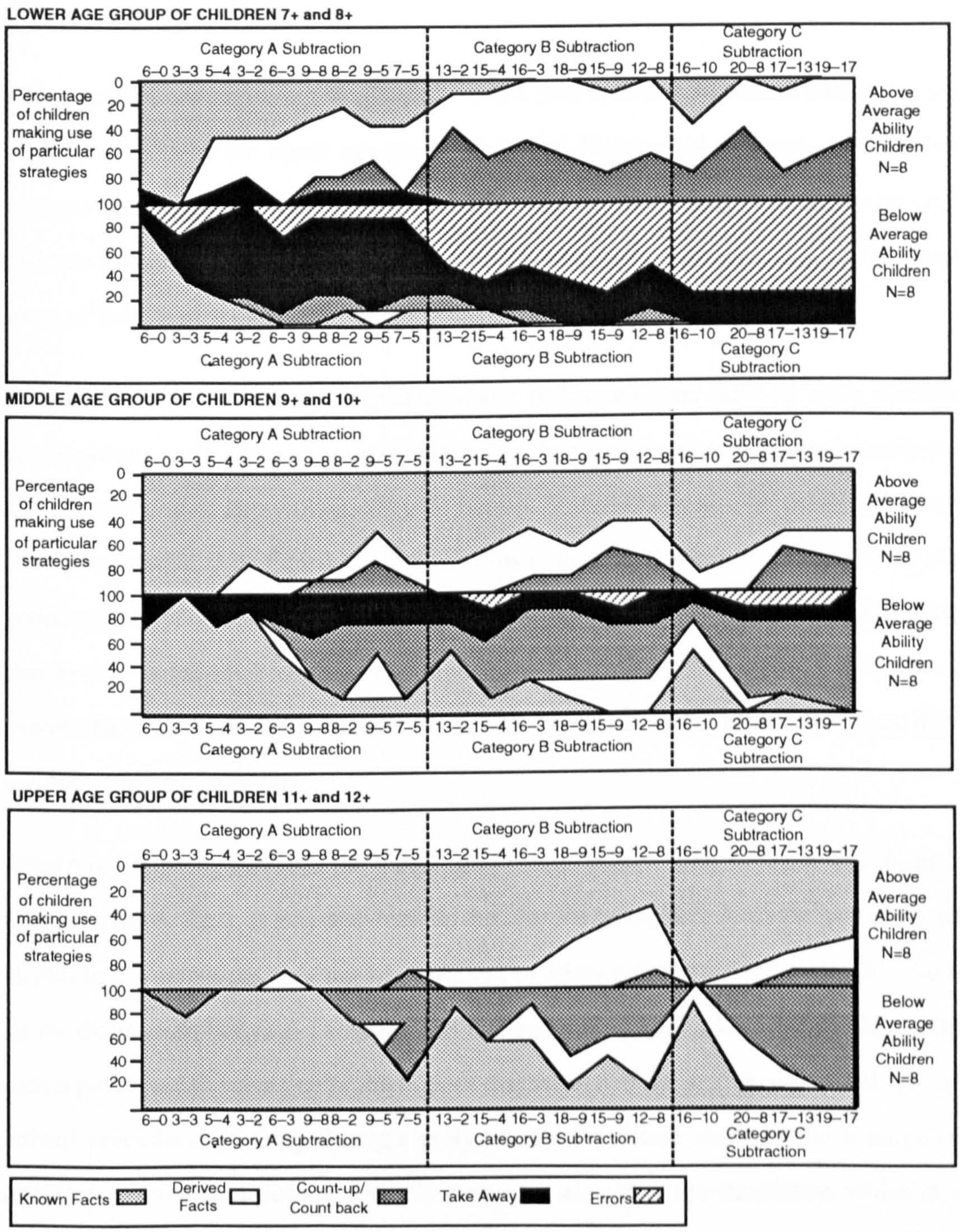


Figure 6.5: Diverging approaches to basic subtraction combinations: Age and ability comparisons.

The graphs not only illustrate differences between the 'below-average' and the 'above-average' children but also how individual combinations evoke different particular responses, and within each group a different mix of integrated strategies. The 'above-average' children show a high incidence of known facts and how much of what they do not know is derived. The 'below-average' rarely use derived facts, instead they almost always count.

The combinations involving single digits and a sum between 10 and 20 evoke the use of derived facts by the upper age group of 11 and 12 year old children. At the other extreme the successful procedural methods used by the youngest 'below-average' children to obtain solutions to number combinations to ten fail to generalise and provide some of them with a means of obtaining solutions to combinations to twenty.

An additional feature of the graphs indicates that within the field of basic number combinations, even when a group of 'below-average' children know approximately the same proportion of known facts they do not use derived facts as a prominent back-up strategy to obtain facts they do not know. This feature is illustrated if we compare the youngest 'above-average' group with the eldest 'below-average' children. Even when they know a substantial number of combinations the 'below-average' make little use of derived facts. It would seem that even when the children close the achievement gap they are doing things differently.

Proceptual thinking includes the meaningful use of known facts to arrive at solutions through derived facts. It may also include the use of a procedure. A single item analysis of children's responses may not be sufficient to allow a distinction to be made – how do we distinguish between a rote learned fact and a meaningful known fact? If a child uses a procedure to solve one problem does that mean that the next problem will also be solved procedurally? Only through analysis of the solution strategies to a range of problems may we get our answer. We may then see either the flexibility, which is a keynote of the proceptual interplay between conceptual and procedural methods, or the

limitations imposed by the reliance on fixed counting procedures. It is conjectured that the latter may provide considerable success at one level but may ultimately lack the generality to lead to success in more sophisticated problems.

The empirical evidence obtained from the snapshots presented within Chapter 5 therefore tends to support the hypothesis that there is a qualitatively different approach to simple arithmetic between children who tend to display the use of more flexible proceptual techniques (including the selection of more appropriate procedures) and those who rely upon flexible procedural methods of counting.

Proceptual thinking appears to be a quality demonstrated by the more successful. Flexible back-up strategies used by them produce new known facts from old, giving a built-in feedback loop which acts as an autonomous knowledge generator. Many of the least successful have only a back-up strategy which is a procedure of counting. This can grow ever more lengthy as the problems grow more complex. In between these extremes, the less able who *do* attempt to derive facts from a limited range of known facts may end up following an inventive but tortuous route that succeeds only with the greatest effort. The high sense of risk generated may then lead to such a child falling back to the security of counting. It is therefore hypothesised that what might be a continuous spectrum of performance tends to become a dichotomy in which those who begin to fail are consigned to become procedural. It is believed that this bifurcation of strategy – between flexible use of number as object or process and fixation on procedural counting – is one of the most significant factors in the difference between success and failure. The divergence that stems from such a bifurcation is termed *the proceptual divide.*

6.7 CHAPTER SUMMARY

Throughout history the mathematics of those cultures that emphasised the use of procedures without an effective symbolism to communicate their ideas has been

doomed to failure. The invention of symbolism provided mathematicians with the means of representing process/object ambiguity; the outcome of procedures used to assess, compare and generalise the logic and order of the environment was represented within a symbol which also provided a sense of how the outcome was obtained. This ambiguity is seen in symbolism that represents, for example, number, the numerical processes, measurement and algebra. Those fortunate enough to maintain sight of this ambiguity continue to develop and utilise the power of symbolism. It is hypothesised that the notion of procept provides a new way of interpreting the mathematical proficiency of the more able whilst at the same time providing some explanation for the difficulties of the less able. It provides a means of expressing the cognitive development which gives power to the more able and exhibits their diversion away from the procedural cul-de-sac that, it is suggested, is the inevitable destination of the less able.

It is hypothesised the divergence between those who think proceptually and those who think procedurally stems from the qualitative difference that may arise through the use of count-on. Although it may be the source for short term success a longer term prognosis may lead to a very different outcome.

CHAPTER 7

POINT OF DEPARTURE: A QUALITATIVE ANALYSIS OF ARITHMETIC ACHIEVEMENT

7.1 INTRODUCTION

Within the previous chapter the qualitative differences in the theoretical distinction between procedural and proceptual thinking were introduced. The notion of *proceptual* was ascribed to thinking which was characterised through the flexible use of symbols used to denote process or object. Within the framework of this form of thinking, count-on may be viewed as process which can become encapsulated in the concept of sum. Within in the context of procedural thinking, count-on is identified as a procedure which though it may provide a successful means of resolving basic number combinations it may not be encapsulated into the concept of sum.

Similar distinctions in thinking may be applied to the process of subtraction. In proceptual terms subtraction is another way of viewing addition facts. In procedural terms, the reverse of the count-on procedure is count-back. This is a much more difficult operation to carry out.

The experiment reported within this chapter is partially designed to replicate the study reported within Chapter 5 but more particularly it is designed to consider the issues raised within Chapter 5 within the context of children's level of mathematical achievement. Within the chapter the theoretical distinctions between procedural and proceptual thinking are used to analyse the arithmetical achievement of a class of children. The children were initially interviewed at the age of approximately seven, and then, they were interviewed 10 months later. It is hypothesised that distinctions made between children in terms of arithmetical achievement may ignore distinctions that may be determined from the quality of their arithmetical thinking. The chapter provides evidence which highlights the freedom that is available to children who may be

described as *proceptual thinkers*, and contrasts this with the sometimes hard-won success or failure that may be achieved by children for whom arithmetic is a sequence of procedures. By considering the responses made by a class of children to the items within the Mathematics Assessment Tasks used in 1992, the evidence indicates that at the age of seven, children who achieved Level 2 may have done so through procedural or proceptual thinking. Those who achieved Level 3 did so through using the flexibility attached to the notion of proceptual thinking.

7.2 CONTEXT OF THE ANALYSIS

Children who are around seven years of age, identified as Y2 within the English and Welsh school system, now face Standard Attainment Tasks (SAT's) designed to measure their level of attainment in a range of Mathematics Attainment Targets (MAT's). Depending on their degree of success or otherwise, the result of a SAT provides an indication of a child's level of competence at a particular task. After the analysis of their results, children within the current sample were placed at one of three levels as a result of achievement on a criterion linked SAT – thus a child may be identified as having achieved Level 1, Level 2, or Level 3. This scale identifies the levels of competence expected of the average seven year old who would normally be expected to achieve a Level 2 standard of attainment. However, it is worth noting that specification of these levels of attainment does not arise from the standardisation normally associated with the use of objective tests.The levels of attainment must be seen as manifestations of a child's ability to match a set of criteria that are established subjectively within a somewhat traditional framework (e.g. Board of Education, 1905). Within the current discussion, the notion of achievement is restricted to an analysis of some of the numerical components of the Key Stage 1 series of SAT's. A child who satisfied the Level 2 requirement within these components will be considered to achieve the 'average' level of attainment. Those who do not achieve Level 2 will be identified as

'below-average', whilst those who achieve higher than level 2 will be deemed 'above-average'.

Although the MAT's cover a range of mathematical areas – Number, Shape and Space, Data Handling, Algebra and Measurement – a compulsory component of the MAT's is the numerical component. One Statement of Attainment (SoA) required for the achievement of Level 2 is that the child "knows and uses the addition and subtraction facts to ten" (D.E.S., 1989, Ma3/2a, p. 9), whilst an equivalent statement of attainment for Level 3 indicates that the child should demonstrate that (s)he 'knows and uses the addition and subtraction number facts to twenty' (D.E.S., 1989, Ma3/3a, p.9).

7.3 METHOD

The responses made by children within a mixed ability class (N=29), aged between 6 years 8 months and 7 years 7 months (7+), to the numerical components of a series of Standard Assessment Tasks (SAT), (SEAC, 1992a, 1992b, 1992c) were considered. The children's level of attainment within the formal element of the SAT was matched with their responses in individual interviews carried out:

(i) in May 1992 and within a period of three weeks of the formal testing.

(ii) over a three week period during March 1993.

In the course of the discussion the children's age as a result of the second series of interviews will be deemed as 8+

The tests were administered at the end of April, 1992. The numerical aspects, compulsory components of the 1991 Mathematics Assessment Tasks (MAT), which also included the option of Data Handling or Probability, included addition and subtraction number combinations, within a contextual situation and in a context free situation. Assessment through the context free SAT's was aimed at the children's ability to recall number facts without calculating (S.E.A.C, 1992a). It is the children's responses to the four context free components that form the basis for the ensuing

discussion. Both the formal testing, carried out by the class teacher, and the individual interviews, conducted by myself, were recorded on video. During the individual interviews the children were presented with the same combinations under similar conditions to the formal testing but they were then invited to consider how particular solutions were obtained. The strategies used by the children were noted in the way outlined in Section 4.5. Validation of many of the strategies used by individual children was through the independent assessment of colleagues.

On both occasions the combinations were presented orally. During the individual interviews this oral presentation matched that of the class teacher who carried out the formal testing. Addition combinations such as 5 + 4, were presented as "5 add 4" whilst subtraction combinations of the form 4 – 3 were presented as "4 take away 3". During the formal testing solutions were written, during individual interviews solutions were given orally. In the manner of the formal test, children were not provided with counting aids.

During the formal test it was envisaged that maximum time allowed for each item was to be five seconds – in practice this turned out to be eight seconds. During the individual interviews, since the time to respond to each item between its delivery and the child's first response was recorded, breaches of the five second limit could be identified.

For a child to achieve a level of attainment only one error in addition and one error in subtraction at a particular level was allowed. To attempt the Level 2 tasks children had to satisfy the criteria for the achievement of Level 1 – "they could add and subtract objects where the numbers involved were no greater than ten". As a result of their responses children were identified as having the following levels of achievement in the statements of attainment under consideration:

Level 2 (L2): children had achieved Level 1 and illustrated that they were able to "recall the number combinations to ten without calculation" (S.E.A.C. 1992a, pp 36-37; S.E.A.C. 1992b, pp 30-31)

The combinations considered as part of this SAT were:

Addition: 1 + 6; 5 + 4; 3 + 7; 4 + 3; 1 + 5; 6 + 2; 4 + 4

Subtraction: 9 – 6; 8 – 2; 5 – 3; 7 – 5; 9 – 4; 4 – 3; 10 – 7

Level 3 (L3): the children achieved Level 2 and they illustrated that they were able to "recall the number combinations to twenty without calculation" (S.E.A.C. 1992a, pp 39-40; 1992b, pp 34-35).

The combinations considered as part of this SAT were:

Addition: 9 + 6; 8 + 10; 4 + 11; 7 + 7; 15 + 2; 17 + 0

Subtraction: 19 - 13; 18 - 10; 20 - 5; 17 - 6; 11 - 9; 15 - 0

7.4 SUCCESS AND FAILURE AT SEVEN

The analysis of both test video and interview video indicates that for many of the children, even though the purpose of the time limit was to prevent calculation, counting was the dominant means through which solutions were obtained. The overt actions of many of the children during the formal test and the individual interview included perceptual representation of numerical equivalents and finger counting. Arguably, such actions should have negated any responses within the formal test. Motor acts or abstract counting would not have been noticed by the class teacher. Only through the ensuing individual interviews was it possible to identify those children who obtained solutions through "knowing" and those who obtained solutions through a less overt form of counting. Overall there was very little evidence of count-all amongst the children

7.4.1 Failure to Achieve 'Average' Attainment

The five children who were unsuccessful at level 2, not only failed to recall the solutions to most of the number combinations but they attempted to use a procedure

which also failed; it was either inefficient or too lengthy to satisfy the criteria that solutions must be given within five seconds. For example, Simon during the formal test component attempted to obtain solutions using his fingers to support a count-all procedure. His procedure was so inefficient and lengthy that he not only ran out of time to obtain a solution but his concentration on the application of the procedure to solve one combination inevitably meant that he also failed to hear the first part of a subsequent one. Joanne used her fingers to support her efforts to obtain the solution to 3+7 by counting-on seven from three. She was very slow and illustrated no overall pattern when tagging her extended fingers. Although she recited the number words in sequence each finger was indicated (through a slight movement) and tagged in a quite arbitrary way. Her lengthy procedure gave the solution "8". Joseph, on the other hand, tried to carry out all of the counting in his head with no external physical support. He complained during the formal test because he couldn't keep up. There is no evidence to indicate whether he was attempting count-all or count-on. Whichever, he found his strategy very hard and would sit for extended periods with no obvious sign of action but he,"... liked trying to do things my head. I like them to be harder because when I grow up I will be able to do harder things." Interestingly, during the 1993 interviews Joseph continued to use the same procedures for the same reasons. In 1992 he failed to obtain the solution to any of the L2 addition or subtraction combinations but he was a little more successful in 1993.

7.4.2 The Success of the 'Average' Child

Relatively few of the combinations were known as facts by seventeen children who achieved Level 2 in 1992. Addition combinations were only completely known by one child, two knowing three or less, the remainder knowing between three and six out of the total of seven. No child knew all of the subtraction combinations, seven knew one or less – five none of them – and the remainder knowing between 2 and 5, again out of a total of seven. Their competent use of procedures allowed these children to achieve

Level 2. The procedures were generally compatible with those used by children who had failed to achieve level 2, the essential difference being that they were more efficient. Count-on was the dominant procedure used for addition and usually fingers were used very effectively as referents to support the counting. Most frequently a set of fingers equivalent to the amount to be counted-on was subitised prior to incremental tagging from the value of the first set.

A complementary approach was frequently used for the subtraction combinations – children used fingers to aid counting back from the value of the larger set and, in a very few cases, to check the amount counted up from the small set to the large set. However, many children used an approach which involved a subitised display of the large amount (minuend) on their fingers, subitising and bending the number of fingers equivalent to the value of the smaller set (subtrahend) and subitising of the value of the remainder. This strategy is termed *enactive subitising* following Bruner's (1962) notion of enactive. Within this approach, which involved no actual counting, a child seemed to need visual support of the numbers in a concrete form; to see, for example, that four take away three is one. Modifications of this approach, not termed enactive subitising, included counting the value of the subtrahend prior to subitising the remainder.

Whereas 'below-average' children always tried to use the same procedure, without the necessary procedural competence, amongst the 'average' group of children there was more evidence an integrated combination of strategies. For example Rebecca and Tara used an integrated combination of strategies which involved counting, enactive subitising and knowing. Rebecca very quickly used her fingers to count-on and obtain solutions to the addition number combinations she could not recall. She completed each problem within 5 seconds. For those subtraction solutions she could not recall she used enactive subitising. Tara knew most of the addition combinations to ten but again used enactive subitising for the subtraction ones.

An indication that a child had achieved Level 2 standard of attainment in basic arithmetic takes no account of the means through which the children reached this level of achievement. Success at Level 2 does not distinguish children such as the above from children who solved every Level 2 combination by recalling the related addition and subtraction facts. The procedural efficiency of many of the children within the 'average' group had the potential of clouding the whole issue of achievement at Level 2. It was only when the children began to attempt the level 3 stage that the consequences of these differences began to emerge.

7.4.3 Success and Failure at Level 3

For many children, failure to evoke the recall of a number combination to twenty evoked highly idiosyncratic procedures. Some were so inefficient that even when they provided a successful outcome they generously exceeded the time constraints. The general pattern that emerged from all of the children who achieved success with the number combinations to ten but failed to achieve success at level 3 was that they recalled solutions to combinations such as 17 + 0, 7 + 7 and 15 – 0 but for most of the others i.e. 9 + 6, 4 + 11,15 + 2, 17 – 6, 11 – 9, and even 18 – 10 they attempted to use a counting procedure.

The success of those children who achieved Level 3 did not arise immediately from the combinations they knew but from their flexibility in obtaining a solution. Indeed, one child failed to directly recall any Level 2 subtraction combination–he used related addition combinations. He had known six out of seven of those. Another knew most of the combinations at both levels but if immediate retrieval failed him he displayed a considerable degree of flexibility, particularly to solve the Level 3 combinations:

- For 9 + 6 he described his solution by saying,

  "You get nine, add one to make ten, and then add five."
- For 19 – 13 he described his approach by saying,

"You have thirteen and count on to the nineteen–you add some of the nineteen onto the thirteen."

- For 11–9 he,

  "... took one away from the eleven. That leaves ten. You take one away from that and that leaves nine."

Simon and Jacob also demonstrated some flexibility. Simon, for example, recalled all of the solutions to the level 2 combinations. To obtain the solution to 17 – 6 he counted back six from seventeen When attempting 20–5 he knew that "fifteen add five is twenty so twenty take away five is fifteen".

7.4.4 An Interim Conclusion

In the absence of the ability to recall a combination some children attempted to use counting procedures that totally failed them, they did not achieve Level 2. Many children had developed procedures that though they proved successful at Level 2, the children were not able to generalise them to the Level 3 combinations. Others who had developed procedural competence to achieve Level 2 failed at Level 3 because their procedure was not efficient. Those who did achieve Level 3 appeared to do so with relative ease even though, on one or two occasions, they too used counting. Some children were able to recognise the limitations of their counting procedure within the framework of the time constraints and consequently did not use it if they felt it was an unreasonable approach. Jacob didn't do 19–13 because "*...it was a bit too hard and I knew I couldn't count it in quickly*".

The distinctions in achievement between procedural and proceptual thinking were clouded at a level where procedural thinkers could succeed through procedural competence. At the next higher level it was those who had demonstrated a flexible approach at Level 2, the proceptual thinkers, who achieved success at Level 3.

7.5 THE USE OF PROCEDURES THAT ARE COUNT BASED

Almost every child who responded to the series of attainment tasks used some form of counting to solve at least one combination within either the level 2 or the level 3 components.

Figure 7.1 illustrates the overall percentage of number combinations attempted through the use of a counting procedure during the individual interviews in May 1992. Figure 7.2 shows the percentage of occasions when these attempts provided a successful outcome in that the solution to a combination was obtained within 5 seconds. Had this time limit been strictly adhered to during the formal aspect of the test it is suggested that five children who achieved level 2 may not have done so. Those who achieved level 3 obtained at least 6 of the 7 solutions for the Level 2 combinations within 5 seconds

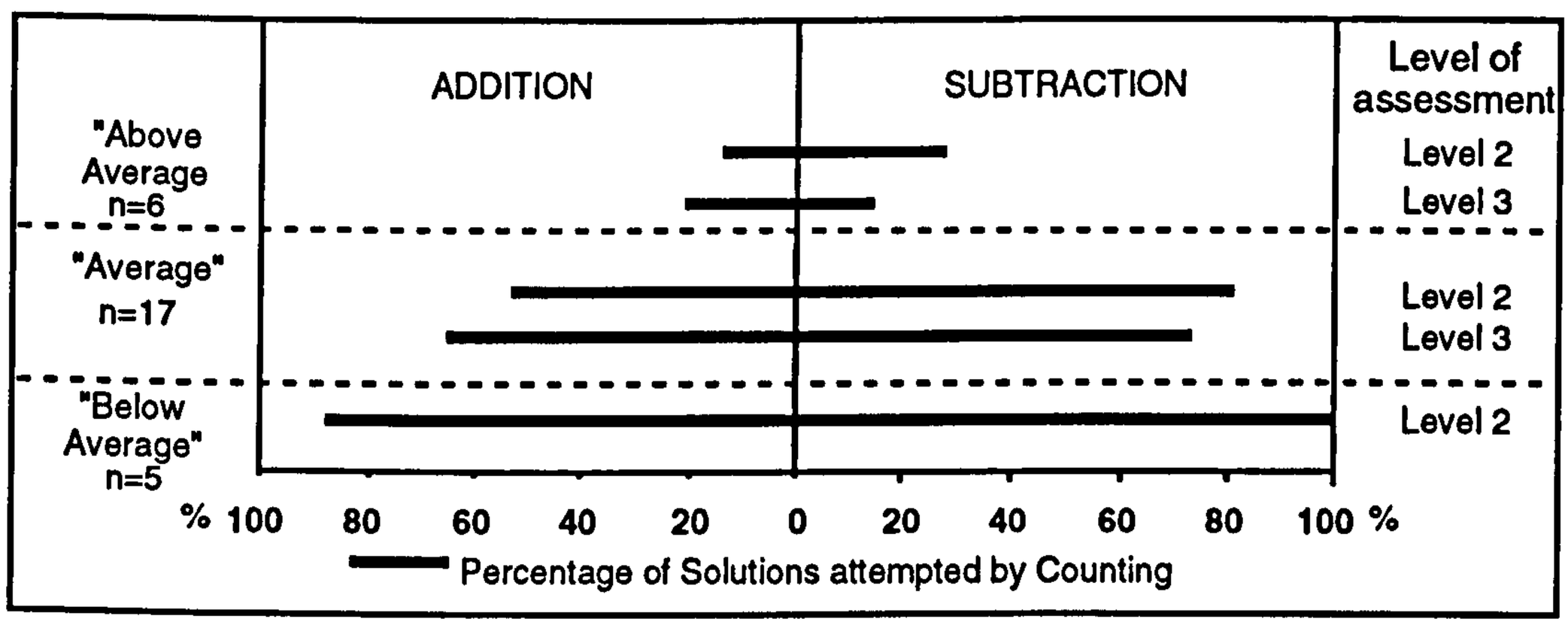


Figure 7.1: Percentage of solutions to basic number combinations attempted through counting

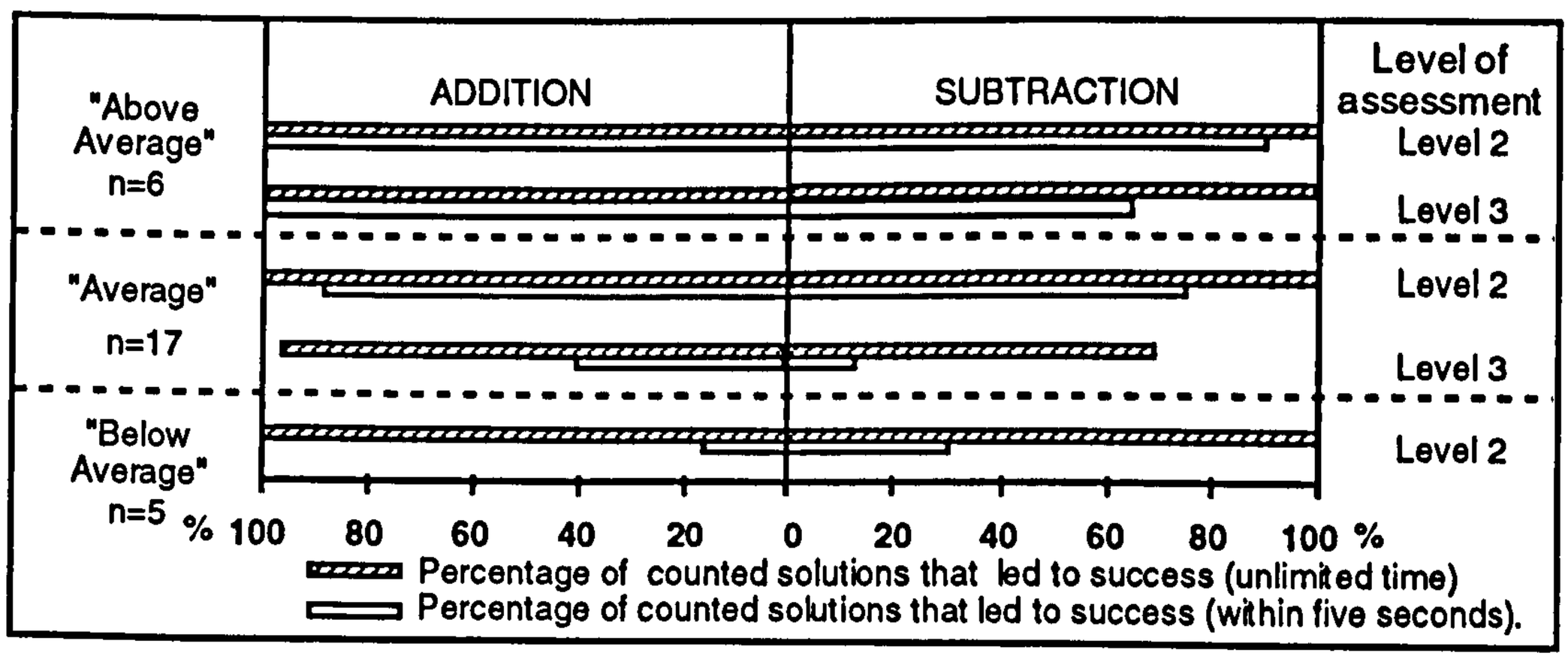


Figure 7.2: The percentage of counted solutions that led to success.

Although the percentages presented in the above graphs can be somewhat misleading – contrast the 66% success rate within the time limit of the 'above-average' children solving the level 3 subtraction combinations (representing four successful procedures out of six that were used) with the 74% success rate of the 'average' group on the level 2 subtraction combinations (representing 73 successful procedures out of 98 used) – of particular interest is the overall extent in which counting was used and the extent with which it led to success. The difference in the extent to which counting was used at Level 2 between those who did not achieve L2, those who achieved Level 2 but not Level 3, and those who achieved L3, is particularly striking. The use of a counting procedure not only declined the higher the level of attainment the child achieved, but the procedural efficiency and generalisability of procedures used by children who achieved Level 3 was likely to be an improvement upon the procedures of the children who failed to achieve this level. Not only did children who achieved level 3 use counting less frequently than the other children but when they used it was more likely to lead to success. Compared to those children who did not achieve Level 3, the 'above average' children demonstrated that generally they possessed a procedural competence that transcended arithmetical difficulty.

7.5.1 Procedural Efficiency–Addition and Subtraction Combinations to Ten.

Figures 7.3 and 7.4 show the approximate periods of time that it took during the individual interviews each child who counted to complete their procedure. Periods of time were taken from the end of the interviewers verbal presentation of the combination to the start of the first spoken response of the child. They were recorded on the video. Children had been told to respond to each problem as quickly as they could. It is not claimed that these times are necessarily accurate, particularly those that are very short, for example, those up to three seconds. Margins of error could be such that to make too much of the distinctions between them would lead to erroneous conclusions. However,

as the length of time taken to resolve a combination increases such errors will become marginalised. It is a sense of the time taken by different groups to provided a solution to a combination that it is wished to communicate within the discussion.

In both figures the number combinations are arranged from left to right to reflect the numbers of children that solved the particular combination through the use of a counting procedure.

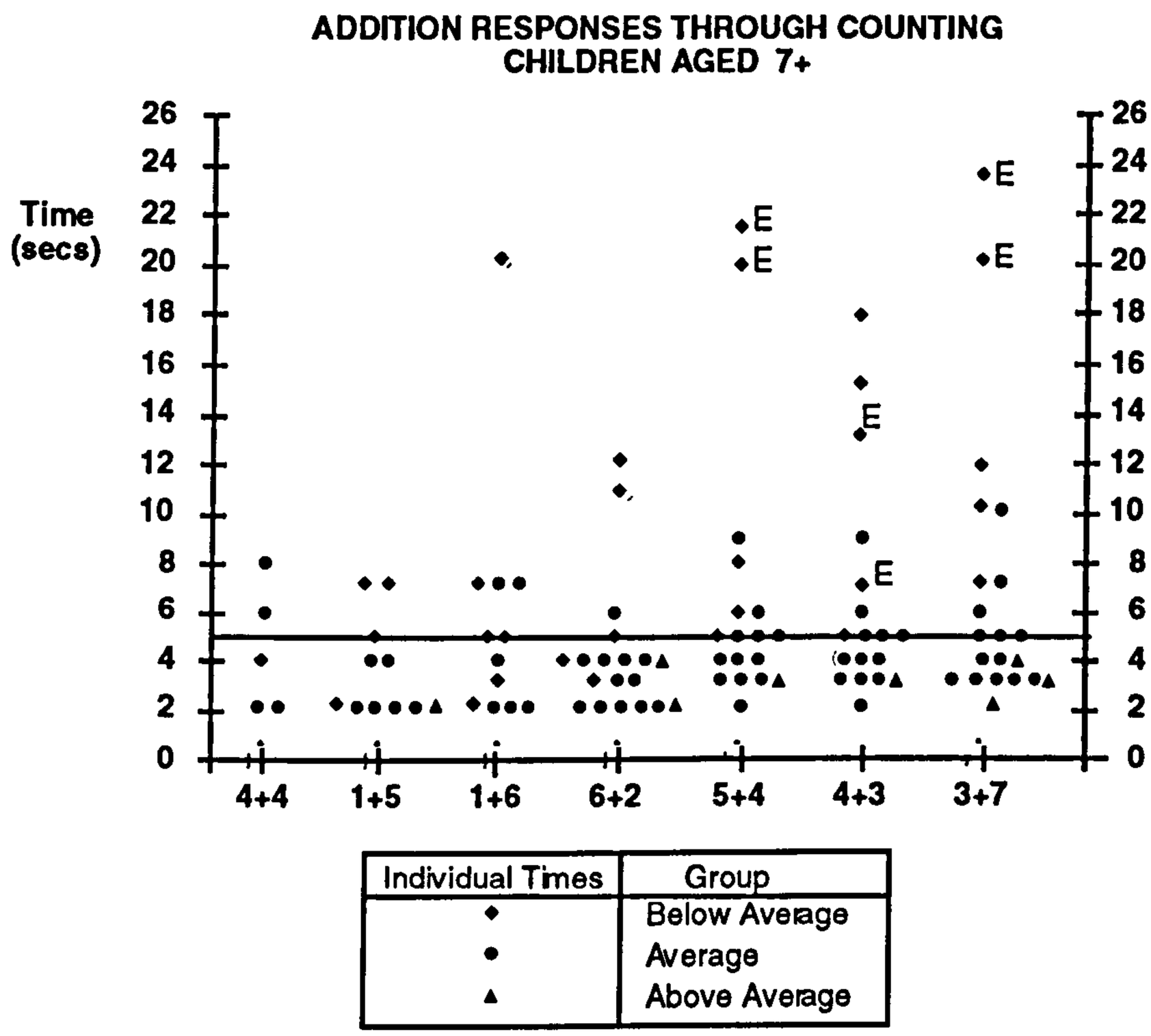


Figure 7.3: Time taken for seven year old children to complete addition procedures.

We see immediately the excessive period of time it took the children who did not succeed at level 2 to use their procedure. In contrast, only in two instances could the length of time used with an addition procedure have had an affect on the success or otherwise formally recorded for children who achieved level 2. It is interesting too to note the time that those who achieved Level 3 took over their count-up procedure.

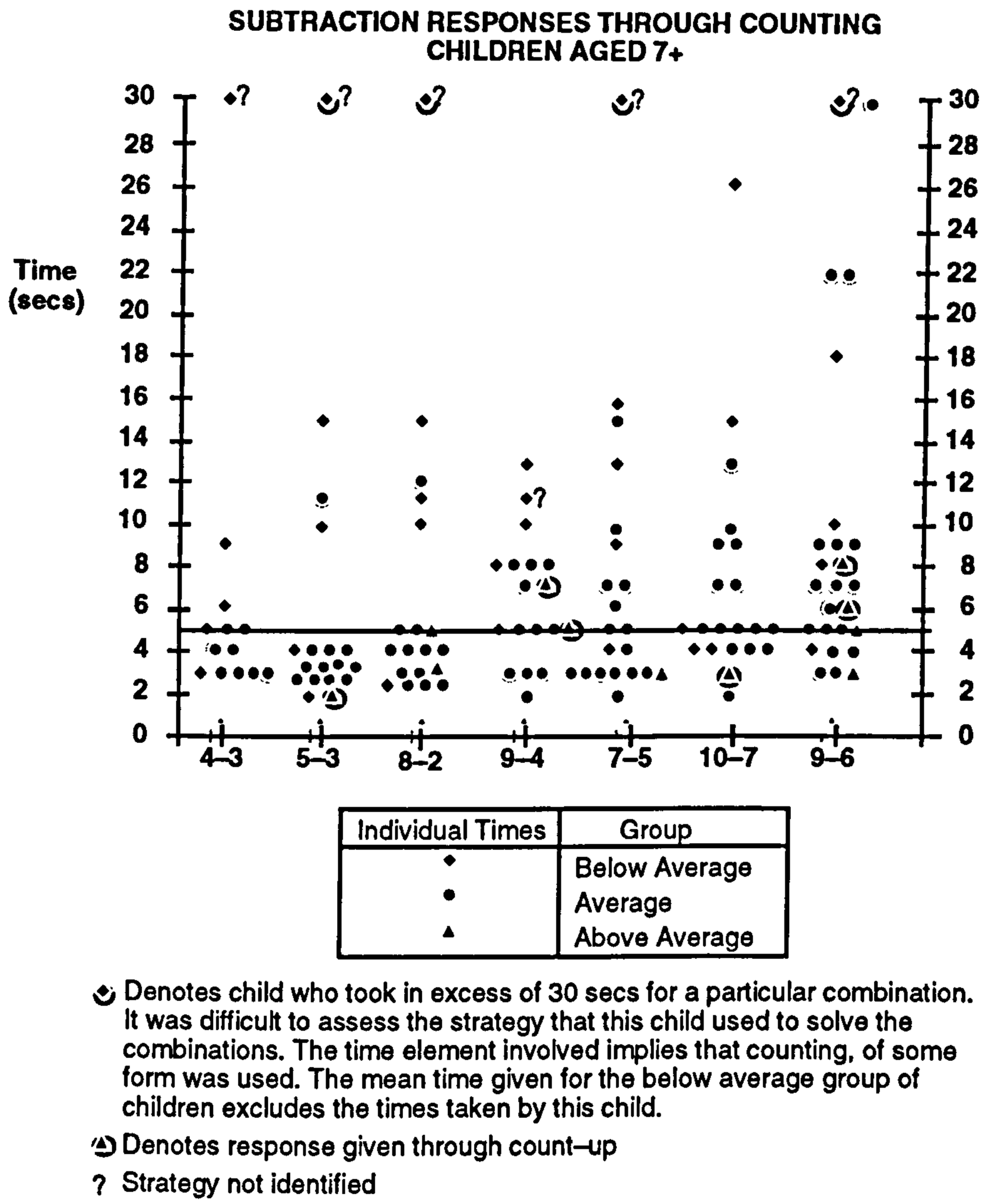


Figure 7.4: Time taken for seven year old children to carry out subtraction procedures

Table 7.5 indicates the mean times taken for each group to solve individual combinations.

It is not appropriate to consider whether differences between them are significant or not–it is the trend of the differences, together with a notion of the extent to wish procedures were used by each group that it is wished to communicate.

		ADDITION LEVEL 2							SUBTRACTION LEVEL 2						
		4+4	1+5	6+2	1+6	4+3	5+4	3+7	4−3	5−3	8−2	9−4	7−5	10−7	9−6
Below Average	Total count	1	4	5	5	5	5	5	5	5	5	5	5	5	5
	Mean (secs)	4	5 3	7	8	12	12	15	10 6	15 2	11 8	9 4	18 4	10 8	25 6
Average	Total count	6	11	13	12	4	13	7	12	12	13	14	15	15	17
	Mean (secs)	2 7	3 3	4 3	4 4	4 5	4 7	5	3 2	3 9	4 1	4.9	5 3	6 3	9 2
Above Average	Total count	0	0	1	1	2	3	2	0	2	2	2	1	2	4
	Mean (secs)	0	0	2	3	3	3	3.5	0	2	4	6	3	3	5.5

Table 7.5. Chromatographic summary of counting procedures used at Level 2 by 7+ children

It has already been indicated that use of external referents characterised the counting procedures used by most of the 'average' and 'below-average' children but there was very little evidence if their use amongst the 'above-average' children. In one way it is their use and non use that may account for time differences. Amongst the 'below-average' group of children there seemed to be a lengthy period of time devoted to a procedural search. Time was expended looking for and consolidating the use of appropriate referents, even if such referents were always fingers. Initial tagging of the selected referents was always slow and frequently children had to be reminded of the combination they were dealing with. It would appear that the limits of short term memory were reached during the initial counting procedure. It is suggested, for example, that when some of the 'below-average' children attempted 9 – 6 the limits of short term memory were expended during the initial count of the nine. It is suggested that the children then had to spend time refocusing on the problem or requested that the problem be repeated.

The lower mean times taken by the 'average' group can generally explained by their more efficient procedures, count-on in the case of addition and either count-back or take-away for subtraction. Enactive subitising proved to be very efficient. Tara, for example, used this procedure to obtain four of the seven solutions to the subtraction combinations within three seconds. Combinations that could be resolved through enactive subitising making use of only one hand e.g. 4 – 3, 5 – 3, were solved quicker than those that involved two hands. Even the use of count-back meant that some of the combinations were solved within the time limit. 10 – 7 and 9 – 6 are exceptions to these

generalisations. For example, some children who used enactive subitising seemed to have some motor difficulty in representing the nine and then concentrating on that without being distracted by the extra digit, usually the thumb. As a count-back procedure will influence the time taken to carry out the procedure (Woods *et al* 1975) we would expect these combinations to take the longest time.

Where procedures were required amongst the 'above-average' children they proved to be efficient and unsupported by external referents. 'Choice' strategies identified by Woods *et al* were noted amongst the children within this group. There was no evidence of 'choice' amongst the 'average children'.

7.5.2 Procedural Efficiency–Addition and Subtraction Combinations to Twenty.

Children who did not achieve level 2 were not required to attempt level 3.

At level 3, 67% of the solutions were attempted through the use of procedures by the 'average' group of children. Figure 7.1 indicates the extent to which these procedures led to success or failure. Although given an appropriate amount of time during the individual interviews many children demonstrated that their counting procedure could lead to a successful outcome, the "success" rate, given the time restriction, dropped dramatically. However, in only 40% of the cases where counting was used for addition was the outcome achieved within five seconds. Almost 50% of these cases were accounted for by a count-on procedure used for 15+2. In these successful cases the average success time was 3 seconds. Children who failed to resolve this combination within the time limit took up to 18 seconds which not only implies that the child used a counting procedure which involved at least 17 counts, that is, "one, two, three.....fourteen, fifteen" and then "sixteen seventeen" but the child's explanation supported this view.

At least half of the 'average' children who used a counting procedure for any one combination apart from 15 + 2 took in excess of 5 seconds. (17 + 0 was known by all of the children). The mean times for these "slower" counting procedures being as in table 7.6.

Counting procedures used for addition at level 2 were applied to the combinations at level 3. They were often were applied considerably slower.

	4 + 11	9 + 6	7 +7	8 + 10
Total Counted	15	15	4	13
"slow count"	10	8	2	9
Mean slow count	15.7	11.75	12.5	12.8

Table 7.6: Means of slower procedures used by children to obtain solutions to Level 3 addition combinations

In some cases children illustrated similar difficulties to those experienced by the 'below-average' children at level 2. Some of the 'average' group of children, particularly those who used count-all, had difficulty nominating items beyond ten that were to be tagged, a factor that caused difficulties for even more children when they attempted the subtraction combinations.

73% of the solutions to subtraction combinations were attempted through the use of a counting procedure. Of these approximately 34% did not lead to a correct outcome, whilst 80% of those that did only did so by being carried out over an excessive length of time. For example, only two solutions provided for the two combinations 17 – 6 and 20 – 5 were given in less than five seconds. A more detailed analysis of the length of time taken to obtain solutions indicates the difficulty children have with subtraction if it is seen in procedural terms.

Table 7.7 indicates the extent to which procedural methods were used to obtain solutions to the level 3 subtraction combinations during the individual interviews.

	17–6	19–13	20 –5	11–9	18–10
Total Counted	15	14	14	17	15
Errors	3	8	3	4	5
"slow count"	11	6	10	10	10
Mean slow count	21.6	19.8	16	15.1	21

Table 7.7: Analysis of the success of procedures used by children to obtain solutions to Level 3 subtraction combinations

17 – 0 is not included in table 7.7 because all of the children knew the solution.

Eleven children had successfully completed the requirement to have at least 5 of the 6 Level 3 addition combination correct during the formal aspect of the SAT but the difficulty caused by the subtraction combinations for the 'average' children was indicated by the large number of incorrect or missing solutions to particular combinations.

During the individual interviews almost half of the 'average' group attempted a take-away procedure for at least one combination. This presented particular difficulties for those who had successfully used enactive subitising at level 2. They appeared to have no effective strategy to resolve the level 3 subtraction combinations. Children who had successfully obtained the solution to 10 – 7 using enactive subitising, could not take the next step and obtain the solution to 11 – 2. *Enactive subitising, as used by these children, is a procedure that does not generalise.*

By far the most extensively used back up procedure for these subtraction combinations by the 'average' group was count-back. None of the 'average' ability children used count-up although two children used count-back-to on one occasion.

We have discussed the cognitive complexity of count-back. There were many examples to illustrate its difficulties:

- Richard tried to extend his count back procedure from the number combinations to ten to those between ten and twenty. He used his fingers to keep check of the amount counted but he could not remember the count back sequence so that his solution to 18 – 10 was 6 because:

"17,16,15,14,13,12,.....9,8,7,6"

- Christopher tried 19 – 13 through count back. His count back sequence was, "19, 18, 16, 14.....no ...15.....14.....14 .....13, 12, ...".

  He gave the correct solution "six" but his procedure took 37 seconds. At the point where he made his error he had to create a loop; he moved from a count back sequence to a count up sequence and then to a count-back sequence.

Many of those children who had achieved Level 2 had used an integrated strategy approach to solve the combinations which may have involved counting, subitising and knowing. Such a combinations was efficient enough to provide them with a level of achievement which satisfied the criteria for level 2. However, the procedures they used proved to be either too inefficient or lacked the generalisability to enable the children to obtain a spectrum of achievement which would enable them to satisfy criteria for level 3.

The problem with these children being recognised as "successful" at level 2 was that their route to success did not distinguish them from those children who eventually went on to succeed at level 3.

7.6 A PROCEPTUAL APPROACH

As identified within the sample considered within Chapter 5 a distinguishing factor between those who were identified as the 'above-average' and those who were identified as the 'average' was flexibility. Flexibility of approach was the feature distinguishing of the children eventually identified as 'above average' amongst this sample. This flexibility extended to the use of derived facts, extensively at level 3, and the use of the efficient counting procedures. These procedures do not extend to the use of enactive subitising.

In comparison to children within the other two groups, whose integrated strategy approach could be summarised as knowing together with the use of a procedure,

children within the 'above-average' group tempered their use of procedural approaches by using a 'choice' model and through the use of derived facts. There was no evidence of the use of the latter amongst 'below-average' children and very limited evidence of its use amongst the 'average' children. The evidence of the use of derived facts amongst the 'average' children was restricted to five, only two of whom derived more than one solution.

7.7 STRATEGY INTEGRATION: GROUP DIFFERENCES

Figure 7.8 illustrate the cumulative percentages of strategies used by children in each of the three groups. The graphs for 1992 indicate the overall use of each strategy identified as a result of the first series of interviews, that is when the children ages ranged from between 6 years 8 months and 7 years 7 months. Those for 1993 indicate the use of strategies by groups containing the same children 10 months later–when the children were aged 7 years 6 months to 8 years 5 months. It is worth noting at this point that the children's level of achievement bore no relationship to their age.

The trends that are apparent from the snap shots of the seven and eight-year-old children shown within figures 5.5 and 5.6 are also apparent amongst the seven and eight-year-old children within the current sample as seen in figure 7.8. Firstly, we see the extensive numbers of difficulties, indicated by errors, amongst the 'below-average' children and to a lesser extent the 'average' children. Secondly, there is the extensive use of take-away amongst the 'average' and 'below-average' children. Thirdly, only the 'above-average' children illustrate the extensive use of derived facts.

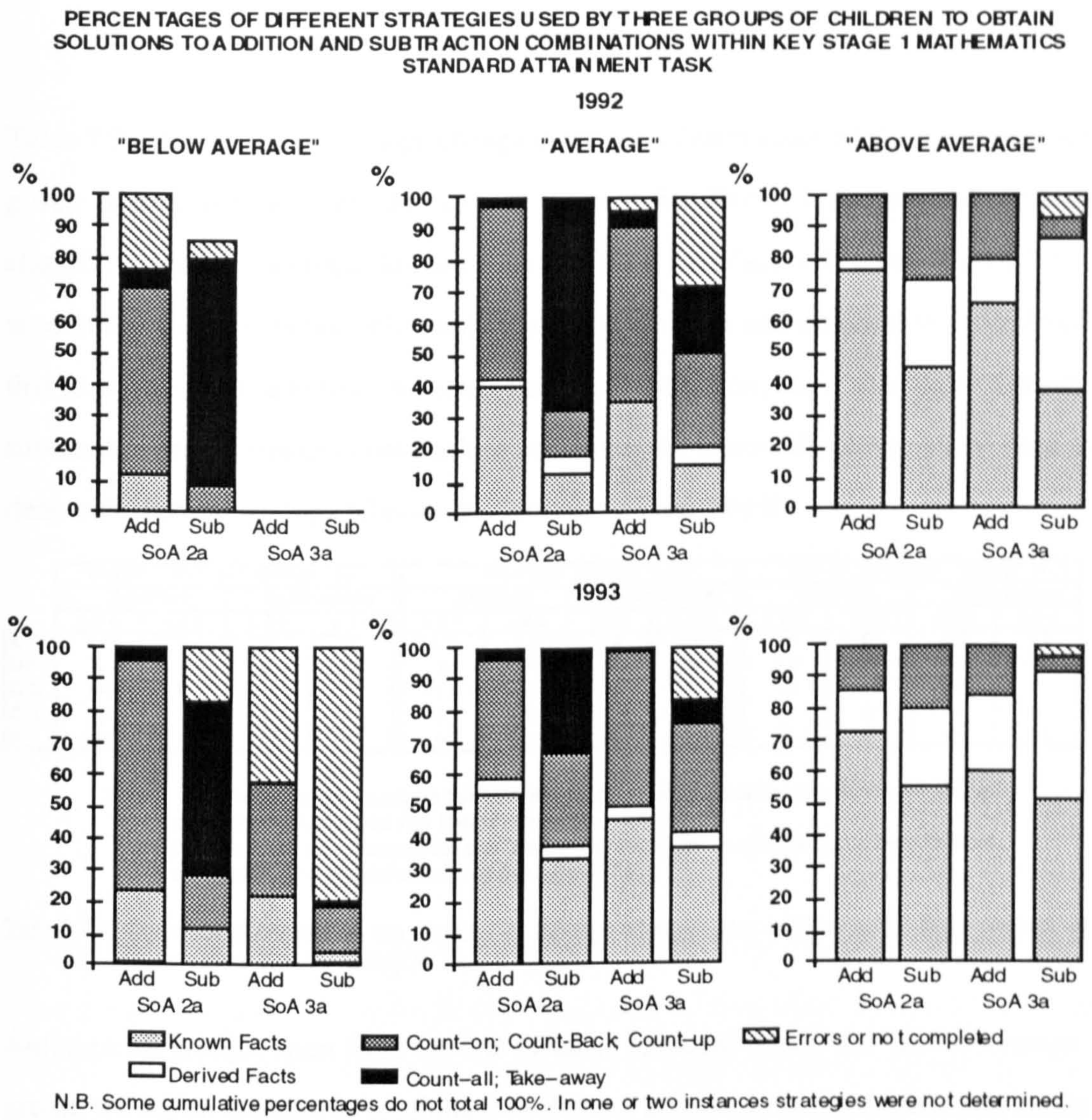


Figure 7.8: Cumulative strategy percentages: seven and eight year old children

At the age of 7+ every child within the 'above-average' group used a derived fact to obtain the solution to at least one of the range of number combinations that were presented. At this age no child who used count-all and/or take-away used a derived fact. We see that procedures tended to dominate the strategies used by the 'average' group at 7+ and tend to be dominant again at 8+. This is the case even though in some instances by 8+ there seem to be conditions available which could promote the use of derived facts. For example, compare the use of known facts and derived facts by the 'above-average' group at 7+ with the known facts and the use of derived facts at the age of 8+ by the 'average' children.

7.8 THE PARTING OF THE WAYS

Table 7.9 indicates the percentage change in the use of each successful strategy by each group of children between the ages of 7+ and 8+. Each of the general strategy classifications are specified; known facts (KF), derived facts (DF), count-on (CO) – note in this case CO is not only used for inverse addition and subtraction procedures, that is, count-on for addition and count back for subtraction, but it also used to denote subtraction procedures of count-back-to and count-up. Count-all, (CA), is also used to denote the inverse strategy take-away. Errors are denoted by E.

	'Below Average' Group: n=5				'Average' Group: n=17				'Above Average' Group: n=6			
	Addition		Subtraction		Addition		Subtraction		Addition		Subtraction	
	L2a	L3a	L2a	L3a	L2a	L3a	L2a	L3a	L2a	L3a	L2a	L3a
KF	12	0	9	0	12	12	19	20	-5	0	13	15
DF	0	0	0	0	2	3	0	3	12	6	-4	-10
CO	14	0	14	0	-15	-5	14	1	-7	-6	-9	0
CA	-3	0	-17	0	0	-5	-34	-12	0	0	0	0
E	-23	0	11	0	0	-5	0	-13	0	0	0	-4

Note: a) Percentage changes for L2a subtraction within the 'below average' group includes strategies not identified at 7+
b) There are no changes for the "below average' group at L3a since they did not attempt the combinations at the age of 7+

Table 7.9: Percentage change in ten months in the use of strategies and errors of three groups of children identified by ability at 7+

Amongst all groups, apart from the one instance apparent within the 'above-average' group, we see an increase in combinations that are known. In the exception previously known combinations were derived.

7.8.1 Procedural Stagnation and Procedural Change

Generally, amongst the 'below-average' and the 'average' groups, the shift to known facts was accompanied by the decline in procedural methods. There were some exceptions: the combinations 5 + 4 and 9 – 5, based as they are on doubles, encouraged some children to derive solutions but the proportions were small, 23% and 18% respectively; only one child derived them both.

Although there was some slight increase in the use of known facts amongst the 'below-average' group – whilst attempting the Level 2 combinations, 11% to 23% for addition and 0% to 9% for subtraction – their strategies were once again dominated by procedures, particularly count-on and take-away. Simon, continued to use enactive subitising for subtraction, Joseph continued his efforts to compute solutions mentally. Lee, who had mixed success with the subtraction combinations, always used count-back. Sometimes he tried this mentally, sometimes with his fingers. He exceeded the time limit to solve every combination, marginally when he counted back mentally, excessively when he used fingers. With mental counting there were no verbal utterances, with figurative counting there were and they were often repeated. He explained that the method he used depended upon how short the 'take-away' was; subtract 2 or 3 appeared to be short, 4, 5, 6 and 7 long. He wanted to do them all in his head, "...when I do them on my fingers somebody might hear me, and will copy me...."

Even though there was a decline in the use of procedures amongst the 'average' group of children and thus a increase in the combinations that were known, the use of procedures still played a large part in the children's solutions to the number combinations to ten. The evidence of the interviews with the children at the age of 8+ indicated that, whilst all of the 'average' group may have satisfied Level 2 criteria for addition, four of the children exceeded the time criteria quite excessively for at least two of the subtraction combinations. Their achievement of Level 2 may be described as 'marginal'.

It is particularly interesting to note the effect of the decrease in the use of 'take-away' at level 2 amongst the 'average' group. 'Take-away' comprised 69% of the procedures the 7+ children used for the subtraction combinations to ten. It accounted for 48% of the procedures amongst the 8+, overall a decline of 34%. 36% of this decline is accounted for by children who moved from take–away to known fact–one child accounting for one third of this percentage–whilst the balance was accounted for by

children who changed from 'take-away' to 'count-back', 46%, or to count-up, 17%. One child continued to solve all of these subtraction problems using 'take-away' whilst a further 5 continued to solve over half of the combinations through this procedure. Only two children indicated any evidence of moving from count–back to a known fact.

Overall, at eight the children were more successful with the Level 3 combinations than they had been the previous year. We can see from figure 7.8 and table 7.9 that the number of errors declined to the extent that there were none amongst the 'average' ability children for addition and they were almost halved for subtraction. However, although the percentage of Level 3 combinations known by the 'average' group increased within the year – addition, 34% to 46%, subtraction 16% to 36% – the addition increases were accounted for by strategy changes of less than half of the children. Where there was a change it was accounted for in the solutions given for 7 + 7, 8 + 10 and 15 + 2. Overall, when solving the L3 addition combinations there was a remarkable consistency in the solution procedures the 'average' children used at 7+ and 8+. Generally the improvement in success was accounted for by the more efficient use of a procedure. Once again we are facing the situation where procedural efficiency can cloud the issues of differences in qualitative thinking. Nine of the children within the average group now obtain at least six addition solutions within the time limit.

Consider Shelley when asked to add 4 + 11:

Age 7 +	Age 8+
There were several moments hesitation. Finally four fingers were extended on the left hand without counting. Incremental counting continued on the her remaining fingers accompanied by verbal utterances "*five, six, seven, eight, nine, ten,*" until each finger of both hands had been tagged. A pencil on the table was tagged as eleven. There is a pause.....extended fingers of both hands are slowly but continually flexed. Finally the five of the right hand are very positively extended. The five fingers of the left hand are incremented to ten. Again the pencil is pointed to on the utterance of eleven. Eleven is given as a solution. Time 14 seconds	Several moments hesitation. head resting on left hand. Left hand moved to the desk top. Hand rested on the desk top with pressure on the little finger. The other four fingers are suspended. Left thumb placed on top of desk where it remains as the remaining three raised fingers are one at a time placed on the desk top. Each finger movement is accompanied by verbal utterances, "*twelve, thirteen, fourteen, fifteen.....fifteen*" the final statement signifies the solution. Time 4 seconds.

Interestingly Shelley had only known one of the addition combinations to ten when she was seven. She had used count-on to solve the remainder, one of which, 3 + 7, had been incorrect. At eight she knew five combination and counted the other two within five seconds. At seven she had used 'take-away' to obtain solutions to the subtraction combinations within 5 seconds. She had not been able to develop an appropriate strategy for the combinations to 20. At eight she still extensively used 'take-away' for the combinations to ten but relatively successfully used count-back for the combinations to twenty. However, some of these took an excessive amount of time.

If procedural efficiency had marked the change in addition for the 'average' children over the ten months, procedural difficulty continued to be a hall-mark of some of the

children in subtraction. Even if, as eight year olds, they displayed procedural competence they remained unable to complete the procedure within 5 seconds. Some children still took considerably longer, but others only need some improvement before being able to achieve Level 3.

Some children continued to perpetuate the same mathematical errors as those identified the previous year although there was a qualitative change in the use of referents. Rebecca, for example, apart from knowing 15 – 0, failed to obtain a correct solution to any of the other combinations to twenty in both years.

During 1992, when solving 20 – 5, she subitised 10 on her fingers, and then objects around the room were tagged from 11 to 20. A similar procedure had worked reasonably well for addition; Rebecca showing extraordinary ability to remember the 'external' items that were tagged. Her subtraction procedure was take-away; after having tagged the ten 'external' items, she started with the last one nominated and tagging the items in reverse order renamed them 1 to five. The next item in reverse order she renamed as one and continued until the remaining five 'external' items had also been renamed 1 to 5. Five was given as a solution. Rebecca had shown considerable powers of memory, but it is suggested that considerable aspect of her thinking had been expended first in finding and tagging the additional ten items, and then remembering them in reverse order. The consequence was that the ten was forgotten. A similar procedure was used in the more recent interview. Ten was identified in the same way but this time her ten fingers were used to count-on from ten to twenty. Take-away was carried out in the same way as previously, the fingers were now renamed in reverse order. However the solution was once again given as five.

Successful take-away procedures at seven were used at eight. Fiona, when attempting 11 – 9 in 1992 proffered ten extended fingers and uttered "ten". The right thumb was then extended backwards as she uttered "eleven". Concentrating on this thumb she then moved it twice and uttered "one, two". On this second occasion she folded it across her

palm whilst she renamed the other fingers 1 to 9 staring from the right index finger. At the final tagging, "nine', she looked up and said "nine". The same procedure was repeated in 1993.

Two 8+ children from the 'average' group, both of whom had shown some evidence of the use of derived facts at 7+, reached a level of attainment to achieve Level 3 as a result of the 1993 interviews. They achieved this level of attainment by using a combination of known facts, derived facts and "choice". They were the two 'average' ability children who had solved at least two of the combinations at the age of 7+ through the use of derived facts.

7.8.2 Proceptual Competence

There are instances where subtraction combinations to ten, described as known in 1992, are in 1993 claimed to have been solved through related addition combinations. At the start of the study the problems of interpreting children's mental thinking from verbal statements are indicated. Perhaps this is a case in point. At 7+ the children may have resolved the solutions to some of the subtraction combinations through a related addition combination but their thinking may be so automatic that a difference between the two may not have occurred to them. Two points emerge: (i) there may have been a greater use of derived facts amongst these children at 7+ than is described and (ii) the relationship between a known fact and a derived fact for the children may be so linked that at times they are unable to articulate the difference.

Such difference seem to be clearer at level 3 not least because of the consistency of the responses that the children gave to the subtraction combinations. Five of the six children within the group used a derived fact to obtain a solution to at least one of these combinations when age 7+. At 8+ the children had 'turned' some of these derived facts into known facts, three of the children continued to derive some of the solutions and some instances gave different derivations. Consider some responses:

Solutions to 17 – 6

Vincenzo (7+): "Seven take away six is one so 11 because you add the ten"

Vincenzo (8+): "If you take away seven it's ten, so add one more is eleven"

Solutions to 20 – 5

Jonathan (7+): "That five from ten is five add the other ten is fifteen"

Jonathan (8+): "Its three fives..fifteen"

For these children the duality inherent in the symbolism of simple arithmetic is treated with ambiguity to give the flexibility that leads to their success in simple arithmetic. They fit the mould of the proceptual thinker.

7.9 SUMMARY

Clearly the 1992 SATs relating to knowing number combinations achieved their purpose of differentiating between children over three levels of achievement. However, a longer term prognosis, which only takes note of a child's current level of attainment as a starting point without noting how the attainment was achieved, may lead to a very different outcome.

Distinctions between the 'below-average' children and the 'above-average' children are illustrated by the attempts of the former to use procedures whilst for the latter they appear to be 'an optional extra'.

Clearly, asking children to attempt what is considered to be the same range of problems presents each child with a different level of difficulty. This is not only dependant upon what they know and the way in which they use what they know but it is also a function of their procedural approach and particularly:

- the child's competence with the procedure,
- the efficiency of the procedure,
- the ease with which the procedure can be generalised.

In the context of simple arithmetic, the procedures used qualify as counting procedures in the sense that counting is defined by Steffe *et al* (1981). We see amongst the 'below-average' children evidence, that with few exceptions, appears to support Cobb (1987); the children are only able to construct enactive relationships between the numbers. To do this they generally require external referents. These may provide some support which enhances the child's procedural competence but the child does need to recognise that they require such a support. We see that the actions of the children in both addition and subtraction are far from the point where in Piaget's terms they may become internalised as the concepts of 'sum' and 'difference'. In the short term an addition procedure for, for example, 3 + 7, is not encapsulated into the object 10, and this leaves the children very much within a procedural paradigm when dealing with subtraction.

But this paradigm is one in which we are able to detect the more expanded procedures of count-all and take-away contracted into the more 'efficient' procedures of count-on and .....and count-back? Efficiency in count-back is dependant on, in the first instance, competence with Fuson *et al*'s (1982) 'elaboration phase' for advanced backward counting skills. The absence of this alone seems to indicate that in subtraction procedural competence is diminished for the 'below-average' group. As such the procedures still remain so protracted that any sense of the notion of encapsulation is itself diminished. Thus, although difficult to implement, the children rely on their procedures and any sense in which they may develop a notion of the part-whole relationship is undermined.

As we consider the 'average' group of children we may see two different groups emerging; those whose procedural competence remains relatively unsophisticated and is thus suspect, and those whose procedural competence is increasing in efficiency to the point that it has the potential to further cloud the distinction between procedural thinkers and proceptual thinkers in the context of simple arithmetic.

The significant decline in the use of procedures and the increase in known facts for the combinations to ten provides many of these children with the potential to derive solutions for combinations to twenty. It could be expected that as their number knowledge increases these children would think in the same way that the 'above-average' group of children had done almost a year before. In fact, apart from two instances, this does not happen. This suggests that there is a different quality attached to the combinations that are known. In one case they are isolated, in the other they are part of an expanding relationship between number combinations. A known addition combination can be used to solve a subtraction combination. The flexibility this attaches to the child provides an enormous step forward. It is a flexibility that those 'average-ability' children who know the combinations to ten do not appear to possess at this time.

Isolated knowledge can cloud theoretical issues within the context of the number combinations. So too can the efficient use of procedures cloud the issues of achievement. It is believed that the notions of procedural and proceptual thinking go some way towards clarifying distinctions between the ways in which children think about their mathematics. The qualitative evidence indicates that the thinking is qualitatively different.

CHAPTER 8
QUALITATIVELY DIFFERENT THINKING

8.1 REFLECTIONS

Through the qualitative analysis of the integrated strategies that various groups of young children use to solve simple arithmetic tasks, this study has examined the different approaches perceived in children of various ages and abilities. It was seen that whilst some children resolved combinations they did not know through the use of procedures, others demonstrated flexibility; at one moment they may use a procedure to compute a solution whilst at another they used the arithmetical object produced through a process.

It was suggested that such differences in behaviour were manifestations of the interpretation of properties possessed by mathematical symbolism which represents notions of process and concept at one and the same time. The cognitive representation of process and concept represented by symbolism was given the name *procept* which is seen as an extremely powerful form of concept – it embodies a procedure to be able to compute the product and a mental object that can be manipulated at a higher level.

From the theoretical construct of procept it was hypothesised that there is a qualitative difference in children's arithmetical thinking which is:

(i) on the one hand manifest as a spectrum of performance in the operations on numbers as a procedure that relates to counting and

(ii) on the other, the flexible manipulation of procepts.

This chapter reviews the evidence that supports this hypothesis, provides a hypothesised development of the theory of procepts and in the light of this hypothesised development makes suggestions for future research.

8.2 A REVIEW OF THE EVIDENCE

Two experiments were set up to consider the evidence to support the hypothesis. The first considered the evidence of a divergence in thinking taken from a series of snapshots of different groups of children over a variety of ages. From this evidence we saw that children acquired knowledge of basic number facts over a period of years, and that groups of different ability acquired known facts at a different rate. However, there were no indication that 'below-average' children would acquire the same facility in the subtraction combinations to twenty as the 'above-average' children by the time they left the combined school. What was of particular interest though, was the form of the back-up strategies that the children used if they did not know the number combinations.

We saw amongst all of the younger children the use of procedural methods but there was a qualitative difference in their use. Whereas, for example, below average children used count-back as the natural inverse of count-on, we see the 'above-average' children of seven using either count-back or count-up, whichever is appropriate. Later we see all of the 'above-average' using count-up if they needed to use a procedure. Through their choice of procedures the children demonstrated the difference between making things easy for themselves or making things difficult. But there were not just procedural differences. There were also difference in the quality of the 'units' used for counting. Whereas the 'below-average' children used perceptual items, the 'above-average' used 'motor' or 'abstract' units. Thus, we see the more successful children who use procedures using them more efficiently, and more particularly, with some flexibility.

Amongst the older children we saw a general decline in the use of procedural methods. This decline was not matched amongst the 'above-average' children with an immediate increase in known facts, but the emergence of significant use of derived facts. The use of this strategy was hardly apparent amongst the 'below-average' children, even when older 'below-average' children knew the same number of combinations as a group of younger 'above-average' children.

The evidence from the first experiment indicated that when more successful children used procedures they used them competently, efficiently and flexibly. The 'below-average' may have demonstrated procedural competence at one level but this only provided procedural efficiency after a period of development and they seldom displayed procedural flexibility.

Such qualities as those possessed by children whose arithmetic is seen in terms of procedures is quite distinct from the qualities displayed by those whose thinking is manifest in the flexible use of procedures, and derived facts. More able children were doing a qualitatively different arithmetic to the less able.

The difference in thinking between the 'below-average' and the 'above-average' in the first experiment may be expressed in procedural and conceptual terms. However, two issues clouded any distinctions that it was wished to make. Firstly most of the children knew some number facts whilst many of those who were considered to be flexible also used procedures. We have seen that conceptual knowledge about simple arithmetic is identified as knowledge about the relationship between numbers, but there is no explicit statement in this notion that children who possess conceptual knowledge also have the flexibility for the compression and decompression of procedures.

If we look at all children and attempt to draw conclusions from our observations we frequently conclude by formulating a particular theory. This has been consistently apparent in the review of literature. At times there then may appear to be a conflict between the theory and reality. As a case in point we may consider the general conclusions that can be drawn from some chronometric studies (e.g. Ashcraft, 1982, who based on his predictions concluded that 6 and 7 year old children in America consistently use count-on for addition) and some of the protocol based studies (e.g. Carpenter & Moser, 1982, who indicated that children used a variety of approaches).

By looking at two extremes within the first investigation rather than at one whole, a theory of two components was established which does not carry the apparent

dichotomies that may be seen in the development of two theories established through considering different wholes. Such a theory stems from the notion of procept.

Establishing a theory of procepts centres upon a close analysis of the interpretations that may be placed upon mathematical symbolism. Rather than perpetuating a potential dichotomy between object and process, the duality of mathematical symbolism encompasses the two in an ambiguous way. Recognising this ambiguity provides the user of symbolism with considerable flexibility.

The second experiment was designed to relate this postulated theory with a widely used standard assessments task and to trace the development within specific children over a period to compare this development with that seen in the snapshots.

Once again flexibility of approach was a distinguishing feature of the children eventually identified as 'above-average' within this second sample. Again the children used derived facts, the only group to extensively do so. The evidence shows that the children's use of procedures proved to be efficient, flexible, unsupported by external referents and supported by the use of derived facts. The 'above-average' children within the second sample display similar qualities to those displayed by their 'above-average' peers in the first sample. Both groups display the flexibility of thought consistent with proceptual thinking. For these children the duality and ambiguity inherent in the symbolism of simple arithmetic is reflected in the flexibility that leads to their success. They fit the mould of the proceptual thinker. Thus, although the samples were drawn in two different ways, evidence of proceptual thinking occurred in seven and eight-year-old children within the 'above-average' groups of both samples.

In the second experiment the results were clouded somewhat by the nature of the assessment mechanism used to assign children to the various groups. As a measuring instrument the SAT it is too blunt an instrument to distinguish between children within the average group, i.e. those achieving Level 2 but not Level 3, although teacher

assessment may be equally blunt if it does not take into account how attainment is achieved.

The procedural competence of the 'average' group within the second experiment was such that any deficiency in their knowledge of the number combinations was compensated for by the use of a procedure efficient enough to satisfy the constraints of the time limit. Technically, had the application of the test be rigorously applied – successful achievement restricted to non-counted responses – none of the 'average' group of children would have achieved Level 2. A more rigorous application of the test procedure would have been difficult. It has already been indicated that motor counters and abstract counters would not have been easily identified. As it was, the head teacher and the senior advisor for the County were both present during the administration of the test to one group and neither questioned the teacher's administrative procedure. Such was the procedural competence of many of the 'average' ability children that their achievement at this level was hard to distinguish from the performance of the 'above-average'. It was only when their procedures failed them at a higher level that the distinction could be made – which of course was part of the overall testing intention.

Some consequences of the procedural approaches used by the 'average' children were apparent at the end of the ten month period. Although several of the children had failed to improve their procedural competence, the greater proportion had improved it to the point where they were now almost achieving the same level of attainment in the basic number combinations as the 'above average' group. Only two children had demonstrated any flexibility and these were the only children to achieve Level 3 after the ten months.

8.3 A CONCLUSION

What we seem to be seeing amongst the 'average' ability children is a demonstration of the point of bifurcation of the proceptual divide. Although many of them by Y3 demonstrated that they had learned some number combinations, most were not using

the number knowledge they possessed to derive combinations they did not know. Instead they had developed efficient count-on procedures for addition and were in the process of developing procedural competence for subtraction based on the use of count-back. In the opinion of Steffe *et al* (1982) they would not meet the crucial preconditions to enable them to identify the benefits of the part-whole relationship. This would be a major step forward in a move towards a proceptual view of subtraction.

Amongst 'average' children who used procedures three main groups may now be identified,

- those who are procedurally competent and efficient,
- those who are procedurally competent and inefficient,
- those whose procedure is unsound.

This last group would also contain the 'below-average' children for whom the procedures generally remain so protracted that any sense of the notion of encapsulation is itself diminished. The children rely so much on their procedures, even if they are difficult to implement, that any sense in which they may develop a notion of the part-whole relationship is undermined.

The qualitative differences that exist in the ways children think about in simple arithmetic may be viewed from a proceptual perspective. It is hypothesised that successful mathematical thinker uses a mental structure which is manifest in the ability to think proceptually which, it is conjectured, explains qualitatively different thinking between those who are successful and those who are not successful in arithmetic. The notion of using count-on, on the one hand as a procedure, and on the other as a means of obtaining flexible number knowledge, may be the parting of the ways between an increasingly inflexible method in which arithmetic becomes more and more difficult, and a flexible method which can lead to long-term success.

8.4 COUNT–ON: 'THE PARTING OF THE WAYS'

This second experiment provides some indication of what may be happening at the point of bifurcation which leads to either the development of competence in a compressed procedure or procedural encapsulation. It is concluded that count-on is a point of bifurcation in simple arithmetic (Gray, 1993).

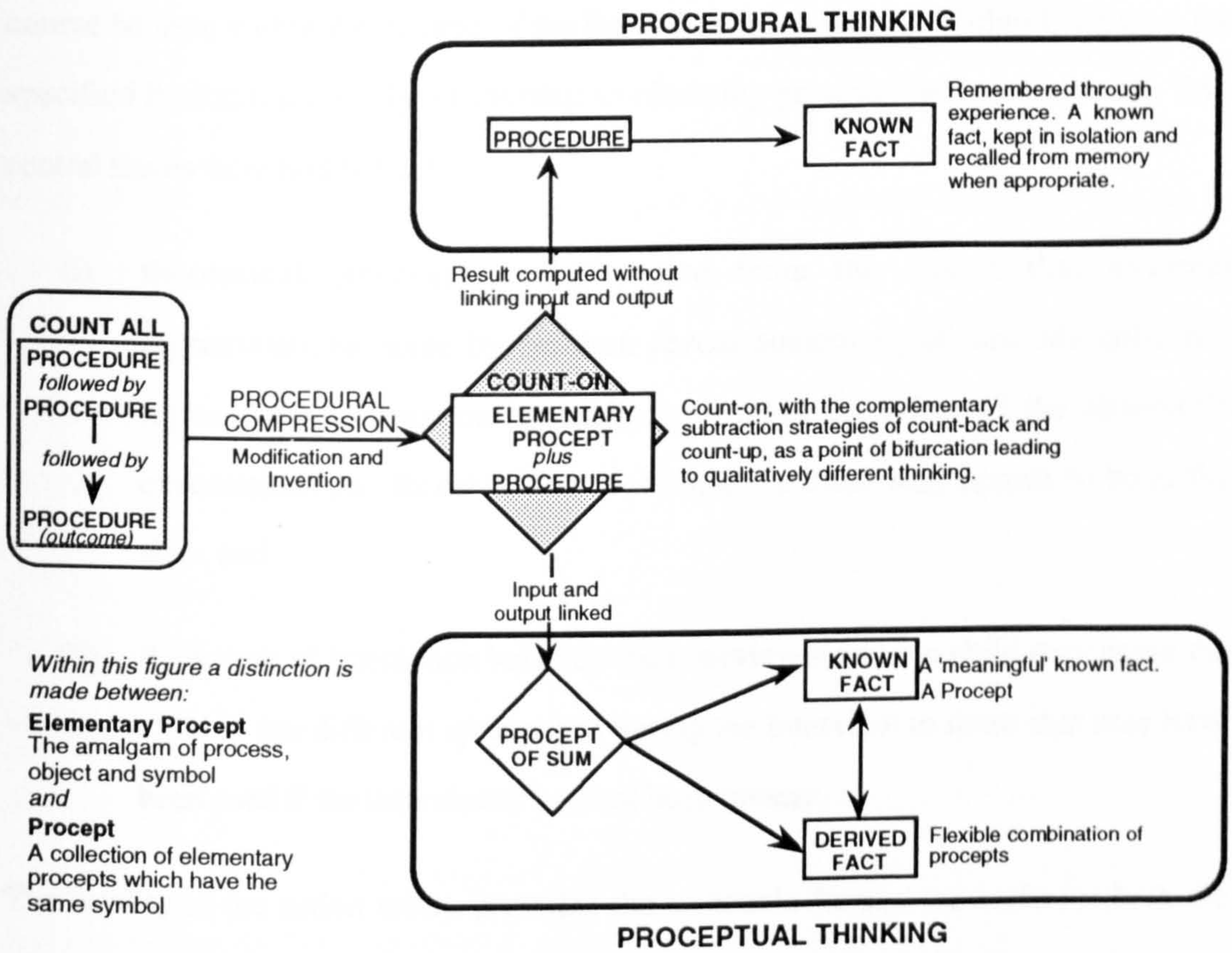


Figure 8.1: Count-on: A pivot for a proceptual divide in simple arithmetic

Figure 8.1 (from Gray, 1993) provides an indication of how count-on may be seen as such a point. It provides children with a compressed counting procedure which is potentially efficient. However, it also appears to act as a springboard to a different quality of thinking. In this sense it acts as a "junction box": it can cause a bifurcation that leads to a 'parting of the ways' between those who are successful and those who are not successful.

From a procedural point of view the essence of strategies such as count-on, count-back or count-up is that they may be refined to such a degree that though they may become

very efficient at one level they may not only mitigate against reflection but also against success at the next higher level. One of the consequences is the development of qualitatively different approaches to simple arithmetic.

8.5 LIMITATIONS OF STUDY

The theoretical argument established through the analysis of the protocols must of course be seen within the context of the limitations of the study. Within Chapter 4 the specified limitations directly attributable to obtaining protocols were considered. Two central issues were highlighted:

(i) theoretical constructs, which arise from the notion that external representations made by children reveal something of how the child has represented information internally, are not entirely free of the observer's expectations and theories, however objective these may appear to be at the time, and

(ii) the nature of interaction between the interviewer and the child may cause the child to use different approaches during the interview to those that may have been used if the interviewer had not been present.

The totality of the action which provided the protocols formed the basis for both the analysis of the data and the resulting theory. However, further limitations are imposed by the sampling method and the level of reporting subsequently presented. The study itself is concerned with a coarsely grained analysis of behaviour and this must be placed within a context that accepts a third form of limitation; the selection of samples and their size.

Although it has been indicated earlier that the nature of the SAT used for the second series of interviews must be seen as a blunt instrument in terms of its ability to differentiate between children, the nature of the selection of the first sample must also be seen as a blunt instrument. Twelve teachers were involved in making an assessment

of children's ability, but teacher's notions of a child's ability may not be restricted to the actual quality being considered (Secada, 1992) and they may be reflected in the child's level of achievement in the sense that the child may rise or fall to the teacher's level of expectation (Nash, 1973).

The small discrete samples identified within the first series of interviews may serve to be both the weakness and strength of the initial analysis of the data and the later theoretical development. The weakness stems from the relatively small differences that are particularly apparent between the average and the above-average groups of children whilst the strength lies in the observation that different forms of thinking appear to be consistent over several different samples of children. Such differences are again apparent within the second sample, which, although taken from only one class within one school, does appear to support the notion that children display different qualities of thinking.

It has not been the intention of this study to indicate proportions of children that reflect different styles of thinking. Neither has it been the intention to stereotype children through their quality of thinking to establish a long term prognosis which predicts different levels of success in mathematics. However, it was the intention to indicate whether or not children may display qualitatively different thinking and then consider the consequences for achievement as the immediate level of difficulty increased. Although the results of the first series of interviews do not represent the effects of differences that may be seen within the same children taken over a period of time, they suggest that there are qualities that are worthy of further investigation. In the second sample, from a different school, each child is allocated to a group as a result of their level of achievement. The analysis of the results of this second sample suggests that the features identified through the snapshots of the seven and eight year olds within the first sample may be consistent over a period of one year. However, although short term effects and changes that may arise through qualitatively different thinking are considered within the context of this second sample, a conclusive long term prognosis

for these children would be conjecture. There may well be cumulative longer term effects but, for the moment, these remain in the realm of a new set of hypotheses.

8.6 A HYPOTHESISED CUMULATIVE EFFECT OF THE PROCEPTUAL DIVIDE

Proceptual encapsulation occurs at various stages throughout arithmetic: repeated counting becoming addition, repeated addition becoming multiplication and so on, giving what is usually considered to be as a complex hierarchy of relationships (figure 8.2).

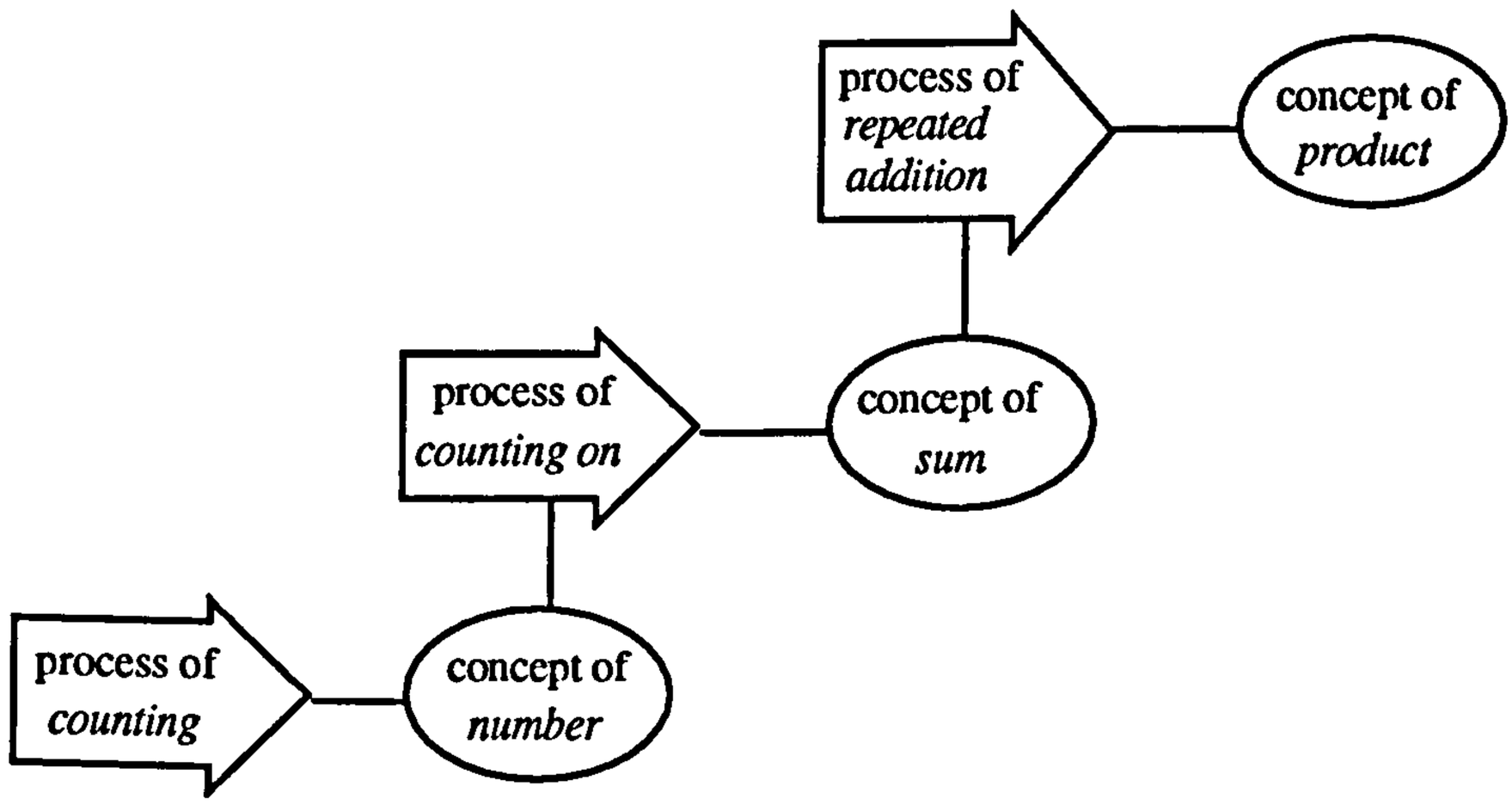


Figure 8.2 : Higher order encapsulations

Such a theory is the one proposed by Sfard (1991). Using this theory we may explain the encapsulation process; the operational aspect of counting may be reified into the structural aspect of number. This is then used as the operational aspect of count–on which is then reified into the structural aspect of sum. It is conjectured that this is the way that the less able see simple arithmetic. This is, for them, an extremely difficult process. If they are faced with a problem two levels up, then the structure will almost certainly be too burdensome for them to support (see Linchevski & Sfard, 1991). Multiplication facts are almost impossible for them to coordinate whilst they are having

difficulty with addition. Even the process of reversing addition to give subtraction is seen by them as a new process ("count-back" instead of "count-up").

It is conjectured that the more able, proceptual thinker is faced with an easier task. The symbols for sum and product again represent *numbers*. Thus counting, addition and multiplication are operating on the same procept which can be decomposed into process for calculation purposes whenever desired. A proceptual view which amalgamates process and concept through the use of the same notation therefore *collapses the hierarchy* into a single level in which arithmetic operations (processes) act on numbers (procepts) (figure 8.3).

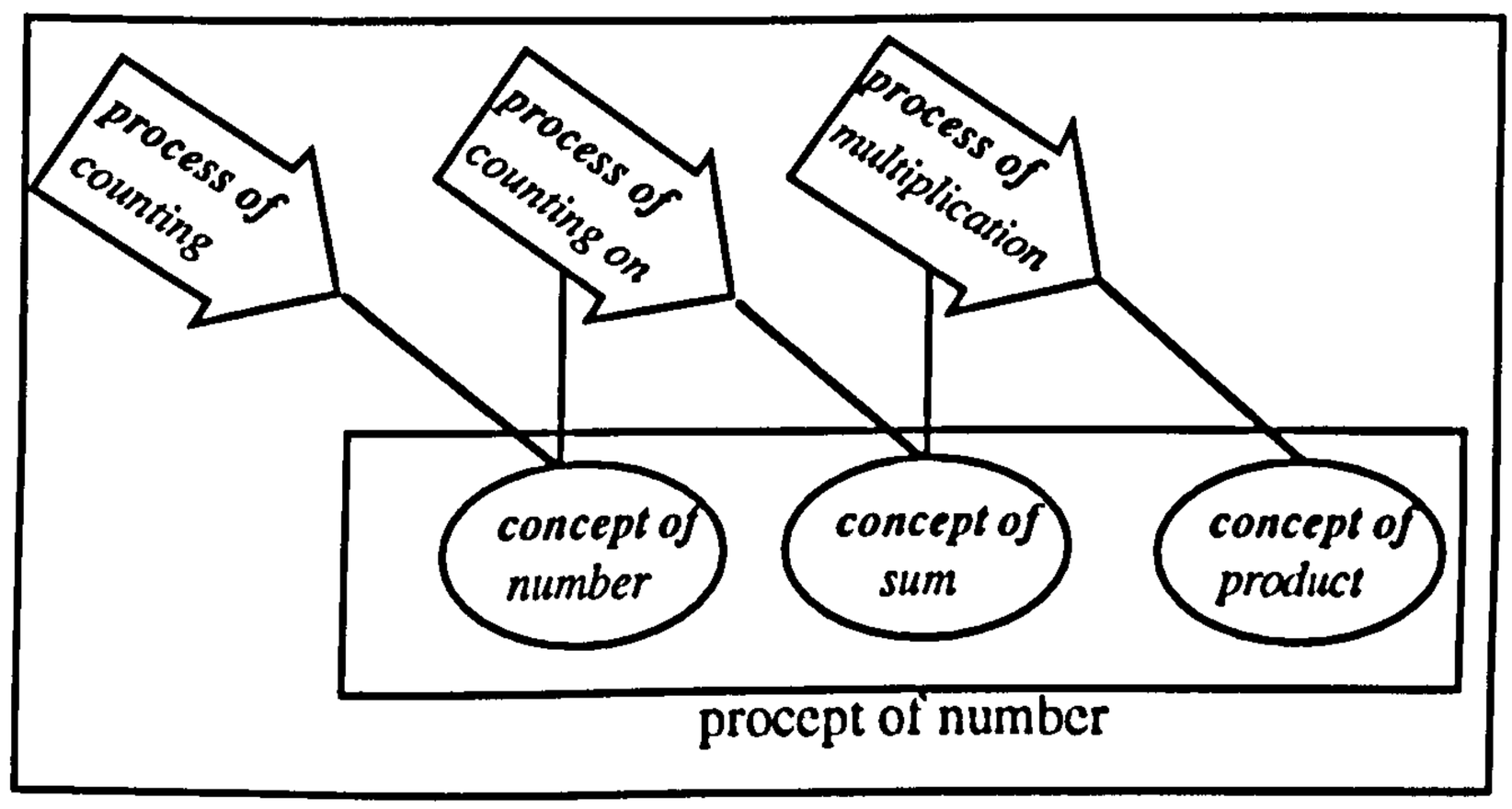


Figure 8.3: Collapse of hierarchy into operations on numbers

It is conjectured that this is the development by which a more able thinker develops a flexible relational understanding in simple arithmetic, which is seen as a meaningful relationship between notions at the same level. We have seem some examples of this in the way that some of the children used multiplication as the derived fact to solve some of the addition and subtraction combinations. The less able are faced with a hierarchical ladder which is more difficult to climb. It is suggested that children across a wide spectrum of performance face this challenge at each stage of encapsulation, and that at each stage more children fail.

This provides an insight into why the practising expert sees mathematics as such a simple subject and may find it difficult to appreciate the difficulties faced by the novice. It is conjectured that as proceptual thinking grows in conceptual richness, procepts can be manipulated as simple symbols at a higher level or opened up to perform computations, to be decomposed and recomposed at will. Such forms of thinking become entirely unattainable for the procedural thinker who fails to develop a rich proceptual structure.

The difficulties that children have in establishing a proceptual view of simple arithmetic should be a signal us to the difficulties they may have in more complex areas of arithmetic. How may we reasonably expect children to understand the multiplication and division if in simple addition and subtraction that are still procedural? It is conjectured that the problem of the proceptual divide that occurs in simple arithmetic is a microcosm of the problems that occur as mathematics becomes more complex; at each higher level a proceptual divide occurs. Some children take to the fast route fairly easily to become successful, others take the slower, procedural route to achieve success at one level only to be faced with another parting of the ways through which they take an even slower route which eventually leads to failure.

The ability to use the proceptual nature of mathematical symbolism would seem to provide those who are able to do mathematics with a very powerful tool. The wonder is that such mathematicians do not consciously realise that they make use of the tool; they possess tremendous flexibility because of the inherent ambiguity of the symbolism, they are able to select the quality appropriate to the moment without conscious thought. Lesser beings who rely on memory and a procedure, and I include myself in this category, only see half a picture and for them the same flexibility is not an option.

8.7 SUGGESTIONS FOR FUTURE RESEARCH

In highlighting the qualitative different thinking that children apply in simple arithmetic I believe several issues arise which question values which provide acclaim to extensive

use of procedures and give some concern for some of the children's progress through the wider mathematics curriculum. It is the latter that I believe will lead to fruitful research.

1. To what extent would a longitudinal study of mixed-ability children confirm or deny the conclusion of this study and provide further evidence of the proceptual divide within arithmetic?

The difficulties that arise from the proceptual divide in simple arithmetic would seem to indicate that a longitudinal study establishing the veracity of the divide with a wider group of children is called for.

From the evidence presented the notion of the proceptual divide within simple arithmetic calls for consideration of the consequences in more complex areas of arithmetic. How may we reasonably expect children to understand the multiplication and division if in simple addition and subtraction that are still procedural? How can children who remain procedural within whole number arithmetic develop a proceptual view of fractions. It may be hypothesised that, whilst in simple arithmetic, it is *advantageous* for the child to think proceptually, when operating with rational numbers it is *incumbent* upon them to do so. For the child to operate the addition and subtraction of fractions successfully they need to be able to see the same fraction in many different ways. The notion of fraction may be a significant point of bifurcation in the same way that count-on appears to be.

2. What is the role of procepts in other areas of mathematics?

Recognising the proceptual nature of mathematical symbolism implies that the proceptual divide will be a recurring theme in mathematical development. At the level of school mathematics, children's difficulties in algebra (Tall & Thomas, 1991) arise from a procedural interpretation of the symbolism and can be explained by the inability of the children to see algebraic expressions as procepts. Similar conclusions may be drawn from an examination of efforts in trigonometry (Blackett 1990, Blackett & Tall, 1991).

The conception of trigonometric ratio as a process of calculation (opposite over hypotenuse) and not a flexible procept causes difficulties. Process application involves reliance on an encapsulated hierarchy in which misconception, inconsistency and confusion is bound to appear. It would seem that the examination of the use of procepts within these and other areas would also be profitable. An interesting question may consider whether procedural thinkers in arithmetic may become proceptual thinkers in algebra.

Other areas of the mathematics curriculum lend themselves to an in-depth study of the proceptual divide and an examination of the relationship between process and object orientations of mathematical thinking. We have seen that the notion of procept pervades the whole of mathematics. It what way does this proceptual nature of mathematics effect learning and understanding particularly if through design or otherwise the nature of the input to the learner is procedural.

E.M. Gray,
April, 1993

GLOSSARY

Conceptual Knowledge:

This is used to denote knowledge that is understood. It makes use of the underlying relationships which exist within, or between, mathematical objects themselves:

> a unit of conceptual knowledge cannot be an isolated piece of information; by definition it is part of conceptual knowledge only if the holder recognises its relationship to other pieces of information.
>
> (Hiebert and Lefevre 1986, pp. 3-4)

Conceptual Thinking:

Thinking that reflects the use of conceptual knowledge:

Elementary procept:

The amalgam of three components; the *process* which produces a mathematical *object* and the *symbol* which is used to represent either the process or object.

Enactive representation:

A mode of representing past events through appropriate motor response and it "is based, it seems, upon a learning of responses and forms of habituation" (Bruner, 1968, p. 11).

Enactive Subitising:

The representation, without counting, and usually with fingers, of numbers in a concrete form to support addition and subtraction procedures.

The term follows Bruner's notion of enactive representation and it is conjectured that the habitual use of fingers as a counting aid leads to their use as a means of displaying amounts less than ten without counting.

Encapsulation (Following Dubinskey, 1991) **:**

The cognitive process of forming a (static) conceptual entity from a (dynamic) process. Such a process has variously been called "entification" (Kaput, 1982) and "reification" (Sfard, 1989, 1991).

Object (of thinking):

The encapsulated entity created through a mathematical process. The notion of object means that we can refer to the entity as if it were a real thing; we can recognise it at a glance and manipulate it as a whole. Thus we may see how the process of counting five leads to the encapsulated entity "five" which qualifies as an object when it is permanently available for manipulation.

Procedural Thinking:

Thinking that reflects the use of procedural knowledge.

Procept:

A collection of elementary procepts which have the same object.

Proceptual Divide:

The divergence in thinking that stems from the flexible use of procepts as object or process on the one hand and the excessive reliance on procedures on the other. It is conjectured that this is one of the most significant factors in the difference between success and failure.

Proceptual Thinking:

Thinking which portrays the flexibility to view symbolism either as a trigger for carrying out a procedure or as the representation of a mental object which may be decomposed, recomposed and manipulated at a higher level. Proceptual thinking includes thinking that demonstrates, where appropriate, the use of procedural knowledge. It is conjectured that the possession of such a quality is at the root of powerful mathematical thinking.

Within the context of simple arithmetic proceptual thinking includes the meaningful use of known facts to arrive at solutions through derived facts and it may include the use of procedures.

Process (Theme of action)

Within this study the term "process" is used in a general sense, as in the "process of addition", the "process of multiplication", the "process of solving an equation". It means the cognitive representation of a mathematical operation. It need not be a process that is currently being carried out in thought, for instance we may speak of the process of addition without actually performing it

and neither is there any implication that the process must be carried out in a unique manner

Procedure:

The term "procedure" is used in the sense of Davis (1983, p.257); it is a specific algorithm for implementing a process.

Procedural Knowledge:

Procedural knowledge is seen as knowledge which focuses on doing through applying. It is usually hierarchically arranged and prescribes instructions on how to complete tasks which operate on objects which are represented either by concrete materials, spoken words, written symbols, or mental images. It may be seen as the knowledge of the sequence of actions or procedures which may allow mathematical tasks to be completed efficiently.

Although such knowledge may be applied mechanically without reference to related conceptual knowledge, for the successful mathematician it is conjectured that the compression of processes, such as count-all to count-on, is manifest through procedural competence reflecting more sophisticated forms of procedural knowledge. Such compression forms a route which has the potential to lead to the encapsulation of the process. For example, the process of count-all is first compressed into count-on which, implemented through increasingly sophisticated procedural knowledge, has the potential to be encapsulated as the concept of 'sum'.

Subitising:

The immediate quantification of a set of objects without counting.

*

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QUESTIONNAIRE: STAGE 1 PROBLEMS

NAME..... AGE: YEARS.....MONTHS.....

SCHOOL TYPE.....YEAR GROUP.....

NUMBER COMBINATIONS TO 10

ADDITION	CA	CO	DF	KF	SUBTRACTION	TA	CU	DF	KF
2 + 1					3 – 2				
3 + 5					5 – 4				
8 + 2					6 – 3				
4 + 4					9 – 5				
0 + 2					3 – 3				
6 + 3					6 – 0				
5 + 4					8 – 2				
5 + 0					9 – 8				
7 + 2					7 – 5				

COMMENTS

QUESTIONNAIRE: Combinations to twenty

NAME..... AGE: YEARS.....MONTHS.....

SCHOOL TYPE.....YEAR GROUP.....

NUMBER COMBINATIONS TO 20

ADDITION	CA	CO	DF	KF	SUBTRACTION	TA	CU	DF	KF
12 + 1					13 – 2				
13 + 5					15 – 4				
18 + 2					16 – 3				
14 + 4					15 – 9				
10 + 2					16 – 10				
3 + 16					12 – 8				
15 + 4					18 – 9				
9 + 8					20 – 8				
4 + 7					17 – 13				
8 + 6					19 – 17				

COMMENTS

RAW SCORES: GROUP INTERVIEWS

BELOW AVERAGE ABILITY CHILDREN: ADDITION SOLUTION STRATEGIES

			ADDITION COMBINATIONS TO TEN									TOTALS			
Child	AGE	SEX	2+1	3+5	8+2	4+4	0+2	6+3	4+5	5+0	7+2	CA	CO	DF	KF
1		F	1	1	1	1	1	1	1	1	1	9	0	0	0
2		F	1	1	1	1	1	1	1	1	1	9	0	0	0
3	7	F	1	1	1	4	4	1	1	4	1	6	0	0	3
4		F	4	2	2	4	4	2	3	4	2	0	4	1	4
13		F	1	1	1	4	4	1	2	4	1	5	1	0	3
14		M	4	2	2	4	4	4	1	4	1	2	2	0	5
15	8	M	1	1	1	1	1	1	1	4	1	8	0	0	1
16		M	1	2	4	4	4	2	2	4	2	1	4	0	4
25		M	4	2	2	1	4	1	1	1	1	5	2	0	2
26		F	4	2	2	4	4	4	2	4	4	0	3	0	6
27	9	F	4	4	4	2	2	4	4	4	4	0	2	0	7
28		M	4	4	4	4	4	2	2	4	2	0	3	0	6
37		F	4	2	2	4	4	2	4	4	2	0	4	0	5
38		M	4	2	2	4	4	4	2	4	2	0	4	0	5
39	10	F	4	2	4	4	4	4	3	4	4	0	1	1	7
40		M	4	2	4	4	4	4	3	4	2	0	2	1	6
49		M	4	2	2	4	4	4	2	4	4	0	3	0	6
50		F	4	4	4	4	4	4	4	4	4	0	0	0	9
51	11	F	4	2	4	4	4	2	2	4	4	0	3	0	6
52		F	4	4	4	4	4	4	2	4	4	0	1	0	8
61		F	4	2	4	4	4	4	4	4	4	0	1	0	8
62	12	M	4	2	4	4	4	4	2	4	4	0	2	0	7
63		M	4	2	4	4	4	4	2	4	4	0	2	0	7

		ADDITION COMBINATIONS TO TWENTY										TOTALS			
Child	AGE	12+1	13+5	18+2	14+4	10+2	3+16	15+4	9+8	4+7	8+6	CA	CO	DF	KF
1		0	0	0	0	0	0	0	0	0	0	0	0	0	0
2		1	0	0	0	0	0	0	0	0	0	1	0	0	0
3	7	1	0	2	0	1	0	0	0	0	0	2	1	0	0
4		2	2	0	0	2	0	2	0	2	2	0	6	0	0
13		2	2	2	2	2	2	2	2	2	2	0	10	0	0
14		2	1	1	1	2	2	2	2	1	1	5	5	0	0
15	8	1	1	2	2	2	2	1	1	1	1	6	4	0	0
16		2	2	2	2	4	2	2	2	2	2	0	9	0	1
25		1	1	1	1	1	1	1	1	1	1	10	0	0	0
26		4	2	4	3	4	2	2	2	2	2	0	6	1	3
27	9	4	2	4	4	4	2	2	3	2	2	0	5	1	4
28		4	2	2	4	2	2	2	2	2	2	0	8	0	2
37		4	2	2	2	4	2	2	2	2	2	0	2	0	8
38		4	2	2	3	4	2	2	2	2	2	0	7	1	2
39	10	4	2	4	2	4	2	2	2	2	2	0	7	0	3
40		4	2	4	3	4	2	2	2	2	2	0	6	1	3
49		4	4	4	4	4	2	2	2	2	2	0	5	0	5
50		4	4	4	4	4	4	4	2	3	2	0	2	1	7
51	11	4	2	4	2	2	2	2	3	3	3	0	5	3	2
52		4	2	4	2	4	4	2	2	2	2	0	6	0	4
61		4	2	4	3	4	4	4	3	2	2	0	3	2	5
62	12	4	2	4	2	4	4	2	3	2	2	0	5	1	4
63		4	2	4	2	4	4	2	3	2	2	0	5	1	4

4 KNOWN FACT (KF)	2 COUNT-ON(CO)
3 DERIVED FACT(DF)	1 COUNT-ALL(CA)

RAW SCORES: GROUP INTERVIEWS

AVERAGE ABILITY CHILDREN :ADDITION SOLUTION STRATEGIES

			ADDITION COMBINATIONS TO TEN									TOTALS				ADDITION COMBINATIONS TO TWENTY										TOTALS			
Child	AGE	SEX	2+1	3+5	8+2	4+4	0+2	6+3	4+5	5+0	7+2	CA	CO	DF	KF	12+1	13+5	18+2	14+4	10+2	3+16	15+4	9+8	4+7	8+6	CA	CO	DF	KF
5		M	4	2	2	4	4	2	3	4	2	0	4	1	4	4	2	2	3	4	3	3	2	2	2	0	5	3	2
6		F	4	2	4	4	4	4	3	4	2	0	2	1	6	4	2	2	2	2	2	2	2	2	2	0	9	0	1
7	7	F	4	2	4	4	4	2	4	4	2	0	3	0	6	4	2	2	2	2	2	2	2	2	2	0	9	0	1
8		M	4	2	4	4	4	2	4	4	2	0	3	0	6	2	2	2	3	4	2	3	4	2	2	0	6	2	2
17		M	4	2	2	4	4	2	3	4	2	0	4	1	4	2	2	2	3	2	2	2	2	2	2	0	9	1	0
18		F	2	4	4	4	4	2	3	4	4	0	2	1	6	4	2	4	2	4	2	3	2	2	2	O	6	1	3
19	8	F	2	2	2	4	4	2	3	4	2	0	5	1	3	2	2	4	2	2	2	2	2	2	2	0	9	0	1
20		F	4	2	2	4	4	2	2	4	2	0	5	0	4	4	2	2	2	4	2	2	2	2	2	0	8	0	2
29		M	4	2	4	4	4	2	3	4	3	0	2	2	5	4	3	4	3	4	3	3	3	3	3	0	0	7	3
30		F	4	4	4	4	4	3	4	4	4	0	0	1	8	4	2	4	3	4	3	3	3	3	3	0	1	7	3
31	9	M	4	2	2	4	4	4	4	4	4	0	2	0	7	4	2	2	3	4	2	4	3	3	2	0	4	3	3
32		M	4	2	4	4	2	3	4	3	4	0	2	2	5	2	4	2	4	2	3	3	3	2	2	0	5	3	2
41		M	4	2	2	4	4	2	2	4	2	0	5	0	4	2	2	2	2	2	2	4	2	2	2	0	9	0	1
42		F	4	2	2	4	4	2	3	4	2	0	4	1	4	2	2	3	3	3	3	3	3	2	3	0	3	7	0
43	10	M	4	4	4	4	4	4	3	4	3	0	0	2	7	4	4	3	4	4	4	4	3	3	3	0	0	4	6
44		M	4	3	4	4	4	4	4	4	4	0	0	1	8	4	2	4	4	4	4	4	4	4	4	0	1	0	9
53		M	4	4	4	4	4	3	3	4	2	0	1	2	6	4	3	4	3	4	3	3	3	3	3	0	0	7	3
54		M	4	4	4	4	4	4	4	4	4	0	0	0	9	4	4	4	4	4	4	4	3	4	4	0	0	1	9
55	11	M	4	4	4	4	4	4	4	4	4	0	0	0	9	4	4	4	4	4	4	4	3	3	3	0	0	3	7
56		M	4	4	4	4	4	4	4	4	4	0	0	0	9	4	4	4	4	4	4	4	3	3	3	0	0	3	7
64		F	4	4	4	4	4	4	4	4	4	0	0	0	9	4	3	4	4	4	4	4	3	3	3	0	0	4	6
65	12	M	4	4	4	4	4	4	4	4	4	0	0	0	9	4	4	4	4	4	4	4	3	3	3	0	0	3	7
66		M	4	4	4	4	4	4	4	4	4	0	0	0	9	4	4	4	4	4	4	4	3	3	3	0	0	3	7

4 KNOWN FACT (KF)	2 COUNT-ON(CO)
3 DERIVED FACT(DF)	1 COUNT-ALL(CA)

RAW SCORES: GROUP INTERVIEWS

ABOVE AVERAGE ABILITY CHILDREN: ADDITION SOLUTION STRATEGIES

			ADDITION COMBINATIONS TO TEN									TOTALS				ADDITION COMBINATIONS TO TWENTY										TOTALS			
Child	AGE	SEX	2+1	3+5	8+2	4+4	0+2	6+3	4+5	5+0	7+2	CA	CO	DF	KF	12+1	13+5	18+2	14+4	10+2	3+16	15+4	9+8	4+7	8+6	CA	CO	DF	KF
9		M	2	2	2	4	4	4	3	4	2	0	4	1	4	2	2	2	3	2	3	2	3	2	2	0	7	3	0
10		M	4	4	4	4	4	3	3	4	2	0	1	2	6	4	2	4	3	4	2	3	3	2	2	0	4	3	3
11	7	F	4	4	4	4	4	2	2	4	4	0	2	0	7	4	2	4	2	2	2	2	2	2	2	0	8	0	2
12		M	4	2	4	4	4	2	2	4	2	0	4	0	5	4	2	4	2	4	2	2	2	2	2	0	7	0	3
21		M	4	3	3	3	4	3	3	4	3	0	0	6	3	3	3	3	3	3	3	3	3	3	3	0	0	10	0
22		M	4	3	4	4	4	3	4	4	3	0	0	3	6	3	3	4	4	4	4	3	3	3	2	0	1	5	4
23	8	M	4	4	4	4	4	4	3	4	4	0	0	1	8	4	3	3	4	4	3	3	3	3	2	0	1	6	3
24		F	4	4	2	4	4	4	4	4	2	0	2	0	7	4	2	2	3	4	3	3	2	4	2	0	4	3	3
33		M	4	4	4	4	4	4	4	4	4	0	0	0	9	4	3	4	3	4	3	3	3	3	3	0	0	7	3
34		F	4	2	4	4	4	2	2	4	2	0	4	0	5	2	2	2	4	4	2	3	2	2	2	0	7	1	2
35	9	F	4	2	4	4	4	4	4	4	4	0	1	0	8	4	2	4	4	4	4	4	3	2	3	0	2	2	6
36		F	4	4	4	4	4	3	4	4	4	0	0	1	8	4	4	4	4	4	3	3	2	4	4	0	1	2	7
45		F	4	4	4	4	4	4	4	4	4	0	0	0	9	4	2	4	4	4	4	4	2	2	2	0	4	0	6
46		M	4	4	4	4	4	4	3	4	4	0	0	1	8	4	4	4	4	4	4	4	3	3	3	0	0	3	7
47	10	F	4	4	4	4	4	4	4	4	4	0	0	0	9	4	4	4	4	4	4	4	4	4	4	0	0	0	10
48		F	4	2	4	4	4	4	4	4	4	0	1	0	8	4	4	4	4	4	4	4	3	2	4	0	1	1	8
57		F	4	2	4	4	4	4	4	4	4	0	1	0	8	4	3	4	4	3	3	3	3	2	3	0	1	6	3
58		M	4	4	4	4	4	4	4	4	4	0	0	0	9	4	4	4	4	4	4	4	4	4	4	0	0	0	10
59	11	F	4	4	4	4	4	4	4	4	4	0	0	0	9	4	4	4	4	4	4	4	4	4	4	0	0	0	10
60		M	4	4	4	4	4	4	4	4	4	0	0	0	9	4	4	4	4	4	4	4	4	4	4	0	0	0	10
67		M	4	4	4	4	4	4	4	4	4	0	0	0	9	4	4	4	4	4	4	4	3	3	3	0	0	3	7
68		F	4	4	4	4	4	4	4	4	4	0	0	0	9	4	4	4	4	4	4	4	3	3	3	0	0	3	7
69	12	F	4	4	4	4	4	4	4	4	4	0	0	0	9	4	4	4	4	4	4	4	3	3	3	0	0	3	7
70		M	4	4	4	4	4	4	4	4	4	0	0	0	9	4	4	4	4	4	4	4	3	3	3	0	0	3	7

4 KNOWN FACT (KF)	2 COUNT–ON(CO)
3 DERIVED FACT(DF)	1 COUNT–ALL(CA)

RAW SCORES: GROUP INTERVIEWS

BELOW AVERAGE ABILITY CHILDREN: SUBTRACTION SOLUTION STRATEGIES

			SUBTRACTION COMBINATIONS TO TEN									TOTALS			
Child	AGE	SEX	3–2	5–4	6–3	9–5	3–3	6–0	8–2	9–8	7–5	TA	CO	DF	KF
1		F	1	1	1	1	4	4	0	0	0	4	0	0	2
2		F	1	1	1	1	1	4	1	1	1	8	0	0	1
3	7	F	4	4	2*	3	4	4	2*	2*	4	0	3	1	5
4		F	1	1	0(1)	1	0(4)	4	4	1	1	5	0	0	2
13		F	1	1	1	1	1	1	1	1	1	9	0	0	0
14		M	1	4	1	1	1	4	1	2	2	5	2	0	2
15	8	M	2	0	0	1	0	4	1	1	1	3	1	0	1
16		M	1	1	1	1	4	4	1	1	1	7	0	0	2
25		M	1	1	1	1	4	1	1	1	1	8	0	0	1
26		F	4	4	3	2*	4	4	2*	4	2*	0	3	1	5
27	9	F	4	4	4	1	4	4	2	2	2	1	3	0	5
28		M	4	4	2	3	4	4	2	2	2	0	4	1	4
37		F	4	4	4	4	4	1	1	1	1	4	0	0	5
38		M	4	4	4	2	4	4	2	4	2	0	2	0	7
39	10	F	4	4	4	3	4	4	4	2	4	0	1	1	7
40		M	4	1	1	3	4	4	2	1	2	3	2	1	3
49		M	4	4	4	4	4	4	4	4	2	0	1	0	8
50		F	4	4	4	4	4	4	4	4	4	0	0	0	9
51	11	F	4	4	4	3	4	4	4	4	4	0	0	1	8
52		F	4	4	4	2	4	4	2	4	0(2)	0	2	0	6
61		F	4	4	4	4	4	4	2	4	2	0	2	0	7
62	12	M	4	4	4	2	2	4	4	4	2	0	3	0	6
63		M	4	4	4	4	2	4	4	4	2	0	2	0	7

	SUBTRACTION COMBINATIONS TO TWENTY										TOTALS			
Child	13–2	15–4	16–3	15–9	16–10	12–8	18–9	20–8	17–13	19–17	TA	CO	DF	KF
1											0	0	0	0
2											0	0	0	0
3											0	0	0	0
4											0	0	0	0
13	1	1	1	0(1)	0(1)	1	1	0(1)	0(1)	0(1)	5	0	0	0
14	1	0(I)	2	0(2)	0(2)	2	0(2)	0(2)	0(2)	0(2)	1	2	0	0
15	4	4	1	1	1	1	1	1	1	1	8	0	0	2
16	2	1	1	1	1	1	1	1	1	1	9	1	0	0
25	1	1	2	0	1	1	2	2	1	1	6	3	7	3
26	1	1	1	1	4	2*	2*	2*	2*	2*	4	5	0	1
27	4	2	2	2	3	2	2	2	4	2	0	7	1	1
28	4	0	4	3	3	3	3	2	2	2	0	3	4	2
37	2	2	2	2	2	2	2	0	2	2	0	9	0	0
38	4	2	2	3	4	3	4	3	0(2)	0(2)	0	2	3	3
39	4	4	4	2	4	2	3	2	2	2	0	5	1	4
40	2	2	2	2	4	1	1	1	2	2*	3	6	0	1
49	4	4	4	2	4	2	2	2	3	2	0	5	1	4
50	4	2	2	3	4	2	3	2	2	2	0	6	2	2
51	4	4	4	4	4	4	4	4	4	4	0	0	0	10
52	4	2	4	2	2	2	2	2	2	2	0	9	0	1
61	2	2	4	4	4	3	2	3	2	2	0	5	2	3
62	4	4	3	2	4	3	3	3	2	2	0	3	4	3
63	4	4	3	4	4	3	2	3	2	2*	0	3	3	4

4 KNOWN FACT(KF	2 COUNT–BACK (CB):2* COUNT–UP CU
3 DERIVED FACT(D	1 TAKE–AWAY (TA)
0 Error or no Strategy	(n) Strategy that led to error (where Identified)

RAW SCORES: GROUP INTERVIEWS

AVERAGE ABILITY CHILDREN: SUBTRACTION SOLUTION STRATEGIES

			SUBTRACTION COMBINATIONS TO TEN									TOTALS				SUBTRACTION COMBINATIONS TO TWENTY										TOTALS			
Child	AGE	SEX	3–2	5–4	6–3	9–5	3–3	6–0	8–2	9–8	7–5	TA	CO	DF	KF	13–2	15–4	16–3	15–9	16–10	12–8	18–9	20–8	17–13	19–17	TA	CO	DF	KF
5		M	4	4	4	3	4	4	2	2	2	0	3	1	5	4	2	0	2	4	2	2	4	2	2	0	6	0	3
6		F	2	4	3	3	4	4	2	3	2*	0	3	3	3	2	2	2	2	3	2	3	2	2*	2*	0	8	2	0
7	7	F	4	1	3	2	4	4	2	2	2	1	4	1	3	2	0(2)	0(2)	2	0(2)	2	2	2	0(2)	0(2)	0	5	0	0
8		M	4	4	4	2*	4	4	4	2*	2	0	0	3	6	2	2	3	2	4	2	2	2	3	3	0	6	3	1
17		M	1	1	1	2	1	1	2	2	2	5	4	0	0	2	2	3	2	4	2	0	3	2	2	0	6	2	1
18		F	4	4	4	2	4	4	4	4	2	0	2	0	7	4	2	2	2	4	2	3	2	2	2	0	7	1	2
19	8	F	1	2*	2*	1	1	4	2*	2*	2*	3	5	0	1	2*	2*	2*	2*	4	2*	2*	2*	2*	2*	0	9	0	1
20		F	1	4	1	1	4	4	1	1	1	6	0	0	3	1	1	1	1	1	1	1	1	1	1	10	0	0	0
29	9	M	4	3	3	3	4	4	4	3	3	0	0	5	4	3	3	3	3	4	4	3	3	2	2	0	2	6	2
30		F	2	4	3	3	4	4	2	4	2*	0	3	2	4	3	3	3	3	4	2*	3	3	3	3	0	1	8	1
31		M	4	4	4	4	4	4	2*	4	2*	0	2	0	7	4	2	4	3	4	4	3	3	2	2	0	4	3	3
32		M	4	4	4	3	4	4	2	4	3	0	1	2	6	4	3	2	2	2	2	2	2	2	2	0	8	1	1
41		M	4	4	3	3	4	4	2	2	3	0	2	3	4	2*	2*	3	4	4	2	3	2*	3	2	0	5	3	2
42		F	4	2*	2*	3	4	4	3	3	3	0	2	4	3	3	3	3	3	4	3	3	3	2*	2*	0	2	7	1
43	10	M	4	4	4	4	4	4	4	4	4	0	0	0	9	4	4	4	3	4	3	3	4	3	3	0	0	5	5
44		M	4	4	4	4	4	4	4	4	4	0	0	0	9	4	4	4	3	4	4	4	4	2*	2*	0	2	1	7
53		M	4	4	4	4	4	4	2	3	3	0	1	2	6	4	4	3	3	4	3	3	3	3	3	0	0	7	3
54		M	4	4	4	4	4	4	4	4	2	0	1	0	8	4	4	4	3	4	4	4	3	4	4	0	0	2	8
55	11	M	4	4	4	4	4	4	4	4	4	0	0	0	9	4	4	4	3	4	3	4	4	4	4	0	0	2	8
56		M	4	4	2	4	4	4	4	4	4	0	1	0	8	4	4	4	3	4	3	3	3	4	4	0	0	4	6
64		F	4	4	4	3	4	4	4	4	4	0	0	1	8	4	4	4	3	4	3	4	3	4	4	0	0	3	7
65	12	M	4	4	4	4	4	4	4	4	4	0	0	0	9	4	4	4	3	4	3	3	3	4	4	0	0	4	6
66		M	4	4	4	4	4	4	4	4	4	0	0	0	9	4	4	4	4	4	3	3	3	4	4	0	0	3	7

4 KNOWN FACT(KF	2 COUNT–BACK (CB):2* COUNT–UP CU
3 DERIVED FACT(D	1 TAKE–AWAY (TA)
0 Error or no Strategy	(n) Strategy that led to error (where Identified)

RAW SCORES: GROUP INTERVIEWS

ABOVE AVERAGE ABILITY CHILDREN: SUBTRACTION SOLUTION STRATEGIES

			SUBTRACTION COMBINATIONS TO TEN									TOTALS				SUBTRACTION COMBINATIONS TO TWENTY										TOTALS			
Child	AGE	SEX	3–2	5–4	6–3	9–5	3–3	6–0	8–2	9–8	7–5	TA	CO	DF	KF	13–2	15–4	16–3	15–9	16–10	12–8	18–9	20–8	17–13	19–17	TA	CO	DF	KF
9		M	3	3	3	2*	4	4	4	3	3	0	1	5	3	2*	2*	2	3	3	2	2*	2*	3	3	0	6	4	0
10		M	1	4	3	3	4	4	2	4	3	1	1	3	4	2	3	3	3	3	3	3	2	3	2	0	3	7	0
11	7	F	4	4	3	2*	4	4	3	4	4	0	1	2	6	2*	2*	2*	2*	2*	2*	2*	2*	2*	2*	0	10	0	0
12		M	1	1	4	1	4	4	1	1	1	6	0	0	3	2	2	2	2	2	2	2	2	2	2	0	10	0	0
21		M	4	3	3	3	4	4	3	3	3	0	0	6	3	3	3	3	3	4	3	3	3	3	3	0	0	9	1
22		M	3	3	4	4	4	4	3	3	4	0	0	4	5	3	3	3	3	3	3	3	3	3	3	0	0	10	0
23	8	M	4	4	4	4	4	2	4	2	4	0	2	0	7	2	3	2	3	4	3	3	3	2	3	0	3	6	1
24		F	4	4	4	3	4	4	3	4	3	0	0	3	6	4	4	3	4	4	3	3	3	4	2	0	1	4	5
33		M	3	4	3	3	4	4	3	4	3	0	0	5	4	3	3	3	3	4	4	4	4	4	4	0	0	4	6
34		F	3	4	4	2*	4	4	2*	2*	2*	0	4	1	4	3	3	2*	2*	4	2*	2*	3	2*	2*	0	6	3	1
35	9	F	4	4	4	2*	4	4	4	4	4	0	1	0	8	4	4	4	3	4	3	4	4	4	4	0	0	2	8
36		F	4	4	4	3	4	4	4	4	4	0	0	1	8	4	4	3	4	3	3	3	4	2	2	0	2	4	4
45		F	4	4	4	4	4	4	4	4	4	0	0	0	9	4	4	4	2*	4	3	3	3	3	3	0	1	5	4
46		M	4	4	4	4	4	4	4	4	4	0	0	0	9	4	4	4	4	4	4	4	4	4	4	0	0	0	10
47	10	F	4	4	4	4	4	4	4	4	4	0	0	0	9	4	4	4	4	4	4	4	4	4	4	0	0	0	10
48		F	4	4	4	4	4	4	4	4	4	0	0	0	9	4	3	3	2*	4	2	4	4	2*	3	0	3	3	4
57		F	4	4	4	4	4	4	4	4	4	0	0	0	9	4	4	4	4	4	4	4	4	4	4	0	0	0	10
58		M	4	4	3	4	4	4	4	4	4	0	0	1	8	3	3	3	3	4	2	3	3	4	4	0	1	6	3
59	11	F	4	4	4	4	4	4	4	4	4	0	0	0	9	4	4	4	4	4	4	4	4	4	4	0	0	0	10
60		M	4	4	4	4	4	4	4	4	4	0	0	0	9	4	4	4	4	4	4	4	4	4	4	0	0	0	10
67		M	4	4	4	4	4	4	4	4	4	0	0	0	9	4	4	4	3	4	3	4	4	3	3	0	0	4	6
68		F	4	4	4	4	4	4	4	4	4	0	0	0	9	4	4	4	3	4	3	4	4	2	2	0	2	2	6
69	12	F	4	4	4	4	4	4	4	4	4	0	0	0	9	4	4	4	4	4	3	3	4	4	4	0	0	2	8
70		M	4	4	4	4	4	4	4	4	4	0	0	0	9	4	4	4	3	4	3	3	4	4	4	0	0	3	7

4 KNOWN FACT(KF	2 COUNT–BACK (CB):2* COUNT–UP CU
3 DERIVED FACT(D	1 TAKE–AWAY (TA)

STRATEGY USED AND TIME TAKEN:ADDITION SoT 3/2a

	ADDITION LEVEL 2														Ma 3 2a
Child	4+4		1+5		3+7		4+3		6+2		5+4		1+6		
	1992	1993	1992	1993	1992	1993	992	1993	1992	1993	1992	1993	1992	993	Att. L
1	4	4	2	4	2	2	1	2	2	2	2	2	1	2	1
	5	*2*	*5*	*3*	*12*	*7*	*18*	*5*	*12*	*5*	*8*	*14*	*20*	*3*	
2	0	4	0	4	0	2	0	2	0	2	0	2	0	2	1
	(4)6	*3*	*(2)7*	*2*	*(0)25*	*4*	*(0)13*	*6*	*(2)4*	*3*	*(?)20*	*5*	*(2)7*	*2*	
3	2	4	2	4	2	2	2	2	2	2	0	2	0	2	1
	4	*2*	*4*	*1*	*7*	*4*	*5*	*6*	*5*	*3*	*(2)5*	*4*	*(2)5*	*10*	
4	4	4	2	4	0	2	0	2	2	2	0	2	0	1	1
	2	*1*	*2*	*1*	*(0)20*	*12*	*(0)7*	*4*	*3*	*1*	*6*	*10*	*3*	*1*	
5	4	2	2	2	2	2	2	2	2	2	0	2	0	2	1
	3	*4*	*7*	*4*	*10*	*6*	*15*	*7*	*11*	*10*	*(0)22*	*23*	*(2)5*	*13*	
6	4	4	2	4	2	2	2	2	4	4	2	3	2	4	2
	1	*1*	*2*	*1*	*3*	*3*	*3*	*3*	*3*	*1*	*3*	*4*	*2*	*1*	
7	4	4	4	4	4	2	2	2	4	4	4	4	4	2	2
	2	*3*	*2*	*1*	*2*	*5*	*4*	*4*	*2*	*2*	*2*	*2*	*2*	*2*	
8	4	4	2	4	0	2	2	4	2	4	2	2	2	1	2
	2	*1*	*4*	*1*	*7*	*4*	*4*	*3*	*4*	*3*	*5*	*4*	*7*	*5*	
9	4	4	4	4	4	4	4	4	4	4	4	4	4	4	2
	2	*1*	*2*	*1*	*4*	*3*	*2*	*2*	*2*	*2*	*2*	*1*	*2*	*1*	
10	2	2	0	1	2	2	2	2	2	2	2	2	1	1	2
	6	*4*	*4*	*5*	*5*	*4*	*3*	*3*	*2*	*3*	*6*	*3*	*11*	*5*	
11	2	4	4	4	2	2	3	4	2	4	2	4	4	4	2
	2	*1*	*3*	*1*	*4*	*2*	*4*	*2*	*4*	*2*	*4*	*4*	*2*	*1*	
12	2	4	2	4	2	2	2	2	2	2	2	2	2	4	2
	2	*3*	*2*	*1*	*5*	*3*	*3*	*3*	*2*	*4*	*4*	*3*	*2*	*2*	
13	4	4	4	4	2	2	4	3	2	2	4	3	4	4	2
	2	*2*	*2*	*2*	*4*	*3*	*2*	*3*	*4*	*5*	*2*	*3*	*2*	*4*	
14	4	4	4	4	2	4	2	2	2	2	2	3	2	4	2
	2	*2*	*2*	*1*	*5*	*2*	*4*	*3*	*2*	*2*	*3*	*3*	*2*	*2*	
15	4	4	4	4	2	2	2	2	4	2	2	2	4	4	2
	2	*2*	*1*	*2*	*10*	*4*	*5*	*3*	*3*	*4*	*5*	*5*	*2*	*3*	
16	4	4	2	4	2	2	4	2	4	2	4	4	4	2	2
	1	*1*	*2*	*1*	*3*	*2*	*2*	*3*	*2*	*2*	*2*	*4*	*2*	*3*	
17	4	4	4	4	2	2	2	2	4	4	3	3	4	4	2
	2	*1*	*2*	*2*	*3*	*3*	*12*	*7*	*2*	*2*	*4*	*2*	*2*	*2*	
18	4	4	4	4	4	2	2	4	2	4	2	2	2	4	2
	2	*2*	*2*	*1*	*4*	*3*	*4*	*7*	*4*	*2*	*9*	*3*	*4*	*2*	
19	3	2	4	4	3	2	2	2	2	2	2	2	4	2	2
	2	*4*	*2*	*2*	*4*	*3*	*4*	*3*	*3*	*2*	*5*	*3*	*3*	*2*	
20	2	4	2	4	2	2	2	2	2	2	2	4	2	4	2
	8	*2*	*2*	*2*	*6*	*5*	*5*	*4*	*3*	*2*	*4*	*2*	*7*	*1*	
21	4	4	4	4	2	2	2	2	2	2	2	4	4	4	2
	4	*1*	*2*	*1*	*3*	*3*	*2*	*3*	*6*	*2*	*2*	*2*	*2*	*1*	
22	4	4	4	4	2	2	2	2	2	4	2	4	4	4	2
	2	*2*	*2*	*2*	*3*	*3*	*3*	*3*	*2*	*2*	*3*	*2*	*4*	*2*	
23	4	4	4	4	4	4	4	4	4	4	4	4	4	4	3
	1	*1*	*1*	*1*	*1*	*2*	*1*	*1*	*2*	*1*	*1*	*1*	*1*	*1*	
24	4	4	4	4	4	4	4	3	4	4	4	4	4	3	3
	2	*1*	*1*	*1*	*1*	*1*	*1*	*2*	*1*	*1*	*3*	*1*	*2*	*2*	
25	4	4	2	4	2	4	4	4	4	2	4	4	4	4	3
	2	*1*	*2*	*1*	*4*	*2*	*2*	*1*	*2*	*2*	*3*	*1*	*2*	*1*	
26	4	4	4	4	2	3	2	2	2	4	2	3	4	2	3
	2	*2*	*2*	*1*	*3*	*2*	*3*	*3*	*4*	*2*	*3*	*3*	*2*	*1*	
27	4	4	4	4	2	4	4	4	2	4	4	2	4	4	3
	2	*2*	*2*	*2*	*2*	*3*	*2*	*2*	*2*	*2*	*2*	*4*	*2*	*2*	
28	4	4	4	4	4	2	2	3	4	3	3	2	4	4	3
	2	*1*	*1*	*1*	*2*	*3*	*4*	*2*	*2*	*2*	*4*	*3*	*2*	*2*	

4 Known Fact
3 Derived Fact
2 Count-on
1 Count-all
0 Error InSAT
Time (seconds) given in italics
(1)7 Strategy and time at interview

STRATEGY USED AND TIME TAKEN: SUBTRACTION SoT 3/2a

Child	SUBTRACTION LEVEL 2														Ma 3 2a Att. L
	9–4		7–5		4–3		10–7		9–6		8–2		5–3		
	1992	1993	1992	1993	1992	1993	1992	1993	1992	1993	1992	1993	1992	1993	
1	1	0	0	0	0	0	0	2	0	0	0	2	0	2	1
	(1)8	*34*	*1)13*	*42*	*(1)5*	*(1)7*	*(1)26*	*26*	*1)16*	*59*	*1)11*	*12*	*1)10*	*8*	
2	0	1	0	1	0	1	0	1	0	1	0	2	0	0	1
	(?)11	*24*	*?)50*	*4*	*?)30*	*11*	*(?)15*	*21*	*?)90*	*30*	*?)20*	*5*	*?)45*	*30*	
3	1	1	1	1	1	1	1	1	0	1	1	0	1	1	1
	5	*5*	*4*	*5*	*3*	*10*	*4*	*5*	*4*	*4*	*3*	*30*	*2*	*3*	
4	0	2	0	0	1	4	1	2	0	2	2	4	0	4	1
	(1)10	*8*	*(1)9*	*12*	*6*	*3*	*4*	*9*	*1)10*	*8*	*10*	*3*	*(1)4*	*2*	
5	0	1	0	1	0	1	1	1	0	1	0	1	0	1	1
	(1)13	*9*	*1)16*	*4*	*(1)3*	*5*	*(1)5*	*7*	*(1)8*	*5*	*1)15*	*6*	*1)15*	*5*	
6	2	1	2	4	2	4	2	4	2	1	3	2	2	1	2
	5	*5*	*3*	*1*	*2*	*1*	*5*	*2*	*9*	*6*	*3*	*5*	*4*	*5*	
7	4	3	1	4	4	4	1	2	1	4	4	4	4	4	2
	2	*5*	*5*	*1*	*2*	*3*	*5*	*3*	*5*	*5*	*5*	*4*	*5*	*3*	
8	2	1	1	1	1	4	1	1	0	1	1	2	1	2	2
	3	*4*	*2*	*4*	*2*	*3*	*5*	*5*	*(1)7*	*8*	*2*	*4*	*3*	*3*	
9	3	4	4	4	4	4	2	2•	2	2•	4	2•	4	2•	2
	2	*4*	*2*	*2*	*4*	*1*	*2*	*3*	*22*	*3*	*3*	*4*	*2*	*3*	
10	1	1	1	1	1	1	1	1	1	1	1	1	1	1	2
	8	*3*	*7*	*5*	*5*	*2*	*7*	*12*	*5*	*5*	*4*	*3*	*3*	*3*	
11	2	2•	2	4	4	4	2	2•	2	2•	0	2	4	4	2
	4	*4*	*4*	*3*	*2*	*1*	*4*	*2*	*3*	*4*	*(2)5*	*5*	*3*	*2*	
12	4	2	4	2•	1	4	1	2	1	2•	1	2	1	1	2
	4	*9*	*2*	*8*	*5*	*2*	*5*	*3*	*9*	*7*	*3*	*7*	*4*	*3*	
13	1	2	0	3	4	2	4	4	2	4	3	4	3	4	2
	8	*5*	*1)10*	*5*	*1*	*9*	*2*	*5*	*9*	*4*	*3*	*3*	*2*	*1*	
14	1	3	2	2	1	4	2	2	1	3	1	2	1	2•	2
	3	*2*	*3*	*3*	*3*	*1*	*5*	*3*	*6*	*4*	*3*	*5*	*3*	*4*	
15	1	2	1	1	1	1	1	1	0	1	1	1	1	1	2
	5	*3*	*3*	*10*	*3*	*3*	*4*	*6*	*(1)4*	*5*	*4*	*9*	*3*	*3*	
16	1	1	1	1	1	1	1	1	1	2	1	2	1	2	2
	7	*9*	*7*	*5*	*4*	*5*	*9*	*5*	*7*	*20*	*4*	*11*	*3*	*3*	
17	2	2	2	4	2	4	2	4	0	2•	2	4	3	4	2
	4	*5*	*3*	*5*	*2*	*2*	*10*	*3*	*30*	*4*	*2*	*2*	*4*	*2*	
18	4	2	1	2	1	1	1	1	1	1	1	1	1	1	2
	5	*3*	*5*	*10*	*4*	*4*	*9*	*3*	*4*	*4*	*4*	*3*	*4*	*3*	
19	1	1	4	1	1	2	1	1	1	1	1	2	1	1	2
	4	*7*	*3*	*5*	*3*	*4*	*4*	*5*	*7*	*5*	*2*	*5*	*3*	*5*	
20	1	4	1	1	1	4	1	1	1	1	1	2	1	4	2
	2	*1*	*15*	*7*	*2*	*1*	*13*	*11*	*22*	*9*	*12*	*2*	*11*	*1*	
21	1	3	1	4	4	4	1	4	1	1	1	4	1	4	2
	8	*11*	*6*	*2*	*3*	*2*	*7*	*3*	*5*	*14*	*5*	*3*	*3*	*2*	
22	1	4	1	4	1	4	1	4	1	2•	1	2•	1	2•	2
	3	*3*	*3*	*2*	*3*	*2*	*4*	*2*	*3*	*4*	*3*	*2*	*3*	*2*	
23	4	3	4	4	4	4	4	4	4	4	4	4	4	4	3
	2	*3*	*1*	*1*	*1*	*1*	*1*	*1*	*3*	*1*	*2*	*1*	*3*	*1*	
24	4	4	4	4	4	4	4	4	4	4	4	4	4	4	3
	3	*1*	*5*	*1*	*3*	*1*	*2*	*1*	*4*	*1*	*2*	*1*	*2*	*1*	
25	3	4	3	4	3	4	3	3	2	2	3	4	3	4	3
	3	*3*	*2*	*3*	*3*	*1*	*3*	*2*	*6*	*5*	*3*	*2*	*4*	*2*	
26	2	3	2	2•	4	3	4	4	2	2•	2	2	2	3	3
	5	*2*	*3*	*5*	*4*	*1*	*5*	*2*	*3*	*3*	*3*	*3*	*2*	*4*	
27	.2	4	4	3	4	4	2	3	3	3	3	3	3	4	3
	7	*1*	*3*	*2*	*2*	*1*	*3*	*1*	*5*	*2*	*3*	*3*	*2*	*2*	
28	3	2	4	2	3	4	2•	2•	2•	3	4	3	2	4	3
	3	*3*	*1*	*3*	*1*	*2*	*3*	*3*	*8*	*3*	*2*	*3*	*2*	*1*	

KF Known Fact 2 Count-Back 1 Count-all 0 Error inSAT
DF Derived Fact 2• Count-up Time (seconds) given in italics
(1)7 Strategy and time at interview

STRATEGY USED AND TIME TAKEN:ADDITION SoT 3/3a

	ADDITION LEVEL 3											
Child	9+6		17+0		7+7		4+11		8+10		15+2	
	1992	1993	1992	1993	1992	1993	1992	1993	1992	1993	1992	1993
1												
2												
3		2		4		2		2		2		0
		17		*1*		*29*		*60*		*9*		*0*
4		2		4		4		2		2		4
		10		*2*		*3*		*12*		*10*		*2*
5		2		4		2		2		4		2
		16		*2*		*14*		*15*		*8*		*2*
6	0	2	4	4	4	4	2	2	2	2	2	2
	8	*10*	*2*	*2*	*2*	*2*	*5*	*4*	*6*	*20*	*2*	*6*
7	0	3	4	4	4	4	0	2	4	4	2	2
	counters	*5*	*2*	*1*	*2*	*1*	counters	*13*	*2*	*2*	*4*	*4*
8	2	2	4	4	4	4	0	2	0	2	2	2
	9	*5*	*2*	*1*	*2*	*3*	*(2)14*	*4*	*2(14)*	*4*	*10*	*2*
9	2	2	4	4	4	4	2	2	0	2	2	4
	5	*9*	*2*	*1*	*2*	*2*	*10*	*4*	*11*	*6*	*2*	*2*
10	0	2	4	4	2	2	0	2	0	1	0	2
	0	*5*	*2*	*4*	*2*	*4*	*(1)15*	*13*	*(1)22*	*32*	*(2)18*	*9*
11	2	2	4	4	4	4	2	2	2	4	2	4
	6	*4*	*2*	*1*	*4*	*1*	*5*	*3*	*4*	*1*	*2*	*2*
12	2	2	4	4	2	4	2	2	2	4	2	4
	4	*4*	*2*	*2*	*5*	*1*	*5*	*2*	*5*	*2*	*2*	*2*
13	2	2	4	4	4	4	2	2	2	4	2	4
	5	*5*	*2*	*2*	*2*	*2*	*4*	*15*	*5*	*2*	*3*	*2*
14	0	2	4	4	0	4	2	3	2	4	2	4
	(2)8	*5*	*2*	*2*	*(4)2*	*1*	*7*	*4*	*13*	*3*	*3*	*2*
15	0	2	4	4	4	2	0	2	0	2	2	2
	(1)22	*6*	*3*	*1*	*8*	*6*	*(1)38*	*5*	*0*	*6*	*5*	*2*
16	2	2	4	4	2	4	2	2	2	2	2	2
	3	*12*	*2*	*1*	*4*	*2*	*6*	*3*	*8*	*9*	*2*	*3*
17	2	2	4	4	4	4	2	2	0	2	4	2
	3	*5*	*2*	*1*	*2*	*2*	*8*	*3*	*12*	*7*	*2*	*2*
18	0	2	4	4	0	4	0	2	0	2	2	2
	(1)21	*7*	*3*	*1*	*(1)15*	*2*	*2(23)*	*4*	*(2)19*	*9*	*5*	*2*
19	2	2	4	4	4	4	4	2	4	2	2	4
	6	*7*	*1*	*2*	*1*	*2*	*6*	*4*	*2*	*6*	*3*	*1*
20	2	2	4	4	2	4	2	2	0	4	2	4
	8	*10*	*2*	*2*	*5*	*2*	*23*	*6*	*(4)3*	*1*	*6*	*1*
21	2	2	4	4	2	4	2	2	2	3	2	2
	5	*7*	*2*	*1*	*10*	*2*	*13*	*7*	*11*	*9*	*3*	*2*
22	0	2	4	4	4	4	2	2	2	4	2	4
	(2)5	*8*	*2*	*2*	*2*	*2*	*4*	*5*	*5*	*3*	*3*	*3*
23	4	4	4	4	4	4	3	4	4	4	2	4
	2	*1*	*1*	*1*	*2*	*1*	*2*	*1*	*2*	*1*	*3*	*1*
24	3	3	4	4	4	4	4	4	4	4	4	2
	5	*2*	*1*	*1*	*1*	*1*	*1*	*1*	*2*	*1*	*2*	*2*
25	3	3	4	4	4	4	2	3	3	4	3	3
	3	*2*	*1*	*1*	*3*	*1*	*4*	*3*	*2*	*1*	*2*	*2*
26	2	2	4	4	4	4	2	2	4	4	2	2
	5	*5*	*2*	*2*	*3*	*2*	*4*	*5*	*2*	*1*	*3*	*2*
27	4	3	4	4	4	4	4	4	4	4	4	3
	3	*3*	*2*	*1*	*2*	*1*	*4*	*4*	*2*	*2*	*3*	*4*
28	2	2	4	4	4	4	2	3	4	4	2	2
	5	*3*	*2*	*2*	*2*	*2*	*3*	*3*	*3*	*2*	*2*	*2*

4 Known Fact
3 Derived Fact
2 Count-on
1 Count-all
0 Error inSAT
Time (seconds) given in italics
(1)7 Strategy and time at interview

STRATEGY USED AND TIME TAKEN: SUBTRACTION SoT 3/3a

Child	SUBTRACTION LEVEL 3/3a												Ma 3 3a Att. L
	17–6		19–13		20–5		11–9		18–10		15–0		
	1992	1993	1992	1993	1992	1993	1992	1993	1992	1993	1992	1993	
1													1
2													1
3													1
4		2		0		2		2		2		4	1
		12		*45*		*7*		*9*		*13*		*3*	
5		1		0		0							1
		42		*(1)40*		*(1)24*							
6	0	2•	0	0	0	2•	2	2•	0	2•	4	4	2
	(2)11	*60*	*(2)29*	*(2)29*	*(2)21*	*10*	*8*	*4*	*(2)23*	*33*	*3*	*2*	
7	?	3	0	2•	3	4	1	2	0	4	4	4	2
	13	*2*	*0*	*29*	*3*	*4*	*5*	*10*	*(3)3*	*4*	*2*	*2*	
8	1	2	0	2	0	2	0	2	0	2	4	4	2
	20	*14*	*[1]*	*18*	*(1)20*	*5*	*[1]*	*6*	*(1)23*	*10*	*5*	*1*	
9	0	0	0	2•	0	2•	2	4	0	4	4	4	2
	(2)18	*(2)11*	*(2)25*	*12*	*(2)12*	*12*	*2*	*1*	*(2)15*	*1*	*2*	*1*	
10	0	0	0	0	0	0	0	0	0	0	0	1	2
	[1]	*[1]*	*[1]*	*[1]*	*[1]*	*[1]*	*[1]*	*[1]*	*[1]*	*[1]*	*[1]*	*3*	
11	0	2	0	2•	3	3	0	2•	0	4	4	4	2
	[2]	*21*	*[2]*	*7*	*13*	*3*	*(2to)6*	*4*	*(2•)9*	*1*	*2*	*1*	
12	2	0	0	0	0	4	0	0	0	4	4	4	2
	15	*[1]*	*[2]*	*[1]*	*(2)15*	*4*	*(2)15*	*[1]*	*(2)39*	*1*	*2*	*1*	
13	2	2	0	2	0	4	0	2	0	2	4	4	2
	12	*21*	*[2]*	*10*	*(2)6*	*3*	*(2)26*	*12*	*[2]*	*3*	*2*	*2*	
14	0	3	0	2•	2	4	2	2•	2	4	4	4	2
	(2)62	*9*	*(2)38*	*20*	*13*	*2*	*(2)47*	*4*	*(2)36*	*4*	*2*	*1*	
15	0	1	0	1	0	4	0	1	0	4	4	4	2
	0	*63*	*0*	*10*	*(1)0*	*2*	*(1)0*	*9*	*(1)0*	*2*	*2*	*2*	
16	0	2	0	0	2	4	2	2	2	0	4	4	2
	[2]	*24*	*[2]*	*[2]*	*[2]*	*4*	*[2]*	*43*	*[2]*	*[2]*	*2*	*2*	
17	2	2	0	2•	3	3	2	4	0	4	4	4	2
	58	*6*	*(2•)14*	*5*	*3*	*3*	*7*	*3*	*(2•)17*	*2*	*2*	*2*	
18	0	0	0	0	0	0	0	0	0	1	0	4	2
	counters	*[1]*	*counters*	*[1]*	*(1)15*	*[2]*	*(1)15*	*[1]*	*[1]*	*40*	*(4)3*	*2*	
19	0	3	0	1	2	3	1	2	0	4	4	4	2
	0	*7*	*0*	*[1]*	*(2)13*	*8*	*(1)14*	*38*	*0*	*3*	*3*	*2*	
20	1	2	0	1	0	2•	1	1	1	4	4	4	2
	9	*15*	*[1]*	*12*	*(1)36*	*10*	*7*	*7*	*12*	*4*	*7*	*2*	
21	1	0	0	2	0	3	0	4	1	4	4	4	2
	4	*[2]*	*[2]*	*13*	*(3)32*	*2*	*(1)3*	*2*	*(1)23*	*2*	*8*	*2*	
22	2	2	0	2•	0	2	0	2•	0	4	4	4	2
	10	*15*	*(2to)8*	*5*	*(2) 9*	*7*	*2(7)*	*4*	*(2) 11*	*2*		*2*	
23	3	3	3	4	3	4	2to	4	3	4	4	4	3
	3	*4*	*5*	*3*	*2*	*2*	*2*	*2*	*2*	*3*	*2*	*2*	
24	4	4	0	3	4	4	2	4	4	4	4	4	3
	2	*2*	*(2•)10*	*3*	*2*	*2*	*4*	*2*	*2*	*2*	*1*	*1*	
25	3	3	0	0	3	3	3	3	4	4	4	4	3
	3	*2*	*[3]*	*[2]*	*4*	*2*	*10*	*3*	*2*	*3*	*2*	*1*	
26	3	3	0	2•	3	3	3	2•	3	4	4	4	3
	2	*4*	*0*	*5*	*3*	*3*	*4*	*4*	*3*	*2*	*2*	*1*	
27	3	3	0	0	3	3	3	3	3	3	4	4	3
	2	*2*	*0*	*3*	*12*	*2*	*2*	*5*	*2*	*2*	*2*	*2*	
28	3	4	3	3	2to	3	4	4	3	4	4	4	3
	2	*2*	*5*	*3*	*12*	*3*	*2*	*2*	*2*	*2*	*2*	*2*	

[] Denotes method used but incorrect solution
(2)10 Denotes method used during interview with time taken

KF	Known Fact	2	Count-Back	0 Error inSAT
DF	Derived Fact	2•	Count-up	Time (seconds) given in italics
		1	Count-all	*(1)7* Strategy and time at interview