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Social Network Theory, Broadband and the Future of the World Wide Web

Abstract

This paper aims to predict some possible futures for the World Wide Web based on several key network parameters: size, complexity, cost and increasing connection speed through the uptake of broadband technology. This is done through the production of a taxonomy specifically evaluating the stability properties of the fully-connected star and complete networks, based on the Jackson and Wolinsky (1996) connections model, modified to incorporate complexity concerns. The taxonomy demonstrates that when connection speeds are low neither the star nor complete networks are stable, and when connection speeds are high the star network is usually stable, while the complete network is never stable. For intermediate speed levels much depends upon the other parameters. Under plausible assumptions about the future, the taxonomy suggests that the Web may be increasingly dominated by a single intermediate site, perhaps best described as a search engine.

1   Introduction

Recently in economics, a literature has emerged looking at how networks develop in society. A good survey can be found in Jackson (2003) focusing on social networks, and Bloch (2003) which provides a useful list of applications to industrial organization. The potential range of application is considerable, from social networks, information networks in firms, through to physical networks, such as the Internet or broadband networks, though the Internet has so far received surprisingly sparse attention. Possible exceptions exist, for example an analysis of peering and transit in the Internet by Badasyan and Chakrabarti (2003), but still not as much as the size and importance of the Internet might warrant.[1] A key feature of the current literature on social network formation is that networks are not necessarily planned centrally, or even prone to much control, they often simply emerge and evolve, which is a characteristic of the Internet and the World Wide Web. A second related literature looks at what makes a network optimal, which represents a possible end-state to this process of evolution. Perhaps the key

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[1] That is not to say that there are not many excellent studies of the Internet or World Wide Web which use empirical methods, for example the study of the diameter of the Web by Albert et al (1999).

criterion linking these two literatures is the notion of network stability. Should an individual user be able to create a new link in a network and should this improve his or her utility then it seems reasonable to assume that it will be done, at least in the medium to long-run. Eventually however a state might be reached where no one would benefit from a change to the network structure and this seems a good candidate for the long-term resting point of the network formation process.

A dichotomy exists between the literature on social network theory which tends to deal with links between individuals or firms which are often cheap to initiate, and so the *de facto* cost may come in terms of congestion or complexity, and physical networks where links are often very expensive to build and maintain and hence have a high direct cost. Take for instance the decision to make a work colleague aware of your area of expertise, versus the decision to build a new road, gas pipeline, or railway line. The big cost in the first case might be the concern that once identified you might face a greater work-load, in terms of the second case there is a considerable cost even to building the link. The Internet provides an interesting hybrid. It is a genuinely physical network, but one where connecting to a given existing network is relatively cheap in terms of direct cost, with much of the analysis taking place in terms of associated costs like complexity or network externality effects on others which might seem more relevant concerns for the social network literature.

This paper seeks to use some of the existing tools of social network theory to analyze the World Wide Web and broadband access. A taxonomy of predictions for the future of the Web is developed in which the impact of high speed broadband access can be judged. To give an overview, first the Jackson and Wolinsky (1996) connections model, an archetypal social network model, is modified to incorporate differing connection speeds and complexity costs. Secondly, this new model is applied to the World Wide Web and a taxonomy of predictions for the future of the Web is formulated. Finally, a residual, but important, aim of the paper is to show how social network theory can assist in assessing the future impact of broadband access, and is by no means an exhaustive attempt to complete the task, rather it presents a number of starting points for research. Section 2 develops a connections model for the Web, defining notation, and examining the archetypal networks to be used throughout, and ends with a particularly important list of limitations and potential extensions to the underlying model used in this paper. Section 3 examines the key concept of stability, and sections 4 and 5 apply this concept to the star network and complete network in turn. Section 6 summarizes the main findings of the paper and section 7 concludes.

2   A Connections Model of the Web

A user accessing the World Wide Web can derive utility by accessing the web-sites of others, or more specifically the information contained on those sites. The Internet allows connection to distant resources, which raises the value of the Internet as more and more users join and allow further interconnection. For simplicity, consider web-sites to offer two basic services: either they provide information; or they provide access to another web-site which in turn may provide information, or further access. Then it might seem reasonable to model the Web as a series of connected nodes, each of which is of value for the information contained in each node, and the access to further nodes which is granted. In this way the Web is well modelled as a collection of nodes connected by edges on a graph: the classic *modus operandi* of the social network literature.

Crucially in the social network literature the cost of forming links is relatively low, and hence there is a clear distinction between road and rail networks for which the cost of establishing a link is so significant that notions of stability are very different. In particular it may not be possible for users to connect to each other; the network may be under the direct control of others who are not themselves nodes. The World Wide Web with its series of interconnected web-sites is much more like a social network in practice, perhaps typified by a network of familial or social connections, or a firm with a series of connections across different workplaces. On the Web, users can and do control the links on their own web-sites, and can and will alter those links if they wish to re-optimize, making notions of individual stability important at the level of individual nodes.

The model begins with some simple definitions of what is meant by nodes, edges, links, etc. applied to the Internet, and also what is meant by the Internet, World Wide Web, broadband, etc. Then there are two examples of how social network theory can be applied to Internet usage, and how the move from slow connection speeds to faster broadband links alter these decisions.

The paper directly applies a modified version of the model of Jackson and Wolinsky (1996) to the World Wide Web. The web is simply characterized as a complex graph in which nodes represent web-sites, and edges represent hyper-links between sites. One major issue for a Web user is how quickly the relevant information can be obtained. Here the focus is on the number of clicks required to reach a page containing the information needed, and the level of complexity a user must negotiate when searching for information on the Web. In each case these considerations are proxied using the standard tools of social network theory plus a few modifications.

Consider a user who wishes to find a certain piece of information and so decides to search the Web.

For simplification assume that each user begins his search on his own home-page, which may itself contain some information which might make it the end point of someone else's quest for knowledge. The user is assumed to know exactly where the information rests and indeed the shortest route to travel. The user then has several considerations: firstly the number of pages which have to be traversed; secondly, the level of complexity of each page which might slow the search, and finally the cost of maintaining any links which enable other web-pages to be accessed. To summarize the key parameters:

1. The distance between nodes is measured as the minimum number of clicks between a user and the information he seeks;

2. Complexity is measured in terms of the number of superfluous hyper-links on each site, which can be time-consuming to navigate;

3. Direct cost is a constant measure of the cost associated with keeping and maintaining direct links from the user's home-page to other sites;

4. Finally, connection speed between nodes is of considerable importance and acts together with the distance between nodes.[2]

Each of these measures will be introduced and explained. It will be shown that this model can be used to address questions about the network, such as how broadband technology might impact on the network, through a unilateral increase in the speed parameter. Note that throughout there is the assumption of the perfect reliability of links, so the notion of superfluous links does not factor in any benefits from redundancy.

Figure 1 gives an example of a simple representation through which a single user can gain a required piece of information. Any site with many links, especially the centre of the star like structure in Figure 1 might represent a search engine, which potentially links to all sites on the network. This is complex to navigate but ensures a link to virtually anywhere a user wishes to go.

[insert **FIGURE 1** here]

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[2] On the Internet if a user wishes to travel from one node to another it is quite possible that the information he wishes to access will actually travel via an alternative route. However, the focus here is on the actual experience of the user. To give an example, consider a user wishing to navigate from his home-page, node A, to node C, via node B. First, the user finds the link to node B on his home-page and clicks this link. Then the user waits for node B, the new webpage, to appear, and must navigate this site searching for the link to node C. Once this is found the user clicks the link and waits for node C to appear. So the user has traversed from A to C via B, and even if the connection speeds when clicking the links to B and C were such that connection was instantaneous, the user still had to search through the websites to find the links to B and C, and it is this searching time which provides one of the novel aspect of this paper. It is important to note that this paper is mainly concerned with these metaphorical travels by users through the Web, and not necessarily the flow of information from node to node.

2.1 The Basic Model

The model starts with the network model given in Jackson and Wolinsky (1996). Let $N = \{1, 2, ..., N^S\}$ be the finite set of nodes on the World Wide Web (or simply the Web) with the cardinality of this set of nodes (or network size) as N^S. Examples of such nodes include web-sites containing information of use, the home-pages of users themselves, ISP home-sites to which users subscribing to those services initially connect, and search engines, which are represented as nodes with a particularly large number of edges or *central stars* (representing their aim to allow connection to a wide variety of sites, for example on 10th May 2004, Google claimed to be able to search 4,285,199,774 web-pages). The next definitions consider the graph and edges.

Definition 1 *The complete graph, denoted g^N is the set of all subsets of N of size 2. The set of all possible graphs on N is then $\{g \mid g \subset g^N\}$.*

Definition 2 *Denote the subset of N containing nodes i and j as the (hyper)link or edge ij.*

So if $ij \in g$ then the nodes or sites i and j are directly connected while if $ij \notin g$ then the nodes or sites are not directly linked, though they may be connected indirectly (via other sites). Denote the addition or subtraction of a node or site from a network via the notation $g + ij = g \cup \{ij\}$ or $g - ij = g \backslash \{ij\}$ respectively. Let $N(g) = \{i \mid \exists j \text{ s.t. } ij \in g\}$. The next important concept is the path connecting nodes.

Definition 3 *A path in g connecting i_1 and i_n is a set of nodes $\{i_1, i_2, ..., i_n\} \subset N(g)$ s.t. $\{i_1i_2, i_2i_3, ..., i_{n-1}i_n\} \subset g$. The graph $g' \subset g$ is a component of g, if for all $i \in N(g')$, $i \neq j$, there exists a path in g' connecting i and j, and for any $i \in N(g')$ and $j \in N(g)$, $ij \in g$ implies that $ij \in g'$.*

Finally, the following definitions consider individual stability and individual defeat.[3]

Definition 4 *The graph g is individually stable if: (i) for all $ij \in g$, $u_i(g) \geq u_i(g - ij)$; and (ii) for all $ij \notin g$, $u_i(g) \geq u_i(g + ij)$.*

This is simply the requirement that any individual node i in the network does not strictly benefit from breaking a link or establishing a link with any other node $j \neq i$.

Definition 5 *The graph g is individually defeated by g' if $g' = g - ij$ and (i) is violated for ij, or if $g' = g + ij$ and (ii) is violated for ij.*

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[3] Jackson and Wolinksy specifically focus on pairwise stability, efficiency, value and allocation functions, but since the focus here is individual stability there is no need to define these terms.

Both individual stability and individual defeat are more relevant concepts for the network than the concept of pairwise stability as defined in Jackson and Wolinsky (1996), since in the application each individual i can make or break a link ij without the express permission of individual j. An alternative methodology might be to consider the concept of Nash networks which typically refer to networks that result when agents are free to delete multiple links rather than a single link, and can be used to analyze static and dynamic settings. For example, Bala and Goyal (2000) use Nash networks, and specifically assume that agents choose links in response to what happened in the previous period, and further assume that there is very little discounting of time by agents. For more discussion on the stability condition see section 2.4.2.

2.2 Costs and Benefits

Next, the utility function in the Jackson and Wolinsky model needs to be modified to take account of the complexity of web-browsing. Furthermore, since the model considers mainly directed links, so the network will look slightly different from each user's perspective, which partly explains the use of individual stability as indicated in later sections.

Consider the shortest distance t_{ij} between i and j to be the main measure of distance, and $\delta \in (0, 1)$ to be a connection speed variable, so as δ rises the connection speed rises, rendering distance less important, but as δ falls sites several links away become hard to reach and provide relatively low utility. This is represented by the function $\delta^{t_{ij}}$. In particular while the speed variable is restricted to lie in the unit interval under normal circumstances, a broadband high speed connection is represented in the model by simply allowing $\delta = 1$ and comparing the impact with lower levels of δ achievable under a dialup connection.[4] Note that if there is no path between i and j then $t_{ij} = \infty$ and $\delta^{t_{ij}} = 0$.

The notion of the cost of setting up links is a more complex procedure than simply setting a fixed cost per link of c_{ij} or c, as the main cost comes in terms of the ensuing complexity of sites with large numbers of links, which add text and time to negotiate the site. Even fast connection times will not alter the need to spend time and effort negotiating a complex site. On the other hand the direct cost to setting up a link is small, though positive. Therefore for simplicity the model incorporates a fixed cost of $c_{ij} = c \in (0, 1)$ for all links, but adds to this a complexity cost $(1 - \beta^{h_{ij}})$ where $\beta \in (0, 1)$ and h_{ij} is a direct measure of the number of irrelevant links on intermediate sites (including i) on the path

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[4] Note of course that $\delta = 0$ literally means zero connection speed, and $\delta = 1$ literally means instantaneous connection speed. However, by considering the limits values of δ at 0 and 1 the impact of slowing connection speeds right down, or moving towards instantaneous connections can be proxied, and that is the approach taken in this paper. This paper also considers intermediate speeds throughout, but the extremes are often useful in revealing the ultimate impact of increasing or decreasing speeds.

between i and j. For example, a simple line from i to j with no other links contains no complexity, so $h_{ij} = 0$, and $\beta^{h_{ij}} = 1$, therefore producing an overall complexity cost of 0. However, adding a link to node k at node i renders the node i more complex to negotiate, so the complexity cost rises to $1 - \beta$, and so on as links are added from intermediate nodes on the geodesic between i and j.[5] Figure 2 gives an example of the minimally complex route between i and k in network A, and a much more complex route between i and k in network B, even though the distance, $t_{ij} = 2$, remains the same across both networks. Note that the model does not include "doubling-back" in the measure of complexity since it seems reasonable that a user can recognize their own site, or where they have come from, so need not worry about doubling-back in error.[6] Hence for network A there is no complexity cost as $h_{ij} = 0$ and so $\left(1 - \beta^0\right) = 0$, but for network B there is positive complexity as $h_{ij} = 4$ and so $\left(1 - \beta^4\right) > 0$.

[insert **FIGURE 2** here]

Denote the total number of links from i as N_i^L and $N^L = \sum_{i \in N(g)} N_i^L$ as the total number of links in the network (which is a possible proxy measure of the overall complexity of the network), and the total number of nodes or sites in g as N^S (which is defined earlier as the cardinality or total size of the network).[7] Superfluous links can then be identified as the number of links along the shortest route from i to j minus the minimal distance t_{ij}, so

$$h_{ij} = \left(N_i^L - 1\right) + \left[\left(N_{k_1}^L - 2\right) + ... + \left(N_{k_{t_{ij}-1}}^L - 2\right)\right] = N_i^L + 1 - 2t_{ij} + \sum_{z=1}^{t_{ij}-1} N_{k_z}^L \qquad (1)$$

Where $k_1, ..., k_{t_{ij}-1}$ are the $t_{ij} - 1$ intermediate nodes which lie on the shortest path between i and j.[8] Let $\psi_i(g) \subset N$ be the set of nodes to which a path from i exists. The total utility to i from network

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[5] Note that if the perfect reliability of links is assumed there is no gain from the existence of irrelevant links. It is of course feasible that an alternative route, which would require superfluous links, would be worthwhile, but for simplicity this is ignored.

[6] Allowing users to return along their path of travel trivially seems appropriate given the existence of a "back" key on most browsers. This explains why the model does not consider the return path back to the last site as a superfluous link, and therefore excludes going back as part of the complexity cost, since the user can differentiate hitting the "back" button, rather than following a new link. It is also clearly not superfluous in any sense. For a search engine, a user might of course encounter the site he came from, but we can reasonably assume he would recognize this and instantly hit "back".

[7] The formulation of complexity costs given below is here concave in N, and so once the number of linked sites becomes large, the increase in complexity costs will be small.

[8] Note that where required the model assumes a lexicographic ordering whereby firstly a shortest distance is determined and secondly complexity is calculated, no user is assumed to opt for a longer route no matter how much lower the complexity cost. If there are multiple shortest distances then the least complex is used. Note that this assumption can be justified on two grounds. Firstly, users may have some idea of the distance, but not necessarily the complexity of the route. Secondly and far more significantly, note that under the two key architectures to be used this requirement is not important. For the star network there is effectively no alternative route, and when a new route is constructed through an additional link ij it provides both a shorter and less complex route from i to j. For a complete network since there are multiple routes from i to j when the direct link ij is removed, any one of the shortest remaining routes is always no more complex than any other.

g can therefore be given as:

$$u_i(g) = \sum_{j \neq i} \delta^{t_{ij}} - N_i^L c - \sum_{j \in \psi_i(g)} \left(1 - \beta^{h_{ij}}\right) \tag{2}$$

2.3 Specific Network Architectures

This section defines some standard network architectures which will form the bulk of the analysis.

Definition 6 *In a (fully-connected) star all sites connect to the centre star (cs) and no other node, so $N_i^L = 1$ for all $i \neq cs$, and $N_{cs}^L = N^S - 1$. The shortest distances between i and j is*

$$t_{ij} = \begin{cases} 2 & \text{for } i \neq j, cs; j \neq cs \\ 1 & \text{for } i = cs; j \neq cs \\ 1 & \text{for } i \neq cs; j = cs \end{cases} \tag{3}$$

and by equation 1 the number of superfluous links is

$$h_{ij} = \begin{cases} N^S - 3 & \text{for } i \neq j, cs; j \neq cs \\ N^S - 2 & \text{for } i = cs; j \neq cs \\ 0 & \text{for } i \neq cs; j = cs \end{cases} \tag{4}$$

So for a star, movement to *cs* is very short and not complex, movement via (or from) *cs* is also short but complex. Next consider a symmetric network.

Definition 7 *In a symmetric network every node (site) has the same number edges (links) of, $N_i^L \equiv \overline{N}$ for all i, and therefore by equation 1 the number of superfluous links between the shortest distance between i and j are simply given by*

$$h_{ij} = N_i^L + 1 - 2t_{ij} + \sum_{z=1}^{t_{ij}-1} N_{k_z}^L = \left(\overline{N} - 2\right) t_{ij} + 1 \tag{5}$$

One example of a symmetric network is a circle, where every node has two edges, another special type of symmetric network called the complete network is one in which every node is connected to every other node, so in particular $N_i^L = N^S - 1$.

Definition 8 *In a complete network every node (site) links to every other node (site). Therefore for all $i \neq j$ the shortest distance between i and j is $t_{ij} = 1$ and by equation 1 the number of superfluous links*

is

$$h_{ij} = N_i^L - 1 = N^S - 2$$

In a complete network all sites are very close, but direct costs are high.

2.4 Limitations and Extensions

Before going on to examine stability, it is important to examine the limitations of this paper, which are addressed in a series of inter-linked sections.

2.4.1 Network Architectures

This paper is very focused on predicting the shape of the Web, and in particular limiting itself to the two most well-known possible resting places for the evolution of any network: the complete network and the fully-connected star. It is of course feasible that other long-term solutions can and will exist. It is also clear that changes in technology may render a network unstable from an initial point of stability. Should technological change occur at a rate greater than the speed of movement towards a stable architecture, then stability may never occur, with changes constantly altering the end-point of the process, so it can never be reached.

2.4.2 Pairwise and Individual Stability

Simply removing complexity as an issue does not reduce this paper to a retelling of Jackson and Wolinsky (1996) in a different setting, as the stability conditions are different. This paper assumes that anyone can link to anyone else's site, or browse the information contained on that site. This may seem a reasonable approximation, but many sites can and do restrict access through passwords, or the blocking of access from certain IP addresses. If this is increasingly the case, then it makes sense to move to *pairwise stability* as in Jackson and Wolinsky (1996) rather than individual stability as defined in section 2. Alternatively there is the concept of Nash networks which typically refer to networks that result when agents are free to delete multiple links rather than a single link, and can be further refined to analyze static and dynamic settings. For example, Bala and Goyal (2000) use Nash networks, allowing agents with a very low rate of time-discounting to best respond to what occurred in the previous time period.

2.4.3 The Centre of the Star

In many ways the central node in a fully-connected star network is very different in this paper, than simply a node that happens to be at the centre of the network. Consider the argument that especially

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with a network of huge size why should the central node wish to link to every single web-site? Costs might become prohibitive, and traffic through the site might also reach prohibitive levels. The answer for the purposes of this paper is that the central node is implicitly considered to be different from all other nodes: it has a rationale for existing that is not based on sharing information with other nodes, rather it is itself a firm attempting to win business as the central node and it has a profit function that is increasing in the number of sites it links. The various candidate central nodes such as Google, Yahoo and others, provide ample justification for the quest to have a higher searching capacity than rivals. For this reason the model does not specifically model the stability properties of the central node when it forms the centre of a star network, rather it might be more sensible to treat its rationale as very distinct from other typical nodes.

Future work might include looking more closely at the rationale behind the actions of the central node in a star network, or "search engines", and model their actions directly. For example, a future model might explicitly examine a profit function that is positively related to the number of links, in an effort to capture advertising revenue, direct revenue through services which attract fees or even issues of content control.

Additionally, a future model might include an explanation for the appearance of such a site in a network, either through some evolution of a "standard node", or through the direct insertion of a search engine into the network. Historically there have been many such sites, though often only a small number have been dominant. In many respects the simplicity of this paper cannot justify the long-run existence of more than one such site.

2.4.4   Efficiency

The main issue for this paper concerns stability, as the focus is on making long-run predictions. However efficiency is traditionally at least as important a concern. One major feasible point (made in Jackson and Wolinsky) concerns the contradictions between these two concepts. In particular the most efficient network may not be stable, or the only stable network may not be efficient. Future work will and should examine the implications for the shape of the World Wide Web. Since the Web evolves, rather than being planned, stability may well be the more important predictive concept, but if a final prediction involves an inefficient structure there may be important policy implications.

2.4.5 Small Worlds

Recent related work (Goyal, 2004), provides an alternative framework from Jackson and Wolinsky (1996). Goyal starts with the notion of co-authors working together, with some highly regarded authors who link to others providing something like the central nodes of star networks. This provides some interesting results, which recall the small world argument, that no two people (or authors, or nodes) are likely to be too distant in equilibrium, because of the existence of highly regarded authors. Even lowly economists might co-author with someone who has co-authored with someone who has co-authored with a highly regarded economist. This might equally well be a good basis for a model of web-links, and in particular incorporates a form of congestion cost (based on the limit of what the central authors can hope to achieve) that might relate well to web-browsing.

2.4.6 Speed and Content

This paper makes an assumption that broadband is well-modelled in terms of a large speed increase. Add to this the assumption that broadband will become increasingly popular and an assumption could be made that speeds will rise over time. Taken to its logical extreme, the paper moves towards predicting what the Web will look like when connection times are instantaneous. However, throughout there is an assumption that content does not change, or at least if it does, not sufficiently quickly to counterbalance increases in speed. It may well be that this is incorrect. Greater use of video streaming, online gaming and other high bandwidth content, can undoubtedly limit the rises in connection speed produced by greater use of broadband. To attempt to second guess the advances of such content is beyond the scope of this paper, but it may provide an upper limit to effective speed increases.

2.4.7 Heterogenous Nodes and Users

In this paper it is assumed for simplicity that nodes are homogenous, however, in reality nodes on the Web are in fact heterogeneous. Hence, the value of information contained in a node differs between one node and another and with regard to one user relative to another. Users may not be sure where they wish to go, and may even resort to random browsing, and this is not considered in the model. Secondly, users themselves are treated homogeneously in this paper (with the marked exception of the discussion of the central node in a star network), however, again, in reality there is heterogeneity between users. In particular, there is a significant difference between users who provide content and users who browse the Web. Here the focus is on the latter.

3 Stability in an Example Network

The model examines network stability properties under the assumption that any node can break or make additional links, making the relevant notion of stability, individual stability. This paper argues that stability should be a very important consideration for the World Wide Web, even though the Web is necessarily changing and growing as time progresses. The primary rationale is that, especially in terms of the impact of faster connection speeds, stability might provide a reference point from which to begin predicting the long-term future shape of the Web.

In this section the method for determining the stability of an example network is examined before going on in the following sections to examine the stability properties of the star network and the complete network.

Figure 3 shows a network g_1. Define $g_2 = g_1 + ij$ (i.e. g_2 is the network g_1 in addition to the dotted link ij).

[insert FIGURE 3 here]

To determine the individual stability requirement for g_1 consider the addition of a single link at ij. The gain in terms of proximity to j for i would be given by the difference $\delta - \delta^2$, while there is an extra cost both direct, c, and indirect through added complexity. Denote the gain in terms of distance from the change from g_1 to g_2 as $\Delta D(g_1, g_2)$. In this case the gain in distance is simply

$$\Delta D(g_1, g_2) = \delta^{t_{ij}(g_2)} - \delta^{t_{ij}(g_1)} = \delta - \delta^2$$

Denote the change in direct cost and complexity as $\Delta C(g_1, g_2)$ which is

$$
\begin{aligned}
\Delta C(g_1, g_2) &= c + \left[\left(1 - \beta^{h_{ij}(g_2)}\right) + \left(1 - \beta^{h_{ik}(g_2)}\right) + \left(1 - \beta^{h_{il}(g_2)}\right) \right. \\
&\quad \left. + \left(1 - \beta^{h_{im}(g_2)}\right) \right] - \left[\left(1 - \beta^{h_{ij}(g_1)}\right) + \left(1 - \beta^{h_{ik}(g_1)}\right) + \left(1 - \beta^{h_{il}(g_1)}\right) + \left(1 - \beta^{h_{im}(g_1)}\right) \right] \\
&= c + 3\left(\beta^2 - \beta^3\right)
\end{aligned}
$$

The net stability condition requires that the change in cost outweighs the benefit in reduced distance, so yielding the condition

$$g_1 \text{ is stable relative to } g_2 \ \Leftrightarrow \Delta C(g_1, g_2) > \Delta D(g_1, g_2) \Leftrightarrow c + 3\left(\beta^2 - \beta^3\right) > \delta - \delta^2$$

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The first term c is just the direct cost, which added to the term $3(\beta^2 - \beta^3)$ gives the additional complexity for i of the new network. Put simply, when i wishes to travel to any node other than j, a slight increase in complexity is faced as a new superfluous link has been added. On the other hand, j is closer by $(\delta - \delta^2)$, and the route to j has remained as complex as before and so is not considered. Now note that if $\beta = 1$ and $\delta = 0$ so the connection speed becomes instantaneous and concern for complexity falls away, then g_1 is stable for any $c > 0$. Similarly for all other extreme combinations such as $\{\beta = 0, \delta = 0\}$, $\{\beta = 1, \delta = 1\}$ and $\{\beta = 0, \delta = 1\}$. For intermediate values of β and δ the network can become unstable for low values of c. Given that c (the direct cost of establishing a hyper-link on a web-site) is likely to be quite low, stability in the network is only ensured by extreme values of β and δ. Note that by switching from dialup to broadband, the approximation of $\delta = 1$ might be used, and will move the network some way towards stability.

To complete the discussion of stability from node i's perspective consider the prospect of *removing* links. Assume that node m has sufficient worth to i since this would remove m from the network altogether and so the model rules out removing link im by assumption.[9] That leaves removing links ik and il, which are assessed in turn. Consider $g_1 - ik$ to be the g_1 network minus the link ik. In this case the reduction in distance applies to both the shortest route between i and k and between i and j and equals

$$\Delta D\left(g_1 - ik, g_1\right) = \delta^{t_{ik}(g_1)} + \delta^{t_{ij}(g_1)} - \delta^{t_{ik}(g_1 - ik)} - \delta^{t_{ij}(g_1 - ik)} = \delta - \delta^3$$

Now denote the reduction in direct cost and complexity as $\Delta C\left(g_1 - ik, g_1\right)$ which is

$$
\begin{aligned}
\Delta C\left(g_1 - ik, g_1\right) &= c - \left[\left(1 - \beta^{h_{ij}(g_1 - ik)}\right) + \left(1 - \beta^{h_{ik}(g_1 - ik)}\right) + \left(1 - \beta^{h_{il}(g_1 - ik)}\right) - \left(1 - \beta^{h_{im}(g_1 - ik)}\right)\right] \\
&\quad + \left[\left(1 - \beta^{h_{ij}(g_1)}\right) + \left(1 - \beta^{h_{ik}(g_1)}\right) + \left(1 - \beta^{h_{il}(g_1)}\right) + \left(1 - \beta^{h_{im}(g_1)}\right)\right] \\
&= 4\beta - 3\beta^2 - \beta^3 + c
\end{aligned}
$$

So from the point of view of deleting link ik, the reduction in costs needs to be less significant than the reduced distance:

$$g_1 \text{ is stable relative to } g_1 - ik \iff 4\beta - 3\beta^2 - \beta^3 + c < \delta - \delta^3$$

Finally, consider $g_1 - il$ to be the g_1 network minus the link il. In this case the reduction in distance

[9] This amounts to assuming that $\delta > c + 2\beta - 2\beta^2 - \beta^3 + 1$ since the utility lost from removing link im is $\delta - c - \left(1 - \beta^2\right)$ while there is a gain in terms of the reduction in complexity in paths from i to nodes j, k and l of $2\beta - \beta^2 - \beta^3$.

applies only to both the shortest route between i and l and equals

$$\Delta D\left(g_1 - il, g_1\right) = \delta^{t_{il}(g_1-il)} - \delta^{t_{il}(g_1)} = \delta - \delta^2$$

Now denote the reduction in direct cost and complexity as $\Delta C\left(g_1 - il, g_1\right)$ which is

$$\begin{aligned}
\Delta C\left(g_1 - il, g_1\right) &= c - \left[\left(1 - \beta^{h_{ij}(g_1-il)}\right) + \left(1 - \beta^{h_{ik}(g_1-il)}\right) + \left(1 - \beta^{h_{il}(g_1-il)}\right) - \left(1 - \beta^{h_{im}(g_1-il)}\right)\right] \\
&\quad + \left[\left(1 - \beta^{h_{ij}(g_1)}\right) + \left(1 - \beta^{h_{ik}(g_1)}\right) + \left(1 - \beta^{h_{il}(g_1)}\right) + \left(1 - \beta^{h_{im}(g_1)}\right)\right] \\
&= 2\beta - \beta^2 - \beta^3 + c
\end{aligned}$$

So from the point of view of deleting link il, the reduction in costs needs to be less significant than the reduced distance:

$$g_1 \text{ is stable relative to } g_1 - il \Leftrightarrow 2\beta - \beta^2 - \beta^3 + c < \delta - \delta^2$$

So stability from i's perspective is only obtained if it makes no sense to add a link to any node to which i is already connected, and if it makes no sense to cut any links to any nodes to which i is already connected. In this network each node would have to go through the same process and in each case decide not to add or subtract any links before it is certain that the network g_1 is stable.

4   Stability in the Star Network

Figure 4 below shows a network g_3, and define $g_4 = g_3 + ij$. Here and unless otherwise stated, consider the star network to have only one component and to encompass everyone.

[insert **FIGURE 4** here]

This section considers whether the network is individually stable. First, the section considers the impact of adding a new link, and then follows with the impact of deleting a link. Should both of these actions result in a lower utility for the node considering the addition or subtraction of a link, then the network is individually stable.

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4.1 Adding a Link

Starting with the addition of a new link ij results in a shorter distance between the nodes i and j, so

$$\Delta D\left(g_3, g_4\right) = \delta^{t_{ij}(g_4)} - \delta^{t_{ij}(g_3)} = \delta - \delta^2$$

However, the addition of ij also adds a direct cost and a greater complexity

$$
\begin{aligned}
\Delta C\left(g_3, g_4\right) &= c + \left[\left(1 - \beta^{h_{ij}(g_4)}\right) + \left(1 - \beta^{h_{ik}(g_4)}\right) + \left(1 - \beta^{h_{il}(g_4)}\right) + \left(1 - \beta^{h_{im}(g_4)}\right)\right] \\
&\quad - \left[\left(1 - \beta^{h_{ij}(g_3)}\right) + \left(1 - \beta^{h_{ik}(g_3)}\right) + \left(1 - \beta^{h_{il}(g_3)}\right) + \left(1 - \beta^{h_{im}(g_3)}\right)\right] \\
&= c + 1 - 2\beta + 3\beta^2 - 2\beta^3
\end{aligned}
$$

The total impact of the change from g_3 to g_4 is therefore

$$\Delta D\left(g_3, g_4\right) - \Delta C\left(g_3, g_4\right) = \left(\delta - \delta^2\right) - c - \left(1 - 2\beta + 3\beta^2 - 2\beta^3\right)$$

The first term represents the shorter distance, the second the direct cost c, the third term the extra complexity cost of the new network. Before continuing, note the following useful lemma.

Lemma 1 $0.5p^{n-1} + 0.5p^{n+1} > p^n$ for all $n > 0$ and $p \in (0, 1)$.

Proof. Since $dp^n/dn = p^n \ln p < 0$ and $d^2p^n/dn^2 = p^n\left(\ln p\right)^2 > 0$ therefore $p^n - p^{n+1} < p^{n-1} - p^n$ which immediately implies the result. ∎

Now, since $\beta \in (0, 1)$, it is the case that $2\beta^2 > 2\beta^3$, and from lemma 1 it must be that $1 + \beta^2 > 2\beta$, and therefore it must be that $\left(1 - 2\beta + 3\beta^2 - 2\beta^3\right) > 0$. Hence g_4 is definitely more complex than g_3, as is expected with the addition of a new link at i. Now the condition for stability is

$$g_3 \text{ is stable} \Rightarrow c + 1 - 2\beta + 3\beta^2 - 2\beta^3 > \left(\delta - \delta^2\right) \tag{6}$$

Next, generalize expression 6 to give stability conditions for any star network. Any new link ij where neither i nor j are the centres of the star, necessitate a direct cost rise of c and a rise in complexity costs of $\left(1 - \beta^{N^S-2}\right) - \left(1 - \beta^{N^S-3}\right)$ for roots between i and $k \neq j, cs$. There are $\left(N^S - 3\right)$ such nodes. Complexity costs fall between i and j by $\left(1 - \beta^{N^S-3}\right) - (1 - \beta)$ and rise between i and the cs by $(1 - \beta)$,

leaving a final stability condition for a general star network of

$$
\begin{aligned}
\text{stable} \quad \Rightarrow \quad & c + \left(N^S - 3\right)\left[\left(1 - \beta^{N^S-2}\right) - \left(1 - \beta^{N^S-3}\right)\right] + (1 - \beta) \\
> \quad & \left(\delta - \delta^2\right) + \left[\left(1 - \beta^{N^S-3}\right) - (1 - \beta)\right] \\
\Rightarrow \quad & c + \left(N^S - 3\right)\left(1 - \beta^{N^S-2}\right) + 2(1 - \beta) > \left(\delta - \delta^2\right) + \left(N^S - 2\right)\left(1 - \beta^{N^S-3}\right) \\
\Rightarrow \quad & c + 1 - 2\beta + \left(N^S - 2\right)\beta^{N^S-3} - \left(N^S - 3\right)\beta^{N^S-2} > \left(\delta - \delta^2\right)
\end{aligned}
\tag{7}
$$

While this is only a necessary condition, and not sufficient for stability, the paper will consider some extreme values of the relevant parameters in brief now, and look in depth again when the necessary and sufficient condition is produced below. Note that this collapses to expression 6 when $N^S = 5$.

As complexity concerns grow very important, for instance when $\beta = 0$ then condition 7 becomes a requirement for $1 + c > \delta - \delta^2$, which must be true as $1 > \delta - \delta^2$, for $\delta \in (0, 1)$. As complexity becomes unimportant, so $\beta = 1$, then the required condition becomes $c > \delta - \delta^2$, which is recognizably the same condition as in Jackson and Wolinsky (1996) which makes sense as their model is similar, but with no complexity concerns. Note that when $\beta = 1$ and $\delta = 1$, i.e. instantaneous connection, stability is trivially satisfied for any c. As network size shrinks down to a small size, for example $N^S = 4$, then the stability condition becomes $1 + c > \delta - \delta^2 + \beta^2$ which is certainly satisfied for $\delta = 1$.

Usually $N^S = 4$ is considered to be the smallest non-trivial network in this paper, since $N^S = 3$ shifts from star to complete with the addition of a link, and $N^S = 2$ shifts to a the empty network with the subtraction of a link. However, just to check the example for $N^S = 3$, the condition becomes $2 + c > \delta - \delta^2 + 2\beta$ which is also trivially satisfied for $\delta = 1$. For $\beta < 1$ and for an arbitrarily large network (approximated by $N^S \to \infty$) the condition becomes $1 + c > \left(\delta - \delta^2\right) + 2\beta$ for intermediate δ, and $1 + c > 2\beta$ for extreme values of δ.

4.2 Subtracting a Link

The impact of subtracting a link is immediate for $i \neq cs$, since this will remove i from the network, so this section need only consider whether the network itself provides a positive utility.[10] In effect then $u_i(g) > 0$ is the second criterion for stability. In general this gives $u_i(g) = \sum_{j \neq i} \delta^{t_{ij}} - N_i^L c - \sum_{j \in \psi_i(g)}(1 - \beta^{h_{ij}})$.

[10] It might be prudent to also consider the prospect of the central node removing a link. However, it seems reasonable to make the claim that the central node for a network designed to model the World Wide Web might face very different incentives than a typical node. In particular it is later argued that the closest real-world fit might be something akin to a search engine or other core content provider. With this in mind the direct modelling of the motivation of the central node is beyond the scope of this paper, but a profit function might be envisaged that is positively related to the number of connections, perhaps through issues like advertising revenue, direct revenue through services which attract a fee or issues of content control. See section 2.4.3 for more on the special properties of the central node.

In the star network there is only a single link at $i \neq cs$, so $N_i^L = 1$. Furthermore, the distance to any node $j \neq cs$, is simply 2, while the distance to $j = cs$ is simply 1, so

$$\sum_{j \neq i} \delta^{t_{ij}} = \left(N^S - 2\right) \delta^2 + \delta$$

Finally, since there is only one link at $i \neq cs$ but $N^S - 1$ links at cs then $h_{ij} = N^S - 3$ for $i \neq cs$, and since it costs nothing in terms of complexity to reach cs, the total complexity costs are

$$\sum_{j \in \psi_i(g)} \left(1 - \beta^{h_{ij}}\right) = \left(N^S - 2\right)\left(1 - \beta^{N^S - 3}\right)$$

Totalling these yields a second stability condition

$$\text{stable} \Rightarrow u_i\left(g\right) > 0 \Rightarrow \left(N^S - 2\right)\delta^2 + \delta - c - \left(N^S - 2\right)\left(1 - \beta^{N^S - 3}\right) > 0 \tag{8}$$

Now once again consider some extreme cases, noting of course that this is a necessary but not sufficient condition for stability. Firstly, consider $\delta \in (0,1)$, $\beta \in (0,1)$ and $c \in (0,1)$. For an arbitrarily large network (approximated by $N^S \to \infty$) this gives $(1 - \beta^{N^S - 3}) \uparrow 1$ for all $\beta < 1$, so $u_i\left(g\right) \to -\infty\left(1 - \delta^2\right) + \delta - c < 0$. Put simply, as the network size grows the complexity of the central star becomes so great that journeys via the central star are simply too costly. However, since all journeys must go via the central star (aside from visiting the centre itself) the entire network becomes too costly for each node, and each node will have an incentive to drop out.

4.3   Overall Stability

In order to be stable a star network needs to meet both condition 7 and condition 8, and hence the overall stability condition incorporating both addition and subtraction of a link is

$$\begin{aligned} \text{stable} \quad \Leftrightarrow \quad & 0 < \min\{\left(N^S - 2\right)\left(\delta^2 + \beta^{N^S - 3} - 1\right) + \delta - c, \\ & 1 + c - 2\beta + \left(N^S - 3\right)\left(\beta^{N^S - 3} - \beta^{N^S - 2}\right) + \beta^{N^S - 3} - \left(\delta - \delta^2\right)\} \end{aligned} \tag{9}$$

Now it is possible to examine what this final necessary and sufficient condition looks like for various extreme parameter values.

Appendix 1 works through all the various possible combinations, and show how the stability condition responds to various parameter values, and here is presented a brief summary of the main results. If speeds

are low, $\delta = 0$, stability is never possible. If speeds are high, $\delta = 1$, then stability is usually assured. Stability *may* not be assured for high speeds if both complexity concerns are intermediate, $\beta \in (0, 1)$, and network size is sufficiently large. For intermediate speed levels, $\delta \in (0, 1)$, then much depends upon parameter values, though the network is less likely to be stable for large network sizes.[11]

5   Stability in the Complete Network

Figure 5 shows a network g_5 and also the slightly larger network $g_6 = g_1 + ij$. Once again the section considers whether the network is individually stable.

[insert **FIGURE 5** here]

5.1   Adding a Link

For the complete network the impact for i of adding a link must result in the duplication of an existing link, which adds an additional cost of c, and additional complexity, but with no gain, and is therefore clearly not a beneficial change for any parameter values except for $c = 0$, and $\beta = 1$, which would render the analysis trivial.

5.2   Subtracting a Link

More importantly, consider the impact on the utility to i of cutting the link ij. The gain here is in terms of a reduced direct cost, c, a change in overall complexity, and a fall in the shortest distance between i and j. Note that all other nodes have a direct link, so the complexity cost across the entire network would be relatively easy to calculate, since it only impacts on journeys starting from i and the journey from j to i. Focusing on i the shortest distance to j falls from δ to δ^2, and direct cost falls by c.

The complexity of the link ij changes from $1 - \beta^{N^S-2}$ to $1 - \beta^{2(N^S-3)}$. Initially, the route involves a direct movement from i to j. At i there are $N^S - 1$ links (the network is complete, but there is no link from i to i) and one of these links is essential, so there are $N^S - 2$ superfluous links. When the ij path is cut movement from i to j is forced to go via an intermediate site k. At i there are now $N^S - 3$ superfluous links since one further link has been removed. At k there are also $N^S - 3$ superfluous links, since of the $N^S - 1$ links at k, one is essential, and one would involve doubling-back. So there are $2N^S - 6$ superfluous links for the route ij in $g - ij$, as opposed to $N^S - 2$ in g.

---

[11] When dealing $N^S \to \infty$ limit results are needed and in some cases the use of L'Hopital's Rule. Appendix 1 goes into more detail and footnotes 12 and 13 below give some examples.

Finally, there is an impact on all other routes from i, which fall in complexity from $1 - \beta^{N^S-2}$ to $1 - \beta^{N^S-3}$ for all but the ij link (so for $N^S - 2$ other routes). Overall for a general complete network it makes sense to cut the ij link if:

$$c + \left(N^S - 2\right)\left(\beta^{N^S-3} - \beta^{N^S-2}\right) > \left(\delta - \delta^2\right) + \left(\beta^{N^S-2} - \beta^{2N^S-6}\right)$$

Which gives a stability condition

$$\text{stable} \Rightarrow \left(\delta - \delta^2\right) - c + \left(\beta^{N^S-2} - \beta^{2N^S-6}\right) - \left(N^S - 2\right)\left(\beta^{N^S-3} - \beta^{N^S-2}\right) > 0 \qquad (10)$$

5.3 Overall Stability

Since adding a link is not sensible, the conditions relating to removing a link are both necessary and sufficient to establish individual stability and hence condition 10 can be replaced with the necessary and sufficient condition:

$$\text{stable} \Leftrightarrow \left(\delta - \delta^2\right) - c + \left(\beta^{N^S-2} - \beta^{2N^S-6}\right) - \left(N^S - 2\right)\left(\beta^{N^S-3} - \beta^{N^S-2}\right) > 0 \qquad (11)$$

Of course, $\delta - \delta^2 > 0$ and $c > 0$. Clearly, since $\beta \in (0,1)$ it must be the case that $\beta^{N^S-3} > \beta^{N^S-2}$, for all $N^S > 2$. Finally, $\beta^{N^S-2} = \beta^{2N^S-6}$ for $N^S = 4$, and $\beta^{N^S-2} > \beta^{2N^S-6}$ for all $N^S > 4$. By considering restrictions on β, δ and N^S the set of parameter values for which expression 11 is met and the complete network is stable can be determined.

A full examination of how stability changes for different parameter values is found in Appendix 2. To provide a brief summary, when $\beta < 1$, the complete network is never stable for either very fast speeds, $\delta = 1$, or very slow speeds, $\delta = 0$, but may be stable for intermediate speeds depending upon the exact value of the other parameters. To see this consider $\beta < 1$ and apply L'Hopital's Rule to take limits when $N^S \to \infty$ leaving a stability condition of $\left(\delta - \delta^2\right) > c$.[12]

---

[12] Consider any function of the form $N^S \beta^{N^S}$ and take the limit as $N^S \to \infty$. First note that:

$$\lim_{N^S \to \infty} N^S \beta^{N^S} = \lim_{N^S \to \infty} \frac{N^S}{\beta^{-N}}$$

The applying L'Hopital's Rule yields

$$\lim_{N^S \to \infty} \frac{N^S}{\beta^{-N}} = \lim_{N^S \to \infty} \frac{1}{(-\ln\beta)\beta^{-N^S}} = 0$$

And hence in expression 11 taking $N^S \to \infty$ for $\beta < 1$, yields the condition $\left(\delta - \delta^2\right) > c$.

6 A Taxonomy of Stability Results

Here are summarized the stability results of the last two sections together with a taxonomy of predicted network architectures, including the implications for the move towards broadband access for all Internet users. The findings are also compared with those of the unmodified Jackson and Wolinsky model.

Of main concern in this paper is the long-run stability of certain archetypal networks: the star network and the complete network, which have received considerable attention in the literature as possible long-run stable architectures.

In each case these propositions have already been proved in the main text of sections 4 and 5 or in Appendices 1 and 2. For a Star Network:

Proposition 1 *For $\beta \leq 1$, a star network encompassing everyone is stable if and only if $\left(N^S - 2\right) \delta^2 + \delta - c - \left(N^S - 2\right) \left(1 - \beta^{N^S - 3}\right) \geq 0$ and $1 + c - 2\beta + (N^S - 2)\beta^{N^S - 3} - (N^S - 3)\beta^{N^S - 2} - \left(\delta - \delta^2\right) \geq 0$.*

The following propositions give some of the sharpest results for certain extreme parameter values of δ and network size for the star network, all proved in section 4.

Proposition 2 *For $\beta = 0$, a star network is never stable if $\delta = 0$, and always stable if $\delta = 1$, regardless of network size. If $\beta = 0$ and $\delta \in (0, 1)$ then the star network may be stable though this becomes less feasible as N^S rises.*

Proposition 3 *For $\beta = 1$, a star network is never stable if $\delta = 0$, and always stable if $\delta = 1$, regardless of network size. If $\beta = 1$ and $\delta \in (0, 1)$ then the star network may be stable depending upon parameter values.*

Proposition 4 *For $\beta \in (0, 1)$, a star network is never stable if $\delta = 0$. When $\delta = 1$, the star is: (i) stable for small networks; and (ii) for intermediate and large networks will be stable if $1 + c > 2\beta$. If $\beta \in (0, 1)$ and $\delta \in (0, 1)$ a star network will not be stable if $N^S \to \infty$, and may be stable for lower N^S depending upon parameter values.*

For a complete network the main result as proved in section 5, part C is:

Proposition 5 *For $\beta \leq 1$, a complete network is stable if and only if:*
$$\left(\delta - \delta^2\right) - c + \left(\beta^{N^S - 2} - \beta^{2N^S - 6}\right) - \left(N^S - 2\right) \left(\beta^{N^S - 3} - \beta^{N^S - 2}\right) > 0.$$

The following propositions give some results for certain extreme values of δ and network size for the complete network, all proved in section 5.

Proposition 6 *For $\beta = 0$ a complete network will be stable if and only if $c < \delta - \delta^2$, and hence is certainly not stable for $\delta = 0$ or $\delta = 1$.*

Proposition 7 *For $\beta = 1$ a complete network will be stable if and only if $c < \delta - \delta^2$, and hence is certainly not stable for $\delta = 0$ or $\delta = 1$.*

Proposition 8 *For $\beta \in (0,1)$ a complete network is not stable if $\delta = 0$ or $\delta = 1$, and for $\delta \in (0,1)$ may be stable depending upon parameter values.*

[**insert TABLE 1 here**]

Finally, summarizing these results produces a taxonomy as described in Table 1. The table lists both the stability results for different combinations of parameter values and also the location in the appendices which fully describes each case. Among the most interesting features present in Table 1 are the fact that in some cases only one of the two standard structures looks likely to survive into the long-run for a given set of parameter values. For example, when considering extreme complexity concerns, $\beta = 0$, while both networks are unstable for low speeds, the star network cannot be stable for intermediate speeds and $N^S \to \infty$, but only the star network is sure to be stable when $\delta = 1$. For intermediate complexity concerns neither network is stable for low speeds, $\delta = 0$, but at higher speeds much depends upon the network size. For example for small networks with intermediate complexity concerns and fast networks, the star network is stable, while the complete network is not whereas for large networks with intermediate speeds, the star network structure is not stable, while the complete network might be stable. When there are no complexity concerns the star network is stable for fast speeds, $\delta = 1$, while the complete network is not. For intermediate speeds it all rides on parameter values, while for low speeds neither is stable.

The best use for the table is if some idea of the likely parameter values can be estimated. For example, if the number of nodes in the system is likely to become extremely large (noting that there are already many billions of nodes, then $N^S \to \infty$ might provide a good approximation). Next one prediction might be that with the widespread adoption of broadband technology connection speeds will only get higher and so $\delta > 0$, but not so high that $\delta = 1$ is reasonable and so $\delta \in (0,1)$ seems reasonable. Finally, there might be some concern for complexity in the future, so $\beta \in (0,1)$. Glancing at the relevant element in the table reveals that the star network is not stable, though the complete network might be, and so something closer to the complete than star network, with large numbers of inter-linked nodes, might be a good prediction based on the predicted parameters. On the other hand predictions for the

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future might be very different, broadband technology might enable almost instantaneous connection speeds, and complexity concerns might fall as users become more and more used to the complexities of web-navigation, then the taxonomy would work best with $\delta = 1$ and $\beta = 1$, and so favor the star network over the complete network.

7 Conclusions and Policy Implications

Social network theory often using fairly restrictive assumptions can provide some insights into the stability properties of networks, and the Web is one such network that can be analyzed. The seminal paper by Jackson and Wolinsky (1996), with the addition of complexity costs, is a useful starting point for a discussion of the long-term shape of the World Wide Web, together with predictions about the future size of the Web, and the future speed of Internet connections. This paper considers a variety of possible predictions and modifies the model to incorporate complexity costs and thereby generates a taxonomy of predictions for the future of the structure of the Web.

To give some examples of how this taxonomy might be used, consider some of the results restated in more emotive language. Consider a network to be "large" where the number of nodes is finite, but so great that it can be approximated by considering the limit as $N^S \rightarrow \infty$, a network where $\beta < 1$ involves concern for complexity, but not if $\beta = 1$, and a network where $\delta = 1$ as having a fast connection (say, broadband), and otherwise slow (say, dialup). Now, the results show that as networks grow large in size and connection speeds increase (say everyone has access to broadband, and broadband speeds continue to rise) social network theory under the assumptions detailed in this paper would predict that with concern for complexity a star is a more likely end result. With very low connection speeds neither the complete network nor the star seem plausible stable structures.

What this means for telecommunications policy in practice, requires an examination of what is meant by stars and complete networks. A star is given here as a model of the Web based around many users connecting to each other's web-sites via a central clearing house, well represented by a single ISP or an all-encompassing search engine, with no need for any other hyper-linking between pages, or even use of favorites or book-marking. A complete network is very different, a much more decentralized structure with many large sites containing hyper-links to all pages of interest. Users might travel via hyper-links on their own pages or choices from their favorites menus directly to where they want to go, with minimal use of search engines, or centralized sites. The implications of whether either of these archetypes become dominant are considerable given the role which major online presences such as ISPs, search engines and

all other forms of centralized structures have to play. The more centralized and star-like the structure, the more power rests in the hands of fewer sites, while the more complete a network the less controlling influence such sites can wield over the majority of users.

Perhaps the single most powerful prediction is that *as speeds increase with the greater adoption of broadband technology, the star network will be the more likely architecture to survive into the long-run, and so there might well be an increasingly centralized structure.* Whether this is seen as a problem from a policy perspective, depends upon a subjective set of priorities, such as whether policy-makers are concerned about whether it is important that a small number of sites (potentially a single site) hold positions of vital importance.

At a basic level, what has been provided is a clear tool for predicting whether and when potential monopoly problems might be expected, or whether the structure of Web can resolve such problems naturally. At a first pass this paper argues that, if a monopolistic structure is not stable given reasonable predictions about the likely size, speed and complexity of the Web in the future, then policy-makers do not need to be as concerned as when a monopolistic star-like structure is indeed stable. The problem for policy-makers then becomes a simpler one: rather than trying to predict the structure of the World Wide Web, they need only focus on parameters like size and speed, which are much less complex to predict.

Though in large part well beyond the scope of this paper, it is important to examine the relationship a monopoly search engine might have with policy-makers. It is normal in economics to be concerned about the scope for monopoly power between the provider of a resource and its users which would typically involve issues about pricing, restriction of supply, quality or choice. For example, the most famous recent case in the computing world, is perhaps the United States vs Microsoft (1998), in which the Department of Justice alleged that Microsoft had abused monopoly power in its handling of the Windows operating system and the web browser Internet Explorer. However, care should be taken since there might also be uses from a policy perspective of having a small number of pivotal internet sites with which to interact, since negotiating with every user on the Web would be all but impossible.

Finally, and especially important when considering policy implications, it must be stressed that the model used in this paper has several limitations which need to be understood before accepting the final results. The final part of section 3 provides a list of such issues, together with some possible extensions to the model.

REFERENCES

Albert, R., Jeong, H., and Barabasi, A.-L. (1999) 'Diameter of the World-Wide Web', *Nature* 401, pp. 130–131.

Badasyan, N. and Chakrabarti, S. (2003). 'Private Peering among Internet Backbone Providers', Mimeo: Virginia Tech.

Bala, V. and Goyal, S. (2000). 'Self-organization in Communication Networks', Econometrica 68, pp. 1181-1230.

Bloch, F. (2003). 'Group and Network Formation in Industrial Organisation: A Survey', Mimeo: GREQAM.

Goyal, S. and van der Leij, M. (2004). 'Economics: An emerging small world', Mimeo: University of Essex.

Jackson, M. O. (2003). 'A Survey of Models of Network Formation: Stability and Efficiency', forthcoming in Group Formation in Economics: Networks, Clubs, and Coalitions, edited by G. Demange and M. Wooders, Cambridge University Press: Cambridge.

Jackson, M. O. and Wolinsky, A. (1996). 'A Strategic Model of Social and Economic Networks', *Journal of Economic Theory* 71, pp. 44-74.

Here are presented the full impact of different assumed parameter values on the stability of the star network. Note that this appendix considers general network sizes (finite values of N^S above 4) but for contrast will also consider a "small" network to be approximated by $N^S = 4$, and a "large" network to have an arbitrarily large though finite number of nodes, which for simplicity will be approximated by $N^S \to \infty$ to aid intuition.[13]

Part A: No Concern for Complexity

Start with the model with no complexity concerns, so $\beta = 1$. Now condition 9 becomes

$$\text{stable }|_{\beta=1} \Leftrightarrow 0 < \min\left\{\left(N^S - 2\right)\delta^2 + \delta - c, c - \delta + \delta^2\right\} \tag{12}$$

Next consider condition 12 for some variations of the other parameter values.

(i) $\delta = 0$. When connection speeds are very slow, condition 12 becomes

$$\text{stable }|_{\beta=1,\delta=0} \Leftrightarrow 0 < \min\left\{-c, c\right\}$$

Which cannot hold, so the network is definitely not stable. This is immediate given the nature of the star network: with speeds at zero, a user will wish to build a direct link to j since it is simply to slow to travel via the central node.

(ii) $\delta \in (0,1)$. Next consider an intermediate level of speed, firstly, combined with a small network, which produces the stability condition

$$\text{stable }|_{\beta=1,\delta\in(0,1),N^S=4} \Leftrightarrow 0 < \min\left\{2\delta^2 + \delta - c, c - \delta + \delta^2\right\}$$

Note that the condition requires $c < \delta + 2\delta^2$ and $c > \delta - \delta^2$, so stability is attained only if $c \in \left(\delta - \delta^2, \delta + 2\delta^2\right)$. This is feasible but depends upon parameter values. Considering intermediate speed levels and an intermediate network size yields condition 12 but ruling out extremes

$$\text{stable }|_{\beta=1,\delta\in(0,1)} \Leftrightarrow 0 < \min\left\{\left(N^S - 2\right)\delta^2 + \delta - c, c - \left(\delta - \delta^2\right)\right\} \tag{13}$$

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[13] Note that for the $N^S \to \infty$ case limit results are needed, and these are used throughout this and the following appendix. For example, while $0.\infty$ is undefined, $\lim_{x\to\infty}\{0.x\}$ is well defined to be zero. Similarly $\lim_{x\to\infty} 1^x$ is defined to be 1, etc. Footnote 12 shows how L'Hopital's Rule can be sued to deal with the more complex problem of $\lim_{N^S\to\infty} N^S \beta^{N^S}$ for $\beta < 1$, to again produce a well defined answer.

Note that since network size is likely to be considerable, the results for $N^S \to \infty$ might be the most compelling, which gives

$$\text{stable } |_{\beta=1, \delta \in (0,1), N^S \to \infty} \Leftrightarrow 0 < c - \left(\delta - \delta^2\right)$$

Therefore stability requires $c > \delta - \delta^2$. However, even for lower network sizes the second half of condition 13 is likely to be the binding constraint, and so $c > \delta - \delta^2$ is likely needed for stability. In effect then, for intermediate network sizes the star network will be stable, if direct costs are high relative to the savings to be gained from building a direct route from i to j.

(iii) $\delta = 1$. With instantaneous connection speeds, and the smallest non-trivial network, condition 12 becomes

$$\text{stable } |_{\beta=1, \delta=1, N^S=4} \Leftrightarrow 0 < \min\{3 - c, c\}$$

Again, with immediate connection, and an intermediate network condition 12 becomes

$$\text{stable } |_{\beta=1, \delta=1} \Leftrightarrow 0 < \min\{N^S - 1 - c, c\}$$

Finally, with a combination of immediate connection times and an extremely large network condition 12 becomes

$$\text{stable } |_{\beta=1, \delta=1, N^S \to \infty} \Leftrightarrow 0 < c$$

All three conditions are trivially satisfied since $c \in (0, 1)$, so the network must be stable when $\delta = 1$, if there is no concern for complexity. Put simply, with instantaneous connection, there is very little to be gained from building a direct route from i to j as going via the central node is so quick, but there is a direct cost c to be incurred. Hence the star network is stable when speeds are extremely high.

Part B: Some Concern for Complexity

When there are some complexity concerns, consider $\beta \in (0, 1)$, and especially values of β which are not too close to either extreme. The basic stability condition is not refined any further here, so starting with condition 9, and refining it further with additional constraints produces various cases.

(i) $\delta = 0, N^S = 4$. When connection speeds are zero and the network is extremely small condition 9 becomes

$$\text{stable } |_{\beta \in (0,1), \delta=0, N^S=4} \Leftrightarrow 0 < \min\{2\beta - c - 2, 1 + c - \beta^2\}$$

Clearly since $\beta < 1$ and $c < 1$, then $2\beta - c - 2 < 0$, and so the network is not stable.

(ii) $\delta = 0, N^S$ intermediate. Once again connection speeds are zero, but now consider intermediate

network size. Condition 9 becomes

$$\text{stable}_{\beta \in (0,1), \delta=0} \quad \Leftrightarrow \quad 0 < \min\{-c - \left(N^S - 2\right)(1 - \beta^{N^S-3}),$$
$$1 + c - 2\beta + (N^S - 2)\beta^{N^S-3} - (N^S - 3)\beta^{N^S-2}\}$$

Now consider the first half of the condition. $-c - \left(N^S - 2\right)(1 - \beta^{N^S-3}) > 0$ is necessary for stability. However for $N^S > 4$ this will be negative, and so the network is not stable.

(iii) $\delta = 0, N^S \to \infty$. Now consider $N^S \to \infty$, with $\delta = 0$ and therefore the stability condition becomes

$$\text{stable} \mid {}_{\beta \in (0,1), \delta=0, N^S=\infty} \Leftrightarrow 0 < \min \lim_{N^S \to \infty} \{-c - \left(N^S - 2\right)(1 - \beta^{N^S-3}),$$
$$1 + c - 2\beta + (N^S - 2)\beta^{N^S-3} - (N^S - 3)\beta^{N^S-2}\}$$

The first part of this expression must be negative and so results in a network which is not stable.

(iv) $\delta \in (0,1), N^S = 4$. Next consider an intermediate level of speed, combined with a small network, which produces the stability condition

$$\text{stable} \mid_{\beta \in (0,1), \delta \in (0,1), N^S=4} \Leftrightarrow 0 < \min\{2\delta^2 + \delta + 2\beta - c - 2, 1 + c - \beta^2 - \delta + \delta^2\}$$

A stable network looks unlikely here, but it is possible. For example if $c > 1/4 \geq \left(\delta - \delta^2\right)$, then the second half of the condition must be satisfied, and with a high enough value of δ and β the first condition is met, so the network is stable.

(v) $\delta \in (0,1), N^S$ intermediate. Considering both intermediate speed levels and an intermediate network size simply reproduces condition 9. Stability is possible or not, depending upon the parameter values. For example, as just seen in (iv) with a low enough value of N^S, and various other conditions in the other parameters, stability is possible, though as is shown next, as N^S becomes large stability is ruled out.

(vi) $\delta \in (0,1), N^S \to \infty$. With intermediate speed and an infinite network size condition 9 becomes

$$\text{stable} \mid {}_{\beta \in (0,1), \delta \in (0,1), N^S \to \infty} \Leftrightarrow 0 < \min \lim_{N^S \to \infty} \{\delta - c - \left(N^S - 2\right)(1 - \delta^2 - \beta^{N^S-3}),$$
$$1 + c - 2\beta + (N^S - 2)\beta^{N^S-3} - (N^S - 3)\beta^{N^S-2} - \left(\delta - \delta^2\right)\}$$

Now since the first half of the condition tends towards negative infinity, the network is not stable.

(vii) $\delta = 1, N^S = 4$. With instantaneous connection speeds, and the smallest non-trivial network, condition 9 becomes

$$\text{stable } |_{\beta \in (0,1), \delta = 1, N^S = 4} \Leftrightarrow 0 < \min\{1 - c + 2\beta, 1 + c - \beta^2\}$$

Since $c > 0$, it must be that $1 - c + 2\beta > 0$, so the condition becomes

$$\text{stable } |_{\beta \in (0,1), \delta = 1, N^S = 4} \Leftrightarrow 0 < 1 + c - \beta^2$$

Since $\beta < 1$ the network must be stable.

(viii) $\delta = 1, N^S$ intermediate. With virtually instantaneous connection speeds, and an intermediate network condition 9 becomes

$$\text{stable } \Big|_{\beta \in (0,1), \delta = 1} \Leftrightarrow 0 < \min\{1 - c + \left(N^S - 2\right)\beta^{N^S - 3},$$
$$1 + c - 2\beta + (N^S - 2)\beta^{N^S - 3} - (N^S - 3)\beta^{N^S - 2}\}$$

The first term, $1 - c + \left(N^S - 2\right)\beta^{N^S - 3}$, must be positive, so focusing on the second yields

$$\text{stable } |_{\beta \in (0,1), \delta = 1} \Leftrightarrow 0 < 1 + c - 2\beta + (N^S - 2)\beta^{N^S - 3} - (N^S - 3)\beta^{N^S - 2}$$

For low values of N^S, the result in (vii) shows that the network will be stable, but for larger values of N^S, it is not certain. Since $(N^S - 2)\beta^{N^S - 3} - (N^S - 3)\beta^{N^S - 2} > 0$, it is not certain whether $1 + c > 2\beta$ provides a sufficient condition for stability regardless of the value of N^S. If $\beta < 1/2$ the network will certainly be stable. This reveals that a low value of β or a high value of c can ensure stability.

(ix) $\delta = 1, N^S \to \infty$. Finally, with the combination of instantaneous connection speeds and an infinite network, condition 9 becomes

$$\text{stable } \Big|_{\beta \in (0,1), \delta = 1, N^S \to \infty} \Leftrightarrow 0 < \min \lim_{N^S \to \infty}\{1 - c + \left(N^S - 2\right)(\beta^{N^S - 3}),$$
$$1 + c - 2\beta + (N^S - 2)\beta^{N^S - 3} - (N^S - 3)\beta^{N^S - 2}\}$$

Since the first half is trivially satisfied the condition becomes

$$\text{stable } |_{\beta \in (0,1), \delta = 1, N^S \to \infty} \Leftrightarrow 0 < \lim_{N^S \to \infty}\{1 + c - 2\beta + (N^S - 2)\beta^{N^S - 3} - (N^S - 3)\beta^{N^S - 2}\}$$

Once again, as in (viii) it is clear that $(N^S - 2)\beta^{N^S - 3} > (N^S - 3)\beta^{N^S - 2}$ though the two tend towards each other as $N^S \to \infty$, so as in (viii) $1 + c > 2\beta$ ensures stability, though here it is a necessary condition as well as being sufficient.

Part C: Extreme Concern for Complexity

Now consider an extreme concern for complexity, which is examined in the model by setting $\beta = 0$ in condition 9 which gives

$$\text{stable} \Leftrightarrow 0 < \min\{\delta - c - \left(1 - \delta^2\right)\left(N^S - 2\right), 1 + c - \left(\delta - \delta^2\right)\} \tag{14}$$

(i) $\delta = 0, N^S = 4$. When connection speeds are zero and the network is extremely small condition 14 becomes

$$\text{stable}\,|_{\beta=0, \delta=0, N^S=4} \Leftrightarrow 0 < \min\left\{-c - 2, 1 + c\right\}$$

This condition fails and the network is not stable.

(ii) $\delta = 0, N^S$ intermediate. Once again connection speeds are zero, but now consider intermediate network size. The condition 14 becomes

$$\text{stable}\,|_{\beta=0, \delta=0} \Leftrightarrow 0 < \min\left\{2 - c - N^S, 1 + c\right\}$$

And so for $N^S > 4$, the network cannot be stable.

(iii) $\delta = 0, N^S \to \infty$. Now consider $N^S \to \infty$, with $\delta = 0$ and so the stability condition becomes

$$\text{stable}\,|_{\beta=0, \delta=0, N^S \to \infty} \Leftrightarrow 0 < \min \lim_{N^S \to \infty}\left\{2 - c - N^S, 1 + c\right\}$$

With the condition trivially failing and so the network is not stable.

(iv) $\delta \in (0, 1), N^S = 4$. Next consider an intermediate level of speed, combined with a small network, which produces the stability condition

$$\text{stable}\,|_{\beta=0, \delta\in(0,1), N^S=4} \Leftrightarrow 0 < \min\{\delta + 2\delta^2 - c - 2, 1 + c - \left(\delta - \delta^2\right)\}$$

This condition can be met, though it very much depends upon parameter values and requires a very high value of δ, and low value of c.

(v) $\delta \in (0, 1), N^S$ intermediate. Considering both intermediate speed levels and an intermediate network size yields condition 14 and so for stability it was shown in (iv) that stability is possible for very

high values of δ, but difficult otherwise, and becomes increasingly difficult as N^S rises.

(vi) $\delta \in (0,1)$, $N^S \to \infty$. With intermediate speed and an infinite network size condition 14 becomes

$$\text{stable} \mid_{\beta=0, \delta\in(0,1), N^S\to\infty} \Leftrightarrow 0 < \min \lim_{N^S\to\infty} \left\{ \delta - c - \left(1 - \delta^2\right)\left(N^S - 2\right), 1 + c - \left(\delta - \delta^2\right) \right\}$$

Which will not hold.

(vii) $\delta = 1$. With instantaneous connection speeds, condition 14 becomes

$$\text{stable} \mid_{\beta=0, \delta=1, N^S=4} \Leftrightarrow 0 < \min\{1 - c, 1 + c\}$$

This is not dependent upon N^S and is easily met since $c < 1$, and so the network is stable for any size if $\delta = 1$.

Appendix 2: The Complete Network

Here is presented the full impact of different assumed parameter values on the stability of the complete network. Again, the focus will be on general network sizes (finite values of N^S above 4) but for contrast the appendix will also consider a "small" network to be approximated by the assumption that $N^S = 4$, and a "large" network to have an arbitrarily large though finite number of nodes, which for simplicity will be approximated by $N^S \to \infty$ to aid intuition.[14]

Part A: No Concern for Complexity

Start with the model with no complexity concerns, so $\beta = 1$. Now condition 11 becomes

$$\text{stable} \mid_{\beta=1} \Leftrightarrow \delta - \delta^2 > c \tag{15}$$

Note that this is not dependent upon N^S, so condition 15 need only be considered for variations of δ.

(i) $\delta = 0$. When connection speeds are zero, condition 15 becomes

$$\text{stable} \mid_{\beta=1, \delta=0} \Leftrightarrow 0 > c$$

This is not the case, and hence the network is not stable under extremely slow connection speeds. Simply put, connection speeds are so low that being connected to another link is too costly.

(ii) $\delta \in (0, 1)$. Next consider an intermediate level of speed, combined with a small network, which produces the stability condition

$$\text{stable } |_{\beta=1, \delta \in (0,1)} \Leftrightarrow \delta - \delta^2 > c$$

Since $\inf_{\delta} (\delta - \delta^2) = 0$, for $\delta \in (0, 1)$, it is the case that as δ approaches the extreme values of 0 and 1 there will be no level of cost sufficiently small to enable the network to be stable. However, $\max_{\delta} (\delta - \delta^2) = 1/4$, when $\delta = 1/2$, so for values of δ approaching $1/2$, there are levels of cost sufficiently low to enable the network to be stable.

(iii) $\delta = 1$. With instantaneous connection speeds, condition 15 becomes

$$\text{stable } |_{\beta=1, \delta=1} \Leftrightarrow 0 > c$$

Since $c > 0$, this condition cannot hold, and the network is not stable. Here the speed is so high that lengthening the path from i to j makes no effective difference, but since cutting the direct link saves the cost c, it is worthwhile.

Part B: Some Concern for Complexity

The model examines complexity concerns when $\beta \in (0, 1)$, and more especially values of β are set not too close to either extreme. The basic stability condition is not refined any further here, so starting with condition 11 and refining it further with additional constraints yields the following cases.

(i) $\delta = 0$. When connection speeds are zero and the network is extremely small (say $N^S = 4$) condition 11 becomes

$$\text{stable } |_{\beta \in (0,1), \delta=0, N^S=4} \Leftrightarrow -c - 2 \left(\beta - \beta^2 \right) > 0$$

This will fail as $\beta \in (0, 1)$ and $c > 0$. Considering larger networks, results in a condition

$$\text{stable } |_{\beta \in (0,1), \delta=0} \Leftrightarrow -c + \left(\beta^{N^S-2} - \beta^{2N^S-6} \right) - \left(N^S - 2 \right) \left(\beta^{N^S-3} - \beta^{N^S-2} \right) > 0$$

Which again, fails. Finally, when considering $N^S \to \infty$, with $\delta = 0$, failure is even more clear, since stability requires

$$\text{stable } |_{\beta \in (0,1), \delta=0, N^S \to \infty} \Leftrightarrow -c - \left(N^S - 2 \right) \left(\beta^{N^S-3} - \beta^{N^S-2} \right) > 0$$

Therefore with some concern for complexity, and a network where connection speeds equal zero, the complete network will not be stable, irrespective of network size. To understand this consider the great

advantage of the complete network: every node is close to every other. This becomes irrelevant when connection speeds are so slow that even a single link is prohibitively slow to navigate.

(ii) $\delta \in (0,1)$. Next consider an intermediate level of speed, combined with a small network, which produces the stability condition

$$\text{stable} \mid_{\beta \in (0,1), \delta \in (0,1), N^S = 4} \Leftrightarrow \left(\delta - \delta^2\right) - c - 2\left(\beta - \beta^2\right) > 0$$

A stable network is possible for low enough values of β and c, if δ is sufficiently close to the middle of the range. The best case scenario for stability is where $\delta = 1/2$, where the stability condition is

$$\text{stable} \mid_{\beta \in (0,1), \delta = 1/2, N^S = 4} \Leftrightarrow 1/4 > c + 2\left(\beta - \beta^2\right)$$

Considering both intermediate speed levels, so $\delta \in (0,1)$, and an intermediate network size, yields condition 11. Stability is possible or not, depending upon the parameter values. For example, as just seen for $N^s = 4$, with a low enough value of N^S, β and c stability is possible. As seen in (i), and (iii) if $\delta = 0$ or $\delta = 1$ then stability will fail, so much relies on a connection speed which is not too fast or too slow. Considering intermediate speed levels, so $\delta \in (0,1)$, but an infinite network, so $N^S \to \infty$, changes condition 11 to become

$$\text{stable} \mid_{\beta \in (0,1), \delta \in (0,1), N^S \to \infty} \Leftrightarrow \left(\delta - \delta^2\right) - c - \left(N^S - 2\right)\left(\beta^{N^S - 3} - \beta^{N^S - 2}\right) > 0$$

Hence, with some concern for complexity and intermediate speeds, the complete network may be stable, but much depends upon the parameter values.

(iii) $\delta = 1$. Note that since the only expression in δ is $\delta - \delta^2$, the case where $\delta = 1$, is exactly as in the case where $\delta = 0$. In particular, when $\delta = 1$ and with the smallest non-trivial network $N^S = 4$ condition 11 becomes

$$\text{stable} \mid_{\beta \in (0,1), \delta = 1, N^S = 4} \Leftrightarrow -c - 2\left(\beta - \beta^2\right) > 0$$

As in case (i) stability fails. For intermediate network sizes, the condition becomes

$$\text{stable} \mid_{\beta \in (0,1), \delta = 1} \Leftrightarrow -c + \left(\beta^{N^S - 2} - \beta^{2N^S - 6}\right) - \left(N^S - 2\right)\left(\beta^{N^S - 3} - \beta^{N^S - 2}\right) > 0$$

So, once again, as in case (i) stability will fail for intermediate values of N^S. Finally, when $N^S \to \infty$,

condition 11 becomes

$$\text{stable} \mid_{\beta \in (0,1), \delta=1, N^S \to \infty} \Leftrightarrow -c - \left(N^S - 2\right)\left(\beta^{N^S-3} - \beta^{N^S-2}\right) > 0$$

So, as in case (i) stability fails. Irrespective of the size of the network, if there is some concern for complexity, then stability will always fail when connection speeds are instantaneous. This is perfectly reasonable given that the major advantage of the complete network is the proximity of each node to each other, which becomes less and less important as speeds increase.

Part C: Extreme Concern for Complexity

Now consider extreme concern for complexity, which is examined in the model by setting $\beta = 0$, in condition 11 to yield a new stability condition

$$\text{stable} \mid_{\beta=0} \Leftrightarrow \left(\delta - \delta^2\right) - c > 0 \tag{16}$$

Note that this is exactly the same as condition 15. This condition is most likely to be met where δ is not too close to either 0 or 1, in particular if $\delta = 1/2$, then stability is achieved if $c < 1/4$. For $\delta = 0$ or $\delta = 1$, stability is impossible for all $c > 0$, and if $c > 1/4$ then stability is impossible for any $\delta \in (0, 1)$. Note also that N^S is irrelevant when there is extreme concern for complexity.

| Complexity Concern	Network Type	"Small" Network: $N^S=4$			"Intermediate" Network: $N^S\in(4,\infty)$			"Large" Network: $N^S\to\infty$		
		$\delta=0$ (slow)	$\delta\in(0,1)$	$\delta=1$ (fast)	$\delta=0$ (slow)	$\delta\in(0,1)$	$\delta=1$ (fast)	$\delta=0$ (slow)	$\delta\in(0,1)$	$\delta=1$ (fast)
$\beta=0$ (high)	Star	No App 1C(i)	Maybe App 1C(iv)	Yes App 1C(vii)	No App 1C(ii)	Maybe App 1C(v)	Yes App 1C(vii)	No App 1C(iii)	No App 1C(vi)	Yes App 1C(vii)
	Complete	No App 2C	Maybe App 2C	No App 2C	No App 2C	Maybe App 2C	No App 2C	No App 2C	Maybe App 2C	No App 2C
$\beta\in(0,1)$ (some)	Star	No App 1B(i)	Maybe App 1B(iv)	Yes App 1B(vii)	No App 1B(ii)	Maybe App 1B(v)	Maybe App 1B(viii)	No App 1B(iii)	No App 1B(vi)	Maybe App 1B(ix)
	Complete	No App 2B(i)	Maybe App 2B(ii)	No App 2B(iii)	No App 2B(i)	Maybe App 2B(ii)	No App 2B(iii)	No App 2B(i)	Maybe App 2B(ii)	No App 2B(iii)
$\beta=1$ (none)	Star	No App 1A(i)	Maybe App 1A(ii)	Yes App 1A(iii)	No App 1A(i)	Maybe App 1A(ii)	Yes App 1A(iii)	No App 1A(i)	Maybe App 1A(ii)	Yes App 1A(iii)
	Complete	No App 2A(i)	Maybe App 2A(ii)	No App 2A(iii)	No App 2A(i)	Maybe App 2A(ii)	No App 2A(iii)	No App 2A(i)	Maybe App 2A(ii)	No App 2A(iii)

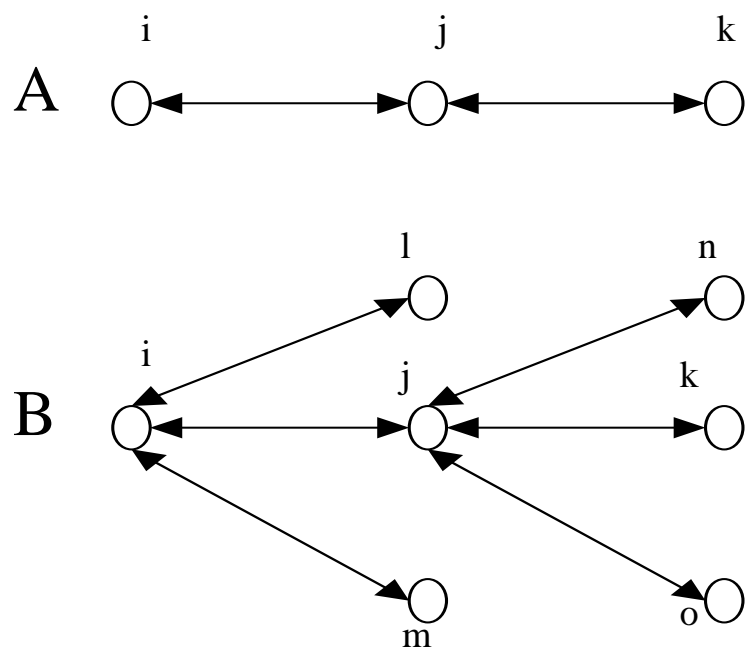Table 1: A Taxonomy of Stability Results
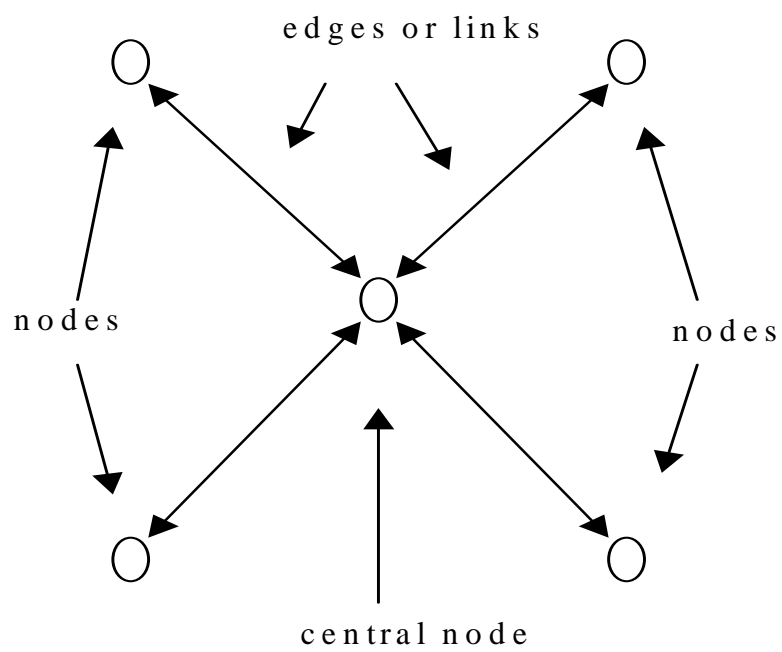
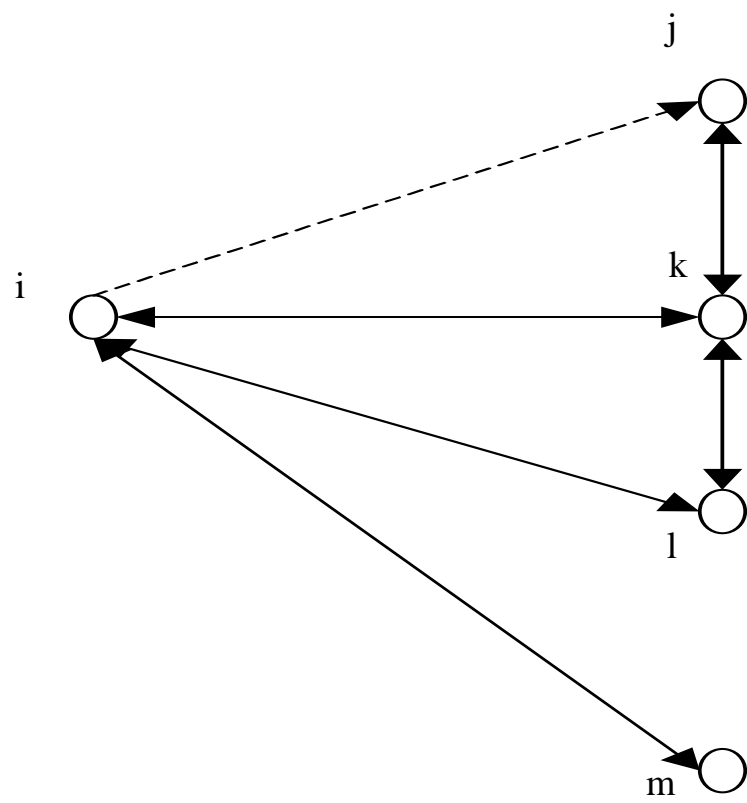Figure 1: Example Network

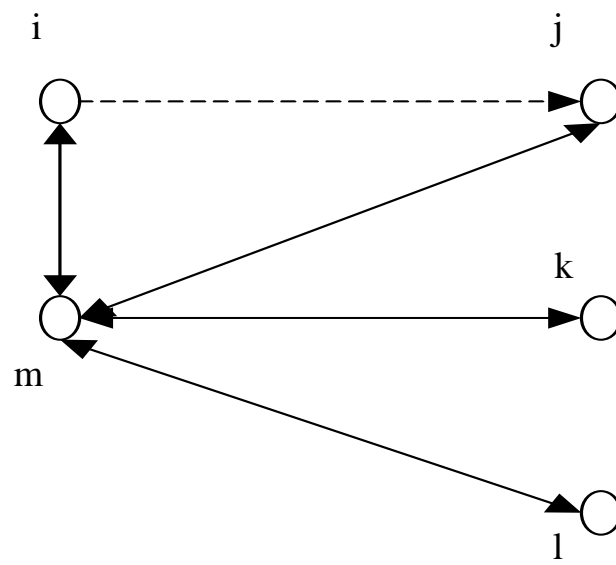Figure 2: Complexity

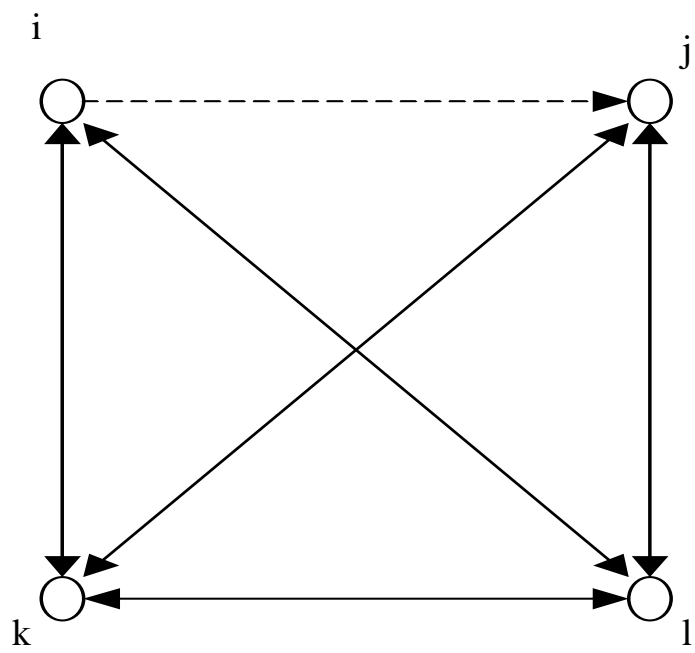Figure 3: Stability in an Example Network

Figure 4: Stability in the Star Network

Figure 5: Stability in the Complete Network