NONLINEAR CONTROLLABILITY AND OBSERVABILITY WITH APPLICATIONS TO GRADIENT SYSTEMS

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SUMMARY

We extend the theory of nonlinear observability due to Hermann-Krener [5] to the non-regular case, in which the observability co-distribution is not constant dimensional, and we obtain results in some sense dual of the ones already known for accessibility.

We discuss a conjecture of P. Varaya [15], namely that the isomorphism of two locally controllable gradient systems is an isometry for the underlying pseudo Riemannian manifolds, proving it to be false without further, or different, assumptions; we also prove some positive results, and the analogue of the above for Hamiltonian systems, with weaker conditions: an isomorphism of reachable Hamiltonian systems is a symplectomorphism.

Finally we prove that a Hamiltonian system with finite-dimensional Lie algebra, satisfying standard conditions, has an accessible Hamiltonian realization, constructed in a canonical way.

INTRODUCTION

The basic definitions, results and notation follow, as much as possible, those adopted in [5] and [11-13], when concerning control theory, and in [1] where relative to differential geometry.

The construction of a locally weakly observable system, made in chapter II, is based on a result (proposition II.2.3) that can be formulated without reference to control systems: If $\mathcal{F}$ is a codimension q foliation on M , and D is a family of complete vector fields on M leaving $\mathcal{F}$ invariant and spanning the tangent space at every point, the leaf space $M_1 = M/\mathcal{F}$ can be given the structure of a paracompact Hausdorff manifolds of dimension q such that the canonical projection $\pi_1 : M \to M_1$ is a submersion, and moreover a fibre map (a fibration in the C^∞ case).

In chapter IV, the construction of an accessible realization of a Hamiltonian system is inspired by the use made in Mechanics of the "method of orbits" introduced in [7] to study Hamiltonian systems with symmetry and obtain a reduced phase space.

CHAPTER I - NONLINEAR ACCESSIBILITY

In this chapter we present the known results on nonlinear accessibility that are needed later, essentially based on the work by H. Sussmann [11, 13].

1 - Control Systems

A C^K $(K = \infty,\omega)$ control system is a 4-tuple $\Sigma = (M,\Omega,f,\mathcal{U})$ verifying:

i) The state space M is a n-dimensional C^K differentiable manifold, which we assume to be Hausdorff, paracompact and connected.

ii) The control space Ω is a metric space, usually a m-dimensional Euclidian space or discrete.

iii) The dynamics $f:M \times \Omega \to TM$ is such that, for any $u \in \Omega$, $X^u = f(.,u) : M \to TM$ is a C^K vector field.

iv) The class of admissable controls $\mathcal{U}$ is a family of functions $\mathbb{R} \to \Omega$ defined on intervals of the form $[0,T]$ with $T \in \mathbb{R}^+$, and closed under concatenation, i.e., if $u : [0,T_1] \to \Omega$ and $v : [0,T_2] \to \Omega$ are in $\mathcal{U}$, the map $u*v : [0,T_1 + T_2] \to \Omega$ defined by $u*v|_{[0,T_1]} = u$, $u*v|_{[T_1,T_1+T_2]} = v(. - T_1)$ is also in $\mathcal{U}$.

Let $\mathcal{U}_m$ be the set of functions $\mathbb{R} \to \Omega$ with domains on above, such that every $u \in \mathcal{U}_m$ is the limit almost everywhere of a sequence of piecewise constant K-valued functions, where $K \subset \Omega$ is compact, and let U_{pc} be the set of piecewise constant Ω-valued functions with the same type of domain. We assume $\mathcal{U}_{pc} \subset \mathcal{U} \subset \mathcal{U}_m$.

In order to ensure the existence and uniqueness of solution of $\dot{x} = f(x,u)$, $x(0) = x_0$ for every $x_0 \in M$ and $u \in \mathcal{U}$, we assume that

the family of associated vector fields $D = \{X^u\}_{u\in\Omega}$ satisfies a local Lipschitz condition uniformly in u over any compact subset K of Ω, i.e., given $x_0 \in M$ and a compact $K \subset \Omega$ there is a chart U around x_0 and a constant $C \in \mathbb{R}^+$ such that every component of the local representatives in U of $\{X^u\}_{u\in K}$ is Lipschitz with constant C.

We denote by $\pi(x_0,u;t)$ the solution at time t of the equation $\dot{x} = f(x,u)$, $x(0) = x_0$.

The following approximation lemma [12] allows us, from now on, to consider $\mathcal{U} = \mathcal{U}_{pc}$ in every case:

<u>Lemma I.1.1.</u>

Let Σ be a control system and $\{u_i\}$ a sequence of admissible controls defined on $[0,T]$ with values in the compact set $K \subset \Omega$. If $\{u_i\} \to u$ almost everywhere and $\pi(x_0,u;t)$ is defined on $[0,T]$ then, for sufficiently large i, $\pi(x_0,u_i;t)$ is defined on $[0,T]$ and $\lim_{i\to\infty} \pi(x_0,u_i;t) = \pi(x_0,u;t)$.

2 - <u>Distributions</u>

A distribution (codistribution) Δ on a manifold M is the assignment to each $x \in M$ of a subspace $\Delta(x)$ of T_xM (T_x^*M).

Δ is a C^k (co)distribution if it is spanned by a family D of C^k vector fields (1-forms), i.e., if $\Delta(x)$ is the linear subspace of

T_xM (T_x^*M) spanned by the family $\{X(x)\}$ with $X \in D$ for every $x \in M$.

A C^k distribution Δ is involutive if whenever the vector fields X^1, X^2 belong to Δ (i.e. $X^i(x) \in \Delta(x)$ for $i = 1,2$ and $x \in M$) so does $[X^1, X^2]$.

We say Δ is r-dimensional if $\Delta(x)$ has dimension r for each $x \in M$.

Let D be a family of vector fields; the pseudo group G_D (pseudo semigroup S_D) of D is the set of all finite products of elements belonging to the pseudo-groups of flows (pseudo-semigroups of flows for non-negative time) of vector fields in D.

We say that Δ is D-invariant if for every $g \in G_D$ and $x \in \mathrm{dom}(g)$ we have $g_*(\Delta(x)) \subset \Delta(gx)$, or in the case of codistributions $g^* \Delta(gx) \subset \Delta(x)$.

We denote by Δ_D the distribution spanned by D, by $\mathcal{T}(D)$ the smallest involutive distribution containing Δ_D, and by P_D the smallest D-invariant distribution containing Δ_D.

A submanifold S of M is said to be an integral submanifold of Δ if $T_xS = \Delta(x)$ for $x \in S$, and a maximal integral submanifold of Δ is a connected integral submanifold of Δ maximal for the relation of inclusion.

Δ is integrable if through every point of M there passes a maximal

integral submanifold of Δ .

The orbit of D containing $x \in M$ is the set $G_D x$.

Theorem I.2.1[11]:

P_D is an integrable distribution, and the maximal integral submanifolds are the orbits of D .

This in particular implies $\mathcal{T}(D) \subset P_D$.

Theorem I.2.2[11]:

Let Δ be a C^K distribution spanned by a family D of vector fields. Δ is integrable iff $\Delta = P_D$.

As corollaries we have

Nagano Theorem I.2.3 [11,8]:

An analytic involutive distribution is integrable.

Frobenius Theorem I.2.4:

A C^∞ constant dimensional involutive distribution is integrable.

3 - Reachability and accessibility

Let Σ be a control system, and D the family of associated vector fields.

Σ is said to have the reachability (controllability) property if $G_D x = M$ $(S_D x = M)$ for any $x \in M$.

Σ is accessible at $x \in M$ if $S_D x$ has non-empty interior in M, and locally accessible at x if for any neighbourhood U of x the set of points attainable from x, i.e. of the form $y = gx$ with $g \in S_D$, with trajectories not leaving U has non-empty interior in M. Equivalently, we can say Σ is locally accessible at $x \in M$ if its restriction to any neighbourhood of x is accessible at x.

Accessibility and local accessibility are defined in the natural way; and clearly local accessibility implies accessibility.

Chow Theorem [11] I.3.1:

Σ is reachable iff P_D has dimension n.

We say that Σ satisfies the controllability rank condition at $x \in M$ if $\dim \mathcal{J}(D)(x) = n$.

Theorem I.3.2 [5]:

If Σ satisfies the controllability rank condition at $x \in M$, then it is locally accessible at x.

We have a partial converse for the above theorem:

Theorem I.3.3 [5]:

If Σ is locally accessible, the controllability rank condition is satisfied on an open dense set in M.

In the analytic case $P_D = \mathcal{T}(D)$ (Nagano theorem) and the same is true if $\mathcal{T}(D)$ is constant dimensional, from Frobenius theorem.

Thus we get

Theorem I.3.4:

If Σ is an analytic system, or $\mathcal{T}(D)$ has constant dimension, reachability, accessibility and local accessibility are equivalent.

It is quite easy to construct counter examples to the above result for the C^∞ case, where $\mathcal{T}(D)$ has not constant dimension.

Example I.3.5:

Let $M = \mathbb{R}^2$ and D be $\{\frac{\partial}{\partial x}, \phi(x)\frac{\partial}{\partial y}\}$ where $\phi(x) = 0$ if $x \geq 0$ and $\phi(x) = \exp(-1/x)$ if $x < 0$.

Then the corresponding control system has the reachability property, but Σ is not accessible at a point $x = (x_1, x_2)$ if $x_1 \geq 0$ (and a it is not locally accessible at the same points).

Example I.3.6:

Consider now the above example changing ϕ into ψ, defined by $\psi(x) = \phi(-x)$.

The corresponding system is then accessible but it is not locally accessible at points $x = (x_1, x_2)$ if $x_1 < 0$.

4 - Strong accessibility

A system is said to be strongly accessible at $x \in M$ if, for some $t \in \mathbb{R}^+$, the set of points attainable from x in time t has non-empty interior. Strong accessibility is defined in the natural way.

Let $\mathcal{T}_0(D)$ be the distribution spanned by all sums $Z_1 + Z_2$, where Z_1 is a linear combination of vectors in D such that the sum of the coefficients is zero, and Z_2 is an element in the derived Lie algebra of $\mathcal{T}(D)$, i.e. $Z_2 \in [\mathcal{T}(D), \mathcal{T}(D)]$ if we interpret $\mathcal{T}(D)$ as a Lie algebra of vector fields.

Lemma I.4.1 [13]:

$\mathcal{T}_0(D)$ is an ideal of $\mathcal{T}(D)$ of codimension one or zero, as Lie algebras, and at each $x \in M$, $\mathcal{T}(D)(x)$ and $\mathcal{T}_0(D)(x)$ are subspaces of T_xM such that the codimension of $\mathcal{T}_0(D)(x)$ in $\mathcal{T}(D)(x)$ is zero or one.

Define P_D^0 as the distribution spanned by the differences of vector fields belong to P_D .

Lemma I.4.2. [4]:

P_D^0 is a D-invariant distribution, and its codimension in P_D is zero or one. If the system is analytic P_D^0 is the distribution spanned by

$\mathcal{T}_0(D)$, in general $\mathcal{T}_0(D) \subset P_D^0$.

Theorem I.4.3 [13]:

An analytic system isstrongly accessible iff $\mathcal{T}_0(D)$ is n-dimensional.

Theorem I.4.4 [4]:

If Σ is strongly accessible, P_D^0 is n-dimensional.

A particular case is of interest: suppose we are given a C^k system, in which $f(x,u) = X^0(x) + \sum_{i=1}^{\ell} u_i X^i(x)$, where X^i , $i = 0,1,\ldots,n$ are complete vector fields in M , and $\mathcal{T}(D)$ is a finite dimensional Lie algebra.

Theorem I.4.5 [6]:

i) $\mathcal{T}(D)$ is the Lie algebra generated by $\{X^0,X^1,\ldots,X^\ell\}$, and every element of $\mathcal{T}(D)$ is a complete vector field.

ii) $\mathcal{T}_0(D)$ is the ideal of $\mathcal{T}(D)$ generated by $\{X^1,\ldots,X^\ell\}$.

iii) If we define $ad^K_{X^0}$ by $ad^0_{X^0} X = X$, $ad^K_{X^0} X = [X^0, ad^{K-1}_{X^0} X]$ for $K > 0$, $\mathcal{T}_0(D)$ is the Lie algebra generated by $\{ad^r_{X^0} X^i , \ i = 1,2,\ldots,K: \ r = 0,1,\ldots\}$.

From this theorem, it follows that control systems as above are strongly accessible iff they are accessible and $X^0(x) \in \mathcal{T}_0(D)(x)$ at every $x \in M$.

CHAPTER II - NONLINEAR OBSERVABILITY

Our aim is to extend the results of Hermann-Krener [5] to the non-regular case, obtaining some kind of duality of the known results about accessibility, presented in the previous chapter.

After the basic definitions in the first section, section two consists of the construction of strongly equivalent complete locally weakly observable and weakly observable systems from a given complete reachable system.

The construction is used in section three to give criteria for observability, and section four presents the duality with corresponding results for accessibility.

In section five we give some examples of the application of these results, and section six contains the basic notions about minimal realizations that are used later.

1 - Systems : basic definitions

A C^K system is a σ-tuple $\Sigma = (M,\Omega,f,\mathcal{U},N,h)$ where

i) $(M,\Omega,f,\mathcal{U})$ is a C^K control system

ii) N is a ℓ-dimensional Euclidean space

iii) $h:M \to N$ is a C^K map.

N is called the output space, and h the output map. Properties of the associated control system, also denoted by Σ, will be attributed to the system Σ.

We say two points x_0,x_1 in M are (weakly) indistinguishable if for every $\gamma \in S_D$ $(\gamma \in G_D)$, such that γx_0 and γx_1 are defined, we have $h\circ\gamma\, X_0 = h\circ\gamma\, X_1$. We denote by $S_D^* h$ $(G_D^* h)$ the set of functions of the form $h\circ\gamma$, $\gamma \in S_D(\gamma \in G_D)$.

Σ is (weakly) observable if no two points are (weakly) indistinguishable

Σ is strictly locally (weakly) observable at $x \in M$ if there exists an open nhd U of x such that the restriction of Σ to any nhd of x $V \subset U$ is (weakly) observable.

Σ is locally (weakly) observable at x if there exists a nhd U of x such that there are no two points in U (weakly) indistinguishable.

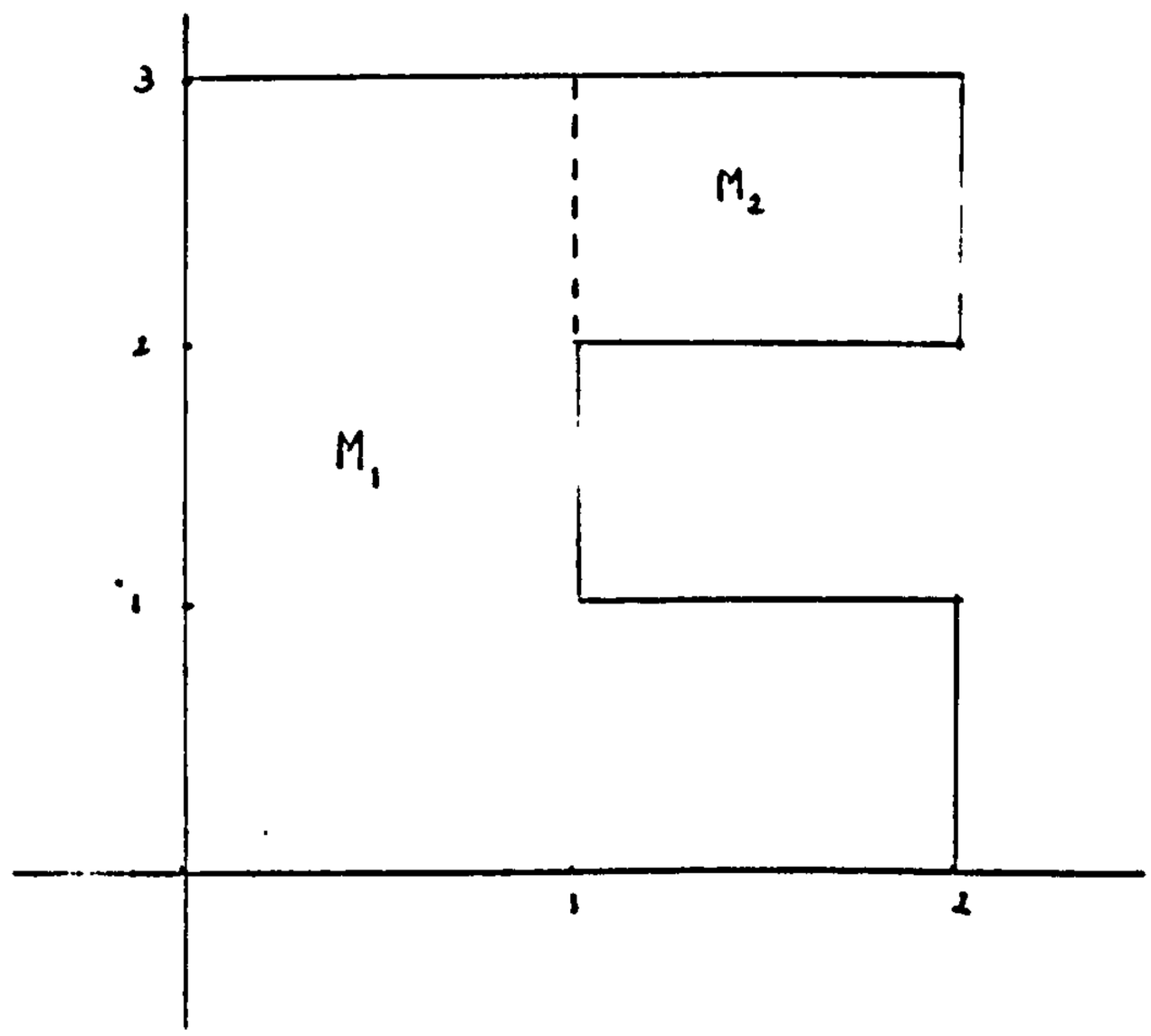
Clearly the weak definitions are equivalent to the usual ones if Σ is

symmetric, i.e. $X \varepsilon D \Rightarrow -X \varepsilon D$, or analytic.

If the system is complete (every $X \varepsilon D$ is a complete vector field) the domain of any $\gamma \varepsilon G_D$ is the whole state space; in that case, or if the system is analytic, indistinguishability and weak indistinguishability are equivalence relations [12].

Example II.1.1.

Let M be the bounded open set enclosed by the following polygonal:



Let $h:M \to \mathbb{R}$ be defined by $h|_{M_1} = 0$ and $h(x,y) = \exp(-1/(x-1)$ if $(x,y) \varepsilon M_2$. Let $D = \{\frac{\partial}{\partial x_1}, \frac{\partial}{\partial y}\}$.

Then, if $x_0 = (1/2, 1/2)$, $x_1 = (1/2, 3/2)$ and $x_2 = (1/2, 5/2)$ we have x_0, x_1 and x_1, x_2 indistinguishable, but x_0 and x_2 are not indistinguishable.

Given two systems Σ_1 and Σ_2 , with the same control class and output space, we say $x_1 \varepsilon M_1$ and $x_2 \varepsilon M_2$ are (weakly) indistinguishable if $h_1 o\, g_1\, x_1 = h_2 o\, g_2\, x_2$ where $g_1 \varepsilon S_D$ $(g_1 \varepsilon G_D)$ and g_2 is obtained

using the same piecewise constant control as the one involved in g_1 . If Σ_1 and Σ_2 are not complete, the above equality has to be satisfied when $x_1 \in \mathrm{dom}\, g_1$ and $x_2 \in \mathrm{dom}\, g_2$.

Σ_1 and Σ_2 are strongly equivalent if every state in Σ_1 is weakly indistinguishable from some state in Σ_2 and vice-versa.

2 - Construction of strongly equivalent weakly observable system

We assume Σ is a complete, reachable system with output map $h:M \to \mathbb{R}^\ell$, $h = (h_1,\ldots,h_\ell)$. We construct the two following codistributions:

i) $\mathcal{H}(D)$ is the smallest codistribution containing dh_i , $i = 1,2,\ldots,\ell$ and closed for Lie differentiation by elements of D .

ii) $\mathcal{P}(D)$ is the smallest D-invariant codistribution containing dh_i , $i = 1,2,\ldots,\ell$.

From the definition of Lie derivative, it is clear that $\mathcal{H}(D) \subset \mathcal{P}(D)$.

Since Σ is reachable, i.e. M is an orbit of D , $\mathcal{P}(D)$ is constant dimensional. We denote by q its dimension.

Let F_D be the smallest subspace (over $\mathbb{R}$) of $C^K(M)$ containing h_i , $i = 1,\ldots,\ell$ and closed under Lie differentiation by elements of D . Since the exterior differentiation commutes with Lie derivative on functions, $\mathcal{H}(D) = dF_D$ in the sense that $\mathcal{H}(D)$ is the codistribution spanned by the differentials of the functions in F_D .

Let E_D be the smallest subspace of $C^K(M)$ containing h_i, $i = 1,\ldots,\ell$ and closed for the action of G_D on $C^K(M)$ defined by $(g,f) \to g^*f$. Then $\mathcal{P}(D) = dE_D$, as above, since $dg^*f = g^*df$.

Let Λ be the C^K distribution defined by $\Lambda = \mathrm{Ker}\mathcal{P}(D)$, i.e. Λ is spanned by the vector fields $X \in V^K(M)$ such that, for any differential form α belonging to $\mathcal{P}(D)$, $\alpha\cdot X = 0$. Equivalently, we can characterize X by $X\psi \equiv 0$ for every $\psi \in E_D$.

Lemma II.2.1:

Λ is an integrable distribution.

Proof:

Since $\mathcal{P}(D)$ is constant dimensional, so is Λ.

Let X,Y be two vector fields belonging to Λ, and $\psi \in E_D$. Then:

$$[X,Y]\psi = X(Y\psi) - Y(X\psi) = X.o - Y.o = 0$$

and thus $[X,Y]$ is in Λ, therefore Λ is involutive. By Frobenius theorem, it is integrable.

Lemma II.2.2:

The maximal integral submanifolds of Λ are closed, and any two of them are diffeomorphic.

Proof:

It follows from the definition of Λ that its maximal integral submanifold containing some point $x \in M$ is the connected component of the set $\{x' \in M : \psi(x) = \psi(x') \quad \psi \in E_D\}$ containing x. As the connected component of a closed set is closed, the first part of the lemma is proved.

Now let L, L' be two maximal integral submanifolds of Λ passing through x, x' respectively. As the system is reachable, there exists $g \in G$ such that $gx = x'$.

For every $g \in G$, if X is a vector field in Λ, g_*X is also in Λ (i.e. Λ is D-invariant):

$$g_*X.\ \psi = d\psi.g_*X = (g^*d\psi)X = X.(g^*\psi) = 0$$

since $\psi \in E_D \Rightarrow g^*\psi \in E_D$.

Thus g takes maximal integral submanifolds into maximal integral submanifolds, and as $gx = x'$ we have $gL = L'$.

Proposition II.2.3:

There exists a Hausdorff paracompact C^K manifold M_1 of dimension q, and a map $\pi_1 : M \to M_1$ such that:

i) π_1 is a submersion.

ii) The fibres of π_1 are the maximal integral submanifolds of Λ.

iii) π_1 is a fibre map, and in the C^∞ case it is a fibration.

Given two manifolds E,M and a map $p:E \to M$ we say p is a fibre map if M has an open cover of sets $\{U_i\}$ such that, for every $_i$ there exists a diffeomorphism $\phi_i : p^{-1}(U_i) \to U_i \times F$ making the following diagram commutative

$$\begin{tikzcd} p^{-1}(U_i) \arrow[r, "\phi_i"] \arrow[d, "p"'] & U_i \times F \arrow[dl, "p_1"] \\ U_i & \end{tikzcd}$$

where p_1 is the projection on the first factor, F is a manifold called the fibre.

<u>Proof</u>:

Consider the equivalence relation R defined on M by $x \sim y \Leftrightarrow x$ and y belong to the same maximal integral submanifold of Λ. Let $M_1 = M/R$ and π be the natural projection.

Take $[x] \in M_1$, $[x] = \pi_1(x)$. We can find $\phi_1,\ldots,\phi_q \in E_D$ such that $d\phi_1(x),\ldots,d\phi_q(x)$ span $\mathcal{P}(D)(x)$. Then this is also true for an open nhd U' of x.

Let $U_1 = \pi_1(\mathcal{U}')$ and $U = \pi_1^{-1}(\mathcal{U}_1)$. We want to show that $d\phi_1,\ldots,d\phi_q$ span $\mathcal{P}(D)$ on U ; it is enough to show that if $d\phi(y) \neq 0$ for some $\phi \in E_D$ and $y \in M$, then $d\phi$ is non-zero in every point of the maximal integral submanifold L_y passing though y .

Let $y' \in L_y$ and suppose $d\phi(y') \doteq 0$. Then for every vector field X in $D' = G_*D$ (i.e. $X \in D'$ iff there exists $g \in G$ and $Y \in D$ such that $X = g_*Y$) we have $L_X\phi(y') = d\phi(y').X(y') = 0$.

We remark that if Λ is D-invariant it is also D'-invariant, and as the system is reachable $P(D)$ and therefore D' span the tangent space at all points. Clearly if Λ is D'-invariant so is $\mathcal{P}(D)$.

Now $L_X\phi$ is constant on the maximal integral submanifolds of Λ , since $d(L_X\phi)$ is contained in $\mathcal{P}(D)$

$$d(L_X\phi) = L_X d\phi$$

and from D'-invariance we have $L_X d\phi \in \mathcal{P}(D)$. Thus $L_X\phi(y) = 0$ for every $X \in D'$, or equivalently $d\phi(y) \,.\, X(y) = 0$; as $D'(y)$ spans T_yM we conclude $d\phi(y) = 0$, and the contradiction proves our claim.

If we denote by $\phi:M \to \mathbb{R}^q$ the map $\phi(y) = (\phi_1(y),\ldots,\phi_q(y))$ we see that ϕ is a submersion at every point of U . Moreover the equivalence relation R considered on U is the same as $x \simeq y \iff \phi(x) = (\phi(y)$.

Then, by [3, pg. 91] we have the following commutative diagram

$$\begin{tikzcd} U \arrow[r] \arrow[d, "\pi_1"'] & \phi(U) \subset \mathbb{R}^q \\ U_1 \arrow[ur, "\phi_1"'] & \end{tikzcd}$$

where $\phi(U) = \phi_1(U_1)$ is an open set, since ϕ is an open map and $\phi(U) = \phi(U')$, ϕ_1 is a diffeomorphism onto its image and π_1 is a submersion on U .

This way, when we vary $[x]$ in M_1 we construct an atlas on M_1 , with charts (U_1,ϕ_1) , and if M_1 is given the C^K structure defined by this atlas π_1 is a submersion, with the maximal integral submanifolds of Λ as fibres.

As the induced topology on M_1 is the quotient topology, M_1 is paracompact [3].

Let $[x] \in M_1$, and consider a chart (U_1,ϕ_1) around $[x]$ constructed as above. Take $x \in U$.

We have vector fields $X^1,\ldots,X^q \in D'$ such that the linear subspace $S(x)$ spanned by $X^1(x),\ldots,X^q(x)$ verifies $S(x) \oplus \Lambda(x) = T_xM$. Define the map $\Phi:\mathbb{R}^q \times L_x \to M$ by $\Phi(t_1,\ldots,t_q,y) = X^1_{t_1} o \ldots o\, X^q_{t_q} q$; clearly this map has rank n at $(0,x)$, so it is a diffeomorphism of some open set $A \subset \mathbb{R}^q \times L_x$ containing $(0,x)$, onto an open neighbourhood N of x .

We can take $N \subset U$, and A with the form $C \times W$, when C is an open cubic neighbourhood of the origin in $\mathbb{R}^q$ and W is an open neighbourhood of x in L_x .

Then $\sigma = \pi_1 \circ \Phi/_A$ is a submersion, and by D'-invariance the fibres of that map have the form $\{s\} \times W$ with $s \in C$. So we have: C is diffeomorphic to A/σ , then to $\sigma(A) = V_1$, i.e. we have the commutative diagram:

$$\begin{tikzcd}
A = C\times W \arrow[r, "\Phi/_A"] \arrow[d, "p_1"'] \arrow[dr, "\sigma"] & M \arrow[d, "\pi_1"] \\
C \arrow[r, "\sigma_1"'] & M_1
\end{tikzcd}$$

where σ_1 is a diffeomorphism onto its image.

Define the map $\Psi : V_1 \times L_x \to M$ by taking $\Psi = \Phi \circ (\sigma_1^{-1}, \mathrm{id})$; it is a C^K map, and its image is contained in $V = \pi_1^{-1}(V_1)$.

Define $\Theta : V \to V_1 \times L_x$ by $\Theta(y) = (\pi_1(y), X^q_{-s_q} \circ \ldots \circ X^1_{-s_1} y)$ where $(s_1, \ldots, s_q) = \sigma_1^{-1} \circ \pi_1(y)$. Clearly Θ is a C^K map.

It is trivial to check that $\Psi \circ \Theta = \mathrm{id}_V$ and $\Theta \circ \Psi = \mathrm{id}_{V_1 \times L_x}$; therefore Θ is a diffeomorphism and if $p_1 : V_1 \times L_x \to V_1$ denotes the projection on the first factor, we have the commutative diagram:

$$\begin{tikzcd} V = \pi_1^{-1}(V_1) \arrow[r, "\Theta"] \arrow[d, "\pi_1"'] & V_1 \times L_x \arrow[dl, "p_1"] \\ V_1 & \end{tikzcd}$$

Since we can obtain a cover of M_1 by this process, we have proved that π_1 is a fibre map.

Now let $[x] = \pi_1(x)$, $[y] = \pi_1(y)$ be two distinct points in M_1 .

We can find an open nhd U' of x such that $\bar{U}' \cap L_y = \emptyset$. Let V_1 be an open uhd of $[x]$ over which we have as before

$$\begin{tikzcd} V = \pi_1^{-1}(V_1) \arrow[r, "\Theta"] \arrow[d, "\pi_1"] & V_1 \times L \arrow[r, "(\sigma_1^{-1},\ id)"] \arrow[dl, "p_1"] & C \times L \\ V_1 & & \end{tikzcd}$$

We can assume $\bar{U}' \subset V$; if $U = \pi_1^{-1}(\pi_1(U'))$ we can see, using the above diffeomorphism onto $C \times L$, that $\bar{U}$ is the union of the maximal integral submanifolds of Λ passing through $\bar{U}'$, and hence $\bar{U} \cap L_y = \emptyset$.

Then we can find an open nhd W' of y such that $\bar{W}' \cap \bar{U} = \emptyset$; as before, if we take $W = \pi_1^{-1}(\pi_1(W'))$, we can assume $\bar{W} \cap \bar{U} = \emptyset$. As U, W are open sets, $\pi_1(U)$ and $\pi_1(W)$ are open uhds of $[x]$ and $[y]$ respectively, and moreover they are disjoint. Therefore M_1 is Hausdorff.

In the C^∞ case, if $\pi_1 : M \to M_1$ is a fibre map and M_1 is a paracompact Hausdorff manifold, then π_1 is a fibration [10].

Corollary II.2.4:

The maximal integral submanifolds of Λ are regular submanifolds of $\underline{M}$, of dimension $n-q$.

Proof:

Since π_1 is a submersion, and the maximal integral submanifolds of Λ are the inverse image of (then regular) points in M_1, the result follows.

Λ is then called a regular submanifold, from the regularity properties of its maximal integral submanifolds.

Theorem II.2.5:

Let Σ be a complete system Then we can construct a locally weakly observable system Σ_1, complete, such that Σ and Σ_1 are strongly equivalent.

Proof:

Consider, as in the previous proposition, the fibre map $\pi_1 : M \to M_1$. Define a family of vector fields D_1 by taking $X_1 \in D_1 \iff X_1 = \pi_{1*} X$; this is well defined, since as Λ is D-invariant the projection $X_1([x])$ is independent of the point taken on the fibre above $[x]$. Moreover if $X = f(\ ,u)$ with $u \in \mathcal{U}$, then $X_1 = f_1(\ ,u)$ for the same $u \in \mathcal{U}$ with

$f_1 = \pi_{1*}f$, which is similarly well defined.

Since the output map is constant along maximal integral submanifolds of Λ we have

$$\begin{tikzcd} M \arrow[r, "h"] \arrow[d, "\pi_1"'] & N \\ M_1 \arrow[ur, "h^1"'] & \end{tikzcd}$$

Let $\Sigma_1 = (M_1,\Omega,f_1,\mathcal{U},N,h^1)$; it is a complete system, from the construction of the associated vector fields.

Since $X_t^* h_i$ with $X \in D$ is constant on the fibres of π_1 we have

$$\begin{tikzcd} M \arrow[r, "X_t^* h_i"] \arrow[d, "\pi_1"'] & R \\ M_1 \arrow[ur, "\eta"'] & \end{tikzcd}$$

Clearly $\eta = X_{1_t}^* h_i^1$, and hence the way we construct the charts (U_1,ϕ^1) around each point $[x]$ in M_1 shows that Σ_1 is locally weakly observable (the points in U_1 can be distinguished).

From the construction we see that x and $\pi_1(x)$ are indistinguishable, concluding the proof.

The proof that Σ_1 is defined on a Hausdorff manifold does not assume local accessibility, or better, the controllability rank condition

satisfied at every point, for the system Σ .

In the proof that Σ_1 is Hausdorff is involved a family of vector fields for which the distribution Λ is invariant; the family D is not rich enough, in general, but as Hermann-Krener used it in [5] they had to assume the controllability rank condition at every point to do the job.

In our proof, instead of using D we used D' which trivially satisfies the controllability rank condition, since in fact D' spans the tangent space at every point.

Related to this, we can remark that example 3.10 in [5], which is meant as a counter-example for the non-assumption of the controllability rank condition, is misleading: the important fact is that the original system is not reachable in that example, not that it is not locally accessible.

The equivalence relation used for the construction of M_1 has as equivalence classes the connected components of the sets on which all $\phi \in E_D$ are constant; the resulting system Σ_1 is only locally weakly observable.

If we take as equivalence classes the sets (and not their connected components) on which E_D is constant, we will be able to obtain a new system Σ_2 , now weakly observable. Clearly the new manifold M_2 thus defined will be the same as the one obtained from M_1 with the equivalence

relation $[x] \sim [y] \iff E_{D_1}([x]) = E_{D_1}([y])$, i.e. we have the following commutative diagram:

$$\begin{tikzcd} M \arrow[r, "\pi_1"] \arrow[d, "\Pi"'] & M_1 \arrow[dl, "\pi_2"] \\ M_2 & \end{tikzcd}$$

<u>Proposition II.2.6:</u>

There exists a Hausdorff paracompact C^K manifold M_2 of dimension q , and a map $\pi_2 : M_1 \to M_2$ such that:

i) π_2 is a covering projection.

ii) the fibres of π_2 are the equivalence classes for the weak indistinguishability relation $\sim$ on M_1 .

iii) $\pi_2 \circ \pi_1$ is a fibre map, and its fibres are the equivalence classes for the equivalence relation R on M $xR_y \iff \phi \varepsilon E_D \ \phi(x) = \phi(y)$, i.e. $xR_y \Rightarrow x,y$ are weakly indistinguishable.

<u>Proof:</u>

iii) is a consequence of i) and ii) and the previous remarks.

Let $M_2 = M_1/\sim$ and π_2 the natural projection.

Take $[[x]] \in M_2$ and $[x] \in M_1$ such that $\pi_2([x]) = [[x]]$. Construct around $[x]$ a chart (U_1, ϕ^1) as in proposition 4.

Then any two points in U_1 are weakly indistinguishable, and therefore $\pi_2|_{U_1}$ is a bijection onto its image.

Let $U_2 = \pi_2(U_1)$. Then $(U_2, \phi^1 o(\pi_2|_{U_1})^{-1})$ is a chart for M_2 ; it is clear that in this way we can construct a C^K structure in M_2 , for which π_2 is a local diffeomorphism; since the induced topology is the quotient topology, M_2 is paracompact.

If we prove π_2 is a covering projection, then it results, as in proposition 4, that M_2 is Hausdorff.

Now, noting that the flows of the vector fields in D_1 take equivalence classes into equivalence classes, i.e. fibres of π_2 , we can use the same reasoning as in the proof of proposition 4 to show that every point $[[x]]$ has an open nhd V_2 such that

$$\begin{tikzcd} V_1 = \pi_2^{-1}(V_2) \arrow[r, "\Theta_1"] \arrow[d, "\pi_2"'] & V_2 \times \pi_2^{-1}([[x]]) \arrow[dl, "p_1"] \\ V_2 & \end{tikzcd}$$

$\pi_2^{-1}([[x]])$ is a discrete space, i.e., its tangent space is $\{0\}$ at every point.

This shows that π_2 is a covering projection, and the proof is complete.

Theorem II.2.7:

Let Σ be a complete system. Then we can construct a weakly observable complete system Σ_2, such that Σ and Σ_2 are strongly equivalent.

Proof:

The proof is in any way the analogue of the one of theorem 5.

3 - Observability and local observability

We assume Σ is a complete C^K system, as previously.

Σ is said to satisfy the observability rank condition at $x \in M$ if $\dim \mathcal{H}(D)(x) = n$.

Theorem II.3.1:

Σ is locally weakly observable iff $\mathcal{P}(D)$ has dimension n.

Proof:

If $\mathcal{P}(D)$ has dimension less than n we can construct, as in the previous chaper, a strongly equivalent system of lower dimension. Then any two points in the same fibre are weakly indistinguishable, and so

the original system is not locally weakly observable (note that the fibre is at least one dimensional).

On the other hand, if $\dim \mathcal{P}(D)(x) = n$ we can take $\phi_1,\ldots,\phi_n$ E_D such that $\{d\phi_i\}_{i=1}^n$ span $\mathcal{P}(D)(x)$ and T_x^*M .

By the inverse function theorem, $\phi:M \to \mathbb{R}^n$ given by $\phi = (\phi_1,\ldots,\phi_n)$ is a diffeomorphism on some neighbourhood U of x ; then if $x_1,x_2 \in U$ we can find ϕ_i such that $\phi_i(x_1) \neq \phi_i(x_2)$.

Since in our definition of weak indistinguishability we can substitute E_D for G_D^*h , it follows that no two points in U are weakly indistinguishable. As the construction can be made for any $x \in M$, Σ is locally weakly observable.

Theorem II.3.2 [5]:

If Σ satisfies the observability rank condition at $x \in M$, then Σ is strictly locally observable at x .

Theorem II.3.3 [5]:

If Σ is strictly locally observable, the observability rank condition is satisfied on an open dense subset of M .

As when considering accessibility, we have the following:

Theorem II.3.4:

If $\mathcal{H}(D)$ has constant dimension, in particular when Σ is analytic,

local observability, local weak observability and strict local observability are equivalent.

Proof:

Let $\Delta = \mathrm{Ker}\mathcal{H}(D)$; if $\mathcal{H}(D)$ is constant dimensional so is Δ . Now, following [9], Δ is D-invariant if and only if, for every locally defined vector field Y contained in Δ and every $X \in D$ we have $[X,Y]$ contained in Δ (note that constant dimensionality is essential).

Take $\phi \in F_D$; then $d\phi[X,Y] = X.(d\phi Y) - Y.(d\phi X) - dd\phi(X,Y) = 0$, and so $[X,Y]$ is in Δ .

If Δ is D-invariant, so is $\mathcal{H}(D)$ and then $\mathcal{P}(D) = \mathcal{H}(D)$.

This proves the equivalence stated above. The proof of theorem 3.12 in [5] shows that if the system is analytic $\mathcal{H}(D)$ is constant dimensional.

4 - Duality

Comparing the results of the previous section with those in I.3 we can see we have some kind of duality, in the sense that theorems II.3.1, II.3.2, II.3.3, II.3.4 can be obtained from the ones in the first chapter (and vice-versa) using a table of correspondences as follows:

controllability rank condition	observability rank condition
(I.3.2)$\Downarrow$ $\Uparrow$(I.3.3)	(II.3.2)$\Downarrow$ $\Uparrow$(II.3.3)
local accessibility	strict local observability
$\Downarrow$ $\Uparrow$(I.3.4)	$\Downarrow$ $\Uparrow$(II.3.4)
accessibility	local observability
$\Downarrow$ $\Uparrow$(I.3.4)	$\Downarrow$ $\Uparrow$(II.3.4)
reachability	local weak observability
$\Updownarrow$(I.3.1)	$\Updownarrow$(II.3.1)
$\dim P_D = n$	$\dim \mathcal{P}(D) = n$

Note that $\mathcal{P}(D)$ and $\mathcal{H}(D)$, apart from having dual rotes of P_D and $\mathcal{T}(D)$ as shown above, have dual constructions, in the sense that:

$\mathcal{T}(D)$ $(\mathcal{H}(D))$ is the (co)distribution spanned by all the Lie derivatives with respect to vector fields in D of vector fields in D (of differentials of the output map components).

$P_D(\mathcal{P}(D))$ is the smallest D-invariant (co)distribution containing D (dh).

Thus, in a certain sense, the vector fields in D are "dual" of the differentials of the components of the output map.

As another case of duality we have the following result:

Proposition II.4.1:

Whenever $S_D = G_D$, controllability and reachability are equivalent, and so are weak observability and observability. In particular, that is true for systems defined on compact Lie groups.

5 - Examples

Example II.5.1: Consider the system defined on $\mathbb{R}^2$ the family of vector fields $D = \{\frac{\partial}{\partial x}, \phi(x)\frac{\partial}{\partial y}\}$, where $\phi(x) = e^{-1/x^2}$ for $x \leq 0$, and $\phi(x) = 0$ for $x > 0$, and $h : \mathbb{R}^2 \to \mathbb{R}$, $h(x,y) = \sin x$.

This system is reachable, but not accessible, and it is not weakly observable, since any two points in $\mathbb{R}^2$ with the same first coordinate are weakly indistinguishable.

If we construct $\mathcal{H}(D)$ we see it is spanned by $\cos x\, dx$, i.e., it is not constant dimensional, since it has dimension 1 on $\mathbb{R}^2 - A$, where $A = \{(x,y) \in \mathbb{R}^2 , x = \pi/2 + k$ with $k \in Z\}$, and dimension zero on A . On the other hand $\mathcal{P}(D)$ is spanned by dx , and has constant dimension, equal to 1.

The construction in proposition 2.4 gives rise to the system Σ_1 , define on $\mathbb{R}$, with $D_1 = \{\frac{\partial}{\partial x}\}$ and $h_1 : \mathbb{R} \to \mathbb{R}$, $h_1(x) = \sin x$. This new system is accessible, and locally observable : we can distinguish any two points x_1, x_2 in positive turn, if $x_1 - x_2 \neq 2K\pi$ with $K \in \mathbb{Z}$.

The construction in proposition 2.6 gives the system Σ_2 , defined on $M_2 = S^1$, with D_2 consisting of the unit tangent vector, and

$h_2 : S^1 \to \mathbb{R}$ given by the sinus. This system is accessible and observable.

The maps π_1 and π_2 are the usual projections $\mathbb{R}^2 \to \mathbb{R}$ and $\mathbb{R} \to S^1$, respectively.

Example II.5.2: In general, the final system Σ_2 we obtain is not accessible or observable.

Let Σ be defined on $\mathbb{R}^3$ by $D = \{-\frac{\partial}{\partial x}, \phi(-x)\frac{\partial}{\partial y}, \frac{\partial}{\partial z}\}$ with ϕ as above, and $h:\mathbb{R}^3 \to \mathbb{R}^2$, $h(x,y,z) = (\phi(-x),y)$. Then $\Sigma_1 = \Sigma_2$ is defined on $\mathbb{R}^2$ by $D_1 = D_2 = \{-\frac{\partial}{\partial x}, \phi(-x)\frac{\partial}{\partial y}\}$ and $h_1 = h_2 : \mathbb{R}^2 \to \mathbb{R}^2$ the map $(x,y) \to (\phi(-x),y)$.

Σ_1 is not accessible, because the set of points attainable from (x,y) with $x < 0$ is $\{(x',y'), x' \leq x \text{ and } y' = y\}$, which has empty interior in any nhd of (x,y). But it is reachable.

Also, it is not observable, since the points (x_1,y) and (x_2,y) with $x_1,x_2 < 0$ are indistinguishable, but it is weakly observable because that type of pair can be distinguished if we allow the negative trajectories of $-\frac{\partial}{\partial x}$.

6 - Minimal Realizations

Given an initialized system (Σ,x), i.e. a pair where Σ is a system and x some point in M, its input-output map $\mu(\Sigma,x)$ is the map from $\mathcal{U}$ into the set of curves in N defined as follows if $u \in \mathcal{U}$, with domain $[0,T]$, $\mu(\Sigma,x)(u)$ is the curve $t \to h\circ\pi(x,u;t)$, which has a domain $\mathcal{J}$ such that $0 \in \mathcal{J} \subset [0,T]$.

An input output map, with inputs $\mathcal{U}$ and outputs in N, is a map μ such that, if $u \in \mathcal{U}$ is as above, $\mu(u)$ is some curve in N, with domain $\mathcal{J} \subset [0,T]$ and $0 \in \mathcal{J}$.

We say two input-output maps are equivalent, $\mu \simeq \nu$, if for every $u \in \mathcal{U}$, $\mu(u)$ and $\nu(u)$ coincide in the common domain of definition.

An initialized system (Σ,x) is a realization of the input-output map μ if $\mu(\Sigma_1 x) \simeq \mu$. A realization is called minimal if Σ is reachable and weakly observable, and quasi-minimal if Σ is reachable and locally weakly observable.

Given two systems Σ_1, Σ_2 we say $F:M_1 \to M_2$ is an isomorphism of systems if

i) F is a C^K diffeomorphism

ii) $f_2 = F_* f_1$

iii) $h_1 = F^* h_2$

F is an isomorphism of initialized systems (Σ_1,x_1), (Σ_2,x_2) if it is an isomorphism of the systems Σ_1 and Σ_2, and moreover $F(x_1) = x_2$.

<u>Theorem II . 6.1:</u>

Given a complete initialized system (Σ,x) there exist a minimal

and a quasiminimal equivalent complete realization. Moreover the complete minimal realizations are unique up to isomorphism.

Proof:

To prove existence, we first obtain a reachable realization of (Σ,x) : if S is the orbit of x under D , the system (Σ_S,x) , obtained by restriction to S , is an equivalent reachable complete realization.

Now if Σ_1 and Σ_2 are the locally weakly observable, respectively weakly observable, systems obtained from Σ by using the constructions of proposition 2.4 and 2.5, $(\Sigma_1, \pi_1(x))$ and $(\Sigma_2, \pi_2(x))$ are the required realizations.

The uniqueness of the complete minimal realizations has been proved by Sussmann [12]. Of course we need a slight adaptation, in view of the differences in the definitions.

CHAPTER III - GRADIENT SYSTEMS

We discuss a conjecture presented by P. Varaya [15], and then proved for the linear case : an isomorphism of two locally controllable gradient systems in an isometry of the underlying pseudo-Riemannian manifolds.

In the first section we present the basic definitions and results, extending the class of systems to allow for the non-symmetric case.

In section two we prove:

i) the conjecture is false (example 2.4), but some positive results can be obtained, with different kinds of conditions.

ii) the above conjecture is true for Hamiltonian systems, assuming only reachability instead of local controllability (theorem 2.3).

Section three is the statement and proof of a positive result for "bilinear" gradient systems.

Following [15], we consider a nonlinear RLC electrical network, with k_1 capacitors, k_2 inductors, s_1 current sources and s_2 voltage sources.

Let $q:\mathbb{R} \to \mathbb{R}^{k_1}$ be the change on the capacitors as function of time, and $\phi:\mathbb{R} \to \mathbb{R}^{k_2}$ be the fluxes through the inductors. We denote by $i:\mathbb{R} \to \mathbb{R}^{k_1} \times \mathbb{R}^{k_2} \times \mathbb{R}^{s_1} \times \mathbb{R}^{s_2}$ and $v:\mathbb{R} \to \mathbb{R}^{k_1} \times \mathbb{R}^{k_2} \times \mathbb{R}^{s_1} \times \mathbb{R}^{s_2}$ the current and voltage.

Assuming the capacitors are voltage controlled and the inductors current-controlled, we have

$$q_\alpha(t) = \bar{q}_\alpha(v_\alpha(t)) \qquad \alpha = 1,\ldots,k_1$$

$$\phi_\beta(t) = \bar{\phi}_\beta(i_{\beta+k_1}(t)) \qquad \beta = 1,\ldots,k_2$$

for some $\bar{q}:\mathbb{R}^{k_1} \to \mathbb{R}^{k_1}$ and $\bar{\phi}:\mathbb{R}^{k_2} \to \mathbb{R}^{k_2}$.

By Maxwell's equations

$$i_\alpha(t) = \frac{d}{dt} q_\alpha(t) = \frac{d}{dv_\alpha} \bar{q}_\alpha(v_\alpha(t)) \; \dot{v}_\alpha(t)$$

$$\dot{v}_{k_1+\beta}(t) = \frac{d}{dt} \phi_\beta(t) = \frac{d}{di_{\beta+k_1}} \bar{\phi}_\beta(i_{\beta+k_1}(t)) \quad i_{\beta+k_1}(t)$$

or denoting $\frac{d}{dv_\alpha} \bar{q}_\alpha$ by C_α , and $\frac{d}{di_{\beta+k_1}} \bar{\phi}_\beta$ by L_β .

$$i_\alpha(t) = C_\alpha(v_\alpha(t))\ \dot{v}_\alpha(t) \qquad \alpha = 1,\ldots,k_1$$

$$v_{k_1+\beta}(t) = L_\beta(i_{\beta+k_1}(t))\ \dot{i}_{k_1+\beta}(t) \qquad \beta = 1,\ldots,k_2 \quad .$$

We assume C_α and L_β are positive. We can think of the network as converting

$$z \overset{T}{=} (v_1,\ldots,v_{k_1}, -i_{k_1+1},\ldots, -i_{k_1+k_2}, i_{k_1+k_2+1},\ldots,i_{k_1+k_2+s_1}, v_{k_1+k_2+s_1+1},\ldots,v_{k_1+k_2+s_1+s_2})$$

into

$$w = (-i_1,\ldots,-i_{k_1}, v_{k_1+1},\ldots,v_{k_1+k_1}, v_{k_1+k_2+1},\ldots,v_{k_1+k_2+s_1}, i_{k_1+k_2+s_1+1},\ldots,i_{k_1+k_2+s_1+s_2})$$

and we assume there is a map $f:\mathbb{R}^{k_1} \times \mathbb{R}^{k_2} \times \mathbb{R}^{s_1} \times \mathbb{R}^{s_2} \to \mathbb{R}^{k_1} \times \mathbb{R}^{k_2} \times \mathbb{R}^{s_1} \times \mathbb{R}^{s_2}$ making the conversion.

If the network is reciprocal, i.e. $\frac{\partial f}{\partial z}$ is a symmetric matrix, there exists a map $P:\mathbb{R}^{k_1} \times \mathbb{R}^{k_2} \times \mathbb{R}^{s_1} \times \mathbb{R}^{s_2} \to \mathbb{R}$ such that $f = \frac{\partial}{\partial z} P$. We can then write

$$-i_\alpha = -C_\alpha(v_\alpha(t))\dot{v}_\alpha(t) = \frac{\partial}{\partial z_\alpha} P \qquad \alpha = 1,\ldots,k_1$$

$$v_{k_1+\beta} = L_\beta(i_{k_1+\beta}(t)\ \dot{i}_{k_1+\beta}(t) = \frac{\partial}{\partial z_{k_1+\beta}} P \qquad \beta = 1,\ldots,k_2$$

$$v_{k_1+k_2+\gamma} = \frac{\partial}{\partial z_{k_1+k_2+\gamma}} P \qquad \gamma = 1,\ldots,s_1$$

$$i_{k_1+k_2+s_1+\delta} = \frac{\partial}{\partial z_{k_1+k_2+s_1+\delta}} P \qquad \delta = 1,\ldots,s_2 \ .$$

If we take $x \in R^{k_1+k_2}$ as $x = (v_1,\ldots,v_{k_1},i_{k_1+1},\ldots,i_{k_1+k_2})$, $u \in R^{s_1+s_2}$ as $u = (i_{k_1+k_2+1},\ldots,$ [illegible] $,\ldots,i_{k_1+k_2+s_1},v_{k_1+k_2+s_1+1},\ldots,$ $v_{k_1+k_2+s_1+s_2})$ and $y \in \mathbb{R}^{s_1+s_2}$ as $y = (v_{k_1+k_2+1},\ldots,v_{k_1+k_2+s_1},$ $i_{k_1+k_2+s_1+1},\ldots,i_{k_1+k_2+s_1+s_2})$ the above equations can be written:

$$A(x)\dot{x} = \frac{\partial}{\partial x} P(x,u)$$

$$y = \frac{\partial}{\partial u} P(x,u)$$

where $A = \text{diag}\ (-C_1,\ldots,-C_{k_1},L_1,\ldots,L_{k_2})$.

We can generalize the above model assuming it is only a local description, and allowing for mutual inductance we substitute a diagonal matrix by a non-singular symmetric matrix.

Then we can take the state space as M , a k_1+k_2 dimensional

matrix, ξ as a nondegenerate symmetric bilinear tensor field on M, and assuming P is globally defined we have

$$\begin{cases} \dot{x} = f(x,u) \\ y = h(x,u) \end{cases}$$

where

$$\begin{tikzcd} M \times \mathbb{R}^{\ell} \arrow[rr, "f"] \arrow[dr, "p_1"'] & & TM \arrow[dl] \\ & M & \end{tikzcd}$$

is a commutative diagram, $f(\cdot,u) \lrcorner \xi = dP(\cdot,u)$ and $h(x,u) := \frac{\partial}{\partial u} P(x,u)$.

Systems of this form, with ξ non-degenerate but not necessarily symmetric, will be called gradient systems, and if ξ is a symplectic form, i.e. skew symmetric and closed, the system will be said to be Hamiltonian.

1 - Gradient systems

Gradient systems are defined on C^k manifolds having additional structure, namely a non-singular bilinear tensor field ξ . This means that at every $x \in M$, $\xi(x)$ is a non-singular bilinear map $T_xM \times T_xM \to R$.

If $z:M \to R$ is a C^K map, we can define $\text{grad}_\xi z$ as the vector field X such that $\xi(x)\ (X(x),v) = dz(x)\cdot v$, for every $x \in M$ and $v \in T_xM$, or equivalently $X \lrcorner \xi = dz$.

The systems we will consider are defined as follows. Let $P:M \times \mathbb{R}^m \to \mathbb{R}$ be a C^K map, and denote by P_u the map $x \mapsto P(x,u)$. Define $f(x,u) = \text{grad}_\xi P_u(x)$.

The control system will be $\Sigma = (M,R^m,f,\mathcal{U}_{pc})$. Some difficulty arises with the definition of the output space and output map. We want the output to be $y(x,u) = \frac{\partial}{\partial u} P(x,u)$; if we consider it as being a family of maps $\{y_u\}_{u \in R^m}$ with $y_u:M \to R^m$, $y_u(x) = y(x,u)$ we see that the output space is not finite dimensional.

The results of the previous chapter can be carried through, if we adopt the following definition of indistinguishability: two points x_1,x_2 in M are (weakly) indistinguishable if for every $\gamma \in S_D$ $(\gamma \in G_D)$ such that γx_1 and γx_2 are defined, we have $y_u \circ \gamma\, x_1 = y_u \circ \gamma\, x_2$ for every $u \in \mathbb{R}^m$.

It is clear that the results in chapter II do not depend on the system

having just one output map or having a family of output maps. Thus our system will be $\Sigma = (M,\mathbb{R}^m,f,\mathcal{U}_{pc},\mathbb{R}^m,\{y_u\})$.

Thus give an intuitive meaning to the above definition of indistinguishability we can prove:

<u>Lemma III.1.1</u>: x_1 and x_2 are indistinguishable iff, for any $u \in \mathcal{U}_{pc}$, $y(\pi(x_1,u;t),u(t)) = y(\pi(x_2,u;t), u(t))$, whenever t is in the common domain of definition of $\pi(x_1,u;\cdot)$ and $\pi(x_2,u;\cdot)$.

<u>Proof</u>:

It is clear that indistinguishability implies the above result: for simplicity assume u assumes two values, u_1 in $[o,t_1]$ and u_2 in $]t_1,t_2]$.

Then for $t \in [o,t_1]$, $y(\pi(x_i,u;t),u(t)) = y_{u_1}(\pi(x_i,u;t))$ and thus the property is verified on $[o,t_1]$. Similarly, if $t \in]t_1,t_2]$ $y(\pi(x_i,u;t) , u(t)) = y_{u_2}(\pi(x_i,u;t))$ and our claim follows. Clearly the reasoning is independent of the number of values u assumes.

Consider $\gamma \in S_D$ and let $v \in \mathcal{U}_{pc}$ be the corresponding control, defined on $[o,T]$. Consider the control v' such that $v' = v$ on $[o,T[$ and $v' = u$ on $[T,T']$.

By assumption $y(\pi(x_1;v',t),v'(t)) = y(\pi(x_2,v',t), v'(t))$ for every $t \in [o,T']$. Then on $[T,T']$ we have $y_u(\pi(x_1,v';t) = y_u(\pi(x_2,v';t)$ and

by continuity, if we let $t \to T$ we get $y_u(\gamma x_1) = y_u(\gamma x_2)$.

Since u can be chosen arbitrarily, this concludes the proof.

The two main cases we are interested in are: i) ξ is symmetric; ii) ξ is skew-symmetric, thus a 2-form.

In case ii) if we assume ξ is closed, we obtain what we will call a Hamiltonian system. We list now some properties and definitions about Hamiltonian vector fields, that are used in the sequel.

Let (M,w) be a symplectic manifold, i.e. w is a closed non-degenerate 2-form on M . A vector field X on M is said to be Hamiltonian if $X \lrcorner w = df$ for some function f on M , which we call the Hamiltonian of X .

Since w is non degenerate, given any function f we can find a vector field X_f such that $X_f \lrcorner w = df$.

<u>Proposition III.1.2:</u>

i) X is Hamiltonianian iff $L_X w = 0$.

ii) The set of Hamiltonian vector fields is a subalgebra of the Lie algebra of vector fields.

Denote by $\{f,g\} = w(X_f,X_g)$. Then we have

<u>Proposition III.1.3:</u>

i) $X_{\{f,g\}} = [X_g,X_f]$

A map $\phi:M_1 \to M_2$ between two symplectic manifolds is a symplectomorphism iff $\phi^* w_2 = w_1$. The diffeomorphisms associated to a Hamiltonian vector field are symplectomorphisms.

2 - Nonlinear equivalence

Assume we are given two equivalent gradient systems Σ_1 and Σ_2 , i.e. there exists an isomorphism Φ between them. We are interested in knowing how Φ acts on ξ_1 , more precisely, we want to study conditions under which $\xi_1 = \Phi^* \xi_2$.

There are two specially interesting cases:

i) ξ_1 and ξ_2 are pseudo-Riemannian metrics, and we want Φ to be an isometry;

ii) ξ_1 and ξ_2 are symplectic forms, i.e. are skew-symetric, thus differential 2-forms, and closed, $d\xi_1 = d\xi_2 = 0$, and then we want Φ to be a symplectomorphism.

Proposition III.2.1 [15]:

Let Σ_1, Σ_2 be two equivalent systems, and Φ an isomorphism $\Phi:M_1 \to M_2$ between them. Then if $\xi' = \Phi^* \xi_2 - \xi_1$, there exists a map $S:M_1 \to \mathbb{R}$ such that $f_1(\ ,u) \lrcorner \ \xi' = dS$, for every $u \in \mathbb{R}^m$.

Proof:

If Σ_1 and Σ_2 are equivalent, we have $\frac{\partial}{\partial u} P_1(x,u) = \frac{\partial}{\partial u} P_2(\Phi(x),u)$

and so there exists $S:M \to \mathbb{R}$ such that $P_2(\Phi(x),u) = P_1(x,u) + S(x)$.

Differentiating with respect to x , we obtain

$$\Phi^* dP_{2u} = dP_{1u} + dS = f_1(\ ,u) \lrcorner \xi_1 + dS \quad .$$

On the other hand $\Phi^* dP_{2u} = \Phi^*(f_2(\ ,u) \lrcorner \xi_2)$ and as $f_2(\ ,u) = \Phi_* f_1(\ ,u)$ we get

$$\Phi^*(\Phi_* f_1(\ ,u) \lrcorner \xi_2) = f_1(\ ,u) \lrcorner \xi_1 + dS \quad .$$

The first member is $f_1(\ ,u) \lrcorner \Phi^* \xi_2$ and so

$$f_1(\ ,u) \lrcorner \Phi^* . \xi_2 = f_1(\ ,u) \lrcorner \xi_1 + dS$$

or

$$f_1(\ ,u) \lrcorner \xi' = dS \quad .$$

A system Σ is said to be locally controllable at $x \in M$ if, given any neighbourhood U of x , we can find a neighbourhood V of x such that any point in V can be attained from x , following a trajectory not leaving U .

We remark that local controllability implies, but is not implied by, controllability.

Lemma III.2.2:

If one of the following conditions is verified, then $dS = 0$, i.e. S is constant:

i) ξ_1, ξ_2 are skew-symmetric, and Σ_1 is reachable.

ii) [15] ξ_1, ξ_2 are symmetric and Σ_1 is locally controllable.

Proof:

Assume i) is verified. Then if $X \in D_1$ we have $L_X S = \xi'(X,X) = 0$, since ξ' is skew-symmetric when both ξ_1 and ξ_2 are.

This means S is constant along trajectories of vector fields in D_1 , and as Σ_1 is reachable, S is constant.

Now suppose ii) is verified. Then $L_X S$ is independent of the vector field X in D_1 ; let $X,Y \in D_1$ then $L_Y S = dS.Y = \xi'(X,Y) = \xi'(Y,X) = dS.X = L_X S$ since the symmetry of ξ_1 and ξ_2 implies the symmetry of ξ' .

If $L_X S(x) > 0$, let U be an open neighbourhood of x on which $L_X S$ is still positive, we can find V as in the definition of local controllability.

A point $x_1 \in V$ is attainable from x_1 and therefore $S(x_1) > S(x)$. This is a contradiction, if we choose x_1 in the non-empty set $\{x' | x' \in V, S(x') = S(x)\}$.

Similarly we prove that $L_X S$ cannot be negative, therefore $L_X S = 0$ for every $X \in D_1$. As local controllability clearly implies local accessibility and therefore reachability, we can conclude, as above, that S is constant.

Theorem III.2.3:

Let Σ_1 and Σ_2 be equivalent Hamiltonian systems; then if Σ_1 is reachable, Φ is a symplectomorphism.

Proof:

Let $\gamma_1 \in G_{D_1}$; since all elements of G_{D_1} are symplectomorphisms, $\gamma_1^* w_1 = w_1$.

As Σ_1 and Σ_2 are equivalent, there exists γ_2 in G_{D_2} such that $\Phi o \gamma_1 = \gamma_2 o \Phi$: Thus we have $\gamma_1^* \Phi^* w_2 = (\Phi o \gamma_1)^* w_2 = (\gamma_2 o \Phi)^* w_2 = \Phi^* \gamma_2^* w_2 = \Phi^* w_2$ and therefore $\gamma_1^* w' = w'$.

From lemma 2, we have $X \lrcorner w' = 0$ for every $X \in D_1$. If $\gamma_1 \in G_{D_1}$, $((\gamma_{1*}X) \lrcorner w')(x).v = w'(x)\ (\gamma_{1*}X(\gamma_1^{-1}(x)),v) =$
$= w'(\gamma_1^{-1}(x))\ (X(\gamma_1^{-1}(x)),\ \gamma_{1*}^{-1}v) = 0$, and thus $\gamma_{1*}X \lrcorner w' = 0$.

Since the system Σ_1 is reachable, the set of vector fields of the form $\gamma_{1*}X$ as above spans the tangent space of M_1 at every point, therefore $w' \equiv 0$, and the result is proved.

We would like to have a result as above for the symmetric case, with the condition ii) in lemma 2, as it was conjectured in [15], but

the following example shows that is impossible:

Example III.2.4:

Let $M_1 = \mathbb{R}^4$, and define $P_1(x_1,x_2,x_3,x_4,u_1,u_2) = u_1x_1 + u_2\,(x_2+x_3+x_4)$. Let $\xi_1(x) = \mathrm{diag}\,(1,e^{-x_4},e^{-x_1},e^{-x_3})$.

An easy computation shows that the vector fields in D_1 are linear combinations of the vector fields $X^1 = \frac{\partial}{\partial x_1}$ $(u_1 = 1,\ u_2 = 0)$ and $X^2 = e^{x_4}\frac{\partial}{\partial x_2} + e^{x_1}\frac{\partial}{\partial x_3} + e^{x_3}\frac{\partial}{\partial x_4}$ $(u_1 = 0,\ u_2 = 1)$.

Then $X_3^3 = [X^1,X^2] = e^{x_1}\frac{\partial}{\partial x_3}$ and $X^4 = [X^3,X^2] = e^{x_1+x_3}\frac{\partial}{\partial x_4}$. This shows that $\mathcal{T}(D_1)$ spans the tangent space of M_1 at every point, so the system is locally accessible, and as it is symmetric, it is locally controllable.

The output functions are $y_1 = \frac{\partial P}{\partial u_1} = x_1$ and $y_2 = \frac{\partial P}{\partial u_2} = x_2+x_3+x_4$. Take $X^5 = [X^4,X^2] = e^{x_4(x_1+x_3)}\frac{\partial}{\partial x_2} - e^{(x_1+x_3)}\,e^{x_1}\frac{\partial}{\partial x_3}$.

Then $dy_1 = (1,0,0,0)$, $dy_2 = (0,1,1,1)$ in the basis (dx_1,dx_2,dx_3,dx_4) , and so we have $d(dy_2.X^2) = (e^{x_1},0,e^{x_3},0)$ and $d(dy_2.X^5) = (e^{x_1+x_3}(e^{x_4}-2e^{x_1})\ ,\ 0\ ,\ (x_4-x_1)e^{x_3(x_4-x_1)}\ ;\ x_1e^{x_1x_4})$.

Taking $x_1 = x_3 = x_4 = 1$, the corresponding covectors

$$\begin{matrix} (1 & 0 & 0 & 0) \\ (0 & 1 & 1 & 1) \\ (e & 0 & e & 0) \\ (-e^3 & 0 & 0 & e) \end{matrix}$$

and linearly independent. As the system is analytic and accessible, that means the observability rank condition is verified everywhere, and Σ_1 is observable.

Now consider $M_2 = M_1$, $P_2 = P_1$ and let ξ_2 be defined by

$$\xi_2(x) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & e^{-x_4} & 0 & 0 \\ 0 & 0 & e^{-x_1}+e^{x_3} & -e^{x_1} \\ 0 & 0 & -e^{x_1} & e^{-x_3}(1+e^{2x_1}) \end{bmatrix}$$

Computation shows us that this gives rise to the same system, more precisely the identity is an isomorphism between Σ_1 and Σ_2 , but $\xi' = \xi_2 - \xi_1 \neq 0$.

Σ_1 is analytic complete, locally controllable, observable, and therefore we cannot expect to prove any positive result without further assumptions.

We remark that in the skew-symmetric case we had to assume the 2-form to be closed, in theorem 3, and then $L_X w = 0$ for every $X \in D$.

If ξ is a (pseudo) Riemannian metric on M , and $L_X \xi = 0$, X is called a Killing vector field, and we can prove:

<u>Theorem III.2.5</u>:

Let Σ_1 and Σ_2 be equivalent gradient systems, ξ_1 and ξ_2

(pseudo) Riemannian metrics, and assume the elements of D_i are Killing vector fields for ξ_i. Then, if Σ_1 is locally controllable, Φ is an isometry.

Proof:

If $L_X\xi = 0$ and γ is a diffeomorphism associated to X, then $\gamma^*\xi = \xi$ i.e. γ is an isometry. This was the condition involved in the proof of theorem 3 and since local controllability implies reachability, we have just to reproduce the argument of that proof to get our result.

In his Ph.D. thesis [15], Verma states the following:

Theorem III.2.6:

Assume $M_1 = M_2 = \mathbb{R}^n$, and $\xi_1 = \xi_2$ is the usual Riemannian metric. If Σ_1 is locally controllable and

i) these systems have the form

$$\dot{x} = Fg(F^T x) + FBu$$
$$y = (FB)^T x$$

where F and B are matrices, and $g:\mathbb{R}^n \to \mathbb{R}^n$ is such that g_i depends only of x_i, i.e. $\frac{\partial g_i}{\partial x_j} = 0$ if $i \neq j$.

ii) $\mathcal{F}(D_i)$ can be generated as a vector space by the associated vector fields and Lie brackets of them in which at least one of the vectors is constant.

Then Φ is an isometry.

When trying to generalize this result, we found that its proof is (at least) uncomplete, since it fails to consider the case in which the constant vector field is not an input vector field. The proof uses the fact that Φ takes a constant vector field into another constant vector field, which is true for input vector fields, but is not proved in general.

If instead of condition ii) we assume a stronger condition avoiding that situation, we can obtain:

Theorem III.2.7:

Σ_1 is locally controllable, and Σ_1, Σ_2 are such that:

i) $f_i(.,u) = X_i^0 + \sum_{j=1}^{e} u_j X_i^j$, where X_i^r are C^K vector fields, and

$$\nabla^i X_i^j = 0 \ , \quad j = 1,2,\ldots,\ell$$

where ∇^i is the Levi-Civita connection associated to ξ_i .

ii) $\mathcal{T}(D_i)$ can be generated as a vector space by the associated vector fields and by successive Lie bracketting with input vector fields.

Then Φ is an isometry.

Proof:

If (M,ξ) is a manifold, and ∇ its Levi-Civita connection, we have

$$2\xi(\nabla_X Y,Z) = X.\xi(Y,Z) + Y.\xi(X,Z) - Z.\xi(X.Y) +$$
$$+ \xi([Z,X],Y) + \xi([Z,Y],X) + \xi([X,Y],Z) \quad .$$

Also $X \lrcorner \xi' = 0$ for any vector field $X \in D_1$, we shall prove by induction that the same is true for the vector fields in $\mathcal{T}(D_1)$ generating it, as in ii).

The induction is on the order of the brackets: it is already verified for order zero. Now assume it has been proved up to order p .

Let $X \in \mathcal{T}(D_1)$ be a vector field of the considered type of order $p + 1$, i.e. $X = [Y,Z]$ where Y is an input vector field and Z is the order p .

We have

$$Y \lrcorner \xi' = 0 \quad \text{(local controllability)}$$
$$Z \lrcorner \xi' = 0 \quad \text{(induction)}$$

and also $\nabla^1_Z Y = 0$ since $\nabla^1 Y = 0$.

Then, by the above formula:

$$\begin{aligned}
\xi_1(X,A) &= Z.\xi_1(Y,A) + Y.\xi_1(Z,A) - A.\xi_1(Y,Z) + \\
&\quad + \xi_1([A,Y],Z) + \xi_1([A,Z],Y) \\
&= \Phi^*(\Phi_* Z.\xi_2(\Phi_* Y,\Phi_* A) + \Phi_* Y.\xi_2(\Phi_* Z,\Phi_* A) \\
&\quad - \Phi_* A.\xi_2(\Phi_* Y,\Phi_* Z) + \xi_2([\Phi_* A,\Phi_* Y],\Phi_* Z) \\
&\quad + \xi_2([\Phi_* A,\Phi_* Z],\Phi_* Y) \quad .
\end{aligned}$$

As $\nabla^2_{\Phi_* 2}\, \Phi_* Y = 0$ since $\nabla^2\, \Phi_* Y = 0$ (note $\Phi_* Y$ is an input vector field for Σ_1) we have

$$\xi_1(X,A) = \Phi^* \xi_2([\Phi_* Y, \Phi_* Z], \Phi_* A) =$$
$$= \Phi^* \xi_2(\Phi_* X, \Phi_* A) \quad .$$

As A is an arbitrary vector field, this means $X \lrcorner\, \xi' = 0$, and the proof is complete.

Instead of assuming certain properties of the family of associated vector fields, or relations of them with the pseudo Riemannian metric, we can assume convenient properties of the diffeomorphism Φ in order to obtain positive results.

<u>Theorem III.2.8</u>:

If Φ is an affine diffeomorphism, i.e. if $\Phi_* \nabla^1_X Y = \nabla^2_{\Phi_* X}\, \Phi_* Y$ and Σ_1 is strictly locally observable, Φ is an isometry.

<u>Proof</u>:

If Σ_1 is strictly locally observable, the observability rank condition is satisfied on an open dense subset of M . As will be seen from the proof, we can assume it is satisfied everywhere.

Let $\mathscr{D}_1$ be the family of vector fields which are gradient of functions in F_{D_1} ; then, since we are assuming $\mathscr{H}(D_1)$ spans the cotangent space at every point, $\mathscr{D}_1$ spans the tangent space at every point.

From proposition III.2.1 and taking derivatives with respect to u, we have $X \lrcorner \xi' = 0$ for every X which is the gradient in (M_1, ξ_1) of a component of the output function.

The other vector fields in $\mathcal{D}_1$ are gradient of Lie derivatives of components of output maps with respect to vector fields in D_1. We can prove that $Y \lrcorner \xi' = 0$ for every $Y \in \mathcal{D}_1$ by induction on the order of the Lie derivatives.

The above remark proves the case of 0-th order derivatives. Assume we have proved our claim for vector fields gradient of Lie derivatives of order p.

Let $Y = \mathrm{grad}_{\xi_1} L_X\phi$, where ϕ is a Lie derivative of order p, and denote by Z the gradient of ϕ.

Claim, $Y = \nabla^1_X Z + \nabla^1_Z X$.

Denote the second member by Y'; then we have, from the formula stated in the beginning of the proof of the previous theorem:

$$2\xi_1(Y',A) = 2(X.\xi_1(Z,A) + Z.\xi_1(X,A) - A.\xi_1(X,Z) + \\ + \xi_1([A,X],Z) + \xi_1([A,Z],X))$$

by just substitution, for any vector field A.

Now

$$\xi_1([A,X],Z) = d\phi[A,X] = A(d\phi.X) - X.(d\phi A)$$

and if $X = \mathrm{grad}_{\xi_1} \eta$

$$\xi_1([A,Z],X) = d\eta|A,Z| = A.(d\eta.Z) - Z.(d\eta.A) \quad .$$

By substitution

$$\xi_1(Y',A) = -A\ \xi_1(Z,X) + A.\xi_1(Z,X) + A\ \xi_1(X,Z)$$
$$= A.\xi_1(X,Z)$$

since $X.(d\phi A) = X\ \xi_1(Z,A)$ and $Z(d\eta.A) = Z.\xi_1(X,A)$.

Thus we have proved:

$$Y' \lrcorner\ \xi_1 = d\xi_1(X,Z)$$

and since $\xi_1(X,Z) = d\phi.X = L_X\phi$, we have

$$Y' \lrcorner\ \xi_1 = dL_X\phi$$

and Y' is the gradient of $L_X\phi$, therefore $Y = Y'$.

From the equivalence of Σ_1 and Σ_1 we have $\Phi^* L_{\Phi_* X}(\Phi^{-1})^* \phi = L_X\phi$.

Consider the vector field $\Phi_* Y$.

Since Φ preserves the covariant derivative, we have

$$\Phi_* Y = \nabla^2_{\Phi_* X}\ \Phi_* Z + \nabla^2_{\Phi_* Z}\ \Phi_* X$$

and therefore

$$\Phi_* Y \lrcorner\ \xi_2 = d\ \xi_2(\Phi_* X, \Phi_* Z) \quad .$$

As $Z \lrcorner \xi' = 0$ by the induction hypothesis, we have

$$\Phi_* Y \lrcorner \xi_2 = d(\Phi^{-1})^* \xi_1(X,Z) = (\Phi^{-1})^* d\xi_1(X,Z)$$
$$= (\Phi^{-1})^* (Y \lrcorner \xi_1)$$

or equivalently

$$\Phi^*(\Phi_* Y \lrcorner \xi_2) = Y \lrcorner \xi_1$$
$$Y \lrcorner \Phi^* \xi_2 = Y \lrcorner \xi_1 \quad .$$

Thus $Y \lrcorner \xi' = 0$, and this will be true for every Y in $\mathcal{D}_1$. As they span all the tangent space, $\xi' = 0$ i. e. Φ is an isometry.

We can look for other conditions by making assumptions on the metrics and type of diffeomorphisms allowed: consider gradient systems of the form

$$f_i(x_i)\dot{x}_i = \frac{\partial P}{\partial x_i}(x,u)$$
$$y_j(x,u) = \frac{\partial P}{\partial u_j}(x,u)$$

defined on $M = \mathbb{R}^n$, and restrict the diffeomorphisms $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ to have the form

$$\Phi(x_1,\ldots,x_n) = (\phi_1(x_1),\ldots,\phi_n(x_n)) \quad .$$

Working inside this class of systems we obtain:

Proposition III.2.9:

Assume Σ_1, Σ_2, Φ are as above, and Σ_1 is locally controllable. Then Φ is an isometry.

Proof:

We have $X \lrcorner \xi' = 0$, for every X in D_1, or in coordinates

$$\xi'_{i_j}(x)\, X_j(x) \equiv 0 \quad .$$

Now, if $Z = [X,Y]$, where $X,Y \in D_1$, we have

$$Z \lrcorner \xi' = [X,Y] \lrcorner \xi'$$

which we can write in coordinates as

$$\xi'_{ij} \frac{\partial}{\partial x_K}(Y_j)X_K - \xi'_{ij} \frac{\partial}{\partial x_K}(X_j)Y_K$$

for the i-th component, summing over K, and j .

From the form of ξ_1, ξ_2 and Φ, it is clear that

$i \neq j \Rightarrow (\xi'_{ij} \equiv 0$, and $\frac{\partial}{\partial x_j} \xi'_{ii} \equiv 0)$.

Therefore, we have

$$(Z \lrcorner \xi')_i = \xi'_{ii} \frac{\partial}{\partial x_K}(Y_i)X_K - \xi'_{ii} \frac{\partial}{\partial x_K}(X_i)Y_K =$$

$$= \frac{\partial}{\partial x_K}(\xi'_{ii}Y_i)X_K - \frac{d}{dx_i}(\xi'_{ii})Y_i\ X_i -$$

$$- \frac{\partial}{\partial x_K}(\xi'_{ii}X_i)Y_K + \frac{d}{dx_i}(\xi'_{ii})X_i\ Y_i$$

(summing over K) , and as $\xi'_{ii}X_i \equiv 0$, $\xi'_{ii}\ Y_i \equiv 0$ we have

$$(Z \lrcorner \xi')_i = - \frac{d}{dx_i}(\xi'_{ii})X_i\ X_i + \frac{d}{dx_i}(\xi'_{ii})X_i\ Y_i \equiv 0$$

Thus, we have $X \lrcorner \xi' = 0$, $Y \lrcorner \xi' = 0 \Rightarrow [X,Y] \lrcorner \xi' = 0$, and by induction on the order of the brackets, we get

$$X \in \mathcal{T}(D_1) \Rightarrow X \lrcorner \xi' \equiv 0$$

and therefore $\xi' = 0$, i.e. Φ is an isometry.

3 - Equivalence of "bilinear" gradient systems

We shall consider a special class of systems, by taking $M = \mathbb{R}^n$, $\xi(v_1,v_2) = \langle v_1,Av_2\rangle$ when A is a symmetric non-singular matrix ($\langle,\rangle$ is the usual inner product in $\mathbb{R}^n$) and $P:\mathbb{R}^n \times \mathbb{R}^\ell \to \mathbb{R}$ is given by $P(x,u) = \langle x,Fx\rangle + \sum_{k=1}^{\ell} u_K\langle x,N_Kx\rangle + \langle x,Gu\rangle$ where F,N_K and

G are matrices, F and N_K symmetric.

Then Σ has the form

$$A\dot{x} = 2Fx + 2(\Sigma u_K N_K)x + Gu$$

$$y_K = \langle G^K, x\rangle + \langle x, N_K x\rangle \quad K = 1,2,\ldots,\ell$$

when G^K is a vector with components the elements of the k-th column of G.

Σ is not bilinear, since the output is not linear in x, and the system obtained by eliminating those terms in the output is called the associated bilinear system.

We say Σ is irreducible if the associated system is a minimal bilinear realization.

Theorem III.3.1:

A linear isomorphism Φ is an equivalence of two irreducible gradient systems Σ_1 and Σ_2 iff it is an isomorphism of the associated bilinear systems and then it is an isometry.

Proof:

Let $Z = \Phi X$ be a linear equivalence of Σ_1 and Σ_2. It is clear that $G_2^T\Phi = G_1^T$, from the condition on the outputs of equivalent systems.

Since Φ takes the dynamics of the first system into the dynamics of the second, it does the same to the associated bilinear systems.

together with $G_2^T = G_1^T$ means that Φ is an isomorphism of the associated bilinear systems.

Now suppose Φ is an isomorphism of the a.b.s. , then Φ is the unique solutions of the equations:

i) $\Phi^{-1}A_2^{-1}F_2\Phi = A_1^{-1}F_1$

ii) $\Phi^{-1}A_2^{-1}N_{2K}\Phi = A_1^{-1}N_{1K}$

iii) $\Phi^{-1}A_2^{-1}G_2 = A_1^{-1}G_1$

iv) $G_2^T\Phi = G_1^T$.

An easy computation shows that $A_2^{-1}(\Phi^{-1})^TA_1$ satisfies the same set of equations:

i) $= F_1A_1^{-1} = \Phi^TF_2A_2^{-1}\Phi^{-1})^T \Rightarrow$

$$A_1^{-1}F = A_1^{-1}\Phi^TA_2^{-}\ A_2^{-1}F_2\ A_2^{-1}(\Phi^{-1})^TA_1$$

$$A_1^{-1}F = (A_2^{-1}(\Phi^{-1})^TA_1)^{-1}A_2^{-1}F_2\ (A_2^{-1}(\Phi^{-1})^TA_1) \ .$$

Similarly, from ii) we can obtain the equation

$$A_1^{-1}N_{1K} = (A_2^{-1}(\Phi^{-1})^TA_1)^{-1}\ A_2^{-1}N_{2K}\ (A_2^{-1}(\Phi^{-1})^TA_1) \ .$$

Now iii) => $A_1\Phi^{-1}A_2^{-1}\ G_2 = G_1$

and we obtain

$$G_1^T = G_2^T\ (A_2^{-1}(\Phi^{-1})^T A_1)\quad .$$

From iv) we get $G_1 = \Phi^T\ G_2$, and then $A_1^{-1}G_1 = A_1^{-1}\Phi^T$,

thus

$$A_1^{-1}G_1 = (A_2^{-1}(\Phi^{-1})^T A_1)^{-1}\ A_2^{-1}G_2\quad .$$

Since Φ is the unique solution of these equations $\Phi = A_2^{-1}(\Phi^{-1})^T\ A_1$ or equivalently $A_1 = \Phi^T\ A_2\ \Phi$.

Now $\Phi^T N_{2K}\Phi = A_1\Phi^{-1}A_2^{-1}N_{2K} = A_1A_1^{-1}N_{1K}$ and therefore

$\Phi^T N_{2K}\Phi = N_{1K}$ $\therefore$ This together with the other equations shows that Φ is an equivalence of the gradient systems, and as $A_1 = \Phi^T\ A_2\Phi$, Φ is an isometry.

CHAPTER IV - HAMILTONIAN SYSTEMS

Section one presents the background material, and in section two we prove that an initialized complete analytic Hamiltonian system, with finite dimensional Lie algebra and satisfying quite standard assumptions, has an equivalent accessible Hamiltonian realization defined on an orbit in the dual of the Lie algebra of the coadjoint action; moreover, if that realization is strongly accessible it is quasi-minimal.

Section three consists of an example for the above results.

1 - Symplectic actions

Let G be a connected Lie group acting on a symplectic manifold M by symplectormorphisms. Then, to each vector v in $\mathcal{G}$, the Lie algebra of G, there is a vector field X_v in M, given by $X_v(x) = (\Phi_x)_* v$, where $\Phi: G \times M \to M$ is the action and $\Phi_x(\gamma) = \Phi(\gamma, x)$; the derivative is taken at the identity in G.

$X_v \lrcorner w$ is a closed form, since G acts by symplectormorphisms [1, pg 189].

The action Φ is said to be Poisson if the two following conditions are verified:

i) $X_v \lrcorner w$ is exact, i.e. there exists a function f_v such that $X_v \lrcorner w = df_v$.

ii) It is possible to define a map $\beta: \mathcal{G} \to C^\omega(M)$ which is a homomorphism of Lie algebras and such that $X_v \lrcorner w = d\beta(v)$.

If Φ is a Poisson action, we can define a map $\mathcal{J}: M \to \mathcal{G}^*$, the dual of $\mathcal{G}$, by $\mathcal{J}(x).v = \beta(v)(x)$. $\mathcal{J}$ is called the moment map.

We denote by Ad, respectively Ad^*, the adjoint, respectively coadjoint, action of G on $\mathcal{G}$, respectively $\mathcal{G}^*$. [7].

Then we have:

Theorem IV.1.1 [2].

The following diagram is commutative:

$$\begin{tikzcd} M \arrow[r, "\gamma"] \arrow[d, "\mathcal{J}"'] & M \arrow[d, "\mathcal{J}"] \\ \mathcal{G}^* \arrow[r, "\mathrm{Ad}^*_\gamma"] & \mathcal{G}^* \end{tikzcd}$$

Theorem IV.1.2 [7]:

Let S be an orbit of the coadjoint action on $\mathcal{G}^*$. Then S has a canonical symplectic form w_S defined as follows: let ξ_{v_1} and ξ_{v_2} be the two vector fields in S corresponding to $v_1, v_2 \in \mathcal{G}$; then $w_S(\xi_{v_1}, \xi_{v_2})(\alpha) = \alpha([v_1, v_2])$, $\alpha \in S$. Moreover, ξ_v is Hamiltonian, corresponding to the restriction to S of the function $\alpha \to \alpha(v)$.

We remark that if $v \in \mathcal{G}$, the vector field X_v is tangent to the orbits of G on M, and thus, by theorem 1, induces a vector field ξ_v on $\mathcal{G}^*$ tangent to the orbits of the coadjoint action; it is the restriction of this vector field to S that we call the vector field in S corresponding to the vector v in $\mathcal{G}$.

Corollary IV.1.3:

The Hamiltonian of ξ_v is the restriction of $\overline{\beta}(v)$ to S, where $\overline{\beta}(v)$ satisfies $\beta(v) = \overline{\beta}(v) \circ \mathcal{J}$.

Proof:

It is enough to show that $\alpha \to \alpha(v)$ is the same as $\overline{\beta}(v)$.

Now, if we denote the above map by $\bar{f}_v$, we have $\bar{f}_v(\mathcal{J}(x)) = \mathcal{J}(x)(v) = \beta(v)(x)$, i.e. $\bar{f}_v$ satisfies the relation $\bar{f}_v \circ \mathcal{J} = \beta(v)$ and thus $\bar{f}_v = \bar{\beta}(v)$.

2 - Realizations of Hamiltonian systems

Let (M,w) be a symplectic manifold; we assume we have defined on M a Hamiltonian system Σ , as in the previous chapter, such that

i) Σ is analytic and complete.

ii) $\mathcal{T}(D)$ is a finite dimensional Lie algebra.

From [6] we know that G is a Lie group, the Lie algebra $\mathcal{G}$ of G is isomorphic to $\mathcal{T}(D)$, and all the vector fields in $\mathcal{T}(D)$ are complete.

We shall assume the action of G on M is Poisson, with moment $\mathcal{J}$.

Lemma IV.2.1:

If $\mathcal{J}(x_1) = \mathcal{J}(x_2)$ then $y_u(x_1) = y_u(x_2)$ for every $u \in R^\ell$.

Proof:

$\mathcal{J}(x_1) = \mathcal{J}(x_2)$ in particular means that $P_u(x_1) = P_u(x_2)$ for every $u \in R^\ell$: let v_u be the vector in $\mathcal{G}$ corresponding to the associated vector field X^u ; then $\mathcal{J}(x_1).v_u = \mathcal{J}(x_2).v_u$ or equivalently

$\mathcal{P}_u(x_1) = \mathcal{P}_u(x_2)$ since $\mathcal{P}_u$ is the Hamiltonian of X^u .

So the two maps $u \to P(x_1,u)$ and $u \to P(x_2,u)$ are equal at every point, so $\frac{\partial}{\partial u} P(x_1,u) = \frac{\partial}{\partial u} P(x_2,u)$ or $y_u(x_1) = y_u(x_2)$ for every $u \in R^\ell$.

From this lemma and theorem IV.1.1 we see that we can define a strongly equivalent system Σ' on $\mathcal{G}^*$, since both the dynamics and the output commute with $\mathcal{J}$.

Suppose we have an initialized system (Σ,x) ; and let S be the orbit in $\mathcal{G}^*$ of $\mathcal{J}(x)$ by the coadjoint action of G . Let Σ_S be the restriction of Σ' to S .

<u>Theorem IV.2.2.</u>

$(\Sigma_S, \mathcal{J}(x))$ is strongly equivalent to (Σ,x) , and moreover it is an accessible Hamiltonian system.

<u>Proof:</u>

It is clear that Σ_S is accessible, since S is an orbit, and strongly equivalent (as an initialized system) to Σ .

As remarked on the previous lemma, we can define $\bar{P}: \mathcal{G}^* \times R^\ell \to R$ such that $\mathcal{P}(x,u) = \bar{P}(\mathcal{J}(x),u)$ for every $(x,u) \in M \times R^\ell$. Let P_S be its restriction to $S \times R^\ell$.

By theorem IV.1.3 we know that the associated vector field ξ^u

corresponding to the control $u \in R^{\ell}$ (and to X^u) has Hamiltonian given by $P_{S_u}:S \to R^{\ell}$, $P_{S_u}(\alpha) = P_S(\alpha,u)$.

The output y_u is given by the restriction to S of the map $\bar{y}_u$ which satisfies $y_u(x) = \bar{y}(\mathcal{J}(x))$ for every $x \in M$ (the existence of $\bar{y}_u$ is proved by the previous lemma). Therefore, to finish the proof it is enough to show that

$$y_u^S(\alpha) = \frac{\partial}{\partial u} P_S(\alpha,u) \quad .$$

From $\bar{P}(\mathcal{J}(x),u) = P(x,u)$ for every $(x,u) \in M \times R^{\ell}$ we get $\frac{\partial}{\partial u} \bar{P}(\mathcal{J}(x),u) = \frac{\partial}{\partial u} P(x,u)$, i.e. $\bar{y}_u(\alpha)$ is given by $\frac{\partial}{\partial u} \bar{P}(\alpha,u)$.

Now if $\alpha \in S$ $\frac{\partial}{\partial u} P_S(\alpha,u) = \frac{\partial}{\partial u} \bar{P}(\alpha,u) = \bar{y}_u(\alpha) = y_u^S(\alpha)$ and the proof is complete.

We remark that Σ_S is accessible, but not necessarily observable or locally observable.

From the results in [14] we can derive:

<u>Theorem IV.2.3:</u>

If Σ_S is strongly accessible it is locally observable, thus a quasiminimal realization.

Proof:

Since strong accessibility and local observability are local conditions, we can assume one system to be defined $\mathbb{R}^n$:

$$\dot{x} = w^{-1}(x)\,\frac{\partial P}{\partial x}(x,u)$$

$$y = \frac{\partial}{\partial u}\,P(x,u)$$

where $P : R^n \times \mathbb{R}^\ell \to M$ and $w(x)$ is an invertible skew-symmetric matrix.

As in [14] we define the associated extended system as being

$$\dot{x} = w^{-1}(x)\,\frac{\partial}{\partial x}\,P(x,y)$$

$$\dot{v} = u$$

$$y(x,v) = \frac{\partial}{\partial v}\,P(x,v)$$

on the phase space $\mathbb{R}^n \times \mathbb{R}^\ell$.

In [14] it has been proved:

i) strong accessibility of the extended system is equivalent to strong accessibility of the original system.

ii) The strong accessibility distribution at (x,v) for the extended system is given by $\{0\} \times \mathbb{R}^\ell + \mathcal{V} \times \{0\}$, where $\mathcal{V}$ is a distribution on $\mathbb{R}^n$ such that $\mathcal{V} \lrcorner\, w$ is the observability codistribution for the original system at x , for a fixed $u = v$.

Therefore, from i) and ii), we see that strong accessibility implies local observability.

3 - Example

Let $M = R^6$, with w the usual symplectic form and $P(x,u) = g_0(x) + u_1 g_1(x) + u_2 g_2(x)$, thus $P: M \times R^2 \to R$, where $g_0(x) = x_3x_5 - x_2x_6$, $g_1(x) = x_1x_5 - x_2x_4$, $g_2(x) = x_1x_6 - x_3x_4$.

In this case $w = \begin{bmatrix} 0 & I_3 \\ -I_3 & 0 \end{bmatrix}$

An easy computation shows that, if we define X^i by $X^i \lrcorner\, w = dg_i$ we have

$$X^0 = \begin{bmatrix} 0 \\ x_3 \\ -x_2 \\ 0 \\ x_6 \\ -x_5 \end{bmatrix} \qquad X^1 = \begin{bmatrix} -x_2 \\ x_1 \\ 0 \\ -x_5 \\ x_4 \\ 0 \end{bmatrix} \qquad X^2 = \begin{bmatrix} -x_3 \\ 0 \\ x_1 \\ -x_6 \\ 0 \\ x_4 \end{bmatrix}$$

and also $[X^1,X^2] = X^0$, $[X^2,X^0] = X^1$, $[X^0,X^1] = X^2$.

Therefore $\mathcal{T}(D) = \mathcal{T}_0(D)$ and finite dimensional; from the above relations we see that $\mathcal{T}(D)$ is isomorphic to $so(3)$, the Lie algebra of the skew symmetric 3×3 matrices, with the isomorphism given by

$$X^0 \to A_0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$X^1 \to A_1 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$X^2 \to A_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

Moreover, we can identify so(3) with R^3 be means of

$$(x_1,x_2,x_3) \to \begin{bmatrix} 0 & x_1 & x_2 \\ -x_1 & 0 & x_3 \\ -x_2 & -x_3 & 0 \end{bmatrix}$$

Then the bracket in so(3) corresponds to the usual vector product in R^3.

If we define an action of SO(3) on R^6 by means of

$$(A,x) \to \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix} x = A \cdot x$$

and take $Y^i(x) = \frac{d}{dt}\Big|_{t=0} (\exp tA_i \cdot x)$ we see that $Y^i \equiv X^i$ $i=0,1,2$.

Thus we can assume we have been given an action $SO(3) \times R^6 \to R^6$; the maps $x \to A \cdot x$ are symplectormorphisms for every $A \in SO(3)$, since they are the same as $x \to \alpha x$ for some $\alpha \in G_D$. (α depends on A) , which follows from [6, pg 52-53].

If we identify R^6 with T^*R^3 , we see the above action on R^6 is the one induced by the usual action of $SO(3)$ on R^3 , and thus it is a Poisson action [2, pg 377].

Having identified $so(3)$ with R^3 , we can identify $so(3)^*$ with R^3 as well, using the Euclidean structure.

It is easy to see that the adjoint action of $SO(3)$ on R^3 (identified with $so(3)$) is equivalent, but not the same, to the usual action of $SO(3)$ on R^3 , for instance $Ad_{\exp t A_1} x = (\exp t A_0)x$, and so the orbits are as follows the orbit of $x \in R^3$ is the set $\{y \in R^3 \mid ||y|| = ||x||\}$.

From the definition of coadjoint action, its orbits are exactly the same as the ones above.

As we know that the moment map does exist, we can compute it as follows:

$$\mathcal{J}(x_1,x_2,x_3,x_4,x_5,x_6)\ (1,0,0) = g_1(x) = x_1x_5 - x_2x_4$$
$$\mathcal{J}(x_1,x_2,x_3,x_4,x_5,x_6)\ (0,1,0) = g_2(x) = x_1x_6 - x_3x_4$$
$$\mathcal{J}(x_1,x_2,x_3,x_4,x_5,x_6)\ (0,0,1) = g_0(x) = x_2x_6 - x_3x_5$$

and therefore

$$\mathcal{J} : \mathbb{R}^6 \to \mathbb{R}^3 \ , \mathcal{J}(x) = (x_1x_5 - x_2x_4, x_1x_6 - x_3x_4, x_2x_6 - x_3x_5)$$

If we denote by p_i the projection on the i-th factor of $\mathbb{R}^3 = \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ (i=1,2,3) we have

$$g_1(x) = p_1 \circ \mathcal{J}(x)$$
$$g_2(x) = p_2 \circ \mathcal{J}(x)$$
$$g_0(x) = -p_3 \circ \mathcal{J}(x)$$

and from the original system

$$\dot{x} = X^0 + u_1 X^1 + u_2 X^2$$
$$y_1 = g_1(x)$$
$$y_2 = g_2(x)$$

we obtain a strongly equivalent system in $\mathbb{R}^3$, in which the output-maps are the projections on the first and second factor of $\mathbb{R}^3$.

To obtain an accessible realization, in this case a minimal realization and strongly accessible since $[X^1, X^2] = X^0$, we have to restrict it to an orbit of the coadjoint action, i.e. if we initialize the system at $x \in \mathbb{R}^3$ the state space of the minimal realization will be $M_1 = \{y \in \mathbb{R}^3 \mid ||y|| = ||(x)||\}$, the output functions will be the

restriction to M_1 of p_1 and p_2, and the associated vector fields will be obtained from p_1, p_2 and $-p_3$ using the canonical symplectic form on the sphere.

Therefore, beginning with $x \in \mathbb{R}^6$, we obtain a new Hamiltonian system on M_1, where $P_1 : M_1 \times \mathbb{R}^2 \to \mathbb{R}$ is given by $P_1(x_1,x_2,x_3, u_1,u_2) = -x_3 + u_1x_1 + u_2x_2$. The symplectic form is defined as follows: if $v_1,v_2 \in \mathbb{R}^3$ are tangent to M_1 $w(v_1,v_2)(x) = (x, v_1 \wedge v_2)$ where $(\, , \,)$ denotes the usual inner product in $\mathbb{R}^3$. The associated vector fields are tangent to the sets $\{y \in M_1, p_i(y) = \text{const}\}$, so the corresponding flows are rotations around the axes.

Of course, if $\mathcal{J}(x) = 0$, then the orbit is just the origin, and the analysis above is not needed, nor valid.

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