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Categorical and Geometric Aspects of Noncommutative Algebras: Mutations, Tails and Perversities

by

Jorge Vitória

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Contents

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Gastam-se as palavras, ficam as pessoas e as memórias. Perto.

Obrigado.

Declarations

I declare to the best of my knowledge, except where otherwise stated, cited or commonly known, that this thesis is my own original work. I confirm that this thesis has not been submitted for a degree at another university.

Part of the work in chapter 3 has been published ([Vit09]).

Abstract

This thesis concerns some interactions between algebraic geometry and noncommutative algebra in a categorical language. This interplay allows noncommutative constructions of geometric motivation and we explore their structure.

In chapters 1 and 2 we survey the main ideas, contextualising this area and introducing the main concepts and results used later in the thesis. These include Morita theory for derived categories, tilting t-structures with respect to torsion theories and generalities on noncommutative projective geometry.

Chapter 3 is devoted to prove that, under certain conditions, mutations for quivers with potentials induce derived equivalences on the corresponding Jacobian algebras. We give examples of such Jacobian algebras and show how they occur naturally in geometry.

In chapter 4 we turn our attention to to a class of skew-polynomial algebras and explore ways of classifying their noncommutative projective geometry, studying graded Morita equivalences, point varieties and birational equivalences.

Finally, chapter 5 contains an algebraic description of perverse coherent t-structures for the derived category of coherent sheaves on a complex projective variety. Furthermore we define analogous structures in adequate noncommutative settings.

Chapter 1

Introduction

The search for noncommutative spaces is a relatively old quest. The far-reaching comprehension of commutative algebra through geometric techniques, and the generalisations of the concept of 'space' through these algebraic tools lead naturally to question whether one can formulate such a theory without the commutativity assumption. Since the middle of the twentieth century, either with the intent of providing a better understanding of new developments in the classical theory or with the aim of building a new theory of noncommutative structures, noncommutative algebra phenomena have been recurrent in modern algebro-geometric contexts.

1.1 Noncommutative algebra in classical algebraic geometry

Homological algebra plays an essential role in the understanding of algebraic varieties. The work developed by Grothendieck and Verdier in the 60's on a new categorical formalism for the techniques of homological algebra has become an

important tool in modern algebraic geometry. Triangulated categories are now omnipresent in the literature, having provided new techniques not only in geometry but also in representation theory and other related areas. The examples most relevant for this thesis are those of derived categories of the following two types of abelian categories: modules over an algebra and coherent sheaves over a projective scheme.

In 1978, the work of Beilinson ([Bei78]) created a innovative new approach to the study of derived categories of coherent sheaves. In this paper Beilinson proves the existence of strong exceptional sequences on projective spaces and uses them to study properties of the correspondent derived categories. These sequences are ordered sets of sheaves satisfying certain Ext-vanishing conditions and generating, as a triangulated category (by allowing shifts, cones and direct summands), the derived category. They are useful to decompose the given category into simpler ones ([BO95]). Other examples of exceptional sequences were given by Kulsehov and Orlov for del Pezzo surfaces ([KO95]). All these examples consist of collections of line bundles and it was conjectured by King in 1997 ([Kin97]) that smooth complete toric varieties have strong exceptional sequences of this form. This conjecture was, however, disproved in 2006 by Hille and Perling ([HP06]).

In 1988 Baer ([Bae88]) produced the most relevant use of exceptional sequences for the purposes of this thesis. In his work, Baer defines a tilting sheaf T on a projective space as one that produces a derived equivalence between the bounded derived category of coherent sheaves and the bounded derived category of finitely generated modules over the algebra of endomorphisms of T. Moreover, Baer describes the cohomological and generation conditions needed for a coherent sheaf T to be tilting. It is then clear, as observed by Bondal in 1989 ([Bon90]) that, given a strong exceptional sequence on a projective variety X, the direct sum

of its elements is a tilting sheaf. This reduces, at the derived level, the study of coherent sheaves to the study of representations of an associative algebra. Furthermore, it turns out that such associative algebra is in fact a finite dimensional algebra that can be presented as a path algebra of a quiver with relations.

Path algebras and their quotients span an interesting class of noncommutative algebras. For example it is well known that all finite dimensional basic connected algebras belong to this class. These are associative algebras generated as vector spaces by the paths of a directed graph, a quiver. The multiplication is given by concatenation of paths, when possible, and zero otherwise. The representation theory of such algebras is, in many cases, well-known. For instance, Gabriel ([Gab62]) proved that the only path algebras of finite representation type are those arising from oriented Dynkin diagrams. More relevant to our work, however, is their representation theory at a derived level.

In 1973, Bernstein, Gelfand and Ponomarev ([BGP73]) introduced the reflection functors at sources and sinks of a quiver. The purpose was to present a condensed proof of Gabriel's result on algebras of finite representation type. These functors between categories of modules over the respective path algebras, later known as BGP-reflections, became important instruments and origin of several generalisations. This can be considered one of the starting points of tilting theory. The first appearance of tilting modules goes back to 1979 to the work of Auslander, Platzeck and Reiten ([APR79]). In 1980 the work of Brenner and Butler ([BB80]) brings an axiomatic approach to tilting, laying the foundations of the theory. Such tilting objects were then used to relate the structure of categories of modules over different rings. It is in 1986 that Happel ([Hap87]) proves that such objects induce equivalences, not at the level of abelian categories but at a derived level. This constituted a first step towards a Morita theory for derived categories

that got finally settled by Rickard in 1989 ([Ric89], [Ric91]). The notion of tilting complex became then a central one.

A more recent generalisation of BGP-reflections shows up in the theory of cluster algebras: mutations. Initially developed by Fomin and Zelevinsky ([FZ02]), this theory started to gain shape in 2002 and is now intimately connected with tilting theory through the work of Buan, Caldero, Keller, Marsh, Reineke, Reiten, Todorov, and others ([BMR$^+$06], [CK08],...). Relevant to this thesis is the concept of mutation of a quiver, a notion that shows up when defining cluster algebras and that one would hope suitable for establishing derived equivalences. Indeed, following the paper of Derksen, Weyman and Zelevinsky ([DWZ08]) of 2007, concerning mutations of quivers with potential, work towards such result has been done ([KY10], [Vit09]) and we present some of it in this thesis. Quivers with potential arise naturally in many contexts such as toric geometry ([UY07]) and 3-Calabi-Yau algebras ([Gin06]). Bocklandt proved in 2008 that graded 3-Calabi-Yau algebras can be presented in this way. Indeed, a very recent result (2010) by Van den Bergh [VdB10] proves that all complete 3-Calabi-Yau algebras are complete Jacobian algebras of quivers with potential. Also, Keller proved in 2009 ([KY10]) that Ginzburg's construction of a differential graded algebra from the Jacobian algebra of a quiver with potential is fundamental to understand the role of mutations as derived equivalences. The setting of Calabi-Yau algebras is special since in that case the cohomology of the differential graded algebra is concentrated in one degree.

The development of tilting theory in categories of modules and the connections with categories of coherent sheaves ([Bae88], [Bon90],...) has established a fruitful area of research concerning the interactions between representation theory and algebraic geometry.

From a rather different perspective, Beilinson, Bernstein and Deligne created in 1982 the notion of t-structure on a triangulated category ([BBD82]). A t-structure is a pair of full subcategories satisfying some orthogonality conditions such that each object of the triangulated category can fit as the middle term of a triangle whose other two vertices lie in appropriate subcategories related to this pair by at most a shift. One of their main results ([BBD82]) states that given a t-structure, the intersection of the two subcategories is abelian. The authors then use this fact to construct a new abelian category lying inside the derived category of constructible sheaves on a stratified scheme: the category of perverse sheaves. Perverse sheaves play an important role both in the theory of D-modules and in the theory of intersection cohomology for singular spaces ([BBD82], [Bei87]). The theory of perverse sheaves was later adapted to the derived category of coherent sheaves by Bezrukavnikov and Arinkin, following ideas of Deligne ([Bez00], [AB10]). This development has particular relevance for the study of equivariant sheaves under the action of algebraic groups. Similar ideas were recently used by Achar to construct other significant abelian categories in the representation theory of algebraic groups, the staggering sheaves ([Ach09]).

The notion of t-structure turns up in tilting theory as well namely through the work of Happel, Reiten and Smalø in 1996 ([HRS96]). They proved that torsion theories in an abelian category produce t-structures in the correspondent derived category. This was the starting point to further results on derived categories of hereditary abelian categories and of quasitilted algebras.

On the other hand, the study of t-structures in a derived category of coherent sheaves is of geometric significance. The work of Bridgeland on the theory of stability conditions ([Bri02],[Bri07],[Bri05]) is an example of that. Using these ideas, in 2004 Gorodentsev, Kuleshov and Rudakov classified all t-structures in the

derived category of coherent sheaves over the projective line and over a smooth elliptic curve ([GKR04]).

One other way of constructing t-structures is through autoequivalences of the derived category. As hinted above, the existence of exceptional sequences provides us with a way of constructing such t-structures. A lot of work within the past 10 years has concerned related topics. Bondal and Orlov ([BO01]), for example, described the autoequivalences of the derived categories of projective spaces. On the other hand Bridgeland constructed t-structures for local Calabi-Yau varieties (see [Bri05]), results that were later generalised together with David Stern ([BS09]). This work relates the theory of helices, developed by Beilinson, Gorodentsev, Rudakov and others ([R$^+$90]) with changes of t-structure.

Further developments in derived algebraic geometry have recently been attained. Bondal and Orlov proved that derived categories are an interesting geometric invariant ([BO01]). In fact if the canonical bundle of a projective variety is ample or anti-ample we can then recover the variety, up to isomorphism, from its derived category of coherent sheaves. Also, since the work of Mukai in the early 80's ([Muk81]) that Fourier-Mukai transforms provide interesting examples in this area. It is however through the work of Bridgeland in 1999 ([Bri99]) that we see a general treatment of this fundamental tool, where conditions are given for a Fourier-Mukai transform to be an equivalence of derived categories. And through the work of Orlov ([Orl97], [Orl03]) we know that all such equivalences have to be of this form. Fourier-Mukai transforms have been used in several contexts: in the study of flips and flops by Bondal, Bridgeland and Orlov ([BO95],[Bri02]), in the search for relations between birational geometry and derived geometry (see, for example, Kawamata's work, [Kaw02]) or in the context of the McKay correspondence through the work of Bridgeland, King and Reid ([BKR01]). In this

paper suitable Fourier-Mukai transforms establish equivalences between derived categories of crepant resolutions of quotient singularities (by the action of a finite subgroup G of automorphisms) and derived categories of G-equivariant sheaves on the quotient variety. A great source of material on Fourier-Mukai transforms is Huybrechts' book, [Huy06].

1.2 The advent of noncommutative projective geometry

The title 'noncommutative algebraic geometry' is rather ambiguous as it can be understood through many different flavours. This thesis explores one of them in particular. The foundations for noncommutative projective geometry were laid out in detail by Artin and Zhang in 1994 ([AZ94]) but the subject can be traced back to the work of Artin, Tate and Van den Bergh in 1990 ([ATVdB90]) and subsequent work in 1991 and 1992 ([Art92], [ATVdB91]). These papers explore the geometry within Artin-Schelter regular algebras, defined in 1987 by Artin and Schelter ([AS87]), namely through the notion of point scheme. These algebras modell the properties that the homogeneous coordinate ring of a putative noncommutative projective space should have. Twisted homogeneous coordinate rings appeared naturally in these papers and were extensively studied by Artin and Van den Bergh ([AVdB90]) and work thereafter.

Given that, in general, localisation is not available in the noncommutative world, one turns to the categorical tools of algebraic geometry rather than the sheaf theoretic ones. Such categorical methods are justified by the work of Gabriel ([Gab62]) that shows that a noetherian scheme is completely determined by its category of coherent sheaves. An affine noncommutative space would then be a category of modules and a projective noncommutative space a category of 'tails'

over suitable noncommutative algebras, following the work of Serre ([Ser55]). A tail means, in this context, an object of the Grothendieck category obtained as the quotient of graded modules modulo torsion modules over a graded algebra. While in the affine case the study of isomorphisms within these varieties falls on classical Morita theory, the projective case reveals to be rather more intricate and we dedicate part of this thesis to it. More generally speaking, as stated in the survey by Stafford and Van den Bergh ([SVdB01]), these categorical methods are more varied and seem to better apply to the projective context and this is the starting point for Artin and Zhang's seminal paper ([AZ94]) where a suitable cohomology theory for these objects is also built.

Even though the concept of noncommutative projective scheme was only formalised in 1994, a lot of work in this direction was already being done for the preceding seven years with particular emphasis on the relations between Artin-Schelter regular algebras and algebraic geometry through the notion of point scheme. A point module over an $\mathbb{N}_0$-graded (or positively graded as we shall refer to henceforth) ring R is first defined in [ATVdB90] as a graded module generated in degree zero such that each graded piece is one dimensional. This definition makes such a module resemble a skyscraper sheaf in the commutative case. The point scheme is a scheme E parametrising point modules and, in the case of AS-regular algebras of dimension 3, R has as a factor a twisted homogeneous coordinate ring of E. In 2006, Mori gave sufficient conditions for the point variety to be invariant under equivalences of categories of tails preserving the structure sheaf ([Mor06]). This is an important invariant and, in some interesting cases, easy to compute.

Twisted coordinate rings seem to be central in this theory. For example, every graded domain R of quadratic growth (i.e., Gelfand-Kirilov dimension 2) generated in degree 1 is a twisted coordinate ring of a projective curve E. Moreover,

the category of tails of R is equivalent to the category of coherent sheaves over E ([AS95]). Such a phenomenon happens for a twisted coordinate algebra whenever the line bundle considered is ample with respect to a certain automorphism of the space ([AVdB90]). As a consequence, many noncommutative algebras actually produce commutative schemes rather than genuine noncommutative ones.

The development of geometric tools in this theory for the past decade has been vast. These include Serre duality ([Jør97], [Yek99]), intersection theories ([MS01][Jør00]), the study of Grothendieck groups ([MS06]), a Riemann-Roch like theorem ([Mor04b]), blow-ups and the study of birational geometry ([KRS05], [VdB01]) and many others. Birational classification of noncommutative projective schemes is a problem of strong relevance and it is unknown even for the simpler nontrivial case: noncommutative surfaces. Indeed Artin conjectures ([Art97]) that the isomorphism classes of division rings of fractions for these algebras are of three possible types. They should be finite over a central subfield of transcendence degree two, fractions over a skew polynomial extension of a fraction field of a commutative curve or fraction division rings of 3-dimensional Sklyanin algebras.

The interest in studying derived categories of noncommutative projective schemes has also been growing. The work of Mori ([Mor04b], [Mor04a]) has contributed for a reinterpretation of the problem of classifying derived categories of tails in terms of stable equivalences between Frobenius algebras of fixed dimension. In fact the derived categories of tails of two noetherian Koszul algebras A and B are equivalent if and only if the stable categories of graded modules over the Koszul duals of A and B are equivalent. This provides a strong link of noncommutative projective geometry with representation theory of finite dimensional algebras. Moreover, Minamoto introduced the concept of quasi-Fano algebras ([Min09]). Then, Minamoto and Mori ([MM10]) proved that the isomorphism

classes of these algebras are strongly related with the graded Morita equivalence classes of certain Artin-Schelter regular algebras associated to them. We observe this phenomenon in a restricted class later in this thesis. In fact, in many cases, a quasi-Fano algebra arises as the endomorphism algebra of the direct sum of some strong exceptional sequence.

In parallel to these technical features, new classes of algebras have been born and intensively studied. A significant example is that of Sklyanin algebras. Sklyanin algebras were first defined by Odesskii and Feigin ([FO89]), later followed by Tate and Van den Bergh on the study of their basic homological properties ([TvdB96]). These algebras are AS-regular with the same Hilbert polynomial of the polynomial algebra and their construction depends on a triple formed by an elliptic curve, an automorphism and a line bundle. A lot of work has been carried out for Sklyanin algebras of certain fixed global dimensions, namely 3 and 4.

From a distinct perspective, the theory of quantum groups has been a strong area of research since its roots in the 80's. Quantized coordinate rings of algebraic varieties have then since been regarded as noncommutative analogues of such varieties and many of their features, such as their prime spectrum, have produced thousands of pages of research. A good account for this theory is the book by Brown and Goodearl ([BG02]). For the purpose of this thesis, multiparameter quantisations of the coordinate ring of the affine space (i.e., the ring of polynomials) will be of particular interest. These are graded AS-regular of global dimension equal to the number of variables and therefore can be regarded through the above theory as noncommutative projective spaces.

Chapter 2

Preliminaries

In this chapter we fix some notation and present some results that will be used in the later parts of this thesis. Further notation shall be introduced as needed.

For a ring R (always unital in this thesis), $Mod(R)$ denotes the category of right R-modules and $mod(R)$ the subcategory of finitely generated right R-modules. In fact we use this distinction often: lower case means the subcategory of finitely generated objects. If R is graded, $Gr(R)$ denotes the category of graded right R-modules. In general, unless otherwise stated, modules are right modules and ideals are two-sided ideals. If X is a scheme, $Qcoh(X)$ and $coh(X)$ denote the categories of quasicoherent and coherent O_X-modules, respectively, where O_X is the structure sheaf on X. In general, rings are $\mathbb{K}$-algebras and schemes are $\mathbb{K}$-schemes, where $\mathbb{K}$ is an algebraically closed field of characteristic zero.

For an abelian category $\mathcal{A}$, $K^b(\mathcal{A})$ ($K^-(\mathcal{A})$, $K^+(\mathcal{A})$) denotes the bounded (right bounded, left bounded respectively) homotopy category of $\mathcal{A}$ and $D^b(\mathcal{A})$ ($D^-(\mathcal{A})$, $D^+(\mathcal{A})$) the bounded (right bounded, left bounded respectively) derived category of $\mathcal{A}$. All subcategories (of either abelian or triangulated categories) are considered to be closed under isomorphisms.

2.1 Derived equivalences

Our main concern in this section (and in the next one) is the structure of some derived categories. We start with those of coherent sheaves over a projective variety X over $\mathbb{K}$. The attempt to give an algebraic treatment of geometric objects will be frequent in this thesis. This perspective is crucial when working with some ideas of noncommutative algebraic geometry, as explained in the introduction. Indeed, we start by observing that, under certain conditions, we can have a better insight into $D^b(coh(X))$ through representations of a quiver with relations. This happens, for instance, when strong exceptional sequences exist.

Definition 2.1.1. *An* ***exceptional sequence*** *over X is an ordered collection of coherent sheaves $(E_1, ..., E_n)$ such that:*

- *Ext$^k(E_i, E_i) = 0$ for all $k > 0$ and Hom$(E_i, E_i) = \mathbb{K}$;*
- *Ext$^k(E_i, E_j) = 0$ for all $1 \leq j < i \leq n$ and $k \geq 0$;*
- *The complexes formed by these sheaves in degree zero and zero in every other degree (the stalk complexes) generate $D^b(coh(X))$ as a triangulated category.*

The sequence is said to be ***strong exceptional*** *if, furthermore, Ext$^k(E_i, E_j) = 0$, for all $k > 0$ and $1 \leq i, j \leq n$.*

The existence of strong exceptional sequences in $D^b(coh(X))$ is known for some varieties X ([Bei78], [KO95]), for instance, for projective spaces and del Pezzo surfaces.

Example 2.1.2. *The sequence $(O, O(1), O(2), ..., O(n))$ over $\mathbb{P}^n$ is strong exceptional ([Bei78]).*

Example 2.1.3. *If X is a blow-up of $\mathbb{P}^2$ at one point, then there is a strong exceptional sequence in $D^b(coh(X))$ ([KO95]). For example, if E is the exceptional curve of the blow-up, then $(O, O(E), O(1), O(2))$ is such a sequence over X ([Per09],[HP08]).*

Definition 2.1.4. *Let X be a nonsingular projective variety over $\mathbb{K}$. A coherent sheaf T is said to be* ***tilting*** *if:*

- *Ext*$^k(T, T) = 0,\ \forall k > 0$;
- *T generates $D^b(coh(X))$ as triangulated category;*
- *$B =$End$_{D^b(coh(X))}(T)$ has finite global dimension.*

Remark 2.1.5. *Hille and Van den Bergh have proved that the nonsingularity of X is enough to prove that B has finite global dimension ([HVdB07]) and hence this condition is superfluous in the definition above. This is, however, the classic definition and the most commonly found in the literature.*

Given a strong exceptional sequence of sheaves, its direct sum is a tilting sheaf. Tilting sheaves provide derived equivalences to categories of modules over $\mathbb{K}$-algebras. When obtained from a strong exceptional sequence it turns out that such algebra is a path algebra of a quiver with relations.

Theorem 2.1.6 (Baer, [Bae88]). *Let X be a nonsingular projective variety over $\mathbb{K}$, T a coherent sheaf over X and $B =$End(T). The following are equivalent:*

1. *T is tilting;*
2. *There is an equivalence $\Phi : D^b(coh(X)) \to D^b(mod(B))$ of triangulated categories with $\Phi(F) =$RHom(T, F).*

Example 2.1.7. *Let X and E be as in example 2.1.3 and consider the strong exceptional sequence therein. Then $T = O \oplus O(E) \oplus O(1) \oplus O(2)$ is a tilting sheaf over X and its endomorphisms algebra can be presented as the quotient of the path algebra of the quiver*

$$\begin{tikzcd}
\bullet^1 \arrow[r, "a"] \arrow[dr, "b"] & \bullet^2 \arrow[d, shift right, "c_1"'] \arrow[d, shift left, "c_2"] \\
\bullet^4 & \bullet^3 \arrow[l, shift right, "d_1"'] \arrow[l, "d_2"] \arrow[l, shift left, "d_3"]
\end{tikzcd}$$

by the ideal generated by:

$$d_3c_1 - d_1c_2,\ d_2c_1a - d_1b,\ d_3b - d_2c_2a.$$

The previous theorem tells us that this algebra is derived equivalent to X.

The theorem above turns our attention to the problem of a Morita-type classification of derived categories of modules. Rickard's theorems from 1989 and 1991 give a complete answer to this. The key concept is that of a tilting complex. Note that a full subcategory $\mathcal{A}'$ of an abelian category $\mathcal{A}$ induces full subcategories $K^b(\mathcal{A}')$ and $D^b(\mathcal{A}')$ of $K^b(\mathcal{A})$ and $D^b(\mathcal{A})$ respectively. For a ring R we denote by $P(R)$ the full subcategory of $mod(R)$ of finitely generated projective R-modules.

Definition 2.1.8. *A **tilting complex** over a ring R is an object $T^\bullet$ of $K^b(P(R))$ such that:*

1. *$\forall i \neq 0$, $Hom_{K^b(P(R))}(T^\bullet, T^\bullet[i]) = 0$;*
2. *$T^\bullet$ generates $K^b(P(R))$ as a triangulated category.*

Rickard proved the following key result.

Theorem 2.1.9 (Rickard, [Ric89]). *Given two rings R and S, $D^b(Mod(R))$ is equivalent to $D^b(Mod(S))$ if and only if there is a tilting complex $T^\bullet$ over R such that $S \cong End_{K^b(Mod(R))}(T^\bullet)$.*

Remark 2.1.10. *Recall that, for any bounded complexes of projective objects $X^\bullet$ and $Y^\bullet$ in an abelian category $\mathcal{A}$, we have ([GM03]):*

$$\mathrm{Hom}_{\mathrm{D^b}(\mathcal{A})}(\mathrm{X}^\bullet, \mathrm{Y}^\bullet) = \mathrm{Hom}_{\mathrm{K^b}(\mathcal{A})}(\mathrm{X}^\bullet, \mathrm{Y}^\bullet).$$

Despite the simple statement above, the proof of this theorem does not construct such an equivalence explicitly. This was fixed in 1991 as follows.

Theorem 2.1.11 (Rickard, [Ric91]). *If A and B are two derived equivalent algebras over a field $\mathbb{K}$ then there is a bounded complex of A-B-bimodules $M^\bullet$ such that each M^i is a finitely generated projective A-module and*

$$\begin{aligned} \Phi : D^b(Mod(A)) &\rightarrow D^b(Mod(B)) \\ X^\bullet &\mapsto X^\bullet \bigotimes_A^L M^\bullet \end{aligned}$$

is an equivalence.

Note that a derived equivalence between two abelian categories $\mathcal{A}$ and $\mathcal{A}'$ allows us to find an abelian subcategory of $D^b(\mathcal{A}')$ equivalent to $\mathcal{A}$. A natural question is thus which other abelian categories can be embedded in a fixed derived category $D^b(\mathcal{A}')$ and how do they relate to $\mathcal{A}'$. The concept of t-structure is a step in that direction.

2.2 t-structures

In this section we discuss t-structures for general triangulated categories. Our examples will, however, be mostly concerned with derived categories of abelian categories. We shall include some proofs for the sake of completeness.

Definition 2.2.1. *A t-**structure** on a triangulated category $\mathcal{D}$ is a pair of full subcategories $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ of $\mathcal{D}$ such that, for $\mathcal{D}^{\leq n} := \mathcal{D}^{\leq 0}[-n]$ and $\mathcal{D}^{\geq n} := \mathcal{D}^{\geq 0}[-n]$, $n \in \mathbb{Z}$, we have:*

1. *Hom$(X, Y) = 0$, $\forall X \in \mathcal{D}^{\leq 0}$, $Y \in \mathcal{D}^{\geq 1}$*
2. *$\mathcal{D}^{\leq 0} \subseteq \mathcal{D}^{\leq 1}$*
3. *For all $X \in \mathcal{D}$, there is a distinguished triangle*

$$A \longrightarrow X \longrightarrow B \longrightarrow A[1]$$

such that $A \in \mathcal{D}^{\leq 0}$ and $B \in \mathcal{D}^{\geq 1}$.

*$\mathcal{D}^{\leq 0} \cap \mathcal{D}^{\geq 0}$ is called the **heart** and $\mathcal{D}^{\leq 0}$ the **aisle** of the t-structure.*

The following result gives us the link between t-structures and embedded abelian categories.

Theorem 2.2.2 (Beilinson, Bernstein, Deligne, [BBD82]). *The heart of a t-structure on a triangulated category is an abelian category.*

It is also well known ([KV88]) that the aisle determines the whole t-structure by setting $\mathcal{D}^{\geq 0} = (\mathcal{D}^{\leq 0})^{\perp}[1]$.

Example 2.2.3. *The simplest example of a t-structure is the one occurring naturally in any derived category. Let $\mathcal{D} = D(\mathcal{A})$ where $\mathcal{A}$ is abelian. The **standard t-structure** in $\mathcal{D}$ is given by:*

$$\mathcal{D}^{\leq 0} = \left\{ X^{\bullet} \in \ \mathcal{D} : \ H^i(X^{\bullet}) = 0, \ \forall i > 0 \right\},$$

$$\mathcal{D}^{\geq 0} = \left\{ X^{\bullet} \in \ \mathcal{D} : \ H^i(X^{\bullet}) = 0, \ \forall i < 0 \right\}$$

whose heart is $\mathcal{A}$ itself.

Associated with a t-structure $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ in a triangulated category $\mathcal{D}$ there are truncation functors $t_{\leq i}$ and $t_{\geq i}$ for all $i \in \mathbb{Z}$ from $\mathcal{D}$ to $\mathcal{D}^{\leq i}$ and $\mathcal{D}^{\geq i}$, respectively. These allow the construction of cohomological functors

$$H^i(X) := t_{\geq i} t_{\leq i}(X)[i]$$

from $\mathcal{D}$ to the heart of the t-structure, $\mathcal{A}$.

The following are useful properties of a t-structure.

Definition 2.2.4. *A t-structure is said to be* ***nondegenerate*** *if*

$$\bigcap_{n \in \mathbb{Z}} \mathcal{D}^{\leq n} = \{0\} \quad \text{and} \quad \bigcap_{n \in \mathbb{Z}} \mathcal{D}^{\geq n} = \{0\}$$

and ***bounded*** *if, furthermore,*

$$\bigcup_{n \in \mathbb{Z}} \mathcal{D}^{\leq n} = \mathcal{D} \quad \text{and} \quad \bigcup_{n \in \mathbb{Z}} \mathcal{D}^{\geq n} = \mathcal{D}.$$

Clearly, the standard t-structure in $D(\mathcal{A})$ is nondegenerate. Moreover, in $D^b(\mathcal{A})$ the standard t-structure is also bounded.

Remark 2.2.5. *A t-structure is nondegenerate if for any object X such that $H^i(X) = 0$, for all integers i, then $X = 0$. It is bounded if, furthermore, for every object X, there are only finitely many integers i such that $H^i(X) \neq 0$ ([Mil]).*

One can fully characterise which full additive subcategories are hearts of some t-structures. We present a proof for such a statement.

Lemma 2.2.6 (Bridgeland, [Bri05]). *Let $\mathcal{D}$ be a triangulated category and $\mathcal{A}$ a full additive subcategory of $\mathcal{D}$. Then $\mathcal{A}$ is the heart of a bounded t-structure in $\mathcal{D}$ if and only if:*

1. *for all A,B in $\mathcal{A}$ and $k < 0$, Hom$_{\mathcal{D}}(A, B[k]) = 0$;*

2. *for all E in $\mathcal{D}$, there is an HN-filtration, i.e., a collection of distinguished triangles*

$$\begin{tikzcd}
E_m = 0 \arrow[rr] & & E_{m+1} \arrow[r] \arrow[dl] \arrow[dr] & \cdots \arrow[r] & E_{n-1} \arrow[rr] & & E_n = E \arrow[dl] \\
& A_{m+1} \arrow[ul] & & \cdots \arrow[ur] & & A_n \arrow[ul] &
\end{tikzcd}$$

for some integers $m < n$ and with $A_i[i]$ in $\mathcal{A}$.

Proof. If $\mathcal{A}$ is the heart of a bounded t-structure then clearly property 1 holds. Suppose now $E \in \mathcal{D}$. Since the t-structure is bounded we can take

$$m + 1 = \min\left\{i \in \mathbb{Z} : H^i(E) \neq 0\right\} \quad \text{and} \quad n = \max\left\{i \in \mathbb{Z} : H^i(E) \neq 0\right\}.$$

We define the length of an HN-filtration (or HN-length) as $n - m$.

Define $E_i = t_{\leq i}(E)$ where the $t_{\leq i}$'s are the truncation functors associated with the t-structure. Clearly $E_m = 0$ and $E_n = E$. Also one has

$$H^k(E) = t_{\geq k} t_{\leq k}(E)[k] \cong \operatorname{cone}(t_{\leq k-1}(E) \longrightarrow t_{\leq k}(E))[k] \cong A_k[k]$$

for all k, thus proving that the objects $A_k[k]$ lie in $\mathcal{A}$.

Conversely, suppose 1 and 2. Define $\mathcal{D}^{\leq 0}$ as the full subcategory of $\mathcal{D}$ formed by the objects E such that there is an HN-filtration of E with $A_i = 0$, for all $i > 0$. Similarly, define $\mathcal{D}^{\geq 0}$ as the full subcategory of $\mathcal{D}$ formed by the objects E such that there is an HN-filtration of E with $A_i = 0$, for all $i < 0$. By definition, condition 2 of t-structure axioms is satisfied.

Let us start by checking axiom 1, i.e., $\mathrm{Hom}_{\mathcal{D}}(E, F) = 0$ for all $E \in \mathcal{D}^{\leq 0}$, $F \in \mathcal{D}^{\geq 1}$. We prove this using a double induction on the lengths of HN-filtrations of E and F (denote them by l and k respectively). Suppose $l = 1$ and $k = 1$. Then $E[s] \in \mathcal{A}$ and $F[r] \in \mathcal{A}$, for some $s \leq 0$ and $r \geq 1$. By condition 1, it is done. Suppose the result is valid for $l = 1$ and for F with HN-length less than

$k \in \mathbb{N}$. Let f be a map from E to $F = F_{m+k}$ for some $m \geq 0$ ($F_m = 0$ and $F_{m+1} \neq 0$) and let B_i be the element of $\mathcal{D}$ such that the distinguished triangle

$$\begin{tikzcd} F_{i-1} \arrow[rr] & & F_i \arrow[dl] \\ & B_i \arrow[ul, "{[1]}"] & \end{tikzcd}$$

is on a fixed HN-filtration of F. Then condition 1 implies that the induced map from E to B_{m+k} is zero. Thus, as it is well known from the general theory of triangulated categories ([Mil]), f factors through F_{m+k-1} and therefore we have, by induction hypothesis, $f = 0$. Suppose now the result is valid for E of HN-length less than $l \in \mathbb{N}$ and for F of arbitrary finite HN-length. Let E be of HN-length l and A_i be such that

$$\begin{tikzcd} E_{i-1} \arrow[rr] & & E_i \arrow[dl] \\ & A_i \arrow[ul, "{[1]}"] & \end{tikzcd}$$

is a distinguished triangle on a fixed HN-filtration of E, with $m' + l \leq 0$ where m' is such that $E_{m'} = 0$ and $E_{m'+1} \neq 0$. Suppose f a nonzero map from $E = E_{m'+l}$ to $F = F_{m+k}$. The previous argument shows that there is $m+1 \leq s \leq m+k$ such that f is a nonzero map from E to B_s. Now, if the induced map from $E_{m'+l-1}$ to B_s is zero, then the same argument as before shows that there is a nonzero a map from $A_{m'+l}$ to B_s which is a contradiction ($m' + l \leq 0$ and $s \geq 1$). But iterating this argument one observes that moving back through the filtration of E, a contradiction like the one above will always be achieved - in the worst case scenario once we reach $E_{m'+1} \in \mathcal{A}[-m' - 1]$. Hence $f = 0$.

Now we want to check axiom 3 of the definition of t-structure. Let $E \in \mathcal{D}$ and suppose it is not in $\mathcal{D}^{\leq 0}$ (otherwise it is done). Fix an HN-filtration of E. Clearly $E_0 \in \mathcal{D}^{\leq 0}$ and we have a map from E_0 to E. Consider a distinguished

triangle of the form:

$$\begin{tikzcd} E_0 \arrow[rr] & & E \arrow[dl] \\ & \hat{E} \arrow[ul, "{[1]}"] & \end{tikzcd}$$

for some $\hat{E} \in \mathcal{D}$. We want to prove that $\hat{E} \in \mathcal{D}^{\geq 1}$. In fact we can construct an HN-filtration for $\hat{E}$ by defining $\hat{E}_k$ $(k \geq 1)$ to be an object in $\mathcal{D}$ completing a triangle as follows:

$$\begin{tikzcd} E_0 \arrow[rr] & & E_k \arrow[dl] \\ & \hat{E}_k \arrow[ul, "{[1]}"] & \end{tikzcd} .$$

This is indeed an HN-filtration as we shall now see, using the octahedral axiom to prove that triangles are as expected. Consider the diagram

$$\begin{tikzcd} E_0 \arrow[r] \arrow[d] & E_{k-1} \arrow[r] \arrow[d] & \hat{E}_{k-1} \arrow[r] & E_0[1] \arrow[d] \\ E_0 \arrow[r] \arrow[d] & E_k \arrow[r] \arrow[d] & \hat{E}_k \arrow[r] & E_0[1] \arrow[d] \\ E_{k-1} \arrow[r] & E_k \arrow[r] & A_k \arrow[r] & E_{k-1}[1] \end{tikzcd} \tag{2.2.1}$$

in which the rows are the previously known distinguished triangles and the vertical morphisms are the obvious ones. Then the octahedral axiom states that this

diagram can be completed to the following:

$$\begin{tikzcd}
E_0 \arrow[r] \arrow[d] & E_{k-1} \arrow[r] \arrow[d] & \hat{E}_{k-1} \arrow[r] \arrow[d] & E_0[1] \arrow[d] \\
E_0 \arrow[r] \arrow[d] & E_k \arrow[r] \arrow[d] & \hat{E}_k \arrow[r] \arrow[d] & E_0[1] \arrow[d] \\
E_{k-1} \arrow[r] \arrow[d] & E_k \arrow[r] \arrow[d] & A_k \arrow[r] \arrow[d] & E_{k-1}[1] \arrow[d] \\
\hat{E}_{k-1} \arrow[r] & \hat{E}_k \arrow[r] & A_k \arrow[r] & \hat{E}_{k-1}[1]
\end{tikzcd} \tag{2.2.2}$$

where, again, rows are distinguished triangles. The last row gives us the expected triangles appearing in the HN-filtration of $\hat{E}$, thus showing that the HN-filtration above construction has $\hat{E}_0 = 0$, thus stopping at zero. Hence $\hat{E} \in \mathcal{D}^{\geq 1}$. $\square$

There is a counterpart of the notion of t-structure for abelian categories: torsion theory. These concepts are related in a relevant way and this will be of essential nature in chapter 5.

Definition 2.2.7. *Let $\mathcal{A}$ be an abelian category. A pair of full subcategories ($\mathcal{T}$,$\mathcal{F}$) is said to be a* ***torsion theory*** *if:*

1. *Hom$(T, F) = 0$, for all $T \in \mathcal{T}$ and $F \in \mathcal{F}$,*

2. *For all $M \in \mathcal{A}$ there is an exact sequence*

$$0 \longrightarrow \tau(M) \longrightarrow M \longrightarrow M/\tau(M) \longrightarrow 0$$

where $\tau(M) \in \mathcal{T}$ and $M/\tau(M) \in \mathcal{F}$. We call $\mathcal{T}$ the ***torsion class*** *and $\mathcal{F}$ the* ***torsion-free class****.*

A torsion theory is ***hereditary*** *if the torsion class is closed under subobjects.*

To a torsion theory we have an associated torsion radical τ (i.e. a subfunctor of the identity functor). This functor associates to an object X of $\mathcal{A}$ the maximal subobject of X that lies in $\mathcal{T}$ ([Ste75]). This resembles the idea of the truncation functors in the triangulated setting.

If $\mathcal{A}$ is a complete and cocomplete abelian category (i.e., it admits arbitrary products and coproducts) then we can give necessary and sufficient conditions for a subcategory $\mathcal{T}$ to be a torsion class in $\mathcal{A}$ (check [Ste75] for details).

Proposition 2.2.8. *A full subcategory $\mathcal{T}$ of a complete and cocomplete abelian category $\mathcal{C}$ is a torsion class if and only if $\mathcal{T}$ is closed under epimorphic images, coproducts and extensions.*

The following theorem ([Bri05]) is a slight generalisation of the original idea of Happel, Reiten and Smalø ([HRS96]). It establishes a way of getting a new t-structure out of a torsion theory on a heart. This result is of significant relevance for this thesis and hence we present a proof for it.

Theorem 2.2.9 (Happel, Reiten, Smalø, Bridgeland, [HRS96], [Bri05]). *Let $\mathcal{A}$ be the heart of a bounded t-structure in $\mathcal{D}$, a triangulated category. Suppose that $(\mathcal{T}, \mathcal{F})$ is a torsion theory on $\mathcal{A}$ and that H^i denotes the i-th cohomology functor with respect to $\mathcal{A}$. Then $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ is a t-structure on $\mathcal{D}$, where*

$$\mathcal{D}^{\leq 0} = \left\{ E \in \mathcal{D} : H^i(E) = 0,\ \forall i > 0, H^0(E) \in \mathcal{T} \right\}$$

and

$$\mathcal{D}^{\geq 0} = \left\{ E \in \mathcal{D} : H^i(E) = 0,\ \forall i < -1, H^{-1}(E) \in \mathcal{F} \right\}.$$

Proof. Let us check the three axioms of t-structure. Again, axiom 2 is clear. Let $E \in \mathcal{D}^{\leq 0}$ and $F \in \mathcal{D}^{\geq 1}$ and suppose that f is a nonzero map from E to F. Let $t_{\leq 0}$ be the truncation functor of the t-structure associated to $\mathcal{A}$. Consider

HN-filtrations (E_i) and (F_j) for E and F respectively, where $E_i = t_{\leq i}(E)$ and $F_j = t_{\leq j}(F)$. Clearly by definition of $\mathcal{D}^{\leq 0}$ and $\mathcal{D}^{\geq 0}$ one has $E_0 = E$ and $F_{-1} = 0$. Now $t_{\leq 0}(f) \neq 0$ given that $f = \iota t_{\leq 0}(f)$ where ι is the natural map from $t_{\leq 0}(F)$ to F. Then one has a morphism of triangles as follows:

$$\begin{tikzcd}
E_{-1} \arrow[r] \arrow[d, "t_{\leq -1}(f)"'] & E_0 \arrow[r] \arrow[d, "t_{\leq 0}(f)"'] & H^0(E) \arrow[r] \arrow[d, "h"'] & E_{-1}[1] \arrow[d] \\
F_{-1} \arrow[r] & F_0 \arrow[r] & H^0(F) \arrow[r] & F_{-1}[1]
\end{tikzcd}$$

for some h determined by $t_{\leq -1}(f)$ and $t_{\leq 0}(f)$. But now $h = 0$ as it is a morphism from a torsion object to a torsion free one. Since $F_{-1} = 0$, $F_0 \cong H^0(F)$ and thus $t_{\leq 0}(f) = 0$, yielding a contradiction.

Finally we shall check axiom 3 of t-structure. Let τ denote the torsion radical associated to the torsion theory $(\mathcal{T}, \mathcal{F})$. Define $\widetilde{E_0}$ as an object in $\mathcal{D}$ completing the following triangle:

$$\begin{tikzcd}
E_{-1} \arrow[rr] & & \widetilde{E}_0 \arrow[dl] \\
& \tau(H^0(E)) \arrow[ul, "{[1]}"] &
\end{tikzcd} .$$

By definition, $\widetilde{E}_0 \in D^{\leq 0}$. Now, there is a map from $\widetilde{E}_0$ to E_0 (and thus to E) completing the diagram:

$$\begin{tikzcd}
E_{-1} \arrow[r] \arrow[d, "id"'] & \widetilde{E}_0 \arrow[r] & \tau(H^0(E)) \arrow[r] \arrow[d] & E_{-1}[1] \arrow[d] \\
E_{-1} \arrow[r] & E_0 \arrow[r] & H^0(E) \arrow[r] & E_{-1}[1]
\end{tikzcd} .$$

Define $\hat{E}$ by completing the triangle

$$\begin{tikzcd} \widetilde{E}_0 \arrow[rr] & & E \arrow[dl] \\ & \hat{E} \arrow[ul, "{[1]}"] & \end{tikzcd} .$$

We want to prove $\hat{E} \in D^{\geq 1}$. For this we consider a diagram as in (2.2.1) and use the octahedral axiom to complete it as in (2.2.2), the only difference being on considering $\widetilde{E_0}$ rather than E_0 in our diagram computation. This, however, leaves a problem to solve at zero, given that $\hat{E}_0 := t_{\leq 0}(\hat{E})$ may not be 0. This can be solved using the following diagram

$$\begin{tikzcd} E_{-1} \arrow[r] \arrow[d] & \widetilde{E}_0 \arrow[r] \arrow[d] & \tau(H^0(E)) \arrow[r] & E_{-1}[1] \arrow[d] \\ E_{-1} \arrow[r] \arrow[d] & E_0 \arrow[r] \arrow[d] & H^0(E) \arrow[r] & E_{-1}[1] \arrow[d] \\ \widetilde{E_0} \arrow[r] & E_0 \arrow[r] & \hat{E}_0 \arrow[r] & \widetilde{E_0}[1] \end{tikzcd} \tag{2.2.3}$$

which we complete to

$$\begin{tikzcd} E_{-1} \arrow[r] \arrow[d] & \widetilde{E}_0 \arrow[r] \arrow[d] & \tau(H^0(E)) \arrow[r] \arrow[d] & E_{-1}[1] \arrow[d] \\ E_{-1} \arrow[r] \arrow[d] & E_0 \arrow[r] \arrow[d] & H^0(E) \arrow[r] \arrow[d] & E_{-1}[1] \arrow[d] \\ \widetilde{E_0} \arrow[r] \arrow[d] & E_0 \arrow[r] \arrow[d] & \hat{E}_0 \arrow[r] \arrow[d] & \widetilde{E_0}[1] \arrow[d] \\ \tau(H^0(E)) \arrow[r] & H^0(E) \arrow[r] & \hat{E}_0 \arrow[r] & \tau(H^0(E))[1] \end{tikzcd} \tag{2.2.4}$$

thus proving that $\hat{E}_0 \in \mathcal{F}$ and hence axiom 3 of t-structure. $\square$

Other nonstandard t-structures which we will be using are the perverse coherent t-structures. These are t-structures in the derived category of coher-

ent sheaves on a scheme. They were first defined by Deligne and recovered by Bezrukavnikov, later together with Arinkin ([Bez00], [AB10]). We shall introduce these in chapter 5.

2.3 Noncommutative projective geometry

The starting point for the Artin-Zhang approach to noncommutative geometry is the famous theorem of Serre below ([Ser55]). It uses the machinery of quotient categories developed by Gabriel ([Gab62]). We fix an embedding of a projective variety X into a projective space $\mathbb{P}^m$, for some $m \in \mathbb{N}$, and we denote its homogeneous coordinate ring by $\bar{\Gamma}(X)$. By definition, this is the graded ring $\bigoplus_{n\in\mathbb{Z}} \Gamma(X, O_X(n))$, where Γ is the functor of gobal sections and $O_X(n)$ is the pullback of $O_{\mathbb{P}^m}(n)$ via the embedding.

Theorem 2.3.1 (Serre, [Ser55]). *Let X be a complex projective variety with $R = \bar{\Gamma}(X)$ a noetherian $\mathbb{C}$-algebra. Then there is an equivalence of categories between $Qcoh(X)$ and $Tails(R) := Gr(R)/Tors(R)$ where $Tors(R)$ is the full subcategory of torsion right modules of the category $Gr(R)$.*

Note that, when written in the lower case, $tails(R) = gr(R)/tors(R)$ is the full subcategory of $Tails(R)$ of finitely generated objects and thus we get a category equivalent to $coh(X)$. The implicit torsion theory in the theorem is the one defined by the torsion class formed by modules such that each cyclic submodule is right bounded, i.e., modules M such that for every element $m \in M$ there is an integer $n \geq 1$ for which $mR_{\geq n} = 0$. In the finitely generated setting this amounts to the class of right bounded modules.

The general idea of noncommutative projective geometry is to study such categories for noncommutative graded algebras R. Unless stated otherwise, we

will be concerned with connected noetherian $\mathbb{N}_0$-graded $\mathbb{K}$-algebras generated in degree 1.

Recall from the general theory of quotient categories ([Gab62]) that there is an adjoint pair of functors $(\pi, \bar{\Gamma})$:

$$\pi : Gr(R) \longrightarrow Tails(R), \quad \bar{\Gamma} : Tails(R) \longrightarrow Gr(R)$$

and they induce, by restriction to the corresponding subcategories, an adjoint pair on the categories of finitely generated objects. We shall keep the same notation for both these functors and their restrictions to $gr(R)$ and $tails(R)$. $\bar{\Gamma}$ can be defined by

$$\bar{\Gamma}(\pi M) = \bigoplus_{i=0}^{\infty} \mathrm{Hom}_{tails(R)}(\pi R, \pi M(i)).$$

Also, we can describe homomorphisms (see [AZ94]) by

$$\mathrm{Hom}_{Tails(R)}(\pi M, \pi N) := \varinjlim_{M' \leq M : M/M' \text{torsion}} \mathrm{Hom}_{Gr(R)}(M', N/\tau(N))$$

where $\tau(N)$ is the torsion part of N. In fact homomorphisms can be described more simply if M is finitely generated.

Lemma 2.3.2 (Artin, Zhang, [AZ94]). *If M is a finitely generated graded module over R, there is a natural isomorphism*

$$Hom_{Tails(R)}(\pi M, \pi N) \cong \varinjlim_{n} \mathrm{Hom}_{Gr(R)}(M_{\geq n}, N).$$

We now define noncommutative projective schemes following Artin and Zhang ([AZ94]). The definition of morphisms has been extensively discussed by Smith (see, for example, [Smi01] and [Smi04]). The definition below is adapted from the one by Artin and Zhang ([AZ94]) and can be found, for example, in the work of Mori ([Mor06]).

Definition 2.3.3. *The* ***noncommutative projective scheme*** *associated with an* $\mathbb{N}_0-$*graded* $\mathbb{K}$*-algebra* R *is the pair* $(tails(R), \pi R)$*, denoted by* $Proj(R)$*.*

A ***morphism*** *of noncommutative projective schemes, from* $Proj(R)$ *to* $Proj(R')$ *is a functor*

$$F : tails(R) \longrightarrow tails(R')$$

such that $F(\pi R) \cong \pi R'$*. A morphism is an* ***isomorphism*** *if* F *is an equivalence of categories.*

When R is implicit, we shall denote πR by O, as it is the structure sheaf of the noncommutative projective scheme.

An interesting class of noncommutative algebras in this context is the class of Artin-Schelter (AS for short) regular algebras. These algebras share many homological properties with rings of polynomials and therefore will provide us with analogues of projective spaces.

Definition 2.3.4. *A connected positively graded* $\mathbb{K}$*-algebra* R *is* ***Artin-Schelter regular of dimension*** d *if:*

1. R *is of global dimension* d*;*

2. R *has finite Gelfand-Kirillov dimension, i.e., dim* $R_j \leq cj^k$*, for some* $c \in \mathbb{R}$ *and* $k \in \mathbb{Z}$*;*

3. R *is Gorenstein, i.e., if we consider* $\mathbb{K}$ *as an* R*-module (with trivial action of elements of strictly positive degree) then* $\bigoplus_{n=-\infty}^{+\infty} \mathrm{Ext}^i_{Gr(R)}(\mathbb{K}, R(n)) = 0$*, for all* $i \neq d$*, and* $\bigoplus_{n=-\infty}^{+\infty} \mathrm{Ext}^d_{Gr(R)}(\mathbb{K}, R(n)) \cong \mathbb{K}(l)$ *as graded* R*-modules for some* $l \in \mathbb{Z}$ *(which we call the Gorenstein parameter of* R*).*

Note that, in the case that R is generated in degree 1, its Gorenstein parameter l actually equals the global dimension d ([AKO08]).

The category $Tails(R)$ is a complete and cocomplete abelian category and it has enough injectives. One can thus define cohomology in $Tails(R)$ through the right derived functors of $\bar{\Gamma}$ (which is only left exact). It turns out that, for AS-regular algebras of global dimension n, the cohomology spaces for the graded shifts of the ring (i.e., $\pi R(k)$) coincide with those of the twists of the structure sheaf over the projective space $\mathbb{P}^{n-1}$ ([AZ94]).

Example 2.3.5. *Let $F^n = \mathbb{K}\langle X_1, ..., X_n\rangle$ be the free $\mathbb{K}$-algebra in n variables and consider I_ω the ideal generated by $X_jX_i - \omega_{ij}X_iX_j$, for all $1 \leq i, j \leq n$, where $\omega_{ij} \in \mathbb{K}^*$. Define $S^n_\omega := F^n/I_\omega$. These algebras are Artin-Schelter regular of dimension n and will be explored in chapter 4 of this thesis. Moreover, using some higher cohomology vanishing conditions for this class of algebras, Auroux, Katzarkov and Orlov proved that $(O, O(1), ..., O(n-1))$ is a strong exceptional sequence in $Proj(S^n_\omega)$ ([AKO08]).*

We shall henceforth assume R to be an $\mathbb{N}_0$-graded AS-regular algebra of global dimension n. An important geometric tool in this theory is the notion of point schemes. We start with point modules.

Definition 2.3.6. *A graded R-module M is said to be a* ***point module*** *if:*

- *M is generated in degree zero;*
- *$M_0 = \mathbb{K}$;*
- *$\dim M_i = 1$, $\forall i \geq 0$.*

It turns out that the point modules of AS-regular algebras of dimension 3 can be parametrized in a natural way by the point scheme ([ATVdB90]). We now recall how the construction of such scheme works in general. Given R as before,

let I be such that $R \cong T(R_1)/I$. Recall that R is finitely generated in degree 1. Note that, for each $i \in \mathbb{N}_0$, I_i can be seen as a set of multilinear functions on $(V^*)^i$, where V is the vector space spanned by R_1. Thus we can define projective schemes associated with I_i,

$$\Omega_i := \left\{(p_1, ..., p_i) \in \mathbb{P}(V^*)^i : f(p_1, ..., p_i) = 0,\ \forall f \in I_i\right\},$$

and clearly we have, for $i \leq j$, a map

$$pr_i^j : \Omega_j \longrightarrow \Omega_i$$

which is the restriction of the projection from $\mathbb{P}(V^*)^j$ to $\mathbb{P}(V^*)^i$ on the first i coordinates. $\left\{\Omega_i, pr_i^j\right\}$ forms an inverse system of projective schemes ([ATVdB90]).

Definition 2.3.7. *Assume the inverse limit of the system $\left\{\Omega_i, pr_i^j\right\}$ is a scheme. Then it is called the* ***point scheme*** *of R and it is denoted by Ω_R (or just Ω if no confusion arises). We shall refer to the* ***point variety*** *when considering the reduced structure of the point scheme.*

In many cases, more relevantly in the ones we shall be studying, the inverse system is eventually constant and thus the point scheme is defined. Moreover, the point variety can actually be computed as a subvariety of the commutative projective space of corresponding dimension, i.e., the point variety of an Artin-Schelter regular algebra of dimension n can be embedded into $\mathbb{P}^{n-1}$.

AS-regular algebras of dimension 3 have been classified ([ATVdB90]). Each of them can be seen as a quotient of the free algebra in r generators by r relations of degree s, where $(r, s) \in \{(2, 3), (3, 2)\}$. Furthermore, to each AS-regular algebra of dimension 3 we can associate a triple (E, σ, L) where E is a subscheme of $\mathbb{P}^2$, $\sigma \in Aut(E)$ and L is an invertible sheaf on E ([ATVdB90]). In the case where $r = 3$ the algebra is said to be **elliptic** if E is a divisor of degree 3 in $\mathbb{P}^2$ and L is

the restriction of $O_{\mathbb{P}^2}(1)$. The only other case to consider is when $E = \mathbb{P}^2$ - and then we say the algebra is **linear**. It can be proven that if R is linear then $A \cong B$ where B is a twisted coordinate ring of $\mathbb{P}^2$ and therefore $tails(R) \cong coh(\mathbb{P}^2)$. In fact E is the point scheme of R.

Point varieties turn out to be, in certain cases, invariants under isomorphism of noncommutative projective schemes.

Theorem 2.3.8 (Mori, [Mor06]). *Let R and R' be graded quotients of quantum polynomial rings of global dimension $n \geq 1$ and Gorenstein parameters r and r' in $\mathbb{Z} \setminus \{0\}$ respectively. If $Proj(R) \cong Proj(R')$ then $\Omega_R \cong \Omega_{R'}$.*

A quantum polynomial ring is, in this context, an Artin-Schelter regular algebra of global dimension n generated in degree 1, with the Hilbert polynomial of the polynomial ring in n variables and satisfying the Cohen-Macaulay property with respect to the Gelfand-Kirillov dimension.

In chapter 4 we use this theorem to give examples of non-isomorphic noncommutative projective spaces as it applies to the algebras S^n_ω of example 2.3.5.

Chapter 3

Mutations of quivers with potentials and derived equivalences

3.1 Introduction

For a quiver with potential, Derksen, Weyman and Zelevinsky defined in 2008 ([DWZ08]) a combinatorial transformation - mutations. Mukhopadhyay and Ray, on the other hand, tell us how to compute Seiberg dual quivers for some quivers with potential through a tilting procedure, thus obtaining derived equivalent algebras ([MR04]). In this chapter, we compare mutations with this approach to Seiberg duality, concluding that for a certain class of potentials and under certain conditions they coincide. Therefore mutations provide us with some derived equivalences.

A broad class of noncommutative algebras can be presented as a path algebra of a quiver with relations. We shall be studying the derived categories of

some of these algebras, namely when their relations can be suitably encoded on a potential via cyclic derivatives as follows. Q_0 and Q_1 denotes the sets of vertices and arrows, respectively, of a quiver Q. $\mathbb{K}Q$ is the path algebra of the quiver Q over $\mathbb{K}$ and our convention is to write concatenation of paths as composition of functions. The following definitions are due to Derksen, Weyman and Zelevisnky ([DWZ08]).

Definition 3.1.1. *A **potential** on a quiver is an element of the vector space spanned by the cycles of the quiver (denote it by $\mathbb{K}Q_{cyc}$).*

Remark 3.1.2. *We will assume throughout this chapter, unless otherwise stated, that every cycle in any potential S is simple, i.e., it does not pass through the same vertex twice.*

Definition 3.1.3. *Let $A = \langle Q_1 \rangle$, i.e., the vector space spanned by all arrows. For each $\xi \in A^*$ (the dual of A), define a **cyclic derivative**:*

$$\partial/\partial\xi : \begin{array}{ccc} \mathbb{K}Q_{cyc} & \to & \mathbb{K}Q \\ a_1 \dots a_n & \mapsto & \sum_{k=1}^{n} \xi(a_k) a_{k+1} \dots a_n a_1 \dots a_{k-1} \end{array} .$$

If $x \in Q_1$, we will denote by $\partial/\partial x$ the cyclic derivative correspondent to the element of A^* which is the dual of x in the dual basis of A. Potentials are regarded as a way to encode the relations of certain path algebras, when the relations are precisely given by the ideal generated by all the cyclic derivatives. Different potentials can, however, define the same set of relations. For example, the same cycle can be written with different starting vertices even though its cyclic derivatives do not depend on such choices. To identify these, the following equivalence relation is introduced.

Definition 3.1.4. *Two potentials are **cyclically equivalent** if $S - S'$ lies in the span of elements of the form $a_1 \dots a_{n-1} a_n - a_2 \dots a_n a_1$. A pair (Q, S) is said to*

be a ***quiver with potential*** *if Q has no loops and no two cyclically equivalent paths appear on S.*

The following notion of (strong) right equivalence will be central in our discussion. However, one needs at this point to introduce the notion of complete path algebra. Recall that $\mathbb{K}Q$ can be seen as $\bigoplus_{i=0}^{\infty} A^i$.

Definition 3.1.5. *The* ***complete path algebra*** *is defined as $\widehat{\mathbb{K}Q} := \prod_{i=0}^{\infty} A^i$.*

Definition 3.1.6. *Two quivers with potentials (Q, S) and (Q', S') are said to be* ***right equivalent*** *if there is isomorphism ϕ between $\widehat{\mathbb{K}Q}$ and $\widehat{\mathbb{K}Q'}$ such that $\phi(S)$ is cyclically equivalent to S'. We shall say that they are* ***strongly right equivalent*** *if we can take ϕ to be an isomorphism between $\mathbb{K}Q$ and $\mathbb{K}Q'$ such that $\phi(S)$ is cyclically equivalent to S'.*

In particular it is clear that strong right equivalence implies right equivalence. We now introduce the algebras of our focus in this chapter.

Definition 3.1.7. *Given a quiver with potential (Q, S), define the* ***Jacobian algebra*** *of (Q, S) as $J(Q, S) = \mathbb{K}Q / \langle J(S) \rangle$, where $J(S) = (\partial S / \partial x)_{x \in Q_1}$. We call $\widehat{J(Q, S)} = \widehat{\mathbb{K}Q} / \langle\langle J(S) \rangle\rangle$ the* ***complete Jacobian algebra****, where $\langle\langle J(S) \rangle\rangle$ is the closure of the ideal generated by $J(S)$ in $\widehat{\mathbb{K}Q}$ in the m-adic topology, for m the maximal in $\widehat{\mathbb{K}Q}$ generated by all arrows.*

Remark 3.1.8. *Note that two strongly right equivalent quivers with potentials have isomorphic Jacobian algebras while two right equivalent ones have isomorphic complete Jacobian algebras ([DWZ08]).*

A very interesting class of examples arises naturally in toric geometry and homological mirror symmetry ([UY07]). These examples are constructed from

bipartite graphs on the torus as we now explain. Let G be a bipartite graph embedded on a torus T, with the two sets of vertices being called W (white) and B (black). We can construct a quiver Q and a potential S as follows:

- The vertices of Q are the faces of G, i.e., the connected component of $T \setminus G$;
- There is an arrow between two vertices of Q if the corresponding faces of G share a common edge;
- The direction of the arrow a in Q is determined by the convention that the white vertex of the corresponding edge in G lies on the right side of a;
- The terms of the potential are the cycles that go around each vertex of G, assigning positive signs to those coming from white vertices and negative sign otherwise.

In some cases the quivers with potential obtained in this way are derived equivalent to toric varieties combinatorially related to the bipartite graphs ([UY07]). To get a quiver with potential we must ensure that no loops are allowed. For this we require the embedding of G to be such that each edge separates two distinct faces.

In the next section we will define mutation and Seiberg duality for a quiver with potential followed by some results on the links between them in section 3.3. Section 3.4 explores an example of algebro-geometric nature and we end this chapter by discussing the results of 3.3 in the 3-Calabi-Yau context.

3.2 Mutation and Seiberg Duality

For a quiver with potential (Q, S), $K^b(Q, S)$ and $D^b(Q, S)$ will denote, respectively, the bounded homotopy category and the bounded derived category of right

modules over $J(Q,S)$. Given an arrow $\alpha \in Q_1$, let $t(\alpha)$ denote the **target** of α and $s(\alpha)$ the **source** of α (i.e., the arrival and departure vertices, respectively).

It is well known (see [Hap87]) that given a path algebra, reflection functors on vertices that are either sources (i.e., vertices with no incoming arrows) or sinks (i.e., vertices with no outgoing arrows) provide us with derived equivalences. Our aim is to identify some derived equivalent algebras and hence we shall consider a generalisation of these reflection functors, DWZ-mutations, for which we need first the following definition and theorem ([DWZ08]).

Definition 3.2.1. *A potential S (or a quiver with potential (Q,S)) is said to be* ***trivial*** *if it is homogeneous of degree 2, i.e., if it is constituted only by 2-cycles. A potential S (or a quiver with potential (Q,S)) is said to be* ***reduced*** *if it has no 2-cycles. For a quiver with potential (Q,S), if m is the ideal generated by the arrows in $\mathbb{K}Q$, we define m_{triv} as the ideal generated by arrows appearing in the two-cycles of the potential and $m_{red} = m/m_{triv}$.*

Note that, since we assume that all cycles in the potential are simple (i.e., no cycle in the potential passes through the same vertex twice), each 2-cycle of S is a summand of S.

Theorem 3.2.2 (Derksen, Weyman, Zelevinsky). *For a quiver with potential (Q,S), there exist a trivial quiver with potential (Q_{triv}, S_{triv}) (where the arrows of Q_{triv} generate m_{triv}) and a reduced quiver with potential (Q_{red}, S_{red}) (where the arrows in Q_{red} generate m_{red}) such that (Q,S) is right equivalent to $(Q_{triv} \oplus Q_{red}, S_{triv} + S_{red})$ (where the arrows in $Q_{triv} \oplus Q_{red}$ generate $m_{triv} \oplus m_{red}$).*

We can now describe the procedure of mutation of a quiver with potential (Q,S) on a vertex k (denote it by $\mu_k(Q,S)$).

1. Suppose k does not belong to any 2-cycle and that S does not have any cycle starting and finishing on k (if it does, substitute it by a cyclically equivalent potential that does not).

2. Change the quiver in the following way:

   - Reflect arrows starting or ending at k. Denote reflected arrows by $(.)^*$;
   - Create one new arrow for each path $\beta\alpha$ of length two, with $\alpha, \beta \in Q_1$ such that $t(\alpha) = s(\beta) = k$ and denote it by $[\beta\alpha]$.

   We denote the resulting quiver by $\tilde{Q}$.

3. Change the potential in the following way:

   - Substitute factors appearing in S of the form $\beta\alpha$ with middle vertex k by the new arrow $[\beta\alpha]$ and denote it by $[S]$;
   - Define $\tilde{S} := \Delta_k + [S]$ where $\Delta_k = \sum_{t(\alpha)=s(\beta)=k} [\beta\alpha]\alpha^*\beta^*$.

4. The mutation at k of (Q, S) is $\mu_k(Q, S) = (\bar{Q}, \bar{S}) := (\tilde{Q}_{red}, \tilde{S}_{red})$.

We proceed to define Seiberg duality ([MR04]). This is a tilting procedure and therefore it is an equivalence of derived categories. To check if a complex is tilting we will have to compute homomorphisms in the derived category between (finitely generated) projective modules. For this we will use remark 2.1.10.

From now on, we will assume that (Q, S) is a quiver with potential with n vertices such that every vertex is contained in some cycle. Let P_i be the projective right module over $J(Q, S)$ associated to the vertex i of Q, i.e., $P_i = e_i J(Q, S)$ where e_i is the stationary path on vertex i. For each vertex k, consider the following complex:

$$T^k = \bigoplus_{i=1}^{n} T_i^k$$

where

$$T_i^k = 0 \longrightarrow P_i \longrightarrow 0, \text{ if } i \neq k$$

(P_i is in degree zero of the complex) and

$$T_k^k = 0 \longrightarrow \bigoplus_{t(\alpha)=k} P_{s(\alpha)} \xrightarrow{(\alpha)} P_k \longrightarrow 0$$

($\bigoplus_{t(\alpha)=k} P_{s(\alpha)}$ is in degree zero of the complex), where (α) denotes the morphism defined in each component of the direct sum by

$$P_{s(\alpha)} \longrightarrow P_k : \ u \mapsto \alpha u.$$

Remark 3.2.3. *We observe that the projective modules $P_i = e_i J(Q,S)$ are indecomposable. This argument is due to Dong Yang and the result follows as a consequence of a lemma proved by Keller and Yang ([KY10]). In their paper, they observe that the projective modules $e_i\Gamma(Q,S)$ associated with the Ginzburg algebra $\Gamma(Q,S)$ - a differential graded algebra defined such that $H^0\Gamma(Q,S) = \widehat{J(Q,S)}$ - are indecomposable (indeed, they prove more: the perfect derived category, $per(\Gamma)$, is Krull-Schmidt). Hence, since*

$$\mathrm{Hom}_{D^b(\Gamma)}(e_i\Gamma, e_i\Gamma) = e_i H^0\Gamma e_i = \mathrm{Hom}_{H^0\Gamma}(e_i H^0\Gamma, e_i H^0\Gamma),$$

the endomorphism algebra of $\hat{P}_i = e_i\widehat{J(Q,S)} = P_i \otimes_{J(Q,S)} \widehat{J(Q,S)}$ is local and hence $\hat{P}_i$ is indecomposable. This implies that P_i is also indecomposable.

Lemma 3.2.4. *T^k is a tilting complex over the Jacobian algebra of (Q,S) if and only if Hom$_{K^b(P(J(Q,S)))}(T_k^k, T_s^k[-1]) = 0$, $\forall s \in Q_0$.*

Proof. We split the proof into two parts: homomorphism vanishing and generation.

First we prove that Hom$_{K^b(P(J(Q,S)))}(T_r^k, T_s^k[i]) = 0$, for all $i \neq 0$ if $r \neq k$ and for all $i > 0$ if $r = k$.

It is clear that if $r, s \neq k$, then $\mathrm{Hom}_{K^b(P(J(Q,S)))}(T_r^k, T_s^k[i]) = 0$, for all $i \neq 0$ (as this is some higher Ext-group between projectives). Now, suppose $s = k$ and $r \neq k$. We only have to check that $\mathrm{Hom}_{K^b(P(J(Q,S)))}(T_r^k, T_k^k[1])$ is trivial. Note that, since a homomorphism between P_r to P_k is identified with an element of the path algebra with each term being a path from r to k, every such homomorphism factors through $\bigoplus_{t(\alpha)=k} P_{s(\alpha)}$.

$$\begin{tikzcd}
 & & 0 \arrow[r] & P_i \arrow[r] \arrow[d] \arrow[dl, dotted] & 0 \\
0 \arrow[r] & \bigoplus_{t(\alpha)=k} P_{s(\alpha)} \arrow[rr] & & P_k \arrow[r] & 0
\end{tikzcd}$$

This factorisation implies that such maps are homotopic to zero, thus zero in the homotopy category.

If $s = r = k$ then we also have such a homotopy just by taking identity maps.

$$\begin{tikzcd}
 & 0 \arrow[r] & \bigoplus_{t(\alpha)=k} P_{s(\alpha)} \arrow[r] \arrow[d] \arrow[dl, dotted] & P_k \arrow[r] \arrow[dl, dotted] & 0 \\
0 \arrow[r] & \bigoplus_{t(\alpha)=k} P_{s(\alpha)} \arrow[r] & P_k \arrow[r] & 0 &
\end{tikzcd}$$

Secondly we check that T^k generates $K^b(P(J(Q,S)))$ as a triangulated category.

It is enough to prove that the stalk complexes of indecomposable projective modules are generated by the direct summands of T^k.

Consider the direct summands of T^k and take the cone of the map from

T_k^k to $\bigoplus_{t(\alpha)=k} T_{s(\alpha)}^k$ defined by:

$$\begin{tikzcd}
0 \arrow[r] & \bigoplus_{t(\alpha)=k} P_{s(\alpha)} \arrow[r] \arrow[d, "id"'] & P_k \arrow[r] & 0 \\
0 \arrow[r] & \bigoplus_{t(\alpha)=k} P_{s(\alpha)} \arrow[r] & 0 &
\end{tikzcd}$$

That cone is just the following complex (the underlined term is in degree zero):

$$0 \longrightarrow \bigoplus_{t(\alpha)=k} P_{s(\alpha)} \xrightarrow{((\alpha),id)} \underline{P_k \oplus (\bigoplus_{t(\alpha)=k} P_{s(\alpha)})} \longrightarrow 0 \tag{3.2.1}$$

Consider the map from the complex (3.2.1) to the stalk complex of P_k in degree zero defined by identity in the first component and $-(\alpha)$ in the second component. Consider also the map from this same stalk complex to (3.2.1) defined by the inclusion of P_k. We observe that the composition of these maps is homotopic to the identity map, hence proving that these complexes are isomorphic in the derived category. In fact, that follows from the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & \bigoplus_{t(\alpha)=k} P_{s(\alpha)} \arrow[r, "((\alpha),id)"] \arrow[d] \arrow[ddl, dotted] & \underline{P_k \oplus (\bigoplus_{t(\alpha)=k} P_{s(\alpha)})} \arrow[r] \arrow[d, "(id,-(\alpha))"] \arrow[ddl, dotted, "(0,id)"'] & 0 . \arrow[ddl, dotted] \\
 & 0 \arrow[r] \arrow[d] & P_k \arrow[r] \arrow[d, "(id,0)"'] & 0 \\
0 \arrow[r] & \bigoplus_{t(\alpha)=k} P_{s(\alpha)} \arrow[r, "((\alpha),id)"] & \underline{P_k \oplus (\bigoplus_{t(\alpha)=k} P_{s(\alpha)})} \arrow[r] & 0
\end{tikzcd}$$

Similarly we can see the same phenomenon for the reverse composition and therefore (3.2.1) is isomorphic to the stalk complex P_k in degree zero.

Therefore, the complex is tilting if and only if the remaining conditions, i.e., $\mathrm{Hom}_{K^b(P(J(Q,S)))}(T_k^k, T_s^k[-1]) = 0$, for all $s \in Q_0$, are verified. $\square$

Definition 3.2.5. *Given a quiver with potential (Q,S), define $\delta(Q,S)$ as the set of vertices for which the complex above is tilting over $J(Q,S)$, i.e.,*

$$\delta(Q,S) = \big\{k \in Q_0 : Hom_{K^b(P(J(Q,S)))}(T_k^k, T_s^k[-1]) = 0,\ \forall s\big\}.$$

If $\delta(Q,S) \neq \emptyset$, then we say that (Q,S) is ***locally dualisable*** *in $\delta(Q,S)$. Furthermore, if $\delta(Q,S) = Q_0$ then we say that (Q,S) is* ***globally dualisable****.*

Remark 3.2.6. *Note that to check whether the complex is tilting we just need to check that, for any $s \neq k$, there is no element $f \neq 0$ in the path algebra such that*

$$\begin{tikzcd} \bigoplus_{t(\alpha)=k} P_{s(\alpha)} \arrow[r, "(\alpha)"] \arrow[d] & P_k \arrow[d, "f"] \\ 0 \arrow[r] & P_s \end{tikzcd}$$

commutes. The existence of such an f implies that the ideal of relations must contain the set $\{f\alpha : t(\alpha) = k\}$. This allows us, given a potential S for Q, to determine $\delta(Q,S)$.

Moreover, observe that if such f exists, then $fJ(Q,S) \cong S_k$, where S_k is the simple module at the vertex k. This means that $soc(P_s) \neq 0$. So, if Hom$(S_k, P_s) = 0$ for all $s \neq k$ then T^k is tilting.

From now on we will drop the superscript on T whenever the vertex with respect to which we are considering the tilting complex is fixed.

Definition 3.2.7. *The* ***Seiberg dual algebra*** *of a quiver Q with potential S (or of its Jacobian algebra) at the vertex $k \in \delta(Q,S)$ is End$_{D^b(Q,S)}(T^k)$, the endomorphism algebra of T^k.*

Rickard's theorem then asserts that Seiberg dual algebras have derived equivalent categories of modules. For an illustrative example see section 3.4.

3.3 Seiberg duality for good potentials

Let us consider the following class of potentials:

Definition 3.3.1. *A potential on a quiver Q is a* ***good potential*** *if its cycles are simple (i.e., do not pass through the same vertex twice), each arrow of Q appears at least twice and no subpath of length two appears repeated in any two distinct cycles of the potential.*

Note that, in particular, a quiver with a good potential has the property that every arrow is contained in at least two distinct cycles.

Proposition 3.3.2. *A quiver with good potential is globally dualisable.*

Proof. This is immediate from the definition of good potential. In fact, the generators for the ideal of relations are of the form $\partial S/\partial a = \sum_{i=1}^{d} \lambda_i v_i$, where $\lambda_i \in \mathbb{K}$. Hence, $d \geq 2$ and the v_i's are paths starting with different arrows thanks to the requirement that no subpath of length two should be repeated in two distinct terms of the potential. Thus, the ideal cannot contain any element of the form $u\alpha$ where u is not a relation and $\alpha \in Q_1$. Therefore $\delta(Q, S) = Q_0$. □

Remark 3.3.3. *Let G be a bipartite graph embedded on a torus (such that each edge separates two distinct faces) and (Q,S) the quiver with potential associated to it as explained in the introduction of this chapter. Under very mild assumptions on G, S is a good potential. In fact, it is always true that each arrow appears exactly twice in S since there are no loops in G and thus each edge of G has two vertices (thus, dually, each arrow appears in two terms of the potential). To guarantee that no subpath of length two appears repeated we just have to ensure that no face of G is limited by only two edges.*

Let (Q, S) be a quiver with good potential. We want to give a presentation of its Seiberg dual algebra at a fixed vertex k. We will see that this algebra is in fact the Jacobian algebra of a quiver with potential, which we shall call the **Seiberg dual quiver**.

First we compute the quiver. It has the same number of vertices as the initial quiver (since the indecomposable projectives of $\mathrm{End}_{D^b(Q,S)}(T)$ correspond to the direct summands of T) and, for each irreducible homomorphism between the T_i's, we draw an arrow between the correspondent vertices. As we shall see later, those irreducible homomorphisms are of three types (the terminology below, used for simplicity of language, is inspired by Mukhopadhyay and Ray's work, [MR04]). Also theorem 3.3.7 shows that our repeated choice of notation below is adequate since the procedure to get the of the Seiberg dual quiver is the same as mutation of the initial quiver.

- Arrows of the form a, where a is also an arrow in Q, will be called **internal arrows**. These arrows correspond to morphisms between T_i^k and T_j^k (which are stalk complexes of projective modules over $J(Q, S)$), for $i \neq k$, that do not factor through the stalk complex of P_k;

- Arrows of the form α^* will be called **dual arrows**. These arrows correspond to morphims either from or to T_k^k;

- Arrows of the form $[\beta\alpha]$ will be called **mesonic arrows**. These arrows correspond to morphisms between T_i^k and T_j^k (which are stalk complexes of projective modules over $J(Q, S)$), for $i \neq k$, that factor through the stalk complex of P_k.

Similarly to the mutation process, we will do Seiberg duality in two main steps:

obtain a quiver $\tilde{Q}$ that may contain more arrows than the irreducible homomorphisms and then, looking at relations, eliminate the appropriate arrows that do not correspond to irreducible ones (those will be the arrows lying in the 2-cycles of the potential). It turns out that relations on the Seiberg dual quiver can also be encoded in a potential (see proposition 3.3.9) and it will be determined as follows:

1. Determine $\tilde{S} := [S] + \sum_{t(\alpha)=s(\beta)=k} [\beta\alpha]\alpha^*\beta^*$ (eventually containing some arrows representing non-irreducible homomorphisms);

2. For every arrow a in a two cycle ab, take the relation $\partial\tilde{S}/\partial a = 0$ and substitute b in $\tilde{S}$ using this equality (and thus eliminate b from the quiver, since b is not irreducible as it can be written as a composition of arrows). Call $\bar{S}$ to the potential thus obtained.

Remark 3.3.4. *Again, for simplicity of language, arrows appearing in two cycles will be called* ***massive arrows*** *and the process described on item 2 of the algorithm above will be called* ***integration over massive arrows****.*

Definition 3.3.5. *If one massive arrow a appears in two or more different 2-cycles of $\tilde{S}$, that is, if we get an expression of the form:*

$$\tilde{S} = \sum_{i=1}^{d} \lambda_i a b_i + \sum_{j=1}^{l} a u_j + W$$

where $\lambda_i \in \mathbb{K}$, each b_i is an arrow, $d \geq 2$, each u_i is a path of length greater or equal than 2 and a does not appear in W, then we say that the b_i's are ***related arrows*** *(by a).*

Given a quiver Q with good potential S, suppose that no related arrows occur in $\tilde{S}$. Then $\tilde{S}$ can be written as follows:

$$\tilde{S} = \sum_{i=1}^{N} (\lambda_i a_i b_i + \sum_{j} \sigma_{i,j} a_i u_{i,j} + b_i v_i) + W \tag{3.3.1}$$

where $\sigma_{i,j}$, $\lambda_i \in \mathbb{K}$, each $a_i b_i$ is a 2-cycle (i.e., a_i and b_i are massive arrows), each b_i is mesonic (thus the coefficient of $b_i v_i$ is 1), u_{ij} does not contain any arrow b_k, v_i is a composition of dual arrows and W does not have any term involving massive arrows. Since there are no related arrows we have $a_i \neq a_j$ and since each b_i is mesonic (and S, being good, does not have repeated subpaths of length two) $b_i \neq b_j$, for all $i \neq j$.

Theorem 3.3.6. *Let Q be a quiver with a good potential S. If k is a vertex such that no related arrows occur in the mutation, there is a strong right equivalence ϕ from $(\tilde{Q}, \tilde{S})$ to $(\tilde{Q}, S' + \bar{S})$, where S' is trivial and $\bar{S}$ is obtained by Seiberg duality.*

Proof. Since there are no related arrows, let us assume that $\tilde{S}$ is of the form (3.3.1). Take the family of homomorphisms given by

$$\begin{aligned} \phi_i : \mathbb{K}\tilde{Q} &\rightarrow \mathbb{K}\tilde{Q} \\ a_i &\mapsto a_i - \tfrac{1}{\lambda_i} v_i \\ b_i &\mapsto b_i - \tfrac{1}{\lambda_i} \textstyle\sum_j \sigma_{i,j} u_{i,j} \\ z &\mapsto z \quad \text{if } z \neq a_i, b_i,\ z \in Q_1 \end{aligned}$$

where i ranges from 1 to N, the number of 2-cycles in $\tilde{S}$. Note that ϕ_i is in fact an automorphism for all $1 \leq i \leq N$: injectivity is clear and all arrows lie in the image since

$$\phi_i(a_i + \frac{1}{\lambda_i} v_i) = a_i \quad \text{and} \quad \phi_i(b_i + \frac{1}{\lambda_i} \sum_j \sigma_{i,j} u_{i,j}) = b_i$$

and since they generate the algebra, surjectivity holds.

Let ϕ be the composition of all ϕ_i's. Then we may compute $\phi(\tilde{S})$ thus getting

$$\phi(\tilde{S}) = \sum_{i=1}^{N} (\lambda_i a_i b_i - \frac{1}{\lambda_i} \sum_j \sigma_{i,j} u_{i,j} v_i) + W$$

whose reduced part is exactly

$$\sum_{i=1}^{N}(-\frac{1}{\lambda_i}\sum_j \sigma_{i,j}u_{i,j}v_i) + W.$$

Now, if we integrate over massive arrows in 3.3.1, taking in account that

$$\partial\tilde{S}/\partial a_i = \lambda_i b_i + \sum_j \sigma_{i,j}u_{i,j} \qquad \partial\tilde{S}/\partial b_i = \lambda_i a_i + v_i$$

and using the relations $\partial\tilde{S}/\partial a_i = 0$ and $\partial\tilde{S}/\partial b_i = 0$ in $\tilde{S}$ we get

$$\sum_{i=1}^{N}(-\frac{1}{\lambda_i}\sum_j \sigma_{i,j}u_{i,j}v_i) + W$$

which is the same as $\phi(\tilde{S})_{red}$. Thus ϕ is such a strong right equivalence. □

The following theorem establishes the desired comparison between the mutated quiver and the Seiberg dual quiver.

Theorem 3.3.7. *Let Q be a quiver with a good potential S such that no related arrows occur in the mutation at a vertex k. Then the jacobian algebra of the quiver with potential obtained by mutation at k is isomorphic to with the Seiberg dual algebra of (Q, S) at k.*

Proof. We start by looking at the shape of the quiver.

1. First we prove that Seiberg duality at k inverts incoming arrows to k. The complex T_k has in degree zero one copy of P_j for every arrow from j to k. Therefore, for each such arrow we get one projection map from the direct sum to P_j and thus an irreducible homomorphism from T_k to T_j, which translates into an arrow from k to j in the dual quiver. For each arrow α_j from j to k, denote the correspondent homomorphism from T_k to T_j by α_j^*. There are no more irreducible homomorphims from T_k to T_j: any other homomorphism factors through some factor of the direct sum first.

2. Now we prove that Seiberg duality at k inverts outgoing arrows from k. This requires the commutativity of a diagram like the following:

$$\begin{tikzcd} 0 \arrow[r] & P_i \arrow[r] \arrow[d, "f"] & 0 \arrow[d] & \\ 0 \arrow[r] & \bigoplus_{t(\alpha)=k} P_{s(\alpha)} \arrow[r, "(\alpha)"] & P_k \arrow[r] & 0 \end{tikzcd} .$$

The diagram commutes if and only if $(\alpha)f = 0$ and so we have to check the relations in the quiver to obtain such a condition. Fix an arrow β from k to i and take the (cyclic) derivative of the potential with respect to β. Since S is a good potential, $\partial S/\partial\beta = \sum_{q=1}^{d} \lambda_q v_q$ where the v_q's are paths from i to k (since βv_q is a cycle for all q) and $d \geq 2$. To give a homomorphism from P_i to $\bigoplus_{t(\alpha)=k} P_{s(\alpha)}$ we just need to give a homomorphism from P_i to each $P_{s(\alpha)}$. Let α_q be the arrow such that $t(\alpha_q) = k$ and it is on the path v_q. Observe that $v_q = \alpha_q \widetilde{v_q}$, where $\widetilde{v_q}$ is a path from i to $s(\alpha_q)$ as in the picture.

$$\begin{tikzcd} & k \arrow[dr, "\beta"] & \\ s(\alpha_q) \arrow[ur, "\alpha_q"] & & i \arrow[ll, dotted, "\widetilde{v_q}"] \end{tikzcd} .$$

Set a homomorphism from P_i to $P_{s(\alpha)}$, for α such that $t(\alpha) = k$, as follows:

- zero if $\alpha \neq \alpha_q$ for some q;
- $\lambda_q \widetilde{v_q}$ if $\alpha = \alpha_q$ for some q;

and set β^* to be the homomorphism induced by this set of homomorphisms to the direct sum and therefore to the complex T_k. Clearly this map makes the diagram above commute. Now we need to prove that it is irreducible. If not, then it factors through other T_r via an element $u \in e_r J(Q,S) e_i$. This would imply that $\widetilde{v_q} = w_q u$ for some $w_q \in e_{s(\alpha_q)} J(Q,S) e_r$, for all $1 \leq q \leq d$,

which cannot happen since the potential is good. Hence β^* is irreducible. By construction, these homomorphisms are the only irreducible ones from T_i to T_k.

3. For each path of length two $\beta\alpha$ such that $t(\alpha) = s(\beta) = k$ we clearly get a homomorphism from $T_{s(\alpha)}$ to $T_{t(\beta)}$. Denote this homomorphism by $[\beta\alpha]$. We show that it is irreducible if and only it is not contained in a two cycle of the potential $\tilde{S}$. Suppose a is an arrow such that $[\beta\alpha]a$ is a 2-cycle of $\tilde{S}$. Then $\partial\tilde{S}/\partial a$ gives an explicit factorisation of the mesonic arrow. On the other hand, if it is not contained in a 2-cycle of $\tilde{S}$ then it is irreducible since it could only factor through the stalk complex of P_k which does not correspond to an indecomposable projective module over $\mathrm{End}_{K^b(Q,S)}(T)$.

4. Finally, if none of the previous cases apply, then the homomorphisms between T_j and T_i that can be irreducible are just arrows from j to i. Again, they are in fact irreducible if and only if they are not contained in a 2-cycle of $\tilde{S}$ and a similar argument to the one above applies to this case.

Let $\tilde{Q}$ be the quiver obtained by taking all homomorphisms above considered (α^* for every arrow α with target k, β^* for every arrow β with source k, $[\beta\alpha]$ for every path $\beta\alpha$ with middle vertex k, and a for every arrow a not starting or ending at k), even if they are not irreducible. Determining $\tilde{Q}$ is clearly the same procedure either via mutations or via Seiberg duality. Now, by 3.3.6, we see that the reduced part of $(\tilde{Q}, \tilde{S})$ can be found by eliminating the 2-cycles of $\tilde{Q}$ appearing in $\tilde{S}$ and taking the potential obtained through integration over those massive arrows (thus eliminating the non-irreducible morphisms). Thus the result follows. □

Corollary 3.3.8. *If Q is a quiver with a good potential S and if k is a vertex such that no related arrows arise in the mutation procedure, then mutation at k*

produces a derived equivalence between the Jacobian algebras of (Q, S) *and of* $\mu_k(Q, S)$.

Proof. From the previous theorem we have $J(\mu_k(Q, S)) \cong \mathrm{End}_{D^b(Q,S)}(T^k)$. Then, given that T^k is a tilting complex over $J(Q, S)$ (by lemma 3.3.2), Rickard's theorem 2.1.9 gives the desired derived equivalence. □

To finish this section, we shall prove that the algorithm previously described actually computes the Seiberg dual potential of a quiver with potential (Q, S) at a fixed vertex k.

Proposition 3.3.9. *The algorithm described in the beginning of this section computes a potential for the Seiberg dual quiver such that its Jacobian algebra is* $\mathrm{End}_{D^b(Q,S)}(T^k)$*, for a quiver with a good potential* (Q, S).

Proof. Let the homomorphisms represented by dual arrows of outgoing arrows be as it is described in the proof of theorem 3.3.7 and keep the notation therein. Denote by $\tau_{\beta\alpha}$ the coefficient of $[\beta\alpha]$ in $[S]$. We will first prove that the relations induced by the potential $\tilde{S}$ obtained through the algorithm above are satisfied in $\mathrm{End}_{D^b(Q,S)}(T)$. Case by case, we analyse relations coming from differentiating:

- with respect to β^* (dual of an outgoing arrow):

$$\partial\tilde{S}/\partial\beta^* = \sum_{t(\alpha)=k} [\beta\alpha]\alpha^* = \beta(\alpha) = 0,$$

  since it is homotopic to zero in the category of complexes;

- with respect to α^* (dual of an incoming arrow):

$$\partial\tilde{S}/\partial\alpha^* = \sum_{s(\beta)=k} \beta^*[\beta\alpha] = (\sum_{s(\beta)=k} \beta^*\beta)\alpha.$$

Let us check that $\sum_{s(\beta)=k} \beta^*\beta = 0$. For this we compute each component of this vector by looking at the occurrences of a fixed γ incoming to k in S. We have in $[S]$ some sub expression of the form

$$\sum_{q=1}^{d} \tau_{\beta_i\gamma}[\beta_i\gamma]\widetilde{v_i}$$

for some β_i's with source k, where each $\tilde{v_i}$ completes the corresponding cycle and $\tau_{\beta_i\alpha} \neq 0$. Then we have the corresponding entry of $\sum_{s(\beta)=k} \beta^*\beta$ given by

$$\sum_{q=1}^{d} \tau_{\beta_i\alpha}\widetilde{v_i}\beta_i$$

which is zero since it equals $\partial S/\partial\gamma$;

- with respect to a, an internal arrow:

$$\partial\tilde{S}/\partial a = \partial[S]/\partial a = 0,$$

since this is essentially the same as $\partial S/\partial a$ (with some extra square brackets);

- and, finally, with respect to $[\beta_i\alpha_j]$ (mesonic arrow):

$$\partial\tilde{S}/\partial[\beta_i\alpha_j] = \alpha_j^*\beta_i^* = 0$$

this follows from the definition of α_j^* and β_i^* as homomorphisms (see proof of theorem 3.3.7).

Integration over massive arrows does not change the relations induced by the potential since the expressions obtained by differentiating with respect to a massive arrow are zero in the Jacobian algebra, according to the proof above.

The last thing we need to check is that this potential $\tilde{S}$ gives generators for the ideal of relations. Let r be a nonzero relation in the new quiver such that

none of its factors are relations (i.e., if $r = uv$ then neither u nor v lie in the ideal of relations). We prove that this relation has already been contemplated. We split the proof into several cases.

- r **does not pass by** k**.** Observe that if r does not involve morphisms to or from T_k^k then it can be expressed as linear combinations of elements of the path algebra of Q from j to i, for some vertices i and j (where we identify such a path with the corresponding endomorphism of T^k). Therefore there are some internal arrows $a_1, ..., a_n$ such that a linear combination of $\partial S/\partial a_1$,...,$\partial S/\partial a_n$ equals r up to square brackets.

- r **passes by** k **and** $t(r), s(r) \neq k$**.** If r involves morphisms both to T_k^k and from T_k^k, then each of its terms involve both duals of arrows β outgoing from k (β^* is the natural map from T_i^k to T_k^k defined by the relation $\partial S/\partial \beta$ - see proof of 3.3.7) and duals of arrows α incoming to k (α^* is just a projection map - see proof of theorem 3.3.7). Then r can be also be identified with some linear combination of paths in Q, as the factor involving dual arrows can be read as the projection of a component of β^*, which is an element of $\mathbb{K}Q$. Since r is a zero morphism and none of its factors are relations, it can be identified with a linear combination of terms of the form $\partial S/\partial a_i$ for some internal arrows a_i from $t(r)$ to $s(r)$. Now, each a_i is also an arrow in $\tilde{Q}$ and therefore r is a linear combination of $\partial \tilde{S}/\partial a$.

- $t(r) = k$ **and** $s(r) = l \neq k$**.** Suppose $r = \sum_i \beta_i^* r_i$ where each r_i is an element of $\mathbb{K}\tilde{Q}$. Then, as a map from T_l^k to T_k^k, it is identified with n elements of $J(Q, S)$, $u_{s(\alpha)}$, starting at l and ending at some $s(\alpha)$ (where $t(\alpha) = k$), n being the number of terms in the direct sum $\bigoplus_{\alpha: t(\alpha)=k} P_{s(\alpha)}$ (see proof of theorem 3.3.7). Each $\beta_i^* r_i$ appears in at least two components of the

direct sum, by construction of β_i^*, since S is good. In each such component, $\beta_i^* r_i$ provides a summand of $u_{s(\alpha)}$ (and $u_{s(\alpha)}$ yields a zero morphism from P_l to $P_{s(\alpha)}$). Also we identify that summand with an element of $\mathbb{K}Q$ by definition of β_i^*. In order to be zero, $u_{s(\alpha)}$ must have as a factor some relation in $J(Q,S)$ and thus the summand mentioned above contains as a factor some terms of this factor. Furthermore, this factor of the summand must contain the terms coming from β_i^* otherwise r would not be irreducible (factoring through the projective corresponding to the target of this factor).Now, in order to be able to read a relation involving the terms from the morphisms β_i^*, r_i must pass through the vertex k (and the relation is the factor of that component which starts at k). This follows from the definition of β_i^* as a morphism and from the fact that the potential S is good, not allowing repetition of subpaths of length two. Therefore all terms in each component $P_{s(\alpha)}$ begin with a common nontrivial path from $l \neq k$ to k. This path must be the same in every component since β_i^* appears at least in two components and such pairs of components do not coincide for any given two indices i, j (this also follows from S being good). Since r is irreducible (in the sense above), this path can only be an arrow to k (or a scalar multiple of it), otherwise r would factor through some T_m^k. This is because arrows to k in Q are no longer arrows of $\tilde{Q}$.Denote this arrow by γ. Hence, $r_i = r_i'\gamma$ for some paths r_i' starting at k with different arrows (otherwise, again, r would split) and, as an element of $J(Q,S)$, $\sum_i \beta_i^* r_i' = 0$, i.e., $\sum_i \beta_i^* r_i'$ is a n-tuple of relations in $J(Q,S)$ from k to all the vertices of the form $s(\alpha)$ with $t(\alpha) = k$. Therefore, by construction of β_i^*, $r_i' = \beta_i$ for all i (again because S is good) and the sum needs to run over all arrows β_i starting at k. Thus $r = \sum_{s(\beta)=k} \beta^*[\beta\gamma]$ which is precisely $\partial\tilde{S}/\partial\gamma^*$.

- $s(r) = k$ **and** $t(r) = l \neq k$**.** If r starts at k, the first arrow appearing in each term is a dual of an arrow α incoming to k (again, α^* is just a projection map). This is a situation different in nature to the previous ones: we are looking at a map from the direct sum $\bigoplus_{\alpha:t(\alpha)=k} P_{s(\alpha)}$ to some projective P_l and hence we are not able to identify r with a relation of $J(Q,S)$. Therefore the fact that r is zero must come from the fact that the map is homotopic to zero in the category of complexes over $J(Q,S)$. That is equivalent to the existence of a linear combination of paths in Q, call it u, from k to l such that $u(\alpha) = r$ in $J(Q,S)$. Since r is minimal, u must irreducible - and the space of irreducible maps from P_k to P_l has the arrows between k and l as a basis. Therefore r must be a linear combination of terms of the form $\sum_{t(\alpha)=k}[\beta\alpha]\alpha^*$ (which is precisely $\partial\tilde{S}/\partial\beta^*$) for each β a summand of u.

To complete the proof we need to show that no such relation r can both start and end at k. Suppose we have such a map r from T_k^k to T_k^k, i.e.,

$$\begin{tikzcd}
0 \arrow[r] & \bigoplus_{\alpha:t(\alpha)=k} P_{s(\alpha)} \arrow[r, "(\alpha)"] \arrow[d, "r_0"] & P_k \arrow[r] \arrow[d, "r_1"] & 0 \,. \\
0 \arrow[r] & \bigoplus_{\alpha:t(\alpha)=k} P_{s(\alpha)} \arrow[r, "(\alpha)"] & P_k \arrow[r] & 0
\end{tikzcd}$$

It can not be identified with an element of $J(Q,S)$ so there is a homotopy to zero h such that $h(\alpha) = r_0$. On the other hand, this is also a homotopy to zero for r_0 as a map from T_k^k to $\bigoplus_{\alpha:t(\alpha)=k} T_{s(\alpha)}^k$ and r_0 lies in the ideal of relations of $J(\tilde{Q},\tilde{S})$ and it is covered by the cases above. Since r_1 is determined from r_0 by the commutation of the diagram as a linear combination of cycles in $J(Q,S)$, this means that r is generated by the relations contemplated above.

This completes the proof.

$\square$

3.4 An example

We shall exemplify mutation on a quiver with potential arising in derived algebraic geometry. Given a Del Pezzo surface, we can study its derived category of coherent sheaves using the existence of a strong exceptional sequence.

Theorem 3.4.1 (Kuleshov, Orlov, Hille, Perling, [KO95], [HP08]). *If X is a Del Pezzo Surface, we have strong exceptional sequences of sheaves given by:*

- $\{O, O(1), O(2)\}$ *if* $X = \mathbb{P}^2$
- $\{O, O(1,0), O(0,1), O(1,1)\}$ *if* $S = \mathbb{P}^1 \times \mathbb{P}^1$
- $\{O, O(E_1), ..., O(E_r), O(1), O(2)\}$ *if X is dP_r with $r \leq 8$, where each E_i is an exceptional curve of the blow up and dP_r is the Del Pezzo obtained by blowing up $1 \leq r \leq 8$ points in $\mathbb{P}^2$.*

As mentioned in the introduction, the direct sum of a strong exceptional sequence over a projective variety X is a tilting sheaf, yielding a derived equivalence between $coh(X)$ and $mod(\mathbb{K}Q/I)$ for some quiver Q and some ideal of relations I. These are determined by looking at the irreducible homomorphisms between the sheaves in the sequence and taking relations between those homomorphisms.

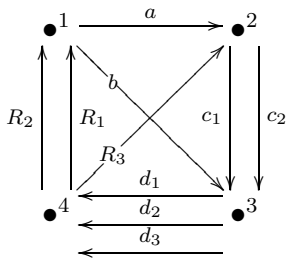
We shall focus on X as in example 2.1.7, i.e., a blow-up of $\mathbb{P}^2$ at one point. To get a derived equivalence to a Jacobian algebra of a quiver with potential, we ought to consider not X itself but $Y = \omega_X$ - the total space of the canonical bundle of X - instead. This is a local Calabi-Yau three-fold. If we let $\pi : Y \to X$ be the natural projection, we get that $\tilde{B} = End_Y(\oplus_i \pi^* E_i)$ is derived equivalent to $coh(Y)$, whenever the exceptional sequence $(E_i)_i$ is geometric over X ([Bri05]). Geometric in this context means that its associated helix satisfies some extra

Ext-vanishing conditions ([Bri05]), but we will not explore this. We proceed to characterise the algebra $\tilde{B}$ and that is enough for our purposes.

The algebra $\tilde{B}$ can be seen as the path algebra of a quiver with relations. It can be obtained from the correspondent quiver of a geometric exceptional sequence $(E_i)_i$ adding one arrow for each generator of the ideal of relations in the opposite direction of the composition of arrows in that relation ([Seg08]). This will be a quiver with potential, where the potential is the sum of the cycles obtained through the composition of each new arrow with the correspondent relations. This process is also described in [ABS08]. In fact it is easy to observe in our concrete example that the quiver with potential obtained via this construction is the same whether we consider the exceptional sequence of example 2.1.7 or other sequences frequently found in the literature ([Kin97],[Per09])

Example 3.4.2. *The algebra $\tilde{B}$ associated to X, $\mathbb{P}^2$ blown-up at one point, with exceptional sequence $\{O, O(E_1), O(1), O(2)\}$ is:*

$Q =$



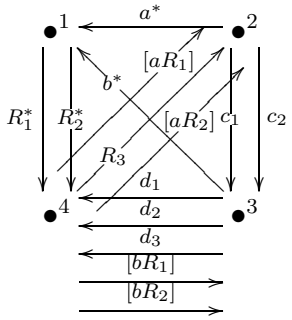
with potential:

$$S = R_3(d_3c_1 - d_1c_2) + R_1(d_1b - d_2c_1a) + R_2(d_2c_2a - d_3b)$$

Note that this is a good potential. For any fixed vertex k, it is easy to check that no related arrows occur in the mutation. Hence, the results in this section yield that mutations give derived equivalent path algebras. Since the one

above is derived equivalent to $coh(Y)$ *([Bri05], [BS09]), so will be* $J(\mu_k(Q,S))$*. Let us present* $\mu_1(Q,S)$*.*

$\tilde{Q} =$



We take a cyclically equivalent potential since there are terms on it starting and ending at 1*. Then we substitute paths of length two passing through* 1 *by new arrows and add* Δ_1*.*

$$\tilde{S} = R_3d_3c_1 - R_3d_1c_2 - d_2c_1[aR_1] + d_1[bR_1] - d_3[bR_2] + d_2c_2[aR_2]$$

$$+ [aR_1]R_1^*a^* + [aR_2]R_2^*a^* + [bR_1]R_1^*b^* + [bR_2]R_2^*b^*$$

Clearly this potential is not reduced. Following the proof of theorem 3.3.6, let us condsider the following (strong) right equivalence:

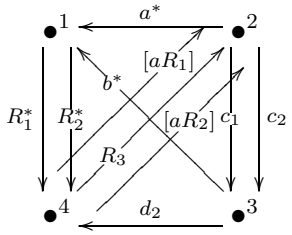
$$\begin{array}{rcl} \phi : \mathbb{K}\tilde{Q} & \to & \mathbb{K}\tilde{Q} \\ d_1 & \mapsto & d_1 - R_1^*b^* \\ d_3 & \mapsto & -d_3 + R_2^*b^* \\ [bR_1] & \mapsto & [bR_1] + c_2R_3 \\ [bR_2] & \mapsto & [bR_2] + c_1R_3 \\ u & \mapsto & u \ \ \textit{if}\ u \neq d_1, d_3, [bR_1], [bR_2], \ \ u \in Q_1 \end{array}.$$

If we compute $\phi(\tilde{S})$*, it is of the form* $S' + \bar{S}$ *and thus we can take the reduced part. More simply, we can integrate over massive arrows by taking the relations:*

$$[bR_1] = c_2R_3, \ [bR_2] = c_1R_3.$$

In any case, as proved in theorem 3.3.6, we get the same result which is:

$\bar{Q} =$



with potential

$$\bar{S} = c_2R_3R_1^*b^* + c_1R_3R_2^*b^* + d_2c_2[aR_2] - d_2c_1[aR_1] + [aR_1]R_1^*a^* + [aR_2]R_2^*a^*.$$

3.5 3-Calabi-Yau algebras

Suggested by the example of section 3.4, we investigate ideas of the previous sections in the 3-Calabi-Yau context. Here, the restrictions on the Jacobian algebras will be of homological rather than of combinatorial nature. 3-Calabi-Yau (3-CY for short) algebras are, in general, quotients of smooth algebras by ideals of relations coming from potentials ([Gin06]). In fact very recent results of Van den Bergh ([VdB10]) show that complete 3-CY algebras come from quivers with potential. We use the following definition.

Definition 3.5.1. *A $\mathbb{K}$-algebra R is said to be n-Calabi-Yau ($n \geq 1$) if:*

1. *R is homologically smooth, i.e., as an $R^{op} \otimes R$-module it has a finite resolution by finitely generated projective modules;*

2. *RHom$_{R^{op}\otimes R}(R, R^{op} \otimes R) \cong R[-d]$ in $D^b(R^{op} \otimes R)$.*

This definition is due to Ginzburg ([Gin06]). The following lemma is crucial to our approach. In fact it is common to find in the literature definitions of Calabi-Yau algebra based on the duality of the lemma.

Lemma 3.5.2 (Keller, [Kel08]). *Let R be an n-Calabi-Yau algebra. Suppose $X, Y \in D^b(Mod(R))$ such that $X \in D^b(fd(R))$, i.e., X is a complex of finite dimensional modules over $\mathbb{K}$. Then we have a canonical isomorphism*

$$\mathrm{Hom}_{D^b(Mod(R))}(X, Y)^* \cong \mathrm{Hom}_{D^b(Mod(R))}(Y, X[n]), \tag{3.5.1}$$

where $*$ denotes $\mathbb{K}$-duality.

Remark 3.5.3. *For an n-Calabi-Yau algebra R, it is clear that $D^b(fd(R))$ is Hom-finite, i.e., the Hom-spaces are finite dimensional over $\mathbb{K}$. Indeed, the duality in lemma 3.5.2 applied twice (which is possible when both X and Y are elements of $D^b(fd(R))$) shows that*

$$\mathrm{Hom}_{D^b(Mod(R))}(X, Y)^{**} = \mathrm{Hom}_{D^b(Mod(R))}(X[n], Y[n]) = \mathrm{Hom}_{D^b(Mod(R))}(X, Y).$$

The results obtained by Keller and Yang on the relations between mutations and derived equivalences ([KY10]) are far more general than the remarks we present here. There, it is proven that mutations hold derived equivalences between the dg-algebras obtained through Ginzburg's construction ([Gin06]) over the complete Jacobian algebra. 3-CY complete Jacobian algebras are such that the associated Ginzburg dg-algebras have their cohomology concentrated in degree zero (and equal to the original algebra). Our approach, however, will be as before, not working on the complete setting nor making use of Ginzburg's differential graded construction. Also Iyama and Reiten have obtained similar results for mutations of quivers without potentials ([IR08]).

Let R be a 3-Calabi-Yau algebra such that there is (Q, S) quiver with potential satisfying $J(Q, S) = R$. Let every vertex of Q be contained in some cycle (this seems to be a reasonable assumption as we can see from the graded 3-Calabi-Yau case - [Boc08]) and let Q be without loops or two cycles.

Fix a vertex k in Q. We want to prove that T^k is tilting for any vertex k of Q. It is enough to prove that Hom$(S_k, P_s) = 0$ for all $s \neq k$ (see 3.2.6). Indeed, as a consequence of 3.5.2 we have the following result

Corollary 3.5.4. *If R is n-CY algebra, then Hom$(S_k, P_s) = 0$ for all $s \neq k$ and hence T^k is tilting for any vertex k of Q.*

Proof. Lemma 3.5.2 shows that

$$Hom_{D^b(mod(R))}(S_k, P_s)^* = Hom_{D^b(mod(R))}(P_s, S_k[n]) = Ext^n(P_s, S_k) = 0$$

and thus the result follows. □

Remark 3.5.5. *If we take as definition of a Calabi-Yau algebra the existence of a duality (3.5.1) in $D^b(fd(R))$, then it is still possible to prove corollary 3.5.4 through a result proved by Iyama and Reiten ([IR08]). Indeed, they prove that for such algebras the duality can be extended to work also when one of the variables is a complex in $K^b(P(R))$. Even though their results are primarily concerned with finite dimensional algebras, the result is true in this generality as well.*

We are now able to prove similar results to the ones obtained in previous sections.

Theorem 3.5.6. *If $J(Q, S)$ is a 3-Calabi-Yau algebra, then End$_{K^b(J(Q,S))}(T^k) \cong J(\tilde{Q}, \tilde{S})$ where $(\tilde{Q}, \tilde{S})$ are obtained in the process of mutation at k before reduction. Furthermore, $\widehat{\mathrm{End}_{\mathrm{K^b(J(Q,S))}}}(\mathrm{T^k}) \cong J(\widehat{\mu_k(Q, S)})$, where $\mu_k(Q, S)$ is the reduced part of $(\tilde{Q}, \tilde{S})$.*

Proof. Let us fix a vertex k and drop the superscript on T^k for simplicity. First we take the indecomposable projective modules T_i of End$_{K^b(J(Q,S))}(T)$ and determine 'candidates' to irreducible homomorphisms between them. This gives us a quiver (call it G). We'll start by proving that $G = \tilde{Q}$

1. *Inversion of incoming arrow*: The argument on item 1 of proof 3.3.7 works here;

2. *Inversion of outgoing arrows*: This requires the commutativity of a diagram of the form:

$$\begin{tikzcd} 0 \arrow[r] & P_i \arrow[r] \arrow[d, "f"] & 0 \arrow[d] & \\ 0 \arrow[r] & \bigoplus_{t(\alpha)=k} P_{s(\alpha)} \arrow[r, "(\alpha)"] & P_k \arrow[r] & 0 \end{tikzcd}$$

i.e. the existence of an f such that $(\alpha)f = 0$. Thus we have to look for relations on the quiver that may allow us to obtain such f. Fix an arrow β from k to i and differentiate the potential with respect to β getting $\partial S/\partial\beta = \sum_{t=1}^{d} \lambda_t v_t$ where the v_t's are paths from i to k (since βv_t is a cycle for all t). To give a homomorphism from P_i to $\bigoplus_{t(\alpha)=k} P_{s(\alpha)}$ we just need to give a homomorphism from P_i to each $P_{s(\alpha)}$, by the universal property of the direct sum. Define $\alpha^{-1}\gamma$ for any path γ to be zero if γ does not end with the arrow α and to be u if $\gamma = \alpha u$ for some $u \in \mathbb{K}Q$. Then we can define the following maps:

$$\begin{aligned} \beta^*_\alpha : P_i &\rightarrow P_{s(\alpha)} \\ \gamma &\mapsto \alpha^{-1}\tfrac{\partial S}{\partial \beta}\gamma \end{aligned}$$

and set β^* to be the homomorphism induced by this set of homomorphisms in the direct sum and therefore to the complex T_k. Clearly this map makes the diagram above commute, as

$$(\alpha)\beta^* = \sum_{t(\alpha)=k} \alpha\beta^*_\alpha = \sum_{t(\alpha)=k} \alpha(\alpha^{-1}\frac{\partial S}{\partial \beta}) = \frac{\partial S}{\partial \beta}$$

which is zero in the Jacobian algebra. Now we need to prove that this is irreducible.

Suppose this homomorphism is not irreducible, factoring through T_l for some $l \in Q_0$. Then we have the following diagram:

$$\begin{tikzcd}
P_i \arrow[r] \arrow[d, "h"'] & 0 \arrow[d] \\
P_l \arrow[r] \arrow[d, "g"'] & 0 \arrow[d] \\
\bigoplus_{t(\alpha)=k} P_{s(\alpha)} \arrow[r, "(\alpha)"] & P_k
\end{tikzcd}$$

where $\beta^* = gh$ and each square commutes. The commutativity of the bottom diagram requires the existence of such relation in $J(Q,S)$. If we denote this relation by θ, then

$$\frac{\partial S}{\partial \beta} = (\alpha)\beta^* = (\alpha)gh = \theta h.$$

Now, let R be a minimal set of generators of the ideal of $\mathbb{K}Q$ generated by all the cyclic derivatives of the potential S. We recall that the dimension of $\text{Ext}^1(S_j, S_l)$ (respectively $\text{Ext}^2(S_j, S_l)$), for $j, l \in Q_0$, measure the number of arrows from l to j (respectively the number of elements in R from l to j). This can be understood by computing a projective resolution for S_j. Then, since $J(Q,S)$ is 3-CY, we have:

$$|\{r \in R : t(r) = k,\ s(r) = i\}| = \dim \text{Ext}^2(S_k, S_i) =_{3-CY}$$

$$= \dim \text{Ext}^1(S_i, S_k) = |\{a \in Q_1 : t(a) = i,\ s(a) = k\}|,$$

However this yields a contradiction since, by the equation above, the relation induced by β is not in R (θ is, and θ is not induced by β as $l \neq i$). Thus β^* is irreducible.

3. *Gluing arrows* The argument on item 3 of proof 3.3.7 works here.

4. Finally, if none of the previous cases apply, then homomorphisms between T_j and T_i are just arrows from j to i. Again, these homomorphisms are irreducible if and only if they are not contained in a 3-cycle of the potential going through k and a similar argument to the one above applies to this case.

Let G then be the quiver obtained by taking all the homomorphisms considered in the cases above, even if they are not irreducible. We just proved that this quiver is the same as $\tilde{Q}$. Using proposition 3.3.9 we have that $\mathrm{End}_{K^b(J(Q,S))}(T) \cong J(G, \tilde{S}) = J(\tilde{Q}, \tilde{S})$. Now, since $(\tilde{Q}, \tilde{S})$ is right equivalent to $\mu_k(Q, S)$, we have an isomorphism of complete path algebras as stated. □

Remark 3.5.7. *Note that we need to consider completions because, in general, the removal of 2-cycles in the mutation procedure is not guaranteed. Derksen, Weyman and Zelevinsky have produced examples of such phenomenon ([DWZ08]). Indeed, we can only produce a strong right equivalence using the techniques of section 3.3 when the mutated quiver has no 2-cycles. Therefore, we have a derived equivalence between $J(Q, S)$ and $J(\tilde{Q}, \tilde{S})$ but we cannot guarantee the existence of a strong right equivalence between $(\tilde{Q}, \tilde{S})$ and $\mu_k(Q, S)$.*

Chapter 4

Equivalences for noncommutative projective spaces

4.1 Introduction

In this chapter we look closely at some noncommutative projective spaces. Consider the family of quadratic graded algebras, with deg$(X_i) = 1$:

$$S_\omega^n = \mathbb{C}\langle X_1, ..., X_n\rangle / \langle X_j X_i - \omega_{ij} X_i X_j,\ i, j \in \{1, ..., n\}\rangle$$

where $\omega_{ij} \in \mathbb{C}^*$, for all i and j, and $\omega_{ij}\omega_{ji} = 1$.

We will study the family of noncommutative projective spaces associated to these algebras, $\mathbb{P}_\omega^{n-1} := Proj(S_\omega^n) = (tails(S_\omega^n), \pi S_\omega^n)$. We denote the structure sheaf of $\mathbb{P}_\omega^{n-1}$ by O_ω^n instead of πS_ω^n. This follows Artin and Zhang's ([AZ94]) formulation of noncommutative projective geometry. Our target is to study graded Morita equivalences, birational equivalences and point varieties in this family of spaces.

First, we observe that no noncommutative projective spaces of different dimensions are isomorphic in the sense of definition 2.3.3. We provide an argument for this straight away, as it will ease the notation throughout the chapter.

As in example 2.3.5, we recall the following result ([AKO08]).

Proposition 4.1.1. *The categories $D^b(tails(S^n_\omega))$ are generated by a strong exceptional sequence of length n, namely $(O^n_\omega, O^n_\omega(1), ..., O^n_\omega(n-1))$.*

Such strong exceptional sequence gives us a derived equivalence between the categories $tails(S^n_\omega)$ and $mod(B^n_\omega)$ ([AKO08]) where

$$B^n_\omega := End_{D^b(tails(S^n_\omega))}(\bigoplus_{i=0}^{n-1} O^n_\omega(i)).$$

This algebra B^n_ω can be presented as a path algebra with relations, the quiver being the Beilinson quiver (n ordered vertices, n arrows between any two consecutive vertices) and the relations being described as follows. Let α^k_l be the i-th arrow starting at vertex k (i.e., representing multiplication by X_l, mapping $O^n_\omega(k-1)$ to $O^n_\omega(k)$), then the ideal of relations is

$$\left\langle \alpha^k_j \alpha^{k-1}_i - \omega_{ij} \alpha^k_i \alpha^{k-1}_j,\ 1 \leq i, j \leq n, 1 \leq k \leq n-1 \right\rangle.$$

For example, B^4_ω has a presentation as the path algebras of

$$\begin{tikzcd}
\bullet^1 \arrow[r, "\alpha^1_1"] \arrow[r, "\alpha^1_2"'] \arrow[r, shift left=3, "\alpha^1_3"] \arrow[r, shift right=3, "\alpha^1_4"'] & \bullet^2 \arrow[r, "\alpha^2_1"] \arrow[r, "\alpha^2_2"'] \arrow[r, shift left=3, "\alpha^2_3"] \arrow[r, shift right=3, "\alpha^2_4"'] & \bullet^3 \arrow[r, "\alpha^3_1"] \arrow[r, "\alpha^3_2"'] \arrow[r, shift left=3, "\alpha^3_3"] \arrow[r, shift right=3, "\alpha^3_4"'] & \bullet^4
\end{tikzcd}$$

with relations $\alpha^k_j \alpha^{k-1}_i = \omega_{ij} \alpha^k_i \alpha^{k-1}_j$, where $k \in \{2, 3\}$ and $1 \leq i \neq j \leq 4$.

It is easy to see that such a strong exceptional sequence forms a basis for the Grothendieck group of the derived category (see, for example, [BS09]). Therefore, the number of elements in such sequence is preserved via derived equivalence.

Lemma 4.1.2. *If $D^b(tails(S^n_\omega)) \cong D^b(tails(S^m_{\omega'}))$ then $m = n$.*

In particular if the categories themselves are equivalent (and thus derived equivalent) we get $m = n$. From now on we are thus interested in comparing the categories associated to S^n_ω and $S^n_{\omega'}$. The superscript n will be dropped when understood and assumed to be fixed.

We shall start by looking at the categories of graded modules. Useful in this context is the theory of twisting systems for graded algebras, introduced by Zhang ([Zha96]), which we now recall.

Let R and S be connected $\mathbb{N}_0$-graded right noetherian $\mathbb{C}$-algebras.

Definition 4.1.3. *A **twisting system** is a set $\tau = \{\tau_n : n \in \mathbb{N}_0\}$ of $\mathbb{C}$-linear, degree preserving bijections from R to R satisfying:*

$$\tau_n(y\tau_m(z)) = \tau_n(y)\tau_{n+m}(z),\ \forall y \in R_m, \forall z \in R_l, \forall l, m, n \in \mathbb{N}_0, .$$

*Given a twisting system over R we can construct a new algebra structure on the underlying vector space R, the **twisted algebra**, by defining new multiplication $\bullet$ as:*

$$y \bullet z := y\tau_m(z),\ \forall y \in R_m, \forall z \in R_l.$$

These two algebra structures are related by the theorem below.

Theorem 4.1.4. *[Zhang,[Zha96]] For R and S as above such that $R_1 \neq 0$, the following are equivalent:*

1. *R is isomorphic (as a graded algebra) to a twist of S;*
2. *$Gr(R)$ is equivalent to $Gr(S)$;*
3. *There is an equivalence Φ between $Tails(R)$ and $Tails(S)$ such that shifts of the structure sheaf are preserved, i.e., $\Phi(\pi R(n)) = \pi S(n)$, for all $n \in \mathbb{Z}$.*

This result motivates our approach. The next section is dedicated to the study of birational and graded Morita equivalences between the algebras S^n_ω while in section 4.3 focus on the computation of the point varieties. Finally section 4.4 contains some interesting examples that show how certain properties of spaces of the form $\mathbb{P}^2_\omega$ do not generalise to spaces of the form $\mathbb{P}^3_\omega$.

4.2 Graded modules and birational equivalence

We start by looking at this family of algebras for a fixed dimension and classify them up to graded Morita equivalence. Later in this section we look at their birational classification. Throughout this chapter, Σ_n will denote the symmetric group in n letters.

Lemma 4.2.1. *S_ω is isomorphic as a graded algebra to $S_{\omega'}$ if and only if there is $\sigma \in \Sigma_n$ such that $\omega'_{\sigma(i)\sigma(j)} = \omega_{ij}$, for all $i, j \in \{1, ..., n\}$.*

Proof. Let Ψ be a graded isomorphism from S_ω to $S_{\omega'}$ and let C be the corresponding element in $GL_n(\mathbb{C})$ such that $\Psi(X_j) = \sum_i c_{ij} X_i$. The matrix C induces a graded endomorphism Φ of the free algebra F in n variables, thus making the following diagram commute:

$$\begin{tikzcd} F \arrow[r, "\Phi"] \arrow[d, "p_\omega"'] & F \arrow[d, "p_{\omega'}"] \\ F/I_\omega = S_\omega \arrow[r, "\Psi"'] & S_{\omega'} = F/I_{\omega'} \end{tikzcd} .$$

Thus, we have $\Phi(I_\omega) = I_{\omega'}$. Given that Φ is graded, the images of the standard generators of I_ω are of degree two and thus they are linear combinations of the standard generators of $I_{\omega'}$. In particular the coefficients of the image of such a relation at X_k^2 have to be zero, for all k. From similar simple observations,

we get the following equations:

$$c_{ki} = 0 \vee c_{kj} = 0 \vee \omega_{ij} = 1, \ \forall 1 \leq i < j \leq n, \ \forall 1 \leq k \leq n \tag{4.2.1}$$

$$c_{kj}c_{li} - \omega_{ij}c_{ki}c_{lj} = \omega'_{kl}(\omega_{ij}c_{li}c_{kj} - c_{lj}c_{ki}). \tag{4.2.2}$$

Suppose that $\omega_{ij} \neq 1$, for all $1 \leq i < j \leq n$. Then for any fixed row, equations (4.2.1) guarantee that given any two entries, one of them is necessarily zero. Thus in each row there is a unique nonzero entry and, since the matrix is invertible, the same applies to columns. Thus, there is $\sigma \in \Sigma_n$ such that $\Phi(X_i) = c_{\sigma(i)i}X_{\sigma(i)}$. So we get:

$$\Phi(X_jX_i - \omega_{ij}X_iX_j) = c_{\sigma(i)i}c_{\sigma(j)j}(X_{\sigma(j)}X_{\sigma(i)} - \omega_{ij}X_{\sigma(i)}X_{\sigma(j)})$$

and therefore $\omega'_{\sigma(i)\sigma(j)} = \omega_{ij}$, since Φ is linear.

If some of the parameters ω_{ij} are 1, the matrix C might not be of the form above described. Note however that, for $\omega_{ij} = 1$, equation (4.2.2) is the same as $M_{k,l,i,j} = \omega'_{kl}M_{k,l,i,j}$, where $M_{k,l,i,j}$ stands for the 2×2 minor given by rows k, l and columns i, j. As C is invertible, for each i, j such that $\omega_{ij} = 1$, there is k, l such that $M_{k,l,i,j} \neq 0$. Moreover, the nonsingularity of C assures that we can make these choices so that they do not coincide for distinct pairs i, j with $\omega_{ij} = 1$.

This means that we can create a new matrix C' by deleting some entries of C (i.e., setting them to be zero), leaving the chosen nonzero 2×2 minors, in such a way that $C' \in GL_n(\mathbb{C})$ and satisfies:

$$c'_{ki} = 0 \vee c'_{kj} = 0, \ \forall 1 \leq i < j \leq n, \ \forall 1 \leq k \leq n.$$

Hence, equations (4.2.1) and (4.2.2) hold for C' and thus C' provides an isomorphism $\tilde{\Psi}$ from S_ω to $S_{\omega'}$ of the form $\tilde{\Psi}(X_i) = c'_{\sigma(i)i}X_{\sigma(i)}$, as above. This proves that $\omega'_{\sigma(i)\sigma(j)} = \omega_{ij}$. The converse follows from the construction of $\tilde{\Psi}$.

□

The following definition will prove useful and, later, natural.

Definition 4.2.2. *Given $\tau \in \Sigma_n$ a k-cycle, $k \leq n$, define the τ-cyclic q-number by $q_\tau(\omega) := \prod_{i=1}^{k} \omega_{\tau^{i-1}(v)\tau^i(v)}$, for any $1 \leq v \leq n$ which is not fixed by τ.*

It is clear that the definition does not depend on the choice of v.

Proposition 4.2.3. *The following conditions are equivalent:*

1. $\exists \sigma \in \Sigma_n,\ \exists (m_1, ..., m_n) \in \mathbb{C}^{*n} :\ \omega'_{\sigma(i)\sigma(j)} = m_i m_j^{-1} \omega_{ij}$;
2. $Gr(S_\omega) \cong Gr(S_{\omega'})$;
3. *There is an equivalence between $Tails(S_\omega)$ and $Tails(S_{\omega'})$ preserving the shifts of the structure sheaf;*
4. $B_\omega \cong B_{\omega'}$;
5. $\exists \sigma \in \Sigma_n : \forall \tau \in \Sigma_n$, τ *k-cycle:* $q_\tau(\omega') = q_{\sigma^{-1}\tau\sigma}(\omega)$;
6. $Mod(B_\omega) \cong Mod(B_{\omega'})$.

Remark 4.2.4. *Note that it is sufficient to check condition (5) for cycles of length 3. In fact, if $k < n$, $\tau = (a_1 a_2 ... a_k)$, and b is fixed by τ, then*

$$q_\tau(\omega) = (\prod_{i=1}^{k} q_{(b a_i a_{i+1})}) q_{(b a_k a_1)}.$$

If τ is of maximal length (i.e., $k = n$) we can write

$$q_\tau(\omega) = q_{(a_1 ... a_{n-1})}(\omega) q_{(a_1 a_{n-1} a_n)}(\omega)$$

and then repeat the previous step to write $q_{(a_1 ... a_{n-1})}(\omega)$ (and thus $q_\tau(\omega)$) as a product of q-numbers of length 3. This argument shows that we only need the information provided by the cyclic q-numbers of length 3, $q_{(abc)}(\omega)$, with fixed a.

Let us now prove the proposition.

Proof. Theorem 4.1.4 tells us that (2)⇔(3). We shall prove (1)⇔(2), (1)⇔(5), (1)⇒(4), (4)⇒(3) and (4)⇔(6).

Let us start with (1)⇒ (2). Suppose that (1) holds. Let f be the algebra automorphism of S_ω defined by $f(X_i) = m_i X_i$, for all $1 \leq i \leq n$. Consider the twisting system $\{f^n : n \in \mathbb{Z}\}$. The twisted algebra is such that $X_i \bullet X_j = m_j X_i X_j$ and $X_j \bullet X_i = m_i X_j X_i$. Since $X_j X_i = \omega_{ij} X_i X_j$ we get:

$$X_j \bullet X_i = m_i \omega_{ij} X_i X_j = m_i m_j^{-1} \omega_{ij} X_i \bullet X_j = \tilde{\omega}_{ij} X_i \bullet X_j$$

and thus the twisted algebra is just $S_{\tilde{\omega}}$ which, by the previous lemma, is isomorphic to $S_{\omega'}$. By Zhang's theorem 4.1.4 we have (2).

To prove the converse, (2)⇒ (1), we start by observing that, since S_ω^n is generated in degree 1, for a twisting system τ we have

$$(S_\omega^n)^\tau = \mathbb{C}\langle X_1, ..., X_n \rangle / \left\langle X_j \tau_1^{-1}(X_i) - \omega_{ij} X_i \tau_1^{-1}(X_j),\ 1 \leq i < j \leq n \right\rangle \tag{4.2.3}$$

and thus the twisted algebra is completely determined by how τ_1 acts in the degree 1 component of S_ω^n, $(S_\omega^n)_1$. Let τ_1 act in $(S_\omega^n)_1$ by a matrix $C \in GL_n(\mathbb{C})$. We observe that the twisted algebra induced by any conjugate of C in $GL_n(\mathbb{C})$ is isomorphic to the twisted algebra induced by C. Indeed, given $P = (p_{ij})_{1 \leq i,j \leq n} \in GL_n(\mathbb{C})$ let $Y_1, ..., Y_n$ be the basis of $(S_\omega^n)_1$ such that $Y_i = \sum_{j=1}^n p_{ij} X_j$ and let A be an algebra such that P induces an algebra isomorphism between S_ω^n and A (A can be presented as the algebra generated by $Y_1, ..., Y_n$ with relations obtained by rewriting the each X_i in the standard relations of S_ω^n in the new basis). Now let $\tilde{\tau}_1$ be such that its action in A_1 is determined by PCP^{-1}. Thus, as linear maps on $A_1 \cong (S_\omega)_1$, τ_1 and $\tilde{\tau}_1$ coincide. Hence, if $\tilde{\tau}$ is a twisting system of A containing

$\tilde{\tau}_1$ we have $A^{\tilde{\tau}} \cong S^{\tau}_{\omega}$. But since S^n_ω and A are isomorphic as graded algebras, we also have $S^{\tau}_{\omega} \cong S^{\tilde{\tau}}_{\omega}$. Hence, we may assume without loss of generality that C is in Jordan normal form, i.e.,

$$C = \begin{pmatrix} m_1 & 0 & 0 & ... & 0 \\ \delta_1 & m_2 & 0 & ... & 0 \\ 0 & \delta_2 & m_3 & ... & 0 \\ ... & ... & ... & ... & ... \\ 0 & ... & 0 & \delta_{n-1} & m_n \end{pmatrix}, \tag{4.2.4}$$

where $m_1, ..., m_n \in \mathbb{C}^*$ and $\delta_1, ..., \delta_{n-1} \in \{0, 1\}$. Note also that, for a matrix of this form, $\tau_1^{-1}(X_i)$ is a linear combination of $X_1, ..., X_{i-1}$.

We say that the set of parameters ω is non-repetitive if $\omega_{ij} \neq \omega_{kl}$ for any $1 \leq i < j \leq n$ and $1 \leq k < l \leq n$. It is easy to see that for a non-repetitive set of parameters, the only normal elements (i.e., elements x of S_ω such that $xS_\omega = S_\omega x$) of degree 1 are scalar multiples of $X_1, X_2, ..., X_n$. For that purpose we recall the following result from [LLR06].

Lemma 4.2.5 (Launois, Lenagan, Rigal,[LLR06]). *Let R be a prime noetherian ring and suppose that d, s are normal elements of R such that dR is prime and $s \notin dR$. Then, there is a unit $v \in R$ such that $sd = vds$.*

It is clear that X_k generates a (completely) prime ideal since the respective factor algebra is a domain (it is of the form $S^{n-1}_{\omega|}$ for some $\omega|$ a restriction of parameters). Since all invertible elements are scalars, if $\sum_{i=1}^n a_i X_i$ is normal then, by the lemma above,

$$X_k \sum_{i=1}^n a_i X_i = (\sum_{i=0}^n a_i \omega_{ik} X_i) X_k = \lambda_k (\sum_{i=1}^n a_i X_i) X_k$$

which implies that for all $a_i \neq 0$, $\omega_{ik} = \lambda_k$. Since ω is non-repetitive, all but one of the coefficients a_i are equal to zero.

Note that for every S_ω there is some non-repetitive set of parameters $\tilde{\omega}$ such that $Gr(S_\omega) \cong Gr(S_{\tilde{\omega}})$. In fact we can obtain such an algebra $S_{\tilde{\omega}}$ by choosing a suitable matrix C of the form (4.2.4) with $\delta_1 = ... = \delta_{n-1} = 0$ (i.e., a diagonal matrix) determining a twisting system. To do this, consider G to be the finitely generated subgroup of $\mathbb{C}^*$ generated by ω_{ij}, $1 \leq i, j \leq n$. Clearly one can choose n elements of $\mathbb{C}^*$, $a_1, ..., a_n$, such that they generate a subgroup H of $\mathbb{C}^*$ of rank n intersecting G trivially (if this was not the case, G would necessarily be infinitely generated). Define C as the diagonal matrix formed by these numbers. The twisted algebra is such that (see $(1) \Rightarrow (2)$) $\tilde{\omega}_{ij} = a_i a_j^{-1} \omega_{ij}$. If $\tilde{\omega}_{ij} = \tilde{\omega}_{kl}$ then $a_i a_j^{-1} a_k a_l^{-1} = \omega_{ij}^{-1} \omega kl \in G$ and since H has rank n, $k = j$ and $l = i$ and therefore $\tilde{\omega}$ is non-repetitive.

Now if $Gr(S_\omega) \cong Gr(S_{\omega'})$, then $Gr(S_\omega) \cong Gr(S_{\tilde{\omega}})$ for some non-repetitive set of parameters $\tilde{\omega}$ defined by $\tilde{\omega}_{ij} = a_i a_j^{-1} \omega'_{ij}$ for some $a_1, ..., a_n \in \mathbb{C}^*$. Therefore, if we prove that there is $\sigma \in \Sigma_n$ such that $\tilde{\omega}_{\sigma(i)\sigma(j)} = b_i b_j^{-1} \omega_{ij}$ then we also have

$$\omega'_{\sigma(i)\sigma(j)} = a_{\sigma(j)} a_{\sigma(i)}^{-1} \tilde{\omega}_{\sigma(i)\sigma(j)} = a_{\sigma(j)} a_{\sigma(i)}^{-1} b_i b_j^{-1} \omega_{ij}$$

which implies that, by taking $m_i = a_{\sigma(i)}^{-1} b_i$, we get the wanted result. It is then enough to prove the result for $S_{\omega'}$ with non-repetitive ω'.

Suppose $Gr(S_\omega) \cong Gr(S_{\omega'})$ with non-repetitive ω'. By theorem 4.1.4 $S_{\omega'} \cong (S_\omega)^\tau$ for some twisting system τ and, moreover, by our argument above τ can be taken to be determined by a matrix C in Jordan normal form (4.2.4). This implies, in particular, that $X_1, ..., X_n$ form a polynormal sequence in $(S_\omega)^\tau$, i.e., X_1 is normal and X_i is normal in the quotient ring $(S_\omega)^\tau / \langle X_1, ..., X_{i-1} \rangle$. This is due to the fact that $\tau_1^{-1}(X_i) = m_i^{-1} X_i + f_i$ where f_i is a linear combination

of $X_1, ..., X_{i-1}$. By the observations above, since ω' is non-repetitive, $(S_\omega)^\tau$ has, up to scalar multiples, exactly n normal elements of degree 1. Therefore, if ψ is an isomorphism from $(S_\omega^n)^\tau$ to $S_{\omega'}$ there is $1 \leq k \leq n$ such $\psi X_1 = \lambda X_k$ for some $\lambda \in \mathbb{C}^*$ and thus ψ establishes an isomorphism from $(S_\omega^n)^\tau / X_1 (S_\omega^n)^\tau$ to $S_{\omega'|}^{n-1}$ where $\omega'|$ is the set of parameters naturally obtained by restriction, i.e., by forgetting ω_{ik} and ω_{kj} for all $i < k \leq n$ and $n \geq j > k$ (and thus $\omega'|$ is non-repetitive). Similarly, each factor ring $(S_\omega)^\tau / \langle X_1, ..., X_{i-1} \rangle$ is isomorphic to $S_{\omega'|}^{n-i+1}$ for some suitable restriction of ω' and therefore it has, up to scalar multiples, precisely $n - i + 1$ normal elements of degree 1. Since the projections of the normal elements of degree 1 of $((S_\omega)^\tau)_1$ in such a factor ring are normal as well (and there are $n-i+1$ of them), we conclude that these projections coincide, up to scalar multiples, with the normal elements of degree 1 in the factor ring. Therefore, in $((S_\omega)^\tau)_1$ the normal elements must be of the form:

$$\begin{aligned} Y_1 &= X_1 \\ Y_2 &= X_2 + \lambda_{21} X_1 \\ Y_3 &= X_3 + \lambda_{32} X_2 + \lambda_{31} X_1 \ . \\ &\dots \\ Y_n &= X_n + \textstyle\sum_{k=1}^{n-1} \lambda_{nk} X_k \end{aligned} \tag{4.2.5}$$

Clearly we must have $Y_j Y_i = \omega'_{\sigma(i)\sigma(j)} Y_i Y_j$ for some $\sigma \in \Sigma_n$ by lemma 4.2.1.

We are now ready to prove that, if $(S_\omega)^\tau \cong S_{\omega'}$ with τ_1 acting in $(S_\omega)_1$ by a matrix C of the form 4.2.4 and with ω' non-repetitive then there is $\sigma \in \Sigma_n$ such that $\omega'_{\sigma(i)\sigma(j)} = m_i m_j^{-1}$, the $m_1, ..., m_n$ being the elements in the diagonal of C. For this we use induction on the number n of variables. For $n = 2$ it is well-known that S_ω^2 is a always twist of the polynomial ring in two variables given by the $\tau_1(X_1) = X_1$ and $\tau_1(X_2) = \omega_{12}^{-1} X_2$. More generally, one can twist S_ω^2 to $S_{\omega'}^2$ by defining $\tau_1(X_1) = X_1$ and $\tau_1(X_2) = \omega'_{12} \omega_{12}^{-1} X_2$. Moreover, if the twisted

algebra is determined by a non-diagonalisable matrix, an easy calculation shows that the parameter of the twisted algebra remains the same (i.e., $\omega'_{12} = mm^{-1}\omega_{12}$ where m is the element appearing twice in the diagonal of the Jordan normal form). Thus the result holds for $n = 2$. Suppose now the result is valid for $n-1$ and let ψ be an isomorphism from $(S^n_\omega)^\tau$ to $S^n_{\omega'}$ with τ and ω' are as before. Then, applying our induction hypothesis to the quotient $(S^n_\omega)^\tau / \langle X_1 \rangle \cong S^{n-1}_{\omega'|}$, where $\omega'|$ is a suitable restriction of ω' (forgetting $X_k = \psi(X_1)$), we conclude that there is a bijection $\tilde{\sigma}$ from $\{2, ..., n\}$ to $\left\{1, ..., \hat{k}, ..., n\right\}$ such that $\omega'_{\tilde{\sigma}(i)\tilde{\sigma}(j)} = m_i m_j^{-1}\omega_{ij}$, where $m_2, ..., m_n$ are the elements in the diagonal of (4.2.4). Now, consider the basis of normal elements of $(S^n_\omega)^\tau$ given by $Y_1, ..., Y_n$ as in (4.2.5). We have

$$Y_1 Y_j = \omega'_{\sigma(j)\sigma(1)} Y_j Y_1$$

where σ is determined by $\tilde{\sigma}$ such that $\sigma(1) = k$. Expanding this equation we get

$$X_1(X_j + \sum_{k=1}^{j-1} \lambda_{jk} X_k) = \omega'_{\sigma(j)\sigma(1)}(X_j + \sum_{k=1}^{j-1} \lambda_{jk} X_k) X_1$$

which by using the relations in $(S^n_\omega)^\tau$ means that

$$X_1(X_j + \sum_{j=1}^{j-1} \lambda_{jk} X_k) = \omega'_{\sigma(j)\sigma(1)} m_1 X_1(\omega_{1j} m_j^{-1} X_j + f_j + \sum_{k=1}^{j-1} \omega_{1k} m_k^{-1} \lambda_{jk} X_k + f_k)$$

where each f_k is a linear combination of $X_1, ..., X_{k-1}$. Therefore, by linear independence of the terms, looking at the coefficients of $X_1 X_n$ we conclude that

$$1 = \omega'_{\sigma(j)\sigma(1)} m_1 m_j^{-1} \omega_{ij}$$

which finishes our proof.

Auroux, Katzarkov and Orlov proved (4)⇒(3) and we recall the main ingredients of their argument ([AKO08]). Suppose $B_\omega \cong B_{\omega'}$ via Φ and consider the chain of equivalences given by

$$D^b(tails(S_\omega)) \longrightarrow D^b(mod(B_\omega)) \longrightarrow D^b(mod(B_{\omega'})) \longrightarrow D^b(tails(S_{\omega'})).$$

They prove that such chain takes $O_\omega(i)$ to $O_{\omega'}(i)$, for all $i \in \mathbb{Z}$ (this fact is clear for $0 \leq i \leq n-1$ since the middle equivalence is induced from Φ). Observing that this sequence is ample ([AKO08]), and that it is preserved by the functor, a previous result by Bondal and Orlov ([BO01], [Orl97]) regarding autoequivalences preserving an ample sequence yields the result. A good account of this result can also be found in Huybrechts' book ([Huy06]).

We will now prove (1)⇒(4). Let $\sigma \in \Sigma_n$ and $m_1, ..., m_n$ as in (1). Choose a_i^1 and a_i^2 such that $m_i = \frac{a_i^2}{a_i^1}$ and inductively define $a_i^k = \frac{a_i^{k-2}}{a_i^{k-1}}$. Further define $\Phi : B_\omega \longrightarrow B_{\omega'}$ by $\Phi(\alpha_l^k) = a_l^k \alpha_{\sigma(l)}^k$ and (1) easily implies that the ideal of relations in B_ω is mapped to the ideal of relations in $B_{\omega'}$, thus making Φ an isomorphism.

It is straightforward to check that (1)⇒(5): in fact, following remark 4.2.4 take σ as in (1) and, given a 3-cycle $\tau = (abc)$,

$$q_\tau(\omega') = \omega'_{ab}\omega'_{bc}\omega'_{ca}$$

$$= \frac{m_a}{m_b}\omega_{\sigma^{-1}(a)\sigma^{-1}(b)}\frac{m_b}{m_c}\omega_{\sigma^{-1}(b)\sigma^{-1}(c)}\frac{m_c}{m_a}\omega_{\sigma^{-1}(c)\sigma^{-1}(a)}$$

$$= \omega_{\sigma^{-1}(a)\sigma^{-1}(b)}\omega_{\sigma^{-1}(b)\sigma^{-1}(c)}\omega_{\sigma^{-1}(c)\sigma^{-1}(a)} = q_{\sigma^{-1}\tau\sigma}(\omega).$$

Conversely, note first that the existence of a collection of $\{m_i\}_{1\leq i\leq n}$ as in (1) is equivalent to the existence of a collection $\{\lambda_{ij}\}_{1\leq i,j\leq n}$ of nonzero complex numbers such that

$$\lambda_{ij}\lambda_{jk} = \lambda_{ik} \quad \text{and} \quad \lambda_{ij}\lambda_{ji} = 1 \tag{4.2.6}$$

where $\lambda_{ij} = m_i m_j^{-1}$ (and given such collection, we can choose m_1 and set $m_j = m_1\lambda_{1j}^{-1} = m_1\lambda_{j1}$). Let us assume (5). To prove (1), we shall fix σ as in (5) and find appropriate λ_{ij}'s satisfying (4.2.6) such that $\omega'_{\sigma(i)\sigma(j)} = \lambda_{ij}\omega_{ij}$. By remark 4.2.4 it is enough to consider 3-cycles (which simplifies notation a great deal).

Fix $1 < k < l < s \leq n$ and let $\sigma(i) = k$, $\sigma(j) = l$ and $\sigma(t) = s$. Consider the cycle of length 3 $(1kl)$. By hypothesis we have $q_{(1kl)}(\omega') = q_{\sigma^{-1}(1kl)\sigma}(\omega)$ and thus we can write

$$\omega'_{kl} = \underbrace{\frac{\omega_{\sigma^{-1}(1)i}\omega_{j\sigma^{-1}(1)}}{\omega'_{1k}\omega'_{l1}}}_{=:\lambda_{ij}} \omega_{ij}. \tag{4.2.7}$$

Similarly, by considering the cycle $(1ls)$, one gets

$$\lambda_{jt} = \frac{\omega_{\sigma^{-1}(1)j}\omega_{t\sigma^{-1}(1)}}{\omega'_{1l}\omega'_{s1}} \tag{4.2.8}$$

and, by considering $(1ks)$,

$$\lambda_{it} = \frac{\omega_{\sigma^{-1}(1)i}\omega_{t\sigma^{-1}(1)}}{\omega'_{1k}\omega'_{s1}}. \tag{4.2.9}$$

We need to prove that the definition of λ_{ij} does not depend on the choice of the cycle and that they satisfy equation (4.2.6). This is the same as showing that

$$\frac{\omega_{\sigma^{-1}(1)i}\omega_{j\sigma^{-1}(1)}}{\omega'_{1k}\omega'_{l1}} = \frac{\omega_{\sigma^{-1}(v)i}\omega_{j\sigma^{-1}(v)}}{\omega'_{vk}\omega'_{lv}}$$

for any $v \notin \{1, k, l\}$. This is clear since the equation above is equivalent to

$$\frac{q_{(vkl)}(\omega')/\omega'_{kl}}{q_{(1kl)}(\omega')/\omega'_{kl}} = \frac{q_{(\sigma^{-1}(v)ij)}(\omega)/\omega_{ij}}{q_{(\sigma^{-1}(1)ij)}(\omega)/\omega_{ij}}$$

given that, by hypothesis, $q_{(vkl)}(\omega') = q_{(\sigma^{-1}(v)ij)}(\omega)$ and $q_{(1kl)}(\omega') = q_{(\sigma^{-1}(1)ij)}(\omega)$.

Finally we see that

$$\lambda_{ij}\lambda_{jt} = \frac{\omega_{\sigma^{-1}(1)i}\omega_{j\sigma^{-1}(1)}}{\omega'_{1k}\omega'_{l1}} \frac{\omega_{\sigma^{-1}(1)j}\omega_{t\sigma^{-1}(1)}}{\omega'_{1l}\omega'_{s1}} = \frac{\omega_{\sigma^{-1}(1)i}\omega_{t\sigma^{-1}(1)}}{\omega'_{1k}\omega'_{s1}} = \lambda_{it}$$

since $\omega_{ab}\omega_{ba} = 1$, for all a and b. Thus we get (1).

Clearly (4) implies (6). The converse follows from the fact that B_ω is a basic algebra. Indeed, a progenerator in $mod(B_\omega)$ such that its endomorphism algebra is $B_{\omega'}$ (thus basic as well) has to be B_ω. Thus Morita equivalence in this case implies isomorphism (see [Erd90] for more details). $\square$

Remark 4.2.6. *Minamoto and Mori have recently showed that the graded Morita equivalence classes within certain families of algebras (S^n_ω in our case) depend only on the isomorphism classes of certain related finite dimensional algebras (B^n_ω in our case). In that sense, their results generalise the previous proposition ([MM10]).*

The context in which these cylic numbers appear naturally will be explored below. They are essential ingredients for the birational classification of these spaces.

Definition 4.2.7. *The* ***function division ring*** *of $\mathbb{P}^{n-1}_\omega$ is*

$$\mathbb{C}(\mathbb{P}^{n-1}_\omega) := Frac_{Gr}(S^n_\omega)_0 = \left\{ fg^{-1} : f, g \in h(S^n_\omega), g \neq 0, deg(f) = deg(g) \right\}.$$

$\mathbb{P}^{n-1}_\omega$ and $\mathbb{P}^{n-1}_{\omega'}$ are said to be ***birationally equivalent*** *if $\mathbb{C}(\mathbb{P}^{n-1}_\omega) \cong \mathbb{C}(\mathbb{P}^{n-1}_{\omega'})$.*

Let T_ω be the algebra of quantum Laurent polynomials containing S_ω as a subalgebra, i.e., $T_\omega = \mathbb{C}\left\langle X_1^{\pm 1}, ..., X_n^{\pm 1} \right\rangle / \left\langle X_j X_i - \omega_{ij} X_i X_j,\ i, j \in \{1, ..., n\} \right\rangle$. Richard classified, up to isomorphism, this family of algebras, which are called quantum tori ([Ric02]).

Lemma 4.2.8 (Richard, [Ric02]).

1. *$T_\omega \cong T_{\omega'}$ if and only if there is a matrix $A = (a_{ij}) \in GL_n(\mathbb{Z})$ such that $\omega'_{ij} = \prod_{1 \leq k,l \leq n} \omega_{kt}^{a_{ki} a_{tj}}$;*
2. *T_ω is simple if and only if there is not a nonzero vector $a = (a_1, ..., a_n) \in \mathbb{Z}^n$ such that, for all $1 \leq j \leq n$, $\omega_{1j}^{a_1} ... \omega_{nj}^{a_n} = 1$;*
3. *If T_ω is simple, then $T_\omega \cong T_{\omega'}$ if and only if $Frac(T_\omega) \cong Frac(T_{\omega'})$, where $Frac(T_\omega)$ is the division ring of right fractions of T_ω.*

We shall refer to ω as *generic* whenever T_ω is simple.

Remark 4.2.9. *Observe an important fact about q-cyclic numbers: they show up when commuting fractions as follows:*

$$X_jX_k^{-1}X_iX_k^{-1} = \omega_{ki}X_jX_iX_k^{-2} = \omega_{ki}\omega_{ij}X_iX_jX_k^{-2} = \tag{4.2.10}$$

$$= \omega_{ki}\omega_{ij}\omega_{jk}X_iX_k^{-1}X_jX_k^{-1} = q_{(ijk)}(\omega)X_iX_k^{-1}X_jX_k^{-1}.$$

Define ${}^kS_q := (S_\omega[X_k^{-1}])_0$. By the above, It is easy to see that

$${}^kS_q \cong \mathbb{C}\left\langle Y_1, ..., \hat{Y}_k, ..., Y_n \right\rangle / \left\langle Y_jY_i - q_{(kij)}(\omega)Y_iY_j,\ \forall 1 \leq i,j \leq n, i,j \neq k \right\rangle$$

where $Y_i = X_iX_k^{-1}$. Let T_q be the torus associated with ${}^kS_q$, i.e., the algebra of Laurent polynomials containing ${}^kS_q$ as a subalgebra.

Remark 4.2.10. *As the notation suggests, the isomorphism class of T_q does not depend on k. This can be shown using lemma 4.2.8 and keeping in mind relations of the type $q_{(234)} = q_{(123)}q_{(134)}q_{(142)}$. In practice, we will take $k = 1$.*

In the following proposition we denote $q_{(1ij)}(\omega)$ by q_{ij} and $q_{(1ij)}(\omega')$ by q'_{ij}.

Proposition 4.2.11. *If there is a matrix $A = (a_{ij})_{2\leq i,j\leq n}$ in $GL_{n-1}(\mathbb{Z})$ such that $q'_{ij} = \prod_{2\leq k,l\leq n} q_{kl}^{a_{ki}a_{lj}}$, then $\mathbb{P}^{n-1}_\omega$ and $\mathbb{P}^{n-1}_{\omega'}$ are birationally equivalent. Moreover, if $q = (q_{ij})_{1\leq i,j\leq n-1}$ is generic then the converse is also true.*

Proof. We shall prove that $Frac(T_q) \cong \mathbb{C}(\mathbb{P}^{n-1}_\omega)$, where T_q is as previously defined. Consider the map $\phi : Frac(T_q) \longrightarrow \mathbb{C}(\mathbb{P}^{n-1}_\omega)$ such that $\phi(Y_i) = X_iX_1^{-1}$. Note that this is well defined, as $X_jX_1^{-1}X_iX_1^{-1} = q_{ij}X_iX_1^{-1}X_jX_1^{-1}$ (see equation (4.2.10)). Clearly ϕ is injective as it is a nonzero map defined on a division ring, $Frac(T_q)$. Now, suppose $fg^{-1} \in \mathbb{C}(\mathbb{P}^{n-1}_\omega)$, with f monomial. Then we can write

$$fg^{-1} = (gf^{-1})^{-1} = (\sum_i g_if^{-1})^{-1}$$

where the g_i's are monomials as well and deg(g_i) =deg(f). Thus $g_i f^{-1}$ is a product of powers of some $X_{i_j} X_1^{-1}$'s and therefore fg^{-1} lies in the image of $Frac(T_q)$, proving surjectivity of ϕ.

If q is generic, T_q is simple. By proposition 4.2.8, $\mathbb{C}(\mathbb{P}^{n-1}_\omega) \cong \mathbb{C}(\mathbb{P}^{n-1}_{\omega'})$ if and only if $T_q \cong T'_q$ (and clearly we only need generic q for the nontrivial implication). The same result also shows that this is the case if and only if there is $A \in GL_{n-1}(\mathbb{Z})$ such that $q'_{ij} = \prod_{1 \le k,l \le n-1} q_{kl}^{a_{ki}a_{lj}}$, hence finishing the proof. □

4.3 The point variety

In this section we compute the point variety (recall definition 2.3.7) of $\mathbb{P}^n_\omega$ in terms of its parameters ω_{ij}. Once again, this depends only on the q-cyclic numbers. First we recall a well-known corollary of Artin, Tate and Van den Bergh's work ([ATVdB90]) as observed by Mori ([Mor06]).

Lemma 4.3.1 (Artin, Tate, Van den Bergh, [ATVdB90]). *Let $R = T(V)/I$ be a quadratic graded $\mathbb{C}$-algebra, where V is a finite dimensional complex vector space. If $V(I) \subset \mathbb{P}(V^*) \times \mathbb{P}(V^*)$ is the graph of an automorphism of a closed $\mathbb{C}$-subscheme, E, then E is the point scheme of R.*

Proof. Since I is generated in degree 2, we have $I_3 = T_1 I_2 + I_2 T_1$ and therefore $\Omega_3 = \Omega_2 \times \mathbb{P}(V^*) \cap \mathbb{P}(V^*) \times \Omega_2$ ([ATVdB90]). Since $\Omega_2 = \{(x, \sigma(x)) : x \in E\}$ for some automorphism σ of E, we get

$$\Omega_3 = \left\{(x, \sigma(x), \sigma^2(x)) : x \in E\right\}.$$

Clearly $E \cong \Omega_2 \cong \Omega_3$ and, by induction, $E \cong \Omega_d$ for all $d \ge 2$. Thus the point scheme is E as the inverse system is constant. □

We can now compute the point varieties of S^n_ω.

Proposition 4.3.2. *The point variety of S^n_ω is the subvariety of $\mathbb{P}^{n-1}$ defined by $\bigcap_{q_{(ijk)}(\omega)\neq 1} V(X_iX_jX_k)$, where $V(X_iX_jX_k)$ is the zero locus of $X_iX_jX_k$ in $\mathbb{P}^{n-1}$.*

Proof. This proof was first sketched by Hattori ([Hat09]). It also uses some ideas from Vancliff ([Van94]). For practical purposes we shall consider the generators of I^n_ω rewritten in the form $\theta_{ji}X_jX_i - \theta_{ij}X_iX_j$ where θ is a fixed $n \times n$ matrix of parameters in $\mathbb{C}^*$ such that $\theta_{ij}\theta_{ji}^{-1} = \omega_{ij}$.

Following the notation set up in section 2.3, we have $\Omega_2 \subset \mathbb{P}^{n-1} \times \mathbb{P}^{n-1}$ and let us consider E_1 (respectively E_2) to be the image of Ω_2 under the projection map on the first (respectively second) component from $\mathbb{P}^{n-1} \times \mathbb{P}^{n-1}$ to $\mathbb{P}^{n-1}$. The first step of this proof will be to determine E_1. We have

$$E_1 = \left\{x \in \mathbb{P}^{n-1} : \exists y \in \mathbb{P}^{n-1} : (x, y) \in \Omega_2\right\}$$

and the defining condition can be rewritten in the form $A_x\bar{y} = 0$ where A_x is a $\binom{n}{2} \times n$ matrix in the coordinates of x and $\bar{y}$ an n-dimensional column vector with entries the coordinates of y. Index lines of the matrix A_x by pairs (i, j) with $1 \leq i < j \leq n$ and columns by $1 \leq k \leq n$. Then we can describe the matrix $A_x = (a_{(i,j)k})$ (where $x = (x_1, ..., x_n)$ is) by setting

$$a_{(i,j)i} = \theta_{ji}x_j,\ a_{(i,j)j} = -\theta_{ij}x_i,\ a_{(i,j)k} = 0, \forall k \neq i, j. \tag{4.3.1}$$

Note that we can recover the generators of I^n_ω in the algebra S^n_ω by taking the entries of the vector A_XX, where $X = (X_1, ..., X_n)$ is the vector formed by the generators of S^n_ω. This is because I^n_ω is generated in degree 2. A similar matrix appears in the original work of Artin and Schelter ([AS87]).

For each $x \in E_1$, we need at least a 1-dimensional space of solutions for the equation $A_x\bar{y} = 0$ so that we get a point in the projective space. This happens if and only if the rank of A_x is less or equal than $n-1$. The rank of A_x is at least

$n-1$ since, given a nonzero x_i we have a diagonal $(n-1)\times(n-1)$ submatrix with entries $\theta_{ij}x_i$, where $1 \leq j \leq n$ and $j \neq i$. Therefore $x \in E_1$ if and only if the rank of A_x is exactly $n-1$, i.e., if the $n \times n$ minors of A_x equal to zero.

The focus is then on computing such minors for A_X. For this we use Hattori's technique. Note that A_X is such that each row has exactly two nonzero entries (see description (4.3.1)). Let $\tilde{A}_X$ be the a submatrix formed by n rows of A_X. We want to compute $det(\tilde{A}_X)$. To $\tilde{A}_X$ we associate a graph G with n vertices, numbered 1 to n, such that there is an edge between two vertices i and j if and only if the line (i,j) is in $\tilde{A}_X$ or, equivalently, if X_i and X_j appear in one row of $\tilde{A}_X$. We denote by $E(G)$ the set of edges of a graph G. Each row corresponds to an edge, each column to a vertex and vice-versa. We permute first the rows and then the columns of $\tilde{A}_X$ (but keep the notation) in such a way that, whenever possible, adjacent rows and adjacent columns correspond, respectively to edges and vertices lying in the same connected component of G.

If $det(\tilde{A}_X) \neq 0$, then each connected component of G has exactly the same number of edges and vertices. This is equivalent to each connected component containing exactly one cycle. G itself clearly has this property (i.e., there are n rows and n variables). Indeed, if there is a connected component D with less edges than vertices, we can find a rectangular block in $\tilde{A}_X$, and thus $det(\tilde{A}_X) = 0$ since the column vectors forming this block would be linearly dependent.

So we have a square block decomposition of $\tilde{A}_X$ whose blocks are in bijection with the connected components of G. We focus on a given block B with connected graph D. To compute the determinant of B, we regroup the rows and then the columns (but keep the notation) such that the first rows correspond to the edges and the first columns to the vertices on the cycle of D - call it C. Then

we get a matrix

$$B = \begin{pmatrix} B_1 & 0 \\ * & B_2 \end{pmatrix}$$

where B_1 is the square matrix with the information from C and B_2 is the square matrix with the information from the *legs* - denote them by L - of D (i.e., acyclic paths with one vertex but no edges on C). Also, by permutation of rows and columns, B_2 can be made lower triangular and thus its determinant will be the product of the elements in the diagonal. If we orient the legs such that the source of an edge e (denoted by $s(e)$) is closer to the cycle than the target of e (denoted by $t(e)$), then we can easily compute this product by

$$det(B_2) = \prod_{e \in \mathrm{E(L)}} \theta_{s(e)t(e)} X_{s(e)}.$$

The determinant of B_1 can be computed by choosing a vertex of the cycle, v, and using the Laplace rule along the corresponding column. We orient the cycle clockwise. It is easy to see that we get

$$det(B_1) = \prod_{e \in \mathrm{E(C)}} \theta_{s(e)t(e)} X_{s(e)} - \prod_{e \in \mathrm{E(C)}} \theta_{t(e)s(e)} X_{t(e)}.$$

Note that $det(B) = det(B_1)det(B_2) = 0$ if and only if

$$\prod_{e \in \mathrm{E(C)}} \theta_{s(e)t(e)} = \prod_{e \in \mathrm{E(C)}} \theta_{t(e)s(e)} \quad \vee \quad \prod_{e \in E(D)} X_{s(e)} = 0$$

or equivalently, if $i_1, ..., i_l$ are the vertices of the cycle of C,

$$q_{(i_1 ... i_l)}(\omega) = 1 \vee \prod_{e \in E(D)} X_{s(e)} = 0.$$

Thus the determinant of $\tilde{A_X}$ is nonzero whenever, for any cycle $(i_1 ... i_l)$ in the graph G, $q_{(i_1 ... i_l)}(\omega) \neq 1$. Let Δ_ω be the set of graphs with n vertices and

n edges such that each connected component contains exactly one cycle and for each such cycle $(i_1...i_l)$ we have $q_{(i_1...i_l)}(\omega) \neq 1$. Then, by the above, we get

$$E_1 = \bigcap_{G \in \Delta_\omega} V(\prod_{e \in E(G)} X_{s(e)}).$$

We will now show that $(E_1)_{red}$, i.e., the reduced structure of E_1, is in fact the same as $\bigcap_{q_{(ijk)}(\omega) \neq 1} V(X_i X_j X_k)$. For this effect we ignore multiplicities in the formula above. Suppose $x \in (E_1)_{red}$ and choose (ijk) such that $q_{(ijk)}(\omega) \neq 1$. Consider G the graph consisting of the triangle with vertices i, j and k and with $n-3$ edges with source on k and target on the remaining $n-3$ vertices. This graph is clearly in Δ_ω and therefore $x \in V(X_i X_j X_k)$. Conversely, suppose $x \in \bigcap_{q_{(ijk)}(\omega) \neq 1} V(X_i X_j X_k)$. Since any q-cyclic number can be written as a product of q-cyclic numbers of length 3 (see 4.2.4), for any graph $G \in \Delta_\omega$ we have some triple (ijk) such that $q_{(ijk)}(\omega) \neq 1$ and i, j, k are contained in some cycle of G. Hence $x \in (E_1)_{red}$.

It remains to prove that $\Omega = E_1$ and that, therefore, Ω_{red} can be described as wanted. With a similar argument to the above on the right, we can easily see that $(E_2)_{red} = (E_1)_{red}$. Thus, the right projection following the inverse of the left projection can be regarded as an isomorphism whose graph is $(\Omega_2)_{red}$.

Lemma 4.3.3 (Le Bruyn, Smith, Van den Bergh, [LBSVdB96]). *If $(\Omega_2)_{red}$ is the graph of an isomorphism between $(E_1)_{red}$ and $(E_2)_{red}$ then Ω_2 is the graph of an isomorphism between E_1 and E_2.*

This lemma proves that Ω_2 can be regarded as the graph of an automorphism of E_1. Then by lemma 4.3.1 we have that $\Omega = E_1$.

□

4.4 Noncommutative projective spaces in dimensions 2 and 3

In this section we shall study $\mathbb{P}^3_\omega$ and compare the results with the ones obtained for $\mathbb{P}^2_\omega$ by Mori ([Mor06]). We also use proposition 4.3.2 to provide examples of noncommutative projective spaces which are not isomorphic (see theorem 2.3.8). It will be useful to determine the possible point varieties of $\mathbb{P}^3_\omega$. This is summarised in the following corollary, an easy application of proposition 4.3.2.

Corollary 4.4.1. *The point variety of $\mathbb{P}^3_\omega$ is isomorphic to one of the following:*

1. $\mathbb{P}^3$ *if* $q_{(123)} = q_{(124)} = q_{(134)} = 1$;
2. $V(X_1, X_2) \cup V(X_3) \cup V(X_4)$ *if one of the following holds:*
   - $q_{(123)} = q_{(124)} = 1$ *and* $q_{(134)} \neq 1$;
   - $q_{(123)} = q_{(134)} = 1$ *and* $q_{(124)} \neq 1$;
   - $q_{(124)} = q_{(134)} = 1$ *and* $q_{(123)} \neq 1$;
   - $q_{(123)} = 1$ *and* $q_{(124)} = q_{(134)} \neq 1$;
   - $q_{(124)} = 1$ *and* $q_{(123)} = q_{(134)} \neq 1$;
   - $q_{(134)} = 1$ *and* $q_{(123)} = q_{(124)} \neq 1$.
3. $V(X_1) \cup V(X_2, X_3) \cup V(X_2, X_4) \cup V(X_3, X_4)$ *if one of the following holds*
   - $q_{(123)} = 1$ *and* $1 \neq q_{(124)} \neq q_{(134)} \neq 1$;
   - $q_{(124)} = 1$ *and* $1 \neq q_{(123)} \neq q_{(134)}^{-1} = q_{(143)} \neq 1$;
   - $q_{(134)} = 1$ *and* $1 \neq q_{(123)} \neq q_{(124)} \neq 1$;
   - $q_{(123)}, q_{(124)}, q_{(134)} \neq 1$ *and* $q_{(123)}q_{(134)} = q_{(124)}^{-1} = q_{(142)}$.

4. $V(X_1, X_2) \cup V(X_1, X_3) \cup V(X_1, X_4) \cup V(X_2, X_3) \cup V(X_2, X_4) \cup V(X_3, X_4)$ *otherwise.*

Observe that the description above tells us that the point varieties in case 2 ore formed by one line and two hyperplanes, in case 3 by one hyperplane and three lines and in case 4 by six lines.

Concerning $\mathbb{P}^2_\omega$ we have the following result. Note that it also solves the classification problem for the family $\mathbb{P}^2_\omega$.

Theorem 4.4.2 (Mori, [Mor06]). *The following are equivalent:*

1. $Gr(S^3_\omega)$ *is equivalent to* $Gr(S^3_{\omega'})$*;*
2. $\mathbb{P}^2_\omega$ *is isomorphic to* $\mathbb{P}^2_{\omega'}$*;*
3. $\mathbb{C}(S^3_\omega)$ *is isomorphic to* $\mathbb{C}(S^3_{\omega'})$*.*

Remark 4.4.3. *This theorem clearly shows that point varieties of isomorphic spaces within the family* $\mathbb{P}^2_\omega$ *are the same. There are only two possibilities for the point variety: either* $\mathbb{P}^2$ *or the triangle formed by the lines* $x = 0$, $y = 0$ *and* $z = 0$ *in* $\mathbb{P}^2$ *with coordinates* $(x : y : z)$*. In the first case we are talking about a linear* $\mathbb{P}^2_\omega$*, i.e.,* S^3_ω *is a twisted coordinate ring and thus, by theorem 4.4.2, its category of graded modules is equivalent to the one of commutative polynomials. Hence, an isomorphic noncommutative projective space would also be isomorphic to the commutative one and therefore it would have the same point variety. On the other hand, if the point variety is a triangle, then the noncommutative projective space is not isomorphic to the commutative one and thus the point variety has to be preserved under equivalence.*

Remark 4.4.4. *For generic* q *(in this case, just meaning that* $q_{12} := q_{(123)}(\omega)$ *is not a root of unit), the theorem above is also a corollary of proposition 4.2.11.*

In fact, suppose that $\mathbb{P}^2_\omega$ and $\mathbb{P}^2_{\omega'}$ are birationally equivalent. Then there is $A \in GL_2(\mathbb{Z})$ such that $q'_{12} = q_{12}^{det(A)}$ and thus, by theorem 4.2.3, $Gr(S_\omega)$ is equivalent to $Gr(S_{\omega'})$.

The theorem above is not true for higher dimensional $\mathbb{P}^n_\omega$'s as the following example shows:

Example 4.4.5. *Let $X = \mathbb{P}^3_\omega$ and $X' = \mathbb{P}^3_{\omega'}$ where:*

$$\omega = (\omega_{ij})_{1\leq i,j\leq 4} = \begin{pmatrix} 1 & 1 & 1 & 1/2 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & 1 \end{pmatrix}$$

and

$$\omega' = (\omega'_{ij})_{1\leq i,j\leq 4} = \begin{pmatrix} 1 & 1 & 2 & 1/2 \\ 1 & 1 & 1 & 1 \\ 1/2 & 1 & 1 & 1/8 \\ 2 & 1 & 8 & 1 \end{pmatrix}.$$

We will use proposition 4.2.11 to check that X and X' are birational. We use the notation, as in the proposition, $q_{ij} := q_{(1ij)}(\omega)$ and $q'_{ij} := q_{(1ij)}(\omega')$. Consider the matrix

$$A = \begin{pmatrix} a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ -1 & 0 & 1 \end{pmatrix}.$$

We now check that this matrix relates q'_{ij}'s and q_{ij}'s as expected. Since

$$q_{23} = 1, \quad q_{24} = q_{34} = 2 \quad \text{and} \quad q'_{23} = q'_{34} = 1/2, \quad q'_{24} = 2,$$

by using the fact that $q_{ij} = q_{ji}^{-1}$ we have:

$$q'_{23} = q_{23}^{a_{22}a_{33}-a_{32}a_{23}} q_{24}^{a_{22}a_{43}-a_{42}a_{23}} q_{34}^{a_{32}a_{43}-a_{42}a_{33}} = 1.2^0.2^{-1} = 1/2$$

$$q'_{24} = q_{23}^{a_{22}a_{34}-a_{32}a_{24}} q_{24}^{a_{22}a_{44}-a_{42}a_{24}} q_{34}^{a_{32}a_{44}-a_{42}a_{34}} = 1.2^1.2^0 = 2$$

$$q'_{34} = q_{23}^{a_{23}a_{34}-a_{33}a_{24}} q_{24}^{a_{23}a_{44}-a_{43}a_{24}} q_{34}^{a_{33}a_{44}-a_{43}a_{34}} = 1.2^0.2^{-1} = 1/2$$

as expected, thus proving the birational equivalence of X and X'.

However, using corollary 4.4.1, we can easily see that their point varieties are not isomorphic: the point variety of X is formed by two hyperplanes and one line while the point variety of X' is formed by six lines. Thus X cannot be isomorphic to X'.

One may ask, however, how far theorem 4.4.2 is from being true for $\mathbb{P}^3_\omega$. The previous example shows that (3) does not imply (2) (and thus neither (1)) because some birational equivalences fail to preserve the point variety. The natural question is then: what can we say about birationally equivalent $\mathbb{P}^3_\omega$'s with isomorphic point varieties?

Example 4.4.6. *Let $X = \mathbb{P}^3_\omega$ and $X' = \mathbb{P}^3_{\omega'}$ where:*

$$\omega = (\omega_{ij})_{1\leq i,j\leq 4} = \begin{pmatrix} 1 & 1 & 1 & 1/2 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 4 \\ 2 & 1 & 1/4 & 1 \end{pmatrix}$$

and

$$\omega' = (\omega'_{ij})_{1\leq i,j\leq 4} = \begin{pmatrix} 1 & 1 & 1 & 1/4 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1/2 \\ 4 & 1 & 2 & 1 \end{pmatrix}$$

Then it is easy to see that X and X' are birational since the matrix:

$$B = \begin{pmatrix} b_{22} & b_{23} & b_{24} \\ b_{32} & b_{33} & b_{34} \\ b_{42} & b_{43} & b_{44} \end{pmatrix} = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix}$$

is relating $q_{(1ij)}(\omega') =: q'_{ij}$ and $q_{(1ij)}(\omega) =: q_{ij}$ as in proposition 4.2.11 (see previous example for similar computations). It is also clear, by corollary 4.4.1, that they have the same point variety (one hyperplane and three lines). However, looking at the q-cyclic numbers

$$q_{(123)}(\omega) = 1, \quad q_{(124)}(\omega) = 2, \quad q_{(134)}(\omega) = 8, \quad q_{(234)}(\omega) = 4 \text{ and}$$

$$q_{(123)}(\omega') = 1, \quad q_{(124)}(\omega') = 4, \quad q_{(134)}(\omega') = 2, \quad q_{(234)}(\omega') = 1/2$$

and using proposition 4.2.3 we conclude that $Gr(S_\omega)$ and $Gr(S_{\omega'})$ are not equivalent, hence showing that, in general, $(3) \implies (1)$ of theorem 4.4.2 fails for $\mathbb{P}^3$ even with the additional condition of isomorphic point varieties.

The question of whether $(2) \Rightarrow (1)$ of theorem 4.4.2 holds for higher dimensions remains without an answer. We, however, conjecture it to be true.

Conjecture 4.4.7. *If $\mathbb{P}^{n-1}_\omega \cong \mathbb{P}^{n-1}_{\omega'}$ then $Gr(S_\omega) \cong Gr(S_{\omega'})$.*

Chapter 5

Perverse coherent t-structures via torsion theories

5.1 Introduction

Perverse t-structures for the derived category of coherent sheaves on a projective scheme were first defined by Deligne in some unpublished manuscript that was eventually made public through the exposition by Bezrukavnikov ([Bez00]). Arinkin and Bezrukavnikov have later further developed this topic ([AB10]). On the other hand, given a torsion theory on the heart of a t-structure we can construct a new t-structure ([HRS96], [Bri05]) - see theorem 2.2.9. In this chapter, we shall use an iteration of this process to obtain the perverse coherent t-structure from the standard t-structure on $D^b(coh(X)) = D^b(tails(R))$ when R is a commutative noetherian connected positively graded $\mathbb{K}$-algebra generated in degree one. This description allows similar constructions for some noncommutative projective planes. Similar ideas on constructing t-structures for coherent sheaves were explored by Kashiwara ([Kas04]).

We start by recalling the perverse coherent construction.

Definition 5.1.1. *Let X be a scheme and X^{top} the set of generic points of all its closed subschemes. A* ***perversity*** *is a map $p : X^{top} \longrightarrow \mathbb{Z}$ satisfying the monotone and comonotone properties:*

- *monotone:* $y \in \bar{x} \Rightarrow p(y) \geq p(x)$
- *comonotone:* $y \in \bar{x} \Rightarrow p(x) \geq p(y) - (dim(x) - dim(y))$

Note that the image of a perversity on an n-dimensional scheme has at most $n+1$ elements. Given a perversity p we define a **perverse coherent t-structure** by taking

$$D^{p,\leq 0} = \left\{F^{\bullet} \in D^b(coh(X)) : \forall x \in X^{top},\ Li_x^*(F^{\bullet}) \in D^{\leq p(x)}(O_x - mod)\right\}$$

as an aisle ([Bez00], [AB10]), where $i_x : \{x\} \longrightarrow X$ is the inclusion map, Li_x^* is the left derived functor of the pullback functor by i_x and $D^{\leq p(x)}(O_x - mod)$ is the standard aisle of $D^b(O_x - mod)$ shifted by $-p(x)$.

The chapter is outlined as follows: section 5.2 presents some basic facts on torsion theories for categories of graded modules; section 5.3 shows how to obtain a t-structure by iterating tilts by torsion theories on a complete and cocomplete abelian category; section 5.4 shows how the construction of 5.3 can describe perverse coherent t-structures and section 5.5 applies section 5.3 to some noncommutative projective planes where we define perverse quasi-coherent t-structures.

5.2 Torsion theories for graded modules

In this section we prove some results concerning torsion theories for categories of graded modules.

Let R be a, not necessarily commutative, noetherian graded ring. $Gr(R)$ is a Grothendieck category admitting injective envelopes which, for a graded module M, we will denote by $E^g(M)$. We shall use $\mathrm{Hom}_{Gr(R)}(M, N)$ for homomorphisms in this category (i.e., R-linear, grading preserving) between graded modules M and N. $h(M)$ shall denote the subset of homogeneous elements of M. It is clear that M is generated by $h(M)$ as a right R-module. Also, for a prime ideal P, define $C^g(P) = C(P) \cap h(R)$, where $C(P)$ is the set of regular elements mod P, i.e., the set of elements x of R such that $x + P$ is neither left nor right zero divisor in R/P. If R is commutative, then $C(P) = R \setminus P$.

Remark 5.2.1. *Given a connected positively graded ring R generated in degree one and a homogeneous prime ideal $P \neq R_+ := \bigoplus_{i\geq 1} R_i$, we have $P_n \neq R_n$ for all $n > 1$. In fact, suppose there is $n_0 > 1$ such that $P_{n_0} = R_{n_0}$. Then, since the ring is generated in degree one, we have $P_n = R_n$ for all $n > n_0$. Now, if $x_1 \in R_1 \setminus P_1$ then, since P is prime, there is $\tilde{r}_1 \in R$ such that $x_1\tilde{r}_1x_1 \notin P$. Thus there is also $r_1 \in h(R)$ such that $x_2 := x_1r_1x_1 \notin P$. Now, $\deg(x_2) \geq 2$ since R is positively graded. We can then inductively construct a sequence of elements $(x_n)_{n\in\mathbb{N}}$ none of them lying in P and such that $\deg(x_n) > \deg(x_{n-1})$, thus yielding a contradiction.*

Definition 5.2.2. *To an injective object E in $Gr(R)$ we associate a natural torsion theory in $Gr(R)$, for which a module M is torsion if $\mathrm{Hom}_{Gr(R)}(M, E) = 0$. This torsion theory is said to be* ***cogenerated by*** *E in $Gr(R)$.*

Since R is noetherian, $gr(R)$ is closed under taking subobjects. It is then clear that the torsion theories above defined in $Gr(R)$ restrict to torsion theories in $gr(R)$. In fact, given M finitely generated graded R-module and τ the torsion radical functor induced by a torsion theory in $Gr(R)$, $\tau(M)$ and $M/\tau(M)$ are

finitely generated. Therefore axiom (2) in definition 2.2.7 is satisfied by considering in $gr(R)$ the same exact sequence as in $Gr(R)$ (axiom (1) is, in its turn, automatic).

The following lemma proves a useful criterion for graded modules to belong to the torsion class associated with an injective object. The arguments of the proof mimic the ungraded case ([LM74]).

Lemma 5.2.3. *Given graded modules T and F over a graded ring R, the following conditions are equivalent:*

1. $\mathrm{Hom}_{Gr(R)}(T, E^g(F)) = 0$;

2. $\forall t \in h(T)$, $\forall f \in h(F) \setminus 0$, $\deg(f) = \deg(t)$, $\exists r \in h(R)$: $tr = 0 \wedge fr \neq 0$.

Proof. Suppose $\mathrm{Hom}_{Gr(R)}(T, E^g(F)) \neq 0$. Let α be one of its nonzero elements. Choose $u \in h(T)$ such that $\alpha(u) \neq 0$. F is a graded essential submodule of $E^g(F)$, i.e., given any nontrivial graded submodule of $E^g(F)$, its intersection with F is nontrivial. Hence there is $s \in h(R)$ such that $0 \neq \alpha(u)s = \alpha(us) \in F$. If we choose $t = us$ and $f = \alpha(us)$ they are homogeneous of the same degree and clearly, given $r \in R$, if $tr = 0$ then $fr = 0$.

Suppose now that (2) is false, i.e., there are $t \in T$ and $f \in F \setminus \{0\}$ homogeneous of the same degree such that for all $r \in h(R)$, if $tr = 0$ then $fr = 0$. Then, there is a well defined nonzero graded homomorphism

$$tR \longrightarrow F,\ tr \mapsto fr$$

since $\langle h(R) \rangle = R$. Since $E^g(F)$ is an injective object in the category of graded modules, we can find a nonzero graded homomorphism from T to $E^g(F)$. $\square$

The following corollary shows that, even though in general it does not make

sense to talk about the *degree zero of localisation* (simply because localisation may not exist), we can reformulate such statement it in terms of torsion.

Corollary 5.2.4. *Let R be a commutative graded ring, P a homogeneous prime ideal in R and $S = h(R \setminus P)$. Given M a graded R-module then $(S^{-1}M)_0 = 0$ if and only if $\mathrm{Hom}_{Gr(R)}(M, E^g(R/P)) = 0$.*

Proof. This follows from the fact $(S^{-1}M)_0 = 0$ is equivalent, by definition of graded localisation, to condition (2) of the above lemma. □

We discuss now rigid torsion theories ([NVO82]). Consider the following subset of $\mathrm{Hom}_R(M, N)$: $\overline{\mathrm{Hom}}(M, N) := \bigoplus_{i \in \mathbb{Z}} \mathrm{Hom}_{Gr(R)}(M, N(i))$.

Definition 5.2.5. *We say that a torsion theory in $Gr(R)$ is **rigid** if the class of torsion modules (equivalently, the class of torsion-free modules) is closed under shifts of the grading. The rigid torsion theory associated to an injective object E in $Gr(R)$ is defined so that a module M is torsion if $\overline{\mathrm{Hom}}(M, E) = 0$.*

We easily get a lemma similar to 5.2.3.

Lemma 5.2.6. *Given graded modules T and F over a graded ring R, the following conditions are equivalent:*

1. $\overline{\mathrm{Hom}}(T, E^g(F)) = 0$;
2. $\forall t \in h(T)$, $\forall f \in h(F) \setminus 0$, $\exists r \in h(R)$ *such that* $tr = 0$ *and* $fr \neq 0$.

Proof. The argument is the same as before. Observe that not having a relation between the degrees of t and f we can get homomorphisms of any degree from tR to F. Since $\overline{\mathrm{Hom}}(T, E^g(F)) = \bigoplus_i \mathrm{Hom}_{Gr(R)}(T, E^g(F)(i))$, the result follows. □

As a consequence, one gets a result similar to 5.2.4.

Corollary 5.2.7. *Let R be a commutative graded ring, P a homogeneous prime ideal in R and $S = h(R \setminus P)$. Given a graded R-module M, $S^{-1}M = 0$ if and only if $\overline{\mathrm{Hom}}(M, E^g(R/P)) = 0$.*

Proof. This follows from the fact that $S^{-1}M = 0$ is equivalent, by definition of graded localisation, to condition (2) of the above lemma. □

One can, in fact, get a more general statement, including some noncommutative rings. This can be done by comparing this rigid torsion theory with the torsion theory associated to a multiplicative set. For a right homogeneous ideal J of a graded ring R we use the notation $J \lhd_{rg} R$ and, given $r \in R$ we define a right ideal

$$r^{-1}J := \{a \in R : ra \in J\}.$$

Recall the following result ([NVO82]).

Proposition 5.2.8. *Let R be a graded ring, S a multiplicative subset contained in $h(R)$. Then the class of modules M such that there is $J \in \mathbb{L}_S$ with $MJ = 0$, where*

$$\mathbb{L}_S = \left\{J \lhd_{rg} R : r^{-1}J \cap S \neq \emptyset,\ \forall r \in h(R)\right\},$$

is a torsion class for a rigid torsion theory in $Gr(R)$.

$\mathbb{L}_S$ as above is said to be a graded Gabriel filter for that torsion theory. If $S = C^g(P)$ for some homogeneous prime ideal P, then we denote the filter by $\mathbb{L}_P$. The rigid torsion theory associated to an injective graded module E also has an associated graded Gabriel filter given by:

$$L_E^r = \{J \lhd_{rg} R : \mathrm{Hom}_R(R/J, E) = 0\}.$$

.

In fact hereditary rigid torsion theories are in bijection with graded Gabriel filters ([NVO82]). Thus the graded Gabriel filter determines the torsion theory and vice-versa.

Theorem 5.2.9. *Let P be a homogeneous prime ideal of a graded ring R with R/P right noetherian. Let M be a graded right R-module. Then M is torsion with respect to $C^g(P)$ if and only if M is torsion with respect to the rigid torsion theory associated to $E^g(R/P)$.*

Proof. We will prove that the Gabriel filters of both torsion theories are the same. Let $E = E^g(R/P)$ and $J \in \mathbb{L}_E^r$, i.e., $J \lhd_{rg} R$ such that $\overline{\mathrm{Hom}}(R/J, E) = 0$. By lemma 5.2.6, for all $a \in h(R)$ and $b \in h(R) \setminus P$, there is $c \in h(R)$ such that $ac \in J$ and $bc \notin P$. Since P is a two-sided ideal, $c \notin P$. Thus we conclude that for all $a \in h(R)$, $a^{-1}J$ is not contained in P. This means that $(a^{-1}J+P)/P \lhd_{rg} R/P$ is nontrivial and since P is prime, it is essential. By Goldie's theorem for graded rings ([GS00]) we have that $(a^{-1}J + P)/P$ has a homogeneous regular element and thus, for all $a \in h(R)$, $a^{-1}J \cap C^g(P) \neq \emptyset$ which means that $J \in \mathbb{L}_P$.

Conversely, suppose $J \in \mathbb{L}_P$ and let $a, b \in h(R),\ b \notin P$. By hypothesis, $a^{-1}J \cap C^g(P) \neq \emptyset$. Let z be one of its elements. Then, clearly, $az \in J$ and $bz \notin P$. Again, by lemma 5.2.6, the result follows. □

In the commutative case, however, the torsion theories discussed above coincide.

Proposition 5.2.10. *Let R be a commutative noetherian positively graded connected $\mathbb{K}$-algebra generated in degree 1, P a homogeneous prime ideal in R not equal to the irrelevant ideal and M a graded R-module. Then, for $S = h(R \setminus P)$, $S^{-1}M = 0$ if and only if $(S^{-1}M)_0 = 0$.*

Proof. One direction is clear. For the converse, suppose that $(S^{-1}M)_0 = 0$. Let $m \in M$ such that $\deg(m) > 0$. Then from remark 5.2.1 one concludes that there is $s \in S$ such that $\deg(m) = \deg(s)$, and therefore, since $(S^{-1}M)_0 = 0$, we get $\frac{m}{s} = 0$, i.e., there is $r \in S$ such that $mr = 0$. Thus for any $s' \in S$ we have $\frac{m}{s'} = 0$ and hence $S^{-1}M_{\geq 0} = 0$.

Note that since R is positively graded, $M_{\geq 0}$ is a submodule of M. Consider the following short exact sequence:

$$0 \longrightarrow M_{\geq 0} \longrightarrow M \longrightarrow M/M_{\geq 0} \longrightarrow 0$$

and apply to it the exact localisation functor, thus obtaining

$$0 \longrightarrow S^{-1}M_{\geq 0} \longrightarrow S^{-1}M \longrightarrow S^{-1}(M/M_{\geq 0}) \longrightarrow 0.$$

Let $x \in M/M_{\geq 0}$. Using again remark 5.2.1, there is $r \in S$ such that $\deg(r) = -\deg(x)$. This means that $xr = 0$ and thus $\frac{x}{s} = 0$ for all $s \in S$. This shows that $S^{-1}(M/M_{\geq 0}) = 0$ and since we already had $S^{-1}M_{\geq 0} = 0$, it shows that $S^{-1}M = 0$, finishing the proof. □

This shows that under the conditions of the proposition above, the torsion theory cogenerated by $E^g(R/P)$ is rigid. This statement can, however, be reproduced by dropping the commutativity assumption.

Lemma 5.2.11. *Let R be a noetherian positively graded connected $\mathbb{K}$-algebra generated in degree 1, P a homogeneous prime ideal in R not equal to the irrelevant ideal and M a graded right R-module. Then, $\mathrm{Hom}_{Gr(R)}(M, E^g(R/P)) = 0$ if and only if $\overline{\mathrm{Hom}}(M, E^g(R/P)) = 0$.*

Proof. One direction is clear. Suppose $\overline{\mathrm{Hom}}(M, E^g(R/P)) \neq 0$. Then by lemma 5.2.6 there is $m \in h(M)$ such that $Ann(m) \cap C_l^g(P) = \emptyset$, where $Ann(m)$

stands for right annihilator of m and $C_l^g(P)$ stands for homogeneous left regular elements mod P. We want to prove that $\mathrm{Hom}_{Gr(R)}(M, E^g(R/P)) \neq 0$ which, by 5.2.3 and remark 5.2.1, is equivalent to the existence of $\tilde{m} \in h(M_{\geq 0})$ such that $Ann(\tilde{m}) \cap C_l^g(P) = \emptyset$.

Note that the irrelevant ideal, R_+, is a homogeneous maximal ideal containing P. So, by graded Goldie's theorem ([GS00]), R_+/P is an essential ideal in the graded prime Goldie ring R/P, thus containing a regular element. This means that there is a homogeneous regular element of positive degree in R/P and thus $C^g(P)_{\geq k} \neq \emptyset$ for all $k \in \mathbb{N}$. Choose $s \in C^g(P)$ such that $\deg(ms) \geq 0$. Note that if there is $a \in Ann(ms) \cap C_l^g(P)$, then $sa \in Ann(m) \cap C_l^g(P)$ yielding a contradiction. Therefore, take $\tilde{m} = ms$ and we are done. □

Remark 5.2.12. *We summarise the results of this section. If R is a noetherian positively graded connected $\mathbb{K}$-algebra generated in degree 1, P a homogeneous prime ideal not equal to the irrelevant ideal R_+ and M a graded R-module, then the following are equivalent:*

1. $\mathrm{Hom}_{Gr(R)}(M, E^g(R/P)) = 0$;
2. $\overline{\mathrm{Hom}}(M, E^g(R/P)) = 0$;
3. *M is torsion with respect to $C^g(P)$.*

If, furthermore, R is commutative and $S = h(R \setminus P)$, then (1), (2) and (3) are equivalent to $S^{-1}M = 0$ and to $(S^{-1}M)_0 = 0$.

5.3 t-structures via torsion theories

Let $\mathcal{A}$ be a complete and cocomplete abelian category. Fix $a \in \mathbb{Z}$, $n \in \mathbb{N}$ and an ordered set (indexed by a string of integers of length n starting at a) of hereditary

torsion classes $S = \{T_a, T_{a+1}..., T_{a+n-1}\}$ such that $\bigcap_{i=a}^{a+n-1} T_i = 0$.

Our target is to prove that the following subcategory is the aisle of a t-structure:

$$D^{S,\leq 0} = \left\{X^\bullet \in D(\mathcal{A}) : \ H^i(X^\bullet) \in T_j, \ \forall i > j\right\}.$$

Remark 5.3.1. *Clearly such a category is a subcategory of $D^{\leq a+n-1}$, a shift of the aisle of the standard t-structure. This is because the intersection of all torsion classes is zero.*

We will start by proving a simple but useful lemma regarding how truncations between two t-structures are related when their aisles are subject to a similar relation of the remark above.

Lemma 5.3.2. *Suppose $(\mathcal{D}_A^{\leq 0}, \mathcal{D}_A^{\geq 0})$ and $(\mathcal{D}_B^{\leq 0}, \mathcal{D}_B^{\geq 0})$ are two t-structures in a triangulated category $\mathcal{D}$ with truncation functors $t_A^{\leq 0}$ and $t_B^{\leq 0}$, respectively. If $\mathcal{D}_A^{\leq 0} \subset \mathcal{D}_B^{\leq 0}$, then for all $X \in \mathcal{D}$ there is a triangle:*

$$t_A^{\leq 0}(X) \longrightarrow t_B^{\leq 0}(X) \longrightarrow Y \longrightarrow t_A^{\leq 0}(X)[1] \tag{5.3.1}$$

such that $Y \in D_A^{\geq 1} \cap D_B^{\leq 0}$.

Proof. First note that, since $\mathcal{D}_A^{\leq 0} \subset \mathcal{D}_B^{\leq 0}$, we have $\mathcal{D}_B^{\geq 1} \subset \mathcal{D}_A^{\geq 1}$. The triangle

$$t_B^{\leq 0}(X) \longrightarrow X \longrightarrow t_B^{\geq 1}(X) \longrightarrow t_B^{\leq 0}(X)[1]$$

then shows that the natural map $t_A^{\leq 0}(X) \longrightarrow X$ must factor through $t_B^{\leq 0}(X)$ (since Hom$(t_A^{\leq 0}(X), t_B^{\geq 1}(X)) = 0$). Let Y be defined by the following triangle

$$t_A^{\leq 0}(X) \longrightarrow t_B^{\leq 0}(X) \longrightarrow Y \longrightarrow t_A^{\leq 0}(X)[1].$$

Since aisles are closed under taking cones and $t_A^{\leq 0}(X) \in D_B^{\leq 0}$, we have that $Y \in D_B^{\leq 0}$. We want to prove $Y \in D_A^{\geq 1}$. Consider the diagram

$$\begin{tikzcd}
t_A^{\leq 0}(X) \arrow[r] \arrow[d] & t_B^{\leq 0}(X) \arrow[r] \arrow[d] & Y \arrow[r] & t_A^{\leq 0}(X)[1] \arrow[d] \\
t_A^{\leq 0}(X) \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & t_A^{\geq 1}(X) \arrow[r] & t_A^{\leq 0}(X)[1] \arrow[d] \\
t_B^{\leq 0}(X) \arrow[r] & X \arrow[r] & t_B^{\geq 1}(X) \arrow[r] & t_B^{\leq 0}(X)[1]
\end{tikzcd}$$

where rows are triangles and the squares commute by the observation above. Then, the octahedral axiom gives us a new triangle

$$Y \longrightarrow t_A^{\geq 1}(X) \longrightarrow t_B^{\geq 1}(X) \longrightarrow Y[1].$$

Since $t_B^{\geq 1}(X) \in D_A^{\geq 1}$, so is $t_B^{\geq 1}(X)[-1]$. By the long exact sequence of cohomology induced from this triangle, it is easy to see that this shows that $Y \in D_A^{\geq 1}$. $\square$

Theorem 5.3.3. *If $n \geq 2$, then $D^{S,\leq 0}$ is the aisle of a t-structure in $D(\mathcal{A})$.*

Proof. We shall use induction on n. Without loss of generality we shall assume $a = -n + 1$.

Suppose $n = 2$ and $S = \{T_{-1}, T_0\}$. Thus we have

$$D^{S,\leq 0} = \left\{X^\bullet \in D(\mathcal{A}) : \ H^0(X^\bullet) \in T_{-1}, \ H^i(X^\bullet) = 0, \forall i > 0\right\}.$$

To see this note that for $X \in D^{S,\leq 0}$, we have $H^i(X^\bullet) = 0$ for all $i > 0$ since, for such i, $H^i(X^\bullet) \in T_{-1} \cap T_0 = 0$. This is the aisle of a t-structure: it is obtained via tilting with respect to T_{-1} - see theorem 2.2.9.

Suppose the result is valid for any ordered set of n hereditary torsion classes with zero intersection. Let S be such a set with $n + 1$ elements, i.e., $S = \{T_{-n}, T_{-n+1}, ..., T_0\}$. We want to prove that $D^{S,\leq 0}$ defines a t-structure on $D(\mathcal{A})$.

First let us consider $\bar{S} = \left\{\bar{T}_{-n+1}, \bar{T}_{-n+2}, ..., \bar{T}_0\right\}$ where $\bar{T}_i = T_{i-1}$ for $i < 0$ and $\bar{T}_0 = T_{-1} \cap T_0$. Clearly, by assumption on S, this is also an ordered set of torsion classes with zero intersection in $\mathcal{A}$. We fall into the case of n torsion classes and by the induction hypothesis we have an associated t-structure whose heart will be denoted by $\mathcal{B}$. The corresponding cohomological functor will be denoted by $H^0_{\bar{S}} := t^{\geq 0}_{\bar{S}} t^{\leq 0}_{\bar{S}}$, where the $t_{\bar{S}}$'s are the associated truncation functors.

Consider now the following subcategory of $\mathcal{B}$:

$$\mathcal{W} = \left\{X^\bullet \in \mathcal{B} : \ H^0(X^\bullet) \in T_{-1}, \ H^i(X^\bullet) = 0, \ \forall i < 0\right\}.$$

$\mathcal{W}$ can be seen as the *stalk subcategory* $0 \to 0 \to \bigcap_{i=-n}^{-1} T_i \to 0$ (since it is a subcategory of $\mathcal{B}$ and therefore all positive cohomologies vanish). Since $\mathcal{A}$ is complete and cocomplete, proposition 2.2.8 allows us to observe that $\mathcal{W}$ is a torsion class inside $\mathcal{B}$. Indeed, since homomorphisms in $\mathcal{W}$ can be seen as homomorphisms in $\mathcal{A}$ and $\mathcal{W}$ is a torsion class in $\mathcal{A}$, $\mathcal{W}$ is closed under epimorphic images and coproducts. It is also closed under extensions since exact sequences in $\mathcal{B}$ are precisely the distinguished triangles of $D(\mathcal{A})$ that lie in $\mathcal{B}$ and the result follows from the long exact sequence of cohomology of a distinguished triangle.

Now the crucial observation is the following lemma:

Lemma 5.3.4. *$X^\bullet \in D^{S,\leq 0}$ if and only if $H^0_{\bar{S}}(X^\bullet) \in \mathcal{W}$ and $H^i_{\bar{S}}(X^\bullet) = 0$ for all $i > 0$.*

Proof. Note first that $H^i_{\bar{S}}(X^\bullet) = 0$, for all $i > 0$, is equivalent to $X^\bullet \in D^{\bar{S},\leq 0}$.

Suppose $X^\bullet \in D^{S,\leq 0}$. It is clear from the definition of the perversity $\bar{S}$ that $D^{S,\leq 0} \subset D^{\bar{S},\leq 0}$, thus proving the vanishing of positive $\bar{S}$-cohomologies.

Now, we can fit $\bar{S}$-cohomology in the following distinguished triangle:

$$t^{\leq -1}_{\bar{S}}(X^\bullet) \longrightarrow t^{\leq 0}_{\bar{S}}(X^\bullet) \longrightarrow H^0_{\bar{S}}(X^\bullet) \longrightarrow t^{\leq -1}_{\bar{S}}(X^\bullet)[1]$$

which, again due to the fact that $D^{S,\leq 0} \subset D^{\bar{S},\leq 0}$, amounts to the distinguished triangle

$$t_{\bar{S}}^{\leq -1}(X^\bullet) \longrightarrow X^\bullet \longrightarrow H_{\bar{S}}^0(X^\bullet) \longrightarrow t_{\bar{S}}^{\leq -1}(X^\bullet)[1].$$

Now, lemma 5.3.2 applied to $D^{\bar{S},\leq -1} \subset D^{\leq -1}$ (see remark 5.3.1) shows that

$$t_{\bar{S}}^{\leq -1}(X^\bullet) \longrightarrow t^{\leq -1}(X^\bullet) \longrightarrow Y \longrightarrow t_{\bar{S}}^{\leq 0}(X^\bullet)[1]$$

where $Y \in D^{\bar{S},\geq 0} \cap D^{\leq -1}$. Since $X^\bullet \in D^{S,\leq 0}$, we have that $t^{\leq -1}(X^\bullet) \in D^{\bar{S},\leq -1}$ and thus $Y = 0$. Therefore, $t_{\bar{S}}^{\leq -1}(X^\bullet) \cong t^{\leq -1}(X^\bullet)$. Since in any distinguished triangle two of the vertices determine the third one up to isomorphism, we have

$$H_{\bar{S}}^0(X^\bullet) = H^0(X^\bullet)$$

which, by definition of $D^{S,\leq 0}$, tells us that $H_{\bar{S}}^0(X^\bullet) \in \mathcal{W}$.

Conversely, suppose $X^\bullet \in D^{\bar{S},\leq 0}$ and $H_{\bar{S}}^0(X^\bullet) \in \mathcal{W}$. As before, we have an exact triangle

$$t_{\bar{S}}^{\leq -1}(X^\bullet) \longrightarrow X^\bullet \longrightarrow H_{\bar{S}}^0(X^\bullet) \longrightarrow t_{\bar{S}}^{\leq -1}(X^\bullet)[1]$$

whose long exact sequence of cohomology (for the standard cohomology functor) tells us that $H^i(t_{\bar{S}}^{\leq -1}(X^\bullet)) \cong H^i(X^\bullet)$ for all $i < 0$ (since negative cohomologies vanish for $H_{\bar{S}}^0(X^\bullet)$) and that $H^0(X^\bullet) \cong H^0(H_{\bar{S}}^0(X^\bullet)) \in \mathcal{W}$. Note that we will have $H^i(t_{\bar{S}}^{\leq -1}(X^\bullet)) \in T_{i-1}$ and thus $H^i(X^\bullet) \in T_{i-1}$ for all $-n+1 \leq i < 0$. This is because $D^{\bar{S},\leq -1} = D^{\bar{S},\leq 0}[1]$. On the other hand $H^0(X^\bullet) \cong H^0(H_{\bar{S}}^0(X^\bullet)) \in \mathcal{W}$ proving that $H^0(X^\bullet) \in T_{-1}$. This is precisely the additional conditions that an element in $D^{\bar{S},\leq 0}$ needs to satisfy to be in $D^{S,\leq 0}$, thus finishing the proof. □

$D^{S,\leq 0}$ can then be obtained by tilting the heart $\mathcal{B}$, defined earlier in this proof, with respect to the torsion theory whose torsion class is $\mathcal{W}$. Therefore it is the aisle of a t-structure. □

Remark 5.3.5. *Note that the assumption that $\mathcal{A}$ is complete and cocomplete was important to prove that $\mathcal{W}$ is a torsion class in $\mathcal{B}$. This fact would also be true if $\mathcal{B}$ were noetherian (since proposition 2.2.8 also holds in this context). However, it is well known that $\mathcal{B}$ might not be noetherian even when $\mathcal{A}$ is (see [Pol07] for a discussion of this topic), hence our assumption.*

Recall the definition 2.2.4 of nondegenerate t-structure.

Lemma 5.3.6. *The t-structure associated to an ordered set of hereditary torsion classes S with zero intersection, as defined above, is nondegenerate.*

Proof. Suppose without loss of generality that the maximal index in S is zero. Then, as before, $D^{S,\leq 0} \subseteq D^{\leq 0}$. The standard t-structure is nondegenerate and thus $\bigcap_{n\in\mathbb{Z}} D^{S,\leq n} = 0$.

On the other hand

$$\bigcap_{n\in\mathbb{Z}} D^{S,\geq n} = \bigcap_{n\in\mathbb{Z}} (D^{S,\leq n-1})^{\perp}$$
$$= (\bigcup_{n\in\mathbb{Z}} D^{S,\leq n-1})^{\perp}.$$

Since $D^{\leq n-1} \subseteq D^{\leq n-1+k}$ for any $k \geq 0$ and $\bigcup_{n\in\mathbb{Z}} D^{\leq n} = D$ we also have $\bigcup_{n\in\mathbb{Z}} D^{S,\leq n-1} = D$ and thus $\bigcup_{n\in\mathbb{Z}} D^{S,\geq n} = D$. □

Remark 5.3.7. *A nondegenerate t-structure restricts well to the bounded derived category ([Mil]). Thus, the construction above can be restricted to bounded derived categories.*

5.4 Perverse coherent t-structures via torsion theories

In this section we prove the main theorem of this chapter. Let X be a smooth projective scheme such that $R = \bar{\Gamma}(X)$ is a commutative noetherian positively graded

$\mathbb{K}$-algebra generated in degree 1, where $\mathbb{K}$ is algebraically closed of characteristic zero. π shall denote, as before, the projection functor from $Gr(R)$ to its quotient $Tails(R)$ (and the corresponding restriction to $gr(R)$) and $\bar{\Gamma}$ its right adjoint (see chapters 2 and 4). Let $p : X^{top} \longrightarrow \mathbb{Z}$ be a perversity as in the introduction to this chapter. Suppose that the perversity has n values and that, without loss of generality, the maximal value of the perversity is zero. Set $E_i = \prod_{\{x \in X : p(x)=i\}} E^g(R/I_x)$, $i \in Im(p)$, where I_x is the homogeneous ideal of functions vanishing at x.

Lemma 5.4.1. *Let A and B be graded modules over R, B torsion-free and injective. Then Hom$_{Gr(R)}(A, B) = 0$ if and only if Hom$_{Tails(R)}(\pi A, \pi B) = 0$*

Proof. Suppose $f \in$ Hom$_{Tails(R)}(\pi A, \pi B) \neq 0$. B is torsion-free and so we have

$$\mathrm{Hom}_{Tails(R)}(\pi A, \pi B) = \varinjlim_{A' \leq A : A/A' \text{torsion}} \mathrm{Hom}_{Gr(R)}(A', B).$$

Let A' be such that there is $\tilde{f} \in$ Hom$_{Gr(R)}(A', B)$ such that $\pi \tilde{f} = f$. Then, since B is injective, $\tilde{f}$ can be extended to A, proving that Hom$_{Gr(R)}(A, B) \neq 0$.

Conversely, suppose $f \in$ Hom$_{Gr(R)}(A, B) \neq 0$. Since B is torsion-free, we have that for all $A' \leq A$ such that A/A' is torsion, $f_{|A'} \neq 0$ (since otherwise we would have a nonzero map from A/A' to B, which is not allowed by definition of torsion). Thus $\pi f \neq 0$ in Hom$_{Tails(R)}(\pi A, \pi B)$. □

Recall that a set of injective objects $\{I_1, ..., I_n\}$ in an abelian category $\mathcal{A}$ is a cogenerating set for $\mathcal{A}$ if, for any $X \in \mathcal{A}$, Hom$(X, I_j) = 0$ for all j implies $X = 0$. Clearly, if T_i is the torsion class in $\mathcal{A}$ associated to I_i (i.e., the set of objects X such that Hom$_{\mathcal{A}}(X, I_i) = 0$) then $\{I_1, ..., I_n\}$ is a cogenerating set if and only if $\bigcap_i T_i = 0$.

Remark 5.4.2. *Note that given R positively graded noetherian connected $\mathbb{K}$-algebra, R/P is torsion-free for any homogeneous prime ideal P not equal to the*

irrelevant ideal. Indeed if, for $x \notin P$, $xR_{\geq n} = 0$ then $(RxR)(R_{\geq n}) \subset P$ and hence, by 5.2.1, $RxR \subset P$ which yields a contradiction.

Note that all modules E_i are torsion-free by the remark above. Indeed, since the torsion-free class of a hereditary torsion theory is closed under taking injective envelopes, $E^g(R/P)$ is torsion-free. Also, the objects πE_i are clearly injective in $Tails(R)$.

Corollary 5.4.3. *The objects πE_i cogenerate $Tails(R)$, where $R = \bar{\Gamma}(X)$ is as above.*

Proof. Suppose that M is not torsion, i.e., that there is an element $m \in h(M)$ such that $Ann(m) \neq R_{\geq n}$ for any $n > 1$. We prove that $Ann(m)$ is contained in a homogeneous prime ideal. It is clear that, since m is not torsion, the radical of $Ann(m)$, which we shall denote by $\sqrt{Ann(m)}$, is not the augmentation ideal R_+. Thus we can choose $f \in R_1$ such that $f \notin \sqrt{Ann(m)}$. Applying Zorn's lemma to the set $S = \left\{ J \supset Ann(m) \text{ homogeneous} : f \notin \sqrt{J} \right\}$ (which is nonempty since $Ann(m) \in S$) we get a maximal element - call it P. We prove that P is prime. In fact, for $a, b \in h(R)$, if $ab \in P$ and $a \notin P$, then there is an integer l such that $f^l \in aR + P$ (since P is maximal in S). If there is an integer s such that $f^s \in bR + P$, then $f^{l+s} \in (aR+P)(bR+P) \subset P$, a contradiction. Hence $b \in P$. This proves that P is a homogeneous gr-prime ideal.

To complete the proof we need the lemma below. Recall that in noncommutative ring theory primality of an ideal P is defined in terms of products of ideals, i.e., if $IJ \subset P$ for some ideals I and J, then $I \subset P$ or $J \subset P$. If this property holds at the level of elements (i.e., if $ab \in P$ for some elements a, b of the ring, then $a \in P$ or $b \in P$) then we say P is strongly prime. There are obvious graded counterparts of these notions and the following property holds.

Lemma 5.4.4 (Nastasescu, Van Oystaeyen, [NVO84]). *For a $\mathbb{Z}$-graded ring, a homogeneous ideal is gr-strongly prime if and only if it is strongly prime.*

Since R is commutative, the notions of prime and strongly prime are the same and hence P is prime.

Note now that there is a graded isomorphism from $R/Ann(m)(deg(m))$ to mR and thus a graded injection from $R/Ann(m)(deg(m))$ to M. Since $Ann(m)$ is contained in some prime ideal P, $R/Ann(m)(deg(m))$ maps nontrivially to $E^g(R/P)(deg(m))$ and thus so does M. Since R satisfies the hypothesis of lemma 5.2.11, one has that M maps nontrivially to $E^g(R/P)$ and thus, by the previous lemma, $\mathrm{Hom}_{tails}(\pi M, \pi E^g(R/P)) \neq 0$. □

Before stating the main theorem let us prove a useful lemma.

Lemma 5.4.5. *Suppose R is a commutative local ring with maximal ideal m. Given $X^\bullet$ a bounded complex of finitely generated free R-modules, define a complex $Y^\bullet := R/m \otimes_R X^\bullet$. If, for some fixed integer α, $H^j(Y^\bullet) = 0$ for all $j \geq \alpha$, then $H^j(X^\bullet) = 0$ for all $j \geq \alpha$.*

Proof. Suppose $H^j(Y^\bullet) = 0$ for all $j \geq \alpha$. Suppose $X^k = 0$ for all $k \geq p$ ($X^\bullet$ is bounded). If $\alpha > p$ then it is done. If $\alpha \leq p$, consider the following exact sequence:

$$X^{p-1} \longrightarrow X^p \longrightarrow coker(d_X^{p-1}) \longrightarrow 0$$

and apply to it the functor $F := R/m \otimes_R \_$ thus getting another exact sequence

$$Y^{p-1} \longrightarrow Y^p \longrightarrow R/m \otimes_R coker(d_X^{p-1}) \longrightarrow 0,$$

since F is right exact. By definition of $Y^\bullet$, the first map of the sequence is d_Y^{p-1}. Since $\alpha \leq p$, $H^p(Y^\bullet) = 0$, thus proving that d_Y^{p-1} is surjective ($Y^{p+1} = 0$ by

definition). Therefore $R/m \otimes_R coker(d_X^{p-1}) = 0$ which, by Nakayama's lemma (since R is local and $coker(d_X^{p-1})$ is a finitely generated R-module), implies that $coker(d_X^{p-1}) = 0$. Hence d_X^{p-1} is surjective, thus proving that $H^p(X^\bullet) = 0$.

If $\alpha = p$ it is done. Otherwise assume $H^{p-1}(Y^\bullet) = 0$ and we prove that $H^{p-1}(X^\bullet) = 0$ as well. Note that then the result follows by iterating this process a finite number of times (the difference between α and p). First, since X^p is free, the short exact sequence

$$0 \longrightarrow Ker(d_X^{p-1}) \longrightarrow X^{p-1} \longrightarrow X^p \longrightarrow 0$$

splits and thus $Ker(d_X^{p-1})$ is a summand of the free module X^{p-1}, i.e., a projective module (it is exact because $H^p(X^\bullet) = 0$ and $X^{p+1} = 0$). However it is well-known (Kaplansky's theorem) that projective modules over local rings are free.

Now we observe that $Ker(d_X^{p-1}) \cap mX^{p-1} = mKer(d_X^{p-1})$. In fact take $z_1, ..., z_t, z_{t+1}, ..., z_n$ a basis for X^{p-1} such that the first t elements form a basis for $Ker(d_X^{p-1})$. Given $x \in Ker(d_X^{p-1}) \cap mX^{p-1}$ we have, on one hand $x = \sum_{i=1}^t a_i z_i$ with $a_i \in R$ and on the other hand $x = \sum_{i=1}^n b_i z_i$ with $b_i \in m$. Linear independence of the elements of the basis assure $b_i = 0$ for $i > t$ and $a_i = b_i$ for $i \leq t$, thus proving that $x \in mKer(d_X^{p-1})$. The converse inclusion is trivial. This allows us to see, by considering the natural map from $Ker(d_X^{p-1})$ to $(Ker(d_X^{p-1}) + mX^{p-1})/mX^{p-1}$, that

$$Ker(d_Y^{p-1}) = \frac{Ker(d_X^{p-1}) + mX^{p-1}}{mX^{p-1}} \cong \frac{Ker(d_X^{p-1})}{mKer(d_X^{p-1})} = \frac{R}{m} \otimes_R Ker(d_X^{p-1}).$$

This means that, by definition of truncation, $\tilde{Y}^\bullet := t_{\leq p-1}(Y^\bullet) = R/m \otimes_R \tilde{X}^\bullet$ where $\tilde{X}^\bullet = t_{\leq p-1}(X^\bullet)$. Note that, since $H^{p-1}(Y^\bullet) = 0$, $Ker(d_Y^{p-1}) = Im(d_Y^{p-2})$ and thus

$$\tilde{Y}^{p-2} = Y^{p-2} \longrightarrow Ker(d_Y^{p-1}) = \tilde{Y}^{p-1}$$

is surjective, allowing us to conclude, by repeating first argument of the proof, that $H^{p-1}(\tilde{X}^\bullet) = 0$, which concludes the proof since $\tilde{X}^\bullet$ is quasi-isomorphic to $X^\bullet$, thus having the same cohomology.

□

Corollary 5.4.6. *Suppose R is a commutative local $\mathbb{K}$-algebra. Given $X^\bullet$ a bounded complex of finitely generated free R-modules and $Z^\bullet = \mathbb{K} \otimes_R X^\bullet$, if $H^j(Z^\bullet) = 0$ for all $j \geq \alpha$, then $H^j(X^\bullet) = 0$ for all $j \geq \alpha$, for some fixed integer α.*

Proof. Let m denote the unique maximal ideal of R. Note that

$$Y^\bullet = R/m \otimes_R X^\bullet = (R/m \otimes_{\mathbb{K}} \mathbb{K}) \otimes_R X^\bullet = R/m \otimes_{\mathbb{K}} Z^\bullet.$$

Since tensoring over $\mathbb{K}$ is exact, we have that $H^j(Z^\bullet) = 0$ for all $j \geq \alpha$ implies $H^j(Y^\bullet) = 0$ for all $j \geq \alpha$ and hence, by the previous lemma, we have the result. □

Finally we prove the main theorem of the chapter.

Theorem 5.4.7. *Given a perversity p, denote by T_i the torsion theory cogenerated by $\pi E_i = \pi \prod_{\{x \in X : p(x) = i\}} E^g(R/I_x)$ in $Tails(R)$. Let $S := \{T_i : i \in im(p)\}$. Then $D^{p,\leq 0} = D^{S,\leq 0} \cap D^b(coh(X))$.*

Proof. Let us denote by $\widehat{T_i}$ the torsion theory cogenerated by E_i in $Gr(R)$. We start by rewriting the conditions defining the aisle $D^{S,\leq 0}$. By definition, we have

$$D^{S,\leq 0} = \left\{ F^\bullet \in D^b(Tails(R)) : \ H^j(F^\bullet) \in T_k, \ \forall j > k \right\}$$

and given that the objects E_k are torsion-free injective objects, by lemma 5.4.1 we have

$$D^{S,\leq 0} = \left\{ F^\bullet \in D^b(Tails(R)) : \ \bar{\Gamma}(H^j(F^\bullet)) \in \widehat{T_k}, \ \forall j > k \right\} =$$

$$\left\{F^\bullet \in D^b(Qcoh(X)) : \forall x \in X^{top},\ \mathrm{Hom}_{Gr(R)}(\bar{\Gamma}(H^j(F^\bullet)), E^g(R/I_x)) = 0\ \forall j > p(x)\right\}.$$

Now we intersect with $D^b(coh(X))$ (i.e., pass from the quasi-coherent setting to the coherent one). For simplicity, define $D^{s,\leq 0} := D^{S,\leq 0} \cap D^b(coh(X))$. Using corollary 5.2.4 we get

$$D^{s,\leq 0} = \left\{F^\bullet \in D^b(coh(X)) : \forall x \in X^{top}, \bar{\Gamma}(H^j(F^\bullet))_{(x)} = 0, \forall j > p(x)\right\}$$

where $\bar{\Gamma}(H^j(F^\bullet))_{(x)}$ is the degree zero part of the localisation of $\bar{\Gamma}(H^j(F^\bullet))$ in complement of I_x, which is the same as stalk at x of the sheaf $H^j(F^\bullet)$. Since taking stalks is an exact functor (thus t-exact for the standard t-structure and therefore commuting with cohomology functors) we get

$$D^{s,\leq 0} = \left\{F^\bullet \in D^b(coh(X)) : \forall x \in X^{top},\ H^j(F^\bullet_x) = 0\ \forall j > p(x)\right\}.$$

Recall that

$$D^{p,\leq 0} = \left\{F^\bullet \in D^b(coh(X)) : \forall x \in X^{top},\ Li^*_x(F^\bullet) \in D^{\leq p(x)}(O_x - mod)\right\}$$

which is clearly the same as

$$\left\{F^\bullet \in D^b(coh(X)) : \forall x \in X^{top},\ H^j(Li^*_x(F^\bullet)) = 0,\ \forall j > p(x)\right\}.$$

So, it suffices to prove that $H^j(F^\bullet_x) = 0$ for all $j > p(x)$ is equivalent to $L_j i^*_x(F^\bullet) = 0$, for all $j > p(x)$.

Suppose $F^\bullet$ such that $H^j(F^\bullet_x) = 0$ for all $j > p(x)$. By definition of the pullback functor ($i^*_x(V) = V_x \otimes_{O_{X,x}} \mathbb{K}$ for any coherent sheaf V), there is a spectral sequence of Grothendieck type of the following form:

$$E_2^{ab} = Tor_a^{O_{X,x}}(\mathbb{K}_x, H^b(F^\bullet_x)) \Longrightarrow L_{a+b} i^*_x(F^\bullet),$$

where $\mathbb{K}_x$ is the skyscraper sheaf over x. Our hypothesis assures that $E_2^{ab} = 0$ for all $a < 0$ or $b > p(x)$ and thus $E_\infty^{ab} = 0$ for all $a < 0$ or $b > p(x)$. Let $\mathbb{F}^i$

denote the i-th part of the decreasing filtration assumed to exist (by definition of convergent spectral sequence) on the limit object $\Omega^{a+b} := L_{a+b}i_x^*(F^\bullet)$. Then, for $q > p(x)$ we get

$$... = \mathbb{F}^{-2}\Omega^{-2+(q+2)} = \mathbb{F}^{-1}\Omega^{-1+(q+1)} = \mathbb{F}^0\Omega^q = \mathbb{F}^1\Omega^q$$

and thus they are all equal to zero, proving that $\Omega^q = L_q i_x^*(F^\bullet) = 0$ for all $q > p(x)$.

Conversely, suppose we have $F^\bullet$ such that $L_j i_x^*(F^\bullet) = 0$ for all $j > p(x)$. Since X is smooth, let $G^\bullet$ be a complex of locally free sheaves such that $G^\bullet$ is quasi-isomorphic to $F^\bullet$ (thus isomorphic in the derived category) - [Huy06]. Then $L_j i_x^*(F^\bullet) = 0$ means that $H^j((i_x^*G)^\bullet)$, where $(i_x^*G)^\bullet$ denotes the complex resulting from applying i_x^* componentwise to $G^\bullet$. Take now $X^\bullet = G_x^\bullet$ and $Y^\bullet = (i_x^*G)^\bullet$ and recall that $G_x^\bullet$ is a complex of free modules over the local ring $O_{X,x}$. This leaves us in the context of corollary 5.4.5, thus proving that $H^j(G_x^\bullet) = H^j(G^\bullet)_x = 0$ for all $j > p(x)$. Finally we have $H^j(F_x^\bullet) = H^j(F^\bullet)_x = H^j(G^\bullet)_x = 0$ for all $j > p(x)$, hence finishing the proof. □

Remark 5.4.8. *Note that as a consequence we get that, for S defined as above, the t-structure constructed in section 5.3 restricts well to the derived category of finitely generated objects (coherent sheaves). An interesting problem to address would be that of finding necessary and sufficient conditions on a general ordered set of hereditary torsion classes so that this phenomena of restriction of t-structures holds in the general case.*

5.5 Perverse quasi-coherent t-structures for noncommutative projective planes

The aim of this section is to use the construction of section 5.3 to create an analogue of perverse coherent t-structures in the derived categories of certain noncommutative projective planes. This entails finding a cogenerating set of injective objects in $Tails(R)$ for a suitable class of $\mathbb{K}$-algebras R and set up a definition of perversity that generalises the commutative one.

Remark 5.5.1. *$tails(R)$ is not complete nor cocomplete and therefore, in this section, we can only do the construction of section 5.3 in $Tails(R)$ (hence the word quasi-coherent rather than coherent in the title). However, taking into account theorem 5.4.7, we conjecture that indeed the constructions in this section restrict well to $D^b(tails(R))$.*

We shall focus on the case where R is a graded elliptic 3-dimensional Artin-Schelter regular algebra which is finite over its centre. These algebras are interesting for our purposes since they are fully bounded noetherian (more than that, they are PI - [ATVdB91]). Also, a graded noetherian algebra which is fully bounded is graded fully bounded ([VOV81]). This is important for the following result that allows us to parametrise a useful collection of injective objects via prime ideals. In this sense, although these examples are noncommutative, we are still very close to the commutative setting ([Mat58]).

Recall that ther is a map from the set of indecomposable injective graded modules to the set of homogeneous prime ideals given by assigning to an injective E its homogeneous assassinator ideal, $Ass(E)$. The assassinator ideal of an indecomposable object is the only prime ideal associated to E, i.e., the only prime ideal which is maximal among the annihilators of nonzero submodules of E (and

there is a natural graded version of this concept - see [NVO82] and [VOV81]).

Proposition 5.5.2 (Natasescu, Van Oystaeyen, [NVO82]). *Let R be a positively graded noetherian ring. Then R is graded fully bounded if and only if the map that assigns the corresponding assassinator ideal to an indecomposable injective graded modules induces a bijection between indecomposable injective modules in $Gr(R)$ (up to isomorphism and graded shift) and homogeneous prime ideals of R.*

Remark 5.5.3. *In the context of the proposition, the indecomposable injective associated with a homogeneous prime P is the unique (up to isomorphism and shifts) direct summand of $E^g(R/P)$ ([NVO82]), thus establishing an inverse map.*

This result brings us closer to the desired cogenerating set. Its significance in our context comes from the work of Matlis on the decomposition of injective modules over noetherian rings. Matlis proved that R is (right) noetherian if and only if every injective (right) modules is the direct sum of indecomposable injective (right) modules ([Mat58]). This shows in particular that the set of indecomposable injective objects cogenerates the category of modules over a noetherian ring. One may hope a similar phenomena for graded rings and indeed one has the following ([Mah96]).

Proposition 5.5.4 (Mahmoud,[Mah96]). *Let M be a finitely generated graded module over a graded noetherian ring R. Then $E^g(M)$ is a finite direct sum of indecomposable injective objects in $Gr(R)$.*

These two results yield a useful set of torsion classes parametrised by the homogeneous prime ideals of R. Indeed, for an indecomposable injective E, denote the corresponding torsion class cogenerated by E in $Gr(R)$ by $\mathcal{T}_E$, i.e.,

$$\mathcal{T}_E := \left\{ M \in Gr(R) : \ Hom_{Gr(R)}(M, E) = 0 \right\}.$$

Let $\widehat{Y}$ denote the set of all such classes where E runs over indecomposable injective objects, up to isomorphism and graded shift, such that its assassinator ideal is not the irrelevant ideal, i.e., $Ass(E) \neq R_+$. Analogously define Y to be the set of the torsion classes of the form $\mathcal{T}_{\pi E}$ in $Tails(R)$.

Corollary 5.5.5. *Let R be a positively graded fully bounded connected noetherian $\mathbb{K}$-algebra generated in degree 1. Then the intersection (in $Tails(R)$) of the torsion classes in Y is zero.*

Proof. Suppose πM lies in the intersection of the torsion classes of Y. Then by lemma 5.4.1, M lies in the intersection of the torsion classes in $\hat{Y}$. By propostion 5.5.4, the indecomposable injective objects cogenerate $Gr(R)$ and thus $E^g(M)$ must be a finite direct sum of the indecomposable injective associated with R_+. This indecomposable is a direct summand of $E^g(R/R_+)$ (see remark 5.5.3), whose projection in $Tails(R)$ is therefore zero. Thus $\pi E^g(M) = 0$ and so is πM. $\square$

We proceed now to do the desired construction. Let R be a positively graded Artin-Schelter regular algebra of dimension 3 generated in degree one which is finitely generated over its centre. As discussed before, it is graded fully bounded noetherian. We need to define a perversity in Y, where Y is as above.

Definition 5.5.6. *A perversity is a map $p : Y \longrightarrow \mathbb{Z}$ such that, given $\mathcal{T}_{\pi E_1}$, $\mathcal{T}_{\pi E_2}$ in S, if there is a nonzero homomorphism from πE_2 to πE_1 then*

$$p(\mathcal{T}_{\pi E_1}) - (GKdim(R/Ass(E_2)) - GKdim(R/Ass(E_1))) \leq p(\mathcal{T}_{\pi E_2}) \leq p(\mathcal{T}_{\pi E_1}).$$

We aim to prove that this definition of perversity coincides, when the algebra is commutative, with the definition of perversity of section 5.1. We start by a supporting lemma.

Lemma 5.5.7. *Let R be a positively graded commutative noetherian $\mathbb{K}$-algebra and $X = Proj(R)$. The following are equivalent.*

1. *For $x_1, x_2 \in X^{top}$, $x_1 \in \bar{x_2}$;*
2. *$P_2 := Ann(x_2) \subset Ann(x_1) =: P_1$, where $Ann(x_i)$ denotes the homogeneous ideal of functions vanishing in x_i;*
3. *There is a nonzero homomorphism from R/P_2 to R/P_1;*
4. *There is a nonzero homomorphism from $E^g(R/P_2)$ to $E^g(R/P_1)$.*

Proof. It is clear that $(1) \Leftrightarrow (2) \Rightarrow (3) \Rightarrow (4)$.

We only need to prove $(4) \Rightarrow (2)$. Let f be a homomorphism from $E^g(R/P_2)$ to $E^g(R/P_1)$ and $a \in h(P_2) \setminus h(P_1)$. Clearly $N := R/P_1 \cap im(f) \neq 0$ since R/P_1 is a graded essential submodule of $E^g(R/P_1)$. Now, $N \cap (a+P_1)R \neq 0$ since any homogeneous ideal of a commutative graded domain is graded essential (the product of two nonzero ideals is nonzero and it is contained in the intersection). Hence, $0 \neq (a+P_1)R \cap N \subset (a+P_1)R \cap im(f)$. Let then b be a nonzero element in $(a+P_1)R \cap im(f)$ and $y \in E^g(R/P_2)$ such that $b = ar + P_1 = f(y)$. Note that $ya \in P_2 E^g(R/P_2)$ and P_2 annihilates $E^g(R/P_2)$, thus $0 = f(ya) = a^2 r + P_1$ and $r \in P_1$ since P_1 is prime. Hence $b = 0$ in R/P_1, reaching a contradiction and proving the result. □

Proposition 5.5.8. *If R is a positively graded noetherian connected commutative $\mathbb{K}$-algebra generated in degree 1, the definition of perversity above is equivalent to the commutative definition of perversity of section 5.1.*

Proof. Note that points $x \in X^{top}$ are in bijection with homogeneous prime ideals not equal to the irrelevant ideal of R and these are in bijection with graded torsion-free indecomposable injectives in $Gr(R)$ (up to isomorphisms and shifts). Suppose

$x_1, x_2 \in X^{top}$, P_1, P_2 the associated homogeneous prime ideals and E_1, E_2 the corresponding injectives. The condition $x_1 \in \bar{x_2}$ translates to the existence of a nonzero map from E_2 to E_1 by the lemma above (note that, in this case, $E_i = E^g(R/P_i)$ since R/P is indecomposable in $Gr(R)$ and hence so is its injective envelope) and by lemma 5.4.1 this is equivalent to the existence of a map from πE_2 to πE_1.

Since R is finitely generated over $\mathbb{K}$ (as it is noetherian), and hence are all its quotients, it is known the Krull dimension of R/P_i (which is the same as $dim(x_i)$ in the geometric definition of perversity - see 5.1.1) coincides with the Gelfand-Kirillov dimension of R/P_i ([KL85]).

The result then follows by making the adequate substitutions in the geometric definition of perversity (see definition 5.1.1). □

Recall that 3-dimensional Artin-Schelter regular algebras are noetherian domains - in particular, they are prime rings ([ATVdB90], [ATVdB91]). This allows us to prove the following useful lemma.

Lemma 5.5.9. *Let R be a positively graded connected 3-dimensional Artin-Schelter regular algebra generated in degree 1 which is finitely generated over its centre. Then the image of a perversity p as defined above is finite.*

Proof. Since R is prime, (0) is a prime ideal not equal to the irrelevant ideal. Thus it corresponds to an indecomposable injective object which we shall denote by E_0. Furthermore, as a consequence of remark 5.5.3, $E^g(R)$ is a finite direct sum of copies of E_0. Similarly, $E^g(R/P)$ is a finite direct sum of copies of E_P, the indecomposable injective object associated to the homogeneous prime ideal P. We observe that for any such P, there is a map from $E^g(R)$ to $E^g(R/P)$ induced by the canonical projection from R to R/P. Therefore, there is a nontrivial map

from E_0 to E_P. The perversity condition then assures that:

$$p(\mathcal{T}_{\pi E_P}) - (GKdim(R) - GKdim(R/P)) \leq p(\mathcal{T}_{\pi E_0}) \leq p(\mathcal{T}_{\pi E_P}).$$

Since, by definition, the Gelfand-Kirillov dimension of R is finite (and so is the dimension of any of its quotients - [KL85]) we have that, for a fixed value of $p(\mathcal{T}_{\pi E_0})$, $p(\mathcal{T}_{\pi E_P})$ is an integer that differs at most $GKdim(R)$ from it. Hence the image of p is finite. $\square$

Thus, for R Artin-Schelter regular algebra of dimension 3 and finite over its centre, we can form a finite set of hereditary torsion classes

$$S := \left\{ T_i := \bigcap_{T:p(T)=i} T, min(p) \leq i \leq max(p) \right\}.$$

By corollary 5.5.5, the intersection of all its elements is zero. Finally, section 5.3 provides a way of building a perverse quasi-coherent t-structure with respect to p by defining its aisle to be $D^{S,\leq 0}$ in $D^b(Tails(R))$. As mentioned in remark 5.5.1, in light of section 5.4 we conjecture that these t-structures restrict to $D^b(tails(R))$.

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