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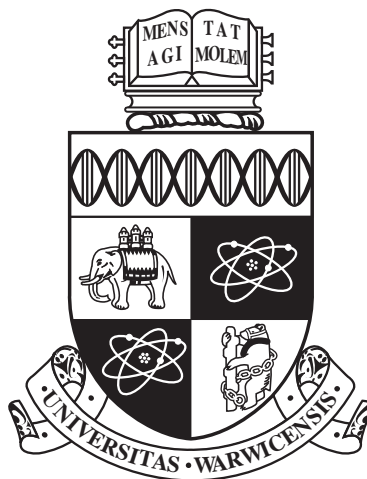
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# Monte Carlo Methods in Derivative Modelling

by

**Kai Zhang**

**Thesis**

Submitted to the University of Warwick

for the degree of

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THE UNIVERSITY OF  
**WARWICK**

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# Declarations

The work submitted in this thesis is the result of my own investigation, except where otherwise stated. It has not already been accepted for any degree at another university, and is also not being concurrently submitted for any other degree at another university.



# Abstract

This thesis addresses issues in discretization and variance reduction methods for Monte Carlo simulation.

For the discretization methods, we investigate the convergence properties of various Itô-Taylor schemes and the strong Taylor expansion (Siopacha and Teichmann [77]) for the LIBOR market model. We also provide an improvement on the strong Taylor expansion method which produces lower pricing bias.

For the variance reduction methods, we have four contributions. Firstly, we formulate a general stochastic volatility model nesting many existing models in the literature. Secondly, we construct a correlation control variate for this model. Thirdly, we apply the model as well as the new control variate to pricing average rate and barrier options. Numerical results demonstrate the improvement over using old control variates alone. Last but not least, with the help of our model and control variate, we explore the variations in barrier option pricing consistent with the implied volatility surface.

# Chapter 1

## Introduction

In derivative modelling, the asset process is normally described in the form of a stochastic differential equation (SDE). The valuation problem is generally in high dimensions, that is, the derivative values depend on multiple factors such as several assets, several points along the sample path or multi-factor stochastic volatility models. Due to this reason, Monte Carlo becomes a universal valuation tool. Two distinct concerns of financial practitioners are: 1) how to do Monte Carlo with low pricing bias; and 2) how to make it run faster.

### 1.1 Accuracy and Speed in Monte Carlo Methods

The first concern (the bias issue) is of the discrete approximation of continuous SDEs. Very often the underlying SDE does not have an explicit solution; one therefore must resort to Itô-Taylor approximations. These approximations introduce biases in the estimation. We can reduce those biases by increasing the number of time steps, but not the number of sample paths. The biases are defined in different aspects with respect to the SDE. An introduction of this material is given by Kloeden and Platen [55]. They describe weak and strong approximations of different orders of convergence.

The second concern (the speed issue) is of the standard error of the result. Monte Carlo is a way of estimating the expectation of some random variables whose explicit solutions are unavailable. The estimation is supplied with the standard error as a measure of convergence. A good estimator achieves a certain level of standard error with as little time as possible. This is the place where various variance reduction methods come in. Good introductions on this topic are in Jäckel [47] and also

Glasserman [36].

In this thesis, we address both issues in different modelling frameworks.

## 1.2 Summary of Contributions

The contributions of this thesis cover both the discretization and variance reduction schemes. We summarize them as following.

### **Discretization schemes (chapter 2):**

1. We provide numerical comparisons (section 2.6) of Itô-Taylor methods (reviewed in section 2.3) and strong Taylor expansion using the method of Siopacha and Teichmann [77]) (reviewed in section 2.4) for the LIBOR market model.
2. We propose an improvement on the strong Taylor expansion method. (Section 2.5)
3. We find that the iterative predictor-corrector method by Joshi and Stacey [51] has the best performance in a single long step simulation. In the multiple steps case, the order 2 Weak Itô-Taylor scheme produces the lowest pricing biases. The improved strong Taylor expansion method has lower pricing biases than the original method in a single step Monte Carlo, but neither of them converges as the number of time step increases.

### **Variance reduction methods (chapter 4):**

1. We formulate a general stochastic volatility model which nests many existing models in the literature. (Section 4.2)
2. We construct a new control variate (correlation control variate) for this model. (Section 4.4 and 4.5)
3. We apply the model and control variate to pricing average rate (section 4.6) and barrier options (section 4.7).
4. Extensive numerical evaluation is performed in chapter 5 and 6. We find significant improvements on the efficiency gains by including our new control variates in Monte Carlo.

5. With the help of our new control variates, we explore variations in barrier option pricing in our models in consistency with the implied volatility surface. (chapter 7)

The rest of this thesis is organised as following. In chapter 2, we address the issues in discretizing the LIBOR market model. In chapter 3, we review some basics for one of the most important variance reduction methods - the control variate method. In chapter 4, we propose new control variates for a general class of stochastic volatility models. Chapter 5 and 6 provide thorough numerical results for valuing average rate and barrier options using our proposed control variates. In chapter 7, we explore differences in barrier option prices in our calibrated models with the help of new control variates.

## Chapter 2

# Discretizing the LIBOR Market Model

### 2.1 Introduction

Market models are widely used in the market to price interest rate products. One reason for their popularity lies in their compatibility with market compliant formula for vanilla interest rate derivatives. The LIBOR market model (LMM) prices caplets with Black's [13] caplet formula while the Swap market model (SMM) prices swaptions with Black's swaption formula. A second reason is their ability to calibrate to the caplet or swaption volatility term structures in the market (see Rebonato [69]). For detailed and background description of these models, we refer to the papers by Brace *et al.* [15], Miltersen *et al.* [61], Jamshidian [48], *et cetera*. The book by Brigo and Mercurio [16] is also an excellent reference on this material.

There are various issues regarding numerical implementation of the LIBOR market model. Among others the three most important ones are: (i) pricing caps and swaptions simultaneously; (ii) tenor mismatch; and (iii) the joint discretization of forward LIBOR SDEs under a common probability measure. The third issue is crucial to any implementation that involves or requires Monte Carlo simulation and is the main concern of this chapter.

The dynamics of each forward LIBOR rate in any probability measure has a state independent diffusion coefficient but generally has a drift term depending on other forward LIBOR rates (except under its own forward measure). This feature prohibits the use of exact simulation. Most work on this topic (Balland [8], Hunter *et*

*al.* [46], Pietersz *et al.* [66], *et cetera*,) focuses on approximating the drift term. Another approach by Glasserman and Zhao [40] maps forward LIBOR rates to bonds which maintain a martingale property in discretization. A full comparison of these methods can be found in Beveridge *et al.* [12] (using the spot-LIBOR measure) or Joshi and Stacey [51] (using the terminal measure for very long step discretization).

In this chapter, we investigate the convergence properties of seven Itô-Taylor schemes including two new methods based on strong Taylor approximations to perturbed SDE. For Itô-Taylor approximations we apply results in Kloeden and Platen [55] tailored to the LIBOR market model. For strong Taylor approximations we review the framework introduced by Siopacha and Teichmann [77] and propose an improvement on their formulation. We point out that the improved version is just a special case of order 1.5 strong Itô-Taylor method and does not converge as the number of time steps increases. We argue that strong Taylor approximations should not be used for discretization. Instead, we recommend the use of iterative predictor-corrector method for single step discretization and order 2 weak Itô-Taylor scheme for multiple steps discretization.

The rest of this chapter is organised as follows. In Section 2.2, we review the basics of the LIBOR market model and show that forward LIBOR SDEs admit strong solutions. In Section 2.3, we review weak and strong convergence and derive formulae for various Itô-Taylor approximations. In Section 2.4, we introduce concepts of strong Taylor approximation to perturbed SDEs and review the original formulation of the LIBOR market model for drift approximation. An improved formulation is given in section 2.5. In Section 2.6, we provide numerical results for various discretization schemes. Section 2.7 concludes.

## 2.2 The LIBOR Market Model

The LIBOR market model (BGM/LMM) considers the dynamics of term structure of forward LIBOR rates. For detail derivations of the joint arbitrage free dynamics, we refer to Brace *et al.* [15], Miltersen *et al.* [61] and Jamshidian [48].

Assume we have a trading period  $\mathcal{T} = [0, T_{N+1}]$  with current time  $t \in \mathcal{T}$  and reset dates  $t < T_0 < T_1 < T_2 < \dots < T_{N+1}$ , where  $\tau_i = T_{i+1} - T_i$ ,  $i = 0, 1, 2, \dots, N$ . Let  $P(t, T)$  be the price at time  $t \leq T$  of the zero-coupon bond maturing at time

$T$ . The forward LIBOR rate at time  $t$  over period  $[T_i, T_{i+1}]$  is:

$$L_t^i := \frac{P(t, T_i) - P(t, T_{i+1})}{\tau_i P(t, T_{i+1})}, \quad i = 0, \dots, N \quad (2.1)$$

Each forward LIBOR rate has the dynamics

$$dL_t^i = L_t^i \left[ \mu_t^i dt + (\underline{\sigma}_t^i)' \mathbf{dW}_t^\mathbb{Q} \right], \quad (2.2)$$

where  $\mathbf{dW}_t = (dW_t^1, \dots, dW_t^n)'$  is a  $n$ -dimensional Brownian motion driving all forward LIBOR rates under a generic probability measure  $\mathbb{Q}$  and  $\underline{\sigma}_t^i = (\sigma_t^{i1}, \dots, \sigma_t^{in})'$  is a vector of factor loadings for  $L_t^i$  not dependent on  $L_t^i$ . The covariance between  $L_t^i$  and  $L_t^j$  is therefore  $(\underline{\sigma}_t^i)' \underline{\sigma}_t^j$ . For calibration purpose, it is convenient to assume functional forms for volatility and correlation separately. As a result, we may reinterpret the covariance as  $(\underline{\sigma}_t^i)' \underline{\sigma}_t^j = \sigma_t^i \sigma_t^j \rho_{ij}$  where  $\sigma_t^i = \|\underline{\sigma}_t^i\|$  can be a piecewise constant or deterministic function taking the form suggested by Rebonato [69]. In practice, the number of factors  $n$  is normally chosen to be 2 to 4 by reducing the rank of the correlation matrix  $\{\rho\}_{ij} = \rho_{ij}, \forall i, j$  (see, for example, Brigo and Mercurio [16]).

The drift term  $\mu_t^i$  depends on the probability measure under which  $L_t^i$  is modelled. Let  $\gamma(t) = i$  if  $T_{i-1} \leq t < T_i$ ,  $i = 1, 2, \dots, N$ . Table 2.1 summarizes the numeraire  $N_t$  and drift  $\mu_t^i$  under the terminal measure and spot-LIBOR measure (see also Brigo and Mercurio [16]).

Measure, $\mathbb{Q}$	Numerator, $N_t$	Drift, $\mu_t^i$
<b>Terminal</b>	$P(t, T_{N+1}) = \frac{P(t, T_{\gamma(t)})}{\prod_{i=\gamma(t)}^N (1 + \tau_i L_t^i)}$	$-\sum_{j=i+1}^N \frac{\tau_j L_t^j}{1 + \tau_j L_t^j} (\underline{\sigma}_t^i)' \underline{\sigma}_t^j$
<b>Spot-LIBOR</b>	$B_t = P(t, T_{\gamma(t)}) \prod_{i=0}^{\gamma(t)-1} (1 + \tau_i L_{T_i}^i)$	$\sum_{j=\gamma(t)}^i \frac{\tau_j L_t^j}{1 + \tau_j L_t^j} (\underline{\sigma}_t^i)' \underline{\sigma}_t^j$

Table 2.1: Summary of Probability Measures

We describe in sections 2.3 and 2.4 some long step Monte Carlo simulation schemes for LIBOR Market model. All these schemes evolve forward LIBOR,  $L_t^i$ , from time  $S$  up to  $T$ , for  $0 \leq S < T \leq T_i$  where  $T_i$  is the maturity of  $L_t^i$ . The interval  $[S, T]$  can stride over the earlier reset dates,  $T_0, \dots, T_{i-1}$ , on which  $L_t^0, \dots, L_t^{i-1}$  expire. In order to simulate  $L_t^i$  using forward rates that are still alive until time  $T$ , we evolve the LIBOR market model in the terminal measure  $\mathbb{Q}(N+1)$  because  $\mu_t^i$  is then a

function of alive rates only. In the terminal measure  $\ln L_T^i$  has the dynamics

$$d \ln L_t^i = \left( \mu_t^i - \frac{1}{2} \|\underline{\sigma}_t^i\|^2 \right) dt + (\underline{\sigma}_t^i)' d\mathbf{W}_t^{\mathbb{Q}(N+1)}, \quad (2.3)$$

where the drift term  $\mu_t^i$  given in Table 2.1 depends only on  $L_t^j$ ,  $j > i$ . None of these rates expire by time  $T_i$ .

Since  $\mu_t^i$  depends on other forward LIBOR rates, it is hard to discretize equation (2.3) analytically. Theorem 2.2.1 ensures that it admits strong solution and therefore we can write

$$\ln L_T^i = \ln L_S^i + \int_S^T \mu_t^i dt - \frac{1}{2} \int_S^T \|\underline{\sigma}_t^i\|^2 dt + \int_S^T (\underline{\sigma}_t^i)' d\mathbf{W}_t. \quad (2.4)$$

**Theorem 2.2.1** (*Existence and Uniqueness of Strong Solution (Oksendal [64])*). Suppose a stochastic differential equation has the form

$$d\mathbf{X}_t = a(\mathbf{X}_t, t)dt + b(\mathbf{X}_t, t)d\mathbf{W}_t, \quad (2.5)$$

where  $\mathbf{X}_t = (X_t^1, \dots, X_t^Q)'$  is the  $Q$ -dimensional process and  $\mathbf{W}_t = (W_t^1, \dots, W_t^M)'$  is the  $M$ -dimensional Brownian motion with  $a: \mathbb{R}^Q \times [0, \infty) \mapsto \mathbb{R}^Q$  and  $b: \mathbb{R}^Q \times [0, \infty) \mapsto \mathbb{R}^{Q \times M}$ . There exists a unique strong solution for Equation (2.5) if: (i)  $\mathbb{E}(\|\mathbf{X}_0\|^2) < \infty$ ; (ii)  $a$  and  $b$  satisfy Lipschitz condition and linear growth bound. That is there exists a constant  $K$  such that

**Lipschitz condition:** For all  $t \in [0, T]$  and  $\mathbf{x}, \mathbf{y} \in \mathbb{R}^Q$

$$\|a(\mathbf{x}, t) - a(\mathbf{y}, t)\| + \|b(\mathbf{x}, t) - b(\mathbf{y}, t)\| \leq K \|\mathbf{x} - \mathbf{y}\|. \quad (2.6)$$

**Linear Growth Bound:** For all  $t \in [0, T]$  and  $x \in \mathbb{R}^Q$

$$\|a(\mathbf{x}, t)\| + \|b(\mathbf{x}, t)\| \leq K(1 + \|\mathbf{x}\|) \quad (2.7)$$

where  $\|\cdot\|$  denotes the Euclidian norm of a vector or the Frobenius norm of a matrix. The solution is unique in the sense that if  $\tilde{\mathbf{X}}_t$  is another solution, then  $\mathbb{P}(\mathbf{X}_t = \tilde{\mathbf{X}}_t, \forall t \in [0, T]) = 1$ . The solution satisfies  $\mathbb{E}(\|\mathbf{X}_t\|^2) < \infty$  for all  $t \in [0, T]$ .  $\square$

These conditions place very severe restrictions on drift and diffusion coefficients. For example, the Lipschitz condition implies that the first derivative of the coefficients



must be bounded. This condition fails in a large class of process as in finance known as square root processes (such as CIR [24] and Heston [44]). Fortunately, for the LIBOR market model, we can show that these conditions are met. The Lipschitz condition is satisfied because

$$\|\mu(t, \mathbf{x}) - \mu(t, \mathbf{y})\| \leq \sum_{i=0}^N \left| \sum_{j=i+1}^N \left( \frac{\tau_j x_j}{1 + \tau_j x_j} - \frac{\tau_j y_j}{1 + \tau_j y_j} \right) (\underline{\sigma}_t^i)' \underline{\sigma}_t^j \right| \quad (2.8)$$

$$\leq NC_1 \sum_{j=1}^N \left| \frac{\tau_j x_j}{1 + \tau_j x_j} - \frac{\tau_j y_j}{1 + \tau_j y_j} \right| \quad (2.9)$$

$$\leq NC_1 \sum_{j=1}^N |x_j - y_j| \leq N^{\frac{3}{2}} C_1 \|\mathbf{x} - \mathbf{y}\| = C \|\mathbf{x} - \mathbf{y}\|, \quad (2.10)$$

where  $\mu(\cdot, \cdot) = (\mu_t^1, \dots, \mu_t^N)'$ . Equation (2.9) to (2.10) follow from the property of Lipschitz function  $\frac{\tau_j x}{1 + \tau_j x}$ .

We can also check for the linear growth bound condition

$$\left\| \mu(t, \mathbf{x}) - \frac{1}{2} (\|\underline{\sigma}_t^1\|^2, \dots, \|\underline{\sigma}_t^N\|^2)' \right\| + \|\sigma_t\| \leq \sum_{i=0}^N \left| \mu_t^i - \frac{1}{2} \|\sigma_t^i\|^2 \right| + \|\sigma_t\| \quad (2.11)$$

$$\leq \sum_{i=0}^N \left| \sum_{j=i+1}^N \frac{\tau_j x_j}{1 + \tau_j x_j} (\underline{\sigma}_t^i)' \underline{\sigma}_t^j \right| + \left( \frac{1}{2} \sum_{i=1}^N \|\sigma_t^i\|^2 + \|\sigma_t\| \right) \quad (2.12)$$

$$\leq NC_1 \left| \sum_{j=1}^N \frac{x_j}{1 + \tau_j x_j} \right| + C_2 \leq NC_1 \sum_{j=1}^N |x_j| + C_2 \quad (2.13)$$

$$\leq N^{\frac{3}{2}} C_1 \|\mathbf{x}\| + C_2 \leq C (1 + \|\mathbf{x}\|), \quad (2.14)$$

where  $\sigma_t = (\underline{\sigma}_t^1, \dots, \underline{\sigma}_t^N)'$  and  $C = \max(N^{\frac{3}{2}} C_1, C_2)$ . The initial condition is satisfied automatically as  $L_0^i$  is constant.

For the LIBOR market model, the diffusion part is a time changed Brownian motion. Schemes approximating the  $\int_S^T \mu_t^i dt$  term in equation (2.4) are generally referred to as drift approximations. Due to the existence of a strong solution, weak and strong Itô-Taylor approximations apply to the LIBOR SDE. We look at these schemes in turn.

## 2.3 A Review of Itô-Taylor Approximations

We review in sections 2.3.1-2.3.5 six Itô-Taylor schemes for the LIBOR market model: the Euler, Milstein, predictor-corrector (Hunter *et al.* [46]), order 2 weak Itô-Taylor (Glasserman and Merener [38]), order 1.5 strong Itô-Taylor and order 2 predictor-corrector. We include a seventh scheme, a variant of predictor-corrector method introduced by Joshi and Stacey [51], known as the iterative predictor-corrector. This scheme applies to the LIBOR market model only in the terminal measure. We include it here as an improvement to the predictor-corrector method. These schemes require that the simulated value  $\tilde{L}_T^i$  is a good approximation to continuous time value  $L_T^i$  according to some criterion. The most widely used are strong and weak convergence criteria.

**Definition** (Strong and Weak Convergence). Suppose we are given a set of discretized dates  $t_0, t_1, \dots, t_n$ , with  $t_i < t_{i+1}$  for  $i = 0, \dots, n-1$ ,  $t_0 = 0$  and  $t_n = T$ . Write  $\delta_i = t_{i+1} - t_i$  and let  $\delta = \max_i \{\delta_i\}$  be the mesh size. Given a continuous time Wiener sample path  $W = \{W_t, t \in [0, T]\}$ ,  $X_{t_i}^\delta$ ,  $i = 0, \dots, N$ , is a discretization of  $X_t$  with mesh size  $\delta$ .  $X_{t_N}^\delta = X_T^\delta$  is said to converge to the exact solution  $X_T$  with order  $\beta$  if

**Strong Convergence:**  $\mathbb{E}(|X_T^\delta - X_T|) \leq C\delta^\beta$

**Weak Convergence:**  $|\mathbb{E}[f(X_T^\delta)] - \mathbb{E}[f(X_T)]| \leq C\delta^\beta$  for  $f \in \mathcal{C}^{2(\beta+1)}$  is  $2(\beta+1)$  times differentiable function with polynomial growth,

where  $C$  is constant and independent of  $\delta$ . (See Oksendal [64])  $\square$

In finance the weak approximation is more important because in the risk-neutral framework the option price is given by  $v = \mathbb{E}[f(X_T)]$  with some numeraire re-based payoff function  $f$ . Unfortunately, option payoff functions are usually not differentiable everywhere and thus fail the conditions for weak convergence. In practice, however, we sometimes obtain the theoretical weak order of convergence even if  $f$  is not differentiable at some points. We apply both weak and strong Itô-Taylor approximations to the LIBOR market model.

We consider a single step from time  $S$  to time  $T$  with  $\Delta t = T - S$ . If  $\Delta t$  is intended to be small the scheme is called “short stepping”; if  $\Delta t$  is large the scheme is called “long stepping”. Suppose we have a  $n$ -dimensional Wiener sample path  $\mathbf{W}_t$ , we define the Wiener increment from time  $S$  to time  $T$  as  $\Delta \mathbf{W}_{S,T} = \mathbf{W}_T - \mathbf{W}_S$ . We use the Wiener increment at each time step.

### 2.3.1 Euler Stepping

In the Euler scheme (or Euler-Maruyama [58] scheme) the state variables in  $\mu_t^i$  are all set to their initial values at the start of the step. For a general SDE like equation (2.5), it has the form

$$\Delta_{S,T}^E(\mathbf{X}_S) = a(\mathbf{X}_S, S) \Delta t + b(\mathbf{X}_S, S) \Delta \mathbf{W}_{S,T}. \quad (2.15)$$

In the LIBOR market model, we apply the scheme to  $\ln L_t^i$ . The discretization is

$$\Delta_{S,T}^E(\ln L_S^i) = \mu_S^i \Delta t - \frac{1}{2} (\underline{v}_{S,T}^i)' (\underline{v}_{S,T}^i) + (\underline{v}_{S,T}^i)' \Delta \mathbf{W}_{S,T}, \quad (2.16)$$

where  $\underline{v}_{S,T}^i = \left( \sqrt{\int_S^T (\sigma_t^{i,1})^2 dt}, \dots, \sqrt{\int_S^T (\sigma_t^{i,n})^2 dt} \right)'$ . This is the simplest approximation achieving order 1 weak convergence (see Kloeden and Platen [55]). When  $S = 0$  and  $T = T_i$ , this long stepping scheme is also known as “drift freezing”.

**Milstein.** A Milstein scheme has strong convergence of order 1 (see Kloeden and Platen [55]). For each dimension  $X_t^i$  of SDE (2.5), the scheme is

$$\Delta^M(X_S^i) = \Delta^E(X_S^i) + \sum_{l,m,n} \frac{\partial b_{k,l}}{\partial X_t^n} b_{n,m} \Big|_{t=S} I_{m,l}, \quad (2.17)$$

where  $I_{m,l} = \int_{t_i}^{t_{i+1}} \int_{t_i}^t dW_s^m dW_t^l$ . If we set  $X_t^i = \ln L_t^i$ ,  $\forall i$ , in the LIBOR market model, then  $b(\mathbf{X}_t, t) = (\underline{\sigma}_t^1, \dots, \underline{\sigma}_t^n)'$  is independent of  $\mathbf{X}_t$ . Therefore,  $\frac{\partial \sigma_t^i}{\partial X_t^j} \equiv \mathbf{0}$ ,  $\forall i, j$ , the Milstein scheme for  $\ln L_t^i$  is the same as the Euler scheme in equation (2.16) and therefore Euler stepping for the LIBOR market model automatically achieves strong convergence of order 1.

This approximation, however, is too crude for large  $\Delta t$  (see Jäckel [47]). It is essentially a short-stepping scheme. An improvement on this scheme is the predictor-corrector scheme.

### 2.3.2 Order 1 Predictor-Corrector

The predictor-corrector scheme for a general SDE is

$$\begin{aligned} \Delta_{S,T}^{pc}(\mathbf{X}_S) &= [p \cdot \underline{\mathbf{a}}(T, \underline{\mathbf{X}}_T) + (1-p) \cdot \underline{\mathbf{a}}(S, \mathbf{X}_S)] \Delta t \\ &\quad + [q \cdot \underline{\mathbf{b}}(T, \underline{\mathbf{X}}_T) + (1-q) \cdot \underline{\mathbf{b}}(S, \mathbf{X}_S)] \Delta \mathbf{W}_{S,T}, \end{aligned} \quad (2.18)$$

where

$$\underline{\mathbf{X}}_T = \mathbf{X}_S + a(\mathbf{X}_S, S) \Delta t + b(\mathbf{X}_S, S) \Delta \mathbf{W}_{S,T}, \quad (2.19)$$

and

$$\underline{a}_i = a_i - q \sum_{j,k} b_{j,k} \frac{\partial b_{i,k}}{\partial X^j}, \quad p, q \in (0, 1). \quad (2.20)$$

The order 1 weak convergence predictor-corrector method is easy to implement when the derivatives of diffusion coefficients with respect to all state variables are zero (see Kloeden and Platen [55]). This is the case for the LIBOR market model in equation (2.3) where  $\underline{a}_t^i$ 's are deterministic so that all  $\frac{\partial \underline{a}_t^i}{\partial L_t^j} = \mathbf{0}$ . Hunter *et al.* [46] apply this scheme to the standard LIBOR market model.

The predictor-corrector scheme is a two-step method. The first step is to obtain realizations,  $\tilde{L}_T^{i+1}, \dots, \tilde{L}_T^N$ , using an Euler scheme (2.16) and  $\Delta \mathbf{W}_{S,T}$ . These realizations are called “predictors”. Then the drift term is approximated as

$$\tilde{\mu}^i = \frac{1}{2} (\mu_S^i + \mu_T^i), \quad (2.21)$$

where  $\mu_T^i$  are calculated using  $\tilde{L}_T^{i+1}, \dots, \tilde{L}_T^N$ . The Wiener increment  $\Delta \mathbf{W}_{S,T}$  used in the first step are then reused to evolve the “corrector”,

$$\ln L_T^i = \ln L_S^i + \tilde{\mu}^i \Delta t - \frac{1}{2} (\underline{v}_{S,T}^i)' (\underline{v}_{S,T}^i) + (\underline{v}_{S,T}^i)' \Delta \mathbf{W}_{S,T}. \quad (2.22)$$

Jäckel [47] found this scheme is accurate for very long time steps in the context of Bermudan swaption valuation. Beveridge *et al.* [12] compared it with arbitrage-free discretization (Glasserman and Zhao [40]) in spot-LIBOR measure and claimed it to be superior.

### 2.3.3 Iterative Predictor-Corrector

In the terminal measure, Joshi and Stacey [51] improved the predictor-corrector scheme so that there is no need for a corrector step. Since  $L_t^N$  has no drift, it can be discretized analytically. The drift term of  $L_t^{N-1}$  depends only on  $L_t^N$ . Given the realization  $\tilde{L}_T^N$  we can approximate the drift of  $L_t^{N-1}$  in the same way as (2.21). Then we can evolve to obtain realization,  $\tilde{L}_T^{N-1}$ , using the approximated drift. All other rates,  $L_T^{N-2}, \dots, L_T^i$ , are simulated backwards in turn iteratively. This scheme is not an Itô-Taylor method in a strict sense because its “predictors” are not obtained through Itô-Taylor approximations. It is unclear whether this scheme increases the order of convergence over the predictor-corrector method but it reduces the

computation cost since “predictors” are obtained from previous iterations without the need to recompute them in the current one.

### 2.3.4 Order 2 Weak Itô-Taylor

The second order weak scheme was introduced first by Milstein [60]. It is discussed in Glasserman [36]. Glasserman and Merener [38] applied it to the jump diffusion LIBOR market model. This scheme improves upon the first order Milstein scheme by keeping more terms in the Itô-Taylor expansion. It is equivalent to first obtaining the SDEs for both  $a(\mathbf{X}_t, t)$  and  $b(\mathbf{X}_t, t)$  in equation (2.5) using Itô formula and then applying the Euler method to them. The one step approximation is given by

$$\begin{aligned}\Delta_{S,T}^{2w}(\mathbf{X}_S) &= \int_S^T [a(\mathbf{X}_S, S) + \Delta_{S,t}^E a(\mathbf{X}_S, S)] dt + \int_S^T [b(\mathbf{X}_S, S) + \Delta_{S,t}^E b(\mathbf{X}_S, S)] d\mathbf{W}_t \\ &= \Delta_{S,T}^E(\mathbf{X}_S) + \int_S^T \Delta_{S,t}^E a(\mathbf{X}_S, S) dt + \int_S^T \Delta_{S,t}^E b(\mathbf{X}_S, S) d\mathbf{W}_t, \quad (2.23)\end{aligned}$$

for  $t \in (S, T)$ . The last two integrals can be expressed in terms of  $\int_S^T \int_S^t du dt$ ,  $\int_S^T \int_S^t d\mathbf{W}_u dt$ ,  $\int_S^T \int_S^t du d\mathbf{W}_t$  and  $\int_S^T \int_S^t d\mathbf{W}_u (d\mathbf{W}_t)'$ .

We introduce a quick derivation of the discretization formula based on equation (2.23). We first define the forward discounted payoff and then base our scheme on its dynamics.

**Forward Discounted Payoff.** From Table 2.1 we notice the term  $\frac{\tau_i L_t^i}{1+\tau_i L_t^i}$  appears in  $\mu_t^i$  in the forward LIBOR SDE. This term has the intuition of a forward discounted payoff at a reset date. By investing £1 at time  $T_i$ , the interest  $\tau_i L_t^i$  is paid at time  $T_{i+1}$ . This is equivalent to receiving  $\frac{\tau_i L_t^i}{1+\tau_i L_t^i}$  at time  $T_i$ .

Define  $X_t^i = \frac{\tau_i L_t^i}{1+\tau_i L_t^i}$ .  $X_t^i$  has the dynamics

$$\frac{dX_t^i}{X_t^i(1-X_t^i)} = -\sum_{j=i}^N (\underline{\sigma}_t^j)' X_t^j dt + (\underline{\sigma}_t^i)' d\mathbf{W}_t. \quad (2.24)$$

We can re-write equation (2.4), the strong solution for  $L_t^i$ , as

$$\ln L_T^i = \ln L_S^i - \sum_{j=i+1}^N \int_S^T X_t^j (\underline{\sigma}_t^j)' \underline{\sigma}_t^j dt - \frac{1}{2} \int_S^T \|\underline{\sigma}_t^i\|^2 dt + \int_S^T (\underline{\sigma}_t^i)' d\mathbf{W}_t. \quad (2.25)$$

The dynamics of  $X_t^j (\underline{\sigma}_t^i)' \underline{\sigma}_t^j$  is

$$\begin{aligned} d \left[ X_t^j (\underline{\sigma}_t^i)' \underline{\sigma}_t^j \right] &= X_t^j \frac{\partial}{\partial t} \left[ (\underline{\sigma}_t^i)' \underline{\sigma}_t^j \right] dt + (\underline{\sigma}_t^i)' \underline{\sigma}_t^j dX_t^j \\ &= X_t^j \frac{\partial}{\partial t} \left[ (\underline{\sigma}_t^i)' \underline{\sigma}_t^j \right] dt + X_t^j (1 - X_t^j) (\underline{\sigma}_t^i)' \underline{\sigma}_t^j \left[ - \left( \sum_{k=j}^N (\underline{\sigma}_t^j)' \underline{\sigma}_t^k X_t^k dt - (\underline{\sigma}_t^j)' d\mathbf{W}_t \right) \right]. \end{aligned} \quad (2.26)$$

The Euler approximation for  $X_t^j (\underline{\sigma}_t^i)' \underline{\sigma}_t^j$ , for  $t \in (S, T)$ , is then

$$X_t^j (\underline{\sigma}_t^i)' \underline{\sigma}_t^j \approx X_S^j (\underline{\sigma}_S^i)' \underline{\sigma}_S^j + \Delta_{S,t}^E \left[ X_S^j (\underline{\sigma}_S^i)' \underline{\sigma}_S^j \right] \quad (2.27)$$

$$\begin{aligned} &= X_S^j \frac{\partial}{\partial u} \left[ (\underline{\sigma}_u^i)' \underline{\sigma}_u^j \right] \Big|_{u=S} \int_S^t du + X_S^j (\underline{\sigma}_S^i)' \underline{\sigma}_S^j \\ &\quad \times \left[ 1 - X_S^j (1 - X_S^j) \left( \sum_{k=j}^N (\underline{\sigma}_S^j)' \underline{\sigma}_S^k X_S^k \int_S^t du - (\underline{\sigma}_S^j)' \int_S^t d\mathbf{W}_u \right) \right]. \end{aligned} \quad (2.28)$$

The terms in the summation in Equation (2.25) can be approximated as

$$\begin{aligned} &\int_S^T X_t^j (\underline{\sigma}_t^i)' \underline{\sigma}_t^j dt \\ &\approx \int_S^T \left\{ X_S^j (\underline{\sigma}_S^i)' \underline{\sigma}_S^j + \Delta_{S,t}^E \left[ X_S^j (\underline{\sigma}_S^i)' \underline{\sigma}_S^j \right] \right\} dt \\ &= \frac{\Delta t^2}{2} X_S^j \frac{\partial}{\partial u} \left[ (\underline{\sigma}_u^i)' \underline{\sigma}_u^j \right] \Big|_{u=S} + X_S^j (\underline{\sigma}_S^i)' \underline{\sigma}_S^j \\ &\quad \times \left[ \Delta t - (1 - X_S^j) \left( \frac{\Delta t^2}{2} \sum_{k=j}^N (\underline{\sigma}_S^j)' \underline{\sigma}_S^k X_S^k - (\underline{\sigma}_S^j)' \mathbf{I} \right) \right], \end{aligned} \quad (2.29)$$

where  $\mathbf{I} = (I_{(1,0)}, \dots, I_{(n,0)})'$  with  $I_{(i,0)} = \int_S^T \int_S^t dW_u^i dt$ .

We approximate the diffusion term in equation (2.25) by taking a first order Taylor expansion of  $\underline{\sigma}_t^i$  around  $t = S$ .

$$\begin{aligned} \int_S^T (\underline{\sigma}_t^i)' d\mathbf{W}_t &\approx \int_S^T \left[ (\underline{\sigma}_S^i)' + \frac{\partial}{\partial u} (\underline{\sigma}_u^i)' \Big|_{u=S} \int_S^t du \right] d\mathbf{W}_t \\ &= (\underline{\sigma}_S^i)' \Delta \mathbf{W}_{S,T} + \frac{\partial}{\partial u} (\underline{\sigma}_u^i)' \Big|_{u=S} \bar{\mathbf{I}}, \end{aligned} \quad (2.30)$$

where  $\bar{\mathbf{I}} = (I_{(0,1)}, \dots, I_{(0,n)})'$  with  $I_{(0,i)} = \int_S^T \int_S^t du dW_t^i$ . The discretization then reads

$$\ln L_T^i \approx \ln L_S^i + \sum_{j=i+1}^N \tilde{\mu}_{S,T}^{i,j} - \frac{1}{2} \int_S^T \|\underline{\sigma}_t^i\|^2 dt + (\underline{\sigma}_S^i)' \Delta \mathbf{W}_{S,T} + \frac{\partial}{\partial u} (\underline{\sigma}_u^i)' \Big|_{u=S} \bar{\mathbf{I}}, \quad (2.31)$$

where  $\tilde{\mu}_{S,T}^{i,j}$  denotes the expression on the right hand side of equation (2.29).

Glasserman [36] suggests that weak second order convergence still holds if one replaces  $I_{(i,0)}$  and  $I_{(0,i)}$  with their conditional expectation  $\frac{1}{2} \Delta t \Delta W_{S,T}^i$ . By doing so, the second order scheme does not require any additional random numbers to the Wiener increments  $\Delta W_{S,T}$ . The second order weak scheme theoretically has convergence of weak order 2 under certain conditions on drift and diffusion coefficients: both must be six times continuously differentiable, with all derivatives uniformly bounded, and pairwise products of diffusion coefficients have a linear growth bound (see Talay [80], Kloeden and Platen [55]).

**Order 1.5 Strong Itô-Taylor.** The order 1.5 strong scheme for equation (2.5) adds only an additional term to order 2 weak scheme which involves a triple iterated Itô integral. For each dimension  $q$ ,

$$\Delta^{1.5s} X_t^q = \Delta^{2w} X_t^q + \sum_{j_1, j_2, j_3=1}^M \sum_{k, l=1}^Q b_{lj_1} \left( \frac{\partial b_{kj_2}}{\partial X_t^l} \frac{\partial b_{qj_3}}{\partial X_t^k} + b_{kj_2} \frac{\partial^2 b_{qj_2}}{\partial X_t^k \partial X_t^l} \right) I_{(j_1, j_2, j_3)}, \quad (2.32)$$

where  $I_{(j_1, j_2, j_3)} = \int_S^T \int_S^{t_2} \int_S^{t_1} dW_u^{j_1} dW_{t_1}^{j_2} dW_{t_2}^{j_3}$  and all the partial derivatives are evaluated at  $t = S$  (see Kloeden and Platen [55]). Fortunately, since in the LIBOR market model of equation (2.3)  $\frac{\partial \sigma_S^i}{\partial L_t^j} \equiv \mathbf{0}, \forall i, j$ , the coefficients in front of  $I_{(j_1, j_2, j_3)}$  vanish which leaves order 1.5 strong scheme having the same expression as the order 2 weak scheme. The only difference is that  $I_{(j,0)}$  and  $I_{(0,j)}, j \geq i$ , must be simulated, rather than being replaced by their conditional expectations. Kloeden and Platen [55] showed that they can be simulated exactly as

$$I_{(i,0)} = \frac{1}{2} \Delta t \left( \Delta W_S^i + \frac{1}{\sqrt{3}} \Delta y_i \right), \quad (2.33)$$

$$I_{(0,i)} = \Delta W_S^i \Delta t - I_{(i,0)}, \quad (2.34)$$

where  $\Delta y_i \sim N(0, \Delta t)$ .

### 2.3.5 Order 2 Predictor-Corrector

The second order predictor-corrector scheme differs from the first order predictor-corrector in two aspects: 1) it uses order 2 weak scheme to obtain predictors; 2) the diffusion coefficient  $b(\mathbf{X}_t, t)$  is approximated by the Euler method instead of a linear combination of  $b(\mathbf{X}_S, S)$  and  $b(\mathbf{X}_T, T)$  as is the case of order 1 predictor-corrector in equation (2.18). Specifically, the discretization form for equation (2.5) reads (see Kloeden and Platen [55])

$$\begin{aligned}\Delta_{S,T}^{2pc}(\mathbf{X}_S) &= \frac{1}{2} [a(\mathbf{X}_S, S) + a(\mathbf{X}_S + \Delta_{S,T}^{2w} \mathbf{X}_S, T)] \Delta t \\ &\quad + \int_S^T [b(\mathbf{X}_S, S) + \Delta_{S,t}^E b(\mathbf{X}_S, S)] d\mathbf{W}_t \\ &= \frac{1}{2} [a(\mathbf{X}_S, S) + a(\mathbf{X}_S + \Delta_{S,T}^{2w} \mathbf{X}_S, T)] \Delta t \\ &\quad + b(\mathbf{X}_S, S) \Delta \mathbf{W}_{S,T} + \int_S^T \Delta_{S,t}^E b(\mathbf{X}_S, S) d\mathbf{W}_t\end{aligned}\quad (2.35)$$

For LIBOR market model of equation (2.3) in each dimension,

$$\ln L_T^i \approx \ln L_S^i + \frac{1}{2} (\mu_S^i + \tilde{\mu}_T^i) - \frac{1}{2} \int_S^T \|\underline{\sigma}_u^i\|^2 dt + (\underline{\sigma}_S^i)' \Delta \mathbf{W}_S + \frac{\partial}{\partial u} (\underline{\sigma}_u^i)' \Big|_{u=S} \bar{\mathbf{I}}, \quad (2.36)$$

where  $\tilde{\mu}_T^i$  are evaluated at the predictors  $\tilde{L}_T^j$ 's which are obtained through second order weak Taylor scheme given by Equation (2.31).  $I_{(i,0)}$  can be approximated by  $\frac{1}{2} \Delta t \Delta W_S^i$  as before.

## 2.4 A Review of Strong Taylor Expansion

In this section, we introduce two new schemes for drift approximation. They both adopt the same technique of *strong Taylor approximation* for perturbed stochastic processes, but with different formulations. The difference with previous schemes is that they introduce new SDEs which are very close to the LIBOR market model but easier to implement.

Strong Taylor approximation is a recent scheme based on expansion of a perturbed SDE around a point where the explicit solution is known. It was first introduced by Siopacha and Teichmann [77]. Papapantoleon and Siopacha [65] derived formulas for simulating more general Lévy LIBOR model. The main contributions of their methods are: 1) using Taylor expansion for each perturbed forward LIBOR SDE



and 2) providing a new framework for drift approximation. However, we notice that the original formulation introduces two sources of error and is far away from a satisfactory discretization scheme. An improvement on this formulation is proposed which turns the strong Taylor approximation into a practical implementable method.

Our method differs from theirs in that we consider the perturbation of the forward discounted payoff, rather than  $L_t^i$ ,  $\forall i$ , and apply strong Taylor approximation to it. By doing so, we are able to simulate forward LIBOR rates in a single long step and introduce only one source of error. Nevertheless, we shall see that the strong Taylor approximation methods do not converge towards the continuous time solution of the standard LIBOR market model as the number of time steps increases.

We first introduce the concepts of strong Taylor approximation.

**Strong Taylor Approximation.** We adopt the definition from Siopacha and Teichmann [77]. Given a set of smooth functions  $F_\epsilon: \mathbb{R} \mapsto \mathbb{R}$  with  $F_\epsilon \in L^2(\Omega, \mathbb{R})$  square integrable. Set

$$\mathbf{T}_\epsilon^n(F_\epsilon) = \sum_{i=0}^n \frac{\epsilon^i}{i!} \frac{\partial^i F_\epsilon}{\partial \epsilon^i} \Big|_{\epsilon=0}. \quad (2.37)$$

$F_\epsilon$  is a strong Taylor approximation of order  $n$  if

$$\mathbb{E} [|F_\epsilon - \mathbf{T}_\epsilon^n(F_\epsilon)|] = o(\epsilon^n). \quad (2.38)$$

When  $\epsilon \rightarrow 0$ ,  $\mathbf{T}_\epsilon^n(F_\epsilon) \rightarrow F_\epsilon$  in the strong sense. We refer to the error coming from truncating at level  $n$  as *truncation error*.

The application of the Taylor expansion to the perturbed LIBOR market model was introduced by Siopacha and Teichmann [77]. They applied the perturbation to the forward LIBOR rate.<sup>1</sup> The perturbed version of  $L_t^i$  is

$$\frac{d\hat{L}_t^{(i,\epsilon)}}{\hat{L}_t^{(i,\epsilon)}} = \epsilon \left( - \sum_{j=i+1}^N \frac{\tau_j \hat{L}_t^{(j,\epsilon)} (\underline{\sigma}_t^i)' \underline{\sigma}_t^j}{1 + \tau_j \hat{L}_t^{(j,\epsilon)}} dt + (\underline{\sigma}_t^i)' d\mathbf{W}_t \right). \quad (2.39)$$

The first order strong Taylor approximation for  $\hat{L}_t^{(i,\epsilon)}$  around  $\epsilon = 0$  is given by

$$\hat{L}_t^{(i,\epsilon)} \approx \mathbf{T}_\epsilon^1(\hat{L}_t^{(i,\epsilon)}) = L_0^i + \epsilon \frac{\partial \hat{L}_t^{(i,\epsilon)}}{\partial \epsilon} \Big|_{\epsilon=0}, \quad (2.40)$$

---

<sup>1</sup>It could be better to apply perturbation to  $\ln L_t^i$  though.

where

$$\left. \frac{\partial \hat{L}_t^{(i,\epsilon)}}{\partial \epsilon} \right|_{\epsilon=0} = L_0^i \left[ - \sum_{j=i+1}^N \left( \frac{\tau_j L_0^j}{1 + \tau_j L_0^j} \int_0^t (\underline{\sigma}_t^i)' \underline{\sigma}_t^j dt \right) + \int_0^t (\underline{\sigma}_s^i)' d\mathbf{W}_s \right]. \quad (2.41)$$

The stochastic process

$$\frac{dL_t^{(i,\epsilon)}}{L_t^{(i,\epsilon)}} = - \sum_{j=i+1}^N \frac{\tau_j \hat{L}_t^{(j,\epsilon)} (\underline{\sigma}_t^i)' \underline{\sigma}_t^j}{1 + \tau_j \hat{L}_t^{(j,\epsilon)}} dt + (\underline{\sigma}_t^i)' d\mathbf{W}_t \quad (2.42)$$

returns to the standard LIBOR market model when  $\epsilon = 1$  and is the frozen drift approximation when  $\epsilon = 0$ . By replacing  $\hat{L}_t^{(j,\epsilon)}$  with  $\left( \mathbf{T}_\epsilon^1 \left( \hat{L}_t^{(i,\epsilon)} \right) \right)_+$ , the SDE for simulation is

$$d \ln L_t^{(i,\epsilon)} = \left[ - \sum_{j=i+1}^N \frac{\tau_j \left( \mathbf{T}_\epsilon^1 \left( \hat{L}_t^{(i,\epsilon)} \right) \right)_+ (\underline{\sigma}_t^i)' \underline{\sigma}_t^j}{1 + \tau_j \left( \mathbf{T}_\epsilon^1 \left( \hat{L}_t^{(i,\epsilon)} \right) \right)_+} - \frac{1}{2} \|\underline{\sigma}_t^i\|^2 \right] dt + (\underline{\sigma}_t^i)' d\mathbf{W}_t, \quad (2.43)$$

or in integrated form

$$\begin{aligned} \ln L_T^{(i,\epsilon)} = \ln L_S^{(i,\epsilon)} &- \sum_{j=i+1}^N \int_S^T \frac{\tau_j \left( \mathbf{T}_\epsilon^1 \left( \hat{L}_t^{(i,\epsilon)} \right) \right)_+ (\underline{\sigma}_t^i)' \underline{\sigma}_t^j}{1 + \tau_j \left( \mathbf{T}_\epsilon^1 \left( \hat{L}_t^{(i,\epsilon)} \right) \right)_+} dt \\ &- \frac{1}{2} \int_S^T \|\underline{\sigma}_t^i\|^2 dt + \int_S^T (\underline{\sigma}_t^i)' d\mathbf{W}_t, \end{aligned} \quad (2.44)$$

where  $\mathbf{T}_\epsilon^1 \left( \hat{L}_t^{(i,\epsilon)} \right)$  is a function of  $t$  and  $\int_S^T (\underline{\sigma}_t^i)' d\mathbf{W}_t$ .

To apply the discretization we choose  $\epsilon = 1$  as  $L_t^{i,1}$  has the dynamics of the standard LIBOR market model. The problem with Equation (2.44) is that the drift integral is not capable of being evaluated analytically. One needs to resort to the trapezium method to evaluate this integral using discrete points on the sample path of  $L_t^{(i,\epsilon)}$ . This introduces an additional source of error which we refer to as *discretization error*. This can be reduced only when  $\Delta t \rightarrow 0$ . Together with the truncation error from the strong Taylor approximation, this formulation is subject to two source of errors.

In Section 2.5, we propose another formulation which only incurs one type of error, the truncation error.

## 2.5 An Improvement on Strong Taylor Expansion

Instead of applying a perturbation to  $\ln L_t^i$ , we apply it to the forward discounted payoff  $X_t^i$ , obtaining

$$\frac{d\hat{X}_t^{(i,\epsilon)}}{\hat{X}_t^{(i,\epsilon)}(1 - \hat{X}_t^{(i,\epsilon)})} = \epsilon \left( - \sum_{j=i}^N (\underline{\sigma}_t^i)' \underline{\sigma}_t^j \hat{X}_t^{(j,\epsilon)} dt + (\underline{\sigma}_t^i)' d\mathbf{W}_t \right), \quad (2.45)$$

where  $\epsilon \in \mathbb{R}$ . If  $\epsilon = 0$ ,  $\hat{X}_t^{(i,\epsilon)} \equiv X_0^i$ . If  $\epsilon = 1$  we obtain the unperturbed version of forward discounted payoff,  $\hat{X}_t^{(i,\epsilon)} = X_t^i$ . It is not difficult to prove that for  $\epsilon > 0$ ,  $\mathbb{E} \left[ \hat{X}_t^{(i,\epsilon)} \right]^2 < \infty$ , that is,  $\hat{X}_t^{(i,\epsilon)} \in L^2(\Omega, \mathfrak{R})$ , so that  $\mathbf{T}_\epsilon^n \left( \hat{X}_t^{(i,\epsilon)} \right) \rightarrow \hat{X}_t^{(i,\epsilon)}$  as  $\epsilon \downarrow 0$ .

**Taylor Expansion.** The first order strong Taylor approximation for  $\hat{X}_t^{(i,\epsilon)}$  is

$$\hat{X}_t^{(i,\epsilon)} \approx \mathbf{T}_\epsilon^1 \left( \hat{X}_t^{(i,\epsilon)} \right) = X_0^i + \epsilon \left. \frac{\partial \hat{X}_t^{(i,\epsilon)}}{\partial \epsilon} \right|_{\epsilon=0} := X_0^i + \epsilon Y_t^i. \quad (2.46)$$

where

$$Y_t^i = Y_S^i + X_0^i (1 - X_0^i) \left( - \sum_{j=i}^N X_0^j \int_S^t (\underline{\sigma}_s^i)' \underline{\sigma}_s^j ds + \int_S^t (\underline{\sigma}_s^i)' d\mathbf{W}_s \right). \quad (2.47)$$

with  $Y_0^i = 0$ . Now we formulate the LIBOR market model in terms of the parameter  $\epsilon$ .

$$\ln L_T^{(i,\epsilon)} = \ln L_S^{(i,\epsilon)} - \sum_{j=i+1}^N \int_S^T \hat{X}_t^{(j,\epsilon)} (\underline{\sigma}_t^i)' \underline{\sigma}_t^j dt - \frac{1}{2} \int_S^T \|\underline{\sigma}_t^i\|^2 dt + \int_S^T (\underline{\sigma}_t^i)' d\mathbf{W}_t. \quad (2.48)$$

If  $\epsilon = 0$  Equation (2.48) is the same as the frozen drift forward LIBOR SDE. If  $\epsilon = 1$  it is the standard LIBOR market model. Using the first order strong Taylor approximation for  $\hat{X}_t^{(i,\epsilon)}$ , we derive another drift approximation formula.

**Drift Approximation.** We replace  $\hat{X}_t^{(j,\epsilon)}$  in  $\int_0^{T_i} \hat{X}_t^{(j,\epsilon)} (\underline{\sigma}_t^i)' \underline{\sigma}_t^j dt$ ,  $j > i$ , with its first order strong Taylor approximation.

$$\begin{aligned} \int_S^T \hat{X}_t^{(j,\epsilon)} (\underline{\sigma}_t^i)' \underline{\sigma}_t^j dt &= \int_S^T \hat{X}_t^{(j,\epsilon)} dI_{S,t}^{i,j} \approx \left[ \int_S^T \mathbf{T}_\epsilon^1 \left( \hat{X}_t^{(j,\epsilon)} \right) dI_{S,t}^{i,j} \right]_+ \\ &= \left[ X_0^j I_{S,T}^{i,j} + \epsilon \int_S^T Y_t^j dI_{S,t}^{i,j} \right]_+, \end{aligned} \quad (2.49)$$

where  $I_{S,t}^{i,j} = \int_S^t (\underline{\sigma}_s^i)' \underline{\sigma}_s^j ds$ .

**Truncation Error.** We estimate the error term from using the strong Taylor approximation. Denote the approximated forward LIBOR SDE using Equation (2.49) by  $\hat{L}_t^{(i,\epsilon)}$ . The absolute error is given by

$$\begin{aligned} \left| \ln \hat{L}_{T_i}^{(i,\epsilon)} - \ln L_{T_i}^{(i,\epsilon)} \right| &= \left| \sum_{j=i+1}^N \left( \left[ \int_0^{T_i} \mathbf{T}_\epsilon^1 \left( \hat{X}_t^{(j,\epsilon)} \right) dI_t^{i,j} \right]_+ - \int_0^{T_i} \hat{X}_t^{(i,\epsilon)} dI_t^{i,j} \right) \right| \\ &\leq \sum_{j=i+1}^N \int_0^{T_i} \left| \mathbf{T}_\epsilon^1 \left( \hat{X}_t^{(j,\epsilon)} \right) - \hat{X}_t^{(i,\epsilon)} \right| dI_t^{i,j} \\ &= \sum_{j=i+1}^N \int_0^{T_i} \left| \mathbf{T}_\epsilon^1 \left( \hat{X}_t^{(j,\epsilon)} \right) - \hat{X}_t^{(i,\epsilon)} \right| (\underline{\sigma}_t^i)' \underline{\sigma}_t^j dt \\ &\leq M \sum_{j=i+1}^N \int_0^{T_i} \left| \mathbf{T}_\epsilon^1 \left( \hat{X}_t^{(j,\epsilon)} \right) - \hat{X}_t^{(i,\epsilon)} \right| dt, \end{aligned} \quad (2.50)$$

where  $M = \max_{0 \leq t \leq T_i, \forall i,j} \left( (\underline{\sigma}_t^i)' \underline{\sigma}_t^j \right) < 1$ .

**Expression for  $\int_S^T Y_t^j dI_{S,t}^{i,j}$ .** We define  $A_{S,t}^i$  and  $Z_{S,t}^i$  as

$$A_{S,t}^i := \sum_{j=i}^N X_0^j \int_S^t (\underline{\sigma}_s^i)' \underline{\sigma}_s^j ds = \sum_{j=i}^N X_0^j I_{S,t}^{i,j}, \quad Z_{S,t}^i := \int_S^t (\underline{\sigma}_s^i)' d\mathbf{W}_s, \quad (t \geq S) \quad (2.51)$$

$A_{S,t}^i$  is deterministic and  $Z_{S,t}^i \sim N \left( 0, \int_0^t \|\underline{\sigma}_s^i\|^2 ds \right)$ . We re-express  $Y_t^i$  in terms of  $A_{S,t}^i$  and  $Z_{S,t}^i$  as

$$Y_t^i = Y_S^i + X_0^i (1 - X_0^i) (-A_{S,t}^i + Z_{S,t}^i). \quad (2.52)$$

The integral term on the right hand side of Equation (2.49) becomes

$$\begin{aligned} \int_S^T Y_t^j dI_{S,t}^{i,j} &= Y_S^j I_{S,T}^{i,j} + X_0^j (1 - X_0^j) \left( - \int_S^T A_{S,t}^j dI_{S,t}^{i,j} + \int_S^T Z_{S,t}^j dI_{S,t}^{i,j} \right) \\ &= Y_S^j I_{S,T}^{i,j} + X_0^j (1 - X_0^j) \left( - \int_S^T A_{S,t}^j (\underline{\sigma}_t^i)' \underline{\sigma}_t^j dt + \int_S^T Z_{S,t}^j dI_{S,t}^{i,j} \right). \end{aligned} \quad (2.53)$$

The first integral on the right hand side of Equation (2.53) is deterministic and can be pre-computed before the simulation. Applying integration by parts to the second integral, we have

$$\int_S^T Z_{S,t}^j dI_{S,t}^{i,j} = I_{S,T}^{i,j} Z_{S,T}^j - \int_S^T I_{S,t}^{i,j} dZ_{S,t}^j. \quad (2.54)$$

To simulate  $L_T^{(i,\epsilon)} \mid L_S^{(i,\epsilon)}$  we need  $2(N-i)+1$  correlated normal random variables,

$$Z = \left( Z_{S,T}^i, Z_{S,T}^{i+1}, \dots, Z_{S,T}^N, \int_S^T I_{S,t}^{i,i+1} dZ_{S,t}^{i+1}, \dots, \int_S^T I_{S,t}^{i,N} dZ_{S,t}^N \right)'. \quad (2.55)$$

We have the following proposition for the variance covariance matrix of  $Z$ .

**Theorem 2.5.1** *The variance covariance matrix of  $\mathbf{Z}$  is given by*

$$\mathbf{C} := \text{Var } \mathbf{Z} = \begin{pmatrix} \Phi & \Sigma' \\ \Sigma & \Omega \end{pmatrix}, \quad (2.56)$$

where

$$\Phi_{jk} = \text{Cov}(Z_{S,T}^j, Z_{S,T}^k) = \int_S^T (\underline{\sigma}_t^j)' \underline{\sigma}_t^k dt, \quad (2.57)$$

$$\Sigma_{jk} = \text{Cov} \left( Z_{S,T}^j, \int_S^T I_{S,t}^{i,k} dZ_{S,t}^k \right) = \int_S^T I_{S,t}^{i,k} (\underline{\sigma}_t^j)' \underline{\sigma}_t^k dt, \quad (2.58)$$

$$\Omega_{jk} = \text{Cov} \left( \int_S^T I_{S,t}^{i,j} dZ_{S,t}^j, \int_S^T I_{S,t}^{i,k} dZ_{S,t}^k \right) = \int_S^T I_{S,t}^{i,j} I_{S,t}^{i,k} (\underline{\sigma}_t^j)' \underline{\sigma}_t^k dt. \quad (2.59)$$

**Proof** We first look at the matrix  $\Phi$ . By applying Itô's formula to  $Z_t^j Z_t^k$ , we have

$$d(Z_{S,t}^j Z_{S,t}^k) = Z_{S,t}^j dZ_{S,t}^k + Z_{S,t}^k dZ_{S,t}^j + dZ_{S,t}^j dZ_{S,t}^k. \quad (2.60)$$

Taking the expectation of both sides, we get

$$\mathbb{E} \left[ d(Z_{S,t}^j Z_{S,t}^k) \right] = \mathbb{E} \left( dZ_{S,t}^j dZ_{S,t}^k \right) = (\underline{\sigma}_t^j)' \underline{\sigma}_t^k dt. \quad (2.61)$$

Dynkin's formula (see Klebaner [54]) gives us

$$\Phi_{jk} = \text{Cov} \left( Z_{S,T}^j, Z_{S,T}^k \right) = \mathbb{E} \left( Z_{S,T}^j Z_{S,T}^k \right) = \int_S^T (\underline{\sigma}_t^j)' \underline{\sigma}_t^k dt. \quad (2.62)$$

Using the same techniques, we derive for other two matrices

$$\Sigma_{jk} = \text{Cov} \left( Z_{S,T}^j, \int_S^T I_{S,t}^{i,k} dZ_{S,t}^k \right) = \mathbb{E} \left( Z_{S,T}^j \int_S^T I_{S,t}^{i,k} dZ_{S,t}^k \right) = \int_S^T I_{S,t}^{i,k} (\underline{\sigma}_t^j)' \underline{\sigma}_t^k dt, \quad (2.63)$$

$$\begin{aligned} \Omega_{jk} &= \text{Cov} \left( \int_S^T I_{S,t}^{i,j} dZ_{S,t}^j, \int_S^T I_{S,t}^{i,k} dZ_{S,t}^k \right) = \mathbb{E} \left( \int_S^T I_{S,t}^{i,j} dZ_{S,t}^j \int_S^T I_{S,t}^{i,k} dZ_{S,t}^k \right) \\ &= \int_S^T I_{S,t}^{i,j} I_{S,t}^{i,k} (\underline{\sigma}_t^j)' \underline{\sigma}_t^k dt. \end{aligned} \quad (2.64)$$

□

There is no discretization error in equation (2.48) since the drift integral approximated by Equation (2.49) can be evaluated explicitly. The only source of error is the truncation error which can be estimated by Equation (2.50) and reduced by increasing the terms in the Taylor expansion (2.37).

Although the new formulation removes the discretization error, it still involves truncation error. This cannot be reduced by increasing the number of time steps unless one adds more terms to the strong Taylor approximation. But this requires knowledge of higher order derivatives which are not as straightforward as the first order one. This is an undesirable feature. We can also see that Equation (2.49) for the strong Taylor approximation is a special case of Equation (2.29) when one uses only one single step and the volatilities are constant.

In Table 2.2 we give a brief summary of the orders of error for each scheme we have so far investigated.

Scheme	Error Source	Reference
Euler (Milstein)	Weak $\mathcal{O}(\Delta t)$	Maruyama [58]
Order 1 PC	Weak $\mathcal{O}(\Delta t)$	Hunter <i>et al.</i> [46]
IPC	Weak $\mathcal{O}(\Delta t)$	Joshi and Stacey [51]
Order 1.5 Strong	Strong $\mathcal{O}\left(\Delta t^{\frac{3}{2}}\right)$	Kloeden and Platen [55]
Order 2 Weak	Weak $\mathcal{O}(\Delta t^2)$	Glasserman and Merener [38]
Order 2 PC	Weak $\mathcal{O}(\Delta t^2)$	Kloeden and Platen [55]
ST1	Weak $\mathcal{O}(\Delta t)$ , Strong $\mathcal{O}(\epsilon)$	Siopacha and Teichmann [77]
ST2	Strong $\mathcal{O}(\epsilon)$	Proposed

Table 2.2: Schemes Summary

## 2.6 Numerical Results

We use two vanilla instruments, caplet and swaption, in our example. A caplet is a European call option on a single LIBOR rate. A swaption is an option to enter into

a swap contract and its payoff depends on more than one forward LIBOR rate.

$$i^{th} \text{ Caplet Payoff: } V_{T_i} = \frac{\tau_i(L_{T_i}^i - K)_+}{1 + \tau_i L_{T_i}^i} \quad (2.65)$$

$$i \times l \text{ Swaption Payoff: } V_{T_i} = Y_{T_i}^{i,i+l}(S_{T_i}^{i,i+l} - K)_+ \quad (2.66)$$

where  $K$  is the strike,  $Y_t^{a,b} = \sum_{i=a+1}^b \tau_i P(t, T_i)$  and  $S_t^{a,b} = \frac{P(t, T_a) - P(t, T_b)}{\sum_{i=a+1}^b \tau_i P(t, T_i)}$ . Notice that we use the caplet payoff discounted to  $T_i$ . The actual payoff  $\tau_i(L_{T_i}^i - K)_+$  takes place at  $T_{i+1}$ . Using the formula for any contingent claim under the no-arbitrage assumption, we have

$$V_t = P(t, T_{N+1}) \mathbb{E} \left[ \frac{V_{T_i}}{P(T_i, T_{N+1})} \right] \quad (2.67)$$

where  $V_t$  is the value of caplet or swaption.

**Parameters Specifications.** We use a flat initial term structure of 20 forward LIBOR rates  $L_0^i = 0.06$  with tenor  $\tau_i \equiv \tau = 1$ , for  $i = 0, 1, \dots, 19$ , and  $T_0 = 1$ . Each forward LIBOR rate has the same constant volatility  $\sigma_t^i \equiv \sigma = 0.3$ ,  $\forall i$ .  $L_t^i$  and  $L_t^j$  have the instantaneous correlation  $\rho_{i,j}$ ,  $\forall i, j$ . We use the correlation form introduced by Rebonato [69].

$$\rho_{i,j} = \rho_\infty + (1 - \rho_\infty)e^{-\beta|i-j|}, \quad (2.68)$$

where  $\rho_\infty = 0.2$  and  $\beta = 0.5$ . Such a choice makes  $\rho_{i,j}$  decrease rapidly as they move away from the diagonal, but stay relatively flat at  $\rho_\infty$  when  $|i - j| > 10$ . As a result,  $(\underline{\sigma}_t^i)' \underline{\sigma}_t^i = \sigma_t^i \sigma_t^j \rho_{i,j} = \sigma^2 \rho_{i,j} \approx \sigma^2 \rho_\infty$  when  $|i - j| > 10$ . The drift term  $\mu_t^i$  can be viewed as a weighted sum of  $X_t^j = \frac{\tau_j L_t^j}{1 + \tau_j L_t^j}$  with weights equal to  $(\underline{\sigma}_t^i)' \underline{\sigma}_t^i \approx \sigma^2 \rho_\infty$ . Therefore, each term  $X_t^j$  has an equal contribution towards to  $\mu_t^i$ . The drift term reflects the biases in all realised  $X_t^j$ ,  $j > i$ , instead of only a few. This gives a clearer picture of approximation errors in all the discretization methods.

We price two vanilla instruments, caplets and swaptions. The caplets pay off against  $L_{T_i}^i$  with maturities  $T_i$ ,  $i = 0, \dots, 19$ , and each with strike  $K = 0.06$  (ATM),  $K = 0.04$  (ITM) and  $K = 0.08$  (OTM). We considered swaptions with maturity dates  $T_0, T_1, \dots, T_9$ , all terminating at  $T_{20}$ , and each with strike  $K = 0.06$ . These swaptions are also known as co-terminal swaptions.

For reasons of clarity, we first define some abbreviations of methods in Table 2.3.

Abbreviation	Method
PC	Predictor-corrector
IPC	Iterative predictor-corrector
ST1	Strong Taylor expansion in equation (2.44)
ST2	Strong Taylor expansion in equation (2.48)

Table 2.3: Abbreviations of Methods

### 2.6.1 Caplets

We use the caplet as a benchmark case since its value under the LIBOR market model can be obtained in closed-form (Black’s formula [13]). The pricing error is defined as

$$\text{Err} = |C_{MC} - C_B|, \quad (2.69)$$

where  $C_{MC}$  and  $C_B$  denote the Monte Carlo and Black’s caplet prices. We simulate each  $L_t^i$  from 0 up to  $T_i$  using a single long step. We compare the pricing errors and computation time for ATM caplets using various methods. Since the volatilities are constant, in a single step discretization ST1 is the same as Euler method. We do not list both schemes as the pricing errors are too large compared with other methods.

ST2 is the same as order 1.5 strong Itô-Taylor approximation so that we only give the results for the latter.

The number of sample paths is  $10^7$  so that the standard errors are below 1 basis point. For the ATM case, we fill up the Wiener sample path using normal random numbers in order to obtain estimates of standard errors. For the ITM and OTM cases, we plot only the pricing errors against different caplet maturities. We do not report the standard errors since they are much smaller than the pricing errors for most caplets. We use inverse transform of Sobol numbers to generate normal variables for both the ITM and OTM cases.

**Single Step Pricing Errors.** The single step pricing errors for various methods are given in Tabel 2.4 (page 24). The caplets have maturities from 1 year to 20 years. Explicit caplet values are given by the Black’s formula. The computation time (CPU) is the simulation time only. The platform was a Dell D620 latitude.

The pricing errors from every method are first increasing and then decreasing with maturity. When the maturity is short (first few caplets) the time step is not long enough for the drift term to cause large pricing errors. When the maturity is long



		Order 1 PC			IPC		
Maturity	Explicit	Err	SE	CPU	Err	SE	CPU
1	63.671	0.002	(0.041)	154.5	0.002	(0.041)	103.5
2	84.632	0.048	(0.063)	147.1	0.018	(0.063)	99.6
3	97.422	0.201	(0.084)	134.2	0.104	(0.084)	90.6
4	105.732	0.424	(0.103)	127.6	0.206	(0.103)	87.3
5	111.109	0.620	(0.122)	116.1	0.215	(0.123)	80.2
6	114.402	1.011	(0.141)	112.5	0.361	(0.144)	78.5
7	116.146	1.768	(0.156)	98.0	0.832	(0.162)	70.2
8	116.708	2.215	(0.167)	92.5	0.956	(0.175)	66.3
9	116.355	2.732	(0.180)	81.7	1.131	(0.192)	58.6
10	115.285	3.116	(0.192)	77.3	1.193	(0.209)	56.4
11	113.655	3.742	(0.211)	67.3	1.531	(0.235)	49.4
12	111.584	4.206	(0.209)	62.4	1.795	(0.234)	46.6
13	109.171	4.465	(0.225)	53.7	1.931	(0.265)	40.7
14	106.496	4.648	(0.233)	49.2	2.127	(0.274)	38.2
15	103.621	4.407	(0.211)	40.5	2.048	(0.236)	32.0
16	100.601	4.009	(0.235)	37.0	1.976	(0.256)	30.2
17	97.479	3.205	(0.251)	29.1	1.729	(0.266)	24.2
18	94.292	2.106	(0.218)	25.7	1.375	(0.223)	22.7
19	91.069	0.870	(0.163)	18.9	0.870	(0.163)	16.7
20	87.835	0.115	(0.118)	15.6	0.115	(0.118)	14.7

Order 1.5 Strong			Order 2 Weak			Order 2 PC		
Err	SE	CPU	Err	SE	CPU	Err	SE	CPU
0.091	(0.041)	180.3	0.013	(0.041)	123.3	0.002	(0.041)	215.5
0.012	(0.064)	171.4	0.051	(0.064)	114.5	0.019	(0.063)	201.5
0.137	(0.085)	160.5	0.100	(0.085)	103.4	0.112	(0.084)	185.0
0.376	(0.107)	143.6	0.233	(0.106)	97.7	0.230	(0.103)	171.4
0.545	(0.128)	133.4	0.591	(0.131)	87.1	0.270	(0.123)	151.9
0.945	(0.155)	121.4	0.924	(0.160)	82.6	0.463	(0.143)	140.4
1.482	(0.213)	111.1	0.993	(0.194)	74.4	0.994	(0.160)	126.2
1.662	(0.199)	100.4	1.550	(0.222)	69.1	1.193	(0.172)	114.6
2.032	(0.263)	90.6	2.132	(0.256)	61.0	1.456	(0.187)	100.4
2.202	(0.309)	81.1	2.874	(0.305)	57.1	1.611	(0.201)	91.8
3.037	(0.439)	72.2	3.413	(0.421)	49.3	2.044	(0.223)	78.8
4.142	(0.537)	63.6	3.853	(0.414)	45.9	2.380	(0.221)	70.9
4.427	(0.396)	55.8	4.850	(0.703)	38.3	2.584	(0.242)	59.3
4.365	(0.366)	48.2	5.286	(0.780)	35.8	2.810	(0.250)	53.3
4.857	(0.468)	41.2	5.228	(0.465)	28.9	2.717	(0.222)	43.0
5.292	(0.505)	34.3	5.506	(0.505)	27.1	2.590	(0.244)	37.7
5.137	(0.438)	28.0	5.312	(0.592)	21.0	2.203	(0.257)	28.7
4.302	(0.406)	22.2	4.203	(0.367)	18.9	1.625	(0.220)	24.8
2.636	(0.198)	16.3	2.397	(0.196)	13.1	0.870	(0.163)	17.2
0.157	(0.118)	10.8	0.115	(0.118)	11.3	0.115	(0.118)	13.8

Table 2.4: Absolute Pricing Errors (in Basis Points), CPU (in Seconds), for ATM Caplets,  $10^7$  Paths

(the last few caplets) there are very few stochastic terms in the drift summation ( $L_t^N$  is driftless) so that the error caused by the drift term is not significant.

The simple iterative predictor-corrector method (IPC) outperforms all the other methods in terms of both caplet pricing errors and computation time. The only method which is close to IPC in terms of the pricing error is order 2 predictor-corrector (Order 2 PC). Their standard errors (SE) are more or less the same. However, the computation time required by Order 2 PC is staggering for the first several caplets (almost twice as much as that for IPC) because of the path re-evolution and updating of state variables required at each time step. This discrepancy in computation time narrows as the caplet maturity increases.

The pricing errors for order 2 weak and order 1.5 strong Itô-Taylor approximations are close to each other. This is reasonable because the only difference between them is whether or not the double Wiener integrals,  $I_{(0,j)}$  and  $I_{(j,0)}$ , are simulated. The number of normal random variables needed for order 1.5 strong is twice as many as needed by the IPC method but unfortunately the former method performs much worse than IPC in this single step case in terms of pricing error, standard error and computation time.

The order 1 predictor-corrector method (Order 1 PC) produces higher pricing errors for the first several caplets, but lower pricing errors later on, compared to both Order 2 Weak and Order 1.5 Strong. The computation time needed for Order 1 PC is about 1.5 times that needed for IPC but it seems the standard errors of the former method are slightly lower.

The computation time and standard errors in the ITM and OTM cases are similar to those in the ATM case. We plot the pricing errors only for ITM and OTM caplets in Figure 2.1 and 2.2. They are obtained using the same set of Sobol numbers. The results are consistent with the ATM case except that in Order 1.5 Strong pricing errors for caplets on  $L_t^{14}$  and  $L_t^{15}$  are much higher than those in Order 2 Weak.

### 2.6.2 Co-Terminal Swaptions

Co-terminal swaptions are options on swap contracts with different starting dates but all terminating on the same final date. That means in Equation (2.66)  $l = N + 1 - i$  for all  $i$ . Co-terminal swaps are fundamental rates in the swap market

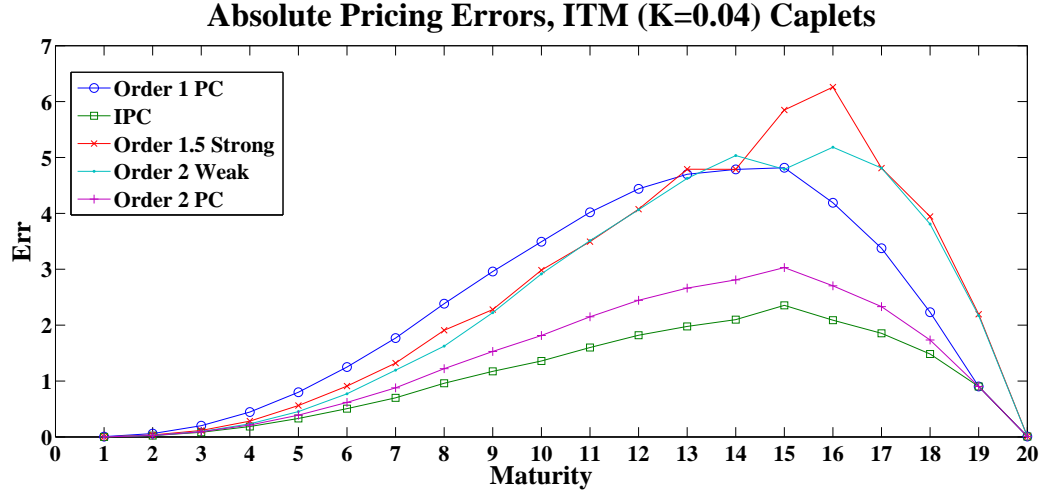


Figure 2.1: Pricing Errors (in Basis Points), ITM ( $K = 0.04$ ) Caplets,  $10^7$  Paths with Inverse Transform of Sobol Sequence

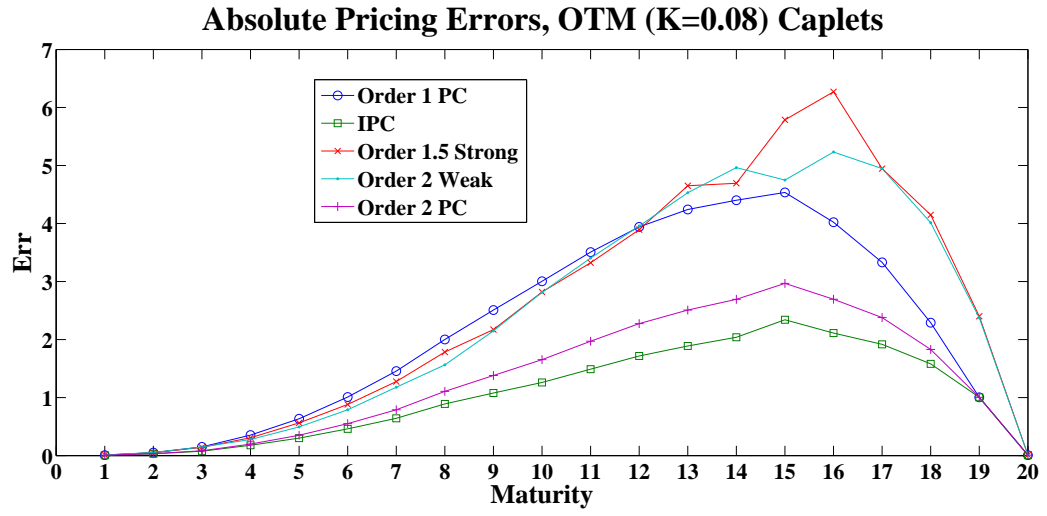


Figure 2.2: Pricing Errors (in Basis Points), OTM ( $K = 0.08$ ) Caplets,  $10^7$  Paths with Inverse Transform of Sobol Sequence

model. The concept was introduced by Jamshidian [48]. Co-terminal swaptions are important components of a Bermudan swaption whose holder has the right to enter into swap contracts on a set of pre-determined reset dates. On each of these dates, the holder either receives a co-terminal swaption payoff or keeps the continuation value of the Bermudan swaption.

Since the swaption value in the LIBOR market model is not known in the closed-form, we obtain benchmark results using iterative predictor-corrector method with two steps each year (as we will see in Section 2.6.3 that this prices caplets with bias less than 0.1 basis points). Co-terminal swaption values are given in Table 2.5. Forward LIBOR rates (and therefore the swap rates) are simulated in a single long

$T_0$	$T_1$	$T_2$	$T_3$	$T_4$	$T_5$	$T_6$	$T_7$	$T_8$	$T_9$
907.84	998.78	1016.35	1000.31	966.26	922.10	872.19	818.81	764.55	709.61

Table 2.5: Benchmark Results for Co-Terminal Swaptions (in Basis Points)

step from time 0 to the starting date of the underlying swap using every method with  $10^7$  paths and Sobol numbers.

For the same reason in the caplet case, neither ST1 nor ST2 is included in the result. Pricing error is defined as

$$\text{Err} = |C_{MC} - C_0|, \quad (2.70)$$

where  $C_0$  is given in Table 2.5. The pricing errors for swaptions are displayed in Figure 2.3. As in the caplet case, IPC outperforms all other methods. It is followed by Order 2 PC. Order 2 Weak is better than Order 1.5 Strong. Both are much better than Order 1 PC for all the first ten co-terminal swaptions.

So far, we have investigated the single step discretization errors. In practice, people are also interested in how various method perform with short step discretization. We therefore look at how they converge with an increasing number of time steps.

### 2.6.3 Weak Order of Convergence

We compare the weak order of convergence for each method. Weak convergence is defined with respect to the expectation of some function. It is important for a method to converge to the theoretical expectation as quickly as possible. In finance, the function is normally chosen to be the option payoff. However, option payoff

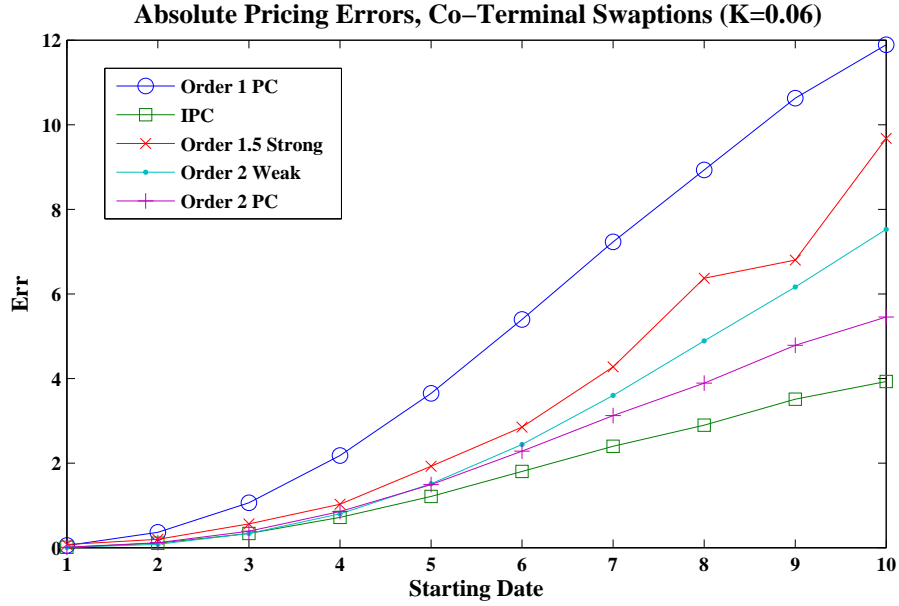


Figure 2.3: Pricing Errors (in Basis Points), Co-Terminal Swaptions ( $K = 0.06$ ),  $10^7$  Paths with Inverse Transform of Sobol Sequence

functions often fail the conditions under which the theoretical order of weak convergence occurs. In practice, Webber [83] finds that convergence sometimes occurs despite technical disqualifications, with rates lower than (at least not higher than) the theoretical rates of convergence.

If we plot  $\log_2(\text{Err})$  against  $\log_2(\Delta t)$ , the theoretical slope is -1 for Euler and PC, -2 for Order 1.5 Strong, Order 2 Weak and Order 2 PC. IPC is not Itô-Taylor in the strict sense and thus its theoretical order of weak convergence is unclear. Pricing errors for ST1 come from both  $\Delta t$  and  $\epsilon$ . ST2 has error only from  $\epsilon$  so its pricing error does not decrease with  $\Delta t$ .

We choose the function in the weak criteria as the numeraire re-based payoff function of the ATM caplet on  $L_t^{13}$  so simulation is up to  $T_{13} = 14$ . Its theoretical expectation is known. The methods considered are ST1, ST2 and all the Itô-Taylor approximations considered in previous sections including Euler. We fill up the whole Wiener sample path using a Brownian bridge. Each draw in the Brownian bridge construction is an inverse transform of one dimension in the Sobol sequence. We then back out the Wiener increments from the Wiener sample path. Although this Wiener path construction makes convergence faster, Webber [83] finds that there is

bias in this method. This bias is small compared to the pricing errors and therefore does not have significant effect in our analysis. We do not discuss this effect further.

We vary the number of time steps from 1 to 32 from time 0 up to  $T_{13}$  so that  $\Delta t$  decreases each time by half from 14 to 0.4375. The error curves for all methods are displayed in Figure 2.4. The Euler method has slope equal to -1. ST1 is equiv-

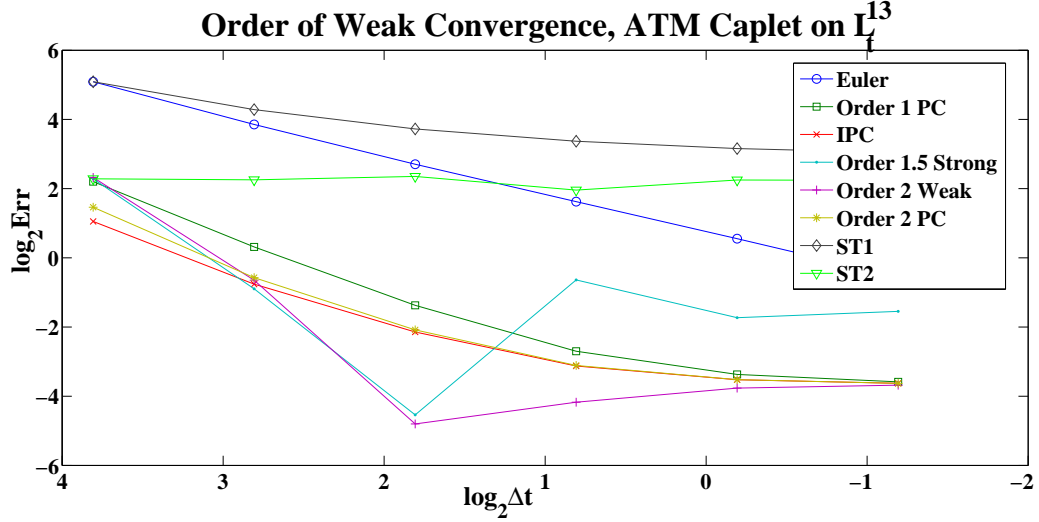


Figure 2.4: Convergence with Steps,  $10^7$  Paths with Inverse Transform of Sobol Sequence

alent to Euler if one uses only one time step (in fact this is drift freezing method). However, as  $\Delta t$  decreases only the discretization error of ST1 will be corrected but not the truncation error. It is converging to the same level of the ST2 whose dynamics can be discretized analytically in a single long step. Because there is no error coming from  $\Delta t$  in ST2, increasing the number of time steps does not reduce the pricing error and therefore the error curve stays flat. Both ST1 and ST2 produce pricing errors much larger than other method in the multiple steps case and therefore neither should be used in practice.

The curves of three predictor-corrector methods, Order 1 PC, IPC and Order 2 PC, are relatively close to each other. This is reasonable because when the volatilities are constant the only difference among those three is how the “predictors” are obtained. The IPC method has the best performance in the multiple steps discretization among those three. The Order 2 Weak has the best performance in this case. Its pricing errors are lower than those in all the other methods when the

number of time steps is more than one. It nevertheless has a “kink” in the error curve and does not achieve the expected order 2 weak convergence.

The error curve for Order 1.5 Strong is rather erratic. It has similar performance to Order 2 Weak up to 4 time steps but the errors “bounce” back above predictor-corrector methods afterwards.

## 2.7 Conclusion

We investigated various numerical discretization schemes for the LIBOR market models. The methods we compared are Itô-Taylor approximations and strong Taylor approximations to the perturbed SDE. We conclude that the latter methods, even the improved formulation, should not be used for drift approximation in the LIBOR market model. Iterative predictor-corrector method outperforms all the other methods in terms of single step pricing error and computation time. It is simple and recommended for long step discretizations. The order 2 weak Itô-Taylor scheme has an edge over IPC in the multiple steps case with only trivial additional cost. The order 2 predictor-corrector has performance closest to IPC in terms of pricing errors but is costly. These findings pose an interesting question of which the best discretization scheme is in other probability measure where IPC cannot be applied.

Apart from discretization schemes, another important technique in Monte Carlo is the variance reduction method. Having chosen a certain discretization scheme for an SDE, we want to reduce standard errors with as little additional effort as possible. In the next two chapters, we discuss variance reduction schemes, in particular, the control variate method.

## Chapter 3

# A Review of Control Variate Methods

### 3.1 Introduction

A bank's option book contains thousands of options which, to reduce computation time, practitioners want to price all at once. The payoffs of some options depends on the whole discrete sample path. Examples are average rate options, barrier options, lookback options, *et cetera*. In many cases, however, these instruments do not have explicit solutions, and the natural choice of valuation method is Monte Carlo simulation. The problem is how to make the result converges as fast as possible without loss of accuracy.

By convergence, we mean the level of standard error.<sup>1</sup> One can reduce the standard error simply by increasing the number of sample paths, but this comes at the cost of a significant increase in computation time. Suppose we want to reduce the standard error by a factor of a tenth. In a plain Monte Carlo method we would need to use 100 times the number of sample paths. If we assume each sample path takes roughly the same amount of time to simulate, the computation time is then 100 times greater than before. In a time crucial application in a bank, this is certainly not acceptable. A good Monte Carlo method gets the same standard error as plain method with as little computation time as possible. This is where variance reduction methods come in.

There are many different variance reduction methods. Three standard methods

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<sup>1</sup>Bias is also important but we are here concerned chiefly with variance reduction.



are antithetic sampling, stratified sampling and low discrepancy sampling. They improve the construction of a set of Wiener sample paths. A fourth method is importance sampling. This method values derivatives by simulating a modified process (by virtual of the Girsanov theorem (see for example Shreve [76])). By doing so appropriately, one can obtain a standard error lower than if valuing derivatives in the original model. The value then needs to be adjusted back to the result in the original process. The crucial thing is the Randon-Nikodým derivative must be able to be computed explicitly.

The final method is to use a control variate. When valuing a derivative which has no explicit value in a particular model, it is possible that there is some other quantity, similar to the value of the derivative instrument we are interested in, that can be evaluated explicitly. The control variate method computes simultaneously the value of the instrument we are interested in and that of the quantity whose value is known explicitly. Then the errors in the known quantity are used to correct the errors in the unknown instrument. In practice, the control variate method is a very efficient technique of variance reduction. A good introduction on this topic is in the book by Glasserman [36] (see also Webber [83]). We will review standard control variate approaches in this chapter.

The rest of this chapter is organised as follows. In section 3.2, we discuss the amount of variance reduction that can be achieved by control variates. In section 3.3, we introduce the concept of an efficiency measure. In section 3.4, we review three different ways of constructing control variates. In section 3.5, we discuss three standard control variates. In section 3.6, we focus on tailored control variates for average rate and barrier options.

These are the preliminaries for our contributions in chapter 4. We start by discussing the amount of variance reduction given by control variates.

## 3.2 The Achievable Degree of Variance Reduction

The common feature of all control variates is to have a theoretical expectation equal to zero. Suppose we want to compute a derivative value  $C$  and we find a quantity  $CV$ . For each sample path  $i$ ,  $i = 1, \dots, M$ , we compute the term  $Z^{(i)} = C^{(i)} - \beta \cdot CV^{(i)}$  where the superscript  $(i)$  denotes a value computed on the  $i^{th}$  sample path, and  $\beta$  is a constant coefficient chosen to achieve the highest amount

of variance reduction. Since by choice  $\mathbb{E}[CV^{(i)}] = 0$ , we have

$$\mathbb{E}[Z^{(i)}] = \mathbb{E}[C^{(i)}] - \beta \mathbb{E}[CV^{(i)}] = \mathbb{E}[C^{(i)}] = C. \quad (3.1)$$

The control variate does not introduce bias into the theoretical mean. However, the variance of  $Z^{(i)}$  is

$$\text{Var}[Z^{(i)}] = \text{Var}[C^{(i)}] + \beta^2 \text{Var}[CV^{(i)}] - 2\beta \text{Cov}[C^{(i)}, CV^{(i)}]. \quad (3.2)$$

$\text{Var}[Z^{(i)}]$  is minimized by setting  $\beta = \frac{\text{Cov}[C^{(i)}, CV^{(i)}]}{\text{Var}[CV^{(i)}]}$ . With the value of  $\beta$ , the minimum variance achieved is

$$\text{Var}[Z^{(i)}] = \text{Var}[C^{(i)}] (1 - \rho_{C,CV}^2) \leq \text{Var}[C^{(i)}], \quad (3.3)$$

where  $\rho_{C,CV}$  is the correlation between  $C$  and  $CV$ . The higher the correlation is, the greater the variance reduction achieved.

In practice, the optimal value of  $\beta$  is usually not available in explicit form. When implementing a control variate method, one may estimate  $\beta$  by regression. Given the pairs  $\{(C^{(i)}, CV^{(i)})\}_{i=1, \dots, M}$ , we regress  $C^{(i)}$  against  $CV^{(i)}$  and obtain

$$C^{(i)} = a + b \cdot CV^{(i)} + \epsilon_i, \quad (3.4)$$

where the coefficient

$$b = \frac{\sum_{i=1}^M C^{(i)} CV^{(i)} - \frac{1}{M} \sum_{i=1}^M C^{(i)} \sum_{j=1}^M CV^{(j)}}{\sum_{i=1}^M (CV^{(i)})^2 - \frac{1}{M} \left( \sum_{i=1}^M CV^{(i)} \right)^2} \quad (3.5)$$

is an estimator of  $\beta$  and  $\mathbb{E}(\epsilon_i) = 0$ . From equation (3.1), the intercept  $a$  is the estimator of the derivative value.

A problem arises when we use  $b$  to replace  $\beta$  in equation (3.4) as it depends on  $C^{(i)}$  and  $CV^{(i)}$ . This dependence can be ignored when  $M$  is large. But care must be taken for small samples. The small-sample issues are discussed at great length in Glasserman [36]. We do not elaborate on this any further.

If one wishes to use multiple control variates, they can just be added in to the regression. Suppose we have  $l$  control variates  $CV_1, \dots, CV_l$ . Then the regression

reads

$$C^{(i)} = a + \sum_{j=1}^l b_j \cdot CV_j^{(i)} + \epsilon_i, \quad (3.6)$$

and  $a$  is the estimation of the derivative value.

The amount of variance reduction must justify the additional time taken to use the control variate. We measure the benefits obtained from the control variate by computing the efficiency gain.

### 3.3 The Efficiency Measure

We adopt the definitions from Webber [83]. Suppose we use two Monte Carlo methods to value the same derivative. The first one gives a standard error estimate  $se_1$  in time  $\tau_1$  and the second a standard error  $se_2$  in time  $\tau_2$ . The relative efficiency gain (or loss)  $E = E_{1,2}$  of the second method over the first one is

$$E = \frac{se_2^2 \tau_2}{se_1^2 \tau_1}. \quad (3.7)$$

$E$  is the multiple of the time taken by the first method over the second one needed to achieve the same standard error. In this sense, speed-up is a better term than variance reduction to describe the benefit of a method.

The efficiency gain is related to the correlation between the CV and option. By equation (3.3), we see the theoretical variance of the CV estimator is a fraction  $\frac{1}{1-\rho_{C,CV}^2}$  of the plain estimator. Taking into account the computation time,

$$E = \frac{\tau_2}{\tau_1} \frac{1}{1 - \rho_{C,CV}^2}. \quad (3.8)$$

If  $\frac{\tau_1}{\tau_2} \approx 5$ ,  $\rho_{C,CV} \approx 0.99$  gives  $E \approx 10$  and  $\rho_{C,CV} \approx 0.999$  gives  $E \approx 100$ .

There are many ways that we can construct a CV such that it has mean zero.

### 3.4 Different Control Variate Approaches

The control variate can be chosen in different ways. A control variate (CV) can be: 1) a different instrument (auxiliary instrument) priced in the same model; 2) the same instrument priced in a different model (auxiliary model); or 3) a different

instrument priced in a different model (auxiliary instrument and auxiliary model). In each way, one needs to be able to compute the value of the auxiliary instrument explicitly, or at least by a quick and accurate numerical procedure, in order to use it as a control variate.

We discuss these three approaches in turn.

**Auxiliary instrument CV.** Suppose we have an instrument such that: 1) its value  $p$  is known explicitly in the pricing model: and 2) along each sample path  $i$ , its payoff  $p^{(i)}$  is highly correlated with that of the option  $C^{(i)}$  we want to value. Then

$$CV^{(i)} = p^{(i)} - p \quad (3.9)$$

is an auxiliary instrument CV.

**Auxiliary model CV.** We write  $\mathcal{M}$  for the pricing model. Suppose we have an auxiliary model  $\widetilde{\mathcal{M}}$  such that: 1)  $\widetilde{\mathcal{M}}$  is similar to  $\mathcal{M}$ ; 2) the same set of Wiener sample paths can be used for both  $\mathcal{M}$  and  $\widetilde{\mathcal{M}}$ ; and 3) the option  $C$  has an explicit value  $\widetilde{C}$  in  $\widetilde{\mathcal{M}}$ . Write  $\widetilde{C}^{(i)}$  for the discounted payoff of the option on sample path  $i$  in the auxiliary model  $\widetilde{\mathcal{M}}$ . Then

$$CV^{(i)} = \widetilde{C}^{(i)} - \widetilde{C} \quad (3.10)$$

is an auxiliary model CV.

**Auxiliary instrument and auxiliary model CV.** This type of CV is a combination of the previous two. Suppose we have an auxiliary model  $\widetilde{\mathcal{M}}$  and an auxiliary instrument whose value is known explicitly as  $\widetilde{p}$  in  $\widetilde{\mathcal{M}}$ . Write  $\widetilde{p}^{(i)}$  for the payoff of the auxiliary instrument along sample path  $i$  in  $\widetilde{\mathcal{M}}$ . Then

$$CV^{(i)} = \widetilde{p}^{(i)} - \widetilde{p} \quad (3.11)$$

is an auxiliary instrument and auxiliary model CV.

The underlying asset, a European call or put option and a discretely rebalanced delta hedging portfolio are standard auxiliary instruments. They have universal application to most options we want to value. Other control variates are tailored to specific instruments. A classical example is the geometric average rate option control variate for the arithmetic one in the Black-Scholes model (Kemman and Vorst

[53]). We first look at those standard auxiliary instruments.

## 3.5 Standard Auxiliary Instruments

Define a set of equally spaced discretization dates  $0 = t_0 < t_1 < \dots < t_N = T$  with  $t_i - t_{i-1} = \Delta t$ ,  $i = 1, \dots, N$ , where  $T$  is the option maturity. These are the dates where the state variables in a model are sampled with a chosen discretization scheme. We borrow the following sections from Webber [83].

### 3.5.1 The Underlying Asset Control Variate

In any arbitrage-free model, the discounted expectation of the asset value at any future point  $S_{t_j}$  is equal to the current asset price  $S_0$ . This gives us a number of control variates

$$CV_j^{(i)} = e^{-rT} S_{t_j}^{(i)} - S_0, \quad j = 1, \dots, N. \quad (3.12)$$

In addition, given a set of weights  $w = \{w_j\}_{j=1, \dots, N}$ ,

$$CV_w^{(i)} = \sum_{j=1}^N w_j CV_j^{(i)} \quad (3.13)$$

is also a CV candidate. A proper choice of  $w$  can yield a higher correlation with the option payoff.

### 3.5.2 The European Call Control Variate

The derivative instrument we are interested in is often more complicated than the European call option. But it may happen that there is an explicit solution for the European call option in the pricing model, for example, the Heston model. When the instrument also has a call-type payoff, the European call option can be a CV candidate. If we denote  $S_t$  as the stock price,  $r$  the risk free interest rate and  $K$  the strike, then the European call CV along sample path  $i$  reads,

$$CV^{(i)} = e^{-rT} \left( S_T^{(i)} - K \right)^+ - \mathbb{E} \left[ e^{-rT} \left( S_T^{(i)} - K \right)^+ \right], \quad (3.14)$$

where we assume the expectation is explicitly available.

### 3.5.3 The Delta Control Variate

The delta CV is a discretely rebalanced delta hedging portfolio for the instrument. We write  $\Delta_t = \frac{\partial C_t}{\partial S_t}$  for the delta of the instrument at time  $t$ . Suppose at  $t_0$  we hedge by buying  $\Delta_{t_0}$  units of stock and rebalance the portfolio at  $t_1, \dots, t_{N-1}$ . At  $t_N$  we close the position by selling  $\Delta_{t_N}$  units of stock. The present value of the sum of the discounted cash flows at  $t_1, \dots, t_{N-1}$  is

$$\begin{aligned}
CV^{(i)} &= \Delta_{t_0}^{(i)} S_{t_0}^{(i)} + \sum_{j=1}^{N-1} e^{-rt_j} \left( \Delta_{t_{j-1}}^{(i)} - \Delta_{t_j}^{(i)} \right) S_{t_j}^{(i)} - e^{-rt_N} \Delta_{t_N}^{(i)} S_{t_N}^{(i)} \\
&= \sum_{j=0}^{N-1} e^{-rt_j} \Delta_{t_j}^{(i)} \left( S_{t_j}^{(i)} - e^{-r\Delta t} S_{t_{j+1}}^{(i)} \right) \\
&= \sum_{j=0}^{N-1} e^{-rt_{j+1}} \Delta_{t_j}^{(i)} \left( e^{r\Delta t} S_{t_j}^{(i)} - S_{t_{j+1}}^{(i)} \right). \tag{3.15}
\end{aligned}$$

Since  $\mathbb{E} \left( e^{r\Delta t} S_{t_j}^{(i)} - S_{t_{j+1}}^{(i)} \right) = e^{rt_{j+1}} S_0^{(i)} - e^{rt_{j+1}} S_0^{(i)} = 0$ ,  $\mathbb{E} [CV^{(i)}] = 0$  so it is a CV candidate. We can re-arrange equation (3.15) as

$$\begin{aligned}
CV^{(i)} &= \sum_{j=0}^{N-1} e^{-rt_{j+1}} \Delta_{t_j}^{(i)} \left( e^{r\Delta t} S_{t_j}^{(i)} - S_{t_{j+1}}^{(i)} \right) \\
&= \sum_{j=0}^{N-1} e^{-rt_{j+1}} \Delta_{t_j}^{(i)} \left( e^{rt_{j+1}} S_0^{(i)} - S_{t_{j+1}}^{(i)} \right) + \sum_{j=0}^{N-1} e^{-rt_{j+1}} \Delta_{t_j}^{(i)} \left( e^{r\Delta t} S_{t_j}^{(i)} - e^{rt_{j+1}} S_0^{(i)} \right) \\
&= \sum_{j=0}^{N-1} \Delta_{t_j}^{(i)} \left( CV_j^{(i)} - CV_{j+1}^{(i)} \right) \\
&= \sum_{j=1}^{N-1} \left( \Delta_{t_j}^{(i)} - \Delta_{t_{j-1}}^{(i)} \right) CV_j^{(i)} - \Delta_{t_{N-1}}^{(i)} CV_N^{(i)}, \tag{3.16}
\end{aligned}$$

where  $CV_j^{(i)}$  is given in equation (3.12). Equation (3.16) is a special case of equation (3.13) if we set  $w_j = \Delta_{t_j}^{(i)} - \Delta_{t_{j-1}}^{(i)}$ ,  $j = 1, \dots, N-1$ , and  $w_N = -\Delta_{t_{N-1}}^{(i)}$ .

If the portfolio is rebalanced frequently, we expect the delta CV to be highly correlated with the option. In practice, however, the cost of computing the delta at each discrete date along each sample path is non-trivial and one must take this into account.

The underlying CV, European call/put CV and the delta CV are applicable to

most options. For specific types of instruments, such as arithmetic average rate options and barrier options, we can use tailored CVs which may be much more correlated to the option than the standard CVs. In the following sections, we discuss these CVs for arithmetic average rate and barrier options.

## 3.6 Instrument Specific Control Variates

We discuss control variates for two instruments, the arithmetic average rate and barrier option. The standard control variates can also apply but with detailed implementations tailored to instrument being priced. There are also CVs limited to specific instruments that outperform the standard CVs. We introduce both types for each instrument.

### 3.6.1 Average Rate Options

We start with some useful definitions for arithmetic average rate options. Define an index vector  $\iota = (\iota_1, \dots, \iota_n)$  representing the indexes of reset dates, where  $n$  is the total number of reset dates, and  $\delta = \iota_i - \iota_{i-1}$ ,  $i = 1, \dots, n$  with  $\iota_0 = 0$  and  $\iota_n = N$ . The asset values along a sample path are then averaged at dates  $t_{\iota_i}$ ,  $i = 1, \dots, n$ . We can apply standard CVs which we introduced in section 3.5.

#### Tailored Standard Control Variates for Average Rate Options

We discuss the underlying asset CV, call CV and delta CV for arithmetic average rate options in turn.

**Underlying asset CV.** Webber [83] applies the underlying asset CV to average rate option. He sets

$$w_i = \begin{cases} \frac{1}{n} e^{rt_{\iota_i}}, & i = \iota_1, \dots, \iota_n, \\ 0, & \text{otherwise.} \end{cases} \quad (3.17)$$

in equation (3.13), we have

$$CV_w^{(i)} = \sum_{j=1}^n \frac{1}{n} e^{rt_{\iota_j}} CV_{\iota_j}^{(i)} = \frac{1}{n} \sum_{j=1}^n \left( S_{t_{\iota_j}}^{(i)} - e^{rt_{\iota_j}} S_0 \right) = \frac{1}{n} \sum_{j=1}^n S_{t_{\iota_j}}^{(i)} - \frac{S_0}{n} \sum_{j=1}^n e^{rt_{\iota_j}}. \quad (3.18)$$

$CV_w^{(i)}$  should have decent correlation with the arithmetic average rate option because the first term is the path statistics that the option pays off against.

**Call CV.** The payoff of the arithmetic average rate option

$$\left( \frac{1}{n} \sum_{i=1}^n S_{t_{\iota_i}} - K \right)^+ \rightarrow (S_T - K)^+, \quad (3.19)$$

as  $n \rightarrow 1$ . We therefore expect the call CV to work better for options with few reset dates.

**Delta CV.** When we apply the delta CV, a question is how do we know the delta of an instrument if its value is unknown? We address this problem by using the delta in an auxiliary model. Since the delta of the geometric average rate option is known in the GBM process (see Webber [83]), we use it as an approximation.

The classical choice of tailored control variate for the arithmetic average rate option is its geometric counterpart whose explicit value is available in the Black-Scholes model (Kemma and Vorst [53]).

### The Geometric Average Rate Option Control Variate

The CV has the form

$$CV^{(i)} = e^{-rT} \left[ \left( \prod_{j=1}^n S_{t_{\iota_j}}^{(i)} \right)^{\frac{1}{n}} - K \right]^+ - \mathbb{E} \left\{ e^{-rT} \left[ \left( \prod_{j=1}^n S_{t_{\iota_j}}^{(i)} \right)^{\frac{1}{n}} - K \right]^+ \right\}, \quad (3.20)$$

where the expectation can be computed explicitly if the asset price follows a GBM process (Kemma and Vorst [53]). Denote by  $C_{BS}(K, T, S_0, y, r, \sigma)$  the Black-Scholes call formula with strike  $K$ , time to maturity  $T$ , initial asset price  $S_0$ , dividend yield  $y$ , riskless rate  $r$  and volatility  $\sigma$ . The value of the geometric average rate option has the following form

$$\mathbb{E} \left\{ e^{-rT} \left[ \left( \prod_{j=1}^n S_{t_{\iota_j}}^{(i)} \right)^{\frac{1}{n}} - K \right]^+ \right\} = e^{r(\bar{T}-T)} C_{BS}(K, T, S_0, y, r, \sigma), \quad (3.21)$$

where

$$\bar{T} = \frac{1}{n} \sum_{i=1}^n t_{\iota_i}, \quad \bar{\sigma} = \frac{\sigma^2}{n^2 \bar{T}} \sum_{i=1}^n (n-i) t_{\iota_{n-i}}, \quad y = \frac{1}{2} (\sigma^2 - \bar{\sigma}^2). \quad (3.22)$$

We refer to this control variate in our numerical examples in chapter 5 as the GBM auxiliary CV.



In the next section, we first discuss some valuation problems in barrier options. A GBM auxiliary model and instrument CV is then introduced.

### 3.6.2 Barrier Options

Barrier options are derivative securities whose payoffs depend on the whole sample path. Two types of barrier options are knock-in options and knock-out options. The knock-in option gets a vanilla option payoff if the asset price ever reaches a certain region before maturity, usually below or above a barrier, or a zero rebate otherwise. The knock-out option, on the other hand, has a vanilla option payoff if the asset price never goes in to the region and zero payoff otherwise. The value of a knock-in or knock-out option is lower than the vanilla option since they have the same payoff only when a certain condition is met. In the GBM model, there is an explicit solution for knock-in or knock-out options with a single and continuous barrier. In other cases, for example, stochastic volatility models, double barriers or if the barrier is discrete, one often has to resort to Monte Carlo simulation.

There are two main issues in valuing barrier options with Monte Carlo: 1) How to deal with the problem of knocking out between two discretization dates at which both sampled asset prices are below the barrier (for the up and out barrier option); and 2) How to improve the efficiency of the Monte Carlo method?

The first issue arises only when the barrier is continuous. If one checks the barrier condition only at discretization dates, one ends up with a high-bias estimate of the barrier option price because there is a possibility of knocking out between reset dates. This is called barrier discretization bias. Fortunately, Beaglehole *et al.* [11] and El Babsiri and Noel [29] designed a method to eliminate the bias if the asset price follows a GBM process. In fact, their method applies to any case where each one-step evolution follows a GBM process, even if each step has a different constant volatility. But in the stochastic volatility model barrier discretization bias remains an issue. One needs to use a large number of time step to approximate the payoff of a continuous barrier option. In this section, we only look at discrete barrier options.

The second issue is not for barrier options *per se*. But certain variance reduction methods, for example, control variates, are very much tailored to instruments. The barrier option differs from the vanilla option in that it has an additional uncertainty as the hitting time distribution, that is, the time when the option knocks out.

Glasserman and Staum [39] propose an importance sampling method to remove this uncertainty. They force each sample path to survive until maturity. We focus on control variate methods for barrier options. Since the continuous barrier options in the GBM model has an explicit value, it is a CV candidate. We illustrate this specific CV for up and out barrier options. The arguments are analogous for other types of barrier options with an explicit solution in the GBM model.

### The Up and Out Barrier Call Option

Define  $T$  as the maturity of the option. The barrier  $U$  is monitored discretely at  $\mathcal{T} = \{t_0, t_1, \dots, t_N\}$  where  $t_0 = 0$ ,  $t_N = T$  and  $\Delta t = t_{i+1} - t_i$ ,  $i = 0, 1, \dots, N-1$ . The value of the discrete barrier option is

$$v = e^{-rT} \mathbb{E} [h(S_t, t \in \mathcal{T})], \quad (3.23)$$

where

$$h(S_t, t \in \mathcal{T}) = \begin{cases} (S_T - K)^+ & \max_{t \in \mathcal{T}} \{S_t\} \leq U, \\ 0 & \text{otherwise.} \end{cases} \quad (3.24)$$

Its continuous counterpart is

$$H(S_t, t \leq T) = \begin{cases} (S_T - K)^+ & \max_{t \leq T} \{S_t\} \leq U, \\ 0 & \text{otherwise.} \end{cases} \quad (3.25)$$

If the number of reset dates  $N$  is large, the continuous barrier option price should be close to the discrete price. In addition, the value of the former option is analytically available in the Black-Scholes model. This gives us a candidate for a control variate.

### The Continuous Barrier Option Control Variate

We use the continuous barrier option in the GBM model as a CV. One must be very careful when evaluating the payoff  $H$ . Because we have only a discrete path  $\{\tilde{S}_{t_0}, \dots, \tilde{S}_{t_N}\}$ , it is possible that  $\tilde{S}_t > U$  for  $t \in (t_i, t_{i+1})$  even if  $\tilde{S}_{t_i} \leq U$  and  $\tilde{S}_{t_{i+1}} \leq U$ . Therefore the result has a high bias because the probability of knocking out in continuous time is higher. Beaglehole *et al.* [11] and El Babsiri and Noel [29] introduce a method to eliminate the bias if  $S_t$  follows a GBM. If we know the survival probabilities  $p_i = \mathbb{P}(S_t \leq U, \forall t \in [t_i, t_{i+1}] | S_{t_i} \leq U, S_{t_{i+1}} \leq U)$ ,  $i = 0, \dots, N$ , the

unbiased evaluation of  $H$  is

$$H'(S_t, t \in \mathcal{T}) = \begin{cases} (S_T - K)^+ \prod_{i=0}^{N-1} p_i, & \max_{t \in \mathcal{T}} \{S_t\} \leq U, \\ 0, & \text{Otherwise.} \end{cases} \quad (3.26)$$

The CV then has the form

$$CV^{(i)} = H'(S_t^{(i)}, t \in \mathcal{T}) - \mathbb{E} \left[ H(S_t^{(i)}, t \leq T) \right], \quad (3.27)$$

where the expectation is explicit in the GBM model.

The GBM auxiliary CV has a constant volatility. In many pricing models, however, the volatility follows a separate stochastic process. Webber [83] finds that stochastic volatility makes the GBM auxiliary CV less efficient when valuing arithmetic average rate options. This leads us to think whether we can do better than the GBM auxiliary CV under stochastic volatility. In the next chapter, we propose CVs for a fairly general class of stochastic volatility models.

## Chapter 4

# Control Variates for Stochastic Volatility Models

### 4.1 Introduction

Control variates, as we discussed in chapter 3, make use of the analytical tractability of auxiliary instruments or auxiliary models to reduce the standard error. In sophisticated modelling frameworks, such as stochastic volatility models, we may need to use both in combination. How to find good auxiliary models and instruments under stochastic volatility is the main question we want to answer. In this chapter, we make the following contributions:

1. We formulate a very general class of stochastic volatility models which nest many existing models in the literature.
2. We construct new control variates for this class of models.
3. We apply the model as well as our new control variates to price average rate and barrier options.

There are papers on variance reduction methods in the stochastic volatility environment. Fouque and Tullie [33] introduce a method based on importance sampling in a two-factor stochastic volatility model. Fouque and Han [30] extend the method to apply importance sampling in two-factor stochastic volatility models. They also explore variance reduction methods for continuous average rate options. Fouque and Han [31] propose a control variate method for European option pricing in a multi-factor stochastic volatility models and find that it is more stable and efficient than the importance sampling method of their earlier paper. Fouque and Han [32]

introduce a generic control variate, which they refer to as a martingale control variate, under stochastic volatility. They consider the pricing problem of European, barrier and American options. Han and Lai [42] investigate further the martingale control variate method for the average rate options by means of an option price approximation.

In this chapter, we propose new control variates, which we refer to as correlation control variates, in a general class of stochastic volatility models. The correlation control variates are constructed for two instruments, the arithmetic average rate option and barrier option.

The rest of this chapter is organized as follows. In section 4.2, we formulate a general class of stochastic volatility models which our methods apply to. In section 4.3, we provide an overview of our correlation CVs. In section 4.4, we construct auxiliary models for instruments with linear path statistics. In section 4.5, we propose auxiliary models for instruments with particular non-linear path statistics, the maximum or minimum functions. Auxiliary instruments for two examples of linear and non-linear path statistics, the average rate and barrier option, are discussed in section 4.6 and 4.7. Section 4.8 concludes. Numerical results and analysis are given in chapter 5 for average rate option and in chapter 6 for barrier options.

We start by formulating a family of stochastic volatility model in order to nest as many existing stochastic volatility models as possible. Our correlation control variates are introduced in this context.

## 4.2 A Class of Stochastic Volatility Models

We consider a general class of two dimensional stochastic volatility model. Let  $(S)_{t \geq 0}$  be an asset process and  $(V)_{t \geq 0}$  be a volatility process. Their joint SDE is

$$dS_t = rS_t dt + \sigma f(V_t) S_t^\beta dW_t^S, \quad (4.1)$$

$$dV_t = \alpha(\mu - V_t)dt + \eta V_t^\gamma dW_t^V, \quad (4.2)$$

$$dW_t^X dW_t^V = \rho dt, \quad (4.3)$$

where  $f(v) = v^\xi$  and

$\xi, \beta \geq 0$  determine the structure of volatility for  $S_t$ ,

$\gamma \geq 0$  determines the process for  $V_t$ ,

$\gamma > 0$  when  $\xi \neq \mathbb{Z}$ ,

$\sigma > 0$  is included for generality,

$\alpha, \mu \in \mathbb{R}$  are not required to be positive.

When  $\xi \notin \mathbb{Z}$ ,  $V_t$  is required to remain positive. When  $\xi \in \mathbb{Z}$ ,  $V_t$  is permitted to be negative. We discuss a few characteristics of the processes.

#### 4.2.1 Characteristics of the Processes

The parameter  $\beta$  determines the type of the process for  $S_t$ :

$\beta = 0$ :  $S_t$  is an absolute diffusion process with stochastic volatility.

$\beta \in (0, 1)$ :  $S_t$  is a CEV process with stochastic volatility.

$\beta = 1$ :  $S_t$  is a GBM process with stochastic volatility.

The parameter  $\xi$  controls the process for  $V_t$ :

$\xi = 0$ :  $S_t$  is a GBM process with deterministic volatility.

$\xi = \frac{1}{2}$ :  $V_t$  is a variance process.

$\xi = 1$ :  $V_t$  is a volatility process.

When  $\xi = 1$ , we do not require  $V_t$  to remain positive. When  $\beta < \frac{1}{2}$ , one needs to impose an absorbing boundary at 0 so that SDE (4.1) has a unique solution (Andersen and Andreasen [3]). Our model nests many existing stochastic volatility models in the literature.

#### 4.2.2 Nested Stochastic Volatility Models

Two nested and widely used stochastic volatility models are Heston [44] and SABR (Hagan *et al.* [41]). The Heston model ( $\beta = 1$ ,  $\xi = \gamma = \frac{1}{2}$ ) has been widely used because it allows the option price to have an implied volatility smile. In addition, there is an explicit formula for European option value. The SABR model is another widely used stochastic volatility model. Its popularity is due both to the existence of a closed-form approximation of the European option implied volatility, and its consistency with the dynamics of market implied volatility smile. The latter characteristic renders an advantage over local volatility models in the context of trading

and hedging. To nest the SABR model, we require some manipulations of the original process.

Let  $F_t$  be the forward price on asset  $S_t$  and let  $\sigma_t$  be the volatility. The SABR model has the dynamics

$$dF_t = \sigma_t F_t^\beta dW_t^S, \quad (4.4)$$

$$d\sigma_t = \eta \sigma_t dW_t^V, \quad (4.5)$$

$$\mathbb{E}(dW_t^S dW_t^V) = \rho dt. \quad (4.6)$$

Since  $F_t = e^{r(T-t)} S_t$ , we can express the SABR dynamics in terms of  $S_t$  and  $\sigma_t$ .

$$dS_t = r S_t dt + \sigma_t e^{(\beta-1)r(T-t)} S_t^\beta dW_t^S. \quad (4.7)$$

Fixing  $T$ , we set  $V_t = \sigma_t e^{(\beta-1)r(T-t)}$  so that, for  $t \leq T$

$$dV_t = r(1 - \beta)V_t dt + \eta V_t dW_t^V. \quad (4.8)$$

We arrive at alternative dynamics for the SABR model in terms of  $S_t$  and  $V_t$

$$dS_t = r S_t dt + V_t S_t^\beta dW_t^S, \quad (4.9)$$

$$dV_t = r(1 - \beta)V_t dt + \eta V_t dW_t^V. \quad (4.10)$$

This has the form of our stochastic volatility model which  $\xi = 1$ ,  $\gamma = 1$ ,  $\mu = 0$ ,  $\alpha = r(\beta - 1)$  and  $V_0 = \sigma_0 e^{(\beta-1)rT}$ .

Another useful model is the GARCH diffusion model (Nelson [63]). It has the parameters  $\beta = 1$ ,  $\xi = \frac{1}{2}$  and  $\gamma = 1$ . We list other nested stochastic volatility models in the literature in Table 4.1.

$\beta = 0$			$\beta \in (0, 1)$			$\beta = 1$		
Model	$\xi$	$\gamma$	Model	$\xi$	$\gamma$	Model	$\xi$	$\gamma$
Absolute diffusion models	1	1	SABR	1	1	H&W	1	1
	1	$\frac{1}{2}$	SABR- like	1	$\frac{1}{2}$	J&S	1	$\frac{1}{2}$
	1	0		1	0	S&Z, S&S	1	0
	$\frac{1}{2}$	1	LKD	$\frac{1}{2}$	1	GARCH	$\frac{1}{2}$	1
	$\frac{1}{2}$	$\frac{1}{2}$		$\frac{1}{2}$	$\frac{1}{2}$	Heston	$\frac{1}{2}$	$\frac{1}{2}$
n/a	$\frac{1}{2}$	0	n/a	$\frac{1}{2}$	0	n/a	$\frac{1}{2}$	0

Table 4.1: Nested Stochastic Volatility Models

where  $(\beta, \xi, \gamma) =$

$(\beta, 1, \gamma)$ : Johnson and Shanno [50] (J&S).

$(\beta, 1, 1)$ : Hagan *et al.* [41] (SABR).

$(\beta, \frac{1}{2}, \gamma)$ : Lord, Koekkoek and Dijk [57] (LKD).

$(1, 1, 1)$ : Wiggins [84]; Hull and White [45] (H&W) ( $\mu = 0, \rho = 0$ ).

$(1, 1, 0)$ : Stein and Stein [78] ( $\rho = 0$ ) (S&S); Schöbel and Zhu [71] ( $\rho \neq 0$ ) (S&Z).

$(1, \frac{1}{2}, 1)$ : Nelson [63] (GARCH); Barone-Adesi *et al.* [9]; (and H&W, as  $\mu = 0$ ).

$(1, \frac{1}{2}, \frac{1}{2})$ : Heston [44]; Ball and Roma [7].

$(1, \frac{1}{2}, \gamma)$ : Andersen and Piterbarg [6]; Lewis [56]; Ait-Sahalia and Kimmel [1].

$(1, 2, 0)$ : Sbai and Jourdain [70] (special case).

There are also models, however, not nested in our class. There is an exponential volatility model where  $(\beta, \xi, \gamma) = (1, 1, 0)$  with  $f(v) = \exp(v^\xi)$  by Scott [72], Chesney and Scott [22] and Melino and Turnbull [59], jump diffusion models such as Bates [10], general Lévy models (VG, NIG, CGMY, *et cetera*) and stochastic interest rate and higher factor models (Haastrecht *et al.* [82], *et cetera*). There are also papers which propose more general specifications, for example, Sbai and Jourdain [70] with  $(\beta, f(v), g(v))$  and Bourgade and Croissant [14] with  $(h(S), \xi, \gamma)$ .

In general, there is no explicit solution to option values in the model (4.1)-(4.3). To value derivatives, we always resort to numerical methods. PDE and lattice methods are difficult to get prices out quickly when there are more than two factors. In Monte Carlo, there are also issues with bias and convergence. The square-root process, in particular, is likely to be problematical. The CIR process, for example, admits no strong solution. If it is used to model the asset variance as in the Heston model, zero is accessible when the model is calibrated to equity options. Although exact simulation is possible (Scott [73]), it is expensive to use.

Simulation of the Heston model has been studied extensively both from the theoretical and practical point of view (Lord *et al.* [57], *et cetera*). Despite that, effective valuation of derivatives whose payoffs depend on the whole sample path remains a challenging research question. Exact simulation methods (Broadie and Kaya [17], Glasserman and Kim [37], *et cetera*) are expensive. These methods are



plausible for long-step Monte Carlo but infeasible to do short stepping. Approximate solutions are poor particularly when zero is accessible for  $V_t$  (Webber [83]). Therefore, fast simulation methods are necessary for the Heston model.

We propose new control variates, which we refer to as correlation CVs, in our model. The idea is to circumvent the explicit solutions for the CV candidates as they are usually unavailable other than under the simple geometric Brownian motion. To fix the idea, we first give an overview of our correlation CV method.

### 4.3 Overview of Correlation Control Variates

We introduced the concept of auxiliary instrument and auxiliary model CVs in section 3.4. In this section, we propose a variation of it for the class of stochastic volatility models that we have formulated. Not only do we allow the payoff of the auxiliary instrument to vary for each sample path, but also allow the auxiliary model to change so that it reflects the variation in the stochastic volatility.

#### 4.3.1 Path-dependent Auxiliary Model Control Variate

We write  $\widetilde{\mathcal{M}}^{(i)}$  for the auxiliary model along sample path  $i$ . The auxiliary instrument has payoff  $\widetilde{p}^{(i)}$  and explicit value  $\widetilde{p}_e^{(i)}$  in  $\widetilde{\mathcal{M}}^{(i)}$ . Then

$$CV^{(i)} = \widetilde{p}^{(i)} - \widetilde{p}_e^{(i)} \quad (4.11)$$

is a path-dependent auxiliary model CV. For this CV to work, we require that  $\mathbb{E}[\widetilde{p}^{(i)}] = \mathbb{E}[\widetilde{p}_e^{(i)}]$  so that  $\mathbb{E}[CV^{(i)}] = 0$ . This is true if we choose  $\widetilde{\mathcal{M}}^{(i)}$  to be conditioned on a realisation of the  $i^{th}$  sample path of the volatility  $\widetilde{V}_t^{(i)}$ . The explicit value of the auxiliary instrument in  $\widetilde{\mathcal{M}}^{(i)}$  is

$$\widetilde{p}_e^{(i)} = \mathbb{E}[\widetilde{p}^{(i)} \mid \widetilde{V}_t^{(i)}]. \quad (4.12)$$

By iterated expectation,

$$\mathbb{E}[\widetilde{p}_e^{(i)}] = \mathbb{E}[\mathbb{E}[\widetilde{p}^{(i)} \mid \widetilde{V}_t^{(i)}]] = \mathbb{E}[\widetilde{p}^{(i)}]. \quad (4.13)$$

Good control variates are often tailored to specific instruments. Auxiliary processes should allow CVs to have explicit solutions. As a result, it is practically impossible that we can use the same auxiliary model for all instruments. However, many instruments have common features. One important part of instruments' payoffs is

the path statistics, or in other words, which quantities that the instruments pays off against. We put them into two groups, linear path statistics and non-linear path statistics.

**Linear path statistics:** the path statistics is a linear function of observations made along the sample path. Examples are European options (final value), arithmetic average rate options (average), caps, swaps, *et cetera*.

**Non-linear path statistics:** the path statistics is a non-linear function of observations made along the sample path. We are particularly interested in maximum and minimum functions. Many instruments' payoffs depend jointly on them and the final value along the sample path. Examples are barrier options, lookback options, *et cetera*.

We propose auxiliary process for each category so that it applies to any instrument within this category. Relating to the path-dependent auxiliary model CV, the question is how to find  $\widetilde{\mathcal{M}}^{(i)}$ . To answer it, it is crucial to first re-write our model with two dimensional Brownian motion.

#### 4.3.2 Preliminary Re-write

Set  $\tilde{\rho} = \sqrt{1 - \rho^2}$ . We write processes (4.1) - (4.3) in the following form.

$$dS_t = rS_t + V_t^\xi S_t^\beta (\rho dW_t^1 + \tilde{\rho} dW_t^2), \quad (4.14)$$

$$dV_t = \alpha(\mu - V_t)dt + \eta V_t^\gamma dW_t^1, \quad (4.15)$$

$$dW_t^1 dW_t^2 = 0, \quad (4.16)$$

where  $W_t^S = \rho W_t^1 + \tilde{\rho} W_t^2$  and  $\sigma$  has been absorbed into  $V_t$ . We develop our auxiliary models for both categories with this formulation.

### 4.4 An Auxiliary Model for Linear Path Statistics

We first transform the SDE of  $S_t$  to make the volatility independent of  $S_t$ . The transformations are different for  $\beta = 1$  and  $\beta \in (0, 1)$ .

**The  $\beta = 1$  case.** Set  $Y_t = \ln S_t$ , then

$$dY_t = \left( r - \frac{1}{2} V_t^{2\xi} \right) dt + V_t^\xi (\rho dW_t^1 + \tilde{\rho} dW_t^2). \quad (4.17)$$

**The  $\beta \in (0, 1)$  case.** Set  $Y_t = \frac{1}{1-\beta} S_t^{1-\beta}$ , then

$$dY_t = \left[ r(1-\beta)Y_t - \frac{\beta}{2(1-\beta)Y_t} V_t^{2\xi} \right] dt + V_t^\xi (\rho dW_t^1 + \tilde{\rho} dW_t^2). \quad (4.18)$$

Our auxiliary processes essentially depend on the discretization of the transformed processes,  $Y_t$ .

#### 4.4.1 Discretization

The typical way of discretizing the asset process in the stochastic volatility models, for example the Heston model, is to discretize  $Y_t$  as if  $V_t$  were piecewise constant over the intervals  $[t_{i-1}, t_i)$ ,  $i = 1, \dots, N$  (Gatheral [35]). Therefore, it is convenient to define the piecewise-constant approximation  $\tilde{V}_t$  to  $V_t$  as

$$\tilde{V}_t \equiv \tilde{V}_{t_i}, \quad t \in [t_i, t_{i+1}), \quad i = 0, \dots, N-1, \quad (4.19)$$

where  $\tilde{V}_{t_i}$  is determined by some discretization scheme. One should take care that discretization of  $V_t$  must not allow  $\tilde{V}_{t_i}$  to be negative because otherwise  $\tilde{V}_{t_i}^\xi$  is not in general well-defined. Conditional on  $\tilde{V}_t$ , we then discretize  $Y_t$  with some approximation  $\tilde{Y}_t$ . For reasons of clarity, we often write  $\tilde{Y}_i = \tilde{Y}_{t_i}$ ,  $\tilde{V}_i = \tilde{V}_{t_i}$  and  $\tilde{V} = (\tilde{V}_0, \dots, \tilde{V}_N)$ . In the auxiliary model, discretization bias is not so important as in the pricing model. However, we want to choose a approximation such that auxiliary instruments are more likely to have explicit solutions. To our best knowledge, the normal approximation is a tractable option.

The one-step discretization for  $Y_t$  is

$$\tilde{Y}_{i+1} = \tilde{Y}_i + \mu^Y(\tilde{Y}_i, \tilde{V}_i) + \sigma^Y(\tilde{Y}_i, \tilde{V}_i) \epsilon_i^Y, \quad \epsilon_i^Y \sim N(0, 1), \quad (4.20)$$

where  $\mu^Y$  and  $\sigma^Y$  are formed below for specific cases. This discretization determines our auxiliary model  $\tilde{\mathcal{M}}^{(j)}$  along each sample path  $j$ . Conditional on  $\tilde{V}_t$ ,  $\tilde{Y}_i$  has normal increments and therefore is more likely that the auxiliary instruments have an explicit solution in  $\tilde{\mathcal{M}}^{(j)}$ .

#### 4.4.2 The Auxiliary Process

It is convenient to start with some definitions. Define

$$I_i = \int_{t_i}^{t_{i+1}} V_s^{2\xi} ds, \quad (4.21)$$

$$J_i = \int_{t_i}^{t_{i+1}} V_s^{\xi-\gamma} \left[ \alpha(\mu - V_s) + \frac{1}{2}(\xi - \gamma)\eta^2 V_s^{2\gamma-1} \right] ds. \quad (4.22)$$

$I_i$  and  $J_i$  depend only on  $V_t$ . They are determined by the pricing models. Table 4.2 lists their forms, obtained by plugging in values of  $\xi$ ,  $\gamma$ ,  $\alpha$  and  $\mu$ , in four specific models. These forms are needed for applying our CVs to these four models in

Model	$I_i$	$J_i$
Heston	$\int_{t_i}^{t_{i+1}} V_s ds$	$\alpha(\mu\Delta t - I_i)$
GARCH	$\int_{t_i}^{t_{i+1}} V_s ds$	$\int_{t_i}^{t_{i+1}} V_s^{-\frac{1}{2}} \left[ \alpha(\mu - V_s) - \frac{1}{4}\eta^2 V_s \right] ds$
SABR	$\int_{t_i}^{t_{i+1}} V_s^2 ds$	$\alpha\mu\Delta t - \alpha \int_{t_i}^{t_{i+1}} V_s ds$
J&S $(\beta, 1, \frac{1}{2})$	$\int_{t_i}^{t_{i+1}} V_s^2 ds$	$\frac{1}{4}\eta^2\Delta t + \int_{t_i}^{t_{i+1}} V_s^{\frac{1}{2}} \alpha(\mu - V_s) ds$

Table 4.2: Forms of  $I_i$  and  $J_i$

chapter 5. We denote the realisations of them as  $\tilde{I}_i$  and  $\tilde{J}_i$ . In this chapter, we approximate them by a single-step trapezium rule, for example,

$$\int_{t_i}^{t_{i+1}} V_s^2 ds \approx \frac{1}{2} (\tilde{V}_i^2 + \tilde{V}_{i+1}^2) \Delta t. \quad (4.23)$$

We discuss the auxiliary model separately for  $\beta = 1$  and  $\beta \in (0, 1)$  cases.

##### The $\beta = 1$ Case

We integrate equation (4.17) to obtain

$$\begin{aligned} Y_{t_{i+1}} = & Y_{t_i} + r\Delta t - \frac{1}{2} \int_{t_i}^{t_{i+1}} V_s^{2\xi} ds + \tilde{\rho} \int_{t_i}^{t_{i+1}} V_s^\xi dW_s^2 \\ & + \frac{\rho}{\eta} \left\{ \frac{V_{t_{i+1}}^{\xi-\gamma+1} - V_{t_i}^{\xi-\gamma+1}}{\xi - \gamma + 1} - \int_{t_i}^{t_{i+1}} V_s^{\xi-\gamma} \left[ \alpha(\mu - V_s) + \frac{1}{2}(\xi - \gamma)\eta^2 V_s^{2\gamma-1} \right] ds \right\}. \end{aligned} \quad (4.24)$$

Using the definitions of  $I_i$  and  $J_i$ , we have

$$Y_{t_{i+1}} = Y_{t_i} + r\Delta t + \frac{\rho}{\eta} \left\{ \frac{V_{t_{i+1}}^{\xi-\gamma+1} - V_{t_i}^{\xi-\gamma+1}}{\xi - \gamma + 1} - J_i \right\} - \frac{1}{2} I_i + \tilde{\rho} \sqrt{I_i} \epsilon_i^Y. \quad (4.25)$$

Hence we have

$$\mu^Y(\tilde{Y}_i, \tilde{V}_i) = r\Delta t + \frac{\rho}{\eta} \left\{ \frac{\tilde{V}_{i+1}^{\xi-\gamma+1} - \tilde{V}_i^{\xi-\gamma+1}}{\xi - \gamma + 1} - \tilde{J}_i \right\} - \frac{1}{2}\tilde{I}_i, \quad (4.26)$$

$$\sigma^Y(\tilde{Y}_i, \tilde{V}_i) = \tilde{\rho}\sqrt{\tilde{I}_i} \quad (4.27)$$

in equation (4.20).

### The $\beta \in (0, 1)$ Case

We integrate equation (4.18) to get

$$\begin{aligned} Y_{t_{i+1}} = Y_{t_i} &+ \theta(1 - \beta) \int_{t_i}^{t_{i+1}} Y_s ds - \frac{\beta}{2(1 - \beta)} \int_{t_i}^{t_{i+1}} \frac{V_s^{2\xi}}{Y_s} ds + \tilde{\rho} \int_{t_i}^{t_{i+1}} V_s^\xi dW_s^2 \\ &+ \frac{\rho}{\eta} \left\{ \frac{V_{t_{i+1}}^{\xi-\gamma+1} - V_{t_i}^{\xi-\gamma+1}}{\xi - \gamma + 1} - \int_{t_i}^{t_{i+1}} V_s^{\xi-\gamma} \left[ \alpha(\mu - V_s) + \frac{1}{2}(\xi - \gamma)\eta^2 V_s^{2\gamma-1} \right] ds \right\}. \end{aligned} \quad (4.28)$$

Discretizing  $Y_t$  requires more approximations than in the  $\beta = 1$  case. Equation (4.28) is untractable because of the appearance of  $Y_s$  in the integrals. One may want to fix  $Y_s$  to its value at time  $t_i$  and arrive at a normal approximation of the form (4.20). If we do so, we will have both  $\tilde{Y}_i$  and  $\frac{1}{\tilde{Y}_i}$  in  $\mu^Y(\tilde{Y}_i, \tilde{V}_i)$ . This is an undesirable feature because we will have problems in determining the mean and variance of the distribution of any linear function in  $\tilde{Y}_i$ , for example, the arithmetic average. The knowledge of the path statistics' distribution is crucial in constructing our CVs.

We may approximation  $Y_t$  in such a way that  $\mu^Y(\tilde{Y}_i, \tilde{V}_i)$  and  $\sigma^Y(\tilde{Y}_i, \tilde{V}_i)$  do not depend on  $\tilde{Y}_i$  and this makes our life easier when we derive the distribution of a linear path statistics. There are two alternatives we can think of, approximating  $Y_s$  by  $Y_0$  (freezing), or by  $E_i = \mathbb{E}(\tilde{Y}_i | \tilde{V}, \tilde{I}, \tilde{J})$ .

**Freeze  $Y_s$ .** If we freeze  $Y_s$  to its initial value  $Y_0$ , then equation (4.28) becomes

$$\tilde{Y}_{i+1} = \tilde{Y}_i + r(1 - \beta)Y_0\Delta t - \frac{\beta}{2(1 - \beta)Y_0}\tilde{I}_i + \frac{\rho}{\eta} \left( \frac{\tilde{V}_{i+1}^{\xi-\gamma+1} - \tilde{V}_i^{\xi-\gamma+1}}{\xi - \gamma + 1} - \tilde{J}_i \right) + \tilde{\rho}\sqrt{\tilde{I}_i}\epsilon_i^Y, \quad (4.29)$$

so that

$$\mu^Y \left( \tilde{Y}_i, \tilde{V}_i \right) = r(1 - \beta)Y_0\Delta t - \frac{\beta}{2(1 - \beta)Y_0}\tilde{I}_i + \frac{\rho}{\eta} \left( \frac{\tilde{V}_{i+1}^{\xi-\gamma+1} - \tilde{V}_i^{\xi-\gamma+1}}{\xi - \gamma + 1} - \tilde{J}_i \right), \quad (4.30)$$

$$\sigma^Y \left( \tilde{Y}_i, \tilde{V}_i \right) = \tilde{\rho}\sqrt{\tilde{I}_i}. \quad (4.31)$$

Note that  $\mu^Y \left( \tilde{Y}_i, \tilde{V}_i \right)$  and  $\sigma^Y \left( \tilde{Y}_i, \tilde{V}_i \right)$  do not depend on  $\tilde{Y}_i$ .

**Approximate  $Y_s$  by its expectation.** The Euler discretization for equation (4.28) is

$$\tilde{Y}_{i+1} - \tilde{Y}_i = \theta(1 - \beta)\tilde{Y}_i\Delta t - \frac{\beta}{2(1 - \beta)\tilde{Y}_i}\tilde{I}_i + \frac{\rho}{\eta} \left( \frac{\tilde{V}_{i+1}^{\xi-\gamma+1} - \tilde{V}_i^{\xi-\gamma+1}}{\xi - \gamma + 1} - \tilde{J}_i \right) + \tilde{\rho}\sqrt{\tilde{I}_i}\epsilon_i^Y. \quad (4.32)$$

Taking the expectation  $\mathbb{E} \left( \cdot \mid \tilde{V}, \tilde{I}, \tilde{J} \right)$  on both sides of equation (4.32),

$$\begin{aligned} \mathbb{E} \left( \tilde{Y}_{i+1} \mid \tilde{V}, \tilde{I}, \tilde{J} \right) &= \mathbb{E} \left( \tilde{Y}_i \mid \tilde{V}, \tilde{I}, \tilde{J} \right) [1 + \theta(1 - \beta)\Delta t] \\ &\quad - \frac{\beta\tilde{I}_i}{2(1 - \beta)} \mathbb{E} \left( \frac{1}{\tilde{Y}_i} \mid \tilde{V}, \tilde{I}, \tilde{J} \right) + \frac{\rho}{\eta} \left( \frac{\tilde{V}_{i+1}^{\xi-\gamma+1} - \tilde{V}_i^{\xi-\gamma+1}}{\xi - \gamma + 1} - \tilde{J}_i \right). \end{aligned} \quad (4.33)$$

The expectation  $\mathbb{E} \left( \frac{1}{\tilde{Y}_i} \mid \tilde{V}, \tilde{I}, \tilde{J} \right)$  does not exist because  $\tilde{Y}_i$  can be zero. One approximation (see for example Qiao *et al.* [68]) is

$$\mathbb{E} \left( \frac{1}{\tilde{Y}_i} \mid \tilde{V}, \tilde{I}, \tilde{J} \right) \approx \frac{1}{\mathbb{E} \left( \tilde{Y}_i \mid \tilde{V}, \tilde{I}, \tilde{J} \right)}. \quad (4.34)$$

Set  $E_i = \mathbb{E} \left( \tilde{Y}_i \mid \tilde{V}, \tilde{I}, \tilde{J} \right)$ . From equation (4.33) and (4.34),  $E_i$  can be computed approximately by the following recursion.

$$E_{i+1} \approx E_i + \theta(1 - \beta)E_i\Delta t - \frac{\beta}{2(1 - \beta)E_i}\tilde{I}_i + \frac{\rho}{\eta} \left( \frac{\tilde{V}_{i+1}^{\xi-\gamma+1} - \tilde{V}_i^{\xi-\gamma+1}}{\xi - \gamma + 1} - \tilde{J}_i \right), \quad (4.35)$$

where  $E_0 = \tilde{Y}_0$ . We then replace  $\tilde{Y}_i$  on the right hand side of equation (4.32) by  $E_i$  and obtain

$$\tilde{Y}_{i+1} = \tilde{Y}_i + \theta(1 - \beta)E_i\Delta t - \frac{\beta}{2(1 - \beta)E_i}\tilde{I}_i + \frac{\rho}{\eta} \left( \frac{\tilde{V}_{i+1}^{\xi-\gamma+1} - \tilde{V}_i^{\xi-\gamma+1}}{\xi - \gamma + 1} - \tilde{J}_i \right) + \tilde{\rho}\sqrt{\tilde{I}_i}\epsilon_i^Y. \quad (4.36)$$

Hence we have

$$\mu^Y(\tilde{Y}_i, \tilde{V}_i) = r(1 - \beta)E_i\Delta t - \frac{\beta}{2(1 - \beta)E_i}\tilde{I}_i + \frac{\rho}{\eta} \left( \frac{\tilde{V}_{i+1}^{\xi-\gamma+1} - \tilde{V}_i^{\xi-\gamma+1}}{\xi - \gamma + 1} - \tilde{J}_i \right), \quad (4.37)$$

$$\sigma^Y(\tilde{Y}_i, \tilde{V}_i) = \tilde{\rho}\sqrt{\tilde{I}_i}. \quad (4.38)$$

We provide a comparison between using equation (4.29) versus equation (4.36) as the auxiliary process in section 5.6 and find that their empirical correlations are very similar. There is no clear advantage of using equation (4.36) over equation (4.29). We therefore use equation (4.29) throughout other numerical examples.

In the next section, we introduce auxiliary models for instruments with non-linear path statistics. Payoffs of these instruments often depend on the maximum or minimum along each sample path. Examples of these instruments are barrier options, lookback options, *et cetera*.

## 4.5 An Auxiliary Model for Non-linear Path Statistics

Many instruments' payoffs depend jointly on the maximum (minimum) value and the final value along each sample path. The auxiliary instruments, which have similar payoffs, must have explicit solutions in the auxiliary model. Due to this requirement, any auxiliary model should have an explicit joint distribution of the maximum (minimum) value and final value. To our best knowledge, we can derive it only when  $\beta = 1$  and therefore must restrict the pricing model to  $\beta = 1$  case. Hence this auxiliary process does not apply to CEV-type models such as the SABR model.

### 4.5.1 The Auxiliary Process

Obtaining the auxiliary process requires more manipulation than in the linear path statistics case. Firstly, we only consider the zero-correlation case, that is  $\rho = 0$  in the

auxiliary model. Secondly, the drift term in the asset process is chosen deliberately as a function of the volatility. We first look at the process.

$$dZ_t = \lambda V_t^{2\xi} Z_t dt + V_t^\xi Z_t dW_t^S, \quad (4.39)$$

$$dV_t = \alpha(\mu - V_t)dt + \eta V_t^\gamma dW_t^V, \quad (4.40)$$

$$dW_t^S dW_t^V = 0. \quad (4.41)$$

Later in this section we discuss the choice of  $\lambda$  so that it makes the auxiliary process as close to the pricing model as possible. To see why we manipulate the auxiliary model in such a way, we first state the following lemma.

**Lemma 4.5.1** *Suppose  $\sigma_t$  is some deterministic function. Define  $B_t = \int_0^t \sigma_s dW_s$ ,  $\widehat{B}_t = B_t + a \int_0^t \sigma_s^2 ds$  for some constant  $a$  and  $\widehat{M}_t = \max_{s \in [0, t]} \widehat{B}_s$ . Then we have the joint density function of  $\widehat{B}_t$  and  $\widehat{M}_t$  as*

$$f_{\widehat{B}_T, \widehat{M}_T}(b, m) = \frac{2(2m - b)}{\overline{T} \sqrt{2\pi \overline{T}}} \exp \left[ ab - \frac{1}{2} a^2 \overline{T} - \frac{1}{2\overline{T}} (2m - b)^2 \right], \quad (4.42)$$

where  $\overline{T} = \int_0^T \sigma_t^2 dt$ .

See proof in Appendix A.1 (page 207).  $\square$

This is a simple extension of the result from standard Brownian motion (Karatzas and Shreve [52]) to the case with a deterministic time change.  $Z_t \mid V_t$  is a GBM process with deterministic volatility. Lemma 4.5.1 opens up the possibility of deriving analytical joint distribution of  $\max_{t \leq T} (Z_t \mid V_t)$  and  $Z_T \mid V_{t \leq T}$ .

#### 4.5.2 The Joint Distribution

Define the transformation  $Y_t = \ln Z_t$ .

$$dY_t = \left( \lambda - \frac{1}{2} \right) V_t^{2\xi} dt + V_t^\xi dW_t^S. \quad (4.43)$$

Integrating equation (4.43), we get

$$\begin{aligned} Y_T &= Y_0 + \left( \lambda - \frac{1}{2} \right) \int_0^T V_s^{2\xi} ds + \int_0^T V_s^\xi dW_s^S \\ &= Y_0 + \left( \lambda - \frac{1}{2} \right) \int_0^T V_s^{2\xi} ds + B_T \\ &= Y_0 + \widehat{B}_T, \end{aligned} \quad (4.44)$$



where  $B_T = \int_0^T V_s^\xi dW_s^S$  and  $\widehat{B}_T = (\lambda - \frac{1}{2}) \int_0^T V_s^{2\xi} ds + B_T$ . Then

$$\begin{aligned} \mathbb{P} \left( Z_T \leq z, \max_{t \leq T} Z_t \leq m \right) &= \mathbb{P} \left( Y_T \leq \ln z, \max_{t \leq T} Y_t \leq \ln m \right) \\ &= \mathbb{P} \left( \widehat{B}_T \leq \ln z - Y_0, \max_{t \leq T} \widehat{B}_t \leq \ln m - Y_0 \right). \end{aligned} \quad (4.45)$$

If we assume in equation (4.44) that  $V_t$  is deterministic, the joint cumulative probability (4.45) can be computed explicitly by Lemma 4.5.1. As a result, any instrument depending on the joint distribution is more likely to be valued in closed form.

Our final task is to choose the parameter value  $\lambda$  in the auxiliary process (4.39).

**Choice of  $\lambda$ .** We want equation (4.39) to be as close to the pricing model as possible. This means we need to choose  $\lambda$  in such a way that  $\lambda V_t^{2\xi}$  is around the constant value  $r$  throughout the path.  $V_t$  is always reverting to the long term mean  $\mu$  and we may also have  $V_0 \approx \mu$ .  $V_t$  is then roughly around  $\mu$ . We set  $\lambda V_t^{2\xi} \approx \lambda \mu^{2\xi} = r$  and therefore  $\lambda = \frac{r}{\mu^{2\xi}}$ .

We have constructed auxiliary processes for instruments with linear and non-linear path statistics. The auxiliary instruments, however, are tailored. In section 4.6, we discuss the auxiliary instruments for arithmetic average rate options as an example of linear path statistics. In section 4.7, we discuss the auxiliary instrument for barrier options as an example of instruments with non-linear path statistics.

## 4.6 Auxiliary Instruments for Average Rate Options

Instruments with linear path statistics can share a common auxiliary model. However, good auxiliary instruments are always tailored and resemble those we want to price. In addition, we want the auxiliary instrument to have an explicit solution  $\widehat{p}_e^{(j)}$  in the auxiliary model  $\widetilde{\mathcal{M}}^{(j)}$  along sample path  $j$ . In this section, we propose auxiliary instruments for an important instrument with linear path statistics, the arithmetic average rate option. We adopt all notation from section 3.6.1.

The arithmetic average rate option has the discounted payoff along sample path  $j$

$$C^{(j)} = e^{-rT} \left( A_a^{(j)} - K \right)^+, \quad (4.46)$$

where  $A_a^{(j)}$  is the arithmetic average of asset values at reset dates along the  $j^{th}$  sample path

$$A_a^{(j)} = \frac{1}{n} \sum_{i=1}^n S_{t_{\iota_i}}^{(j)} \quad (4.47)$$

We use separate auxiliary instruments for the  $\beta = 1$  and  $\beta \in (0, 1)$  cases.

For the  $\beta = 1$  case, we use the option on discrete geometric average,  $A_g$ ,

$$A_g = \left( \prod_{i=1}^n S_{t_{\iota_i}} \right)^{\frac{1}{n}}. \quad (4.48)$$

The discounted payoff is  $\tilde{p}^{(j)} = e^{-rT} (A_g^{(j)} - K)^+$ .

For the  $\beta \in (0, 1)$  case, we use the option on the discrete  $\beta$ -average,  $A_\beta$ ,

$$A_\beta = \left( \frac{1}{n} \sum_{i=1}^n S_{t_{\iota_i}}^{1-\beta} \right)^{\frac{1}{1-\beta}} \quad (4.49)$$

The discounted payoff is  $\tilde{p}^{(j)} = e^{-rT} (A_\beta^{(j)} - K)^+$ .

In both cases, we need to compute  $\tilde{p}_e^{(j)}$ . We discuss the two cases separately.

#### 4.6.1 Auxiliary Instrument for $\beta = 1$ Case

From equation (4.25), we have

$$Y_{t_i} = Y_0 + rt_i + \frac{\rho}{\eta} \left( \frac{V_{t_i}^{\xi-\gamma+1} - V_0^{\xi-\gamma+1}}{\xi - \gamma + 1} - \sum_{j=0}^{i-1} J_j \right) - \frac{1}{2} \sum_{j=0}^{i-1} I_j + \tilde{\rho} \sum_{j=0}^{i-1} \epsilon_j^Y \sqrt{I_j}. \quad (4.50)$$

Define  $g = \frac{1}{n} \sum_{i=1}^n \tilde{Y}_{\iota_i}$  so that  $A_g = e^g$ .

$$g = Y_0 + \frac{r}{n} \sum_{i=1}^n t_{\iota_i} - \frac{1}{2} H_1 + \varepsilon \sqrt{H_2}, \quad \varepsilon \sim N(0, 1), \quad (4.51)$$

where

$$H_1 = \frac{1}{n} \sum_{i=0}^{n-1} (n-i) \sum_{j=\iota_i}^{\iota_{i+1}-1} \tilde{I}_j - \frac{2\rho}{n\eta} \left[ \frac{\sum_{i=1}^n \tilde{V}_{\iota_i}^{\xi-\gamma+1} - n\tilde{V}_0^{\xi-\gamma+1}}{\xi - \gamma + 1} - \sum_{i=0}^{n-1} (n-i) \sum_{j=\iota_i}^{\iota_{i+1}-1} \tilde{J}_j \right], \quad (4.52)$$

$$H_2 = \frac{\tilde{\rho}^2}{n^2} \sum_{i=0}^{n-1} (n-i)^2 \sum_{j=\iota_i}^{\iota_{i+1}-1} \tilde{I}_j. \quad (4.53)$$

Write  $\tilde{I} = (\tilde{I}_0, \dots, \tilde{I}_{N-1})$  and  $\tilde{J} = (\tilde{J}_0, \dots, \tilde{J}_{N-1})$ . The value of the geometric average rate option conditional on  $\tilde{V}$ ,  $\tilde{I}$  and  $\tilde{J}$  is given by

$$\begin{aligned} \tilde{p}_e^{(j)} &= \mathbb{E} \left[ e^{-rT} (A_g - K)^+ \mid \tilde{V}^{(j)}, \tilde{I}^{(j)}, \tilde{J}^{(j)} \right] = \mathbb{E} \left[ e^{-rT} (e^g - K)^+ \mid \tilde{V}^{(j)}, \tilde{I}^{(j)}, \tilde{J}^{(j)} \right] \\ &= e^{r(\bar{T}-T)} C_{BS}(K, \bar{T}, S_0, y, r, \bar{\sigma}), \end{aligned} \quad (4.54)$$

where

$$\bar{T} = \frac{1}{n} \sum_{i=1}^n t_{\iota_i}, \quad \bar{\sigma}^2 = \frac{H_2}{T}, \quad y = \frac{1}{2\bar{T}}(H_1 - H_2). \quad (4.55)$$

The control variate for each sample path  $j$  is then chosen as

$$CV^{(j)} = e^{-rT} \left( A_g^{(j)} - K \right)^+ - \tilde{p}_e^{(j)}. \quad (4.56)$$

We now propose an auxiliary instrument for the  $\beta \in (0, 1)$  case.

#### 4.6.2 Auxiliary Instrument for $\beta \in (0, 1)$ Case

Each  $\tilde{Y}_i$  in equation (4.29) can be simulated as

$$\begin{aligned} \tilde{Y}_i &= \tilde{Y}_0 + r(1 - \beta)\tilde{Y}_0 t_i + \frac{\rho}{\eta} \left( \frac{\tilde{V}_i^{\xi-\gamma+1} - \tilde{V}_0^{\xi-\gamma+1}}{\xi - \gamma + 1} - \sum_{j=0}^{i-1} \tilde{J}_j \right) - \frac{\beta}{2(1 - \beta)\tilde{Y}_0} \sum_{j=0}^{i-1} \tilde{I}_j \\ &\quad + \tilde{\rho} \sum_{j=0}^{i-1} \epsilon_j^Y \sqrt{\tilde{I}_j}. \end{aligned} \quad (4.57)$$

We notice that  $\tilde{Y}_i$  has a positive probability of becoming negative. Since this is not in pricing model, we do not impose an absorbing or reflection boundary. Define  $g_\beta = \frac{1-\beta}{n} \sum_{i=1}^n \tilde{Y}_{\iota_i}$ . From equation (4.57),  $g_\beta$  is normal

$$g_\beta = (1 - \beta)(m + s\varepsilon), \quad \varepsilon \sim N(0, 1), \quad (4.58)$$

where

$$m = \tilde{Y}_0 + \frac{r(1-\beta)\tilde{Y}_0}{n} \sum_{i=1}^n t_{\iota_i} + \frac{\rho}{n\eta} \left( \frac{\sum_{i=1}^n \tilde{V}_{\iota_i}^{\xi-\gamma+1} - n\tilde{V}_0^{\xi-\gamma+1}}{\xi-\gamma+1} - \sum_{i=0}^{n-1} (n-i) \sum_{j=\iota_i}^{\iota_{i+1}-1} \tilde{J}_j \right) - \frac{\beta}{2n(1-\beta)\tilde{Y}_0} \sum_{i=0}^{n-1} (n-i) \sum_{j=\iota_i}^{\iota_{i+1}-1} \tilde{I}_j, \quad (4.59)$$

$$s^2 = \frac{\tilde{\rho}^2}{n^2} \sum_{i=0}^{n-1} (n-i)^2 \sum_{j=\iota_i}^{\iota_{i+1}-1} \tilde{I}_j. \quad (4.60)$$

By definition,  $A_\beta = \left( \frac{1-\beta}{n} \sum_{i=1}^n Y_{t_{\iota_i}} \right)^{\frac{1}{1-\beta}} \approx \left( g_\beta^+ \right)^{\frac{1}{1-\beta}}$  where we apply a zero truncation to  $g_\beta$  because it has a normal distribution and is likely to be negative. Since  $g_\beta$  is normal, we can compute

$$\tilde{p}_e^{(j)} = e^{-rT} \mathbb{E} \left[ \left( \left( g_\beta^+ \right)^{\frac{1}{1-\beta}} - K \right)^+ \middle| \tilde{V}^{(j)}, \tilde{I}^{(j)}, \tilde{J}^{(j)} \right]. \quad (4.61)$$

We derive an accurate approximation to the conditional expectation above in terms of a series expansion.

**The conditional expectation.** For brevity, we suppress the terms  $\tilde{V}^{(i)}, \tilde{I}^{(i)}, \tilde{J}^{(i)}$  in the expectation. All the expectations are actually conditional on them. Define  $\lambda = 1 - \beta$  and  $K' = K^\lambda > 0$ . We have

$$\begin{aligned} \mathbb{E} \left[ \left( \left( g_\beta^+ \right)^{\frac{1}{1-\beta}} - K \right)^+ \right] &= \mathbb{E} \left( g_\beta^{\frac{1}{\lambda}} - K \middle| g_\beta > K' \right) \cdot \mathbb{P}(g_\beta > K') \\ &= \left[ \mathbb{E} \left( g_\beta^{\frac{1}{\lambda}} \middle| g_\beta > K' \right) - K \right] \cdot \mathbb{P}(g_\beta > K'). \end{aligned} \quad (4.62)$$

If we define  $z \sim N(m, s^2)$ ,  $g_\beta = \lambda z$ . Then

$$\mathbb{E} \left( g_\beta^{\frac{1}{\lambda}} \middle| g_\beta > \kappa \right) = \mathbb{E} \left[ (\lambda z)^{\frac{1}{\lambda}} \middle| z > \kappa \right], \quad (4.63)$$

where  $\kappa = \frac{K'}{\lambda}$ . We state two lemmas for equation (4.63).

**Lemma 4.6.1** Suppose  $Y \sim N(m, s^2)$ . For  $\lambda, \kappa > 0$ , define  $X = (\lambda Y^+)^{\frac{1}{\lambda}}$  and

$b = \frac{\kappa - m}{s}$ . We have

$$\mathbb{E}(X | Y > \kappa) = (\lambda m)^{\frac{1}{\lambda}} + \sum_{i=1}^{\infty} \frac{1}{i!} (\lambda m)^{\frac{1}{\lambda} - i} s^i \mathbb{E}(Z^i | Z > b) \prod_{j=0}^{i-1} (1 - j\lambda), \quad Z \sim N(0, 1). \quad (4.64)$$

See proof in Appendix A.2 (page 208).  $\square$

The following lemma gives moments of truncated normal distribution.

**Lemma 4.6.2** Suppose  $Z \sim N(0, 1)$  and define  $M_i = \mathbb{E}(Z^i | Z > b)$ . Let  $\phi(y, m, s^2 | y > \kappa)$  denote the density function of normal distribution  $N(m, s^2)$  truncated at  $\kappa$ .  $M_i$  has the following recursive representation

$$M_i = b^{i-1} \phi(b, 0, 1 | Z > b) + (i-1)M_{i-2}, \quad (4.65)$$

with the initial conditions

$$M_0 = \int_b^{+\infty} z^0 \phi(z, 0, 1 | z > b) dz = 1, \quad (4.66)$$

$$M_1 = \int_b^{+\infty} z \phi(z, 0, 1 | z > b) dz = \phi(b, 0, 1 | z > b) = \frac{\phi(b)}{1 - \Phi(b)}, \quad (4.67)$$

where  $\phi(\cdot)$  and  $\Phi(\cdot)$  denote the probability and cumulative density functions of standard normal distribution respectively. See Dhrymes [28].  $\square$

Define the double factorial  $n!! = \prod_{i \in \mathbb{Z}, 0 \leq 2i < n} (n - 2i)$ . We have the general expression for  $M_i$ ,

$$M_i = \begin{cases} \left[ 1 + \phi(b, 0, 1 | z > b) \sum_{j=1}^n \frac{b^{2j-1}}{(2j-1)!!} \right] \cdot (2n)!! & \text{if } i = 2n \\ \phi(b, 0, 1 | z > b) \left[ 1 + \sum_{j=1}^n \frac{b^{2j-1}}{(2j)!!} \right] \cdot (2n+1)!! & \text{if } i = 2n+1 \end{cases} \quad (4.68)$$

where  $n \in \mathbb{Z}^+$ .

From the computational point of view, we are still in favour of using the recursive representation (4.65) to compute  $M_i$  as our priority is to compute the expectations quickly.

With lemmas 4.6.1 and 4.6.2, equation (4.63) can be approximated accurately (within a tolerance level  $\epsilon$ ) by truncating the infinite sum (4.64) at a certain level

$N_{\max}$  so that

$$\left| \tilde{\mathbb{E}}(X|Y > \kappa) - \mathbb{E}(X|Y > \kappa) \right| = \left| \sum_{i=N_{\max}}^{\infty} \frac{1}{i!} (\lambda m)^{\frac{1}{\lambda}-i} s^i \mathbb{E}(Z^i|Z > b) \prod_{j=0}^{i-1} (1-j\lambda) \right| < \epsilon, \quad (4.69)$$

where  $\tilde{\mathbb{E}}(X|Y > \kappa)$  is the truncated expectation. In practice, however, the truncated sum can blow up when  $N_{\max}$  is too large due to the numerical instability caused by the data representation in any programming language, for example, when the number is too large or small.

**Numerical stability.** Let  $A_i = \frac{1}{i!} (\lambda m)^{\frac{1}{\lambda}-i} s^i \mathbb{E}(Z^i|Z > b) \prod_{j=0}^{i-1} (1-j\lambda)$ .  $\{A_i\}$  has alternating signs. According to the Leibniz rule, if  $\{|A_i|\}$  is monotone decreasing and has zero limit at infinity, then  $\sum_{i=1}^{\infty} A_i$  converges. For  $|A_i|$  to be decreasing, we need

$$\left| \frac{A_i}{A_{i-1}} \right| = \frac{s \mathbb{E}(Z^i|Z > b)}{i \lambda m \mathbb{E}(Z^{i-1}|Z > b)} |1 - (i-1)\lambda| < 1. \quad (4.70)$$

It is difficult to determine the conditions from inequality (4.70) for parameters  $s$ ,  $\lambda$  and  $m$  to satisfy. There are occasions when it is violated and  $A_i$  oscillates in increasing amplitude. As a result,  $\sum_{i=1}^{\infty} A_i$  blows up. We can see this numerically in Figure 4.1. In the case of instability,  $\left| \frac{A_i}{A_{i-1}} \right| > 1$  when  $N_{\max} > 20$ . In fact, we

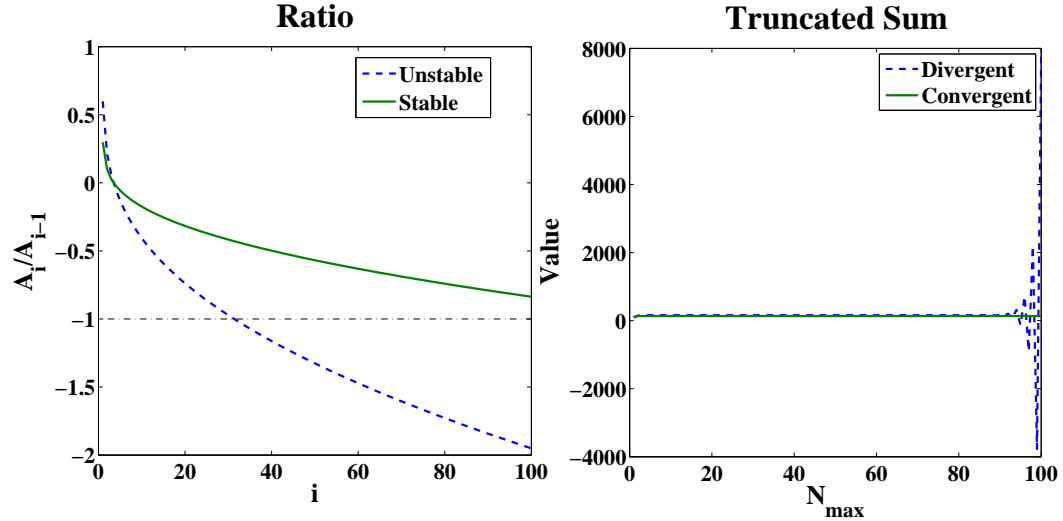


Figure 4.1: Convergence Analysis

find that usually the truncated sum has already converged to an acceptable level (6-7 decimal places) when  $N_{\max} = 10$  in which case the numerical instability does not happen in our numerical example. In our implementation we use  $N_{\max} = 10$ .

In  $\beta \in (0, 1)$  case, we choose the control variate to be

$$CV^{(j)} = e^{-rT} \left[ \left( g_\beta^{+(j)} \right)^{\frac{1}{1-\beta}} - K \right]^+ - \tilde{p}_e^{(j)} \quad (4.71)$$

along the  $j^{th}$  sample path.

The derivations of CVs so far take into account the correlation  $\rho$  between  $S_t$  and  $V_t$ . We refer to them as correlation CVs (SV ( $\rho \neq 0$ )). Our CVs are sensitive to the correlation between  $X_t$  and  $V_t$  because of the way they are constructed. We expect them to achieve high variance reduction in low correlation cases. We explain the reason in the next section.

#### 4.6.3 The Effect of Correlation between $S_t$ and $V_t$

We show that our methods reduce only the variance coming from the Wiener path driving the asset process which we assume accounts for a large part of the standard error. By computing the CV along each sample path conditional on the variance process, we effectively ignore the randomness coming from the variance path. If it is highly correlated with the asset path, making it deterministic also makes the asset path close to deterministic and loses the advantage of our methods. We explain the reason more carefully as follows.

The Wiener paths driving the auxiliary process are

$$W_t^S = \rho W_t^1 + \tilde{\rho} W_t^2, \quad W_t^V = W_t^1, \quad (4.72)$$

or equivalently we can write

$$W_t^S = \rho W_t^V + \tilde{\rho} W_t^2. \quad (4.73)$$

By conditioning on  $V_t$ , we equivalently assume  $W_t^V$  to be a deterministic process. When  $|\rho| \rightarrow 1$  (and therefore  $\tilde{\rho} \rightarrow 0$ ),  $W_t^S \approx W_t^V$ .  $W_t^S \mid W_t^V$  is then also close to deterministic and our methods barely have any advantage in this case. In chapter 5, we examine the performance of correlation CVs for highly positive and negative correlation in individual models.

The issue of correlation CVs motivates us to consider a special case of our cor-

relation CV. By setting  $\rho = 0$  in the auxiliary process, we may obtain another auxiliary model  $\widetilde{M}_0^{(j)}$ . We refer to it as zero-correlation CV (SV ( $\rho = 0$ )).

#### 4.6.4 Special Case: Zero-correlation Control Variate

There are two reasons for us to consider this special CV. Firstly, it is cheaper to compute because there is no  $J_i$ ,  $i = 0, \dots, N-1$  in  $\widetilde{M}_0^{(j)}$ . Secondly, it can also be used when  $f(v) = e^v$  in our model (4.1). The most important reason is it may yield higher empirical correlations between the option and CV.

In the zero-correlation CV, the driving Wiener processes are

$$W_t^S = W_t^1, \quad W_t^V = W_t^2. \quad (4.74)$$

Unlike in the correlation CVs, when  $|\rho| \rightarrow 1$ ,  $W_t^S \mid W_t^V$  remains highly random although the auxiliary process becomes less related to the pricing model by setting  $\rho = 0$ . This makes the zero-correlation CV to be an effective CV. The formulas can be easily obtained by setting  $\rho = 0$  in the correlation CV.

**The  $\beta = 1$  case.** Set  $\rho = 0$  in equation (4.50), (4.51), (4.52) and (4.53).

$$\tilde{Y}_i = \tilde{Y}_0 + rt_i - \frac{1}{2} \cdot \sum_{j=0}^{i-1} \tilde{I}_j + \sum_{j=0}^{i-1} \epsilon_j^Y \sqrt{\tilde{I}_j}, \quad \epsilon_j \sim N(0, 1), \quad (4.75)$$

and

$$g = Y_0 + r\bar{T} - \frac{1}{2} \cdot h_1 + \varepsilon \sqrt{h_2}, \quad \varepsilon \sim N(0, 1), \quad (4.76)$$

where

$$h_1 = \frac{1}{n} \sum_{i=0}^{n-1} (n-i) \sum_{j=\iota_i}^{\iota_{i+1}-1} \tilde{I}_j, \quad h_2 = \frac{1}{n^2} \sum_{i=0}^{n-1} (n-i)^2 \sum_{j=\iota_i}^{\iota_{i+1}-1} \tilde{I}_j. \quad (4.77)$$

$g$  is normally distributed and

$$p_e^{(j)} = e^{-rT} \mathbb{E} \left[ (e^g - K)^+ \mid \tilde{I} \right] = e^{r(\bar{T}-T)} C_{BS}(K, \bar{T}, S_0, y, r, \bar{\sigma}), \quad (4.78)$$

where

$$\sigma^2 = \frac{h_1}{T}, \quad \bar{\sigma}^2 = \frac{h_2}{T}, \quad y = \frac{1}{2}(\sigma^2 - \bar{\sigma}^2). \quad (4.79)$$



**The  $\beta \in (0, 1)$  case.** Set  $\rho = 0$  in equation (4.59) and (4.60). Then

$$m = Y_0 + r(1 - \beta)Y_0\bar{T} - \frac{\beta}{2n(1 - \beta)Y_0} \sum_{i=0}^{n-1} (n - i) \sum_{j=\iota_i}^{\iota_{i+1}-1} \tilde{I}_j, \quad (4.80)$$

$$s^2 = \frac{1}{n^2} \sum_{i=0}^{n-1} (n - i)^2 \sum_{j=\iota_i}^{\iota_{i+1}-1} \tilde{I}_j. \quad (4.81)$$

The formula for  $p_e^{(j)}$  follows from equation (4.64) as before.

There is a subtlety, however, in implementing SV ( $\rho = 0$ ). In the pricing model, we set

$$W_t^S = W_t^1, \quad W_t^V = \rho W_t^1 + \tilde{\rho} W_t^2, \quad (4.82)$$

so that we keep  $W_t^S$  to be the same in both the pricing and auxiliary model. We shall see in chapter 5 that SV ( $\rho = 0$ ) has high efficiency gains in most cases.

## 4.7 An Auxiliary Instrument for Barrier Options

Very few papers address the issue of valuing barrier options with Monte Carlo simulations in the stochastic volatility model. Heath and Platen [43] investigate this issue in the Heston model [44]. They use the continuous barrier option in the GBM model as a control variate. Although they claim their method to be fast and accurate, they did not fully examine the real speed-ups of the control variate. Glasserman and Staum [39] propose an importance sampling method for barrier options. It achieves a certain degree of variance reduction with unusual parameter values. However, we find in chapter 6 that their method barely has any speed-up with any reasonable parameter values.

In this section, we propose a new control variate method for barrier options in our stochastic volatility models. We claim that our method is much more efficient than the importance sampling method and also has higher efficiency gains than standard control variate methods in most cases.

Our CV is a path-dependent auxiliary model CV. The auxiliary model  $\widetilde{M}^{(j)}$  is given as equation (4.39) but conditional on  $\tilde{V}_t$ . We derive the explicit value  $\tilde{p}_e^{(j)}$  of our auxiliary instrument, the continuous barrier option, in the auxiliary model. To illustrate the idea, we use up and out barrier call option as an example. The

arguments apply equally well to up and in, down and out and down and in calls and puts. We adopt all definitions from section 3.6.2.

**Idea.** We are interested in the value of equation (3.23). A control variate estimator for the barrier option price is

$$\tilde{v} = e^{-rT} \frac{1}{M} \sum_{j=1}^M h\left(\tilde{S}_t^{(j)}, t \in \mathcal{T}\right) + b \cdot CV^{(j)}. \quad (4.83)$$

We use the continuous barrier option in the auxiliary model as a control variate. If  $Z_t$  is given by SDE (4.39), we construct our CV as

$$CV^{(j)} = e^{-rT} H'\left(Z_t^{(j)}, t \in \mathcal{T}\right) - \tilde{p}_e^{(j)}, \quad (4.84)$$

where

$$\tilde{p}_e^{(j)} = e^{-rT} \mathbb{E}\left[H(Z_t, t \leq T) \mid V_t^{(j)}\right]. \quad (4.85)$$

$H(Z_t, t \leq T)$  depends on  $\max_{t \leq T} Z_t$  and  $Z_T$ . We have seen in section 4.5 that their joint distribution is available by conditioning on  $V_t$ . It is therefore straightforward to derive the explicit form of  $\tilde{p}_e^{(j)}$ . We supply the following lemma for the explicit value of our CV.

**Lemma 4.7.1** *Suppose  $Z_t$  has the following dynamics.*

$$dZ_t = \lambda \sigma_t^2 Z_t dt + \sigma_t Z_t dW_t, \quad (4.86)$$

*where  $\lambda$  is some constant and  $\sigma_t$  is some deterministic function. Then*

$$\mathbb{E}[H(Z_t, t \leq T)] = Z_0(I_1 - I_3) - K(I_2 - I_4), \quad (4.87)$$

where

$$I_1 = e^{\lambda \bar{T}} \left[ \Phi \left( d_+ \left( \bar{T}, \frac{Z_0}{K} \right) \right) - \Phi \left( d_+ \left( \bar{T}, \frac{Z_0}{U} \right) \right) \right], \quad (4.88)$$

$$I_2 = \Phi \left( d_- \left( \bar{T}, \frac{Z_0}{K} \right) \right) - \Phi \left( d_- \left( \bar{T}, \frac{Z_0}{U} \right) \right), \quad (4.89)$$

$$I_3 = e^{\lambda \bar{T}} \left( \frac{U}{Z_0} \right)^{2\lambda+1} \left[ \Phi \left( d_+ \left( \bar{T}, \frac{U^2}{K Z_0} \right) \right) - \Phi \left( d_+ \left( \bar{T}, \frac{U}{Z_0} \right) \right) \right], \quad (4.90)$$

$$I_4 = \left( \frac{U}{Z_0} \right)^{2\lambda-1} \left[ \Phi \left( d_- \left( \bar{T}, \frac{U^2}{K Z_0} \right) \right) - \Phi \left( d_- \left( \bar{T}, \frac{U}{Z_0} \right) \right) \right], \quad (4.91)$$

$$d_{\pm}(\tau, x) = \frac{1}{\sqrt{\tau}} \left[ \ln x + \left( \lambda \pm \frac{1}{2} \right) \tau \right], \quad (4.92)$$

$$\bar{T} = \int_0^T \sigma_s^2 ds, \quad (4.93)$$

and  $\Phi(\cdot)$  is the cumulative density function of standard normal distribution.

See proof in Appendix A.3 (page 209).  $\square$

This is a simple extension of the result from the Black-Scholes model with constant volatility (Shreve [76]) to the case with a deterministic volatility. We see from Lemma 4.7.1 that we can have the explicit value of the barrier option in the auxiliary model by setting  $\sigma_t = V_t^\xi$ . By piece-wise constant approximation to  $V_t$ , equation (4.86) has the discrete form

$$d\tilde{Z}_t = \lambda \tilde{V}_{t_i}^{2\xi} \tilde{Z}_t dt + \tilde{V}_{t_i}^\xi \tilde{Z}_t dW_t^S, \quad t \in [t_i, t_{i+1}). \quad (4.94)$$

with  $\bar{T}$  given by

$$\bar{T} = \int_0^T \tilde{V}_t^{2\xi} dt = \sum_{i=0}^{N-1} \tilde{V}_{t_i}^{2\xi} \Delta t. \quad (4.95)$$

We have  $\mathbb{E} \left[ H \left( \tilde{Z}_t, t \leq T \right) \middle| \tilde{V}_t^{(j)} \right]$  available. But we are not yet done. The final thing left in equation (4.84) is  $H' \left( \tilde{Z}_t^{(j)}, t \in \mathcal{T} \right)$ . This requires knowledge of the survival probabilities.

### The Survival Probabilities

In order to correct for simulation bias from the discrete sample path, we also need to know the survival probabilities  $p_i$  between discrete points on the path of  $\tilde{Z}_t$ . We state the following lemma.

**Lemma 4.7.2** Define  $M_T = \max_{t \leq s \leq T} \left\{ \widehat{W}_s \mid \widehat{W}_T \right\}$  as the maximum of the bridge distribution over the interval  $[t, T]$  where  $\widehat{W}_t = W_t + at$  is a Brownian motion with some constant drift  $a$ . Then

$$\mathbb{P} \left( M_T \leq m \mid \widehat{W}_T \right) = 1 - \exp \left[ -\frac{2}{T} m \left( m - \widehat{W}_T \right) \right], \quad m \geq \max \left\{ 0, \widehat{W}_T \right\}. \quad (4.96)$$

(See Karatzas and Shreve [52]).  $\square$

Note that the result does not depend on  $a$ .

We apply a logarithmic transformation to equation (4.94).

$$d \ln \tilde{Z}_t = \left( \lambda - \frac{1}{2} \right) \tilde{V}_{t_i}^{2\xi} dt + \tilde{V}_{t_i}^\xi dW_t^S = \tilde{V}_{t_i}^\xi d\widehat{W}_t^S, \quad t \in [t_i, t_{i+1}). \quad (4.97)$$

This SDE has constant drift and diffusion coefficient over the interval  $t \in [t_i, t_{i+1})$ . The survival probability can be easily computed as

$$\begin{aligned} p_i &= \mathbb{P} \left( \max_{t \in [t_i, t_{i+1}]} \left\{ \tilde{Z}_t \right\} \leq U \mid \tilde{Z}_{t_i} \leq U, \tilde{Z}_{t_{i+1}} \leq U \right) \\ &= \mathbb{P} \left( \max_{t \in [t_i, t_{i+1}]} \left\{ \frac{\ln \tilde{Z}_t - \ln \tilde{Z}_{t_i}}{\tilde{V}_{t_i}^\xi} \right\} \leq \frac{\ln U - \ln \tilde{Z}_{t_i}}{\tilde{V}_{t_i}^\xi} \mid \tilde{Z}_{t_i} \leq U, \tilde{Z}_{t_{i+1}} \leq U \right) \\ &= \mathbb{P} \left( M_{\Delta t} \leq \frac{\ln U - \ln \tilde{Z}_{t_i}}{\tilde{V}_{t_i}^\xi} \mid \widehat{W}_{\Delta t} \leq \frac{\ln U - \ln \tilde{Z}_{t_i}}{\tilde{V}_{t_i}^\xi} \right) \\ &= 1 - \exp \left[ -\frac{2 \left( \ln U - \ln \tilde{Z}_{t_i} \right) \left( \ln U - \ln \tilde{Z}_{t_{i+1}} \right)}{\Delta t \tilde{V}_{t_i}^{2\xi}} \right]. \end{aligned} \quad (4.98)$$

where the last equality is a direct application of Lemma 4.7.2. The values of  $p_i$  are then substituted into  $H' \left( \tilde{Z}_t^{(j)}, t \in \mathcal{T} \right)$  to complete the computation of equation (4.84).

## 4.8 Conclusion

We constructed CVs for a general stochastic volatility models. The instruments are divided into two broad categories, those with linear and non-linear path statistics. We proposed auxiliary models in each category. We discussed the tailored auxiliary instruments for the arithmetic average rate and barrier option as examples in both categories. What we have not done is the auxiliary model for non-linear path statistics when  $\beta \in (0, 1)$  due to our inability to derive the joint distribution of path

statistics. We leave the discovery of efficient CVs in that case as a future research topic.

The new CVs buy us more than efficient valuation of instruments. In fact, they also make the model calibration faster. They help us to investigate the variations in barrier option pricing in various stochastic volatility models calibrated to the same vanilla option prices. In chapter 5 and 6, we provide numerical results for average rate and barrier options. Performances of various CVs are compared. In chapter 7, we discuss the variations in barrier option pricing under stochastic volatility.

## Chapter 5

# Numerical Results for Average Rate Options

### 5.1 Introduction

We provide comparisons of various control variates for arithmetic average rate options in different models. The result for each Monte Carlo valuation is often given in the following form.

$$\begin{pmatrix} v \\ se \\ \tau \\ E \end{pmatrix} = \begin{pmatrix} 3.98 \\ (0.048) \\ [7.15] \\ \mathbf{5.3} \end{pmatrix} \quad (5.1)$$

The top entry is the Monte Carlo estimate of the option value. The second entry in the round bracket is the standard error. The third entry in the square bracket is the computation time. The bottom entry in bold is the efficiency gain that we defined in section 3.3.

We compare five CVs in our numerical examples. These are three standard CVs, namely, the European call CV (if applicable), the GBM auxiliary CV and the GBM delta CV, plus the two correlation CVs, SV ( $\rho = 0$ ) and SV ( $\rho \neq 0$ ), that we introduced in section 4.6. We use them to price three arithmetic average rate options with 4, 16, and 64 reset dates up to the maturity  $T = 1$ .

The GBM auxiliary process has a constant volatility  $\sigma^*$ . A subtle question is how to choose  $\sigma$  to make the auxiliary model as close to the pricing model as possible.

If we compare the diffusion terms in both models, we have

$$\sigma^* S_t = V_t^\xi S_t^\beta \approx V_0^\xi S_0^{\beta-1} S_t \Rightarrow \sigma^* \approx V_0^\xi S_0^{\beta-1}. \quad (5.2)$$

We use this  $\sigma^*$  for the GBM auxiliary CV. For the delta CV, we can do even better. The delta in the GBM process at  $t_i$  is a function of the current stock price  $S_{t_i}$  and the volatility  $\sigma_{t_i}^*$  which we allow to change at each time step  $t_i$ . By equating diffusion terms,

$$\sigma_t^* S_t = V_t^\xi S_t^\beta \Rightarrow \sigma_t^* = V_t^\xi S_t^{\beta-1}. \quad (5.3)$$

Then at each time  $t_i$ , set  $\sigma_{t_i}^* = V_{t_i}^\xi S_{t_i}^{\beta-1}$ .

We apply CVs with four different models listed in Table 5.1. These are Heston,

	$\beta$	$\xi$	$\gamma$
Heston	1	$\frac{1}{2}$	$\frac{1}{2}$
SABR	(0, 1)	1	1
GARCH	1	$\frac{1}{2}$	1
J&S	(0, 1)	1	$\frac{1}{2}$

Table 5.1: Stochastic Volatility Models

SABR, GARCH and a special case of Johnson and Shanno [50] (J&S). Our selected models include cases where  $\beta = 1$ ,  $0 < \beta < 1$ ,  $\xi = \gamma$  and  $\xi \neq \gamma$ . Note that the European call CV can be applied only to the Heston and SABR model. In the Heston model, we compute the explicit CV value using the formula given by Albrecher *et al.* [2]. We refer to it as Heston call later. In the SABR model, we use the approximation for the implied volatility in Hagan *et al.* [41]. We refer to it as SABR call.

We use a large number of sample paths ( $M = 10^6$ ) for the plain Monte Carlo method (without any variance reduction method) to obtain a low standard error. This is a the sanity check to ensure that the CVs do not introduce pricing bias. The number of sample paths we use for Monte Carlo methods with CVs is  $M = 10000$ . There is no stratification applied to the random numbers. The code is written in VBA 6.5 and run on a Dell desktop with Duo core processor.

The rest of this chapter is organised as following. In sections 5.2-5.5, we present empirical correlations as well as efficiency gains for various CVs for the four models that are listed in Table 5.1. A guideline for choosing the best CVs is provided at the end of each section. In section 5.6, we compare the efficiency gains under the

two alternative auxiliary processes for  $\beta \in (0, 1)$  that we discussed in section 4.4. In section 5.7, we investigate the efficiency of correlation CVs under different values of  $\beta$ . In section 5.8, we summarize the improvements of using our new CVs over using the existing ones alone. Section 5.9 concludes.

We find that including our new CVs always gives extra efficiency gains to using standard CVs only.

## 5.2 Valuation in the Heston Model

We describe our discretization method in the pricing model.

**Discretization.** To discretize  $V_t$ , we apply a log-normal approximation suggested by Andersen and Brotherton-Ratcliffe [5]. Given the conditional mean and variance of  $V_t$

$$m = \mathbb{E}(V_{t_{i+1}} | V_{t_i}) = \mu + (V_{t_i} - \mu)e^{-\alpha\Delta t}, \quad (5.4)$$

$$s^2 = \text{Var}(V_{t_{i+1}} | V_{t_i}) = \frac{V_{t_i}\eta^2 e^{-\alpha\Delta t}}{\alpha} (1 - e^{-\alpha\Delta t}) + \frac{\mu\eta^2}{2\alpha} (1 - e^{-\alpha\Delta t})^2, \quad (5.5)$$

the value of  $V_{t_{i+1}}$  can be approximated with a log-normal variable

$$\tilde{V}_{i+1} = m \exp\left(-\frac{1}{2}\Lambda_i^2 + \Lambda_i z_i\right), \quad z_i \sim N(0, 1), \quad (5.6)$$

where  $\Lambda_i = \ln\left(1 + \frac{s^2}{m^2}\right)$  so that  $\tilde{V}_{i+1}$  matches the conditional mean and variance. Although there are exact simulation methods, for example, Broadie and Kaya [17], Glasserman and Kim [37], *et cetera*, they are relatively slow. They can be used in a single step Monte Carlo simulation. But in the case of average rate option whose valuation by Monte Carlo requires many points on a sample path the computation cost of exact simulation is considerable. We therefore apply the following one-step discretization for  $S_t$

$$\tilde{S}_{i+1} = \tilde{S}_i \exp\left[\left(r - \frac{1}{2}\tilde{V}_i^{2\xi}\right)\Delta t + \tilde{V}_i^\xi \Delta W_i^S\right]. \quad (5.7)$$

**Parameters.** We use the following two parameter sets for the Heston model as the base parameter setting, corresponding to the cases when the Feller condition  $d = 2\alpha\mu/\eta^2 < 1$  (Case 1) and  $d > 1$  (Case 2). In Case 1, we use the parameters in the paper by Albrecher *et al.* [2]. In Case 2, parameter values are taken from



Webber [83]. Webber [83] found the efficiency gain from using standard CVs when

	Case 1	Case 2
$S_0 =$	100	100
$r =$	0.0250	0.05
$V_0 =$	0.0175	0.04
$\alpha =$	1.5768	0.20
$\mu =$	0.0398	0.05
$\eta =$	0.5751	0.10
$\rho =$	-0.5711	-0.50
$d =$	0.3795	2

Table 5.2: Heston Parameters

$d < 1$  is much lower than when  $d > 1$ . We show in our results that our proposed CVs have significant improvements in both cases, particularly Case 2. We consider the ATM options, ITM options, OTM options and the cases of highly positive and negative correlations and high volatility.

### The ATM Options

We start by comparing the empirical correlations  $\hat{\rho}$  between various CVs and the discounted payoff of arithmetic average rate options.

**Empirical correlations.** Table 5.3 lists the empirical correlations (in absolute values) achieved by all CVs. We recall that when computation times are identical

		Case 1			Case 2		
CV Type		Av, 64	Av, 16	Av, 4	Av, 64	Av, 16	Av, 4
GBM	Auxiliary	0.73	0.72	0.72	0.991	0.990	0.988
	Delta	0.96	0.96	0.93	0.998	0.996	0.98
Heston	Call	0.83	0.85	0.89	0.84	0.85	0.89
SV	$(\rho = 0)$	0.89	0.89	0.87	0.997	0.997	0.997
	$(\rho \neq 0)$	0.96	0.96	0.96	0.91	0.91	0.92

Table 5.3: Correlations  $\hat{\rho}$ : CVs with Discounted Option Payoffs (ATM) in the Heston Model

a correlation  $\hat{\rho}$  gives a variance reduction of  $\frac{1}{1-\hat{\rho}^2}$  so that  $\hat{\rho} = 0.9$  has a variance reduction of 5 and  $\hat{\rho} = 0.99$  of 20. Two CVs achieve a correlation larger than 0.9 in both cases: the GBM delta CV and non-zero correlation CV. In Case 1, the non-zero correlation CV even has a  $\hat{\rho}$  slightly higher than GBM delta CV and significantly higher than the other CVs. The zero correlation CV is comparable to (or slightly

better than) the Heston European call CV.

In Case 2, the empirical correlations are in general much higher than in Case 1 with the only exception being the non-zero correlation CV, which is unexpected. The top two performers in terms of  $\hat{\rho}$  in this case are the GBM delta CV and zero correlation CV, followed by the GBM auxiliary CV.

Looking at the empirical correlation, the GBM delta CV achieves the highest overall performance. However, it is not the whole picture. The additional work required by the GBM delta CV largely offsets the gain from variance reduction. We see next the efficiency gains of the various CVs.

**Efficiency gains.** Table 5.36 (page 101) gives the efficiency gains of all CVs considered in both Case 1 and Case 2. The second column corresponds to the plain Monte Carlo simulation (with no variance reduction technique). Column 2 to 6 are results for individual CVs. The final column is the result of using our two correlation CVs together. We first look at individual CVs. We notice that although the GBM delta CV significantly reduces the standard error, the computation time required is more than triple that needed for the plain Monte Carlo. This is reflected in the low efficiency gain in Case 1. In this case, our non-zero correlation CV has an efficiency gain at least double the other CVs while the zero correlation CV has a performance similar to the others.

In Case 2 the efficiency gain from using the GBM delta CV is very sensitive to the number of reset dates  $n$ . When  $n = 64$ , it has an efficiency gain of 64.8 and outperforms the GBM auxiliary CV by approximately 10. The efficiency gains of the GBM delta CV decreases significantly as the number of reset dates gets smaller. When  $n = 4$ , it only has an efficiency gain of 8.1. Although both the GBM auxiliary CV and the delta CV achieve high efficiency gains, their efficiency gains are only half of those of the zero correlation CV. It is considerably better than any other CV while the non-zero correlation CV is not as good as expected. Their complementary performances in Case 1 and Case 2 leads us to use both our new CVs in combination in order to come up with a method which applies well in both cases. This indeed works well, particularly in Case 2, we see some extra 30-50 efficiency gains to using SV ( $\rho = 0$ ) alone. Although in Case 1 the efficiency gains are not as large as in Case 2, this still gives a speed-up which is almost five times as standard CVs can do.

## ITM and OTM Options

We also look at the ITM ( $K = 80$ ) and OTM ( $K = 120$ ) cases. Table 5.4 and 5.5 display the empirical correlations. In the ITM case, the delta CV has the highest empirical correlations in Case 1. In Case 2, its correlation decreases as the number of reset dates increases. For options with  $n \leq 16$ , SV ( $\rho = 0$ ) has the highest correlations. In the OTM case, SV ( $\rho \neq 0$ ) works well in Case 1 while other CVs have very low empirical correlations, especially the GBM auxiliary CV. In Case 2, SV ( $\rho = 0$ ) has the highest correlations.

		Case 1			Case 2		
CV Type		Av, 64	Av, 16	Av, 4	Av, 64	Av, 16	Av, 4
GBM	Auxiliary	0.83	0.83	0.82	0.994	0.993	0.992
	Delta	0.9986	0.997	0.98	0.9996	0.9986	0.986
Heston	Call	0.84	0.86	0.89	0.86	0.87	0.91
SV	( $\rho = 0$ )	0.94	0.94	0.93	0.9990	0.9989	0.9986
	( $\rho \neq 0$ )	0.84	0.85	0.85	0.87	0.87	0.88

Table 5.4: Correlations  $\hat{\rho}$ : CVs with Discounted Option Payoffs (ITM) in the Heston Model

		Case 1			Case 2		
CV Type		Av, 64	Av, 16	Av, 4	Av, 64	Av, 16	Av, 4
GBM	Auxiliary	0.02	0.18	0.10	0.96	0.96	0.95
	Delta	0.79	0.89	0.76	0.987	0.98	0.93
Heston	Call	0.56	0.54	0.72	0.68	0.69	0.81
SV	( $\rho = 0$ )	0.67	0.74	0.70	0.996	0.996	0.995
	( $\rho \neq 0$ )	0.991	0.990	0.990	0.98	0.98	0.98

Table 5.5: Correlations  $\hat{\rho}$ : CVs with Discounted Option Payoffs (OTM) in the Heston Model

Table 5.37 (page 102) gives efficiency gains for ITM options. In the ITM case in Case 1, our proposed CVs are much worse than the delta CV but comparable to other CVs. The more reset dates, the more efficient the delta CV is. With 64 reset dates, its efficiency gain is over 100 while our proposed CVs only achieve efficiency gains of 3-7. This pattern of the delta CV is somewhat repeated in the ITM case in Case 2 although our proposed CVs are much more efficient than in Case 1. With 4 and 16 reset dates, SV ( $\rho = 0$ ) largely outperforms the delta CV, though using both SV ( $\rho = 0$ ) and ( $\rho \neq 0$ ) gives only marginal improvement to using SV ( $\rho = 0$ ) along. With 64 reset dates, the delta CV overtakes SV ( $\rho = 0$ ) but its edge is not big.

Table 5.38 (page 103) presents the results for OTM options. The OTM case is similar to the ATM case. In Case 1, SV ( $\rho \neq 0$ ) is much better than the standard CVs. The delta CV this time proves to be too costly. It is barely more efficient than the plain Monte Carlo method. With 4 reset dates, its efficiency gain is only 0.8. In Case 2, SV ( $\rho = 0$ ) is significantly better than other CVs and it combines well with SV ( $\rho \neq 0$ ). When using SV ( $\rho = 0$ ) and ( $\rho \neq 0$ ) together, we see efficiency gains around 130.

### Highly Positive and Negative $\rho$

We now let the correlation  $\rho$  to be 0.9 or  $-0.9$  in both Case 1 and Case 2. Our intention is to examine our methods in the extreme correlation cases in which our methods are expected to be less efficient. The empirical correlations are given in Table 5.6 and 5.7. In both extreme correlation cases, the delta CV provides much

		Case 1 ( $\rho = 0.9$ )			Case 2 ( $\rho = 0.9$ )		
CV Type		Av, 64	Av, 16	Av, 4	Av, 64	Av, 16	Av, 4
GBM	Auxiliary	0.72	0.83	0.82	0.992	0.992	0.991
	Delta	0.993	0.991	0.97	0.999	0.997	0.98
Heston	Call	0.82	0.82	0.90	0.84	0.85	0.90
SV	( $\rho = 0$ )	0.80	0.80	0.79	0.990	0.991	0.990
	( $\rho \neq 0$ )	0.43	0.39	0.37	0.48	0.46	0.44

Table 5.6: Correlations  $\hat{\rho}$ : CVs with Discounted Option Payoffs ( $\rho = 0.9$ ) in the Heston Model

		Case 1 ( $\rho = -0.9$ )			Case 2 ( $\rho = -0.9$ )		
CV Type		Av, 64	Av, 16	Av, 4	Av, 64	Av, 16	Av, 4
GBM	Auxiliary	0.73	0.74	0.73	0.991	0.992	0.990
	Delta	0.97	0.97	0.94	0.998	0.997	0.98
Heston	Call	0.82	0.84	0.89	0.84	0.85	0.89
SV	( $\rho = 0$ )	0.74	0.73	0.72	0.991	0.992	0.989
	( $\rho \neq 0$ )	0.67	0.67	0.67	0.51	0.52	0.51

Table 5.7: Correlations  $\hat{\rho}$ : CVs with Discounted Option Payoffs ( $\rho = -0.9$ ) in the Heston Model

higher empirical correlations than other CVs. In Case 2, all correlations are in general higher than Case 1 (except for those of SV ( $\rho \neq 0$ )). The delta CV still has the highest correlations while SV ( $\rho = 0$ ) and the GBM auxiliary CV give high and similar correlations.

The efficiency gains for  $\rho = 0.9$  and  $\rho = -0.9$  are presented in Table 5.39 (page 104) and 5.40 (page 105). No CV has significant efficiency gains in Case 1. In Case 2, the delta CV has very high efficiency gains when the number of reset dates is above 16. SV ( $\rho \neq 0$ ) barely has any advantage over the standard CVs as we have explained in section 4.6.3. However, SV ( $\rho = 0$ ) has good performance although it is not as good as the delta CV in the case of 16 and 64 reset dates. It greatly reduces the variance from the Wiener process driving the asset process even though the auxiliary process is less related to the pricing model than for SV ( $\rho \neq 0$ ).

### High Volatility

We test our methods when volatility is high. In both Case 1 and Case 2, we set  $V_0 = \mu = 0.25$ . Since  $d$  increases with  $\mu$  (from 0.38 to 2.38 in Case 1 and from 2 to 10 in Case 2), it is no longer less than one in Case 1.

The empirical correlations are displayed in Table 5.8. They are generally higher

		Case 1 ( $V_0 = \mu = 0.25$ )			Case 2 ( $V_0 = \mu = 0.25$ )		
CV Type		Av, 64	Av, 16	Av, 4	Av, 64	Av, 16	Av, 4
GBM	Auxiliary	0.95	0.96	0.95	0.996	0.996	0.996
	Delta	0.994	0.992	0.97	0.997	0.995	0.98
Heston	Call	0.84	0.86	0.90	0.84	0.85	0.89
SV	( $\rho = 0$ )	0.985	0.98	0.98	0.998	0.998	0.998
	( $\rho \neq 0$ )	0.91	0.91	0.91	0.91	0.91	0.91

Table 5.8: Correlations  $\hat{\rho}$ : CVs with Discounted Option Payoffs ( $V_0 = \mu = 0.25$ ) in the Heston Model

than those for the ATM case with lower volatilities in Table 5.3. The empirical correlation of the delta CV is decreasing with the number of reset dates, as is seen in all other cases. SV ( $\rho = 0$ ) has the highest correlations in Case 2 and for options with  $n = 4$  in Case 1. In other places, the delta CV gives the highest empirical correlations.

The efficiency gains are shown in Table 5.41 (page 106). As is seen in the ATM case in Table 5.36 (page 101), when  $d$  is high the GBM auxiliary CV and SV ( $\rho = 0$ ) work well while SV ( $\rho \neq 0$ ) is less efficient. SV ( $\rho = 0$ ) is more efficient than the GBM auxiliary CV in all options. The efficiency gain of the delta CV increases with the number of resets in both cases.

Practitioners in a quant group are interested in how to automate the Monte Carlo valuation system so that it can choose suitable CVs without needing them to be specified. In Table 5.9, we summarise the best individual CVs in the Heston model in different cases. In the table, “both” refers to SV ( $\rho = 0$ )+SV ( $\rho \neq 0$ ), “GBM” stands for the GBM auxiliary CV. We have considered only individual CVs. Using

	$d < 1$		$d > 1$	
Reset dates	$n \leq 16$	$n > 16$	$n \leq 16$	$n > 16$
ATM	both			
ITM	delta		both	delta
OTM	SV ( $\rho = 0$ )		both	
High $ \rho $	delta		GBM	delta
High Vol	both		SV ( $\rho = 0$ )	

Table 5.9: Best Individual CVs in the Heston Model

CVs together may provide additional efficiency gains. We discuss combination in the next section.

### Combinations of Correlation CVs with Standard Ones

The results for ATM options are given in Table 5.42 (page 107). In Case 1, using all CVs together enhances the performance, beating the highest efficiency gains in Table 5.36 (page 101). Using the auxiliary CV with the other CVs is not too much different from using each CV individually. In Case 2, combining CVs with each other is not significantly more efficient than using them individually.

Table 5.43 (page 108) presents the results in the ITM case. In Case 1, our proposed CVs do not perform well on their own. However, using them with the delta CV gives great efficiency gains. The largest improvement on individual CVs reaches a factor over 20. When using all CVs together, the efficiency gains are even higher. The situation is even better in Case 2 where our proposed CVs work well on their own. There is a very impressive efficiency gain of 897.4 for average rate option with 64 reset dates when using all CVs together.

The results for the OTM case are given in Table 5.44 (page 109). In the OTM case, our proposed CVs by themselves outperform standard CVs, while using them with the delta CV makes things worse and using them with other standard CVs does not make too much difference.

The results when  $\rho = 0.9$  are displayed in Table 5.45 (page 110). As we have explained in section 4.6.3, our proposed CVs do not have an advantage over standard CVs in this case. However, combining them with all the other standard CVs makes a big difference. This is the case in both Case 1 and Case 2. Efficiency gains for  $\rho = -0.9$  in Table 5.46 (page 111) are very similar to when  $\rho = 0.9$ .

The results for high volatility are presented in Table 5.47 (page 112). In Case 1, combining the delta CV with SV ( $\rho = 0$ ) and ( $\rho \neq 0$ ) slightly enhances the performance, and using all CVs together is even better. In case 2, on the other hand, using delta CV with our proposed ones reduces the efficiency gain.

In summary, we find that in Case 1, except for the OTM case, the combination of all CVs has the top performance. In the OTM case, the best combinations are the GBM auxiliary CV plus SV ( $\rho \neq 0$ ) in Case 1 and the GBM auxiliary CV with SV ( $\rho = 0$ ) and ( $\rho \neq 0$ ) in Case 2. They both include SV ( $\rho \neq 0$ ) which has high efficiency gains individually. Due to the low efficiency gains of the delta CV in the OTM case, it does not combine well with any other CV.

In Case 2, we suggest using the GBM auxiliary CV with SV ( $\rho = 0$ ) and ( $\rho \neq 0$ ) for options with small number of reset dates (4 or 16). For options with 64 reset dates, the best CV depends on the model parameters. It can either be the combination of all CVs or just the GBM auxiliary CV with SV ( $\rho = 0$ ).

### 5.3 Valuation in the SABR Model

Unlike in the Heston model where the variance follows a mean-reverting process, the SABR model considers volatility itself as log-normal.

**Discretization.** To discretize the SABR model, we first use the transformation  $Y_t = \frac{1}{1-\beta} F_t^{1-\beta}$  and  $v_t = \ln V_t$ .

$$dY_t = -\frac{\beta}{2(1-\beta)Y_t} e^{2v_t} dt + e^{v_t} dW_t^S, \quad (5.8)$$

$$dv_t = -\frac{1}{2}\eta^2 dt + \eta dW_t^V, \quad dW_t^S dW_t^V = \rho \quad (5.9)$$

We then apply the Euler scheme to the two dimensional process  $(Y_t, v_t)'$ . The discretization of the transformed process (5.8) can result in  $Y_t$  attaining zero or negative values. In fact, the CEV-type process has an attainable boundary at zero

for  $0 < \beta < 1$ . We fix the Euler scheme for (5.8) by assigning an absorbing boundary at zero. Jäckel [47] claims this yields a good approximation of the probability of absorption at zero for the CEV process for  $\beta < \frac{1}{2}$ . Finally, we recover  $\tilde{F}_t = \left[ (1 - \beta) \tilde{Y}_t \right]^{\frac{1}{1-\beta}}$  and  $\tilde{S}_t = \tilde{F}_t e^{r(t-T)}$ .

**Parameters.** We use the following parameters for the SABR model as our base case. The initial stock price  $S_0 = 100$ , risk free interest rate  $r = 0.05$ , initial volatility  $V_0 = 2$ , the correlation between asset and variance processes is  $\rho = -0.5$ , the volatility of volatility  $\eta = 0.4$ . Since 0.5 is the critical value of  $\beta$  below which the solution to the CEV process is not unique unless a boundary condition at zero is specified, we consider two cases,  $\beta = 0.4$  (Case 1) and  $\beta = 0.6$  (Case 2). The implied volatilities from these two cases can be computed using the expansion given by Hagan *et al.* [41]. Figure 5.1 shows the implied volatility skews for both cases. In Case 1, the level of the implied volatility is around 0.15. It is seen in foreign

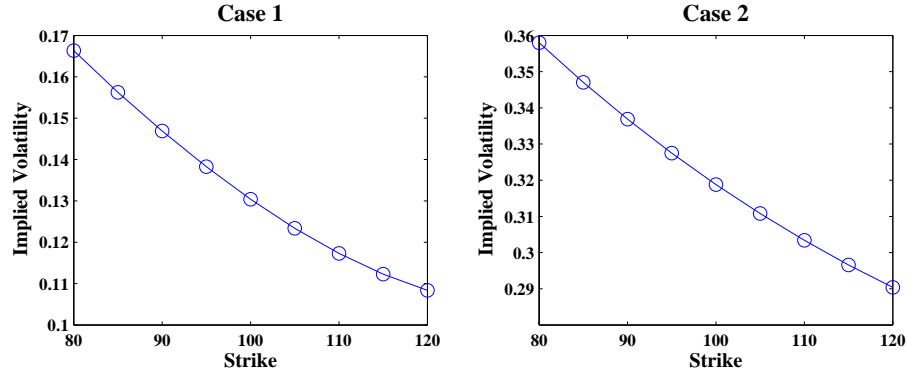


Figure 5.1: Implied Volatility Skews in the SABR Model

exchange options. Case 2 has an implied volatility around 0.3. This can be seen in equity options.

We compare five CVs, namely, the GBM auxiliary CV, the GBM delta CV, the SABR call CV and SV ( $\rho = 0$ ) and SV ( $\rho \neq 0$ ).

### The ATM Options

As before, we first compare the empirical correlations  $\hat{\rho}$  between the CVs and discounted option payoffs. Table 5.10 contains the empirical correlations of the CVs with different option payoffs in the SABR model. Unlike in the Heston model, results are fairly consistent between the different sets of parameters. In both Cases,



		Case 1			Case 2		
CV Type		Av, 64	Av, 16	Av, 4	Av, 64	Av, 16	Av, 4
GBM	Auxiliary	0.98	0.98	0.97	0.97	0.97	0.96
	Delta	0.995	0.994	0.98	0.994	0.992	0.98
SABR	Call	0.84	0.85	0.89	0.84	0.84	0.89
SV	$(\rho = 0)$	0.993	0.993	0.992	0.992	0.992	0.990
	$(\rho \neq 0)$	0.92	0.92	0.92	0.93	0.93	0.93

Table 5.10: Correlations  $\hat{\rho}$ : CVs with Discounted Option Payoffs in the SABR Model (ATM)

the delta CV and SV ( $\rho = 0$ ) have very high empirical correlations. Those of SV ( $\rho = 0$ ) are roughly at the same level for options with different number of reset dates while those of the delta CV increase with the number of reset dates, particularly from 4 to 16 reset dates. The GBM auxiliary CV also has very high correlations but not as high as those of the delta CV and SV ( $\rho = 0$ ). The possible reason why the empirical correlations of SV ( $\rho \neq 0$ ) are not as high as those of SV ( $\rho = 0$ ) is that the way the  $\beta$ -average rate option is priced only reduces the standard error coming from the Wiener path driving the asset process,  $W_t^S$ , as we have explained in section 4.6.3. Building in correlation between the asset and variance processes reduces the proportion of the option price explained by  $W_t^S$  along.

Table 5.48 (page 113) lists the efficiency gains from using different CVs. We see that SV ( $\rho = 0$ ) performs much better than the other CVs by at least a factor of 2 and it combines well with SV ( $\rho \neq 0$ ). Using them together gains an extra amount of efficiency of around 20. The GBM auxiliary CV comes second because it has high empirical correlations and is cheap to use. The delta CV has good efficiency gains for options with high reset frequency but is not so good for options with 4 reset dates.

### ITM and OTM Options

We then look at the performance of the CVs in the ITM ( $K = 80$ ) and OTM ( $K = 120$ ) cases. Empirical correlations are listed in Table 5.11 and 5.12. In the ITM case, the delta CV has the highest empirical correlations, followed by SV ( $\rho = 0$ ). The auxiliary CV has all empirical correlations around 0.98 which is consistent with the ATM case in Table 5.10. SV ( $\rho \neq 0$ ) has similar performance to the SABR call CV. Their correlations are all around 0.9. In the OTM case, SV ( $\rho \neq 0$ ) has much higher correlations than the other CVs in Case 1, but in Case 2 has lower

		Case 1			Case 2		
CV Type		Av, 64	Av, 16	Av, 4	Av, 64	Av, 16	Av, 4
GBM	Auxiliary	0.98	0.98	0.98	0.98	0.98	0.98
	Delta	0.9997	0.9988	0.987	0.998	0.997	0.98
SABR	Call	0.86	0.88	0.91	0.85	0.87	0.90
SV	$(\rho = 0)$	0.995	0.996	0.995	0.997	0.996	0.994
	$(\rho \neq 0)$	0.86	0.87	0.87	0.89	0.89	0.89

Table 5.11: Correlations  $\hat{\rho}$ : CVs with Discounted Option Payoffs in the SABR Model (ITM)

		Case 1			Case 2		
CV Type		Av, 64	Av, 16	Av, 4	Av, 64	Av, 16	Av, 4
GBM	Auxiliary	0.60	0.53	0.72	0.93	0.93	0.93
	Delta	0.95	0.95	0.79	0.98	0.98	0.94
SABR	Call	0.37	0.39	0.59	0.74	0.77	0.84
SV	$(\rho = 0)$	0.92	0.95	0.97	0.989	0.988	0.987
	$(\rho \neq 0)$	0.997	0.998	0.997	0.98	0.97	0.97

Table 5.12: Correlations  $\hat{\rho}$ : CVs with Discounted Option Payoffs in the SABR Model (OTM)

correlations than SV ( $\rho = 0$ ) and similar ones to those of the delta CV. Gains for the auxiliary CV and SABR call CV are very low in Case 1, although pick up in Case 2.

Efficiency gains for the ITM options are given in Table 5.49 (page 114). In the ITM case, the efficiency gain of the delta CV increases sharply with the number of reset dates. It reaches over 450 for options with 64 reset dates in Case 1 and is more than 5 times as high as that of the second performer, SV ( $\rho = 0$ ). However, for options with 4 reset dates, the efficiency gain of the delta CV is only 1/7 of that of SV ( $\rho = 0$ ). In Case 2, SV ( $\rho = 0$ ) has higher performance than all other CVs for all options. In both cases, the auxiliary CV has efficiency gains all around 20. Combining SV ( $\rho = 0$ ) with SV ( $\rho \neq 0$ ) does not give higher efficiency gains than using SV ( $\rho = 0$ ) along.

Efficiency gains for OTM options are presented in Table 5.50 (page 115). In Case 1 of the OTM case, SV ( $\rho \neq 0$ ) obtains efficiency gains over 100 and is much better than the other CVs. This is consistent with the OTM options in the Heston model in Table 5.38. SV ( $\rho = 0$ ) is better than standard CVs although not as good as SV ( $\rho \neq 0$ ). In Case 2, SV ( $\rho = 0$ ) performs better than SV ( $\rho \neq 0$ ) and both are

much better than standard CVs. In both Case 1 and 2, using them together gains an additional efficiency of over 30 and is very worthwhile. The delta CV, unlike in the ITM case, does not have high efficiency gains. All its gains are below 10.

### Highly Positive and Negative $\rho$

Table 5.13 and 5.14 display the empirical correlations in the cases when  $\rho = 0.9$  and  $\rho = -0.9$ . Recall that in both cases, our proposed methods are not expected to work well. In both tables, the delta CV gives the highest empirical correlation.

		Case 1 ( $\rho = 0.9$ )			Case 2 ( $\rho = 0.9$ )		
CV Type		Av, 64	Av, 16	Av, 4	Av, 64	Av, 16	Av, 4
GBM	Auxiliary	0.98	0.986	0.98	0.98	0.98	0.98
	Delta	0.9987	0.998	0.98	0.997	0.996	0.98
SABR	Call	0.84	0.87	0.90	0.83	0.84	0.89
SV	( $\rho = 0$ )	0.98	0.98	0.98	0.97	0.97	0.97
	( $\rho \neq 0$ )	0.45	0.45	0.42	0.46	0.44	0.45

Table 5.13: Correlations  $\hat{\rho}$ : CVs with Discounted Option Payoffs in the SABR Model ( $\rho = 0.9$ )

		Case 1 ( $\rho = -0.9$ )			Case 2 ( $\rho = -0.9$ )		
CV Type		Av, 64	Av, 16	Av, 4	Av, 64	Av, 16	Av, 4
GBM	Auxiliary	0.98	0.98	0.98	0.97	0.97	0.97
	Delta	0.997	0.996	0.98	0.997	0.995	0.98
SABR	Call	0.85	0.85	0.89	0.84	0.85	0.89
SV	( $\rho = 0$ )	0.98	0.98	0.98	0.98	0.98	0.97
	( $\rho \neq 0$ )	0.50	0.52	0.51	0.53	0.55	0.53

Table 5.14: Correlations  $\hat{\rho}$ : CVs with Discounted Option Payoffs in the SABR Model ( $\rho = -0.9$ )

SV ( $\rho = 0$ ) is similar to the auxiliary CV. Both have high correlations. SV ( $\rho \neq 0$ ) performs badly. None of its empirical correlations are above 0.5 (for the reason we explained in section 4.6.3).

Efficiency gains are listed in Table 5.51 (page 116) for  $\rho = 0.9$  and Table 5.52 (page 117) for  $\rho = -0.9$ . In both tables, the delta CV performs better as the number of option reset dates increases, which is consistent with all previous cases. It has higher efficiency gains than the other CVs in all cases except for options with 4 reset dates. SV ( $\rho = 0$ ) has very similar performance to the auxiliary CV. This is because they have similar empirical correlations and have similar computation cost.

SV ( $\rho \neq 0$ ), however, barely has any variance reduction effect as its efficiency gains are all around 1.0 (the same as the plain Monte Carlo method). Combining them drags down the performance of SV ( $\rho = 0$ ) slightly.

### High Volatility

We set  $V_0 = 8$  in Case 1 and  $V_0 = 3$  in Case 2. This gives a model implied volatility of around 0.5 both cases. In practice, it is very rare that the volatility is as high as 0.5 but we include this case for completeness. To avoid numerical instability caused by too many terms in the series expansion, we set the number of terms equal to 10. This does not affect the result as we find that the numerical value has converged at this level. Empirical correlations are given in Table 5.15. The empirical correlations

		Case 1 ( $V_0 = 8$ )			Case 2 ( $V_0 = 3$ )		
CV Type		Av, 64	Av, 16	Av, 4	Av, 64	Av, 16	Av, 4
GBM	Auxiliary	0.95	0.95	0.95	0.96	0.96	0.95
	Delta	0.993	0.991	0.97	0.993	0.992	0.97
SABR	Call	0.84	0.85	0.89	0.84	0.85	0.89
SV	( $\rho = 0$ )	0.991	0.991	0.990	0.991	0.991	0.989
	( $\rho \neq 0$ )	0.93	0.93	0.93	0.93	0.93	0.93

Table 5.15: Correlations  $\hat{\rho}$ : CVs with Discounted Option Payoffs in the SABR Model (High Volatility)

of the delta CV are close to those of SV ( $\rho = 0$ ) except that the former are lower for options with 4 reset dates. Both the delta CV and SV ( $\rho = 0$ ) have much higher than the correlations of the other CVs.

Efficiency gains are given in Table 5.53 (page 118). Those of SV ( $\rho = 0$ ) are much higher than the other CVs. Although the delta CV has similar empirical correlations to SV ( $\rho = 0$ ), its benefit is offset by its high computation cost. Combining SV ( $\rho = 0$ ) and ( $\rho \neq 0$ ) gives a further efficiency gain to more than 10.

In Table 5.16, we give a summary of best CVs individually for different cases in the SABR model.

We next look at the combination of our proposed CVs with standard ones.

### Combinations of Correlation CVs with Standard Ones

Table 5.54 (page 119) gives the results for the base case (ATM). Using the auxiliary CV with SV ( $\rho = 0$ ) and ( $\rho \neq 0$ ) slightly increases the efficiency gains in Case 1 from

	$\beta < 0.5$		$\beta > 0.5$	
Reset dates	$n < 16$	$n \geq 16$	$n < 16$	$n \geq 16$
ATM	both			
ITM	SV ( $\rho = 0$ )	delta	SV ( $\rho = 0$ )	both
OTM	both			
High $ \rho $	GBM	delta	GBM	delta
High Vol	both			

Table 5.16: Best Individual CVs in the SABR Model

Table 5.48 but does not change those in Case 2 too much. Although SV ( $\rho = 0$ ) does not perform as well as SV ( $\rho = 0$ ) in the individual case, it combines better with the delta CV than SV ( $\rho = 0$ ) does for options with high reset frequency ( $n \geq 16$ ) in which case the best combination is all CVs together. For options with the number of reset dates less than 4, the best combination is the auxiliary CV with SV ( $\rho = 0$ ) and ( $\rho \neq 0$ ).

Table 5.55 (page 120) displays results for the ITM options. For options with more than 16 reset dates, the delta CV combines much better with the other CVs than the auxiliary CV. For those with 4 reset dates, the best combinations are the auxiliary CV with SV ( $\rho = 0$ ) in Case 1 and that with SV ( $\rho = 0$ ) and ( $\rho \neq 0$ ) in Case 2. For options with 16 or 64 reset dates, the highest efficiency gains come from using all CVs together. Efficiency gain of over 1000 is seen.

Table 5.56 (page 121) lists the efficiency gains of different combinations in the OTM case. The best combination is the auxiliary CV plus SV ( $\rho = 0$ ) and ( $\rho \neq 0$ ). But efficiency gains are not very much different from (if not higher than) those of SV ( $\rho = 0$ ) and ( $\rho \neq 0$ ) together (in Table 5.50 (page 115)). Adding the auxiliary CV makes little difference. The delta CV does not combine as well with other CVs as the auxiliary CV does.

Table 5.57 (page 122) gives the results for highly positive correlation. Combining the auxiliary CV with the new CVs does not give a significant improvement to the efficiency gains. However, in combination with the SABR call CV it provides efficiency gains higher than if using either of them along. For options with high reset frequency ( $n \geq 16$ ), the highest efficiency gains are obtained by combining all CVs together. This phenomenon is consistent with the ATM and ITM cases in Table 5.54 (page 119) and 5.55 (page 120). The reason is that the main computation cost of the combination comes from the delta CV. It does very well in options with many

reset dates, and enhances the overall performance, but it does poorly in options with few reset dates and drags down the efficiency. In the cases of few reset dates, using the auxiliary CV and SABR call CV together gives the highest efficiency gains when  $\rho = 0.9$ .

The results of highly negative correlations are provided in Table 5.58 (page 123). In this case, results are very similar to the case when  $\rho = 0.9$  although the efficiency gains are slightly lower in general.

Table 5.59 (page 124) gives efficiency gains for the cases with high volatility. The best combination overall is the auxiliary CV with SV ( $\rho = 0$ ) and ( $\rho \neq 0$ ). However, the efficiency gains are not very different from those of SV ( $\rho = 0$ ) + ( $\rho \neq 0$ ) in Table 5.53 (page 118) in Case 1. Using all CVs together has a slight edge over the combination of the auxiliary CV with SV ( $\rho = 0$ ) and ( $\rho \neq 0$ ) but this is likely to be due to the noise in different simulations. But the combination of all CVs performs much worse than the auxiliary CV+SV ( $\rho = 0$ )+SV ( $\rho \neq 0$ ) for options with 4 reset dates.

In Table 5.17, we summarise the best CV combinations for the SABR model. In the table, “all” stands for the combination of all CVs, “call” stands for the SABR call CV.

Reset Dates	$n < 16$	$n \geq 16$
ATM	GBM+both	all
ITM	GBM+SV ( $\rho = 0$ )	all
OTM	GBM+both	
High $ \rho $	GBM+call	all
High Vol	GBM+both	all

Table 5.17: Best CV Combinations for the SABR Model

## 5.4 Valuation in the GARCH Diffusion Model

The GARCH diffusion process ensures that the variance is always positive.

**Discretization.** We discretize the GARCH process with a Milstein scheme (Kloeden and Platen [55]). The simple Euler method for the GARCH diffusion process is

$$\tilde{V}_{i+1} = \tilde{V}_i + \alpha \left( \mu - \tilde{V}_i \right) \Delta t + \eta \tilde{V}_i \sqrt{\Delta t} z_i, \quad z_i \sim N(0, 1). \quad (5.10)$$

The Euler discretization, however, can yield a negative variance. This happens when  $\tilde{V}_i$  is close to zero and  $z_i$  is highly negative. We argue that a Milstein scheme can eliminate the negative variance. The Milstein discretization is

$$\begin{aligned}\tilde{V}_{i+1} &= \tilde{V}_i + \alpha (\mu - \tilde{V}_i) \Delta t + \eta \tilde{V}_i \sqrt{\Delta t} z_i + \frac{1}{2} \eta^2 \Delta t \tilde{V}_i (z_i^2 - 1) \\ &= \tilde{V}_i \left[ \frac{1}{2} \left( 1 + \eta \sqrt{\Delta t} Z_i \right)^2 + \frac{1}{2} - \Delta t \left( \alpha + \frac{1}{2} \eta^2 \right) \right].\end{aligned}\quad (5.11)$$

When  $\tilde{V}_i$  is small (but positive), as long as  $\Delta t (2\alpha + \eta^2) < 1$ , the positivity of  $\tilde{V}_{i+1}$  is guaranteed. This condition holds in our parameter specifications. We notice that Barone-Adesi *et al.* [9] also use Milstein method for  $V_t$  in the GARCH diffusion model.

**Parameters.** We use two parameter specifications for  $\alpha$ ,  $\mu$  and  $\eta$ , from Lewis [56], given in Table 5.18. The values of  $S_0$ ,  $r$  and  $V_0$  are the same as those used in Case 1 and 2 in the Heston example. Implied volatility skews are computed from the

	Case 1	Case 2
$S_0 =$	100	100
$r =$	0.025	0.05
$V_0 =$	0.0175	0.04
$\alpha =$	4	2
$\mu =$	0.0225	0.09
$\eta =$	1.2	0.8
$\rho =$	-0.5	-0.5

Table 5.18: GARCH Parameters

European call option prices estimated by Monte Carlo simulation. The purpose is to justify that the parameter choices are reasonable in practice. The implied volatility skews are plotted in Figure 5.2. As in the SABR example, Case 1 (the lower implied volatility case) features the implied volatility of foreign exchange options and Case 2 (the higher implied volatility case) of equity options. Since there is no explicit European option value in the GARCH model, the call CV is not used. We use a large number of sample paths ( $M = 10^6$ ) for the plain Monte Carlo method to obtain a benchmark result. For the CVs,  $M = 10000$ . In the base case we price average rate options with strike  $K = 100$ .

### The ATM Options

Empirical correlations are given in Table 5.19. The auxiliary CV, SV ( $\rho = 0$ ) and

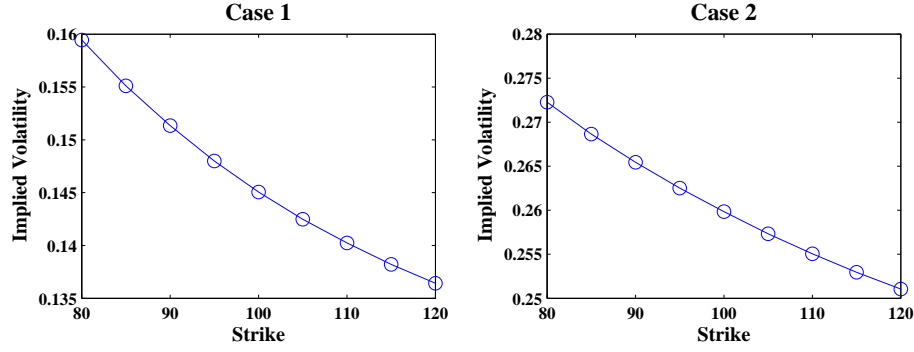


Figure 5.2: Implied Volatility Skews in the GARCH Diffusion Model

		Case 1			Case 2		
CV Type		Av, 64	Av, 16	Av, 4	Av, 64	Av, 16	Av, 4
GBM	Auxiliary	0.98	0.98	0.98	0.98	0.98	0.98
	Delta	0.997	0.995	0.98	0.997	0.995	0.97
SV	$(\rho = 0)$	0.994	0.994	0.994	0.997	0.997	0.996
	$(\rho \neq 0)$	0.92	0.92	0.92	0.91	0.91	0.91

Table 5.19: Correlations  $\hat{\rho}$ : CVs with Discounted Option Payoffs (ATM) in the GARCH Model

the delta CV all have high correlations. SV ( $\rho = 0$ ), in particular, has all correlations above 0.99.

Table 5.60 (page 125) lists efficiency gains for the different CVs. SV ( $\rho = 0$ ) gives the highest efficiency gain. The auxiliary CV also has good performance but not as good as SV ( $\rho = 0$ ). Using SV ( $\rho = 0$ ) and ( $\rho \neq 0$ ) is very much the same with using SV ( $\rho = 0$ ) alone, though there are slight improvements in options with 16 and 64 reset dates.

### ITM and OTM Options

The empirical correlations for ITM options ( $K = 80$ ) and OTM options ( $K = 120$ ) are given in Table 5.20 and 5.21. In the ITM case, we notice that both the delta CV and SV ( $\rho = 0$ ) have extremely high correlations. With 64 reset dates, the delta CV has empirical correlations above 0.999. SV ( $\rho \neq 0$ ) does not have very outstanding performance. Its empirical correlations are below 0.9. In the OTM case, SV ( $\rho = 0$ ) and SV ( $\rho \neq 0$ ) are both highly correlated with the option payoffs. The former performs better in Case 1 while the latter does better in Case 2.



		Case 1			Case 2		
CV Type		Av, 64	Av, 16	Av, 4	Av, 64	Av, 16	Av, 4
GBM	Auxiliary	0.987	0.986	0.986	0.987	0.987	0.986
	Delta	0.9997	0.9989	0.986	0.9990	0.998	0.98
SV	$(\rho = 0)$	0.997	0.997	0.997	0.9988	0.9987	0.998
	$(\rho \neq 0)$	0.86	0.87	0.87	0.87	0.87	0.88

Table 5.20: Correlations  $\hat{\rho}$ : CVs with Discounted Option Payoffs (ITM) in the GARCH Model

		Case 1			Case 2		
CV Type		Av, 64	Av, 16	Av, 4	Av, 64	Av, 16	Av, 4
GBM	Auxiliary	0.84	0.85	0.88	0.92	0.94	0.94
	Delta	0.97	0.95	0.85	0.99	0.98	0.93
SV	$(\rho = 0)$	0.98	0.98	0.98	0.994	0.994	0.994
	$(\rho \neq 0)$	0.991	0.990	0.990	0.97	0.97	0.96

Table 5.21: Correlations  $\hat{\rho}$ : CVs with Discounted Option Payoffs (OTM) in the GARCH Model

Efficiency gains for ITM options are listed in Table 5.61 (page 126). In the ITM case, one extraordinarily high efficiency gain (above 500) appears in the delta CV for the option with 64 reset dates in Case 1. Except for that, SV ( $\rho = 0$ ) is the best CV. This is consistent with the cases of  $\rho = 0.9$  and  $\rho = -0.9$  where the delta CV has higher efficiency gains as option reset dates increase. SV ( $\rho = 0$ ) plus ( $\rho \neq 0$ ) is worse than SV ( $\rho = 0$ ) alone because of the not-so-good individual performance of SV ( $\rho \neq 0$ ).

The efficiency gains for OTM options are given in Table 5.62 (page 127). In the OTM case, both SV ( $\rho = 0$ ) and SV ( $\rho \neq 0$ ) have high efficiency gains. Using both together gives a big boost in Case 1 and a slight improvements in Case 2 (except for the option with 16 reset dates). The delta CV, however, is less efficient than in the ITM case. The only place that its efficiency gain exceeds 10 is the option with 64 reset dates in Case 2.

### Highly Positive and Negative $\rho$

Table 5.22 and 5.23 give empirical correlations when  $\rho$  is set to 0.9 and  $-0.9$ . In both cases, SV ( $\rho \neq 0$ ) has very low empirical correlations (around 0.5). This means it almost has no variance reduction effect. SV ( $\rho = 0$ ) has similar correlations to the GBM auxiliary CV. However, they both have lower correlations than the delta

		Case 1 ( $\rho = 0.9$ )			Case 2 ( $\rho = 0.9$ )		
CV Type		Av, 64	Av, 16	Av, 4	Av, 64	Av, 16	Av, 4
GBM	Auxiliary	0.97	0.98	0.98	0.97	0.97	0.96
	Delta	0.998	0.997	0.98	0.997	0.995	0.97
SV	( $\rho = 0$ )	0.98	0.98	0.98	0.98	0.98	0.98
	( $\rho \neq 0$ )	0.47	0.47	0.47	0.44	0.46	0.45

Table 5.22: Correlations  $\hat{\rho}$ : CVs with Discounted Option Payoffs ( $\rho = 0.9$ ) in the GARCH Model

		Case 1 ( $\rho = -0.9$ )			Case 2 ( $\rho = -0.9$ )		
CV Type		Av, 64	Av, 16	Av, 4	Av, 64	Av, 16	Av, 4
GBM	Auxiliary	0.987	0.987	0.986	0.989	0.988	0.988
	Delta	0.998	0.996	0.98	0.998	0.996	0.98
SV	( $\rho = 0$ )	0.98	0.98	0.98	0.990	0.989	0.989
	( $\rho \neq 0$ )	0.53	0.52	0.51	0.51	0.50	0.52

Table 5.23: Correlations  $\hat{\rho}$ : CVs with Discounted Option Payoffs ( $\rho = -0.9$ ) in the GARCH Model

CV.

Efficiency gains are displayed in Table 5.63 (page 128) for  $\rho = 0.9$  and Table 5.64 (page 129) for  $\rho = -0.9$ . When  $\rho = 0.9$ , the delta CV has the highest efficiency gains when there are 16 and 64 reset dates. SV ( $\rho = 0$ ) also has good performance and is better than the GBM auxiliary CV, while the latter is better when  $\rho = -0.9$ . However, in none of the cases does SV ( $\rho \neq 0$ ) achieve an efficiency gain above 1, which indicates it does not work when  $\rho$  is highly positive or negative.

### High Volatility

As in the Heston example, we set  $V_0 = \mu = 0.25$  in both cases. This gives values of the volatility path at around 0.5. The empirical correlations are in Table 5.24. SV ( $\rho = 0$ ) has all empirical correlations above 0.99. Those of the delta CV is very similar to SV ( $\rho = 0$ ) for options with 64 and 16 reset dates but lower in the case of 4 reset dates.

Efficiency gains are listed in Table 5.65 (page 130). The best CV is SV ( $\rho = 0$ ). It outperforms the delta CV by more than 30 points of efficiency gain. Using SV ( $\rho = 0$ ) and ( $\rho \neq 0$ ) is almost the same as using SV ( $\rho = 0$ ) alone.

		Case 1 ( $V_0 = \mu = 0.25$ )			Case 2 ( $V_0 = \mu = 0.25$ )		
CV Type		Av, 64	Av, 16	Av, 4	Av, 64	Av, 16	Av, 4
GBM	Auxiliary	0.98	0.98	0.98	0.98	0.98	0.98
	Delta	0.995	0.994	0.97	0.996	0.995	0.98
SV	$(\rho = 0)$	0.993	0.993	0.991	0.995	0.995	0.994
	$(\rho \neq 0)$	0.92	0.92	0.93	0.92	0.93	0.92

Table 5.24: Correlations  $\hat{\rho}$ : CVs with Discounted Option Payoffs ( $V_0 = \mu = 0.25$ ) in the GARCH Model

In Table 5.25, we summarise the best individual CVs in the GARCH model.

	Case 1			Case 2		
Reset dates	$n = 4$	$n = 16$	$n = 64$	$n = 4$	$n = 16$	$n = 64$
ATM	both					
ITM	SV ( $\rho = 0$ )		delta	SV ( $\rho = 0$ )		
OTM	both					
High $ \rho $	GBM	delta		SV ( $\rho = 0$ )	delta	
High Vol	both					

Table 5.25: Best Individual CVs in the GARCH Model

### Combinations of Correlation CVs with Standard CVs

We examine the various CVs in combination. Table 5.66 (page 131) gives results for the ATM case. The best combination is the auxiliary CV with SV ( $\rho = 0$ ) and ( $\rho \neq 0$ ). It increases the efficiency gains from the best CV in Table 5.60 (page 125), SV ( $\rho = 0$ ) + ( $\rho \neq 0$ ), by about 10, except for the option with 16 reset dates in Case 2. Using the delta CV with SV ( $\rho = 0$ ) and ( $\rho \neq 0$ ) decreases the efficiency gains in many cases.

The results for the ITM options are given in Table 5.67 (page 132). The auxiliary CV combines well with SV ( $\rho = 0$ ). The improvement on SV ( $\rho = 0$ ) by the auxiliary CV in Case 2 is more than 50. When combining the delta CV with SV ( $\rho = 0$ ), the results for the option with 64 reset dates in Case 1 are extremely good. The efficiency gain is over 850 whenever both of them are in the combination.

Table 5.68 (page 133) displays the results for the OTM options. Using the auxiliary CV with SV ( $\rho = 0$ ) and ( $\rho \neq 0$ ) produces high efficiency gain. This is higher than combining SV ( $\rho = 0$ ) and ( $\rho \neq 0$ ). The delta CV gives no additional gain when it combines with other CVs.

Efficiency gains for  $\rho = 0.9$  and  $\rho = -0.9$  are given in Table 5.69 (page 134) and 5.70 (page 135). In both cases, the auxiliary CV adds a couple of points of efficiency gain to SV ( $\rho = 0$ ). Apart from that, combining different types of CVs has no big advantage over using individual CVs.

Table 5.71 (page 136) displays efficiency gains when the volatility is high. The best combination in this table is the auxiliary CV used with SV ( $\rho = 0$ ) and ( $\rho \neq 0$ ). It improves on SV ( $\rho = 0$ ) and ( $\rho \neq 0$ ) by 5 to 15 points of efficiency gain except for the option with 16 reset dates in Case 2. Combining the delta CV reduces the efficiency gains from the SV ( $\rho = 0$ ) alone, but is more efficient than either the delta CV or SV ( $\rho \neq 0$ ) along when using them together.

In summary, we find that combining the CVs with good performances in the individual cases usually produces higher efficiency, that is, higher than using either of those CVs individually. One CV that always performs well is SV ( $\rho = 0$ ). The performances of the others vary from case to case. In Table 5.26, we summarise the best CV combinations for the GARCH model.

	Case 1		Case 2	
Reset dates	$n < 16$	$n \geq 16$	$n < 16$	$n \geq 16$
ATM	GBM+both			
ITM	GBM+SV ( $\rho = 0$ )	delta+SV ( $\rho = 0$ )	GBM+SV ( $\rho = 0$ )	
OTM	GBM+both			
High $ \rho $	all	GBM+SV ( $\rho = 0$ )	all	GBM+SV ( $\rho = 0$ )
High Vol	GBM+both			

Table 5.26: Best CV Combinations in the GARCH Model

## 5.5 Valuation in the Johnson and Shanno Model

Finally We value arithmetic average rate options in the special case of the Johnson and Shanno model where  $0 < \beta < 1$ ,  $\xi = 1$  and  $\gamma = \frac{1}{2}$ .

**Discretization.** We apply the transformation  $Y_t = \frac{1}{1-\beta} S_t^{1-\beta}$ .

$$dY_t = \left[ r(1-\beta)Y_t - \frac{\beta}{2(1-\beta)Y_t} V_t^{2\xi} \right] dt + V_t^\xi dW_t^S. \quad (5.12)$$

We also impose an absorbing boundary at zero as in the SABR model. The volatility follows a CIR process. We use the same log-normal moment-matched approximation (5.6) for  $V_t$  as in the Heston model. Asset values are then recovered by setting  $\tilde{S}_t = \left[(1 - \beta) \tilde{Y}_t\right]^{\frac{1}{1-\beta}}$ .

**Parameters.** We set  $S_0 = 100$ ,  $r = 0.05$ ,  $V_0 = 2$ ,  $\alpha = 2$ ,  $\mu = 2$ ,  $\eta = 0.1$  and  $\rho = -0.5$ . In Case 1 we take  $\beta = 0.4$  and in Case 2,  $\beta = 0.6$ . This parameter setting is consistent with the SABR example in that the volatilities  $V_t^\xi S_t^\beta$  in both examples are roughly at the same level (about 12 in Case 1 and 30 in Case 2). Instead of having the forward price with a CEV process as in the SABR model, this model uses the stock price. Since the volatility of volatility  $\eta$  is low in our setting, the volatility process is almost deterministic. Therefore, the results are likely to be good.

We test 4 CVs: the GBM auxiliary CV, the delta CV, SV ( $\rho = 0$ ) and SV ( $\rho \neq 0$ ). We price 3 average rate options with 4, 16 and 64 reset dates up to the maturity  $T = 1$ . The number of sample paths for the plain Monte Carlo method is  $10^6$  and for the Monte Carlo with CVs is 10000.

### The ATM Options

The empirical correlations are listed in Table 5.27. We have seen very exciting re-

		Case 1			Case 2		
CV Type		Av, 64	Av, 16	Av, 4	Av, 64	Av, 16	Av, 4
GBM	Auxiliary	0.9988	0.9987	0.9985	0.997	0.997	0.996
	Delta	0.9991	0.998	0.98	0.998	0.997	0.98
SV	( $\rho = 0$ )	0.99991	0.99990	0.99989	0.9997	0.9997	0.9997
	( $\rho \neq 0$ )	0.89	0.89	0.89	0.90	0.90	0.90

Table 5.27: Empirical Correlations  $\hat{\rho}$ : CVs with Discounted Option Payoffs in Johnson and Shanno Model (ATM)

sults. The empirical correlations of SV ( $\rho = 0$ ) are extraordinarily high (more than 0.999). It stands out from other CV candidates as a big winner. The delta CV is similar to the auxiliary CV. Their correlations are also very high. SV ( $\rho \neq 0$ ), however, is far behind the above three CVs as its correlations are all around 0.9 only.

Efficiency gains for the ATM options are given in Table 5.72 (page 137). Those for SV ( $\rho = 0$ ) are not of the same order of magnitude as those of the standard CVs. They are much better. The efficiency gains for SV ( $\rho = 0$ ) are over 3400

in Case 1 and 1200 in Case 2. This level of efficiency gain is not even seen when using geometric average rate option CV for its arithmetic counterpart in the GBM model (Webber [83]). Adding SV ( $\rho \neq 0$ ) to SV ( $\rho = 0$ ) does not further improve the efficiency gain. This is because SV ( $\rho \neq 0$ ) is not even nearly as strong as SV ( $\rho = 0$ ). The auxiliary CV also has very high efficiency gains (over 300 in Case 1 and 130 in Case 2). But they are only 1/10 of those of SV ( $\rho = 0$ ). Although the delta CV has similar empirical correlations to the auxiliary CV, its benefits are partly offset by its high computation cost. Its efficiency gains are increasing with the number of option reset dates which is also the case in the other models.

### ITM and OTM Options

The empirical correlations for the ITM ( $K = 80$ ) and OTM ( $K = 120$ ) cases are given in Table 5.28 and 5.29. In the ITM case, the results are rather consistent

		Case 1			Case 2		
CV Type		Av, 64	Av, 16	Av, 4	Av, 64	Av, 16	Av, 4
GBM	Auxiliary	0.9990	0.9989	0.9987	0.9985	0.998	0.998
	Delta	0.9998	0.9990	0.988	0.9991	0.998	0.98
SV	( $\rho = 0$ )	0.99996	0.99995	0.99994	0.99988	0.99988	0.99985
	( $\rho \neq 0$ )	0.87	0.86	0.87	0.88	0.87	0.88

Table 5.28: Empirical Correlations  $\hat{\rho}$ : CVs with Discounted Option Payoffs in Johnson and Shanno Model (ITM)

		Case 1			Case 2		
CV Type		Av, 64	Av, 16	Av, 4	Av, 64	Av, 16	Av, 4
GBM	Auxiliary	0.993	0.988	0.994	0.996	0.996	0.996
	Delta	0.98	0.97	0.81	0.995	0.991	0.96
SV	( $\rho = 0$ )	0.9994	0.9995	0.9995	0.9990	0.9990	0.9991
	( $\rho \neq 0$ )	0.98	0.98	0.97	0.94	0.94	0.93

Table 5.29: Empirical Correlations  $\hat{\rho}$ : CVs with Discounted Option Payoffs in Johnson and Shanno Model (OTM)

with the ATM case. Again we have seen very high empirical correlations for SV ( $\rho = 0$ ) and they are even higher than in the ATM case. In the OTM case, the empirical correlations for SV ( $\rho = 0$ ) are a bit lower than those in the ITM case but still much higher than those of other CVs. SV ( $\rho \neq 0$ ) has higher correlations than in the ATM and ITM options in Case 1, which is consistent with those in the Heston model (Table 5.5), the SABR model (Table 5.12) and the GARCH model

(Table 5.21). The correlations of the auxiliary CV are higher than those of the delta CV.

Table 5.73 (page 138) displays efficiency gain for ITM options. In the ITM case, the efficiency gains of SV ( $\rho = 0$ ) are almost twice as high as those in the ATM case (Table 5.72). Those of the delta CV increases sharply with the number of reset dates. They reach above 800 for options with 64 reset dates in Case 1 but are still less than a tenth of that of SV ( $\rho = 0$ ). For options with  $n \leq 16$ , the auxiliary CV is more efficient than the delta CV by a factor of more than two.

Table 5.74 (page 139) gives efficiency gains for OTM options. In the OTM case, the efficiency gains of SV ( $\rho = 0$ ) are lower than those in the ITM case by a factor of more than 5. The delta CV is not as efficient as in the ITM case. Although SV ( $\rho \neq 0$ ) has decent efficiency gains, combining it with SV ( $\rho = 0$ ) does not produce higher efficiency gains than using SV ( $\rho = 0$ ) by itself.

### Highly Positive and Negative $\rho$

We set  $\rho$  equal to 0.9 and  $-0.9$ . The empirical correlations are presented in Tables 5.30 and 5.31. In both tables, the auxiliary CV, the delta CV and SV ( $\rho = 0$ )

		Case 1 ( $\rho = 0.9$ )			Case 2 ( $\rho = 0.9$ )		
CV Type		Av, 64	Av, 16	Av, 4	Av, 64	Av, 16	Av, 4
GBM	Auxiliary	0.9997	0.9997	0.9996	0.9987	0.9987	0.998
	Delta	0.9992	0.998	0.98	0.998	0.997	0.98
SV	( $\rho = 0$ )	0.9996	0.9996	0.9996	0.9992	0.9992	0.9992
	( $\rho \neq 0$ )	0.44	0.47	0.46	0.48	0.49	0.49

Table 5.30: Empirical Correlations  $\hat{\rho}$ : CVs with Discounted Option Payoffs in Johnson and Shanno Model ( $\rho = 0.9$ )

		Case 1 ( $\rho = 0.9$ )			Case 2 ( $\rho = 0.9$ )		
CV Type		Av, 64	Av, 16	Av, 4	Av, 64	Av, 16	Av, 4
GBM	Auxiliary	0.9986	0.9986	0.998	0.997	0.996	0.996
	Delta	0.9991	0.998	0.98	0.998	0.997	0.98
SV	( $\rho = 0$ )	0.9997	0.9997	0.9996	0.9995	0.9995	0.9994
	( $\rho \neq 0$ )	0.48	0.48	0.48	0.49	0.50	0.50

Table 5.31: Empirical Correlations  $\hat{\rho}$ : CVs with Discounted Option Payoffs in Johnson and Shanno Model ( $\rho = -0.9$ )

all have very high empirical correlations. In Table 5.30, those of the auxiliary CV are higher than those of SV ( $\rho = 0$ ) in Case 1. This is not seen in other cases. SV ( $\rho \neq 0$ ) has very low correlations (below 0.5) in both tables. This is expected (see section 4.6.3) and consistent with the Heston model (Table 5.6 and 5.7), the SABR model (Table 5.13 and 5.14) and the GARCH model (Table 5.22 and 5.23).

Table 5.75 (page 140) presents efficiency gains for  $\rho = 0.9$ . When  $\rho = 0.9$ , the efficiency gains of the auxiliary CVs are the highest in Case 1 (over 1200), higher than those of SV ( $\rho = 0$ ) by more than 300. But in Case 2 SV ( $\rho = 0$ ) is still the top performer as in the ATM, ITM, OTM cases. SV ( $\rho \neq 0$ ) has efficiency gains all below 1, because of low empirical correlations, and drags down the efficiency gains a lot when used along with SV ( $\rho = 0$ ).

Table 5.76 (page 141) lists efficiency gains for  $\rho = -0.9$ . When  $\rho = -0.9$ , the results are similar to when  $\rho = 0.9$  except that the efficiency gains of SV ( $\rho = 0$ ) are much higher than those of the auxiliary CV in both cases.

### High Volatility

We set  $V_0 = \mu = 8$  in Case 1 and  $V_0 = \mu = 3$  in Case 2 to keep the same level of volatility  $V_t S_t^\beta$  in the high volatility case of the SABR example (roughly 47 in Case 1 and 50 in Case 2). The empirical correlations are given in Table 5.32. Those of

		Case 1 ( $V_0 = \mu = 8$ )			Case 2 ( $V_0 = \mu = 3$ )		
CV Type		Av, 64	Av, 16	Av, 4	Av, 64	Av, 16	Av, 4
GBM	Auxiliary	0.988	0.988	0.985	0.994	0.994	0.993
	Delta	0.997	0.996	0.98	0.998	0.996	0.98
SV	( $\rho = 0$ )	0.9998	0.9998	0.9998	0.9996	0.9996	0.9995
	( $\rho \neq 0$ )	0.90	0.90	0.90	0.91	0.90	0.90

Table 5.32: Empirical Correlations  $\hat{\rho}$ : CVs with Discounted Option Payoffs in Johnson and Shanno Model (High Volatility)

SV ( $\rho = 0$ ) remain high and are the highest. The correlations of SV ( $\rho \neq 0$ ) are all around 0.9, which is similar to the ATM and ITM cases in Table 5.27 (page 92) and 5.28 (page 93).

Efficiency gains are listed in Table 5.77 (page 142). The situation is not very different from the ATM, ITM and OTM cases in Tables 5.72 (page 137), 5.73 (page 138) and 5.74 (page 139). SV ( $\rho = 0$ ) is much better than the other CVs. The auxiliary



CV and the delta CV also give decent efficiency gains.

In summary, throughout all cases SV ( $\rho = 0$ ) achieves extremely high efficiency gains (except for the highly positive correlation case where the auxiliary CV offers higher efficiency gains than SV ( $\rho = 0$ )). This is an example of modelling the stock price as CEV process to complement the SABR example which models the forward price. The results have shown great variance reduction using SV ( $\rho = 0$ ) in both examples, especially the former. Efficiency gains of several thousand are truly exciting.

### **Combinations of Correlation CVs with Standard Ones**

We look at CVs in combinations. Results for the ATM case are presented in Table 5.78 (page 143). Compared with the individual CVs (Table 5.72 on page 137), adding the auxiliary CV with SV ( $\rho = 0$ ) increases the efficiency gains by roughly 600, which offers the highest efficiency gains among all combinations.

The results for the ITM case are given in Table 5.79 (page 144). The highest efficiency gains come from the combination of the GBM auxiliary CV and SV ( $\rho = 0$ ). It increases the efficiency gains from Table 5.73 (page 138) by more than 600 in Case 1. However, gains are slightly lower in Case 2 by 100-200. The delta CV combines well with the auxiliary CV for options with 64 reset dates. The efficiency gains are higher than those of each in the individual case. But they do not combine well for options with lower number of reset dates. The auxiliary CV always gives a boost to any other CV used with it.

The results for the OTM options are listed in Table 5.80 (page 145). The combination of the auxiliary CV with SV ( $\rho = 0$ ) increases the highest efficiency gains from Table 5.74 (page 139) by around 700 in Case 1 and 50 in Case 2. In fact, the auxiliary CV combines well with SV ( $\rho \neq 0$ ) and SV ( $\rho = 0$ ) + ( $\rho \neq 0$ ) too. Those efficiency gains are all higher than in the individual case.

Efficiency gains when  $\rho = 0.9$  are presented in Table 5.81 (page 146). The combination of the auxiliary CV with SV ( $\rho = 0$ ) offers the highest efficiency gains as in the ATM, ITM and OTM cases. Compared with Table 5.75 (page 140), they increase by more than 1500 in Case 1 and 250 in Case 2.

Efficiency gains when  $\rho = -0.9$  are displayed in Table 5.82 (page 147). The highest

efficiency gains come from using the auxiliary CV with SV ( $\rho = 0$ ), as when  $\rho = 0.9$ . However, the efficiency gain for the option with 64 reset dates in Case 1 is slightly lower than that of SV ( $\rho = 0$ ) in Table 5.76. In other cases, the combination improves the efficiency gains but the improvement is not as much as when  $\rho = 0.9$ .

Finally, we look at the efficiency gains for the high volatility case (Table 5.83). The best combination is the auxiliary CV with SV ( $\rho = 0$ ) as in all other cases. The efficiency gains are around 200 more in Case 1 and 50 more in Case 2 than the best of those in Table 5.75 (page 140).

In conclusion, the rule of choosing the best CV in this model is quite simple, that is, to use the combination of the auxiliary CV with SV ( $\rho = 0$ ) for all options and parameter settings.

When  $\beta \in (0, 1)$ , the auxiliary model in section 4.4 has two alternative processes. In the next section, we provide a comparison of them.

## 5.6 The Alternative Auxiliary Process for $\beta \in (0, 1)$

We compare two alternative auxiliary processes (equation (4.29) and (4.36)) for  $\beta \in (0, 1)$ . The instrument we price is the ATM average rate option with 64 reset dates. Table 5.33 displays the empirical correlations in both auxiliary processes for the SABR and Johnson and Shanno model whose parameters are given in section 5.3 and 5.5. In each individual case, the empirical correlations for both SV ( $\rho = 0$ )

$\hat{\rho}$		SABR				Johnson and Shanno			
		SV ( $\rho = 0$ )		SV ( $\rho \neq 0$ )		SV ( $\rho = 0$ )		SV ( $\rho \neq 0$ )	
		Case 1	Case 2	Case 1	Case 2	Case 1	Case 2	Case 1	Case 2
$\tilde{Y}_0$	ITM	0.87	0.9998	0.87	0.87	0.99995	0.99989	0.87	0.87
	ATM	0.91	0.9994	0.89	0.90	0.99990	0.9997	0.89	0.90
	OTM	0.94	0.9986	0.987	0.95	0.9994	0.9990	0.98	0.94
$\mathbb{E}(\tilde{Y}_i)$	ITM	0.87	0.9998	0.86	0.87	0.99995	0.99989	0.86	0.88
	ATM	0.90	0.9994	0.90	0.89	0.99992	0.9997	0.89	0.90
	OTM	0.95	0.9985	0.987	0.94	0.9995	0.9990	0.988	0.94

Table 5.33: Empirical Correlations in the Two Alternative Auxiliary Processes

and ( $\rho \neq 0$ ) are very similar under the auxiliary processes (4.29) and (4.36). Due to the slight additional complexity involved in computing  $E_i$  in order to simulate equation (4.36), we prefer using the drift freezing auxiliary process (4.29).

In the next section, we investigate how  $\beta$  can affect the amount of variance reduction from our CVs.

## 5.7 The $\beta$ Effect

Our methods apply for all  $0 < \beta < 1$ . But we find the amount of variance reduction varies for different  $\beta$ . We expect that the smaller  $\beta$  is, the higher the empirical correlation is between the option payoff and our CV. When  $\beta \rightarrow 0$  the  $\beta$ -average  $g_\beta^{1/(1-\beta)} \rightarrow g_0 = \frac{1}{n} \sum_{i=1}^n S_{t_i}$  which is the arithmetic average. This means when  $\beta$  approaches zero our CV (the payoff on the  $\beta$ -average) gets closer to the payoff of an arithmetic average rate option. When  $\beta = 0$ , both our CVs and the option have the same payoff but with slightly different underlying asset process. We shall see below that this difference is very small and SV ( $\rho = 0$ ) has very high empirical correlations for small  $\beta$ .

We consider a series of models with  $\xi = 1$ ,  $\gamma = \frac{1}{2}$  but  $\beta = 0, 0.1, 0.2, \dots, 0.9$ . In order to keep the volatility roughly at the same level, we vary  $V_0 = \mu$  accordingly (because  $V_t$  will always revert to  $\mu$ ) such that  $V_t^\xi S_t^\beta \approx V_0 S_0^\beta \approx 20$ . We take  $S_0 = 100$  and  $r = 0.05$ . Figure 5.3 displays the relationship between  $V_0$  and  $\beta$ .

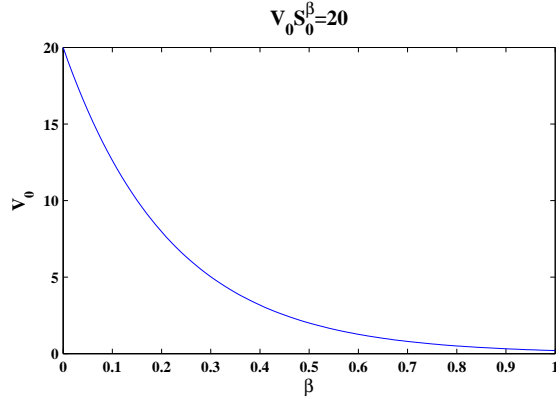


Figure 5.3: Relationship between  $V_0$  and  $\beta$

Table 5.34 gives the empirical correlation between the payoff of the option with 64 reset dates and SV ( $\rho = 0$ ) and ( $\rho \neq 0$ ). We have seen very high empirical correlations from SV ( $\rho = 0$ ) and they are decreasing with  $\beta$ . However, for SV ( $\rho \neq 0$ ), the empirical correlations are shown increasing with  $\beta$ . This is a bit unexpected but it seems that increasing  $\beta$  also increases the closeness of the process for SV ( $\rho \neq 0$ )

$\beta =$	0	0.1	0.2	0.3	0.4
SV ( $\rho = 0$ )	0.99997	0.99998	0.99997	0.99995	0.99992
SV ( $\rho \neq 0$ )	0.888	0.890	0.893	0.893	0.893
$\beta =$	0.5	0.6	0.7	0.8	0.9
SV ( $\rho = 0$ )	0.99987	0.99979	0.99967	0.9995	0.9992
SV ( $\rho \neq 0$ )	0.895	0.894	0.899	0.902	0.900

Table 5.34: Relationship Between  $\beta$  and Empirical Correlations

to the pricing model and this increment overtakes the difference between the option and CV payoffs.

Our final task is to investigate the improvement over existing methods by including our correlation CVs to them.

## 5.8 Relative Improvement over Existing Methods

We define the relative performance (RP) of our correlation methods as the ratio of the best efficiency gain by including correlation CVs to that with standard CVs alone. Table 5.35 summarises the results from our four examples in all cases. Any value over 1 is a success for the new CVs. For ITM, ATM and OTM options with

RP	Heston		GARCH		SABR		J&S	
	Case 1	Case 2	Case 1	Case 2	Case 1	Case 2	Case 1	Case 2
ITM	1.7	2.4	1.7	2.7	1.9	2.7	6.3	9.4
ATM	6.9	2.5	2.3	2.9	3.1	2.8	11	10
OTM	34	10.7	14	5.8	92	8.4	26	3.6
$\rho = 0.9$	1.2	1.2	1.0	1.0	1.3	1.2	1.9	2.1
$\rho = -0.9$	3.1	1.4	1.3	1.2	1.6	1.7	3.8	6.6
High Vol	2.4	1.6	2.1	2.3	3.5	3.0	30	11

Table 5.35: Relative Performances over Existing Methods

ordinary parameter values, our correlation CVs enhance performances by sizable factor. The improvement is particularly large (tens) in Case 1 of OTM options. These are the cases where zero is accessible in the pricing models and the existing methods perform poorly. Generally, as options go OTM, the relative improvements are larger. In the extreme correlation cases, correlation CVs perform less well but still give mild speed-ups. In the extreme volatility cases, the speed-ups are comparable to those in the ATM case with ordinary parameter values.

## 5.9 Conclusion

Our numerical results show that the CV with zero correlation provides significant efficiency gains for arithmetic average rate options in most cases, especially in special case with a stock price process of CEV type where efficiency gains of several thousand are seen. The CV with non-zero correlation is worth using for the OTM options with some particular parameters. Using it in combination with the other CVs also provides improvement. We have also looked at how the correlation control variates combine with standard ones and offered suggestions on how to choose the right control variates in different situations.

In the next chapter, we present numerical results for barrier options.

# Tables

Average Rate Options, Case 1							
Resets	Plain	GBM		Heston	SV		
		Auxiliary	Delta	Call	$(\rho = 0)$	$(\rho \neq 0)$	Both
4	4.69	4.70	4.67	4.70	4.67	4.70	4.70
	(0.006)	(0.039)	(0.020)	(0.025)	(0.027)	(0.016)	(0.012)
	[736.8]	[7.7]	[24.9]	[7.8]	[10.5]	[8.0]	[11.0]
	-	<b>2.0</b>	<b>2.4</b>	<b>4.7</b>	<b>3.0</b>	<b>11.4</b>	<b>14.1</b>
16	4.07	4.09	4.07	4.11	4.07	4.07	4.08
	(0.005)	(0.034)	(0.014)	(0.026)	(0.023)	(0.014)	(0.011)
	[741.0]	[7.7]	[24.7]	[7.8]	[10.6]	[8.0]	[10.9]
	-	<b>2.0</b>	<b>3.4</b>	<b>3.3</b>	<b>3.3</b>	<b>11.4</b>	<b>14.5</b>
64	3.93	3.86	3.91	3.91	3.93	3.93	3.92
	(0.005)	(0.032)	(0.014)	(0.027)	(0.022)	(0.013)	(0.010)
	[738.8]	[7.7]	[24.7]	[7.8]	[10.7]	[8.1]	[11.2]
	-	<b>2.1</b>	<b>3.6</b>	<b>2.9</b>	<b>3.3</b>	<b>11.2</b>	<b>14.4</b>

Average Rate Options, Case 2							
Resets	Plain	GBM		Heston	SV		
		Auxiliary	Delta	Call	$(\rho = 0)$	$(\rho \neq 0)$	Both
4	7.00	7.02	6.99	7.01	6.99	7.00	7.00
	(0.009)	(0.014)	(0.018)	(0.041)	(0.007)	(0.038)	(0.006)
	[733.8]	[7.7]	[24.8]	[7.8]	[10.6]	[7.9]	[11.5]
	-	<b>41.4</b>	<b>8.1</b>	<b>4.7</b>	<b>104.2</b>	<b>5.5</b>	<b>135.3</b>
16	6.10	6.10	6.09	6.14	6.09	6.10	6.09
	(0.008)	(0.011)	(0.007)	(0.042)	(0.006)	(0.033)	(0.005)
	[728.3]	[7.7]	[24.6]	[7.8]	[10.6]	[8.0]	[10.9]
	-	<b>46.8</b>	<b>40.7</b>	<b>3.4</b>	<b>123.7</b>	<b>5.4</b>	<b>171.8</b>
64	5.87	5.89	5.88	5.93	5.88	5.91	5.87
	(0.008)	(0.010)	(0.005)	(0.042)	(0.006)	(0.032)	(0.005)
	[732.3]	[7.8]	[25.7]	[7.9]	[10.6]	[8.1]	[11.1]
	-	<b>55.3</b>	<b>64.8</b>	<b>3.2</b>	<b>128.7</b>	<b>5.3</b>	<b>171.4</b>

Table 5.36: Efficiency Gains in the Heston Model: Average Rate Options (ATM)

Average Rate Options, Case 1							
Resets	Plain	GBM		Heston	SV		
		Auxiliary	Delta	Call	$(\rho = 0)$	$(\rho \neq 0)$	Both
4	21.34	21.43	21.33	21.37	21.29	21.34	21.32
	(0.009)	(0.054)	(0.018)	(0.041)	(0.034)	(0.049)	(0.029)
	[734.1]	[7.6]	[21.9]	[7.7]	[10.8]	[7.9]	[11.0]
	-	<b>2.9</b>	<b>9.1</b>	<b>4.9</b>	<b>5.1</b>	<b>3.3</b>	<b>6.9</b>
16	21.00	20.99	20.99	20.94	20.99	21.09	21.01
	(0.008)	(0.047)	(0.006)	(0.043)	(0.030)	(0.043)	(0.025)
	[735.3]	[7.7]	[21.9]	[7.8]	[10.7]	[8.1]	[11.0]
	-	<b>3.0</b>	<b>64.7</b>	<b>3.6</b>	<b>5.4</b>	<b>3.3</b>	<b>7.5</b>
64	20.91	20.85	20.91	20.98	20.89	20.89	20.88
	(0.008)	(0.045)	(0.004)	(0.043)	(0.028)	(0.043)	(0.025)
	[736.9]	[7.7]	[21.9]	[7.7]	[10.7]	[8.1]	[11.2]
	-	<b>3.0</b>	<b>116.9</b>	<b>3.3</b>	<b>5.6</b>	<b>3.1</b>	<b>7.1</b>

Average Rate Options, Case 2							
Resets	Plain	GBM		Heston	SV		
		Auxiliary	Delta	Call	$(\rho = 0)$	$(\rho \neq 0)$	Both
4	22.26	22.27	22.28	22.30	22.27	22.30	22.27
	(0.013)	(0.016)	(0.022)	(0.055)	(0.007)	(0.064)	(0.007)
	[736.8]	[7.7]	[22.8]	[7.7]	[10.8]	[8.0]	[11.0]
	-	<b>63.3</b>	<b>11.8</b>	<b>5.5</b>	<b>241.8</b>	<b>3.9</b>	<b>245.8</b>
16	21.69	21.70	21.71	21.74	21.70	21.75	21.71
	(0.012)	(0.013)	(0.006)	(0.056)	(0.006)	(0.058)	(0.005)
	[735.4]	[7.7]	[22.8]	[7.8]	[10.7]	[8.0]	[10.9]
	-	<b>71.3</b>	<b>114.0</b>	<b>4.0</b>	<b>303.4</b>	<b>3.7</b>	<b>313.5</b>
64	21.57	21.55	21.57	21.64	21.57	21.53	21.57
	(0.011)	(0.013)	(0.003)	(0.057)	(0.005)	(0.056)	(0.005)
	[734.0]	[7.8]	[22.7]	[7.7]	[10.7]	[8.1]	[11.2]
	-	<b>74.1</b>	<b>361.5</b>	<b>3.6</b>	<b>336.9</b>	<b>3.7</b>	<b>328.9</b>

Table 5.37: Efficiency Gains in the Heston Model: Average Rate Options (ITM)

Average Rate Options, Case 1							
Resets	Plain	GBM		Heston	SV		
		Auxiliary	Delta	Call	$(\rho = 0)$	$(\rho \neq 0)$	Both
4	0.102	0.097	0.103	0.096	0.110	0.099	0.101
	(0.0010)	(0.0098)	(0.0064)	(0.0075)	(0.0072)	(0.0014)	(0.0016)
	[729.8]	[7.8]	[22.3]	[7.8]	[11.4]	[7.9]	[11.1]
	-	<b>1.1</b>	<b>0.8</b>	<b>1.8</b>	<b>1.3</b>	<b>47.1</b>	<b>28.9</b>
16	0.046	0.047	0.050	0.039	0.045	0.046	0.046
	(0.0006)	(0.0059)	(0.0038)	(0.0045)	(0.0044)	(0.0010)	(0.0009)
	[751.1]	[7.8]	[22.5]	[7.8]	[11.8]	[8.1]	[11.1]
	-	<b>1.1</b>	<b>1.0</b>	<b>2.0</b>	<b>1.4</b>	<b>39.7</b>	<b>38.7</b>
64	0.039	0.036	0.031	0.043	0.037	0.036	0.038
	(0.0006)	(0.0054)	(0.0030)	(0.0058)	(0.0046)	(0.0008)	(0.0008)
	[737.0]	[7.8]	[22.6]	[7.8]	[11.0]	[8.2]	[11.4]
	-	<b>1.2</b>	<b>1.3</b>	<b>1.0</b>	<b>1.2</b>	<b>47.8</b>	<b>37.6</b>

Average Rate Options, Case 2							
Resets	Plain	GBM		Heston	SV		
		Auxiliary	Delta	Call	$(\rho = 0)$	$(\rho \neq 0)$	Both
4	0.871	0.879	0.877	0.876	0.877	0.879	0.872
	(0.0033)	(0.0101)	(0.0120)	(0.0186)	(0.0033)	(0.0075)	(0.0024)
	[732.6]	[7.8]	[22.8]	[7.8]	[10.7]	[8.0]	[11.1]
	-	<b>9.9</b>	<b>2.4</b>	<b>2.9</b>	<b>66.7</b>	<b>17.3</b>	<b>118.4</b>
16	0.524	0.517	0.512	0.517	0.523	0.524	0.523
	(0.0023)	(0.0065)	(0.0045)	(0.0174)	(0.0021)	(0.0049)	(0.0016)
	[732.4]	[7.8]	[22.7]	[7.8]	[10.8]	[8.1]	[11.1]
	-	<b>12.1</b>	<b>8.9</b>	<b>1.7</b>	<b>81.4</b>	<b>20.9</b>	<b>142.9</b>
64	0.446	0.457	0.448	0.439	0.447	0.444	0.452
	(0.0021)	(0.0066)	(0.0033)	(0.0149)	(0.0019)	(0.0043)	(0.0014)
	[737.7]	[7.8]	[22.7]	[7.8]	[11.0]	[8.3]	[11.3]
	-	<b>9.6</b>	<b>13.6</b>	<b>1.9</b>	<b>81.5</b>	<b>21.5</b>	<b>143.8</b>

Table 5.38: Efficiency Gains in the Heston Model: Average Rate Options (OTM)



Average Rate Options, Case 1 ( $\rho = 0.9$ )							
Resets	Plain	GBM		Heston	SV		
		Auxiliary	Delta	Call	( $\rho = 0$ )	( $\rho \neq 0$ )	Both
4	4.37	4.34	4.37	4.32	4.37	4.36	4.29
	(0.011)	(0.058)	(0.027)	(0.047)	(0.064)	(0.096)	(0.060)
	[742.5]	[7.8]	[21.9]	[7.9]	[10.6]	[8.0]	[10.9]
	-	<b>3.2</b>	<b>5.2</b>	<b>4.8</b>	<b>1.9</b>	<b>1.1</b>	<b>2.2</b>
16	3.79	3.78	3.80	3.73	3.71	3.84	3.73
	(0.009)	(0.049)	(0.012)	(0.049)	(0.053)	(0.086)	(0.052)
	[742.2]	[7.8]	[23.5]	[7.8]	[10.7]	[8.1]	[11.0]
	-	<b>3.3</b>	<b>18.2</b>	<b>3.3</b>	<b>2.1</b>	<b>1.0</b>	<b>2.1</b>
64	3.65	3.63	3.65	3.65	3.59	3.65	3.64
	(0.009)	(0.062)	(0.011)	(0.050)	(0.050)	(0.080)	(0.049)
	[746.6]	[7.8]	[22.1]	[7.9]	[10.8]	[8.2]	[11.2]
	-	<b>1.9</b>	<b>21.6</b>	<b>2.9</b>	<b>2.1</b>	<b>1.1</b>	<b>2.1</b>

Average Rate Options, Case 2 ( $\rho = 0.9$ )							
Resets	Plain	GBM		Heston	SV		
		Auxiliary	Delta	Call	( $\rho = 0$ )	( $\rho \neq 0$ )	Both
4	6.91	6.89	6.89	6.76	6.87	6.81	6.87
	(0.011)	(0.014)	(0.021)	(0.046)	(0.015)	(0.094)	(0.015)
	[739.8]	[7.8]	[21.9]	[7.8]	[10.6]	[8.1]	[11.0]
	-	<b>51.4</b>	<b>8.9</b>	<b>5.0</b>	<b>35.8</b>	<b>1.2</b>	<b>35.2</b>
16	5.99	6.01	6.02	6.04	5.99	6.05	6.02
	(0.009)	(0.012)	(0.007)	(0.048)	(0.012)	(0.081)	(0.013)
	[742.6]	[7.8]	[24.7]	[7.8]	[10.6]	[8.1]	[11.0]
	-	<b>59.2</b>	<b>51.3</b>	<b>3.5</b>	<b>37.9</b>	<b>1.2</b>	<b>35.9</b>
64	5.79	5.80	5.80	5.81	5.78	5.69	5.77
	(0.009)	(0.011)	(0.005)	(0.049)	(0.012)	(0.076)	(0.012)
	[743.5]	[7.8]	[22.3]	[7.8]	[10.7]	[8.1]	[11.3]
	-	<b>60.1</b>	<b>113.1</b>	<b>3.1</b>	<b>36.0</b>	<b>1.2</b>	<b>37.2</b>

Table 5.39: Efficiency Gains in the Heston Model: Average Rate Options ( $\rho = 0.9$ )

Average Rate Options, Case 1 ( $\rho = -0.9$ )							
Resets	Plain	GBM		Heston	SV		
		Auxiliary	Delta	Call	( $\rho = 0$ )	( $\rho \neq 0$ )	Both
4	4.63	4.60	4.63	4.67	4.63	4.64	4.61
	(0.004)	(0.030)	(0.015)	(0.020)	(0.031)	(0.032)	(0.025)
	[742.1]	[7.8]	[22.0]	[7.8]	[10.7]	[8.0]	[11.2]
	-	<b>2.0</b>	<b>2.8</b>	<b>4.6</b>	<b>1.4</b>	<b>1.7</b>	<b>2.0</b>
16	4.03	4.05	4.03	4.02	4.02	3.98	4.05
	(0.004)	(0.026)	(0.010)	(0.021)	(0.026)	(0.028)	(0.022)
	[742.6]	[7.7]	[21.9]	[7.8]	[10.6]	[8.0]	[11.0]
	-	<b>2.1</b>	<b>5.2</b>	<b>3.2</b>	<b>1.5</b>	<b>1.7</b>	<b>2.1</b>
64	3.88	3.88	3.89	3.87	3.85	3.85	3.86
	(0.004)	(0.025)	(0.009)	(0.021)	(0.025)	(0.027)	(0.021)
	[741.5]	[7.8]	[22.3]	[7.8]	[10.8]	[8.1]	[11.2]
	-	<b>2.0</b>	<b>5.4</b>	<b>2.9</b>	<b>1.5</b>	<b>1.7</b>	<b>2.1</b>

Average Rate Options, Case 2 ( $\rho = -0.9$ )							
Resets	Plain	GBM		Heston	SV		
		Auxiliary	Delta	Call	( $\rho = 0$ )	( $\rho \neq 0$ )	Both
4	7.01	7.02	7.01	6.97	7.02	7.18	6.97
	(0.009)	(0.012)	(0.016)	(0.039)	(0.013)	(0.076)	(0.013)
	[738.2]	[7.8]	[22.1]	[7.7]	[10.6]	[8.0]	[10.9]
	-	<b>48.9</b>	<b>9.4</b>	<b>4.7</b>	<b>33.6</b>	<b>1.2</b>	<b>29.5</b>
16	6.12	6.12	6.12	6.11	6.11	6.12	6.13
	(0.008)	(0.010)	(0.006)	(0.040)	(0.010)	(0.066)	(0.010)
	[745.7]	[7.8]	[22.1]	[7.7]	[10.6]	[8.0]	[11.0]
	-	<b>59.4</b>	<b>57.7</b>	<b>3.6</b>	<b>42.7</b>	<b>1.3</b>	<b>42.5</b>
64	5.89	5.89	5.89	6.00	5.89	5.97	5.89
	(0.007)	(0.010)	(0.004)	(0.040)	(0.010)	(0.065)	(0.009)
	[740.9]	[7.8]	[22.1]	[7.7]	[10.7]	[8.2]	[11.3]
	-	<b>55.3</b>	<b>108.9</b>	<b>3.3</b>	<b>40.6</b>	<b>1.2</b>	<b>40.3</b>

Table 5.40: Efficiency Gains in the Heston Model: Average Rate Options ( $\rho = -0.9$ )

Average Rate Options, Case 1 ( $V_0 = \mu = 0.25$ )							
Resets	Plain	GBM		Heston	SV		
		Auxiliary	Delta	Call	( $\rho = 0$ )	( $\rho \neq 0$ )	Both
4	13.75	13.88	13.74	13.70	13.79	13.65	13.69
	(0.022)	(0.065)	(0.049)	(0.096)	(0.042)	(0.092)	(0.036)
	[736.8]	[7.7]	[22.0]	[7.8]	[10.7]	[8.1]	[11.0]
	-	<b>10.8</b>	<b>6.6</b>	<b>5.0</b>	<b>18.8</b>	<b>5.2</b>	<b>25.0</b>
16	12.13	12.12	12.17	12.07	12.11	12.07	12.09
	(0.019)	(0.057)	(0.024)	(0.099)	(0.033)	(0.081)	(0.028)
	[731.3]	[7.9]	[22.0]	[7.8]	[10.7]	[8.2]	[11.0]
	-	<b>10.5</b>	<b>20.8</b>	<b>3.5</b>	<b>22.3</b>	<b>5.0</b>	<b>31.8</b>
64	11.73	11.71	11.71	11.58	11.75	11.79	11.79
	(0.019)	(0.056)	(0.020)	(0.100)	(0.032)	(0.079)	(0.026)
	[735.5]	[7.8]	[22.0]	[7.8]	[10.8]	[8.2]	[11.3]
	-	<b>10.2</b>	<b>27.9</b>	<b>3.2</b>	<b>22.8</b>	<b>4.9</b>	<b>34.4</b>

Average Rate Options, Case 2 ( $V_0 = \mu = 0.25$ )							
Resets	Plain	GBM		Heston	SV		
		Auxiliary	Delta	Call	( $\rho = 0$ )	( $\rho \neq 0$ )	Both
4	14.62	14.68	14.66	14.84	14.66	14.71	14.69
	(0.025)	(0.023)	(0.053)	(0.112)	(0.017)	(0.103)	(0.017)
	[729.1]	[7.7]	[21.8]	[7.8]	[10.6]	[8.0]	[11.0]
	-	<b>113.3</b>	<b>7.3</b>	<b>4.6</b>	<b>149.3</b>	<b>5.4</b>	<b>135.3</b>
16	12.95	12.90	12.89	12.96	12.88	12.91	12.90
	(0.022)	(0.018)	(0.020)	(0.114)	(0.013)	(0.091)	(0.013)
	[734.5]	[7.7]	[22.0]	[7.8]	[10.7]	[8.0]	[11.1]
	-	<b>133.3</b>	<b>38.7</b>	<b>3.4</b>	<b>183.1</b>	<b>5.2</b>	<b>183.5</b>
64	12.44	12.48	12.44	12.32	12.44	12.46	12.47
	(0.021)	(0.019)	(0.016)	(0.115)	(0.013)	(0.087)	(0.013)
	[740.0]	[7.7]	[22.1]	[7.7]	[10.7]	[8.2]	[11.3]
	-	<b>118.3</b>	<b>56.8</b>	<b>3.2</b>	<b>188.6</b>	<b>5.2</b>	<b>162.5</b>

Table 5.41: Efficiency Gains in the Heston Model: Average Rate Options ( $V_0 = \mu = 0.25$ )

		Case 1			Case 2		
CV Type		Av, 64	Av, 16	Av, 4	Av, 64	Av, 16	Av, 4
No CV (Plain)		3.92 (0.005) [738.8]	4.07 (0.005) [741.0]	4.69 (0.006) [736.8]	5.87 (0.008) [732.3]	6.10 (0.008) [728.3]	7.00 (0.009) [733.8]
Auxiliary	+SV ( $\rho = 0$ )	3.95 (0.022) [10.8] <b>3.2</b>	4.08 (0.023) [10.8] <b>3.2</b>	4.74 (0.027) [11.0] <b>2.9</b>	5.88 (0.006) [11.0] <b>128.7</b>	6.08 (0.006) [10.7] <b>121.3</b>	6.98 (0.007) [11.0] <b>107.6</b>
	+SV ( $\rho \neq 0$ )	3.91 (0.012) [8.3] <b>14.2</b>	4.07 (0.012) [8.2] <b>14.3</b>	4.65 (0.014) [8.1] <b>14.2</b>	5.88 (0.009) [8.4] <b>62.7</b>	6.08 (0.010) [8.2] <b>61.7</b>	6.98 (0.011) [8.1] <b>58.0</b>
	+SV ( $\rho = 0$ ) +SV ( $\rho \neq 0$ )	3.94 (0.010) [11.7] <b>14.3</b>	4.09 (0.010) [11.1] <b>14.8</b>	4.70 (0.012) [11.0] <b>14.2</b>	5.88 (0.005) [11.4] <b>181.5</b>	6.09 (0.005) [11.1] <b>168.0</b>	6.99 (0.006) [11.1] <b>143.2</b>
	+Call	3.96 (0.032) [8.3] <b>1.9</b>	4.05 (0.033) [7.9] <b>2.1</b>	4.67 (0.039) [7.9] <b>1.9</b>	5.87 (0.010) [8.0] <b>50.5</b>	6.10 (0.011) [8.0] <b>52.1</b>	6.99 (0.013) [7.9] <b>48.5</b>
Delta	+Auxiliary	3.94 (0.014) [22.1] <b>4.0</b>	4.09 (0.014) [22.0] <b>3.8</b>	4.71 (0.020) [22.0] <b>2.6</b>	5.88 (0.005) [22.1] <b>74.9</b>	6.10 (0.006) [22.2] <b>50.5</b>	6.96 (0.011) [22.0] <b>22.4</b>
	+Call	3.92 (0.012) [24.6] <b>4.7</b>	4.10 (0.012) [25.9] <b>4.5</b>	4.69 (0.014) [24.7] <b>5.0</b>	5.87 (0.005) [24.9] <b>77.5</b>	6.10 (0.006) [24.8] <b>61.9</b>	6.97 (0.012) [24.9] <b>17.4</b>
	+SV ( $\rho = 0$ )	3.92 (0.013) [25.2] <b>3.7</b>	4.08 (0.014) [24.9] <b>3.6</b>	4.70 (0.019) [25.8] <b>2.5</b>	5.88 (0.005) [25.4] <b>84.5</b>	6.08 (0.005) [25.0] <b>73.1</b>	6.99 (0.007) [24.9] <b>46.0</b>
	+SV ( $\rho \neq 0$ )	3.92 (0.005) [22.4] <b>25.1</b>	4.08 (0.006) [22.4] <b>23.0</b>	4.68 (0.009) [23.0] <b>11.4</b>	5.88 (0.004) [22.4] <b>121.0</b>	6.10 (0.006) [22.5] <b>65.6</b>	7.00 (0.015) [22.8] <b>11.7</b>
	+SV ( $\rho = 0$ ) +SV ( $\rho \neq 0$ )	3.92 (0.005) [25.5] <b>23.2</b>	4.07 (0.006) [25.2] <b>20.3</b>	4.69 (0.009) [25.1] <b>11.1</b>	5.87 (0.003) [25.6] <b>158.9</b>	6.09 (0.004) [25.3] <b>107.3</b>	6.99 (0.006) [25.3] <b>66.5</b>
All		3.92 (0.005) [25.6] <b>25.1</b>	4.08 (0.005) [25.4] <b>24.1</b>	4.67 (0.008) [25.3] <b>15.5</b>	5.88 (0.003) [25.7] <b>187.1</b>	6.09 (0.003) [25.5] <b>154.0</b>	6.98 (0.005) [25.4] <b>83.2</b>

Table 5.42: CVs in Combinations in the Heston Model: Average Rate Options (ATM)

		Case 1			Case 2		
CV Type		Av, 64	Av, 16	Av, 4	Av, 64	Av, 16	Av, 4
No CV (Plain)		20.91 (0.008) [736.9]	21.00 (0.008) [735.3]	21.34 (0.009) [734.1]	21.57 (0.011) [734.0]	21.69 (0.012) [735.4]	22.26 (0.013) [736.8]
Auxiliary	+SV ( $\rho = 0$ )	20.92 (0.027) [10.8] <b>5.9</b>	21.05 (0.028) [10.7] <b>6.1</b>	21.34 (0.032) [10.7] <b>5.7</b>	21.58 (0.005) [11.1] <b>390.0</b>	21.70 (0.005) [11.0] <b>327.6</b>	22.26 (0.007) [10.9] <b>235.3</b>
	+SV ( $\rho \neq 0$ )	20.87 (0.033) [8.5] <b>5.2</b>	21.00 (0.034) [8.0] <b>5.4</b>	21.36 (0.038) [8.0] <b>5.4</b>	21.60 (0.012) [8.2] <b>75.1</b>	21.72 (0.013) [8.0] <b>74.7</b>	22.27 (0.015) [8.0] <b>66.1</b>
	+SV ( $\rho = 0$ ) +SV ( $\rho \neq 0$ )	20.92 (0.023) [11.3] <b>7.7</b>	21.02 (0.024) [11.1] <b>8.2</b>	21.40 (0.027) [11.1] <b>7.9</b>	21.57 (0.005) [11.2] <b>372.0</b>	21.71 (0.005) [11.0] <b>334.6</b>	22.27 (0.007) [10.9] <b>253.1</b>
	+Call	20.95 (0.033) [7.8] <b>5.6</b>	20.97 (0.033) [7.8] <b>6.0</b>	21.35 (0.034) [7.7] <b>7.3</b>	21.57 (0.012) [7.8] <b>77.8</b>	21.70 (0.013) [7.8] <b>72.4</b>	22.26 (0.015) [7.8] <b>69.1</b>
Delta	+Auxiliary	20.91 (0.004) [21.9] <b>120.5</b>	20.99 (0.006) [21.8] <b>61.9</b>	21.32 (0.018) [21.8] <b>9.0</b>	21.58 (0.003) [22.6] <b>372.5</b>	21.70 (0.006) [22.6] <b>144.3</b>	22.28 (0.013) [22.4] <b>32.7</b>
	+Call	20.91 (0.004) [21.9] <b>127.6</b>	20.99 (0.005) [21.8] <b>103.0</b>	21.32 (0.012) [21.8] <b>20.7</b>	21.57 (0.003) [22.7] <b>515.7</b>	21.71 (0.005) [22.5] <b>217.5</b>	22.27 (0.014) [22.6] <b>27.6</b>
	+SV ( $\rho = 0$ )	20.91 (0.004) [25.0] <b>117.4</b>	20.99 (0.006) [25.9] <b>58.4</b>	21.33 (0.016) [24.6] <b>10.5</b>	21.57 (0.002) [25.9] <b>693.3</b>	21.71 (0.004) [25.6] <b>312.7</b>	22.27 (0.006) [25.6] <b>121.0</b>
	+SV ( $\rho \neq 0$ )	20.90 (0.004) [22.2] <b>163.8</b>	21.00 (0.006) [22.1] <b>69.9</b>	21.34 (0.016) [22.1] <b>11.3</b>	21.57 (0.003) [23.8] <b>353.1</b>	21.70 (0.006) [24.2] <b>116.0</b>	22.28 (0.020) [22.7] <b>13.3</b>
	+SV ( $\rho = 0$ ) +SV ( $\rho \neq 0$ )	20.91 (0.003) [25.4] <b>161.6</b>	20.99 (0.005) [25.1] <b>76.2</b>	21.32 (0.014) [24.9] <b>12.7</b>	21.58 (0.002) [26.1] <b>702.2</b>	21.71 (0.004) [25.9] <b>307.2</b>	22.27 (0.006) [25.7] <b>119.7</b>
All		20.91 (0.003) [25.9] <b>197.2</b>	20.99 (0.004) [25.2] <b>123.5</b>	21.35 (0.010) [25.0] <b>23.2</b>	21.58 (0.002) [26.1] <b>897.4</b>	21.71 (0.003) [26.3] <b>428.2</b>	22.27 (0.006) [25.8] <b>153.1</b>

Table 5.43: CVs in Combinations in the Heston Model: Average Rate Options (ITM)

		Case 1			Case 2		
CV Type		Av, 64	Av, 16	Av, 4	Av, 64	Av, 16	Av, 4
No CV (Plain)		0.039 (0.0006) [737.0]	0.046 (0.0006) [751.1]	0.102 (0.0010) [729.8]	0.446 (0.0021) [737.7]	0.524 (0.0023) [732.4]	0.871 (0.0033) [732.6]
Auxiliary	+SV ( $\rho = 0$ )	0.037 (0.0038) [10.8] <b>1.7</b>	0.049 (0.0046) [10.6] <b>1.4</b>	0.122 (0.0073) [10.6] <b>1.4</b>	0.444 (0.0018) [10.9] <b>96.1</b>	0.524 (0.0021) [10.8] <b>86.0</b>	0.877 (0.0031) [10.7] <b>75.2</b>
	+SV ( $\rho \neq 0$ )	0.038 (0.0008) [9.8] <b>38.2</b>	0.047 (0.0010) [9.8] <b>34.1</b>	0.097 (0.0015) [9.8] <b>37.4</b>	0.451 (0.0034) [8.3] <b>35.2</b>	0.521 (0.0036) [8.2] <b>38.5</b>	0.868 (0.0051) [8.2] <b>36.5</b>
	+SV ( $\rho = 0$ ) +SV ( $\rho \neq 0$ )	0.038 (0.0009) [11.2] <b>28.2</b>	0.048 (0.0010) [11.0] <b>27.9</b>	0.098 (0.0015) [10.9] <b>33.9</b>	0.449 (0.0013) [11.5] <b>161.8</b>	0.523 (0.0015) [11.3] <b>152.7</b>	0.878 (0.0023) [11.2] <b>131.3</b>
	+Call	0.037 (0.0056) [7.8] <b>1.1</b>	0.049 (0.0051) [7.8] <b>1.6</b>	0.117 (0.0084) [7.8] <b>1.4</b>	0.449 (0.0063) [7.9] <b>10.4</b>	0.528 (0.0065) [7.8] <b>12.1</b>	0.873 (0.0088) [7.8] <b>13.0</b>
Delta	+Auxiliary	0.037 (0.0029) [22.5] <b>1.4</b>	0.048 (0.0036) [22.5] <b>1.1</b>	0.098 (0.0062) [22.4] <b>0.9</b>	0.450 (0.0034) [22.8] <b>12.8</b>	0.519 (0.0041) [23.0] <b>10.4</b>	0.883 (0.0086) [22.9] <b>4.6</b>
	+Call	0.041 (0.0031) [22.4] <b>1.2</b>	0.046 (0.0034) [22.4] <b>1.2</b>	0.098 (0.0062) [22.4] <b>1.4</b>	0.449 (0.0032) [23.0] <b>14.1</b>	0.527 (0.0039) [23.0] <b>11.7</b>	0.883 (0.0086) [22.9] <b>4.3</b>
	+SV ( $\rho = 0$ )	0.044 (0.0034) [25.3] <b>0.9</b>	0.048 (0.0035) [25.2] <b>1.0</b>	0.103 (0.0058) [25.2] <b>0.9</b>	0.447 (0.0018) [25.6] <b>40.4</b>	0.522 (0.0021) [25.5] <b>34.9</b>	0.878 (0.0032) [26.0] <b>29.3</b>
	+SV ( $\rho \neq 0$ )	0.036 (0.0008) [22.9] <b>17.8</b>	0.046 (0.0009) [22.9] <b>17.8</b>	0.099 (0.0015) [22.7] <b>15.6</b>	0.448 (0.0024) [23.2] <b>25.6</b>	0.519 (0.0032) [23.0] <b>17.0</b>	0.875 (0.0066) [23.0] <b>7.8</b>
	+SV ( $\rho = 0$ ) +SV ( $\rho \neq 0$ )	0.038 (0.0010) [25.7] <b>9.7</b>	0.045 (0.0009) [26.0] <b>16.6</b>	0.102 (0.0017) [25.5] <b>10.2</b>	0.447 (0.0014) [26.2] <b>66.5</b>	0.523 (0.0015) [25.9] <b>66.6</b>	0.877 (0.0021) [25.8] <b>69.7</b>
All		0.038 (0.0008) [25.9] <b>17.2</b>	0.045 (0.0010) [25.7] <b>12.6</b>	0.102 (0.0014) [25.6] <b>15.1</b>	0.448 (0.0012) [26.1] <b>90.1</b>	0.522 (0.0013) [26.0] <b>86.7</b>	0.879 (0.0019) [25.9] <b>81.3</b>

Table 5.44: CVs in Combinations in the Heston Model: Average Rate Options (OTM)

		Case 1 ( $\rho = 0.9$ )			Case 2 ( $\rho = 0.9$ )		
CV Type		Av, 64	Av, 16	Av, 4	Av, 64	Av, 16	Av, 4
No CV (Plain)		3.65 (0.009) [746.6]	3.79 (0.009) [742.2]	4.37 (0.011) [742.5]	5.79 (0.009) [743.5]	5.99 (0.009) [742.6]	6.91 (0.011) [739.8]
Auxiliary	+SV ( $\rho = 0$ )	3.64 (0.041) [10.8] <b>3.1</b>	3.84 (0.048) [10.6] <b>2.5</b>	4.43 (0.058) [10.6] <b>2.4</b>	5.79 (0.010) [10.8] <b>49.4</b>	6.01 (0.011) [10.8] <b>49.6</b>	6.88 (0.013) [10.8] <b>45.7</b>
	+SV ( $\rho \neq 0$ )	3.61 (0.045) [8.2] <b>3.4</b>	3.83 (0.058) [8.3] <b>2.2</b>	4.40 (0.060) [8.1] <b>2.9</b>	5.79 (0.011) [8.7] <b>54.5</b>	6.03 (0.013) [8.5] <b>46.3</b>	6.90 (0.014) [8.5] <b>47.3</b>
	+SV ( $\rho = 0$ ) +SV ( $\rho \neq 0$ )	3.65 (0.042) [11.2] <b>2.9</b>	3.78 (0.049) [10.9] <b>2.4</b>	4.35 (0.052) [11.0] <b>2.9</b>	5.79 (0.011) [11.2] <b>46.0</b>	6.02 (0.012) [11.1] <b>42.6</b>	6.88 (0.012) [10.9] <b>50.1</b>
	+Call	3.60 (0.035) [7.9] <b>5.8</b>	3.84 (0.038) [7.8] <b>5.6</b>	4.37 (0.038) [7.8] <b>7.6</b>	5.80 (0.011) [8.3] <b>61.7</b>	5.98 (0.010) [8.2] <b>70.7</b>	6.88 (0.012) [8.2] <b>70.4</b>
Delta	+Auxiliary	3.66 (0.010) [22.1] <b>24.2</b>	3.81 (0.012) [22.2] <b>18.4</b>	4.34 (0.024) [22.1] <b>6.5</b>	5.80 (0.005) [23.3] <b>111.7</b>	6.01 (0.007) [23.3] <b>61.8</b>	6.87 (0.013) [22.3] <b>21.6</b>
	+Call	3.67 (0.012) [22.0] <b>18.9</b>	3.81 (0.012) [22.2] <b>26.1</b>	4.36 (0.016) [22.1] <b>14.4</b>	5.80 (0.004) [21.9] <b>135.7</b>	6.01 (0.007) [23.3] <b>106.8</b>	6.92 (0.013) [21.9] <b>21.9</b>
	+SV ( $\rho = 0$ )	3.64 (0.011) [24.9] <b>20.0</b>	3.78 (0.012) [24.8] <b>15.9</b>	4.37 (0.024) [24.7] <b>6.0</b>	5.80 (0.005) [24.9] <b>100.2</b>	6.02 (0.007) [24.7] <b>58.5</b>	6.87 (0.014) [24.8] <b>18.3</b>
	+SV ( $\rho \neq 0$ )	3.63 (0.009) [22.5] <b>28.4</b>	3.79 (0.011) [22.4] <b>22.6</b>	4.33 (0.024) [23.3] <b>6.3</b>	5.80 (0.004) [22.3] <b>132.3</b>	6.02 (0.007) [22.2] <b>62.6</b>	6.92 (0.021) [22.1] <b>8.5</b>
	+SV ( $\rho = 0$ ) +SV ( $\rho \neq 0$ )	3.66 (0.010) [25.2] <b>21.6</b>	3.79 (0.011) [25.1] <b>18.9</b>	4.37 (0.025) [25.1] <b>5.3</b>	5.81 (0.005) [25.5] <b>106.2</b>	6.02 (0.007) [25.4] <b>56.6</b>	6.88 (0.013) [25.3] <b>18.6</b>
All		3.64 (0.009) [25.4] <b>28.9</b>	3.80 (0.010) [25.2] <b>23.5</b>	4.38 (0.015) [25.0] <b>14.9</b>	5.79 (0.004) [25.6] <b>153.1</b>	6.01 (0.005) [25.4] <b>101.0</b>	6.89 (0.009) [25.5] <b>39.1</b>

Table 5.45: CVs in Combinations in the Heston Model: Average Rate Options ( $\rho = 0.9$ )

		Case 1 ( $\rho = -0.9$ )			Case 2 ( $\rho = -0.9$ )		
CV Type		Av, 64	Av, 16	Av, 4	Av, 64	Av, 16	Av, 4
No CV (Plain)		3.88 (0.004) [741.5]	4.03 (0.004) [742.6]	4.63 (0.004) [742.1]	5.89 (0.007) [740.9]	6.12 (0.008) [745.7]	7.01 (0.009) [738.2]
Auxiliary	+SV ( $\rho = 0$ )	3.88 (0.024) [10.9] <b>1.7</b>	3.99 (0.024) [10.7] <b>1.8</b>	4.57 (0.028) [10.7] <b>1.7</b>	5.89 (0.008) [10.8] <b>54.2</b>	6.13 (0.009) [10.6] <b>53.9</b>	7.00 (0.011) [10.6] <b>42.4</b>
	+SV ( $\rho \neq 0$ )	3.89 (0.019) [8.2] <b>3.3</b>	4.02 (0.020) [8.1] <b>3.3</b>	4.66 (0.024) [8.0] <b>3.2</b>	5.90 (0.009) [8.2] <b>63.2</b>	6.12 (0.009) [8.2] <b>59.7</b>	6.99 (0.012) [8.0] <b>50.9</b>
	+SV ( $\rho = 0$ ) +SV ( $\rho \neq 0$ )	3.86 (0.019) [11.3] <b>2.5</b>	4.02 (0.019) [11.1] <b>2.6</b>	4.67 (0.023) [11.1] <b>2.5</b>	5.90 (0.008) [11.2] <b>55.1</b>	6.11 (0.009) [11.0] <b>52.2</b>	7.00 (0.010) [10.9] <b>47.0</b>
	+Call	3.87 (0.017) [7.8] <b>4.6</b>	4.03 (0.017) [7.7] <b>4.9</b>	4.63 (0.017) [7.7] <b>6.1</b>	5.90 (0.009) [7.8] <b>64.2</b>	6.12 (0.010) [7.8] <b>57.5</b>	7.02 (0.012) [7.7] <b>53.2</b>
Delta	+Auxiliary	3.87 (0.009) [21.9] <b>5.5</b>	4.04 (0.010) [21.9] <b>5.0</b>	4.65 (0.015) [21.8] <b>3.0</b>	5.90 (0.004) [22.0] <b>125.0</b>	6.12 (0.005) [22.0] <b>78.8</b>	7.02 (0.010) [22.2] <b>24.9</b>
	+Call	3.89 (0.008) [21.9] <b>6.6</b>	4.03 (0.009) [21.9] <b>6.8</b>	4.63 (0.010) [21.9] <b>6.0</b>	5.90 (0.004) [22.0] <b>130.6</b>	6.11 (0.005) [22.0] <b>93.5</b>	6.99 (0.011) [22.0] <b>19.7</b>
	+SV ( $\rho = 0$ )	3.88 (0.009) [25.0] <b>5.0</b>	4.03 (0.010) [24.9] <b>4.6</b>	4.64 (0.014) [24.8] <b>2.8</b>	5.89 (0.004) [24.9] <b>120.9</b>	6.12 (0.005) [24.9] <b>74.7</b>	7.03 (0.010) [24.8] <b>23.0</b>
	+SV ( $\rho \neq 0$ )	3.88 (0.005) [22.2] <b>17.0</b>	4.03 (0.006) [22.1] <b>14.5</b>	4.64 (0.011) [22.0] <b>5.6</b>	5.89 (0.004) [22.5] <b>123.4</b>	6.12 (0.006) [22.3] <b>63.7</b>	7.01 (0.016) [22.4] <b>9.8</b>
	+SV ( $\rho = 0$ ) +SV ( $\rho \neq 0$ )	3.88 (0.005) [25.3] <b>15.7</b>	4.03 (0.006) [25.1] <b>13.0</b>	4.63 (0.011) [25.0] <b>5.0</b>	5.90 (0.003) [25.3] <b>138.0</b>	6.12 (0.005) [25.1] <b>78.8</b>	7.02 (0.010) [25.0] <b>23.4</b>
All		3.87 (0.005) [25.5] <b>17.1</b>	4.04 (0.005) [25.1] <b>16.4</b>	4.65 (0.008) [25.2] <b>8.7</b>	5.89 (0.003) [25.4] <b>178.8</b>	6.13 (0.004) [25.2] <b>128.1</b>	7.02 (0.008) [25.1] <b>38.8</b>

Table 5.46: CVs in Combinations in the Heston Model: Average Rate Options ( $\rho = -0.9$ )



		Case 1 ( $V_0 = \mu = 0.25$ )			Case 2 ( $V_0 = \mu = 0.25$ )		
CV Type		Av, 64	Av, 16	Av, 4	Av, 64	Av, 16	Av, 4
No CV (Plain)		11.73 (0.019) [735.5]	12.13 (0.019) [731.3]	13.75 (0.022) [736.8]	12.44 (0.021) [740.0]	12.95 (0.022) [734.5]	14.62 (0.025) [729.1]
Auxiliary	+SV ( $\rho = 0$ )	11.71 (0.032) [10.8] <b>22.9</b>	12.08 (0.035) [10.6] <b>20.6</b>	13.77 (0.041) [10.6] <b>20.2</b>	12.45 (0.012) [10.9] <b>189.8</b>	12.89 (0.014) [10.8] <b>170.3</b>	14.70 (0.017) [10.6] <b>144.4</b>
	+SV ( $\rho \neq 0$ )	11.69 (0.040) [8.3] <b>19.3</b>	12.06 (0.041) [8.0] <b>19.9</b>	13.79 (0.051) [8.0] <b>17.1</b>	12.44 (0.017) [8.2] <b>142.9</b>	12.88 (0.018) [8.0] <b>133.4</b>	14.67 (0.021) [8.0] <b>127.7</b>
	+SV ( $\rho = 0$ ) +SV ( $\rho \neq 0$ )	11.79 (0.027) [11.2] <b>30.2</b>	12.16 (0.028) [11.0] <b>31.3</b>	13.75 (0.036) [10.9] <b>25.5</b>	12.45 (0.013) [11.3] <b>181.7</b>	12.91 (0.013) [11.0] <b>178.9</b>	14.70 (0.017) [11.1] <b>145.5</b>
	+Call	11.75 (0.047) [7.8] <b>14.6</b>	12.15 (0.051) [7.7] <b>13.4</b>	13.77 (0.057) [7.8] <b>14.2</b>	12.43 (0.015) [7.8] <b>177.7</b>	12.91 (0.016) [7.7] <b>169.7</b>	14.67 (0.020) [7.7] <b>147.7</b>
Delta	+Auxiliary	11.75 (0.021) [21.9] <b>26.2</b>	12.16 (0.024) [21.9] <b>20.7</b>	13.78 (0.044) [21.8] <b>8.3</b>	12.44 (0.015) [21.9] <b>65.0</b>	12.85 (0.016) [21.8] <b>58.7</b>	14.70 (0.021) [21.8] <b>45.1</b>
	+Call	11.75 (0.019) [21.8] <b>33.5</b>	12.17 (0.020) [21.9] <b>31.8</b>	13.79 (0.032) [21.8] <b>16.0</b>	12.44 (0.014) [21.8] <b>79.8</b>	12.91 (0.015) [21.8] <b>68.6</b>	14.68 (0.032) [22.1] <b>20.0</b>
	+SV ( $\rho = 0$ )	11.77 (0.019) [25.0] <b>28.6</b>	12.14 (0.022) [24.8] <b>23.2</b>	13.69 (0.034) [24.7] <b>12.4</b>	12.44 (0.012) [25.0] <b>83.1</b>	12.89 (0.013) [24.9] <b>77.5</b>	14.69 (0.017) [25.0] <b>60.6</b>
	+SV ( $\rho \neq 0$ )	11.75 (0.015) [22.2] <b>49.3</b>	12.17 (0.019) [22.1] <b>33.7</b>	13.74 (0.039) [22.1] <b>10.4</b>	12.45 (0.015) [22.3] <b>60.6</b>	12.91 (0.021) [22.3] <b>37.1</b>	14.70 (0.048) [22.1] <b>8.8</b>
	+SV ( $\rho = 0$ ) +SV ( $\rho \neq 0$ )	11.71 (0.014) [25.4] <b>53.9</b>	12.14 (0.016) [25.1] <b>41.5</b>	13.80 (0.027) [25.0] <b>18.7</b>	12.46 (0.013) [25.5] <b>77.4</b>	12.91 (0.013) [25.2] <b>76.5</b>	14.70 (0.017) [25.2] <b>60.0</b>
All		11.75 (0.012) [25.6] <b>66.3</b>	12.15 (0.013) [25.2] <b>61.5</b>	13.76 (0.021) [25.2] <b>32.2</b>	12.45 (0.011) [25.6] <b>106.1</b>	12.88 (0.011) [25.2] <b>107.7</b>	14.66 (0.013) [25.3] <b>103.5</b>

Table 5.47: CVs in Combinations in the Heston Model: Average Rate Options ( $V_0 = \mu = 0.25$ )

Average Rate Options, Case 1							
Resets	Plain	GBM		SABR	SV		
		Auxiliary	Delta	Call	$(\rho = 0)$	$(\rho \neq 0)$	Both
4	5.10	5.13	5.11	5.13	5.12	5.13	5.12
	(0.006)	(0.013)	(0.011)	(0.025)	(0.007)	(0.022)	(0.005)
	[601.1]	[6.6]	[22.3]	[6.4]	[7.6]	[8.1]	[9.8]
	-	<b>17.5</b>	<b>6.6</b>	<b>4.7</b>	<b>50.0</b>	<b>4.7</b>	<b>64.5</b>
16	4.44	4.41	4.43	4.41	4.43	4.45	4.43
	(0.005)	(0.011)	(0.005)	(0.026)	(0.006)	(0.020)	(0.005)
	[601.2]	[6.6]	[22.0]	[6.4]	[7.6]	[8.1]	[9.8]
	-	<b>19.0</b>	<b>24.0</b>	<b>3.5</b>	<b>56.8</b>	<b>4.7</b>	<b>71.7</b>
64	4.27	4.26	4.26	4.24	4.26	4.26	4.26
	(0.005)	(0.010)	(0.004)	(0.026)	(0.006)	(0.019)	(0.004)
	[603.3]	[6.6]	[22.0]	[6.4]	[7.7]	[8.2]	[10.0]
	-	<b>18.9</b>	<b>31.1</b>	<b>3.2</b>	<b>55.2</b>	<b>4.6</b>	<b>74.4</b>

Average Rate Options, Case 2							
Resets	Plain	GBM		SABR	SV		
		Auxiliary	Delta	Call	$(\rho = 0)$	$(\rho \neq 0)$	Both
4	9.95	9.99	9.97	10.03	9.96	9.92	9.97
	(0.013)	(0.037)	(0.030)	(0.061)	(0.019)	(0.049)	(0.014)
	[603.6]	[6.5]	[22.4]	[6.4]	[7.6]	[8.1]	[9.8]
	-	<b>12.0</b>	<b>5.5</b>	<b>4.6</b>	<b>39.6</b>	<b>5.5</b>	<b>57.5</b>
16	8.70	8.75	8.72	8.64	8.71	8.73	8.71
	(0.012)	(0.029)	(0.014)	(0.063)	(0.015)	(0.044)	(0.011)
	[604.2]	[6.6]	[22.0]	[6.4]	[7.6]	[8.1]	[9.8]
	-	<b>14.6</b>	<b>18.0</b>	<b>3.3</b>	<b>45.5</b>	<b>5.3</b>	<b>64.1</b>
64	8.39	8.40	8.40	8.32	8.39	8.41	8.41
	(0.011)	(0.027)	(0.012)	(0.061)	(0.015)	(0.042)	(0.011)
	[602.7]	[6.6]	[22.0]	[6.4]	[7.7]	[8.2]	[10.0]
	-	<b>15.4</b>	<b>23.8</b>	<b>3.2</b>	<b>47.2</b>	<b>5.2</b>	<b>63.1</b>

Table 5.48: Efficiency Gains in the SABR Model: Average Rate Options (ATM)

Average Rate Options, Case 1							
Resets	Plain	GBM		SABR	SV		
		Auxiliary	Delta	Call	$(\rho = 0)$	$(\rho \neq 0)$	Both
4	22.08	22.09	22.10	22.09	22.09	22.08	22.09
	(0.008)	(0.018)	(0.014)	(0.034)	(0.008)	(0.041)	(0.008)
	[608.4]	[6.6]	[22.3]	[6.4]	[8.0]	[8.0]	[9.7]
	-	<b>19.4</b>	<b>10.2</b>	<b>5.5</b>	<b>73.1</b>	<b>3.1</b>	<b>66.2</b>
16	21.61	21.63	21.62	21.57	21.63	21.58	21.62
	(0.007)	(0.015)	(0.004)	(0.035)	(0.007)	(0.037)	(0.007)
	[607.9]	[6.6]	[22.3]	[6.4]	[8.0]	[8.1]	[9.7]
	-	<b>21.8</b>	<b>115.1</b>	<b>4.0</b>	<b>88.7</b>	<b>3.0</b>	<b>70.8</b>
64	21.48	21.50	21.49	21.50	21.50	21.50	21.50
	(0.007)	(0.014)	(0.002)	(0.035)	(0.007)	(0.035)	(0.007)
	[610.4]	[6.6]	[22.3]	[6.4]	[8.1]	[8.2]	[9.9]
	-	<b>22.9</b>	<b>453.7</b>	<b>3.8</b>	<b>84.1</b>	<b>3.0</b>	<b>70.0</b>

Average Rate Options, Case 2							
Resets	Plain	GBM		SABR	SV		
		Auxiliary	Delta	Call	$(\rho = 0)$	$(\rho \neq 0)$	Both
4	23.63	23.68	23.62	23.76	23.60	23.69	23.63
	(0.018)	(0.041)	(0.034)	(0.079)	(0.021)	(0.084)	(0.016)
	[612.7]	[6.6]	[22.5]	[6.4]	[8.0]	[8.0]	[9.7]
	-	<b>18.5</b>	<b>7.9</b>	<b>5.2</b>	<b>60.2</b>	<b>3.7</b>	<b>79.1</b>
16	22.70	22.67	22.71	22.72	22.72	22.57	22.73
	(0.017)	(0.034)	(0.013)	(0.084)	(0.014)	(0.077)	(0.013)
	[609.8]	[6.6]	[22.5]	[6.4]	[8.0]	[8.1]	[9.7]
	-	<b>21.9</b>	<b>46.0</b>	<b>3.7</b>	<b>104.2</b>	<b>3.5</b>	<b>103.1</b>
64	22.46	22.45	22.48	22.59	22.49	22.42	22.46
	(0.016)	(0.032)	(0.009)	(0.083)	(0.013)	(0.075)	(0.015)
	[611.7]	[6.6]	[22.5]	[6.4]	[8.1]	[8.2]	[9.9]
	-	<b>23.7</b>	<b>82.3</b>	<b>3.6</b>	<b>109.7</b>	<b>3.5</b>	<b>73.3</b>

Table 5.49: Efficiency Gains in the SABR Model: Average Rate Options (ITM)

Average Rate Options, Case 1							
Resets	Plain	GBM		SABR	SV		
		Auxiliary	Delta	Call	$(\rho = 0)$	$(\rho \neq 0)$	Both
4	0.056	0.052	0.052	0.057	0.056	0.055	0.055
	(0.0005)	(0.0034)	(0.0031)	(0.0045)	(0.0013)	(0.0004)	(0.0003)
	[608.1]	[6.6]	[22.4]	[6.5]	[7.6]	[8.2]	[9.8]
	-	<b>2.3</b>	<b>0.8</b>	<b>1.3</b>	<b>14.0</b>	<b>122.8</b>	<b>166.9</b>
16	0.018	0.021	0.016	0.016	0.018	0.017	0.017
	(0.0003)	(0.0027)	(0.0010)	(0.0024)	(0.0009)	(0.0002)	(0.0001)
	[608.8]	[6.6]	[22.9]	[6.5]	[7.6]	[8.3]	[9.8]
	-	<b>1.0</b>	<b>2.1</b>	<b>1.2</b>	<b>7.7</b>	<b>172.1</b>	<b>259.9</b>
64	0.012	0.013	0.012	0.010	0.012	0.012	0.012
	(0.0002)	(0.0018)	(0.0007)	(0.0019)	(0.0007)	(0.0001)	(0.0001)
	[607.3]	[6.6]	[22.5]	[6.6]	[7.7]	[8.4]	[10.0]
	-	<b>1.3</b>	<b>2.5</b>	<b>1.3</b>	<b>7.5</b>	<b>203.6</b>	<b>251.5</b>

Average Rate Options, Case 2							
Resets	Plain	GBM		SABR	SV		
		Auxiliary	Delta	Call	$(\rho = 0)$	$(\rho \neq 0)$	Both
4	2.69	2.68	2.72	2.69	2.70	2.67	2.69
	(0.007)	(0.025)	(0.024)	(0.038)	(0.012)	(0.017)	(0.007)
	[608.1]	[6.6]	[22.4]	[6.5]	[7.6]	[8.2]	[9.8]
	-	<b>7.1</b>	<b>2.4</b>	<b>3.1</b>	<b>29.4</b>	<b>12.9</b>	<b>56.9</b>
16	1.87	1.90	1.89	1.85	1.89	1.88	1.89
	(0.005)	(0.021)	(0.011)	(0.035)	(0.008)	(0.012)	(0.005)
	[605.4]	[6.6]	[22.5]	[6.5]	[7.6]	[8.3]	[9.8]
	-	<b>6.5</b>	<b>6.5</b>	<b>2.3</b>	<b>32.9</b>	<b>14.1</b>	<b>65.6</b>
64	1.70	1.70	1.70	1.63	1.69	1.70	1.70
	(0.005)	(0.019)	(0.010)	(0.033)	(0.008)	(0.011)	(0.005)
	[605.0]	[6.6]	[22.4]	[6.5]	[7.7]	[8.3]	[10.0]
	-	<b>6.8</b>	<b>7.6</b>	<b>2.1</b>	<b>35.2</b>	<b>14.4</b>	<b>64.1</b>

Table 5.50: Efficiency Gains in the SABR Model: Average Rate Options (OTM)

Average Rate Options, Case 1 ( $\rho = 0.9$ )							
Resets	Plain	GBM		SABR	SV		
		Auxiliary	Delta	Call	( $\rho = 0$ )	( $\rho \neq 0$ )	Both
4	4.87	4.85	4.88	4.90	4.85	4.87	4.83
	(0.007)	(0.012)	(0.013)	(0.030)	(0.014)	(0.061)	(0.014)
	[606.6]	[6.6]	[22.0]	[6.5]	[7.5]	[8.1]	[9.7]
	-	<b>28.0</b>	<b>7.7</b>	<b>4.9</b>	<b>18.5</b>	<b>0.9</b>	<b>14.2</b>
16	4.21	4.22	4.23	4.25	4.25	4.33	4.21
	(0.006)	(0.010)	(0.004)	(0.030)	(0.012)	(0.054)	(0.012)
	[607.6]	[6.6]	[22.0]	[6.5]	[7.6]	[8.1]	[9.8]
	-	<b>33.7</b>	<b>57.9</b>	<b>3.6</b>	<b>19.5</b>	<b>0.9</b>	<b>15.9</b>
64	4.07	4.07	4.07	4.07	4.07	3.95	4.07
	(0.006)	(0.010)	(0.003)	(0.031)	(0.011)	(0.050)	(0.011)
	[607.3]	[6.6]	[22.0]	[6.5]	[7.7]	[8.3]	[10.0]
	-	<b>29.4</b>	<b>106.5</b>	<b>3.2</b>	<b>19.3</b>	<b>0.9</b>	<b>16.4</b>

Average Rate Options, Case 2 ( $\rho = 0.9$ )							
Resets	Plain	GBM		SABR	SV		
		Auxiliary	Delta	Call	( $\rho = 0$ )	( $\rho \neq 0$ )	Both
4	9.75	9.82	9.78	9.71	9.78	10.00	9.69
	(0.017)	(0.037)	(0.037)	(0.077)	(0.042)	(0.161)	(0.039)
	[604.8]	[6.6]	[22.0]	[6.5]	[7.6]	[8.1]	[9.7]
	-	<b>20.1</b>	<b>5.9</b>	<b>4.7</b>	<b>13.3</b>	<b>0.9</b>	<b>11.9</b>
16	8.54	8.57	8.55	8.54	8.53	8.37	8.51
	(0.015)	(0.026)	(0.014)	(0.079)	(0.033)	(0.131)	(0.031)
	[604.6]	[6.6]	[21.9]	[6.5]	[7.6]	[8.1]	[9.8]
	-	<b>29.5</b>	<b>30.3</b>	<b>3.3</b>	<b>15.7</b>	<b>0.9</b>	<b>13.9</b>
64	8.24	8.26	8.26	8.33	8.21	8.21	8.28
	(0.014)	(0.027)	(0.012)	(0.080)	(0.032)	(0.127)	(0.030)
	[608.5]	[6.7]	[22.0]	[6.5]	[7.7]	[8.2]	[10.0]
	-	<b>26.1</b>	<b>41.6</b>	<b>3.0</b>	<b>15.6</b>	<b>0.9</b>	<b>13.4</b>

Table 5.51: Efficiency Gains in the SABR Model: Average Rate Options ( $\rho = 0.9$ )

Average Rate Options, Case 1 ( $\rho = -0.9$ )							
Resets	Plain	GBM		SABR	SV		
		Auxiliary	Delta	Call	( $\rho = 0$ )	( $\rho \neq 0$ )	Both
4	5.14	5.16	5.13	5.14	5.14	5.07	5.15
	(0.005)	(0.011)	(0.010)	(0.024)	(0.011)	(0.045)	(0.010)
	[606.5]	[6.6]	[22.5]	[6.5]	[7.6]	[8.1]	[9.7]
	-	<b>19.6</b>	<b>7.1</b>	<b>4.5</b>	<b>18.0</b>	<b>1.0</b>	<b>15.8</b>
16	4.47	4.45	4.46	4.48	4.46	4.41	4.46
	(0.005)	(0.010)	(0.004)	(0.024)	(0.009)	(0.039)	(0.009)
	[601.1]	[6.6]	[22.1]	[6.5]	[7.6]	[8.2]	[9.7]
	-	<b>21.3</b>	<b>35.0</b>	<b>3.3</b>	<b>19.7</b>	<b>1.0</b>	<b>16.3</b>
64	4.29	4.30	4.29	4.29	4.30	4.30	4.29
	(0.004)	(0.009)	(0.003)	(0.024)	(0.009)	(0.038)	(0.009)
	[606.2]	[6.6]	[22.1]	[6.5]	[7.7]	[8.3]	[9.9]
	-	<b>24.3</b>	<b>52.7</b>	<b>3.2</b>	<b>21.5</b>	<b>1.0</b>	<b>16.6</b>

Average Rate Options, Case 2 ( $\rho = -0.9$ )							
Resets	Plain	GBM		SABR	SV		
		Auxiliary	Delta	Call	( $\rho = 0$ )	( $\rho \neq 0$ )	Both
4	9.94	9.95	9.94	9.97	9.94	9.94	9.89
	(0.012)	(0.032)	(0.025)	(0.056)	(0.029)	(0.103)	(0.029)
	[605.8]	[6.6]	[23.1]	[6.4]	[7.6]	[8.1]	[9.7]
	-	<b>13.7</b>	<b>6.2</b>	<b>4.5</b>	<b>14.8</b>	<b>1.1</b>	<b>11.4</b>
16	8.70	8.67	8.68	8.80	8.71	8.69	8.66
	(0.011)	(0.027)	(0.011)	(0.057)	(0.024)	(0.091)	(0.024)
	[604.2]	[6.6]	[22.7]	[6.5]	[7.6]	[8.2]	[9.7]
	-	<b>14.9</b>	<b>27.9</b>	<b>3.4</b>	<b>16.8</b>	<b>1.0</b>	<b>13.0</b>
64	8.40	8.36	8.38	8.40	8.36	8.29	8.39
	(0.010)	(0.025)	(0.008)	(0.057)	(0.022)	(0.087)	(0.021)
	[604.7]	[6.6]	[22.7]	[6.5]	[7.7]	[8.3]	[9.9]
	-	<b>15.6</b>	<b>40.4</b>	<b>3.1</b>	<b>17.2</b>	<b>1.1</b>	<b>14.7</b>

Table 5.52: Efficiency Gains in the SABR Model: Average Rate Options ( $\rho = -0.9$ )

Average Rate Options, Case 1 ( $V_0 = 8$ )							
Resets	Plain	GBM		SABR	SV		
		Auxiliary	Delta	Call	( $\rho = 0$ )	( $\rho \neq 0$ )	Both
4	14.69	14.69	14.76	14.82	14.75	14.73	14.75
	(0.020)	(0.067)	(0.047)	(0.092)	(0.029)	(0.075)	(0.023)
	[603.0]	[6.6]	[22.4]	[6.4]	[7.6]	[8.1]	[9.7]
	-	<b>8.6</b>	<b>5.0</b>	<b>4.6</b>	<b>39.8</b>	<b>5.4</b>	<b>51.1</b>
16	12.94	12.87	12.91	12.84	12.88	12.85	12.88
	(0.018)	(0.056)	(0.024)	(0.095)	(0.023)	(0.066)	(0.018)
	[601.2]	[6.6]	[22.8]	[6.4]	[7.6]	[8.2]	[9.8]
	-	<b>9.4</b>	<b>14.1</b>	<b>3.3</b>	<b>46.8</b>	<b>5.5</b>	<b>59.4</b>
64	12.45	12.46	12.46	12.34	12.50	12.44	12.45
	(0.017)	(0.054)	(0.021)	(0.095)	(0.023)	(0.063)	(0.017)
	[606.0]	[6.6]	[23.0]	[6.5]	[7.7]	[8.3]	[10.0]
	-	<b>9.5</b>	<b>18.1</b>	<b>3.1</b>	<b>45.4</b>	<b>5.4</b>	<b>63.0</b>

Average Rate Options, Case 2 ( $V_0 = 3$ )							
Resets	Plain	GBM		SABR	SV		
		Auxiliary	Delta	Call	( $\rho = 0$ )	( $\rho \neq 0$ )	Both
4	14.00	14.01	13.98	14.13	14.04	14.03	13.99
	(0.020)	(0.061)	(0.045)	(0.091)	(0.029)	(0.073)	(0.022)
	[603.8]	[6.6]	[22.5]	[6.4]	[7.7]	[8.1]	[9.7]
	-	<b>9.8</b>	<b>5.3</b>	<b>4.5</b>	<b>36.6</b>	<b>5.6</b>	<b>49.7</b>
16	12.30	12.26	12.29	12.32	12.28	12.21	12.23
	(0.018)	(0.050)	(0.023)	(0.090)	(0.024)	(0.065)	(0.018)
	[601.7]	[6.6]	[22.5]	[6.4]	[7.7]	[8.2]	[9.7]
	-	<b>11.2</b>	<b>16.1</b>	<b>3.6</b>	<b>42.2</b>	<b>5.4</b>	<b>57.3</b>
64	11.84	11.79	11.87	11.90	11.86	11.90	11.84
	(0.017)	(0.045)	(0.020)	(0.092)	(0.022)	(0.063)	(0.018)
	[605.5]	[6.7]	[22.6]	[6.5]	[7.8]	[8.3]	[9.9]
	-	<b>12.7</b>	<b>19.6</b>	<b>3.2</b>	<b>44.9</b>	<b>5.3</b>	<b>55.7</b>

Table 5.53: Efficiency Gains in the SABR Model: Average Rate Options (High Volatility)

		Case 1			Case 2		
CV Type		Av, 64	Av, 16	Av, 4	Av, 64	Av, 16	Av, 4
No CV (Plain)		4.27 (0.005) [603.3]	4.44 (0.005) [601.2]	5.10 (0.006) [601.1]	8.39 (0.011) [602.7]	8.70 (0.012) [604.2]	9.95 (0.013) [603.6]
Auxiliary	+SV ( $\rho = 0$ )	4.26 (0.005) [8.3] <b>54.2</b>	4.44 (0.006) [8.2] <b>53.7</b>	5.11 (0.007) [8.3] <b>46.9</b>	8.37 (0.014) [8.3] <b>44.5</b>	8.69 (0.015) [8.2] <b>42.6</b>	9.93 (0.019) [8.1] <b>36.8</b>
	+SV ( $\rho \neq 0$ )	4.27 (0.008) [8.4] <b>26.9</b>	4.42 (0.008) [8.2] <b>27.4</b>	5.11 (0.010) [8.2] <b>24.8</b>	8.41 (0.021) [8.4] <b>21.4</b>	8.70 (0.022) [8.3] <b>21.5</b>	9.92 (0.026) [8.2] <b>19.9</b>
	+SV ( $\rho = 0$ ) +SV ( $\rho \neq 0$ )	4.25 (0.004) [10.0] <b>76.0</b>	4.43 (0.004) [9.9] <b>80.5</b>	5.11 (0.005) [9.8] <b>70.2</b>	8.39 (0.011) [10.0] <b>64.4</b>	8.72 (0.011) [9.8] <b>65.6</b>	9.92 (0.015) [9.8] <b>48.8</b>
	+Call	4.25 (0.009) [6.6] <b>22.8</b>	4.43 (0.010) [6.5] <b>22.8</b>	5.12 (0.011) [6.5] <b>22.7</b>	8.38 (0.026) [6.6] <b>17.1</b>	8.67 (0.026) [6.6] <b>18.0</b>	9.95 (0.031) [6.5] <b>16.6</b>
Delta	+Auxiliary	4.26 (0.004) [22.1] <b>30.6</b>	4.43 (0.005) [22.0] <b>24.3</b>	5.10 (0.010) [22.1] <b>9.3</b>	8.40 (0.012) [22.0] <b>24.1</b>	8.73 (0.014) [22.3] <b>18.3</b>	9.95 (0.025) [22.1] <b>7.8</b>
	+Call	4.26 (0.004) [22.0] <b>35.9</b>	4.42 (0.005) [22.3] <b>32.0</b>	5.11 (0.008) [22.0] <b>13.2</b>	8.38 (0.011) [22.2] <b>30.0</b>	8.72 (0.012) [21.9] <b>27.2</b>	9.98 (0.020) [22.3] <b>12.5</b>
	+SV ( $\rho = 0$ )	4.27 (0.004) [24.2] <b>32.8</b>	4.43 (0.005) [23.8] <b>29.3</b>	5.11 (0.007) [24.1] <b>18.5</b>	8.41 (0.011) [24.1] <b>26.5</b>	8.71 (0.012) [24.2] <b>23.2</b>	9.96 (0.017) [23.8] <b>15.9</b>
	+SV ( $\rho \neq 0$ )	4.26 (0.003) [23.8] <b>73.7</b>	4.44 (0.004) [23.7] <b>48.8</b>	5.12 (0.009) [24.0] <b>10.2</b>	8.40 (0.008) [23.9] <b>53.1</b>	8.71 (0.010) [23.8] <b>35.6</b>	9.95 (0.022) [23.7] <b>9.2</b>
	+SV ( $\rho = 0$ ) +SV ( $\rho \neq 0$ )	4.26 (0.002) [26.2] <b>86.2</b>	4.43 (0.003) [25.4] <b>66.2</b>	5.11 (0.005) [25.5] <b>32.2</b>	8.39 (0.007) [25.5] <b>65.3</b>	8.71 (0.008) [25.4] <b>51.9</b>	9.95 (0.012) [25.3] <b>27.1</b>
All		4.26 (0.002) [26.2] <b>96.9</b>	4.43 (0.003) [25.9] <b>79.7</b>	5.11 (0.004) [25.9] <b>39.6</b>	8.40 (0.006) [25.6] <b>73.9</b>	8.72 (0.007) [25.6] <b>67.0</b>	9.96 (0.011) [25.4] <b>35.6</b>

Table 5.54: CVs in Combinations in the SABR Model: Average Rate Options (ATM)



		Case 1			Case 2		
CV Type		Av, 64	Av, 16	Av, 4	Av, 64	Av, 16	Av, 4
No CV (Plain)		21.48 (0.007) [610.4]	21.61 (0.007) [607.9]	22.08 (0.008) [608.4]	22.46 (0.016) [611.7]	22.70 (0.017) [609.8]	23.63 (0.018) [612.7]
Auxiliary	+SV ( $\rho = 0$ )	21.50 (0.007) [8.3] <b>85.3</b>	21.62 (0.007) [8.2] <b>86.0</b>	22.10 (0.008) [8.3] <b>74.9</b>	22.47 (0.013) [8.3] <b>108.6</b>	22.71 (0.014) [8.2] <b>104.7</b>	23.65 (0.018) [8.1] <b>77.7</b>
	+SV ( $\rho \neq 0$ )	21.48 (0.014) [8.3] <b>19.9</b>	21.62 (0.014) [8.2] <b>19.9</b>	22.06 (0.017) [8.2] <b>17.5</b>	22.51 (0.028) [8.3] <b>23.8</b>	22.75 (0.029) [8.2] <b>23.4</b>	23.71 (0.034) [8.2] <b>21.8</b>
	+SV ( $\rho = 0$ ) +SV ( $\rho \neq 0$ )	21.48 (0.007) [10.2] <b>66.7</b>	21.62 (0.007) [9.8] <b>74.4</b>	22.11 (0.008) [9.9] <b>66.3</b>	22.49 (0.012) [10.0] <b>106.1</b>	22.70 (0.013) [10.2] <b>95.4</b>	23.65 (0.016) [9.8] <b>81.5</b>
	+Call	21.50 (0.014) [6.7] <b>24.4</b>	21.57 (0.014) [6.6] <b>24.9</b>	22.08 (0.016) [6.8] <b>24.6</b>	22.48 (0.031) [6.6] <b>25.4</b>	22.69 (0.032) [6.7] <b>24.8</b>	23.59 (0.036) [6.5] <b>24.1</b>
Delta	+Auxiliary	21.50 (0.002) [22.5] <b>529.6</b>	21.61 (0.003) [22.6] <b>122.5</b>	22.10 (0.011) [22.5] <b>15.1</b>	22.48 (0.009) [22.7] <b>80.9</b>	22.69 (0.012) [23.4] <b>50.2</b>	23.68 (0.027) [22.6] <b>13.1</b>
	+Call	21.50 (0.002) [22.8] <b>564.0</b>	21.61 (0.003) [22.4] <b>211.9</b>	22.10 (0.009) [22.9] <b>23.3</b>	22.50 (0.008) [22.6] <b>105.5</b>	22.70 (0.010) [22.9] <b>78.8</b>	23.65 (0.022) [22.9] <b>18.9</b>
	+SV ( $\rho = 0$ )	21.50 (0.001) [24.5] <b>813.4</b>	21.61 (0.003) [24.2] <b>178.8</b>	22.10 (0.007) [24.2] <b>36.7</b>	22.48 (0.007) [24.4] <b>135.8</b>	22.71 (0.008) [24.3] <b>97.4</b>	23.66 (0.014) [24.8] <b>40.1</b>
	+SV ( $\rho \neq 0$ )	21.49 (0.002) [24.7] <b>433.7</b>	21.62 (0.003) [24.1] <b>110.5</b>	22.09 (0.013) [24.1] <b>10.6</b>	22.48 (0.008) [24.6] <b>94.6</b>	22.69 (0.011) [24.3] <b>53.9</b>	23.63 (0.030) [24.2] <b>9.4</b>
	+SV ( $\rho = 0$ ) +SV ( $\rho \neq 0$ )	21.49 (0.001) [26.0] <b>767.2</b>	21.61 (0.003) [25.7] <b>175.8</b>	22.08 (0.007) [26.1] <b>34.9</b>	22.48 (0.006) [26.0] <b>193.2</b>	22.70 (0.007) [25.9] <b>126.6</b>	23.65 (0.013) [26.4] <b>48.1</b>
All		21.50 (0.001) [26.2] <b>1014.1</b>	21.61 (0.002) [25.8] <b>287.3</b>	22.09 (0.006) [26.0] <b>47.5</b>	22.49 (0.005) [26.2] <b>222.0</b>	22.72 (0.006) [26.2] <b>175.2</b>	23.64 (0.012) [26.1] <b>59.7</b>

Table 5.55: CVs in Combinations in the SABR Model: Average Rate Options (ITM)

		Case 1			Case 2		
CV Type		Av, 64	Av, 16	Av, 4	Av, 64	Av, 16	Av, 4
No CV (Plain)		0.012 (0.0002) [607.3]	0.018 (0.0003) [608.8]	0.056 (0.0005) [608.1]	1.70 (0.005) [605.0]	1.87 (0.005) [605.4]	2.69 (0.007) [608.1]
Auxiliary	+SV ( $\rho = 0$ )	0.012 (0.0007) [8.3] <b>6.7</b>	0.019 (0.0009) [8.2] <b>6.6</b>	0.054 (0.0013) [8.1] <b>11.9</b>	1.70 (0.008) [8.3] <b>30.4</b>	1.87 (0.008) [8.1] <b>31.0</b>	2.70 (0.012) [8.1] <b>25.7</b>
	+SV ( $\rho \neq 0$ )	0.012 (0.0002) [8.4] <b>150.1</b>	0.018 (0.0002) [8.2] <b>174.4</b>	0.056 (0.0004) [8.2] <b>129.4</b>	1.71 (0.009) [8.4] <b>22.9</b>	1.88 (0.009) [8.2] <b>25.1</b>	2.70 (0.013) [8.2] <b>23.3</b>
	+SV ( $\rho = 0$ ) +SV ( $\rho \neq 0$ )	0.012 (0.0001) [10.1] <b>238.2</b>	0.017 (0.0001) [9.9] <b>253.7</b>	0.055 (0.0003) [9.8] <b>171.9</b>	1.68 (0.005) [10.3] <b>62.5</b>	1.88 (0.005) [9.8] <b>66.5</b>	2.69 (0.007) [10.0] <b>56.9</b>
	+Call	0.014 (0.0016) [6.6] <b>1.6</b>	0.017 (0.0018) [6.6] <b>2.2</b>	0.064 (0.0038) [6.8] <b>1.8</b>	1.70 (0.017) [6.7] <b>7.7</b>	1.91 (0.018) [6.6] <b>8.5</b>	2.71 (0.021) [6.7] <b>10.2</b>
Delta	+Auxiliary	0.013 (0.0007) [22.8] <b>2.6</b>	0.017 (0.0010) [22.7] <b>2.2</b>	0.053 (0.0028) [22.8] <b>1.0</b>	1.69 (0.010) [22.4] <b>7.6</b>	1.89 (0.011) [22.7] <b>6.2</b>	2.70 (0.020) [22.9] <b>3.1</b>
	+Call	0.011 (0.0007) [22.5] <b>2.5</b>	0.018 (0.0010) [22.5] <b>2.2</b>	0.055 (0.0029) [22.4] <b>0.9</b>	1.68 (0.009) [22.4] <b>8.7</b>	1.88 (0.010) [22.3] <b>8.6</b>	2.69 (0.016) [22.5] <b>5.2</b>
	+SV ( $\rho = 0$ )	0.013 (0.0006) [24.0] <b>3.8</b>	0.017 (0.0007) [24.0] <b>4.1</b>	0.053 (0.0012) [24.5] <b>4.7</b>	1.69 (0.007) [24.0] <b>11.7</b>	1.87 (0.008) [24.1] <b>10.6</b>	2.70 (0.012) [24.3] <b>8.8</b>
	+SV ( $\rho \neq 0$ )	0.012 (0.0001) [24.2] <b>56.3</b>	0.018 (0.0002) [24.0] <b>56.7</b>	0.056 (0.0004) [24.3] <b>44.5</b>	1.70 (0.006) [24.5] <b>20.7</b>	1.88 (0.007) [23.9] <b>16.8</b>	2.68 (0.012) [23.9] <b>8.1</b>
	+SV ( $\rho = 0$ ) +SV ( $\rho \neq 0$ )	0.012 (0.0001) [26.2] <b>90.4</b>	0.017 (0.0001) [26.3] <b>93.8</b>	0.055 (0.0003) [26.1] <b>59.1</b>	1.70 (0.005) [26.2] <b>28.2</b>	1.88 (0.005) [25.9] <b>25.9</b>	2.70 (0.007) [26.3] <b>20.5</b>
All		0.012 (0.0001) [26.5] <b>122.3</b>	0.017 (0.0001) [26.1] <b>102.4</b>	0.055 (0.0003) [26.1] <b>66.3</b>	1.69 (0.004) [26.3] <b>37.1</b>	1.88 (0.005) [26.3] <b>32.7</b>	2.70 (0.007) [26.1] <b>27.2</b>

Table 5.56: CVs in Combinations in the SABR Model: Average Rate Options (OTM)

		Case 1 ( $\rho = 0.9$ )			Case 2 ( $\rho = 0.9$ )		
CV Type		Av, 64	Av, 16	Av, 4	Av, 64	Av, 16	Av, 4
No CV (Plain)		4.07 (0.006) [607.3]	4.21 (0.006) [607.6]	4.87 (0.007) [606.6]	8.24 (0.014) [608.5]	8.54 (0.015) [604.6]	9.75 (0.017) [604.8]
Auxiliary	+SV ( $\rho = 0$ )	4.07 (0.009) [8.3] <b>30.6</b>	4.23 (0.010) [8.2] <b>27.5</b>	4.86 (0.011) [8.1] <b>30.2</b>	8.23 (0.023) [8.3] <b>29.0</b>	8.56 (0.026) [8.1] <b>24.5</b>	9.70 (0.029) [8.1] <b>25.4</b>
	+SV ( $\rho \neq 0$ )	4.08 (0.010) [8.4] <b>22.7</b>	4.23 (0.010) [8.2] <b>25.6</b>	4.87 (0.012) [8.2] <b>23.0</b>	8.17 (0.024) [8.4] <b>26.2</b>	8.55 (0.027) [8.2] <b>22.6</b>	9.80 (0.032) [8.2] <b>21.0</b>
	+SV ( $\rho = 0$ ) +SV ( $\rho \neq 0$ )	4.06 (0.008) [10.0] <b>27.6</b>	4.22 (0.009) [9.8] <b>26.5</b>	4.86 (0.011) [9.8] <b>25.3</b>	8.28 (0.026) [10.0] <b>18.7</b>	8.57 (0.027) [10.0] <b>18.1</b>	9.82 (0.032) [9.8] <b>17.7</b>
	+Call	4.08 (0.009) [6.6] <b>36.4</b>	4.23 (0.009) [6.6] <b>39.0</b>	4.87 (0.011) [6.6] <b>36.6</b>	8.26 (0.024) [6.8] <b>31.2</b>	8.57 (0.025) [6.6] <b>33.4</b>	9.76 (0.028) [6.6] <b>34.1</b>
Delta	+Auxiliary	4.07 (0.003) [22.6] <b>104.8</b>	4.23 (0.004) [22.3] <b>56.8</b>	4.86 (0.010) [22.6] <b>13.6</b>	8.25 (0.011) [22.4] <b>46.6</b>	8.57 (0.015) [22.4] <b>27.0</b>	9.77 (0.029) [22.6] <b>9.6</b>
	+Call	4.07 (0.003) [22.7] <b>119.2</b>	4.23 (0.003) [22.3] <b>84.7</b>	4.85 (0.008) [22.4] <b>18.9</b>	8.26 (0.010) [22.2] <b>57.5</b>	8.55 (0.011) [22.5] <b>52.0</b>	9.81 (0.023) [22.3] <b>15.3</b>
	+SV ( $\rho = 0$ )	4.07 (0.003) [24.3] <b>104.2</b>	4.23 (0.004) [23.8] <b>55.0</b>	4.87 (0.010) [24.0] <b>11.6</b>	8.25 (0.011) [23.8] <b>45.9</b>	8.54 (0.014) [24.2] <b>26.2</b>	9.76 (0.032) [24.0] <b>7.2</b>
	+SV ( $\rho \neq 0$ )	4.07 (0.003) [24.1] <b>120.0</b>	4.22 (0.004) [24.1] <b>55.9</b>	4.87 (0.013) [23.8] <b>7.3</b>	8.26 (0.010) [24.1] <b>46.9</b>	8.56 (0.013) [23.8] <b>31.6</b>	9.80 (0.038) [23.8] <b>5.2</b>
	+SV ( $\rho = 0$ ) +SV ( $\rho \neq 0$ )	4.07 (0.003) [26.0] <b>108.3</b>	4.23 (0.004) [26.1] <b>54.5</b>	4.85 (0.010) [25.8] <b>11.2</b>	8.26 (0.011) [26.1] <b>40.8</b>	8.54 (0.013) [25.4] <b>29.1</b>	9.80 (0.035) [25.7] <b>5.8</b>
All		4.07 (0.002) [26.1] <b>135.4</b>	4.23 (0.003) [25.8] <b>89.1</b>	4.86 (0.007) [25.7] <b>24.5</b>	8.25 (0.009) [25.9] <b>57.9</b>	8.56 (0.010) [25.4] <b>51.6</b>	9.77 (0.019) [25.8] <b>19.2</b>

Table 5.57: CVs in Combinations in the SABR Model: Average Rate Options ( $\rho = 0.9$ )

		Case 1 ( $\rho = -0.9$ )			Case 2 ( $\rho = -0.9$ )		
CV Type		Av, 64	Av, 16	Av, 4	Av, 64	Av, 16	Av, 4
No CV (Plain)		4.29 (0.004) [606.2]	4.47 (0.005) [601.1]	5.14 (0.005) [606.5]	8.40 (0.010) [604.7]	8.70 (0.011) [604.2]	9.94 (0.012) [605.8]
Auxiliary	+SV ( $\rho = 0$ )	4.30 (0.007) [8.3] <b>26.3</b>	4.44 (0.008) [8.1] <b>24.0</b>	5.15 (0.010) [8.1] <b>21.9</b>	8.37 (0.021) [8.4] <b>18.1</b>	8.74 (0.022) [8.2] <b>17.9</b>	9.99 (0.026) [8.1] <b>16.9</b>
	+SV ( $\rho \neq 0$ )	4.30 (0.008) [8.3] <b>21.1</b>	4.46 (0.009) [8.2] <b>20.2</b>	5.16 (0.011) [8.3] <b>17.6</b>	8.41 (0.023) [8.4] <b>14.5</b>	8.71 (0.025) [8.2] <b>13.9</b>	9.96 (0.030) [8.2] <b>12.1</b>
	+SV ( $\rho = 0$ ) +SV ( $\rho \neq 0$ )	4.27 (0.007) [10.2] <b>21.0</b>	4.44 (0.008) [9.9] <b>21.5</b>	5.15 (0.010) [9.8] <b>18.8</b>	8.38 (0.020) [10.0] <b>16.0</b>	8.70 (0.021) [10.0] <b>16.2</b>	9.87 (0.026) [9.8] <b>13.9</b>
	+Call	4.30 (0.008) [6.8] <b>25.6</b>	4.47 (0.009) [6.7] <b>23.6</b>	5.16 (0.010) [6.6] <b>23.5</b>	8.36 (0.023) [6.6] <b>18.1</b>	8.69 (0.024) [6.6] <b>18.7</b>	9.93 (0.029) [6.6] <b>16.8</b>
Delta	+Auxiliary	4.29 (0.003) [22.4] <b>61.0</b>	4.46 (0.004) [22.1] <b>42.2</b>	5.14 (0.008) [22.4] <b>11.9</b>	8.38 (0.008) [22.3] <b>46.0</b>	8.69 (0.010) [22.3] <b>33.3</b>	9.96 (0.020) [22.3] <b>10.0</b>
	+Call	4.29 (0.003) [22.4] <b>60.6</b>	4.46 (0.003) [22.4] <b>48.4</b>	5.16 (0.007) [22.1] <b>14.5</b>	8.38 (0.008) [22.4] <b>48.5</b>	8.68 (0.009) [22.4] <b>41.1</b>	9.96 (0.017) [22.2] <b>13.8</b>
	+SV ( $\rho = 0$ )	4.28 (0.003) [23.9] <b>59.4</b>	4.46 (0.004) [24.0] <b>41.6</b>	5.16 (0.007) [24.0] <b>12.5</b>	8.38 (0.008) [23.9] <b>47.4</b>	8.69 (0.009) [23.5] <b>34.8</b>	9.93 (0.018) [23.5] <b>11.8</b>
	+SV ( $\rho \neq 0$ )	4.29 (0.003) [24.3] <b>65.8</b>	4.46 (0.004) [23.7] <b>41.2</b>	5.15 (0.010) [24.1] <b>7.5</b>	8.38 (0.007) [24.0] <b>51.8</b>	8.68 (0.009) [23.6] <b>34.1</b>	9.97 (0.024) [23.6] <b>6.8</b>
	+SV ( $\rho = 0$ ) +SV ( $\rho \neq 0$ )	4.29 (0.002) [25.5] <b>77.8</b>	4.45 (0.003) [25.3] <b>48.6</b>	5.13 (0.007) [25.8] <b>12.1</b>	8.38 (0.007) [25.7] <b>57.9</b>	8.70 (0.008) [25.2] <b>40.7</b>	9.96 (0.018) [25.7] <b>11.4</b>
All		4.29 (0.002) [25.8] <b>100.4</b>	4.46 (0.003) [25.4] <b>74.2</b>	5.14 (0.006) [26.5] <b>20.7</b>	8.38 (0.006) [25.6] <b>77.8</b>	8.69 (0.007) [25.5] <b>62.2</b>	9.96 (0.014) [25.5] <b>19.1</b>

Table 5.58: CVs in Combinations in the SABR Model: Average Rate Options ( $\rho = -0.9$ )

		Case 1 ( $V_0 = 8$ )			Case 2 ( $V_0 = 3$ )		
CV Type		Av, 64	Av, 16	Av, 4	Av, 64	Av, 16	Av, 4
No CV (Plain)		12.45 (0.017) [606.0]	12.94 (0.018) [601.2]	14.69 (0.020) [603.0]	11.84 (0.017) [605.5]	12.30 (0.018) [601.7]	14.00 (0.020) [603.8]
Auxiliary	+SV ( $\rho = 0$ )	12.48 (0.022) [8.2] <b>44.3</b>	12.87 (0.023) [8.1] <b>44.6</b>	14.69 (0.028) [8.1] <b>38.7</b>	11.82 (0.022) [8.2] <b>44.5</b>	12.24 (0.023) [8.1] <b>42.8</b>	14.04 (0.029) [8.1] <b>34.9</b>
	+SV ( $\rho \neq 0$ )	12.50 (0.038) [8.3] <b>15.4</b>	12.93 (0.038) [8.2] <b>16.0</b>	14.71 (0.046) [8.2] <b>14.4</b>	11.90 (0.034) [8.3] <b>18.5</b>	12.30 (0.036) [8.4] <b>17.6</b>	13.97 (0.042) [8.2] <b>16.7</b>
	+SV ( $\rho = 0$ ) +SV ( $\rho \neq 0$ )	12.44 (0.017) [10.0] <b>66.3</b>	12.91 (0.018) [9.8] <b>61.1</b>	14.67 (0.022) [9.7] <b>53.6</b>	11.81 (0.017) [10.1] <b>56.3</b>	12.28 (0.019) [10.0] <b>54.1</b>	14.03 (0.023) [9.7] <b>48.5</b>
	+Call	12.43 (0.045) [6.6] <b>13.5</b>	12.91 (0.048) [6.6] <b>12.8</b>	14.71 (0.054) [6.7] <b>12.9</b>	11.84 (0.042) [6.8] <b>14.7</b>	12.25 (0.044) [6.6] <b>14.8</b>	13.97 (0.050) [6.7] <b>14.2</b>
Delta	+Auxiliary	12.42 (0.020) [22.7] <b>19.2</b>	12.90 (0.024) [22.3] <b>15.7</b>	14.71 (0.043) [22.7] <b>6.0</b>	11.83 (0.019) [22.7] <b>21.3</b>	12.29 (0.022) [22.3] <b>17.6</b>	14.06 (0.040) [22.5] <b>6.6</b>
	+Call	12.45 (0.018) [22.2] <b>24.1</b>	12.92 (0.019) [22.6] <b>23.3</b>	14.73 (0.031) [22.5] <b>11.8</b>	11.84 (0.017) [22.3] <b>27.8</b>	12.33 (0.018) [22.3] <b>25.7</b>	14.00 (0.029) [22.3] <b>12.7</b>
	+SV ( $\rho = 0$ )	12.44 (0.018) [23.9] <b>24.4</b>	12.90 (0.019) [23.8] <b>22.6</b>	14.70 (0.026) [23.8] <b>15.6</b>	11.84 (0.017) [23.9] <b>25.7</b>	12.26 (0.019) [23.9] <b>22.2</b>	13.98 (0.026) [24.0] <b>15.1</b>
	+SV ( $\rho \neq 0$ )	12.43 (0.014) [24.2] <b>40.3</b>	12.91 (0.017) [24.3] <b>27.2</b>	14.77 (0.035) [24.0] <b>8.7</b>	11.87 (0.013) [24.4] <b>39.3</b>	12.28 (0.016) [24.5] <b>29.8</b>	14.00 (0.034) [24.2] <b>8.7</b>
	+SV ( $\rho = 0$ ) +SV ( $\rho \neq 0$ )	12.44 (0.011) [25.6] <b>55.4</b>	12.91 (0.013) [25.6] <b>44.5</b>	14.70 (0.019) [25.4] <b>27.7</b>	11.84 (0.011) [26.0] <b>51.4</b>	12.29 (0.013) [25.5] <b>43.7</b>	13.95 (0.019) [25.4] <b>25.3</b>
All		12.43 (0.010) [25.7] <b>64.7</b>	12.90 (0.011) [25.5] <b>61.6</b>	14.71 (0.022) [25.9] <b>19.4</b>	11.85 (0.010) [26.1] <b>63.2</b>	12.28 (0.011) [25.6] <b>62.9</b>	13.97 (0.017) [26.0] <b>33.4</b>

Table 5.59: CVs in Combinations in the SABR Model: Average Rate Options (High Volatility)

Average Rate Options, Case 1						
Resets	Plain	GBM		SV		
		Auxiliary	Delta	$(\rho = 0)$	$(\rho \neq 0)$	Both
4	4.64	4.64	4.64	4.65	4.63	4.64
	(0.006)	(0.012)	(0.013)	(0.007)	(0.025)	(0.006)
	[599.0]	[6.5]	[20.9]	[7.6]	[8.0]	[9.6]
	-	<b>24.7</b>	<b>6.5</b>	<b>66.7</b>	<b>4.7</b>	<b>68.9</b>
16	4.06	4.03	4.04	4.05	4.06	4.05
	(0.005)	(0.010)	(0.005)	(0.006)	(0.021)	(0.005)
	[598.4]	[6.5]	[20.9]	[7.6]	[8.0]	[9.7]
	-	<b>26.5</b>	<b>29.2</b>	<b>68.8</b>	<b>4.8</b>	<b>77.9</b>
64	3.90	3.91	3.91	3.90	3.91	3.90
	(0.005)	(0.010)	(0.004)	(0.005)	(0.021)	(0.004)
	[610.8]	[6.6]	[21.0]	[7.7]	[8.3]	[10.0]
	-	<b>26.0</b>	<b>42.7</b>	<b>70.7</b>	<b>4.7</b>	<b>83.5</b>

Average Rate Options, Case 2						
Resets	Plain	GBM		SV		
		Auxiliary	Delta	$(\rho = 0)$	$(\rho \neq 0)$	Both
4	8.04	8.06	8.06	8.04	8.04	8.07
	(0.011)	(0.021)	(0.025)	(0.010)	(0.045)	(0.009)
	[602.1]	[6.5]	[21.0]	[7.6]	[8.1]	[9.6]
	-	<b>26.7</b>	<b>5.8</b>	<b>106.8</b>	<b>4.5</b>	<b>105.5</b>
16	6.96	7.02	6.97	6.98	7.06	6.97
	(0.010)	(0.018)	(0.009)	(0.008)	(0.039)	(0.007)
	[598.0]	[6.5]	[21.0]	[7.6]	[8.1]	[9.7]
	-	<b>27.2</b>	<b>30.0</b>	<b>112.4</b>	<b>4.4</b>	<b>123.5</b>
64	6.70	6.74	6.69	6.71	6.68	6.70
	(0.009)	(0.017)	(0.007)	(0.008)	(0.037)	(0.007)
	[601.0]	[6.6]	[21.0]	[7.7]	[8.3]	[10.0]
	-	<b>26.9</b>	<b>48.1</b>	<b>115.2</b>	<b>4.4</b>	<b>120.5</b>

Table 5.60: Efficiency Gains in the GARCH Model: Average Rate Options (ATM)

Average Rate Options, Case 1						
Resets	Plain	GBM		SV		
		Auxiliary	Delta	$(\rho = 0)$	$(\rho \neq 0)$	Both
4	21.10	21.11	21.09	21.10	21.12	21.10
	(0.009)	(0.016)	(0.016)	(0.007)	(0.047)	(0.007)
	[604.0]	[6.5]	[22.2]	[7.5]	[8.0]	[9.6]
	-	<b>32.5</b>	<b>9.8</b>	<b>133.4</b>	<b>3.1</b>	<b>101.5</b>
16	20.83	20.86	20.84	20.83	20.85	20.83
	(0.008)	(0.014)	(0.004)	(0.006)	(0.041)	(0.006)
	[599.6]	[6.5]	[21.7]	[7.6]	[8.1]	[9.7]
	-	<b>34.4</b>	<b>124.7</b>	<b>135.6</b>	<b>3.1</b>	<b>107.0</b>
64	20.76	20.75	20.77	20.78	20.76	20.78
	(0.008)	(0.013)	(0.002)	(0.006)	(0.041)	(0.006)
	[601.3]	[6.6]	[21.7]	[7.8]	[8.3]	[10.0]
	-	<b>34.0</b>	<b>521.0</b>	<b>141.9</b>	<b>2.9</b>	<b>111.5</b>

Average Rate Options, Case 2						
Resets	Plain	GBM		SV		
		Auxiliary	Delta	$(\rho = 0)$	$(\rho \neq 0)$	Both
4	22.51	22.54	22.60	22.54	22.61	22.54
	(0.015)	(0.026)	(0.030)	(0.009)	(0.074)	(0.008)
	[599.7]	[6.5]	[21.8]	[7.6]	[8.0]	[9.6]
	-	<b>33.8</b>	<b>7.5</b>	<b>237.3</b>	<b>3.3</b>	<b>210.3</b>
16	21.91	21.89	21.87	21.89	21.87	21.87
	(0.014)	(0.022)	(0.009)	(0.007)	(0.066)	(0.006)
	[597.5]	[6.6]	[21.8]	[7.8]	[8.1]	[9.7]
	-	<b>36.0</b>	<b>62.9</b>	<b>309.0</b>	<b>3.1</b>	<b>273.5</b>
64	21.72	21.72	21.72	21.71	21.73	21.71
	(0.013)	(0.021)	(0.006)	(0.006)	(0.064)	(0.006)
	[600.9]	[6.6]	[22.0]	[7.7]	[8.3]	[10.0]
	-	<b>36.7</b>	<b>144.5</b>	<b>337.5</b>	<b>3.1</b>	<b>280.4</b>

Table 5.61: Efficiency Gains in the GARCH Model: Average Rate Options (ITM)

Average Rate Options, Case 1						
Resets	Plain	GBM		SV		
		Auxiliary	Delta	$(\rho = 0)$	$(\rho \neq 0)$	Both
4	0.13	0.12	0.12	0.13	0.13	0.13
	(0.001)	(0.004)	(0.005)	(0.002)	(0.001)	(0.001)
	[600.0]	[6.5]	[21.6]	[7.6]	[8.0]	[9.7]
	-	<b>4.5</b>	<b>1.0</b>	<b>26.7</b>	<b>39.2</b>	<b>70.1</b>
16	0.05	0.05	0.05	0.05	0.05	0.05
	(0.001)	(0.003)	(0.002)	(0.001)	(0.001)	(0.001)
	[601.8]	[6.5]	[21.6]	[7.6]	[8.1]	[9.9]
	-	<b>3.7</b>	<b>3.3</b>	<b>23.7</b>	<b>42.7</b>	<b>67.6</b>
64	0.039	0.046	0.042	0.040	0.040	0.039
	(0.0005)	(0.0027)	(0.0012)	(0.0010)	(0.0006)	(0.0004)
	[602.0]	[6.5]	[21.8]	[7.7]	[8.3]	[10.0]
	-	<b>2.7</b>	<b>4.3</b>	<b>18.7</b>	<b>44.9</b>	<b>66.1</b>

Average Rate Options, Case 2						
Resets	Plain	GBM		SV		
		Auxiliary	Delta	$(\rho = 0)$	$(\rho \neq 0)$	Both
4	1.61	1.61	1.61	1.61	1.59	1.60
	(0.005)	(0.018)	(0.019)	(0.006)	(0.014)	(0.005)
	[610.8]	[6.5]	[21.6]	[7.6]	[8.0]	[9.7]
	-	<b>8.0</b>	<b>2.1</b>	<b>60.9</b>	<b>10.6</b>	<b>65.8</b>
16	1.01	1.01	1.00	1.02	1.03	1.02
	(0.004)	(0.013)	(0.007)	(0.004)	(0.010)	(0.004)
	[605.6]	[6.5]	[21.9]	[7.7]	[8.1]	[9.8]
	-	<b>7.3</b>	<b>7.7</b>	<b>64.1</b>	<b>11.3</b>	<b>61.7</b>
64	0.88	0.88	0.89	0.89	0.90	0.88
	(0.003)	(0.013)	(0.005)	(0.004)	(0.009)	(0.003)
	[601.5]	[6.6]	[21.9]	[7.7]	[8.3]	[10.1]
	-	<b>6.6</b>	<b>11.3</b>	<b>62.2</b>	<b>11.2</b>	<b>67.7</b>

Table 5.62: Efficiency Gains in the GARCH Model: Average Rate Options (OTM)



Average Rate Options, Case 1 ( $\rho = 0.9$ )						
Resets	Plain	GBM		SV		
		Auxiliary	Delta	( $\rho = 0$ )	( $\rho \neq 0$ )	Both
4	4.59	4.60	4.58	4.56	4.50	4.56
	(0.007)	(0.016)	(0.014)	(0.016)	(0.065)	(0.015)
	[599.6]	[6.5]	[21.0]	[7.5]	[8.0]	[9.8]
	-	<b>19.0</b>	<b>7.8</b>	<b>17.3</b>	<b>1.0</b>	<b>13.9</b>
16	3.98	3.97	3.99	3.97	3.98	3.99
	(0.006)	(0.014)	(0.005)	(0.013)	(0.057)	(0.013)
	[600.5]	[6.5]	[20.9]	[7.6]	[8.0]	[9.9]
	-	<b>19.7</b>	<b>45.7</b>	<b>19.9</b>	<b>0.9</b>	<b>13.9</b>
64	3.83	3.82	3.84	3.85	3.82	3.83
	(0.006)	(0.014)	(0.004)	(0.012)	(0.055)	(0.013)
	[604.1]	[6.5]	[20.9]	[7.7]	[8.2]	[10.0]
	-	<b>17.5</b>	<b>79.9</b>	<b>19.1</b>	<b>0.9</b>	<b>13.8</b>

Average Rate Options, Case 2 ( $\rho = 0.9$ )						
Resets	Plain	GBM		SV		
		Auxiliary	Delta	( $\rho = 0$ )	( $\rho \neq 0$ )	Both
4	7.96	7.97	7.92	7.94	8.10	7.96
	(0.013)	(0.035)	(0.029)	(0.023)	(0.117)	(0.025)
	[598.7]	[6.5]	[21.0]	[7.6]	[8.0]	[9.7]
	-	<b>13.0</b>	<b>6.0</b>	<b>24.7</b>	<b>0.9</b>	<b>16.4</b>
16	6.91	6.88	6.89	6.90	6.84	6.92
	(0.011)	(0.028)	(0.011)	(0.022)	(0.100)	(0.021)
	[598.5]	[6.5]	[20.9]	[7.6]	[8.0]	[9.8]
	-	<b>14.9</b>	<b>31.5</b>	<b>20.0</b>	<b>1.0</b>	<b>17.1</b>
64	6.64	6.65	6.63	6.65	6.60	6.64
	(0.011)	(0.027)	(0.009)	(0.023)	(0.096)	(0.020)
	[602.3]	[6.6]	[21.3]	[7.7]	[8.2]	[10.1]
	-	<b>14.6</b>	<b>45.6</b>	<b>17.0</b>	<b>0.9</b>	<b>18.0</b>

Table 5.63: Efficiency Gains in the GARCH Model: Average Rate Options ( $\rho = 0.9$ )

Average Rate Options, Case 1 ( $\rho = -0.9$ )						
Resets	Plain	GBM		SV		
		Auxiliary	Delta	( $\rho = 0$ )	( $\rho \neq 0$ )	Both
4	4.65	4.60	4.63	4.63	4.66	4.62
	(0.006)	(0.010)	(0.012)	(0.011)	(0.051)	(0.011)
	[605.8]	[6.5]	[20.9]	[7.6]	[8.0]	[9.7]
	-	<b>33.0</b>	<b>6.9</b>	<b>21.9</b>	<b>1.0</b>	<b>17.6</b>
16	4.03	4.04	4.04	4.04	4.05	4.03
	(0.005)	(0.008)	(0.004)	(0.009)	(0.043)	(0.010)
	[600.6]	[6.5]	[20.9]	[7.6]	[8.0]	[9.7]
	-	<b>36.1</b>	<b>37.8</b>	<b>24.7</b>	<b>1.0</b>	<b>17.4</b>
64	3.88	3.90	3.89	3.89	3.90	3.91
	(0.005)	(0.008)	(0.003)	(0.009)	(0.042)	(0.009)
	[597.1]	[6.5]	[21.1]	[7.7]	[8.3]	[10.0]
	-	<b>34.7</b>	<b>61.7</b>	<b>23.7</b>	<b>1.0</b>	<b>17.7</b>
Average Rate Options, Case 2 ( $\rho = -0.9$ )						
Resets	Plain	GBM		SV		
		Auxiliary	Delta	( $\rho = 0$ )	( $\rho \neq 0$ )	Both
4	8.03	8.06	8.04	8.06	8.04	8.04
	(0.011)	(0.017)	(0.023)	(0.016)	(0.091)	(0.017)
	[598.3]	[6.5]	[20.9]	[7.5]	[8.0]	[9.6]
	-	<b>37.6</b>	<b>5.9</b>	<b>35.3</b>	<b>1.0</b>	<b>25.2</b>
16	6.94	6.95	6.97	6.96	6.92	6.97
	(0.009)	(0.014)	(0.008)	(0.014)	(0.079)	(0.013)
	[600.1]	[6.5]	[20.9]	[7.6]	[8.0]	[9.7]
	-	<b>37.7</b>	<b>35.3</b>	<b>36.4</b>	<b>1.0</b>	<b>29.8</b>
64	6.70	6.73	6.70	6.68	6.66	6.70
	(0.009)	(0.013)	(0.006)	(0.013)	(0.076)	(0.013)
	[598.9]	[6.5]	[20.9]	[7.6]	[8.3]	[10.0]
	-	<b>41.1</b>	<b>63.8</b>	<b>36.4</b>	<b>1.0</b>	<b>29.2</b>

Table 5.64: Efficiency Gains in the GARCH Model: Average Rate Options ( $\rho = -0.9$ )

Average Rate Options, Case 1 ( $V_0 = \mu = 0.25$ )						
Resets	Plain	GBM		SV		
		Auxiliary	Delta	( $\rho = 0$ )	( $\rho \neq 0$ )	Both
4	13.85	13.89	13.91	13.86	13.89	13.82
	(0.023)	(0.049)	(0.052)	(0.032)	(0.088)	(0.028)
	[600.5]	[6.5]	[20.9]	[7.6]	[8.0]	[9.6]
	-	<b>20.4</b>	<b>5.7</b>	<b>42.8</b>	<b>5.3</b>	<b>43.1</b>
16	12.19	12.24	12.20	12.25	12.20	12.22
	(0.020)	(0.042)	(0.023)	(0.025)	(0.077)	(0.022)
	[602.7]	[6.5]	[21.0]	[7.6]	[8.0]	[9.7]
	-	<b>21.4</b>	<b>22.5</b>	<b>53.3</b>	<b>5.2</b>	<b>54.5</b>
64	11.83	11.80	11.80	11.85	11.88	11.82
	(0.019)	(0.040)	(0.019)	(0.022)	(0.075)	(0.020)
	[600.2]	[6.6]	[21.0]	[7.7]	[8.2]	[10.0]
	-	<b>21.7</b>	<b>30.6</b>	<b>59.5</b>	<b>5.0</b>	<b>57.9</b>

Average Rate Options, Case 2 ( $V_0 = \mu = 0.25$ )						
Resets	Plain	GBM		SV		
		Auxiliary	Delta	( $\rho = 0$ )	( $\rho \neq 0$ )	Both
4	14.54	14.52	14.48	14.56	14.61	14.54
	(0.024)	(0.045)	(0.052)	(0.026)	(0.092)	(0.025)
	[600.6]	[6.5]	[21.0]	[7.6]	[8.0]	[9.6]
	-	<b>25.5</b>	<b>5.9</b>	<b>65.3</b>	<b>5.0</b>	<b>54.8</b>
16	12.78	12.76	12.76	12.78	12.65	12.78
	(0.021)	(0.039)	(0.022)	(0.021)	(0.079)	(0.018)
	[601.7]	[6.5]	[21.0]	[7.6]	[8.0]	[9.7]
	-	<b>25.4</b>	<b>25.9</b>	<b>80.2</b>	<b>5.2</b>	<b>80.0</b>
64	12.33	12.40	12.34	12.31	12.26	12.35
	(0.020)	(0.035)	(0.018)	(0.019)	(0.077)	(0.017)
	[605.2]	[6.6]	[21.0]	[7.7]	[8.2]	[10.0]
	-	<b>29.8</b>	<b>35.9</b>	<b>81.7</b>	<b>4.8</b>	<b>78.4</b>

Table 5.65: Efficiency Gains in the GARCH Model: Average Rate Options ( $V_0 = \mu = 0.25$ )

		Case 1			Case 2		
CV Type		Av, 64	Av, 16	Av, 4	Av, 64	Av, 16	Av, 4
No CV (Plain)		3.90 (0.005) [610.8]	4.06 (0.005) [598.4]	4.64 (0.006) [599.0]	6.70 (0.009) [601.0]	6.96 (0.010) [598.0]	8.04 (0.011) [602.1]
Auxiliary	+SV ( $\rho = 0$ )	3.89 (0.005) [8.3] <b>74.1</b>	4.05 (0.005) [8.2] <b>71.7</b>	4.63 (0.006) [8.1] <b>69.0</b>	6.70 (0.007) [8.5] <b>118.0</b>	6.97 (0.008) [8.2] <b>115.9</b>	8.06 (0.009) [8.2] <b>104.5</b>
		3.89 (0.008) [8.3] <b>34.6</b>	4.04 (0.008) [8.1] <b>32.8</b>	4.63 (0.009) [8.0] <b>32.3</b>	6.69 (0.014) [8.3] <b>31.8</b>	6.95 (0.015) [8.1] <b>31.1</b>	8.03 (0.017) [8.0] <b>31.6</b>
		3.90 (0.004) [10.1] <b>92.3</b>	4.04 (0.004) [9.8] <b>89.9</b>	4.65 (0.005) [9.7] <b>85.6</b>	6.71 (0.006) [10.1] <b>140.7</b>	6.97 (0.007) [9.8] <b>120.5</b>	8.05 (0.008) [9.7] <b>128.0</b>
	+SV ( $\rho \neq 0$ )	3.90 (0.004) [21.0] <b>43.2</b>	4.05 (0.005) [21.2] <b>30.7</b>	4.63 (0.010) [21.0] <b>11.4</b>	6.69 (0.007) [21.3] <b>49.0</b>	6.98 (0.010) [21.0] <b>27.1</b>	8.03 (0.019) [21.1] <b>9.6</b>
		3.90 (0.004) [22.8] <b>47.9</b>	4.05 (0.004) [22.7] <b>39.2</b>	4.64 (0.006) [22.5] <b>23.9</b>	6.70 (0.006) [22.8] <b>60.7</b>	6.97 (0.007) [22.7] <b>50.0</b>	8.05 (0.010) [22.8] <b>33.9</b>
		3.91 (0.003) [22.7] <b>83.3</b>	4.05 (0.004) [22.6] <b>48.4</b>	4.65 (0.010) [22.7] <b>9.4</b>	6.71 (0.006) [22.7] <b>67.0</b>	6.95 (0.008) [22.5] <b>37.9</b>	8.06 (0.021) [22.5] <b>7.4</b>
Delta	+SV ( $\rho = 0$ )	3.90 (0.003) [24.5] <b>101.0</b>	4.05 (0.003) [24.5] <b>67.0</b>	4.63 (0.005) [24.1] <b>33.2</b>	6.70 (0.005) [24.4] <b>88.9</b>	6.97 (0.006) [24.2] <b>66.8</b>	8.05 (0.008) [24.1] <b>45.1</b>
		3.90 (0.003) [24.5] <b>97.5</b>	4.05 (0.003) [24.2] <b>66.6</b>	4.64 (0.005) [24.1] <b>34.8</b>	6.70 (0.005) [24.7] <b>88.8</b>	6.97 (0.006) [24.2] <b>69.2</b>	8.06 (0.008) [24.2] <b>48.9</b>
		3.90 (0.003) [24.5] <b>97.5</b>	4.05 (0.003) [24.2] <b>66.6</b>	4.64 (0.005) [24.1] <b>34.8</b>	6.70 (0.005) [24.7] <b>88.8</b>	6.97 (0.006) [24.2] <b>69.2</b>	8.06 (0.008) [24.2] <b>48.9</b>
	All	3.90 (0.003) [24.5] <b>97.5</b>	4.05 (0.003) [24.2] <b>66.6</b>	4.64 (0.005) [24.1] <b>34.8</b>	6.70 (0.005) [24.7] <b>88.8</b>	6.97 (0.006) [24.2] <b>69.2</b>	8.06 (0.008) [24.2] <b>48.9</b>

Table 5.66: CVs in Combinations in the GARCH Model: Average Rate Options (ATM)

		Case 1			Case 2		
CV Type		Av, 64	Av, 16	Av, 4	Av, 64	Av, 16	Av, 4
No CV (Plain)		20.76 (0.008) [601.3]	20.83 (0.008) [599.6]	21.10 (0.009) [604.0]	21.72 (0.013) [600.9]	21.91 (0.014) [597.5]	22.51 (0.015) [599.7]
Auxiliary	+SV ( $\rho = 0$ )	20.77 (0.006) [8.3] <b>149.1</b>	20.84 (0.006) [8.2] <b>143.1</b>	21.08 (0.007) [8.2] <b>127.8</b>	21.72 (0.006) [8.4] <b>401.0</b>	21.87 (0.006) [8.2] <b>376.7</b>	22.55 (0.008) [8.2] <b>288.4</b>
		20.75 (0.013) [8.4] <b>30.0</b>	20.83 (0.013) [8.1] <b>29.6</b>	21.08 (0.015) [8.1] <b>29.4</b>	21.73 (0.019) [8.4] <b>33.7</b>	21.86 (0.020) [8.2] <b>32.9</b>	22.53 (0.023) [8.4] <b>31.9</b>
		20.77 (0.006) [10.3] <b>119.2</b>	20.84 (0.006) [9.8] <b>127.2</b>	21.10 (0.007) [9.7] <b>116.1</b>	21.72 (0.005) [10.1] <b>350.1</b>	21.87 (0.006) [9.8] <b>306.6</b>	22.54 (0.008) [9.7] <b>237.2</b>
	+SV ( $\rho \neq 0$ )	20.77 (0.002) [22.0] <b>526.5</b>	20.83 (0.004) [21.5] <b>130.3</b>	21.11 (0.012) [21.5] <b>18.3</b>	21.72 (0.006) [21.8] <b>151.4</b>	21.88 (0.009) [21.7] <b>59.7</b>	22.47 (0.023) [21.7] <b>12.0</b>
		20.77 (0.001) [23.4] <b>884.9</b>	20.84 (0.003) [23.1] <b>215.6</b>	21.09 (0.007) [23.1] <b>55.5</b>	21.71 (0.004) [23.6] <b>350.6</b>	21.87 (0.005) [23.3] <b>199.8</b>	22.54 (0.008) [23.4] <b>88.7</b>
		20.77 (0.002) [23.3] <b>494.5</b>	20.84 (0.004) [23.4] <b>115.0</b>	21.09 (0.015) [23.2] <b>10.8</b>	21.72 (0.006) [23.5] <b>139.2</b>	21.87 (0.009) [23.3] <b>58.4</b>	22.55 (0.028) [23.4] <b>8.0</b>
Delta	+SV ( $\rho = 0$ )	20.77 (0.001) [25.2] <b>854.9</b>	20.83 (0.003) [24.8] <b>193.6</b>	21.10 (0.006) [24.7] <b>52.9</b>	21.71 (0.004) [25.4] <b>336.6</b>	21.88 (0.005) [25.4] <b>193.3</b>	22.54 (0.008) [25.2] <b>88.8</b>
		20.77 (0.001) [25.2] <b>854.9</b>	20.83 (0.003) [24.8] <b>193.6</b>	21.10 (0.006) [24.7] <b>52.9</b>	21.71 (0.004) [25.4] <b>336.6</b>	21.88 (0.005) [25.4] <b>193.3</b>	22.54 (0.008) [25.2] <b>88.8</b>
		20.77 (0.001) [25.2] <b>854.9</b>	20.83 (0.003) [24.8] <b>193.6</b>	21.10 (0.006) [24.7] <b>52.9</b>	21.71 (0.004) [25.4] <b>336.6</b>	21.88 (0.005) [25.4] <b>193.3</b>	22.54 (0.008) [25.2] <b>88.8</b>
	+SV ( $\rho \neq 0$ )	20.77 (0.001) [25.2] <b>854.9</b>	20.83 (0.003) [24.8] <b>193.6</b>	21.10 (0.006) [24.7] <b>52.9</b>	21.71 (0.004) [25.4] <b>336.6</b>	21.88 (0.005) [25.4] <b>193.3</b>	22.54 (0.008) [25.2] <b>88.8</b>
All		20.77 (0.001) [25.1] <b>870.3</b>	20.83 (0.003) [24.8] <b>203.2</b>	21.09 (0.006) [24.7] <b>54.4</b>	21.72 (0.003) [25.6] <b>351.6</b>	21.87 (0.005) [25.1] <b>200.7</b>	22.55 (0.007) [24.9] <b>108.5</b>

Table 5.67: CVs in Combinations in the GARCH Model: Average Rate Options (ITM)

		Case 1			Case 2		
CV Type		Av, 64	Av, 16	Av, 4	Av, 64	Av, 16	Av, 4
No CV (Plain)		0.039 (0.0005) [602.0]	0.053 (0.0006) [601.8]	0.128 (0.0010) [600.0]	0.88 (0.003) [601.5]	1.01 (0.004) [605.6]	1.61 (0.005) [610.8]
Auxiliary	+SV ( $\rho = 0$ )	0.040 (0.0009) [8.3] <b>20.0</b>	0.052 (0.0009) [8.1] <b>29.8</b>	0.128 (0.0014) [8.1] <b>33.2</b>	0.89 (0.004) [8.3] <b>62.4</b>	1.02 (0.004) [8.1] <b>61.7</b>	1.62 (0.006) [8.1] <b>56.0</b>
		0.039 (0.0006) [8.3] <b>39.4</b>	0.051 (0.0007) [8.1] <b>50.1</b>	0.125 (0.0012) [8.0] <b>46.2</b>	0.87 (0.007) [8.3] <b>16.5</b>	1.01 (0.008) [8.1] <b>16.3</b>	1.59 (0.011) [8.0] <b>17.3</b>
		0.039 (0.0004) [10.1] <b>82.1</b>	0.051 (0.0005) [9.8] <b>84.8</b>	0.127 (0.0009) [9.7] <b>74.0</b>	0.88 (0.003) [10.1] <b>69.2</b>	1.01 (0.003) [9.8] <b>76.9</b>	1.61 (0.005) [9.7] <b>74.7</b>
	+SV ( $\rho \neq 0$ )	0.039 (0.0010) [21.7] <b>5.9</b>	0.052 (0.0014) [21.7] <b>4.3</b>	0.127 (0.0035) [21.7] <b>2.2</b>	0.89 (0.005) [22.0] <b>11.9</b>	1.01 (0.007) [21.9] <b>7.4</b>	1.61 (0.016) [21.8] <b>2.8</b>
		0.039 (0.0008) [23.7] <b>9.9</b>	0.050 (0.0008) [23.3] <b>12.1</b>	0.126 (0.0015) [23.7] <b>10.1</b>	0.88 (0.004) [23.7] <b>20.1</b>	1.01 (0.004) [23.8] <b>20.6</b>	1.61 (0.005) [23.7] <b>23.7</b>
		0.039 (0.0006) [23.6] <b>17.0</b>	0.051 (0.0007) [23.2] <b>17.6</b>	0.127 (0.0014) [23.3] <b>12.8</b>	0.88 (0.004) [24.0] <b>16.6</b>	1.00 (0.006) [23.3] <b>10.8</b>	1.61 (0.012) [23.2] <b>4.8</b>
Delta	+SV ( $\rho = 0$ )	0.040 (0.0004) [25.7] <b>31.1</b>	0.052 (0.0005) [25.0] <b>27.6</b>	0.125 (0.0009) [24.8] <b>25.6</b>	0.89 (0.003) [25.2] <b>24.3</b>	1.02 (0.004) [24.9] <b>25.9</b>	1.60 (0.004) [25.0] <b>33.5</b>
		0.040 (0.0004) [25.5] <b>40.2</b>	0.051 (0.0005) [25.0] <b>30.4</b>	0.126 (0.0009) [24.8] <b>30.8</b>	0.89 (0.003) [25.3] <b>27.9</b>	1.02 (0.003) [25.2] <b>28.9</b>	1.60 (0.004) [24.8] <b>34.2</b>
		0.040 (0.0004) [25.5] <b>40.2</b>	0.051 (0.0005) [25.0] <b>30.4</b>	0.126 (0.0009) [24.8] <b>30.8</b>	0.89 (0.003) [25.3] <b>27.9</b>	1.02 (0.003) [25.2] <b>28.9</b>	1.60 (0.004) [24.8] <b>34.2</b>
	All	0.040 (0.0004) [25.5] <b>40.2</b>	0.051 (0.0005) [25.0] <b>30.4</b>	0.126 (0.0009) [24.8] <b>30.8</b>	0.89 (0.003) [25.3] <b>27.9</b>	1.02 (0.003) [25.2] <b>28.9</b>	1.60 (0.004) [24.8] <b>34.2</b>

Table 5.68: CVs in Combinations in the GARCH Model: Average Rate Options (OTM)

		Case 1 ( $\rho = 0.9$ )			Case 2 ( $\rho = 0.9$ )		
CV Type		Av, 64	Av, 16	Av, 4	Av, 64	Av, 16	Av, 4
No CV (Plain)		3.83 (0.006) [604.1]	3.98 (0.006) [600.5]	4.59 (0.007) [599.6]	6.64 (0.011) [602.3]	6.91 (0.011) [598.5]	7.96 (0.013) [598.7]
Auxiliary	+SV ( $\rho = 0$ )	3.83 (0.011) [8.3] <b>21.3</b>	3.96 (0.012) [8.2] <b>20.2</b>	4.54 (0.013) [8.2] <b>21.8</b>	6.64 (0.019) [8.3] <b>22.9</b>	6.90 (0.018) [8.2] <b>27.2</b>	7.99 (0.025) [8.2] <b>20.4</b>
	+SV ( $\rho \neq 0$ )	3.82 (0.013) [8.5] <b>15.2</b>	3.98 (0.013) [8.2] <b>16.4</b>	4.59 (0.016) [8.1] <b>15.2</b>	6.70 (0.028) [8.4] <b>10.4</b>	6.90 (0.028) [8.1] <b>12.2</b>	7.98 (0.033) [8.1] <b>11.8</b>
	+SV ( $\rho = 0$ ) +SV ( $\rho \neq 0$ )	3.82 (0.011) [10.1] <b>18.5</b>	3.97 (0.012) [9.8] <b>17.5</b>	4.56 (0.014) [9.8] <b>16.3</b>	6.64 (0.019) [10.2] <b>19.6</b>	6.92 (0.020) [9.8] <b>18.8</b>	7.97 (0.025) [9.8] <b>17.2</b>
Delta	+Auxiliary	3.83 (0.004) [21.0] <b>84.2</b>	3.99 (0.005) [21.3] <b>45.6</b>	4.58 (0.012) [21.1] <b>10.6</b>	6.62 (0.008) [21.2] <b>55.2</b>	6.90 (0.012) [21.3] <b>25.4</b>	7.95 (0.027) [21.0] <b>6.7</b>
	+SV ( $\rho = 0$ )	3.83 (0.004) [22.8] <b>79.4</b>	3.99 (0.005) [22.7] <b>41.7</b>	4.58 (0.011) [22.6] <b>11.2</b>	6.62 (0.007) [22.7] <b>56.0</b>	6.88 (0.010) [22.9] <b>30.2</b>	7.94 (0.020) [22.8] <b>11.4</b>
	+SV ( $\rho \neq 0$ )	3.84 (0.003) [22.9] <b>85.7</b>	3.98 (0.005) [22.6] <b>46.9</b>	4.58 (0.014) [22.5] <b>7.1</b>	6.62 (0.007) [22.8] <b>56.4</b>	6.90 (0.012) [22.6] <b>25.0</b>	7.98 (0.029) [22.6] <b>5.5</b>
	+SV ( $\rho = 0$ ) +SV ( $\rho \neq 0$ )	3.84 (0.003) [24.8] <b>82.6</b>	3.98 (0.005) [24.5] <b>45.6</b>	4.56 (0.012) [24.2] <b>10.0</b>	6.64 (0.008) [24.6] <b>49.3</b>	6.88 (0.012) [24.4] <b>23.4</b>	7.97 (0.022) [24.3] <b>8.4</b>
All		3.83 (0.003) [24.6] <b>88.1</b>	3.97 (0.005) [24.3] <b>45.5</b>	4.56 (0.011) [24.2] <b>10.6</b>	6.64 (0.007) [24.6] <b>53.8</b>	6.88 (0.010) [24.3] <b>33.6</b>	7.98 (0.020) [24.5] <b>10.2</b>

Table 5.69: CVs in Combinations in the GARCH Model: Average Rate Options ( $\rho = 0.9$ )

		Case 1 ( $\rho = -0.9$ )			Case 2 ( $\rho = -0.9$ )		
CV Type		Av, 64	Av, 16	Av, 4	Av, 64	Av, 16	Av, 4
No CV (Plain)		3.88 (0.005) [597.1]	4.03 (0.005) [600.6]	4.65 (0.006) [605.8]	6.70 (0.009) [598.9]	6.94 (0.009) [600.1]	8.03 (0.011) [598.3]
Auxiliary	+SV ( $\rho = 0$ )	3.89 (0.007) [8.2] <b>36.7</b>	4.03 (0.007) [8.1] <b>36.0</b>	4.64 (0.009) [8.0] <b>34.8</b>	6.69 (0.010) [8.2] <b>54.0</b>	6.94 (0.011) [8.1] <b>54.0</b>	8.07 (0.013) [8.1] <b>51.9</b>
	+SV ( $\rho \neq 0$ )	3.90 (0.008) [9.8] <b>25.7</b>	4.03 (0.008) [9.5] <b>27.1</b>	4.63 (0.009) [9.5] <b>24.8</b>	6.68 (0.013) [9.9] <b>28.1</b>	6.97 (0.013) [9.6] <b>29.1</b>	8.04 (0.016) [9.5] <b>28.5</b>
	+SV ( $\rho = 0$ ) +SV ( $\rho \neq 0$ )	3.89 (0.007) [10.0] <b>30.4</b>	4.03 (0.007) [9.7] <b>30.6</b>	4.63 (0.009) [9.6] <b>29.9</b>	6.71 (0.010) [10.0] <b>47.5</b>	6.96 (0.010) [9.7] <b>48.0</b>	8.07 (0.012) [9.6] <b>45.7</b>
Delta	+Auxiliary	3.89 (0.003) [22.5] <b>61.4</b>	4.04 (0.004) [22.4] <b>39.6</b>	4.63 (0.008) [22.5] <b>13.0</b>	6.69 (0.006) [22.7] <b>60.8</b>	6.97 (0.008) [22.5] <b>32.6</b>	8.05 (0.016) [22.6] <b>11.2</b>
	+SV ( $\rho = 0$ )	3.89 (0.003) [22.6] <b>66.2</b>	4.04 (0.004) [22.5] <b>43.0</b>	4.62 (0.009) [22.5] <b>12.7</b>	6.70 (0.005) [22.6] <b>75.4</b>	6.96 (0.007) [22.5] <b>44.9</b>	8.05 (0.013) [22.6] <b>16.9</b>
	+SV ( $\rho \neq 0$ )	3.89 (0.003) [24.2] <b>65.7</b>	4.04 (0.004) [24.0] <b>38.7</b>	4.63 (0.012) [24.0] <b>6.3</b>	6.69 (0.006) [24.3] <b>62.9</b>	6.93 (0.008) [24.2] <b>33.6</b>	8.04 (0.023) [24.0] <b>5.4</b>
	+SV ( $\rho = 0$ ) +SV ( $\rho \neq 0$ )	3.89 (0.003) [24.5] <b>73.0</b>	4.04 (0.004) [24.6] <b>43.6</b>	4.62 (0.009) [24.2] <b>11.4</b>	6.70 (0.005) [24.7] <b>74.8</b>	6.95 (0.007) [24.7] <b>43.1</b>	8.04 (0.013) [24.0] <b>15.4</b>
All		3.89 (0.003) [25.0] <b>76.8</b>	4.04 (0.004) [24.3] <b>45.3</b>	4.63 (0.007) [24.1] <b>16.1</b>	6.69 (0.005) [24.5] <b>77.6</b>	6.95 (0.007) [24.2] <b>43.7</b>	8.04 (0.012) [24.3] <b>18.8</b>

Table 5.70: CVs in Combinations in the GARCH Model: Average Rate Options ( $\rho = -0.9$ )



		Case 1 ( $V_0 = \mu = 0.25$ )			Case 2 ( $V_0 = \mu = 0.25$ )		
CV Type		Av, 64	Av, 16	Av, 4	Av, 64	Av, 16	Av, 4
No CV (Plain)		11.83 (0.019) [600.2]	12.19 (0.020) [602.7]	13.85 (0.023) [600.5]	12.33 (0.020) [605.2]	12.78 (0.021) [601.7]	14.54 (0.024) [600.6]
Auxiliary	+SV ( $\rho = 0$ )	11.82 (0.022) [8.4] <b>57.6</b>	12.24 (0.024) [8.2] <b>51.3</b>	13.89 (0.028) [8.1] <b>50.7</b>	12.33 (0.019) [8.3] <b>81.9</b>	12.79 (0.020) [8.1] <b>81.8</b>	14.55 (0.025) [8.1] <b>65.4</b>
	+SV ( $\rho \neq 0$ )	11.85 (0.032) [8.3] <b>26.8</b>	12.23 (0.033) [8.2] <b>28.0</b>	13.83 (0.040) [8.0] <b>25.0</b>	12.34 (0.029) [8.4] <b>33.9</b>	12.78 (0.031) [8.2] <b>33.7</b>	14.51 (0.037) [8.0] <b>30.8</b>
	+SV ( $\rho = 0$ ) +SV ( $\rho \neq 0$ )	11.81 (0.019) [10.0] <b>64.0</b>	12.21 (0.020) [9.8] <b>62.6</b>	13.86 (0.025) [9.7] <b>54.3</b>	12.35 (0.017) [10.1] <b>84.2</b>	12.78 (0.019) [9.7] <b>75.6</b>	14.56 (0.023) [9.6] <b>68.0</b>
Delta	+Auxiliary	11.81 (0.019) [21.1] <b>30.5</b>	12.25 (0.023) [21.0] <b>22.0</b>	13.87 (0.040) [21.1] <b>9.6</b>	12.33 (0.018) [21.3] <b>34.6</b>	12.78 (0.022) [21.0] <b>24.7</b>	14.54 (0.037) [21.0] <b>11.8</b>
	+SV ( $\rho = 0$ )	11.85 (0.017) [23.0] <b>34.6</b>	12.23 (0.019) [23.0] <b>30.1</b>	13.87 (0.028) [22.6] <b>18.8</b>	12.33 (0.015) [22.9] <b>45.1</b>	12.76 (0.017) [23.0] <b>37.1</b>	14.55 (0.025) [22.6] <b>24.6</b>
	+SV ( $\rho \neq 0$ )	11.80 (0.015) [22.9] <b>42.7</b>	12.23 (0.019) [22.7] <b>30.1</b>	13.93 (0.042) [22.6] <b>8.1</b>	12.30 (0.015) [23.0] <b>49.1</b>	12.76 (0.019) [22.8] <b>32.9</b>	14.54 (0.043) [22.8] <b>8.0</b>
	+SV ( $\rho = 0$ ) +SV ( $\rho \neq 0$ )	11.82 (0.014) [24.4] <b>49.9</b>	12.25 (0.016) [24.3] <b>38.2</b>	13.92 (0.024) [24.0] <b>23.5</b>	12.35 (0.013) [24.5] <b>53.5</b>	12.76 (0.016) [24.1] <b>43.7</b>	14.53 (0.022) [24.0] <b>28.6</b>
All		11.85 (0.014) [24.6] <b>47.2</b>	12.24 (0.016) [24.6] <b>38.8</b>	13.83 (0.023) [24.4] <b>25.3</b>	12.34 (0.014) [24.5] <b>52.4</b>	12.76 (0.015) [24.3] <b>44.9</b>	14.56 (0.022) [24.3] <b>28.1</b>

Table 5.71: CVs in Combinations in the GARCH Model: Average Rate Options ( $V_0 = \mu = 0.25$ )

Average Rate Options, Case 1						
Resets	Plain	GBM		SV		
		Auxiliary	Delta	$(\rho = 0)$	$(\rho \neq 0)$	Both
4	5.09	5.09	5.09	5.08	5.07	5.08
	(0.006)	(0.003)	(0.011)	(0.001)	(0.027)	(0.001)
	[699.0]	[7.5]	[22.3]	[9.1]	[10.8]	[12.8]
	-	<b>306.9</b>	<b>9.7</b>	<b>3460.1</b>	<b>3.1</b>	<b>3164.9</b>
16	4.42	4.41	4.42	4.42	4.40	4.42
	(0.005)	(0.003)	(0.003)	(0.001)	(0.024)	(0.001)
	[699.7]	[7.5]	[22.4]	[9.1]	[10.8]	[12.9]
	-	<b>366.3</b>	<b>74.8</b>	<b>3941.1</b>	<b>3.1</b>	<b>3496.2</b>
64	4.25	4.25	4.25	4.25	4.27	4.25
	(0.005)	(0.002)	(0.002)	(0.001)	(0.023)	(0.001)
	[706.5]	[7.5]	[22.5]	[9.3]	[11.1]	[13.3]
	-	<b>390.5</b>	<b>171.9</b>	<b>4102.9</b>	<b>3.0</b>	<b>3682.2</b>

Average Rate Options, Case 2						
Resets	Plain	GBM		SV		
		Auxiliary	Delta	$(\rho = 0)$	$(\rho \neq 0)$	Both
4	10.00	9.99	10.03	9.99	10.04	9.99
	(0.014)	(0.012)	(0.028)	(0.004)	(0.063)	(0.003)
	[714.2]	[7.5]	[22.3]	[9.1]	[10.8]	[13.0]
	-	<b>131.2</b>	<b>8.6</b>	<b>1215.8</b>	<b>3.4</b>	<b>944.2</b>
16	8.75	8.75	8.74	8.75	8.82	8.75
	(0.013)	(0.010)	(0.010)	(0.003)	(0.054)	(0.003)
	[708.8]	[7.5]	[22.3]	[9.2]	[10.8]	[13.0]
	-	<b>153.6</b>	<b>53.7</b>	<b>1421.8</b>	<b>3.5</b>	<b>1071.4</b>
64	8.43	8.43	8.43	8.44	8.44	8.44
	(0.012)	(0.009)	(0.007)	(0.003)	(0.053)	(0.003)
	[706.8]	[7.5]	[22.3]	[9.3]	[11.1]	[13.5]
	-	<b>155.2</b>	<b>96.3</b>	<b>1464.5</b>	<b>3.4</b>	<b>1087.0</b>

Table 5.72: Efficiency Gains in Johnson and Shanno Model: Average Rate Options (ATM)

Average Rate Options, Case 1						
Resets	Plain	GBM		SV		
		Auxiliary	Delta	$(\rho = 0)$	$(\rho \neq 0)$	Both
4	22.05	22.07	22.06	22.06	22.04	22.06
	(0.008)	(0.004)	(0.013)	(0.001)	(0.042)	(0.001)
	[700.0]	[7.6]	[22.7]	[9.1]	[10.6]	[12.8]
	-	<b>378.7</b>	<b>12.9</b>	<b>6160.0</b>	<b>2.6</b>	<b>4297.0</b>
16	21.58	21.60	21.60	21.60	21.64	21.60
	(0.007)	(0.003)	(0.003)	(0.001)	(0.037)	(0.001)
	[700.8]	[7.7]	[22.7]	[9.1]	[10.7]	[12.8]
	-	<b>425.4</b>	<b>156.5</b>	<b>7875.8</b>	<b>2.6</b>	<b>5848.0</b>
64	21.49	21.48	21.48	21.48	21.44	21.48
	(0.007)	(0.003)	(0.001)	(0.001)	(0.036)	(0.001)
	[707.3]	[7.6]	[22.8]	[9.2]	[11.0]	[13.2]
	-	<b>447.5</b>	<b>803.1</b>	<b>8983.6</b>	<b>2.6</b>	<b>6581.9</b>

Average Rate Options, Case 2						
Resets	Plain	GBM		SV		
		Auxiliary	Delta	$(\rho = 0)$	$(\rho \neq 0)$	Both
4	23.35	23.37	23.33	23.37	23.25	23.36
	(0.019)	(0.013)	(0.034)	(0.003)	(0.093)	(0.003)
	[701.2]	[7.6]	[23.1]	[9.1]	[10.7]	[12.9]
	-	<b>197.2</b>	<b>9.8</b>	<b>2617.8</b>	<b>2.9</b>	<b>1889.5</b>
16	22.44	22.47	22.50	22.48	22.43	22.48
	(0.017)	(0.010)	(0.011)	(0.003)	(0.085)	(0.003)
	[699.5]	[7.6]	[23.1]	[9.1]	[10.8]	[12.8]
	-	<b>274.2</b>	<b>77.1</b>	<b>3184.7</b>	<b>2.7</b>	<b>2462.7</b>
64	22.25	22.26	22.27	22.27	22.23	22.26
	(0.017)	(0.009)	(0.007)	(0.003)	(0.082)	(0.002)
	[703.9]	[7.6]	[22.9]	[9.2]	[11.0]	[13.2]
	-	<b>305.3</b>	<b>175.4</b>	<b>3313.6</b>	<b>2.7</b>	<b>2567.1</b>

Table 5.73: Efficiency Gains in Johnson and Shanno Model: Average Rate Options (ITM)

Average Rate Options, Case 1						
Resets	Plain	GBM		SV		
		Auxiliary	Delta	$(\rho = 0)$	$(\rho \neq 0)$	Both
4	0.105	0.105	0.109	0.104	0.108	0.104
	(0.0008)	(0.0009)	(0.0045)	(0.0003)	(0.0018)	(0.0003)
	[706.0]	[7.5]	[23.3]	[9.2]	[10.8]	[12.9]
	-	<b>74.2</b>	<b>1.0</b>	<b>696.6</b>	<b>12.9</b>	<b>436.1</b>
16	0.040	0.040	0.040	0.040	0.039	0.040
	(0.0004)	(0.0006)	(0.0012)	(0.0001)	(0.0008)	(0.0002)
	[707.9]	[7.5]	[23.0]	[9.2]	[10.8]	[13.0]
	-	<b>55.4</b>	<b>4.5</b>	<b>712.8</b>	<b>20.6</b>	<b>410.4</b>
64	0.030	0.031	0.030	0.030	0.030	0.030
	(0.0004)	(0.0005)	(0.0007)	(0.0001)	(0.0007)	(0.0001)
	[704.4]	[7.6]	[23.1]	[9.3]	[11.1]	[13.3]
	-	<b>53.1</b>	<b>10.0</b>	<b>641.2</b>	<b>21.3</b>	<b>508.5</b>

Average Rate Options, Case 2						
Resets	Plain	GBM		SV		
		Auxiliary	Delta	$(\rho = 0)$	$(\rho \neq 0)$	Both
4	3.04	3.06	3.06	3.05	3.05	3.05
	(0.008)	(0.007)	(0.022)	(0.004)	(0.029)	(0.003)
	[714.0]	[7.5]	[22.9]	[9.2]	[10.8]	[12.9]
	-	<b>112.9</b>	<b>4.1</b>	<b>413.0</b>	<b>5.2</b>	<b>310.3</b>
16	2.19	2.20	2.20	2.20	2.18	2.20
	(0.006)	(0.006)	(0.008)	(0.003)	(0.023)	(0.003)
	[708.1]	[7.5]	[22.8]	[9.2]	[11.0]	[13.0]
	-	<b>110.4</b>	<b>18.9</b>	<b>362.6</b>	<b>5.1</b>	<b>274.3</b>
64	2.00	1.99	1.99	1.99	1.99	2.00
	(0.006)	(0.005)	(0.006)	(0.003)	(0.020)	(0.003)
	[707.4]	[7.6]	[22.9]	[9.3]	[11.1]	[13.6]
	-	<b>110.9</b>	<b>31.4</b>	<b>373.0</b>	<b>5.6</b>	<b>239.1</b>

Table 5.74: Efficiency Gains in Johnson and Shanno Model: Average Rate Options (OTM)

Average Rate Options, Case 1 ( $\rho = 0.9$ )						
Resets	Plain	GBM		SV		
		Auxiliary	Delta	( $\rho = 0$ )	( $\rho \neq 0$ )	Both
4	5.05	5.05	5.08	5.06	5.08	5.06
	(0.006)	(0.002)	(0.011)	(0.002)	(0.054)	(0.002)
	[702.0]	[7.6]	[22.6]	[9.2]	[10.8]	[12.9]
	-	<b>1256.1</b>	<b>9.7</b>	<b>913.4</b>	<b>0.8</b>	<b>599.5</b>
16	4.39	4.40	4.39	4.39	4.41	4.39
	(0.005)	(0.001)	(0.003)	(0.001)	(0.048)	(0.002)
	[704.0]	[7.6]	[22.7]	[9.2]	[10.8]	[13.0]
	-	<b>1553.5</b>	<b>77.5</b>	<b>1000.9</b>	<b>0.8</b>	<b>684.9</b>
64	4.23	4.23	4.23	4.23	4.15	4.23
	(0.005)	(0.001)	(0.002)	(0.001)	(0.045)	(0.001)
	[705.3]	[7.6]	[22.8]	[9.3]	[11.1]	[13.3]
	-	<b>1734.4</b>	<b>184.7</b>	<b>957.0</b>	<b>0.8</b>	<b>650.4</b>

Average Rate Options, Case 2 ( $\rho = 0.9$ )						
Resets	Plain	GBM		SV		
		Auxiliary	Delta	( $\rho = 0$ )	( $\rho \neq 0$ )	Both
4	10.00	9.97	9.99	9.97	10.16	9.97
	(0.015)	(0.008)	(0.028)	(0.006)	(0.130)	(0.006)
	[705.5]	[7.6]	[22.5]	[9.2]	[10.8]	[13.0]
	-	<b>295.5</b>	<b>8.6</b>	<b>504.6</b>	<b>0.8</b>	<b>340.6</b>
16	8.74	8.72	8.73	8.73	8.72	8.73
	(0.013)	(0.007)	(0.010)	(0.005)	(0.112)	(0.005)
	[707.3]	[7.6]	[22.6]	[9.2]	[10.8]	[13.0]
	-	<b>352.4</b>	<b>52.6</b>	<b>503.3</b>	<b>0.9</b>	<b>357.4</b>
64	8.46	8.42	8.42	8.43	8.57	8.42
	(0.012)	(0.006)	(0.007)	(0.005)	(0.110)	(0.005)
	[707.8]	[7.7]	[22.8]	[9.3]	[11.1]	[13.5]
	-	<b>360.9</b>	<b>97.8</b>	<b>471.2</b>	<b>0.8</b>	<b>319.1</b>

Table 5.75: Efficiency Gains in Johnson and Shanno Model: Average Rate Options ( $\rho = 0.9$ )

Average Rate Options, Case 1 ( $\rho = -0.9$ )						
Resets	Plain	GBM		SV		
		Auxiliary	Delta	( $\rho = 0$ )	( $\rho \neq 0$ )	Both
4	5.08	5.08	5.09	5.09	5.06	5.09
	(0.006)	(0.004)	(0.011)	(0.002)	(0.052)	(0.002)
	[706.7]	[7.6]	[22.2]	[9.2]	[10.8]	[12.9]
	-	<b>261.4</b>	<b>9.5</b>	<b>1086.5</b>	<b>0.9</b>	<b>791.1</b>
16	4.43	4.43	4.42	4.42	4.42	4.42
	(0.005)	(0.003)	(0.003)	(0.001)	(0.046)	(0.001)
	[709.5]	[7.6]	[22.3]	[9.2]	[10.8]	[13.0]
	-	<b>334.2</b>	<b>76.7</b>	<b>1180.4</b>	<b>0.8</b>	<b>828.7</b>
64	4.25	4.25	4.26	4.26	4.22	4.26
	(0.005)	(0.003)	(0.002)	(0.001)	(0.044)	(0.001)
	[702.1]	[7.6]	[22.5]	[9.3]	[11.1]	[13.5]
	-	<b>320.3</b>	<b>171.6</b>	<b>1241.4</b>	<b>0.8</b>	<b>842.2</b>

Average Rate Options, Case 2 ( $\rho = -0.9$ )						
Resets	Plain	GBM		SV		
		Auxiliary	Delta	( $\rho = 0$ )	( $\rho \neq 0$ )	Both
4	9.99	9.97	10.02	9.99	10.18	10.00
	(0.014)	(0.013)	(0.027)	(0.005)	(0.125)	(0.005)
	[704.5]	[7.6]	[22.3]	[9.2]	[10.8]	[12.9]
	-	<b>105.2</b>	<b>8.5</b>	<b>648.6</b>	<b>0.8</b>	<b>492.5</b>
16	8.76	8.75	8.76	8.75	8.66	8.75
	(0.012)	(0.010)	(0.010)	(0.004)	(0.106)	(0.004)
	[705.4]	[7.6]	[22.1]	[9.2]	[10.8]	[13.0]
	-	<b>130.6</b>	<b>52.9</b>	<b>782.0</b>	<b>0.9</b>	<b>567.7</b>
64	8.44	8.43	8.43	8.45	8.43	8.44
	(0.012)	(0.010)	(0.007)	(0.004)	(0.105)	(0.004)
	[706.0]	[7.6]	[22.4]	[9.5]	[11.1]	[13.4]
	-	<b>135.9</b>	<b>103.1</b>	<b>816.9</b>	<b>0.8</b>	<b>574.0</b>

Table 5.76: Efficiency Gains in Johnson and Shanno Model: Average Rate Options ( $\rho = -0.9$ )

Average Rate Options, Case 1 ( $V_0 = \mu = 8$ )						
Resets	Plain	GBM		SV		
		Auxiliary	Delta	( $\rho = 0$ )	( $\rho \neq 0$ )	Both
4	14.86	14.90	14.83	14.89	14.95	14.90
	(0.022)	(0.038)	(0.046)	(0.005)	(0.096)	(0.005)
	[702.1]	[7.5]	[23.0]	[9.0]	[10.6]	[12.7]
	-	<b>32.1</b>	<b>7.0</b>	<b>1751.9</b>	<b>3.5</b>	<b>1281.6</b>
16	13.08	13.09	13.10	13.08	13.00	13.08
	(0.019)	(0.030)	(0.018)	(0.004)	(0.085)	(0.004)
	[700.2]	[7.5]	[23.0]	[9.1]	[10.7]	[12.7]
	-	<b>39.5</b>	<b>34.6</b>	<b>2113.3</b>	<b>3.4</b>	<b>1485.0</b>
64	12.61	12.56	12.62	12.62	12.69	12.62
	(0.019)	(0.030)	(0.014)	(0.003)	(0.082)	(0.003)
	[702.6]	[7.6]	[23.0]	[9.2]	[10.9]	[13.1]
	-	<b>36.8</b>	<b>58.8</b>	<b>2226.2</b>	<b>3.4</b>	<b>1569.5</b>

Average Rate Options, Case 2 ( $V_0 = \mu = 3$ )						
Resets	Plain	GBM		SV		
		Auxiliary	Delta	( $\rho = 0$ )	( $\rho \neq 0$ )	Both
4	14.12	14.08	14.15	14.12	14.17	14.11
	(0.022)	(0.026)	(0.045)	(0.007)	(0.095)	(0.007)
	[704.2]	[7.5]	[23.4]	[9.1]	[10.6]	[12.7]
	-	<b>65.4</b>	<b>7.1</b>	<b>745.1</b>	<b>3.4</b>	<b>537.0</b>
16	12.38	12.39	12.34	12.39	12.24	12.39
	(0.019)	(0.022)	(0.017)	(0.006)	(0.081)	(0.006)
	[703.3]	[7.5]	[23.2]	[9.1]	[10.7]	[12.7]
	-	<b>71.8</b>	<b>38.0</b>	<b>849.1</b>	<b>3.6</b>	<b>605.4</b>
64	11.99	11.95	11.97	11.95	11.80	11.95
	(0.018)	(0.020)	(0.013)	(0.005)	(0.078)	(0.005)
	[703.3]	[7.5]	[23.0]	[9.2]	[11.0]	[13.1]
	-	<b>81.6</b>	<b>63.3</b>	<b>930.4</b>	<b>3.5</b>	<b>694.3</b>

Table 5.77: Efficiency Gains in Johnson and Shanno Model: Average Rate Options (High Volatility)

		Case 1			Case 2		
CV Type		Av, 64	Av, 16	Av, 4	Av, 64	Av, 16	Av, 4
No CV (Plain)		4.25 (0.005) [706.5]	4.42 (0.005) [699.7]	5.09 (0.006) [699.0]	8.43 (0.012) [706.8]	8.75 (0.013) [708.8]	10.00 (0.014) [714.2]
Auxiliary	+SV ( $\rho = 0$ )	4.25 (0.001) [9.9] <b>4413.4</b>	4.42 (0.001) [9.7] <b>4407.1</b>	5.08 (0.001) [9.7] <b>3907.6</b>	8.43 (0.003) [9.8] <b>1616.6</b>	8.74 (0.003) [9.9] <b>1643.1</b>	9.98 (0.003) [9.6] <b>1513.1</b>
	+SV ( $\rho \neq 0$ )	4.25 (0.002) [11.8] <b>256.8</b>	4.41 (0.003) [11.3] <b>247.9</b>	5.08 (0.003) [11.3] <b>217.0</b>	8.45 (0.009) [12.0] <b>101.8</b>	8.74 (0.010) [11.3] <b>99.9</b>	9.98 (0.012) [11.5] <b>81.9</b>
	+SV ( $\rho = 0$ ) +SV ( $\rho \neq 0$ )	4.25 (0.001) [13.9] <b>4342.5</b>	4.42 (0.001) [13.5] <b>4206.3</b>	5.08 (0.001) [13.4] <b>3775.9</b>	8.44 (0.002) [13.9] <b>1277.4</b>	8.75 (0.002) [13.8] <b>1345.5</b>	9.99 (0.003) [13.4] <b>1158.8</b>
Delta	+Auxiliary	4.25 (0.002) [22.7] <b>309.8</b>	4.42 (0.002) [22.9] <b>196.6</b>	5.08 (0.003) [22.8] <b>114.5</b>	8.44 (0.006) [23.2] <b>137.7</b>	8.75 (0.007) [22.6] <b>94.7</b>	9.98 (0.012) [22.8] <b>44.2</b>
	+SV ( $\rho = 0$ )	4.25 (0.001) [24.9] <b>1587.2</b>	4.42 (0.001) [24.8] <b>1438.5</b>	5.09 (0.001) [24.8] <b>1248.9</b>	8.44 (0.003) [25.4] <b>534.7</b>	8.75 (0.003) [24.7] <b>552.3</b>	9.99 (0.004) [25.2] <b>451.3</b>
	+SV ( $\rho \neq 0$ )	4.25 (0.002) [26.9] <b>150.4</b>	4.42 (0.003) [26.9] <b>65.7</b>	5.09 (0.010) [26.8] <b>9.1</b>	8.44 (0.007) [26.5] <b>88.2</b>	8.75 (0.010) [26.5] <b>46.3</b>	9.98 (0.025) [26.6] <b>8.8</b>
	+SV ( $\rho = 0$ ) +SV ( $\rho \neq 0$ )	4.25 (0.001) [29.3] <b>1665.4</b>	4.42 (0.001) [28.8] <b>1620.0</b>	5.08 (0.001) [28.9] <b>1370.8</b>	8.43 (0.003) [29.3] <b>533.4</b>	8.75 (0.003) [28.7] <b>509.6</b>	9.98 (0.003) [28.6] <b>419.5</b>
All		4.25 (0.001) [29.1] <b>2198.1</b>	4.42 (0.001) [29.0] <b>2048.6</b>	5.08 (0.001) [28.8] <b>1869.6</b>	8.44 (0.002) [29.1] <b>686.3</b>	8.75 (0.002) [29.0] <b>693.4</b>	9.99 (0.003) [28.8] <b>554.5</b>

Table 5.78: CVs in Combinations in Johnson and Shanno Model: Average Rate Options (ATM)



		Case 1			Case 2		
CV Type		Av, 64	Av, 16	Av, 4	Av, 64	Av, 16	Av, 4
No CV (Plain)		21.49 (0.007) [707.3]	21.58 (0.007) [700.8]	22.05 (0.008) [700.0]	22.25 (0.017) [703.9]	22.44 (0.017) [699.5]	23.35 (0.019) [701.2]
Auxiliary	+SV ( $\rho = 0$ )	21.48 (0.001) [9.8] <b>9524.3</b>	21.60 (0.001) [9.7] <b>8548.0</b>	22.06 (0.001) [9.9] <b>6984.6</b>	22.26 (0.003) [9.8] <b>3133.7</b>	22.48 (0.003) [9.7] <b>3193.4</b>	23.36 (0.003) [9.6] <b>2541.1</b>
		21.48 (0.003) [11.9] <b>288.5</b>	21.60 (0.003) [11.3] <b>297.6</b>	22.06 (0.004) [11.5] <b>241.2</b>	22.27 (0.009) [11.7] <b>218.0</b>	22.48 (0.010) [11.6] <b>194.4</b>	23.35 (0.012) [11.4] <b>148.1</b>
		21.48 (0.001) [14.2] <b>6388.7</b>	21.60 (0.001) [13.5] <b>6477.9</b>	22.06 (0.001) [13.4] <b>5198.3</b>	22.26 (0.002) [14.2] <b>2298.5</b>	22.48 (0.003) [13.9] <b>2181.0</b>	23.36 (0.003) [13.5] <b>1851.8</b>
	+SV ( $\rho \neq 0$ )	21.48 (0.001) [22.8] <b>1531.6</b>	21.60 (0.002) [22.7] <b>382.8</b>	22.06 (0.004) [22.9] <b>140.0</b>	22.26 (0.005) [23.0] <b>347.6</b>	22.49 (0.006) [23.1] <b>223.6</b>	23.37 (0.011) [23.3] <b>92.9</b>
		21.48 (0.0005) [24.9] <b>6039.9</b>	21.60 (0.001) [24.8] <b>3572.8</b>	22.06 (0.001) [24.8] <b>2404.7</b>	22.26 (0.002) [26.2] <b>1382.1</b>	22.48 (0.003) [25.1] <b>1239.9</b>	23.36 (0.003) [24.9] <b>967.1</b>
		21.48 (0.001) [27.3] <b>663.7</b>	21.60 (0.003) [26.5] <b>132.5</b>	22.06 (0.013) [26.9] <b>11.4</b>	22.26 (0.007) [26.9] <b>158.9</b>	22.47 (0.011) [27.3] <b>69.4</b>	23.38 (0.032) [26.6] <b>9.6</b>
Delta	+SV ( $\rho = 0$ )	21.48 (0.0005) [29.4] <b>5076.8</b>	21.60 (0.001) [28.7] <b>3117.7</b>	22.06 (0.001) [28.6] <b>2038.4</b>	22.27 (0.002) [29.1] <b>1258.4</b>	22.48 (0.003) [28.8] <b>1004.5</b>	23.36 (0.003) [28.8] <b>844.8</b>
		21.48 (0.0005) [29.4] <b>5076.8</b>	21.60 (0.001) [28.7] <b>3117.7</b>	22.06 (0.001) [28.6] <b>2038.4</b>	22.27 (0.002) [29.1] <b>1258.4</b>	22.48 (0.003) [28.8] <b>1004.5</b>	23.36 (0.003) [28.8] <b>844.8</b>
		21.48 (0.0005) [29.4] <b>5076.8</b>	21.60 (0.001) [28.7] <b>3117.7</b>	22.06 (0.001) [28.6] <b>2038.4</b>	22.27 (0.002) [29.1] <b>1258.4</b>	22.48 (0.003) [28.8] <b>1004.5</b>	23.36 (0.003) [28.8] <b>844.8</b>
	+SV ( $\rho \neq 0$ )	21.48 (0.0005) [29.4] <b>5076.8</b>	21.60 (0.001) [28.7] <b>3117.7</b>	22.06 (0.001) [28.6] <b>2038.4</b>	22.27 (0.002) [29.1] <b>1258.4</b>	22.48 (0.003) [28.8] <b>1004.5</b>	23.36 (0.003) [28.8] <b>844.8</b>
All		21.48 (0.0005) [29.2] <b>5321.4</b>	21.60 (0.001) [28.8] <b>3457.6</b>	22.06 (0.001) [28.7] <b>2753.4</b>	22.26 (0.002) [29.6] <b>1220.7</b>	22.48 (0.003) [29.1] <b>1099.1</b>	23.36 (0.003) [28.7] <b>884.7</b>

Table 5.79: CVs in Combinations in Johnson and Shanno Model: Average Rate Options (ITM)

		Case 1			Case 2		
CV Type		Av, 64	Av, 16	Av, 4	Av, 64	Av, 16	Av, 4
No CV (Plain)		0.030 (0.0004) [704.4]	0.040 (0.0004) [707.9]	0.105 (0.0008) [706.0]	2.00 (0.006) [707.4]	2.19 (0.006) [708.1]	3.04 (0.008) [714.0]
Auxiliary	+SV ( $\rho = 0$ )	0.030 (0.0001) [9.8] <b>1371.4</b>	0.040 (0.0001) [9.7] <b>1405.6</b>	0.105 (0.0002) [9.6] <b>1503.6</b>	1.99 (0.003) [9.8] <b>393.0</b>	2.20 (0.003) [9.7] <b>401.0</b>	3.05 (0.003) [9.6] <b>504.3</b>
		0.029 (0.0003) [12.1] <b>79.1</b>	0.040 (0.0004) [11.8] <b>91.8</b>	0.104 (0.0006) [11.5] <b>102.4</b>	1.99 (0.005) [12.3] <b>75.8</b>	2.19 (0.005) [11.6] <b>83.0</b>	3.06 (0.007) [11.6] <b>83.0</b>
		0.030 (0.0001) [13.8] <b>1215.8</b>	0.040 (0.0001) [13.5] <b>1359.5</b>	0.105 (0.0002) [13.3] <b>1009.3</b>	2.00 (0.003) [13.8] <b>258.2</b>	2.19 (0.003) [13.4] <b>322.5</b>	3.05 (0.003) [13.5] <b>363.3</b>
	+SV ( $\rho \neq 0$ )	0.030 (0.0003) [23.7] <b>36.4</b>	0.040 (0.0004) [23.4] <b>31.3</b>	0.106 (0.0008) [23.4] <b>29.1</b>	1.99 (0.005) [23.4] <b>48.5</b>	2.19 (0.005) [23.5] <b>42.9</b>	3.08 (0.008) [23.6] <b>32.1</b>
		0.030 (0.0001) [25.4] <b>346.6</b>	0.040 (0.0001) [25.3] <b>367.7</b>	0.105 (0.0002) [25.6] <b>284.2</b>	2.00 (0.003) [25.6] <b>125.1</b>	2.20 (0.003) [25.1] <b>165.1</b>	3.05 (0.003) [25.1] <b>206.2</b>
		0.029 (0.0005) [27.9] <b>14.3</b>	0.039 (0.0007) [27.2] <b>9.5</b>	0.105 (0.0017) [27.2] <b>6.0</b>	2.00 (0.006) [27.6] <b>28.9</b>	2.19 (0.008) [26.8] <b>17.4</b>	3.05 (0.019) [26.7] <b>5.0</b>
Delta	+SV ( $\rho = 0$ ) +SV ( $\rho \neq 0$ )	0.030 (0.0001) [29.5] <b>262.9</b>	0.040 (0.0001) [29.3] <b>254.7</b>	0.104 (0.0002) [29.2] <b>273.0</b>	2.00 (0.003) [29.4] <b>109.8</b>	2.20 (0.003) [29.1] <b>142.7</b>	3.06 (0.003) [28.8] <b>184.1</b>
	All	0.030 (0.0001) [29.7] <b>513.3</b>	0.040 (0.0001) [29.0] <b>721.5</b>	0.105 (0.0002) [29.2] <b>662.8</b>	2.00 (0.003) [29.5] <b>131.8</b>	2.19 (0.002) [29.0] <b>169.1</b>	3.06 (0.003) [29.1] <b>185.6</b>

Table 5.80: CVs in Combinations in Johnson and Shanno Model: Average Rate Options (OTM)

		Case 1 ( $\rho = 0.9$ )			Case 2 ( $\rho = 0.9$ )		
CV Type		Av, 64	Av, 16	Av, 4	Av, 64	Av, 16	Av, 4
No CV (Plain)		4.23 (0.005) [705.3]	4.39 (0.005) [704.0]	5.05 (0.006) [702.0]	8.46 (0.012) [707.8]	8.74 (0.013) [707.3]	10.00 (0.015) [705.5]
Auxiliary	+SV ( $\rho = 0$ )	4.23 (0.001) [9.7] <b>3329.3</b>	4.40 (0.001) [9.5] <b>3103.6</b>	5.06 (0.001) [9.5] <b>2949.0</b>	8.43 (0.004) [9.7] <b>755.2</b>	8.73 (0.004) [9.5] <b>808.2</b>	9.97 (0.005) [9.7] <b>748.5</b>
		4.23 (0.001) [11.8] <b>1183.3</b>	4.40 (0.001) [11.5] <b>1148.6</b>	5.06 (0.002) [11.2] <b>905.3</b>	8.43 (0.006) [11.6] <b>255.7</b>	8.73 (0.007) [11.3] <b>227.2</b>	9.97 (0.008) [11.3] <b>198.1</b>
		4.23 (0.001) [14.0] <b>2383.6</b>	4.40 (0.001) [13.4] <b>2530.6</b>	5.06 (0.001) [13.2] <b>2141.0</b>	8.42 (0.004) [13.7] <b>542.7</b>	8.73 (0.004) [13.3] <b>547.2</b>	9.97 (0.005) [13.4] <b>490.6</b>
	+SV ( $\rho \neq 0$ )	4.23 (0.001) [23.2] <b>669.2</b>	4.39 (0.001) [23.3] <b>558.8</b>	5.06 (0.002) [23.2] <b>441.0</b>	8.42 (0.005) [23.2] <b>163.4</b>	8.73 (0.006) [22.9] <b>139.5</b>	9.97 (0.009) [22.8] <b>90.1</b>
		4.23 (0.001) [25.0] <b>436.9</b>	4.39 (0.001) [25.3] <b>383.5</b>	5.06 (0.002) [24.8] <b>355.1</b>	8.42 (0.005) [25.2] <b>188.2</b>	8.74 (0.005) [25.4] <b>172.1</b>	9.98 (0.006) [25.0] <b>147.5</b>
		4.23 (0.002) [26.8] <b>160.3</b>	4.40 (0.003) [26.5] <b>68.1</b>	5.06 (0.011) [26.6] <b>8.0</b>	8.43 (0.007) [26.7] <b>82.2</b>	8.73 (0.010) [26.8] <b>44.4</b>	10.01 (0.029) [26.4] <b>7.0</b>
Delta	+SV ( $\rho = 0$ )	4.23 (0.001) [29.1] <b>394.8</b>	4.40 (0.001) [28.5] <b>374.4</b>	5.06 (0.002) [28.7] <b>298.3</b>	8.43 (0.005) [29.0] <b>151.6</b>	8.74 (0.005) [28.5] <b>152.6</b>	9.97 (0.006) [28.7] <b>164.1</b>
		4.23 (0.001) [29.1] <b>1176.5</b>	4.40 (0.001) [28.8] <b>1113.7</b>	5.06 (0.001) [28.6] <b>981.9</b>	8.43 (0.004) [29.3] <b>263.8</b>	8.74 (0.004) [28.6] <b>270.2</b>	9.98 (0.005) [29.0] <b>233.7</b>
		4.23 (0.001) [29.1] <b>1176.5</b>	4.40 (0.001) [28.8] <b>1113.7</b>	5.06 (0.001) [28.6] <b>981.9</b>	8.43 (0.004) [29.3] <b>263.8</b>	8.74 (0.004) [28.6] <b>270.2</b>	9.98 (0.005) [29.0] <b>233.7</b>
	All	4.23 (0.001) [29.1] <b>1176.5</b>	4.40 (0.001) [28.8] <b>1113.7</b>	5.06 (0.001) [28.6] <b>981.9</b>	8.43 (0.004) [29.3] <b>263.8</b>	8.74 (0.004) [28.6] <b>270.2</b>	9.98 (0.005) [29.0] <b>233.7</b>

Table 5.81: CVs in Combinations in Johnson and Shanno Model: Average Rate Options ( $\rho = 0.9$ )

		Case 1 ( $\rho = -0.9$ )			Case 2 ( $\rho = -0.9$ )		
CV Type		Av, 64	Av, 16	Av, 4	Av, 64	Av, 16	Av, 4
No CV (Plain)		4.25 (0.005) [702.1]	4.43 (0.005) [709.5]	5.08 (0.006) [706.7]	8.44 (0.012) [706.0]	8.76 (0.012) [705.4]	9.99 (0.014) [704.5]
Auxiliary	+SV ( $\rho = 0$ )	4.26 (0.001) [9.9] <b>1210.4</b>	4.42 (0.001) [9.6] <b>1214.7</b>	5.09 (0.002) [9.5] <b>1119.4</b>	8.44 (0.003) [9.7] <b>929.0</b>	8.75 (0.003) [9.5] <b>977.1</b>	9.99 (0.004) [9.5] <b>820.8</b>
		4.26 (0.003) [11.7] <b>222.2</b>	4.42 (0.003) [11.2] <b>230.3</b>	5.10 (0.003) [11.2] <b>188.1</b>	8.46 (0.009) [11.5] <b>99.1</b>	8.76 (0.010) [11.2] <b>94.7</b>	10.00 (0.013) [11.1] <b>74.9</b>
		4.26 (0.001) [13.7] <b>841.7</b>	4.42 (0.001) [13.5] <b>868.6</b>	5.09 (0.002) [13.2] <b>817.9</b>	8.44 (0.003) [13.8] <b>688.1</b>	8.75 (0.003) [13.2] <b>673.3</b>	9.99 (0.004) [13.4] <b>615.9</b>
	+SV ( $\rho \neq 0$ )	4.26 (0.002) [22.8] <b>318.4</b>	4.42 (0.002) [22.9] <b>190.2</b>	5.09 (0.003) [22.6] <b>102.2</b>	8.44 (0.006) [22.8] <b>140.3</b>	8.76 (0.007) [22.8] <b>94.6</b>	9.99 (0.012) [22.7] <b>43.3</b>
		4.26 (0.001) [24.8] <b>657.0</b>	4.42 (0.001) [25.0] <b>545.8</b>	5.09 (0.002) [24.8] <b>418.4</b>	8.44 (0.003) [25.0] <b>347.6</b>	8.75 (0.004) [24.8] <b>321.4</b>	9.99 (0.005) [24.6] <b>255.3</b>
		4.25 (0.002) [26.4] <b>150.7</b>	4.42 (0.003) [26.5] <b>67.2</b>	5.10 (0.011) [26.4] <b>8.5</b>	8.44 (0.007) [26.3] <b>83.5</b>	8.74 (0.010) [26.6] <b>42.8</b>	10.02 (0.028) [26.3] <b>7.0</b>
Delta	+SV ( $\rho = 0$ )	4.26 (0.001) [28.7] <b>591.9</b>	4.42 (0.001) [28.5] <b>474.4</b>	5.09 (0.002) [28.4] <b>380.7</b>	8.44 (0.003) [28.9] <b>297.6</b>	8.75 (0.004) [28.4] <b>274.7</b>	9.99 (0.005) [28.3] <b>218.5</b>
		4.26 (0.001) [28.7] <b>591.9</b>	4.42 (0.001) [28.5] <b>474.4</b>	5.09 (0.002) [28.4] <b>380.7</b>	8.44 (0.003) [28.9] <b>297.6</b>	8.75 (0.004) [28.4] <b>274.7</b>	9.99 (0.005) [28.3] <b>218.5</b>
		4.26 (0.001) [28.7] <b>591.9</b>	4.42 (0.001) [28.5] <b>474.4</b>	5.09 (0.002) [28.4] <b>380.7</b>	8.44 (0.003) [28.9] <b>297.6</b>	8.75 (0.004) [28.4] <b>274.7</b>	9.99 (0.005) [28.3] <b>218.5</b>
	+SV ( $\rho \neq 0$ )	4.26 (0.001) [28.7] <b>591.9</b>	4.42 (0.001) [28.5] <b>474.4</b>	5.09 (0.002) [28.4] <b>380.7</b>	8.44 (0.003) [28.9] <b>297.6</b>	8.75 (0.004) [28.4] <b>274.7</b>	9.99 (0.005) [28.3] <b>218.5</b>
All		4.25 (0.001) [28.6] <b>580.4</b>	4.42 (0.001) [28.2] <b>498.7</b>	5.09 (0.002) [28.4] <b>383.7</b>	8.44 (0.003) [28.7] <b>332.1</b>	8.75 (0.003) [28.7] <b>319.2</b>	9.99 (0.004) [28.1] <b>265.5</b>

Table 5.82: CVs in Combinations in Johnson and Shanno Model: Average Rate Options ( $\rho = -0.9$ )

		Case 1 ( $V_0 = \mu = 8$ )			Case 2 ( $V_0 = \mu = 3$ )			
CV Type		Av, 64	Av, 16	Av, 4	Av, 64	Av, 16	Av, 4	
No CV (Plain)		12.61	13.08	14.86	11.99	12.38	14.12	
		(0.019)	(0.019)	(0.022)	(0.018)	(0.019)	(0.022)	
		[702.6]	[700.2]	[702.1]	[703.3]	[703.3]	[704.2]	
Auxiliary	+SV ( $\rho = 0$ )	12.62	13.08	14.90	11.94	12.38	14.11	
		(0.003)	(0.003)	(0.004)	(0.005)	(0.005)	(0.007)	
		[9.6]	[9.5]	[9.5]	[9.6]	[9.5]	[9.5]	
		<b>2411.0</b>	<b>2381.1</b>	<b>1992.9</b>	<b>981.4</b>	<b>923.2</b>	<b>808.6</b>	
	+SV ( $\rho \neq 0$ )	12.63	13.07	14.90	11.98	12.39	14.06	
		(0.028)	(0.028)	(0.036)	(0.019)	(0.020)	(0.026)	
		[11.5]	[11.2]	[11.1]	[11.6]	[11.4]	[11.1]	
			<b>28.3</b>	<b>29.4</b>	<b>23.9</b>	<b>56.4</b>	<b>54.0</b>	<b>44.5</b>
	+SV ( $\rho = 0$ ) +SV ( $\rho \neq 0$ )	12.63	13.08	14.89	11.96	12.38	14.11	
(0.003)		(0.003)	(0.004)	(0.005)	(0.005)	(0.006)		
[13.7]		[13.2]	[13.1]	[13.8]	[13.4]	[13.1]		
	<b>1674.5</b>	<b>1801.5</b>	<b>1661.1</b>	<b>706.0</b>	<b>738.0</b>	<b>653.8</b>		
Delta	+Auxiliary	12.64	13.08	14.94	11.95	12.39	14.17	
		(0.012)	(0.015)	(0.027)	(0.011)	(0.014)	(0.022)	
		[22.8]	[23.0]	[22.7]	[22.7]	[22.5]	[22.4]	
		<b>78.5</b>	<b>52.5</b>	<b>20.2</b>	<b>86.9</b>	<b>60.3</b>	<b>31.1</b>	
	+SV ( $\rho = 0$ )	12.62	13.08	14.90	11.95	12.39	14.11	
		(0.003)	(0.004)	(0.005)	(0.005)	(0.006)	(0.007)	
		[24.5]	[24.7]	[24.4]	[24.5]	[24.8]	[25.0]	
		<b>952.8</b>	<b>870.9</b>	<b>666.1</b>	<b>381.2</b>	<b>335.9</b>	<b>266.1</b>	
+SV ( $\rho \neq 0$ )	12.63	13.08	14.99	11.96	12.40	14.17		
	(0.013)	(0.017)	(0.042)	(0.013)	(0.016)	(0.040)		
	[26.9]	[26.1]	[26.1]	[26.8]	[26.1]	[26.2]		
	<b>53.3</b>	<b>33.2</b>	<b>7.6</b>	<b>55.2</b>	<b>36.2</b>	<b>7.9</b>		
	+SV ( $\rho = 0$ ) +SV ( $\rho \neq 0$ )	12.62	13.08	14.89	11.96	12.39	14.11	
		(0.003)	(0.004)	(0.005)	(0.005)	(0.005)	(0.007)	
		[28.9]	[28.5]	[28.6]	[29.0]	[28.6]	[28.5]	
		<b>827.5</b>	<b>711.7</b>	<b>571.4</b>	<b>327.7</b>	<b>297.8</b>	<b>253.0</b>	
All		12.62	13.07	14.90	11.95	12.38	14.10	
		(0.003)	(0.003)	(0.004)	(0.004)	(0.005)	(0.006)	
		[28.8]	[28.5]	[28.1]	[28.9]	[28.4]	[28.3]	
		<b>1150.0</b>	<b>1041.0</b>	<b>682.2</b>	<b>407.3</b>	<b>415.8</b>	<b>310.9</b>	

Table 5.83: CVs in Combinations in Johnson and Shanno Model: Average Rate Options (High Volatility)

## Chapter 6

# Numerical Results for Barrier Options

### 6.1 Introduction

We provide comparisons of using control variates, including our new CVs, for barrier options in different models. We consider ATM, ITM and OTM options with different barrier levels, representing medium, high and low hitting probabilities  $p$ . Table 6.1 lists the option parameters. Note that the barrier levels for the OTM options

		$K = 100$ (ATM)	$K = 80$ (ITM)	$K = 120$ (OTM)
$U$	$p$ Medium	120	120	140
	$p$ High	105	105	130
	$p$ Low	150	150	150

Table 6.1: Barrier Option Parameters

are different from other cases because the option payoff is always zero if  $K \geq U$ . For all options, we fix the time to maturity  $T = 1$ .

We use a large number of sample paths ( $M = 10^6$ ) for the plain Monte Carlo method (without any variance reduction method) to obtain a low standard error. This as a the sanity check to ensure that the CVs do not introduce pricing bias. The number of sample paths we use for Monte Carlo methods with CVs is  $M = 10000$ . There is no stratification applied to the random numbers. The number of steps  $N = 256$ . The code is written in VBA 6.5 and run on a Dell desktop with Duo core processor.

We compare the performances of CVs in the Heston and GARCH diffusion model. We do not do comparisons in the SABR and the Johnson and Shanno model because our method do not apply to the case where  $\beta \in (0, 1)$ .

**Model parameters.** The common feature of the Heston and GARCH diffusion model is that  $\xi = \frac{1}{2}$ . We use four parameter sets in total, two for each model. Table 6.2 displays the values of the parameters. These parameters are the same as in

	Heston		GARCH	
	Case 1	Case 2	Case 3	Case 4
$S_0 =$	100	100	100	100
$r =$	0.025	0.05	0.025	0.05
$V_0 =$	0.0175	0.04	0.0175	0.04
$\alpha =$	1.5768	0.2	4	2
$\mu =$	0.0398	0.05	0.0225	0.09
$\eta =$	0.5751	0.1	1.2	0.8
$\xi =$	0.5	0.5	0.5	0.5
$\gamma =$	0.5	0.5	1	1
$\rho =$	-0.5711	-0.5	-0.5	-0.5

Table 6.2: The Heston and GARCH Parameters

chapter 5.

We provide results for standard and digital barrier options. The standard barrier option has the payoff of the form (3.24). The digital barrier option pays off 1 if  $S_T > K$ , unless it knocks out before time  $T$ . The reason we include the digital barrier option is that Glasserman and Staum [39] claim their importance sampling method works well in the case of high hitting probability. We compare our CV methods with their results.

The rest of this chapter is organised as following. In section 6.2, we review the importance sampling method of Glasserman and Staum [39]. In section 6.3, we provide results for standard barrier options. In section 6.4, we give results for digital barrier options. In section 6.5, we benchmark the importance sampling method with the results in Glasserman and Staum [39]. In section 6.6, we examine the performance of CVs for barrier options with different number of reset dates. Section 6.7 concludes.

We find that our new CV has a slightly higher efficiency gain than the GBM auxiliary CV. However, efficiency gains in barrier option pricing are never as high as those in average rate option pricing in chapter 5.

## 6.2 The Importance Sampling Method

One of the main sources of randomness in barrier option pricing is the hitting time distribution. Glasserman and Staum [39] design a method which forces all sample paths to survive up to maturity so that the randomness in the hitting time is removed. Their method is set in the model with  $\beta = 1$  is based on the following discretization of the model.

$$\tilde{S}_{t_{i+1}} = \tilde{S}_{t_i} \exp \left[ \left( r - \frac{1}{2} \tilde{V}_{t_i}^{2\xi} \right) \Delta t + \tilde{V}_{t_i}^\xi \sqrt{\Delta t} \Phi^{-1}(u_i^1) \right], \quad (6.1)$$

$$\tilde{V}_{t_{i+1}} = f \left( \tilde{V}_{t_i}, \rho \Phi^{-1}(u_i^1) + \tilde{\rho} \Phi^{-1}(u_i^2) \right), \quad (6.2)$$

where  $\tilde{\rho} = \sqrt{1 - \rho^2}$ ,  $u_i^1, u_i^2 \sim U(0, 1)$  and  $f(\cdot)$  is any discretization of  $V_t$  which depends only on  $\tilde{V}_{t_i}$  and  $\Delta W_i^V$ . The one-step survival probability is

$$\pi_i = \mathbb{P} \left( \tilde{S}_{t_{i+1}} \leq U \mid \tilde{S}_{t_i} \right) = \Phi \left( \frac{\ln \frac{U}{\tilde{S}_{t_i}} - \left( r - \frac{1}{2} \tilde{V}_{t_i}^{2\xi} \right) \Delta t}{\tilde{V}_{t_i}^\xi \sqrt{\Delta t}} \right). \quad (6.3)$$

To prevent the path from knocking out, we sample from the region where  $\tilde{S}_{t_{i+1}} \leq U$ , that is  $u_i^1 \sim U(0, \pi_i)$ . Equivalently, we can simulate  $u_i^1$  and  $u_i^2$  as

$$u_i^1 = \pi_i v_i^1, \quad v_i^1 \sim U(0, 1), \quad (6.4)$$

$$u_i^2 = v_i^2, \quad v_i^2 \sim U(0, 1). \quad (6.5)$$

By doing so, no path knocks out and we get a payoff of vanilla call option at maturity but scaled by the product of all one-step survival probabilities

$$\mathcal{H} \left( \tilde{S}_t, t \in \mathcal{T} \right) = \left( \tilde{S}_{t_N} - K \right)^+ \prod_{i=0}^{N-1} \pi_i. \quad (6.6)$$

Glasserman and Staum proved that this estimator is unbiased.

We point out the advantage and disadvantage of their method compared to using CVs:



**Advantage:**

1. Can apply not only to single barrier options but also to double or partial barrier options.

**Disadvantage:**

1. Can only take the discretization of the form (6.1).
2. Requires an evaluation of  $\Phi(\cdot)$  at each time step.
3. Must simulate the whole path until maturity.

We apply the importance sampling method (IS) in both the medium and high hitting probability cases where their method is likely to make a contribution. We compare it with three candidate CVs. They are the GBM European call CV (for options with low probability of hitting barrier only), the GBM up and out call CV (UOC CV) and the correlation CV that we proposed in section 3.6.2 (SV CV). When we present our results, in addition to the four outputs in equation (5.1), we add the empirical correlation of the CV as a fifth entry in curly brackets.

## 6.3 Standard Barrier Options

We start by comparing the empirical correlations of different CVs. The efficiency gains of all individual CVs, CVs in combinations and the importance sampling method (IS) are reported later in section 6.3.2.

### 6.3.1 Empirical Correlations

It is useful to first look at how empirical correlations between CVs and option payoffs change when the barrier level varies. We use the ATM option and vary the barrier level from 105 to 150. The empirical correlations of three CVs in 4 cases are plotted in Figure 6.1. We see that the empirical correlation generally increases with the barrier level for all three CVs. A lower barrier level means higher probability of knocking out. We know that the random hitting time accounts for a large part of the variance in the simulation. Because it is difficult for the auxiliary process to track all the random hitting time in the original process, the UOC CV and SV CV are not highly correlated with the option payoff. The call CV does not even take into account the hitting time so it is not surprising that it is virtually uncorrelated with the option payoff for low  $U$ . When  $U$  is high, knocking out becomes an unlikely event. In these cases, the advantage of UOC CV or SV CV over the call CV

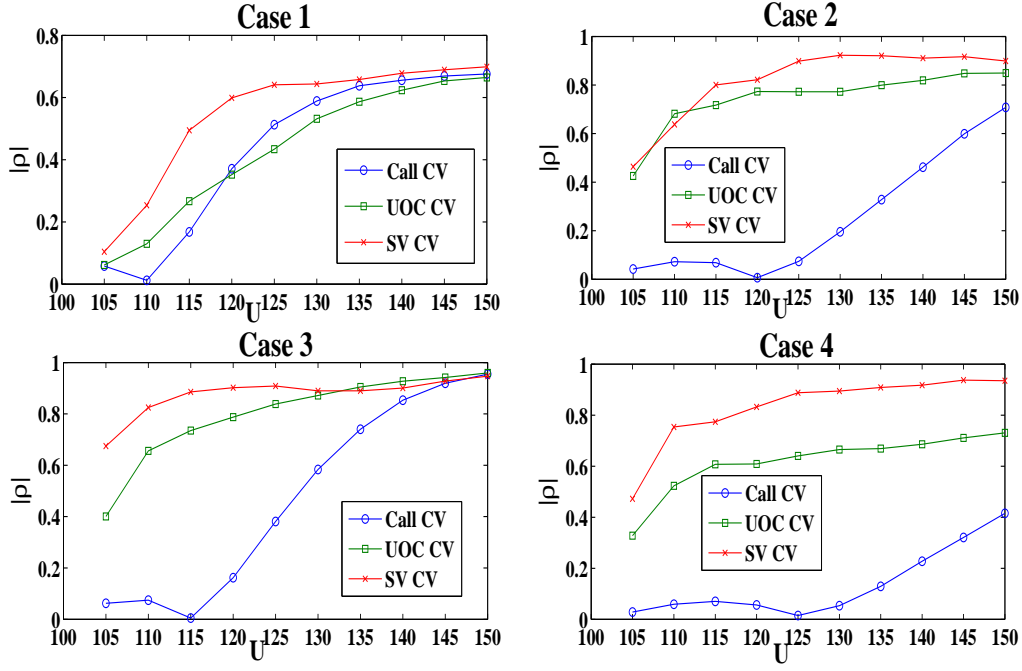


Figure 6.1: Empirical Correlations (Standard Barrier Options, ATM)

is reduced as the barrier option is more like a European option.

In all 4 cases, we can see that our proposed CV generally has higher empirical correlations with the option payoffs than the other CVs, especially in Case 4 where its gap with UOC CV is almost constant at about 0.2. This is evidence that our CV auxiliary process is closer to the original than the GBM model.

We are, however, more interested in the efficiency gains since the amount of reduction in the standard error needs to justify the additional time taken to use the CV.

### 6.3.2 Efficiency Gains

We price the options listed in Table 6.1. We use UOC CV and SV CV for all options and only use the call CV for options with low  $p$  (when it is likely to make a contribution). We discuss ATM, ITM and OTM options separately. In each case, we list the results for medium, high and low hitting probability  $p$ .

**ATM options.** Table 6.5 (page 163) displays the results for ATM options with medium hitting probability. Our proposed CV (SV CV) has higher empirical corre-

lations than the UOC CV at the cost of an additional 30% of the computational time of UOC CV. The efficiency gains of SV CV are higher than those of the UOC CV, although not too much. But considering the difficulty of achieving high variance reduction for barrier options, our proposed CV provides an improvement. Using both CVs together gives slightly higher efficiency gains than using them individually.

The importance sampling method, however, has very poor performance. It takes most computation time because it requires at each time step one evaluation of the cumulative normal density function. But this additional cost is not justified by the variance reduction it provides. In fact, all of its standard errors are higher than those of other CVs. As a result, its efficiency gains are all below one.

We notice that in Case 1 the empirical correlations are much lower than those in other cases. Recall that Case 1 corresponds to the Heston model with  $d < 1$ . The relatively low efficiencies in this case are also seen in other options such as arithmetic average rate options (see Webber [83]).

Table 6.6 (page 164) lists the results for ATM options with high barrier hitting probability. This is a case where all CVs perform poorly. Due to the low empirical correlation in all cases, the efficiency gains are never significantly different from 1, which means in this case we would rather not use any CV at all. The reason for the poor performance is that the randomness of the hitting time forms a large part of the variance in the simulation. The payoff varies only between 0 to 5 while a large number of the sample paths hit the barrier before maturity. When a path gets close to the barrier, the payoff of the option is very sensitive to fluctuations in the path. Since the asset price is close to the barrier at the very beginning, it is very difficult for our auxiliary processes to capture this sensitivity locally even if they can track the original process globally. The importance sampling method provides lower standard error in Cases 1 and 4 as its performance is slightly enhanced relative to other methods in the case of high  $p$ . The efficiency gains are slightly higher than those in the case of medium  $p$  in Table 6.5 (page 163) but still below one.

Table 6.7 (page 165) presents the results for ATM options with low  $p$ . In this case, the options are very much like vanilla European options. Therefore, the European call option becomes a useable CV candidate. The empirical correlations of three CVs are close to each other in Cases 1 and 3 as is seen in Figure 6.1. In both cases, we would rather use the standard CVs (UOC and Call CV) since our

proposed CV has greater computation time and therefore is less efficient than them. We also find that the best combination in Cases 1 and 3 is UOC with the Call CV. In Cases 2 and 4, however, SV CV has higher efficiency gains, especially in Case 4 where its efficiency gains are more than twice of those of others.

**ITM options.** Table 6.8 (page 166) displays results for ITM options with medium hitting probability. SV CV has higher empirical correlations and efficiency gains than UOC CV. The efficiency gains in Case 1 are lower than those in other cases. This is consistent with the ATM cases with medium and high  $p$  in Table 6.5 (page 163) and 6.6 (page 164). The importance sampling method has efficiency gains all around 0.6 as in the ATM case in Table 6.5 (page 163).

Table 6.9 (page 167) gives results for ITM options with high  $p$ . In Case 1 and 2, the efficiency gains of both CVs are very similar while in Case 3 and 4 SV CV has higher efficiency gains than UOC CV. Their combination does not have much improvement to the individual ones. The importance sampling method has low efficiency gains as in the ATM case.

Table 6.10 (page 168) presents results for ITM options with low  $p$ . SV CV performs better than standard CVs in Cases 2 and 4 but worse in Cases 1 and 3. This is consistent with the results for ATM options in Table 6.7 (page 165).

**OTM options.** The results for OTM options with medium  $p$  are given in Table 6.11 (page 169). Note that the barrier levels are different from those in the ATM and ITM cases. In Cases 2 and 4, the empirical correlations of SV CV are much higher than those of UOC CV. This leads to higher efficiency gains of SV CV. Case 1 remains difficult as both empirical correlations are very low, as is seen in all other examples. In Case 3, the correlation of SV CV is slightly lower than that of UOC CV. This causes a lower efficiency gains from our proposed CV. The importance sampling method has efficiency gains all below one as in other examples.

Table 6.12 (page 170) displays results for OTM options with high  $p$ . In all cases, the efficiency gains of both CVs are very similar, though SV CV has higher empirical correlations. The efficiency gains never exceed 2. The importance sampling method still has very poor efficiency gains.

Table 6.13 (page 171) presents results for OTM options with low  $p$ . SV CV provides

more significant efficiency gain than other CVs only in Case 4. In other cases, its efficiency gains are not as high as those of other CVs due to their similar empirical correlations, as seen in Figure 6.1.

In summary, SV CV is more worthwhile to use than standard CVs in the case of medium hitting probability, which is seen from both the higher empirical correlations and efficiency gains. In the case of high hitting probability, SV CV performs better in the ITM and OTM cases but worse in the ATM case than the UOC CV. When the hitting probability is low, our method generally performs better in Cases 2 and 4 than standard CVs but not so good in Cases 1 and 3. The importance sampling method has very low efficiency gains in all cases and therefore is not recommended for standard barrier options.

## 6.4 Digital Barrier Options

The application of our CV is straightforward in the digital option as its conditional value is also explicitly available in the auxiliary process. As in the standard barrier option case, we start investigating the empirical correlations of different CVs as the barrier level changes.

### 6.4.1 Empirical Correlations

We plot the empirical correlations for ATM options with different barrier levels in Figure 6.2. In Cases 1, 3 and 4, SV CV has higher empirical correlations than the other CVs. In Case 2, SV CV has similar correlations to UOC CV. This is different from the standard barrier option case in Figure 6.2.

### 6.4.2 Efficiency Gains

We report efficiency gains for different CVs. In addition, in the case of medium and high  $p$ , we include the importance sampling method in our comparisons because it is only likely to make a difference in these two cases.

**ATM options.** Table 6.14 (page 172) displays results for ATM options with medium  $p$ . SV CV has very similar efficiency gains with UOC CV. Combining them has no significant improvement. The importance sampling method, however, has very poor performance. It is the most expensive but still has higher standard errors than those of the CVs. Its efficiency gains are all less than one. It is definitely not worth using.

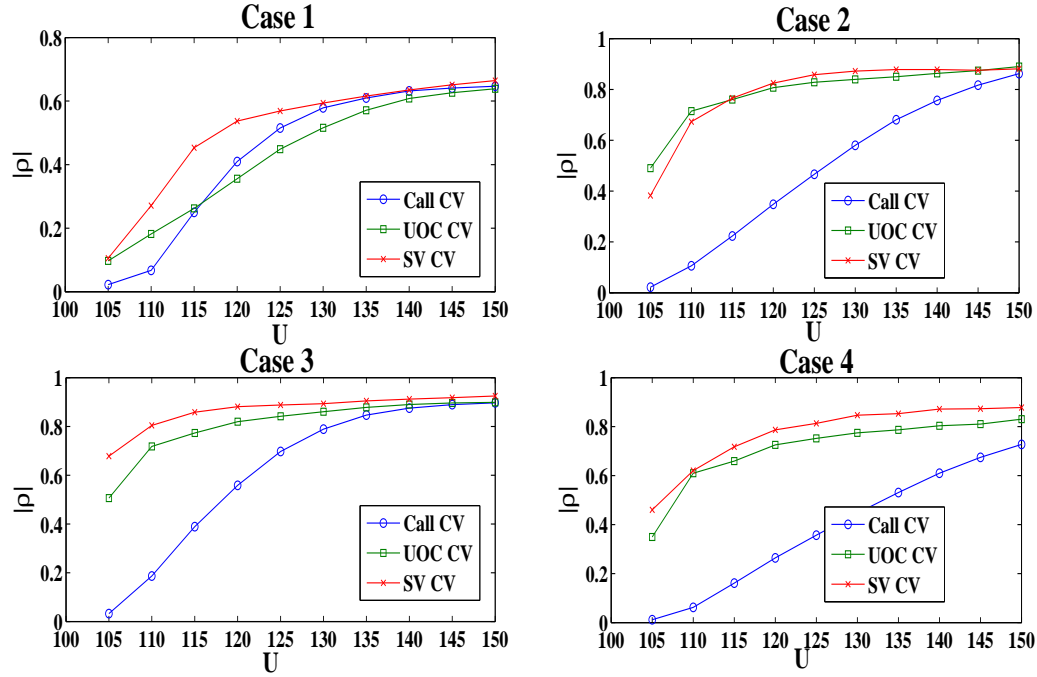


Figure 6.2: Empirical Correlations (Digital Barrier Options, ATM)

Table 6.15 (page 173) presents the results for ATM options with high  $p$ . Neither SV CV nor UOC CV produces significant efficiency gains. Especially in Case 1, the empirical correlations of both CVs are less than 0.1. It remains a difficult case for variance reduction methods. The importance sampling method gives slightly lower standard errors than the CVs do. But due to much larger computational cost (at least about 40% more time) than the CVs, its efficiency gains never exceed those of the CVs.

Table 6.16 (page 174) lists results for ATM options with low  $p$ . SV CV has lower efficiency gains than UOC CV and Call CV except in Case 4. The latter two has very similar efficiency gains. This is because they have similar empirical correlations (as seen in Figure 6.2) and have roughly the same computational cost.

**ITM options.** Table 6.17 (page 175) shows results for ITM options with medium  $p$ . SV CV performs much better than UOC CV in Cases 2, 3 and 4, both in terms of the empirical correlations and efficiency gains. The importance sampling method has very poor performance as in the case of ATM options in Table 6.17 (page 175).

Table 6.18 (page 176) presents results for ITM options with high  $p$ . The situation is very similar as in the case of medium  $p$  in Table 6.17 (page 175). Our proposed CV provides higher efficiency gains than UOC CV. The importance sampling method is still inefficient.

Table 6.19 (page 177) displays results for ITM options with low  $p$ . The efficiency gains of the Call CV and UOC CV are very similar, but lower than those of SV CV in Cases 2, 3 and 4, as is in the case of medium and high  $p$ . High efficiency gains in Case 1 are hard to achieve with CVs. Using CV makes no difference to the plain Monte Carlo method.

**OTM options.** Table 6.20 (page 178) gives results for OTM options with medium  $p$ . Except in Case 1, the efficiency gains of SV CV are slightly higher than those of UOC CV. Combining both further increases the efficiency gains in Cases 2 and 3. Gains for the importance sampling method remain poor.

Table 6.21 (page 179) presents results for OTM options with high  $p$ . The situation is similar to Table 6.20 (page 178), but the efficiency gains are in general lower than those in the case of medium  $p$ .

Table 6.22 (page 180) lists results for OTM options with low  $p$ . The efficiency gains of SV CV are slightly lower than UOC CV and Call CV in Case 1 and 2 (the Heston model) but higher in Case 3 and 4 (the GARCH model).

In summary, SV CV provides higher efficiency gains for ITM and OTM options than standard CVs. Using it is worthwhile. For ATM options, the efficiency gains of SV CV are similar to (or slightly lower than) those of the standard CVs. We also find that in Case 1 (the Heston model with  $d < 1$ ) it is very hard to get high efficiency gains with any method. Using CVs in combinations has no significant improvement on the efficiency gains of individual CVs. Finally, the importance sampling method has very poor performance in all cases. One should use CVs instead of importance sampling method for barrier options.

## 6.5 Benchmark with Glasserman and Staum [39]

The idea of importance sampling method is promising because it removes the randomness from the hitting time distribution by forcing all paths to survive until

maturity. However, we have seen that in section 6.3 and 6.4 it has poor results. In order to be fully confident that we implement their method correctly, we want to replicate the results reported in their paper. We choose Example 1C (page 933). Note that they use a different measure of efficiency gain to equation (3.7).

**The efficiency measure.** To price a knock-out barrier option with plain Monte Carlo method, it is not necessary to simulate the whole path until maturity. We can stop the evolution and return a zero payoff as soon as the barrier is crossed. Due to the natural of their importance sampling method, however, one must simulate each path until maturity. Denote the average number of time steps taken by the plain Monte Carlo method by  $N_1$  so  $N_1 \leq N$  and that taken by the importance sampling method by  $N_2 = N$ . Glasserman and Staum measure the efficiency gain as

$$\hat{E}_{1,2} = \frac{N_1 \cdot se_1^2}{N_2 \cdot se_2^2}. \quad (6.7)$$

This is supposed to represent the multiple of cost required by the plain Monte Carlo method in order to achieve the same level of standard error as the importance sampling method.<sup>1</sup> This measure, however, is not equivalent to the standard definition (3.7) as it assumes the same cost of each step in the plain Monte Carlo method and the importance sampling method. This is obviously not true because the importance sampling method requires at each step one additional evaluation of the cumulative normal density function which is expensive. For benchmark and comparison purposes, we include their measure in the results in this section.

**Parameters specifications.** The underlying asset has a GBM process. The initial stock price  $S_0 = 100$ . The interest rate  $r = 0$ . There is a single down barrier at level  $D$  and it is always equal to the strike  $K$ . We list in Table 6.3 parameter values of all five scenarios in their paper.  $\sigma$  is the volatility of the GBM process and  $n$  is

Scenario	$\sigma$	$T$	$D$	$n$
1	30.0%	0.25	94.30	3
2	73.7%	0.25	94.30	3
3	30.0%	1.50	94.30	3
4	30.0%	0.25	98.62	3
5	30.0%	0.25	94.30	63

Table 6.3: The Parameters and Results in Glasserman and Staum [39]

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<sup>1</sup>Note that Glasserman and Staum report  $1/\hat{E}_{1,2}$  in their paper. We use  $\hat{E}_{1,2}$  as it is comparable with  $E_{1,2}$  in equation (3.7).



the number of reset dates.

Table 6.4 displays the results from our implementation. The numbers in angle

		Scenario 1	Scenario 2	Scenario 3	Scenario 4	Scenario 5
Standard	Plain	8.12	14.06	14.85	5.33	6.42
		(0.116)	(0.279)	(0.296)	(0.098)	(0.114)
		[0.03]	[0.05]	[0.03]	[0.03]	[0.33]
		$\langle 2.3 \rangle$	$\langle 2.0 \rangle$	$\langle 2.0 \rangle$	$\langle 2.0 \rangle$	$\langle 33.8 \rangle$
		8.04	14.12	14.19	5.25	6.15
	IS	(0.075)	(0.156)	(0.160)	(0.051)	(0.101)
		[0.11]	[0.09]	[0.09]	[0.09]	[2.00]
		<b>0.7</b>	<b>1.6</b>	<b>1.1</b>	<b>1.2</b>	<b>0.2</b>
		$\langle 3.0 \rangle$	$\langle 3.0 \rangle$	$\langle 3.0 \rangle$	$\langle 3.0 \rangle$	$\langle 63.0 \rangle$
		<i>1.8</i>	<i>2.1</i>	<i>2.3</i>	<i>2.5</i>	<i>0.7</i>
	GS	<i>1.7</i>	<i>2.1</i>	<i>2.1</i>	<i>2.4</i>	<i>0.6</i>
Digital	Plain	0.50	0.34	0.33	0.34	0.33
		(0.005)	(0.005)	(0.005)	(0.005)	(0.005)
		[0.05]	[0.03]	[0.05]	[0.05]	[0.33]
		$\langle 2.3 \rangle$	$\langle 2.0 \rangle$	$\langle 2.0 \rangle$	$\langle 2.0 \rangle$	$\langle 33.9 \rangle$
		0.50	0.34	0.34	0.34	0.32
	IS	(0.002)	(0.001)	(0.001)	(0.001)	(0.004)
		[0.09]	[0.09]	[0.09]	[0.09]	[2.00]
		<b>5.5</b>	<b>5.5</b>	<b>8.3</b>	<b>9.7</b>	<b>0.2</b>
		$\langle 3.0 \rangle$	$\langle 3.0 \rangle$	$\langle 3.0 \rangle$	$\langle 3.0 \rangle$	$\langle 63.0 \rangle$
		<i>8.5</i>	<i>10.9</i>	<i>10.9</i>	<i>12.6</i>	<i>0.8</i>
	GS	<i>8.3</i>	<i>11.1</i>	<i>11.1</i>	<i>12.5</i>	<i>0.8</i>

Table 6.4: Benchmark with results in Glasserman and Staum [39]

brackets  $\langle \cdot \rangle$  are the actual number of time steps  $N_1$  and  $N_2$ . Those in italic are  $\hat{E}$  computed with Glasserman and Staum’s measure (6.7). “GS” stands for the equivalent results in Glasserman and Staum [39]. They serve as benchmarks of  $\hat{E}$  in our implementation.

We see that our results, for both standard and digital barrier options, benchmark very closely to the results reported by Glasserman and Staum. However, they are considerably and consistently greater than the standard measures reported in bold, suggesting their measure over-estimates the real speed-ups.

The efficiency gains for digital barrier options are much higher than those for standard ones. In those scenarios with very few reset dates, the importance sampling

method performs very well, especially for digital barrier options. However, when  $n$  is high, its real efficiency gains are very low.

## 6.6 Varying the Number of Reset Dates

We price barrier options with 4, 16 and 64 reset dates up to one year. The parameters are the same as in Table 6.2. We only look at the case of medium  $p$  for ATM standard barrier options.

The importance sampling method is difficult to implement in this example. The reason is the following. Suppose the barrier resets at  $t_i$  but not at  $t_{i-1}$  and  $\tilde{S}_{t_{i-1}} \gg U$ . In this case, the conditional probability  $\mathbb{P}(\tilde{S}_{t_i} \leq U \mid \tilde{S}_{t_{i-1}})$  is so small such that VBA regards it as equal to zero. In other words, there is no way to force  $\tilde{S}_{t_i} \leq U$ . Therefore, we use only CVs in this example and not the importance sampling method. Note that our CVs are all continuous barrier options. We expect the empirical correlations and efficiency gains to be lower than those in section 6.3. However, as the number of reset dates increases, the efficiency gains should also go up.

Table 6.23 (page 181) presents results for options with 4 reset dates. SV CV has higher empirical correlations than UOC CV in Case 1 and 2 but similar in Case 3 and 4. The efficiency gains of SV CV are all slightly lower than UOC CV.

Table 6.24 (page 182) displays results for options with 16 reset dates. The empirical correlations and efficiency gains of both CVs are higher than those in Table 6.23. But the efficiency gains of SV CV are lower than those of UOC CV. We also see that the option prices are lower than those with 4 reset dates because the probability of knocking out is higher.

Table 6.25 (page 183) gives results for options with 64 reset dates. There are significant increases in empirical correlations from Table 6.24 except for Case 1. In fact, they are close to the correlations in the case of 256 reset dates in Table 6.5. The efficiency gains of SV CV are higher than those of UOC CV. Using them in combination provides additional efficiency gains.

In summary, we have compared the efficiency gains of CVs and the importance sampling method for standard and digital barrier options with medium, high and low hitting probability and different number of reset dates. We have also bench-

marked our implementation of the importance sampling method by Glasserman and Staum [39]. We find that our proposed CV, SV CV, has higher empirical correlations as well as efficiency gains in general. But we should point out that their efficiency gains are not significant. The importance sampling method works well in the GBM process with high hitting probability and few reset dates. But it fails in the our general class of stochastic volatility models.

We list all the work we have done for barrier option pricing.

1. We compared empirical correlations and efficiency gains of the CVs and the importance sampling method for both standard and digital barrier options.
2. We benchmarked our results with those reported in Glasserman and Staum [39].
3. We compared empirical correlations and efficiency gains of the CVs for standard barrier options with 4, 16 and 64 reset dates.

## 6.7 Conclusion

We have compared the performance of our method with standard CVs and the importance sampling method across different models, parameters for barrier options. Our proposed CV provides slightly higher efficiency gains than GBM auxiliary CVs and the importance sampling method in many cases. But in general none of the methods has significant efficiency gains in the barrier option valuation. The importance sampling method, in particular, has very poor performance and is not recommended to use. We have also benchmarked with the results in Glasserman and Staum [39] to ensure that our implementation of their method is correct. The efficiency measure in their paper is non-standard as it does not take into account the computation time.

We have seen in chapter 5 and 6 that our correlation CVs generally perform better than old CVs. With the help of them, we investigate the variations in barrier option pricing under our class of stochastic volatility models in chapter 7. By applying the best combinations of CVs, it is faster to calibrate our class of models and to distinguish the barrier option prices under different stochastic volatility models.

## Tables

Standard Barrier Options (ATM, Medium $p$ )					
	Plain	GBM Auxiliary	SV Auxiliary	Combinations	IS
		UOC	SV	UOC+SV	
Case 1	4.14	4.15	4.12	4.10	4.07
	(0.005)	(0.051)	(0.043)	(0.042)	(0.051)
	[575.4]	[7.0]	[9.0]	[9.7]	[11.0]
	-	<b>0.9</b> {0.33}	<b>1.0</b> {0.61}	<b>1.0</b> -	<b>0.6</b> -
Case 2	1.56	1.56	1.55	1.53	1.54
	(0.004)	(0.024)	(0.019)	(0.017)	(0.033)
	[587.0]	[6.8]	[9.0]	[9.7]	[11.8]
	-	<b>2.1</b> {0.76}	<b>2.4</b> {0.84}	<b>2.9</b> -	<b>0.6</b> -
Case 3	2.68	2.68	2.66	2.68	2.68
	(0.005)	(0.028)	(0.020)	(0.019)	(0.043)
	[567.3]	[7.0]	[9.1]	[9.9]	[11.8]
	-	<b>2.2</b> {0.80}	<b>3.4</b> {0.90}	<b>3.4</b> -	<b>0.6</b> -
Case 4	0.86	0.87	0.86	0.85	0.83
	(0.003)	(0.022)	(0.016)	(0.015)	(0.024)
	[568.6]	[6.9]	[9.0]	[9.8]	[12.0]
	-	<b>1.4</b> {0.63}	<b>2.0</b> {0.84}	<b>2.1</b> -	<b>0.7</b> -

Table 6.5: Efficiency Gains for Standard Barrier Options (ATM, Medium  $p$ )

Standard Barrier Options (ATM, High $p$ )					
	Plain	GBM Auxiliary	SV Auxiliary	Combinations	IS
		UOC	SV	UOC+SV	
Case 1	0.100	0.102	0.102	0.107	0.094
	(0.0005)	(0.0054)	(0.0053)	(0.0055)	(0.0043)
	[582.0]	[7.0]	[8.8]	[9.3]	[11.5]
	-	<b>0.8</b>	<b>0.7</b>	<b>0.6</b>	<b>0.8</b>
Case 2	-	{0.10}	{0.08}	-	-
	0.016	0.017	0.016	0.017	0.019
	(0.0002)	(0.0019)	(0.0019)	(0.0016)	(0.0017)
	[585.8]	[6.7]	[8.8]	[9.3]	[11.8]
Case 3	-	<b>1.0</b>	<b>0.7</b>	<b>1.0</b>	<b>0.7</b>
	-	{0.42}	{0.40}	-	-
	0.041	0.038	0.039	0.044	0.040
	(0.0003)	(0.0029)	(0.0025)	(0.0024)	(0.0026)
Case 4	[593.5]	[6.8]	[9.0]	[9.4]	[11.8]
	-	<b>1.1</b>	<b>1.1</b>	<b>1.1</b>	<b>0.8</b>
	-	{0.43}	{0.59}	-	-
	0.008	0.009	0.010	0.009	0.010
Case 5	(0.0001)	(0.0016)	(0.0015)	(0.0013)	(0.0012)
	[593.6]	[6.7]	[8.9]	[9.3]	[11.8]
	-	<b>0.8</b>	<b>0.6</b>	<b>0.8</b>	<b>0.8</b>
	-	{0.36}	{0.48}	-	-

Table 6.6: Efficiency Gains for Standard Barrier Options (ATM, High  $p$ )

Standard Barrier Options (ATM, Low $p$ )							
	Plain	GBM Auxiliary		SV Auxiliary	Combinations		
		UOC	Call	SV	UOC+Call	UOC+SV	All
Case 1	7.08	7.14	7.00	7.16	7.06	6.99	7.06
	(0.008)	(0.063)	(0.062)	(0.062)	(0.061)	(0.053)	(0.054)
	[567.7]	[7.0]	[6.2]	[9.1]	[6.4]	[9.9]	[9.9]
	-	<b>1.4</b>	<b>1.7</b>	<b>1.1</b>	<b>1.7</b>	<b>1.4</b>	<b>1.4</b>
Case 2	-	{0.66}	{0.67}	{0.68}	-	-	-
	8.66	8.78	8.61	8.65	8.73	8.69	8.67
	(0.011)	(0.063)	(0.082)	(0.051)	(0.055)	(0.049)	(0.044)
	[566.3]	[7.0]	[6.3]	[9.1]	[6.4]	[9.9]	[10.0]
Case 3	-	<b>2.6</b>	<b>1.8</b>	<b>3.1</b>	<b>3.7</b>	<b>3.1</b>	<b>3.7</b>
	-	{0.84}	{0.69}	{0.90}	-	-	-
	6.91	6.91	6.94	6.87	6.93	6.94	6.91
	(0.009)	(0.028)	(0.030)	(0.030)	(0.024)	(0.023)	(0.020)
Case 4	[566.1]	[7.0]	[6.3]	[9.2]	[6.5]	[10.2]	[10.1]
	-	<b>8.9</b>	<b>8.5</b>	<b>5.8</b>	<b>13.1</b>	<b>9.0</b>	<b>12.1</b>
	-	{0.95}	{0.95}	{0.94}	-	-	-
	7.37	7.37	7.28	7.37	7.30	7.30	7.40
Case 4	(0.011)	(0.078)	(0.102)	(0.043)	(0.077)	(0.039)	(0.040)
	[573.2]	[7.1]	[6.3]	[9.2]	[6.5]	[10.0]	[10.0]
	-	<b>1.7</b>	<b>1.1</b>	<b>4.2</b>	<b>1.9</b>	<b>4.9</b>	<b>4.4</b>
	-	{0.73}	{0.41}	{0.92}	-	-	-

Table 6.7: Efficiency Gains for Standard Barrier Options (ATM, Low  $p$ )

Standard Barrier Options (ITM, Medium $p$ )					
	Plain	GBM Auxiliary	SV Auxiliary	Combinations	IS
		UOC	SV	UOC+SV	
Case 1	17.10	17.00	17.11	17.11	16.99
	(0.012)	(0.116)	(0.100)	(0.099)	(0.117)
	[566.9]	[6.6]	[8.6]	[9.6]	[10.9]
	-	<b>1.0</b>	<b>1.0</b>	<b>0.9</b>	<b>0.6</b>
Case 2	-	{0.33}	{0.59}	-	-
	8.58	8.50	8.53	8.52	8.45
	(0.011)	(0.060)	(0.046)	(0.042)	(0.099)
	[574.9]	[6.4]	[8.6]	[9.4]	[11.7]
Case 3	-	<b>2.9</b>	<b>3.8</b>	<b>4.0</b>	<b>0.6</b>
	-	{0.83}	{0.91}	-	-
	14.00	14.02	14.05	13.97	13.98
	(0.012)	(0.064)	(0.050)	(0.044)	(0.108)
Case 4	[574.5]	[6.6]	[8.8]	[9.8]	[11.6]
	-	<b>2.8</b>	<b>3.4</b>	<b>4.0</b>	<b>0.6</b>
	-	{0.83}	{0.90}	-	-
	5.37	5.41	5.36	5.34	5.32
Case 4	(0.009)	(0.061)	(0.036)	(0.038)	(0.082)
	[568.9]	[6.5]	[8.5]	[9.4]	[11.7]
	-	<b>2.0</b>	<b>4.2</b>	<b>3.6</b>	<b>0.6</b>
	-	{0.75}	{0.92}	-	-

Table 6.8: Efficiency Gains for Standard Barrier Options (ITM, Medium  $p$ )

Standard Barrier Options (ITM, High $p$ )					
	Plain	GBM Auxiliary	SV Auxiliary	Combinations	IS
		UOC	SV	UOC+SV	
Case 1	2.74	2.73	2.66	2.70	2.78
	(0.006)	(0.058)	(0.052)	(0.052)	(0.056)
	[571.9]	[6.4]	[8.4]	[9.3]	[11.5]
	-	<b>1.1</b> {0.37}	<b>1.0</b> {0.54}	<b>0.9</b> -	<b>0.6</b> -
Case 2	1.11	1.12	1.13	1.13	1.11
	(0.004)	(0.021)	(0.018)	(0.016)	(0.031)
	[570.2]	[6.3]	[8.3]	[8.9]	[11.8]
	-	<b>2.9</b> {0.83}	<b>2.9</b> {0.88}	<b>3.4</b> -	<b>0.7</b> -
Case 3	2.42	2.42	2.42	2.45	2.36
	(0.005)	(0.031)	(0.023)	(0.022)	(0.047)
	[574.7]	[6.5]	[8.5]	[9.2]	[11.7]
	-	<b>2.7</b> {0.82}	<b>3.9</b> {0.91}	<b>3.7</b> -	<b>0.7</b> -
Case 4	0.64	0.64	0.62	0.63	0.63
	(0.003)	(0.017)	(0.013)	(0.012)	(0.023)
	[572.1]	[6.4]	[8.4]	[9.0]	[11.9]
	-	<b>2.5</b> {0.78}	<b>3.2</b> {0.88}	<b>3.4</b> -	<b>0.7</b> -

Table 6.9: Efficiency Gains for Standard Barrier Options (ITM, High  $p$ )



Standard Barrier Options (ITM, Low $p$ )							
	Plain	GBM Auxiliary		SV Auxiliary	Combinations		
		UOC	Call	SV	UOC+Call	UOC+SV	All
Case 1	22.73	22.78	22.61	22.81	22.76	22.71	22.63
	(0.013)	(0.084)	(0.084)	(0.087)	(0.083)	(0.072)	(0.069)
	[562.2]	[6.8]	[6.1]	[8.7]	[6.7]	[9.8]	[9.8]
	-	<b>2.0</b>	<b>2.2</b>	<b>1.4</b>	<b>2.1</b>	<b>1.9</b>	<b>2.0</b>
Case 2	-	{0.77}	{0.76}	{0.74}	-	-	-
	22.24	22.41	22.33	22.21	22.07	22.37	22.18
	(0.017)	(0.094)	(0.116)	(0.075)	(0.079)	(0.074)	(0.064)
	[561.5]	[6.6]	[6.1]	[8.8]	[6.6]	[10.0]	[9.8]
Case 3	-	<b>2.7</b>	<b>1.9</b>	<b>3.1</b>	<b>3.8</b>	<b>2.9</b>	<b>3.8</b>
	-	{0.83}	{0.71}	{0.89}	-	-	-
	22.16	22.08	22.14	22.24	22.19	22.21	22.22
	(0.014)	(0.037)	(0.042)	(0.052)	(0.033)	(0.032)	(0.028)
Case 4	[567.9]	[6.7]	[6.1]	[8.9]	[6.8]	[9.9]	[10.0]
	-	<b>11.7</b>	<b>9.6</b>	<b>4.4</b>	<b>14.6</b>	<b>10.1</b>	<b>13.1</b>
	-	{0.96}	{0.95}	{0.93}	-	-	-
	18.48	18.40	18.51	18.56	18.47	18.50	18.38
Case 4	(0.018)	(0.116)	(0.160)	(0.065)	(0.114)	(0.060)	(0.057)
	[567.4]	[6.7]	[6.1]	[8.8]	[6.7]	[10.0]	[9.8]
	-	<b>2.0</b>	<b>1.1</b>	<b>4.8</b>	<b>2.0</b>	<b>5.0</b>	<b>5.5</b>
	-	{0.75}	{0.43}	{0.93}	-	-	-

Table 6.10: Efficiency Gains for Standard Barrier Options (ITM, Low  $p$ )

Standard Barrier Options (OTM, Medium $p$ )					
	Plain	GBM Auxiliary	SV Auxiliary	Combinations	IS
		UOC	SV	UOC+SV	
Case 1	0.45	0.40	0.43	0.42	0.42
	(0.002)	(0.018)	(0.018)	(0.018)	(0.019)
	[572.5]	[6.1]	[8.3]	[8.8]	[10.3]
	-	<b>1.1</b>	<b>0.9</b>	<b>0.8</b>	<b>0.6</b>
Case 2	-	{0.22}	{0.33}	-	-
	0.96	0.94	0.96	1.00	0.98
	(0.003)	(0.021)	(0.015)	(0.015)	(0.027)
	[572.8]	[6.3]	[8.3]	[9.0]	[11.7]
Case 3	-	<b>1.7</b>	<b>2.7</b>	<b>2.3</b>	<b>0.6</b>
	-	{0.67}	{0.86}	-	-
	0.63	0.62	0.63	0.63	0.61
	(0.002)	(0.014)	(0.015)	(0.012)	(0.022)
Case 4	[576.7]	[6.2]	[8.4]	[9.0]	[10.3]
	-	<b>2.6</b>	<b>1.8</b>	<b>2.4</b>	<b>0.6</b>
	-	{0.80}	{0.78}	-	-
	0.72	0.74	0.72	0.72	0.72
Case 4	(0.003)	(0.024)	(0.013)	(0.013)	(0.023)
	[584.5]	[6.2]	[8.4]	[9.0]	[11.7]
	-	<b>1.1</b>	<b>2.8</b>	<b>2.7</b>	<b>0.6</b>
	-	{0.41}	{0.87}	-	-

Table 6.11: Efficiency Gains for Standard Barrier Options (OTM, Medium  $p$ )

Standard Barrier Options (OTM, High $p$ )					
	Plain	GBM Auxiliary	SV Auxiliary	Combinations	IS
		UOC	SV	UOC+SV	
Case 1	0.20	0.20	0.20	0.19	0.20
	(0.001)	(0.010)	(0.010)	(0.010)	(0.009)
	[579.1]	[6.1]	[8.3]	[8.8]	[10.8]
	-	<b>1.0</b>	<b>0.7</b>	<b>0.7</b>	<b>0.6</b>
Case 2	-	{0.09}	{0.23}	-	-
	0.21	0.22	0.21	0.22	0.21
	(0.001)	(0.009)	(0.006)	(0.006)	(0.009)
	[595.0]	[6.1]	[8.3]	[8.8]	[11.8]
Case 3	-	<b>1.3</b>	<b>1.8</b>	<b>1.9</b>	<b>0.7</b>
	-	{0.51}	{0.78}	-	-
	0.22	0.22	0.22	0.23	0.22
	(0.001)	(0.008)	(0.007)	(0.007)	(0.010)
Case 4	[586.2]	[6.2]	[8.4]	[9.1]	[11.2]
	-	<b>1.4</b>	<b>1.5</b>	<b>1.4</b>	<b>0.6</b>
	-	{0.62}	{0.74}	-	-
	0.13	0.11	0.12	0.14	0.11
Case 4	(0.001)	(0.007)	(0.006)	(0.006)	(0.006)
	[587.6]	[6.2]	[8.4]	[8.9]	[12.0]
	-	<b>1.2</b>	<b>1.3</b>	<b>1.3</b>	<b>0.9</b>
	-	{0.23}	{0.64}	-	-

Table 6.12: Efficiency Gains for Standard Barrier Options (OTM, High  $p$ )

Standard Barrier Options (OTM, Low $p$ )							
	Plain	GBM Auxiliary		SV Auxiliary	Combinations		
		UOC	Call	SV	UOC+Call	UOC+SV	All
Case 1	0.58	0.52	0.57	0.60	0.62	0.60	0.54
	(0.003)	(0.023)	(0.024)	(0.025)	(0.026)	(0.024)	(0.022)
	[572.7]	[6.1]	[6.1]	[8.5]	[6.2]	[9.0]	[8.9]
	-	<b>1.2</b>	<b>1.0</b>	<b>0.7</b>	<b>0.9</b>	<b>0.7</b>	<b>0.9</b>
Case 2	-	{0.24}	{0.26}	{0.35}	-	-	-
	1.81	1.76	1.75	1.80	1.81	1.80	1.85
	(0.005)	(0.031)	(0.039)	(0.026)	(0.029)	(0.024)	(0.025)
	[577.3]	[6.2]	[6.1]	[8.5]	[6.3]	[9.2]	[9.0]
Case 3	-	<b>2.2</b>	<b>1.4</b>	<b>2.2</b>	<b>2.5</b>	<b>2.5</b>	<b>2.3</b>
	-	{0.74}	{0.54}	{0.83}	-	-	-
	0.83	0.85	0.82	0.85	0.83	0.85	0.83
	(0.003)	(0.015)	(0.018)	(0.017)	(0.013)	(0.013)	(0.012)
Case 4	[574.7]	[6.2]	[6.1]	[8.6]	[6.2]	[9.1]	[9.0]
	-	<b>3.8</b>	<b>2.6</b>	<b>2.1</b>	<b>5.0</b>	<b>3.7</b>	<b>4.4</b>
	-	{0.87}	{0.81}	{0.82}	-	-	-
	1.69	1.72	1.66	1.70	1.65	1.67	1.70
Case 4	(0.005)	(0.040)	(0.045)	(0.022)	(0.039)	(0.020)	(0.021)
	[587.8]	[6.3]	[6.1]	[8.6]	[6.3]	[9.2]	[9.1]
	-	<b>1.3</b>	<b>1.0</b>	<b>3.2</b>	<b>1.4</b>	<b>3.7</b>	<b>3.2</b>
	-	{0.56}	{0.23}	{0.89}	-	-	-

Table 6.13: Efficiency Gains for Standard Barrier Options (OTM, Low  $p$ )

Digital Barrier Options (ATM, Medium $p$ )					
	Plain	GBM Auxiliary	SV Auxiliary	Combinations	IS
		UOC	SV	UOC+SV	
Case 1	0.485	0.477	0.486	0.484	0.476
	(0.0005)	(0.0046)	(0.0042)	(0.0040)	(0.0048)
	[572.8]	[6.4]	[8.4]	[9.2]	[10.9]
	-	<b>1.0</b>	<b>0.9</b>	<b>0.9</b>	<b>0.6</b>
Case 2	-	{0.36}	{0.52}	-	-
	0.204	0.204	0.205	0.200	0.212
	(0.0004)	(0.0023)	(0.0022)	(0.0019)	(0.0037)
	[570.9]	[6.3]	[8.4]	[9.1]	[11.8]
Case 3	-	<b>2.6</b>	<b>2.2</b>	<b>2.7</b>	<b>0.5</b>
	-	{0.80}	{0.83}	-	-
	0.352	0.352	0.346	0.352	0.358
	(0.0005)	(0.0027)	(0.0021)	(0.0020)	(0.0045)
Case 1	[569.9]	[6.4]	[8.6]	[9.5]	[11.6]
	-	<b>2.7</b>	<b>3.2</b>	<b>3.3</b>	<b>0.5</b>
	-	{0.82}	{0.89}	-	-
	0.118	0.121	0.119	0.118	0.122
Case 1	(0.0003)	(0.0022)	(0.0019)	(0.0018)	(0.0029)
	[569.5]	[6.3]	[8.4]	[9.1]	[11.8]
	-	<b>1.9</b>	<b>1.9</b>	<b>2.0</b>	<b>0.6</b>
	-	{0.73}	{0.80}	-	-

Table 6.14: Efficiency Gains for Digital Barrier Options (ATM, Medium  $p$ )

Digital Barrier Options (ATM, High $p$ )					
	Plain	GBM Auxiliary	SV Auxiliary	Combinations	IS
		UOC	SV	UOC+SV	
Case 1	0.046	0.049	0.049	0.044	0.046
	(0.0002)	(0.0021)	(0.0021)	(0.0020)	(0.0018)
	[565.3]	[6.3]	[8.3]	[8.8]	[11.6]
	-	<b>0.9</b>	<b>0.6</b>	<b>0.7</b>	<b>0.6</b>
Case 2	-	{0.09}	{0.08}	-	-
	0.009	0.009	0.008	0.008	0.009
	(0.0001)	(0.0008)	(0.0008)	(0.0007)	(0.0007)
	[564.5]	[6.3]	[8.3]	[8.9]	[11.7]
Case 3	-	<b>1.2</b>	<b>0.9</b>	<b>1.1</b>	<b>0.8</b>
	-	{0.53}	{0.44}	-	-
	0.022	0.020	0.021	0.022	0.021
	(0.0001)	(0.0012)	(0.0011)	(0.0010)	(0.0012)
Case 4	[570.7]	[6.3]	[8.4]	[8.9]	[11.9]
	-	<b>1.3</b>	<b>1.2</b>	<b>1.2</b>	<b>0.7</b>
	-	{0.52}	{0.67}	-	-
	0.004	0.004	0.004	0.004	0.004
Case 5	(0.0001)	(0.0006)	(0.0005)	(0.0005)	(0.0005)
	[569.9]	[6.3]	[8.4]	[8.9]	[11.9]
	-	<b>1.2</b>	<b>1.0</b>	<b>1.1</b>	<b>0.8</b>
	-	{0.38}	{0.41}	-	-

Table 6.15: Efficiency Gains for Digital Barrier Options (ATM, High  $p$ )

Digital Barrier Options (ATM, Low $p$ )							
	Plain	GBM Auxiliary		SV Auxiliary	Combinations		
		UOC	Call	SV	UOC+Call	UOC+SV	All
Case 1	0.616	0.610	0.612	0.606	0.611	0.615	0.615
	(0.0005)	(0.0036)	(0.0036)	(0.0035)	(0.0036)	(0.0032)	(0.0032)
	[568.4]	[6.4]	[6.1]	[8.5]	[6.6]	[9.3]	[9.4]
	-	<b>1.5</b>	<b>1.6</b>	<b>1.2</b>	<b>1.5</b>	<b>1.3</b>	<b>1.3</b>
Case 2	-	{0.64}	{0.64}	{0.68}	-	-	-
	0.517	0.518	0.522	0.517	0.518	0.516	0.516
	(0.0005)	(0.0022)	(0.0025)	(0.0022)	(0.0021)	(0.0019)	(0.0019)
	[556.9]	[6.4]	[6.1]	[8.6]	[6.5]	[9.4]	[9.6]
Case 3	-	<b>4.0</b>	<b>3.4</b>	<b>2.9</b>	<b>4.5</b>	<b>3.6</b>	<b>3.7</b>
	-	{0.89}	{0.86}	{0.88}	-	-	-
	0.545	0.547	0.543	0.546	0.545	0.546	0.548
	(0.0005)	(0.0021)	(0.0021)	(0.0019)	(0.0020)	(0.0016)	(0.0017)
Case 4	[568.5]	[6.6]	[6.1]	[8.7]	[6.5]	[9.5]	[9.5]
	-	<b>4.8</b>	<b>5.1</b>	<b>4.4</b>	<b>5.1</b>	<b>5.4</b>	<b>4.8</b>
	-	{0.91}	{0.90}	{0.92}	-	-	-
	0.419	0.417	0.420	0.423	0.416	0.415	0.420
Case 4	(0.0005)	(0.0027)	(0.0032)	(0.0022)	(0.0027)	(0.0019)	(0.0019)
	[565.8]	[6.5]	[6.2]	[8.6]	[6.5]	[9.4]	[9.7]
	-	<b>2.7</b>	<b>2.0</b>	<b>2.9</b>	<b>2.7</b>	<b>3.9</b>	<b>3.6</b>
	-	{0.83}	{0.74}	{0.88}	-	-	-

Table 6.16: Efficiency Gains for Digital Barrier Options (ATM, Low  $p$ )

Digital Barrier Options (ITM, Medium $p$ )					
	Plain	GBM Auxiliary	SV Auxiliary	Combinations	IS
		UOC	SV	UOC+SV	
Case 1	0.749	0.753	0.748	0.752	0.745
	(0.0004)	(0.0039)	(0.0035)	(0.0035)	(0.0040)
	[567.6]	[6.6]	[8.6]	[9.6]	[11.0]
	-	<b>0.9</b>	<b>0.9</b>	<b>0.8</b>	<b>0.6</b>
Case 2	-	{0.29}	{0.53}	-	-
	0.475	0.477	0.475	0.475	0.469
	(0.0005)	(0.0024)	(0.0016)	(0.0016)	(0.0045)
	[566.3]	[6.5]	[8.6]	[9.4]	[11.9]
Case 3	-	<b>3.5</b>	<b>6.0</b>	<b>5.5</b>	<b>0.5</b>
	-	{0.87}	{0.94}	-	-
	0.717	0.720	0.714	0.721	0.714
	(0.0004)	(0.0025)	(0.0018)	(0.0017)	(0.0041)
Case 4	[574.7]	[6.7]	[8.8]	[9.8]	[11.7]
	-	<b>2.6</b>	<b>3.8</b>	<b>3.7</b>	<b>0.5</b>
	-	{0.82}	{0.91}	-	-
	0.331	0.331	0.332	0.329	0.334
Case 5	(0.0005)	(0.0030)	(0.0017)	(0.0017)	(0.0042)
	[576.7]	[6.5]	[8.6]	[9.4]	[12.1]
	-	<b>2.0</b>	<b>4.8</b>	<b>4.4</b>	<b>0.5</b>
	-	{0.75}	{0.93}	-	-

Table 6.17: Efficiency Gains for Digital Barrier Options (ITM, Medium  $p$ )



Digital Barrier Options (ITM, High $p$ )					
	Plain	GBM Auxiliary	SV Auxiliary	Combinations	IS
		UOC	SV	UOC+SV	
Case 1	0.190	0.190	0.194	0.191	0.191
	(0.0004)	(0.0034)	(0.0030)	(0.0029)	(0.0035)
	[559.0]	[6.4]	[8.5]	[9.1]	[11.5]
	-	<b>1.1</b> {0.47}	<b>1.1</b> {0.65}	<b>1.1</b> -	<b>0.6</b> -
Case 2	0.108	0.109	0.106	0.108	0.108
	(0.0003)	(0.0017)	(0.0012)	(0.0012)	(0.0026)
	[559.1]	[6.4]	[8.4]	[9.0]	[11.8]
	-	<b>2.9</b> {0.83}	<b>3.9</b> {0.91}	<b>4.0</b> -	<b>0.6</b> -
Case 3	0.207	0.204	0.208	0.207	0.206
	(0.0004)	(0.0021)	(0.0015)	(0.0014)	(0.0036)
	[582.3]	[6.5]	[8.5]	[9.2]	[11.8]
	-	<b>3.1</b> {0.84}	<b>4.7</b> {0.93}	<b>5.2</b> -	<b>0.6</b> -
Case 4	0.067	0.063	0.069	0.069	0.068
	(0.0002)	(0.0016)	(0.0011)	(0.0011)	(0.0021)
	[563.6]	[6.4]	[8.4]	[9.0]	[11.8]
	-	<b>1.9</b> {0.71}	<b>3.2</b> {0.90}	<b>3.2</b> -	<b>0.7</b> -

Table 6.18: Efficiency Gains for Digital Barrier Options (ITM, High  $p$ )

Digital Barrier Options (ITM, Low $p$ )							
	Plain	GBM Auxiliary		SV Auxiliary	Combinations		
		UOC	Call	SV	UOC+Call	UOC+SV	All
Case 1	0.887	0.887	0.885	0.889	0.888	0.884	0.892
	(0.0003)	(0.0025)	(0.0025)	(0.0023)	(0.0025)	(0.0022)	(0.0020)
	[567.6]	[6.8]	[6.2]	[8.8]	[6.8]	[9.9]	[10.0]
	-	<b>1.0</b>	<b>1.2</b>	<b>1.0</b>	<b>1.1</b>	<b>1.0</b>	<b>1.1</b>
Case 2	-	{0.45}	{0.47}	{0.59}	-	-	-
	0.810	0.806	0.811	0.813	0.810	0.807	0.810
	(0.0003)	(0.0021)	(0.0023)	(0.0015)	(0.0019)	(0.0013)	(0.0013)
	[563.1]	[6.8]	[6.2]	[8.9]	[6.8]	[9.9]	[9.9]
Case 3	-	<b>2.2</b>	<b>2.0</b>	<b>3.3</b>	<b>2.7</b>	<b>3.7</b>	<b>3.8</b>
	-	{0.80}	{0.74}	{0.90}	-	-	-
	0.914	0.916	0.913	0.914	0.913	0.915	0.915
	(0.0002)	(0.0016)	(0.0017)	(0.0011)	(0.0016)	(0.0012)	(0.0011)
Case 4	[566.4]	[6.8]	[6.2]	[9.0]	[6.9]	[10.4]	[10.3]
	-	<b>1.9</b>	<b>1.8</b>	<b>2.7</b>	<b>1.7</b>	<b>2.1</b>	<b>2.6</b>
	-	{0.74}	{0.72}	{0.88}	-	-	-
	0.683	0.682	0.681	0.685	0.683	0.686	0.682
Case 4	(0.0004)	(0.0031)	(0.0036)	(0.0017)	(0.0031)	(0.0016)	(0.0015)
	[566.2]	[6.8]	[6.3]	[9.0]	[6.8]	[9.9]	[10.2]
	-	<b>1.6</b>	<b>1.3</b>	<b>4.2</b>	<b>1.6</b>	<b>4.0</b>	<b>4.3</b>
	-	{0.69}	{0.54}	{0.92}	-	-	-

Table 6.19: Efficiency Gains for Digital Barrier Options (ITM, Low  $p$ )

Digital Barrier Options (OTM, Medium $p$ )					
	Plain	GBM Auxiliary	SV Auxiliary	Combinations	IS
		UOC	SV	UOC+SV	
Case 1	0.080	0.072	0.083	0.082	0.077
	(0.0003)	(0.0025)	(0.0024)	(0.0024)	(0.0026)
	[564.1]	[6.2]	[8.2]	[8.7]	[10.1]
	-	<b>1.1</b>	<b>0.9</b>	<b>0.8</b>	<b>0.6</b>
Case 2	-	{0.26}	{0.49}	-	-
	0.137	0.134	0.137	0.137	0.137
	(0.0003)	(0.0022)	(0.0019)	(0.0016)	(0.0032)
	[567.0]	[6.2]	[8.3]	[8.8]	[11.5]
Case 3	-	<b>2.2</b>	<b>2.2</b>	<b>2.6</b>	<b>0.5</b>
	-	{0.75}	{0.83}	-	-
	0.102	0.102	0.104	0.102	0.100
	(0.0003)	(0.0018)	(0.0015)	(0.0013)	(0.0029)
Case 4	[567.0]	[6.2]	[8.3]	[8.9]	[10.3]
	-	<b>2.5</b>	<b>2.7</b>	<b>3.5</b>	<b>0.6</b>
	-	{0.80}	{0.87}	-	-
	0.101	0.107	0.099	0.097	0.103
Case 5	(0.0003)	(0.0026)	(0.0017)	(0.0017)	(0.0028)
	[567.4]	[6.2]	[8.3]	[8.9]	[11.8]
	-	<b>1.2</b>	<b>2.0</b>	<b>2.0</b>	<b>0.5</b>
	-	{0.51}	{0.81}	-	-

Table 6.20: Efficiency Gains for Digital Barrier Options (OTM, Medium  $p$ )

Digital Barrier Options (OTM, High $p$ )					
	Plain	GBM Auxiliary	SV Auxiliary	Combinations	IS
		UOC	SV	UOC+SV	
Case 1	0.057	0.057	0.055	0.058	0.062
	(0.0002)	(0.0022)	(0.0021)	(0.0022)	(0.0023)
	[577.2]	[6.2]	[8.2]	[8.8]	[10.5]
	-	<b>1.0</b>	<b>0.8</b>	<b>0.7</b>	<b>0.5</b>
Case 2	-	{0.19}	{0.33}	-	-
	0.057	0.056	0.057	0.057	0.059
	(0.0002)	(0.0019)	(0.0015)	(0.0014)	(0.0021)
	[579.0]	[6.2]	[8.3]	[8.8]	[11.6]
Case 3	-	<b>1.4</b>	<b>1.5</b>	<b>1.8</b>	<b>0.6</b>
	-	{0.57}	{0.75}	-	-
	0.062	0.061	0.061	0.062	0.064
	(0.0002)	(0.0018)	(0.0015)	(0.0013)	(0.0023)
Case 4	[583.3]	[6.2]	[8.4]	[9.3]	[11.1]
	-	<b>1.6</b>	<b>1.8</b>	<b>2.0</b>	<b>0.6</b>
	-	{0.64}	{0.80}	-	-
	0.035	0.037	0.035	0.035	0.035
Case 5	(0.0002)	(0.0018)	(0.0014)	(0.0014)	(0.0016)
	[583.3]	[6.2]	[8.4]	[8.8]	[11.7]
	-	<b>0.9</b>	<b>1.2</b>	<b>1.1</b>	<b>0.7</b>
	-	{0.26}	{0.64}	-	-

Table 6.21: Efficiency Gains for Digital Barrier Options (OTM, High  $p$ )

Digital Barrier Options (OTM, Low $p$ )							
	Plain	GBM Auxiliary		SV Auxiliary	Combinations		
		UOC	Call	SV	UOC+Call	UOC+SV	All
Case 1	0.088	0.091	0.087	0.085	0.085	0.084	0.088
	(0.0003)	(0.0026)	(0.0026)	(0.0024)	(0.0026)	(0.0023)	(0.0023)
	[580.6]	[6.1]	[6.0]	[8.2]	[6.1]	[8.7]	[8.8]
	-	<b>1.1</b>	<b>1.1</b>	<b>1.0</b>	<b>1.1</b>	<b>1.0</b>	<b>1.0</b>
Case 2	-	{0.35}	{0.34}	{0.51}	-	-	-
	0.185	0.185	0.183	0.185	0.183	0.184	0.186
	(0.0004)	(0.0021)	(0.0022)	(0.0019)	(0.0020)	(0.0016)	(0.0016)
	[581.4]	[6.2]	[6.0]	[8.3]	[6.4]	[8.9]	[8.9]
Case 3	-	<b>3.1</b>	<b>2.8</b>	<b>2.7</b>	<b>3.4</b>	<b>3.4</b>	<b>3.7</b>
	-	{0.83}	{0.80}	{0.85}	-	-	-
	0.113	0.113	0.116	0.115	0.112	0.116	0.113
	(0.0003)	(0.0017)	(0.0018)	(0.0012)	(0.0017)	(0.0012)	(0.0011)
Case 4	[585.7]	[6.2]	[6.1]	[8.3]	[6.2]	[8.9]	[8.8]
	-	<b>3.1</b>	<b>3.1</b>	<b>4.7</b>	<b>3.3</b>	<b>4.8</b>	<b>5.6</b>
	-	{0.84}	{0.83}	{0.92}	-	-	-
	0.162	0.161	0.158	0.162	0.162	0.159	0.162
Case 4	(0.0004)	(0.0027)	(0.0030)	(0.0019)	(0.0027)	(0.0018)	(0.0018)
	[583.9]	[6.3]	[6.1]	[8.3]	[6.3]	[9.0]	[9.1]
	-	<b>1.6</b>	<b>1.4</b>	<b>2.6</b>	<b>1.6</b>	<b>2.6</b>	<b>2.4</b>
	-	{0.64}	{0.52}	{0.85}	-	-	-

Table 6.22: Efficiency Gains for Digital Barrier Options (OTM, Low  $p$ )

Standard Barrier Options (4 Resets)				
	Plain	GBM Auxiliary	SV Auxiliary	Combinations
		UOC	SV	UOC+SV
Case 1	4.60	4.62	4.53	4.59
	(0.006)	(0.054)	(0.046)	(0.046)
	[566.6]	[6.3]	[8.3]	[9.4]
	-	<b>1.0</b> {0.33}	<b>1.0</b> {0.58}	<b>0.9</b> -
Case 2	2.53	2.48	2.55	2.54
	(0.005)	(0.039)	(0.036)	(0.035)
	[575.6]	[6.2]	[8.3]	[9.0]
	-	<b>1.5</b> {0.60}	<b>1.3</b> {0.68}	<b>1.2</b> -
Case 3	3.48	3.47	3.45	3.46
	(0.005)	(0.037)	(0.037)	(0.035)
	[564.5]	[6.4]	[8.5]	[9.4]
	-	<b>1.8</b> {0.71}	<b>1.4</b> {0.71}	<b>1.4</b> -
Case 4	1.75	1.77	1.80	1.74
	(0.004)	(0.032)	(0.034)	(0.031)
	[565.5]	[6.2]	[8.5]	[9.3]
	-	<b>1.5</b> {0.64}	<b>1.0</b> {0.61}	<b>1.2</b> -

Table 6.23: Efficiency Gains for Standard Barrier Options (4 Resets)

Standard Barrier Options (16 Resets)				
	Plain	GBM Auxiliary	SV Auxiliary	Combinations
		UOC	SV	UOC+SV
Case 1	4.38	4.37	4.28	4.30
	(0.006)	(0.052)	(0.044)	(0.043)
	[566.1]	[6.3]	[8.4]	[9.1]
	-	<b>1.0</b> {0.33}	<b>1.1</b> {0.62}	<b>1.0</b> -
Case 2	2.00	2.05	1.99	1.98
	(0.004)	(0.032)	(0.026)	(0.024)
	[575.2]	[6.3]	[8.3]	[9.0]
	-	<b>1.7</b> {0.68}	<b>1.9</b> {0.80}	<b>2.0</b> -
Case 3	3.08	3.06	3.09	3.10
	(0.005)	(0.031)	(0.029)	(0.026)
	[569.6]	[6.4]	[8.5]	[9.5]
	-	<b>2.4</b> {0.79}	<b>2.0</b> {0.81}	<b>2.2</b> -
Case 4	1.25	1.26	1.23	1.26
	(0.004)	(0.025)	(0.023)	(0.021)
	[566.0]	[6.4]	[8.5]	[9.2]
	-	<b>1.7</b> {0.69}	<b>1.6</b> {0.76}	<b>1.7</b> -

Table 6.24: Efficiency Gains for Standard Barrier Options (16 Resets)

Standard Barrier Options (64 Resets)				
	Plain	GBM Auxiliary	SV Auxiliary	Combinations
		UOC	SV	UOC+SV
Case 1	4.22	4.22	4.30	4.22
	(0.005)	(0.051)	(0.044)	(0.043)
	[560.6]	[6.3]	[8.6]	[9.1]
	-	<b>1.0</b> {0.33}	<b>1.0</b> {0.60}	<b>1.0</b> -
Case 2	1.70	1.70	1.72	1.66
	(0.004)	(0.026)	(0.023)	(0.018)
	[571.9]	[6.2]	[8.3]	[9.0]
	-	<b>2.0</b> {0.74}	<b>2.0</b> {0.82}	<b>3.1</b> -
Case 3	2.83	2.77	2.82	2.84
	(0.005)	(0.029)	(0.024)	(0.021)
	[568.5]	[6.4]	[8.5]	[9.3]
	-	<b>2.5</b> {0.80}	<b>2.7</b> {0.87}	<b>3.0</b> -
Case 4	0.98	1.02	1.02	1.00
	(0.003)	(0.023)	(0.018)	(0.017)
	[569.4]	[6.3]	[8.3]	[9.1]
	-	<b>1.6</b> {0.67}	<b>2.1</b> {0.83}	<b>2.0</b> -

Table 6.25: Efficiency Gains for Standard Barrier Options (64 Resets)



## Chapter 7

# Exploring Variations in Barrier Option Pricing

### 7.1 Introduction

In chapter 4 we discussed efficient Monte Carlo method for valuing certain instruments in a general class of stochastic volatility models. In addition to pricing, another primary concern of a financial institution is how to hedge their option book effectively. For effective hedging, there are two issues one needs to address:

1. How to recover prices of hedging instruments?
2. How to model evolution of prices of hedging instruments closely, that is, to match deltas and gammas as closely as possible?

In order to address the first issue, we need to fit our stochastic volatility models to the market implied volatility surface. To do the second, we fit our models to instruments whose values are path-dependent. Calibration to market prices of vanilla options is necessary since any candidate model should price the hedging instruments correctly. Doing only this, however, is insufficient. Vanilla options are priced off the asset density at their maturity time. Their payoffs do not depend on other points along the asset sample path. Even if two stochastic volatility models calibrate to the same market vanilla option prices, they can still misprice other exotic instruments whose payoffs depend on the full sample path. Examples of these instruments are barrier options, average rate options, American or Bermudan options, *et cetera*.

In this chapter, what we are interested in is the discrepancy in the barrier option prices between the Heston model and other models if they all calibrate to the

same vanilla option implied volatility surface. This shows the model has sufficient flexibility to match a variety of barrier option prices, and may give better pricing under issue 2 even if it cannot match delta and gamma exactly.

There are several books and papers discussing the valuation of barrier option in the presence of volatility smiles. Taleb [81] finds the volatility smile has a significant effect on barrier option price. Wilmott [85] observes that barrier option prices are very sensitive to the volatility level.

In order to incorporate the volatility smile into pricing, people propose fast valuation or approximation methods in different modelling frameworks. In local volatility models, Sepp [74] and Zhu [86] compare prices of barrier options in the local volatility model with those in the Black-Scholes model. In these models, Decamps *et al.* [25] provide accurate approximation to the barrier option value. In the stochastic volatility world, Heath and Platen [43] compare the differences of barrier option prices in the Heston and Black-Scholes models. Chiarella *et al.* [23] develop a method of lines approach to evaluate the price as well as delta and gamma of barrier options in the Heston model. Tahani [79] and Shiraya *et al.* [75] propose approximation methods for barrier option prices in a general stochastic volatility model.

Another area in the literature focuses on the static hedging of barrier options. This strand of research is pioneered by Bowie and Carr [18] and Derman *et al.* [26]. Carr and Chou [19] and Carr *et al.* [20] use a simple claim whose payoff, as well as continuous values before maturity, are the same as those of the barrier option. But their result holds only in the Black-Scholes model and it is not clear how to generalize it. Andersen *et al.* [4] and Poulsen [67] present theories of static hedging allowing for the volatility to be a function of time and current asset value (local volatility). Nalholm and Poulsen [62] investigate how sensitive static hedging portfolios are to different pricing models.

On the empirical side, Jessen and Poulsen [49] investigate prediction errors for five models in pricing barrier option while they are calibrated to the vanilla option prices. Gatheral [35] elaborates on how values of certain type of barrier options are highly dependent on pricing models.

Our motivation is to show that even though different combinations of parameter values in our class of models yield roughly the same vanilla option prices, they can

still produce different barrier option prices. This provides us with additional flexibility of fitting our models to barrier as well as vanilla option prices. If we assume the market implied volatility surface is generated by the Heston model, we investigate answers to two questions in the following sections:

1. How to calibrate our class of models to the market implied volatility surface?
2. How the calibrated models price barrier options differently?

With the help of the correlation control variates we proposed in chapter 4, the model calibration is faster and the differences in barrier option prices are discernible in less computation time.

The rest of this chapter is organised as following. In section 7.2, we describe a method of calibrating our models to vanilla option prices. In section 7.3, we examine calibration errors under a variety of models and investigate the variations in barrier option pricing in the models with low calibration errors. Section 7.4 concludes.

## 7.2 Model Calibration

The standard calibration method makes use of a model's analytical tractability. This means the model needs to have analytical solutions for vanilla option prices or at least prices that can be obtained by a quick numerical procedure. By virtue of Carr and Madan [21], we can value a book of options with different strike prices using fast Fourier transform (FFT). However, the pre-requisite is having an explicit form for the characteristic function of  $x_t = \ln S_t$ . We explore the tractability of  $\mathbb{E}(e^{i\phi x_t})$ .

### 7.2.1 The Characteristic Function

We first transform equation (4.1) into  $x_t = \ln S_t$ .

$$dx_t = \left[ r - \frac{1}{2} V_t^{2\xi} e^{2(\beta-1)x_t} \right] dt + V_t^\xi e^{(\beta-1)x_t} dW_t^X. \quad (7.1)$$

We apply the solution technique in the original Heston paper [44] to derive the characteristic function. Define the twice-differentiable function  $f(x, v, t)$  such that

$$f(x, v, t) = \mathbb{E}[g(x_T, V_T) | x_t = x, V_t = v], \quad (7.2)$$

where  $g$  is some other function of  $x_T$  and  $V_T$ . Applying the Itô-formula to  $f$ ,

$$df = \left[ \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x} \left( r - \frac{1}{2} v^{2\xi} e^{2(\beta-1)x} \right) + \frac{\partial f}{\partial v} \alpha(\mu - v) + \frac{1}{2} \frac{\partial^2 f}{\partial x^2} v^{2\xi} e^{2(\beta-1)x} \right. \\ \left. + \frac{1}{2} \frac{\partial^2 f}{\partial v^2} \eta^2 v^{2\gamma} + \frac{\partial^2 f}{\partial x \partial v} \eta \rho v^{\xi+\gamma} e^{(\beta-1)x} \right] dt + \frac{\partial f}{\partial x} v^\xi e^{(\beta-1)x} dW_t^X + \frac{\partial f}{\partial v} \eta v^\gamma dW_t^V. \quad (7.3)$$

From equation (7.1),  $f(x, v, t)$  is a martingale and therefore must have zero drift. From equation (7.3), it yields the following PDE.

$$\frac{\partial f}{\partial t} + \frac{\partial f}{\partial x} \left( r - \frac{1}{2} v^{2\xi} e^{2(\beta-1)x} \right) + \frac{\partial f}{\partial v} \alpha(\mu - v) + \frac{1}{2} \frac{\partial^2 f}{\partial x^2} v^{2\xi} e^{2(\beta-1)x} \\ + \frac{1}{2} \frac{\partial^2 f}{\partial v^2} \eta^2 v^{2\gamma} + \frac{\partial^2 f}{\partial x \partial v} \eta \rho v^{\xi+\gamma} e^{(\beta-1)x} = 0. \quad (7.4)$$

If the terminal condition is  $g(x_T, V_T) = e^{iux_T}$ , the solution to the PDE (7.4) is the characteristic function of  $x_T$ .

Equation (7.4) is highly untractable, due to its non-linear coefficients, unless  $\beta = 1$  and  $\xi = \gamma = \frac{1}{2}$  in which case equation (4.1) to (4.3) reduce to the Heston model. In other cases, the characteristic function is not known analytically and therefore the FFT technique fails to apply to our class of stochastic volatility models. However, equation (7.4) can give us some intuition on what combinations of parameter values give roughly the same solutions and also the option prices and implied volatility surface.

We describe a calibration method based on the coefficients of equation (7.4).

### 7.2.2 Calibration Procedure

We can make the following observation from equation (7.4). The coefficients are made up of the following individual terms:  $v^\xi e^{(\beta-1)x}$ ,  $\alpha\mu$ ,  $\alpha v$ ,  $\eta v^\gamma$ ,  $r$  and  $\rho$ . Note that  $r$  and  $\rho$  are stand-alone and constant. If  $\alpha\mu$ ,  $\alpha v$ ,  $\eta v^\gamma$  and  $v^\xi e^{(\beta-1)x}$  satisfy the following rules:

1.  $\alpha v \approx \alpha\mu = \alpha V_0 \equiv C_1$  for some constant  $C_1$ ,
2.  $\eta v^\gamma \approx \eta\mu^\gamma = \eta V_0^\gamma \equiv C_2$  for some constant  $C_2$ ,
3.  $v^\xi e^{(\beta-1)x} = v^\xi X_t^{\beta-1} \approx \mu^\xi X_0^{\beta-1} = V_0^\xi X_0^{\beta-1} \equiv C_3$  for some constant  $C_3$ ,

we argue that the solutions to PDE (7.4) are close to each other. The first two rules involve only one approximation  $v \approx \mu = V_0$ . Since  $v$  is reverting to the long term mean  $\mu$ , if we set  $V_0 = \mu$ ,  $v$  is around the level  $\mu$ . The third rule involves an additional approximation,  $X_t \approx X_0$  (unless  $\beta = 1$ ). This is less accurate because  $X_t$  has a drift term  $r$  and therefore its mean level at time  $t$  is  $X_0 e^{rt}$ . But if we price options with short maturity then  $e^{rt} \approx 1$ .

We can calibrate our class of stochastic volatility models to the market implied volatility surface by changing parameters  $\gamma$ ,  $\xi$  and  $\beta$  in such a way that  $C_1$ ,  $C_2$  and  $C_3$  have the same values as in the Heston model. We assume that if we vary the parameters according to the above rules, the models will produce roughly the same implied volatility surfaces. We verify our assumption and then price barrier options with these different parameter sets in section 7.3.

### 7.3 Numerical Results

We use the Heston model as the base model (Model 0). We choose  $V_0 = \mu = 0.0225$  so that the volatility is around 15%, a relatively high rate of mean reversion  $\alpha = 4$  so that  $V_t$  stays roughly at the same level (which makes the approximation  $V_t \approx \mu$  more accurate),  $\eta = 0.1333$  to make the diffusion term  $\eta V_t^\gamma \approx C_2 = 0.02$ , and  $\rho = -0.5$  which is the choice in most of our numerical examples in chapter 5 and 6. These setting correspond to  $C_1 = 0.09$ ,  $C_2 = 0.02$  and  $C_3 = 0.15$ . In all models, the initial stock price  $S_0 = 100$  and the interest rate  $r = 0.05$ . The price of the vanilla option is explicit in the Heston model. We value vanilla call options with  $K = 80, 85, \dots, 120$  and  $T = 0.25, 0.5, 0.75, 1$ . The option prices and implied volatilities are displayed in Figure 7.1 with values in the plot given in Table 7.19 (page 204) and Table 7.20 (page 204). These prices and implied volatilities are computed explicitly. We see that the implied volatility decreases as the strike increases. As the time to maturity increases, the skew becomes flatter. This is observed in the real market implied volatility surface.

We examine calibration errors in section 7.3.1. Call option prices are obtained by Monte Carlo simulation with a combination of three control variates described in chapter 4, namely, SV ( $\rho = 0$ ), SV ( $\rho \neq 0$ ), and the GBM call CV. We use the Milstein method on  $V_t$  and Euler method on  $\ln X_t$ . The number of time steps is  $N = 256$  up to one year. The number of sample paths is  $M = 50000$ . If we denote the option price estimate by  $\hat{c}$  and standard error by  $se$ , we invert  $\hat{c}$ ,  $\hat{c}_l = \hat{c} - se$

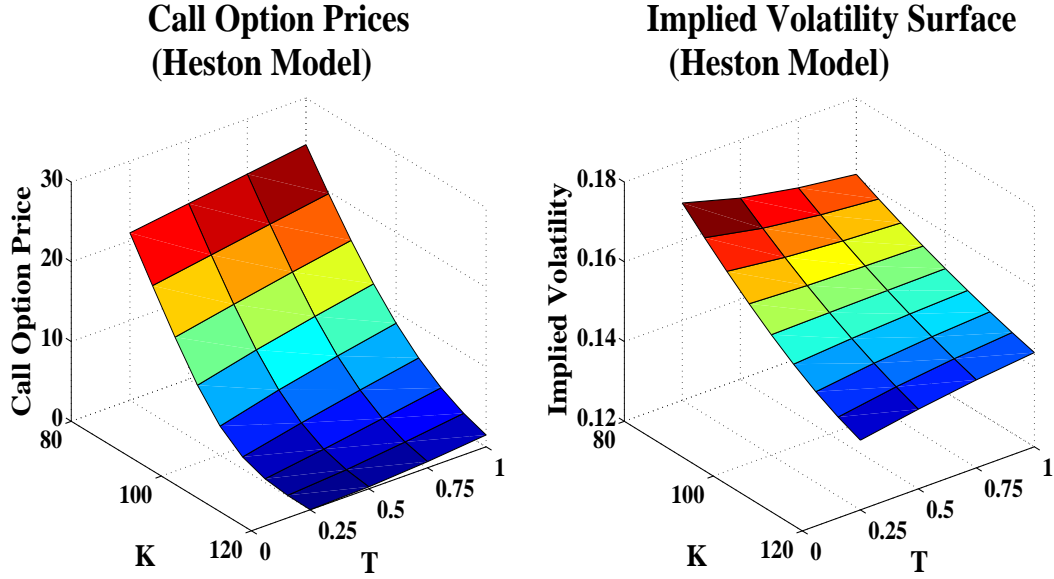


Figure 7.1: Call Option Prices and Implied Volatilities in the Heston Model

and  $\hat{c}_h = \hat{c} + se$  to get their implied volatilities in the Black-Scholes model as  $\hat{\sigma}$ ,  $\hat{\sigma}_l$  and  $\hat{\sigma}_h$ . We use  $\hat{\sigma}_l$  and  $\hat{\sigma}_h$  as error bounds for  $\hat{\sigma}$ . The calibration error is defined as  $\epsilon = \hat{\sigma} - \sigma_0$  where  $\sigma_0$  is the Black-Scholes volatility implied by the Heston model. The lower and upper bound for the calibration error are  $\epsilon_l = \hat{\sigma}_l - \sigma_0$  and  $\epsilon_h = \hat{\sigma}_h - \sigma_0$ . The bound width is  $\epsilon_b = \epsilon_h - \epsilon_l$ . We report the root mean square of  $\epsilon$  and  $\epsilon_b$ ,  $RMS(\epsilon)$  and  $RMS(\epsilon_b)$ , for all options.

### 7.3.1 Calibration Errors

There are a number of ways we can calibrate to the implied volatility surface produced by the Heston model according to the rules in section 7.2.2. We vary the parameters in three different ways: 1) fix  $\xi$  and  $\beta$ , change  $\gamma$  and  $\eta$ ; 2) fix  $\gamma$  and  $\beta$ , change  $\xi$  and so  $\mu$ ,  $\alpha$ ,  $V_0$  and  $\eta$ ; and 3) fix  $\xi$  and  $\gamma$ , change  $\beta$  and so  $\mu$ ,  $\alpha$ ,  $V_0$  and  $\eta$ . We expect calibration in the third way to be less accurate than those in the first and the second for the reason we explained in section 7.2.2.

#### 1. Fix $\xi$ and $\beta$ , Change $\gamma$ and $\eta$

We fix  $\xi = \frac{1}{2}$  and  $\beta = 1$  as in the Heston model.  $\gamma$  takes the values 0.4, 0.6, 0.8 and 1 whereas  $\gamma = \frac{1}{2}$  in the Heston model.  $\eta$  is adapted so that  $C_2 = \eta V^\gamma = 0.02$ .  $C_1 = \alpha\mu = 0.09$  and  $C_3 = V_0^\xi X_0^{\beta-1} = 0.15$  remain the same since  $\alpha$ ,  $\mu$ ,  $V_0$ ,  $X_0$ ,  $\xi$  and  $\beta$  do not change. Table 7.1 lists the parameters of five different models. We

	Model 0	Model 1	Model 2	Model 3	Model 4
$S_0 =$	100	100	100	100	100
$r =$	0.05	0.05	0.05	0.05	0.05
$V_0 =$	0.0225	0.0225	0.0225	0.0225	0.0225
$\alpha =$	4	4	4	4	4
$\mu =$	0.0225	0.0225	0.0225	0.0225	0.0225
$\eta =$	0.1333	0.0912	0.1949	0.4162	0.8889
$\beta =$	1	1	1	1	1
$\xi =$	0.5	0.5	0.5	0.5	0.5
$\gamma =$	0.5	0.4	0.6	0.8	1
$\rho =$	-0.5	-0.5	-0.5	-0.5	-0.5

Table 7.1: Model Parameters, Fix  $\xi$  and  $\beta$ , Change  $\gamma$  and  $\eta$

plot the lower and upper bounds for the calibration errors in Figure 7.2. We see

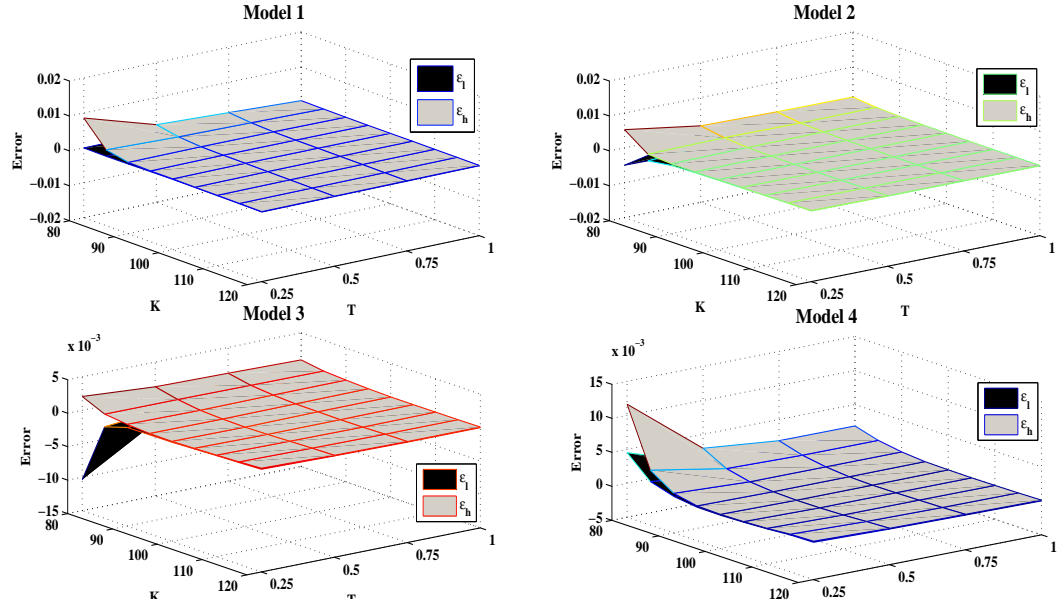


Figure 7.2: Calibration Errors in Model 1-4

that for options deep in the money and close to maturity the calibration errors are highest and the error bounds are widest. As options move towards OTM and time to maturity increases,  $\epsilon_l$  and  $\epsilon_h$  are very close to each other. Table 7.2 presents  $RMS(\epsilon)$  and  $RMS(\epsilon_b)$  for Models 1-4. In Models 1-3,  $RMS(\epsilon) \ll RMS(\epsilon_b)$ . In Model 4, both are roughly at the same level. It means that their implied volatility surfaces are not significantly different from that of the Heston model. In general,  $RMS(\epsilon)$  in all models are lower than 1% of the implied volatilities in the Heston model.

	Model 1	Model 2	Model 3	Model 4
$RMS(\epsilon)$	8.33E-04	3.21E-04	5.67E-04	1.53E-03
$RMS(\epsilon_b)$	1.56E-03	1.80E-03	2.16E-03	1.32E-03

Table 7.2: The Root Mean Square Errors for Models 1-4

## 2. Fix $\gamma$ and $\beta$ , Change $\xi$ , $\mu$ , $\alpha$ , $V_0$ and $\eta$

We set  $\gamma = \frac{1}{2}$  and  $\beta = 1$  to be the same as in the Heston model and let  $\xi$  take the values of 0.4, 0.6, 0.8 and 1.0, compared to  $\xi = \frac{1}{2}$  in the Heston model. In order to keep  $C_3 = 0.02$  we need adapt  $V_0$  and  $\mu$  accordingly. In order to keep  $C_1 = 0.09$  and  $C_2 = 0.02$ , we also need to adjust  $\alpha$  and  $\eta$  since both  $C_1$  and  $C_2$  depend on  $V_0$  and  $\mu$ . Table 7.3 lists the parameters in the models. Figure 7.3 shows the calibration errors.

	Model 0	Model 5	Model 6	Model 7	Model 8
$S_0 =$	100	100	100	100	100
$r =$	0.05	0.05	0.05	0.05	0.05
$V_0 =$	0.0225	0.0087	0.0423	0.0933	0.15
$\alpha =$	4	10.3280	2.1253	0.9641	0.6
$\mu =$	0.0225	0.0087	0.0423	0.0933	0.15
$\eta =$	0.1333	0.2142	0.0972	0.0655	0.0516
$\beta =$	1	1	1	1	1
$\xi =$	0.5	0.4	0.6	0.8	1
$\gamma =$	0.5	0.5	0.5	0.5	0.5
$\rho =$	-0.5	-0.5	-0.5	-0.5	-0.5

Table 7.3: Model Parameters, Fix  $\gamma$  and  $\beta$ , Change  $\xi$ ,  $\mu$ ,  $\alpha$ ,  $V_0$  and  $\eta$

Table 7.4 displays  $RMS(\epsilon)$  and  $RMS(\epsilon_b)$  for the resulting Models 5-8.  $RMS(\epsilon)$  in

	Model 5	Model 6	Model 7	Model 8
$RMS(\epsilon)$	2.33E-03	2.95E-03	3.19E-03	3.90E-03
$RMS(\epsilon_b)$	2.22E-03	2.97E-03	1.20E-03	1.01E-03

Table 7.4: The Root Mean Square Errors for Models 5-8

Models 5-8 are all higher than those in Models 1-4. In Models 5 and 6,  $RMS(\epsilon)$  are at the same level with  $RMS(\epsilon_b)$  but in Models 7 and 8  $RMS(\epsilon)$  are significantly higher  $RMS(\epsilon_b)$ . It means that the implied volatility surfaces produced by Models 7 and 8 differ significantly from that in the Heston model. The  $RMS(\epsilon)$  are all below 3% of the implied volatilities in the Heston model so the differences are not too great.



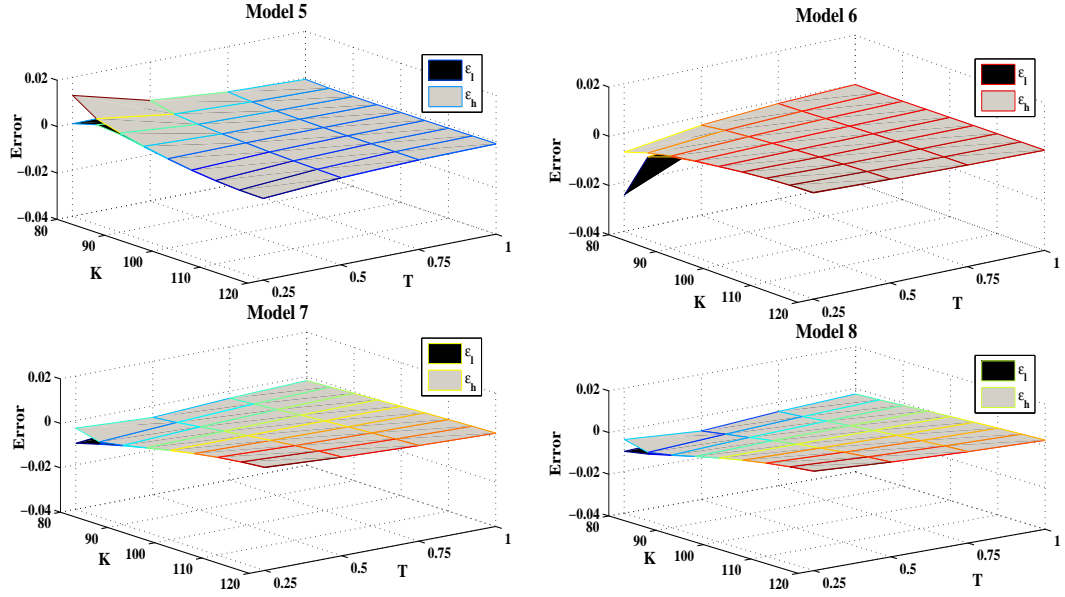


Figure 7.3: Calibration Errors in Model 5-8

### 3. Fix $\xi$ and $\gamma$ , Change $\beta$ , $\mu$ , $\alpha$ , $V_0$ and $\eta$

We fix  $\xi = \gamma = \frac{1}{2}$  as in the Heston model and set  $\beta = 0.2, 0.4, 0.6$  and  $0.8$ , compared with  $\beta = 1$  in the Heston model. Since  $C_3$  depends on  $\beta$ , we need to adapt  $\mu$  and  $V_0$  to fix  $C_3 = 0.15$ . This will cause changes in  $\alpha$  and  $\eta$  in order to keep  $C_1 = 0.09$  and  $C_2 = 0.02$ .

Table 7.5 lists the parameters of the models. Figure 7.4 shows the calibration errors

	Model 0	Model 9	Model 10	Model 11	Model 12
$S_0 =$	100	100	100	100	100
$r =$	0.05	0.05	0.05	0.05	0.05
$V_0 =$	0.0225	35.6601	5.6517	0.8957	0.1420
$\alpha =$	4	0.0025	0.0159	0.1005	0.6340
$\mu =$	0.0225	35.6601	5.6517	0.8957	0.1420
$\eta =$	0.1333	0.0033	0.0084	0.0211	0.0531
$\beta =$	1	0.2	0.4	0.6	0.8
$\xi =$	0.5	0.5	0.5	0.5	0.5
$\gamma =$	0.5	0.5	0.5	0.5	0.5
$\rho =$	-0.5	-0.5	-0.5	-0.5	-0.5

Table 7.5: Model Parameters, Fix  $\xi$  and  $\gamma$ , Change  $\beta$ ,  $\mu$ ,  $\alpha$ ,  $V_0$  and  $\eta$

for the resulting Models 9-12. We see that the surfaces are not as flat at zero level as

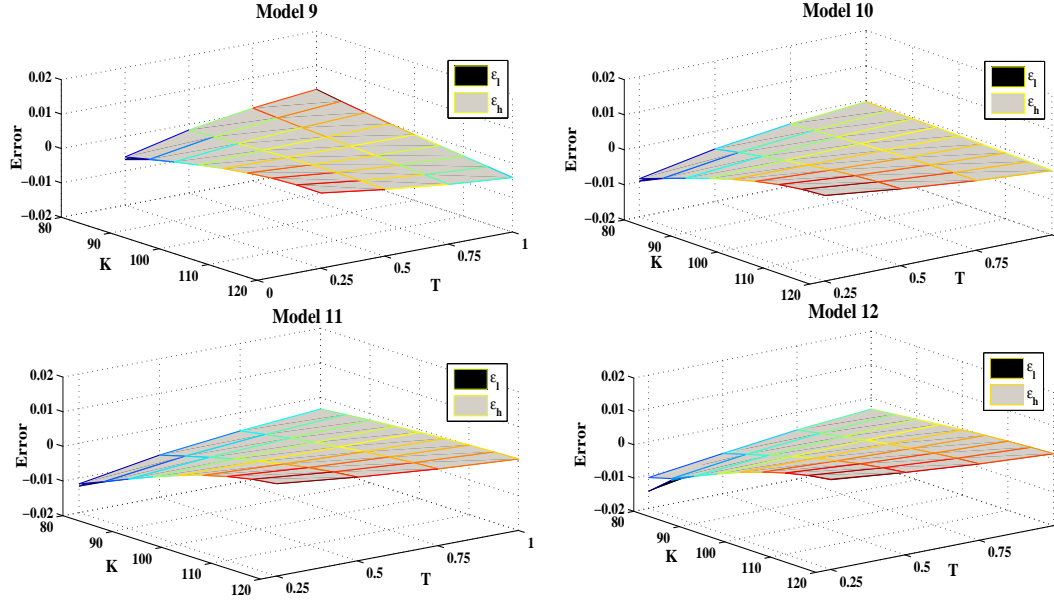


Figure 7.4: Calibration Errors in Models 9-12

those in Figures 7.2 and 7.3 but the error bounds are much narrower. This is due to the high efficiency gains of our proposed control variates when  $0 < \beta < 1$  as we find in chapter 4. Table 7.6 displays  $RMS(\epsilon)$  and  $RMS(\epsilon_b)$  for Models 9-12.  $RMS(\epsilon)$

	Model 9	Model 10	Model 11	Model 12
$RMS(\epsilon)$	2.10E-03	2.82E-03	4.09E-03	4.29E-03
$RMS(\epsilon_b)$	1.47E-04	1.32E-04	1.18E-04	7.10E-04

Table 7.6: The Root Mean Square Errors for Models 9-12

are comparable to those in Table 7.4 but significantly greater than  $RMS(\epsilon_b)$ . However, they are all below 3% of the implied volatilities in the Heston model.

In summary, we find that in Models 1-6  $RMS(\epsilon)$  are less than or around the same level as  $RMS(\epsilon_b)$ . In Models 6-12,  $RMS(\epsilon)$  are much higher than  $RMS(\epsilon_b)$ . Although  $RMS(\epsilon)$  for all 12 models are less than 3% of the market implied volatilities (produced by the Heston model), Models 6-12 produce significantly different implied volatilities with those in the Heston model. In the next section, we explore the difference with the Heston model with respect to barrier option prices in Models 1-6.

### 7.3.2 Variations of Barrier Option Prices

Models 1-6 are calibrated to the market implied volatilities produced by the Heston model. We are interested in how different barrier option prices are in Models 1-6 to those in the Heston model. We consider up and out call options (UOC) with strikes  $K = 80, 85, \dots, 120$  and maturities  $T = 0.25, 0.5, 0.75, 1$  as for the vanilla call options. We use two barrier levels,  $U = 110$  and  $U = 130$ , corresponding to high and low hitting probabilities. Note that when  $U = 110$ , options with  $K \geq 110$  have zero payoffs because they have to knock out to be in the money.

The number of time steps is  $N = 256$  up to one year and the barrier condition is only checked at the discrete points on the sample path. Given the large number of time steps, the discrete barrier option price should be very close to the continuously monitored case.

#### Non-existence of Black-Scholes Implied Volatility

Unlike for vanilla option where there is always a Black-Scholes volatility implied from each of our models, an implied volatility from barrier option prices never exists. We illustrate this problem in Figure 7.5. The solid curve represents the prices

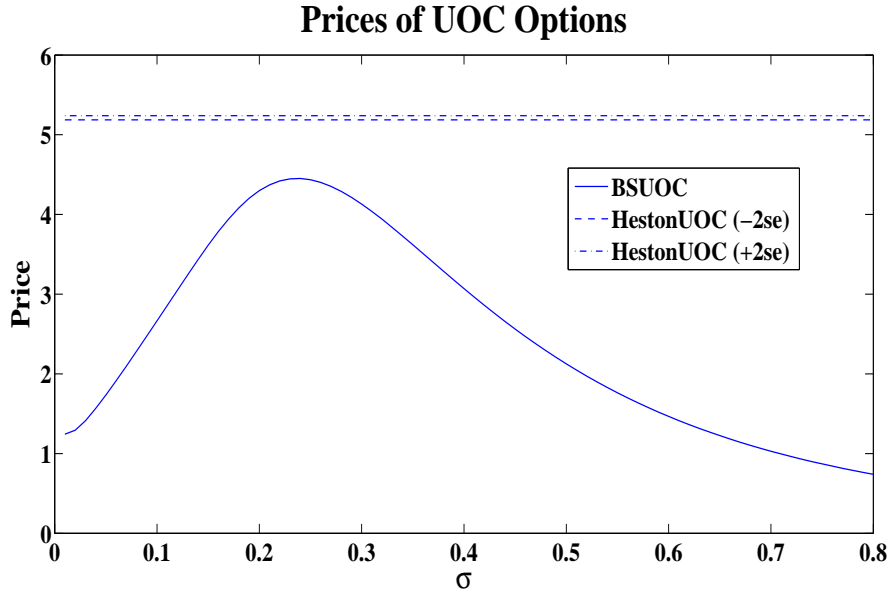


Figure 7.5: UOC Prices with Different Black-Scholes Volatilities

of UOC in the Black-Scholes model with different volatility levels on the horizontal axis. These prices are computed explicitly. The dashed and dash-dotted lines are

lower and upper bounds ( $\pm 2se$ ) of the UOC price in the Heston model with parameters in Table 7.1 or 7.3. We see that the UOC price in the Black-Scholes model first increases and then decreases with the volatility. It reaches its maximum at some  $\sigma^* \in (0.2, 0.3)$ . However, its value is far below the lower bound of the UOC price in the Heston model which means that no  $\sigma$  in the Black-Scholes model can produce the same price of UOC in the Heston model. We have the same findings if we repeat the exercise for options with different strikes or maturities. In fact, Derman *et al.* [27] also report that no Black-Scholes volatility can reproduce the price of UOC in their local volatility model.

Since barrier option prices in the stochastic volatility model are out of the scope of the Black-Scholes model, we cannot use Black-Scholes implied volatilities to differentiate the barrier option prices in various stochastic volatility models as in the case of vanilla options. We instead report the difference between the barrier option prices in our calibrated models and those in the Heston model.

### Differences in Barrier Option Prices

We do not have explicit values for barrier options in the Heston model or any model in our class. In order to investigate the differences of barrier option prices between models, we compute the difference path-wise. Suppose  $c_i$  is the value of the barrier option in Model  $i$ , for  $i = 0, 1, \dots, 6$ . We estimate the difference with the Heston model (Model 0) as

$$d_i^b = c_i^b - c_0^b \approx \frac{1}{M} \sum_{j=1}^M c_i^b(\omega^{(j)}) - c_0^b(\omega^{(j)}), \quad (7.5)$$

where  $c_i(\omega^{(j)})$  denotes the barrier option price in Model  $i$  evaluated on the  $j^{th}$  Wiener sample path  $\omega^{(j)}$ . We report the bias  $T_i^b = \frac{\hat{d}_i^b}{se_i^b}$  where  $\hat{d}_i^b$  is the Monte Carlo estimation of  $d_i^b$  and  $se_i^b$  is the standard error of  $\hat{d}_i^b$ . This measures how far away the difference is from zero. Since the calibration to the vanilla option values is not a perfect match, we also report the bias  $T_i^v$  for vanilla option prices in all models.

$$d_i^v = c_i^v - c_0^v \approx \frac{1}{M} \sum_{j=1}^M c_i^v(\omega^{(j)}) - c_0^v, \quad (7.6)$$

where  $c_0^v$  is the explicit Heston call price. A positive (negative)  $T_i^b$  or  $T_i^v$  indicates that Model  $i$  gives higher (lower) barrier or vanilla option price than the Heston

model. If we find that  $|T_i^b|$  is much higher than  $|T_i^v|$ , we can conclude that Model  $i$  and the Heston model give very different barrier option prices even though they produce similar vanilla option prices.

We use two types of barrier options in our analysis, low *versus* high hitting probabilities ( $U = 130$  and  $U = 110$ ). We look at options with  $T = 0.25, 0.5, 0.75, 1$ . We use options with  $K = 80, 85, \dots, 120$  when  $U = 130$  while when  $U = 110$  we only use options with  $K = 80, 85, \dots, 105$  because those with  $K \geq 110$  have zero payoffs.

We present results in the table form. Each entry in the table has two numbers. The top one is  $T_i^b$  and the bottom one is  $T_i^v$ . That entry represents both the barrier and vanilla option with the same strike and maturity. It shows the difference in the barrier option compared with that in the vanilla option.

**Barrier options with low hitting probabilities.** Table 7.7 displays  $T_1^b$  and  $T_1^v$  for options with  $U = 130$ . For options with  $T = 0.25$ ,  $|T_1^b|$  are all higher than

$K =$	80	85	90	95	100	105	110	115	120
$T = 0.25$	-2.6	-1.7	-0.2	1.7	3.5	4.3	3.9	2.5	0.8
	1.0	0.9	0.7	1.6	2.8	2.8	2.2	1.0	0.2
$T = 0.5$	-2.7	-2.1	-1.3	-0.3	0.7	1.4	1.4	1.0	0.1
	1.0	1.6	1.7	2.3	3.1	2.7	2.3	1.7	0.1
$T = 0.75$	-1.6	-1.3	-0.9	-0.3	0.2	0.6	0.7	0.5	-0.04
	0.9	1.1	1.1	1.4	1.8	2.1	2.0	1.5	0.5
$T = 1$	-0.3	-0.2	-0.04	0.2	0.5	0.7	0.6	0.4	-0.1
	0.4	0.2	0.04	0.2	0.4	1.0	0.7	1.2	0.9

Table 7.7:  $T_1^b$  and  $T_1^v$ ,  $U = 130$

$|T_1^v|$  except for the option with  $K = 90$ . For options with  $T = 0.5$  and  $0.75$ ,  $|T_1^b|$  are larger than  $|T_1^v|$  only when the options are deep in the money ( $K = 80$  and  $85$ ). When  $T = 1$ ,  $|T_1^b|$  are generally lower than  $|T_1^v|$  but they are all lower than 1. It means in that case the differences between Model 1 and the Heston model are not significant. We can also see that  $T_1^b$  becomes more negative when the option moves deeper in the money. That means Model 1 under prices the deep ITM options.

Table 7.8 presents  $T_2^b$  and  $T_2^v$ . For options with  $K = 100, 105$  and  $110$  for all maturities,  $|T_2^b|$  are significantly higher than  $|T_2^v|$ . For options with longer maturities ( $T = 0.75$  and  $1$ ), neither  $|T_2^b|$  nor  $|T_2^v|$  is very high. This is very similar to Model 1 in Table 7.7. Unlike in the Model 1,  $T_2^b$  are most negative at  $K = 105$  and

$K =$	80	85	90	95	100	105	110	115	120
$T = 0.25$	1.3	0.5	-0.7	-2.3	-3.7	-4.4	-4.0	-3.0	-1.7
	0.1	0.5	0.9	1.1	0.4	1.7	3.6	4.4	4.0
$T = 0.5$	1.5	1.0	0.2	-0.7	-1.5	-2.1	-2.1	-1.7	-1.1
	0.7	1.0	0.6	0.2	0.5	0.5	0.1	1.7	2.7
$T = 0.75$	0.6	0.3	-0.1	-0.6	-1.1	-1.4	-1.4	-1.2	-0.8
	1.3	1.0	1.1	0.6	0.4	0.01	0.6	0.8	1.9
$T = 1$	-0.6	-0.7	-0.9	-1.1	-1.3	-1.5	-1.4	-1.1	-0.7
	1.6	1.5	1.4	1.2	1.0	0.5	0.4	0.6	0.6

Table 7.8:  $T_2^b$  and  $T_2^v$ ,  $U = 130$

$|T_2^b|$  becomes smaller as the option moves in either direction of OTM or ITM. It shows Model 2 produces highest biases when the option is close to the money.

Table 7.9 lists  $T_3^b$  and  $T_3^v$ . When  $T = 0.25$ ,  $|T_3^b|$  are much higher than  $|T_3^v|$  for

$K =$	80	85	90	95	100	105	110	115	120
$T = 0.25$	1.7	0.4	-1.7	-4.5	-7.2	-8.7	-8.3	-6.7	-4.5
	0.6	0.4	0.2	1.3	1.9	0.1	3.3	4.5	5.4
$T = 0.5$	2.1	1.2	-0.1	-1.7	-3.2	-4.4	-4.6	-4.1	-3.1
	0.2	0.4	0.5	1.6	2.6	2.2	0.1	2.5	3.8
$T = 0.75$	0.6	0.1	-0.6	-1.5	-2.4	-3.1	-3.2	-2.9	-2.4
	0.7	0.6	0.03	1.0	1.4	1.9	1.6	0.05	1.0
$T = 1$	-1.5	-1.7	-2.0	-2.4	-2.8	-3.1	-3.0	-2.7	-2.2
	0.8	0.5	0.3	0.4	1.1	1.2	0.9	0.7	0.4

Table 7.9:  $T_3^b$  and  $T_3^v$ ,  $U = 130$

those options close to the money ( $95 \leq K \leq 110$ ). When it goes deep in or out of the money, the difference becomes smaller. When  $T = 0.5$ ,  $|T_3^b|$  is significantly higher than  $|T_3^v|$  for deep in the money options ( $K \leq 85$ ) and options not too far out of the money ( $K = 105$  and  $110$ ). When  $T = 0.75$ , barrier options with  $K \geq 100$  have significantly lower values from those in the Heston model. When  $T = 1$ ,  $|T_3^b|$  are higher than  $|T_3^v|$  for all options and  $T_3^b$  are all negative.

Table 7.10 shows the results for Model 4. For options with  $T = 0.25$ ,  $|T_4^b|$  are much higher than  $|T_4^v|$  for options with  $95 \leq K \leq 105$ . In those case,  $|T_4^v|$  are all below 1 which means the vanilla option prices are very close but  $T_4^b$  are highly negative so the barrier option prices are significantly lower than those in the Heston model. When  $T > 0.25$  and  $95 \leq K \leq 105$ ,  $|T_4^b|$  are also higher than  $|T_4^v|$ , but the

$K =$	80	85	90	95	100	105	110	115	120
$T = 0.25$	2.8	1.0	-1.9	-5.7	-9.3	-11.2	-10.7	-8.5	-5.8
	2.0	2.6	2.6	0.3	0.6	1.0	5.7	8.2	7.0
$T = 0.5$	3.3	2.1	0.3	-1.8	-4.0	-5.5	-5.8	-5.2	-4.0
	1.8	1.9	0.8	1.0	2.4	2.3	0.5	3.5	5.6
$T = 0.75$	1.3	0.6	-0.4	-1.6	-2.8	-3.8	-4.0	-3.8	-3.1
	1.9	1.9	0.9	0.9	2.1	1.9	0.9	0.3	3.2
$T = 1$	-1.6	-1.8	-2.2	-2.8	-3.5	-3.9	-3.8	-3.5	-2.8
	2.6	2.2	0.9	0.6	1.2	1.7	1.3	0.6	0.7

Table 7.10:  $T_4^b$  and  $T_4^v$ ,  $U = 130$

differences are not as big as when  $T = 0.25$ . It shows that in Model 4 disagrees with the Heston model in short maturity barrier option prices even though they produce similar vanilla option prices.

Table 7.11 displays the results for Model 5. We notice that ITM vanilla options

$K =$	80	85	90	95	100	105	110	115	120
$T = 0.25$	0.7	2.3	5.0	8.9	12.3	13.7	12.8	9.7	6.1
	1.0	3.1	6.4	2.2	14.9	42.1	75.7	83.9	112.0
$T = 0.5$	-0.2	0.6	1.8	3.4	4.8	5.5	5.7	4.7	3.3
	1.2	0.6	1.1	5.5	11.3	20.1	29.1	36.5	39.2
$T = 0.75$	2.7	2.8	2.8	3.1	3.2	3.0	3.0	2.4	1.8
	0.5	1.8	4.1	6.9	10.3	13.7	16.0	18.0	19.7
$T = 1$	6.7	6.2	5.5	4.9	4.1	3.2	2.8	2.0	1.3
	1.9	3.7	5.9	8.3	10.3	12.5	13.3	13.5	13.0

Table 7.11:  $T_5^b$  and  $T_5^v$ ,  $U = 130$

have very high  $|T_5^v|$ , much higher than  $|T_5^b|$ . It indicates that the vanilla option prices in Model 5 are very different from those in the Heston model. There is hardly any conclusion we can draw on how differently they produce barrier option prices.

Table 7.12 presents the results in Model 6. As in Model 5, the vanilla option prices are significantly different from those in the Heston model. We cannot draw any conclusion in Model 6.

In summary, when  $U = 130$  those models which produce close vanilla option prices (Model 1-4) give different values for barrier options with short maturity ( $T = 0.25$ ). But the difference gets lower for higher maturities. The differences are higher for

$K =$	80	85	90	95	100	105	110	115	120
$T = 0.25$	-3.3	-4.6	-6.5	-9.0	-10.9	-11.6	-10.2	-7.9	-4.9
	2.7	7.3	14.7	15.1	3.1	34.2	75.2	58.8	34.5
$T = 0.5$	-2.9	-3.6	-4.6	-5.7	-6.4	-6.6	-5.9	-4.8	-3.2
	4.7	7.4	9.3	6.3	2.8	15.1	30.6	48.9	51.2
$T = 0.75$	-4.6	-4.9	-5.3	-5.5	-5.4	-5.1	-4.3	-3.5	-2.2
	3.2	4.4	4.1	1.8	2.8	9.4	16.7	24.5	32.0
$T = 1$	-6.8	-6.8	-6.8	-6.5	-5.9	-5.2	-4.2	-3.2	-2.0
	1.1	1.2	0.6	0.7	3.3	5.9	9.0	12.4	16.2

Table 7.12:  $T_6^b$  and  $T_6^v$ ,  $U = 130$

ATM and not-too-deep-ITM options ( $100 \leq K \leq 110$ ) are higher than those options with other strikes. In most cases where the our models calibrate well to the vanilla option prices (low  $|T_i^b|$ ), they under-price the barrier options.

**Barrier options with high hitting probabilities.** Table 7.13 shows  $T_1^b$  and  $T_1^v$  when  $U = 110$ . For options with  $T = 0.25$  and  $K \leq 90$ ,  $|T_1^b|$  are much higher than

$K =$	80	85	90	95	100	105
$T = 0.25$	5.9	5.6	4.8	3.3	1.3	-0.5
	1.0	0.9	0.7	1.6	2.8	2.8
$T = 0.5$	2.4	2.2	1.7	1.0	0.2	-0.5
	1.0	1.6	1.7	2.3	3.1	2.7
$T = 0.75$	0.2	0.1	-0.01	-0.1	-0.2	-0.5
	0.9	1.1	1.1	1.4	1.8	2.1
$T = 1$	-1.2	-1.1	-0.9	-0.7	-0.4	-0.4
	0.4	0.2	0.04	0.2	0.4	1.0

Table 7.13:  $T_1^b$  and  $T_1^v$ ,  $U = 110$

$|T_1^v|$  and  $T_1^b$  are all positive. It indicates that in those cases Model 1 over-prices the barrier options. When  $T = 0.75$  and 1, all  $|T_1^b|$  and  $|T_1^v|$  are low, which means that Model 1 gives similar vanilla as well as barrier option prices to the Heston model. This is also seen in the high hitting probability case in Table 7.7.

Table 7.14 gives results in Model 2 when  $U = 110$ . When  $T = 0.25$  and  $K \leq 90$ ,  $|T_2^b|$  are significantly higher  $|T_2^v|$  as in Model 1 in Table 7.13 but  $T_1^b$  are all negative. It shows Model 3 gives lower values for those barrier options. When  $T = 1$ ,  $|T_2^b|$  are about twice as high as  $|T_2^v|$  while  $|T_2^v|$  are all below 2. It shows that even though there are differences (but not too significant) in the vanilla options, they



$K =$	80	85	90	95	100	105
$T = 0.25$	-3.3	-3.2	-2.6	-1.5	-0.01	-0.2
	0.1	0.5	0.9	1.1	0.4	1.7
$T = 0.5$	-1.5	-1.2	-0.7	0.01	0.9	0.7
	0.7	1.0	0.6	0.2	0.5	0.5
$T = 0.75$	0.8	1.0	1.3	1.5	1.6	1.1
	1.3	1.0	1.1	0.6	0.4	0.01
$T = 1$	3.0	3.0	2.9	2.6	2.1	1.3
	1.6	1.5	1.4	1.2	1.0	0.5

Table 7.14:  $T_2^b$  and  $T_2^v$ ,  $U = 110$

are lower than those in the barrier option price when  $T = 1$ . For  $T = 0.5$  and  $0.75$ , neither  $|T_2^b|$  nor  $|T_2^v|$  is high so that Model 1 and the Heston model produce roughly the same vanilla option price as well as the barrier option price in both cases.

Table 7.15 displays the results for Model 3. We find similar patterns as in Model 2

$K =$	80	85	90	95	100	105
$T = 0.25$	-4.6	-4.7	-4.4	-3.5	-1.2	-0.5
	0.6	0.4	0.2	1.3	1.9	0.1
$T = 0.5$	-1.5	-1.4	-1.1	-0.7	0.3	0.3
	0.2	0.4	0.5	1.6	2.6	2.2
$T = 0.75$	1.2	1.3	1.2	1.1	1.2	0.6
	0.7	0.6	0.03	1.0	1.4	1.9
$T = 1$	3.3	3.2	2.8	2.2	1.7	0.7
	0.8	0.5	0.3	0.4	1.1	1.2

Table 7.15:  $T_3^b$  and  $T_3^v$ ,  $U = 110$

in Table 7.14. The Heston model and Model 3 produce similar prices vanilla options while disagree on the barrier option prices for  $T = 0.25$  and  $1$ .

Table 7.16 presents the results for Model 4. For  $T = 0.25$  and  $0.5$ , especially for  $T = 0.25$ , the discrepancies in the barrier option prices are much larger than those in the vanilla option prices. Model 4 significantly under-prices the barrier options with all strikes. When  $T = 0.75$  and  $1$ ,  $|T_4^b|$  are either low or not significantly higher than  $|T_4^v|$ . In both cases, Model 4 and the Heston model give similar barrier option prices.

Table 7.17 displays the results for Model 5. When  $T = 0.25$  and  $0.5$ ,  $|T_5^b|$  are much higher than  $|T_5^v|$  for options with  $K \leq 90$ . Model 5 completely disagrees with

$K =$	80	85	90	95	100	105
$T = 0.25$	-15.8	-16.4	-15.8	-13.5	-8.8	-3.5
	2.0	2.6	2.6	0.3	0.6	1.0
$T = 0.5$	-6.7	-6.6	-6.0	-5.0	-3.2	-1.4
	1.8	1.9	0.8	1.0	2.4	2.3
$T = 0.75$	-0.6	-0.6	-0.6	-0.7	-0.5	-0.5
	1.9	1.9	0.9	0.9	2.1	1.9
$T = 1$	3.5	3.3	2.7	1.8	0.9	-0.1
	2.6	2.2	0.9	0.6	1.2	1.7

Table 7.16:  $T_4^b$  and  $T_4^v$ ,  $U = 110$

$K =$	80	85	90	95	100	105
$T = 0.25$	-41.2	-39.2	-35.1	-27.2	-16.6	-4.2
	1.0	3.1	6.4	2.2	14.9	42.1
$T = 0.5$	-15.4	-14.0	-11.7	-8.1	-4.0	0.1
	1.2	0.6	1.1	5.5	11.3	20.1
$T = 0.75$	-1.3	-0.7	0.1	1.1	2.0	2.2
	0.5	1.8	4.1	6.9	10.3	13.7
$T = 1$	7.1	7.0	6.7	6.2	5.3	3.4
	1.9	3.7	5.9	8.3	10.3	12.5

Table 7.17:  $T_5^b$  and  $T_5^v$ ,  $U = 110$

the Heston model about the barrier option prices.  $T_5^b$  are very negative in those cases so that in Model 5 the barrier option prices are much lower than those in the Heston model. For options deep ITM ( $K \geq 100$ ),  $|T_5^b| < |T_5^v|$  while  $|T_5^v|$  are usually large. We cannot draw a conclusion about the difference in the barrier option prices because the vanilla option prices are very different.

Table 7.18 presents the results for Model 6. We see  $|T_6^b|$  are much higher than

$K =$	80	85	90	95	100	105
$T = 0.25$	32.4	30.7	26.9	20.5	11.6	2.2
	2.7	7.3	14.7	15.1	3.1	34.2
$T = 0.5$	12.1	10.8	8.5	5.6	2.2	-1.0
	4.7	7.4	9.3	6.3	2.8	15.1
$T = 0.75$	0.9	0.1	-0.8	-1.8	-2.4	-2.7
	3.2	4.4	4.1	1.8	2.8	9.4
$T = 1$	-5.2	-5.6	-5.8	-5.6	-4.8	-3.7
	1.1	1.2	0.6	0.7	3.3	5.9

Table 7.18:  $T_6^b$  and  $T_6^v$ ,  $U = 110$

$|T_6^v|$  for  $T = 0.25$  except for the option with  $K = 105$ . But  $|T_6^v|$  are also high for  $K = 90$  and  $95$ . It means that differences in vanilla option prices are high, but not as high as those differences in barrier option prices. In those cases, Model 6 overprices both barrier and vanilla options (positive  $T_6^b$  and  $T_6^v$ ). When  $T = 1$ ,  $|T_6^v|$  are low for ITM options and much lower than  $|T_6^b|$ . In these cases, Model 6 agrees with the Heston model on the vanilla option prices but gives significantly lower prices of barrier options. For options with  $K = 105$ ,  $|T_6^v| > |T_6^b|$  and we cannot draw conclusion in these cases.

In summary, for barrier options with high hitting probabilities, the discrepancies in barrier option prices are large, much larger than the low hitting probability case, even though the model and the Heston model produce close vanilla option prices (at least not different to the extent as in the barrier option prices). There are, however, cases in Model 5 and 6 where we cannot comment on the differences in barrier option prices because the vanilla option prices are significantly different to those in the Heston model. Generally speaking, our class of stochastic volatility models generates a variety of barrier option prices even though they roughly agree on the vanilla option prices.

## 7.4 Conclusion

We have examined the variations in barrier option prices in the class of models that we formulate in chapter 4. A calibration method is proposed so that the models can produce roughly the same market implied volatility surface which is assumed to be generated by the Heston stochastic volatility model. We have considered, in total, twelve different models by varying one of  $\gamma$ ,  $\xi$  and  $\beta$  and fixing the other two as in the Heston model. The calibration errors in vanilla options are large if  $\beta \neq 1$ . We choose six models whose calibration errors are lowest as our candidate models for barrier option pricing. The differences in barrier *versus* vanilla option prices are reported. Thanks to the low standard errors ensured by our control variate methods, we find that in many cases the variations in barrier option prices are higher than those in vanilla option prices, especially when varying  $\xi$  and fixing  $\gamma$  and  $\beta$ . In some cases, however, the models give significantly different vanilla option prices. We cannot compare the barrier option prices in those cases. A better calibration method in those models will be an interesting future research topic.

## Tables of Values in the Plot

Table 7.19 and 7.20 lists the option values and implied volatilities in Figure 7.1.

$K =$	80	85	90	95	100	105	110	115	120
$T = 0.25$	21.00	16.10	11.35	7.04	3.64	1.48	0.45	0.10	0.02
$T = 0.5$	22.04	17.32	12.84	8.84	5.54	3.11	1.54	0.67	0.25
$T = 0.75$	23.10	18.54	14.26	10.41	7.16	4.60	2.75	1.52	0.78
$T = 1$	24.16	19.73	15.59	11.85	8.63	5.99	3.96	2.48	1.48

Table 7.19: Vanilla Option Values in the Heston Model

$K =$	80	85	90	95	100	105	110	115	120
$T = 0.25$	0.169	0.164	0.159	0.155	0.150	0.146	0.143	0.140	0.137
$T = 0.5$	0.165	0.161	0.157	0.154	0.150	0.147	0.145	0.142	0.140
$T = 0.75$	0.163	0.159	0.156	0.153	0.151	0.148	0.146	0.144	0.142
$T = 1$	0.161	0.158	0.156	0.153	0.151	0.149	0.147	0.145	0.144

Table 7.20: Implied Volatilities in the Heston Model

## Chapter 8

# Conclusion

We addressed issues in discretization and variance reduction methods for Monte Carlo simulation in derivative modelling.

For discretization schemes, we investigated the convergence properties of six Itô-Taylor approximations, the Euler method, the order 1 predictor-corrector, the iterative predictor-corrector, the order 2 weak Itô-Taylor and the order 2 predictor-corrector, for the deterministic volatility LIBOR market model. We also included a seventh method, the strong Taylor expansion, by Siopacha and Teichmann [77]. An improvement on their method was provided.

We compared pricing errors for caplets and swaptions using the various discretization schemes. We find that in a single step Monte Carlo simulation, the iterative predictor-corrector has the lowest pricing error and requires the least computation time. The scheme is simple to implement and is recommended for long step discretization. In a multiple steps case, the order 2 weak Itô-Taylor scheme has lower pricing error than the iterative predictor-corrector scheme with only very little additional cost. The order 2 predictor-corrector has roughly the same pricing errors as the iterative predictor-corrector method, but is costly. We suggest using the order 2 weak Itô-Taylor scheme for multiple steps Monte Carlo.

The strong Taylor expansion by Siopacha and Teichmann [77] has the highest pricing errors. In addition, this scheme does not converge to the explicit value as the number of time steps increases. Although we provided an improvement on the strong Taylor expansion method, its pricing errors cannot be reduced by increasing the number of time steps. One should not use these two methods for discretization.

For variance reduction methods, we investigated control variate methods in stochastic volatility models. We formulated a general stochastic volatility model. This model nests many existing models in the literature. We proposed control variates for this model. The auxiliary processes were constructed for two types of instruments, those with linear path statistics and those with non-linear path statistics. We gave examples for both types. For linear path statistics, we proposed auxiliary instruments for arithmetic average rate options. For non-linear path statistics, we proposed auxiliary instruments for discretely monitored barrier options. Extensive numerical results demonstrate the improvement over using only old control variates. The improvement is large for arithmetic average rate options. Although for barrier options our methods do not have efficiency gains as high as those in the average option case, they still outperform the old methods in most cases.

With the help of our new control variates, we explored variations in barrier option pricing in our model, consistent with the implied volatility surface. Thanks to low standard errors ensured by our control variate methods, we find that in most cases the variations in barrier option prices are higher than those in vanilla option prices with the same strikes and maturities. Our stochastic volatility model is capable of generating a richer set of barrier option prices than the Heston model.

There are future research topics. Firstly, the iterative predictor-corrector scheme cannot be applied in the spot-LIBOR measure. It is interesting to compare Itô-Taylor schemes for the LIBOR market model in this measure. Secondly, it is worth finding an auxiliary model for non-linear path statistics when  $\beta \in (0, 1)$ . Finally, it is important to pursue the possibility of having a better variance reduction method for barrier option pricing.

# Appendix A

## Proof of Lemmas

### A.1 Proof of Lemma 4.5.1.

Define measure  $\mathbb{P}$  under which  $B_t$  has a zero drift and measure  $\widehat{\mathbb{P}}$  under which  $\widehat{B}_t$  has a zero drift.  $\widehat{B}_t$  is a symmetric process under  $\widehat{\mathbb{P}}$ . By reflection principal

$$\widehat{\mathbb{P}}\left(\widehat{M}_T \geq m, \widehat{B}_T \leq b\right) = \widehat{\mathbb{P}}\left(\widehat{B}_T \geq 2m - b\right). \quad (\text{A.1})$$

Define the joint density of  $\widehat{B}_T$  and  $\widehat{M}_T$  under  $\widehat{\mathbb{P}}$  as  $\widehat{f}_{\widehat{B}_T, \widehat{M}_T}(x, y)$ .

$$\widehat{\mathbb{P}}\left(\widehat{M}_T \geq m, \widehat{B}_T \leq b\right) = \int_m^{+\infty} \int_{-\infty}^b \widehat{f}_{\widehat{B}_T, \widehat{M}_T}(x, y) dx dy, \quad (\text{A.2})$$

$$\widehat{\mathbb{P}}\left(\widehat{B}_T \geq 2m - b\right) = \frac{1}{\sqrt{2\pi T}} \int_{2m-b}^{+\infty} e^{-\frac{z^2}{2T}} dz. \quad (\text{A.3})$$

By equation (A.1),

$$\int_m^{+\infty} \int_{-\infty}^b \widehat{f}_{\widehat{B}_T, \widehat{M}_T}(x, y) dx dy = \frac{1}{\sqrt{2\pi T}} \int_{2m-b}^{+\infty} e^{-\frac{z^2}{2T}} dz. \quad (\text{A.4})$$

Differentiating equation (A.4) twice on both sides, first with respect to  $m$  and then with respect to  $b$ , gives us

$$\widehat{f}_{\widehat{B}_T, \widehat{M}_T}(x, y) = \frac{2(2m - b)}{T\sqrt{2\pi T}} e^{-\frac{(2m-b)^2}{2T}}. \quad (\text{A.5})$$

By definition

$$d\widehat{B}_t = \sigma_t d\widehat{W}_t = dB_t + a\sigma_t^2 dt = \sigma_t dW_t + a\sigma_t^2 dt \Rightarrow d\widehat{W}_t = dW_t + a\sigma_t dt. \quad (\text{A.6})$$



By Girsanov Theorem,

$$\begin{aligned} Z_t = \frac{d\widehat{\mathbb{P}}}{d\mathbb{P}} &= \exp \left( -a \int_0^t \sigma_s dW_s - \frac{a^2}{2} \int_0^t \sigma_s^2 ds \right) \\ &= \exp \left( -aB_t - \frac{a^2}{2} \int_0^t \sigma_s^2 ds \right) = \exp \left( -a\widehat{B}_t + \frac{a^2}{2} \int_0^t \sigma_s^2 ds \right). \end{aligned} \quad (\text{A.7})$$

We can use it to work out the joint density of  $\widehat{B}_T$  and  $\widehat{M}_T$  under measure  $\mathbb{P}$ ,  $f_{\widehat{B}_T, \widehat{M}_T}(x, y)$ .

$$\begin{aligned} \mathbb{P} \left( \widehat{B}_T \leq b, \widehat{M}_T \leq m \right) &= \mathbb{E} \left( \mathbb{I}_{\{\widehat{B}_T \leq b, \widehat{M}_T \leq m\}} \right) \\ &= \widehat{\mathbb{E}} \left( \frac{1}{Z_T} \mathbb{I}_{\{\widehat{B}_T \leq b, \widehat{M}_T \leq m\}} \right) \\ &= \widehat{\mathbb{E}} \left[ \exp \left( a\widehat{B}_T - \frac{a^2}{2}\overline{T} \right) \mathbb{I}_{\{\widehat{B}_T \leq b, \widehat{M}_T \leq m\}} \right] \\ &= \int_{-\infty}^m \int_{-\infty}^b \exp \left( ax - \frac{a^2}{2}\overline{T} \right) \widehat{f}_{\widehat{B}_T, \widehat{M}_T}(x, y) dx dy \end{aligned} \quad (\text{A.8})$$

The joint density is then computed as

$$\begin{aligned} f_{\widehat{B}_T, \widehat{M}_T}(b, m) &= \frac{\partial^2}{\partial b \partial m} \mathbb{P} \left( \widehat{B}_T \leq b, \widehat{M}_T \leq m \right) \\ &= \exp \left( ab - \frac{a^2}{2}\overline{T} \right) \widehat{f}_{\widehat{B}_T, \widehat{M}_T}(b, m) \\ &= \frac{2(2m - b)}{\overline{T}\sqrt{2\pi\overline{T}}} \exp \left[ ab - \frac{1}{2}a^2\overline{T} - \frac{1}{2\overline{T}}(2m - b)^2 \right]. \end{aligned} \quad (\text{A.9})$$

End of proof.  $\square$

## A.2 Proof of Lemma 4.6.1

We proof this lemma by using the Taylor series expansion method. Note this the method is used before for the moments of Power Normal distribution (see Freeman and Modarres [34])

$$\mathbb{E}(X|Y > \kappa) = \int_{\kappa}^{+\infty} (\lambda y)^{\frac{1}{\lambda}} \phi(y, \mu, \sigma^2|y > \kappa) dy. \quad (\text{A.10})$$

Define  $S(y) = (\lambda y)^{\frac{1}{\lambda}}$ . By Taylor-expansion around the point  $y = \mu$ , we have

$$S(y) = \sum_{i=0}^{\infty} \frac{1}{i!} S^{(i)}(\mu) (y - \mu)^i, \quad (\text{A.11})$$

where

$$S^{(i)}(y) = \begin{cases} S(y) & \text{if } i = 0 \\ (\lambda y)^{\frac{1}{\lambda}-i} \prod_{j=0}^{i-1} (1 - j\lambda) & \text{otherwise} \end{cases} \quad (\text{A.12})$$

is the  $i^{\text{th}}$  order derivative of  $S(y)$ . Therefore, equation (A.10) becomes

$$\begin{aligned} & \int_{\kappa}^{+\infty} (\lambda y)^{\frac{1}{\lambda}} \phi(y, \mu, \sigma^2 | y > \kappa) dy \\ &= \sum_{i=0}^{\infty} \frac{1}{i!} S^{(i)}(\mu) \int_{\kappa}^{+\infty} (y - \mu)^i \phi(y, \mu, \sigma^2 | y > \kappa) dy \\ &= \sum_{i=0}^{\infty} \frac{1}{i!} S^{(i)}(\mu) \sigma^i \int_{\frac{\kappa-\mu}{\sigma}}^{\infty} z^i \phi\left(z, 0, 1 | z > \frac{\kappa-\mu}{\sigma}\right) dz \quad \left(\text{by transformation } z = \frac{y-\mu}{\sigma}\right) \\ &= (\lambda\mu)^{\frac{1}{\lambda}} + \sum_{i=1}^{\infty} \frac{1}{i!} (\lambda\mu)^{\frac{1}{\lambda}-i} \sigma^i \mathbb{E}(Z^i | Z > b) \prod_{j=0}^{i-1} (1 - j\lambda), \quad Z \sim N(0, 1), \end{aligned} \quad (\text{A.13})$$

where  $b = \frac{\kappa-\mu}{\sigma}$ .

End of proof.  $\square$

### A.3 Proof of Lemma 4.7.1.

$$\begin{aligned} Z_t &= Z_0 \exp \left[ \left( \lambda - \frac{1}{2} \right) \int_0^t \sigma_s^2 ds + \int_0^t \sigma_s dW_s \right] \\ &= Z_0 \exp \left[ \left( \lambda - \frac{1}{2} \right) \int_0^t \sigma_s^2 ds + B_t \right] = Z_0 \exp \left( \widehat{B}_t \right), \end{aligned} \quad (\text{A.14})$$

where  $\widehat{B}_t = a \int_0^t \sigma_s^2 ds + B_t$  and  $a = \lambda - \frac{1}{2}$ .

$$\begin{aligned} \mathbb{E} [H(Z_t, t \leq T)] &= \mathbb{E} \left[ (Z_T - K)^+ \mathbb{I}_{\{Z_T \geq K, \max_{t \leq T} Z_t \leq U\}} \right] \\ &= \mathbb{E} \left[ (Z_T - K)^+ \mathbb{I}_{\{\widehat{B}_T \geq k, \widehat{M}_T \leq u\}} \right], \end{aligned} \quad (\text{A.15})$$

where

$$k = \ln \left( \frac{K}{Z_0} \right), \quad u = \ln \left( \frac{U}{Z_0} \right). \quad (\text{A.16})$$

From Lemma 4.5.1 we know the joint density of  $\widehat{B}_T$  and  $\widehat{M}_T$  under  $\mathbb{P}$ . Equation (A.15) can be computed as

$$\begin{aligned} \mathbb{E} \left[ (Z_T - K)^+ \mathbb{I}_{\{\widehat{B}_T \geq k, \widehat{M}_T \leq u\}} \right] &= \int_k^u \int_{x^+}^u (Z_0 e^x - K) f_{\widehat{B}_T, \widehat{M}_T}(x, y) dy dx \\ &= \int_k^u \int_{x^+}^u (Z_0 e^x - K) \frac{2(2y - x)}{\overline{T} \sqrt{2\pi \overline{T}}} \exp \left[ ax - \frac{1}{2} a^2 \overline{T} - \frac{1}{2\overline{T}} (2y - x)^2 \right] dy dx \\ &= - \int_k^u (Z_0 e^x - K) \frac{1}{\sqrt{2\pi \overline{T}}} \exp \left[ ax - \frac{1}{2} a^2 \overline{T} - \frac{1}{2\overline{T}} (2y - x)^2 \right] \Big|_{y=x^+}^{y=u} dx \\ &= \frac{1}{\sqrt{2\pi \overline{T}}} \int_k^u (Z_0 e^x - K) \exp \left( ax - \frac{1}{2} a^2 \overline{T} - \frac{1}{2\overline{T}} x^2 \right) dx \\ &\quad - \frac{1}{\sqrt{2\pi \overline{T}}} \int_k^u (Z_0 e^x - K) \exp \left[ ax - \frac{1}{2} a^2 \overline{T} - \frac{1}{2\overline{T}} (2u - x)^2 \right] dx \\ &= Z_0 (I_1 - I_3) - K (I_2 - I_4), \end{aligned} \quad (\text{A.17})$$

where

$$I_1 = \frac{1}{\sqrt{2\pi \overline{T}}} \int_k^u \exp \left[ (a+1)x - \frac{1}{2} a^2 \overline{T} - \frac{1}{2\overline{T}} x^2 \right] dx, \quad (\text{A.18})$$

$$I_2 = \frac{1}{\sqrt{2\pi \overline{T}}} \int_k^u \exp \left( ax - \frac{1}{2} a^2 \overline{T} - \frac{1}{2\overline{T}} x^2 \right) dx, \quad (\text{A.19})$$

$$I_3 = \frac{1}{\sqrt{2\pi \overline{T}}} \int_k^u \exp \left[ (a+1)x - \frac{1}{2} a^2 \overline{T} - \frac{1}{2\overline{T}} (2u - x)^2 \right] dx, \quad (\text{A.20})$$

$$I_4 = \frac{1}{\sqrt{2\pi \overline{T}}} \int_k^u \exp \left[ ax - \frac{1}{2} a^2 \overline{T} - \frac{1}{2\overline{T}} (2u - x)^2 \right] dx. \quad (\text{A.21})$$

Each one is of the form

$$\begin{aligned} &\frac{1}{\sqrt{2\pi \overline{T}}} \int_k^u e^{c_1 + c_2 x - \frac{1}{2\overline{T}} x^2} dx \\ &= e^{\frac{1}{2} c_2^2 \overline{T} + c_1} \int_{\frac{1}{\sqrt{\overline{T}}} (k - c_2 \overline{T})}^{\frac{1}{\sqrt{\overline{T}}} (u - c_2 \overline{T})} e^{-\frac{z^2}{2}} dz \\ &= e^{\frac{1}{2} c_2^2 \overline{T} + c_1} \left[ \Phi \left( \frac{u - c_2 \overline{T}}{\sqrt{\overline{T}}} \right) - \Phi \left( \frac{k - c_2 \overline{T}}{\sqrt{\overline{T}}} \right) \right] \\ &= e^{\frac{1}{2} c_2^2 \overline{T} + c_1} \left[ \Phi \left( \frac{\ln \left( \frac{U}{Z_0} \right) - c_2 \overline{T}}{\sqrt{\overline{T}}} \right) - \Phi \left( \frac{\ln \left( \frac{K}{Z_0} \right) - c_2 \overline{T}}{\sqrt{\overline{T}}} \right) \right]. \end{aligned} \quad (\text{A.22})$$

For  $I_1$ ,  $c_1 = -\frac{1}{2}a^2\overline{T} = -\frac{1}{2}(\lambda - \frac{1}{2})^2\overline{T}$  and  $c_2 = a + 1 = \lambda + \frac{1}{2}$ .

$$\begin{aligned} I_1 &= e^{\lambda\overline{T}} \left[ \Phi \left( \frac{\ln \left( \frac{U}{Z_0} \right) - (\lambda + \frac{1}{2})\overline{T}}{\sqrt{\overline{T}}} \right) - \Phi \left( \frac{\ln \left( \frac{K}{Z_0} \right) - (\lambda + \frac{1}{2})\overline{T}}{\sqrt{\overline{T}}} \right) \right] \\ &= e^{\lambda\overline{T}} \left[ \Phi \left( \frac{\ln \left( \frac{Z_0}{K} \right) + (\lambda + \frac{1}{2})\overline{T}}{\sqrt{\overline{T}}} \right) - \Phi \left( \frac{\ln \left( \frac{Z_0}{U} \right) + (\lambda + \frac{1}{2})\overline{T}}{\sqrt{\overline{T}}} \right) \right]. \end{aligned} \quad (\text{A.23})$$

For  $I_2$ ,  $c_1 = -\frac{1}{2}a^2\overline{T} = -\frac{1}{2}(\lambda - \frac{1}{2})^2\overline{T}$  and  $c_2 = a = \lambda - \frac{1}{2}$ .

$$\begin{aligned} I_2 &= \Phi \left( \frac{\ln \left( \frac{U}{Z_0} \right) - (\lambda - \frac{1}{2})\overline{T}}{\sqrt{\overline{T}}} \right) - \Phi \left( \frac{\ln \left( \frac{K}{Z_0} \right) - (\lambda - \frac{1}{2})\overline{T}}{\sqrt{\overline{T}}} \right) \\ &= \Phi \left( \frac{\ln \left( \frac{Z_0}{U} \right) + (\lambda - \frac{1}{2})\overline{T}}{\sqrt{\overline{T}}} \right) - \Phi \left( \frac{\ln \left( \frac{Z_0}{K} \right) + (\lambda - \frac{1}{2})\overline{T}}{\sqrt{\overline{T}}} \right) \end{aligned} \quad (\text{A.24})$$

For  $I_3$ ,  $c_1 = -\frac{1}{2}a^2\overline{T} - \frac{2u^2}{\overline{T}} = -\frac{1}{2}(\lambda - \frac{1}{2})^2\overline{T} - \frac{2u^2}{\overline{T}}$  and  $c_2 = a + 1 + \frac{2u}{\overline{T}} = \lambda + \frac{1}{2} + \frac{2u}{\overline{T}}$ .

$$I_3 = e^{\lambda\overline{T}} \left( \frac{U}{Z_0} \right)^{2\lambda+1} \left[ \Phi \left( \frac{\ln \left( \frac{U^2}{KZ_0} \right) + (\lambda + \frac{1}{2})\overline{T}}{\sqrt{\overline{T}}} \right) - \Phi \left( \frac{\ln \left( \frac{U}{Z_0} \right) + (\lambda + \frac{1}{2})\overline{T}}{\sqrt{\overline{T}}} \right) \right]. \quad (\text{A.25})$$

For  $I_4$ ,  $c_1 = -\frac{1}{2}a^2\overline{T} - \frac{2u^2}{\overline{T}} = -\frac{1}{2}(\lambda - \frac{1}{2})^2\overline{T} - \frac{2u^2}{\overline{T}}$  and  $c_2 = a + \frac{2u}{\overline{T}} = \lambda - \frac{1}{2} + \frac{2u}{\overline{T}}$ .

$$I_4 = \left( \frac{U}{Z_0} \right)^{2\lambda-1} \left[ \Phi \left( \frac{\ln \left( \frac{U^2}{KZ_0} \right) + (\lambda - \frac{1}{2})\overline{T}}{\sqrt{\overline{T}}} \right) - \Phi \left( \frac{\ln \left( \frac{U}{Z_0} \right) + (\lambda - \frac{1}{2})\overline{T}}{\sqrt{\overline{T}}} \right) \right] \quad (\text{A.26})$$

End of proof.  $\square$

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