Holomorphic Curves and Minimal Surfaces in Kähler Manifolds

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0.1 Acknowledgments

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Part of this thesis is devoted to an analysis of the period map for minimal immersions. This part of my work is deeply linked with previous work, also unpublished, of G.P. Pirola. I want to express to him my gratitude for having patiently explained to me his work and for having suggested to me many interesting directions of further investigations.

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0.2 Summary

Our work is concerned with the relation between a complex differential geometric property, namely holomorphicity, and a metric one, namely to be conformal and minimal, of immersions (possibly branched) of Riemann surfaces into Kähler manifolds.

A well known theorem (Wirtinger's Inequality) states that every holomorphic surface inside a Kähler manifold is area minimizing w.r.t. variations with compact support. Of course, the converse is not true in general. However, there are important situations, as in the resolution of the Frankel Conjecture by Siu and Yau, when it is. A first motivation for our research is to understand to which extent is the converse true. In Chapter 1 we discuss this problem after having briefly recalled the basic notions and background material that will be needed in the sequel.

We first tried to prove some general existence result for immersions into riemannian manifolds, which are area minimizing among classes of maps sharing some topological properties. Following the line of the proof of the existence theorem for minimal surfaces incompressible on the fundamental group, due to Sacks-Uhlenbeck and Schoen-Yau, in Chapter 2 we prove existence of minimal surfaces incompressible on the first homology group. We apply this result to the theory of Abelian Varieties, and we present here a new proof, completely based on riemannian techniques, of a classical result about the Schottky Problem, i.e. the characterization of the jacobian locus inside the space of principally polarized abelian varieties of complex dimension 2 and 3.

A crucial step in the proof of this result is the fact, proved by Micallef, that a converse of Wirtinger's Inequality holds for immersions of closed surfaces of genus 2 and 3 into flat T^4 or T^6, respectively. As for the Schottky problem, also for minimal surface theory, the situation becomes more difficult as the dimension of the target torus increases.

In Chapter 3 we give a unified presentation of an unpublished theorem of Micallef (Theorem 3.4.1) with our research work. In particular we give a very explicit way to construct stable minimal immersions of surfaces of genus $r \geq 4$ into flat tori of dimension $2r$ and of genus $r \geq 7$ into flat tori of dimension $2(r-1)$. The existence of such examples represents a major difficulty in the attempt to apply minimal surface theory to the theory of abelian varieties.

In his thesis Micallef proved a converse of Wirtinger's Inequality for isometric stable minimal immersions of complete oriented surfaces into $\mathbb{R}^4$, with the euclidean metric, provided that the Kähler angle of the immersion omits an open set of $[0, \pi]$. In Chapter 4 we show that this result does not depend on the linear structure of $\mathbb{R}^4$, but on a riemannian property of its flat metric, namely the fact that it is hyperkähler. We prove in fact the same theorem replacing $\mathbb{R}^4$ with <u>any</u> hyperkähler 4-manifold.

In the same Chapter we give also a description of known results about the relation between the Kähler angle and the Gauss lift (or the Gauss map, in the case of euclidean space) associated to an immersion.

In the last Chapter we go back to the study of periodic minimal surfaces.

The results we proved in Chapter 2 and 3 pointed out many natural questions about uniqueness and rigidity of periodic minimal surfaces with some topological constraints. In Chapter 5 we describe a framework for the study of this kind of problems that we believe to be very promising in many different situations, and we study in detail this setting for immersions of surfaces of genus r into flat T^{2r}. Our approach makes transparent a deep connection between algebraic properties of an algebraic curve and riemannian properties of the conformal minimal immersions into some flat torus of a fixed closed Riemann surface. Using previous results of Pirola and classical theorems about algebraic curves, such as the Torelli and the Infinitesimal Torelli Theorems, we give fairly complete answers to the problems about uniqueness and rigidity of minimal maps. In particular we see that these minimal immersions do not share the same rigidity properties as holomorphic and harmonic maps, but nevertheless they generically do not come in families.

We are convinced that a deeper study of periodic minimal surfaces in flat tori from the riemmanian point of view could give some new results in the theory of algebraic curves, especially about the structure of the singular locus of the theta divisors. We believe that our approach gives already some new insight on known phenomena.

Chapter 1

Introduction

1.1 Basic notions and definitions

All the material contained in this section is well known. For a detailed study we refer to the books of Osserman ([44]) and Lawson ([31]).

1.1.1 Minimal immersions of surfaces

Consider an immersion $f\colon \Sigma \to (M, g)$ of a topological surface into a riemannian manifold, and a smooth compactly supported section v of the normal bundle to Σ s.t. $supp(v) \cap \partial\Sigma = \emptyset$. Every such variation is induced by a family $F\colon \Sigma \times (-1, 1) \to (M, g)$, with the following properties:

1. $f_t = F(\cdot, t)$ is an immersion for all t.

2. $f_0 = f$.

3. $f_t(\partial\Sigma) = f_{|\partial\Sigma}$ for all t,

4. $F_*(\frac{d}{dt})_{|t=0}(p) = v(p)$ for all $p \in \Sigma$.

We then consider the variation of area induced by v, and we have:

$$\partial A(v) = \frac{dA(f_t(\Sigma))}{dt}_{|t=0} = -\int_\Sigma g(H, v)dVol_\Sigma. \tag{1.1}$$

where H is the mean curvature vector of the immersion f and $dVol_\Sigma$ is the volume form induced by f on the surface.

We say that f is *minimal* if $H = 0$, i.e. if it is a critical point for the Area functional among compactly supported variations.

1.1.2 Isothermal coordinates and conformal harmonic maps

Given any immersion $f: \Sigma \to (M, g)$ of a topological surface into a riemannian manifold, we can consider an atlas $\mathcal{U} = \{U_\alpha, x_\alpha, y_\alpha\}$ of isothermal coordinates for the metric induced by f (see [31]), which define on Σ a conformal structure μ_f. The metric on Σ induced by f is given on U_α by $\lambda(dx_\alpha^2 + dy_\alpha^2)$, where $\lambda = g(f_x, f_x)^{\frac{1}{2}} = g(f_y, f_y)^{\frac{1}{2}}$.

A direct calculation shows that if $f: (\Sigma, h) \to (M, g)$ is an isometric immersion, we have $\triangle_h f = H$. Therefore we have that $f: \Sigma \to (M, g)$ is minimal if and only if $f: (\Sigma, f^*(g)) \to (M, g)$ is harmonic.

On the other hand it is easy to check that, for any positive function λ on Σ, $\triangle_{\lambda h} f = 0$ if and only if $\triangle_h f = 0$. Therefore the space of harmonic maps from a surface is a conformal invariant.

This argument gives an equivalence between the following two data:

1. $f: \Sigma \to (M, g)$ a minimal immersion of a topological surface into a riemannian manifold.

2. $f: (\Sigma, \mu) \to (M, g)$ a conformal harmonic map from a Riemann surface into a riemannian manifold.

The equivalence described above is crucial in the study of minimal immersions of surfaces, because it allows us to put techniques of complex geometry into play.

Our first observation is that holomorphic maps of Riemann surfaces into Kähler manifolds are minimal immersions. Let us recall that a map between complex manifolds $f: (N, J) \to (M, J')$ is holomorphic if $J' \circ df = df \circ J$ at every point in N, i.e. if the tangent space to $f(N)$ is J'-invariant. A simple direct calculation (see [31]) proves the following:

Theorem 1.1.1 *If (M, J', g) is a Kähler manifold, and f as above is holomorphic, then f is minimal.*

Remark 1.1.1 *The above Theorem holds in the more general situation of J-holomorphic submanifolds of symplectic manifolds (M, ω), equipped with an almost complex structure J tamed by ω and metric $\omega(J\cdot, \cdot)$.*

As we will see later, holomorphic submanifolds of Kähler manifolds are not just critical points of the Area functional, but they are in fact minima.

1.1.3 Second variation of Area

As we have seen above minimal immersions are critical points of the Area functional. A way to measure how far a minimal immersion is from being a minimum for this functional is given by the second variation of Area, i.e. the second derivative of the Area in the direction of a compactly supported normal variation. A direct calculation (see [50]) shows that

$$\partial^2\mathcal{A}(v) = \frac{d^2\mathcal{A}(f_t(\Sigma))}{dt^2}_{|t=0} = -\int_\Sigma \{g(\Delta v, v) - |Bv|^2 - \sum_{i=1}^{2} g(R(v, e_i)e_i, v)\} dVol_\Sigma, \quad (1.2)$$

where

$$\Delta(v) = \sum_i (\nabla_{e_i}(\nabla_{e_j}(v))^\perp)^\perp - (\nabla_{\nabla_{e_j} e_j}(v))^\perp ,$$

$\{e_i\}$ is an orthonormal basis of the tangent space to the surface, $\perp$ is the projection on the normal bundle, ∇ is the connection on Σ induced by the Levi-Civita connection of (M, g) and B is the second fundamental form of f.

Definition 1.1.1

1. *A minimal immersion f is* stable *if $\partial^2\mathcal{A}(v) \geq 0$ for every section v of the normal bundle with compact support.*

2. *A section v of the normal bundle with compact support is called a* Jacobi field *along f if $\partial^2\mathcal{A}(v) = 0$. The dimension of the space of Jacobi fields (which is naturally a vector space) is called the* Nullity *of f, and denoted by $Nul(f)$.*

It is in general a very hard problem to calculate the nullity of a minimal immersion. In fact we will see in this thesis that, for minimal immersions into some Kähler manifolds, this space heavily depends on classical algebraic properties of the induced conformal structure.

A first result that we will use in this thesis is the following (see [50]):

Theorem 1.1.2 *If $f: (\Sigma, J) \to (M, J', g)$ is holomorphic and (M, J', g) is Kähler then $Nul(f) = dim_{\mathbb{R}} H^0(\nu)$, where ν is the normal bundle to Σ.*

1.2 Holomorphic curves and minimal surfaces in Kähler manifolds

The starting point of our research is the well known Wirtinger's Inequality (quoted as W-I in the sequel) which states, in the version proved by Federer ([20]) that a complex submanifold of a Kähler manifold minimizes volume in its homology class. This elementary, but remarkable, fact inspired much research in the theory of minimal submanifolds of a Kähler manifold, mainly in the attempt to prove some converse of it. A straight converse in general does not hold; there are mainly two reasons to believe this:

1. to minimize volume is a purely metric property not depending, in general, on a specific complex structure on the target manifold. So, for example, we can consider a manifold with a metric which is Kähler w.r.t. more than one complex structure. In this situation submanifolds which are holomorphic w.r.t. different complex structures are all volume minimizing in their homology classes.

2. there are topological restrictions on the type of classes representable by holomorphic submanifolds; for instance, we recall the following theorem ([22]):

   Theorem 1.2.1 *Analytic submanifolds of complex dimension p of a projective manifold of complex dimension n represent homology classes which are Poincaré dual to cohomology classes of type $(n-p, n-p)$.*

Therefore a reasonable attempt to prove the converse to the Wirtinger's Inequality has to focus on homology classes of special type.

Let us also recall that a refined version to the converse of the above theorem is the outstanding Hodge Conjecture, which has been proved by Lefschetz for $p = 1$ even for integral linear combinations:

Conjecture 1 *On a projective manifold M every rational cohomology class of type (p, p) is a rational linear combination of Poincaré duals of fundamental classes of analytic subvarieties of M.*

We want to underline the fact that even if this conjecture turns out to be true in general, it does not answer the problem of finding volume minimizing submanifolds in these classes. Observe also that the Hodge Conjecture deals with projective manifolds, while we are interested in the larger class of Kähler ones.

There are situations where the above obstructions become empty: the first case is when we have a Kähler manifold s.t. every $2p-$homology class is Poincaré dual to a cohomology class of type $(n-p, n-p)$. This is the case, for example, of the complex projective space with the Fubini-Study metric, where the converse to the Wirtinger's Inequality has been shown to hold, even allowing integral currents as competitors for the volume functional, and replacing the volume minimizing hypothesis with the weaker assumption of stable, by Lawson and Simons ([32]):

Theorem 1.2.2 *A closed integral current in $\mathbf{P}\mathbb{C}^n$ is stable if and only if it is an integral chain of algebraic varieties.*

We also recall that an affirmative answer to the converse to the W-I in the case of area minimizing 2-spheres in Kähler manifolds of positive bisectional curvature, led Siu and Yau ([51]) to the well known solution of the Frankel Conjecture.

There is also another situation, somehow antithetic, when the obstructions above can be overcome: suppose we are dealing with riemannian manifolds which are Kähler w.r.t. a large family of complex structures, so large that every $2p$ cohomology class is of type (p,p) in the Hodge decomposition induced by *some* compatible complex structure. This is the case, for example, of hyperkähler manifolds. A plausible converse to W-I will then be in the case of surfaces:

Problem 1 *Is any area minimizing map of a Riemann surface into a Kähler manifold for which the obstructions described above vanish, holomorphic with respect to* some *complex structure compatible with the metric?*

This is the central problem discussed in this thesis.

The above question has been studied by many authors in the last twenty years and we want now to give a brief account of results related to our work.

A fundamental tool to answer our main problem is the so called *Kähler angle* associated to an immersion of a surface in a Kähler manifold. It appears very clearly in Federer's proof of the W-I, where he in fact shows the following ([31]):

Proposition 1.2.1 *Let Σ be a 2-dimensional oriented real submanifold of a Kähler manifold (M,ω). Then*

$$\omega_{|\Sigma}(p) = \omega(e_1, e_2) dVol_M(p) \ ,$$

where $\{e_i\}$ is an orthonormal basis of $T_p\Sigma$.

It is easy to see that $\omega(e_1, e_2)$ is a real number whose absolute value is not more than 1. Therefore we can define:

Definition 1.2.1 *The Kähler angle of Σ at p is that angle α s.t. $cos(\alpha) = \omega(e_1, e_2)$, with the above notations.*

W-I for surfaces, as stated at the beginning of this section, follows directly from the above proposition, just by a comparison of the area of a holomorphic surface and the area of a homologous one using Stokes's Theorem ([31]).

It is easily seen that, as a function of p, α is a smooth function away from the points where $\alpha = 0$ or $\alpha = \pi$, where it is just Lipschitz.

The notion of Kähler angle was first explicitly introduced by Lichnerovicz and studied in detail by Chern-Wolfson ([13]) and Eells-Wood ([17]) in their study of minimal immersions in the complex projective space. Its relevance in our context is that it gives a measure of how far is Σ from being holomorphic. In fact Σ is a holomorphic submanifold if and only if α is identically zero on it. Points where the Kähler angle vanishes are usually called *complex* points, while points where the Kähler angle is equal to π are called *anticomplex* points. Surfaces with the property that α is identically equal to $\frac{\pi}{2}$ are called *lagrangian* surfaces (observe that this is equivalent to the familiar definition of lagrangian submanifolds in symplectic geometry because $\alpha = \frac{\pi}{2}$ if and only if the restriction of the Kähler form to Σ vanishes identically). A surface without complex and anticomplex points is called *totally real.*

Remark 1.2.1 *A first observation to be made is that a minimal surface in a Kähler surface has only isolated complex and anticomplex points, unless it is holomorphic or antiholomorphic ([53]). A simple proof of this fact is the following: denoting by J the complex structure on M, and z a complex coordinate on Σ induced by isothermal coordinates, it is not difficult to see (we give a complete proof of this claim in Chapter* 4*) that having called:*

$$s = [J(\frac{\partial}{\partial z})]^{\perp} ,$$

where $\perp$ denotes the projection on the complexified normal bundle $\nu_{\mathbb{C}} = \nu \otimes_{\mathbb{R}} \mathbb{C}$ to the surface, we have that

$$s \otimes dz$$

is a global holomorphic section of $\nu_{\mathbb{C}} \otimes \Lambda^{1,0}(\Sigma)$. The claim now follows easily, because complex and anticomplex points are zeros of this section.

Wolfson ([54]) first wrote the Ricci and the Codazzi-Mainardi equations for an immersion in a Kähler 4-manifold in terms of the Kähler angle. He proved, using the method of moving frames, that in such a manifold α satisfies the following two equations:

$$\partial\bar{\partial}(ln(sin^2\alpha)) = -\sqrt{-1}(K + K_\nu)dVol_\Sigma \tag{1.3}$$

$$\partial\bar{\partial}(ln(tan^2\frac{\alpha}{2})) = -\sqrt{-1}Ric \qquad (1.4)$$

Remark 1.2.2 *Equation 1.3 is a direct consequence of the holomorphicity of $s \otimes dz$.*

There are two direct consequences of these equations, due also to Wolfson:

Theorem 1.2.3 *Let Σ be a compact connected surface without boundary.*

1. *If M is a compact Kähler - Einstein 4-manifold of negative scalar curvature, and $f: \Sigma \to M$ is a totally real minimal immersion, possibly branched, then $f(\Sigma)$ is a Lagrangian submanifold of M.*

2. *If M is a hyperkähler 4-manifold and $f: \Sigma \to M$ is a totally real minimal immersion, possibly branched, then $f(\Sigma)$ is a submanifold holomorphic w.r.t. some complex structure on M compatible with the Calabi-Yau metric.*

We want to underline the fact that Wolfson *does not require any stability property of f.*

A careful analysis of the Kähler angle for minimal immersions in the projective plane gave a very detailed description of such maps in [9], [13], [17] and [18].

The Kähler angle plays a role also in Micallef's work on stable minimal surfaces in $\mathbb{R}^4$ ([36]). We recall that the Grassmannian of oriented two-planes in $\mathbb{R}^4$ with its natural symmetric space structure is isometric to $S^2 \times S^2$. Therefore we can look at the Gauss map G of an immersion into $\mathbb{R}^4$ as a map from the surface to $S^2 \times S^2$. In particular it is a classical result ([44]) that this map is holomorphic (taking on the sphere the orientation opposite to the standard one) if and only if the immersion is minimal. In Chapter 4 we will see that these S^2 factors parametrize the space of complex structures on $\mathbb{R}^4$, either oriented or anti-oriented. We will see that if $J \in S^2$ is not in the image of the Gauss map G of a minimal immersion F into $\mathbb{R}^4$, then

$$|\rho(\pi(G(p)))| = |\frac{sin\alpha_J(p)}{1 - cos\alpha_J(p)}| , \qquad (1.5)$$

where ρ is the stereographic projection from J and π is the projection $S^2 \times S^2 \to S^2$ on the factor containing J.

One of the main Micallef's results is then:

Theorem 1.2.4 *Let $F: \Omega \to \mathbb{R}^4$ be a stable minimal isometric immersion of a complete oriented surface, and assume that there exists a complex structure J compatible with the euclidean metric on $\mathbb{R}^4$, s.t. $\alpha_J(\Omega) \subset [\epsilon, \pi]$, $\epsilon > 0$, where α_J is the Kähler angle relative to J. Then α_J is constant and there exists a compatible complex structure $\tilde{J}$ s.t. $\alpha_{\tilde{J}}$ vanishes identically.*

By the equation 1.5 the above statement is equivalent to the following, which is Micallef's original one.

Theorem 1.2.5 *A stable minimal isometric immersion into $\mathbb{R}^4$ of a a complete oriented surface, whose Gauss map has the property that one of the two projections on the two-spheres omits an open set, is holomorphic w.r.t. an orthogonal complex structure.*

For sake of completeness we recall that by a well known theorem due to Chern ([11]) and Osserman ([43]) we know that, in the case of the euclidean 4-space, if both projections on the spheres of the Gauss map omit an open set then M is a plane.

The following results of Micallef ([37]) and Micallef-Wolfson ([38]) give a fairly complete description of the relation between holomorphicity and stability of minimal surfaces in hyperkähler 4-manifolds:

Theorem 1.2.6 *Every full stable minimal immersion of a Riemann surface into any flat 4-torus T^4 is holomorphic w.r.t. some complex structure compatible with the metric on T^4.*

Theorem 1.2.7 *Let (M, ω) be a hyperkähler 4-manifold, not necessarly compact and Σ a closed oriented surface. If $f: \Sigma \to M$ is a stable minimal surface whose normal bundle admits a holomorphic section, then f is holomorphic w.r.t. some complex structure on M compatible with the hyperkähler metric.*

The main tool used for proving the above theorems is a version of the second variation of area in Kähler manifolds first introduced in [36] and [38], sensitive also to the complex geometry of the normal bundle to the immersion.

The proofs of Theorems 1.2.5 and 1.2.6 make deep use of the linear stuctures of $\mathbb{R}^4$ and T^4. As we will see in Chapter 4 it is possible to extend some similar results to hyperkähler 4-manifolds.

These results leave an interesting open question (already asked in [16]) :

Is every stable minimal surface in a K3 surface with the Calabi-Yau Ricci flat metric, holomorphic w.r.t. some compatible complex structure?

Chapter 2

On Minimal Surfaces Incompressible in Homology and Abelian Varieties

2.1 Introduction

In mimimal surface theory one often tries to find immersions of surfaces which are critical points, or even minima, of the Area functional among immersions with some extra topological conditions. For example one can ask the following questions:

Problem 2 *1. Given a 2-homology class $\beta \in H_2(M, \mathbb{Z})$, where M is a riemannian manifold of dimension greater than 2, does there exist a minimal, or area minimizing surface, whose fundamental class represents β ?*

2. Given a continuous map $u: \Sigma \to M$, where M is a riemannian manifold of dimension greater than 2, does there exist an area minimizing map with the same induced action on the fundamental group as u ?

There is an immense literature about these problems, and we do not even attempt to give a description of the known results. Clearly there are no relations between these two problems for a general M; but for manifolds with a particularly simple topology as tori, it is possible to find some connections. In fact, since $H_2(T^n, \mathbb{Z}) = \Lambda^2(H_1(T^n, \mathbb{Z}))$, we have that the images of all maps with the same action on the first homology group represent the same 2-homology class.

About the second problem the most significant result has been found by Schoen-Yau ([49]) and Sacks-Uhlenbeck ([46]), who independently proved that the answer is yes for every riemannian manifold M of dimension at least 3, provided the action of u on the fundamental groups is injective.

Theorem 2.1.1 *Let Σ be a closed topological surface of genus ≥ 1, M a riemannian manifold of dimension at least 3 and let $u: \Sigma \to M$ be a continuous map such that $u_*: \pi_1(\Sigma) \to \pi_1(M)$ is injective. Then there exists a branched minimal immersion $f: \Sigma \to M$ with the same action on $\pi_1(\Sigma)$ as u and such that* $\mathrm{Area}(f) \leq \mathrm{Area}(g)$ *for every C^∞-map $g: \Sigma \to M$ with the same action on $\pi_1(\Sigma)$ as u.*

The minimal surfaces constructed in this way are called, for obvious reasons, *incompressible* on $\pi_1(\Sigma)$.

The above theorem has been used to establish some fundamental connections between geometric and topological properties of riemannian manifolds (see, for example, [49]). Sacks and Uhlenbeck produced examples of non-uniqueness of such maps even when restricting themselves to fixed homotopy classes.

Unfortunately this beautiful theorem does not apply when M is a flat torus, since the fundamental group of a torus is commutative. This suggested us to study the following variation of the second problem above:

Problem 3 *Given a continuous map $u: \Sigma_r \to M$ whose induced action on the first homology group is injective, does there exist an area minimizing map from Σ_r to M with the same induced action as u in homology?*

Problem 3 is the central theme of this Chapter.

Already in the case when M is a torus we see that the answer to this problem cannot be affirmative; in fact if we consider an abelian surface, i.e. a complex 2-dimensional torus endowed with an integral 2-form of type (1.1), there could be, in general, some classes with holomorphic representative given by the sum of two elliptic curves. Taking then an injective homomorphism from $H_1(\Sigma_2, \mathbb{Z})$ to $H_1(T^4, \mathbb{Z})$ which induces such a class in the above sense, we see that the area minimizing map is not defined on a Riemann surface of genus 2, but it is defined on the disjoint union of two Riemann surfaces of genus 1. This suggests that one has to allow some non homotopically trivial curves to collapse in the image to contractible curves (or even points), and that the conformal structure induced by a minimizing sequence for the area functional can degenerate to a conformal structure in the boundary of the Riemann moduli space. Nevertheless, since we are allowing just homologically trivial curves to collapse, it is natural to conjecture that the area minimizing map is defined on the quotient of the original topological surface modulo a finite set of homologically trivial curves. If this happens to be the case, then it would be possible to compare the action on the first homology group of the area minimizing map with the action of our initial continuous map, since they would be defined on two naturally isomorphic groups. In this Chapter we show that this is in fact what happens.

We will use the fact that it is possible to compactify the Riemann moduli space by adding the set of Riemann surfaces with nodes obtained by collapsing a set of admissible simple closed curves. A *Riemann surface with nodes* is a connected complex space $\tilde{\Sigma}$ such that every point p has a neighbourhood isomorphic either to the open disk in the complex plane, or to two disks whose centers are identified with p; in this case we say that p is a node of $\tilde{\Sigma}$. The connected components of $\tilde{\Sigma}\backslash\{nodes\}$ are called the *parts* of the Riemann surface with nodes. We will consider in this Chapter Riemann surfaces with nodes obtained from a smooth surface by collapsing a set of homologically trivial curves. The first homology group of the topological space obtained in this way is naturally isomorphic to the group of the smooth surface; keeping this identification implicit we can compare the action on the first homology group of a map from the smooth surface, with the action on the first homology group of a map from the Riemann surface with nodes.

Our result is then the following:

Theorem 2.1.2 *Let Σ be a closed topological surface of genus $r \geq 1$. M a riemannian manifold of dimension at least 3, and let $u: \Sigma \to M$ be a continous map such that $u_*: H_1(\Sigma, \mathbb{Z}) \to H_1(M, \mathbb{Z})$ is injective. Then there exists a Riemann surface with nodes obtained by collapsing a set of homologically trivial curves $\{\gamma_k\}$, whose parts are $\Sigma_{r_i} \backslash \{p_1^i, \dots, p_{s_i}^i\}$, with the following properties:*

1. *$\forall\, i\, \exists\, f_i: \Sigma_{r_i} \to M$ s.t. f_i is a branched conformal minimal immersion and* $\mathrm{Area}(f_i) \leq \mathrm{Area}(g)$ *for every C^∞-map $g: \Sigma_{r_i} \to M$ with the same action on $H_1(\Sigma_{r_i})$ as the one induced by the restriction of u to Σ_{r_i}.*

2. *If $f: \cup_i\{\Sigma_{r_i}\} \to M$ is the map given by the maps f_i on each Σ_{r_i}, we have $f_* = u_*$ on $H_1(\Sigma, \mathbb{Z})$ in the sense above. Furthermore, after having defined* $\mathrm{Area}(f) = \sum_i \mathrm{Area}(f_i)$, *we get* $\mathrm{Area}(f) \leq \mathrm{Area}(s)$ *for every C^∞-map $s: \Sigma \to M$ with the same action on $H_1(\Sigma, \mathbb{Z})$ as u.*

Beyond the formal difficulty of the statement, the content of our theorem, loosely speaking, is that the programme of Schoen-Yau and Sacks-Uhlenbeck works even in the case where there is a set of homologically trivial curves which collapses to points (or becomes contractible curves) in the image. It is important to underline the fact that, in general, *we lose the connectedness of the image* of the area minimizing map. The strategy of the proof is based on ideas and previous results due to Sacks-Uhlenbeck [46].

We will carry out in the second section an explicit example of a minimizing process in a class of maps whose action on $H_1(\Sigma, \mathbb{Z})$ is not injective, but not zero. The limit map is in this case a constant map, and thus the original action on H_1 is forgotten in the limit. This example suggests that our assumption on the action on homology should be the weakest possible.

Theorem 2.1.2 can be applied in a significantly larger class of situationss than Theorem 2.1.1. For instance, we will apply the previous result in the case of $M =$

$T^{2r}, r \geq 2$, the even dimensional flat tori. Since, by results of Micallef ([36],[37]) a smooth stable minimal immersion of $\Sigma_r \to T^{2r}$ is holomorphic with respect to some complex structure compatible with the metric of the torus, for $r = 2$ and 3, we are able to prove, by riemannian methods, a classical theorem in Algebraic Geometry (see for example [29]):

Theorem 2.1.3 *If (T^4, ω) is a principally polarized abelian surface then, either it is the jacobian of a Riemann surface of genus 2 or, it is the canonically polarized product of two elliptic curves.*

If (T^6, ω) is a principally polarized abelian threefold then, either it is the jacobian of a Riemann surface of genus 3 or, it is the canonically polarized product of three elliptic curves or the canonically polarized product of a jacobian of a Riemann surface of genus 2 and an elliptic curve.

For real dimension greater than 6 there is no analogue of the theorem of Micallef. Actually in the next Chapter we will show that for any nonhyperelliptic Riemann surface of genus $r \geq 4$, there exists a conformal stable minimal immersion $f: \Sigma_r \to (T^{2r}, g)$ into a flat torus, which is not holomorphic with respect to any complex structure compatible with the flat metric g.

As we mentioned at the beginning of this section, Sacks and Uhlenbeck, in [46] found examples of non uniqueness of area minimizing maps among maps with a prescribed action on the homotopy group. A fortiori uniqueness will in general fail among maps with a fixed injective action on homology. We leave open the question (that will be discussed in Chapter 5) about the uniqueness of area minimizing maps among those maps from Σ_r to T^{2r} with a fixed injective action on the first homology group. As we will explain in Chapter 5, the reason for which it seems plausible to us that such a uniqueness result may hold is the fact that both homotopic harmonic maps in flat tori and homotopic holomorphic maps in complex tori are rigid by classical results in these subjects.

This problem seems to us particularly interesting since it is the right analogue of the Torelli theorem for minimal immersions.

2.2 Proof of the Main Result

As we mentioned in the introduction we will adapt the strategy used by Sacks and Uhlenbeck for the case of injectivity on $\pi_1(\Sigma)$. We will need many classical theorems about harmonic maps and minimal immersions and for this reason we will adopt the same notation as [46]. We recall that if $f: \Sigma \to (M, g)$ is a smooth map, (M, g) a riemannian manifold and Σ a Riemann surface with conformal structure μ, then we define the *energy* of f to be

$$\mathcal{E}(f, \mu) = \int_\Sigma trace_h(g_{f(x)}(df, df)) *_h 1 .$$

where h is any metric on Σ compatible with μ.

We say that f is *harmonic* with respect to μ if f is a critical point of $\mathcal{E}(\cdot, \mu)$. On the other hand f is called *minimal* if it is a critical point of the Area functional:

$$\mathcal{A}(f) = \int_\Sigma det(g_{f(x)}(df, df))^{\frac{1}{2}} * 1.$$

The next theorem (see [47]) indicates the line of our argument:

Theorem 2.2.1 *Let $f: \Sigma \to (M, g)$ be harmonic with respect to the conformal structure μ and suppose μ is a critical point of $\mathcal{E}(f, \cdot)$ with respect to all smooth variations of μ, then f is a conformal branched minimal immersion.*

What the previous theorem suggests, and what Schoen-Yau and Sacks-Uhlenbeck did in their case, is that we need to minimize $\mathcal{E}$ in two steps, first by moving the map in the space of C^1-maps with the same action on $H_1(\Sigma, \mathbb{Z})$ (on $\pi_1(\Sigma)$ in their situation), while keeping the conformal structure fixed and then varying the conformal structure on Σ. This first step was done in the course of proving Theorem 2.1.1, using previous results of Lemaire [33], by proving the following:

Theorem 2.2.2 *Let $u:(\Sigma,\mu) \to (M,g)$ be continuous. Then there exists a map $f:(\Sigma,\mu) \to (M,g)$ which is harmonic with respect to the conformal structure μ, which has the same action on $\pi_1(\Sigma)$ as u, and for which $\mathcal{E}(f,\mu)$ is the minimum of $\mathcal{E}(\cdot,\mu)$ among all such maps.*

Remark 2.2.1 *In the above theorem it is not necessary to assume the injectivity of u_* on $\pi_1(\Sigma)$. We note that if in addition $\mathcal{E}(f,\mu) \le \mathcal{E}(g,\nu)$, for all g s.t. $g_* = f_*$ on $\pi_1(\Sigma)$ and for all conformal structures ν, then $\mathcal{A}(f) \le \mathcal{A}(l)$ for every immersion l such that $f_* = l_*$ on $\pi_1(\Sigma)$. In particular, we have that such f is a stable minimal immersion.*

In order to prove our result we need a stronger theorem since there are infinitely many actions on $\pi_1(\Sigma)$ which give the same action on $H_1(\Sigma,\mathbb{Z})$.

Theorem 2.2.3 *Let $u:\Sigma \to M$ be continuous. Then, for each conformal structure μ on Σ and Riemannian metric g on M, there exists a map $f:(\Sigma,\mu) \to (M,g)$, with the same action on $H_1(\Sigma,\mathbb{Z})$ as u, for which $\mathcal{E}(f,\mu)$ is the minimum among all such C^1 maps. In particular f is harmonic and stable.*

Proof: The existence of a harmonic map with the same action on $H_1(\Sigma,\mathbb{Z})$ as u is clearly assured by theorem 2.2.2 by picking an action on $\pi_1(\Sigma)$ which induces the fixed action on $H_1(\Sigma,\mathbb{Z})$. Of course the minimizing property of this map is not directly given by 2.2.2. Nevertheless the proof of theorem 2.2.2 given in [49] (Lemma 1.1 and 1.4) works also in our case after having noticed that, since it is possible to define an action ρ on $\pi_1(\Sigma)$ of a map in the Sobolev space $H_1^2(\Sigma,M)$, then the induced action $\bar{\rho}$ on $H_1(\Sigma,\mathbb{Z})$ is well defined.

$$\begin{array}{ccc} \pi_1(\Sigma) & \xrightarrow{\rho} & \pi_1(M) \\ \downarrow & & \downarrow \\ H_1(\Sigma,\mathbb{Z}) & \xrightarrow{\bar{\rho}} & H_1(M,\mathbb{Z}) \end{array} \qquad (2.1)$$

□

Remark 2.2.2 *Lemma 1.1 in [49] shows in fact that a sequence with bounded energy* $\{f_i\}$ *always has a subsequence* $\{l_i\}$ *which converges pointwise almost everywhere, and therefore there exists* $i_0 \in \mathbb{N}$ *with the property that* $l_{i*} = l_{j*}$ *on* $\pi_1(\Sigma)$, $\forall i, j \geq i_0$. *This was not a priori required to prove our theorem.*

The content of theorem 2.2.3 is that the first minimizing process works even among maps with a prescribed action on $H_1(\Sigma, \mathbb{Z})$. In order to show that the second procedure also works we need a major result in Riemann surface theory. We just recall that

1. *a Riemann surface with nodes* is a connected complex space $\tilde{\Sigma}$ such that every point p has a neighbourhood isomorphic either to the open disk in the complex plane, or to two disks with centres identified corresponding to p; the connected components of $\tilde{\Sigma}\backslash\{nodes\}$ are called *the parts* of $\tilde{\Sigma}$.

2. Consider a surface Σ and the set of couples (μ, f), where μ is a conformal structure on Σ and $f: \Sigma \to \Sigma$ is a diffeomorphism. We say that (μ, f) is equivalent to (ν, g) if $f \circ g^{-1}: (\Sigma, \nu) \to (\Sigma, \mu)$ is homotopic to a biholomorphic map. The set of equivalence classes $[(\mu, f)]$ is called the *Teichmüller space* $\mathcal{T}(\Sigma)$ with base Σ (see [26] for an introduction to the subject).

3. On $\mathcal{T}(\Sigma)$ there is a canonical group action. Let $Mod(\Sigma)$ be the set of homotopy classes of orientation preserving diffeomorphisms $\phi: \Sigma \to \Sigma$. $Mod(\Sigma)$, called the mapping class group, acts on $\mathcal{T}(\Sigma)$ by pulling back conformal structures on Σ. The quotient $\mathcal{T}(\Sigma)/Mod(\Sigma) = \mathcal{R}(\Sigma)$ is the space of all conformal structures on Σ up to biholomorphic equivalence, and is called the *Riemann moduli space*.

We will make use of the following theorem contained in [1]:

Theorem 2.2.4 *There exists a compactification* $\tilde{\mathcal{R}}(\Sigma)$ *of* $\mathcal{R}(\Sigma)$ *such that the points of* $\tilde{\mathcal{R}}(\Sigma)\backslash\mathcal{R}(\Sigma)$ *correspond to the set of Riemann surfaces with nodes that can be obtained from* Σ *by collapsing to a point a set of admissible (see [1] for a precise explanation of this concept) simple closed curves on* Σ.

Remark 2.2.3 *By the above theorem the equivalence class of a conformal structure μ_∞ s.t. $[\mu_\infty] \in \tilde{\mathcal{R}}(\Sigma)$ can be described as follows: we can see a conformal structure with nodes μ_∞ on a surface Σ as a smooth conformal structure μ on each part of $\Sigma \backslash \cup_{m=1}^{k} \gamma_m$, where γ_m are admissible simple closed curves on Σ. If we have a diffeomorphism $f: \Sigma \to (\Sigma, \mu_\infty)$ we define the conformal stucture with nodes $(f^{-1})^*(\mu_\infty)$ on Σ as the one which has nodes $f^{-1}(\gamma_m)$ and parts the inverse image of the parts of (Σ, μ_∞) with (smooth) conformal structures $(f^{-1})^*(\mu)$.*

Suppose now $\rho: H_1(\Sigma, \mathbb{Z}) \to H_1(M, \mathbb{Z})$ is an injective homomorphism and let us consider the functional $\tilde{\mathcal{E}}_\rho: \mathcal{T}(\Sigma_r) \to \mathbb{R}$ defined by:

$$\tilde{\mathcal{E}}_\rho([(\mu, f)]) = inf\{\mathcal{E}(\psi, \mu) | \psi_* = \rho \circ (f^{-1})_*\} \ .$$

If we consider (ν, g) equivalent to (μ, f) and h a biholomorphic map homotopic to $f \circ g^{-1}$, then, by the conformal invariance of the energy, we have

$$\{\mathcal{E}(\psi, \mu) | \psi_* = \rho \circ (f^{-1})_*\} = \{\mathcal{E}(\psi \circ h, \nu) | (\psi \circ h)_* = \rho \circ (g^{-1})_*\}$$

which shows that $\tilde{\mathcal{E}}_\rho$ is well defined.

In order to prove the main result we need now another theorem due to Sacks and Uhlenbeck ([46]):

Theorem 2.2.5 *Let $s_i: (\Sigma, \mu_i) \to (M, g)$ be harmonic for the conformal structure μ_i on Σ, with $\mu_i \to \mu$ and $\mathcal{E}(s_i, \mu_i) \leq K$. Then there exists a finite set of points $\{p_1, \ldots, p_q\}$ and a subsequence t_j which converges in $C^1(\Sigma \backslash \{p_1, \ldots, p_q\})$ to a map s harmonic w.r.t. μ and $\mathcal{E}(s, \mu) \leq \liminf_{j \to \infty} \mathcal{E}(t_j, \nu_j)$, where $\{\nu_j\}$ is the corresponding subsequence of $\{\mu_i\}$.*

Lemma 2.2.1 *Suppose $[(\mu_l, f_l)]$ is a minimizing sequence for $\tilde{\mathcal{E}}_\rho$ and let $\phi_l: \Sigma \to M$ be such that $\tilde{\mathcal{E}}_\rho([(\mu_l, f_l)]) = \mathcal{E}(\phi_l, \mu_l)$ (ϕ_l exists by Theorem 2.2.3). Then one of the following situations occurs:*

1. *$\exists \{\mu_i\} \subseteq \{\mu_l\}$ and a finite set of points $\mathcal{P}$ s.t. $\mu_i \to \mu$, where μ is a smooth conformal structure on Σ, $\phi_i \to \phi$ in $C^1(\Sigma \setminus \mathcal{P}, M)$ harmonic w.r.t. μ, and*

$\exists i_0 \in \mathbb{N}$ s.t. $f_{i*} = f_{j*}$ on $\pi_1(\Sigma)$, $\forall i, j \geq i_0$. Furthermore $\phi \circ f_{i_0}$ is a branched minimal immersion, conformal w.r.t. $(f_{i_0}^{-1})^*(\mu)$, which minimizes area among all C^1-maps with the same action on $H_1(\Sigma, \mathbb{Z})$ as u.

2. *$\exists \{\mu_i\} \subseteq \{\mu_l\}$ s.t. $\mu_i \to \mu_\infty$, where μ_∞ is a conformal structure with nodes on Σ. Let $\{p_1, \ldots, p_k\}$ and $\{\Sigma_1, \ldots, \Sigma_q\}$ be the nodes and the parts of (Σ, μ_∞) respectively. Then each p_r is obtained by collapsing a homologically trivial curve γ_r; furthermore for every part Σ_m there exists a finite set of points $\mathcal{P}_m$ s.t. $\phi_i \to \phi$ in $C^1(\Sigma \setminus \{\cup_{r=1}^k \{\gamma_r\}, \cup_{m=1}^q \mathcal{P}_m\}, M)$, where ϕ is harmonic on each part of (Σ, μ_∞), and $\exists\, i_0 \in \mathbb{N}$ s.t. $f_{i*} = f_{j*}$ on $\pi_1(\Sigma)$, $\forall i, j \geq i_0$.*

   Furthermore it is possible to extend $\eta = \lim \phi_i \circ f_{i_0}$ to a map $\tilde{\eta}$ defined on the disjoint union of the closures $\bar{\Sigma}_j$ of the parts, which minimizes area among all C^1-maps which induce ρ on $H_1(\Sigma, \mathbb{Z})$.

Proof:

1. Suppose $\mu_i \to \mu$ with μ smooth conformal structure. Then by Theorem 2.2.5 there exists a subsequence of $\{\phi_i\}$ (that we call again $\{\phi_i\}$) such that $\phi_i \to \phi$ harmonic w.r.t. μ away from a finite set of points $\{p_1, \ldots, p_k\}$. By a well known theorem of Sacks-Uhlenbeck ([47]) ϕ can be extended to a harmonic map on Σ that we indicate by $\tilde{\phi}$. Since the convergence is in $C^1(\Sigma \setminus \{\text{finite set of points}\}, M)$ we have that $\exists\, i_0 \in \mathbb{N}$ s.t. $\tilde{\phi}_* = \phi_{i*}$ for $i \geq i_0$. Then $\tilde{\phi}_* = u_* \circ (f_i^{-1})_*$ which proves that the actions on $H_1(\Sigma, \mathbb{Z})$ of the diffeomorphisms f_i stabilize to an isomorphism

$$f_{i_0 *}: H_1(\Sigma, \mathbb{Z}) \to H_1(\Sigma, \mathbb{Z}) \,.$$

Then $\tilde{\phi} \circ f_{i_0}$ has the same action on $H_1(\Sigma, \mathbb{Z})$ as u. Furthermore, suppose that there exists v with the same action on $H_1(\Sigma, \mathbb{Z})$ as u and $\mathcal{A}(v) < \mathcal{A}(\tilde{\phi} \circ f_{i_0})$. Since we can approximate v with immersions (by theorem 2.9 in [25] and because M has dimension greater than 2), we may consider the conformal structure μ_v induced by

ψ on Σ. We would then have $\mathcal{E}(\psi, \mu_\psi) = 2\mathcal{A}(\psi) < 2\mathcal{A}(\tilde{\phi} \circ f_{i_0}) \leq \mathcal{E}(\tilde{\phi} \circ f_{i_0}, f_{i_0}^*(\mu))$ which yields a contradiction.

2. Suppose $\mu_i \to \mu_\infty$, a conformal structure with nodes. Then by Theorem 2.2.4 we have that (Σ, μ_∞) is a Riemann surface with nodes $\{p_1, \ldots, p_k\}$ obtained from Σ by collapsing to p_r a simple closed curve γ_r. Choose for each r a sequence of annular neighbourhoods D_j^r of γ_r s.t. $D_j^r \to p_r$ in $\tilde{\Sigma}$ and the change of conformal structure on Σ from μ_j to μ_{j+1} is restricted to the interior of $\cup_r D_j^r$ (the existence of such D_j^r is assured by a result of Bers [8]). Setting $S_j = \Sigma \setminus \cup_r D_j^r$, then for each j we have a smooth harmonic map $\phi^j: S_j \to (M, g)$ which is the C^1-limit of $\{\phi_{i|S_j}\}$ away from a finite set of points. For $l > j$ it is clear that $S_j \subseteq S_l$ and then ϕ^l extends ϕ^j by unique continuation theorem for harmonic maps (see [48]). Letting $l \to \infty$ we then get a smooth harmonic map $\phi: \Sigma' \to (M, g)$ where $\Sigma' = \Sigma \setminus \{\gamma_1, \ldots, \gamma_k\}$ again as a C^1-limit of ϕ^l away from a finite set of points. Let us consider the harmonic extension $\tilde{\phi}$ of ϕ to $\Sigma' \cup \{(x_1, x_1'), \ldots, (x_k, x_k')\}$, the closed disconnected Riemann surface obtained adding (x_r, x_r') to each node p_r. Since the convergence is C^1 away from a finite set of points and from $\cup_{r=1}^k \{\gamma_r\}$ and since $\tilde{\phi}_*([\gamma_r]) = 0 \,\forall\, r$ we have that $\exists\, i_0 \in \mathbb{N}$ s.t. $\phi_{i*}([\gamma_r]) = 0 \,\forall\, r$ and $\forall\, i \geq i_0$. But $\phi_{i*} = \rho \circ (f_i^{-1})_*$, implies that $[f^{-1}(\gamma_r)] \in \ker\rho$. Since ρ is injective, and f is a diffeomorphism, we have proved that γ_r is homologically trivial for every $r = 1, \ldots, m$. We have then proved that $\tilde{\eta}_* = \rho$, where $\tilde{\eta} = \tilde{\phi} \circ f_{i_0}$. We want now to show that this map minimizes area among all maps with this property. As in the first part of the proof, suppose that there exists ψ with the same action on $H_1(\Sigma, \mathbb{Z})$ as u and $\mathcal{A}(\psi) < \mathcal{A}(\tilde{\eta})$. We would then have, for the conformal structure μ_ψ induced by ψ,

$$\mathcal{E}(\psi, \mu_\psi) = 2\mathcal{A}(\psi) < 2\mathcal{A}(\tilde{\eta}) = \mathcal{E}(\tilde{\eta}, f_{i_0}^*(\mu)) .$$

On the other hand, since ϕ is the C^1-limit of ϕ^l away from a finite set of points and from $\cup_{r=1}^m \{\gamma_r\}$, we have that $\mathcal{E}(\tilde{\eta}, f_{i_0}^*(\mu)) \leq \lim_{l \to \infty} \mathcal{E}(\phi_l, \mu_l)$, which gives a contradiction. □

Lemma 2.2.2 *Let $\tilde{\eta}$ be as in 2.2.1. Then $\tilde{\eta}_{|\overline{\Sigma}_i}$ minimizes area among all maps with its action on $H_1(\overline{\Sigma}_i, \mathbb{Z})$.*

Proof: Let us call for simplicity $\tilde{\eta}_{|\overline{\Sigma}_i} = v$. Suppose suppose that there exists ψ with the same action on $H_1(\overline{\Sigma}_i, \mathbb{Z})$ as v and $\mathcal{A}(\psi) < \mathcal{A}(v)$. By approximating ψ with immersions, we can consider the conformal structure μ_ψ induced on $\overline{\Sigma}_i$ by ψ. We would then have, by theorem 2.2.3, an energy minimizer χ w.r.t. the conformal structure μ_ψ and therefore

$$\mathcal{E}(\chi, \mu_\psi) \leq \mathcal{E}(\psi, \mu_\psi) = 2\mathcal{A}(\psi) < 2\mathcal{A}(v) \leq \mathcal{E}(v, f_{i_0}^*(\mu_{\infty|\overline{\Sigma}_i})) .$$

Let us define $\varepsilon = \mathcal{E}(v, f_{i_0}^*(\mu_{\infty|\overline{\Sigma}_i})) - \mathcal{E}(\chi, \mu_\psi)$. We claim that $\varepsilon > 0$ implies the existence of a smooth map $s: \Sigma \to M$ s.t. $\mathcal{A}(s) < \mathcal{A}(\tilde{\phi} \circ f_{i_0})$ with the same action on $H_1(\Sigma, \mathbb{Z})$ as $\tilde{\phi}$. Lemma 2.2.1 would then give a contradiction. Suppose for simplicity that (Σ, μ_∞) has two parts (then there is necessarly one node, because $H_1(\overline{\Sigma}_1 \cup \overline{\Sigma}_2, \mathbb{Z}) = H_1(\Sigma, \mathbb{Z})$), Σ_1 and Σ_2, and $i = 1$. Then there exists a disk D_1 in $(\overline{\Sigma}_1, \mu_\psi)$ with center at p s.t. $\chi(D_1)$ is contained in a geodesic ball of M, $B(x_0, \frac{\varepsilon}{6})$, and $\mathcal{E}(\chi_{|D_1}, \mu_{\psi|D_1}) < \frac{\varepsilon}{6}$ and a disk D_2 in $(\overline{\Sigma}_2, \mu_{\infty|\overline{\Sigma}_i})$ with center at the node, s.t. $v(D_2)$ is contained in a geodesic ball of M, $B(y_0, \frac{\varepsilon}{6})$, and $\mathcal{E}(v_{|D_2}, f_{i_0}{}^*(\mu_\infty)_{|D_2}) < \frac{\varepsilon}{6}$. Let us call r_1 and r_2 the radii of D_1 and D_2 respectively. Consider in D_1 a complex coordinate z centered at p the disks $U_1^1 = \{z \mid |z| \leq \delta_1\}$ and $U_1^2 = \{z \mid |z| \leq \delta_2\}$, and analogously on D_2, $U_2^1 = \{w \mid |w| \leq \delta_1\}$ and $U_2^2 = \{w \mid |w| \leq \delta_2\}$, with $0 < \delta_1 \leq \delta_2$. Let us call c_i^j the boundary curves of U_i^j. Consider now the Riemann surface

$$S = [(\overline{\Sigma}_1 \backslash U_1^1) \cup (\overline{\Sigma}_2 \backslash U_2^1)] / \sim$$

where $\sim$ is the equivalence relation induced by the map $zw = \delta_1\delta_2$. Geometrically this construction (called by many authors "plumbing" of Riemann surfaces) corresponds to glue the two open Riemann surfaces $\overline{\Sigma}_i \backslash U_i^1$ identifying c_1^1 with c_2^2, and c_2^1 with c_1^2 (see figure below). $D_1 \setminus U_1^2$ in S is conformally equivalent to an annulus

$$A_1 = \{z \mid \delta_2 \leq |z| \leq r_1\}$$

and $D_2 \setminus U_2^2$ is conformally equivalent to an annulus

$$A_2 = \{w \mid \delta_2 \leq |w| \leq r_2\} .$$

Consider in A_1 the curve $c_3^1 = \{z \mid |z| = \delta_3, \delta_3 \in (\delta_2, r_1)\}$ and in A_2 the curve $c_3^2 = \{w \mid |w| = \delta_3, \delta_3 \in (\delta_2, r_2)\}$. We want to construct now two maps $f_1^1: W_1^1 = \{z \mid |z| \in [\delta_2, \delta_3]\} \to M$, and $f_1^2: W_1^2 = \{z \mid |z| \in [\delta_3, r_1]\} \to M$ s.t. $\mathcal{E}(f_1^i, \nu) \leq \frac{\varepsilon}{6}$ for $i = 1, 2$, where ν is the conformal structure on S. Let $\gamma: [0, l] \to M$ be a geodesic joining x_0 and y_0. The function

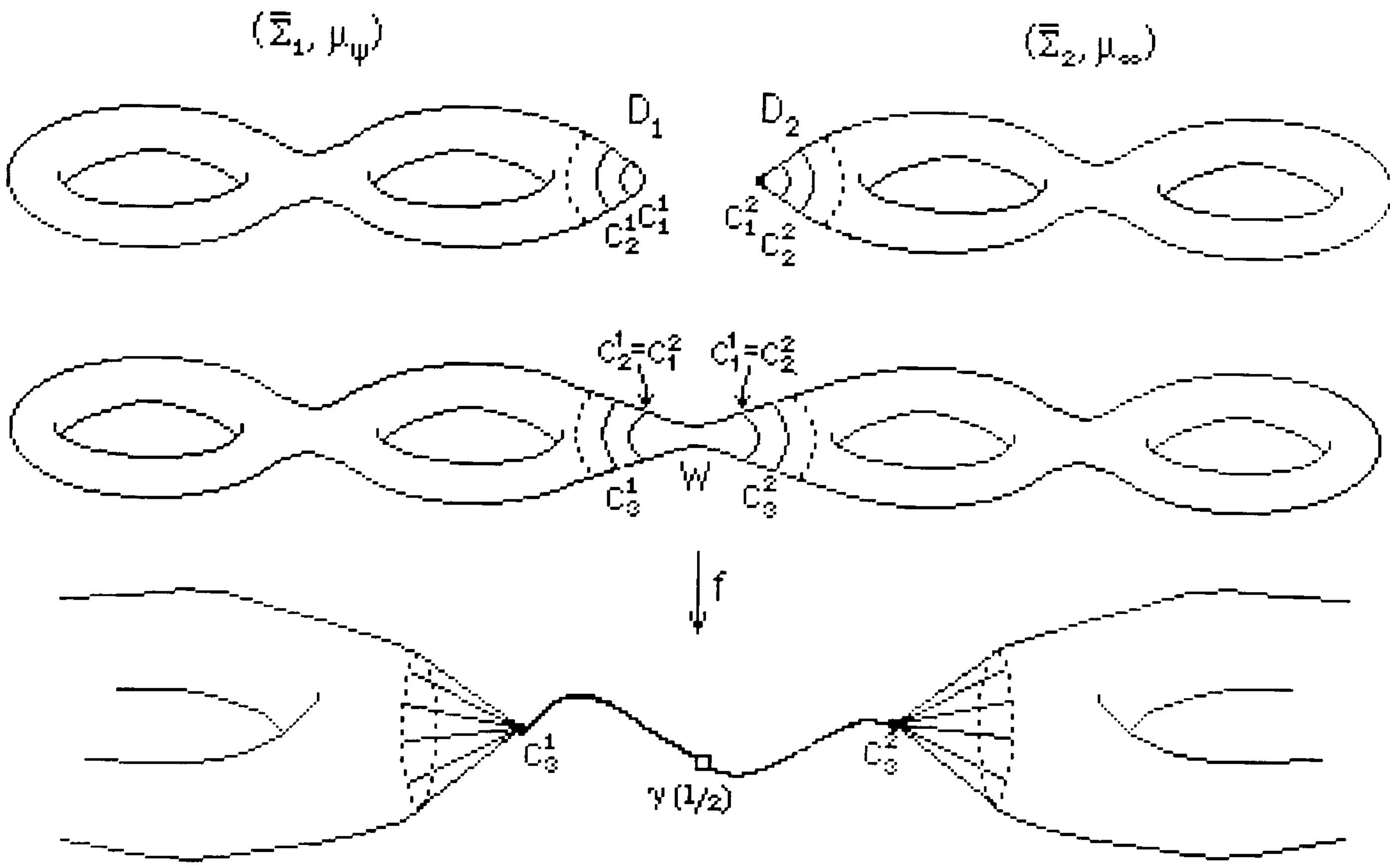
$$f_1^1(r) = \gamma(\frac{l \cdot log(\frac{\delta_3}{r})}{2log(\frac{\delta_3}{\delta_2})})$$

maps c_1^3 on x_0 and c_1^2 on $\gamma(\frac{l}{2})$, and has energy

$$\mathcal{E}(f_1^1, \nu) = 2\pi \frac{dist^2(\gamma(\frac{l}{2}), x_0)}{log(\frac{\delta_3}{\delta_2})} .$$

Then for δ_2 sufficiently small, we have $\mathcal{E}(f_1^1, \nu) \leq \frac{\varepsilon}{6}$. To construct f_1^2 we consider the map $g: W_1^2 \to D_1$ given by $g(\rho e^{i\theta}) = \frac{r_1(\rho - \delta_3)}{r_1 - \delta_3} e^{i\theta}$. g maps the annulus W_1^2 onto D_1 and it is easy to verify that $\mathcal{E}(\chi \circ g, \mu_{\psi|_{W_1^2}}) \leq C\mathcal{E}(\chi, \mu_{\psi|_{D_1}})$, where C is the supremum of all products of partial derivatives of g. A simple calculation gives $C = (\frac{r_1}{r_1 - \delta_3})^2$. Observe that $C = 1 + \lambda$ with $\lambda > 0$ which tends to 0 as δ_3 approaches 0. Finally we map the "plumb" W between c_1^1 and c_2^1 in S constantly on $\gamma(\frac{l}{2})$. Repeating all these constructions on D_2, we can define a map f from S to M in this way:

$$f(x) = \begin{cases} \chi(x) & \text{if } x \in \overline{\Sigma}_1 \backslash D_1 \\ f_i^j(x) & \text{if } x \in W_i^j \\ \gamma(\frac{l}{2}) & \text{if } x \in W \\ \tilde{\phi}(x) & \text{if } x \in \overline{\Sigma}_2 \backslash D_2 \end{cases}$$



f satisfies the following properties:

1. f is a Lipschitz function

2.

$$\mathcal{E}(f,\nu) \le \mathcal{E}(\chi,\mu_\psi) + (1+\lambda)\left(\frac{\varepsilon}{6}+\frac{\varepsilon}{6}\right) + \frac{\varepsilon}{6} + \frac{\varepsilon}{6} + \mathcal{E}(\tilde\eta|_{\bar\Sigma_2 \setminus D_2}, f_{i_0}^*(\mu_\infty)) \le$$

$$\mathcal{E}(\chi,\mu_\psi) + \left(1+\frac{1}{2}\right)\frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \mathcal{E}(\tilde\eta|_{\bar\Sigma_2 \setminus D_2}, f_{i_0}^*(\mu_\infty)) =$$

$$= \mathcal{E}(\chi,\mu_\psi) + \frac{5}{6}\varepsilon + \mathcal{E}(\tilde\eta|_{\bar\Sigma_2 \setminus D_2}, f_{i_0}^*(\mu_\infty)) < \mathcal{E}(\tilde\phi,\mu_\infty)$$

for δ_3 sufficiently small.

It is easy to check that this implies the existence of a C^1-map g homotopic to f s.t. $\mathcal{E}(g,\nu) < \mathcal{E}(\tilde\phi,\mu_\infty)$. This proves the claim.

If (Σ,μ_∞) has more than two parts we can repeat this construction the necessary number of times, obtaining the same contradiction, after having noticed that the intersection of the closures of two parts can be just one point. □

This Lemma clearly concludes the proof of Theorem 2.1.2.

Remark 2.2.4 *There are some observations to be made: the case of injectivity on $\pi_1(\Sigma)$ (i.e. the setting of theorem 2.1.1) is included in case* 1 *of 2.2.1. However, injectivity on $\pi_1(\Sigma)$ is not necessary for case* 1 *of 2.2.1 to arise. For instance, the Jacobi embedding $j: \Sigma_r \to T^{2r}$ is not injective on $\pi_1(\Sigma)$ (since $\pi_1(T^{2r})$ is abelian) but the conformal structure μ of the lemma is clearly smooth. Actually this shows that for any smooth conformal structure there exist infinitely many flat tori (T^{2r}, g), taking any flat metric hermitian w.r.t. the complex structure on the jacobian of Σ, and a map $\Sigma \to (T^{2r}, g)$ which minimizes area among all maps with the same action on $H_1(\Sigma, \mathbb{Z})$ for every g.*

Remark 2.2.5 *As we mentioned in the introduction, Sacks and Uhlenbeck, ([46]), found examples of the failure of uniqueness of area minimizing maps with a fixed injective action on the fundamental group of a surface. A fortiori this means that we cannot hope in general to get uniqueness of area minimizing maps among maps with a fixed injective action on the first homology. Despite this failure of uniqueness, we conjecture the following kind of rigidity for minimal surfaces in flat tori:*

Conjecture 2 *Given any flat torus (T^{2r}, g) and any injective homomorphism ρ from $H_1(\Sigma_r, \mathbb{Z})$ to $H_1(T^{2r}, \mathbb{Z})$, the Riemann surface and the map obtained in Theorem 2.1.2 are unique.*

This conjecture plays the part played by the Torelli Theorem in the theory of holomorphic curves in principally polarized abelian varieties ([4]) as we will see in Chapters 3 *and* 5*. It seems to us particularly intriguing the possibility to reconstruct the conformal structures induced by the area minimizing maps just from two data, as the lattice defining the torus, and the action on the first homology group. Chapter 5 is devoted to the study of the above Conjecture.*

Example: 2.2.1 *It is interesting to analyze in detail what happens to the minimizing process just described in the case of a map which is not injective on the first homology* *group. In this simple example we are able to write explicitly the energy minimizing map, with a fixed (non injective) action on homology, for every fixed conformal structure on the domain, and then to minimize the $\mathcal{E}$-functional. We show that the limiting map is a constant map. Since our starting action on homology was not the zero map, we can conclude that if u collapses a generator of $H_1(\Sigma, \mathbb{Z})$ then the map that we obtain by the second minimizing process collapses another generator, showing that we can not relax our hypothesis in Theorem 2.1.2.*

Our simple example can be constructed as follows: let μ be a complex number of the form $\mu = iL$, with $L \in \mathbb{R}^+$ and let $T_L = \mathbb{C}/\Lambda$ be a one-dimensional complex torus, where Λ is the lattice generated over $\mathbb{Z}$ by $1, iL$. If we consider the standard flat metric on $\mathbb{C}$ and project it onto T_L, we get that T_L is isometric to $S^1_1 \times S^1_L$ where S^1_R is the circle of radius R. Let us call μ_L the conformal structure on T_L obtained in this way. We then consider $T^2 = S^1_1 \times S^1_1$ with the standard metric, and a map $u_L: S^1_1 \times S^1_L \to T^2$ defined by $u_L(\phi, \theta) = (\frac{\theta}{L}, 0, 0)$. This map clearly collapses one of the generators of $H_1(T_L, \mathbb{Z})$. We have that

$$\mathcal{E}(u_L, \mu_L) = \int_{S^1_1 \times S^1_L} \frac{1}{L^2} d\phi d\theta = \frac{4\pi^2 L}{L^2} = \frac{4\pi^2}{L} .$$

This shows that, denoting by ϕ_n any map given by Theorem 2.2.3, $\mathcal{E}(\phi_n, \mu_n) \to 0$ as $n \to \infty$, i.e. as μ_L goes to the boundary of the space of conformal structures of one-dimensional tori. The conformal structure with nodes μ_∞ has precisely one node, obtained by collapsing a simple closed curve representing the kernel of $(u_L)_*$. Removing this point, the topological space remaining is homeomorphic to a sphere with two points removed.

Suppose now $v: S^1_1 \times S^1_L \to T^2$ is a map which induces the same action on $H_1(T_L, \mathbb{Z})$ as u_L. We now claim that $\mathcal{E}(v, \mu_L) \geq \mathcal{E}(u_L, \mu_L)$, with equality iff $v = u_L$ up to translations on T^3: let us call $v(\phi, \theta) = (v_1(\phi, \theta), v_2(\phi, \theta))$. We have then

$$\mathcal{E}(v, \mu_L) = \int_0^{2\pi} \int_0^{2\pi L} [(\frac{\partial v_1}{\partial \theta})^2 + (\frac{\partial v_2}{\partial \theta})^2 + (\frac{\partial v_1}{\partial \phi})^2 + (\frac{\partial v_2}{\partial \phi})^2] d\theta d\phi .$$

We easily have that

$$E(v(\phi,\cdot)) = \int_0^{2\pi L} [(\frac{\partial v_1}{\partial\theta})^2 + (\frac{\partial v_2}{\partial\theta})^2] d\theta \geq \frac{1}{2\pi L}(\int_0^{2\pi L} \frac{\partial v_1}{\partial\theta} d\theta)^2 = \frac{2\pi}{L} .$$

Therefore

$$\int_0^{2\pi}\int_0^{2\pi L} [(\frac{\partial v_1}{\partial\theta})^2 + (\frac{\partial v_2}{\partial\theta})^2 + (\frac{\partial v_1}{\partial o})^2 + (\frac{\partial v_2}{\partial o})^2] d\theta do \geq \frac{4\pi^2}{L}$$

with equality if and only if v differs from u_L by a translation.

The sequence of energy-minimizing maps u_L converges in the C^1-topology to the constant function $\tilde{\phi}(p) = (0,0,0)$, $\forall p \in T_L$ on compact subsets. This agrees with the well known fact that the only harmonic maps from the two sphere to a Riemann surface of genus greater than zero, are the constant functions.

2.3 Minimal surfaces in Abelian Varieties

We will study in detail the case when M is a flat torus $T^{2r}, r > 1$. We will give in the next Chapter a detailed description of the moduli spaces of flat and complex tori. Here we just recall that in this latter space there are two subsets particularly interesting for algebraic geometric reasons. The first is the space of principally polarized tori and the second is the space of jacobians of smooth Riemann surfaces. It is well known that the jacobian locus is contained in the space of principally polarized tori, but the relation between these spaces is, in general, a very hard problem (see for example [4], [5]).

Writing a 2-form ω on T^{2r} w.r.t. a basis of the lattice which defines T^{2r}, we have

$$\omega = \sum_{1\leq i<j\leq 2r} a_{ij} dx_i \wedge dx_j .$$

where x_i are the coordinates given by the lattice and $a_{ij} \in \mathbb{Z}$.

It is a classical fact that the jacobian variety of every Riemann surface carries a natural principal polarization given by the intersection form on the surface. We say that a complex structure J on T^{2r} *is polarized by a 2-form* ω if ω is of type $(1,1)$ w.r.t. J. Furthermore we say that J *is compatible with a riemannian metric* g on T^{2r} if $g(JX, JY) = g(X, Y)\, \forall X, Y \in T(T^{2r})$.

Definition 2.3.1 *A* <u>*principal polarization*</u> ω *of* (T^{2r}, J) *is a 2-form* ω *s.t.* J *is polarized by* ω *and there exists a basis of the lattice w.r.t. which*

$$\omega = \sum_{1 \leq i < j \leq 2r} dx_i \wedge dx_{r+i} .$$

We will use the following results due to Micallef ([37]):

Theorem 2.3.1 *Every full conformal stable minimal immersion of a hyperelliptic Riemann surface into any flat* $2r$*-torus* T^{2r} *is holomorphic w.r.t. some complex structure compatible with the metric on* T^{2r}.

The main scope of this section is to give a new proof, based on our existence result, Theorem 2.1.2, of the following classical results of Algebraic Geometry (see [29]):

Theorem 2.3.2 *If* (T^4, ω) *is a principally polarized abelian surface then either it is the jacobian of a Riemann surface of genus* 2 *or it is the canonically polarized sum of two elliptic curves.*

Theorem 2.3.3 *If* (T^6, ω) *is a principally polarized abelian threefold then either, it is the jacobian of a Riemann surface of genus* 3 *or, it is the canonically polarized sum of three elliptic curves or, the canonically polarized sum of a jacobian of a Riemann surface of genus* 2 *and a elliptic curve.*

Proof of Theorem 2.3.2:

Suppose we have a principal polarization ω on T^4 and a compatible complex structure J. We can then choose a flat metric g s.t. J is compatible with g. Furthermore let us consider the Abel-Jacobi map $j: (\Sigma_2, \mu) \to \mathcal{J}(\Sigma_2, \mu)$ of a Riemann surface of genus 2, where $\mathcal{J}(\Sigma_2, \mu) = (T^4, \bar{J}, \bar{\omega})$ is the jacobian of the Riemann surface. Given a principally polarized 4-torus (T^4, ω) there exists a real linear isomorphism $\phi: T^4 \to T^4$ s.t. $\phi^*(\bar{\omega}) = \omega$ because $\bar{\omega}$ is a principal polarization and both ω and $\bar{\omega}$ can be then written as

$$\omega = \sum_{i=1}^{2} dx_i \wedge dx_{2+i} , \qquad \bar{\omega} = \sum_{i=1}^{2} dx_i \wedge dx_{2+i}$$

w.r.t. some bases of the lattice defining T^4. Of course we will have lost in general any complex property of the composite map $\phi^{-1} \circ j: \Sigma_2 \to (T^4, J, \omega)$ because in general ϕ will be far from holomorphic. The map $u = \phi^{-1} \circ j$ has the property that $u_*: H_1(\Sigma_2, \mathbb{Z}) \to H_1(T^4, \mathbb{Z})$ is an isomorphism: in fact j_* is an isomorphism by construction, and ϕ is a real isomorphism. We can then apply theorem 2.1.2 to the map u. The minimizing process would then give us one of the following possibilities:

1. a map $\tilde{u}: (\Sigma_2, \nu) \to T^4$ which minimizes area among all maps with the same action on $H_1(\Sigma_2, \mathbb{Z})$ as u;

2. two maps $\tilde{u}_i: (\Sigma_1^i, \nu_i) \to T^4$ conformal minimal immersions s.t. the map $\tilde{u}: \Sigma_1^1 \cup \Sigma_1^2 \to T^4$ given by $\tilde{u}(q) = \tilde{u}_i(q)$ if $q \in \Sigma_1^i$, minimizes area among all maps with the same action on $H_1(\Sigma_2, \mathbb{Z})$ as u.

In fact we observe that in general the minimizing procedure could give rise also to minimal spheres in the target manifold; in the case of flat tori it is well known that such maps have to be constants and therefore we get just minimal surfaces of positive genuses.

In case 1 it is easily seen (not necessarly invoking Theorem 2.3.1, as we will show in Chapter 3 Theorem 3.3.3) that such an immersion has to be holomorphic w.r.t. some complex structure $\tilde{J}$ compatible with the metric g. In case 2, using if necessary translations in $\mathbb{R}^4$, since $\tilde{u}_*$ is an isomorphism, and since the only minimal tori of flat tori are 2-dimensional linear subspaces, we have a decomposition

$$T^4 = \tilde{u}(\Sigma_1^1) \oplus \tilde{u}(\Sigma_1^2)$$

in linear subgroups. We now claim that the tangent cone to $\tilde{u}(\Sigma_1^1) \cup \tilde{u}(\Sigma_1^2)$ at the origin is the union of two planes holomorphic w.r.t. the *same* compatible complex structure $\tilde{J}$; we prove this claim in Lemma 2.3.1. Since all compatible complex structures on (T^4, g) are invariant by translations, and $\tilde{u}_i$ are holomorphic at the origin, they have to be holomorphic at every point. This shows that the decomposition above realizes $(T^4, \tilde{J})$ as the sum of two elliptic curves.

Our next claim is that in both cases $\tilde{J} = \pm J$. Suppose again we are in case 1. We prove that (T^4, J, ω) is isomorphic to the jacobian of (Σ_2, ν): we use a theorem due to Calabi ([10]) which states that, given a flat metric g on a torus and a two-form ω, there exists a unique complex structure compatible with g and which makes ω a form of type $(1,1)$. In our case we have that both $\tilde{J}$ and J are compatible with g and by assumption J is polarized by ω. On the other hand, once we see ω as a pairing between vectors of the lattice defining T^4, we have by construction that $\tilde{u}^*(\omega) = \chi$ where χ is the intersection form on Σ_2. The universal property of the Abel-Jacobi map (Theorem 3.3.1) implies then that $(T^4, \tilde{J}, \omega)$ is the jacobian of (Σ_2, ν) and then $\tilde{J}$ is polarized by ω and then we have $\tilde{J} = \pm J$ and the claim follows directly.

The same argument applied in case 2 to each $\tilde{u}_i$ shows directly that in this case (T^4, J) is biholomorhic to $\tilde{u}_1(\Sigma_1^1) \oplus \tilde{u}_2(\Sigma_1^2)$ and that also the principal polarization splits as the sum of the canonical polarizations of the two elliptic curves. □

Remark 2.3.1 *There are classical examples of jacobians of Riemann surfaces of genus 2 which are biholomorphic to the sum of two elliptic curves but we don't get these elliptic curves from the procedure just described since they don't represent the principal polarization which makes this sum a jacobian of a smooth Riemann surface.*

A similar strategy as above works in the case of complex dimension 3. First we need to know that there aren't full minimal conformal immersions of nonhyperelliptic Riemann surfaces in (T^6, g) different from the holomorphic ones and then using Theorem 2.3.1 for the hyperelliptic case. We need the following result which will be proved in the following Chapter (see Theorem 3.3.2):

Theorem 2.3.4 *If $f: \Sigma_3 \to (T^6, g)$ is a stable minimal immersion then it is holomorphic w.r.t. some complex structure compatible with the metric g.*

Proof of Theorem 2.3.3:

We can apply the same argument as in the 4-dimensional case with the extra care due to the fact that a Riemann surface can be pinched by our minimizing process in such a way as to obtain a conformal stable minimal immersion of one of the following:

1. a Riemann surface of genus 3. or

2. a Riemann surface of genus 1 and a Riemann surface of genus 2. or

3. three Riemann surfaces of genus 1.

In case 1 and 3, by Theorems 2.3.4 and 2.3.1. the same argument as in the 4-dimensional case gives either a jacobian of a Riemann surface of genus 3. or the polarized product of three elliptic curves. The second case needs some extra care; first we observe that, having called again $\tilde{u}$ the area minimizing map. $\tilde{u}(\Sigma_2)$ is contained in a 4-dimensional subtorus T^4 of T^6. because the action of $\tilde{u}$ is injective on $H_1(\Sigma_2.\mathbb{Z})$. Therefore, as in Theorem 2.3.2. we know that $\tilde{u}(\Sigma_2)$ is holomorphic w.r.t. some complex structure compatible with the flat metric. By translating the surfaces in the torus, we can assume that $\tilde{u}(\Sigma_2)$ and $\tilde{u}(\Sigma_1)$ intersect at the origin and, since branch points are isolated, we can also assume that the origin is not a branch point of these surfaces.

Let now J_2 be a complex structure on T^4 compatible with the metric $g_{|_{T^4}}$. such that $\tilde{u}\colon \Sigma_2 \to (T^4.J_2)$ is holomorphic.

We first observe that under our assumptions there has to exist a point $p \in \tilde{u}(\Sigma_2)$ s.t., having called P_3 the plane $T_p(\tilde{u}(\Sigma_2))$ translated to the origin, and P_2 the plane $T_o(\tilde{u}(\Sigma_2))$, $P_2 \oplus P_3 = \mathbb{R}^4$: in fact since $\tilde{u}$ is holomorphic w.r.t. J_2 we have that $T_p(\tilde{u}(\Sigma_2)) \cap T_o(\tilde{u}(\Sigma_2))$ is an even dimensional subspace of $\mathbb{R}^4$; moreover it can not be equal to P_2 for all p because $\tilde{u}$ is a full immersion. We then have a splitting

$$T_o^6 = P_1 \oplus P_2 \oplus P_3 \ .$$

Lemma 2.3.1 implies the existence of a complex structure on $span(P_1.P_2)$. compatible with the metric $g_{|_{span(P_1,P_2)}}$. s.t. P_1 and P_2 are complex lines, and the same for the couple $P_1.P_3$. On the other hand J_2 is a complex structure compatible with $g_{|_{span(P_2,P_3)}}$. s.t. P_2 and P_3 are complex lines. This easily implies that there exists a complex structure,

$\tilde{J}$, on T^6, compatible with the metric and s.t. $\tilde{u}(\Sigma_2)$ and $\tilde{u}(\Sigma_1)$ are holomorphic w.r.t. this complex structure. Repeating at this point the same proof as in Theorem 2.3.2, we can conclude the proof also in this case. $\square$

Therefore next lemma concludes the proof of the previous theorems:

Lemma 2.3.1 *Suppose $\tilde{u}: (\Sigma^1_{i_1} \cup \Sigma^2_{i_2}) \to (M, g)$ (notation as in Theorem 2.1.2) is area minimizing and $dim M \geq 4$. Suppose further that $\exists p_j \in \Sigma^j_{i_j}, j = 1, 2$ s.t. $\tilde{u}(p_1) = \tilde{u}(p_2) = q$ and $\tilde{u}$ is an immersion at p_1 and p_2. Then the planes $\Pi_j = \tilde{u}_{j*}(T_{p_j}\Sigma^j_{i_j})$ are simultaneously holomorphic with respect to a complex structure on $\Pi = span\{\Pi_1, \Pi_2\}$ which is orthogonal w.r.t. $g_{|\Pi}$.*

Proof: This Lemma would be an immediate consequence of Corollary 4 in [40] if $\tilde{u}(\Sigma^1_{i_1} \cup \Sigma^2_{i_2})$ were area minimizing *among currents*. However we do not know this. Nevertheless we can still appeal to the following result which is contained in the proof of Theorem 2 in [40]: let B_1 and B_2 be a pair of flat 2-disks in $(\mathbb{R}^4, eucl)$ of radius r, which intersect at q. In [40] Morgan constructs a map f from an annulus A into the ball of $\mathbb{R}^4$ centered at q of radius r, s.t. $\partial f(A) = \partial B_1 \cup \partial B_2$ and $\mathcal{A}(f(A)) \leq \mathcal{A}(B_1) + \mathcal{A}(B_2) - \epsilon r^2$, $\epsilon > 0$, unless B_1, B_2 are simultaneously holomorphic w.r.t. an orthogonal complex structure in $\mathbb{R}^4$. Let us now consider the ball $B^M(r)$ of radius r in $T_{\tilde{u}_1(p_1)}M$, and let $B_j(r)$ be $B^M(r) \cap \Pi_j$. For r sufficiently small we can also assume that $\tilde{u}_i^{-1}$ exists on $exp_{\tilde{u}_i(p_i)}(B_i(r))$.

Let us also denote by $\mathbb{R}^4$ the linear span of Π_1, Π_2 in $T_{\tilde{u}_1(p_1)}M$.

If $B_i(r)$ are not simultaneously holomorphic w.r.t. an orthogonal complex structure, then we have a map

$$f : A \to \mathbb{R}^4 \subset T_{\tilde{u}_1(p_1)}M$$

defined by Morgan using $B_1(r)$ and $B_2(r)$.

To simplify the notation let us also define

$$S_j = \Sigma^j_{i_j} \setminus \tilde{u}_j^{-1}(exp_{\tilde{u}_j(p_j)}(B_j(r))) .$$

We can then define a map $F: S_1 \cup S_2 \cup A \to M$ by

$$F_r(x) = \begin{cases} \tilde{u}_j(x) & \text{if } x \in S_j \\ \\ exp_{\tilde{u}_1(p_1)}(f(x)) & \text{if } x \in A \end{cases}$$

It is easily seen that

- F induces a Lipschitz function on the connected sum of $\Sigma^1_{i_1}$ and $\Sigma^2_{i_2}$

- $\mathcal{A}(F_r(S_1 \cup S_2 \cup A)) =$
$= \mathcal{A}(\tilde{u}_1(\Sigma^1_{i_1})) + \mathcal{A}(\tilde{u}_2(\Sigma^2_{i_2})) + \mathcal{A}(F_r(A)) - \mathcal{A}(\tilde{u}_1(\Sigma^1_{i_1} \setminus S_1)) - \mathcal{A}(\tilde{u}_2(\Sigma^2_{i_2} \setminus S_2))$
$\leq \mathcal{A}(\tilde{u}(\Sigma^1_{i_1} \cup \Sigma^2_{i_2})) + 2\pi r^2 + O(r^3) - \epsilon r^2 - (2\pi r^2 + O(r^3)) =$
$= \mathcal{A}(\tilde{u}(\Sigma^1_{i_1} \cup \Sigma^2_{i_2})) + O(r^3) - \epsilon r^2.$

Therefore, for r sufficiently small, we have a Lipschitz map, and therefore also a $\mathcal{C}^1$-map, that we call F, from a smooth Riemann surface $\Sigma_{i_1+i_2}$ to (M, g) s.t.

- F induces the same action on homology as u;

- $\mathcal{A}(F) < \mathcal{A}(\tilde{u})$,

hence getting a contradiction with Theorem 2.1.2. □

The problem of recognizing jacobians among principally polarized abelian varieties by means of minimal surface theory seems harder to settle because of Theorem 3.4.1, which will be proved in Chapter 3.

Of course Theorem 3.4.1 does not imply that we can't obtain any result, but we would need some analogue of Micallef's results (2.3.1) restricting ourselves to the homology classes which come from principal polarizations since the general result is false by 3.4.1. This will be the subject of further investigations.

Chapter 3

On Stable Minimal Surfaces in Flat Tori

3.1 Introduction

We consider a topological surface Σ_r, a torus $T^{2r} = S^1 \times \ldots \times S^1$, and an isomorphism ρ from $H_1(\Sigma_r, \mathbb{Z})$ to $H_1(T^{2r}, \mathbb{Z})$.

Let us first observe that ρ give rise to a unimodular 2-form on T^{2r} in the following way: consider a symplectic basis for $H_1(\Sigma_r, \mathbb{Z})$, $\{\alpha_i, \beta_i\}$, $i = 1, \ldots, r$, and define

$$\omega_\rho = \sum_{i=1}^{r} \rho(\alpha_i)^* \wedge \rho(\beta_i)^* \ .$$

where * denotes the canonical isomorphism between $H_1(T^{2r}, \mathbb{Z})$ and $H^1(T^{2r}, \mathbb{Z})$.

A crucial simple observation is that if $u \colon \Sigma_r \to T^{2r}$ is a map inducing ρ on homology, then

$$u^*(\omega_\rho) = \lambda_{\Sigma_r} \ .$$

where λ_{Σ_r} is the intersection form on the surface.

We want to study the spaces:

$\mathcal{R}^{Jac} = \{$flat metrics g on T^{2r} | there exists a complex structure J compatible with

g s.t. (T^{2r}, J, ω_ρ) is the jacobian of some smooth Riemann surface of genus $r\}$

$$\mathcal{R}^{\mu} = \{\text{flat metrics } g \text{ on } T^{2r} \mid \text{there exists a conformal stable minimal immersion}$$
$$\phi: (\Sigma_r, \mu) \to (T^{2r}, g) \text{ such that } \phi_* = \rho \text{ and } \phi \text{ is not holomorphic w.r.t. any}$$
$$\text{complex structure compatible with } g\} \ .$$

In this Chapter we want to study the following problem:

Problem 4 *For which μ is $\mathcal{R}^{\mu} \neq \emptyset$?*

The following easy consequence of the Universal Property of the Abel-Jacobi map (Theorem 3.3.1) suggests to study $\mathcal{R}^{Jac}$ in the moduli space of flat structures on the torus:

Proposition 3.1.1 *Let $u: (\Sigma_r, \mu) \to (T^{2r}, g)$ be a map s.t. u is holomorphic w.r.t. a compatible complex structure J, and $u_* = \rho$; then we have*

1. *(T^{2r}, J, ω_ρ) is isomorphic, as principally polarized abelian variety, to $\mathcal{J}(\Sigma_r, \mu)$.*

2. *J is the* unique *complex structure compatible with g and which is positively polarized by ω_ρ.*

As it is well known, the problem of recognizing jacobians among principally polarized abelian varieties is a very hard one (see for example [5] and [29]). We believe it would be very interesting to study whether it is possible to give a riemannian characterization of $\mathcal{R}^{Jac}$ in the moduli space of flat structures.

The second section is devoted to a discussion of moduli spaces of flat, complex, Kähler, and polarized structures on the torus.

A direct dimensional count shows that given any ω_ρ, the set of flat metrics which admit a compatible complex structure J s.t. (T^{2r}, J, ω_ρ) is the jacobian of some Riemann surface has (real) dimension $6r - 6 + r^2$, while the space of flat structures has dimension $2r^2 + r$. Therefore $\mathcal{R}^{Jac}$ is not the whole space of flat structures for $r \geq 4$.

Proposition 3.1.1 and Theorem 2.1.2 in Chapter 2 suggest the following (mentioned also in [34]):

Conjecture 3 1. *For $r = 2,3$, every stable minimal immersion $\Sigma_r \to (T^{2r}, g)$ is holomorphic w.r.t. some compatible complex structure.*

2. *For $r \geq 4$ and for any isomorphism ρ, there exists a flat metric and a stable minimal immersion $\phi: \Sigma_r \to (T^{2r}, g)$ such that $\phi_* = \rho$ and ϕ is not holomorphic w.r.t. any complex structure compatible with g.*

We shall discuss these guesses in section 3, 4 and 5.

A fruitful way to study these problems is to look at the conformal structures induced by stable minimal immersions.

By a mentioned result of Micallef (Theorem 2.3.1) we know that if such a map induces a hyperelliptic conformal structure, then it has to be holomorphic w.r.t. some compatible complex structure. When the surface inherits a nonhyperelliptic structure the same conclusion for $r = 3$ (also for unstable maps) follows from the fact that there aren't non trivial quadrics containing the canonical image of the Riemann surface. The same argument answers affirmatively the above guess for $r = 2$ without using Theorem 2.3.1. On the other hand since there are non trivial quadrics containing the rational normal curve in $\mathbb{C}P^2$, which is the canonical image of any hyperelliptic Riemann surface we have to appeal to Theorem 2.3.1 for $r = 3$. This also implies the existence of unstable minimal maps $\Sigma_3 \to (T^6, g)$ inducing hyperelliptic structures on Σ_3 (see Remark 3.3.2).

Theorem 3.4.1, which we will prove in Section 4, completes the list of possibilities in terms of induced conformal structures.

A natural question is then whether some flat T^4 or T^6 could contain stable minimal surfaces, non holomorphic w.r.t. any compatible complex structure (of course of genus higher than 2 or 3). By a result of Micallef ([36]) mentioned in the introduction (Theorem 1.2.6) we know that this cannot happen in T^4.

The main idea of the proof of Theorem 3.4.1 is to take an holomorphic map, given by the Abel-Jacobi map into the Jacobian of the Riemann surface, and to deform the map and the torus destroying holomorphicity but not stability and conformality. A crucial step in the proof of Theorem 3.4.1 is to find a connection between a classical algebraic geometrical property of nonhyperelliptic Riemann surface, given by Noether's Theorem and the space of Jacobi fields of the Abel-Jacobi map. Noether-type theorems have been the subject also of very recent investigation. In fact using a nice result of Colombo-Pirola ([14]) and of Gieseker ([21]) we show that the proof of Theorem 3.4.1 can be adapted to prove the following result:

Theorem 3.1.1 *1. For $r \geq 7$, there exists a dense subset $\mathcal{D}_1^r$ of the moduli space of Riemann surfaces of genus r, s.t. if $\mu \in \mathcal{D}_1^r$, then there exists a conformal stable minimal immersion $f: (\Sigma_r, \mu) \to (\mathbb{R}^{2(r-1)}/\Lambda, eucl)$ into a flat torus, which is not holomorphic w.r.t. any compatible complex structure.*

2. For $r \geq 9$, there exists a dense subset $\mathcal{D}_2^r$ of the moduli space of Riemann surfaces of genus r, s.t. if $\mu \in \mathcal{D}_2^r$, then there exists a conformal stable minimal immersion $f: (\Sigma_r, \mu) \to (\mathbb{R}^{2(r-2)}/\Lambda, eucl)$ into a flat torus, which is not holomorphic w.r.t. any compatible complex structure.

3. For $r \geq 12$, there exists a dense subset $\mathcal{D}_3^r$ of the moduli space of Riemann surfaces of genus r, s.t. if $\mu \in \mathcal{D}_3^r$, then there exists a conformal stable minimal immersion $f: (\Sigma_r, \mu) \to (\mathbb{R}^{2(r-3)}/\Lambda, eucl)$ into a flat torus, which is not holomorphic w.r.t. any compatible complex structure.

We refer to section 4 for a discussion about the geometry of the sets $\mathcal{D}_k^r$, and in particular about the existence of families of Riemann surfaces in these sets.

In the same section we show that the strategy of the proofs of Theorems 3.4.1 and 3.1.1 can not be adapted to prove the existence of stable minimal immersions in flat tori of dimension 6.

We are actually convinced that the method used to prove theorem 3.1.1 can be adapted to prove similar results in the case of Σ_r in $T^{2(r-k)}$ with $r - k > 3$.

3.2 Moduli of Tori

We shall be interested in 3 different geometric structures on a torus $T^{2n} = \mathbb{R}^{2n}/\mathbb{Z}^{2n} = S^1 \times \ldots \times S^1$, namely a constant complex structure, a flat structure and a Kähler structure. In this section we describe these structures and the relations between them.

3.2.1 Complex Tori

The space of complex tori can be described in two ways, differential geometrically and complex analytically. In this subsection we will give these two classical descriptions and relate them.

- We start with the differential geometrical point of view. In this description a complex torus is the differentiable manifold T^{2n} together with a constant complex structure J on its tangent bundle (where J constant means $dJ = 0$; from now on we will drop the adjective *constant*).

  The space $\mathcal{C}_n$ of complex structures on $\mathbb{R}^{2n}$ can be seen to be the homogeneous space $Gl(2n,\mathbb{R})/Gl(n,\mathbb{C})$ as follows: let J_0 be the standard complex structure on $\mathbb{R}^{2n}$, and consider the map from $Gl(2n,\mathbb{R})$ to $\mathcal{C}_n$, defined by:

  $$A \xmapsto{\phi} A^{-1}J_0A \ .$$

  This map is surjective because for any complex structure we can find a basis of $\mathbb{R}^{2n}$ w.r.t. which it can be represented by J_0. The map ϕ induces an equivalence relation on $Gl(2n,\mathbb{R})$ by $A \sim B$ if and only if $\phi(A) = \phi(B)$. Each equivalence class of this equivalence relation is in $1-1$ correspondence with the invertible matrices which commute with J_0; this space can be identified with $Gl(n,\mathbb{C})$, where an element $B = B_1 + iB_2$ gets identified with $\begin{pmatrix} B_1 & B_2 \\ -B_2 & B_1 \end{pmatrix}$.

  Two complex tori (T^{2n}, J) and (T^{2n}, J') are biholomorphic if and only if there exists $M: \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ s.t. $\{M\mathbb{Z}^{2n}\} = \{\mathbb{Z}^{2n}\}$ and $MJ = J'M$, i.e. $M \in Sl(2n,\mathbb{Z})$ and $MA^{-1}J_0A = B^{-1}J_0BM$. This is equivalent to $B = LAM^{-1}$ for some $L \in$

$Gl(n,\mathbb{C})$ and $M \in Sl(2n,\mathbb{Z})$. We can therefore conclude that the space of complex tori is

$$Gl(n,\mathbb{C})\backslash^{Gl(2n,\mathbb{R})}/_{Sl(2n,\mathbb{Z})} \tag{3.1}$$

with the actions just described.

- In the theory of several complex variables and Algebraic Geometry, a complex torus is defined as $\mathbb{C}^n/\Lambda$, where Λ is a lattice of maximal rank. We can associate to Λ a matrix of $Gl(2n,\mathbb{R})$ by expressing a basis of Λ w.r.t. a basis of $\mathbb{C}^n = \mathbb{R}^{2n}$. We call this matrix Λ again. Of course such a matrix is well defined up to the action on the right of $Sl(2n,\mathbb{Z})$ which corresponds to picking a different basis of the lattice. Two complex tori, $\mathbb{C}^n/\Lambda$ and $\mathbb{C}^n/\Lambda'$, are biholomorphic if there exists a linear map $L:\mathbb{C}^n \to \mathbb{C}^n$ s.t. $[L(\Lambda)] = [\Lambda']$ ($[\ ]$ is the class in $Gl(2n,\mathbb{R})/Sl(2n,\mathbb{Z})$). Thus, the space of complex tori is, once again, described by

$$Gl(n,\mathbb{C})\backslash^{Gl(2n,\mathbb{R})}/_{Sl(2n,\mathbb{Z})} \tag{3.2}$$

It is very clear from the above descriptions that a torus $(\mathbb{R}^{2n}/\Lambda, J_0)$ as seen in complex analytical point of view, corresponds to the torus $(\mathbb{R}^{2n}/\mathbb{Z}^{2n}, \Lambda^{-1}J_0\Lambda)$ in the first picture.

3.2.2 Flat Tori

As explained in the introduction we are interested in Ricci-flat metrics over tori, but since every Ricci-flat riemannian metric on a torus is invariant by translations, we can identify the space of Ricci-flat metrics on T^{2n} with the space of constant flat metrics on $\mathbb{R}^{2n}$.

Again we want to distinguish two ways of proceeding. In the following subsections the relevance of this distinction will appear evident.

- Let T^{2n} be $\mathbb{R}^{2n}/\mathbb{Z}^{2n}$ and $\mathcal{R}_n$ be the space of flat metrics on $\mathbb{R}^{2n}$. By analogy with the case of complex tori, we define a map from $Gl(2n,\mathbb{R})$ to $\mathcal{R}_n$ by

$$A \overset{\psi}{\longmapsto} A^t A \ .$$

The map ψ is onto because if G is the matrix representation of any flat metric w.r.t. any basis of $\mathbb{R}^{2n}$, and A is the matrix relating any basis orthonormal w.r.t. G with the fixed basis, we have $G = A^t A$. The freedom of choosing the orthonormal basis gives directly that each equivalence class of the equivalence relation induced by ψ (defined as in the case of complex tori) is in one to one correspondence with $Gl(2n,\mathbb{R})/O(2n,\mathbb{R})$.

We then get an identification of $\mathcal{R}_n$ with $Gl(2n,\mathbb{R})/O(2n,\mathbb{R})$. Fix now a basis of $\mathbb{Z}^{2n}$ and a metric g on $\mathbb{R}^{2n}$. Let G be the matrix representation of g w.r.t. the fixed basis (again defined up to the action of $SL(2n,\mathbb{Z})$). Now (T^{2n},g) is isometric to (T^{2n},g') if and only if $\exists\ M: T \to T'$ s.t. $M^t G' M = G$, i.e. $M^t B^t B M = A^t A$, and hence $B = OAM$ with $O \in O(2n,\mathbb{R})$ and $M \in SL(2n,\mathbb{Z})$.

- Consider now $T^{2n} = \mathbb{R}^{2n}/\Lambda$ and g_0 the standard metric on $\mathbb{R}^{2n}$. In this case $(\mathbb{R}^{2n}/\Lambda, g_0)$ is isometric to $(\mathbb{R}^{2n}/\Lambda, g_0)$ if and only if there exists $M: \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ s.t. $M^t M = Id$ and (after having fixed a basis of Λ) $[M(\Lambda)] = [\Lambda']$ ($[\,]$ is the class in $Gl(2n,\mathbb{R})/Sl(2n,\mathbb{Z})$ as in the case of complex tori).

  Therefore we get once again:

$$O(2n,\mathbb{R})\backslash Gl(2n,\mathbb{R}) / Sl(2n,\mathbb{Z}) \tag{3.3}$$

The way to pass from one point of view to the other is given by the following:

$$(\mathbb{R}^{2n}/\Lambda, g_0) = (\mathbb{R}^{2n}/\mathbb{Z}^{2n}, g) \text{ with } G = \Lambda^t\Lambda \ .$$

3.2.3 Kähler Tori

We want now to study the space $\mathcal{K}_n$ of Kahler structures on $\mathbb{R}^{2n}$. A Kähler structure is given by a complex structure J and a flat metric g which is hermitian w.r.t. J. Since we have described complex structures and flat structures in different ways it is clear that also for Kähler structures we may take two different points of view. The descriptions of the two possibilities are very similar to the ones given above. and therefore we just indicate the main ideas.

- Consider the map $\phi: Gl(2n.\mathbb{R}) \to \mathcal{K}_n$ defined by

$$A \mapsto (g.J) = (A^tA.A^{-1}J_0A) .$$

  1. ϕ is surjective because every complex structure compatible with g is of the form $O^{-1}JO, O \in O(2n,\mathbb{R})$, for any J compatible with g.
  2. the equivalence classes of the equivalence relation induced by ϕ are given by $O(2n,\mathbb{R}) \cap Gl(n,\mathbb{C}) = U(n)$.

  Therefore $(\mathbb{R}^{2n}/\mathbb{Z}^{2n}.g,J)$ and $(\mathbb{R}^{2n}/\mathbb{Z}^{2n}.g'.J')$ are equivalent as Kähler manifolds if and only if $\exists\ M:\mathbb{R}^{2n} \to \mathbb{R}^{2n}$ s.t. $\{M\mathbb{Z}^{2n}\} = \{\mathbb{Z}^{2n}\}$ and $MA^{-1}J_0A = B^{-1}J_0BM$, and $M^tA^tAM = B^tB$, that we have seen to imply

  1. $M \in Sl(2n,\mathbb{Z})$
  2. $B = LAM = OAM$.

  Therefore the moduli space of Kähler structures on the torus is given by:

$$U(n)\backslash Gl(2n,\mathbb{R})^+ / Sl(2n,\mathbb{Z}) \tag{3.4}$$

- On the other hand if we adopt the algebraic geometric point of view in the description of complex tori, it is more natural to define a Kähler torus as $(\mathbb{C}^n/\Lambda, g_0)$. In this case the equivalence of Kähler structures corresponds to the existence of $L:\mathbb{C}^n \to \mathbb{C}^n$ s.t. $L^tL = Id$ and $[L(\Lambda)] = [\Lambda']$ (where $[\ ]$ is the class in $Gl(2n.\mathbb{R})/Sl(2n.\mathbb{Z})$ as in the case of complex tori). Therefore we get again:

$$U(n)\backslash Gl(2n.\mathbb{R})^+ / Sl(2n.\mathbb{Z}) \tag{3.5}$$

  Once again we directly see that

$$(\mathbb{C}^n/\Lambda, g_0) = (\mathbb{R}^{2n}/\mathbb{Z}^{2n}, J, g) \text{ with } G = \Lambda^t\Lambda \text{ and } \Lambda^{-1}J_0\Lambda = J .$$

3.2.4 Polarized Tori

Given a two form on a complex torus, it is a classical problem whether this gives rise to an embedding of the torus into the projective space such that in cohomology the two form is equivalent to the restriction of a multiple of a generator of $H^2(\mathbf{P}\mathbb{C}^n, \mathbb{Z})$. This question can be answered in a very satisfactory way using Kodaira embedding theorem. If Q is the fixed form, then Q gives an embedding if and only if Q is of type $(1,1)$, is positive definite and represents an integral class. These are the Riemann relations for the complex torus and the form Q. In the case Q is of type $(1,1)$ w.r.t. a complex structure J, we say that Q *polarizes* J. A very natural question to ask is then which is the space of complex structures polarized by a fixed 2-form. It is easily seen that this is the case if and only if $J^tQ = M$ is symmetric (these are part of the so-called *Real Riemann Relations*). For the rest of this section we assume that Q is also non degenerate.

We also say that Q is a *positive polarization* w.r.t. J if Q polarizes J and J^tQ is positive definite.

Let us define $\mathcal{A}_Q$ to be the space of complex structures positively polarized by such Q. We do not try to describe in different ways this space, since it is a classical subject of Algebraic Geometry (see [29] chap.8). We just remark that $dim_{\mathbb{R}} \mathcal{A}_Q = n^2 + n$.

3.2.5 Geometry of the moduli spaces

We want now to give a geometric description of the moduli spaces of complex and flat tori, using the results of the previous subsections.

Let us first define the space $\mathcal{H}_n$ of hermitian matrices and the space $\mathcal{O}_n$ of complex structures compatible with a fixed metric on $\mathbb{R}^{2n}$:

$$\mathcal{H}_n = Gl(n,\mathbb{C})/U(n) \ , \mathcal{O}_n = O(2n,\mathbb{R})/U(n) \ .$$

Therefore we have the following picture:

$$\begin{array}{ccccc} \mathcal{H}_n & & & & \mathcal{O}_n \\ & \searrow & & \swarrow & \\ & & \mathcal{K}_n & & \\ & \swarrow & & \searrow & \\ \mathcal{C}_n & & & & \mathcal{R}_n \end{array} \tag{3.6}$$

A direct dimensional count shows that $dim_{\mathbb{R}}\mathcal{C}_n = 2n^2$ $dim_{\mathbb{R}}\mathcal{K}_n = 3n^2$, $dim_{\mathbb{R}}\mathcal{O}_n = n^2 - n$, $dim_{\mathbb{R}}\mathcal{R}_n = 2n^2 + n$ and $dim_{\mathbb{R}}\mathcal{H}_n = n^2$, but we recall that, by a theorem of Siegel (see Kodaira-Spencer [28]), $\mathcal{C}_n/Sl(2n, \mathbb{Z})$ is not even a topological manifold for $n \geq 2$.

In what follows we are going to disregard the action of $Sl(2n, \mathbb{Z})$ on the above spaces, since, as we will see later in this Chapter, our problems about minimal surfaces will deal with marked *tori.*

Let us first study the space $\mathcal{C}_n$. Let Q be a non degenerate integral 2-form on $\mathbb{R}^{2n}$. For each complex structure J positively polarized by Q, we define F_J to be the set of complex structures $\tilde{J}$ s.t. $\tilde{J}$ is compatible with the (unique) metric g, s.t. (g, J, Q) is a Kähler triple.

Proposition 3.2.1 *1. $\mathcal{C}_n = \cup_{J \in \mathcal{A}_Q} F_J$,*

2. $J \neq J' \Rightarrow F_J \cap F_{J'} = \emptyset$,

3. For each J, $F_J \cap \mathcal{A}_Q = \{J\}$, and the intersection is transversal.

Proof:

1. For any $J' \in \mathcal{C}_n$, consider any metric g hermitian w.r.t. J'. By a theorem of Calabi ([10]) the couple (g, Q) gives rise to a complex structure $J_{g,Q}$ compatible with g and which positively polarizes Q. Therefore $J' \in F_{J_{g,Q}}$.

2. A complex structure $\tilde{J} \in F_J \cap F_{J'}$ has to be compatible with two metrics g and g' s.t. (g, J, Q) and (g', J', Q) are Kähler triples. As we have seen in the previous subsections, this implies that $J = O^t J' O$, for some orthogonal matrix

O. Therefore J and J' are both compatible with g (and g') and then, by the uniqueness part of Calabi's Theorem, $J = J'$.

3. We are going to find canonical models for the tangent spaces of these spaces using the discussion above. First we need to observe that a variation $J(t)$ of complex structures can be given by a family of $\mathbb{C}$-linear map $\theta_t : T^{0,1} \to T^{1,0}$, where these are the eigenspaces of $J(0)$, and $T_t^{0,1}$ are the $-\sqrt{-1}$-eigenspace of $J(t)$: θ_t can be defined by the formula $T_t^{0,1} = \{L + \theta_t(L) \mid L \in T^{0,1}\}$. Choosing complex coordinates $\{z^i\}$ w.r.t. $J(0)$, we can write

$$\theta_t = \sum f_{\bar{i}}^j d\bar{z}^i \otimes \frac{\partial}{\partial z^j}.$$

Suppose now that θ_t describes a family of complex structures all compatible with a fixed metric g. Define now a tensor

$$\phi = \sum f_{\bar{l}\bar{i}} d\bar{z}^l \otimes d\bar{z}^i = \sum g_{\bar{l}j} f_{\bar{i}}^j d\bar{z}^l \otimes d\bar{z}^i \ .$$

Clearly we have $g(L + \theta_t(L), M + \theta_t(M)) = 0 \; \forall L, M \in T^{0,1}$ and then

$$g(\theta_t(L), M) + g(\theta_t(M), L) = 0 \ .$$

Putting $L = \frac{\partial}{\partial \bar{z}^i}, M = \frac{\partial}{\partial \bar{z}^l}$ in this equation we get $g_{\bar{l}j} f_{\bar{i}}^j + g_{\bar{i}j} f_{\bar{l}}^j = 0$ which proves that the tensor ϕ is a skew-symmetric $(0,2)$-tensor.

Suppose on the other hand that θ_t parametrizes a family of complex structures all polarizing a fixed Q. In this case we get

$$Q(\theta_t(L), M) - Q(\theta_t(M), L) = 0 \quad \forall L, M \in T^{0,1}.$$

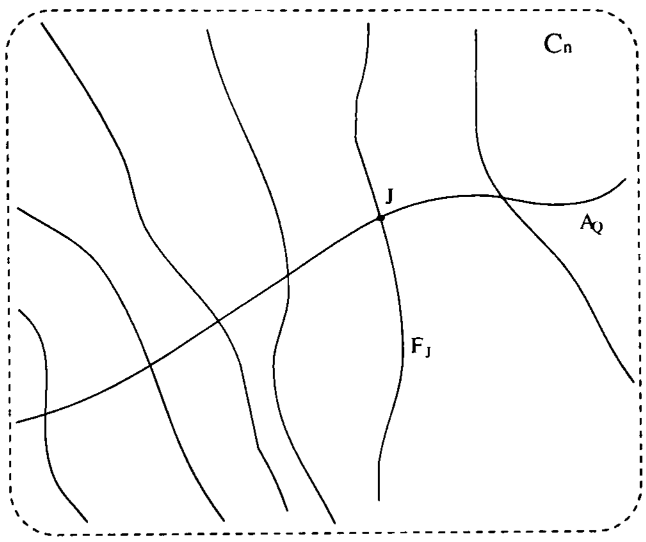
But with the same choice of L and M as above, we find that $Q(\theta_t(L), M) = \sqrt{-1} g_{\bar{l}j} f_{\bar{i}}^j$ and $Q(\theta_t(M), L) = \sqrt{-1} g_{\bar{i}j} f_{\bar{l}}^j$. This shows that the tensor

$$\psi = \sum g_{\bar{l}j} f_{\bar{i}}^j d\bar{z}^l \otimes d\bar{z}^i$$

is symmetric.

We have then in the first case a canonical identification with skew-symmetric $(0,2)$-tensors, while in the second one we get symmetric $(0,2)$-tensors under the same construction; this proves the last stament of the proposition. $\square$

The above proposition justifies then the following picture:



A similar description can be given also of $\mathcal{R}_n$. For this scope we define:

Definition 3.2.1 *1.* $\mathcal{R}_Q = \{$*flat metrics* $g \mid g^{-1}Q$ *is a complex structure*$\}$.

2. $G_g = \{$*flat metrics* $g' \mid g'$ *hermitian w.r.t.* $g^{-1}Q\}$.

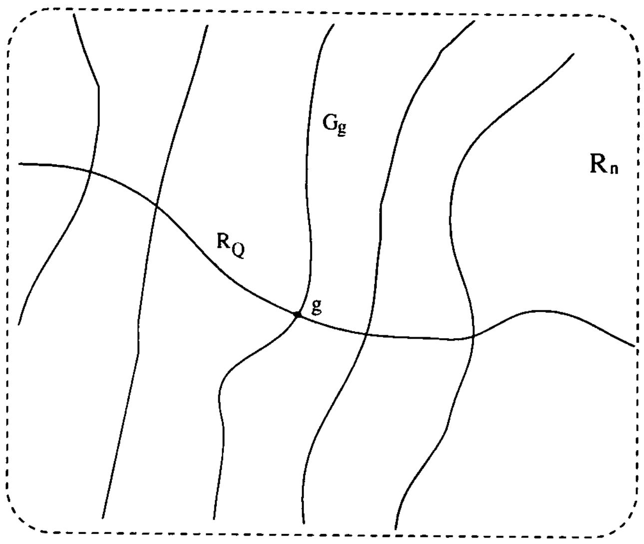
Proposition 3.2.2 *1.* $\mathcal{R}_n = \cup_{g\in\mathcal{R}_Q} G_g$,

2. $g \neq g' \Rightarrow G_g \cap G_{g'} = \emptyset$.

Proof:

1. Given $g' \in \mathcal{R}_n$, there exists (again by Calabi's Theorem) a complex structure $J_{g,Q}$ s.t. $(g', J_{g',Q}, Q)$ is a Kähler triple. Defining $g(\cdot,\cdot) = Q(J_{g',Q}\cdot,\cdot)$, we have $g' \in G_g$.

2. Let $\tilde{g} \in G_g \cap G_{g'}$. By definition $\tilde{g}$ has to be hermitian w.r.t. $J_g = g^{-1}Q$ and $J_{g'} = g'^{-1}Q$. But then $J_g = J_{g'}$ by uniqueness in Calabi's Theorem and therefore $g = g'$. $\square$



3.3 Periodic Minimal Surfaces

From now on we indicate by Σ_r a Riemann surface of genus r and not just a topological surface. As we said in the introduction, given a Kähler manifold, any complex submanifold minimizes volume in its homology class. In the case of complex tori the theory of holomorphic curves in them is a classical subject in algebraic geometry. It is in fact well known that to any Riemann surface Σ_n of genus $n \geq 1$ one can associate a complex torus of real dimension $2n$, called the *Jacobian* of the surface, in the following way: take a basis of the space of holomorphic sections of the canonical bundle $H^0(K) = H^{1,0}(\Sigma_n, \mathbb{C})$, $\{\omega_1, \ldots, \omega_n\}$, and consider $\Lambda = \{Re(\int_\sigma \omega_1, \ldots, \int_\sigma \omega_n, \int_\sigma i\omega_1, \ldots, \int_\sigma i\omega_n)\}$ varying $\sigma \in H_1(\Sigma_n, \mathbb{Z})\}$. The complex torus $(\mathbb{R}^{2n}/\Lambda, J_0)$ is the jacobian of Σ_n, $\mathcal{J}(\Sigma_n)$. The map $j_{p_0}: \Sigma_n \to \mathcal{J}(\Sigma_n)$, defined by $j_{p_0}(p) = \{Re(\int_{p_0}^p \omega_1, \ldots, \int_{p_0}^p \omega_n, \int_{p_0}^p i\omega_1, \ldots, \int_{p_0}^p i\omega_n)\}$ is called the Abel-Jacobi map, and, by classical theorems due to Abel and Jacobi, it is a holomorphic embedding. This map will play a key role in this Chapter. We will base some decisive considerations on the following well known:

Theorem 3.3.1 (Universal property of the Abel-Jacobi map)

If $f\colon \Sigma_r \to (\mathbb{R}^{2k}/\Lambda, J)$ is a full holomorphic map, then f factors through $\mathcal{J}(\Sigma_r)$, i.e. there exists a $\mathbb{C}$-linear map $A\colon \mathcal{J}(\Sigma_r) \to (\mathbb{R}^{2k}/\Lambda, J)$ s.t.

$$\begin{array}{ccc} \Sigma_r & \xrightarrow{f} & (\mathbb{R}^{2k}/\Lambda, J) \\ & {\scriptstyle j_{p_0}} \searrow & \uparrow A \\ & & \mathcal{J}(\Sigma_r) \end{array} \tag{3.7}$$

commutes. In particular $\mathcal{J}(\Sigma_r)$ contains a codimension k complex subtorus, given by the kernel of A.

The basic tool in studying minimal surfaces in flat tori is given by the following:

Theorem 3.3.2 (Generalized Weierstrass Representation)

If $f\colon \Sigma_r \to (\mathbb{R}^{2n}/\Lambda, g_0)$ is a conformal minimal immersion, then, after a translation, f can be represented by $f(p) = Re(\int_{p_0}^{p} \eta_1, \ldots, \int_{p_0}^{p} \eta_{2n})$, where $\eta_i \in H^0(K)$, $\sum_{i=1}^{2n} \eta_i^2 = 0$, and $\{Re(\int_\sigma \eta_1, \ldots, \int_\sigma \eta_{2n}) | \sigma \in H_1(\Sigma_r, \mathbb{Z})\}$ is a sublattice of Λ.

The above theorem is the basis of the whole theory of periodic minimal surfaces (see [34]) which goes back to the end of the last century.

The following discussion and results in this section are due to Micallef; they are included here for sake of completeness.

Suppose now that $f\colon \Sigma_r \to (\mathbb{R}^{2n}/\Lambda, eucl)$ is a minimal immersion. By the above theorem $f(p) = Re(\int_{p_0}^{p} \omega M)$, where M is a $r \times 2n$ complex matrix and $\omega = (\omega_1, \ldots, \omega_r)$ is a basis of $H^0(K)$. Suppose f is holomorphic w.r.t. some complex structure J compatible with $eucl$. We have already observed that, if J_0 is the standard complex structure compatible with $eucl$, then the set of all compatible complex structures is described by $O^t J_0 O$ where O is an orthogonal transformation. By the Universal Property of the Abel-Jacobi map, we then have that there exists a complex linear map $L\colon (\mathbb{R}^{2r}/\Lambda', J_0) \to (\mathbb{R}^{2n}/(\Lambda O^t), J_0)$, where $(\mathbb{R}^{2r}/\Lambda', J_0)$ is the jacobian of Σ_r. Therefore for some $p_0 \in \Sigma_r$, we have that $\tilde{f} = O^t \circ f = L \circ j_{p_0}$, which gives in matrix representation $\tilde{f} = Re(\int_{p_0}^{p} \omega M O^t = Re(\int_{p_0}^{p} \omega L(Id_n \; iId_n))$. This proves the following:

Lemma 3.3.1 *Let $f: \Sigma_r \to (\mathbb{R}^{2n}/\Lambda, eucl)$ be a full minimal immersion in a flat torus, given by $f(p) = Re(\int_{p_0}^{p} \omega M)$, and J a complex structure given by $O^t J_0 O$, with $O \in O(2n, \mathbb{R})$.*

Then J is compatible with the euclidean metric and f is holomorphic w.r.t. the complex structure J if and only if there exists L, a complex $r \times n$ matrix, s.t. $M = L(Id_n \; iId_n)O$.

Corollary 3.3.1 *A full minimal immersion $f: \Sigma_r \to (\mathbb{R}^{2n}/\Lambda, eucl)$ given by $f(p) = Re(\int_{p_0}^{p} \omega M)$ is holomorphic w.r.t. some complex structure compatible with the metric if and only if $MM^t = 0$.*

Proof: By Lemma 3.3.1 we have to prove that $MM^t = 0$ implies $M = L(Id_n \; iId_n)O$ with L a complex $r \times n$ matrix and $O \in \mathcal{O}(2n, \mathbb{R})$. Since f is full, we have $n \leq r$, and then it is clearly sufficient to prove this claim for $r = n$ also for not full immersions, because, under our assumption, f defines a minimal immersion (not full now) into a flat forus of dimension $2r$. We associate to M the $2r \times 2r$ real matrix $\tilde{M}$ given by $\tilde{M} = \begin{pmatrix} A & C \\ B & D \end{pmatrix}$ where $M = (A + iB \;\; C + iD)$. The condition $MM^t = 0$ is then equivalent to

$$\begin{cases} AA^t + CC^t - BB^t - DD^t = 0 \\ AB^t + BA^t + CD^t + DC^t = 0 \, . \end{cases} \tag{3.8}$$

But

$$\tilde{M}\tilde{M}^t = \begin{pmatrix} AA^t + CC^t & BA^t + DC^t \\ AB^t + CD^t & BB^t + DD^t \end{pmatrix} \tag{3.9}$$

and then $MM^t = 0$ if and only if $\tilde{M}\tilde{M}^t$ is an hermitian, semi-positive definite (by Cauchy-Schwarz inequality) matrix of $Gl(r, \mathbb{C})$. This means $\tilde{M}\tilde{M}^t = P^2$, where P is also semipostive definite and $\tilde{M} = PO, O \in \mathcal{O}(2r, \mathbb{R})$.

Then $\tilde{M} = P \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} O$ and therefore $M = P(I_r \; iI_r)O$ as we wanted. □

This simple corollary gives as easy consequence that there aren't full minimal conformal immersions of nonhyperelliptic Riemann surfaces of genus three in $(\mathbb{R}^6/\Lambda, eucl)$

different from the holomorphic ones. The above claim can be proved in the following way: by the Weierstrass representation theorem (see [34]) a minimal conformal immersion f of Σ_3 into a flat $(\mathbb{R}^6/\Lambda, eucl)$ is given by

$$f(p) = Re \int_{p_0}^{p} (\eta_1(z), \ldots, \eta_6(z))dz .$$

where $\eta_1, \ldots, \eta_6$ are $\mathbb{R}$-linearly independent holomorphic differentials on Σ_3 and the conformality assumption translates into

$$\sum_{i=1}^{6} \eta_i^2 = 0. \tag{3.10}$$

But for a nonhyperelliptic surface of genus 3 Noether's theorem shows that the canonical curve in $\mathbb{C}P^2$ is not contained in any quadric and then (3.10) has to be a quadric of rank zero. Choosing $\{\omega_1, \omega_2, \omega_3\}$ a basis of $H^{1,0}(\Sigma_3, \mathbb{C})$ we get $\underline{\eta} = \underline{\omega}M$ where $\underline{\eta}$ and $\underline{\omega}$ are the vectors (η_i) and (ω_i) respectively, $M \in \mathcal{M}(3 \times 6, \mathbb{C})$ and (3.10) becomes $\underline{\omega}MM^t\underline{\omega}^t = 0$ and then $MM^t = 0$. Corollary 3.3.1 then gives:

Corollary 3.3.2 *If $f: \Sigma_3 \to (\mathbb{R}^6/\Lambda, eucl)$ is a minimal immersion inducing a non hyperelliptic conformal structure on Σ_3, then it is holomorphic w.r.t. some complex structure compatible with the metric.*

Remark 3.3.1 *The same argument as in the proof of Corollary 3.3.2 proves the following result:*

Corollary 3.3.3 *Every conformal minimal immersion $f: \Sigma_2 \to (T^4, eucl)$ is holomorphic w.r.t. some compatible complex structure.*

Remark 3.3.2 *The conclusion of Corollary 3.3.2 holds also for stable minimal immersions when the induced conformal structure is hyperelliptic, by Theorem 2.3.1.*

One may wonder whether any full minimal immersion $\Sigma_3 \to (\mathbb{R}^6/\Lambda, eucl)$ is stable; we will show, using once again Corollary 3.3.1, that this is not the case. First we need to observe that the canonical image of a hyperelliptic Riemann surface of genus three *is contained in a non trivial quadric: this follows directly from the fact (see [4]) that such a* Σ_3 *is the Riemann surface of the algebraic function*

$$w^2 = (z - a_1) \cdots (z - a_8) \ .$$

Therefore it has a basis of holomorphic differentials of the form

$$\omega_1 = \frac{dz}{w} \ , \ \omega_2 = z\frac{dz}{w} \ , \ \omega_3 = z^2\frac{dz}{w} \ ,$$

where z, w *are coordinates over* $\mathbb{C}^2$ *(see [4] for a discussion of this classical result). Therefore the quadric of rank three* $X_1^2 = X_0 X_2$ *contains the canonical image of the Riemann surface. By the discussion above it is then clear that, having taken as* M *any* 3×6 *complex matrix with* ReM *and* ImM *of maximal rank, and s.t.*

$$MM^t = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{3.11}$$

then the map

$$f(p) = Re \int_{p_0}^{p} (\underline{\omega} M), (mod \Lambda)$$

is a full conformal minimal immersion not holomorphic w.r.t. any compatible complex structure and therefore unstable *by theorem 2.3.1. For example*

$$M = \frac{i}{\sqrt{2}} \begin{pmatrix} 1-i & 1+i & i & 1 & i & -1 \\ 1+i & 1-i & -i & -1 & i & 1 \\ 0 & -i\sqrt{2} & i\sqrt{2} & 0 & \sqrt{2} & 0 \end{pmatrix} \tag{3.12}$$

satisfies all the required properties.

3.4 Stability of Minimal Surfaces in Flat Tori

In this section we prove the main theorems about the existence of stable minimal surfaces in flat tori not holomorphic w.r.t. any compatible complex structure. Our strategy is the same for both cases: we consider either the Abel-Jacobi map of the

Riemann surface into its jacobian or a projection of this map into a lower dimensional complex torus, on which we put a flat metric compatible with this complex structure. This map is, by Wirtinger's inequality, a conformal stable minimal immersion of the starting Riemann surface in this flat torus. We then deform this map and the torus, but not the conformal structure, in such a way to destroy holomorphicity, but to save all the other properties.

Theorem 3.4.1 *For any nonhyperelliptic Riemann surface of genus $r \geq 4$, there exists a conformal stable minimal immersion $f: \Sigma_r \to (\mathbb{R}^{2r}/\Lambda, eucl)$ into a flat torus, which is not holomorphic with respect to any complex structure compatible with the flat metric.*

The proof of this Theorem is due to Micallef. I want to thank him for having let me include it so as to be able to refer to it in the remaining part of this Chapter. We also remark that this result follows directly from the results in the last Chapter.

Proof: Given a nonhyperelliptic Riemann surface Σ_r we know, by Noether's Theorem (see [4]), that its canonical image is contained in a quadric of rank $k, 3 \leq k \leq r$. Therefore there exists a basis of $H^0(K)$, $\underline{\omega} = (\omega_1, \ldots, \omega_r)$ s.t. $\sum_{i=1}^{k} \omega_i^2 = 0$. We then consider the family of maps $f_s: \Sigma_r \to (\mathbb{R}^{2r}/\Lambda_s, eucl)$ defined for $s \in (-\epsilon, \epsilon)$, by

$$f_s(p) = Re(\int_{p_0}^{p} \underline{\omega} M) = Re(\int_{p_0}^{p} \eta_1, \ldots, \eta_{2r})$$

where

$$M = \begin{pmatrix} Id_k & 0 & ie^s Id_k & 0 \\ 0 & Id_{r-k} & 0 & iId_{r-k} \end{pmatrix}.$$

We observe that f_s is a harmonic map, since it is given by integrals of holomorphic differentials, and moreover we have

$$\sum_{i=1}^{2r} \eta_i^2 = \sum_{i=1}^{k} \omega_i^2 = 0 .$$

Therefore f_s is a conformal minimal immersion for all s.

We still have to prove the stability and the non holomorphicity. This latter is easily settled using the result of the previous section. We notice in fact that

$$MM^t = (1 - e^{2s}) \begin{pmatrix} Id_k & 0 \\ 0 & 0 \end{pmatrix}$$

which is zero if and only if $s = 0$; then, by our version of the Universal property of the Abel-Jacobi map, f_s cannot be holomorphic w.r.t. any compatible complex structure. In order to prove the stability of f_s for small values of s, we need first to calculate the space of Jacobi fields of f_0. Since f_0 is holomorphic, by Simons' Theorem 1.1.2, the space of Jacobi fields is the space of holomorphic sections of the normal bundle $\nu\Sigma_r$.

To calculate this dimension we consider the following exact sequence:

$$0 \to T\Sigma_r \to \mathbb{C}^r \to \nu\Sigma_r \to 0$$

and the associated long exact sequence:

$$0 \to H^0(\mathbb{C}^r) \xrightarrow{\chi} H^0(\nu\Sigma_r) \xrightarrow{\psi} H^1(T\Sigma_r) \xrightarrow{\phi} H^1(\mathbb{C}^r) \to \cdots$$

Since this sequence is exact we have immediately $h^0(\nu\Sigma_r) = dim_{\mathbb{C}} H^0(\nu\Sigma_r) \geq r$. By Serre-Kodaira duality $H^1(T\Sigma_r) = (H^0(2K))^*$ and $H^1(\mathbb{C}^r) = (H^0(\mathbb{C}^r \otimes K))^*$, where K is the canonical bundle of Σ_r. We observe that if ϕ is injective then $\psi = 0$, and therefore $ker\psi = H^0(\nu\Sigma_r) = im\chi = \mathbb{C}^r$. To study when this is the case we consider the dual map of ϕ, $\phi^*: H^0(\mathbb{C}^r \otimes K) \to H^0(2K)$. In the dual sequence

$$0 \to \nu\Sigma_r^* \to (T\mathbb{C}^r)^* \xrightarrow{df_0^*} T\Sigma_r^* \to 0$$

df_0^* is just the pullback, via f_0, of 1-forms from T^{2r} to Σ_r. Therefore if (η_i) is an r-uple of holomorphic differentials on Σ_r, then $\phi^*(\eta_1, \dots, \eta_r) = \eta_1 \cdot \omega_1 + \dots + \eta_r \cdot \omega_r$, where $\cdot$ is the symmetric product between holomorphic differentials.

We want to compare the image of ϕ^* with the image of the classical Noether map $N: H^0(K) \otimes H^0(K) \to H^0(2K)$, defined by $N(\alpha \otimes \beta) = \alpha \cdot \beta$. ImN is clearly the span over the complex of $\{\omega_i \cdot \omega_j\}_{i \leq j}$. But, if

$$\Omega = \sum_{i \leq j} a_{ij}\, \omega_i \cdot \omega_j \ .$$

then we define

$$\eta_i = \sum_{j=i}^{r} \omega_j$$

and we get

$$\phi^*(\eta_1, \ldots \eta_r) = \sum_{i \le j} a_{ij}\ \omega_i \cdot \omega_j\ .$$

Then $ImN = Im\phi^*$. The surjectivity of N (and therefore of ϕ^*) in the nonhyperelliptic case is assured by Noether's Theorem. [4]. We then have that the space of Jacobi fields of the Abel-Jacobi map has the least possible dimension, since translations on the torus clearly induce Jacobi fields on the surface.

The first claim is that the family of normal bundles to $f_s(\Sigma_r)$, $\nu(s)$, is in fact a smooth family of bundles. In order to see this let us first recall that each f_s is an embedding. This follows directly from Abel's Theorem for minimal immersions (see [34]). Since the family f_s is smooth in s it is clear that $f_s^*(T(T^{2r}))$ is a smooth family of bundles; moreover the pull back metrics are given by

$$g(s) = (1 + e^{2s}) \sum_{i=1}^{k} \mid \omega_i \mid^2\ ,$$

which is again smooth in s. This clearly proves the claim, since $\nu(s)$ is the orthogonal complement of $T\Sigma_r$ in $f_s^*(T(T^{2r}))$. The second claim is that the family of operators $\partial^2 \mathcal{A}_{f_s}$ on $\nu(s)$ forms as well a smooth family of operators, i.e. for every ψ_s smooth family of smooth sections of $\nu(s)$, $\partial^2 \mathcal{A}_{f_s}(\psi_s))$ is smooth in s. This follows directly from the formula 1.2 in Chapter 1. A theorem of Kodaira ([27], pag 325) then implies the continuity of the eigenvalues of the operator $\partial^2 \mathcal{A}$. Since at $s = 0$ we have already observed that $\partial^2 \mathcal{A}$ has the least number of Jacobi fields, the continuity of the eigenvalues implies that for s sufficiently small, the space of Jacobi fields has to be the space of translations, since no eigenvector associated to a positive eigenvalue can become a Jacobi field, and the translations cannot become negative eigenvectors. This concludes the proof. □

We want to apply the ideas used in the proof of the previous Theorem also to the case of minimal immersions of surfaces of genus r into flat tori of dimension less than

$2r$.

Theorem 3.4.2 *1. For $r \geq 7$, there exists a dense subset $\mathcal{D}_1^r$ of the moduli space of Riemann surfaces of genus r, s.t. if $\mu \in \mathcal{D}_1^r$, then there exists a conformal stable minimal immersion $f: (\Sigma_r, \mu) \to (\mathbb{R}^{2(r-1)}/\Lambda, eucl)$ into a flat torus, which is not holomorphic w.r.t. any compatible complex structure.*

2. For $r \geq 9$, there exists a dense subset $\mathcal{D}_2^r$ of the moduli space of Riemann surfaces of genus r, s.t. if $\mu \in \mathcal{D}_2^r$, then there exists a conformal stable minimal immersion $f: (\Sigma_r, \mu) \to (\mathbb{R}^{2(r-2)}/\Lambda, eucl)$ into a flat torus, which is not holomorphic w.r.t. any compatible complex structure.

3. For $r \geq 12$, there exists a dense subset $\mathcal{D}_3^r$ of the moduli space of Riemann surfaces of genus r, s.t. if $\mu \in \mathcal{D}_3^r$, then there exists a conformal stable minimal immersion $f: (\Sigma_r, \mu) \to (\mathbb{R}^{2(r-3)}/\Lambda, eucl)$ into a flat torus, which is not holomorphic w.r.t. any compatible complex structure.

Proof: We first prove the following

Lemma 3.4.1 *On the generic Riemann surface of genus $r \geq 6$, there exists an open set of subspaces of complex dimension $r-k$, $k = 1, 2$ and 3, $V_k \subset H^0(K)$, s.t.*

1. $V_k \otimes V_k^{\perp} \to H^0(2K)$ is injective, where $\perp$ means the orthogonal complement w.r.t. the polarization on the Riemann surface

2. $V_k \otimes H^0(K) \to H^0(2K)$ is surjective.

Proof: By a Theorem of Gieseker ([21]) on each nonhyperelliptic Riemann surface there exists an open and dense set of subspaces $V \subset H^0(K)$, with $dim_{\mathbb{C}} V = 3$, $V \otimes H^0(K) \to H^0(2K)$ surjective, and for the generic three dimensional subspace $W \subset V^{\perp}$ also the map $W \otimes H^0(K) \to H^0(2K)$ is surjective. Consider now V_k containing V. 2 holds directly. 1 also holds: in fact for $k = 1$ there is nothing to be proved; for $k = 2$, Condition 1 follows directly from the Base Point Free Pencil Trick ([1]); for $k = 3$ we can choose V_k in such a way that $V_k^\perp \otimes H^0(K) \to H^0(2K)$ is surjective and therefore it has to be an isomorphism. This clearly implies 1. $\square$

As in the proof of Criterion 2 in [15] the above Lemma implies that there exists a dense subset $\mathcal{D}_k^r$ of the Riemann moduli space s.t. if $\mu \in \mathcal{D}_k^r$ then there exists a subspace $V_k \subset H^0(K)$ of dimension $r-k$ s.t.

1. for any basis $\{\eta_1, \ldots, \eta_{r-k}\}$ of V_k, its periods define a sublattice of the jacobian and
2. $V_k \otimes H^0(K) \to H^0(2K)$ is surjective.

We now observe that

$$dim_{\mathbb{C}}(V_k \otimes V_k/\Lambda^2 V_k) = \frac{1}{2}(r-k)(r-k+1) \text{ and } dim_{\mathbb{C}} H^0(2K) = 3r-3\ .$$

Therefore the map $V_k \otimes V_k \to H^0(2K)$ has non trivial kernel for $k=1$ and $r \geq 7$, $k=2$ and $r \geq 9$, $k=3$ and $r \geq 12$. This means that the canonical image of (Σ_r, μ) is contained in a non zero quadric involving just $\{\eta_1, \ldots, \eta_{r-k}\}$, i.e. there exists a basis of V_k, $\underline{\omega} = \{\omega_1, \ldots, \omega_{r-k}\}$, s.t. $\sum_{i=1}^n \omega_i^2 = 0$, $n \leq r-k$.

Now we can adapt the proof of Theorem 3.4.1 to our case.

Let

$$f_s\colon (\Sigma_r, \mu) \to (\mathbb{R}^{2(r-k)}/\Lambda_s, eucl)$$

be defined by

$$f_s(p) = Re \int_{p_0}^{p} \underline{\omega} M = Re \int_{p_0}^{p} (\sigma_1, \ldots, \sigma_{2(r-k)})\ ,$$

where

$$M = \begin{pmatrix} Id_n & 0 & i\epsilon^s Id_n & 0 \\ 0 & Id_{r-k-n} & 0 & iId_{r-k-n} \end{pmatrix}.$$

As in Theorem 3.4.1 f_0 is a holomorphic map with just trivial Jacobi fields, because the map $\phi^*\colon H^0(\mathbb{C}^{r-1} \otimes K) \to H^0(2K)$ is surjective by construction.

Therefore we can again conlcude that f_s is a family of minimal immersions, conformal w.r.t. μ and not holomorphic w.r.t. any compatible complex structure again by Corollary 3.3.1. To prove that f_s is stable for s sufficiently small, we need just to observe that the space V_k is base point free by 2, and therefore f_0 is free of branch points. The same argument used in the proof of Theorem 3.4.1 concludes the proof. □

Remark 3.4.1 *1. The proof of the above theorem requires $r \geq 7$ in order to ensure the existence of a quadric through the canonical curve involving just a set of differentials whose periods form a sublattice of the whole periods lattice. For $r < 7$ this could be false: fix for example $r = 4$ and $k = 1$ (we refer to [14] for a study of $\mathcal{D}_1^4$). If there exists a three dimensional subspace $V \subset H^0(K)$ s.t. $V \otimes H^0(K) \to H^0(2K)$ is surjective, then such a map has to be an isomorphism. In particular $V \otimes V \to H^0(2K)$ cannot have non trivial kernel. This actually shows that the strategy of the above theorems cannot be used to produce stable non holomorphic minimal immersions into flat tori of dimension 6.*

2. A natural question is whether $\mathcal{D}_k^r$ contains families of Riemann surfaces. From Colombo-Pirola's analysis one easily gets that $\mathcal{D}_3^r$ does not contain any family, while $\mathcal{D}_1^r$ and $\mathcal{D}_2^r$ contain families of (real) dimensions $4r-4$ and $2r-2$ respectively.

The strategy of the proofs of theorems 3.4.1 and 3.4.2 deserves some comments. There are two questions we want to study:

- if we keep fixed the conformal structure μ, is it possible to find a smooth family $\phi_t: \Sigma \to (\mathcal{J}(\Sigma), eucl)$ of non congruent minimal immersions all conformal w.r.t. μ and s.t. ϕ_0 is the Abel-Jacobi map?
- Is it possible to characterize the possible families of lattices Λ_t for which there exists a smooth family $\phi_t: \Sigma \to (\mathbb{R}^{2n}/\Lambda_t, eucl)$ as above?

The answer to the first problem is known to be negative. In fact, such a family would form a family of homotopic harmonic maps which, by Hartman's Uniqueness Theorem 5.1.1 (see [23]), has to be constant up to isometries of the torus.

Therefore the lattice Λ_0 has to move, but how? This question is a special case of a problem studied in Chapter 5. What happens is that there exists an open subset V of dimension $2r^2 + 6 - 5r$ of the space of flat tori, s.t. $\Lambda_0 \in V$ and for every $\Lambda_t \subset V$, there exists a smooth family of stable conformal minimal immersions $\phi_t: \Sigma \to (\mathbb{R}^{2n}/\Lambda_t, eucl)$, each of which is not holomorphic w.r.t. any compatible complex structure (apart from ϕ_0 of course). This follows directly from the fact that the period map (see Chapter 5) is a local isomorphism around ϕ_0. In fact the remaining $6r - 6$ directions in the moduli space of flat tori, correspond precisely to the infinitesimal deformations of conformal structures.

3.5 Stable non holomorphic minimal surfaces

In the previous section we have proved the existence of stable non holomorphic minimal surfaces in flat tori. We have remarked that we have to construct also special flat tori to exibit these examples. We want now to prove that the space of flat tori in which these surfaces exist is in fact generic, i.e. it is an open and dense subset of the moduli space of flat tori described in section 2. In order to do this we have to analyze how special are the holomorphic ones. The first observation is the following: consider a map $u: \Sigma_r \to T^{2r}$ s.t. u_* is an isomorphism ρ on $H_1(\Sigma_r, \mathbb{Z})$; as we have seen in the introduction, we can associate to such a map an integral non degenerate 2-form Q on T^{2r} s.t. $u^*(Q) = \chi$, where χ is the intersection form on the surface. We first want to be able to recognize the flat metrics for which there exists an immersion holomorphic w.r.t. a compatible complex structure. In a similar way to the proof of Theorem 2.1.3 we have the following:

Proposition 3.5.1 *Let $u_i: \Sigma_{r_i} \to T^{2r_i}, i = 1, \ldots, p$ be a collection of maps s.t.*

1. *u_i is holomorphic w.r.t. a complex structure J_i.*

2. *$u_{i*} : H_1(\Sigma_{r_i}, \mathbb{Z}) \to H_1(T^{2r_i}, \mathbb{Z})$ is an isomorphism.*

and let Q_i be the integral 2-form on T^{2r_i} constructed from u_{i*} as in the introduction. Then

$$(\oplus_{i=1}^{p} T^{2r_i}, \oplus_{i=1}^{p} J_i, \oplus_{i=1}^{p} Q_i)$$

is isomorphic as principally polarized abelian variety to

$$\oplus_{i=1}^{p} \mathcal{J}(\Sigma_{r_i}) \ .$$

Proof: This propostion follows directly from the Matsusaka-Ran Criterion (see [29]), but we give here a simpler direct proof. We prove that for each i, (T^{2r_i}, J_i, Q_i) is isomorphic to $\mathcal{J}(\Sigma_{r_i})$. First we notice that Q_i polarizes J_i: in fact by the Universal Property of the Abel-Jacobi map j (Theorem 3.3.1) we know that $u_i = M_i \circ j$ where $M_i: \mathcal{J}(\Sigma_{r_i}) \to (T^{2r_i}, J_i)$ is a complex linear map. Moreover, by the construction of j, we know that $j^*(Q') = \chi$, where Q' is the principal polarization on the jacobian of the surface. Therefore $M_i^*(Q) = Q'$, and since M_i is a complex linear map, we have the claim. Therefore the map M_i is the isomorphism we were looking for. □

We are now in position to prove genericity of stable non holomorphic minimal immersions.

Fix a topological surface of genus r, Σ_r, and an isomorphism

$$\rho: H_1(\Sigma_r, \mathbb{Z}) \to H_1(T^{2r}, \mathbb{Z}) \ .$$

Let us define

$\mathcal{M}_\rho = \{$flat tori of dimension $2r$, (T^{2r}, g), | there exists $u: \Sigma_r \to (T^{2r}, g)$ minimal

stable non holomorphic w.r.t. any compatible complex structure with $u_* = \rho\}$.

The results proved in section 3 *show that* $\mathcal{M}_\rho = \emptyset$ *for* $r = 2, 3$.

Theorem 3.5.1 *For all* $r \geq 4$, $\mathcal{M}_\rho$ *is open and dense in the space of flat tori.*

Proof: Let us fix a symplectic basis $\{\alpha_i, \beta_i\}$, $i = 1, \ldots, r$ of $H_1(\Sigma_r, \mathbb{Z})$ and define $\lambda_i = \rho(\alpha_i), \mu_i = \beta_i$. Consider a continuous map u from the surface to the torus with ρ as action on $H_1(\Sigma_r, \mathbb{Z})$. Using the main result in Chapter 2, we know that there exists a stable minimal immersion $\tilde{u}$ of a Riemann surface of genus r, possibly with nodes, s.t. $\tilde{u}_* = \rho$. As above, let Q be the integral non degenerate 2-form on T^{2r} induced by ρ, i.e. $Q = \sum \lambda_i^* \wedge \mu_i^*$. If $\tilde{u}$ is holomorphic w.r.t. J, then by Proposition 3.5.1 we know that (T^{2r}, J, Q) is isomorphic, as polarized abelian variety, to the canonically polarized sum of the jacobians of the closures of the parts of the Riemann surface with nodes, on which $\tilde{u}$ is defined. Moreover, by construction, this isomorphism has to send a symplectic basis for the sum of the jacobians into the symplectic basis of (T^{2r}, J, Q) given by $\{\lambda_i, \mu_i\}$.

Therefore the flat metrics for which such a map exists, form smooth families of (real) dimensions $r^2 + 6(r-k) - 6(p-k) + 2k$ (which is less than $r^2 + r$ for $r \geq 4!$), where k is the number of elliptic curves among the Σ_{r_i}, whose closures intersect in the moduli space of flat metrics. □

Using Proposition 3.5.1 we can moreover describe when nodes occur in the area minimizing process among maps with a fixed injective action in homology. Let us construct an explicit example: consider the flat "square" 4-torus, $T^4 = (\mathbb{R}^4/\mathbb{Z}^4, g_0)$, and a 2-form on it, $Q = dx_1 \wedge dx_3 + dx_2 \wedge dx_4$, where x_i are the standard coordinates on $\mathbb{R}^4$. The set of compatible complex structures is described by

$$J = \begin{pmatrix} 0 & -x & -y & -z \\ x & 0 & z & -y \\ y & -z & 0 & x \\ z & y & -x & 0 \end{pmatrix},$$

where $\{x, y, z \in \mathbb{R} \mid x^2 + y^2 + z^2 = 1\}$.

Consider a closed topological surface of genus 2, Σ, and the isomorphism $\rho: H_1(\Sigma, \mathbb{Z}) \to H_1(T^4, \mathbb{Z})$, defined by $\rho(\alpha_i) = e_i$ $\forall i = 1, \ldots, 4$, where $\{\alpha_i\}$ is a symplectic basis for $H_1(\Sigma, \mathbb{Z})$, and $\{e_i\}$ is the canonical basis of $\mathbb{R}^4$. Suppose now to minimize area among all maps with action ρ on $H_1(\Sigma, \mathbb{Z})$. Observe that

$$\chi(\alpha, \beta) = Q(\rho(\alpha), \rho(\beta)), \ \forall \alpha, \beta \in H_1(\Sigma, \mathbb{Z}) .$$

We are therefore in the situation of the beginning of this section. If, running the minimizing process, we get a stable minimal immersion of a Riemann surface of genus

2, then, by Corollary 3.5.1, we would have that there exists a compatible complex structure J on T^4, s.t. (T^4, J, Q) is isomorphic to a jacobian of some Riemann surface. On the other hand, it easily seen that the only compatible complex structures which polarizes Q are given by $x = z = 0, y = \pm 1$. But in both these cases the abelian surface we get is isomorphic to the canonically polarized product of two elliptic curves, and therefore, by the Matsusaka-Ran criterion ([29] and Chapter 1 for an alternative proof), it can't be the jacobian of a Riemann surface of genus 2. This contradiction implies that nodes have to occur in the limit of the area minimizing process.

This idea is clearly very effective to describe the occurence of nodes in the mentioned minimizing process in flat tori of dimension 4 and 6, where it is relatively simple to distinguish jacobians among all abelian varieties and where by Theorems 2.3.1 and Corollary 3.3.2, we know that we have to end up with a map holomorphic with respect to some compatible complex structure. The main results of this Chapter show that this is not always the case.

Remark 3.5.1 *We want to conclude this Chapter with a discussion of some natural questions arising from our research, which will be left to future investigation.*

Problem 5 *Given a principally polarized abelian variety (T^{2r}, J, Q), is it true that degeneration to a Riemann surface with nodes in the minimizing procedure is independent of choice of g hermitian w.r.t. J?*

Observe that if $r \leq 4$ and the abelian variety is a jacobian of a smooth Riemann surface, then the answer is yes, because if for some g hermitian w.r.t. J we get nodes, then (T^{2r}, J, Q) is the sum of jacobians of Riemann surfaces of lower genus, which is impossible. We believe the answer should be yes in general.

Problem 6 *If (T^{2r}, J, Q) is a principally polarized abelian variety, which does not contain any principally polarized abelian subvariety, then, after having fixed a riemannian metric hermitian w.r.t. J, the minimizing procedure gives a minimal immersion of a Riemann surface with nodes.*

Despite the simplicity of the above statement, we do not know what is a reasonable guess. If $r \leq 7$ we know that if the minimizing procedure degenerates then we would get a minimal immersion of a Riemann surface of genus $k \leq 3$, Σ_k, inside a subtorus $(T^{2k}, g_{|_{T^{2k}}})$, which has to be holomorphic w.r.t. a complex structure J' compatible with $g_{|_{T^{2k}}}$. In order to answer affirmatively Problem 6 in this case, one should prove that the natural inclusion $(T^{2k}, J') \to (T^{2r}, J)$ is holomorphic.

Chapter 4

Stable Complete Minimal Surfaces in Hyperkähler Manifolds

4.1 Introduction

In the previous Chapter we have studied the relation between stability and holomorphicity for minimal immersions of closed surfaces in flat tori; we want now to study the same problem for possibly open surfaces in 4-manifolds. The main question we study is the following:

Problem 7 *Given an isometric stable minimal immersion $F : M \to X$ of a complete oriented surface M into a hyperkähler 4-manifold X, is F holomorphic with respect to some orthogonal complex structure on X ?*

In general the answer to the above problem is negative: Atiyah and Hitchin ([7]) have found an example of a minimal two-sphere in the hyperkähler 4-manifold $\tilde{M}_2^0$, the universal cover of the centered 2-monopoles in $\mathbb{R}^3$ with finite action, which is not holomorphic w.r.t. any compatible complex structure on $\tilde{M}_2^0$, and which has been proved to be stable by Micallef and Wolfson ([38]).

In this Chapter we will find a sufficient condition on the immersion for the problem to have a positive answer.

We recall (see Chapter 1) that for locally embedded submanifolds M in N the property to be a complex submanifold of (N, J) can be expressed by saying that the tangent space T_pM is J-invariant for each $p \in M$. When N has real dimension 4 a way to measure the J-invariance of TM is given by the Kähler angle: it follows from Wirtinger's inequality that if ω is the restriction of the Kähler form of (N, J) to TM, we can write $\omega = cos\alpha\, dVolM$ and that M is a complex submanifold if and only if $\alpha = 0$ on M.

It is possible to express the stability condition in terms of the Kähler angle. Micallef and Wolfson ([38]) proved that if M is stable and σ is a section with compact support of the normal bundle then

$$\int_M \{|\bar{\partial}\sigma|^2 - 2[|d\alpha|^2 + \frac{1}{4}Ssin^2\alpha]|\sigma|^2\}dVol \geq 0 \ ,$$

where S is the scalar curvature of N. Using this formula, they proved (Corollary 5.3 pag. 260) that if N is hyperkähler (see section 2 for the definition), M is compact and the normal bundle admits a holomorphic section, then the immersion F is holomorphic with respect to one of the complex structures of N.

We'll apply the previous formula in the case N is hyperkähler and M not necessarely compact. The crucial problem is then to produce a holomorphic section of the normal bundle with appropriate growth and to do this we'll need some further hypothesis.

To overcome this problem we assume that the composition of the Gauss lift (see section 2 for the definition) with the projection over the sphere S^2 omits an open set. A key observation, due to Eells and Salamon ([16]), is that, under our assumptions, this map is anti-holomorphic, extending the analogy with the Gauss map of minimal surfaces in the euclidean space. This will allow us to prove the main result of this Chapter:

Theorem 4.1.1 *Let $F: M \to N$ be an isometric stable minimal immersion of a complete oriented surface M into a 4-dimensional hyperkähler manifold N. If the Gauss lift $\tilde{F}_+: M \to S_+ = N \times S^2$ omits an open set of S^2, then F is holomorphic with respect to some orthogonal complex structure of N.*

About the assumption on the Gauss lift in the above theorem, we recall that the image of the Gauss lift of the stable two sphere found by Atiyah and Hitchin mentioned before, is the whole S^2.

As we will see in the proof of the main theorem the condition on the Gauss lift is equivalent to the requirement for the Kähler angle to omit an open set of $[0, \pi]$. As we explained in the Chapter 1, this shows that our theorem generalizes Theorem 1.2.5.

4.2 Notations and Definitions

Let N be a riemannian manifold with metric g, M a Riemann surface and $F: M \to N$ a map. Let ∇ denote the Levi-Civita connections on TM and $F^{-1}TN$.

Let now assume that $dimN = 4$ and N is oriented. In this case the Hodge-star operator $*: \Lambda^2(TN) \to \Lambda^2(TN)$, gives rise to a decomposition

$$\Lambda^2(TN) = \Lambda^2_+(TN) \oplus \Lambda^2_-(TN),$$

where $\Lambda^2_\pm(TN)$ are the eigenspaces corresponding to the eigenvalues ± 1. The elements of $\Lambda^2_\pm$ are called *self-dual* and *antiself-dual* forms respectively. Let $S_\pm = S(\Lambda^2_\pm)$ be the two-sphere bundle of unit vectors. The *Grassmann bundle* $\tilde{G}_2$ is the bundle whose fibre at $x \in N$ is $\tilde{G}_2(T_xN)$, the space of oriented two dimensional subspaces of T_xN.

We can associate to an immersion $F: M \to N$ another map, called the *Gauss lift* of F, $\tilde{F}: M \to \tilde{G}_2$ defined by

$$\tilde{F}(p) = F_*(T_pM)$$

which is an element of $\tilde{G}_2(T_xN)$ where $F(p) = x$. In the case of immersions in the euclidean space it is possible to avoid the difficulty of working with bundles in the following way: given $F: M \to \mathbb{R}^n$ define $\gamma_F: M \to \tilde{G}_2(\mathbb{R}^n)$ where $\gamma_F(p)$ is the two plane

$F_*(T_pM)$ translated to the origin. γ_F is called the *Gauss map.* We recall that $\tilde{G}_2(\mathbb{R}^n)$ may be identified with a quadric Q_{n-2} in $\mathbb{C}\mathbf{P}^{n-1}$, and that a conformal immersion is harmonic if and only if $\gamma_F: M \to Q_{n-2}$ is anti-holomorphic (see Chern [11]). It is well known that Q_2 is diffeomorphic to $S^2 \times S^2$ using Plücker coordinates (e.g. see Chern-Spanier [12]). The same happens also in the general case: indeed $\tilde{G}_2(T_xN)$ is isomorphic to $(S_+)_x \times (S_-)_x$ and so we have two projections $p_\pm: \tilde{G}_2(TN) \to S_\pm$ and two new maps $\tilde{F}_\pm: M \to S_\pm$, $\tilde{F}_\pm = p_\pm \circ \tilde{F}$. Hence if $N = \mathbb{R}^n$, $\tilde{F}_+$ is the projection of γ_F onto the first S^2 and this gives the relation between our theorem and Micallef's.

It is possible to give an interpretation of the bundles $S_\pm$ in terms of almost complex structures over N. In fact if $w \in S_\pm$ on the x-fiber, then it is clearly possible to choose an oriented orthonormal basis $\{e_i\}$ of T_xN such that $w = e_1 \wedge e_2 + e_3 \wedge e_4$. Defining $Je_1 = e_2, Je_3 = e_4$ and $J^2 = -1$, we get an almost complex structure over N oriented consistenly with N. If θ_i is the dual basis of e_i then $\omega = \theta_1 \wedge \theta_2 + \theta_3 \wedge \theta_4$ is the fundamental 2-form associated to the almost complex structure J given by $g(JX, Y)$. In the case of S_- we get contrariwise oriented almost complex structures over N.

By definition a riemannian manifold is called *hyperkähler* if it admits a family of compatible complex structures parametrized by S^2, with respect to each of which the manifold is Kähler. In this case $S_+ = N \times S^2$ and every point of the sphere represents a complex structure on N.

Let (N, g, J) be an hermitian manifold, i.e. g is a riemannian metric, J a complex structure such that $g(JX, JY) = g(X, Y)$ for every $X, Y \in TN$, ω the fundamental 2-form and v the fundamental 2-vector of N. If e_i is a unitary basis of $T^{(1,0)}N$, i.e.

$$v = -i \sum_{k=1}^{n} e_k \wedge \bar{e}_k \ .$$

Let us denote by $T_0^{(1,1)}N$ the space of $(1,1)$-vectors orthogonal to v. So we have

$$\Lambda_2^{\mathbb{C}}(TN) = T^{(2,0)}N \oplus T^{(0,2)}N \oplus \mathbb{C}v \oplus T_0^{(1,1)}N \ .$$

It is easy to prove that, if $dim N = 4$,

$$\Lambda_-^{\mathbb{C}}(TN) = T_0^{(1,1)}N \qquad \Lambda_+^{\mathbb{C}}(TN) = T^{(2,0)}N \oplus T^{(0,2)}N \oplus \mathbb{C}v \ . \tag{4.1}$$

If y is a point in the x- fibre of S_+, then $T_yS_+ = \mathcal{V}_y \oplus \mathcal{H}_y$ where $\mathcal{V}_y$ is the space of vertical vectors, i.e. those tangent to the fiber $(S_+)_x$. By (4.1) $\mathcal{V}_y^-$ is isomorphic, via an isomorphism ν, to $T_x^{(2,0)}N \oplus T_x^{(0,2)}N$.

By the above observation y fixes an almost complex structure, $J_{\pi(y)}$, on T_xN. For any y in the x-fibre of S_+, $\pi_{*|\mathcal{H}_y}$ defines an isomorphism between $\mathcal{H}_y$ and T_xN. We will denote this isomorphism with μ. So we can define an almost complex structure J_1 (warning: this is called J_2 in [16]) on S_+ by

$$J_1(v_y, h_y) = (I_y v_y, J_{\pi(y)} h_y) .$$

where I is minus the standard complex structure on S^2. This means that the vectors of type $(1,0)$ with respect to J_1 in T_yS_+ are given by $(T_x^{(1,0)}N)^\mu \oplus (T_x^{(0,2)}N)^\nu$. Let us recall the following (see Eells-Salamon [16]) :

Theorem 4.2.1 *If $F: M \to N$ is a conformal and harmonic immersion, then $\tilde{F}_+$ is J_1- holomorphic.*

Proof: First we observe that if w is the fundamental 2-vector associated to an almost complex structure J of N then for every $X \in TS_+$, $\nabla_X w \in T^{(2,0)}S_+ \oplus T^{(0,2)}S_+$. Then if J is the almost complex structure defined by $\tilde{F}_+$, we have

$$\nabla_{F_*\frac{\partial}{\partial z}} w \in T^{(2,0)} \oplus T^{(0,2)} .$$

At a point $p \in M$,

$$\tilde{F}_+(p) = e_1 \wedge e_2 + e_3 \wedge e_4 = \frac{1}{2i\lambda}(1 + *)(F_*(\frac{\partial}{\partial z})) \wedge (F_*(\frac{\partial}{\partial \bar{z}})) ,$$

where z is a complex coordinate centered at p given by isothermal real coordinates on M and λ is the conformal factor of the immersion, i.e. λ is the square of the length of $F_*(\frac{\partial}{\partial z})$. So we have

$$\nabla_{F_*\frac{\partial}{\partial z}} w = \frac{1}{\lambda}\frac{\partial \lambda}{\partial z} w + \frac{1}{2i\lambda}(1 - *)[(\nabla_{F_*\frac{\partial}{\partial z}}(F_*\frac{\partial}{\partial z})) \wedge (F_*(\frac{\partial}{\partial \bar{z}})) + (F_*\frac{\partial}{\partial z}) \wedge (\nabla_{F_*\frac{\partial}{\partial z}}(F_*\frac{\partial}{\partial \bar{z}}))] .$$

The $(2,0)$ component with respect to J_1 is then zero because $(F_*\frac{\partial}{\partial z}) \wedge (\nabla_{F_*\frac{\partial}{\partial z}}(F_*\frac{\partial}{\partial \bar{z}}))$ vanishes since F is harmonic.

Since

$$\tilde{F}_{+*}(\frac{\partial}{\partial z}) = (F_*\frac{\partial}{\partial z})^\mu + (\nabla_{F_*\frac{\partial}{\partial z}} w)^\nu .$$

we have to show that $\tilde{F}_{+*}(\frac{\partial}{\partial z})$ is a $(1,0)$ vector with respect to J_1. But this follows from the fact that $\tilde{F}_+(p)$ is a complex structure such that p is a complex point for F.

□

Then we are in the following situation: given an hyperkähler manifold N and a minimal isometric immersion $F: M \longrightarrow N$ we have

$$M \xrightarrow{\tilde{F}_+} S_+ = N \times S^2 \xrightarrow{\pi} S^2 \xrightarrow{\rho} \mathbb{C} \cup \{\infty\} .$$

where π is the projection on the second factor and ρ is the stereographic projection. On S^2 we are considering the usual complex structure so that $\pi \circ \tilde{F}_+$ is anti-holomorphic. If the Gauss lift $\tilde{F}_+$ omits an open set of S^2 we have, after composition with a stereographic projection with pole in this open set and conjugation, a bounded holomorphic map from M to $\mathbb{C}$. Let us call this function g. This will be a crucial point in the proof of our theorem.

4.3 Proof of the main result

Micallef and Wolfson ([38]) proved that the stability condition implies that, for every compactly supported section σ of the complexified normal bundle $\nu_{\mathbb{C}}$:

$$\int_M \{|\bar{\partial}\sigma|^2 - 2[|d\alpha|^2 + \frac{1}{4}S sin^2\alpha]|\sigma|^2\} dVol \geq 0$$

where S is the scalar curvature of N. If N is hyperkähler then $S = 0$. So we have

$$\int_M |\bar{\partial}\sigma|^2 \geq 2\int_M |d\alpha|^2|\sigma|^2 . \tag{4.2}$$

Suppose there exists a global holomorphic section σ of $\nu_{\mathbb{C}}$ in L^2. Then, taking a cut-off function f_R such that $f_R = 1$ on $B_R(p)$, $f_R = 0$ outside $B_{2R}(p)$ and $|d(f_R)| < \frac{c}{R}$ everywhere, applying 4.2 to $f\sigma$ we get

$$2\int_M |d\alpha|^2 f_R^2|\sigma|^2 \leq 2\int_M |d(f_R)|^2|\sigma|^2 .$$

Letting $R \to \infty$ we have that, since $\sigma \in L^2$, $d\alpha = 0$ and so α is constant on M. Now, as in [54], choose a point $p \in M$ and a complex structure on TN such that T_pM is a complex subspace of $T_{F(p)}N$. The Kähler angle of this complex structure vanishes at p, but it is still constant on M. *Then the immersion is holomorphic with respect to this complex structure of N.*

So the following lemma concludes the proof of the theorem:

Lemma 4.3.1 *If the hypotheses of the theorem hold then there exists a global holomorphic section σ of the complexified normal bundle such that σ is square integrable.*

Proof: Let W be an open set of S^2 s.t. $W \subset S^2 \setminus \tilde{F}_+(M)$, and J a complex structure on N corresponding, via the discussion in section 2, to a point in W. We then have $1 - cos\alpha_J < 1 - \epsilon$, $\epsilon > 0$, everywhere on M. Since F is conformal there exist $\{e_1, e_2\}$ local real vector fields in $F_*(TM)$ such that $F_*(\frac{\partial}{\partial z}) = \sqrt{\frac{\lambda}{2}}(e_1 - ie_2)$, where λ is the conformal factor of the immersion; then we complete $\{e_1, e_2\}$ to a local orthonormal basis of TN, $\{e_1, \ldots, e_4\}$. Defining

$$\begin{aligned} f_1 &= e_1 \quad , \quad f_2 = Je_1 \\ f_3 &= e_4 \quad , \quad f_4 = -Je_4 \end{aligned} \tag{4.3}$$

we have directly that

$$< \tilde{F}_+(p), f_1 \wedge f_2 + f_3 \wedge f_4 > = cos\alpha_J(p). \tag{4.4}$$

This means, by the discussion in section 2, that the angle between J as a point of the sphere and $\tilde{F}_+(p)$ is precisely the Kähler angle at p and therefore the stereographic projection of $\tilde{F}_+$, from the point corresponding to J in S^2 has norm $\frac{sin\alpha_J}{1-cos\alpha_J}$.

We will indicate the hermitian product of X and Y by $g(X, Y)$ and $X \cdot Y = g(X, \overline{Y})$, so that the $\cdot$ product is *complex bilinear.* Define $s = [JF_*(\frac{\partial}{\partial z})]^\perp$, where J is a complex structure on N and $\perp$ is the projection onto the normal bundle $\nu_{\mathbb{C}}$.

s is a local holomorphic section of $\nu_{\mathbb{C}}$, in fact:

$$\begin{aligned} D_{\bar{z}}s &= D_{\bar{z}}((JF_*(\tfrac{\partial}{\partial z}) - [JF_*(\tfrac{\partial}{\partial z})]^T) = \\ &= (\nabla_{F_*\frac{\partial}{\partial \bar{z}}}(JF_*(\tfrac{\partial}{\partial z})))^\perp - (\nabla_{F_*\frac{\partial}{\partial \bar{z}}}([JF_*(\tfrac{\partial}{\partial z})]^T))^\perp , \end{aligned} \tag{4.5}$$

where D is the covariant derivative in the normal bundle and ∇ is the covariant derivative on N. The first term vanishes because J is parallel and F is harmonic: the second term vanishes also. To prove this first observe that

$$\begin{aligned}[JF_*(\tfrac{\partial}{\partial z})]^T &= \tfrac{1}{\lambda}[(JF_*(\tfrac{\partial}{\partial z})\cdot F_*(\tfrac{\partial}{\partial \bar z}))F_*(\tfrac{\partial}{\partial \bar z})+ \\ &\quad +(JF_*(\tfrac{\partial}{\partial z})\cdot F_*(\tfrac{\partial}{\partial z}))F_*(\tfrac{\partial}{\partial z})]\end{aligned} \tag{4.6}$$

but

$$JF_*(\frac{\partial}{\partial z})\cdot F_*(\frac{\partial}{\partial \bar z}) = 0 \tag{4.7}$$

and then

$$(\nabla_{F_*\frac{\partial}{\partial \bar z}}[JF_*(\frac{\partial}{\partial z})]^T)^\perp = \frac{1}{\lambda}[(JF_*(\frac{\partial}{\partial z})\cdot F_*(\frac{\partial}{\partial z}))(\nabla_{F_*\frac{\partial}{\partial \bar z}}F_*\frac{\partial}{\partial z})]^\perp \tag{4.8}$$

and then again harmonicity (i.e. $\nabla_{F_*\frac{\partial}{\partial \bar z}}F_*\frac{\partial}{\partial z} = 0$) proves our claim.

Then $\frac{\bar s}{|s|^2}$ is a local meromorphic section of $\nu_{\mathbb{C}}$: in fact we have $D_z s = fs$, where f is a complex valued function such that $\frac{\partial(s\cdot\bar s)}{\partial z} = fs\cdot\bar s$ and therefore $f = \frac{\partial log|s|^2}{\partial z}$. So we have

$$\begin{aligned}D_{\bar z}(\tfrac{\bar s}{|s|^2}) &= \tfrac{\partial}{\partial \bar z}(\tfrac{1}{|s|^2})\bar s + \tfrac{1}{|s|^2}D_{\bar z}(\bar s) = (\tfrac{\partial}{\partial \bar z}(\tfrac{1}{|s|^2}) + \tfrac{1}{|s|^2}\bar f)\bar s = \\ &= [\tfrac{\partial}{\partial \bar z}(\tfrac{1}{|s|^2}) + \tfrac{1}{|s|^4}\tfrac{\partial}{\partial \bar z}(|s|^2)]\bar s = 0\end{aligned} \tag{4.9}$$

Since the Gauss lift omits the open set W of S^2 and it is holomorphic, taking $p\in W$, the function h defined by the conjugate of $\rho\circ\pi\circ\tilde F_+$, where ρ is the stereographic projection from the point p, is bounded and holomorphic. So M admits (see Ahlfors-Sario [2] and Springer [52]) a square integrable holomorphic differential β. In a local chart (U,z), $\beta = \zeta dz$. Hence $\sigma = \frac{\bar s}{|s|^2}h\zeta$ is a global meromorphic section of $\nu_{\mathbb{C}}$: in fact, if (V,w) is another local chart such that $U\cap V\neq\emptyset$ we have $\beta = \zeta' dw$, where $\zeta' = \frac{\partial z}{\partial w}\zeta$ and $\frac{\partial}{\partial w} = \frac{\partial z}{\partial w}\frac{\partial}{\partial z}$ on $U\cap V$ and so $\sigma(w) = \sigma(z)$ on $U\cap V$.

To prove that σ is square integrable we look at the zeros of s which are the points where

$$(g(JF_*(\frac{\partial}{\partial z}),\sqrt{\frac{1}{2}}(e_3 - ie_4)),g(JF_*(\frac{\partial}{\partial z}),\sqrt{\frac{1}{2}}(e_3 + ie_4))) = (\sqrt{\lambda}sin\alpha_J,0) = (0,0)\,.$$

Hence they are the anti-complex points of F with respect to J (because there are no complex points w.r.t J by assumption).

We have then

$$|\frac{\bar{s}}{|s|^2}h\zeta| = |\frac{1}{1-cos\alpha}|\frac{|\zeta|}{\sqrt{\lambda}}$$

which is a locally bounded function. Hence

$$|\frac{\bar{s}}{|s|^2}h\zeta|^2 \leq C|\beta|^2$$

and then is in L^2, since β is in L^2. □

Chapter 5

On the Torelli and the Infinitesimal Torelli Theorems for Minimal Immersions

5.1 Introduction

In this Chapter we study rigidity of minimal immersion of closed surfaces in flat tori. First let us recall ([23], [41]):

Theorem 5.1.1 *If f_1 and f_2 are homotopic harmonic maps of a closed Riemann surface in a flat torus, then they differ by a translation.*

This suggested to study the following:

Problem 8 *Given a flat torus $T^n = (\mathbb{R}^n/\Lambda, eucl)$ and two homotopic mimimal immersions of a surface of genus r into T^n, are they congruent?*

For $n = 3$ Meeks ([34]) has shown that the answer is negative for surfaces of genus 3. We want to study this problem for $n = 2r$. An indication that the answer could be affermative comes from a well known theorem in Algebraic Geometry, known as the Torelli Theorem:

Theorem 5.1.2 *The jacobians of two Riemann surfaces are isomorphic as polarized abelian varieties if and only if the two Riemann surfaces are biholomorphic.*

Holomorphic maps in a fixed Abelian variety A are special minimal maps in a fixed flat torus, just by choosing any flat metric hermitian w.r.t. the complex structure on A. The Torelli theorem then states that the equivalence class of a jacobian and a topological information (the polarization) determine uniquely the candidate conformal structure for the Riemann surface to embed holomorphically in it.

For our approach a more sensible way to state Torelli's theorem is the following: consider a symplectic basis of closed curves $\{\alpha_i, \beta_i\}$ $i = 1, \ldots, r$ on the topological surface of genus r, and let us indicate by $\mathcal{T}_r$ the moduli space of Riemann surfaces of genus r marked in this way. We can then define a map, called the *period map*, in this way:

$$\Pi: \mathcal{T}_r \to M(r \times 2r, \mathbb{C})$$

$$\Pi([\Sigma_r, \{\alpha_i, \beta_i\}]) = (\int_{\alpha_i} \omega_j, \int_{\beta_i} \omega_j) \ ,$$

where $\{\omega_i\}$ is the basis of holomorphic differentials on Σ_r dual to the curves $\{\alpha_1, \ldots, \alpha_r\}$. *Another way to state Torelli's theorem is that* Π *is injective.* This suggested to try to see the holomorphic as a special case of Problem 8.

By the Weierstrass Representation Theorem ([31]), we know that a conformal minimal immersion $\phi: \Sigma_r \to (\mathbb{R}^n/\Lambda, eucl)$ of a closed Riemann surface of genus r is given, up to a translation, by

$$\phi(p) = Re(\int_{p_0}^{p} \omega_1, \ldots, \int_{p_0}^{p} \omega_n) \ ,$$

with the following conditions

1. ω_i are holomorphic differentials on Σ_r,

2. $$\sum_{i=1}^{n} \omega_i^2 = 0 \ . \qquad (5.1)$$

3. $\{Re(\int_\sigma \omega_1, \ldots, \int_\sigma \omega_n) | \sigma \in H_1(\Sigma_r, \mathbb{Z})\}$ is a sublattice of Λ.

In order to solve Problem 8 it is convenient to keep track of a fixed marking of the Riemann surface (i.e. of a basis of curves for the first homology of the topological surface of genus r), as we have seen for holomorphic maps. In fact we do not want to know just if two minimal immersions can give rise to the same set of periods, but also if these periods come from the same set of closed curves on the surface. We will therefore consider the space

$$\begin{aligned} \mathcal{M}_r = \{[\Sigma_r, \alpha_i, \beta_i], \omega_1, \dots, \omega_{2r} \mid \ & [\Sigma_r, \alpha_i, \beta_i] \in \mathcal{T}_r \, . \textstyle\sum_{i=1}^{2r} \omega_i^2 = 0, \\ & \{\omega_j\} \text{ are independent over the reals}\} \, . \end{aligned} \tag{5.2}$$

We can now generalize in an obvious way the notion of period mapping for minimal immersions. It is in fact immediate to check that Π is well defined by the following formula:

$$\Pi \colon \mathcal{M}_r \to Gl(2r, \mathbb{R})$$

$$\Pi(p) = Re \begin{pmatrix} \int_{\alpha_1} \omega_1 & \dots & \int_{\alpha_r} \omega_1 & \int_{\beta_1} \omega_1 & \dots & \int_{\beta_r} \omega_1 \\ \int_{\alpha_1} \omega_2 & \dots & \int_{\alpha_r} \omega_2 & \int_{\beta_1} \omega_2 & \dots & \int_{\beta_r} \omega_2 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \int_{\alpha_1} \omega_{2r} & \dots & \int_{\alpha_r} \omega_{2r} & \int_{\beta_1} \omega_{2r} & \dots & \int_{\beta_r} \omega_{2r} \end{pmatrix} \tag{5.3}$$

Our first result is the following:

Theorem 5.1.3 Π *is not injective for $r \geq 3$; by Theorem 5.1.1 the points giving the same periods correspond to distinct Riemann surfaces.*

The proof of Theorem 5.1.3 allows us to refine the information about the induced conformal structures coming from Hartman's Uniqueness Theorem:

Corollary 5.1.1 *There exist two minimal immersions, one inducing a nonhyperelliptic structure and another inducing a hyperelliptic structure, which give rise to the same periods.*

We recall that the moduli space of flat tori (i.e. flat tori up to isometries) is given by (see Chapter 3):

$$O(2r,\mathbb{R})\backslash^{Gl(2r,\mathbb{R})}/_{Sl(2r,\mathbb{Z})} . \tag{5.4}$$

From the geometric point of view it is natural to consider two minimal immersions in two different tori to be the same if one is the composition of the other with an isometry of the torus. Therefore a more sensible definition of period map would be $\overline{\Pi}(p) = [[\Pi(p)]]$, where [[ ]] indicates the equivalence class in (5.4). But the elements of $Sl(2r,\mathbb{Z})$ leave the lattice fixed and then by composing a minimal immersion with such an element (different from the identity) we would get a minimal immersion in the same torus, but with a different action on the first homology groups. We can therefore disregard this action in studying Problem 8.

We observe that the orthogonal group $O(2r,\mathbb{R})$ acts on $\mathcal{M}_r$ too in the following way: since we are looking at full minimal immersions we can write, up to reordering of the coordinates in $\mathbb{R}^{2r}$, a point in $\mathcal{M}_r$ as $p = \{[\Sigma_r,\alpha_i,\beta_i],\underline{\omega}(Id_r\ A)\}$, where $\underline{\omega} = (\omega_1,\ldots,\omega_r)$ is a basis of holomorphic differentials of Σ_r, and A is a complex $r\times r$ matrix, with invertible imaginary part. We can then define $O(p) = \{[\Sigma_r,\alpha_i,\beta_i],\underline{\omega}(Id_r\ A)O\}$.

It is immediate to see that $\Pi(O(p)) = \Pi(p)O$. The above discussion implies the following:

Proposition 5.1.1 *The answer to Problem 8 is affirmative if and only if the induced map* $\tilde{\Pi}\colon \mathcal{M}_r/O(2r,\mathbb{R}) \to Gl(2r,\mathbb{R})/O(2r,\mathbb{R})$ *is injective.*

Theorem 5.1.3 implies therefore that the answer is negative for $r\geq 3$. It became natural then to ask whether the flat metrics for which such minimal immersions exist are special in the moduli space of metrics on a torus. Our next result shows that this is not the case:

Theorem 5.1.4 *For every flat torus* $(T^{2r},eucl)$, $r\geq 3$, *there exists an injective homomorphism* ρ *from* $H_1(\Sigma_r,\mathbb{Z})$ *to* $H_1(T^{2r},\mathbb{Z})$, *and two (non congruent) minimal immersions from* Σ_r *to* $(T^{2r},eucl)$ *inducing* ρ *in homology inducing two different conformal structures on the surface.*

We can also show that one of the two minimal immersions in Theorem 5.1.4 can be choosen among stable ones, but we do not know whether it is possible to choose **both** such maps to be stable.

For $r = 2$ we will show that Π is injective (see below for a discussion of this case) using a very different approach. We will see that this implies also that Problem 8 has an affermative solution in this case.

Because of the negative solution the our basic problem, it became natural to study whether there could exist families of homotopic non congruent minimal immersion in a fixed flat torus. We have therefore studied the following question:

- Is Π an immersion?

Clearly this question makes sense only at smooth point of $\mathcal{M}_r$. In section 2 we study the geometry of this space, which is linked with classical theory of Algebraic Curves. In particular we have that $\mathcal{M}_2$ is a smooth manifold and that for $r \geq 3$, $\mathcal{M}_r$ is an analytic variety of real dimension $4r^2$, and which is the union of two subvarieties, $\mathcal{M}'_r$ and $\mathcal{M}''_r$ of the same dimension as $\mathcal{M}_r$ and which project on the hyperelliptic locus and the nonhyperelliptic locus respectively. We will see that both $\mathcal{M}'_r$ and $\mathcal{M}''_r$ are smooth. We believe it would be interesting on its own to study the locus $\mathcal{L}_r = \overline{\mathcal{M}''_r} \cap \mathcal{M}'_r$, which contains the points corresponding to the Abel-Jacobi maps of hyperelliptic surfaces.

Remark 5.1.1 *It is clearly possible to adopt a similar point of view to study minimal immersions of Σ_r into T^n, defining a suitable moduli space $\mathcal{M}^n_r$, for any n greater or equal to three. The three dimensional case has been extensively studied by Pirola in [45]. The geometry of $\mathcal{M}^3_r$ and $\mathcal{M}^4_r$ is related to very classical problems in the theory of algebraic curves. For example the structure of the singularities of the irreducible components of these moduli spaces for $r > 6$ is still subject of current research (see [4] for $r = 4, 5$), and is closely related to the singularities of the singular locus of the theta divisor.*

Once again when restricting our attention to those minimal maps which are holomorphic w.r.t. a fixed complex structure compatible with the metric, this problem has been extensively studied in Algebraic Geometry and is known in the literature as Infinitesimal Torelli Theorem. In this very special context Oort-Steebrink ([42]) proved the hyperelliptic Riemann surfaces of genus $r \geq 3$ are points where the differential of the classical period map has some non trivial kernel, while they proved it to be an immersion restricting either to the hyperelliptic locus and to the non hyperelliptic one. We believe that the proofs (see section 2) of our theorems give some insight on the reasons of this phenomenon.

A crucial step to prove our main results is to make a deailed study of the period map at the points corresponding to the Abel-Jacobi maps j_{Σ_r} of hyperelliptic Riemann surface. Using Oort-Steebrink's theorem ([42]) we show that $ker(d\Pi_{|_{\mathcal{M}'_r}}) = 0$ at the points corresponding to the Abel-Jacobi maps of hyperelliptic Riemann surfaces. On the other hand we know that at these points there are non trivial Jacobi fields. This shows that the first order variation of conformal structure inducing such Jacobi fields has to be nonhyperelliptic.

Our main result in this direction is the following:

Theorem 5.1.5 *1. Π is an immersion everywhere for $r = 2$,*

2. For $r \geq 3$ Π is an immersion on an open and dense subset of $\mathcal{M}''_r$ containing the Abel-Jacobi maps of the nonhyperelliptic Riemann surfaces,

3. For $r \geq 3$ $\Pi_{|_{\mathcal{M}'_r}}$ is an immersion on an open and dense subset of $\mathcal{M}'_r$ containing the Abel-Jacobi maps of the hyperelliptic Riemann surfaces.

We refer to section 2 for a precise discussion about the above statement.

We also prove that with suitable restrictions of the domain the period map is injective. First let us observe that we can write $\sum_{i=1}^{2r} \omega_i^2 = 0$ as $\underline{\omega} Q \underline{\omega}^t = 0$, where $\underline{\omega} = (\omega_1, \ldots, \omega_r)$ is a basis of $H^0(K)$ (we can assume this by reordering the coordinates in T^{2r}), and Q is a complex $r \times r$ symmetric matrix. We say that the minimal immersion *induces* a quadric through the canonical curve of Σ_r.

As we have proved in Chapter 3 the minimal immersions which induce the quadric $Q = 0$ are *precisely* the maps holomorphic w.r.t. a complex structure compatible with the metric. In the last section, using an argument similar to one we used in Chapter 2 in the proof of Theorem 2.1.3, we prove that the period map is injective when the domain is restricted to this class of maps. By Corollaries 3.3.3 and 3.3.2 we then directly get the following consequences:

Corollary 5.1.2 *1. The period map is injective for $r = 2$.*

2. The period map is injective on $\mathcal{M}_3''$.

Let us finally recall that all full minimal immersion of a surface of genus r in T^{2r} are in fact embeddings. This was remarked in [35] too, and follows easily from the Abel's Theorem for minimal immersions.

We believe this approach to the study of minimal immersions in flat tori or in the euclidean spaces to be extremely promising as far existence is concerned; unfortunately, since it basically relies on implicit function type arguments, one is still hungry for explicit examples.

5.2 Moduli spaces of minimal immersions and period map.

In this section we give a description of a space which parametrizes the set of full minimal immersions of surfaces of genus $r \geq 2$ in flat tori of real dimension $2r$. The basic tool, which has been recalled in the introduction, is the Weierstrass Representation Theorem for such maps. This theorem tells us that a space parametrizing such maps is given by $\mathcal{M}_r$, defined in section 1. This space is clearly an analytic variety, and the first thing we want to know is its dimension:

Lemma 5.2.1 *For $r \geq 2$ $\mathcal{M}_r$ has real dimension $4r^2$, and, for $r \geq 3$, it is the union of two analytic subvarieties of the same dimension, one $\mathcal{M}_r'$ which projects onto the hyperelliptic locus, and the other $\mathcal{M}_r''$ onto the non-hyperelliptic one.*

Proof: Let us first denote by h the natural projection $h: \mathcal{M}_r \to \mathcal{T}_r$, defined by $h([\Sigma_r, \alpha_i, \beta_i], \omega_1, \ldots, \omega_{2r}) = [\Sigma_r, \alpha_i, \beta_i]$. We first observe that we need to consider separately the case of conformal harmonic maps of hyperelliptic Riemann surfaces from the nonhyperelliptic ones: in fact a $2r$-uple of holomorphic differentials defines a conformal map if it induces, in the sense described in the introduction, a quadric containg the canonical image of the Riemann surface. But the space of quadrics containing the canonical image of a Riemann surface is a vector space whose dimension depends only on the fact that the Riemann surface is hyperelliptic or not. For the same reason (which, we recall, does not hold in the case of $\mathcal{M}_r^n, n < 2r$), we have that taking an open set V'' of a nonhyperelliptic surface contained in the nonhyperelliptic locus, and an open set V' of a hyperelliptic surface contained in the hyperelliptic locus, $h^{-1}(V'')$ and $h^{-1}(V')$ are smooth open sets.

- **Case 1: r=2.** The Teichmüller space of Riemann surfaces of genus 2 has real dimension 6. The choices of basis of holomorphic differentials are parametized by $Gl(2, \mathbb{C})$, therefore giving a space of real dimension 8. By the Weierstrass Representation Theorem a minimal immersion is given by

$$\phi(p) = Re(\int_{p_0}^{p} \underline{\omega}, \underline{\omega}A) ,$$

where $\underline{\omega}$ is basis of holomorphic differentials and A is a complex 2×2 matrix. Observe that in this notation the conformality equation becomes

$$\underline{\omega}(Id + AA^t)\underline{\omega}^t = 0 . \tag{5.5}$$

But for $r = 2$ there are no non trivial quadrics containg the canonical curve of a Riemann surface and therefore $Id + AA^t = 0$. This means that we can choose A in the space of orthogonal complex matrices (multiplied by i), which has real dimension 2. Then the dimension of $\mathcal{M}_2$ is $6 + 8 + 2 = 16 = 4 \cdot 2^2$ as claimed.

- **Case 2: Σ_r nonhyperelliptic.** Consider now a fixed nonhyperelliptic Riemann surface of genus $r \geq 3$. It follows from Noether's Theorem (see [4]) that the space of quadrics in P^{r-1} containing the canonical image of the surface is a vector space of real dimension $(r-2)(r-3)$. Therefore, using the same argument as above, around this surface we have an open subset of $\mathcal{M}_r$ of dimension

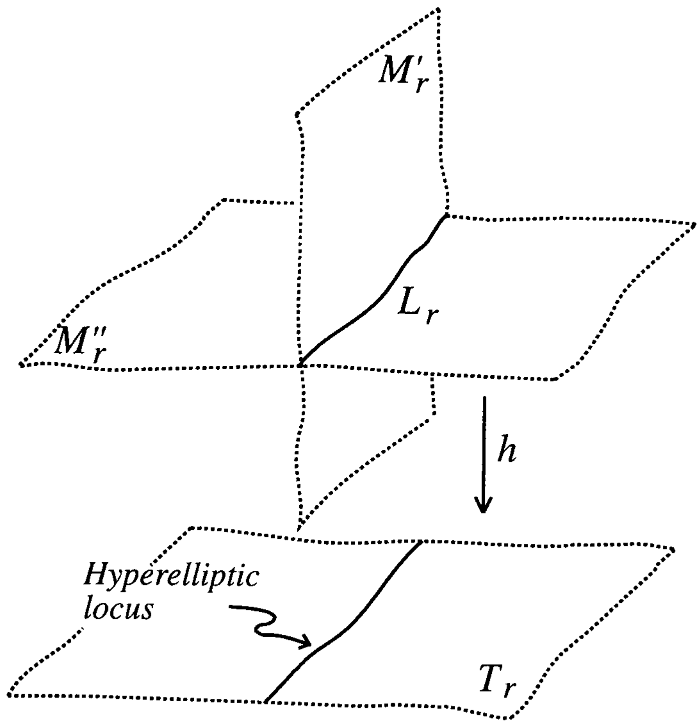
$$6r - 6 + 4r^2 + (r-2)(r-3) - r(r+1) = 4r^2 \ .$$

- **Case 3: Σ_r hyperelliptic.** In the same way we know that the vector space of quadrics in P^{r-1} containg the rational normal curve (i.e. the image of a hyperelliptic surface via the canonical map) is a vector space of real dimension $(r-1)(r-2)$. Once again restricting ourselves to an open set of the hyperelliptic locus we get a manifold of dimension

$$4r - 2 + 4r^2 + (r-2)(r-1) - r(r+1) = 4r^2 \ .$$

The lemma follows now directly. □

Remark 5.2.1 *The geometry of $\mathcal{M}_r$ deserves some comments. By lemma 5.2.1 we have that not every point in $\mathcal{M}'_r$ is a limit of points of $\mathcal{M}''_r$, despite the nonhyperelliptic locus is dense in $\mathcal{T}_r$. The points of $\mathcal{M}'_r$ with this property, i.e. the set $\overline{\mathcal{M}''_r} \cap \mathcal{M}'_r = \mathcal{L}_r$, seems to us to correspond to very special minimal immersions of hyperelliptic Riemann surfaces. For example the Abel-Jacobi maps j_{Σ_r} of hyperelliptic surfaces give points in $\mathcal{L}_r$, but clearly these are not all. We believe a more detailed investigation of this set to be interesting on its own.*



Proof of Theorem 5.1.5: By the above lemma the period map has domain and codomain of the same dimension, and therefore by proving theorem 5.1.3 we will prove that it is a local isomorphism around an open and dense subset of $\mathcal{M}_r$. The crucial step in this direction has been made by Pirola in [45], see also [6], who proved the following:

Theorem 5.2.1 *If $p \in \mathcal{M}_r$, there is an isomorphism ν between $ker d\Pi_p$ and the space of Jacobi vector fields along the minimal immersion corresponding to p, modulo the constant vector fields.*

This theorem is our fundamental tool in the proofs of Theorems 5.1.3 and 5.1.5.

The proof of Theorem 5.1.5 in fact reduces to exibit minimal immersions free of branch points with just $2r$ independent Jacobi fields. The (real) analycity of the period map then shows that if there is a point where it is an immersion, it has to be an immersion on an open and dense subset of the domain. Some care is required since, because of the subtle geometry of $\mathcal{M}_r$ described in Lemma 5.2.1 and in the above remark, we need to find such immersions in $\mathcal{M}'_r$ and in $\mathcal{M}''_r$.

The existence of points in $\mathcal{M}'_r$ corresponding to minimal immersions with the least number of Jacobi fields can be proved showing directly that the Abel-Jacobi map of the nonhyperelliptic Riemann surfaces has just trivial vector fields as we have shown in Chapter 3. This is not the case for the Abel-Jacobi map of a hyperelliptic Riemann surface of genus $r \geq 2$. We want to study in detail what happens at the points of $\mathcal{L}_r$ corresponding to the Abel-Jacobi maps of hyperelliptic surfaces.

By a result of Simons ([50]), we know that the space of Jacobi fields is given by $H^0(\nu(\Sigma_r))$ (see [38] for an alternative simpler proof). We consider now the following exact sequence:

$$0 \to T\Sigma_r \to \mathbb{C}^r \to \nu\Sigma_r \to 0$$

and the associated long exact sequence:

$$0 \to H^0(\mathbb{C}^r) \xrightarrow{\chi} H^0(\nu\Sigma_r) \xrightarrow{\psi} H^1(T\Sigma_r) \xrightarrow{\phi} H^1(\mathbb{C}^r) \to \cdots$$

Since this sequence is exact we have immediately $h^0(\nu\Sigma_r) = dim_{\mathbb{C}} H^0(\nu\Sigma_r) \geq r$. By Serre-Kodaira duality $H^1(T\Sigma_r) = (H^0(2K))^*$ and $H^1(\mathbb{C}^r) = (H^0(\mathbb{C}^r \otimes K))^*$, where K is the canonical bundle of Σ_r. As we have seen in the proof of the main results of Chapter 3, we recall that $im\phi^* = imN$, where $N\colon H^0(K) \otimes H^0(K) \to H^0(2K)$ is the symmetric product of holomorphic differentials. It is a classical result ([19]) that , since Σ_r is hyperelliptic, $dim_{\mathbb{R}} imN = 4r - 2$. This implies that $dim_{\mathbb{R}} ker\phi (= dim_{\mathbb{R}} im\psi) = 2r - 4$.

Therefore $dim_{\mathbb{R}} H^0(\nu\Sigma_r) = 4r - 4$ and then there exist $2r - 4$ non trivial Jacobi fields independent over the reals. By an explicit calculation Oort and Steebrink ([42], pag. 174-177) proved the following result: let Π_h be the restriction of the period map to the set of maps holomorphic w.r.t. a fixed complex structure on $\mathbb{R}^{2r}$. Then $kerd(\Pi_h)_{\Sigma_r}$ is transversal to $(T_{\Sigma_r} H_r)$, where H_r denotes the hyperelliptic locus.

Hence we have

- $ker(d\Pi)_{j_{\Sigma_r}} = kerd(\Pi_h)_{\Sigma_r} = im\psi$,
- $im\psi \oplus T_{\Sigma_r} H_r = T_{\Sigma_r} T_r$ and
- $im\psi \cap T_{\Sigma_r} H_r = \emptyset$.

Suppose now that there exists $\gamma(t) \in \mathcal{M}_r$, s.t. $\gamma(0)$ corresponds to the Abel-Jacobi map of a hyperelliptic Riemann surface Σ_r, j_{Σ_r}, and that $\frac{d}{dt}\Pi(\gamma(t))_{|_{t=0}} = 0$. By the mentioned result of Pirola ([45]), we have that γ induces a non trivial Jacobi vector field

v along j_{Σ_r} (unless $h(\gamma(t))$ does not depend on t, and therefore $\gamma(t)$ would be constant up to translations by Theorem 5.1.1), but, as we observed above, $0 \neq \psi(v) \in Ker\, d\Pi_h$, and therefore the first order deformation of conformal structure induced by γ has to be non hyperelliptic, and then $\frac{d}{dt}\gamma_{|t=0}$ can not be a tangent vector to $\mathcal{M}'_r$ at j_{Σ_r}. This proves that despite the existence of non trivial Jacobi fields along such maps the period map restricted to $\mathcal{M}'_r$ is a local isomorphisms, since the deformations induced by such vector fields push the conformal structure out of the hyperelliptic locus.

This argument and Theorem 5.2.1 prove the Theorem 5.1.5. □

Remark 5.2.2 *By the discussion in the proof of theorem 5.1.5, it follows directly that the generic minimal immersion has the least number of Jacobi fields. It seems plausible to conjecture that the points in $\mathcal{L}_r$ are* precisely *the ones corresponding to mimimal immersions with non trivial Jacobi fields.*

5.3 Non injectivity of the period map.

In this section we use the tools of the previous section to prove that the period map can not be injective.

Proof of Theorem 5.1.3 and of Corollary 5.1.1:

Consider the Abel-Jacobi map ϕ_0 of a hyperelliptic Riemann surface of genus $r \geq 3$. We can clearly find a sequence of nonhyperelliptic Riemann surfaces of the same genus whose Abel-Jacobi maps ϕ_t converge to ϕ_0. As recalled in section 2 at ϕ_t the period map is a local isomorphism since they have just trivial Jacobi fields, and moreover $\Pi_{|\mathcal{M}'_r}$ is a local isomorphism at ϕ_0 too; therefore we get:

- $\Pi(\mathcal{M}'_r) \cap \Pi(\mathcal{M}''_r) \neq \emptyset$.

Since $\mathcal{M}'_r$ and $\mathcal{M}''_r$ fiber over the hyperelliptic and the nonhyperelliptic locus respectively, not just the maps are different but also the induced conformal structures.

□

Remark 5.3.1 *We now observe that if $\Pi(\mathcal{M}'_r) \cap \Pi(\mathcal{M}''_r) \neq \emptyset$ we have that there exists an open set $\mathcal{U}$ of $Gl(2r,\mathbb{R})$ s.t. $\mathcal{U} \subset \Pi(\mathcal{M}'_r) \cap \Pi(\mathcal{M}''_r)$, again because Π is a local isomorphism at ϕ_t for all t and $\Pi_{|\mathcal{M}'_r}$ is a local isomorphism at ϕ_0.*

By the above remark, and since around every ϕ_t all minimal immersions are stable, we have that we can choose $\mathcal{U}$ in such a way that for every $p \in \mathcal{U}$, $\Pi^{-1}(p) \cap \mathcal{M}''_r$ contains a stable minimal immersion. We therefore easily get the following:

Corollary 5.3.1 *For every flat torus $(T^{2r}, eucl)$, there exists an injective homomorphism $\rho: H_1(\Sigma_r, \mathbb{Z}) \to H_1(T^{2r}, \mathbb{Z})$, and two full minimal immersions ϕ, ψ such that*

1. *$\phi_* = \psi_* = \rho$,*

2. *ϕ is stable,*

3. *ϕ induces a nonhyperelliptic conformal structure on the surface, while ψ induces a hyperelliptic one.*

Proof: The set of flat tori wich satisfy the properties of the corollary contains an open set $\mathcal{W}$ by Remark 5.3.1. But given any torus T^{2r}, there exists an isogeny $\mathcal{I}: T^{2r} \to \tilde{T}^{2r}$, where $\tilde{T}^{2r} \in \mathcal{W}$. Therefore $\phi_o\mathcal{I}$ and $\psi_o\mathcal{I}$ are full conformal minimal immersions in T^{2r} with the same action on homology. Moreover, by the above remark, we can choose ϕ to be stable; this easily implies that $\phi_o\mathcal{I}$ is stable too, for every $\mathcal{I}$: in fact every variation of $\phi_o\mathcal{I}(\Sigma_r)$ lifts to a variation of $\phi(\Sigma_r)$ for which $\partial^2\mathcal{A}$ is the same. □

We want now to study the period map for $r = 2, 3$. By the mentioned Corollaries 3.3.3 and 3.3.2, we know that every minimal immersion of a surface of genus 2 into any flat T^4 has to be holomorphic w.r.t. some complex structure compatible with the metric.

We want to prove that the period map Π is injective when we restrict the domain to this class of maps. Suppose we have a minimal immersion $\phi: \Sigma_r \to (T^{2r}, eucl)$ holomorphic w.r.t. J and s.t. it induces an isomorphism on homology. As we have seen in Chpater 3 we can associate to such a map a 2-form ω, which pulls back via

ϕ to the intersection form on the surface. As in the proof of Theorem 2.1.3, we have that (T^{2r}, ω, J) is isomorphic, as principally polarized abelian variety, to the jacobian of Σ_r. Moreover J has to be the *unique* complex strucutre compatible with the flat metric and which is positively polarized by ω. Therefore J is completely determined by the isomorphism on homology and by the the flat metric. It is then impossible, by the classical Torelli Theorem, to have two distinct Riemann surfaces, which give rise to the same periods. This proves the following generalizations of the Torelli Theorem:

Theorem 5.3.1 Π *is injective on the set of minimal immersions holomorphic w.r.t. (also possibly different) compatible complex structures.*

The above theorem leads directly to the following consequences:

Corollary 5.3.2 *1. For $r = 2$, Π is injective.*

2. Π is injective on $\mathcal{M}_3''$.

3. For $r \geq 3$, Π is injective on the subset of $\mathcal{M}_r'$ corresponding to stable minimal immersions.

A simple corollary of Theorem 5.3.1 is that the map φ constructed in Corollary 5.3.1 has to be *unstable* for $r = 3$ since every stable minimal immersion inducing a hyperelliptic structure has to be holomorphic w.r.t. some compatible complex structure.

It would be then interesting to know whether the period map is injective restricting the domain to the set of stable minimal immersion also of non hyperelliptic surfaces. Clearly this question is a special case of the Conjecture 2 in Chapter 2.

Remark 5.3.2 *The infinitesimal study of the period map led us to show that Π is generically an immersion. It is still reasonable to ask whether it is possible to deform the Abel-Jacobi map of a hyperelliptic Riemann surface through minimal immersions in the same torus. Our analysis of the period map does not give an answer to this problem (except of course for $r = 2$). A special situation is when $r = 3$: if such a family $\phi(t)$ exists, then by the above discussion, the induced conformal structures have to change.*

Then by Oort-Steebrink's theorem it has to become nonhyperelliptic, i.e. $\phi(t) \in \mathcal{M}_3''$ *for* $t \neq 0$. *But the period map is injective on this set, then getting a contradiction. For* $r \geq 4$ *the induced conformal structures still should become nonhyperelliptic, but we do not know whether the period map is injective on* $\mathcal{M}_r''$.

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