Symplectic Dirac Operators and Mp^c-structures

Michel Cahen[1], Simone Gutt[1,2] and John Rawnsley[3]

April 2011

Dedication

Michel Cahen met Josh Goldberg fifty years ago in King's College, London. They shared a common interest in the holonomy group of pseudo Riemannian manifolds. Michel enormously appreciated the kindness of an established scientist for a young PostDoc and admired his constant effort for deep understanding of scientific questions. It is with great pleasure that the three authors dedicate this work (partially inspired by physics) to Josh Goldberg.

Abstract

Given a symplectic manifold (M, ω) admitting a metaplectic structure, and choosing a positive ω-compatible almost complex structure J and a linear connection ∇ preserving ω and J, Katharina and Lutz Habermann have constructed two Dirac operators D and $\widetilde{D}$ acting on sections of a bundle of symplectic spinors. They have shown that the commutator $[D, \widetilde{D}]$ is an elliptic operator preserving an infinite number of finite dimensional subbundles. We extend the construction of symplectic Dirac operators to any symplectic manifold, through the use of Mp^c structures. These exist on any symplectic manifold and equivalence classes are parametrized by elements in $H^2(M, \mathbb{Z})$. For any Mp^c structure, choosing J and a linear connection ∇ as before, there are two natural Dirac operators, acting on the sections of a spinor bundle, whose commutator $\mathcal{P}$ is elliptic. Using the Fock description of the spinor space allows the definition of a notion of degree and the construction of a dense family of finite dimensional subbundles; the operator $\mathcal{P}$ stabilizes the sections of each of those.

1 Introduction

Symplectic spinors were introduced by Kostant [8] as a means of constructing half-forms for geometric quantization, a notion similar to half-densities but better suited to the symplectic category. In his paper a metaplectic structure is needed to define his spinors, a notion which is topologically the same as admitting a spin structure. This rules out important examples of symplectic manifolds which are not spin such as $\mathbb{C}P^2$. Instead, in [5, 9] it is shown that symplectic manifolds always admit Mp^c-structures (the symplectic analogue of $Spin^c$) and thus there always exist symplectic spinors (in the Mp^c sense) .

By further choosing suitable connections, there exist Dirac operators on any symplectic manifold. The connections required are a pair consisting of a connection in the tangent bundle preserving the symplectic 2-form (it can have torsion) as in the metaplectic case and additionally a Hermitean connection in an associated complex Hermitean line bundle. In geometric quantization this line bundle is related to the prequantum line bundle and there is a topological constraint on its Chern class. In the case of constructing symplectic spinors on a general symplectic manifold this line bundle can be chosen arbitrarily, and can even be taken to be trivial whilst in the metaplectic case it has to be a square root of the symplectic canonical bundle (which may not have even degree).

Apart from our use of Mp^c structures, there is a second place where our methods differ from those of [6]. The construction of symplectic spinors in [6] is based on the Schrödinger representation of the Heisenberg group whilst we use the Fock picture realised as a holomorphic function Hilbert space. This contains polynomial subspaces and so has a dense subspace graded by degree and this degree transfers to the fibres when a positive compatible almost complex structure is chosen. This makes the bundle of symplectic spinors into a direct sum of finite dimensional subbundles $S^k T'M^* \otimes L$ in a very explicit way. Choosing a connection in the tangent bundle which preserves both the symplectic form and the positive compatible almost complex structure, the Dirac operator D splits $D = D' + D''$ where D' involves $(1,0)$ derivatives and D'' involves $(0,1)$ derivatives. An examination of the combination of Clifford multiplications involved in D' shows that it has coefficients which are creation operators in Fock space and D'' has coefficients which are annihilation operators.

The second Dirac operator $\widetilde{D}$ defined in [6], which is formed using the metric instead of the symplectic form to give the isomorphism of tangents with cotangents, can then be written $\widetilde{D} = -iD' + iD''$. Hence the second order operator $\mathcal{P} = i[\widetilde{D}, D] = 2[D', D'']$ will clearly preserve degrees and acts as $-D''D'$ in degree zero, i.e. in the line bundle L. We show, extending the results of [6] to our context, that the symbol of this second order operator coincides with the symbol of the Laplacian $\nabla^*\nabla$ and we give a Weitzenböck-type formula.

In the framework of Spin geometry on an oriented Riemannian manifold (M, g) each tangent space can be modelled on an oriented Euclidean vector space $(V, \tilde{g})$, and one

needs the following ingredients:

- a spinor space S, which is an irreducible representation cl of the Clifford Algebra $Cl(V,G)$ (the associative unital algebra generated by V with $u \cdot v + v \cdot u = -2\tilde{g}(u,v)1$);

- a group G (the group $Spin$ or $Spin^c$) which is a central extension of the group $SO(V,\tilde{g})$ of linear isometries of $(V,\tilde{g})$, with a surjective homomorphism $\sigma : G \to SO(V,\tilde{g})$, and with a representation ρ on the spinor space S such that

$$\rho(h) \circ cl(v) \cdot \circ\rho(h^{-1}) = cl(\sigma(h)v) \text{ for all } h \in G \text{ and } v \in V;$$

- a G-principal bundle P on M with a map $\phi : P \to B(M,g)$ on the $SO(V,\tilde{g})$-principal bundle of oriented orthonormal frames of (M,g) such that

$$\phi(\xi \cdot h) = \phi(\xi) \cdot \sigma(h) \text{ for all } \xi \in P, \quad h \in G;$$

- spinor fields on M are sections of the associated bundle $\mathcal{S} := P \times_{G,\rho} S$; Clifford multiplication yields a map Cl from the tangent bundle $TM = P \times_{G,\sigma} V$ to the bundle of endomorphisms of $\mathcal{S}$;

- a connection on P (which is a 1-form α on P with values in the Lie algebra $\mathfrak{g}$ of G); this induces a covariant derivative ∇ of spinor fields; one assumes that the projection of α on the Lie algebra of $SO(V,\tilde{g})$ is the pullback by $\phi : P \to B(M,g)$ of the Levi Civita connection on $B(M,g)$;

- one defines the Dirac operator acting on spinor fields as the contraction (using the metric g) of the Clifford multiplication and the covariant derivative:

$$D\psi = \sum_a Cl(e_a)\nabla_{e_a}\psi$$

where $e_a(x)$ is an orthonormal frame at $x \in M$.

In the framework of symplectic geometry and Mp^c structures we present all the corresponding steps. For a symplectic manifold (M,ω) each tangent space has a structure of symplectic vector space (V,Ω) and the symplectic Clifford algebra is the unital associative algebra generated by V with the relations $u \cdot v - v \cdot u = \frac{i}{\hbar}\Omega(u,v)1$. An irreducible representation of this algebra corresponds to an irreducible representation of the Lie algebra of the Heisenberg group with prescribed central character equal to $-\frac{i}{\hbar}$. In Section 2, we describe the Fock representation space of the Heisenberg group and we present possible typical fibres of the symplectic spinor bundle, e.a. smooth vectors of this representation.

In Section 3, we describe the group Mp^c which is a circle extension of the symplectic group. It has a character $\eta : Mp^c \to S^1$ whose kernel is the metaplectic group. We give

explicit formulas for the multiplication in Mp^c and for the representation of Mp^c on the spinor space, in terms of a nice parametrisation of the symplectic group as described in [9] choosing a positive compatible complex structure j on (V, Ω). The subgroup $MU^c(V, \Omega, j)$ lying over the unitary group $U(V, \Omega, j)$ is a trivial circle extension of it.

In Section 4, we recall what are Mp^c structures on a symplectic manifold; these always exist and are parametrized by Hermitean complex line bundles over M. Using a positive compatible almost complex structure J on (M, Ω), we build explicitly those structures from their restriction to the unitary frame bundle on M. Connections on Mp^c structures are described in Section 6 and the symplectic Dirac operator and its properties appear in Section 7. We have tried to give a presentation which is self contained; the content of the first sections is essentially taken from [9]. Although a large part of this paper consists in putting together known results, we believe that the point of view is new and opens some new possibilities in symplectic geometry.

2 The Heisenberg group and its Holomorphic Representation

Let (V, Ω) be a finite-dimensional real symplectic vector space of dimension $2n$. The *Heisenberg group* $H(V, \Omega)$ is the Lie group whose underlying manifold is $V \times \mathbb{R}$ with multiplication

$$(v_1, t_1)(v_2, t_2) = (v_1 + v_2, t_1 + t_2 - \tfrac{1}{2}\Omega(v_1, v_2)).$$

Its Lie algebra $\mathfrak{h}(V, \Omega)$ has underlying vector space $V \oplus \mathbb{R}$ with bracket

$$[(v, \alpha), (w, \beta)] = (0, -\Omega(v, w))$$

and is two-step nilpotent. The exponential map is

$$\exp(v, \alpha) = (v, \alpha).$$

In any (continuous) unitary irreducible representation U of the Heisenberg group on a separable Hilbert space $\mathcal{H}$, the centre $\{0\} \times \mathbb{R}$ will act by multiples of the identity: $(0, t) \mapsto e^{i\lambda t} I_{\mathcal{H}}$ for some real number λ which we call the *central parameter*. When $\lambda = 0$, U arises from a representation of the additive group of V, $U(v, t) = e^{i\Omega(v, \mu)}$, on a one-dimensional space. For non-zero λ the irreducible representation is infinite dimensional. It is known that any two unitary irreducible representations with the same non-zero central parameter are unitarily equivalent (Stone–von Neumann Uniqueness Theorem) and one can change the value of λ by scaling: $U'(v, t) = U(cv, c^2 t)$ has parameter $c^2 \lambda$ if U has parameter λ, whilst the representation with parameter $-\lambda$ is the contragredient of that with λ. Thus up to scaling, complex conjugation and equivalence there is just one infinite dimensional unitary irreducible representation. It is fixed by specifying its parameter which we take as $\lambda = -1/\hbar$ for some positive real number $\hbar$.

This infinite dimensional representation can be constructed in a number of ways, for example on $L^2(V/W)$ with W a Lagrangian subspace of (V, Ω) (Schrödinger picture). For our purposes it is most useful to realise it on a Hilbert space of holomorphic functions (Fock picture [3]) due in various forms to Segal [10], Shale [11], Weil [12] and Bargmann [1]. For this we consider the set of positive compatible complex structures (PCCS) j on V. More precisely, a *compatible complex structure* j is a (real) linear map of V which is symplectic, $\Omega(jv, jw) = \Omega(v, w)$, and satisfies $j^2 = -I_V$. These conditions on j imply that $(v, w) \mapsto \Omega(v, jw)$ is a non-degenerate symmetric bilinear form. We say j is *positive* if this form is positive definite. Let $j_+(V, \Omega)$ denote the set of PCCS.

The compatible complex structures on V form a finite number of orbits under conjugation by elements of $Sp(V, \Omega)$ (see Section 3) which are distinguished by the signature of the quadratic form $\Omega(v, jw)$. The stabilizer of a point in the positive orbit consists of those $g \in Sp(V, \Omega)$ with $gjg^{-1} = j$ or equivalently $gj = jg$ so they are complex linear for the complex vector space structure on V defined by j: $(x + iy)v = xv + yj(v)$. They also preserve $\Omega(v, jw)$ and hence preserve the Hermitean structure

$$\langle v, w \rangle_j = \Omega(v, jw) - i\Omega(v, w), \qquad |v|_j^2 = \langle v, v \rangle_j.$$

Thus picking $j \in j_+(V, \Omega)$ gives a complex Hilbert space (V, Ω, j) of complex dimension $n = \frac{1}{2} \dim_{\mathbb{R}} V$. The stabilizer of j is the unitary group $U(V, \Omega, j)$ of this Hilbert space.

If we put $h = 2\pi\hbar$ we may consider the Hilbert space $\mathcal{H}(V, \Omega, j)$ of holomorphic functions $f(z)$ on (V, Ω, j) which are L^2 in the sense that the norm $\|f\|_j$ given by

$$\|f\|_j^2 = h^{-n} \int_V |f(z)|^2 e^{-\frac{|z|_j^2}{2\hbar}} dz$$

is finite where dz denotes the normalised Lebesgue volume on V for the norm $|\cdot|_j$. This holomorphic function Hilbert space has a reproducing kernel or family of coherent states e_v parametrised by V given by

$$(e_v)(z) = e^{\frac{1}{2\hbar}\langle z, v \rangle_j}$$

such that

$$f(z) = (f, e_z)_j$$

where $(f_1, f_2)_j$ is the inner product in $\mathcal{H}(V, \Omega, j)$ giving the norm $\|f\|_j$.

$H(V, \Omega)$ acts unitarily and irreducibly on $\mathcal{H}(V, \Omega, j)$ by

$$(U_j(v, t)f)(z) = e^{-it/\hbar + \langle z, v \rangle_j/2\hbar - |v|_j^2/4\hbar} f(z - v).$$

The Heisenberg Lie algebra $\mathfrak{h}(V, \Omega)$ then has a skew-Hermitean representation on smooth vectors $\mathcal{H}(V, \Omega, j)^\infty$ of this representation (and these vectors include the coherent states e_v). If $f \in \mathcal{H}(V, \Omega, j)^\infty$ we have

$$(\dot{U}_j(v, \alpha)f)(z) = -i\alpha/\hbar f(z) + \frac{1}{2\hbar}\langle z, v \rangle_j f(z) - (\partial_z f)(v).$$

If we put $cl(v) = \dot{U}_j(v, 0)$ we get operators on the smooth vectors $\mathcal{H}(V, \Omega, j)^\infty$ in $\mathcal{H}(V, \Omega, j)$ by

$$(cl(v)f)(z) = \frac{1}{2\hbar}\langle z, v\rangle_j f(z) - (\partial_z f)(v)$$

which are called Clifford multiplication. They satisfy

$$cl(v)(cl(w)f) - cl(w)(cl(v)f) = \frac{i}{\hbar}\Omega(v, w)f.$$

If we extend the representation of $\mathfrak{h}(V, \Omega)$ to its enveloping algebra then $\mathcal{H}(V, \Omega, j)^\infty$ becomes a Fréchet space with seminorms $f \mapsto \|u \cdot f\|_j$ for u in the enveloping algebra, and its dual $\mathcal{H}(V, \Omega, j)^{-\infty}$ can be viewed as containing $\mathcal{H}(V, \Omega, j)$ so we have a Gelfand triple $\mathcal{H}(V, \Omega, j)^\infty \subset \mathcal{H}(V, \Omega, j) \subset \mathcal{H}(V, \Omega, j)^{-\infty}$ on which the enveloping algebra acts compatibly.

It is convenient to write the Clifford multiplication in terms of creation and annihilation operators:

Definition 1 *For $v \in V$ define operators $c(v), a(v)$ on $\mathcal{H}(V, \Omega, j)$ by*

$$(c(v)f)(z) = \frac{1}{2\hbar}\langle z, v\rangle_j f(z), \qquad (a(v)f)(z) = (\partial_z f)(v), \qquad f \in \mathcal{H}(V, \Omega, j). \quad (1)$$

*$c(v)$ is called the **creation operator** in the direction v and $a(v)$ the **annihilation operator**.*

Note, in this definition f is initially taken in the smooth vectors $\mathcal{H}(V, \Omega, j)^\infty$ but can be viewed as acting on $\mathcal{H}(V, \Omega, j)$ or $\mathcal{H}(V, \Omega, j)^{-\infty}$ in the distributional sense.

Polynomials in z form a dense subspace of $\mathcal{H}(V, \Omega, j)$. The operator $c(v)$ acting on a polynomial multiplies it by a linear form so increases its degree by 1 whilst $a(v)$ performs a directional derivative so reduces it by 1. Easy calculations show:

Proposition 2 *The creation and annihilation operators satisfy*

$$cl(v) = c(v) - a(v)); \quad a(v) = c(v)^*; \quad a(jv) = ia(v); \quad c(jv) = -ic(v);$$

$$[a(u), c(v)] = \frac{1}{2\hbar}\langle u, v\rangle_j; \quad [a(u), a(v)] = 0; \quad [c(u), c(v)] = 0.$$

Annihilation and creation operators are essentially the splitting of Clifford multiplication into j-linear and j-anti-linear parts.

One advantage of using this holomorphic realisation of the basic representation of the Heisenberg group is that the action of linear operators on $\mathcal{H}(V, \Omega, j)$ is not only determined by what they do to the coherent states (since the span of the latter is dense), but there is a formula for the operator on general vectors in terms of an integral kernel constructed from the coherent states. This works just as well for unbounded operators so long as they (and their formal adjoint) are defined on the coherent states.

Definition 3 *Let A be a linear operator defined, along with its formal adjoint, on a dense domain in $\mathcal{H}(V, \Omega, j)$ containing the coherent states. We define its **Berezin kernel** [2] to be*

$$A(z, w) = (Ae_w)(z) = (Ae_w, e_z)_j = (e_w, A^*e_z)_j = \overline{(A^*e_z)(w)}.$$

Proposition 4 *Let A be a linear operator and $A(z, w)$ be its Berezin kernel as in Definition 3. Then $A(z, w)$ is holomorphic in z and anti-holomorphic in w. For any f in the domain of A and A^* we have*

$$(Af)(z) = h^{-n} \int_V f(w) A(z, w) e^{-\frac{|w|_j^2}{2\hbar}} \, dw.$$

Example 5 *The Berezin kernel of the identity map $I_{\mathcal{H}}$ of $\mathcal{H}(V, \Omega, j)$ is $I_{\mathcal{H}}(z, w) = \exp \frac{1}{2\hbar} \langle z, w \rangle_j$, the coherent states themselves.*

3 The symplectic and Mp^{c} groups and their Lie algebras

Let (V, Ω) be a symplectic vector space. We denote by $Sp(V, \Omega)$ the Lie group of invertible linear maps $g \colon V \to V$ such that $\Omega(gv, gw) = \Omega(v, w)$ for all $v, w \in V$. Its Lie algebra $\mathfrak{sp}(V, \Omega)$ consists of linear maps $\xi \colon V \to V$ with $\Omega(\xi v, w) + \Omega(v, \xi w) = 0$ for all $v, w \in V$ or equivalently $(u, v) \mapsto \Omega(u, \xi v)$ is a symmetric bilinear form.

$Sp(V, \Omega)$ acts as a group of automorphisms of the Heisenberg group $H(V, \Omega)$ by

$$g \cdot (v, t) = (g(v), t).$$

By composing the representation U_j of $H(V, \Omega)$ on $\mathcal{H}(V, \Omega, j)$ with an automorphism $g \in Sp(V, \Omega)$ we get a second representation of $H(V, \Omega)$ also on $\mathcal{H}(V, \Omega, j)$:

$$U_j^g(v, t) = U_j(g \cdot (v, t)) = U_j(g(v), t)$$

and evidently the representation U_j^g is still irreducible and has the same central parameter $-\frac{1}{\hbar}$. By the Stone–von Neumann Uniqueness Theorem there is a unitary transformation U of $\mathcal{H}(V, \Omega, j)$ such that

$$U_j^g = U \, U_j \, U^{-1}. \tag{2}$$

Since U_j is irreducible the operator U is determined up to a scalar multiple by the corresponding elements g of $Sp(V, \Omega)$, and it is known to be impossible to make a continuous choice U_g which respects the group multiplication. Instead we view the operators U as forming a new group.

Definition 6 *The group $Mp^c(V, \Omega, j)$ consist of the pairs (U, g) of unitary transformations U of $\mathcal{H}(V, \Omega, j)$ and elements g of $Sp(V, \Omega)$ satisfying (2). The multiplication law in $Mp^c(V, \Omega, j)$ is diagonal.*

The map

$$\sigma(U, g) = g$$

is a surjective homomorphism to $Sp(V, \Omega)$ with kernel of σ consisting of all unitary multiples of the identity, so we have a central extension

$$1 \longrightarrow U(1) \longrightarrow Mp^c(V, \Omega, j) \overset{\sigma}{\longrightarrow} Sp(V, \Omega) \longrightarrow 1 \tag{3}$$

which does not split.

3.1 Parametrising the symplectic group

We now describe a useful parametrisation of the real symplectic group, depending on the triple (V, Ω, j) [so we have now chosen and fixed $j \in j_+(V, \Omega)$]. This description is compatible with the Fock representation and will allow an explicit description of the group $Mp^c(V, \Omega, j)$. Consider $GL(V, j) = \{g \in GL(V) \mid gj = jg\}$ and observe that $U(V, \Omega, j) = Sp(V, \Omega) \cap GL(V, j)$. [Viewing V as a complex vector space using j, then $U(V, \Omega, j)$ is isomorphic to $U(n)$ and $GL(V, j)$ to $GL(n, \mathbb{C})$.]

Any $g \in Sp(V, \Omega)$ decomposes *uniquely* as a sum of a j-linear and j-antilinear part,

$$g = C_g + D_g,$$

where $C_g = \frac{1}{2}(g - jgj)$ and $D_g = \frac{1}{2}(g + jgj)$. It is immediate that for any non-zero $v \in V$

$$4\Omega(C_g v, jC_g v) = 2\Omega(v, jv) + \Omega(gv, jgv) + \Omega(gjv, jgjv) > 0$$

so that C_g is invertible and we have :

Lemma 7 *$C_g \in GL(V, j)$ for all $g \in Sp(V, \Omega)$. If $g \in U(V, \Omega, j)$ we have $C_g = g$.*

Set $Z_g = C_g^{-1} D_g$, then $g = C_g(1 + Z_g)$ and Z_g is $\mathbb{C}$-antilinear.

Write $g^{-1} = C_{g^{-1}}(1 + Z_{g^{-1}})$. Equating $\mathbb{C}$-linear and $\mathbb{C}$-antilinear parts in $1 = g^{-1}g$ gives $1 = C_{g^{-1}}(C_g + Z_{g^{-1}} C_g Z_g)$ and $0 = C_{g^{-1}}(Z_{g^{-1}} C_g + C_g Z_g)$, hence

$$Z_{g^{-1}} = -C_g Z_g C_g^{-1} \quad \text{and} \quad 1 = C_{g^{-1}} C_g (1 - Z_g^2).$$

Thus $1 - Z_g^2$ is invertible with $(1 - Z_g^2)^{-1} = C_{g^{-1}} C_g$. We can also decompose a product

$$C_{g_1 g_2}(1 + Z_{g_1 g_2}) = C_{g_1}(1 + Z_{g_1}) C_{g_2}(1 + Z_{g_2})$$

into linear and anti-linear parts

$$C_{g_1 g_2} = C_{g_1}(C_{g_2} + Z_{g_1} C_{g_2} Z_{g_2}) = C_{g_1}(1 - Z_{g_1} Z_{g_2^{-1}}) C_{g_2} \tag{4}$$

and

$$\begin{aligned}
Z_{g_1 g_2} &= C_{g_2}^{-1}(1 - Z_{g_1} Z_{g_2^{-1}})^{-1}(Z_{g_1} C_{g_2} + C_{g_2} Z_{g_2}) \\
&= C_{g_2}^{-1}(1 - Z_{g_1} Z_{g_2^{-1}})^{-1}(Z_{g_1} - Z_{g_2^{-1}}) C_{g_2}
\end{aligned}$$

allowing us to write the group structure in terms of the C and Z parameters.

We claim that the function

$$(g_1, g_2) \mapsto \det(1 - Z_{g_1} Z_{g_2^{-1}})$$

has a smooth logarithm. To see this we determine that the set where Z_g lives is the Siegel domain.

Since $\langle u, v \rangle_j = \Omega(u, jv) - i\Omega(u, v)$, and $\Omega(gu, v) = \Omega(u, g^{-1}v)$ we have

$$\begin{aligned}
\langle gu, v \rangle_j &= \Omega(u, j(-jg^{-1}jv)) - i\Omega(u, g^{-1}v) \\
\langle jgju, v \rangle_j &= \Omega(u, j(-g^{-1}v)) - i\Omega(u, jg^{-1}jv).
\end{aligned}$$

Subtracting and adding the two relations above give

$$\begin{aligned}
2\langle C_g u, v \rangle_j &= \Omega(u, j(g^{-1} - jg^{-1}j)v) - i\Omega(u, (g^{-1} - jg^{-1}j)v) \\
&= \langle u, (g^{-1} - jg^{-1}j)v \rangle_j = 2\langle u, C_{g^{-1}}v \rangle_j \\
2\langle C_g Z_g u, v \rangle_j &= -\Omega(u, j(g^{-1} + jg^{-1}j)v) - i\Omega(u, (g^{-1} + jg^{-1}j)v) \\
&= -\overline{\langle u, (g^{-1} + jg^{-1}j)v \rangle_j} = -2\langle C_{g^{-1}} Z_{g^{-1}} v, u \rangle_j.
\end{aligned}$$

Therefore

$$C_g^* = C_{g^{-1}}.$$

Moreover,

$$1 - Z_g^2 = (C_{g^{-1}} C_g)^{-1} = (C_g^* C_g)^{-1},$$

which is positive definite. Since Z_g is antilinear and $\langle v, w \rangle_j$ is antilinear in w, the function $(v, w) \mapsto \langle v, Z_g w \rangle_j$ is complex bilinear. We have

$$\langle Z_g u, v \rangle_j = \langle C_g Z_g u, C_g^{*-1} v \rangle_j = \langle C_g Z_g u, C_{g^{-1}}^{-1} v \rangle_j = -\langle C_{g^{-1}} Z_{g^{-1}} C_{g^{-1}}^{-1} v, u \rangle_j = \langle Z_g v, u \rangle_j.$$

Hence

Lemma 8 Z_g *has the three properties: Z_g is $\mathbb{C}$-antilinear; $(v, w) \mapsto \langle v, Z_g w \rangle_j$ is symmetric; $1 - Z_g^2$ is self adjoint and positive definite.*

Let $\mathbb{B}(V, \Omega, j)$ be the *Siegel domain* consisting of $Z \in \operatorname{End}(V)$ such that

$$Zj = -jZ, \quad \langle v, Zw \rangle_j = \langle w, Zv \rangle_j, \text{ and } 1 - Z^2 \text{ is positive definite.}$$

Theorem 9 *We have an injective map*

$$Sp(V, \Omega) \to GL(V, j) \times \mathbb{B}(V, \Omega, j) : g \mapsto (C_g, Z_g),$$

whose image is the set $\{(C, Z) \mid 1 - Z^2 = (C^*C)^{-1}\}$.

Indeed, for any such (C, Z), define $g = C(1 + Z)$. We have

$$
\begin{aligned}
\Omega(gu, gv) &= -\Im \langle C(1+Z)u, C(1+Z)v \rangle_j = -\Im \langle (1+Z)u, C^*C(1+Z)v \rangle_j \\
&= \Omega((1+Z)u, (1-Z)^{-1}v) = \Omega(u, (1-Z)^{-1}v) - \Omega(u, Z(1-Z)^{-1}v) \\
&= \Omega(u, v).
\end{aligned}
$$

In order to parametrise $Mp^{\mathrm{c}}(V, \Omega, j)$ in a similar fashion we observe

Proposition 10 *If* $Z_1, Z_2 \in \mathbb{B}(V, \Omega, j)$, *then* $1 - Z_1 Z_2 \in GL(V, j)$ *and its real part,* $\frac{1}{2}((1 - Z_1 Z_2) + (1 - Z_1 Z_2)^*) = \frac{1}{2}((1 - Z_1 Z_2) + (1 - Z_2 Z_1))$, *is positive definite.*

Indeed, one has $\langle Z_2 Z_1 u, v \rangle_j = \langle Z_2 v, Z_1 u \rangle_j = \langle u, Z_1 Z_2 v \rangle_j$ and

$$
\begin{aligned}
\langle (1 - Z_1 Z_2)v &+ (1 - Z_2 Z_1)v, v \rangle_j \\
&= \langle (1 - Z_1)^2 v, v \rangle_j + \langle (1 - Z_2^2)v, v \rangle_j + \langle (Z_1 - Z_2)^2 v, v \rangle_j \\
&= \langle (1 - Z_1^2)v, v \rangle_j + \langle (1 - Z_2)^2 v, v \rangle_j + \|(Z_1 - Z_2)v\|_j^2.
\end{aligned}
$$

Thus $1 - Z_1 Z_2 \in GL(V, j)_+$ where

$$GL(V, j)_+ = \{g \in GL(V, j) \mid g + g^* \text{ is positive definite}\}.$$

Any $g \in GL(V, j)$ can be written uniquely in the form $X + iY$ with X and Y self-adjoint, and $g \in GL(V, j)_+$ when X is positive definite. Positive definite self adjoint operators X are of the form $X = e^Z$ with Z self-adjoint and $Z \mapsto e^Z$ is a diffeomorphism of all self-adjoint operators with those which are positive definite. Given self-adjoint operators X and Y with X positive definite then $X + iY$ has no kernel, so is in $GL(V, j)$. Thus $GL(V, j)_+$ is an open set in $GL(V, j)$ diffeomorphic to the product of two copies of the real vector space of Hermitean linear maps of (V, Ω, j). In particular $GL(V, j)_+$ is simply-connected. Thus there is a unique smooth function $a \colon GL(V, j)_+ \to \mathbb{C}$ such that

$$\det g = e^{a(g)}, \qquad g \in GL(V, j)_+ \tag{5}$$

and normalised by $a(I) = 0$. Further, since $\det$ is a holomorphic function on $GL(V, j)$, a will be holomorphic on $GL(V, j)_+$. Hence $\det(1 - Z_1 Z_2) = e^{a(1 - Z_1 Z_2)}$ and is holomorphic in Z_1 and anti-holomorphic in Z_2.

3.2 Parametrising the Mp^c group

We defined the group $Mp^c(V, \Omega, j)$ as pairs (U, g) with U a unitary operator on $\mathcal{H}(V, \Omega, j)$ and $g \in Sp(V, \Omega)$ satisfying (2). Here we determine the form of the operator U in terms of the parameters C_g, Z_g of g introduced in the previous paragraph. Fixing a $j \in j_+(V, \Omega)$, any bounded operator A on $\mathcal{H}(V, \Omega, j)$ is determined by its Berezin kernel $A(z, v) = (Ae_v, e_z)_j = (Ae_v)(z)$ as in Definition 3.

Since $(e_v)(z) = e^{\frac{1}{2\hbar}\langle z, v\rangle_j} = e^{\frac{1}{4\hbar}\langle v, v\rangle_j}(U_j(v, 0)e_0)(z)$, the kernel $U(z, w)$ of an operator U with $(U, g) \in Mp^c(V, \Omega, j)$ is such that

$$U(z, v) = e^{\frac{1}{4\hbar}(\langle z, z\rangle_j + \langle v, v\rangle_j)}\left(UU_j(v, 0)e_0, U_j(z, 0)e_0\right)_j;$$

so that, if $v = g^{-1}z$, since $UU_j(g^{-1}z, 0) = U_j(z, 0)U$, we have

$$U(z, g^{-1}z) = e^{\frac{1}{4\hbar}(\langle z, z\rangle_j + \langle g^{-1}z, g^{-1}z\rangle_j)}\left(Ue_0, e_0\right)_j.$$

Now $U(z, v)$ is holomorphic in z and antiholomorphic in v, so it is completely determined by its values on $(z, v = g^{-1}z)$. When $v = g^{-1}z = C_{g^{-1}}(1 + Z_{g^{-1}})z$, one can write $z = C_{g^{-1}}^{-1}v - Z_{g^{-1}}z$ and $v = C_g^{-1}z - Z_g v$ so that

$$U(z, v) = (Ue_0, e_0)_j \exp\frac{1}{4\hbar}\{\langle z, C_{g^{-1}}^{-1}v - Z_{g^{-1}}z\rangle_j + \langle C_g^{-1}z - Z_g v, v\rangle_j\}.$$

Hence

Theorem 11 *If $(U, g) \in Mp^c(V, \Omega, j)$ then the Berezin kernel $U(z, v)$ of U has the form*

$$U(z, v) = \lambda \exp\frac{1}{4\hbar}\{2\langle C_g^{-1}z, v\rangle_j - \langle z, Z_{g^{-1}}z\rangle_j - \langle Z_g v, v\rangle_j\} \tag{6}$$

for some $\lambda \in \mathbb{C}$ with $|\lambda^2 \det C_g| = 1$. Moreover $\lambda = (Ue_0)(0) = (Ue_0, e_0)_j$.

The fact that $|\lambda^2 \det C_g| = 1$ comes from

$$
\begin{aligned}
\|Ue_0\|_j^2 = \|e_0\|_j^2 = 1 &= h^{-n}\int_V U(z, 0)\overline{U(z, 0)}e^{-\frac{|z|_j^2}{2\hbar}}dz \\
&= |\lambda|^2 h^{-n}\int_V e^{-\frac{1}{4\hbar}(\langle z, Z_{g^{-1}}z\rangle_j + \langle Z_{g^{-1}}z, z\rangle_j)}e^{-\frac{|z|_j^2}{2\hbar}}dz \\
&= |\lambda|^2 \det(1 - Z_{g^{-1}}^2)^{-\frac{1}{2}} = |\lambda|^2 \det(C_g C_g^*)^{\frac{1}{2}} = |\lambda^2 \det C_g|.
\end{aligned}
$$

We have used the classical result for Gaussian integrals (see for instance [4]):

$$\int_V e^{-\frac{\pi}{2}(\langle z, Z_1 z\rangle_j + \langle Z_2 z, z\rangle_j)}e^{-\pi|z|_j^2}dz = \det(1 - Z_1 Z_2)^{-\frac{1}{2}} = e^{-\frac{1}{2}a(1 - Z_1 Z_2)}$$

We call g, λ given by Theorem 11 the *parameters* of U. To write the multiplication in $Mp^{\mathrm{c}}(V, \Omega, j)$ in terms of those parameters, we observe that

$$
\begin{aligned}
((U_1 U_2)e_0, e_0)_j &= (U_2 e_0, U_1^* e_0)_j = h^{-n} \int_V (U_2 e_0)(z)\overline{(U_1^* e_0)(z)}e^{-\frac{|z|_j^2}{2\hbar}}\, dw \\
&= h^{-n} \int_V U_1(0, z)U_2(z, 0)e^{-\frac{|z|_j^2}{2\hbar}}\, dz \\
&= \lambda_1 \lambda_2 h^{-n} \int_V e^{-\frac{1}{4\hbar}(\langle z, Z_{g_2^{-1}} z\rangle_j + \langle Z_{g_1} z, z\rangle_j)} e^{-\frac{|z|_j^2}{2\hbar}}\, dz \\
&= \lambda_1 \lambda_2 \det(1 - Z_{g_1} Z_{g_2^{-1}})^{-\frac{1}{2}}
\end{aligned}
$$

so that

Theorem 12 *The product in $Mp^{\mathrm{c}}(V, \Omega, j)$ of (U_i, g_i) with parameters g_i, λ_i, $i = 1, 2$ has parameters $g_1 g_2, \lambda_{12}$ with*

$$
\lambda_{12} = \lambda_1 \lambda_2 e^{-\frac{1}{2}a\left(1 - Z_{g_1} Z_{g_2^{-1}}\right)}.
$$

Corollary 13 *The group $Mp^{\mathrm{c}}(V, \Omega, j)$ is a Lie group. It admits a character η given by*

$$
\eta(U, g) = \lambda^2 \det C_g
$$

if g, λ are the parameters of U. Its restriction to the central $U(1)$ is the squaring map.

The inclusion $U(1) \hookrightarrow Mp^{\mathrm{c}}(V, \Omega, j)$ sends $\lambda \in U(1)$ to $(\lambda I_{\mathcal{H}}, I_V)$ which has parameters I_V, λ.

Definition 14 *The metaplectic group is the kernel of η; it is given by*

$$
Mp(V, \Omega, j) = \{(U, g) \in Mp^{\mathrm{c}}(V, \Omega, j) \mid \lambda^2 \det C_g = 1\}
$$

with the multiplication rule given by Theorem 12. It is a double covering of $Sp(V, \Omega)$.

The sequence $1 \longrightarrow U(1) \longrightarrow Mp^{\mathrm{c}}(V, \Omega, j) \overset{\sigma}{\longrightarrow} Sp(V, \Omega) \longrightarrow 1$ does not split, but splits on the double covering $Mp(V, \Omega, j)$.

We let $MU^{\mathrm{c}}(V, \Omega, j)$ be the inverse image of $U(V, \Omega, j)$ under σ so that (3) induces a corresponding short exact sequence

$$
1 \longrightarrow U(1) \longrightarrow MU^{\mathrm{c}}(V, \Omega, j) \overset{\sigma}{\longrightarrow} U(V, \Omega, j) \longrightarrow 1. \tag{7}
$$

$MU^{\mathrm{c}}(V, \Omega, j)$ is a maximal compact subgroup of $Mp^{\mathrm{c}}(V, \Omega)$.

Proposition 15 *If $(U, k) \in MU^{\mathrm{c}}(V, \Omega, j)$ has parameters k and λ then $\lambda(U, k) = \lambda$ is a character of $MU^{\mathrm{c}}(V, \Omega, j)$. If $f \in \mathcal{H}(V, \Omega, j)$ then $(Uf)(z) = \lambda f(k^{-1}z)$, so unlike the exact sequence (3), (7) does split canonically by means of the homomorphism*

$$
\lambda \colon MU^{\mathrm{c}}(V, \Omega, j) \longrightarrow U(1).
$$

This gives an isomorphism

$$
MU^{\mathrm{c}}(V, \Omega, j) \overset{\sigma \times \lambda}{\longrightarrow} U(V, \Omega, j) \times U(1).
$$

In addition we have the determinant character $\det \colon U(V, \Omega, j) \longrightarrow U(1)$ which can be composed with σ to give a character $\det \circ \sigma$ of $MU^c(V, \Omega, j)$. The three characters η, λ and $\det \circ \sigma$ are related by

$$\eta = \lambda^2 \det \circ \sigma. \tag{8}$$

Let $\mathfrak{mp}^c(V, \Omega, j)$ be the Lie algebra of $Mp^c(V, \Omega, j)$. Differentiating (3) gives an exact sequence of Lie algebras

$$0 \longrightarrow \mathfrak{u}(1) \longrightarrow \mathfrak{mp}^c(V, \Omega, j) \overset{\sigma}{\longrightarrow} \mathfrak{sp}(V, \Omega) \longrightarrow 0. \tag{9}$$

We denote by $\eta_* \colon \mathfrak{mp}^c(V, \Omega, j) \longrightarrow \mathfrak{u}(1)$ the differential of the group homomorphism η, and observe that $\frac{1}{2}\eta_*$ is a map to $\mathfrak{u}(1)$ which is the identity on the central $\mathfrak{u}(1)$ of $\mathfrak{mp}^c(V, \Omega, j)$. Hence (9) splits as a sequence of Lie algebras. We shall refer to the component in $\mathfrak{u}(1)$ of an element ξ of $\mathfrak{mp}^c(V, \Omega, j)$ as its *central component* ξ^c.

To end this section we describe the Lie algebra representation of $\mathfrak{mp}^c(V, \Omega, j)$ on (smooth vectors of) $\mathcal{H}(V, \Omega, j)$. Elements of $\mathfrak{mp}^c(V, \Omega, j)$ are given by pairs (μ, ξ) where $\mu \in \mathfrak{u}(1)$ and $\xi \in \mathfrak{sp}(V, \Omega)$ and

$$[(\mu_1, \xi_1), (\mu_2, \xi_2)] = (0, \xi_1\xi_2 - \xi_2\xi_1).$$

Given $(\mu, \xi) \in \mathfrak{mp}^c(V, \Omega, j)$, let (U_t, g_t) be a curve in $Mp^c(V, \Omega, j)$ with parameters (λ_t, g_t) passing through the identity, $(U_0, g_0) = (I_{\mathcal{H}}, I_V)$, and with tangent (μ, ξ) so that $\dot{\lambda}_0 = \mu$ and $\dot{g}_0 = \xi$. Then if we put $C_t = C_{g_t}$ and $Z_t = Z_{g_t}$ we have $\xi = \dot{C}_0 + \dot{Z}_0 = \eta + \zeta$ with $\dot{C}_0 = \frac{1}{2}(\xi - j\xi j) = \eta$ and $\dot{Z}_0 = \frac{1}{2}(\xi + j\xi j) = \zeta$.

We now differentiate equation (6) at $t = 0$

$$\dot{U}_0(z, v) = \left\{ \mu - \frac{1}{2\hbar}\langle \eta z, v \rangle_j + \frac{1}{4\hbar}\langle z, \zeta z \rangle_j - \frac{1}{4\hbar}\langle \zeta v, v \rangle_j \right\} \exp \frac{1}{2\hbar}\langle z, v \rangle_j$$

which is a kernel for the action of (μ, ξ) on smooth vectors of $\mathcal{H}(V, \Omega, j)$, or on all of $\mathcal{H}(V, \Omega, j)$ if we view the result as in $\mathcal{H}(V, \Omega, j)^{-\infty}$. The final step is to convert this to an action on a general smooth vector $f(z)$.

Obviously μ acts as a multiple of the identity. Let $2n = \dim_{\mathbb{R}} V$. For a j-linear map η of V, the operator with kernel $\frac{1}{2\hbar}\langle \eta z, v \rangle_j \exp \frac{1}{2\hbar}\langle z, v \rangle_j$ is given by $f \mapsto f_1$ where

$$
\begin{aligned}
f_1(z) &= h^{-n} \int_V f(v) \frac{1}{2\hbar}\langle \eta z, v \rangle_j \exp \frac{1}{2\hbar}\langle z, v \rangle_j \exp -\frac{|v|_j^2}{2\hbar} \, dv \\
&= \partial_z \left(h^{-n} \int_V f(v) \exp \frac{1}{2\hbar}\langle z, v \rangle_j \exp -\frac{|v|_j^2}{2\hbar} \, dv \right)(\eta z) = (\partial_z f)(\eta z).
\end{aligned}
$$

Clearly the kernel $\frac{1}{4\hbar}\langle z, \zeta z \rangle_j \exp \frac{1}{2\hbar}\langle x, v \rangle_j$ defines an operator $f \mapsto f_2$ where

$$f_2(z) = \frac{1}{4\hbar}\langle z, \zeta z \rangle_j f(z).$$

Finally, the kernel $\frac{1}{4\hbar}\langle \zeta v, v\rangle_j \exp \frac{1}{2\hbar}\langle z, v\rangle_j$ defines an operator $f \mapsto f_3$ where

$$
\begin{aligned}
f_3(z) &= h^{-n}\int_V f(v)\frac{1}{4\hbar}\langle \zeta v, v\rangle_j \exp \frac{1}{2\hbar}\langle z, v\rangle_j e^{-\frac{|v|_j^2}{2\hbar}}\, dv \\
&= h^{-n}\int_V f(v)\frac{1}{2}(\partial_z \exp \frac{1}{2\hbar}\langle z, v\rangle_j)(\zeta v)e^{-\frac{|v|_j^2}{2\hbar}}\, dv.
\end{aligned}
$$

Take an orthonormal basis $e_1, \ldots, e_n$ for V as a complex Hilbert space then

$$
(\partial_z \exp \frac{1}{2\hbar}\langle z, v\rangle_j)(\zeta v) = \sum_{i=1}^n (\partial_z \exp \frac{1}{2\hbar}\langle z, v\rangle_j)(\zeta e_i)\langle e_i, v\rangle_j
$$

so

$$
\begin{aligned}
f_3(z) &= \sum_{i=1}^n \partial_z \left(h^{-n}\int_V f(v)\frac{1}{2}\langle e_i, v\rangle_j \exp \frac{1}{2\hbar}\langle z, v\rangle_j e^{-\frac{|v|_j^2}{2\hbar}}\, dv \right)(\zeta e_i) \\
&= \hbar\sum_{i=1}^n \partial_z \left(h^{-n}\int_V f(v)\partial_z(\exp \frac{1}{2\hbar}\langle z, v\rangle_j)(e_i)e^{-\frac{|v|_j^2}{2\hbar}}\, dv \right)(\zeta e_i) \\
&= \hbar\sum_{i=1}^n \partial_z \left(\partial_z f(e_i) \right)(\zeta e_i)
\end{aligned}
$$

Hence we have shown

Proposition 16 *If $(\mu, \xi) \in \mathfrak{mp}^c(V, \Omega, j)$ and $\xi = \eta + \zeta$ is the decomposition of ξ into the part, η, which commutes with j and the part, ζ, which anti-commutes then the action of (μ, ξ) on spinors is given by*

$$
((\mu, \xi)f)(z) = \mu f - (\partial_z f)(\eta z) + \frac{1}{4\hbar}\langle z, \zeta z\rangle_j f(z) - \hbar\sum_{i=1}^n \partial_z\left(\partial_z f(e_i)\right)(\zeta e_i).
$$

4 Mp^c structures

Definition 17 *Fix a symplectic vector bundle (E, ω) of rank $2n$ over M and let (V, Ω) be a fixed symplectic vector space of dimension $2n$. Then the symplectic frame bundle $\pi\colon Sp(E, \omega) \longrightarrow M$ of (E, ω) is the bundle whose fibre at $x \in M$ consists of all symplectic isomorphisms $b\colon V \longrightarrow E_x$. By composition on the right it becomes a principal $Sp(V, \Omega)$ bundle.*

Definition 18 *By an Mp^c structure on (E, ω) we mean a principal $Mp^c(V, \Omega, j)$ bundle $\pi^P\colon P \longrightarrow M$ with a fibre-preserving map $\phi\colon P \longrightarrow Sp(E, \omega)$ such that the group actions are compatible:*

$$
\phi(p \cdot g) = \phi(p) \cdot \sigma(g), \quad \forall p \in P,\ g \in Mp^c(V, \Omega, j).
$$

We shall use the following notation: If $\pi^P \colon P \to M$ is a principal G-bundle and $\varphi \colon G \to H$ a homomorphism of Lie groups, then $P \times_G H := (P \times H)_{\sim_G}$ denotes the bundle whose elements are equivalence classes of elements in $P \times H$ for the equivalence defined by G, i.e. $(\xi, h) \sim_G (\xi \cdot g, \varphi(g^{-1})h)$ for any $g \in G$. The bundle $P \times_G H$ has a right free H action making it into a principal H-bundle. Equivalence classes will be denoted by square brackets. Given any $U(1)$-bundle $L^{(1)}$, we denote by L the associated Hermitean complex line bundle $L = L^{(1)} \times_{U(1)} \mathbb{C}$ and reciprocally, given any Hermitean line bundle L over M we denote by $L^{(1)} = \{q \in L \mid |q| = 1\}$ the corresponding $U(1)$-bundle. Given two fibre bundles over M, $\pi^F \colon F \to M$ and $\pi^K \colon K \to M$, we denote by $F \times_M K$ the fibre-wise product bundle $F \times_M K := \{\, (\xi, \xi') \in F \times G \mid \pi^F(\xi) = \pi^K(\xi') \,\}$.

Theorem 19 *Every symplectic vector bundle (E, ω) admits an Mp^c structure, and the isomorphism classes of Mp^c structures on (E, ω) are parametrized by line bundles. Choosing a fibre-wise positive ω-compatible complex structure J on E, those symplectic frames which are also complex linear form a principal $U(V, \Omega, j)$ bundle called the unitary frame bundle which we denote by $U(E, \omega, J)$. If P is an Mp^c structure, we look at the subset P_J of P lying over the unitary frames*

$$P_J := \phi^{-1}(U(E, \omega, J)).$$

This will be a principal $MU^c(V, \Omega, j) \simeq_{\sigma \times \lambda} U(V, \Omega, j) \times U(1)$ bundle. Clearly

$$P \simeq P_J \times_{MU^c(V, \Omega, j)} Mp^c(V, \Omega, j).$$

We denote by $P_J^{(1)}(\lambda) := P_J \times_{MU^c(V, \Omega, j), \lambda} U(1)$ the $U(1)$ principal bundle associated to P_J by the homomorphism λ. We have a map $\tilde{\lambda} : P_J \to P_J^{(1)}(\lambda) : \xi \mapsto [(\xi, 1)]$. Then

$$\phi \times \tilde{\lambda} : P_J \to U(E, \omega, J) \times_M P_J^{(1)}(\lambda) : \xi \mapsto \phi(\xi), [(\xi, 1)]$$

is an isomorphism, with the right action of $MU^c(V, \Omega, j)$ on the right-hand side given via $\sigma \times \lambda$ by the action of $U(V, \Omega, j)$ on the $U(E, \omega, J)$ and of $U(1)$ on $P_J^{(1)}(\lambda)$.
The line bundle associated to P_J by the character λ is denoted by $P_J(\lambda)$; its isomorphism class is independent of the choice of J.

   Reciprocally, given any Hermitean line bundle L over M

$$(U(E, \omega, J) \times_M L^{(1)}) \times_{MU^c(V, \Omega, j)} Mp^c(V, \Omega, j)$$

has an obvious structure of Mp^c structure over M.

Remark that choosing a J is always possible as the bundle of fibrewise positive ω-compatible complex structures has contractible fibres; any two such positive almost complex structures are homotopic. For a given choice of such a J we denote by g_J the corresponding Riemannian metric on M : $g_J(X, Y) := \omega(X, JY)$.

The map which sends the isomorphism class of P to the isomorphism class of $P_J(\lambda)$ is the required parametrisation of Mp^c structures by line bundles.

If P is an Mp^c structure on (E, ω), let $P^{(1)}(\eta) = P \times_{Mp^c, \eta} U(1)$ denote the U(1) bundle associated to P by the homomorphism η of $Mp^c(V, \Omega, j)$. We denote by $\tilde{\eta}$ the map

$$\tilde{\eta} : P \to P^{(1)}(\eta) : \xi \mapsto [(\xi, 1)].$$

The total Chern class of the unitary structure $U(E, \omega, J)$ depends only on (E, ω) and not on the choice of J. We denote by $c_1(E, \omega)$ the first Chern class of the unitary structure determined by picking any positive ω-compatible complex structure in E. Note that $c_1(E, \omega)$ is the first Chern class of the line bundle associated to the determinant character det of $U(V, \Omega, j)$.

From the relationship (8) between the three characters we have

$$c_1(P(\eta)) = 2c_1(P_J(\lambda)) + c_1(E, \omega).$$

Let $\pi^{L^{(1)}} : L^{(1)} \to M$ be a principal $U(1)$ bundle over M and $\pi^c : P \to M$ an Mp^c-structure then the fibre product $L^{(1)} \times_M P$ is obviously a $U(1) \times Mp^c(V, \Omega, j)$ bundle over M and the associated bundle with fibre $Mp^c(V, \Omega, j)$ to the homomorphism

$$(z, (U, g)) \mapsto (zU, g) \quad : \quad U(1) \times Mp^c(V, \Omega, j) \to Mp^c(V, \Omega, j), \qquad (10)$$

gives a principal $Mp^c(V, \Omega, j)$ bundle $(L^{(1)} \times_M P) \times_{U(1) \times Mp^c(V, \Omega, j)} Mp^c(V, \Omega, j)$. We denote it by $L^{(1)} \cdot P$ or by $L \cdot P$.

Note that $(L^{(1)} \cdot P)(\eta) = L^2 \otimes P(\eta)$. We also have $(L^{(1)} \cdot P)_J(\lambda) = L \otimes P_J(\lambda)$. This gives a simply-transitive action of the isomorphism classes of $U(1)$ bundles on the isomorphism classes of Mp^c-structures on a fixed symplectic vector bundle. The map $P \mapsto c(P) := c_1(P_J(\lambda))$ at the level of isomorphism classes is a bijection with $H^2(M, \mathbb{Z})$. Up to isomorphism the unique Mp^c-structure with $c(P) = 0$ is given by

$$P_0(E, \omega, J) := U(E, \omega, J) \times_{U(V, \Omega, j)} Mp^c(V, \Omega, j)$$

and any Mp^c-structure on E is isomorphic to $L^{(1)} \cdot P_0(E, \omega, J)$ for $L^{(1)} = P_J^{(1)}(\lambda)$.

5 Spinors

Definition 20 *Given a principal $Mp^c(V, \Omega, j)$-bundle P we form the associated bundle $\mathcal{S} = \mathcal{S}(P, V, \Omega, j) = P \times_{Mp^c(V, \Omega, j)} \mathcal{H}(V, \Omega, j)$ and similarly for bundles $\mathcal{S}^\infty$ with fibre $\mathcal{H}^\infty(V, \Omega, j)$ and $\mathcal{S}^{-\infty}$ with fibre $\mathcal{H}^{-\infty}(V, \Omega, j)$. Any of these bundles we call a bundle of symplectic spinors associated to P.*

Remark that

$$
\begin{aligned}
(L^{(1)} \cdot P) \times_{Mp^c(V, \Omega, j)} \mathcal{H}(V, \Omega, j) &\cong (L^{(1)} \times_M P_J) \times_{U(1) \times MU^c(V, \Omega, j)} \mathcal{H}(V, \Omega, j) \\
&\cong L \otimes (P_J \times_{MU^c(V, \Omega, j)} \mathcal{H}(V, \Omega, j)) \\
&\cong L \otimes (P \times_{Mp^c(V, \Omega, j)} \mathcal{H}(V, \Omega, j))
\end{aligned}
$$

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and thus
$$\mathcal{S}(L^{(1)} \cdot P, V, \Omega, j) = L \otimes \mathcal{S}(P, V, \Omega, j)$$

with similar statements for the bundles $\mathcal{S}^{\pm\infty}(P, V, \Omega, j)$.

Remark also that for $P_0(E, \omega, J) = U(E, \omega, J) \times_{U(V,\Omega,j)} Mp^c(V, \Omega, j)$ we have

$$\mathcal{S}_0 := \mathcal{S}(P_0(E, \omega, J)) = U(E, \omega, J) \times_{U(V,\Omega,j)} \mathcal{H}(V, \Omega, j),$$

so that, in the general situation, writing $P = L^{(1)} \cdot P_0$ the spinor space is $L \otimes \mathcal{S}_0$.

The spinor bundle inherits a Hermitean structure from the one on the fibre:

$$h(\psi = [\xi, f]), \psi' := [\xi, f']) := (f, f')_j.$$

If P is an Mp^c-structure on (E, ω) then $E = P \times_{Mp^c(V,\Omega,j),\sigma} V$ acts on the space of spinors $\mathcal{S} = P \times_{Mp^c(V,\Omega,j)} \mathcal{H}(V, \Omega, j)$ by Clifford multiplication

$$Cl \colon E \otimes \mathcal{S} \to \mathcal{S} : (e = [\xi, v]) \otimes (\psi = [\xi, f]) \mapsto Cl(e)\psi := [\xi, cl(v)f];$$

this is well defined because $cl(gv)Uf = Ucl(v)f$ for any (U, g) in $Mp^c(V, \Omega, j)$. When one has chosen a positive compatible almost complex structure J on (E, ω), we consider $E = P_J \times_{MU^c(V,\Omega,j),\sigma} V$, $\mathcal{S} = P_J \times_{MU^c(V,\Omega,j)} \mathcal{H}(V, \Omega, j)$ and define in a similar way annihilation and creation operators:

$$A_J \colon E \otimes \mathcal{S} \to \mathcal{S} : (e = [\xi', v]) \otimes (\psi = [\xi', f]) \mapsto A_J(e)\psi := [\xi', a(v)f];$$

$$C_J \colon E \otimes \mathcal{S} \to \mathcal{S} : (e = [\xi', v]) \otimes (\psi = [\xi', f]) \mapsto C_J(e)\psi := [\xi', c(v)f];$$

this is well defined because $a(gv)Uf = Ua(v)f$ for any (U, g) in $MU^c(V, \Omega, j)$. We have (cf prop2) $A_J(Je) = iA_J(e)$, $C_J(Je) = -iC_J(e)$, $h(A_J(e)\psi, \psi') = h(\psi, C_J(e)\psi')$.

Since the centre of MU^c acts trivially on the Heisenberg group and its Lie algebra, the Clifford multiplication will commute with the process of tensoring with a line bundle. If s is a section of L and ψ is a spinor for P then $s \otimes \psi$ is a spinor for $L^{(1)} \cdot P$ and

$$Cl(X)(s \otimes \psi) = s \otimes (Cl(X)\psi)$$

with similar formulas for creation and annihilation operators.

By Proposition 15, when there is a positive compatible almost complex structure J on (E, ω), the grading of $\mathcal{H}(V, \Omega, j)$ by polynomial degree in the complex variable in (V, j) will pass canonically to the bundles associated to P_J giving a dense sub-bundle of the Hilbert and Fréchet bundles isomorphic to

$$L \otimes (\oplus_p S^p((E')^*))$$

where L is the bundle associated to P_J by the character λ, and E' denotes the $(1,0)$ vectors of J on the complexification $E \otimes_{\mathbb{R}} \mathbb{C}$. The term $L \otimes S^0((E')^*) = L$ of degree zero in the grading is a copy of the line bundle L. This can be identified as the common kernel of the annihilation operators, namely the vacuum states in the Fock picture.

6 Mp^{c} Connections

Definition 21 *An Mp^{c}-connection in an Mp^{c} structure P on a symplectic vector bundle (E, ω) is a principal connection α in P.*

At the level of Lie algebras, $\mathfrak{mp}^{\mathrm{c}}(V, \Omega, j)$ splits as a sum $\mathfrak{u}(1) + \mathfrak{sp}(V, \Omega)$ and so a connection 1-form α on P can be split $\alpha = \alpha_0 + \alpha_1$ where α_0 is $\mathfrak{u}(1)$-valued and α_1 is $\mathfrak{sp}(V, \Omega)$-valued. α_0 is then basic for $\tilde{\eta} : P \to P^{(1)}(\eta)$ so is the pull-back of a $\mathfrak{u}(1)$-valued 1-form β_0 on $P^{(1)}(\eta)$ and α_1 is the pull-back under $\phi : P \to Sp(E, \omega)$ of a $\mathfrak{sp}(V, \Omega)$-valued 1-form β_1 on $Sp(E, \omega)$. Each form β_i, $i = 1, 2$, is then a connection form on the corresponding bundle. Thus an Mp^{c}-connection on P induces connections in E and $P(\eta)$. The converse is true – we pull back and add connection 1-forms in $P^{(1)}(\eta)$ and $Sp(E, \omega)$ to get a connection 1-form on P.

Under twisting these connections behave compatibly. If we have a Hermitean connection in $L^{(1)}$ and an Mp^{c}-connection in P then there is an induced Mp^{c}-connection in $L^{(1)} \cdot P$ and the connection induced in E is unchanged.

To calculate the curvature of a connection α in P we observe that $d\alpha = d\alpha_0 + d\alpha_1$ and $[\alpha \wedge \alpha] = [\alpha_1 \wedge \alpha_1]$ so that $d\alpha_0$ is a basic 2-form and descends to an imaginary 2-form $i\omega^\alpha$ on M. We call ω^α the *central curvature* of α.

Another way to proceed is to take the covariant derivative induced in the line bundle $P(\eta)$ by the connection α and take its curvature 2-form, also an imaginary 2-form. To see how these 2-forms are related we choose a local section $p \colon U \to P$ on some open set. Recalling that $P(\eta) = P \times_{Mp^{\mathrm{c}}(V, \Omega, j), \eta} \mathbb{C}$, we get a section s of $P(\eta)$ on U by setting $s(x) = [p(x), 1]$. The covariant derivative ∇s is given on U by $\nabla_X s = (\eta_*)(p^*\alpha(X)) s$ so that the curvature 2-form is then given on U by
$d((\eta_*)(p^*\alpha)) = (\eta_*)(p^* d\alpha) = (\eta_*)(p^* d\alpha_0)$ since η_* vanishes on brackets and thus on $\mathfrak{sp}(V, \Omega)$. On the centre of $Mp^{\mathrm{c}}(V, \Omega, J)$, η is the squaring map, so $\eta_*(\alpha_0) = 2\alpha_0$. Thus $(\eta_*)(p^* d\alpha_0) = 2p^*\pi^*(i\omega^\alpha) = 2i\omega^\alpha$ since $\pi \circ p = \mathrm{Id}_U$. Since the curvature and the right hand side of this equation are globally defined we have shown:

Theorem 22 *If P is an Mp^{c} structure on (E, ω) and α is a connection in P with central curvature ω^α then the connection induced in the associated line bundle $P(\eta)$ has curvature $2i\omega^\alpha$.*

Remark 23 *The derivative of the map (10) is addition so when we take a connection 1-form γ in a $U(1)$-bundle L and an Mp^{c}-connection α, the connection form in the bundle $L \cdot P$ associated to the fibre product will be $\pi_1^* \gamma + \pi_2^* \alpha$, so the net affect is to add γ to the central component α_0. Hence the central curvature of the connection in $L \cdot P$ is the sum of the curvature of L with the central curvature of P.*

An Mp^{c}-connection α on P induces a connection on E (i.e. a covariant derivative on the space of its sections : $\nabla : \Gamma(M, E) \to \Gamma(M, T^*M \otimes E)$) and a connection on $\mathcal{S}$.

Remark that the Clifford multiplication is parallel

$$\nabla_X Cl(e)\psi = Cl(\nabla_X e)\psi + Cl(e)\nabla_X \psi.$$

A connection in the MU^{c}-structure will be an Mp^{c} connection inducing a connection on E which preserves ω and J. Such a connection induces one on the spinor bundle preserving the grading. The maps A_J and C_J of annihilation and creation are parallel under such a MU^{c}-connection and lower and raise degrees by 1, hence Clifford multiplication mixes up degrees.

7 Symplectic Dirac Operators

In [6] a theory of symplectic Dirac operators is developed based on metaplectic structures. However topologically being metaplectic is the same as being spin and many interesting symplectic manifolds such as $\mathbb{C}P^{2n}$ are not spin. We have seen that all symplectic manifolds have Mp^{c} structures, so all have spinors. We shall define Dirac operators in the Mp^{c} context analogous to [6] and in addition make use of the extra structure of the Fock space picture of the symplectic spinors.

In what follows (M, ω) will be a symplectic manifold and we apply the theory of Mp^{c} spinors to the symplectic vector bundle (TM, ω). Let J be a positive almost complex structure on M compatible with ω, $SpFr(M, \omega)$ and $UFr(M, \omega, J)$ the symplectic and unitary frame bundles where we have fixed some symplectic vector space (V, Ω) and $j \in j_+(V, \Omega)$ with $\dim_{\mathbb{R}} V = \dim M$. Fix an Mp^{c} structure P on (TM, ω) and let $P_J \subset P$ be the MU^{c} reduction determined by J. A connection α in P_J determines a connection in P and so covariant derivatives in associated vector bundles such as TM, $L = P_J(\lambda)$ and $\mathcal{S}$ which we denote by ∇. For these covariant derivatives ω and Cl, A_J, C_J are parallel, but the covariant derivative in TM may have torsion which we denote by T^{∇}.

The (symplectic) Dirac operator is a first order differential operator defined on sections of $\mathcal{S}$ as the contraction, using ω, of the Clifford multiplication and of the covariant derivative of spinor fields. Taking a local frame field e_i for TM, we form the dual frame field e^i which satisfies $\omega(e_i, e^j) = \delta_i^j$ i.e. $e^j = -\sum_k \omega^{jk} e_k$ where ω^{ij} are the components of the matrix inverse to $\omega_{ij} := \omega(e_i, e_j)$. For $\psi \in \Gamma(\mathcal{S})$ we set

$$D\psi := \sum_i Cl(e_i)\nabla_{e^i}\psi = -\sum_{ij} \omega^{ij} Cl(e_i)\nabla_{e_j}\psi$$

which is easily seen to be independent of the choice of frame. In [6] a second Dirac operator is defined using the metric to define the contraction of Cl and $\nabla\psi$. Hence

$$\widetilde{D}\psi := \sum_i Cl(Je_i)\nabla_{e^i}\psi = \sum_{ij} g^{ij} Cl(e_i)\nabla_{e_j}\psi.$$

where g^{ij} are the components of the matrix inverse to $g_{ij} := \omega(e_i, Je_j)$. In the presence of an almost complex structure J it is convenient to write derivatives in terms of their $(1,0)$ and $(0,1)$ parts. That is we complexify TM and then decompose $TM^{\mathbb{C}}$ into the $\pm i$ eigenbundles of J which are denoted by $T'M$ and $T''M$. If X is a tangent vector then it decomposes into two pieces $X = X' + X''$ lying in these two subbundles so $JX' = iX'$ and $JX'' = -iX''$. We can then define

$$\nabla'_X := \nabla_{X'}, \qquad \nabla''_X := \nabla_{X''}$$

after extending ∇ by complex linearity to act on complex vector fields. We can now define two partial Dirac operators D' and D'' by using these operators instead of ∇

$$D'\psi = \sum_i Cl(e_i)\nabla'_{e^i}\psi, \qquad D''\psi = \sum_i Cl(e_i)\nabla''_{e^i}\psi.$$

Proposition 24

$$D = D' + D'', \qquad \widetilde{D} = -iD' + iD''.$$

Proof. The first is obvious since $\nabla = \nabla' + \nabla''$. For the second we observe that if e_i is a frame, so is Je_i and since J preserves ω, the dual frame of Je_i is Je^i. Thus $\widetilde{D}\psi = \sum_i Cl(Je_i)\nabla_{e^i}\psi = -\sum_i Cl(e_i)\nabla_{Je^i}\psi = -\sum_i Cl(e_i)(\nabla'_{Je^i}\psi + \nabla''_{Je^i}\psi)$. But $(JX)' = iX'$ and $(JX)'' = -iX''$ giving the result. $\square$ A nice thing happens due to the behaviour of A_J and C_J with respect to J:

Proposition 25

$$D'\psi = \sum_i C_J(e_i)\nabla_{e^i}\psi = -\sum_{kl} \omega^{kl} C_J(e_k)\nabla_{e_l}\psi = \sum_i C_J(e_i)\nabla'_{e^i}\psi,$$

$$D''\psi = -\sum_i A_J(e_i)\nabla_{e^i}\psi = \sum_{rs} \omega^{rs} A_J(e_r)\nabla_{e_s}\psi = -\sum_i A_J(e_i)\nabla''_{e^i}\psi.$$

Proof. We show the result for D'; the second result is similar.

$$\begin{aligned}
D'\psi &= \tfrac{1}{2}\sum_j Cl(e_j)\nabla_{(e^j - iJe^j)}\psi = \frac{1}{2}\Big(\sum_j Cl(e_j)\nabla_{e^j}\psi + i\sum_j Cl(Je_j)\nabla_{e^j}\psi\Big) \\
&= \frac{1}{2}\sum_j \big(Cl(e_j) + iCl(Je_j)\big)\nabla_{e^j}\psi = \sum_i C_J(e_i)\nabla_{e^i}\psi.
\end{aligned}$$

On the other hand $\sum_j C_J(e_j)\nabla_{e^j}\psi = \sum_j C_J(Je_j)\nabla_{Je^j}\psi = -i\sum_j C_J(e_j)\nabla_{Je^j}\psi$ so

$$D'\psi = \frac{1}{2}\left(\sum_j C_J(e_j)\nabla_{e^j}\psi - i\sum_j C_J(e_j)\nabla_{Je^j}\psi\right) = \sum_j C_J(e_j)\nabla'_{e^j}\psi.$$

$\square$

The second order operator $\mathcal{P}$ defined in [6] by $\mathcal{P} = i[\widetilde{D}, D]$ is now given by $\mathcal{P} = i[-iD' + iD'', D' + D''] = 2[D', D'']$. Observing that D' raises the Fock degree by 1 whilst D'' lowers it by 1, it is clear that $\mathcal{P}$ preserves the Fock degree. Thus,

20

Proposition 26 *On the dense subspace of polynomial spinor fields, the operator $\mathcal{P} := 2[D', D'']$ is a direct sum of operators acting on sections of finite rank vector bundles.*

One defines in a natural way a L^2-structure on the space of sections of the spinor bundle $\mathcal{S}$ and on the space of sections of $T^*M \otimes \mathcal{S}$:

$$< \psi, \psi' > := \int_M h(\psi, \psi') \frac{\omega^n}{n!}, \quad < \gamma, \gamma' > := \int_M \sum_{ab} g^{ab} h(\gamma(e_a), \gamma'(e_b)) \frac{\omega^n}{n!}$$

for smooth sections with compact support, where $\{e_a\}$ is a local frame field and where, as before, g^{ab} are the components of the matrix which is the inverse of the matrix (g_{ab}) with $g_{ab} := g(e_a, e_b) := \omega(e_a, Je_b)$.

If X is a vector field on M its divergence is the trace of its covariant derivative: $\div^\nabla X := \text{Trace}[Y \mapsto \nabla_Y X]$. One defines the torsion-vector field τ^∇:

$$\tau^\nabla = \tfrac{1}{2} \sum_k T^\nabla(e_k, e^k) \quad \text{with, as before,} \quad \omega(e_k, e^l) = \delta_k^l.$$

One has $\omega(Z, \tau^\nabla) = \text{Trace}[Y \mapsto T^\nabla(Y, Z)]$. Indeed the sum over cyclic permutations of X, Y, Z of $\omega(T^\nabla(Y, Z), X)$ vanishes and $\text{Trace}[Y \mapsto T^\nabla(Y, Z)] = \sum_k \omega(T^\nabla(e_k, Z), e^k)$ $= \tfrac{1}{2} \sum_k \left(\omega(T^\nabla(e_k, Z), e^k) + \omega(T^\nabla(Z, Xe^k), e_k) \right) = \tfrac{1}{2}\omega(T^\nabla(e_k, e^k), Z)$.

Lemma 27 $\mathcal{L}_X \omega^n = (\div^\nabla X + \omega(X, \tau^\nabla))\omega^n$.

Proof. For any 2-form α, one has $\alpha \wedge \omega^{n-1} = \left(\frac{1}{2n} \sum_k \alpha(e_k, e^k) \right) \omega^n$. On the other hand $\mathcal{L}_X \omega^n = n (\mathcal{L}_X \omega) \wedge \omega^{n-1}$ and

$$
\begin{aligned}
\sum_k \mathcal{L}_X \omega(e_k, e^k) &= -\sum_k \omega([X, e_k], e^k) - \sum_k \omega(e_k, [X, e^k]) \\
&= \sum_k \omega(\nabla_{e_k} X, e^k) + \sum_k \omega(e_k, \nabla_{e^k} X) \\
&\quad + \sum_k \omega(T^\nabla(X, e_k), e^k) + \sum_k \omega(e_k, T^\nabla(X, e^k)) \\
&= 2 \div^\nabla X - \sum_k \omega(T^\nabla(e_k, e^k), X) = 2 \div^\nabla X + 2\omega(X, \tau^\nabla).
\end{aligned}
$$

$\square$

Proposition 28 *Given a MU^c-structure and a MU^c-connection on (M, ω), taking any local frame field $\{e_a\}$ of the tangent bundle, we have, for compactly supported smooth sections:*

$$< D'\psi, \psi' > = < \psi, \left(D'' + A_J(\tau^\nabla)\right) \psi' >$$

$$< \nabla\psi, \beta > = < \psi, \nabla^*\beta > \quad \text{with } \nabla^*\beta := -\sum_{ab} g^{ab}(\nabla_{e_a}\beta)(e_b) + \beta(J\tau^\nabla)$$

The Laplacian on spinors is thus given by $\nabla^*\nabla\psi = -g^{ab}\nabla^2\psi(e_a, e_b) + \nabla_{J\tau^\nabla}\psi$ where $\nabla^2\psi(e_a, e_b) := \nabla_{e_a}(\nabla_{e_b}\psi) - \nabla_{\nabla_{e_a} e_b}\psi$.

Proof. Indeed

$$
\begin{aligned}
< D'\psi, \psi' > \ &= \ < \sum_k C_J(e_k)\nabla_{e^k}\psi, \psi' > = \sum_k \int_M h(C_J(e_k)\nabla_{e^k}\psi, \psi')\frac{\omega^n}{n!} \\
&= \ \sum_k \int_M h(\nabla_{e^k}\psi, A_J(e_k)\psi')\frac{\omega^n}{n!} \\
&= \ -\sum_k \int_M h(\psi, \nabla_{e^k}(A_J(e_k)\psi'))\frac{\omega^n}{n!} - \sum_k \int_M h(\psi, A_J(e_k)\psi')\frac{\mathcal{L}_{e^k}\omega^n}{n!} \\
&= \ < \psi, D''\psi' > \\
&\quad -\sum_k \int_M h(\psi, ((A_J(\nabla_{e^k}e_k) + (\div^\nabla e^k + \omega(e^k, \tau^\nabla))A_J(e_k))\,\psi'))\frac{\omega^n}{n!}.
\end{aligned}
$$

Since $\sum_k \nabla_{e^k}e_k = -\sum_r (\div^\nabla e^r)e_r$, we get the result. To get the formula for ∇^*:

$$
\begin{aligned}
< \nabla\psi, \beta > \ &= \ \int_M \sum_{ab} g^{ab} h(\nabla_{e_a}\psi, \beta(e_b))\frac{\omega^n}{n!} \\
&= \ \int_M \sum_{ab} g^{ab} \left(e_a\left(h(\psi, \beta(e_b))\right) - h(\psi, \nabla_{e_a}(\beta(e_b)))\right)\frac{\omega^n}{n!} \\
&= \ -\int_M \sum_{ab} h(\psi, \beta(e_b))\left(e_a\left(g^{ab}\right)\frac{\omega^n}{n!} + g^{ab}\frac{\mathcal{L}_{e_a}\omega^n}{n!}\right) \\
&\quad - < \psi, \sum_{ab} g^{ab}\nabla_{e_a}(\beta(e_b)) > .
\end{aligned}
$$

But $\mathcal{L}_{e_a}\omega^n = \left(\div^\nabla e_a + \omega(e_a, \tau^\nabla)\right)\omega^n$ and $\div^\nabla e_r = \sum_a(\nabla_{e_a}e_r)^a$; since $\nabla g = 0$ we have $e_a\left(g^{ab}\right) = -\sum_r\left(g^{ar}(\nabla_{e_a}e_r)^b + g^{rb}(\nabla_{e_a}e_r)^a\right)$; hence

$$
\begin{aligned}
< \nabla\psi, \beta > \ &= \ \int_M \sum_{ar} g^{ar} h(\psi, \beta(\sum_b(\nabla_{e_a}e_r)^b e_b))\frac{\omega^n}{n!} \\
&\quad - \int_M \sum_{ab} h(\psi, \beta(e_b))g^{ab}\omega(e_a, \tau^\nabla)\frac{\omega^n}{n!} - < \psi, \sum_{ab} g^{ab}\nabla_{e_a}(\beta(e_b)) > \\
&= \ < \psi, \left(-\sum_{ab} g^{ab}(\nabla_{e_a}\beta)(e_b) + \beta(J\tau^\nabla)\right)\beta > .
\end{aligned}
$$

The formula for the Laplacian follows readily. $\qquad\square$

Proposition 29 *The operator $\mathcal{P} = 2[D', D'']$ is elliptic and one has:*

$$
\begin{aligned}
[D', D''] \ &= \ -\frac{1}{2\hbar}\nabla^*\nabla\psi + \frac{1}{2\hbar}\nabla_{J\tau^\nabla}\psi \\
&\quad + \tfrac{1}{2}\sum \omega^{kl}\omega^{rs}(C_J(e_k)A_J(e_r) - A_J(e_k)C_J(e_r))\left(R^\nabla(e_l, e_s)\psi - \nabla_{T^\nabla(e_l,e_s)}\psi\right)
\end{aligned}
$$

where R^∇ denotes the curvature, i.e. $R^\nabla(X, Y)\psi = \nabla_X(\nabla_Y\psi) - \nabla_Y(\nabla_X\psi) - \nabla_{[X,Y]}\psi$. The last term can be written $+\frac{i}{2}\sum \omega^{kl}\omega^{rs}Cl(e_k)Cl(Je_r)\left(R^\nabla(e_l, e_s)\psi - \nabla_{T^\nabla(e_l,e_s)}\psi\right)$.

Proof. Since $\nabla A_J = 0$ and $\nabla C_J = 0$ we have

$$
\begin{aligned}
[D', D''] &= [-\sum_{kl} \omega^{kl} C_J(e_k) \nabla_{e_l}, \sum_{rs} \omega^{rs} A_J(e_r) \nabla_{e_s}] \\
&= -\sum_{klrs} \omega^{kl} \omega^{rs} \left(C_J(e_k) A_J(e_r) \nabla^2_{e_l e_s} - A_J(e_r) C_J(e_k) \nabla^2_{e_s e_l} \right) \\
&= -\sum_{klrs} \omega^{kl} \omega^{rs} \left(C_J(e_k) A_J(e_r) - A_J(e_k) C_J(e_r) \right) \nabla^2_{e_s e_l} \\
&= -\tfrac{1}{2} \sum_{klrs} \omega^{kl} \omega^{rs} \left(C_J(e_k) A_J(e_r) - A_J(e_k) C_J(e_r) \right) \left(\nabla^2_{e_s e_l} + \nabla^2_{e_l e_s} \right) \\
&\quad -\tfrac{1}{2} \sum_{klrs} \omega^{kl} \omega^{rs} \left(C_J(e_k) A_J(e_r) - A_J(e_k) C_J(e_r) \right) \left(\nabla^2_{e_s e_l} - \nabla^2_{e_l e_s} \right).
\end{aligned}
$$

The first term is also $-\tfrac{1}{2} \sum_{klrs} \omega^{kl} \omega^{rs} \left(C_J(e_k) A_J(e_r) - A_J(e_r) C_J(e_k) \right) \left(\nabla^2_{e_s e_l} + \nabla^2_{e_l e_s} \right)$;
since $-[C_J(e_k), A_J(e_r)] = \frac{1}{2\hbar} \langle e_r, e_k \rangle_j = \frac{1}{2\hbar} (g_{rk} - i\omega_{rk})$ and $\sum_{kr} \omega^{kl} \omega^{rs} g_{rk} = g^{ls}$, it is equal to $\frac{1}{2\hbar} \sum_{ls} g^{ls} \nabla^2_{e_s e_l} = \frac{1}{2\hbar} (-\nabla^* \nabla + \nabla_{J\tau^\nabla})$.
For the second term, we observe that $\nabla^2_{e_s e_l} - \nabla^2_{e_l e_s} = R^\nabla(e_s, e_l) - \nabla_{T^\nabla(e_s, e_l)}$. $\qquad \square$ On

any symplectic manifold with a chosen positive ω-compatible almost complex structure J, there are linear connections such that $\nabla \omega = 0, \nabla J = 0$ and $\tau^\nabla = 0$. Indeed, if ∇^1 is a linear connection such that $\nabla^1 \omega = 0, \nabla^1 J = 0$ we set

$$
\nabla_X Y := \nabla^1_X Y - \frac{1}{2n} \left(\omega(\tau^{\nabla^1}, Y) X + \omega(X, Y) \tau^{\nabla^1} + \omega(J\tau^{\nabla^1}, Y) JX + \omega(JX, Y) J\tau^{\nabla^1} \right).
$$

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