MARTINGALES ON MANIFOLDS

AND

GEOMETRIC ITO CALCULUS

R.W.R. DARLING

Dedicated to André Meyer

Approach the problems from the right end and start with the solutions. Then perhaps, one day you will find the final question.

R.H. Van Gulik

MARTINGALES ON MANIFOLDS AND GEOMETRIC ITO CALCULUS

Summary

This work studies properties of stochastic processes taking values in a differential manifold M with a linear connection Γ, or in a Riemannian manifold with a metric connection.

Part A develops aspects of Ito calculus for semimartingales on M, using stochastic moving frames instead of local co-ordinates. New results include:

- a formula for the Ito integral of a differential form along a semimartingale, in terms of stochastic moving frames and the stochastic development (with many useful corollaries);
- an expression for such an integral as the limit in probability and in L^2 of Riemann sums, constructed using the exponential map;
- an intrinsic stochastic integral expression for the 'geodesic deviation', which measures the difference between the stochastic development and the inverse of the exponential map;
- a new formulation of 'mean forward derivative' for a wide class of processes on M.

Part A also includes an exposition of the construction of non-degenerate diffusions on manifolds from the viewpoint of geometric Ito calculus, and of a Girsanov-type theorem due to Elworthy.

Part B applies the methods of Part A to the study of 'Γ-martingales' on M. It begins with six characterizations of Γ-martingales, of which three are new; the simplest is: a process whose image under every local Γ-convex function is (in a certain sense) a submartingale. However to obtain the other characterizations from this one requires a difficult proof. The behaviour of Γ-martingales under harmonic maps, harmonic morphisms and affine maps is also studied.

On a Riemannian manifold with a metric connection Γ, a Γ-martingale is said to be L^2 if its stochastic development is an L^2 Γ-martingale. We prove that if M is complete, then every such process has an almost sure limit, taking values in the one-point compactification of M. No curvature conditions are required. (After this result was announced, a simpler proof was obtained by P.A.Meyer, and a partial converse by Zheng Wei-an.)

The final chapter consists of a collection of examples of Γ-martingales, e.g. on parallelizable manifolds such as Lie groups, and on surfaces embedded in R^3. The final example is of a Γ-martingale on the torus T^2 (Γ is the Levi-Civita connection for the embedded metric) which is also a martingale in R^3.

Martingales on manifolds and geometric Ito calculus

by

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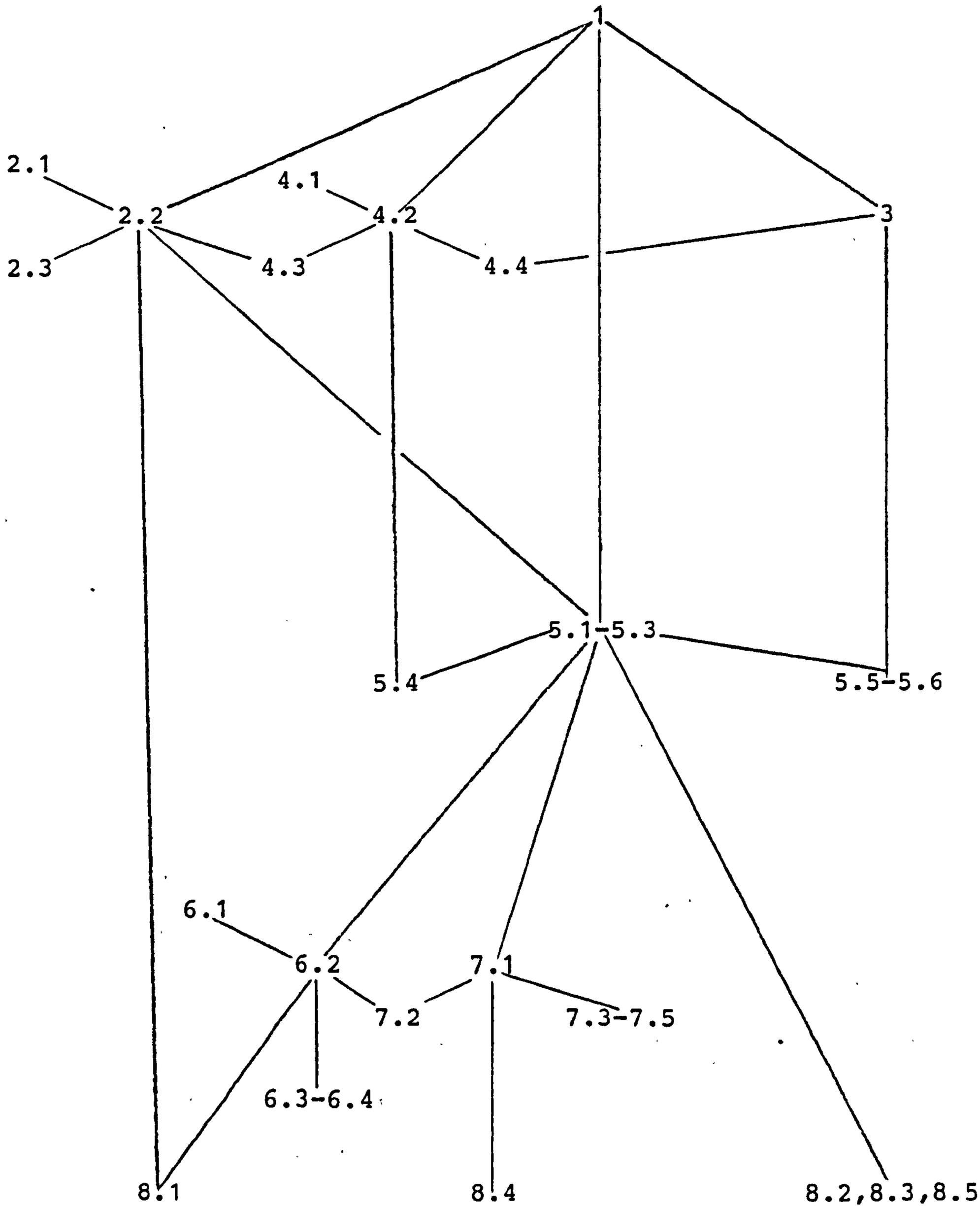
PART B

Martingales on manifolds

INTERDEPENDENCE OF CHAPTERS AND SECTIONS



Acknowledgments

I must begin by thanking my supervisor K.D.Elworthy for his teaching (especially of geometry and manifold theory), his advice and hints on numerous technical points, his readiness to read and discuss my work even when the topic was not of his own choosing, and his general encouragement.His influence on this work is apparent in many places.

The origin of this work was P.A.Meyer's expository lectures on geometric Ito calculus at the Durham symposium in June 1980 [27]. However I did not understand his ideas until J.Eells invited me to give a lecture on them in October 1980. From there on the development of this research was considerably helped by the Warwick Stochastic Analysis Seminar, 1980-1981 (R.W.R.D., K.D.E., S.Rogerson, and L.C.G.Rogers). Remarks of S.Rogerson were very helpful in clearing away early misconceptions. L.C.G.Rogers has generously given many hours of his time to educating me in subtle points of probability theory. His attention to detail is much appreciated. Conversations with J.Rawnsley and J.Eells on geometric topics have also been very helpful.

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Declaration

I declare that no portion of this thesis has been previously submitted for any degree at any university or institute of learning. The contents are my own original work, except for material labelled as expository or attributed to other authors.

Signed ..... Date .....

FREQUENTLY USED NOTATION

I. General notation

A. Manifolds and derivatives

1. R^n means n-dimensional Euclidean space, usually abbreviated to E.

2. $L(E;E')$ means the vector space of linear maps from a Euclidean space E to a Euclidean space E'.

3. $Df(.)$ and $D^2f(.)$, for a C^2 map $f:E \to E'$, stand for the first and second derivatives of f.

4. If M and N are manifolds, and $\varphi:M \to N$, $f:N \to R^1$ (or R^n) are suitable maps, the following terms have their customary meanings, as explained in Lang [23] :

TM , $T_x\varphi$ $(x \in M)$, T_yf $(y \in N)$, $T\varphi$, $f \circ \varphi$ (composition), φ^*f , df , φ^*df

B. Stochastic processes

Regard a stochastic process as a map $X:[0,\infty) \times \Omega \to$ { some measurable space }. Then X may also be written (X_t), and for each t, $X(t,.)$ may be written

X_t , or $X(t)$, or $X_t(.)$

See §0.4 and §0.5 for more notation.

II. Specific notation

Next to each symbol is the section where an explanation may be found.

$Hom(TM;\underline{G})$: §0.1	$Gl(E)$: §0.3
$\Gamma(.)$: §0.2	$Pos(E)$: §0.3
Γ^i_{jk} : §0.2	$\|\|\eta\|\|_x$: §0.3
$\nabla\eta$, ∇df : §0.1,§0.2	$\int Y dZ$: §0.4
$(\nabla df(x))_{ij}$: §0.2	$\int Y \circ dZ$: §0.4

$\int (dX \otimes dX)$: §0.4

$\langle X,Y\rangle_t$: §0.4,6.

$\int J dX^{\varphi}$: §0.4

$\int H(dX \otimes dX)^{\varphi}$: §0.4

$[Z]$: §0.4

$\int (\eta(X) \circ U) dZ$: §0.5

$\int \nabla\eta(X)(UdZ \otimes UdZ)$: §0.5

$\int (Uf) dZ$: §0.5

$(S)\int_{X_o^t} \eta$: §0.6

$(S)\int_{K.X_o^t} \eta$: §0.6

$(S)\int_X \eta$: §0.6

$(S)\int_{K.X} \eta$: §0.6

$(S)\int_X \varphi^*\eta$: §0.6

$(\Gamma)\int_{X_o^t} \eta$: §0.7

$(\Gamma)\int_{K.X_o^t} \eta$: §0.7

$(\Gamma)\int_X \eta$: §0.7

$gl(E)$: §1.1

$p:L(M) \to M$: §1.1

$L_e(.)$: §1.1

L_Γ : §1.1

U : §1.2

Z : §1.2

L_i : §1.2

$p:O(M) \to M$: §1.5

A-metric : §2.1

$M \cup \{\delta\}$: §2.2

$W(\hat{M})$: §2.2

$B(W(\hat{M}))$: §2.2

(C_t) : §2.2

C_t^f : §2.2

$\exp_x^{-1}$: §3.1

V : §3.1

$G(u,u')$: §3.2

θ_u : §3.2

$DH(t)$: §4.1

$DX(t)$: §4.2

(W,W', f) : §5.1

M_c^2 : §7.3

INTRODUCTION

We shall now give a broad idea of the mathematical context and purpose of this work, and of the methods used. The reader is invited to consult the opening section of each chapter for a more detailed review of the chapter's contents.

A. Context

In the last thirty years the theory of stochastic differential equations and martingales in R^1 and in Banach spaces has developed into a powerful and coherent body of knowledge. The state of the art (1982) may be seen in the books of Metivier and Pellaumail [26] on stochastic integration, of Stroock and Varadhan [34] and of Ikeda and Watanabe [15] on diffusion processes, and of Jacod [17] on semimartingale calculus. The concept of martingale, defined in terms of conditional expectation, seems to belong naturally to the context of linear spaces. However there are several reasons for wishing to study martingales on manifolds:

(1) They are a natural generalization of the idea of diffusions without drift. Diffusions on manifolds are already of interest because they are closely related to the study of second-order elliptic operators on manifolds, and so are of interest in differential geometry (harmonic maps) and in mathematical physics (e.g. Quantum Mechanics on manifolds; see [38], [39]).

(2) Manifold- and Lie group-valued processes occur for example in geophysics (geomagnetism), biology [7] , solid-state physics [5] , and control theory [4] . The present work has no direct application to these fields, but it offers an approach to manifold-valued processes which may be useful elsewhere.

B. Purpose

This work arose out of a desire to understand the nature and properties of "Γ-martingales" (martingales on a manifold with a connection Γ). We give below characterizations of these processes (§5), examples (§8), and study their convergence properties (§7) and behaviour under maps (§6).

A tricky proof in §5 necessitated the development of an approximation theorem for the Ito integral of a differential form along a semimartingale. My work in this area forced me to think about "geodesic deviation", and eventually to calculate it without taking a local chart. The results are presented in §3.

Early calculations were all performed in local co-ordinates. It was K.D.Elworthy who suggested that I try to develop a stochastic calculus in which local co-ordinates appear only at the beginning, and intrinsic expressions thereafter. This programme has been successful, and the results appear in §1. (In fact Theorem 1.2.2 (a) is the only result in the whole work whose proof requires the taking of local charts!) The dominant tool in this work is the stochastic moving frame. The power of this calculus is shown in §2, where the construction of non-degenerate diffusions on M reduces to a few easy geometric calculations; we also give Elworthy's version of the Girsanov Theorem. Another spin-off is a concise account of mean forward derivatives for processes on a manifold (§4).

C. Techniques and background

The reader is assumed to know the language of differential manifolds - any textbook will do. As far as geometry is concerned, it suffices to know the following parts of

Kobayashi and Nomizu [20]:

Chapter I, §5; Chapter II, §1, §2, §3; Chapter III, §1, §2, §4, §6, §7, §8; Chapter IV, §1, §2.

Calculations in vector bundles follow the style of Eliasson [10, pp.171-177].

For stochastic integration theory, the first 33 pages of Metivier and Pellaumail's book [26] are highly recommended - a miracle of conciseness, generality and precision. For the calculus of square-integrable martingales in Euclidean spaces, see Kallianpur [18,pp.33-44]; this book is also a good reference for the Ito formula, exponential martingales, and Girsanov's theorem.

For stochastic integration of differential forms on manifolds and for another perspective on geometric Ito calculus, consult Meyer [28,pp.44-62].

A word about the notation: the marriage of probability theory and differential geometry is now a past event, and perhaps now is the time to sweep away the confetti of local co-ordinates! Of course, local co-ordinates are useful in proofs, but we have tried to exclude them from the statement of theorems.

PART A

GEOMETRIC ITO CALCULUS

0. PRELIMINARIES

0.0 Motivation

This chapter is entirely expository. It serves to review basic ideas in differential geometry and stochastic calculus which will be needed in the sequel. The reader will notice that we favour the notations and methods of calculation which are used in the infinite-dimensional theory. There are two reasons for this: first because, as Lang says [23,p. iii], "the exposition gains considerably from the systematic elimination of the indiscriminate use of local co-ordinates"; second because the natural setting for geometric Ito calculus is a Hilbert manifold (see Elworthy [12]). We restrict ourselves to finite-dimensional manifolds in the sequel only because of technicalities surrounding the Ito formula for Hilbert-valued semimartingales (see Metivier and Pellaumail [26 §3.7] .

The reader is advised to pay particular attention to the notations introduced in §0.4 and §0.5.

0.1 Linear connections

Let M be a manifold of class C^{k+2}, $k \geq 0$, modelled on Euclidean n-space E. Whenever $p:F \to M$ is a C^r vector bundle over M, with $0 \leq r \leq k+2$, the C^r sections of p will be denoted $C^r(F)$. If $p':F' \to M$ is another C^r vector bundle, then $\mathrm{Hom}(F';F)$ will denote the vector bundle whose fibre at x in M is the vector space $L(F'_x;F_x)$ of all linear maps from F'_x to F_x. For example if G is a Euclidean space and $\underline{G}$ is the product bundle $M \times G$, then $\mathrm{Hom}(TM;\underline{G})$ denotes the vector bundle whose C^{k+1} sections are the C^{k+1} G-valued differential forms on M, henceforward referred to as <u>G-valued 1-forms</u>, or simply <u>1-forms</u> if $G = R$. Likewise the term <u>vector field</u> on M will normally mean a C^{k+1} vector field, that is, an element of $C^{k+1}(TM)$. Following a common abuse of notation, $C^r(M)$ denotes the C^r functions from M to R.

A C^k <u>linear connection</u> Γ on a C^{k+1} vector bundle $p:F \to M$ will be regarded as an R-linear operator ('<u>covariant derivative</u>')

$$\nabla:C^{k+1}(F) \to C^k(\mathrm{Hom}(TM;F)) \quad , \quad s \to \nabla s$$

such that for all f in $C^{k+2}(M)$, s in $C^{k+1}(F)$, and Y in $C^k(TM)$,

(i) $\nabla_{fY}\, s = f\nabla_Y\, s$

(ii) $\nabla_Y\, fs = f\nabla_Y\, s + (Yf)s$

Often we write $\nabla s(Y)$ instead of $\nabla_Y\, s$.

A C^k linear connection Γ on TM induces a C^k linear connection (also denoted Γ) on $\mathrm{Hom}(TM;\underline{G})$ for each Euclidean space G, and vice versa, by the relation:

(1) $\nabla_Z \eta(Y) + \eta(\nabla_Z Y) = Z(\eta(Y))$

for vector fields Y and Z and G-valued 1-forms η . Often we write $\nabla_Z \eta(Y)$ as $\nabla\eta(Z,Y)$ or as $\nabla\eta(Z\otimes Y)$, identifying $\mathrm{Hom}(TM;\mathrm{Hom}(TM;\underline{G}))$ with $\mathrm{Hom}(TM\otimes TM;\underline{G})$. When we use the expressions "<u>M has a connection Γ</u>" or "<u>Γ is a connection for M</u>", we are referring to a C^k linear connection on TM and an induced C^k linear connection on $\mathrm{Hom}(TM;\underline{G})$ for each G.

Let $\gamma:(-a,b) \to M$ be a C^{k+2} curve on M, for some positive numbers a and b. We say that γ is a <u>geodesic</u> (with respect to Γ) if

$$\nabla_{\dot\gamma(t)} \dot\gamma(t) = 0 , \quad -a < t < b$$

A function f in $C^{k+2}(M)$ is said to be <u>Γ-convex</u> on a subset V of M if for all geodesics γ,

$$\frac{d^2}{dt^2}(f \circ \gamma(t)) \geq 0 , \text{ whenever } \gamma(t) \text{ is in } V .$$

This is eqivalent to saying

(2) $\nabla df(x)(Y,Y) \geq 0$, for all x in V and Y in T_xM .

0.2 Local representations

Let M and E be as in §0.1. Suppose (W,ϕ) is a chart for M. For each vector field Y, define a map $y(.):W \to E$ by $y(x) = T_x\phi(Y(x))$ for x in W. The pair $(\phi(x),y(x))$ is called the local representation of Y at x in W. Likewise a 1-form η has a local representation $(\phi(x),b(x))$ at x in W, where $b(x)$ in E^* is the linear form such that $b(x)z = \eta(x)(T\phi^{-1}(z))$ for z in E.

A connection Γ for M gives rise to a C^k map $\Gamma(.)$ from W to $L(E,E;E)$ called a local connector[1] defined as follows: if vector fields Y and Z have local representations $(\phi(x),y(x))$ and $(\phi(x),z(x))$ respectively at x in W, then the vector field $\nabla_Y Z$ has local representation

$$(\phi(x),Yz(x) + \Gamma(x)(y(x),z(x))$$

where $Yz(.)$ is a vector field acting on a function. Sometimes $\Gamma(x)(y,z)$ is written $\Gamma(x)(y\otimes z)$, identifying $L(E,E;E)$ with $L(E\otimes E;E)$. For a 1-form η as above,

$$(1) \quad \nabla\eta(x)(Y,Z) = Db(x)(y(x),z(x)) - b(x)\Gamma(x)(y(x),z(x))$$

where $Db(x)$ is short for $D(b \circ \phi^{-1})(\phi x)$.

To assist the reader's comprehension, we mention that when E is given a basis $(e_1,\ldots,e_n)$, then $\Gamma(x)(e_i,e_j)$ has k^{th} co-ordinate $\Gamma^k_{ij}(x)$, where $(\Gamma^k_{ij}(.))$ are the familiar Christoffel symbols. When f is a function in $C^{k+2}(M)$ and η is df, the right side of (1) reads

$$(2) \quad (D_{ij}f(x) - D_k f(x)\Gamma^k_{ij}(x))y^i(x)z^j(x)$$

which is often written: $(\nabla df(x))_{ij}y^i(x)z^j(x)$

1. Eliasson [*10*, p.172], would write $\Gamma_\phi(.)$.

0.3 Riemannian metrics

1. M is a C^{k+2} manifold modelled on a Euclidean space E. Gl(E) is the Lie group of all invertible elements of L(E;E) and Pos(E) is the subgroup consisting of all the positive-definite symmetric elements.

A C^{k+1} Riemannian metric on M may be defined in several ways, for example:

(a) as an assignment of an inner product $\langle .,. \rangle_x$ to the tangent space T_xM for each x in M, such that if (W,ϕ) is a chart for M, there is a C^{k+1} map $G:W \to Pos(E)$ with

$$\langle v_1,v_2 \rangle_x = \langle G(x)T_x\phi(v_1),T_x\phi(v_2) \rangle_E \quad v_1,v_2 \in T_xM$$

where $\langle .,. \rangle_E$ is any inner product on E. With respect to a basis for E, G(x) is usually represented by a matrix $(g_{ij}(x))$.

(b) as a section g of Hom(TM;T*M) such that for all x in M and all v_1, v_2 in T_xM, $g(x)(v_1,v_2) = g(x)(v_2,v_1)$, and $g(x)(v_1,v_1) > 0$ if $v_1 \neq 0$.

When we say "let (M,g) be a C^{k+2} Riemannian manifold", we mean that M has a C^{k+1} Riemannian metric g as in (a) or (b).

2. To each C^{k+1} Riemannian metric there is associated a unique C^k linear connection $\bar{\Gamma}$ on TM , called the <u>Levi-Civita connection</u>, such that (i) and (ii) hold for all vector fields Q,Y and Z:

(i) $\bar{\Gamma}$ is a <u>metric</u> connection, i.e.

$$Q(\langle Y,Z\rangle)(x) = \langle \bar{\nabla}_Q Y,Z\rangle_x + \langle Y,\bar{\nabla}_Q Z\rangle_x \quad , x \text{ in } M$$

(ii) $\bar{\Gamma}$ is <u>torsion-free</u>, i.e.

$$\bar{\nabla}_Y Z - \bar{\nabla}_Z Y - [Y,Z] = 0$$

or equivalently, all the local connectors satisfy

$$\bar{\Gamma}(x)(y,z) = \bar{\Gamma}(x)(z,y) \ .$$

3. Using definition (b) of Riemannian metric, we define the <u>Laplacian operator</u> $\Delta : C^{k+2}(M) \to C^k(M)$ by

$$\Delta f(x) = \mathrm{Trace}(\ g(x)^{-1} \circ \bar{\nabla} df(x)\ )$$

noting that $g(x)^{-1} \circ \bar{\nabla} df(x) \in L(T_xM;T_xM)$. When $(e_1,\ldots,e_n)$ is an orthonormal basis for E, take an isometry $u:E \to T_xM$ ("<u>orthonormal frame</u>"), and write

$$(1) \quad \Delta f(x) = \Sigma_i\ \bar{\nabla} df(x)(u(e_i),u(e_i))$$

If $(g^{ij}(x))$ is the matrix of $G(x)^{-1}$ in (a), we may also write

$$(2) \quad \Delta f(x) = g^{ij}(x)(\bar{\nabla} df(x))_{ij}$$

following §0.2,(2).

4. A C^{k+2} function f on M is said to be <u>subharmonic</u> on a subset V of M if $\Delta f(x) \geq 0$ for x in V. It is easy to see that f $\bar{\Gamma}$-convex on V implies f subharmonic on V.

5. A 1-form η on M is said to be <u>bounded</u> if there exists a positive K such that

$$||\eta||_x^2 := \langle \eta,\eta\rangle_x \leq K \quad \text{for all } x \text{ in } M$$

In this case, $\sup\{\ ||\eta||_x : x \in M\}$ will be called the <u>norm</u> of η.

0.4 Stochastic notation

1. Consult Metivier and Pellaumail [26] for the meaning of the following terms:

complete right-continuous probabilized stochastic basis (c.r.c.p.s.b.), stopping-time, adapted process, predictable(= previsible), martingale, local martingale, semimartingale, scalar quadratic variation, tensor quadratic variation, and similar expressions.

This book is our reference for stochastic integration theory. Additional information about continuous semimartingales may be found in the book of Kallianpur [18].

Let $(\Omega, F, P, (F_t)_{t \geq o})$ be a c.r.c.p.s.b., fixed throughout this work. In the sequel, all processes will be adapted to this filtration unless otherwise stated; sometimes we may write "$X = (X_t, F_t)$" and so on, for the sake of emphasis. A continuous process always means one with continuous trajectories.

Suppose E and G are Euclidean spaces, Z is an E-valued process, and Y is an L(E;G)-valued process. Subject to technical conditions on Z and Y, the expressions

$$J = \int Y dZ \ , \ L = \int Y \circ dZ$$

mean that J and L are G-valued processes such that

$$J_t = \int_o^t Y_s dZ_s \quad \text{(Ito)} \ , \ L_t = \int_o^t Y_s \circ dZ_s \quad \text{(Stratonovich)}$$

Sometimes, for the sake of notational clarity, we may write

$$J(t) = \int_0^t Y(s)dZ(s) \quad , \text{ etc.}$$

2. The tensor quadratic variation of a Hilbert-valued semimartingale is defined in [*26*,§3.6].For a continuous E-valued semimartingale X (here $E = R^n$) the tensor quadratic variation process $\int (dX \otimes dX)$ may be characterized as the $(E \otimes E)$-valued process such that: if b and c are in E^* , and $b \otimes c$ is their tensor product regarded as an element of $(E \otimes E)^*$, then

$$\int_0^t b \otimes c(dX \otimes dX)_s = \langle b(X), c(X) \rangle_t$$

where the expression on the right is the angle-brackets process of the real-valued continuous semimartingales $b(X)$ and $c(X)$. When H is an $(E \otimes E)^*$-valued process, it is more convenient in some cases to write $H_s(dX_s, dX_s)$ instead of $H_s(dX \otimes dX)_s$.

3. Since the Ito formula for continuous E-valued semimartingales is used frequently in the sequel, we give here a version from Kallianpur [*18*,Theorem 4.3.1].

<u>Ito Formula</u>: Let E and G be Euclidean spaces, let $f:E \to G$ be a C^2 map, and let X be a continuous E-valued semimartingale. Then f o X is a continuous G-valued semimartingale, and

$$(1) \quad f(X_t) - f(X_o) = \int_0^t Df(X_s)ds + \frac{1}{2}\int_0^t D^2f(X_s)(dX \otimes dX)_s$$

4. Suppose that M is a C^{k+2} manifold, $k \geq 0$, modelled on E. <u>All processes on M will have continuous trajectories</u>. A process X on M will be called a <u>semimartingale</u> if for all f in $C^2(M)$, the image process f o X is a real-valued (continuous) semimartingale. The definition is from L.Schwartz [*33*].

5. Suppose X is a semimartingale on M, and let (W,φ) be a chart for M. Suppose J and H are locally bounded previsible processes taking values in $L(E;G)$ and $L(E,E;G)$ respectively, for some Euclidean space G, such that up to modification

$$J = J\,1_{\{X \in W\}}\ ,\ H = H\,1_{\{X \in W\}}$$

meaning that J_t and H_t are almost surely zero when X_t is not in W. Then the stochastic integrals

$$\int J d(\varphi X)\ ,\ \int H(d(\varphi X) \otimes d(\varphi X))$$

are well-defined. We shorten these expressions to

$$(2)\quad \int J dX^{\varphi}\ ,\ \int H(dX \otimes dX)^{\varphi}$$

If $(X) = (X^1,\ldots,X^n)$ with respect to a basis for E, Meyer [28] would write:

$$\int J^i_j dX^i\ ,\ \int H^i_{jk} d\langle X^j,X^k\rangle$$

6. If Z is a continuous E-valued semimartingale, its scalar quadratic variation will be written $[Z]$. Thus if $(e_1,\ldots,e_n)$ is an orthonormal basis of E, and if $Z_t = (Z^1_t,\ldots,Z^n_t)$ with respect to this basis, then

$$[Z]_t = \Sigma_i \langle Z^i,Z^i\rangle_t =: \Sigma_i \langle Z^i\rangle_t$$

(Recall that if $Z^i = M^i + A^i$ as a sum of local martingale and finite variation processes, then $\langle Z^i,Z^j\rangle$ is the unique process such that $M^iM^j - \langle Z^i,Z^j\rangle$ is a local martingale.)

0.5 Stochastic integrals with moving frames

A linear frame u at x in M means a linear isomorphism from E to T_xM. We shall deal frequently in the sequel with processes U such that U_t is a linear frame at X_t for all t, where X is a semimartingale on M. We would like to explain the appropiate notation in great detail so that probabilists will not be confused.

Let G be a Euclidean space and let η be a G-valued 1-form on M. Then for all t,

$$\eta(X_t) \circ U_t \qquad (\text{'o' denotes composition})$$

is an L(E;G)-valued random variable, since U_t maps linearly from E to $T_{X_t}M$ and $\eta(X_t)$ maps linearly from $T_{X_t}M$ to G. Hence it makes sense to say that

$$\eta(X) \circ U \text{ is an } L(E;G)\text{-valued process.}$$

Consequently if Z is an E-valued process, one can speak of the G-valued process

$$(1) \quad L = \int (\eta(X) \circ U)dZ$$

If $(e_1,\ldots,e_n)$ is a basis for E, and $Z = (Z^1,\ldots,Z^n)$ with respect to this basis, we could write out (1) in full as

$$(1') \quad L_t = \int_0^t (\eta(X_s) \circ U_s(e_i))dZ_s^i$$

Suppose now that M has a connection Γ, yielding a covariant derivative ∇ on Hom(TM;$\underline{G}$) as in §0.1 . Consequently for all x in M and all linear frames u at x, we have a bilinear map

$$\nabla\eta(x)(u(.),u(.)) \in L(E,E;G)$$

therefore we can speak of the stochastic integral

$$J = \int \nabla\eta(X)(U(.),U(.))(dZ\otimes dZ)$$

which is commonly abbreviated to

$$(2) \quad J = \int \nabla\eta(X)(UdZ\otimes UdZ)$$

Corresponding to (1′) we have

$$(2') \quad J_t = \int_o^t \nabla\eta(X_s)(U_s(e_i),U_s(e_j))d\langle Z^i,Z^j\rangle_s \qquad .$$

<u>Note</u>. If $f:M \to G$ is a C^{k+2} function, then df is a G-valued 1-form on M. If v is a tangent vector at x, vf or vf(x) both mean the same thing as df(x)(v). Consequently if $\eta = df$ in (1), we could write

$$(3) \quad L = \int (df(X) \circ U)dZ = \int (Uf)dZ = \int (Uf(X))dZ$$

or in full

$$(3') \quad L_t = \int_o^t (U_s(e_i)f)dZ_s^i$$

0.6 The Stratonovich integral of a differential form along a semimartingale

This subject has been carefully treated by Meyer [*28*, pp.57-62]; an earlier reference is Kohn [*21*]. We give a simplified definition and a list of the basic properties of these integrals. M will be of class C^{k+2}, $k \geq 0$, modelled on a Euclidean space E, and X will be a semimartingale on M. Given a 1-form η and a bounded previsible real-valued process K, we would like to define integrals

$$(1) \quad (S)\int_{X_o^t} \eta \quad , \quad (S)\int_{K.X_o^t} \eta$$

where '$K.X_o^t$' under the integral sign will mean: integrate η along X from 0 to t, with multiplier K.

Suppose first that f is in $C^{k+2}(M)$, and take $\eta = df$. In order to accord with the classical definition of Stratonovich integral when M = E, the Stratonovich integral of the 1-form df along X from 0 to t must satisfy:

$$(2) \quad (S)\int_{X_o^t} df = f(X_t) - f(X_o)$$

Suppose that K is restricted so that (up to modification)

$K = K\, 1_{\{X \in W\}}$, where (W,φ) is some chart for M.

Ito's Formula and equation (2) together imply that

$$(3) \quad (S)\int_{K.X_o^t} df = \int_o^t K_s\{Df(X_s)\,dX_s^\varphi + \tfrac{1}{2}D^2f(X_s)(dX\otimes dX)_s^\varphi\}$$

where dX_s^φ and $(dX\otimes dX)_s^\varphi$ have the meanings explained in §0.4,(2). We would like (1) to have the classical property that for h in $C^{k+2}(M)$,

$$(4)\quad (S)\int_{X_o^t} h\eta = \int_o^t h(X_s)\ o\ dL_s\ ,\quad L_s = (S)\int_{X_o^s} \eta$$

Equations (3) and (4) together imply that the only possible definition of the Stratonovitch integral of η , with multiplier K, is

$$(5)\quad (S)\int_{K.X_o^t} \eta = \int_o^t K_s\{b(X_s)dX_s^\varphi + \tfrac{1}{2}Db(X_s)(dX\otimes dX)_s^\varphi\}$$

where $(\varphi(x),b(x))$ is the local representation of η at x in W, with respect to the chart (W,φ). The right side of (5) is indeed intrinsic, as one checks by Ito's Formula §0.4,(1). Using a partition of unity for M, Meyer [*28*,p.57] is then able to define the integrals in (1), without restriction on K, and to prove that these processes are semimartingales.

Another useful property is the behavior of the (S)-integral under pullbacks. Suppose $\psi:M \to N$ is a C^{k+2} map into another C^{k+2} manifold N. Let $Y_t = \psi(X_t)$. Then for all 1-forms ρ on N,

$$(6)\quad (S)\int_{Y_o^t} \rho = (S)\int_{X_o^t} \psi^*\rho$$

where as usual $\psi^*\rho(x)(v) = \rho(\psi x)(T_x\psi(v))$ for v in T_xM .

<u>Notation</u>. " $L = (S)\int_X \eta$, $J = (S)\int_{K.X} \eta$ "

will mean that L and J are processes such that

$$L_t = (S)\int_{X_o^t} \eta\ ,\quad J = (S)\int_{K.X_o^t} \eta \qquad .$$

0.7 The Ito integral of a differential form with respect to a connection

Let Γ be a C^k linear connection for M. Using the notations of §0.2 and §0.6, we define the Ito integral of η along X from 0 to t, with multiplier K vanishing outside the random set $\{X \in W\}$, by

$$(1) \quad (\Gamma)\int_{K.X_o^t} \eta = \int_o^t K_s b(X_s)\{dX_s^\varphi + \tfrac{1}{2}\Gamma(X_s)(dX \otimes dX)_s^\varphi\}$$

in the previous notation. The right side is indeed invariant under change of chart. (Notice that differentiability of η is not actually needed here.) The integrals

$$(\Gamma)\int_{X_o^t} \eta \ , \quad (\Gamma)\int_{K.X_o^t} \eta$$

without restriction on K, may be constructed as in Meyer [28,p.57], and shown to be semimartingales.

For h in $C^{k+2}(M)$, we have

$$(2) \quad (\Gamma)\int_{X_o^t} h\eta = \int_o^t h(X_s)dN_s \ , \quad N_s = (\Gamma)\int_{X_o^s} \eta$$

If the multiplier K vanishes outside $\{X \in W\}$, then the Ito-Stratonovich correction may be written

$$(3) \quad (S)\int_{K.X} \eta - (\Gamma)\int_{K.X} \eta = \frac{1}{2}\int K\{Db(X) - b(X)\Gamma(X)\}(dX \otimes dX)^\varphi$$

where now and henceforward we use the notation "$N = (\Gamma)\int_X \eta$" to mean that N_t is the Ito integral of η along X from 0 to t,etc.

Remark. If G is another Euclidean space, the constructions of §0.6 and §0.7 extend in an obvious way to give stochastic integrals of G-valued 1-forms.

1 ITO CALCULUS ON FRAME BUNDLES

1.0 Motivation

This chapter is concerned with horizontal lifting and stochastic development of a semimartingale X on M, with respect to a connection Γ. These ideas have been treated in one form or another by many authors, including Ito, Malliavin [25], Eells and Elworthy [12], Ikeda and Watanabe [15], Kunita [22], Bismut [3],and Meyer [28]. There is some overlap with the last reference in particular. The main new results appear to be Theorem 1.2.2 (a) and Corollary 1.4.3 (an absolute Ito formula).

Starting from a semimartingale X on M, we define a 'lift' U to the linear frame bundle and a 'development' Z into E, in an efficient way using Stratonovich integrals of the connection 1-form and the canonical 1-form. We then show how Ito and Stratonovich integrals of differential forms along X may be given intrinsic expressions in terms of U and Z. The use of these formulae will greatly reduce the amount of calculation in subsequent chapters.

The idea to have in mind is the following. Think of E as the plane , and of M as an oblong ball resting on E. Suppose the process X traces out a path in ink on M. A horizontal lift of X to the orthonormal frame bundle is a way of rolling M along E, such that X_t is the point of contact for all t. A stochastic development of X into E is the path on E, traced out in ink by X as M rolls along.(Strictly, this should be called an anti-development.)

1.1 Geometry of the linear frame bundle

We shall briefly review the theory of connections on the linear frame bundle, as explained in Kobayashi and Nomizu [*20*, Ch.II and III].

Let E be a Euclidean space. The Lie group of invertible elements of L(E;E) is denoted Gl(E). When L(E;E) is considered as the Lie algebra of Gl(E), it is denoted gl(E).

Let M be a C^{k+2} manifold, $k \geq 1$, modelled on E. The linear frame bundle $p:L(M) \to M$ is a C^{k+1} principal Gl(E)-bundle over M whose fibre at x in M is the collection of all linear isomorphisms ("linear frames") from E to T_xM. To each C^k linear connection Γ for M there corresponds a unique connection 1-form α on L(M), with values in gl(E); see Kobayashi and Nomizu [*20*,p.64]. This 1-form induces a splitting of the tangent bundle of L(M) into 'horizontal' and 'vertical' sub-bundles H and V, where for each u in L(M), H_u is $\mathrm{Ker}(\alpha(u))$ and V_u is $\mathrm{Ker}(T_up)$. When u is a frame at x, T_up is a linear isomorphism from H_u to T_xM. The '<u>horizontal lift</u>'of a vector v in T_xM, through the connection 1-form α , is defined as the pre-image of v under this isomorphism. Consequently each C^k linear connection Γ for M determines a C^k section L of Hom($\underline{E}$;TL(M)), namely:

(1) $L(u)(e)$ = horizontal lift of $u(e)$, for $u \in L(M)$, $e \in E$

For each fixed e in E, L_e will denote the vector field

(2) $L_e(u) = L(u)(e)$, $u \in L(M)$

The <u>canonical 1-form</u> β on $L(M)$ is the E-valued 1-form defined by

$$\beta(u)(Y) = u^{-1}(T_u p(Y)) \quad , \quad u \in L(M) \quad , \quad Y \in T_u L(M)$$

Let $F = E \times L(E;E)$. Then the F-valued 1-form (β,α) on $L(M)$ induces a trivialization of $TL(M)$, meaning a vector bundle isomorphism of $TL(M)$ onto the product bundle $L(M) \times F$. This yields the <u>trivial connection</u> ${}^L\Gamma$ for $L(M)$ (with respect to Γ), which is the unique connection whose covariant derivative ${}^L\nabla$ satisfies (see §0.1,(1))

(3) $\quad {}^L\nabla\beta = 0 \ , \quad {}^L\nabla\alpha = 0$.

1.2 Horizontal lifting and stochastic development of semimartingales

The notations are those of the previous section. Let X be a semimartingale on M, and let U_o be an F_o-measurable random variable taking values in L(M), such that $p(U_o) = X_o$. The next result is well-known.

PROPOSITION 1.2.1

There is a unique semimartingale on L(M) (i.e. going on for all time) starting at U_o , such that

(1) $(S)\int_U \alpha = 0$, $p(U_t) = X_t$ for all t

Proof. Similar results are proved in Bismut [3, pp 380-404]. A new proof is given in §1.6.

Remark. If X is a semimartingale on M only up to some explosion time, then U will have the same explosion time as X.

DEFINITION

1. We call this U the horizontal lift of X to L(M) through the connection Γ, with initial frame U_o . If U_o is not specified, a horizontal lift of X is a process U satisfying (1).

2. Let β be the canonical 1-form as defined in §1.1. The stochastic development of X into E, through the connection Γ , with initial frame U_o , is the E-valued continuous semimartingale

(2) $Z = (S)\int_U \beta$

THEOREM 1.2.2 (Semimartingale lifting formulae)

For all 1-forms η on M and σ on L(M) (with notation from §0.5)

(a) $(\Gamma)\int_X \eta = \int (\eta(X) \circ U)dZ$

(b) $(S)\int_X \eta = \int (\eta(X) \circ U) \circ dZ$ (Strat.)

(c) $({}^L\Gamma)\int_U \sigma = \int (\sigma(U) \circ L(U))dZ$ (see §1.1 for ${}^L\Gamma$)

(d) $(S)\int_U \sigma = \int (\sigma(U) \circ L(U)) \circ dZ$ (Strat.)

where L is the section of Hom($\underline{E}$;TL(M)) defined in §1.1 .

Remarks.

(i) If the notation seems abstract, notice that in terms of a basis $(e_1,\ldots,e_n)$ for E , the differential of (c) could be written out as

$$\sigma(U_s)(L_{e_i}(U_s))dZ_s^i$$

(ii) Formulae (b) and (d) are already well-known in differential notation, as we shall now explain. Let L_i be short for the vector field L_{e_i} defined in §1.1,(2). Suppose f belongs to $C^{k+1}(L(M))$. Put df in place of σ, apply §0.6,(2) and differentiate to obtain:

$$d(f \circ U) = L_i f(U) \circ dZ^i \quad \text{(Strat.)}$$

or more simply

$$dU = L_i(U) \circ dZ^i$$

in the notation of Ikeda and Watanabe [15,p.234]. Likewise if h is in $C^{k+2}(M)$, put dh in place of η in (b), apply §0.6 (2), and differentiate to obtain: $dX = U(e_i) \circ dZ^i$ (Strat.) - see [39 , p. 2145,(5)]

1.3 Proof of the semimartingale lifting formulae

Using the trivialization of TL(M) noted in §1.1, any 1-form σ on L(M) may be written uniquely as a sum

(1) $\sigma(u) = b(u) \circ \beta(u) + C(u) \circ \alpha(u)$

where $b:L(M) \to E^*$ and $C:L(M) \to gl(E)^*$ are C^k maps. Notice that

(2) $b(u)(e) = \sigma(u)(L(u)(e))$, $e \in E$

Proof of (d).

It is immediate from §0.6,(4) that

$$(S)\int_U \sigma = \int b(U) \circ dZ + \int C(U) \circ dY$$

where $Z = (S)\int_U \beta$ as in §1.2,(2) and $Y = (S)\int_U \alpha$. By the defining property of U, §1.2,(1), $Y = 0$. Hence by (2)

$$(S)\int_U \sigma = \int b(U) \circ dZ = \int (\sigma(U) \circ L(u)) \circ dZ \qquad \square$$

Proof of (b).

By the pullback formula §0.6,(6), and formula (d) above

$$(S)\int_X \eta = (S)\int_U p^*\eta = \int (\eta(X) \circ Tp \circ L(U)) \circ dZ$$
$$= \int (\eta(X) \circ U) \circ dZ$$

(Here p is the projection map $L(M) \to M$) $\square$

Proof of (c).

Combine the defining property §1.1,(3) of the trivial connection ${}^L\Gamma$ for L(M) with the formula §0.7,(3) for the Ito-Strat. correction, to obtain

(3) $({}^L\Gamma)\int_U \beta = (S)\int_U \beta = Z$, by §1.2,(2)

$({}^L\Gamma)\int_U \alpha = (S)\int_U \alpha = 0$, by §1.2,(1)

Express the 1-form σ as in (1) . Applying a property of the Ito integral mentioned in §0.7,(2),

$$({}^{L}\Gamma)\int_U \sigma = \int b(U)dZ = \int (\sigma(U) \circ L(U))dZ \quad \text{by (2)} \quad \square$$

<u>Proof of (a)</u>.

Since the Ito integral on the left side was defined in terms of charts and local connectors in §0.7, it will be necessary to take a chart (W,φ) for M. This gives a chart $(p^{-1}(W),\bar{\varphi})$ for L(M), where for x in W and u in $p^{-1}(x)$,

$$\bar{\varphi}(u) := (\varphi(x), T_x\varphi \circ u) \in E \times Gl(E)$$

Define for each vector e in E a map $A_e : p^{-1}(W) \to E$,

$$A_e(u) = (T_x\varphi \circ u)(e)$$

and for a basis $(e_1,\ldots,e_n)$ for E, abbreviate A_{e_i} to A_i . Suppose the 1-form η on M has local representation $(\varphi(x),b(x))$ at x in W, according to §0.2. The formula to be proved is:

$$\int Kb(X)\{dX^{\varphi} + \tfrac{1}{2}\Gamma(X)(dX\theta dX)^{\varphi}\} = \int Kb(X)A_i(U)dZ^i$$

where K is any bounded previsible process vanishing outside the random set $\{X \in W\}$. Working in differential notation, it suffices to prove formally that

$$(4) \quad dX^{\varphi} + \tfrac{1}{2}\Gamma(X)(dX\theta dX)^{\varphi} = A_i(U)dZ^i$$

It follows from formula (d) proved above that

$$(5) \quad (S)\int_U d\bar{\varphi} = \int L_i(U)\bar{\varphi} \circ dZ^i$$

where the integrand on the right is a vector field acting on the C^{k+1} function $\bar{\varphi}:L(M) \to E \times L(E;E)$. It follows from a formula in Kobayashi and Nomizu [20, p.142] that for x in W, u in $p^{-1}(x)$:

$$(6) \quad L_e(u)\bar{\varphi} = (A_e(u), -\Gamma(x)(A_e(u),A_\cdot(u))$$

where the second entry on the right means the element of L(E;E) given by

$$v \to -\Gamma(x)(A_e(u),A_v(u))$$

It follows from (5), (6) and property §0.6,(2) of the Stratonovich integral that

$$\left.\begin{aligned} (7) \qquad dX^\varphi &= A_i(U) \circ dZ^i \\ (8) \qquad d(A_i(U)) &= -\Gamma(X)(A_j(U),A_i(U)) \circ dZ^j \end{aligned}\right\} \text{(Strat.)}$$

From (7),

$$dX^\varphi = A_i(U)dZ^i + \tfrac{1}{2}d\langle A_i(U),Z^i\rangle$$

where the last term is actually an n-vector with k^{th} entry $\frac{1}{2}d\langle A_i^k(U),Z^i\rangle$. By (8),

$$\begin{aligned} dX^\varphi &= A_i(U)dZ^i - \tfrac{1}{2}\Gamma(X)(A_j(U),A_i(U))d\langle Z^j,Z^i\rangle \\ &= A_i(U)dZ^i - \tfrac{1}{2}\Gamma(X)(dX\otimes dX)^\varphi \quad \text{by (7)} . \end{aligned}$$

□

1.4 Corollaries and an absolute Ito formula

The first two corollaries are similar to some expressions in Meyer [28,p.88].

COROLLARY 1.4.1 (Angle brackets process)

For functions f and h in $C^{k+2}(M)$,

$$\langle f(X),h(X)\rangle = \int (df\otimes dh)(X)(UdZ\otimes UdZ)$$

in the notation of §0.5,(2); with respect to a basis for E,

$$\langle f(X),h(X)\rangle_t = \int_0^t U_s(e_i)f(X_s)\ U_s(e_j)h(X_s)\ d\langle Z^i,Z^j\rangle_s$$

Proof. Apply formula (b) to the 1-form df . By property §0.6,(2) of the Stratonovich integral,

$$d(f(X)) = (df(X)\ o\ U)\ o\ dZ$$

Do the same for h , and the result is immediate by the usual formula for the angle brackets process of two real-valued semimartingales. □

COROLLARY 1.4.2 (Ito-Stratonovich correction)

For all 1-forms η on M, using the notation of §0.5,(2)

$$(S)\int_X \eta - (I)\int_X \eta = \frac{1}{2}\int \nabla\eta(X)(UdZ\otimes UdZ)$$

Proof. Take charts for M and L(M) and a local representation for η as in the proof of (a) in §1.3. For clarity, we also take a basis $(e_1,\ldots,e_n)$ for E. Then

$$\frac{1}{2}\nabla\eta(X)(UdZ\otimes UdZ) = \frac{1}{2}\nabla\eta(X)(U(e_i),U(e_j))d\langle Z^i,Z^j\rangle$$

$= \frac{1}{2}(Db(X) - b(X)\Gamma(X))(A_i(U), A_j(U))d\langle Z^i, Z^j\rangle$ by §0.2,(1)

$= \frac{1}{2}(Db(X) - b(X)\Gamma(X))(dX \otimes dX)^{\varphi}$ by §1.3,(7)

$= dR$

by §0.7,(3), where $R = (S)\int_X \eta - (\Gamma)\int_X \eta$ □

COROLLARY 1.4.3 (An absolute[1] Ito formula)

For any Euclidean space G, and any C^2 function $f:M \to G$,

$$f(X_t) - f(X_o) = \int_o^t \{(df(X) \circ U)dZ + \frac{1}{2}\nabla df(X)(UdZ \otimes UdZ)\}$$

(Notation as in §0.5,(2))

<u>Proof</u>. Combine Theorem 1.2.2(a) and Corollary 1.4.2, for $\eta = df$. □

COROLLARY 1.4.4 (An extended Ito formula)

Suppose G is a Euclidean space, and u and v are bounded stopping-times such that $u \leq v$. Let $F:\Omega \times M \to G$ be a map such that:

(a) for all x in M, $F(.,x)$ is an F_u-measurable random variable with values in G.

(b) for all ω in Ω, $F(\omega,.)$ is a C^2 map from M to G .

Then

$$F(X_v) - F(X_u) = \int_u^v \{(dF(X) \circ U)dZ + \frac{1}{2}\nabla dF(X)(UdZ \otimes UdZ)\}$$

<u>Proof</u>. When M = E , the assertion is a special case of

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1. The name is inspired by Levi-Civita's book [24] .

a theorem of Kunita [22,p.119]. The geometric constructions which were used to prove Corollary 1.4.3 apply equally to the case where the deterministic function f is replaced by the random function F, so the result follows. □

1.5 The Riemannian case - lifting to the orthonormal frame bundle

Suppose that (M,g) is a C^{k+2} Riemannian manifold, $k \geq 1$, with a metric connection Γ (possibly with torsion). The construction of the orthonormal frame bundle $p:O(M) \to M$ is described very clearly in Elworthy [12, Appendix B]; for x in M, the elements of $p^{-1}(x)$ are the isometries $u:E,\langle .,. \rangle_E \to T_xM,\langle .,. \rangle_x$. The connection Γ corresponds to a connection 1-form α on the orthonormal frame bundle. The whole of sections §1.1 to §1.4 may be carried through with $L(M)$ replaced by $O(M)$, $Gl(E)$ by $O(E)$ (= the Lie group of all orthogonal transformations of E) and $gl(E)$ by $o(E)$ (= all skew-symmetric linear transformations of E). A minor point is that the map $\bar{\varphi}:p^{-1}(W) \to E \times O(E)$ in the proof of (a), §1.3, must now be redefined as

$$\bar{\varphi}(u) = (\varphi(x), G(x)^{\frac{1}{2}} \circ T_x\varphi \circ u)$$

where $G(.)$ is as in §0.3,(a).

The following pair of results are included for reference; the first is in Meyer [28,p.89] and the second is well-known.

COROLLARY 1.5.1 (Scalar quadratic variation)

Let (W,φ) be a chart for M, $G(.)$ the local representative for the Riemannian metric as in §0.3,(a), and K a bounded previsible process vanishing outside $\{X \in W\}$. Then for every stochastic development Z (whatever the initial frame),

(1) $\int Kd[Z] = \int KG(X)(dX \otimes dX)^{\varphi}$

where $[Z]$ is the scalar quadratic variation of Z, as in Metivier and Pellaumail [26,§3.2] .

<u>Remark</u>. $G(X_t)(dX \otimes dX)^{\varphi}_t$ could be called a 'stochastic differential of the scalar quadratic variation of X at t , with respect to the Riemannian metric g'.

<u>Proof</u>. Let $(e_1,\ldots,e_n)$ be an orthonormal basis for E. Then formally

$$\begin{aligned} d[Z] &= \Sigma_k \, d\langle Z^k, Z^k\rangle \\ &= \langle U(e_k), U(e_j)\rangle_X \, d\langle Z^k, Z^j\rangle \\ &= \langle G(X)A_k(U), A_j(U)\rangle_X \, d\langle Z^k, Z^j\rangle \end{aligned}$$

where for u in $p^{-1}(x)$, $A_i(u) = T_x\varphi \circ u(e_i)$. By §1.3,(7), this implies that

$$d[Z] = G(X)(dX \otimes dX)^{\varphi}$$ □

COROLLARY 1.5.2

Suppose the stochastic development Z is a Brownian motion on E, and Γ is the Levi-Civita connection. Then the Ito-Stratonovich correction for exact 1-forms df becomes:

(2) $(S)\int_X df - (\Gamma)\int_X df = \frac{1}{2}\int \Delta f(X)\,dt$

<u>Proof</u>. Immediate from §0.3,(1) and Corollary 1.4.2 (write it out in local co-ordinates) □

COROLLARY 1.5.3 (Square-integrability)

Let $0 < t' < \infty$, and let $\Omega' = \Omega \times (0,t']$. Suppose there is a positive finite measure γ on the predictable σ-algebra of Ω' such that for some (hence for all) stochastic developments[1] Z with respect to the metric connection Γ,

$$(3) \quad E[(\int_0^{t'} |Y|dZ)^2] \leq \int_{\Omega'} |Y|^2 d\gamma$$

for all bounded predictable E^*-valued processes Y on $[0,t']$. Then whenever η is a bounded 1-form with respect to the Riemannian metric, with norm c, we have

$$E[((\Gamma)\int_{X_0^{t'}} \eta)^2] \leq c^2\gamma(\Omega') < \infty$$

<u>Proof</u>. By Theorem 1.2.2 (a),

$$(\Gamma)\int_{X_0^{t'}} \eta = \int_0^{t'} YdZ \quad , \quad \text{where} \quad Y_t = \eta(X_t) \circ U_t$$

Since U_t is an orthonormal frame for each t, $|Y_t| \leq c$. The result now follows from assumption (3). □

[1] To each initial frame U_0 there corresponds a different Z.

1.6 Construction of the horizontal lift

This section purports to be the first[1] rigorous proof of Proposition 1.2.1. Note that local charts for M are not used. The proof is by Hasminskii's non-explosion criterion, compact embedding, and the machinery of principal fibre bundles.

Step 1. (Proof when M = E)

Consider first the case when M = E, and L(M) is therefore the trivial principal fibre bundle $M \times Gl(E)$. Let $\Gamma(.)$ be the local connector for Γ in the chart $id:E \to E$. We shall assume for the sake of simplicity that $\Gamma(.)$ and $D\Gamma(.)$ are bounded.

By the local formula for the connection 1-form α, found in Kobayashi and Nomizu [20,p.142], the equations §1.2,(1) are equivalent to:

(1) $U_t = (X_t, A(t))$, (X = the given semimartingale on E)

where A is an L(E;E)-valued process such that for all $e \in E$

(2) $dA_e(t) = -\Gamma(X_t)(A_e(t), \circ\, dX_t)$, (Strat.)

where $A_e(t)$ is the E-vector A(t)(e). The Ito form of (1), suppressing t, is:

$$(3)\begin{cases} dA_e = -\Gamma(X)(A_e, dX) + \frac{1}{2}\Gamma(X)(\Gamma(X)(A_e, dX), dX) \\ \qquad - \frac{1}{2}D\Gamma(X)(dX)(A_e, dX) \quad , \quad e \in E \end{cases}$$

Note that the coefficients of A_e are bounded and globally Lipschitz. The non-explosion criterion, found for example

[1] A similar result is proved by Bismut [3, pp 390-393].

in Metivier and Pellaumail [26,§7.2], applies and shows that for a given starting value $A_e(0)$, (3) has a unique solution which <u>goes on for all time</u>. Define the semi-martingale U by (1). This completes the proof in this special case.

<u>Step 2</u>. (A calculation in principal fibre bundles)
Suppose next that M is compact. By Whitney's embedding theorem, there exists a Euclidean space F and a C^{k+2} embedding $\psi: M \to F$, such that $\psi(M)$ is a <u>closed</u> submanifold of F. Henceforward we will identify M with $\psi(M)$, and regard M as a closed submanifold of F. We are going to construct a series of principal fibre bundles, connection 1-forms, and homomorphisms, summarized in the following diagram:

$$\begin{tikzcd}
\overset{\alpha}{L(M)} \arrow[d, "p_3", "Gl(E)"'] & = L(F,M)/Gl(E') & \overset{\omega'}{L(F,M)} \arrow[l, "h'"'] \arrow[r, "i"] \arrow[d, "p_2", "Gl(E)\times Gl(E')"'] & \overset{\omega}{L(F)}|_M \arrow[r, "j"] \arrow[d, "p_1", "Gl(F)"'] & \overset{F_\omega}{L(F)} \arrow[d, "p_1", "Gl(F)"'] \\
M \arrow[rrr, leftrightarrow] & & M \arrow[r, leftrightarrow] & M \arrow[r] & F
\end{tikzcd}$$

The explanation goes as follows:

1. $p_1: L(F) \to F$ is the linear frame bundle of F, which is simply the trivial principal fibre bundle $\mathrm{proj}: F \times Gl(F) \to F$.

2. $L(F)|_M$ denotes $\{v \in L(F) : p_1(v) \in M\}$.

3. Regard E as a subspace of F; let E' be a complementary subspace of E in F.

4. Let $q: H \to M$ denote the normal bundle of M in F; thus $T_yF = T_yM \oplus H_y$ for all y in M. A frame $v \in L(F)_M$ at y in M

is said to be <u>adapted</u> if the following holds: if $w \in F$ is expressible as $e + e'$ for e in E and e' in E', then $v(e) \in T_yM$ and $v(e') \in H_y$. In the spirit of Kobayashi and Nomizu [20, Vol. II, p.2], we form the <u>bundle of adapted frames</u> $p_2:L(F,M) \to M$ with group $Gl(E) \times Gl(E')$.

5. There is a natural homomorphism $h':L(F,M) \to L(M)$ corresponding to the natural homomorphism $Gl(E) \times Gl(E') \to Gl(E)$, namely: if v is an adapted frame at y in M, then $h'(v)(e) = v(e)$ for e in E. Given a connection 1-form α on $L(M)$, we obtain a connection 1-form $\omega' = (h')^*(\alpha)$ on $L(F,M)$.

6. The maps i and j are the natural inclusion maps. A theorem in Kobayashi and Nomizu [20,p.79] shows that the inclusion $i:L(F,M) \to L(F)|_M$ induces a unique connection 1-form ω on $L(F)|_M$ such that Ti maps horizontal subspaces of ω' to horizontal subspaces of ω, and $i^*\omega = \omega'$ (we always regard the Lie algebras $gl(E)$ and $gl(E) \times gl(E')$ as subalgebras of $gl(F)$). Moreover since M is a closed subset of F, the theorem on existence of connections in Kobayashi and Nomizu [20,p.67] shows that ω may be extended to a connection 1-form ${}^F\omega$ on $L(F)$, which moreover has the following property:

(4) ${}^F\omega$ is trivial over the complement of some W_1 in $L(F)$, with $p_1(W_1)$ compact in F.

We recapitulate for future reference the following formulae:

(5) $\omega' = (h')^*(\alpha)$

(6) $(j \circ i)^*({}^F\omega) = i^*\omega = \omega'$

Step 3. (Explicit form for ${}^F\Gamma$)

Let Γ and ${}^F\Gamma$ be the C^k linear connections for M and F induced by the connection 1-forms α and ${}^F\omega$ respectively. We continue to regard M as an embedded submanifold of F. Therefore at each x in M, we may take a chart (V,φ) for F at x with the following properties:

(i) $(M\cap V,\varphi|_M)$ is a chart for M at x, and the range of $\varphi|_M$ lies in E. Thus $T\varphi$ identifies $TM|_{M\cap V}$ with $(M\cap V) \times E$.

(ii) $T\varphi$ identifies $H|_{M\cap V}$ with $(M\cap V) \times E'$, where $q:H \to M$ is the normal bundle of M in F.

For w_i in F, write $w_i = e_i + e_i'$ with e_i in E and e_i' in E'. It is elementary but tedious to verify that the local connectors $\Gamma(.):M\cap V \to L(E,E;E)$ and ${}^F\Gamma(.):V \to L(F,F;F)$ satisfy:

(7) $\quad {}^F\Gamma(y)(w_1,w_2) = \Gamma(y)(e_1,e_2) \quad , \; y \in M\cap V$

Step 4. (Proof of Proposition 1.2.1 for compact M)

M is still assumed to be compact. We now construct the horizontal lift U of X through Γ, with given initial frame U_o. Let ${}^FX$ denote the process X, viewed as a process in F by means of the inclusion map $M \to F$. Let U_o' be an adapted frame such that $h'(U_o') = U_o$. Define $A(0) \in Gl(F)$ by

$$({}^FX_o, A(0)) = i(U_o')$$

Let ${}^F\Gamma_1(.)$ be the local connector for ${}^F\Gamma$ in the chart $id:F \to F$. By (4), ${}^F\Gamma_1(.)$ has compact support in F, so ${}^F\Gamma_1(.)$ and $D({}^F\Gamma_1(.))$ are bounded; hence the procedure of Step 1 may be

applied to construct the unique process A in Gl(F), with initial value A(O), such that ${}^F U = ({}^F X, A)$ is the horizontal lift of ${}^F X$ to $F \times Gl(F)$ through the connection ${}^F\Gamma$; in other words

(8) $(S)\int_{{}^F U} {}^F\omega = 0$

We shall now prove that ${}^F U_t$ is an <u>adapted</u> frame for all t. Suppose (V,φ) is a chart for F of the special kind described in Step 3. On the random set $\{X \in V\}$, let $X_t^\varphi = \varphi(X_t)$ and $A_e^\varphi(t) = T\varphi({}^F U_t(e))$ for e in E, so $A_e^\varphi(t)$ belongs to F. Let $\Gamma(.)$ and ${}^F\Gamma(.)$ be local connectors with respect to the chart (V,φ). By (2), we have (on $\{X \in V\}$)

$$dA_e^\varphi(t) = - {}^F\Gamma(X_t)(A_e^\varphi(t), \circ\, dX_t^\varphi)$$

By (7),

(9) $dA_e^\varphi(t) = - \Gamma(X_t)(\bar{A}_e^\varphi(t), \circ\, dX_t^\varphi)$

where $\bar{A}_e^\varphi(t)$ is the projection of $A_e^\varphi(t)$ onto E along E'. However ${}^F U_o$ is an adapted frame, and $T\varphi$ has the property that, when restricted to TM, its range lies in E; therefore $A_e^\varphi(0)$ belongs to E. The right side of (9) shows that $dA_e^\varphi(t)$ lies in E, as a Stratonovich differential. Consequently $A_e^\varphi(t)$ lies in E for all t. Since $T\varphi$ is an isomorphism from $TM\restriction_{M\cap V}$ to E, ${}^F U_t(e)$ lies in TM for all t. The same kind of reasoning, using the special property (ii) of the chart (V,φ), shows that for e' in E', $dA_{e'}^\varphi(t) = 0$ for all t. Hence $A_{e'}^\varphi(t)$ is a constant vector in E', and so ${}^F U_t(e')$ lies in H for all t. This verifies that ${}^F U_t$ is an adapted frame for all t.

Let U' denote the process in $L(F,M)$ such that $i \circ j(U'_t) = {}^F U_t$. By (6), (8), and §0.6,(6)

$$(S)\int_{U'} \omega' = (S)\int_{U'} (j \circ i)^*({}^F\omega) = (S)\int_{F_U} {}^F\omega = 0$$

Define the process U on $L(M)$ by $U_t = h'(U'_t)$. Then $p_3(U_t) = X_t$ and by (5) and §0.6,(6)

$$(S)\int_U \alpha = (S)\int_{U'} (h')^*(\alpha) = (S)\int_{U'} \omega' = 0$$

This completes the proof for compact M. □

<u>Step 5</u>. (Proof for general M)

Drop the compactness assumption. Let M be the union of an increasing sequence of compact sets $(C_k : k \in \mathbb{N})$. Define an increasing sequence of stopping-times $(u(k): k \in \mathbb{N})$ by

$$u(k) = \inf\{t : X_t \notin C_k\}$$

Let $X^{u(k)}$ denote the process X, stopped at $u(k)$. Then for $\omega \in \{X_o \in C_k\}$, the trajectory $X^{u(k)}(.,\omega)$ of $X^{u(k)}$ takes values in the compact manifold with boundary C_k. The manifold boundary will make no difference to our calculations, since $X^{u(k)}$ stops there. For each k, the procedure of Steps 1. to 4. may be used to construct a unique semimartingale $U = U^{u(k)}$ on $L(M)$, which is constant on the random interval $[u(k),\infty)$, and such that

$$(S)\int_{1_{[0,u(k)]}\cdot U} \alpha = 0 \ , \ p(U_t) = X_t \ , \text{ for all } t.$$

(The notation was explained in §0.6). By continuity of the trajectories of $U^{u(k)}$, the semimartingales $U^{u(k)}$ and $U^{u(k+j)}$ are P-equivalent on $[0,u(k)]$ for all positive j. Moreover, by the continuity of the trajectories of X,

$$\lim_{k\to\infty} P(u(k) < t) = 0 \text{ for all } t$$

Hence we may define U_t as $\lim_{k\to\infty} U_t^{u(k)}$, and §1.2,(1) follows. □

2 NON-DEGENERATE DIFFUSIONS AND A GIRSANOV THEOREM

2.0 Motivation

The results of this chapter are not new. We present here the method of constructing non-degenerate diffusions on a manifold, developed by Eells and Elworthy [*12* VII.1] and Ikeda and Watanabe [*15*, V.4]. Then we derive the Girsanov measure transformation which corresponds to changing the drift term in the differential generator - a formula previously given by Elworthy [*11*,p.86]. The reason for including this chapter is:

(a) the constructions are of central importance in stochastic differential geometry,

(b) the geometric Ito calculus developed in §1 provides a very direct and simple route, and

(c) these diffusions will furnish examples of processes with mean forward derivatives in §4.3, and of Γ-martingales in §8.1 .

In this chapter, the technique of §1 is applied in reverse: think of M as an oblong ball resting at zero on the plane E, and suppose W is a Brownian motion traced out in ink on E. An elliptic operator A on M prescribes a special way of rolling, so that if we roll M along the path of W, then the path X traced out on M is that of a diffusion associated to A.

2.1 Second-order elliptic operators and metric connections

Let M be a manifold of class C^{k+2}, $k \geq 3$. We are interested in second-order differential operators A on M which are representable in local co-ordinates (x^i) on a chart (W,φ) by

$$(1) \quad Af(x) = b^i(x)D_i f(x) + \tfrac{1}{2}a^{ij}(x)D_{ij}f(x) \quad , x \in W$$

where $b(.):W \to E$ and $a(.):W \to Pos(E)$ are C^k maps, and Pos(E) denotes all positive-definite symmetric elements of Gl(E) . A more elegant prescription would be:

$A:C^{k+2}(M) \to C^k(M)$ is a local operator and the equation

$$(2) \quad \langle df,dh\rangle_x = A(fh)(x) - f(x)Ah(x) - h(x)Af(x)$$

for f and h in $C^{k+2}(M)$ and x in M, specifies a Riemannian metric $\{\langle .,.\rangle_x : x \in M\}$ on T*M

Notes.

1. The metric will be called the A-metric.
2. The prescription implies that: $f(x) = 0 \Rightarrow A(f^3h)(x) = 0$, showing that A is of order two.
3. We shall refer to such operators as elliptic C^k differential operators of order two on M, or simply as strictly elliptic operators.
4. The A-metric on T*M induces a Riemannian metric g on TM (also denoted $\{\langle .,.\rangle_x : x \in M\}$, a Levi-Civita connection $\bar{\Gamma}$ with covariant derivative $\bar{\nabla}$, and a Laplacian Δ.

LEMMA 2.1.1

(i) For all f in $C^{k+2}(M)$, $\Delta(f^2)(x) = 2f(x)\Delta f(x) + 2\langle df,df\rangle_x$

(ii) $A - \frac{1}{2}\Delta$ is a vector field on M.

Proof. $\Delta(f^2) = \text{div grad } f^2 = \text{div } (2f\ \nabla f)$, which is the right side of (i). As for (ii), it suffices to verify that the operator C obeys the liebniz rule, where

(3) $C = A - \frac{1}{2}\Delta$.

For this, it suffices to check that $C(f^2) = 2f\ Cf$. This is immediate from (2) and the result of part (i). □

The following idea comes from Ikeda and Watanabe [*15*,p.271].

PROPOSITION 2.1.2

Let $C = A - \frac{1}{2}\Delta$, and let $\bar{\nabla}$ denote the Levi-Civita connection associated with the A-metric. Then

(i) For vector fields Q and Z on M, the equation

(4) $\nabla_Q Z(x) = \bar{\nabla}_Q Z(x) + \frac{2}{n-1}(\langle Z,C\rangle_x Q(x) - \langle Q,Z\rangle_x C(x))$

defines a metric connection ∇ (or Γ) for M (see §0.3,2.).

(ii) $Af(x) = \frac{1}{2}\ \text{Trace}(g(x)^{-1} \circ \nabla df(x))$

where g(.) is the section of Hom(TM,T*M) which sends v in T_xM to $\langle v,.\rangle_x$ in T_x^*M.

Remark. In local co-ordinates, $Af(x) = \frac{1}{2}g^{ij}(\nabla df)_{ij}$, where (g^{ij}) denotes the inverse matrix of (g_{ij}). Observe that A is a kind of "Laplacian", with respect to the new metric connection.

Proof. It is elementary to check that ∇ satisfies the axioms in §0.1 for a linear connection, and that the metric property holds. As for (ii), it suffices, by the definitions of C and of Δ to prove that

$$(5) \quad Cf(x) = \frac{1}{2}\,\mathrm{Trace}(g(x)^{-1} \circ (\nabla df - \bar{\nabla} df)(x))$$

The connection Γ gives rise to a connection on T^*M, whose action on exact 1-forms is given by the equation

$$\nabla_Q\, df(Z) - \bar{\nabla}_Q\, df(Z) = \frac{2}{n-1}\{\langle Q,Z\rangle Cf - \langle Z,C\rangle Qf\}$$

for vector fields Q and Z, and for f in $C^{k+2}(M)$.

Let x be a point of M, and let $Z_1,\ldots,Z_n$ be an orthonormal basis of T_xM. If $C(x) = C^jZ_j$, then

$$\Sigma_i\ (\nabla df - \bar{\nabla} df)(x)(Z_i,Z_i) = \frac{2}{n-1}\,\Sigma_i\ \{\langle Z_i,Z_i\rangle_x Cf(x) - \langle Z_i,C^jZ_j\rangle_x Z_i f\}$$

$$= \frac{2}{n-1}\{nCf(x) - C^iZ_if\} = 2Cf(x)$$

This verifies (5). □

2.2 Construction of a diffusion

The following procedure for constructing a diffusion on M corresponding to the operator A is well-known. It was invented by Eells and Elworthy [12,VII] when A is the Laplacian on a Riemannian manifold, and modified by Ikeda and Watanabe [15, Ch.V.4] so as to work for any strictly elliptic A . We repeat it here to illustrate the power of the calculus presented in §1.

Let $\hat{M} = M$ if M is compact, and $M \cup \{\delta\}$ (the one-point compactification of M, which is not usually a differential manifold) if not. Let $W(\hat{M})$ be the path space defined by

$$W(\hat{M}) = \{w: w \text{ is a continuous mapping } [0,\infty) \to \hat{M} \text{ such that } w(0) \in M \text{ and if } w(t) = \delta \text{ then } w(t') = \delta \text{ for all } t' > t\} .$$

The reader will find in Ikeda and Watanabe [15,Ch.IV.5] , a careful description of how to construct the Borel σ-algebra $\mathcal{B}(W(\hat{M}))$ and its canonical filtration, and how to obtain a right-continuous filtration (C_t).

Let A be a strictly elliptic C^k differential operator of order two on M. Construct the orthonormal frame bundle $p:O(M) \to M$ with respect to the A-metric. Since the metric is of class C^k , $O(M)$ is a C^k manifold (not C^{k+1} as in §1.5). Let Γ be the metric connection on M introduced in §2.1, to which there corresponds a unique connection 1-form on $O(M)$, and a section L of $\mathrm{Hom}(\underline{E};TO(M))$ as in §1.1 .

Let $W = (W_t, F_t)$ be Brownian motion on E . For any x in M and u in $p^{-1}(x)$, we may solve the Stratonovich dynamical system on O(M) :

(1) $dU = L(U) \circ dW \quad , \quad U_o = u$

Since $k \geq 3$, L is a C^2 section of $\mathrm{Hom}(\underline{E};TO(M))$, so (1) has a unique solution (which may explode), as shown in Elworthy [12,VII§1] .Define

$$X_t = p(U_t)$$

PROPOSITION 2.2.1

(a) The law of the process X on M depends only on x , not on the initial frame u in $p^{-1}(x)$. Consequently we may speak of the family of measures $\{P_x : x \in M\}$ on $(W(\hat{M}), B(W(\hat{M})))$ induced by X .

(b) The family $\{P_x : x \in M\}$ is a <u>diffusion measure generated by</u> A , meaning that

(i) It is a strongly Markovian system (see Ikeda and Watanabe [15,p.190] for the definition).

(ii) For all f in $C^{k+2}(M)$ with compact support, $C^f = (C_t^f, F_t)$ is a martingale, where

$$C_t^f = f(X_t) - f(X_o) - \int_o^t Af(X_s)ds$$

In fact

(2) $C^f = (\Gamma)\int_X df = \int (df(X) \circ U)dW$

Proof.

(a) Suppose u and u' are elements of $p^{-1}(x)$. There exists $a \in O(E)$ (the orthogonal group of E) such that $u' = u \cdot a$. Suppose U is the solution of (1) with starting value u. It suffices to prove that the process U', where $U'_t = U_t \cdot a$ is the solution of (1) with starting value u', for then $p(U'_t) = p(U_t)$ for all t, and X_t does not depend on the choice of u in $p^{-1}(x)$.

It is straightforward to verify from §1.1, (1) and the invariance property $H_{u\cdot a} = T_u R_a(H_u)$ of the horizontal subspaces, that

$$L(u\cdot a)(a^{-1}e) = T_u R_a(L(u)e)$$

By the usual rules for Stratonovich differentials,

$$\begin{aligned} d(U_t\cdot a) &= d(R_a(U_t)) = T_{U_t} R_a(L(U_t) \cdot dW_t) \\ &= L(U_t\cdot a) \cdot (a^{-1}dW_t) \end{aligned}$$

Since the law of Brownian motion on E is invariant under the orthogonal transformation a, there is a Brownian motion W' on E such that

$$d(U'_t) = L(U'_t) \cdot dB'_t$$

This verifies that U' satisfies (1) with initial value u'.

(b) The proof of (i) follows from (a); see Ikeda and Watanabe [15, Remark 4.1, p.269].

Take an orthonormal basis $(e_1,\ldots,e_n)$ for E. Suppose $W_t = (W_t^1,\ldots,W_t^n)$ with respect to this basis. By Proposition 2.1.2, (ii)

$$Af(X_s)ds = \frac{1}{2}\Sigma_p \nabla df(X_s)(U_s(e_p),U_s(e_p))ds$$

since $(U_s(e_1),\ldots,U_s(e_n))$ is an orthonormal basis for T_xM. So

$$Af(X_s)ds = \frac{1}{2}\nabla df(X_s)(U_s(e_p),U_s(e_q))d\langle W^p,W^q\rangle_s$$

by an elementary property of Brownian motion. By Corollary 1.4.2, we deduce

$$(S)\int_X df - (\Gamma)\int_X df = \int Af(X)dt$$

and, by §0.6,(2), the value of the left side at time t is

$$f(X_t) - f(X_o) - (\Gamma)\int_{X_o^t} df$$

Consequently

$$C^f = (\Gamma)\int_X df$$

and the second identity of (2) follows from Theorem 1.2.1 part (a). Equation (2) shows at once that C^f is a local martingale. On the other hand the definition of C^f shows that

$$\left|C_t^f\right| \leq 2K_1 + tK_2$$

where $K_1 = \sup\{\left|f(x)\right| : x \in M\}$ and $K_2 = \sup\{\left|Af(x)\right| : x \in M\}$. Hence C^f is a martingale. □

2.3 Girsanov theorem for diffusions

We began with two lemmas to help with computations. The first is a well-known result of martingale calculus.

LEMMA 2.3.1

Conditions (I) and (II) are equivalent:

(I) N is a continuous real-valued local martingale.

(II) For all real b, N^b is a continuous local martingale, where

$$N_t^b = \exp(bN_t - \frac{1}{2} b^2 \langle N \rangle_t)$$

Proof. See Kallianpur [18,p.166].

LEMMA 2.3.2

Let A, Γ, and X be as in §2.2, and let $N = (\Gamma)\int_X \rho$ for some 1-form ρ. Then

$$\langle N \rangle_t = \int_o^t ||\rho||_{X_s}^2 \, ds$$

where the norm is taken with respect to the A-metric.

Moreover if ρ is bounded by K, then

$$\mathbf{E}\,[(\exp(N_t - \frac{1}{2}\langle N \rangle_t))^2] \leq e^{K^2 t}$$

Proof. By theorem 1.2.2(a),

$$(1) \qquad N_t = \int_o^t (\rho(x) \cdot U) dW$$

$$\Rightarrow \qquad d\langle N \rangle_t = \Sigma_j (\rho(X_t) \cdot U_t(e_j))^2 \, dt$$

But $U_t(e_1), \ldots, U_t(e_n)$ is an orthonormal basis of $T_{X_t} M$, so the first formula follows. As for the second, let

$$R_t = \exp(N_t - \frac{1}{2} N_t)$$

By Ito's formula,

$$R_t = \int_0^t R_s \, dN_s = \int_0^t R_s(\rho(X_s) \cdot U_s)dW_s$$

$$\Rightarrow \qquad \mathbb{E}(R_t^2) = \mathbb{E}(\int_0^t R_s^2 \, ||\rho||_{X_s}^2 \, ds) \leq K^2 \, \mathbb{E}(\int_0^t R_s^2 \, ds)$$

The desired result now follows from Gronwall's inequality. □

THEOREM 2.3.3 (Girsanov theorem)

Let A be a elliptic operator on M , inducing a metric $\{<.,.>_x : x \in M\}$ and a metric connection Γ as in §2.1. Let V be a vector field on M which is bounded with respect to the A-metric. Let ρ be the 1-form $\rho(x) = <V,.>_x$. For each x in M, construct a process X on M starting at x, and a measure P_x on $(W(\hat{M}),\mathcal{B}(W(\hat{M})))$ as in §2.2. Define

(2) $R_x(t) = \exp\{N_t - \frac{1}{2}<N>_t\}$, where $N = (\Gamma)\int_X \rho$

Then

(i) There is a measure Q_x on $(W(\hat{M}),\mathcal{B}(W(\hat{M})))$ such that $Q_x << P_x$, and

(3) $dQ_x = R_x(t)dP_x$ on C_t , $t \geq 0$

for all x in M. Moreover

(ii) The family $\{Q_x : x \in M\}$ is a diffusion measure associated with the operator

(4) $\bar{A} = A + V$

<u>Remark</u>. By equation (1) and Lemma 2.3.2, we could write (2) as

$$R_x(t) = \exp\{\int_0^t (\rho(X_s) \circ U_s)dW_s - \frac{1}{2}\int_0^t ||V||^2_{X_s} ds\}$$

since $||V||_x = ||\rho||_x$. This agrees with the formula of Elworthy [*11*,p.86] . See [*6* ,p.165] for a fuller account.

Proof.

(i) By formula (1), N is a local martingale. The integrand in (1) is bounded by $||\rho||_{X_s} = ||V||_{X_s}$, which is bounded by some constant. Hence N is a martingale. By the two Lemmas, $(R_x(t), F_t)$ is a non-negative P-martingale. By a well-known result (see e.g. Kallianpur [18, p.19]), $R_x(\infty) = \lim_{t\to\infty} R_x(t)$ exists a.s. and $E\, R_x(\infty) = 1$. Define a measure Q on (Ω, F) by

$$dQ = R_x(\infty)\, dP$$

Since $R_x(t) = E^P[R_x(\infty) | F_t]$ a.s. [P], it is easy to check that $dQ = R_x(t)\, dP$ on F_t. Define Q_x (for $x \in M$, $X_o = x$) on $(W(\hat{M}), B(W(\hat{M})))$ to be the image of Q under X. It is immediate that (3) is satisfied.

(ii) The family $\{Q_x : x \in M\}$ inherits the strong Markov property from the family $\{P_x : x \in M\}$. It remains to check that for all f in $C^{k+2}(M)$ with compact support, $(\bar{C}_t^f, F_t)$ is a Q-martingale, where

$$\bar{C}_t^f = f(X_t) - f(X_o) - \int_o^t (Af + Vf)(X_s)\, ds$$

or equivalently,

$$(5) \quad \bar{C}_t^f = C_t^f - \int_o^t Vf(X_s)\, ds$$

where C^f is the P-martingale introduced in Proposition 2.2.1. By Lemma 2.3.1, it suffices to show that for all $s < t$, all F in F_s, and all real numbers b,

$$(6)\begin{cases} E^Q[1_F \exp\{b\bar{C}^f_t - \frac{1}{2}b^2\langle\bar{C}^f\rangle_t\}] = \\ \qquad E^Q[1_F \exp\{b\bar{C}^f_s - \frac{1}{2}b^2\langle\bar{C}^f\rangle_s\}] \end{cases}$$

The left side can be written, using (2), (3), and (5), as

$$(7) \quad E^P[1_F \exp\{N_t + bC^f_t - \tfrac{1}{2}\langle N\rangle_t - b\int_o^t Vf(X_s)ds - \tfrac{1}{2}b^2\langle C^f\rangle_t\}]$$

since $\bar{C}^f$ has the same increasing process as C^f. Let H denote the P-martingale $(N + bC^f)$. Then by martingale calculus,

$$\langle H\rangle_t = \langle N\rangle_t + b^2\langle C^f\rangle_t + 2\langle N, bC^f\rangle_t \quad .$$

However

$$N = (\Gamma)\int_X \rho = \int (\rho(X) \circ U)dW$$

$$C^f = (\Gamma)\int_X df = \int (df(X) \circ U)dW$$

and by the method of Lemma 2.3.2,

$$\langle N, C^f\rangle_t = \int_o^t \langle\rho, df\rangle_{X_s} ds = \int_o^t Vf(X_s)ds$$

Consequently (7) can be expressed as

$$(8) \quad E^P[1_F \exp(H_t - \tfrac{1}{2}\langle H\rangle_t)]$$

By the Lemmas, $(\exp(H_t - \frac{1}{2}\langle H\rangle_t))$ is a P-martingale. So (8) equals

$$E^P[1_F \exp(H_s - \tfrac{1}{2}\langle H\rangle_s)]$$

which is the right side of (6), by the foregoing calculation. This verifies (6) and completes the proof. □

3 . APPROXIMATING ITO INTEGRALS OF DIFFERENTIAL FORMS

3.0 Motivation

The stochastic integral of a continous real-valued process K with respect to a continuous semimartingale X can be obtained as the limit in probability of Riemann sums of the form $\Sigma_j K(t_j)(X(t_{j+1}) - X(t_j))$, as is well known. Suppose now that X is a continuous semimartingale on a manifold M with a linear connection Γ, and η is a first-order differential form on M. A natural way to construct a stochastic integral of η along the path of X is as follows. Let v be a bounded stopping-time and $(0 = v(0) < v(1) < \ldots < v(q) = v)$ a partition of the interval $[0,v]$ by increasing stopping-times, chosen according to certain technical criteria. Write X_j instead of $X(v(j))$. Let γ_j be the geodesic on M (assumed unique) with $\gamma_j(0) = X_j$, $\gamma_j(1) = X_{j+1}$. The derivative of γ_j at zero is an element of the tangent space at X_j , and is usually denoted

$$\exp_{X_j}^{-1}(X_{j+1})$$

We approximate the Ito integral of η along X from 0 to v , with respect to the connection Γ, by the Riemann sum

$$\Sigma_j \ \eta(X_j)(\exp_{X_j}^{-1}(X_{j+1}))$$

We prove that as the mesh of the partition tends to zero, such Riemann sums converge in probability (and on a Riemannian manifold, in L^2) to the Ito integral

$$(\Gamma)\int_{X_0^v} \eta$$

3.1 The exponential map in differential geometry

Let M be a C^{k+2} manifold, $k \geq 2$, modelled on E, and denote its tangent bundle by $\tau : TM \to M$. Let Γ be a C^k connection for M, so that the notion of geodesic is defined. If x is a point of M, $\exp_x$ is the map from a neighbourhood of zero in T_xM to M, which takes a tangent vector v to the point $\gamma(1)$, where γ is the geodesic through x with $\dot{\gamma}(0) = v$. Let Λ denote the diagonal $\{(x,x) : x \in M\}$, a closed submanifold of $M \times M$. Define $\zeta : \Lambda \to TM$ by $\zeta(x,x) = 0_x$, the zero vector in T_xM. From the tubular neighbourhood theorem (see Lang [23,IV§5]) it follows that there exists an open neighbourhood Q of $\zeta(\Lambda)$ in TM, and an open neighbourhood V_o of Λ in $M \times M$, such that the map $f : Q \to V_o$ given by $v \to (\tau(v), \exp_{\tau(v)}(v))$ is a C^k diffeomorphism. When (x,x') belongs to V_o , $f^{-1}(x,x')$ will be written as $\exp_x^{-1}(x')$, which is a tangent vector at x. Finally take V to be an open neighbourhood of Λ in $M \times M$ such that $\bar{V} \subset V_o$.

Let (W,φ) be a chart at some x in M. Let W' be the set of x' such that (x,x') belongs to the set V defined above. Let u be a linear frame at x, that is, a linear isomorphism from E to T_xM. Define a map $\psi : \varphi(W') \to E$ by

$$\psi = (\varphi \circ \exp_x \circ u)^{-1}$$

Since the connection Γ is of class $C^k, k \geq 2$, ψ is a C^k map and $D\psi$ and $D^2\psi$ are well-defined on the open set $\varphi(W')$. The following elementary results will be needed later:

LEMMA 3.1.1 (Derivatives of the exponential map)

(i) $D\psi^{-1}(0) = T_x\varphi \circ u$

or in intrinsic notation, $d(u^{-1} \circ \exp_x^{-1})(u(e)) = e$, $e \in E$.

(ii) $D^2\psi(\varphi x)(e,e') - D\psi(\varphi x)(\Gamma(x)(e,e')) = 0$, $e,e' \in E$

or in intrinsic notation, $\nabla d(u^{-1} \circ \exp_x^{-1})(x) = 0$

Proof.

Equation (i) is Corollary 3.1 in Eliasson [*10*,p.180]. As for (ii), let $\bar{\Gamma}(.)$ denote the local connector associated with the chart (W',θ) at x, where θ is the map $(\exp_x \circ u)^{-1}$. The usual transformation formula for local connectors (see for example Eliasson [*10*,p.172]) reads

$$D\psi(\varphi x')(\bar{\Gamma}(x')(e,e')) = D^2\psi(\varphi x')(e,e') + \bar{\Gamma}(x')(D\psi(\varphi x')(e),D\psi(\varphi x')(e'))$$

for all x' in W'. Put $x' = x$. By the well-known property of normal co-ordinates, $\bar{\Gamma}(x) = 0$. Equation (ii) follows.□

3.2 Geodesic deviation

The notion of geodesic correction to the stochastic development occurs in the work of Dohrn and Guerra [8] (for Brownian motion) and in a forthcoming article of Meyer [29]. The treatment here begins from first principles.

Let M and Γ be as in §3.1, and let X be a semimartingale on M. Let U be a horizontal lift of X to L(M) through Γ, and let Z be the corresponding stochastic development of X into E, as in §1.2. Let u and u′ be bounded stopping-times such that $u \leq u'$, and the pair (X_u, X_t) belongs to V (see §3.1) for all t in the stochastic interval $[u,u']$. Hence $\exp_{X_u}^{-1}(X_{u'})$ is well-defined.

DEFINITION

The geodesic deviation of X from time u to time u′, using the horizontal lift U, is the E-valued random variable

$$(1)\quad G(u,u') = Z_{u'} - Z_u - (U_u^{-1} \circ \exp_{X_u}^{-1})(X_{u'})$$

The main aim of this section is to express $G(u,u')$ as a stochastic integral with respect to Z and the tensor quadratic variation of Z.

THEOREM 3.2.1 (Geodesic deviation formula)

$$(2)\quad \begin{cases} G(u,u') = \int_u^{u'} \{[I - d\theta_u(X_t) \circ U_t]dZ_t - \\ \qquad\qquad \frac{1}{2}\nabla d\theta_u(X_t)(UdZ \otimes UdZ)_t\} \end{cases}$$

→

where $\theta_u = (U_u^{-1} \circ \exp_{X_u}^{-1})$.

If $(e_1,\ldots,e_n)$ is a basis for E, and $Z_t = (Z_t^1,\ldots,Z_t^n)$ with respect to this basis, we could also write

$$(2') \begin{cases} G(u,u') = \int_u^{u'} \{dZ_t - d\theta_u(X_t)(U_t(e_p))dZ_t^p \\ \qquad - \frac{1}{2}\nabla d\theta_u(X_t)(U_t(e_p),U_t(e_q))d\langle Z^p,Z^q\rangle_t\} \end{cases}$$

If we abbreviate (2) to

$$(3) \quad G(u,u') = \int_u^{u'} \{R(u,t)dZ_t + Q(u,t)(dZ\otimes dZ)_t\}$$

then $R(u,t)$ and $Q(u,t)$ are continuous in t on the stochastic interval $[u,u']$, and

$$(4) \quad R(u,u) = 0 \ , \ Q(u,u) = 0$$

<u>Proof</u>. Regard the stopping-time u as fixed. There exists a map $F:\Omega \times M \to E$, satisfying the conditions of Corollary 1.4.4, such that for each ω ,$F(\omega,x) = \theta_{u(\omega)}(\omega,x)$ for all x in a suitable neighbourhood $W(\omega)$ of $X_{u(\omega)}(\omega)$. The fact that $F(\omega,.)$ is C^2 follows from the fact that the map $(x,x') \to \exp_x^{-1}(x')$ is C^2 on the domain V. Formula (2) is now immediate from the definition (1), and Corollary 1.4.4.

The continuity in t of $R(u,t)$ and $Q(u,t)$ follows from the continuity (for each ω) of $d\theta_u$ and $\nabla d\theta_u$, and the fact that X is a continuous process. Assertion (4) follows from Lemma 3.1.1 . □

3.3 L^2 primitive processes and stochastic monotone convergence

The following ideas are borrowed entirely from Metivier and Pellaumail [26]. Let J and K be Hilbert spaces, and let L be the space of all bounded linear operators from J to K with the operator norm. Take the time interval T to be $[0,\infty)$ or $[0,t_m]$, some $t_m < \infty$, and let $\Omega' = \Omega \times (T\setminus\{0\})$. For the notion of predictable σ-algebra on Ω', consult [26 ,§1.7] . Let X be a J-valued stochastic process indexed by T. The notion of (L,J,K)-L^2-primitive process, or simply L^2-process when J and K are fixed, is defined in [26 ,§2.3] . Such a process X has the following property (this is almost the definition):

(1) there exists a finite positive measure γ on predictable sets, vanishing on evanescent sets, and such that for every bounded L-valued predictable process Y,

$$E(||\int_T Y dX\ ||_K^2) \leq \int_{\Omega'} ||Y||_L^2\, d\gamma$$

we say in this case that the measure γ <u>dominates</u> X (for the bilinear mapping $L \times J \to K$).

<u>Remark</u>. All Hilbert-valued processes of finite variation and all Hilbert-valued square integrable martingales on T are (L,J,K)-L^2-primitive processes for all Hilbert spaces K, as shown in [26 ,§2.7, §2.6] .

The following result is a straightforward extension of [26 ,§2.11].

PROPOSITION 3.3.1 (Stochastic monotone convergence)

Let X be a J-valued semimartingale and let K be any Hilbert space. Let (Y_n) be a sequence of L-valued predictable processes such that almost surely the sequence $(||Y_n(t,\omega)||_L : n \in N)$ converges monotonically to zero for all t. Then for every bounded stopping-time v ,

$$\lim_{n \to \infty} \text{prob} \int_0^v Y_n dX = 0$$

<u>Proof</u>. Suppose t_m is a positive real number which is an upper bound for v. Let $\bar{X}$ denote the restriction of X to the time interval $[0,t_m]$. By [26 ,§10.9 and §2.9] , $\bar{X}$ is a local (L,J,K)-L^2-primitive process. Let $(u(k))$ be a localizing sequence for $\bar{X}$, in the sense of [26 ,§2.9]. Let

$$l(k) = \inf\{t: ||Y_1(t)||_L > k\} ,$$

$$\bar{u}(k) = u(k) \wedge l(k)$$

One can check that $(\bar{u}(k))$ is a localizing sequence for $\bar{X}$. Clearly the sequence (Y_n) is uniformly bounded on $[0,\bar{u}(k)]$ for each k. The result now follows from [26,§2.11] . (Actually it is almost immediate from the definition (1) and Lebesgue's bounded convergence theorem for the local dominating measures.) □

3.4 Approximation theorems for Ito integrals of 1-forms

THEOREM 3.4.1 (Approximation in probability)

Let M be a C^{k+2} manifold with a C^k connection Γ, $k \geq 2$. Let X be a semimartingale on M and let v be a bounded stopping-time. Suppose that for each natural number n, $(v(n,m): m \in N)$ is an increasing sequence of bounded stopping-times such that:

(i) For each n, the $(v(n,m): m \in N)$ are a subset of the $(v(n+1,m): m \in N)$.

(ii) $\lim_{n\to\infty} \sup_m (v(n,m+1) - v(n,m)) = 0$ almost surely

(iii) $\lim_{m\to\infty} P(v(n,m) < v) = 0$ for every n

(iv) $(X_{n,m}, X_{n,m+1}) \in V$ for all n,m, where V is the subset of $M \times M$ defined in §3.1, and $X_{n,m}$ is short for $X(v(n,m) \wedge v)$. In other words, $\exp^{-1}_{X_{n,m}}(X_{n,m+1})$ is a well-defined random vector in the tangent space at $X_{n,m}$ for all n,m.

Then for all 1-forms η on M,

(1) $$(\Gamma)\int_{X_0^v} \eta = \lim_{n\to\infty} \text{prob } \Sigma_m\, \eta(X_{n,m})(\exp^{-1}_{X_{n,m}}(X_{n,m+1}))$$

where $X_{n,m} = X(v(n,m) \wedge v)$; (by assumption (iii) the right side is well-defined, because (iii) implies that for every n, $P(\lim_{m\to\infty} v(n,m) < v) = 0$, so almost surely the sum in (1) has only finitely many terms.)

THEOREM 3.4.2 (Approximation in L^2, for Riemannian M)

Let (M,g) be a C^{k+2} Riemannian manifold, with a C^k metric connection (possibly with torsion) Γ. Let $0 < t_m < \infty$ and let v be a stopping-time with values in $[0,t_m]$. Let X be a semimartingale on M with the property (‡):

(‡) some (and hence every) stochastic development Z of X into E, with respect to Γ, is an (E^*,E,R^1)-L^2-primitive process on $[0,t_m]$, in the sense of §3.3.

Let (v(n,m)) be a family of stopping-times satisfying the conditions of Theorem 3.4.1. Then for all 1-forms η of compact support,

(a) $(\Gamma)\int_{X_0^v} \eta$ is in $L^2(P)$

(b) If S_n denotes $\Sigma_m\, \eta(X_{n,m})(\exp^{-1}_{X_{n,m}}(X_{n,m+1}))$, then S_n is in $L^2(P)$ for each n.

(c) $(\Gamma)\int_{X_0^v} \eta = \underset{n\to\infty}{L^2\text{-lim}}\ \Sigma_m\, \eta(X_{n,m})(\exp^{-1}_{X_{n,m}}(X_{n,m+1}))$

where L^2-lim means limit in $L^2(P)$.

<u>Note</u>. If Z is the sum of a square-integrable martingale on $[0,t_m]$ and a process of finite variation, then X satisfies (‡). See §3.3.

3.5 Proofs

The first step of the proof is common to both theorems.

Step 1.

We introduce the following notations:

$$(1) \quad T_n(v) = (\Gamma)\int_{X_o}^{v} \eta - \Sigma_m \, \eta(X_{n,m})(\exp_{X_{n,m}}^{-1}(X_{n,m+1}))$$

$$G_{n,m+1} = G(v(n,m)\wedge v, v(n,m+1)\wedge v)$$

where the last expression is the geodesic deviation with respect to a chosen horizontal lift U (to L(M) in the case of Theorem 3.4.1, or to O(M) in the case of Theorem 3.4.2); this was defined in §3.2.

$$F_{n,m} =]v(n,m)\wedge v, v(n,m+1)\wedge v]$$

$$(2) \quad a_t = \eta(X_t) \circ U_t \ , \ a_{n,m} = a_{v(n,m)\wedge v}$$

Notice that the process (a_t) takes values in E^*. Theorem 1.2.2 (a) says that

$$(\Gamma)\int_{X_o}^{v} \eta = \int_o^v a_t dZ_t$$

The definition of geodesic deviation in §3.2,(1) shows that

$$\eta(X_{n,m})(\exp_{X_{n,m}}^{-1}(X_{n,m+1})) = a_{n,m}(Z_{n,m+1} - Z_{n,m}) - a_{n,m}(G_{n,m+1})$$

where $Z_{n,m} = Z_{v(n,m)\wedge v}$. Consequently (1) gives

$$T_n(v) = \int_0^v J_n(t)dZ_t + \Sigma_m a_{n,m}(G_{n,m+1})$$

where

$$(3) \quad J_n(t) = \Sigma_m 1_{F_{n,m}}(t)(a_t - a_{n,m})$$

The geodesic deviation formula, §3.2,(2), shows that

$$\Sigma_m a_{n,m}(G_{n,m+1}) = \int_0^v \{R_n(t)dZ_t - Q_n(t)(dZ\otimes dZ)_t\}$$

where

$$(4) \quad R_n(t) = \Sigma_m 1_{F_{n,m}}(t)a_{n,m}(I - d\theta_{n,m}(X_t) \circ U_t)$$

$$(5) \quad Q_n(t) = \tfrac{1}{2}\Sigma_m 1_{F_{n,m}}(t)a_{n,m}(\nabla d\theta_{n,m}(X_t)(U_t(.),U_t(.)))$$

(The maps θ_u were defined in §3.2, and $\theta_{n,m} = \theta_{v(n,m)\wedge v}$.)

We arrive at the formula:

$$(6) \quad T_n(v) = \int_0^v \{(J_n + R_n)dZ - Q_n(dZ\otimes dZ)\}$$

<u>Step 2</u>. <u>Proof of Theorem 3.4.1</u>

We are required to prove that $T_n(v)$ tends to zero in probability as n tends to infinity. The result will follow from (6) and Proposition 3.3.1 provided we can show that, almost surely, the sequences $|J_n(t)|$, $|R_n(t)|$, and $||Q_n(t)||$ converge monotonically to zero. First of all

$$|J_n(t)| = \Sigma_m 1_{F_{n,m}}(t)|a_t - a_{n,m}| \to 0 \quad \text{a.s. as } n \to \infty$$

using assumption §3.4,(ii) and the fact that (a_t) has continuous trajectories; monotonicity follows from

assumption §3.4,(i). Next

$$|R_n(t)| \leq \Sigma_m 1_{F_{n,m}}(t)|a(v)|^*|I - d\theta_{n,m}(X_t) \circ U_t|$$

where $|a(v)|^* = \sup\{|a(t)| : 0 \leq t \leq v\}$. Almost surely, this converges monotonically to zero using §3.4,(1) and (ii) and the last part of Theorem 3.2.1. The conclusion for $Q_n(t)$ follows in the same way. □

Step 3. Proof of Theorem 3.4.2

The expressions in Step 1 now have a slightly different meaning, because U is now a horizontal lift through Γ to the bundle of orthonormal frames, and η is a 1-form with compact support. Consequently there is a constant c_1 (= the norm of η with respect to the Riemannian metric) such that

(7) $|a_t| = |\eta(X_t) \circ U_t| \leq c_1$ for all t

Let A be a compact subset of M containing the support of η. There exists constants c_2 and c_3 such that for all x and y in A, all orthonormal frames u at x, and all e,e′ in E,

(8) $|e - d(u^{-1} \circ \exp_x^{-1})(y)(u(e))| \leq c_2|e|$

(9) $|\nabla d(u^{-1} \circ \exp_x^{-1})(y)(u(e),u(e'))| \leq c_3|e|\cdot|e'|$

Equations (3) and (7) show that

(10) $|J_n(t)| \leq 2c_1$

Equations (4), (7), and (8) show that

(11) $|R_n(t)| \leq c_1 c_2$

and equations (5),(7), and (9) show that

(12) $||Q_n(t)|| \leq c_1 c_3$

Let α be a dominating measure for Z and β a dominating measure for the tensor quadratic variation of Z, in the sense of §3.3. Then

$$E(|(\Gamma)\int_{X_o^v} \eta\ |^2) = E(|\int_o^v a_t dZ_t|^2) \leq (c_1)^2 \alpha(\Omega')$$

by (7) and §3.3,(1). This verifies assumption (a) of the theorem. Moreover by (6) and §3.3,(1)

(13) $E(|T_n(v)|^2) \leq 2\int_{\Omega'} |J_n + R_n|^2\, d\alpha + 2\int_{\Omega'} ||Q_n||^2\, d\beta$

The inequalities (10) to (12) show that the right side of (13) is bounded by a constant, so $T_n(v)$ is square integrable. Equation (1) and the result of part (a) lead directly to assertion (b).

Finally, we saw in Step 2 that J_n , R_n , and Q_n all converge to zero almost surely. Inequalities (10) to (12) show that they are also bounded by constants. Hence Lebesgue's bounded convergence theorem may be applied to (13) to deduce that

$$\lim_{n\to\infty} E(|T_n(v)|^2) = 0$$

which is assertion (c). □

4 MEAN FORWARD DERIVATIVE OF A SEMIMARTINGALE ON A MANIFOLD

4.0 Motivation

This chapter presents yet another application of the techniques of horizontal lifting and stochastic development of semimartingales. The notion of mean forward derivative for real-valued processes was introduced by Nelson [*30*,§11] for his theory of Stochastic Mechanics. Generalizations to certain classes of processes on Riemannian manifolds were given by Dankel [*5*] and Dohrn and Guerra [*8*].

We begin by defining the mean forward derivative of Banach-valued processes in a new way, inspired partly by a lecture by Pierre Bremaud (Paris) at Bielefeld University in October 1981. We deliberately avoid expressions such as $\frac{1}{\varepsilon} E[X_{t+\varepsilon} - X_t | F_t]$ for reasons explained in §4.1, Remark (i). When X takes values in a manifold M with a connection Γ, we can (under certain conditions) define the mean forward derivative DX with respect to Γ. We check that when M is Riemannian and X is a diffusion corresponding to a strictly elliptic operator $A = \frac{1}{2}\Delta + C$, where C is a vector field, then the mean forward derivative of X with respect to the Levi-Civita connection is the TM-valued process C(X).

Finally we indicate how such mean forward derivatives might be calculated, using the Ito integral approximation theorem 3.4.2 .

4.1 Mean forward derivative of a process in a Banach space

Suppose H and λ are a pair of processes taking values in a Banach space J. As usual $L^1_J(P)$ denotes the vector space of J-valued P-integrable random variables. Consider the properties:

(a) the increments $\{(H_t - H_s) : 0 \leq s < t < \infty\}$ belong to $L^1_J(P)$,

(b) λ is cadlag (i.e. has trajectories which are right-continuous with left limits) and the stochastic integral

$$\int_0^t \lambda(s)\,ds$$

exists in $L^1_J(P)$ for all t ,

(c) for all $s < t$ and all A in F_s ,

$$(1) \quad E[1_A\,(H_t - H_s)] = E[1_A \int_s^t \lambda(u)\,du]$$

DEFINITION

A J-valued process H will be called an (MFD)-process with mean forward derivative DH if (H,DH) is a pair satisfying (a), (b), and (c) above.

Trivial example.

Let N be a real-valued martingale, of which we choose a cadlag version, and let b be a bounded continuous function. Then the composite process b o N is cadlag, and

$\int_0^t b(N_s)\,ds$ exists, and is in $L^1(P)$ by Fubini's Theorem.

Let

$H_t = N_t + \int_0^t b(N_s)ds$. Then H is an (MFD)-process with mean forward derivative b o N .

Remarks.

(i) The definition of (MFD)-process is modelled on Nelson's definition of an (R1)-process in [30 ,§11]. However the condition that $t \to DH(t)$ be continuous into $L^1_J(P)$ is dropped. The formula (1) is felt to be better than

$$" \ DH(t) = \underset{\varepsilon \to 0}{\text{l.i.m.}} \left(\frac{1}{\varepsilon} E[H_{t+\varepsilon} - H_t | F_t] \right) "$$

because it is more convenient both for theoretical analysis and for practical calculation.

(ii) The requirement that DH is cadlag is included because it is useful to write $DH(t,\omega)$ as $\lim_{\varepsilon \to 0} DH(t+\varepsilon,\omega)$ for all ω. Right- (rather than left-) continuity corresponds to a forward (rather than backward) derivative.

(iii) The definition could be stated in terms of the Doleans function of H; see Metivier and Pellaumail [26,§1.15] .

(iv) The mean forward derivative is unique up to modification, at least if $J = R^n$; this can be deduced from Lemma 4.1.1.

(v) A J-valued process H is a martingale if and only if it is an (MFD)-process such that $H_0 \in L^1_J(P)$ and $DH = 0$.

(vi) Suppose J and K are Banach spaces and L is a Banach space embedded in the space of bounded linear operators

from J to K, with the operator norm. If H is a J-valued (MFD)-process and also a semimartingale, then for all bounded predictable L-valued processes Q,

$$(2) \quad E[\int_0^t Q_s dH_s] = E[\int_0^t Q_s DH(s)ds] \text{ , for all t.}$$

The proof may be outlined as follows. Define a J-valued semimartingale C by

$$C_t = H_t - \int_0^t DH(s)ds$$

Let $\mathcal{E}(L)$ denote the L-valued simple predictable processes in the sense of Metivier and Pellaumail [26,§2.2]. By (1), for each t

$$(3) \quad E[\int_0^t Y_s dC_s] = 0 \text{ , } Y \in \mathcal{E}(L) \text{ .}$$

Let us fix t. By [26,§2,9,§10.9] there is a localizing sequence of stopping times (u(n)) for C, and a dominating measure α_n for each of the stopped processes $C^{u(n)}$, defined on the predictable σ-algebra $\mathcal{P}$ of $\Omega' = \Omega\times]0,t]$; see [26,§2.3,§1.13]. For each n, $\mathcal{E}(L)$ is dense in $L^2_L(\Omega',\mathcal{P},\alpha_n)$, which contains all the bounded predictable processes. Hence (3) holds for all such processes. This verifies (2).□

LEMMA 4.1.1

Let Y be an E (= R^n)-valued cadlag process such that

$$\int_0^t Y_s ds \text{ exists and belongs to } L^1_E(P) \text{ for all t}$$

If

$$E[1_A\int_a^b Y_s ds] = 0 \text{ for all } 0 \leq a < b \leq \infty, \text{ and all } A \in F_a$$

then Y is zero up to modification.

<u>Proof.</u> Let R be the process: $R_t = \int_0^t Y_s ds$. The condition implies that R is a martingale. However it is also of finite variation, hence null by a well-known theorem; see, for example, Kallianpur [*18*,p.41]. Hence for (Lebesgue) almost all t , $Y_t = 0$ a.s. By right continuity of the paths of Y, $Y_t = 0$ a.s. for all t; in other words Y is zero up to modification. □

4.2 Mean forward derivative on a manifold with a connection

Let M and Γ be as in §1.1. Let X be a semimartingale on M, let U be a horizontal lift of X to L(M) through Γ, and let Z be the corresponding stochastic development of X into E as in §1.2.

DEFINITION

X will be called an (MFD)-semimartingale on M, with respect to Γ, whenever Z is an (MFD)-process on E, in the sense of §4.1. The mean forward derivative of X, denoted DX, is the TM-valued process

$$(1) \quad DX(t) = U_t(DZ(t)) \in T_{X_t}M$$

Remarks.

(a) The choice of horizontal lift U (i.e. the choice of U_o) does not affect whether Z is an (MFD)-process or not.

(b) The left side of (1) is the same, whichever horizontal lift is chosen.

(c) The process DX is not in general continuous; its "horizontal" part X is continuous, but the vertical part may jump. This kind of process has been studied by Duncan [37]. See also Rogerson [32].

LEMMA 4.2.1 (Mean forward derivatives and Ito integrals)

Let X be an (MFD)-semimartingale with respect to Γ. Let η be a 1-form on M and let $N = (\Gamma)\int_X \eta$. Let v be

a bounded stopping-time such that $N_{v\wedge t}$ is in $L^1(P)$ for all t. Let N^v denote the process N stopped at v, and likewise for X^v and DX^v. Then N^v is an (MFD)-process and

$$(2) \quad \eta(X_t^v)(DX^v(t)) = DN^v(t)$$

<u>Proof</u>. By Theorem 1.2.2, (a),

$$N^v(t) - N^v(s) = \int_{s\wedge v}^{t\wedge v} (\eta(X_u) \circ U_u) dZ_u \ , \ s \leq t$$

Hence for all A in F_s, Remark (vi) of §4.1 shows that

$$E[1_A (N^v(t) - N^v(s))] = E[1_A \int_{s\wedge v}^{t\wedge v} (\eta(X_u) \circ U_u) DZ(u) du$$

$$= E[1_A \int_{s\wedge v}^{t\wedge v} \eta(X_u)(DX(u)) du \quad \text{by (1)}$$

which proves (2). □

<u>Remark</u>. Formula (2) gives a convenient way of evaluating DX ; simply evaluate DN for sufficiently many 1-forms η.

4.3 Non-degenerate diffusions as examples of (MFD)-semimartingales

Let A be an elliptic operator on M , as defined in §2.1. As before, we construct an A-metric on M, a Levi-Civita connection $\bar{\Gamma}$, a Laplacian Δ, and a vector field $C = A - \frac{1}{2}\Delta$ on M. Then the special connection associated to A, written Γ, may be defined. Using this we construct a diffusion process X as in §2.2. Whatever the initial frame U_o, the stochastic development of X into E, through the connection Γ, is of course a Brownian motion W, whose mean forward derivative is zero by §4.1, Remark (v). Hence with respect to the connection Γ, X is an (MFD)- semimartingale with mean forward derivative DX = O.

Consider instead the Levi-Civita connection $\bar{\Gamma}$.

PROPOSITION 4.3.1

X is an (MFD)-semimartingale with respect to $\bar{\Gamma}$, with mean forward derivative

(1) $DX(t) = C(X_t)$

Proof.

Let f be a C^{k+2} function on M with compact support. Let

$$Y = (\bar{\Gamma})\int_X df - (\Gamma)\int_X df$$

Combining Corollary 1.5.2 and equation §2.2,(2), we see that

$$Y_t = \int_o^t Cf(X_s)ds$$

Let $N = (\bar{\Gamma})\int_X df$. By Proposition 2.2.1, $(\Gamma)\int_X df$ is a martingale. Consequently the mean forward derivative of

N is the same as that of Y, namely

$$DN(t) = Cf(X_t)$$

More generally, one may calculate from §0.7,(2) that if η is any 1-form with compact support and $N = (\bar{\Gamma})\int_X \eta$, then

$$DN(t) = \eta(X_t)(C(X_t))$$

It is easy to check from first principles that a stochastic development of X through $\bar{\Gamma}$ is an Ito diffusion process with a mean forward derivative; hence DX exists. By the last line and Lemma 4.2.1, we must have

$$DX(t) = C(X_t)$$

as desired. □

<u>Remark</u>. This verifies that the present definition of mean forward derivative coincides with those given by Dankel [5] and by Dohrn and Guerra [8] for diffusions on Riemannian manifolds.

4.4 Calculating the mean forward derivative of a diffusion with unknown coefficients

This section is merely heuristic, not rigorous. Suppose an applied mathematician is making discrete-time observations of a diffusion process X, with unknown coefficients, on a Riemannian manifold M. We would like to indicate how he might calculate an approximation to the mean forward derivative DX.

Suppose for simplicity that M is embedded in a Euclidean space G, and $F:M \to G$ is the embedding. Let $\bar{\Gamma}$ be the Levi-Civita connection for M, and define a G-valued process N by

$$N = (\bar{\Gamma})\int_X dF$$

By Lemma 4.2.1, the mean forward derivative DX is completely specified by the equation

$$dF(X_t)(DX(t)) = DN(t)$$

Let $(X_1, X_2, \ldots)$ denote the values of X at successive time-points $(t_1, t_2, \ldots)$. Forgetting certain technicalities, Theorem 3.4.2 shows that at each time t

$$N_t = \underset{\delta \to 0}{L^2\text{-}\lim} \Sigma_{j:t_j<t}\, dF(X_j)(\exp_{X_j}^{-1}(X_{j+1})), \quad \delta = \sup_k\{t_{k+1}-t_k\}$$

Suppose we fix a time t_k and an event A in the σ-algebra F_k generated by the past observations $\{X_j\ ,\ j \leq k\}$. It is natural from the fore-going formulas to say:

"approximate the value of $DX(t_k)$, conditional on A, by $(t_{k+1} - t_k)^{-1}$ times the average in $T_{X_k}M$ of $\exp_{X_k}^{-1}(X_{k+1})$ over all sample trajectories X for which A holds."

The properties of this estimate are a topic for future study.

PART B

MARTINGALES ON MANIFOLDS

5 Γ-MARTINGALES

5.0 Motivation

The chapter begins with an intrinsic defintion of a martingale on a manifold with a connection Γ , in terms of Γ-convex functions. We go on to give five other characterizations, in terms of: the horizontal lift to L(M), the stochastic development, the Ito integral of a differential form, the local connectors, and the mean forward derivative. All the proofs are easy, with the exception of one, which is extremely hard, and which uses most of the machinery of §3.

A Γ-martingale may perhaps best be understood as a semimartingale whose stochastic development is a local martingale in the model space(Meyer[*28*,p.94]).

5.1 Definition in terms of Γ-convex functions

Let M be C^{k+2} manifold, $k \geq 0$, modelled on Euclidean n-space E, with a C^k linear connection Γ. (The same definition would work for an infinite-dimensional manifold). We shall give the simplest intrinsic definition of a Γ-martingale, which arose from a suggestion of J.Eells and K.D.Elworthy in 1980.

DEFINITION

A triple (W,W', f) will be called a Γ-martingale tester (or simply tester) if W and W′ are open sets in M with $\bar{W} \subset W'$, and $f:W' \to R$ is a C^2 Γ-convex function (see §0.1,(2)).

A process $X = (X_t, F_t)$ on M will be called a Γ-martingale if it is a (continuous) semimartingale, and for all Γ-martingale testers (W,W',f), the process Y is a local submartingale, where

$$Y = \int 1_F \, d(f \circ X) \quad , \quad F = \{(t,\omega) : X_t(\omega) \in W\}$$

Remarks.

(i) A natural question is: are there enough local Γ-convex functions to make the definition meaningful ? Given any x in M, we can take a normal co-ordinate system in a neighbourhood of x; then within a smaller neighbourhood, it is possible to take a local co-ordinate system consisting of Γ-convex functions which are quadratic functions of the normal co-ordinates; see Ishihara [16,p.219].

(ii) Strictly speaking, what we have defined is a continuous <u>local</u> Γ-martingale. Since the notion of distance is not defined on M, we have no notion of integrability (for random variables), hence no notion of martingale.

(iii) The random variable Y_t is always well-defined; for given a C^2 convex function $f:W' \to R$, there exists a C^2 function $f_1:M \to R$ such that $f = f_1$ on $\bar{W}$. By the definition of semimartingale on M, $f_1 \circ X$ is a real-valued semimartingale. Also 1_F is a bounded predictable process since X is continuous and W is open. Consequently the integral

$$Y_t = \int_0^t 1_F(s)d(f_1 \circ X_s) = \int_0^t 1_F(s)d(f \circ X_s)$$

is well-defined.

A multitude of examples will be given in §8. We give two elementary examples straight away (one in §5.2) to verify that the definition of Γ-martingale is a suitable one.

<u>Euclidean continuous local martingales</u>

Let M = E with the trivial connection Γ, whose local connector in the chart id:E $\to$ E is zero. Let X be a continuous semimartingale on E. If X is a local martingale in the usual sense and (W,W',f) is a tester, then Ito's formula (§0.4,(1)) shows that

$$\int 1_F \, d(f \circ X) = \int 1_F \, Df(X)dX + \frac{1}{2}\int 1_F \, D^2f(X)(dX \otimes dX)$$

The first integral on the right is a local martingale, and the second is an increasing process since $D^2f(.)$ is positive semi-definite on W'. So the left side is a local submartingale, verifying that X is a Γ-martingale.

<u>Conversely</u>, by taking $W = W' = E$, $f(x) = x^i$, and then $f(x) = - x^i$, we can verify that a Γ-martingale is a continuous local martingale in the usual sense.

5.2 Brownian motion

Let (M,g) be a C^{k+2} Riemannian manifold, $k \geq 2$, with Levi-Civita connection Γ, and Laplacian Δ. Define Brownian motion B on (M,g) to be a diffusion with generator $A = \frac{1}{2}\Delta$, in the sense of §2.2. A result of this is that for all f in $C^2(M)$ with compact support, C^f is a martingale, where

$$(1) \quad C_t^f = f(B_t) - f(B_o) - \frac{1}{2}\int_o^t \Delta f(B_s)ds$$

Hence for all f in $C^2(M)$, whether compactly supported or not, C^f is a local martingale.

LEMMA 5.2.1

Brownian motion is a Γ-martingale, with respect to the Levi-Civita connection.

Proof. Let (V,W,f) be any Γ-martingale tester. Then f is necessarily subharmonic on V, and the result follows from the easier part of the following Lemma.

LEMMA 5.2.2

Suppose V is open in M and $f:V \to R$ is a C^{k+2} map. Let L be the process

$$(2) \quad L = \int 1_F \, d(f \circ B) \ , \quad F = \{B \in V\}$$

Then the following two assertions are equivalent:

(a) f is subharmonic on V ,

(b) L is a local submartingale.

Remark. '(a) ⇒ (b)' is of course well-known (Schwartz [33,p106]

Proof. Assume (a). Then $1_F(s)\Delta f(B_s) \geq 0$ for all s. Rearrange (1) to give:

$$L_t = \int_0^t 1_F(s)\,dC_s^f + \frac{1}{2}\int_0^t 1_F(s)\Delta f(B_s)\,ds$$

which is a local martingale plus an increasing process, in other words a local submartingale, verifying (b).

Assume (b). This implies that for all predictable sets G such that $G \subset \{B \in V\}$, $\int 1_G\, d(f \circ B)$ is a local submartingale. We have to prove that $\Delta f(x) \geq 0$ for x in V. Suppose that for some $\varepsilon > 0$ and some y in V, $\Delta f(y) \leq -2\varepsilon$. Then there is a neighbourhood V′ of y, lying inside V, on which f is bounded, and such that

$$\Delta f(x) \leq -\varepsilon \quad \text{for all } x \text{ in } V'.$$

Start the Brownian motion B at $B_0 = y$, and let u be the first exit time of B from V′. Define a semimartingale Q by

$$Q_t = \int_0^t 1_{[0,u]}(s)\,d(f \circ B_s) = f(B_{t\wedge u}) - f(y)$$

Then Q is a local submartingale by the comment at the beginning. Since f is bounded on V′, Q is bounded, and therefore Q is actually a submartingale. Formula (1) may be rearranged to give the Doob-Meyer decomposition of Q, namely

$$Q_t = C_{t\wedge u}^f + \frac{1}{2}\int_0^{t\wedge u} \Delta f(B_s)\,ds$$

By the definition of V',

(2) $Q_t \leq C^f_{t\wedge u} - \frac{\varepsilon}{2}(t \wedge u)$

C^f is a martingale with mean zero, so $E(C^f_{t\wedge u}) = 0$ for all t, by the Optional Stopping Theorem. On the other hand, since Q is a submartingale,

$$E(Q_t) \geq E(Q_o) = 0 \text{ for all } t.$$

Combined with (2), this imples $E(t \wedge u) = 0$ for all t. This would imply that the exit time of B from V' is zero almost surely, which is false since B has continuous trajectories. Hence no such y exists, and (a) is proved. □

5.3 Other characterizations

The following characterizations are indispensable for the study of Γ-martingales:

THEOREM 5.3.1

Let M be of class C^{k+2}, $k \geq 2$, and let Γ be a C^k connection for M. Let X be a semimartingale on M. The following five statements are equivalent:

(I) X is a Γ-martingale, in the sense of §5.1.

(II) Every horizontal lift U of X to L(M) through Γ is a ${}^L\Gamma$-martingale on L(M). (U was defined in §1.2, and ${}^L\Gamma$ is the trivial connection for L(M) defined in §1.1.)

(III) The stochastic development Z of X into E, through the connection Γ, with any initial frame, is a (continuous) local martingale on E.

(IV) For all 1-forms η on M, $(\Gamma)\int_X \eta$ is a local martingale.

(V) If (W,φ) is any chart for M, if $\Gamma(.)$ is the local connector in W, and $F = \{X \in W\}$, then

$$\int 1_F \left(dX + \tfrac{1}{2}\Gamma(X)(dX \otimes dX)\right)$$

is a local martingale.

Remarks.

(i) Condition (V) is the definition of Γ-martingale

given by Bismut, and exposed by Meyer [*28*,p.54]. The equivalence (IV) $\Leftrightarrow$ (V) is trivial, as noted by Meyer [*28*,p.62]: The equivalence (III) $\Leftrightarrow$ (IV) is due to Bismut and proved in Meyer [*28* ,p.94]. The main difficulty lies in proving (I) $\Rightarrow$ (IV) .

(ii) Condition (IV) is <u>strictly weaker</u> than:

"$(\Gamma)\int_X \eta$ is a martingale for all 1-forms η of compact support"

This fact has been proved by Rogers [*31*].

(iii) We shall frequently omit the adjective 'continuous' when referring to local martingales in the sequel.

<u>Proof</u>. By Remark (i), it will suffice to prove:
(I) $\Leftrightarrow$ (IV), (V) $\Rightarrow$ (III) $\Rightarrow$ (II) $\Rightarrow$ (III) $\Rightarrow$ (IV).

"(IV) $\Rightarrow$ (I)":
Suppose $(\Gamma)\int_X \eta$ is a local martingale for all 1-forms η on M. Let (W,W',f) be a tester, and let $N = (\Gamma)\int_X df$.
By §0.6,(2), taking $F = \{X \in W\}$,

$$\int 1_F \, d(f \circ X) = \int 1_F \, dL, \quad \text{where } L = (S)\int_X df$$

By Corollary 1.4.2,

$$L = N + \frac{1}{2}\int \nabla df(X)(UdZ \otimes UdZ)$$

where U is any horizontal lift of X, and Z is the corresponding stochastic development. Hence

$$(1) \quad \int 1_F \, d(f \circ X) = \int 1_F \, dN + \frac{1}{2}\int 1_F \, \nabla df(X)(UdZ \otimes UdZ)$$

The first term on the right is a local martingale, since N is. Since f is Γ-convex on W,

$1_F(t)\nabla df(X_t)$ is positive semi-definite, for all t;

so the second term on the right of (1) is an increasing process. This shows that the left side of (1) is a local submartingale, completing the proof. □

"(I) ⇒ (IV)": The proof is outlined in §5.5, and given in detail in §5.6, in the remainder of this section, we suppose that it is already proved.

"(V) ⇒ (III)": This follows from §1.3,(4) (The matrix $A_{\cdot}(u) = (A_1(u),\ldots,A_n(u))$ is invertible.).

"(III) ⇒ (II)": Since "(IV) ⇒ (I)" is proved, it suffices to show that for all 1-forms σ on L(M), $({}^L\Gamma)\int_{\underline{U}} \sigma$ is a local martingale. This follows from Theorem 1.2.2 (c) and the assumption that Z is a local martingale.

"(II) ⇒ (III)": If U is a ${}^L\Gamma$-martingale, then $({}^L\Gamma)\int_U \beta$ is an E-valued local martingale, where β is the canonical 1-form defined in §1.1 (here we use characterization (IV)). This is just Z, by §1.3,(3).

"(III) ⇒ (IV)": This is immediate from Theorem 1.2.2 (a).

5.4 Γ-martingales and processes of zero mean forward derivative

COROLLARY 5.4.1

(i) Properties (a) and (b) are equivalent:

(a) X is an (MFD)-semimartingale on M with respect to Γ (see §4.2), such that DX(t) is the zero vector in $T_{X_t}M$ for all t.

(b) X is a semimartingale on M such that any stochastic development of X with respect to Γ is a martingale on E.

(ii) Each of the above conditions implies that X is a Γ-martingale on M.

Proof.

Immediate from the definitions, and Theorem 5.3.1, (III).□

5.5 Summary of the proof that (I) ⇒ (IV)

We proceed as follows. Select a 1-form η and denote $(\Gamma)\int_X \eta$ by N. To show that N is a local martingale, we split up the time interval [0,t] by a stochastic partition $(v(m) : m \in N)$. At each stopping-time v(m) we take normal co-ordinates (x^i) at $X_{v(m)}$. For any positive ε, there is a neighbourhood of $X_{v(m)}$ on which the functions

$$x^i + \varepsilon\Sigma_j (x^j)^2 \quad , \quad - x^i + \varepsilon\Sigma_j (x^j)^2$$

are Γ-convex. If v(m+1) is close enough to v(m), then both $x^i(X_t)$ and $- x^i(X_t)$ are 'nearly' submartingales on]v(m),v(m+1)] ; consequently N is 'nearly' a martingale on this stochastic interval. In order to make this precise, we need to show that the quadratic correction terms can be ignored in the limit. This requires calculations involving the geodesic deviation.

5.6 Proof that (I) ⇒ (IV)

Step 1. Suppose X is a Γ-martingale on M. Given a 1-form η on M, it is required to prove that $(\Gamma)\int_X \eta$ is a local martingale. Denote the last process by N. Notice that we do not need to take a chart for M. We must construct an increasing sequence of stopping-times $(u(k))$ such that $\lim_k P(u(k) < t) = 0$ for all t, and for all $t_1 < t_2$

(1) $E[N^{u(k)}(t_2) - N^{u(k)}(t_1) \mid F_{t_1}] = 0$ a.s.

where $N^{u(k)}$ is the process N stopped at u(k).

The notations will be as follows. Let U be a horizontal lift of X to L(M) and Z the stochastic development of X into E with initial frame U_o. For each positive integer k define

$$T_k^2 = \{(u,u') : u,u' \text{ are bounded stopping-times, } u \leq u' \leq u + k^{-1}, \text{ and } (X_u, X_t) \in V \text{ for } u \leq t \leq u'\}$$

For each (u,u') in T_k^2, the geodesic deviation $G(u,u')$ may be defined (with respect to U) as above . Then

$$G(u,u') = \int_u^{u'} \{R(u,t)dZ_t + Q(u,t)(dZ\otimes dZ)_t\}$$

according to equation §3.2,(3) . Use the abbreviation

$$\theta_u = (\exp_{X_u} \circ U_u)^{-1}$$

Then $\theta_u(X_{u'})$ is well-defined for all (u,u') in T_k^2, for all k. Finally introduce the following norm on E ($= R^n$) :

if v in E has co-ordinates $(v^1,\ldots,v^n)$ as usual, take $|v|_s = \frac{1}{n}\Sigma_i\ |v^i|$. The symbol $|.|_s$ also denotes the corresponding norm on E*.

We now define the sequence of stopping-times (u(k)) as follows. For each positive integer k, u(k) is the first time t at which <u>any</u> of the following occurs:-

(2) $|N_t| > k$

(3) $|\eta(X_t) \circ U_t|_s > k$ ($|.|_s$ is the norm defined above)

(4) $\sup\{\ ||R(u,u')||\ :\ (u,u') \in T_k^2\ ,\ u' \leq t\} > k/4n$

(5) $\sup\{|\theta_u(X_{u'})|:\ (u,u') \in T_k^2\ ,\ u' \leq t\} > k$

(6) $[Z]_t > k$

(7) $t = k$

where [Z] is the <u>scalar</u> quadratic variation process of Z. Notice that $\lim_k P(u(k) < t) = 0$ for all t . The other consequences of this choice of u(k) will be seen later.

<u>Step 2</u>. The next step is a simple result about normal co-ordinates, also noted by Ishihara [*16* ,p.219].

LEMMA 5.6.1 (Constructing local Γ-convex functions)

Fix a in M and a positive number ε. Let (x^i) be a normal co-ordinate system about a, with respect to the connection Γ . Then there exists a neighbourhood $N_\varepsilon(a)$ of a on which each of the 2n functions

$$h^k = x^k + \varepsilon\Sigma_i\ (x^i)^2\ ,\quad \bar{h}^k = -\ x^k + \varepsilon\Sigma_i\ (x^i)^2$$

is Γ-convex.

<u>Proof</u>. (see Note) $D_j h^k = \delta^k_j + 2\varepsilon x^j$

$$(\nabla dh^k)_{pq} = D_{pq}h^k - \Gamma^j_{pq} D_j h^k = -\Gamma^k_{pq} + 2\varepsilon(\delta^p_q - \Sigma_j \Gamma^j_{pq} x^j)$$

Since $x^j(a) = 0$, $\Gamma^j_{pq}(a) = 0$, and $\Gamma^j_{pq}(x)$ is continuous in x, there is a neighbourhood of a on which $\nabla dh^k(.)$ is positive definite; likewise for $\nabla d\bar{h}^k(.)$. Finally $N_\varepsilon(a)$ is taken to be the intersection of these neighbourhoods. □

<u>Note</u>. The Christoffel symbols Γ^j_{pq} are related to the local connector $\Gamma(.)$ thus: $\Gamma^j_{pq}(x)$ is the jth co-ordinate of $\Gamma(x)(e_p, e_q)$. See §0.2.

<u>Step 3</u>. Let us fix a positive integer k, and times $t_1 < t_2$ to serve in (1). For each $\varepsilon \in \{\frac{1}{n} : n \in \mathbb{Z}, n \geq k\}$, define an increasing sequence of bounded stopping-times $(v(\varepsilon,m) : m \in N)$ as follows. Let $v(\varepsilon,0) = t_1 \wedge u(k)$. Whenever $v(\varepsilon,m)$ is defined, abbreviate $X(v(\varepsilon,m))$ to $X_{\varepsilon,m}$ and $\theta_{v(\varepsilon,m)}$ to $\theta_{\varepsilon,m}$. Let $\theta^i_{\varepsilon,m}$ denote the ith co-ordinate function of $\theta_{\varepsilon,m}$. The preceding Lemma shows that there is a neighbourhood $N_\varepsilon(X_{\varepsilon,m})$ of $X_{\varepsilon,m}$ on which each of the 2n functions

$$(8)\begin{cases} h^i(.) = \theta^i_{\varepsilon,m}(.) + \varepsilon|\theta_{\varepsilon,m}(.)|^2 \\ \bar{h}^i(.) = -\theta^i_{\varepsilon,m}(.) + \varepsilon|\theta_{\varepsilon,m}(.)|^2 \end{cases}$$

is Γ-convex; notice that $N_\varepsilon(X_{\varepsilon,m})$ is contained in the domain of the map $\theta_{\varepsilon,m}$. Now define $v(\varepsilon,m+1)$ to be $t_2 \wedge u(k) \wedge \tau_1 \wedge \tau_2 \wedge \tau_3$, where $\tau_1 = v(\varepsilon,m) + \varepsilon$; τ_2 is the first exit time of X from $N_\varepsilon(X_{\varepsilon,m})$ after $v(\varepsilon,m)$; and if $\varepsilon = \frac{1}{n+1}$, $n \geq k$, and $\varepsilon' = \frac{1}{n}$, and q is such that $v(\varepsilon',q) \leq v(\varepsilon,m) < v(\varepsilon',q+1)$, then $\tau_3 = v(\varepsilon',q+1)$.

Since X has continuous trajectories, each trajectory is covered by a finite number of $N_\varepsilon(.)$ sets on the compact time interval $[t_1,t_2]$. It follows that for all $t < t_2$

$$\lim_m P(v(\varepsilon,m) < t \wedge u(k)) = 0$$

The choice of τ_2 and τ_3 ensures that the family $(v(\varepsilon,m))$ satisfies the conditions of Theorem 3.4.1 (with $n = \frac{1}{\varepsilon}$).

Let $F_{\varepsilon,m}$ denote the random interval $]v(\varepsilon,m),v(\varepsilon,m+1)]$. By (5) and (8),

$$(9)\begin{cases} |\theta_{\varepsilon,m}(X_t)| \leq k \quad , \quad |\theta^i_{\varepsilon,m}(X_t)| \leq k \\ |h^i(X_t)| \leq 2k \quad , \quad |\bar{h}^i(X_t)| \leq 2k \end{cases}$$

for all (t,ω) in $F_{\varepsilon,m}$. Since X is a Γ-martingale, the processes

$$\int 1_{F_{\varepsilon,m}} d(h^i \circ X) \quad , \quad \int 1_{F_{\varepsilon,m}} d(\bar{h}^i \circ X)$$

are local submartingales, and indeed submartingales by (9). In particular, since $h^i(X_{\varepsilon,m}) = 0 = \bar{h}^i(X_{\varepsilon,m})$,

$$(10)\begin{cases} E[h^i(X_{\varepsilon,m+1}) \mid F_{\varepsilon,m}] \geq 0 \quad , \\ E[\bar{h}^i(X_{\varepsilon,m+1}) \mid F_{\varepsilon,m}] \geq 0 \end{cases}$$

where $F_{\varepsilon,m}$ denotes $F_{v(\varepsilon,m)}$. It is convenient to abbreviate as follows:

$$J^i_{\varepsilon,m+1} = \theta^i_{\varepsilon,m}(X_{\varepsilon,m+1}) \quad , \quad S_{\varepsilon,m+1} = |\theta_{\varepsilon,m}(X_{\varepsilon,m+1})|^2$$

Hence by (8),

$$h^i(X_{\varepsilon,m+1}) - \varepsilon S_{\varepsilon,m+1} = J^i_{\varepsilon,m+1} = -\bar{h}^i(X_{\varepsilon,m+1}) + \varepsilon S_{\varepsilon,m+1}$$

The inequalities (10) imply

$$(11)\begin{cases} -\varepsilon E[S_{\varepsilon,m+1} \mid F_{\varepsilon,m}] \leq E[J^i_{\varepsilon,m+1} \mid F_{\varepsilon,m}] \leq \\ \qquad\qquad \varepsilon E[S_{\varepsilon,m+1} \mid F_{\varepsilon,m}] \end{cases}$$

Now apply (3), abbreviating $\eta(X_{\varepsilon,m}) \circ U_{\varepsilon,m}$ to $a(X_{\varepsilon,m})$:-

$$k\varepsilon E\{S_{\varepsilon,m+1} \mid F_{t_1}\} \geq \varepsilon E\{ |a(X_{\varepsilon,m})|_s \, S_{\varepsilon,m+1} \mid F_{t_1}\}$$

$$= \varepsilon E\{ |a(X_{\varepsilon,m})|_s \, E[S_{\varepsilon,m+1} \mid F_{\varepsilon,m}] \mid F_{t_1}\}$$

$$\geq E\{ |a(X_{\varepsilon,m})|_s (\Sigma_i \tfrac{1}{n}|E[J^i_{\varepsilon,m+1} \mid F_{\varepsilon,m}]|) \mid F_{t_1}\} \text{ by (11)}$$

$$\geq E\{(|a(X_{\varepsilon,m})(\theta_{\varepsilon,m}(X_{\varepsilon,m+1}))|) \mid F_{t_1}\}$$

by definition of $|.|_s$,

$$\geq \left|E\{\eta(X_{\varepsilon,m})(\exp^{-1}_{X_{\varepsilon,m}}(X_{\varepsilon,m+1})) \mid F_{t_1}\}\right|$$

since $a(X_t) \circ (\exp_{X_t} \circ U_t)^{-1} = \eta(X_t) \circ \exp^{-1}_{X_t}$

<u>Step 4</u>.The next step is to find bounds for $\Sigma_m S_{\varepsilon,m+1}$.

Observe that

$$S_{\varepsilon,m+1} = |Z_{\varepsilon,m+1} - Z_{\varepsilon,m} - G(v(\varepsilon,m),v(\varepsilon,m+1))|^2$$

by the definition of geodesic deviation in §3.1,(1) . Hence

$$S_{\varepsilon,m+1} \leq 2|Z_{\varepsilon,m+1} - Z_{\varepsilon,m}|^2 + 2|G(v(\varepsilon,m),v(\varepsilon,m+1))|^2$$

By the properties of scalar quadratic variation, and §3.2,(3) this implies:-

$$\Sigma_m S_{\varepsilon,m+1} \leq 2\langle Z\rangle_{u(k)} + 2\Sigma_m \int_{v(\varepsilon,m)}^{v(\varepsilon,m+1)} ||R(v(\varepsilon,m),t)||^2 4n^{-1} d[Z]_t$$

By (4) and (6),

$$\Sigma_m S_{\varepsilon,m+1} \leq 2k + 2k^2[Z]_{u(k)} \leq 2(k + k^3)$$

The calculations in step 3 show that

$$(12)\begin{cases} |\Sigma_m E\{\eta(X_{\varepsilon,m})(\exp^{-1}_{X_{\varepsilon,m}}(X_{\varepsilon,m+1})) \mid F_{t_1}\}| \\ \quad \leq k\varepsilon E\{\Sigma_m S_{\varepsilon,m+1} \mid F_{t_1}\} \leq 2\varepsilon(k^2 + k^4) \end{cases}$$

<u>Step 5</u>.Our progress so far may be summarized thus. Use the abbreviations:-

$$Y = N^{u(k)}(t_2) - N^{u(k)}(t_1)$$

$$Y^\varepsilon = \Sigma_m \eta(X_{\varepsilon,m})(\exp^{-1}_{X_{\varepsilon,m}}(X_{\varepsilon,m+1}))$$

$$G = F_{t_1}$$

We know that

$$(13)\begin{cases} |Y| \leq 2k & \text{by (2)} \\ |E[Y^\varepsilon \mid G]| \leq 2\varepsilon(k^2 + k^4) & \text{by (12)} \\ Y = \lim_{\varepsilon \to 0} \text{prob } Y^\varepsilon & \text{by Theorem 3.4.1.} \end{cases}$$

We are required to prove (1), namely

(14) $E[Y \mid G] = 0$

This follows from a simple Lemma :-

LEMMA 5.6.2

If Y and $(Y^\varepsilon : \varepsilon > 0)$ are random variables satisfying (13), then (14) holds.

<u>Proof</u>. Suppose $A \in G$ and $\delta > 0$ are given. Choose ε_0 sufficiently small so that for ε less than ε_0,

$$P(\ |Y^\varepsilon - Y| > \frac{\delta}{4}\ ) < \frac{\delta}{4k}$$

For all such ε,

$$\begin{aligned} |E(1_A\ Y\ )| &\leq |E[1_A\ Y\ ;\{|Y^\varepsilon - Y| > \tfrac{\delta}{4}\}]| \\ &\quad + |E[1_A(Y^\varepsilon - Y) + 1_A\ Y^\varepsilon\ ;\{\ |Y^\varepsilon - Y| \leq \tfrac{\delta}{4}\}]| \\ &\leq 2k(\tfrac{\delta}{4k}) + \tfrac{\delta}{4} + 2\varepsilon(k^2 + k^4) \end{aligned}$$

which is no greater than δ when $\varepsilon < \min(\varepsilon_0\ ,\ \delta/8(k^2 + k^4))$. □

Thus equation (1) is true, and the Theorem is proved. □

6 BEHAVIOUR UNDER MAPS

6.0 Motivation

Out of the many possible characterizations of Γ-martingales, the one concerning Γ-convex functions was chosen as the definition because it gives most insight into the relationship of Γ-martingales with various kinds of maps from one manifold to another. The results of the present chapter are not deep, with the possible exception of Corollary 6.2.2. However we hope that the summary, laid out as a table in §6.1, will give the reader an intuitive idea of the relationship of Γ-martingales to concepts in differential geometry.

6.1 Some classes of maps between manifolds

This section is entirely expository. Let M and N be C^{k+2} manifolds, $k \geq 0$, with C^k connections Γ^M and ${}^N\Gamma$ respectively. In some cases we will endow M, or M and N, with C^{k+1} Riemannian metrics, in which case the connection will be the Levi-Civita connection. Let $\varphi: M \to N$ be a C^{k+2} map. The nature and properties of harmonic maps and harmonic morphisms are discussed at length in the survey by Eells and Lemaire [*9*]; see also Ishihara [*16*]. Affine maps are described in Kobayashi and Nomizu [*20* , Ch. VI], and in Vilms [*35*]. For present purposes it is expedient to use the characterizations of harmonic map and harmonic morphism given in Ishihara [*16* ,pp.220 - 221].

An expression such as

"φ pulls back local ${}^N\Gamma$-convex functions to local ${}^M\Gamma$-convex functions"

means that for all open sets V in N and all C^{k+2} ${}^N\Gamma$-convex functions $f: V \to R$, the map $f \circ \varphi$ is a ${}^M\Gamma$-convex function from $\varphi^{-1}(V)$ to R. Using such language, four classes of maps are defined by the table on the following page. The 'stochastic characterizations' will be proved later in this chapter.

Type of map φ	We require: on M	on N	Functional characterization: φ pulls back local ... functions to local ... f'ns		Stochastic characterization: The image under φ of ... is a ...	
affine map	M_Γ	N_Γ	N_Γ-convex	M_Γ-convex	every M_Γ-martingale	N_Γ-martingale
harmonic map	Riem.[1]	N_Γ	N_Γ-convex	subharmonic	Brownian motion started anywhere	N_Γ-martingale
harmonic morphism	Riem.	Riem.	harmonic	harmonic	Brownian motion started anywhere	time-changed Brownian motion
			implied by[3] (harmonic	subharmonic)		
Riemannian submersion with minimal fibres[2]	Riem.	Riem.	$^M\Delta(f \circ \varphi) = (^N\Delta f) \circ \varphi$		Brownian motion started anywhere	Brownian motion

1 "Riem." means that the manifold is Riemannian with the Levi-Civita connection.

2 See Wallach [36]. We include this only for the sake of completeness.

3 See Ishihara [16 ,p.226].

Note. The affine maps and the harmonic morphisms are each a subset of the harmonic maps, whenever the relevant classes are defined.

6.2 Harmonic maps send Brownian motion to a Γ-martingale

The following result is due to Meyer [27,p.265].

PROPOSITION 6.2.1

A C^{k+2} map $\varphi:M \to N$ is harmonic if and only if for all y in M, the process $\varphi \circ B$ is a ${}^N\Gamma$-martingale, where B is Brownian motion on M started at y.

Proof. Immediate from Lemma 5.2.2 and the functional definition of harmonic map in §6.1. □

COROLLARY 6.2.2

Let M be any connected, non-compact Riemannian manifold of dimension n. There is a proper embedding $i:M \to R^{2n+1}$ such that $i \circ B$ is a local martingale in R^{2n+1}, where B denotes Brownian motion on M.

Proof.

The result follows from the last Proposition, and a theorem of Greene and Wu [14,p.231], which says that such an M has a proper embedding by harmonic functions in R^{2n+1}; (we use here the result obtained in §5.1, Example, namely that Γ-martingales in a Euclidean space with the trivial connection are the same as continuous local martingales). □

Remark. For other uses of stochastic methods in harmonic maps, see Kendall [19], and Elworthy [40].

6.3 Harmonic morphisms preserve Brownian paths

Let M and N be Riemannian manifolds of dimensions m and n respectively. The following characterization of harmonic morphisms is due to Fuglede [*13*,p.116].

LEMMA 6.3.1

A C^{k+2} map $\varphi:M \to N$ is a harmonic morphism if and only if there exists a function $\gamma \geq 0$ on M (necessarily unique and such that γ^2 is C^k) with the property that

$$^M\Delta(f \circ \varphi) = \gamma^2 \, [(^N\Delta f) \circ \varphi]$$

for all f in $C^{k+2}(N)$.

This enables us to characterize harmonic morphisms as those C^{k+2} maps which preserve the paths of Brownian motion, as proved below. This result is related to those of Bernard, Campbell and Davie [2] , and was suggested by J.Eells in 1980.

THEOREM 6.3.2

A C^{k+2} map $\varphi:M \to N$ of Riemannian manifolds is a harmonic morphism if and only if (*) holds for all $a \in M$:

(*) Let B be Brownian motion on M with $B_0 = a$. There exists a continuous increasing process A and a Brownian motion $\bar{B}$ on N such that

$$\bar{B} \circ A = \varphi \circ B \ .$$

($\bar{B} \circ A$ means the process whose value at time t is $\bar{B}(A_t)$.)

Remark.

If $m \leq n$, then a harmonic morphism φ is necessarily constant, and so $A_t = 0$ almost surely.

Proof. ' $\Rightarrow$ '

Fix $a \in M$. Let B be Brownian motion on M with $B_o = a$. Let γ be as in the Lemma. Define a continuous increasing process A by:

$$A_t = \int_o^t \gamma(B_s)^2 ds$$

and define its inverse by

$$C_t = \inf \{u : A_u > t\}$$

Denote $\varphi \circ B$ by Y, which is a semimartingale on N. We shall prove that $Y \circ C$ is a Brownian motion $\bar{B}$ on N. Then $\bar{B} \circ A = Y$ as desired.

It suffices to show that for all C^{k+2} functions $f:N \to R$ the process L^f is a local martingale, where

$$L_t^f = f(Y \circ C_t) - f(Y_o) - \frac{1}{2}\int_o^t N_{\Delta f}(Y \circ C_s)ds$$

However under the change of variables $s \to A_u$,

$$\int_o^t N_{\Delta f}(Y \circ C_s)ds = \int_o^{C_t} N_{\Delta f}(Y_u)dA_u = \int_o^{C_t} \gamma(B_u)^2 N_{\Delta f}(Y_u)du$$

$$= \int_o^{C_t} M_{\Delta(f \circ \varphi)}(B_u)du$$

by the Lemma. By definition of B, the process H^f is a local martingale, where

$$H_t^f = f \circ \varphi(B_t) - f \circ \varphi(a) - \frac{1}{2}\int_0^t M_\Delta(f \circ \varphi)(B_s)ds$$

Hence $(H^f \circ C)$ is also a local martingale; but

$$L_t^f = H^f \circ C_t$$

so the result follows.

' ← '

Let V be open in N and let $f:V \to R$ be harmonic. Fix $a \in M$ with $\varphi(a) \in V$, and assume (*) holds. Let u be the first exit time of B from $\varphi^{-1}(V)$; then $(f \circ \bar{B}_{t\wedge u})$ is a continuous local martingale, and so is $(f \circ \bar{B} \circ A_{t\wedge u}) = (f \circ \varphi \circ B_{t\wedge u})$. Consider $f \circ \varphi$ as a C^2 map $f^{-1}(V) \to R$. It sends Brownian motion stopped at u to a local martingale. and hence is a harmonic map by Proposition 6.2.1. This proves that $f \circ \varphi$ is a local harmonic function on M whenever f is a local harmonic function on N. So φ is a harmonic morphism. □

6.4 Affine maps preserve the Γ-martingale property

Recall that if $\varphi:E \to G$ is a map between Euclidean spaces, local martingales on E are sent to local martingales on G if and only if φ is linear.

The class of affine maps was characterized in §6.1 as the maps which pull back ${}^N\Gamma$-convex functions to ${}^M\Gamma$-convex functions (Riemannian structures are not needed); this follows a result of Ishihara [*16*p.220]. A more usual definition would be : $\varphi:M \to N$ is <u>affine</u> (or totally <u>geodesic</u>) if and only if ϕ sends parallel vector fields along a curve to parallel vector fields along a curve or equivently, if ϕ sends ${}^M\Gamma$-geodesics to ${}^N\Gamma$-geodesics.

PROPOSITION 6.4.1

A C^{k+2} map $\varphi:M \to N$ is affine if and only if φ sends ${}^M\Gamma$-martingales to ${}^N\Gamma$-martingales.

<u>Proof</u>. ' $\Rightarrow$ ': Let (W,W',f) be a ${}^N\Gamma$-martingale tester on N. Then $(f \circ \varphi)$ is ${}^M\Gamma$-convex on $\varphi^{-1}(W')$, so $(\varphi^{-1}W,\varphi^{-1}W',f \circ \varphi)$ is a ${}^M\Gamma$-martingale tester on M. The result now follows immediately from the definition of Γ-martingale in §5.1.

' $\Leftarrow$ ': (The idea of this proof comes from K.D.Elworthy) It suffices to show that if $a,b > 0$ and $\gamma:(-a,b) \to M$ is a ${}^M\Gamma$-geodesic, then $\varphi \circ \gamma:(-a,b) \to N$ is a ${}^N\Gamma$-geodesic. Let (W_t) be one-dimensional Brownian motion with $W_0 = 0$, stopped at the first t at which $|W_t| = \min(a,b)$. The geodesic γ is a harmonic map $(-a,b) \to M$, so by Proposition

6.2.1, $(\varphi \circ W_t)^{\cdot}$ is a ${}^M\Gamma$-martingale. Hence $(\varphi \circ \gamma \circ W_t)$ is a ${}^N\Gamma$-martingale. Apply the 'if' part of Proposition 6.2.1 to conclude that $\varphi \circ \gamma$ is harmonic, hence a geodesic on N. □

Immersed submanifolds

Suppose $i:M \to (N,g)$ is a Riemannian immersion, and ${}^M\Gamma$, ${}^N\Gamma$ are the induced Levi-Civita connections on M and N. The condition for all M-valued ${}^M\Gamma$-martingales to be ${}^N\Gamma$-martingales is that M be a totally geodesic submanifold of N, by the preceding Proposition. What are the possibilities when i is not totally geodesic? We take as examples the sphere and the torus embedded in Euclidean space.

Examples.

(a) Let M be the sphere S^{n-1} embedded in R^n, with the induced metric and Levi-Civita connection Γ. Let $a \in S^{n-1}$ have co-ordinates $(a_1,\ldots,a_n)$. Suppose X is a Γ-martingale on S^{n-1} with $X_o = a$. Then X cannot be a local martingale in R^n, unless it is constant a.s. For if so, then $Y_t = a_1X^1(t) + \ldots + a_nX^n(t)$ would be a local martingale, and in fact a martingale (since X is bounded), with $Y_o = 1$. Hence $E(Y_t) = 1$ for all t. But for any $x = (x^1,\ldots,x^n)$ in S^{n-1} with $x \neq a$, $a_1x^1 + \ldots + a_nx^n < 1$. So $X_t = a$ a.s..

(b) Consider the torus example of §8.5; the embedding $i:T^2 \to R^3$ is not totally geodesic, and yet we construct a Γ-martingale on T^2 which is also an R^3 martingale.

7 L^2 Γ-MARTINGALES ON RIEMANNIAN MANIFOLDS

7.0 Motivation

Suppose (M,g) is Riemannian with a metric connection Γ (possibly with torsion). Suppose X is a Γ-martingale on M with a horizontal lift U to O(M) and a stochastic development Z, as in §1. We know by Theorem 5.3.1,(III), that Z is a local martingale. Impose the condition that Z is an L^2 martingale. Using the fact that U_t is an orthonormal frame for each t, the L^2 property of Z is transferred to X, in the sense of Corollary 1.5.1. In this case X is called an L^2 Γ-martingale. Provided M is complete, we are able to prove that X_t converges almost surely as t tends to infinity, to a random variable in the one-point compactification of M - without any curvature assumptions!

The chapter begins with a remark about representation of L^2 Γ-martingales in terms of Brownian motion, which is not used in the sequel.

7.1 Definition and representation

Let M be a manifold of class C^{k+2}, $k \geq 1$, let g be a C^{k+1} Riemannian metric on M, and let Γ be any C^k metric connection for M (possibly with torsion).

DEFINITION ·

A Γ-martingale X on (M,g) will be called an L^2 Γ-martingaale if for any horizontal lift U of X to O(M),and stochastic development Z with initial frame U_o , Z is an L^2 martingale on E in the usual sense, that is

(1) $\sup_t E(|Z_t|^2) < \infty$

Remarks.

(i) The left side of (1) is the same, whatever initial frame U_o is used to obtain Z. To see this, note that

$$E(|Z_t|^2) = \Sigma_k E((Z_t^k)^2) = \Sigma_k E(\langle Z^k\rangle_t) = E([Z]_t)$$

and the scalar quadratic variation [Z] is given in Corollary 1.5.1 in terms of X and g only.

(ii) An example of an L^2 Γ-martingale (essentially a slowed-down diffusion) is given in §8.4.

Suppose now that $W = (W_t, F_t^W)$ is a p-dimensional Brownian motion and $X = (X_t, F_t^W)$ is an L^2 Γ-martingale on (M,g), adapted to the filtration induced by W. Let U be a horizontal lift of X to O(M) through Γ, and let Z be the stochastic development of X into E with initial frame U_o.

Then Z is also an L^2 Γ-martingale adapted to the filtration (F_t^W). By the usual L^2 martingale representation theorem - see for example Kallianpur [*18*.p.157] - there is an $L(R^p;E)$-valued process $\Phi = (\Phi_s, F_s^W)$ such that $\Phi(s,\omega)$ is jointly measurable,

$$\int_0^\infty E(\mathrm{Trace}(\Phi_s^*\Phi_s))ds < \infty \qquad \text{, and}$$

$$(2) \quad Z = \int \Phi dW = \int (\Phi_1 dW^1 + \ldots + \Phi_p dW^p)$$

where $\Phi_1,\ldots,\Phi_p$ are the vectors in E constituting the columns of Φ.

THEOREM 7.1.1 (L^2 Γ-martingale representation theorem)

If X is an L^2 Γ-martingale adapted to the filtration of a p-dimensional Brownian motion W, then for all 1-forms η on M, we have:

$$(3) \quad (\Gamma)\int_X \eta = \int \eta(X)(U(\Phi_m))dW^m \quad \text{(m is summed from 1 to p)}$$

where U and Φ_m are as above. For $f \in C^{k+2}(M)$, (3) can be written

$$(4) \quad f(X_t) - f(X_o) = \int_o^t (U_s(\Phi_m(s))f)dW_s^m + \frac{1}{2}\Sigma_q \int_o^t \nabla df(X_s)(U_s(\Phi_q(s)),U_s(\Phi_q(s)))ds$$

<u>Remarks</u>

If another horizontal lift U′ were used, giving another set of vectors $\Phi_1',\ldots,\Phi_p'$, then $U_s'(\Phi_m'(s)) = U_s(\Phi_m(s))$, so the representation is unchanged.

For the notation in (4), see §0.5, (2′) and (3′).

Proof.

Recall from Theorem 1.2.2(a) that

$$(\Gamma)\int_X \eta = \int \eta(X)(U(e_i))dZ^i$$

Here, of course, $(e_1,\ldots,e_n)$ is an orthonormal basis of E. Combine this with (2), and note that

$$U(e_i)dZ^i = U(\Phi_m)dW^m$$

to give formula (3). Formula (4) is a combination of the absolute Ito formula in Corollary 1.4.3, and formula (2).□

7.2 Example of Γ-martingale representation

A representation such as (3) arises naturally when we construct a Γ-martingale using a harmonic map (see §6.1). We proceed as follows. Suppose (M,g) is Riemannian, Γ is the Levi-Civita connection, and N is another manifold with a connection ${}^N\Gamma$. Construct Brownian motion B on M, according to the Eells/Elworthy method of §2.2; see §5.2. We proved in §6.2 the result of Meyer, that if $\varphi:(M,g) \to (N,{}^N\Gamma)$ is a harmonic map, then $Y = \varphi \circ B$ is a ${}^N\Gamma$-martingale.

Let σ be a 1-form on N. The usual pullback formula for Stratonovich integrals, §0.4,(6), says that

$$(1) \quad (S)\int_Y \sigma = (S)\int_B \varphi^*\sigma$$

<u>In this special case</u> (not in general!), the formula also works for Ito integrals, namely

$$(2) \quad ({}^N\Gamma)\int_Y \sigma = (\Gamma)\int_B \varphi^*\sigma$$

To see that this is so, note that by (1) and the Ito-Strat. correction formula in Corollary 1.4.2, the left and right sides of (2) differ only by a process of finite variation. But both are local martingales, by Theorem 5.3.1, (IV), so they are equal. As a consequence of (2) and §2.2,(2),

$$({}^N\Gamma)\int_Y \sigma = \int \sigma(Y)(T\varphi \circ U)dW$$

where $W = (W^1,\ldots,W^n)$ is the n (= dim(M))-dimensional Brownian motion on E from which B was constructed. (Of course Y cannot be an $L^2{}^N\Gamma$-martingale here, unless we put a Riemannian metric on N, and place conditions on Tφ - e.g. boundedness)

7.3 Almost sure convergence of L^2 Γ-martingales

THEOREM 7.3.1

Let (M,g) be a complete Riemannian manifold of class C^{k+2}, $k \geq 1$, with a C^k metric connection Γ. Let X be an L^2 Γ-martingale on M, as defined in §7.1. Then there exists a random variable X_∞ with values in $M \cup \{\delta\}$ (= the one-point compactification of M) such that $X(\infty,\omega) = \lim_{t\to\infty} X(t,\omega)$ for almost all ω.

Remarks.

(a) No curvature assumptions are needed.

(b) X may well have 'drift' with respect to the Levi-Civita connection on M.

Notation. The space of continuous square-integrable real-valued martingales is denoted M_c^2.

Proof outline. First we prove two Lemmas, and proceed to prove the Theorem when M is compact. After another Lemma, we prove the Theorem in the general case.

LEMMA 7.3.2

Let (M,g) and Γ be as above.

(a) If X is an L^2 Γ-martingale, then

(1) $(\Gamma)\int_X \eta \in M_c^2$ for all bounded 1-forms η.

(b) The converse also holds, at least if M is compact(possibly with boundary).

Proof. (a) is immediate from Corollary 1.5.3 and the calculus of square integrable martingales. As for (b), let U be a horizontal lift of X to O(M) through Γ, and Z the corresponding stochastic development. By Whitney's embedding theorem and the compactness of M, there is an integer q and an embedding $F:M \to R^q$ of class C^{k+2}. Let $F^1,\ldots,F^q$ be the co-ordinate functions of F; each dF^i is a bounded 1-form because it has compact support. Each U_t is an orthonormal frame, so there exist bounded continuous E-valued processes $(K_1(t)),\ldots,(K_q(t))$, such that

$$U_t^{-1}(.) = K_i(t)dF^i(X_t)(.) \qquad \text{(i is summed from 1 to q)}$$

Using Theorem 1.2.2,(a) and §0.7,(2)

$$(2) \quad Z_t = \int_0^t (U_s^{-1} \circ U_s)dZ_s = \int_0^t K_i(s)dN^i(s)$$

where $N^i = (\Gamma)\int_X dF^i$. By assumption (1), each $N^i \in M_c^2$. Since each K_i is a bounded process, the right side of (2) is a process in M_c^2. Hence $Z \in M_c^2$. □

LEMMA 7.3.3

Let M and M′ belong to M_c^2 , and let R be a bounded predictable process with $|R_t| \leq c$, for some constant c. Then

$$\lim_{t\to\infty} \int_0^t R_s d\langle M,M'\rangle_s \quad \text{exists almost surely.}$$

<u>Proof</u>. Since $\langle M,M'\rangle_s = \frac{1}{4}\{\langle M + M'\rangle_s - \langle M - M'\rangle_s\}$,

$$\left|\int_t^\infty R_s d\langle M,M'\rangle_s\right| \leq \frac{1}{4}\int_t^\infty |R_s|\{d\langle M + M'\rangle_s + d\langle M - M'\rangle_s\}$$

$$\leq \frac{c}{4}\{\langle M + M'\rangle_\infty + \langle M - M'\rangle_\infty - \langle M + M'\rangle_t - \langle M - M'\rangle_t\}$$

The last expression is well-defined (see for instance Kallianpur [*18*,p.34] and tends to zero almost surely as $t \to \infty$. □

7.4 Proof when the manifold is compact

Assume M is compact. All notations continue from the previous section. Let f belong to $C^{k+2}(M)$. By Corollary 1.4.3,

$$f(X_t) - f(X_o) = M^f(t) + A^f(t) \quad , \text{ where}$$

$$M^f = \int (df(X) \circ U)dZ = (\Gamma)\int_X df$$

$$A^f = \frac{1}{2}\int \nabla df(X)(UdZ \otimes UdZ)$$

Since M is compact, df is a bounded 1-form, so $M^f \in \mathcal{M}_c^2$ by Lemma 7.3.2. Hence $M_\infty^f = \lim_{t\to\infty} M^f(t)$ exists almost surely, by the theory of square-integrable martingales. The operator ∇df is also bounded, in the sense that for some K' greater than zero,

$$|\nabla df(x)(v,v')| \leq K'||v||_x||v'||_x \quad \text{for all } v,v' \text{ in } T_xM$$

Since each U_t is an orthonormal frame, the integrand in the definition of A^f is bounded by K'. Each Z^i belongs to $\mathcal{M}_c^2$, so Lemma 7.3.3 applies to show that $A_\infty^f = \lim_{t\to\infty} A^f(t)$ exists almost surely. Hence

(1) $\lim_{t\to\infty} f(X_t)$ exists almost surely for each $f \in C^{k+2}(M)$.

By Whitney's embedding theorem, there exists a positive integer q and a C^{k+2} closed embedding $F:M \to R^q$. Let $(F^1,\ldots,F^q)$ be the local co-ordinate functions of F. By (1), $F^i(X_t)$ converges almost surely to a real-valued random variable, for each i. Since F is 1-1, X_t converges almost surely to a random variable X_∞. □

7.5 Proof in the non-compact case

The assumptions of Theorem 7.3.1 are still in force; in particular, X is an L^2 Γ-martingale.

LEMMA 7.5.1

Suppose x_o is in M, r is a positive number, and B is the closed ball in M with centre x_o and radius r. Let G denote the set $\{\omega : X(0,\omega) \in B\}$ and let u be the first exit time of X from B. Define a process $\bar{X}$ on M by

$$(1) \quad \bar{X}(t,\omega) = \begin{cases} X(t\wedge u(\omega),\omega) & \text{if } \omega \in G \\ x_o & \text{if } \omega \notin G \end{cases}$$

Then $\bar{X}$ is an L^2 Γ-martingale with values in B.

Proof.

Since B is a compact manifold with boundary, it suffices by Lemma 7.3.2,(b) to prove that for all bounded 1-forms η on M, $(\Gamma)\int_{\bar{X}} \eta$ is in M_c^2 (note that every 1-form on B can be extended to a bounded 1-form on M). Denote the last process by $\bar{L}$, and let $L = (\Gamma)\int_X \eta$. Then for G and u as above,

$$\bar{L}_t = 1_G\, L_{t\wedge u}$$

$$(2) \quad \sup_t E((\bar{L}_t)^2) \leq \sup_t E((L_t)^2) < \infty$$

where the last inequality follows from Lemma 7.3.2,(a). Moreover for any $s < t$ and any A in F_s , it follows that $A\cap G \in F_s$ (since $G \in F_o$), and the Optional Sampling Theorem (Metivier and Pellaumail [26 ,§8.6].) shows that

$$E\{1_A\,(\bar{L}_t - \bar{L}_s)\} = E\{1_{A\cap G}\,(L_{t\wedge u} - L_{s\wedge u})\} = 0$$

Combined with (2) this shows that $\bar{L} \in M_c^2$. □

Proof of Theorem 7.3.1

Fix x_o in M, and for each positive integer p , define B_p to be the closed ball in M with centre x_o and radius p . For each p, define a process $\bar{X}^p$ as in Lemma 7.5.1, with B_p in place of B. By the result of Lemma 7.5.1, $\bar{X}^p$ is an L^2 Γ-martingale on the compact manifold (with boundary) B_p. (Note that the completeness of M is used here) By the proof in the compact case (which also applies to compact manifolds with boundary) there exists a set C_p in Ω of probability zero, and a random variable $\bar{X}^p_\infty$ with values in B_p such that for ω not in C_p , $\bar{X}^p(t,\omega) \to \bar{X}^p_\infty$ as $t \to \infty$.

Define $C = \cup_p C_p$, which is of probability 0 . Define another set

$Q = \{\omega : \sup_t \rho(X(0,\omega),X(t,\omega)) = \infty\}$ where $\rho(.,.)$ is the distance function derived from the Riemannian metric. If ω is in $(Q \cup C)^c$, then there exists an integer $q = q(\omega)$ with $X(t,\omega) \in B_q$ for all t ; moreover for all $p > q$, $\bar{X}^p(\infty,\omega) = \bar{X}^q(\infty,\omega)$. Hence for ω in $(Q \cup C)^c$,

$$X(\infty,\omega) = \lim_{p\to\infty} \bar{X}^p(\infty,\omega) \quad \text{exists almost surely, and}$$

$$X(\infty,\omega) = \lim_{t\to\infty} X(t,\omega) \quad , \ \omega \in (Q \cup C)^c$$

Next, for all ω in Q we define $X(\infty,\omega) = \delta$ (the point at infinity in the one-point compactification of M). Finally the set C has probability zero, so we may define $X(\infty,\omega) = \delta$ also for $\omega \in C$. This completes the construction of X_∞ .□

Remark. Since the first announcement of Theorem 7.3.1 in late 1981, P. A. Meyer has communicated a simpler proof to the author. A converse result has been obtained by Zheng [41].

8 EXAMPLES

8.0 Motivation

We look first at some 'degenerate' martingales lying on geodesics. Then we recall the diffusions constructed in §2, and notice that they are Γ-martingales with respect to their 'own' connections Γ(with torsion). We carry out some general constructions of Γ-martingales on parallelizable manifolds, and on Lie groups G in particular; we show that if you apply the exponential map to a (continuous) local martingale in the Lie algebra of G, you obtain a Γ-martingale with respect to the trivial connection. The remaining three examples are diffusions on surfaces M embedded in R^3. We take the embedded metric on M and the induced Levi-Civita connection Γ, and select the diffusion coefficients so that the processes are Γ-martingales. The example on the torus is the most interesting. Think of the torus as a doughnut resting on a table. The 'inner half' of the torus has both positive and negative sectional curvatures: negative if you move in the horizontal plane, negative if you move in any vertical plane. We construct a Γ-martingale on the torus which is so 'balanced' that it is also a martingale in R^3.

8.1 Review of some earlier examples

(a) Continuous local martingales in a Euclidean space E, with the trivial connection; see §5.1.

(b) The image of Brownian motion under a harmonic map; see §6.2.

(c) Suppose $\gamma:(-a,b) \to M$ is a geodesic, for some $a,b > 0$. Then γ is also an affine map, so by §6.4, the image under γ of any continuous local martingale is a Γ-martingale on M. The same is true if $\gamma : \mathbb{R}^1 \to S^1 \to M$ is a closed geodesic, applied to any continuous local martingale on R^1.

(d) Let A be a strictly elliptic operator of order two on M, as in §2.1, and let Γ be the special metric connection (with torsion) constructed in §2.2:

LEMMA 8.1.1

If X is the A-diffusion process constructed in §2.2, then X is a Γ-martingale (assuming nonexplosion).

Proof. Let η be a 1-form on M. Let $W = (W^1,\ldots,W^n)$ be the E-valued Brownian motion from which X was constructed in §2.2. In the notation of that section,

$$(\Gamma)\int_X \eta = \int (\eta(X) \circ U)\,dW$$

by Theorem 1.2.2(a). Hence the left side is a local martingale. Using Theorem 5.3.1, (IV), this completes the proof. □

Remark.

We could summarize this result by saying that "every non-degenerate diffusion is a Γ-martingale with respect to its own special metric connection". This suggests that Γ-martingales may be too general a class of objects to consider as a whole. It may be better to specify a Riemannian metric on M and to study Γ-martingales with respect to the Levi-Civita connection. An example of the latter is Brownian motion, as we saw in §5.2.

8.2 Examples on parallelizable manifolds and Lie groups

Recall that a manifold M modelled on a Euclidean space E is said to be parallelizable if there is a vector bundle isomorphism (called a parallelization)

$$\begin{tikzcd} TM \arrow[rr, "\gamma"] \arrow[dr] & & E \arrow[dl] \\ & M & \end{tikzcd} \tag{1}$$

where $\underline{E}$ is the product bundle $M \times E$. Given such an isomorphism, the trivial connection Γ is the one such that $\nabla\gamma = 0$, regarding γ as an E-valued 1-form on M; to be precise, $\nabla\gamma(Y,Z) = 0$ for all vector fields Y and Z on M. For any semimartingale X on M, it follows from Corollary 1.4.2 that

$$(2) \quad (S)\int_X \gamma = (\Gamma)\int_X \gamma$$

where both sides are E-valued processes. When M is of class C^{k+2}, $k \geq 1$, and γ is of class C^{k+1}, it follows from Elworthy [*12*,VII§2] that the Stratonovich stochastic dynamical system (in the sense of Elworthy)

$$(3) \quad dX = \gamma(X)^{-1} \circ dY$$

has a unique solution (assumed non-explosive) for any continuous semimartingale Y on E. Another way of writing (3) is: X is a solution of

$$(4) \quad (S)\int_X \gamma = Y$$

Combining (2) and (4), we see that if Y is an E-valued continuous local martingale, then any solution X of (3) satisfies:

$$(\Gamma)\int_X \gamma = Y = \text{local martingale}$$

To each 1-form η on M there corresponds a map $b:M \to E^*$ such that for x in M, $\eta(x) = b(x) \circ \gamma(x)$. By a basic property of the Ito integral (see §0.7,(2))

$$(\Gamma)\int_X \eta = \int b(X)dY$$

The right side is a local martingale. From Theorem 5.3.2, (IV), X is therefore a Γ-martingale on M. To summarize:

PROPOSITION 8.2.1 (Parallelizable manifolds)

If M is a parallelizable manifold of class $C^{k+2}, k \geq 1$, if $\gamma:TM \to \underline{E}$ is a parallelization, and Γ the associated trivial connection connection for M, then for any E-valued continuous local martingale Y, the solution X (for given initial value) of

$$dX = \gamma(X)^{-1} \circ dY \quad \text{(Strat.)}$$

is a Γ-martingale on M.

<u>Special case - Lie groups</u>

Let G be a Lie group with Lie algebra g, which is identified with T_eG in the usual way. For g in G, $L_g:G \to G$ is the left translation map $h \to g.h$. A parallelization $\gamma:TG \to \underline{g}$ for G is given by:

$$\gamma(g) = T_g(L_g)^{-1}:T_gG \to T_eG = g$$

Let Γ be the associated trivial connection for G, constructed as above. Let $\exp:T_eG \to G$ be the usual exponential

map on a Lie group, such that for all A in T_eG and all real t, the curve $g(t) = \exp(At)$ on G satisfies:

$$\dot{g}(t) = T_eL_{g(t)}(A) \quad , \quad g(0) = e$$

PROPOSITION 8.2.2

Let Y be a local martingale in the Lie algebra $\mathfrak{g}$ of G, and let X_0 be an $\mathcal{F}_0$-measurable random variable on G. Then the solution X on G of

$$dX = \gamma(X)^{-1} \circ dY = T_eL_X \circ dY \qquad \text{(Strat.)} \tag{5}$$

starting at X_0, is a Γ-martingale on G, with respect to the trivial connection Γ. In the case where M is a real-valued local martingale with $M_0 = 0$, A is an $\mathcal{F}_0$-measurable random variable on $\mathfrak{g}$, and $Y_t = A\,M_t$, then the solution to (5) is

$$X_t = \exp\,(A.M_t)\,X_0 \tag{6}$$

<u>Proof.</u> The existence and uniqueness of the solution of (5) follows from the fact that T_eL_g is C^∞ in g. The first assertion follows from Proposition 8.2.1. As for the second, apply Ito's formula to the function $t \mapsto f(\exp(At).X_0)$ where $f : G \to \mathbb{R}$ is any C^∞ function. We see that X, defined by (6) satisfies

$$d(f(X_t)) = (T_{X_t}f \circ T_eL_{X_t}) \circ AdM \qquad \text{(Strat.)}$$

which shows that X solves (5). □

<u>Example.</u> Let $G = Gl(n)$, so that exp is the usual exponential of matrices. Let (w_t) be a one-dimensional Brownian motion, and let A be an $\mathcal{F}_0$-measurable random variable in $\mathfrak{g}$. Let $X = \exp(Aw_t)$. Then X is a Γ-martingale on $Gl(n)$.

8.3 A Γ-martingale on a surface of revolution whose local co-ordinate processes are martingales

Let $r:R^1 \to (0,\infty)$ be a C^2 convex function, i.e. with non-negative second derivative. Let M be a surface of revolution in R^3, defined by

$$M = \{(x,y,z) : x^2 + y^2 = r(z)^2 , z > 0\}$$

and let g be the embedded metric on M and Γ the Levi-Civita connection. Take co-ordinates (z,θ) on M, so that

$$x = r(z)\cos\theta \quad ,y = r(z)\sin\theta \quad .$$

If we denote $\frac{dr}{dz}$ by $\dot{r}$ and $\frac{d^2r}{dz^2}$ by $\ddot{r}$, g can be expressed as the matrix

$$(g_{ij}) = \begin{pmatrix} 1+\dot{r}^2 & 0 \\ 0 & r^2 \end{pmatrix} \quad x^1 = z \ , \ x^2 = \theta$$

The Christoffel symbols can be computed from the usual formula in Kobayashi and Nomizu [*20* , p.160], and one finds that

$$\Gamma^1_{11} = \dot{r}\ddot{r}/(1+\dot{r}^2) \ , \ \Gamma^1_{22} = - r\dot{r}/(1+\dot{r}^2) \ , \ \Gamma^2_{12} = \dot{r}/r = \Gamma^2_{12}$$

and all other Γ^i_{jk} are zero.

Let $(X_t) = (z_t, \theta_t)$ be a semimartingale on M. Decompose the semimartingales (z_t) and (θ_t) into their local martingale and finite variation parts thus:

$$z_t = M_t + A_t \ , \ \theta_t = N_t + C_t$$

The condition for X to be a Γ-martingale can be written

out using Theorem 5.3.1, (V) as:

$$dA + \frac{1}{2}(\Gamma^1_{11}d\langle M\rangle + \Gamma^1_{22}d\langle N\rangle) = 0$$

$$dC + \frac{1}{2}(\Gamma^2_{12}d\langle M,N\rangle + \Gamma^2_{21}d\langle N,M\rangle) = 0$$

In other words,

$$(1)\quad \begin{cases} dA + \frac{1}{2}(1+\dot{r}^2)^{-1}(\dot{r}\ddot{r}d\langle M\rangle - r\dot{r}d\langle N\rangle) = 0 \\ dC + (\dot{r}/r)d\langle M,N\rangle = 0 \end{cases}$$

A simple but interesting example may be constructed as follows. Let (W_t, W'_t) be two-dimensional Brownian motion, and define

$$(2)\quad \begin{cases} r(z) = z^2 \\ z_t = M_t = \exp(W_t - \frac{1}{2}t)\ ,\ A_t = 0 \\ \theta_t = N_t = \sqrt{2}\ W'_t\ ,\ C_t = 0 \end{cases}$$

Then

$$dz_t = z_t dW_t = \sqrt{r(z_t)}dW_t$$
$$d\theta_t = \sqrt{\ddot{r}(z_t)}dW'_t$$
$$d\langle M\rangle_t = z_t^2 dt = r(z_t)dt$$
$$d\langle N\rangle_t = 2dt = \ddot{r}(z_t)dt$$
$$d\langle M,N\rangle_t = 0$$

Hence

$$\dot{r}\ddot{r}d\langle M\rangle = r\dot{r}\ddot{r}dt = r\dot{r}d\langle N\rangle$$

Therefore equations (1) are satisfied, with $A = C = 0$. Hence $X_t = (z_t, \theta_t) = (\exp(W_t - \frac{1}{2}t), \sqrt{2}\ W'_t)$ is a Γ-martingale on M for which both the <u>local co-ordinate processes are real-valued martingales</u> (this property does <u>not</u> hold in general).

8.4 Example of an L^2 Γ-martingale

We shall modify the previous example slightly. Take $\alpha > 0$ and define $a:[0,\infty) \to [0,\infty)$ by

$$a(s) = \sqrt{(\alpha(1+s)^{-\alpha-1})}$$

Taking $r(z) = z^2$ as before, define

$$z_t = M_t = \exp\left(\int_0^t a(s)\,dW_s - \frac{1}{2}\int_0^t a(s)^2 ds\right)$$

$$\theta_t = N_t = \int_0^t a(s)\ \sqrt{2}\ dW'_s$$

One may check by the method of §8.3 that $(X_t) = (z_t, \theta_t)$ is a Γ-martingale on M. We claim that X is an L^2 Γ-martingale. for this, we must prove that for any stochastic development $\bar{Z}$ of X through Γ, the scalar quadratic variation $[\bar{Z}]$ satisfies:

(1) $\sup_t E([\bar{Z}]_t) < \infty$

By Corollary 1.5.1,

$$[\bar{Z}]_t = \int_0^t g_{ij}(X_s)\,d\langle X^i, X^j\rangle_s$$

where $X_t^1 = z_t$ and $X_t^2 = \theta_t$, and the metric tensor (g_{ij}) is

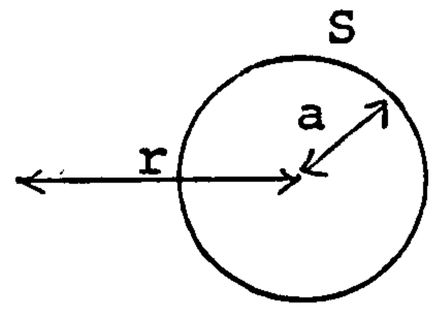
$$(g_{ij}) = \begin{pmatrix} 1+4z^2 & 0 \\ 0 & z^4 \end{pmatrix}$$

One may readily calculate that for every $\alpha > 0$,

$$\sup_t E([\bar{Z}]_t) = \exp(\tfrac{3}{2}) + 6\exp(\tfrac{15}{2})$$

which proves (1).

8.5 A Γ-martingale on the torus T^2 which is also a martingale in R^3



Let $r > a$ be positive numbers, and let S be the circle $\{(x,0,z) : (x-r)^2 + z^2 = a^2\}$ in the (x,z)-plane. Rotate S about $\underline{O}$ in the (x,y)-plane to form a torus T^2, 'resting' on the (x,y)-plane. We parametrize T^2 by angles (γ,φ). where γ is measured in the (x,y)-plane, and φ is measured 'around S'. Thus the (x,y,z)-co-ordinates of the point (γ,φ) are:

$$(1) \quad x = (r - a\cos\varphi)\cos\gamma \ , \ y = (r - a\cos\varphi)\sin\gamma \ , \ z = a\sin\varphi \ .$$

It is easy to compute that the embedded metric g on T^2 has the following matrix, where γ is co-ordinate 1 and φ is co-ordinate 2 :

$$(g_{ij}) = \begin{pmatrix} (r-a\cos\varphi)^2 & 0 \\ 0 & a^2 \end{pmatrix}$$

The Riemannian connection Γ corresponding to g can be calculated from the usual formula in Kobayashi and Nomizu [20 ,p.160] , and is found to have

$$(2) \quad \begin{cases} \Gamma^1_{12} = \Gamma^1_{21} = \sin\phi/(\beta - \cos\phi) \ , \ \Gamma^2_{11} = -\sin\phi(\beta - \cos\phi) \\ \text{all other } \Gamma^i_{jk} = 0 \ , \text{ where } \beta = \frac{r}{a} > 1 \end{cases}$$

Suppose Φ is a semimartingale on T^2 . The condition for Φ to be a Γ -martingale can be written out in terms of the local co-ordinate processes (see Theorem 5.3.1, (V)):

$$(3) \quad \begin{cases} d\gamma_t + \Gamma^1_{12}\, d\langle\gamma , \phi\rangle_t \ \text{ and } \ d\phi_t + \frac{1}{2}\Gamma^2_{11}\, d\langle\gamma,\gamma\rangle_t \\ \text{are the differentials of real local martingales.} \end{cases}$$

Let us decompose the co-ordinate processes (γ_t) and (ϕ_t) into their martingale and bounded variation parts. In differential notation,

$$(4)\quad \begin{cases} d\gamma_t = dM_t + dA_t \\ d\phi_t = dN_t + dC_t \end{cases}$$

Assume Φ is a Γ - martingale. By equations (3) ,

$$dA_t = -\Gamma^1_{12}\, d\langle M,N\rangle_t \ , \ dC_t = -\frac{1}{2}\Gamma^2_{11}\, d\langle M\rangle_t$$

Using equations (2), we can rewrite (4) as:

$$(5)\quad \begin{cases} d\gamma_t = dM_t - [\sin\phi_t/(\beta - \cos\phi_t)]d\langle M,N\rangle_t \\ d\phi_t = dN_t + \frac{1}{2}\sin\phi_t(\beta - \cos\phi_t)d\langle M\rangle_t \end{cases}$$

Naturally Φ is also a process in $\mathbb{R}^3$, and the condition for it to be a martingale therein is that the Euclidean co-ordinate processes (x_t, y_t, z_t) are martingales.

LEMMA 8.5.1

If the co-ordinate processes (γ_t, ϕ_t) in (5) have martingale parts (M_t, N_t) satisfying

$$(6)\qquad d\langle N\rangle_t = \cos\phi_t(\beta - \cos\phi_t)d\langle M\rangle_t$$

then the Euclidean co-ordinate processes (x_t, y_t, z_t) are martingales.

<u>Proof.</u> First we apply Ito's formula to the z co-ordinate process $z_t = a\sin\phi_t$. We obtain

$$\begin{aligned} dz_t &= a\cos\phi_t\, d\phi_t - \frac{1}{2}a\sin\phi_t\, d\langle\phi,\phi\rangle_t \\ &= a\cos\phi_t\, dN_t + a\cos\phi_t\, \frac{1}{2}\sin\phi_t(\beta - \cos\phi_t)\, d\langle M\rangle_t \\ &\quad - \frac{1}{2}a\sin\phi_t\, d\langle N\rangle_t \\ &= a\cos\phi_t\, dN_t \end{aligned}$$

by (6), verifying that (z_t) is a martingale.

It remains to examine (x_t) and (y_t) ; actually it suffices to examine (x_t) , since $y(\gamma,\phi) = x(\gamma - \frac{1}{2}\pi,\phi)$. Applying Ito's formula to the expression

$$x_t = a(\beta - \cos\phi_t)\cos\phi_t$$

we obtain (omitting the suffix t)

$$\begin{aligned} a^{-1}dx = &- (\beta - \cos\phi)\sin\gamma\, d\gamma + \sin\phi\, \cos\gamma\, d\phi \\ &- \frac{1}{2}(\beta - \cos\phi)\cos\gamma\, d\langle\gamma,\gamma) - \sin\phi\, \sin\gamma\, d\langle\gamma,\phi\rangle \\ &+ \frac{1}{2}\cos\phi\, \cos\gamma\, d\langle\phi,\phi\rangle \end{aligned}$$

Using (5) and (6), this reduces to

$$a^{-1}dx = - (\beta - \cos\phi)\sin\gamma\, dM + \sin\phi\, \cos\gamma\, dN$$

which verifies that (x_t) is a martingale. □

We are now ready to construct a process Φ which is simultaneously a Γ-martingale on T^2 and a martingale in $\mathbb{R}^3$.

PROPOSITION 8.5.2

Define $h:S^1 \to \mathbb{R}$ by

$$h(\phi) = \max\,(\cos\phi(\beta - \cos\phi)\ ,\ 0)$$

Let (W_t,W'_t) be 2-dimensional Brownian motion. For a given initial value $(\gamma_\bullet,\phi_\bullet)$, the system of stochastic differential equations

$$(7)\quad \begin{cases} d\gamma = dW \\ d\phi = \sqrt{h(\phi)}\ dW' + \frac{1}{2}\sin\phi(\beta - \cos\phi)\ dt \end{cases}$$

defines a unique process (Φ_t) on T^2 . Let $E = \{(\gamma,\phi) \in T^2 : -\frac{\pi}{2} < \phi < \frac{\pi}{2}\}$ = "the inner half of the torus". Suppose $(\gamma_\bullet,\phi_\bullet) \in E$, and let τ be the first exit time of Φ from E . Then (Φ_t) stopped at τ is simultaneously a Γ-martingale on T^2 and a martingale in $\mathbb{R}^3$.

Proof. A unique solution to (7) exists and goes on for all time, because the coefficients are bounded and Lipschitz. The equations (7) conform to the requirements (5) for Φ to be a Γ-martingale, with

$$M_t = W_t \ , \ N_t = \int_0^t \sqrt{h}(\phi_s)dW'_s$$

Up to time τ , $h(\phi_s) = \cos\phi_s(\beta - \cos\phi_s)$ and so (6) holds, showing that Φ stopped at τ is a martingale in $\mathbb{R}^3$. □

Remark. Such an example could also be constructed using a 1-dimensional driving Brownian motion (W_t) , namely:

$$(7') \quad \begin{cases} d\gamma = dW - [\sin\phi/(\beta - \cos\phi)]\sqrt{h}(\phi)\ dt \\ \\ d\phi = \sqrt{h}(\phi)\ dW + \frac{1}{2}\sin\phi(\beta - \cos\phi)\ dt \end{cases}$$

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