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RECOVERING A TIME-HOMOGENEOUS STOCK PRICE PROCESS FROM PERPETUAL OPTION PRICES

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It is well known how to determine the price of perpetual American options if the underlying stock price is a time-homogeneous diffusion. In the present paper we consider the inverse problem, that is, given prices of perpetual American options for different strikes, we show how to construct a time-homogeneous stock price model which reproduces the given option prices.

1. Introduction. In the classical Black–Scholes model, there is a one-to-one correspondence between the price of an option and the volatility of the underlying stock. If the volatility σ is assumed to be given (e.g., by estimation from historical data), then the arbitrage free option price can be calculated using the Black–Scholes formula. Conversely, if an option price is given, then the implied volatility can be obtained as the unique σ that would produce this option price if inserted in the Black–Scholes formula. It has been well documented that if the implied volatility is inferred from real market data for option prices with the same maturity date but with different strike prices, then, typically, a nonconstant implied volatility is obtained. Since the implied volatility often resembles a smile if plotted against the strike price, this phenomenon is referred to as the *smile effect*. The smile effect is one indication that the Black–Scholes assumption of normally distributed log-returns is too simplistic.

A wealth of different stock price models have been proposed in order to overcome the shortcomings of the standard Black–Scholes model, of which the most popular are jump models and stochastic volatility models. Given a model, option prices can be determined as risk-neutral expectations. However, models are typically governed by a small number of parameters, and only in exceptional circumstances can they be calibrated to perfectly fit the full range of options data.

Instead, there is a growing literature which tries to reverse the procedure, using option prices to make inferences about the underlying price process. At one extreme, models exist which take a price surface as the initial value of a Markov process on a space of functions. In this way the Heath–Jarrow–Morton [5] interest rate models can be made to perfectly fit an initial term structure. Such ideas inspired Dupire [4] to introduce the local volatility model which calibrates perfectly

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to an initial volatility surface. For a local volatility model, Dupire derived the PDE

$$C_T(T,K) + rKC_K(T,K) = \frac{1}{2}\sigma^2(T,K)K^2C_{KK}(T,K),$$

where $C(T,K)$ is the European call option price, T is time to maturity and K is the strike price. Solving for the (unknown) local volatility $\sigma(T,K)$ gives a formula for the time-inhomogeneous local volatility in terms of derivatives of the observed European call option prices.

The local volatility model gives the unique martingale diffusion which is consistent with observed call prices (alternative, nondiffusion models also exist; see, e.g., Madan and Yor [10]). The recent literature (e.g., Schweizer and Wissel [14]) has included attempts to extend the theory to allow for a stochastic local volatility surface. However, it relies on the knowledge of a double continuum of option prices, which are smooth. In contrast, Hobson [6] constructs models which are consistent with a continuum of strikes, but at a single maturity, in which case there is no uniqueness.

In the current article we present a method to recover a time-homogeneous local volatility function from perpetual American option prices. More precisely, we assume that perpetual put option prices are observed for all different values of the strike price, and we derive a time-homogeneous stock price process for which theoretical option prices coincide with the observed ones.

No-arbitrage enforces some fundamental convexity and monotonicity conditions on the put prices, and if these fail, then no model can support the observed prices. If the observed put prices are smooth, then we can use the theory of differential equations to determine a diffusion process for which the theoretical perpetual put prices agree with the observed prices, and our key contribution in this case is to give an expression for the diffusion coefficient of the underlying model in terms of the put prices. It turns out that this expression uniquely determines the volatility coefficient at price levels below the current stock price, but there is some freedom in the choice of the volatility function above the current stock price level. The key idea is to construct a dual function to the perpetual put price, and then the diffusion coefficient can be easily found by taking derivatives of this dual.

The second contribution of this paper is to give time-homogeneous models which are consistent with a given set of perpetual put prices, even when those put prices are not twice differentiable or not strictly convex in the continuation region where it is not optimal to exercise immediately or not strictly convex in the continuation region. Again, the key is the dual function, coupled with a change of scale and a time change. We give a construction of a time-homogeneous process consistent with put prices, which we assume to satisfy the no-arbitrage conditions, but which otherwise has no regularity properties.

One should perhaps note that in reality, put prices are only given in the market for a discrete set of strike prices. Therefore, as a first step one needs to interpolate between the strikes. If a stock price is modeled as the solution to a stochastic differential equation with a continuous volatility function, then the perpetual put price exhibits certain regularity properties with respect to the strike price. Therefore, if one aims to recover a continuous volatility, then one has to use an interpolation method that produces option prices exhibiting this regularity. On the other hand, if a linear spline method is used, then a continuous volatility cannot be recovered. This is one of the motivations for searching for price processes which are consistent with a general perpetual put price function (which is convex, but may be neither strictly convex nor smooth).

While preparing this manuscript we came across a preprint by Alfonsi and Jourdain, now published as [1]. The aim of [1], as in this article, is to construct a time-homogeneous process which is consistent with observed put prices. However, the method is different and considerably less direct. Alfonsi and Jourdain [1] construct a parallel model such that the put price function in the original model (expressed as a function of strike) becomes a call price function expressed as a function of the initial value of the stock. They then solve the perpetual pricing problem for this parallel model and, subject to solving a differential equation for the optimal exercise boundaries in this model, give an analytic formula for the volatility coefficient. In contrast, the approach in this paper is much simpler and, unlike the method of Alfonsi and Jourdain, extends to the irregular case.

2. The forward problem. Assume that the stock price process X is modeled under the pricing measure as the solution to the stochastic differential equation

$$dX_t = rX_t\,dt + \sigma(X_t)X_t\,dW_t, \qquad X_0 = x_0.$$

Here, the interest rate r is a positive constant, the level-dependent volatility $\sigma : (0,\infty) \to (0,\infty)$ is a given continuous function and W is a standard Brownian motion. We assume that the stock pays no dividends, and we let zero be an absorbing barrier for X. If the current stock price is x_0, then the price of a perpetual put option with strike price $K > 0$ is

$$\hat{P}(K) = \sup_{\tau} \mathbb{E}^{x_0}[e^{-r\tau}(K - X_\tau)^+], \tag{1}$$

where the supremum is taken over random times τ that are stopping times with respect to the filtration generated by W. From the boundedness, monotonicity and convexity of the payoff, we have the following.

PROPOSITION 2.1. *The function* $\hat{P} : (0,\infty) \to [0,\infty)$ *satisfies*:

(i) $(K - x_0)^+ \le \hat{P}(K) \le K$ *for all* K;
(ii) $\hat{P}$ *is nondecreasing and convex.*

EXAMPLE. If σ is constant, that is, if X is a geometric Brownian motion, then

$$\hat{P}(K) = \begin{cases} \dfrac{K}{\beta+1}\big(\beta K / x_0(\beta+1)\big)^{\beta}, & \text{if } K < \hat{K}, \\ K - x_0, & \text{if } K \ge \hat{K}, \end{cases} \tag{2}$$

where $\beta = 2r/\sigma^2$ and $\hat{K} = x_0(\beta+1)/\beta$.

Intimately connected with the solution of the optimal stopping problem (1) is the ordinary differential equation

$$\frac{1}{2}\sigma(x)^2x^2u_{xx}+rxu_x-ru=0 \tag{3}$$

for $x>0$. This equation has two linearly independent positive solutions which are uniquely determined (up to multiplication with positive constants) if one requires one of them to be increasing and the other decreasing; see, for example, Borodin and Salminen [3], page 18. We denote the increasing solution by $\hat{\psi}$ and the decreasing one by $\hat{\varphi}$. In the current setting, $\hat{\psi}$ and $\hat{\varphi}$ are given by

$$\hat{\psi}(x)=Cx$$

and

$$\hat{\varphi}(x)=Dx\int_x^\infty \frac{1}{y^2}\exp\biggl\{-\int_{x_0}^y \frac{2r}{z\sigma(z)^2}\,dz\biggr\}\,dy \tag{4}$$

for some arbitrary positive constants C and D. For simplicity, and without loss of generality, we choose

$$D=\biggl(x_0\int_{x_0}^\infty \frac{1}{y^2}\exp\biggl\{-\int_{x_0}^y \frac{2r}{z\sigma(z)^2}\,dz\biggr\}\,dy\biggr)^{-1}$$

so that $\hat{\varphi}(x_0)=1$.

LEMMA 2.2. *The function $\hat{\varphi}$ is strictly decreasing and strictly convex.*

PROOF. Straightforward differentiation yields

$$\hat{\varphi}'(x)=D\int_x^\infty \frac{1}{y^2}\biggl(\exp\biggl\{-\int_{x_0}^y \frac{2r}{z\sigma(z)^2}\,dz\biggr\}-\exp\biggl\{-\int_{x_0}^x \frac{2r}{z\sigma(z)^2}\,dz\biggr\}\biggr)\,dy, \tag{5}$$

so $\hat{\varphi}'(x)<0$. Similarly,

$$\hat{\varphi}''(x)=\frac{2Dr}{x^2\sigma(x)^2}\exp\biggl\{-\int_{x_0}^x \frac{2r}{z\sigma(z)^2}\,dz\biggr\}>0,$$

so $\hat{\varphi}$ is strictly convex. □

It is well known that with $H_z=\inf\{t\geq 0: X_t=z\}$, we have

$$\mathbb{E}^x[e^{-rH_z}]=\begin{cases}\hat{\varphi}(x)/\hat{\varphi}(z), & \text{if } z<x,\\ \hat{\psi}(x)/\hat{\psi}(z), & \text{if } z>x,\end{cases} \tag{6}$$

where the superindex x denotes that the expected value is calculated using $X_0=x$. [This result is easy to check by considering $e^{-rt}\hat{\varphi}(X_t)$ and $e^{-rt}\hat{\psi}(X_t)$, which, since they involve solutions to (3), are local martingales.] Given the assumed time-homogeneity of the process X, it is natural to consider stopping times in (1) that are hitting times. Define

$$
\begin{aligned}
\tilde{P}(K) &:= \sup_{z:z\le x_0\wedge K} \mathbb{E}^{x_0}[e^{-rH_z}(K-X_{H_z})^+] \\
&= \sup_{z:z\le x_0\wedge K} (K-z)\mathbb{E}^{x_0}[e^{-rH_z}] \\
&= \sup_{z:z\le x_0\wedge K} \frac{K-z}{\hat{\varphi}(z)},
\end{aligned} \tag{7}
$$

where the last equality follows from (6). Clearly $\hat{P}(K)\ge\tilde{P}(K)$, and, of course, as we show below, there is equality. Since the function $\hat{\varphi}$ is strictly convex, for each fixed K there exists a unique $z=z(K)\le x_0$ for which the supremum in (7) is attained, that is,

$$
\tilde{P}(K)=\frac{K-z(K)}{\hat{\varphi}(z(K))}. \tag{8}
$$

Geometrically, $z=z(K)$ is the unique value (less than or equal to x_0) which makes the negative slope of the line through $(K,0)$ and $(z,\hat{\varphi}(z))$ as large as possible; see Figure 1.

Define

$$
\hat{K}:=x_0-1/\hat{\varphi}'(x_0).
$$

From the strict convexity of $\hat{\varphi}$ it follows that if $K\ge\hat{K}$, then

$$
\tilde{P}(K)=\sup_{z:z\le x_0}\frac{K-z}{\hat{\varphi}(z)}=\frac{K-x_0}{\hat{\varphi}(x_0)}=K-x_0,
$$

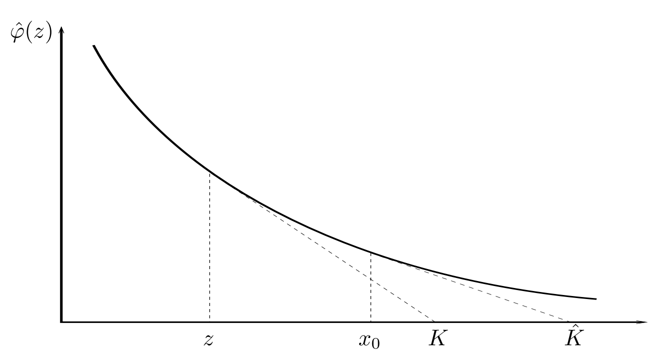

FIG. 1. *For a given $K\le\hat{K}$ the price $\hat{P}(K)$ is minus the reciprocal of the slope of the tangent line to $\hat{\varphi}$ which passes through the point $(K,0)$.*

and if $K \le \hat{K}$, then

$$\tilde{P}(K) = \sup_{z:z \le x_0} \frac{K - z}{\hat{\varphi}(z)} = \sup_{z} \frac{K - z}{\hat{\varphi}(z)}. \tag{9}$$

Moreover, for $K < \hat{K}$ we have $\tilde{P}(K) > (K - x_0)^+$.

LEMMA 2.3. *The functions $\hat{P}$ and $\tilde{P}$ coincide, that is,*

$$\hat{P}(K) = \sup_{z:z \le x_0} \frac{K - z}{\hat{\varphi}(z)}. \tag{10}$$

PROOF. As noted above, we clearly have $\hat{P} \ge \tilde{P}$ since the supremum over all stopping times is at least as large as the supremum over first hitting times.

For the reverse implication, suppose first that $K \le \hat{K}$. In that case $\hat{\varphi}(z) \ge (K - z)^+/\tilde{P}(K)$, by (9). Further, $e^{-rt}\hat{\varphi}(X_t)$ is a nonnegative local martingale and hence a supermartingale. Thus, for any stopping time τ we have

$$1 \ge \mathbb{E}^{x_0}[e^{-r\tau}\hat{\varphi}(X_\tau)] \ge \mathbb{E}^{x_0}[e^{-r\tau}(K - X_\tau)^+/\tilde{P}(K)].$$

Hence, $\tilde{P}(K) \ge \sup_\tau \mathbb{E}^{x_0}[e^{-r\tau}(K - X_\tau)^+] = \hat{P}(K)$.

Finally, let $K > \hat{K}$. It follows from the first part that $\hat{P}(\hat{K}) = \hat{K} - x_0$, so Proposition 2.1 implies that $\hat{P}(K) = K - x_0 = \tilde{P}(K)$, which completes the proof. $\square$

EXAMPLE. If σ is constant, that is, if X is a geometric Brownian motion, then

$$\hat{\varphi}(x) = \left(\frac{x_0}{x}\right)^\beta,$$

where $\beta = 2r/\sigma^2$. Consequently, the put option price is given by

$$\hat{P}(K) = x_0^{-\beta} \sup_{z:z \le x_0} (K - z)z^\beta.$$

Straightforward differentiation shows that the supremum is attained for

$$z = z^* := \frac{\beta K}{\beta + 1}$$

if $z^* < x_0$, and for $z = x_0$ if $z^* \ge x_0$. Consequently, $\hat{P}(K)$ is given by (2).

Under our current assumptions it is not possible to rule out the case where the diffusion X hits zero in finite time, although we then insist that zero is absorbing. Note that X hits zero in finite time if and only if $\hat{\varphi}(0) < \infty$, in which case we set $\underline{K} = -\hat{\varphi}(0)/\hat{\varphi}'(0)$. When $\hat{\varphi}'(0)$ is finite we have $\underline{K} > 0$ and for $K < \underline{K}$, $z(K) = 0$ and $\hat{P}(K) = K/\hat{\varphi}(0)$. By the strict concavity of $\hat{\varphi}$, $\lim_{K \downarrow \underline{K}} z(K) = 0$.

PROPOSITION 2.4. *In addition to the properties described in Proposition 2.1, the following statements about the function $\hat{P}:[0,\infty)\to[0,\infty)$ hold*:

(i) *$\hat{P}$ satisfies $\hat{P}(K) > (K-x_0)^+$ for all $K \in (0,\hat{K})$ and $\hat{P}(K) = K - x_0$ for all $K \geq \hat{K}$*;
(ii) *$\hat{P}$ is continuously differentiable on $(0,\infty)$ and twice continuously differentiable on $(0,\infty)\setminus\{\underline{K},\hat{K}\}$*;
(iii) *$\hat{P}$ is strictly increasing on $(0,\infty)$ with a strictly positive second derivative on $(\underline{K},\hat{K})$.*

PROOF. Statement (i) follows from Lemma 2.3 and the fact that (i) is true for $\tilde{P}$.

Next, consider $\underline{K} < K < \hat{K}$. By (9) we have

$$\hat{P}(K) = \sup_z \frac{K-z}{\hat{\varphi}(z)} = \frac{K-z(K)}{\hat{\varphi}(z(K))}$$

for some $z(K) \in (0,x_0)$. Since $z = z(K)$ maximizes the quotient $(K-z)/\hat{\varphi}(z)$, we have

$$\big(K - z(K)\big)\hat{\varphi}'(z(K)) + \hat{\varphi}(z(K)) = 0. \tag{11}$$

It follows from (11) and the implicit function theorem that $z(K)$ is continuously differentiable for $\underline{K} < K < \hat{K}$. Therefore, differentiating (8) gives

$$\begin{aligned}\hat{P}'(K) &= \frac{(1-z'(K))\hat{\varphi}(z(K)) - (K-z(K))z'(K)\hat{\varphi}'(z(K))}{(\hat{\varphi}(z(K)))^2} \\ &= \frac{1}{\hat{\varphi}(z(K))},\end{aligned} \tag{12}$$

where the second equality follows from (11). Equation (12) shows that $\hat{P}'(\hat{K}-) = 1/\hat{\varphi}(x_0) = 1$, so $\hat{P}$ is C^1 at $\hat{K}$, and, again, when $\underline{K} > 0$ we have $\hat{P}'(\underline{K}+) = 1/\hat{\varphi}(0+)$, so $\hat{P}$ is C^1 also at $\underline{K}$. Moreover, since $\hat{\varphi}(z)$ is C^1 and $z(K)$ is C^1 away from $\hat{K}$, it follows that $\hat{P}(K)$ is C^2 on $(0,\infty)\setminus\{\underline{K},\hat{K}\}$. In fact, for $\underline{K} < K < \hat{K}$ we have

$$\begin{aligned}\hat{P}''(K) &= \frac{-z'(K)\hat{\varphi}'(z(K))}{(\hat{\varphi}(z(K)))^2} \\ &= \frac{(\hat{\varphi}'(z(K)))^2}{(K-z(K))(\hat{\varphi}(z(K)))^2\hat{\varphi}''(z(K))} > 0,\end{aligned}$$

where the second equality follows by differentiating (11). Thus, $\hat{P}$ has a strictly positive second derivative on $(\underline{K},\hat{K})$, which completes the proof. $\square$

REMARK. Note that $\hat{P}'(0+) \geq 0$ with equality if and only if $\hat{\varphi}(0+) = \infty$.

We end this section by showing that $\hat{\varphi}$ can be recovered directly from the put option prices $\hat{P}(K)$, at least on the domain $(0, x_0]$. To do this, we define the function $\varphi : (0, x_0] \to (0, \infty)$ by

$$\varphi(z) = \sup_{K:K\geq z} \frac{K - z}{\hat{P}(K)}, \tag{13}$$

where $\hat{P}$ is given by (10).

LEMMA 2.5.

(a) *Suppose* $f : (0, z_0] \to [1, \infty]$ *is a nonnegative, decreasing convex function on* $(0, z_0]$ *with* $f(z_0) = 1$ *and* $f'(z_0) < 0$. *Define* $g : (0, \infty) \to [0, \infty)$ *by*

$$g(k) = \sup_{z:z\leq z_0} \frac{k - z}{f(z)}. \tag{14}$$

(i) $g(k)$ *is then a nonnegative, nondecreasing convex function with* $(k - z_0)^+ \leq g(k) \leq k$ *and* $g(k) = k - z_0$ *for* $k \geq k^* = z_0 - 1/f'(z_0)$.

(ii) f *and* g *are self-dual in the sense that if, for* $z \leq z_0$, *we define*

$$F(z) = \sup_{k:k\geq z} \frac{k - z}{g(k)},$$

then $F \equiv f$ *on* $(0, z_0]$.

(b) *Similarly, assume that* $g : (0, \infty) \to [0, \infty)$ *is a nonnegative, nondecreasing convex function with* $(k - z_0)^+ \leq g(k) \leq k$ *for all* k. *Also, assume that there exists a point* $k^* > z_0$ *such that* $g(k) = k - z_0$ *for* $k \geq k^*$ *and* $g(k) > k - z_0$ *for* $0 \leq k < k^*$. *Define*

$$f(z) = \sup_{k:k\geq z} \frac{k - z}{g(k)}$$

for $z \leq z_0$.

(i) $f : (0, z_0] \to [0, \infty]$ *is then a decreasing convex function with* $f(z_0) = 1$ *and* $f'(z_0) < 0$.

(ii) g *and* f *are self-dual in the sense that if we define*

$$G(k) = \sup_{z:z\leq z_0} \frac{k - z}{f(z)},$$

then $G = g$ *on* $(0, \infty)$.

PROOF. See Appendix A.1. $\square$

COROLLARY 2.6. *The function* φ *coincides with the decreasing fundamental solution* $\hat{\varphi}$ *on* $(0, x_0]$.

3. The inverse problem: The regular case. We now consider the inverse problem. Let $P(K)$ be observed perpetual put prices for all nonnegative values of the strike K. The idea is that since $\hat{\varphi}$ satisfies the Black–Scholes equation (3), Corollary 2.6 provides a way to recover the volatility $\sigma(x)$ for $x \in (0, x_0]$ from perpetual put prices. In this section we provide the details for the case where the observed put prices are sufficiently regular. We assume that the observed put option price $P : [0, \infty) \to [0, \infty)$ satisfies the following conditions (cf. Propositions 2.1 and 2.4 above).

HYPOTHESIS 3.1.

(i) $(K - x_0)^+ \le P(K) \le K$ *for all* K.
(ii) *There exists a strike price* K^* *such that* $P(K) > (K - x_0)^+$ *for all* $K < K^*$ *and* $P(K) = K - x_0$ *for all* $K \ge K^*$.
(iii) P *is continuously differentiable on* $(0, \infty)$ *and twice continuously differentiable on* $(0, \infty) \setminus \{K^*\}$.
(iv) P *is strictly increasing on* $(0, \infty)$ *with a strictly positive second derivative on* $(0, K^*)$. *Moreover,* $P''(K^*-) := \lim_{K \uparrow K^*} P''(K)$ *exists and satisfies* $P''(K^*-) \in (0, \infty)$.

Motivated by Corollary 2.6, we define the function $\varphi : (0, x_0] \to (0, \infty)$ by

$$\varphi(z) = \sup_{K : K \ge z} \frac{K - z}{P(K)}. \tag{15}$$

PROPOSITION 3.2. *The function* P *can be recovered from* φ *by*

$$P(K) = \sup_{z : z \le z_0} \frac{K - z}{\varphi(z)}.$$

PROOF. This is a consequence of part (iii) of Lemma 2.5. $\square$

PROPOSITION 3.3. *The function* $\varphi : (0, x_0] \to (0, \infty)$ *is twice continuously differentiable with a positive second derivative, and it satisfies* $\varphi(x_0) = 1$ *and* $\varphi'(x_0) = -1/(K^* - x_0)$.

PROOF. For each $z \le x_0$ there exists a unique $K = K(z) \in (z, K^*]$ for which the supremum in (15) is attained. Geometrically, K is the unique value which minimizes the slope of the line through $(z, 0)$ and $(K, P(K))$ (cf. Figure 2). Clearly, $K = K(z)$ satisfies the relation

$$(K - z)P'(K) = P(K). \tag{16}$$

Reasoning as in the proof of Proposition 2.4, one finds that $K(z)$ is continuously differentiable on $(0, x_0]$ with

$$\varphi'(z) = \frac{-1}{P(K(z))} \tag{17}$$

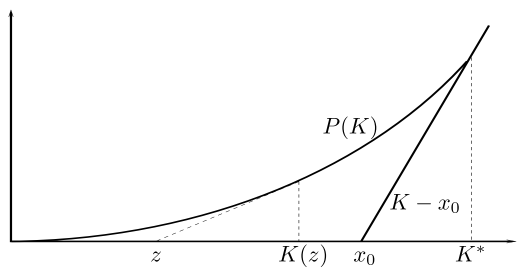

FIG. 2. *For a given $z \le x_0$ the value $\varphi(z)$ is given by the slope of the tangent line to P which passes through the point $(z, 0)$.*

and $\varphi'(x_0) = -1/(K^* - x_0)$. Differentiating (17) with respect to z gives

$$\varphi''(z) = \frac{K'(z)P'(K(z))}{P^2(K(z))} = \frac{(P'(K(z)))^2}{(K(z) - z)P^2(K(z))P''(K(z))}, \tag{18}$$

where the second equality follows by differentiating (16). It follows that $\varphi''(z)$ is continuous and positive, which completes the proof. $\square$

Next, we extend the function φ to the whole positive real axis so that φ is convex, strictly positive, twice continuously differentiable with a strictly positive second derivative and satisfies $\varphi(\infty) = 0$. We also define $\sigma^2(x)$ so that φ is a solution to the corresponding Black–Scholes equation, that is,

$$\sigma^2(x) = 2r\frac{\varphi(x) - x\varphi'(x)}{x^2\varphi''(x)}. \tag{19}$$

Now, given this volatility function $\sigma(\cdot)$, we are in the situation of Section 2 and can thus define $\hat{\varphi}$ to be the decreasing fundamental solution to the corresponding Black–Scholes equation scaled so that $\hat{\varphi}(x_0) = 1$. Moreover, let $\hat{P}(K)$ be the corresponding perpetual put option price as given by (10).

THEOREM 3.4. *Assume that Hypothesis 3.1 holds. The functions $\hat{P}$ and P then coincide. Consequently, the volatility $\sigma(x)$ defined by (19) solves the inverse problem.*

PROOF. Since the decreasing fundamental solution is unique up to a multiplicative constant and $\varphi(x_0) = \hat{\varphi}(x_0)$, we have $\varphi \equiv \hat{\varphi}$. Proposition 3.2 then yields

$$\hat{P}(K) = \sup_{z:z\le x_0} \frac{K - z}{\hat{\varphi}(z)} = \sup_{z:z\le x_0} \frac{K - z}{\varphi(z)} = P(K),$$

which completes the proof. $\square$

REMARK. The inverse problem does not have a unique solution. Indeed, there is plenty of freedom when extending φ (and thereby also σ) for $x > x_0$. Note, however, that the volatility σ is completely determined by the given option prices for values below x_0.

We next show how to calculate the volatility that solves the inverse problem directly from the observed option prices $P(K)$. To do that, note that for each fixed $z \le x_0$, the supremum in (15) is attained at some $K = K(z)$ for which

$$\varphi(z) = \frac{K - z}{P(K)}, \tag{20}$$

$$\varphi'(z) = \frac{-1}{P(K)} \tag{21}$$

and

$$\varphi''(z) = \frac{(P'(K))^2}{(K - z)P^2(K)P''(K)} \tag{22}$$

[cf. (17) and (18)]. Since φ satisfies the Black–Scholes equation, we get

$$\sigma(z)^2 z^2 = 2r\frac{\varphi(z) - z\varphi'(z)}{\varphi''} = \frac{2rKP^2(K)P''(K)}{(P'(K))^3}. \tag{23}$$

Consequently, to solve the inverse problem we first determine z by

$$z = K - \frac{P(K)}{P'(K)},$$

and then, for this z, we determines $\sigma(z)$ from (23).

4. The inverse problem: The irregular case. Again, suppose we are given perpetual put prices $P(K)$ and a constant interest rate $r > 0$. Our goal is to construct a time-homogeneous process which is consistent with the given prices. Unlike in the regular case discussed in Section 3, we now impose no regularity assumptions on the function P beyond the necessary conditions stated in Proposition 2.1 and condition (i) of Proposition 2.4. For a discussion of the necessity of condition (i) of Proposition 2.4, see Section 9.1.

HYPOTHESIS 4.1.

(i) *For all K we have $(K - x_0)^+ \le P(K) \le K$.*
(ii) *P is nondecreasing and convex.*
(iii) *There exists $K^* \in (x_0, \infty)$ such that $P(K) = K - x_0$ for $K \ge K^*$ and $P(K) > K - x_0$ for $K \in [x_0, K^*)$.*

THEOREM 4.2. *Given $P(K)$ satisfying Hypothesis 4.1 and given $r > 0$, there exists a right-continuous (for $t > 0$), time-homogeneous Markov process X_t with $X_0 = x_0$ such that*

$$\sup_{\tau} \mathbb{E}^{x_0}[e^{-r\tau}(K - X_\tau)^+] = P(K) \qquad \forall K > 0$$

and such that $(e^{-rt}X_t)_{t\geq 0}$ is a local martingale.

REMARK. Although we wish to work in the standard framework with right-continuous processes, in some circumstances we have to allow for an immediate jump. We do this by making the process right-continuous, except possibly at $t = 0$. At $t = 0$ we allow a jump subject to the martingale condition $\mathbb{E}[X_0] = x_0$.

Note that condition (iii) of Hypothesis 4.1 excludes the completely degenerate case where $P(x_0) = 0$. If $P(x_0) = 0$, then, necessarily, to preclude arbitrage, $P(K) = (K - x_0)^+$ and $X_t = x_0 e^{rt}$ is consistent with the prices $P(K)$. In this case $\tau \equiv 0$ is an optimal stopping time for every K.

Given $P(K)$ satisfying Hypothesis 4.1, we define φ by

$$\varphi(x) = \sup_{K:K\geq x} \frac{K - x}{P(K)} \tag{24}$$

for $x \in (0, x_0]$. For some values of x, the supremum in (24) may be infinite since P may vanish on a nonempty interval $(0, \underline{K}]$, where $\underline{K} = \sup\{K : P(K) = 0\}$. By Lemma 2.5, $\varphi : (0, x_0] \to [1, \infty]$ is a convex, decreasing, nonnegative function with $\varphi(x_0) = 1$. Further,

$$\varphi(x_0) - \varphi(x_0 - \varepsilon) \leq 1 - \frac{K^* - x_0 + \varepsilon}{K^* - x_0} = \frac{-\varepsilon}{K^* - x_0}, \tag{25}$$

so $\varphi'(x_0) \leq -1/(K^* - x_0) < 0$. We define

$$\underline{x} = \inf\{x > 0 : \varphi(x) < \infty\},$$

and in the case where $\underline{x} > 0$ we see that $\varphi(x) = \infty$ for $x < \underline{x}$. In fact, $\underline{x} > 0$ if and only if $\underline{K} > 0$, and it is then easy to see that these two quantities are equal.

We extend the definition of φ to (x_0, ∞) in any way which is consistent with the convexity, monotonicity and nonnegativity properties and such that $\lim_{x\to\infty} \varphi(x) = 0$. It is convenient to use $\varphi(x) = (x/x_0)^{\varphi'(x_0-)x_0}$, for $\varphi'(x)$ is then continuous at x_0, and φ is twice continuously differentiable and positive on (x_0, ∞).

Given φ, define $s : (\underline{x}, \infty) \mapsto (-\infty, \infty)$ via

$$s(x) = 2\int_{x_0}^{x} \varphi(y)\, dy + x_0 - x\varphi(x)$$

so that if φ is differentiable we have $s'(x) = \varphi(x) - x\varphi'(x)$. Then, s is a concave, increasing function, which is continuous on $(\underline{x}, \infty)$. (It will turn out that s is the scale function, which explains the choice of label.) The function s has a well-defined inverse $g : (s(\underline{x}), s(\infty)) \to (\underline{x}, \infty)$, and if $s(\underline{x}) > -\infty$, then we extend the definition of g so that $g(y) = \underline{x}$ for $y \le s(\underline{x})$. Note that $g : (-\infty, s(\infty)) \to [\underline{x}, \infty)$ is a convex, nondecreasing function with $g(0) = x_0$. Also, define $f(y) = \varphi(g(y))$. Then, f is decreasing and convex with $f(0) = \varphi(x_0) = 1$.

EXAMPLE. For geometric Brownian motion we have $s(0) = -x_0(1+\beta)/(1-\beta)$ and $s(\infty) = \infty$ if $\beta < 1$, and $s(0) = -\infty$ and $s(\infty) = x_0(1+\beta)/(\beta - 1)$ if $\beta > 1$. Moreover,

$$s(x) = x_0^{\beta}(x^{1-\beta} - x_0^{1-\beta})(1+\beta)/(1-\beta),$$
$$g(y) = x_0\left[1 + \frac{y(1-\beta)}{x_0(1+\beta)}\right]^{1/(1-\beta)}$$

and

$$f(y) = \left[1 + \frac{y(1-\beta)}{x_0(1+\beta)}\right]^{-\beta/(1-\beta)}$$

for $\beta \ne 1$. If $\beta = 1$, then the corresponding formulae are $s(0) = -\infty$, $s(\infty) = \infty$, $s(x) = 2x_0 \ln(x/x_0)$, $g(y) = x_0 e^{y/(2x_0)}$ and $f(y) = e^{-y/(2x_0)}$.

REMARK. Recall that a scale function is only determined up to a linear transformation. The choice $s(x_0) = 0$ is arbitrary, but extremely convenient, as it allows us to start the process Z, defined below, at zero. The choice $s'(x) = \varphi(x) - x\varphi'(x)$ is simple, but a case could be made for the alternative normalization $s'(x) = (\varphi(x) - x\varphi'(x))/(1 - x_0\varphi'(x_0))$ for which $s'(x_0) = 1$. Multiplying s by a constant has the effect of modifying the construction defined in the next section, but only by the introduction of a constant factor into the time changes. It is easy to check that this leaves the final model X_t unchanged.

Our goal is to construct a time-homogeneous process which is consistent with observed put prices and such that $e^{-rt}X_t$ is a (local) martingale. In the regular case we have seen how to construct a diffusion with these properties. We now have to allow for more general processes, perhaps processes which jump over intervals, or perhaps processes which have "sticky" points. One very powerful construction method for time-homogeneous, martingale diffusions is via a time change of Brownian motion, and it is this approach which we exploit.

5. Constructing time-homogeneous processes as time changes of Brownian motion. In this section we extend the construction in Rogers and Williams [13], Section V.47, of martingale diffusions as time changes of Brownian motion; see also Itô and McKean [7], Section 5.1. The difference from the classical setting is that the processes defined below may have "sticky" points and may jump over intervals. Since diffusions are continuous by definition, the resulting processes are not diffusions, but one might think of them as "generalized diffusions" ([9], or "gap diffusions" [8]), and they are "as continuous as possible."

Let ν be a Borel measure on $\mathbb{R}$ and let $\mathbb{F}^B = (\mathcal{F}_u^B)_{u\geq 0}$ be a filtration supporting a Brownian motion B started at 0 with local time process L_u^z. Define Γ to be the left-continuous increasing additive functional

$$\Gamma_u = \int_{\mathbb{R}} L_u^z \nu(dz), \qquad \Gamma_0 = 0, \tag{26}$$

and let A be the right-continuous inverse of Γ, that is,

$$A_t = \inf\{u : \Gamma_u > t\}.$$

Note that Γ is a nondecreasing process, so A is well defined, and A_t is an $\mathbb{F}^B$-stopping time for each time t. Set $Z_0 = 0$ and, for $t > 0$, set $Z_t = B_{A_t}$ and $\mathcal{F}_t = \mathcal{F}_{A_t}^B$. Note that Z is right-continuous, except possibly at $t = 0$. The process Z_t is a time-changed Brownian motion adapted to the filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ and subject to mild nondegeneracy conditions on ν (see Lemma 5.1 below), and the processes Z_t and $Z_t^2 - A_t$ are local martingales. Further, if $\nu(dy) = dy/\gamma^2(y)$, then $\Gamma_u = \int_0^u \gamma^{-2}(B_r)\,dr$ and $A_t = \int_0^t \gamma^2(Z_s)\,ds$, so that Z_t is a weak solution to $dZ_t = \gamma(Z_t)\,dW_t$, and Z is a diffusion in natural scale. Similarly, if $\nu(dy) = dy/\gamma^2(y)$ in an interval, then Z solves $dZ_t = \gamma(Z_t)\,dW_t$ in this interval. The measure ν is called the *speed measure* of Z, although, as pointed out by Rogers and Williams, ν is large when Z moves slowly.

The measure ν may have atoms, and it may have intervals on which it places no mass. If there is an atom at $\hat{z}$, then $d\Gamma_u/du > 0$ whenever $B_u = \hat{z}$, and then the time-changed process is "sticky" there. Conversely, if ν places no mass in (α, β), then Γ is constant on any time-periods that B spends in this interval, and the inverse time change A has a jump. In particular, Z_t spends no time in this interval. If $\nu(\{\tilde{z}\}) = \infty$, then $\Gamma_u = \infty$ for any u greater than the first hitting time $H_{\tilde{z}}^B$ by B of level $\tilde{z}$. In that case, $A_\infty \leq H_{\tilde{z}}^B$ so that if Z hits $\tilde{z}$, then $\tilde{z}$ is absorbing for Z. The other possibility is that Z tends to this level without reaching it in finite time.

Define $\overline{z}_\nu \in (0, \infty]$ and $\underline{z}_\nu \in [-\infty, 0)$ via

$$\overline{z}_\nu = \inf\{z > 0 : \nu((0, z]) = \infty\} \quad \text{and}$$
$$\underline{z}_\nu = \sup\{z < 0 : \nu([z, 0)) = \infty\}.$$

The cases where $\overline{z}_\nu = 0$ or $\underline{z}_\nu = 0$ correspond to the degenerate case $X_t = x_0 e^{rt}$ mentioned in the previous section, and we exclude them. The following lemma provides a guide to sufficient conditions for a time change of Brownian motion to be a local martingale and therefore provides insight into the constructions of local martingales via time change that we develop in the next section.

LEMMA 5.1. *Suppose that either $\overline{z}_\nu < \infty$ or ν charges (a,∞) for each a, and, further, that either $\underline{z}_\nu > -\infty$ or ν charges $(-\infty,a)$ for each a. Then, $Z_t = B_{A_t}$ is a local martingale.*

PROOF. See Appendix A.2. $\square$

6. Constructing the model. We now show how to choose the measure ν which gives the process we want. Define ν via

$$\nu(dy) = \frac{1}{2r}\frac{g''(dy)}{g(y)}, \tag{27}$$

where $g''(dy)$ is the measure defined by the second order distribution derivative of g, and let $\nu(\{y\}) = \infty$ for $y \le s(0)$ in the case where $\underline{x} = 0$ and $s(0) > -\infty$. Similarly, in the case where $s(\infty) < \infty$, we set $\nu(\{y\}) = \infty$ for $y \ge s(\infty)$. Where g' is absolutely continuous it follows that ν has a density with respect to Lebesgue measure, but, more generally, (27) can be interpreted in a distributional sense.

Now, for this ν we can use the construction of the previous section to give a process Z_t. If we set $X_t = g(Z_t)$, then, subject to the hypotheses of Lemma 5.1, $Z_t = s(X_t)$ is a local martingale, so that s is a scale function for X. The process X is our candidate process for which the associated put prices are given by P.

EXAMPLE. For geometric Brownian motion,

$$\frac{\nu(dy)}{dy} = \frac{1}{2r}\frac{g''(y)}{g(y)} = \frac{1}{2rx_0^2}\frac{\beta}{(1+\beta)^2}\left[1 + \frac{y(1-\beta)}{x_0(1+\beta)}\right]^{-2}$$

for $y \in (s(0), s(\infty))$. In the case $\beta = 1$ this simplifies to

$$\frac{\nu(dy)}{dy} = \frac{1}{8x_0^2 r} = \frac{1}{C^2},$$

where $C = 2x_0\sqrt{2r}$. Then, $\Gamma_u = uC^{-2}$, $A_t = tC^2$, $Z_t = B_{tC^2} \stackrel{\mathcal{D}}{=} C\tilde{B}_t$ for a Brownian motion $\tilde{B}$ and $X_t = x_0 e^{Z_t/2x_0} = x_0 e^{\sqrt{2r}\tilde{B}_t}$.

REMARK. If the put price $P(K)$ satisfies the regularity conditions of Hypothesis 3.1, then the scale function s and its inverse g are C^2 and satisfy

$$g'(y)s'(g(y)) = 1$$

and

$$g''(y)s'(g(y)) + (g'(y))^2 s''(g(y)) = 0.$$

Moreover, $\sigma^2(x)x^2 s''(x) + 2rxs'(x) = 0$ so that

$$\frac{g(y)^2\sigma(g(y))^2 g''(y)}{2g'(y)^2} = rg(y). \tag{28}$$

Consequently, the speed measure ν is given by

$$\begin{aligned}\nu(dy) &= \frac{1}{2r}\frac{g''(y)}{g(y)}\,dy \\ &= \frac{1}{2r}\frac{-(g'(y))^2 s''(g(y))}{s'(g(y))g(y)}\,dy \\ &= \frac{(g'(y))^2}{\sigma^2(g(y))g^2(y)}\,dy,\end{aligned}$$

and the diffusion Z is the solution to the stochastic differential equation

$$dZ_t = \frac{\sigma(g(Z_t))g(Z_t)}{g'(Z_t)}\,dW_t.$$

Applying Itô's formula to $X_t = g(Z_t)$ yields

$$\begin{aligned}dX_t &= g'(Z_t)\,dZ_t + \frac{1}{2}g''(Z_t)(dZ_t)^2 \\ &= \frac{\sigma^2(X_t)X_t^2 g''(Z_t)}{2(g'(Z_t))^2}\,dt + \sigma(X_t)X_t\,dW_t \\ &= rX_t\,dt + \sigma(X_t)X_t\,dW_t,\end{aligned}$$

where we use (28) for the final equality. We thus recover the diffusion model from the regular case described in Section 3.

Recall that $\Gamma_u = \int_{\mathbb{R}} L_u^z \nu(dz)$ and let ξ be the first explosion time of Γ. Note that by construction Γ is continuous for $t < \xi$ and left-continuous at $t = \xi$. Since ν is infinite outside the interval $[s(0), s(\infty)]$, we also have the expression $\xi = \inf\{u : B_u \notin [s(0), s(\infty)]\} = H^B_{s(0)} \wedge H^B_{s(\infty)}$. The inverse scale function g is convex on $(s(0), s(\infty))$, but may have a jump (from a finite to an infinite value) at $s(\infty)$. In that case we take it to be left-continuous at $s(\infty)$ so that we may have $\bar{g} := \lim_{z\uparrow\infty} g(s(z))$ is finite.

For $0 \le u < \xi$, define $M_u = e^{-r\Gamma_u} g(B_u)$ and $N_u = e^{-r\Gamma_u} f(B_u)$.

LEMMA 6.1. *$M = (M_u)_{0\le u<\xi}$ and $N = (N_u)_{0\le u<\xi}$ are $\mathbb{F}^B$-local martingales.*

SKETCH OF PROOF. Suppose that φ is twice continuously differentiable with a positive second derivative. Then, g is twice continuously differentiable. For $u < \xi$, applying Itô's formula to $M_u = e^{-r\Gamma_u} g(B_u)$ gives

$$e^{r\Gamma_u}\,dM_u = g'(B_u)\,dB_u + \left[-r\frac{d\Gamma_u}{du}g(B_u) + \frac{1}{2}g''(B_u)\right]du.$$

But, by definition, $d\Gamma_u/du = g''(B_u)/(2rg(B_u))$, so M is a local martingale, as required.

A similar argument can be provided for the process N. For the general case, see the Appendix. $\square$

Since M and N are nonnegative local martingales on $[0, \xi)$, they converge almost surely to finite values, which we label M_ξ and N_ξ. In particular, if $\xi = H^B_{s(0)}$, then $M_\xi = 0$. However, if $\xi = H^B_{s(\infty)}$, then there are several cases. The fact that a nonnegative local martingale converges means that we cannot have both $\Gamma_\xi < \infty$ and $\bar{g} = \lim_{z\uparrow\infty} g(s(z)) = \infty$. Instead, if $\Gamma_\xi < \infty$, then $\bar{g} < \infty$ and $M_\xi = e^{-r\Gamma_\xi}\bar{g}$. If $\Gamma_\xi = \infty$ and $\bar{g} < \infty$, then $M_\xi = 0$, whereas if $\Gamma_\xi = \infty$ and $\bar{g} = \infty$, then $(M_u)_{u<\xi}$ typically has a nontrivial limit. Similar considerations apply to N.

Recall that A is the right-continuous inverse of Γ and define the time-changed processes $\tilde{M}_t = M_{A_t}$ and $\tilde{N}_t = N_{A_t}$. Note that these processes are adapted to $\mathbb{F}$ and that, at least for $t < \Gamma_\xi$, we have $\Gamma_{A_t} = t$, $\tilde{M}_t = e^{-rt}g(Z_t) = e^{-rt}X_t$ and $\tilde{N}_t = e^{-rt}f(Z_t) = e^{-rt}\varphi(X_t)$.

If $(s(0), s(\infty)) = \mathbb{R}$, then $\xi = \infty$, $\Gamma_\xi = \infty$ and $\tilde{M}$ is defined for all t.

If $s(0) > -\infty$, then we may have $\xi = H^B_{s(0)}$. In this case either $\Gamma_\xi = \Gamma_{H^B_{s(0)}} = \infty$, whence $\tilde{M}$ is defined for all t as before, or $\Gamma_\xi < \infty$. Then, $\tilde{M}_{\Gamma_\xi} = M_\xi = e^{-r\Gamma_\xi}g(B_\xi) = 0$, and we set $\tilde{M}_t = 0$ for all $t > \Gamma_\xi$. It follows that $X_t = 0$ for all $t \geq \Gamma_\xi$, and 0 is an absorbing state.

Similarly, if $s(\infty) < \infty$, then we may have $\xi = H^B_{s(\infty)}$. Then, either $\Gamma_\xi = \infty$, whence $\tilde{M}$ is defined for all t, or $\Gamma_\xi < \infty$. In the latter case, if $\xi = H^B_{s(\infty)} < \infty$ and $\Gamma_\xi < \infty$, then $\tilde{M}_{\Gamma_\xi} = M_\xi = e^{-r\Gamma_\xi}\bar{g}$. We set $\tilde{M}_t = \tilde{M}_{\Gamma_\xi}$ for all $t > \Gamma_\xi$, and it follows that for $t > \Gamma_\xi$, $X_t := e^{rt}\tilde{M}_t = e^{r(t-\Gamma_\xi)}\bar{g}$. Thus, for $t > \Gamma_\xi$, X grows deterministically. An example of this situation is given in Example 8.4 below. (In fact, the case where $\bar{g} < \infty$, which depends on the behavior of the scale function s to the right of x_0, can always be avoided by a suitable choice of the extension to φ.)

We want to show how $\tilde{M}$ and X inherit properties from M. The key idea below is that, loosely speaking, a time change of a martingale is again a martingale. Of course, to make this statement precise we need strong control on the time change. (Without such control the resulting process can have arbitrary drift. Indeed, as Monroe [11] has shown, any semimartingale can be constructed from Brownian motion via a time change.) We have the following result, the proof of which is given in the Appendix.

COROLLARY 6.2. *The process $(e^{-rt}X_t)_{t\geq 0}$ is a local martingale.*

We can perform a similar analysis on N and $\tilde{N}$ and use similar ideas to ensure that $\tilde{N}$ is defined on $\mathbb{R}_+$. The proof that $\tilde{N}$ is a local martingale mirrors that of Corollary 6.2.

COROLLARY 6.3. *The process* $(e^{-rt}\varphi(X_t))_{t\geq 0}$ *is a local martingale.*

7. Determining the put prices for the candidate process. Recall the definitions of s, g and ν via $s'(x) = \varphi(x) - x\varphi'(x)$, $g \equiv s^{-1}$ and $\nu(dy) = g''(dy)/(2rg(y))$. Suppose that Z is constructed from ν and a Brownian motion using the time change Γ and construct the candidate price process via $X_t = g(Z_t)$. By Corollary 6.2, the discounted price $e^{-rt}X_t$ is a (local) martingale. To complete the proof of Theorem 4.2 we need to show that for the candidate process X_t, the function

$$\hat{P}(K) := \sup_\tau \mathbb{E}[e^{-r\tau}(K - X_\tau)^+]$$

is such that $\hat{P}(K) \equiv P(K)$ for all $K \geq 0$.

Unlike the regular case, the process X that we have constructed may have jumps. For this reason, for $x < x_0$ we modify the definition of the first hitting time so that $H_x = \inf\{u > 0 : X_u \leq x\}$.

THEOREM 7.1. *The perpetual put prices for* X *are given by* P.

PROOF. Fix $x \in (\underline{x}, x_0)$. Suppose first that x is such that Γ is strictly increasing whenever the Brownian motion B takes the value $s(x)$. Then, $X_{H_x} = x$. More generally, the same is true whenever $\nu((s(x) - \delta, s(x)]) > 0$ for every $\delta > 0$. By Corollary 6.3 we have that $(e^{-rt}\varphi(X_t))_{t\leq H_x}$ is a local martingale, and φ is bounded on $[x, \infty)$, so it follows that $e^{-r(t\wedge H_x)}\varphi(X_{t\wedge H_x})$ is a bounded martingale and $\varphi(x_0) = \mathbb{E}^{x_0}[e^{-rH_x}\varphi(x)]$. Hence,

$$\hat{P}(K) \geq \mathbb{E}^{x_0}[e^{-rH_x}(K - x)] = (K - x)\frac{\varphi(x_0)}{\varphi(x)} = \frac{K - x}{\varphi(x)}.$$

Otherwise, fix $x^-(x) = \inf\{w < x : \nu((s(w), s(x)]) = 0\}$ and $x^+(x) = \sup\{w > x : \nu([s(x), s(w)) = 0\}$. It must be the case that φ is linear on $(x^-(x), x^+(x))$, bounded on $[x^-(x), \infty)$ and

$$\begin{aligned}\hat{P}(K) &\geq \max_{w\in\{x^-,x^+\}} \mathbb{E}^{x_0}[e^{-rH_w}(K - w)] \\ &= \max_{w\in\{x^-,x^+\}} \frac{K - w}{\varphi(w)} \geq \frac{K - x}{\varphi(x)}.\end{aligned}$$

It follows that

$$\hat{P}(K) \geq \sup_{x:x\leq x_0} \frac{K - x}{\varphi(x)} = P(K). \tag{29}$$

[Clearly, if $x < \underline{x}$, then $(K - x)/\varphi(x) = 0$, so the supremum cannot be attained for such an x.]

To prove the reverse inequality, we first claim that the left derivative $D^-\varphi$ of the convex function φ satisfies

$$D^-\varphi(x_0) := \lim_{\varepsilon\downarrow 0} \frac{\varphi(x_0)-\varphi(x_0-\varepsilon)}{\varepsilon} = \frac{-1}{K^*-x_0}. \tag{30}$$

To prove (30), first note that it follows from (25) that

$$D^-\varphi(x_0) \le \frac{-1}{K^*-x_0}.$$

Conversely, note that for each $\delta > 0$ there exists a nonempty interval $(x_0-\varepsilon, x_0)$, on which

$$\varphi(x) \le \frac{K^*-\delta-x}{K^*-\delta-x_0}. \tag{31}$$

To see this, let $\delta > 0$ be small and draw the tangent line to P that passes through the point $(K^*-\delta, K^*-x_0-\delta)$. Let $x_0-\varepsilon$ be the x-coordinate of the point of intersection between the tangent line and the x-axis. Then, for all $x \in [x_0-\varepsilon, x_0]$ we have that the line through $(x, 0)$ and $(K^*-\delta, K^*-x_0-\delta)$ is below the graph of P. Consequently, (31) holds. Therefore, for $x \in (x_0-\varepsilon, x_0)$ we have

$$\frac{\varphi(x_0)-\varphi(x)}{x_0-x} \ge \frac{-1}{K^*-\delta-x_0}.$$

Thus, $D^-\varphi(x_0) \ge -1/(K^*-x_0)$ since $\delta > 0$ is arbitrary, so (30) follows.

We next claim that for each fixed $K \le K^*$ we have

$$\varphi(x) \ge (K-x)^+/P(K) \tag{32}$$

for all x. Clearly, this holds for $x \ge K$ and for $x \le x_0$. Similarly, if $x_0 < x < K$, then it follows from (30) and the convexity of φ that

$$\varphi(x) \ge \frac{K^*-x}{K^*-x_0} \ge \frac{K-x}{K-x_0} \ge \frac{K-x}{P(K)}.$$

It follows from (32) and Corollary 6.3 that for any stopping rule τ we have

$$\mathbb{E}^{x_0}[e^{-r\tau}(K-X_\tau)^+] \le P(K)\mathbb{E}^{x_0}[e^{-r\tau}\varphi(X_\tau)] \le P(K)\varphi(x_0) = P(K).$$

Hence, $\hat{P}(K) \le P(K)$ for $K \le K^*$ and, in view of (29), $\hat{P}(K) = P(K)$.

For $K > K^*$ it follows from $\hat{P}(K^*) = P(K^*) = K^* - x_0$, the convexity of $\hat{P}$ and Hypothesis 4.1 that $\hat{P}(K) = K - x_0 = P(K)$, which completes the proof. □

8. Examples. The following examples illustrate the construction of the previous sections. The list of examples is not intended to be exhaustive, but rather indicative of the types of behavior that can arise. In each example we assume $x_0 = 1$.

8.1. *The smooth case*. We have studied the case of exponential Brownian motion throughout. It is very easy to generate other examples, for example, by choosing a smooth decreasing convex function [with $\varphi(x_0) = 1$ and $\lim_{x\uparrow\infty}\varphi(x) = 0$] and defining other quantities from φ.

EXAMPLE 8.1. Suppose $\varphi(x) = (x + 1)/(2x^2)$. Then, from (3) we obtain

$$\sigma^2(x) = r\frac{2x + 3}{x + 3}, \qquad x > 0,$$

and from (10),

$$P(K) = \frac{(K + 9)^{3/2}(K + 1)^{1/2} - (27 + 18K - K^2)}{4}, \qquad K \le 5/3,$$

with $P(K) = (K - 1)$ for $K \ge 5/3$.

8.2. *Kinks in P*. If the first derivative of P is not continuous, then we find that φ is linear over an interval (α, β), say. Then, s' is constant on this interval and g is linear over the interval $(s(\alpha), s(\beta))$. It follows that ν does not charge this interval, so Γ_u is constant whenever $B_u \in (s(\alpha), s(\beta))$, and A_t has a jump. Z_t then jumps over the interval $(s(\alpha), s(\beta))$, and X_t spends no time in (α, β).

EXAMPLE 8.2. Suppose that $P(K)$ satisfying Hypothesis 4.1 is given by

$$P(K) = \begin{cases} K^2/8, & 0 < K \le 27/32, \\ 4K^3/27, & 27/32 \le K \le 3/2, \\ (K - 1), & 3/2 \le K. \end{cases}$$

P is then continuous, but P' has a jump at $K = 27/32$.

Using (24) we find that

$$\varphi(x) = \begin{cases} 2x^{-1}, & 0 < x \le 27/64, \\ x^{-2}, & x > 9/16 \end{cases}$$

(strictly speaking, there is some freedom in the choice of φ for $x \ge x_0 \equiv 1$, but the power function x^{-2} is a natural choice). Over the region $I = [27/64, 9/16]$, φ is given by linear interpolation. The corresponding scale function is linear on I and in the construction of Z, ν assigns no mass to $s(I)$. The process X is a generalized diffusion with diffusion coefficient given by $\sigma(x) = \sqrt{2r}$ for $x \le 27/64$, $\sigma(x) = \sqrt{r}$ for $x \ge 9/16$ and $\sigma(x) = \infty$ for $x \in I$.

8.3. *Linear parts to P*. In this case, the derivative of $\varphi(x)$ is discontinuous at a point γ, say. Then, s' is also discontinuous at this point, and g' is discontinuous at $s(\gamma)$. It follows that ν has a point mass at $s(\gamma)$, and that Γ_u includes a multiple of the local time at $s(\gamma)$.

EXAMPLE 8.3. Suppose that $P(K)$ satisfying Hypothesis 4.1 is given by

$$P(K) = \begin{cases} 8K^3/27, & 0 < K \le 3/4, \\ (2K-1)/4, & 3/4 \le K \le 1, \\ K^2/4, & 1 \le K \le 2, \\ (K-1), & 2 \le K. \end{cases}$$

P is then convex, but is linear on the interval [3/4, 1]. We have

$$\varphi(x) = \begin{cases} x^{-2}/2, & 0 < \mathrm{x} \le 1/2, \\ x^{-1}, & x > 1/2, \end{cases}$$

where we have chosen to extend the definition of φ to $(1,\infty)$ in the natural way. Then, $s(x) = 3 - 2\ln 2 - 3/2x$ for $x < 1/2$ and $s(x) = 2\ln x$ otherwise. It follows that g is everywhere convex, but has a discontinuous first derivative at $z = -2\ln 2$, and that the corresponding measure ν has a positive density with respect to Lebesgue measure *and* an atom of size $r^{-1}/12$ at $-2\ln 2$. In the terminology of stochastic processes, the process Z is "sticky" at this point; for a discussion of sticky Brownian motion, see Amit [2] or, for the one-sided case, see Warren [15].

If P is piecewise linear (e.g., if P is obtained by linear interpolation from a finite number of options), then φ is piecewise linear, s is piecewise linear, g is piecewise linear and ν consists of a series of atoms. As a consequence the process Z_t is a continuous-time Markov process on a countable state space [at least while $Z_t < s(x_0) \equiv 0$], in which transitions are to nearest neighbors only. Holding times in states are exponential and the jump probabilities are such that Z_t is a martingale.

In turn this means that X_t is a continuous-time Markov process on a countable set of points (at least while $X_t < x_0$).

EXAMPLE 8.4. Suppose

$$P(K) = \begin{cases} K/3, & K \le 1, \\ (2K-1)/3, & 1 \le K \le 2, \\ (K-1), & K \ge 2. \end{cases}$$

This is consistent with a situation in which only two perpetual American put options trade, with strikes 1 and 3/2, and prices 1/3 and 2/3, in which case we may assume that we have extrapolated from the traded prices to a put pricing function $P(K)$ which is consistent with the traded prices. The function

$$\varphi(x) = \begin{cases} 3 - 3x, & x < 1/2, \\ 2 - x, & 1/2 \le x \le 2, \\ 0, & x > 2 \end{cases}$$

is a possible choice of φ. Then,

$$s(x) = \begin{cases} 3x - 5/2, & x < 1/2, \\ 2x - 2, & 1/2 \le x \le 2, \\ 2, & x > 2. \end{cases}$$

The inverse of s is given by

$$g(y) = \begin{cases} y/3 + 5/6, & -5/2 \le y < -1, \\ y/2 + 1, & -1 \le y \le 2. \end{cases}$$

The corresponding measure ν assigns no mass to the intervals $(-5/2, -1)$ and $(-1, 2)$, but has a point mass of size $1/(6r)$ at -1. The corresponding process X has state space $\{0\} \cup \{1/2\} \cup [2, \infty)$ and is such that:

- at $t = 0+$, X jumps to $1/2$ or 2 with probabilities $2/3$ and $1/3$, respectively;
- if ever $X_{t_0} \ge 2$, then $X_t = X_{t_0} e^{r(t-t_0)}$ thereafter;
- zero is an absorbing state for X.

To examine what happens if X ever reaches $1/2$, note that $\xi = H_{-5/2} \wedge H_2$ and $\Gamma_\xi = (1/6r) L_\xi^{-1}$ (where the superscript denotes local time at -1 rather than an inverse) and then

$$\mathbb{P}(A_t < \xi) = \mathbb{P}\left(t < \frac{1}{6r} L_\xi^{-1}\right) = \int_{6rt}^{\infty} \frac{1}{2} e^{-y/2}\, dy = e^{-3rt},$$

where we have used the known density of $L^{-1}_{H_{-5/2} \wedge H_2}$ (cf. page 213 in [3]). This implies that if X ever reaches $1/2$, then it stays there for an exponential length of time, rate $3r$, and jumps to 2 with probability $1/3$ and zero with probability $2/3$.

Note that for the continuous-time Markov process X_t, conditional on $X_t = 1/2$, we have

$$\lim_{\Delta \downarrow 0} \frac{1}{\Delta} \mathbb{E}[X_{t+\Delta} - X_t] = 3r\left[\frac{1}{3}(2 - X_t) + \frac{2}{3}(-X_t)\right] = \frac{r}{2} = r X_t.$$

Also, for this process

$$\begin{aligned} P(K) &= \max_{\tau = 0, H_{1/2}, H_0} \mathbb{E}[e^{-r\tau}(K - X_\tau)^+] \\ &= \max\{K - 1, (2K - 1)/3, K/3\}, \end{aligned}$$

so we recover the put price function given at the start of the example.

8.4. *Positive gradient of P at zero* [*i.e.*, $P'(0) > 0$]. In this case $\lim_{x \downarrow 0} \varphi(x) < \infty$. It follows that $s(0) > -\infty$ and the resulting diffusion X_t can hit zero in finite time. Recall that the diffusion X is constructed so that 0 is an absorbing endpoint.

EXAMPLE 8.5. Suppose that

$$P(K) = \begin{cases} K/2, & K < 1, \\ (K+1)^2/8, & 1 \le K \le 3, \\ K - 1, & K \ge 3. \end{cases}$$

Then, $\varphi(x) = 2(x+1)^{-1}$ and $x^2 \sigma(x)^2 = r(x+1)(2x+1)$ so that $dX_t = r X_t\, dt + \sqrt{r(X_t + 1)(2X_t + 1)}\, dB_t$.

The following example covers the case of mixed linear and smooth parts of $P(K)$ and shows an example where reflection, local times and jumps all form part of the construction.

EXAMPLE 8.6. Suppose $P(K)$ satisfies

$$P = \begin{cases} K/4, & K \leq 1, \\ K^2/4, & 1 \leq K \leq 2, \\ (K-1), & K \geq 2. \end{cases}$$

Note that P' has a jump at $K = 1$. We have $\varphi(x) = 4 - 4x$ for $x < 1/2$ and $\varphi(x) = 1/x$ for $1/2 \leq x \leq 1$. We assume this formula also applies on $[1, \infty)$. Then,

$$s(x) = \begin{cases} 2 \ln x, & x \geq 1/2, \\ 4x - 2 - 2\ln 2, & x < 1/2 \end{cases}$$

and

$$g(y) = \begin{cases} e^{y/2}, & y \geq -2\ln 2, \\ y/4 + (1 + \ln 2)/2, & -2 - 2\ln 2 < y < -2\ln 2. \end{cases}$$

Consequently, $\nu(dy) = \frac{1}{8r}\, dy$ for $y \geq -2\ln 2$, no mass is assigned to the interval $(-2 - 2\ln 2, -2\ln 2)$ and $\upsilon(\{y\}) = \infty$ for $y \leq -2 - 2\ln 2$. It follows that for $x \geq 1/2$ we have $\sigma^2(x) = 2r$, and then

$$dX_t = rX_t\, dt + \sqrt{2r} X_t\, dB_t, \tag{33}$$

at least until the first hitting time of $1/2$. To allow for behavior at $1/2$ the general construction includes a local time reflection *and* a compensating downward jump are added at instants when $X_t = 1/2$. The jump takes the process to zero, where it is absorbed.

Alternatively, the process can be formalized as follows. Let I_t be the infimum process given by $I_t = -\inf_{u \leq t}\{(B_u + \ln 2/\sqrt{2r}) \wedge 0\}$. By Skorokhod's lemma, $B_t + I_t$ is then a reflected Brownian motion [reflected at the level $-(\ln 2/\sqrt{2r})$] and $e^{\sqrt{2r}(B_t + I_t)} \geq 1/2$.

Let N^λ be a Poisson process with rate λ, independent of B, and let T^λ be the first event time. The compensated Poisson process $(N_t^\lambda - \lambda t)_{t \geq 0}$ and the compensated Poisson process stopped at the first jump $(N_{t \wedge T^\lambda}^\lambda - \lambda(t \wedge T^\lambda))_{t \geq 0}$ are then martingales. The time change $(N_{I_t \wedge T^\lambda}^\lambda - \lambda(I_t \wedge T^\lambda))_{t \geq 0}$ is also a martingale.

Take $\lambda = \sqrt{2r}$ and define X via $X_0 = 1$ and

$$dX_t = rX_t\, dt + \sqrt{2r} X_t \left(dB_t + dI_t - \frac{dN_{I_t}^{\sqrt{2r}}}{\sqrt{2r}} \right), \qquad t : I_t \leq T^{\sqrt{2r}}.$$

Note that at the first jump time of the time-changed Poisson process, X jumps from $1/2$ to zero.

By construction, $(e^{-rt} X_t)_{t \geq 0}$ is a martingale.

8.5. *P is zero on an interval.* Now, consider the case where $P(K) = 0$ for $K \le \underline{K}$. We then find that $\varphi(x) = \infty$ for $x \le \underline{x}$, where $\underline{x} = \underline{K}$. Depending on whether the right derivative $P'(\underline{K}+)$ is zero or positive, $\varphi(\underline{x}+)$ may be infinite or finite. In the former case we have that X_t does not reach $\underline{x}$ in finite time. In the latter case X_t does hit $\underline{x}$ in finite time.

The first example is typical of the case where $\varphi(\underline{x}+) = \infty$ or, equivalently, where there is smooth fit of P at $\underline{K}$.

EXAMPLE 8.7. Suppose $X_0 = 1$ and that $P(K)$ solves

$$P(K) = \begin{cases} 0, & K \le 1/2, \\ (2K-1)^2/8, & 1/2 \le K \le 3/2, \\ K-1, & K \ge 3/2. \end{cases}$$

P' is then continuous and for $1/2 < x < 1$ we have $\varphi(x) = (2x-1)^{-1}$. We also have $\varphi(x) = \infty$ for $x \le 1/2$. As usual, there is some freedom when extending φ to $(1, \infty)$, but for definiteness we assume that the formula $\varphi(x) = (2x-1)^{-1}$ applies there as well.

It follows that $\eta(x)^2 \equiv (x\sigma(x))^2 = r(2x-1)(4x-1)/4$. Note that since $\varphi(1/2) = \infty$ we have that $H_{1/2}$ (the first hitting time of $1/2$) is infinite. Hence,

$$dX_t = rX_t\,dt + \sqrt{\frac{r(2X_t-1)(4X_t-1)}{4}}\,dB_t, \qquad t \le H_{1/2},$$

is consistent with the observed put prices, and since the process never hits $1/2$ it is not necessary to describe it beyond $H_{1/2}$.

Now, consider the other case where $P'(\underline{K}) > 0$.

EXAMPLE 8.8. Suppose $X_0 = 1$ and that $P(K)$ solves

$$P(K) = \begin{cases} 0, & K \le 1/2, \\ (2K-1)/4, & 1/2 \le K \le 1, \\ K^2/4, & 1 \le K \le 2, \\ K-1, & K \ge 2. \end{cases}$$

P' then has a jump at $K = 1/2$.

We have $\varphi(x) = \infty$ for $x < 1/2$ and $\varphi(x) = 1/x$ for $1/2 \le x \le 1$, and we assume that this formula also applies on $[1, \infty)$. Then, $s(x) = 2\ln x$ for $x > 1/2$, and

$$g(y) = \begin{cases} e^{y/2}, & y > -2\ln 2, \\ 1/2, & y \le -2\ln 2. \end{cases}$$

Then, $\nu(dy) = dy/(8r)$ for $y > -2\ln 2$, $\nu(\{-2\ln 2\}) = 1/(4r)$ and $\nu(dy) = 0$ for $y < -2\ln 2$. Consequently, the time change $\Gamma_u = \frac{1}{8r}O_u^+ + \frac{1}{4r}L_u^{-2\ln 2}$ is a linear combination of O_u^+ and $L_u^{-2\ln 2}$, where O_u^+ is the amount of time spent by the Brownian motion above $s(1/2) = -2\ln 2$ before time u.

It follows that for $x \geq 1/2$, $\eta(x)^2 \equiv (x\sigma(x))^2 = 2rx^2$. As before we have

$$dX_t = rX_t\,dt + \sqrt{2r}X_t\,dB_t, \qquad t \leq H_{1/2}. \tag{34}$$

It is easy to check using Itô's formula that $e^{-r\Gamma_u}g(B_u)$ is a martingale in this case. The process $Z_t = B_{A_t}$ is "sticky" at $s(1/2)$ (this time in the sense of a one-sided sticky Brownian motion; see Warren [15]) and this property is inherited by $X = s(Z)$.

There is a third case, where $\varphi(\underline{x}+) < \infty$, but $\varphi'(x+) = \infty$.

EXAMPLE 8.9. Suppose $\varphi(x) = 2 - \sqrt{2x-1}$ for $1/2 \leq x \leq 5/2$ [and $\varphi(x) = \infty$ for $x < 1/2$]. Equivalently,

$$P(K) = \begin{cases} 0, & K \leq 1/2, \\ 2 - \sqrt{5-2K}, & 1/2 \leq K \leq 2, \\ K-1, & K \geq 2. \end{cases}$$

For $1/2 < x < 5/2$, we then have

$$\eta(x)^2 = (x\sigma(x))^2 = 2r(2x-1)\big(2\sqrt{2x-1} + 1 - x\big).$$

It follows that although X_t can hit $1/2$, the volatility at this level is zero, and the drift alone is sufficient to keep $X_t \geq 1/2$.

8.6. *Kink in P at K^*: $K^* < \infty$ and $P'(K^*-) < 1$.* In this case $\varphi'(x)$ is constant on an interval $(\hat{x}, x_0)$. This case is analogous to the one discussed in Section 8.2.

9. Extensions.

9.1. *No options exercised immediately.* In Hypothesis 4.1, in addition to (i) and (ii), which are enforceable by no-arbitrage considerations, we also assumed (iii) that there exists a finite strike K^* such that for all strikes $K \geq K^*$ the put option is exercised immediately. Since $K^* < \infty$ is equivalent to $\varphi'(x_0) < 0$, it is apparent from the expression in (5) that provided σ is finite on some interval (x_1, x_2) where $x_0 < x_1 < x_2$ or, equivalently, ν gives mass to some interval (y_1, y_2) where $0 < y_1 < y_2$, this property will hold. However, it is interesting to consider what happens when this fails.

Suppose that $P(K) > K - x_0$ for all K and that $\lim_{K\to\infty} P(K) - (K - x_0) = 0$. Then, $\varphi'(x_0) = 0$, but φ is strictly decreasing on $(\underline{x}, x_0)$. The measure ν places no mass on $(0, \infty)$, the process Z_t spends no time on $(0, \infty)$ and X_t never takes values above x_0. In particular, X_t is reflected (downward) at x_0. The resulting model is consistent with observed option prices, but not with the assumption that the discounted price process is a (local) martingale. However, by allowing nonzero dividend rates, we can find a model for which the ex-dividend price process is a martingale and for which the model prices are given by $P(K)$; see Section 9.3 below.

Now, suppose $\lim_K P(K) - (K - x_0) = \delta > 0$. If $P(x_0) = x_0$, then $P(K) \equiv K$ and we have an extreme example which falls into this setting. For P as specified above we have that $\varphi(x) = 1$ on $(x_0 - \delta, \infty)$. The measure ν places no mass on $(-\delta, \infty)$ and $s(x) = x - x_0$ on this region. Except for time 0, the process Z_t spends no time in $(-\delta, \infty)$ and X_t jumps instantly to $x_0 - \delta$, and thereafter spends no time above this point. Alternatively, if x_0 is not specified, then this case can be reduced to the previous case by assuming $x_0 = K - \lim_K P(K)$.

9.2. *Nonzero dividend processes*. Until now, we have assumed that dividend rates are zero. However, if dividend rates are a prespecified function of the asset price, then our method adapts in a straightforward manner.

Given put prices $P(K)$, we recover φ exactly as before from the representation (15). The unknown volatility σ and φ are then related via the modified version of (3):

$$\tfrac{1}{2}\sigma(x)^2 x^2 \varphi_{xx} + \big(rx - q(x)\big)\varphi_x - r\varphi = 0, \tag{35}$$

where q denotes the dividend rate. [We are assuming that under the pricing measure, X is governed by the stochastic differential equation $dX_t = (rX_t - q(X_t))\,dt + \sigma(X_t)X_t\,dB_t$, where q is a known function.] The candidate σ is then given by

$$\sigma^2(x) = 2\frac{r\varphi - (rx - q(x))\varphi_x}{x^2\varphi_{xx}}, \tag{36}$$

at least where this quantity exists.

Since φ is convex by construction, a necessary condition for the prices $P(K)$ to be consistent with some model with dividend rate q is that $r\varphi - (rx - q(x))\varphi_x \geq 0$. Then, in smooth cases, where the existence of the diffusion with volatility σ can be guaranteed, the analysis is complete. However, if φ is not strictly convex and twice differentiable, then some care may be needed to define the diffusion associated with the candidate σ given in (36).

In keeping with our analysis in the previous sections, the most natural approach for defining the (potentially generalized) one-dimensional diffusion X is via scale and speed. Note that if σ is sufficiently regular and L_σ is the operator

$$L_\sigma u = \tfrac{1}{2}\sigma(x)^2 x^2 u_{xx} + \big(rx - q(x)\big)u_x - ru,$$

then $L_\sigma \varphi = 0$. Moreover, we can find a second linearly independent solution ψ of $L_\sigma u = 0$ by the ansatz $\psi = \varphi v$. This leads to the ODE

$$\tfrac{1}{2}\sigma(x)^2 x^2 v_{xx}\varphi + \sigma(x)^2 x^2 v_x \varphi_x + \big(rx - q(x)\big)\varphi v_x = 0,$$

which gives the unknown v_x and its derivative in terms of φ and σ, and which has solution

$$v_x = \frac{A}{\varphi^2(x)} \exp\biggl(-\int_{x_0}^{x} \frac{2(rz - q(z))}{z^2\sigma(z)^2}\,dz\biggr) \tag{37}$$
$$= \frac{A}{\varphi^2(x)} \exp\biggl(-\int_{x_0}^{x} \frac{\varphi_{zz}(rz - q(z))}{r\varphi - (rz - q(z))\varphi_z}\,dz\biggr).$$

Note that the last expression is in terms of the dual function φ and does not involve σ directly.

It is easily checked that the derivative of the scale function is given by the Wronskian, so $s'(x) = \varphi\psi_x - \varphi_x\psi = \varphi^2 v_x$. As before, the scale function can be used to determine the inverse scale function g and measure ν. In turn, ν can be used to determine the time change Γ, and X is given by the formula $X_t = s(B_{A_t})$, where A is inverse to Γ. Thus, in principle, the methods of this article extend directly to the case with dividends, even in the irregular case [although further work is necessary if P is not strictly convex below K^*, whence φ_x is not continuous, and the integral in (37) is not well defined]. However, we will not complete the analysis in this case and instead will just make a remark and give a couple of examples.

REMARK. Whereas when dividends are zero we have (e.g., from Lemma 2.2) that φ is convex, this is not always the case when dividends are positive. This means that the duality between P and φ is more subtle. A convex P will lead to a convex φ and thence to a model which is consistent with the perpetual put prices $P(K)$. However, starting with a model for which φ is not convex, we can still derive option prices P from expressions such as (7), but if we now try to recover the model from those prices, the duality lemma will lead to a function $\tilde{\varphi} \neq \varphi$. Expressed differently, in the case with dividends it is possible to have many time-homogeneous diffusion models for which put prices are identical.

EXAMPLE 9.1. Suppose dividends are proportional so that $q(x) = \overline{q}x$ with $\overline{q} \le r$. Suppose further that $X_0 = 1$, and $P(K)$ is given by

$$\hat{P}(K) = \begin{cases} \dfrac{K}{\beta+1}\bigl(\beta K/(\beta+1)\bigr)^{\beta}, & \text{if } K < K^*, \\ K - 1, & \text{if } K \ge K^*, \end{cases} \tag{38}$$

where $K^* = (\beta+1)/\beta$ for some positive β.

Then, $\varphi(x) = x^{-\beta}$. It follows that $\sigma^2(x) = 2(r + (r - \overline{q})\beta)/(\beta(\beta+1))$. In this case it is clear that X is exponential Brownian motion and it is not necessary to calculate the scale function. However, a scale function can easily be computed and is given by $s(x) = x^c - 1$, where $c = (r - (r - \overline{q})\beta^2)/(r + (r - \overline{q})\beta)$.

EXAMPLE 9.2. Suppose that, as in Example 8.2, $X_0 = 1$ and $P(K)$ is given by

$$P(K) = \begin{cases} K^2/8, & 0 < K \le 27/32, \\ 4K^3/27, & 27/32 \le K \le 3/2, \\ (K-1), & 3/2 \le K. \end{cases}$$

This time, however, we assume that there are proportional dividends with constant of proportionality $\overline{q}$ (with $\overline{q} < r$). As in Example 8.2, we find that (with $\lambda = 4/3$)

$$\varphi(x) = \begin{cases} 2x^{-1}, & 0 < x \le 27/64 = \lambda^{-3}, \\ 4\lambda^3 - 2\lambda^6 x, & 27/64 = \lambda^{-3} < x \le 9/16 = \lambda^{-2}, \\ x^{-2}, & x > 9/16 = \lambda^{-2}. \end{cases}$$

Then, from $s' = \varphi^2 v'$ and (37) we find that s is linear over $I = [27/64, 9/16]$ and, more generally, a choice of s can be obtained by integrating

$$s'(x) = \begin{cases} \lambda^b x^{-2(r-\overline{q})/(2r-\overline{q})}, & 0 < x \le 27/64, \\ \lambda^{2c}, & 27/64 < x \le 9/16, \\ x^{-c}, & x > 9/16, \end{cases}$$

where now $c = 6(r-\overline{q})/(3r-2\overline{q})$ and $b = 6r(r-\overline{q})/[(3r-2\overline{q})(2r-\overline{q})]$.

9.3. *Time-homogeneous processes with known volatility and unknown interest rate or unknown dividend processes.* In the main body of the paper we have assumed that the interest rate r is a given positive constant, that dividend rates are zero and that σ is a function to be determined. In the last section we generalized this analysis to allow for a known, nonzero, dividend rate. We will now argue that the same ideas can be used to find other time-homogeneous models consistent with observed perpetual put prices, whereby the volatility function is given, and either a state-dependent dividend rate or a state-dependent interest rate is inferred.

Suppose X has dynamics $dX_t = (r(X_t)X_t - q(X_t))\,dt + \sigma(X_t)X_t\,dB_t$. Given put prices $P(K)$ as before, define φ via $\varphi(z) = \inf_{K:K\ge z}(K-z)/P(K)$. The relationship between φ and the characteristics of the price process X are then such that φ solves $L\varphi = 0$, where L is given by

$$Lu = \tfrac{1}{2}\sigma(x)^2 x^2 u_{xx} + \bigl(xr(x) - q(x)\bigr)u_x - r(x)u.$$

Note that we now allow for any of σ, q or r to depend on x. Until now we have assumed that r is constant and q is zero (except in the last section, where q was known but nonzero) and solved for σ, but we can alternatively assume that $\sigma(x)$ is a given function and r is a positive constant, and solve for q, or assume that q and σ are given, and solve for r.

For example, if r and σ are given constants, then the dividend rate process is given by

$$q(x) = xr + \frac{x^2\sigma^2\varphi_{xx} - 2r\varphi}{2\varphi_x}.$$

If q is negative, this should be thought of as a convenience yield.

By allowing for dividend processes which are singular with respect to calendar time and which are instead related to the local time of X at level x_0, it is possible to construct candidate price processes which spend no time above x_0. For example, if $\tilde{L}$ is the local time at 1 of X, and if

$$\frac{dX_t}{X_t} = dB_t + r\,dt - \frac{d\tilde{L}_t}{2},$$

then X_t reflects at 1, and if $\hat{\varphi}(x) = (\mathbb{E}^1[e^{-rH_x}])^{-1}$ for $x < 1$, then $\hat{\varphi}'(1-) = 0$. This gives an example of a model consistent with the class of option prices described in Section 9.1.

9.4. *Recovering the model from perpetual calls.* The perpetual American call price function $C : [0, \infty) \to [0, x_0]$ must be nonincreasing and convex as a function of the strike K, and must satisfy the no-arbitrage bounds $(x_0 - K)^+ \leq C(K) \leq x_0$.

If there are no dividends (and if $e^{-rt}X_t$ is a martingale), then the perpetual call prices are given by the trivial function $C(K) = x_0$.

So, suppose instead that the (proportional) dividend rate $\overline{q}$ is positive. Let $\hat{\psi}$ be the increasing positive solution to

$$\tfrac{1}{2}x^2\sigma(x)^2\hat{\psi}'' + (r - \overline{q})x\hat{\psi}' - r\hat{\psi} = 0,$$

normalized so that $\hat{\psi}(x_0) = 1$. Then, for $z > x$, $\mathbb{E}^x[e^{-rH_z}] = \hat{\psi}(x)/\hat{\psi}(z)$, and call prices in a model where $dX_t = (r - \overline{q})X_t\,dt + \sigma(X_t)X_t\,dB_t$ are given by

$$\hat{C}(K) = \sup_\tau \mathbb{E}^{x_0}[e^{-r\tau}(X_\tau - K)^+] = \sup_{x : x \geq x_0} \frac{(x - K)}{\hat{\psi}(x)}.$$

EXAMPLE. Suppose X solves $(dX_t/X_t) = (r - \overline{q})\,dt + \sigma\,dB_t$ with $X_0 = x_0$. Then, $\hat{\psi}(x) = (x/x_0)^\gamma$, where $\gamma = \beta_+$ and

$$\beta_\pm = -\left(\frac{r - \overline{q}}{\sigma^2} - \frac{1}{2}\right) \pm \sqrt{\left(\frac{r - \overline{q}}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2r}{\sigma^2}}.$$

Note that since $\overline{q} > 0$ we have $\gamma > 1$. Note also that $\hat{\varphi}(x) = (x/x_0)^{\beta_-}$.

The corresponding call prices are given by

$$\hat{C}(K) = x_0^\gamma \sup_{x : x \geq x_0} \{(x - K)x^{-\gamma}\},$$

which for $K \leq (\gamma - 1)x_0/\gamma$ gives $\hat{C}(K) = (x_0 - K)$, and for $K > (\gamma - 1)x_0/\gamma$ gives

$$\hat{C}(K) = x_0^\gamma \gamma^{-\gamma}(\gamma - 1)^{\gamma - 1}K^{1-\gamma}.$$

The example discusses the forward problem, but the discussion of the inverse problem is similar to that in the put case. Given perpetual call prices $C(K)$, for $x > x_0$ we can define ψ via $\psi(x) = \inf_{K:K\leq x}(x-K)/C(K)$ and then construct a triple $\sigma(x), q(x), r(x)$ so that

$$\tfrac{1}{2}x^2\sigma(x)^2\psi'' + \big(xr(x) - q(x)\big)\psi' - r(x)\psi = 0.$$

By combining information from put and call prices, it is possible to determine a candidate model which simultaneously matches both puts and calls. The information contained in the perpetual puts determines the volatility below x_0, and the information contained in the perpetual calls determines the volatility above x_0. However, for this candidate model to return the put and call prices, there is an additional consistency condition. For a discussion of this condition in the smooth case, see Alfonsi and Jourdain [1], Proposition 4.6.

APPENDIX: PROOFS

A.1. Duality.

PROOF OF LEMMA 2.5. It is clear that g is nonnegative and nondecreasing since f is positive and nonincreasing. The lower bound on g follows from choosing $z = z_0 \wedge k$ in (14), and the upper bound follows since f is nonincreasing. To show that g is convex, first note that $g(k)$ is minus the reciprocal of the slope of the tangent of the function f which passes through the point $(k, 0)$.

For two given points k_1 and k_2 with $k_1 < k_2$, let $l_1(z)$ and $l_2(z)$ be the corresponding tangent lines. Let $k = \lambda k_1 + (1-\lambda)k_2$ for some $\lambda \in (0, 1)$, and let $l(z)$ be the line through the point $(0, k)$ and the intersection point of l_1 and l_2 (cf. Figure 3). If the intersection point is denoted $(z, l(z))$, then the convexity of f guarantees that

$$g(k) \leq \frac{k-z}{l(z)} = \frac{(1-\lambda)k_1 - (1-\lambda)z}{l_1(z)} + \frac{\lambda k_2 - \lambda z}{l_2(z)} = (1-\lambda)g(k_1) + \lambda g(k_2),$$

which proves that g is convex.

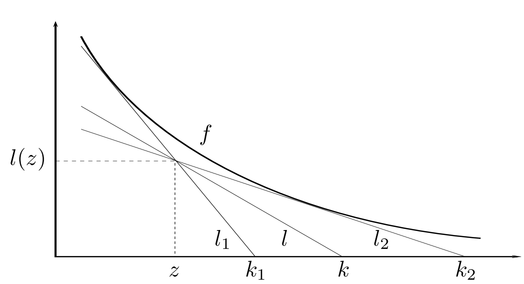

FIG. 3. *The lines l_1, l_2 and l.*

To prove the self-duality, let $z \le z_0$. By the definition of g, we have that $g(k) \ge (k-z)/f(z)$ for all $k \ge z$. Consequently,

$$F(z) = \sup_{k \ge z} \frac{k-z}{g(k)} \le f(z).$$

For the reverse inequality, let $z \le z_0$ and let l be a tangent line to f through the point $(z, f(z))$ (such a tangent is not necessarily unique if f has a kink at z). Assume that the point where l intersects the z-axis is given by $(k', 0)$. Then, $g(k') = (k'-z)/f(z)$, so

$$F(z) = \sup_{k \ge z} \frac{k-z}{g(k)} \ge \frac{k'-z}{g(k')} = f(z),$$

which completes the proof of (ii). The proof of (b) can be constructed along the same lines. □

A.2. Time changes of local martingales.

PROPOSITION A.1. *Suppose $(\gamma_u)_{u\ge 0}$ is a martingale with respect to the filtration $\mathbb{G} = (\mathcal{G}_u)_{u\ge 0}$, and A_t is an increasing process such that A_t is a finite stopping time with respect to $\mathbb{G}$ for each t. Define $\tilde{\gamma}_t = \gamma_{A_t}$ and $\tilde{\mathcal{G}}_t = \mathcal{G}_{A_t}$. In general $(\tilde{\gamma}_t)_{t\ge 0}$ is not a martingale. However, if γ is a bounded martingale, then $\tilde{\gamma}$ is a bounded martingale.*

PROOF. Given a Brownian motion B, for $b > 0$, let H_b^B be the first hitting time of level b. Then, $(\tilde{B}_b)_{b\ge 0}$ defined via $\tilde{B}_b \equiv B_{H_b^B}$ is not a martingale.

However, if γ is bounded, then $\mathbb{E}[\tilde{\gamma}_t | \tilde{\mathcal{G}}_s] = \mathbb{E}[\gamma_{A_t} | \mathcal{G}_{A_s}] = \gamma_{A_s} = \tilde{\gamma}_s$, by optional sampling. □

Suppose now that we are in the setting of Section 5, where Z_t is constructed from the Brownian motion B. In particular, Γ_u is an increasing additive functional of B, and A is the right-continuous inverse to Γ.

PROOF OF LEMMA 5.1. Intuitively, a time change of Brownian motion is a local martingale, but if the additive functional Γ is constant when B is in $[a, \infty)$, then the resulting process spends no time above a and reflects there. To maintain the local martingale property we need either that the time-changed process never gets to a, or that there are arbitrarily large values at which Γ is strictly increasing.

If $[\underline{z}_\nu, \overline{z}_\nu]$ is a bounded interval, then $A_\infty \le H^B_{\underline{z}_\nu} \wedge H^B_{\overline{z}_\nu}$ and $(Z_t)_{0\le t<\infty} = (B_{A_t})_{0\le A_t < A_\infty}$ is a bounded martingale, by Proposition A.1.

Now, suppose $(\underline{z}_\nu, \overline{z}_\nu) = \mathbb{R}$ and suppose that for each a, ν assigns mass to every set (a, ∞) and $(-\infty, -a)$.

We have $A_{\Gamma_t} \geq t$ with equality when Γ is strictly increasing at t. Let $\{a_n^+\}$ and $\{a_n^-\}$ be two sequences converging to $+\infty$ and $-\infty$, respectively, so that ν assigns mass to any neighborhood of a_n^+ and a_n^-, and set $H_n = \inf\{u : B_u \notin (a_n^-, a_n^+)\}$. Then, Γ is strictly increasing at H_n. Set $T_n = \Gamma_{H_n}$. Then, $A_{T_n} = H_n$. Note that Γ_u increases to infinity almost surely, and hence $\Gamma_{H_n} \uparrow \infty$. Under our hypothesis, $(Z_t^{T_n})_{t\geq 0}$ given by

$$Z_t^{T_n} := Z_{t\wedge T_n} = B_{A_{t\wedge T_n}} = B_{A_t \wedge H_n}$$

is a bounded martingale. Hence, T_n is a localization sequence for Z.

The mixed case can be treated similarly. □

PROOF OF LEMMA 6.1. For $y \in (s(0), s(\infty))$, set $H(y) = \ln(g(y)/x_0)$ and write $h(y) = H'(y) = g'(y)/g(y)$. If g is not differentiable at y, then we take the right derivative, which exists since g is convex. (We use a similar convention for f, h and j defined below.) Then,

$$\nu(dy) = \frac{1}{2r}\frac{g''(dy)}{g(y)} = \frac{1}{2r}\big(h'(dy) + h(y)^2\,dy\big)$$

and, as usual, $\nu(\{y\}) = \infty$ for $y \notin [s(0), s(\infty)]$. Note that in the case where g is not twice differentiable, we have $H''(dy) \equiv h'(dy) = g''(dy)/g(y) - h(y)^2\,dy$ so that H'' exists in a distributional sense.

We have $H(y) = \int_0^y h(v)\,dv = \ln(g(y)/x_0)$ and

$$\Gamma_u = \frac{1}{2r}\int_{\mathbb{R}} L_u^y\big(H''(dy) + H'(y)^2\,dy\big).$$

Let ξ be the first explosion time of Γ. Then, by the Itô–Tanaka formula (e.g., Revuz and Yor [12], Theorem VI.1.5), for $u < \xi$,

$$\begin{aligned} H(B_u) &= \int_0^u H'(B_s)\,dB_s + \frac{1}{2}\int_{\mathbb{R}} L_u^y H''(dy) \\ &= \int_0^u h(B_s)\,dB_s - \frac{1}{2}\int_{\mathbb{R}} L_u^y h(y)^2\,dy + r\Gamma_u. \end{aligned}$$

Thus, $g(B_u) = x_0 e^{H(B_u)} = \mathcal{E}(h(B)\cdot B)_u e^{r\Gamma_u}$, where $\mathcal{E}$ denotes the Doléans exponential, and $e^{-r\Gamma_u} g(B_u)$ is a local martingale. It follows that M is a local martingale.

Now, define $J(y) = \int_0^y j(v)\,dv = \ln f(y)$ and

$$\tilde{\Gamma}_u = \frac{1}{2r}\int_{\mathbb{R}} L_u^y\big(J''(dy) + J'(y)^2\,dy\big).$$

Again, $J''(dy) = f''(dy)/f(y) - j(y)^2\,dy$ exists in the distributional sense, even if $j(y)$ is not continuous. By exactly the same argument as above, we find that $f(B_u) = e^{J(B_u)} = \mathcal{E}(j(B)\cdot B)_u e^{r\tilde{\Gamma}_u}$ and $e^{-r\tilde{\Gamma}_u} f(B_u)$ is a local martingale.

It remains to show that $\Gamma_u = \tilde{\Gamma}_u$. Define $L(y) = (f(y)g(y))^{-1}$ so that L is continuous and right-differentiable. [We write $L'(y)$ for this right-derivative when the derivative is not well defined.] Then, $L'(y)/L(y) = -g'(y)/g(y) - f'(y)/f(y) = -(H'(y) + J'(y))$ and

$$J'(y) - H'(y) = \frac{\varphi'(g(y))g'(y)}{\varphi(g(y))} - \frac{g'(y)}{g(y)} = \frac{g'(y)[g(y)\varphi'(g(y)) - \varphi(g(y))]}{g(y)\varphi(g(y))}$$
$$= -\frac{g'(y)s'(g(y))}{g(y)f(y)} = -L(y).$$

We have that $J'(y) - H'(y)$ is (right-) differentiable, even if separately J' and H' are not, and

$$\big(J'(y) - H'(y)\big)' = \big(H'(y) + J'(y)\big)(L(y)) = H'(y)^2 - J'(y)^2.$$

Finally, since L_u^y is a bounded continuous function with compact support for each fixed u, we conclude that $\Gamma_u = \tilde{\Gamma}_u$. $\square$

PROOF OF COROLLARY 6.2. Recall that in our setting, Γ defined via (26) grows without bound and is continuous, at least until B hits $s(0)$ or $s(\infty)$. Thus, if ξ denotes the first explosion time of Γ, then the inverse function A is defined for every t, and $A_t = \xi$ for $t \geq \Gamma_\xi$. Then, using the extension of the definition of $\tilde{M}$ beyond Γ_ξ as necessary, we have

$$\tilde{M}_t = e^{-rt} X_t = \begin{cases} M_{A_t}, & t \leq \Gamma_\xi, \\ M_\xi, & t > \Gamma_\xi. \end{cases}$$

Recall that φ is extended to (x_0, ∞) in such a way that $\lim_{x\uparrow\infty} \varphi(x) = 0$. Therefore, either $s(\infty) < \infty$ and ν assigns infinite mass to all points $z > s(\infty) = \overline{z}_\nu$, or $s(\infty) = \infty$ and there exists a sequence $a_n \uparrow \infty$ such that ν assigns mass to any neighborhood of a_n.

Suppose that the second case obtains. If $s(0) > -\infty$, then $H^B_{s(0)} = \xi < \infty$, otherwise $\xi = \infty$. On $H^B_{a_n} < H^B_{s(0)} = \xi$, Γ_u is strictly increasing at $u = H^B_{a_n}$ and $A_{\Gamma_{H^B_{a_n}}} = H^B_{a_n}$. Set

$$T_n = \begin{cases} \Gamma_{H^B_{a_n}}, & H^B_{a_n} < H^B_{s(0)}, \\ \infty, & H^B_{a_n} > H^B_{s(0)}, \end{cases} \tag{39}$$

where the second line is redundant if $s(0) = -\infty$. Then, $A_{T_n} = H^B_{a_n} \wedge \xi$ is such that $\tilde{M}_t^{T_n} := \tilde{M}_{t\wedge T_n} = M_{A_t \wedge \xi \wedge H^B_{a_n}} \leq g(a_n)$ and T_n is a reducing sequence for $\tilde{M}$.

Now, suppose $s(\infty) < \infty$ and $\bar{g} = \infty$. Choose $a_n \uparrow s(\infty)$ such that ν assigns mass to any neighborhood of a_n. Then, on $H^B_{s(\infty)} < H^B_{s(0)}$, we have, by the argument after Lemma 6.1, that $\Gamma_{H^B_{a_n}} \uparrow \infty$ almost surely, and the argument proceeds as before with T_n given by (39) being a reducing sequence.

Finally, suppose $s(\infty) < \infty$ and $\bar{g} < \infty$. Then, M is bounded by $\bar{g}$ and $\tilde{M}$ is a martingale. $\square$

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