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Constructing Time-Homogeneous Generalised Diffusions Consistent with Optimal Stopping Values

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Abstract

Consider a set of discounted optimal stopping problems for a one-parameter family of objective functions and a fixed diffusion process, started at a fixed point. A standard problem in stochastic control/optimal stopping is to solve for the problem value in this setting.

In this article we consider an inverse problem; given the set of problem values for a family of objective functions, we aim to recover the diffusion. Under a natural assumption on the family of objective functions we can characterise existence and uniqueness of a diffusion for which the optimal stopping problems have the specified values. The solution of the problem relies on techniques from generalised convexity theory.

Keywords: *optimal stopping, generalised convexity, generalised diffusions, inverse American option problem*

1 Introduction

Consider a classical optimal stopping problem in which we are given a discount parameter, an objective function and a time-homogeneous diffusion process started at a fixed point, and we are asked to maximise the expected discounted payoff. Here the payoff is the objective function evaluated at the value of the diffusion at a suitably chosen stopping time. We call this problem the forward optimal stopping problem, and the expected payoff under the optimal stopping rule the (forward) problem value.

The set-up can be generalised to a one-parameter family of objective functions to give a one-parameter family of problem values. In this article we are interested in an associated inverse problem. The inverse problem is, given a one-parameter family of objective functions and associated optimal values, to recover the underlying diffusion, or family of diffusions, for which the family of forward stopping problems yield the given values.

The approach of this article is to exploit the structure of the optimal control problem and the theory of generalised convexity from convex analysis to obtain a duality relation between the Laplace transform of the first hitting time and the set of problem values. The Laplace transform can then be inverted to give the diffusion process.

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The generalised convexity approach sets this article apart from previous work on this problem, see [2, 1, 6]. All these papers are set in the realm of mathematical finance where the values of the stopping problems can be identified with the prices of perpetual American options, and the diffusion process is the underlying stock process. In that context, it is a natural question to ask: Given a set of perpetual American option prices from the market, parameterised by the strike, is it possible to identify a model consistent with all those prices simultaneously? In this article we abstract from the finance setting and ask a more general question: When can we identify a time-homogeneous diffusion for which the values of a parameterised family of optimal stopping problems coincide with a pre-specified function of the parameter.

Under restrictive smoothness assumptions on the volatility coefficients, Alfonsi and Jourdain [2] develop a 'put-call parity' which relates the prices of perpetual American puts (as a function of strike) under one model to the prices of perpetual American calls (as a function of the initial value of the underlying asset) under another model. This correspondence is extended to other payoffs in [1]. The result is then applied to solve the inverse problem described above. In both papers the idea is to find a coupled pair of free-boundary problems, the solutions of which can be used to give a relationship between the pair of model volatilities.

In contrast, in Ekström and Hobson [6] the idea is to solve the inverse problem by exploiting a duality between the put price and the Laplace transform of the first hitting time. This duality gives a direct approach to the inverse problem. It is based on a convex duality which requires no smoothness on the volatilities or option prices.

In this article we consider a general inverse problem of how to recover a diffusion which is consistent with a given set of values for a family of optimal stopping problems. The solution requires the use of generalised, or u-convexity (Carlier [4], Villani [14], Rachev and Rüschendorf [11]). The log-value function is the u-convex dual of the log-eigenfunction of the generator (and vice-versa) and the u-subdifferential corresponds to the optimal stopping threshold. These simple concepts give a direct and probabilistic approach to the inverse problem which contrasts with the involved calculations in [2, 1] in which pdes play a key role.

A major advantage of the dual approach is that there are no smoothness conditions on the value function or on the diffusion. In particular, it is convenient to work with generalised diffusions which are specified by the speed measure (which may have atoms, and intervals which have zero mass).

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2 The Forward and the Inverse Problems

Let $\mathcal{X}$ be a class of diffusion processes, let ρ be a discount parameter, and let $\mathcal{G} = \{G(x,\theta); \theta \in \Theta\}$ be a family of non-negative objective functions, parameterised by a real parameter θ which lies in an interval Θ. The forward problem, which is standard in optimal stopping, is for a given $X \in \mathcal{X}$, to calculate for each $\theta \in \Theta$, the problem value

$$V(\theta) \equiv V_X(\theta) = \sup_{\tau} \mathbb{E}_0[e^{-\rho\tau} G(X_\tau, \theta)], \tag{2.1}$$

where the supremum is taken over finite stopping times τ, and $\mathbb{E}_0$ denotes the fact that $X_0 = 0$. The inverse problem is, given a fixed ρ and the family $\mathcal{G}$, to determine whether $V \equiv \{V(\theta) : \theta \in \Theta\}$ could have arisen as a solution to the family of problems (2.1) and if so, to characterise those

elements $X \in \mathcal{X}$ which would lead to the value function V. The inverse problem, which is the main object of our analysis, is much less standard than the forward problem, but has recently been the subject of some studies ([2, 1, 6]) in the context of perpetual American options. In these papers the space of candidate diffusions is $\mathcal{X}_{stock}$, where $\mathcal{X}_{stock}$ is the set of price processes which, when discounted, are martingales and $G(x, \theta) = (\theta - x)^+$ is the put option payoff (slightly more general payoffs are considered in [1]). The aim is to find a stochastic model which is consistent with an observed continuum of perpetual put prices.

In fact it will be convenient in this article to extend the set $\mathcal{X}$ to include the set of generalised diffusions in the sense of Itô and McKean [8]. These diffusions are generalised in the sense that the speed measure may include atoms, or regions with zero or infinite mass. Generalised diffusions can be constructed as time changes of Brownian Motion, see Section 5.1 below, and also [8], [10], [13], and for a setup related to the one considered here, [6].

We will concentrate on the set of generalised diffusions started and reflected at 0, which are local martingales (at least when away from zero). We denote this class $\mathcal{X}_0$. (Alternatively we can think of an element X as the modulus of a local martingale Y whose characteristics are symmetric about the initial point zero.) The twin reasons for focusing on $\mathcal{X}_0$ rather than $\mathcal{X}$, are that the optimal stopping problem is guaranteed to become one-sided rather than two-sided, and that within $\mathcal{X}_0$ there is some hope of finding a unique solution to the inverse problem. The former reason is more fundamental (we will comment in Section 6.2 below on other plausible choices of subsets of $\mathcal{X}$ for which a similar approach is equally fruitful). For $X \in \mathcal{X}_0$, 0 is a reflecting boundary and we assume a natural right boundary but we do not exclude the possibility that it is absorbing. Away from zero the process is in natural scale and can be characterised by its speed measure, and in the case of a classical diffusion by the diffusion coefficient σ. In that case we may consider $X \in \mathcal{X}_0$ to be a solution of the SDE (with reflection)

$$dX_t = \sigma(X_t)dB_t + dL_t \qquad\qquad X_0 = 0,$$

where L is the local time at zero.

We return to the (forward) optimal stopping problem: For fixed X define $\varphi(x) = \varphi_X(x) = \mathbb{E}_0[e^{-\rho H_x}]^{-1}$, where H_x is the first hitting time of level x. Let

$$\hat{V}(\theta) = \sup_{x:\varphi(x)<\infty} \left[G(x, \theta)\, \mathbb{E}_0[e^{-\rho H_x}]\right] = \sup_{x:\varphi(x)<\infty} \left[\frac{G(x, \theta)}{\varphi(x)}\right]. \tag{2.2}$$

Clearly $V \geq \hat{V}$. Indeed, as the following lemma shows, there is equality and for the forward problem (2.1), the search over all stopping times can be reduced to a search over first hitting times.

Lemma 2.1. *V and $\hat{V}$ coincide.*

Proof. See Appendix. □

The first step in our approach will be to take logarithms which converts a multiplicative problem into an additive one. Introduce the notation

$$\begin{aligned} v(\theta) &= \log(V(\theta)), \\ g(x, \theta) &= \log(G(x, \theta)), \\ \psi(x) &= \log(\mathbb{E}_0[e^{-\rho H_x}]^{-1}) = \log \varphi(x). \end{aligned}$$

Then the equivalent log-transformed problem (compare (2.2)) is

$$v(\theta) = \sup_x [g(x,\theta) - \psi(x)], \tag{2.3}$$

where the supremum is taken over those x for which $\psi(x)$ is finite. To each of these quantities we may attach the superscript X if we wish to associate the solution of the forward problem to a particular diffusion. For reasons which will become apparent, see Equation (2.5) below, we call φ_X the eigenfunction (and ψ_X the log-eigenfunction) associated with X.

In the case where $g(x,\theta) = \theta x$, v and ψ are convex duals. More generally the relationship between v and ψ is that of u-convexity ([4], [14], [11]). (In Section 3 we give the definition of the u-convex dual f^u of a function f, and derive those properties that we will need.) For our setting, and under mild regularity assumptions on the functions g, see Assumption 3.6 below, we will show that there is a duality relation between v and ψ via the log-payoff function g which can be exploited to solve both the forward and inverse problems. In particular our main results (see Proposition 4.4 and Theorems 5.1 and 5.4 for precise statements) include:

Forward Problem: Given a diffusion $X \in \mathcal{X}_0$, let $\varphi_X(x) = (\mathbb{E}_0[e^{-\rho H_x}])^{-1}$ and $\psi_X(x) = \log(\varphi_X(x))$. Set $\psi^g(\theta) = \sup_x\{g(x,\theta) - \psi(x)\}$. Then the solution to the forward problem is given by $V(\theta) = \exp(\psi^g(\theta))$, at least for those θ for which there is an optimal, finite stopping rule. We also find that V is locally Lipschitz over the same range of θ.

Inverse Problem: For $v = \{v(\theta) : \theta \in \Theta = [\theta_-, \theta_+]\}$ to be logarithm of the solution of (2.1) for some $X \in \mathcal{X}_0$ it is sufficient that the g-convex dual (given by $v^g(x) = \sup_\theta\{g(x,\theta) - v(\theta)\}$) satisfies $v^g(0) = 0$, $e^{v^g(x)}$ is convex and increasing, and $v^g(x) > \{g(x,\theta_-) - g(0,\theta_-)\}$ for all $x > 0$.

Note that in stating the result for the inverse problem we have assumed that Θ contains its endpoints, but this is not necessary, and our theory will allow for Θ to be open and/or unbounded at either end.

If X is a solution of the inverse problem then we will say that X is consistent with $\{V(\theta); \theta \in \Theta\}$. By abuse of notation we will say that φ_X (or ψ_X) is consistent with V (or $v = \log V$) if, when solving the optimal stopping problem (2.1) for the diffusion with eigenfunction φ_X, we obtain the problem values $V(\theta)$ for each $\theta \in \Theta$.

The main technique in the proofs of these results is to exploit (2.3) to relate the fundamental solution φ with V. Then there is a second part of the problem which is to relate φ to an element of $\mathcal{X}$. In the case where we restrict attention to $\mathcal{X}_0$, each increasing convex φ with $\varphi(0) = 1$ is associated with a unique generalised diffusion $X \in \mathcal{X}_0$. Other choices of subclasses of $\mathcal{X}$ may or may not have this uniqueness property. See the discussion in Section 5.6.

The following examples give an idea of the scope of the problem:

Example 2.2. *Forward Problem: Suppose $G(x,\theta) = e^{x\theta}$. Let $m > 1$ and suppose that $X \in \mathcal{X}_0$ solves $dX = \sigma(X)dW + dL$ for $\sigma(x)^{-2} = (x^{2(m-1)} + (m-1)x^{m-2})/(2\rho)$. For such a diffusion $\varphi(x) = \exp(\frac{1}{m}x^m), x \geq 0$. Then for $\theta \in \Theta = (0,\infty)$, $V(\theta) = \exp(\frac{m-1}{m}\theta^{\frac{m}{m-1}})$.*

Example 2.3. *Forward Problem: Let X be reflecting Brownian Motion on the positive half-line with a natural boundary at ∞. Then $\varphi(x) = \cosh(x\sqrt{2\rho})$. Let $g(x,\theta) = \theta x$ so that g-convexity is standard convexity, and suppose $\Theta = (0,\infty)$. Then*

$$v(\theta) = \sup_x [\theta x - \log(\cosh(x\sqrt{2\rho}))].$$

It is easy to ascertain that the supremum is attained at $x = x^*(\theta)$ where

$$x^*(\theta) = \frac{1}{\sqrt{2\rho}} \tanh^{-1}\left(\frac{\theta}{\sqrt{2\rho}}\right) \tag{2.4}$$

for $\theta \in [0, \sqrt{2\rho})$. Hence, for $\theta \in (0, \sqrt{2\rho})$

$$\begin{aligned} v(\theta) &= \frac{\theta}{\sqrt{2\rho}} \tanh^{-1}\left(\frac{\theta}{\sqrt{2\rho}}\right) - \log\left(\cosh \tanh^{-1}\left(\frac{\theta}{\sqrt{2\rho}}\right)\right) \\ &= \frac{\theta}{\sqrt{2\rho}} \tanh^{-1}\left(\frac{\theta}{\sqrt{2\rho}}\right) + \frac{1}{2}\log\left(1 - \frac{\theta^2}{2\rho}\right), \end{aligned}$$

with limits $v(0) = 0$ and $v(\sqrt{2\rho}) = \log 2$. For $\theta > \sqrt{2\rho}$ we have $v(\theta) = \infty$.

Example 2.4. *Inverse Problem: Suppose that $g(x, \theta) = \theta x$ and $\Theta = (0, \sqrt{2\rho})$. Suppose also that for $\theta \in \Theta$*

$$V(\theta) = \exp\left(\frac{\theta}{\sqrt{2\rho}} \tanh^{-1}\left(\frac{\theta}{\sqrt{2\rho}}\right) + \frac{1}{2}\log\left(1 - \frac{\theta^2}{2\rho}\right)\right).$$

Then X is reflecting Brownian Motion.

Note that $X \in \mathcal{X}_0$ is uniquely determined, and its diffusion coefficient is specified on $\mathbb{R}^+$. In particular, if we expand the domain of definition of Θ to $(0, \infty)$ then for consistency we must have $V(\Theta) = \infty$ for $\theta > \sqrt{2\rho}$.

Example 2.5. *Inverse Problem: Suppose $G(x, \theta) = x^\theta$ and $V(\theta) = \{\frac{\theta^{\frac{\theta}{2}}(2-\theta)^{\frac{2-\theta}{2}}}{2} : \theta \in (1, 2)\}$. Then $\varphi(x) = 1 + x^2$ for $x > 1$ and, at least whilst $X_t > 1$, X solves the SDE $dX = \rho(1+X)^2 dW$. In particular, V does not contain enough information to determine a unique consistent diffusion in $\mathcal{X}_0$ since there is some indeterminacy of the diffusion co-efficient on $(0, 1)$.*

Example 2.6. *Inverse Problem: Suppose $g(x, \theta) = -\theta^2/(2\{1 + x\})$, $\Theta = [1, \infty)$ and $v(\theta) = \{-1/2 - \log\theta : \theta \geq 1\}$. Then the g-dual of v is given by $v^g(x) = \log(1 + x)/2$, $x \geq 0$ and is a candidate for ψ. However $e^{v^g(x)} = \sqrt{1 + x}$ is not convex. There is no diffusion in $\mathcal{X}_0$ consistent with V.*

Example 2.7. *Forward and Inverse Problem: In special cases, the optimal strategy in the forward problem may be to 'stop at the first hitting time of infinity' or to 'wait forever'. Nonetheless, it is possible to solve the forward and inverse problems.*

Let h be an increasing, differentiable function on $[0, \infty)$ with $h(0) = 1$, such that e^h is convex; let f be a positive, increasing, differentiable function on $[0, \infty)$ such that $\lim_{x\to\infty} f(x) = 1$; and let $w(\theta)$ be a non-negative, increasing and differentiable function on $\Theta = [\theta_-, \theta_+]$ with $w(\theta_-) = 0$.

Suppose that

$$g(x, \theta) = h(x) + f(x)w(\theta).$$

Note that the cross-derivative $g_{x\theta}(x, \theta) = f'(x)w'(\theta)$ is non-negative.

Consider the forward problem. Suppose we are given a diffusion in $\mathcal{X}_0$ with log-eigenfunction $\psi = h$. Then the log-problem value v is given by

$$v(\theta) = \psi^g(\theta) = \sup_{x\geq 0}\{g(x, \theta) - \psi(x)\} = \limsup_{x\to\infty}\{f(x)w(\theta)\} = w(\theta).$$

Conversely, suppose we are given the value function $V = e^w$ on Θ. Then

$$w^g(x) = \sup_{\theta\in\Theta}\{g(x, \theta) - w(\theta)\} = \sup_{\theta\in\Theta}\{h(x) + (f(x) - 1)w(\theta)\} = h(x)$$

is the log-eigenfunction of a diffusion $X \in \mathcal{X}_0$ which solves the inverse problem.

A generalised diffusion $X \in \mathcal{X}_0$ can be identified by its speed measure m. Let m be a non-negative, non-decreasing and right-continuous function which defines a measure on $\mathbb{R}^+$, and let m be identically zero on $\mathbb{R}^-$. We call x a point of growth of m if $m(x_1) < m(x_2)$ whenever $x_1 < x < x_2$ and denote the closed set of points of growth by E. Then m may assign mass to 0 or not, but in either case we assume $0 \in E$. We also assume that if $\xi = \sup\{x : x \in E\}$ then $\xi + m(\xi+) = \infty$. If $\xi < \infty$ then either ξ is an absorbing endpoint, or X does not reach ξ in finite time.

The diffusion X with speed measure m is defined on $[0, \xi)$ and is constructed via a time-change of Brownian motion as follows.

Let $\mathbb{F}^B = (\mathcal{F}_u^B)_{u \geq 0}$ be a filtration supporting a Brownian Motion B started at 0 with a local time process $\{L_u^z; u \geq 0, z \in \mathbb{R}\}$. Define Γ to be the left-continuous, increasing, additive functional

$$\Gamma_u = \int_{\mathbb{R}} L_u^z m(dz),$$

and define its right-continuous inverse by

$$A_t = \inf\{u : \Gamma_u > t\}.$$

If we set $X_t = B(A_t)$ then X_t is a generalised diffusion which is a local martingale away from 0, and which is absorbed the first time that B hits ξ.

For a given diffusion $X \in \mathcal{X}_0$ recall that $\varphi(x) \equiv \varphi_X(x)$ is defined via $\varphi_X(x) = (\mathbb{E}_0[e^{-\rho H_x}])^{-1}$. It is well known (see for example [13, V.50] and [5, pp 147-152]) that φ_X is the unique increasing, convex solution to the differential equation

$$\frac{1}{2}\frac{d^2 f}{dmdx} = \rho f; \qquad f(0) = 1, \qquad f'(0-) = 0. \tag{2.5}$$

Conversely, given an increasing convex function φ with $\varphi(0) = 1$ and $\varphi'(0+) \geq 0$, (2.5) can be used to define a measure m which in turn is the speed measure of a generalised diffusion $X \in \mathcal{X}_0$.

If $m(\{x\}) > 0$ then the process X spends a positive amount of time at x. If $x \in E$ is an isolated point, then there is a positive holding time at x, conversely, if for each neighbourhood N_x of x, m also assigns positive mass to $N_x \setminus \{x\}$, then x is a sticky point.

If $X \in \mathcal{X}_0$ and m has a density, then $m(dx) = \sigma(x)^{-2}dx$ where σ is the diffusion coefficient of X and the differential equation (2.5) becomes

$$\frac{1}{2}\sigma(x)^2 f''(x) - \rho f(x) = 0. \tag{2.6}$$

In this case, depending on the smoothness of g, v will also inherit smoothness properties. Conversely, 'nice' v will be associated with processes solving (2.6) for a smooth σ. However, rather than pursuing issues of regularity, we prefer to work with generalised diffusions.

3 u-convex Analysis

In the following we will consider u-convex functions for $u = u(y, z)$ a function of two variables y and z. There will be complete symmetry in role between y and z so that although we will

discuss u-convexity for functions of y, the same ideas apply immediately to u-convexity in the variable z. Then, in the sequel we will apply these results for the function g, and we will apply them for g-convex functions of both x and θ.

For a more detailed development of u-convexity, see [11], [14], [4] and the references therein. Proofs of the results below are included in the Appendix.

Let D_y and D_z be sub-intervals of $\mathbb{R}$. We suppose that $u : D_y \times D_z \mapsto \bar{\mathbb{R}}$ is well defined, though possibly infinite valued.

Definition 3.1. $f : D_y \to \mathbb{R}^+$ is u-convex iff there exists a non-empty $S \subset D_z \times \mathbb{R}$ such that for all $y \in D_y$

$$f(y) = \sup_{(z,a)\in S} [u(y, z) + a].$$

Definition 3.2. The u-dual of f is the u-convex function on D_z given by

$$f^u(z) = \sup_{y\in D_y} [u(y, z) - f(y)].$$

A fundamental fact from the theory of u-convexity is the following:

Lemma 3.3. *A function f is u-convex iff $(f^u)^u = f$.*

The function $(f^u)^u$ (the u-convexification of f) is the greatest u-convex minorant of f (see the Appendix). The condition $(f^u)^u = f$ provides an alternative definition of a u-convex function, and is often preferred; checking whether $(f^u)^u = f$ is usually more natural than trying to identify the set S.

Diagrammatically (see Figure 1.), we can think of $-(f^u)(z) = \inf_y[f(y) - u(y, z)]$ as the vertical distance between f and $u(., z)$. Thus $f^u(z) \leq 0$ when $f(y) \geq u(y, z)$ for all $y \in D_y$.

The following description due to Villani [14] is helpful in visualising what is going on: f is u-convex if at every point y we can find a parameter z so that we can *caress* f from below with $u(., z)$.

The definition of the u-dual implies a generalised version of the Young inequality (familiar from convex analysis, e.g [12]),

$$f(y) + f^u(z) \geq u(y, z)$$

for all $(y, z) \in D_y \times D_z$. Equality holds at pairs (y, z) where the supremum

$$\sup_z [u(y, z) - f^u(z)]$$

is achieved.

Definition 3.4. The u-subdifferential of f at y is defined by

$$\partial^u f(y) = \{z \in D_z : f(y) + f^u(z) = u(y, z)\},$$

or equivalently

$$\partial^u f(y) = \{z \in D_z : u(y, z) - f(y) \geq u(\hat{y}, z) - f(\hat{y}), \forall \hat{y} \in D_y\}.$$

If U is a subset of D_y then we define $\partial^u f(U)$ to be the union of u-subdifferentials of f over all points in U.

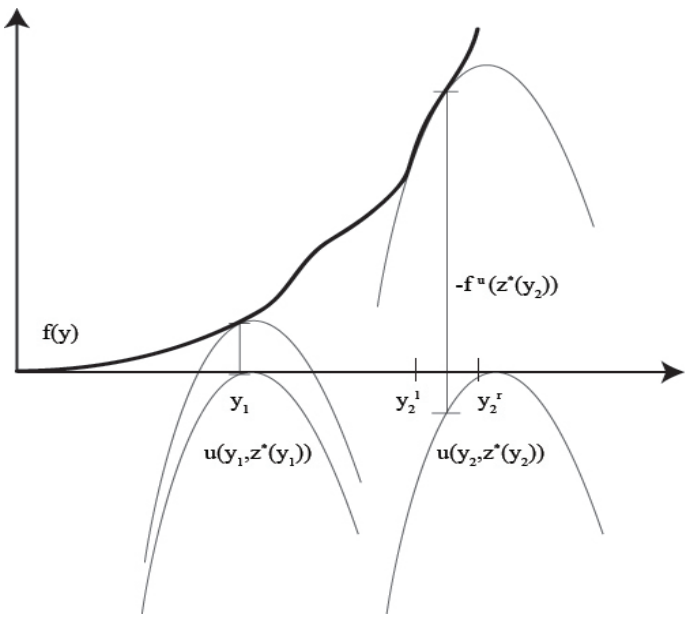

Figure 1: f is u-subdifferentiable. $\partial^u f(y_1) = z^*(y_1)$ and $\partial^u(y_2) = z^*(y_2)$ for $y_2 \in (y_2^l, y_2^r)$. The distance between $u(., z)$ and f is equal to -$f^u(z)$. Note that the u-subdifferential is constant over the interval (y_2^l, y_2^r)

Definition 3.5. f is u-subdifferentiable at y if $\partial^u f(y) \neq \emptyset$. f is u-subdifferentiable on U if it is u-subdifferentiable for all $y \in U$, and f is u-subdifferentiable if it is u-subdifferentiable on $U = D_y$.

In what follows it will be assumed that the function $u(y, z)$ is satisfies the following 'regularity conditions'.

Assumption 3.6. (a) $u(y, z)$ is continuously twice differentiable.

(b) $u_y(y, z) = \frac{\partial}{\partial y} u(y, z)$ as a function of z, and $u_z(y, z) = \frac{\partial}{\partial z} u(y, z)$ as a function of y, are strictly increasing.

Remark 3.7. We will see below that by assuming 3.6(a) irregularities in the value function (2.1) can be identified with extremal behaviour of the diffusion.

Remark 3.8. Condition 3.6(b) is known as the single crossing property and as the Spence-Mirrlees condition ([4]). If instead we have the 'Reverse Spence-Mirrlees condition':
(bb) $u_y(y, z)$ as a function of z, and $u_z(y, z)$ as a function of y, are strictly decreasing,
then there is a parallel theory, see Remark 3.12.

The following results from u-convex analysis will be fundamental in our application of u-convex analysis to finding the solutions of the forward and inverse problems.

Lemma 3.9. *Suppose f is u-subdifferentiable, and u satisfies Assumption 3.6. Then $\partial^u f$ is monotone in the following sense:*
Let $y, \hat{y} \in D_y$, $\hat{y} > y$. Suppose $\hat{z} \in \partial^u f(\hat{y})$ and $z \in \partial^u f(y)$. Then $\hat{z} \geq z$.

Definition 3.10. We say that a function is strictly u-convex, when its u-subdifferential is strictly monotone.

Proposition 3.11. *Suppose that u satisfies Assumption 3.6.*

Suppose f is a.e differentiable and u-subdifferentiable. Then there exists a map $z^* : D_y \to D_z$ such that if f is differentiable at y then $f(y) = u(y, z^*(y)) - f^u(z^*(y))$ and

$$f'(y) = u_y(y, z^*(y)). \tag{3.1}$$

Moreover, z^* is such that $z^*(y)$ is non-decreasing.

Conversely, suppose that f is a.e differentiable and equal to the integral of its derivative. If (3.1) holds for a non-decreasing function $z^*(y)$, then f is u-convex and u-subdifferentiable with $f(y) = u(y, z^*(y)) - f^u(z^*(y))$.

Note that the subdifferential $\partial^u f(y)$ may be an interval in which case $z^*(y)$ may be taken to be any element in that interval. Under Assumption 3.6, $z^*(y)$ is non-decreasing

We observe that since $u(y, z^*(y)) = f(y) + f^u(z^*(y))$ we have $u(y^*(z), z) = f(y^*(z)) + f(z)$ and $y^*(z) \in \partial^u f^u(z)$ so that y^* may be defined directly as an element of $\partial^u f^u$. If z^* is strictly increasing then y^* is just the inverse of z^*.

Remark 3.12. If u satisfies the 'Reverse Spence-Mirrlees' condition, the conclusion of Lemma 3.9 is unchanged except that now '$z \geq \hat{z}$'. Similarly, Proposition 3.11 remains true, except that $z^*(y)$ and $y^*(z)$ are non-increasing.

Proposition 3.13. *Suppose that u satisfies Assumption 3.6.*

Suppose f is u-subdifferentiable in a neighbourhood of y. Then f is continuously differentiable at y if and only if z^* is continuous at y.

4 Application of u-convex analysis to the Forward Problems

Now we return to the context of the family of optimal control problems (2.1) and the representation (2.3).

Lemma 4.1. *Let $X \in \mathcal{X}_0$ be a diffusion in natural scale reflected at the origin with a finite or infinite right boundary point ξ. Then the increasing log-eigenfunction of the generator*

$$\psi_X(x) = -\log(\mathbb{E}[e^{-\rho H_x}]^{-1})$$

is locally Lipschitz continuous on $(0, \xi)$.

Proof. $\varphi_X(x)$ is increasing, convex and finite and therefore locally Lipschitz on $(0, \xi)$. $\varphi(0) = 1$, and since log is locally Lipschitz on $[1, \infty)$, $\psi = \log(\varphi)$ is locally Lipschitz on $(0, \xi)$. □

Henceforth we assume that g satisfies Assumption 3.6, so that g is twice differentiable and satisfies the Spence-Mirrlees condition. We assume further that $G(x, \theta)$ is non-decreasing in x. Note that this is without loss of generality since it can never be optimal to stop at $x' > x$ if $G(x', \theta) < G(x, \theta)$, since to wait until the first hitting time of x' involves greater discounting and a lower payoff.

Consider the forward problem. Suppose the aim is to solve (2.3) for a given $X \in \mathcal{X}_0$ with associated log-eigenfunction $\psi(x) = \psi_X(x) = -\log \mathbb{E}_0[e^{-\rho H_x}]$ for the family of objective functions

$\{G(x,\theta) : \theta \in \Theta\}$. Here Θ is assumed to be an interval with endpoints θ_- and θ_+, such that $\Theta \subseteq D_\theta$.

Now let

$$v(\theta) = \sup_{x:\psi(x)<\infty} [g(x,\theta) - \psi(x).] \tag{4.1}$$

Then $v = \psi^g$ is the g-convex dual of ψ.

By definition $\partial^g v(\theta) = \{x : v(\theta) = g(x,\theta) - \psi(x)\}$ is the (set of) level(s) at which it is optimal to stop for the problem parameterised by θ. If $\partial^g v(\theta)$ is empty then there is no optimal stopping strategy in the sense that for any finite stopping rule there is another which involves waiting longer and gives a higher problem value.

Let θ_R be the infimum of those values of $\theta \in \Theta$ such that

$\partial^g v(\theta) = \emptyset$. If v is nowhere g-subdifferentiable then we set $\theta_R = \theta_-$.

Lemma 4.2. *The set where v is g-subdifferentiable forms an interval with endpoints θ_- and θ_R.*

Proof. Suppose v is g-subdifferentiable at $\hat{\theta}$, and suppose $\theta \in (\theta_-, \hat{\theta})$. We claim that v is g-subdifferentiable at θ.

Fix $\hat{x} \in \partial^g v(\hat{\theta})$. Then $v(\hat{\theta}) = g(\hat{x},\hat{\theta}) - \psi(\hat{x})$ and

$$g(\hat{x},\hat{\theta}) - \psi(\hat{x}) \geq g(x,\hat{\theta}) - \psi(x), \qquad \forall x < \xi, \tag{4.2}$$

and for $x = \xi$ if $\xi < \infty$. We write the remainder of the proof as if we are in the case $\xi < \infty$; the case $\xi = \infty$ involves replacing $x \leq \xi$ with $x < \xi$.

Fix $\theta < \hat{\theta}$. We want to show

$$g(\hat{x},\theta) - \psi(\hat{x}) \geq g(x,\theta) - \psi(x), \qquad \forall x \in (\hat{x},\xi], \tag{4.3}$$

for then

$$\sup_{x\leq\xi}\{g(x,\theta) - \psi(x)\} = \sup_{x\leq\hat{x}}\{g(x,\theta) - \psi(x)\},$$

and since $g(x,\theta) - \psi(x)$ is continuous in x the supremum is attained.

By assumption, $g_\theta(x,t)$ is increasing in x, and so for $x \in (\hat{x},\xi]$

$$\int_\theta^{\hat{\theta}} [g_\theta(\hat{x},t) - g_\theta(x,t)]dt \leq 0$$

or equivalently,

$$g(\hat{x},\hat{\theta}) - g(\hat{x},\theta) \leq g(x,\hat{\theta}) - g(x,\theta). \tag{4.4}$$

Subtracting (4.4) from (4.2) gives (4.3). □

Lemma 4.3. *v is locally Lipschitz on (θ_-, θ_R).*

Proof. On (θ_-, θ_R) $v(\theta)$ is g-convex, g-subdifferentiable and $x^*(\theta)$ is monotone increasing.

Fix θ', θ'' such that $\theta_- < \theta'' < \theta' < \theta_R$. Choose $x' \in \partial^g v(\theta')$ and $x'' \in \partial^g v(\theta'')$ and suppose g has Lipschitz constant K' (with respect to θ) in a neighbourhood of (x', θ').

Then $v(\theta') = g(x', \theta') - \psi(x')$ and $v(\theta'') \geq g(x', \theta'') - \psi(x')$ so that

$$v(\theta') - v(\theta'') \leq g(x', \theta') - g(x', \theta'') \leq K'(\theta' - \theta'')$$

and a reverse inequality follows from considering $v(\theta'') = g(x'', \theta'') - \psi(x'')$. □

Note that it is not possible under our assumptions to date (g satisfying Assumption 3.6, and g monotonic in x) to conclude that v is continuous at θ_-, or even that $v(\theta_-)$ exists. Monotonicity guarantees that even if $\theta_- \notin \Theta$ we can still define $x^*(\theta_-) := \lim_{\theta \downarrow \theta_-} x^*(\theta)$. For example, suppose $\Theta = (0, \infty)$ and for $\epsilon \in (0, 1)$ let $g_\epsilon(x, \theta) = g(x, \theta) + \epsilon f(\theta)$. Then if $v_\epsilon(\theta)$ is the g_ϵ-convex dual of ψ we have $v_\epsilon(\theta) = v(\theta) + \epsilon f(\theta)$, where $v(\theta) = v_0(\theta)$. If g and ψ are such that $\lim_{\theta \downarrow 0} v(\theta)$ exists and is finite, then choosing any bounded f for which $\lim_{\theta \downarrow 0} f(\theta)$ does not exist gives an example for which $\lim_{\theta \downarrow 0} v_\epsilon(\theta)$ does not exist. It is even easier to construct modified examples such that $v(\theta_-)$ is infinite.

Denote $\Sigma(\theta, \xi) = \limsup_{x \uparrow \xi} \{g(x, \theta) - \psi(x)\}$. Then for $\theta_R < \theta < \theta_+$, $\psi^g(\theta) = \Sigma(\theta, \xi)$. We have shown:

Proposition 4.4. *If g satisfies Assumption 3.6, g is increasing in x and if X is a reflecting diffusion in natural scale then the solution to the forward problem is $V(\theta) = \exp(\psi^g(\theta))$.*

Remark 4.5. Suppose now that $g_x(x, .)$ is strictly decreasing (the reverse Spence-Mirrlees condition). The arguments above apply with the obvious modifications. Let θ_L be the supremum of those values $\theta \in \Theta$ such that $x^*(\theta) = \emptyset$. Then the analogues to Lemmas 4.2 and 4.3 show that v is g-subdifferentiable and locally Lipschitz on (θ_L, θ_+) and that for $\theta_- < \theta < \theta_L$

$$V(\theta) = \exp(\Sigma(\theta, \xi)).$$

We close this section with some examples.

Example 4.6. *Recall Example 2.5, but note that in that example θ was restricted to take values in $\Theta = (1, 2)$. Suppose $\Theta = [0, \infty)$, $g(x, \theta) = \theta \log x$ and $\psi(x) = \log(1 + x^2)$. Then $\theta_R = 2$ and for $\theta < \theta_R$, $x^*(\theta) = (\theta/(2 - \theta))^{1/2}$. Further, for $\theta \leq 2$*

$$v(\theta) = \frac{\theta}{2} \log(\theta) + \frac{2 - \theta}{2} \log(2 - \theta) - \log 2,$$

and $v(\theta) = \infty$ for $\theta > 2$.

Note that v is continuous on $[0, \theta_R]$, but not on Θ.

Example 4.7. *Suppose $g(x, \theta) = x\theta$ and $\Theta = (0, \infty)$. Suppose X is a diffusion on $[0, 1)$, with 1 a natural boundary and diffusion coefficient $\sigma(x)^2 = \frac{\rho(1-x^2)^2}{1+x^2}$. Then $\varphi(x) = \frac{1}{1-x^2}$ and*

$$v(\theta) = \sup_{x<1} [\theta x + \log(1 - x^2)].$$

It is straightforward to calculate that $x^*(\theta) = \sqrt{1 + \theta^{-2}} - 1/\theta$ and then that $v(\theta) : (0, \infty) \to \mathbb{R}$ is given by

$$v(\theta) = \sqrt{1 + \theta^2} - 1 - \log\left(\frac{\theta^2}{2(\sqrt{1 + \theta^2} - 1)}\right). \tag{4.5}$$

5 Application of u-convex analysis to the Inverse Problem

Given an interval $\Theta \subseteq \overline{\mathbb{R}}$ with endpoints θ_- and θ_+ and a value function V defined on Θ we now discuss how to determine whether or not there exists a diffusion in $\mathcal{X}_0$ that solves the inverse problem for V. Theorem 5.1 gives a necessary and sufficient condition for existence. This condition is rather indirect, so in Theorem 5.4 we give some sufficient conditions in terms of the g-convex dual v^g and associated objects.

Then, given existence, a supplementary question is whether $\{V(\theta) : \theta \in \Theta\}$ contains enough information to determine the diffusion uniquely. In Sections 5.3, 5.4 and 5.5 we consider three different phenomena which lead to non-uniqueness. Finally in Section 5.6 we give a simple sufficient condition for uniqueness.

Two key quantities in this section are the lower and upper bound for the range of the g-subdifferential of v on Θ. Recall that we are assuming that the Spence-Mirrlees condition holds so that x^* is increasing on Θ. Then, if v is somewhere g-subdifferentiable we set $x_- = \sup\{x \in \partial^g v(\theta_-)\}$, or if $\theta_- \notin \Theta$, $x_- = \lim_{\theta \downarrow \theta_-} x^*(\theta)$. Similarly, we define $x_+ = \inf\{x \in \partial^g v(\theta_+)\}$, or if $\theta_+ \notin \Theta$, $x_+ = \lim_{\theta \uparrow \theta_+} x^*(\theta)$, and $x_R = \lim_{\theta \uparrow \theta_R} x^*(\theta)$. If v is nowhere g-subdifferentiable then we set $x_- = x_R = x_+ = \infty$.

5.1 Existence

In the following we assume that v is g-convex on $\mathbb{R}_+ \times \Theta$, which means that for all $\theta \in \Theta$,

$$v(\theta) = v^{gg}(\theta) = \sup_{x \geq 0}\{g(x, \theta) - v^g(x)\}.$$

Trivially this is a necessary condition for the existence of a diffusion such that the solution of the optimal stopping problems are given by V. Recall that we are also assuming that g is increasing in x and that it satisfies Assumption 3.6.

The following fundamental theorem provides necessary and sufficient conditions for existence of a consistent diffusion.

Theorem 5.1. *There exists $X \in \mathcal{X}_0$ such that $V_X = V$ if and only if there exists $\phi : [0, \infty) \to [1, \infty]$ such that $\phi(0) = 1$, ϕ is increasing and convex and ϕ is such that $(\log \phi)^g = v$ on Θ.*

Proof. If $X \in \mathcal{X}_0$ then $\phi_X(0) = 1$ and ϕ_X is increasing and convex. Set $\psi_X = \log \phi_X$. If $V_X = V$ then

$$v(\theta) = v_X(\theta) = \sup_x\{g(x, \theta) - \psi_X(x)\} = \psi_X^g.$$

Conversely, suppose ϕ satisfies the conditions of the theorem, and set $\psi = \log \phi$. Let $\xi = \sup\{x : \phi(x) < \infty\}$. Note that if $\xi < \infty$ then

$$(\log \phi)^g(\theta) = \sup_{x \geq 0}\{g(x, \theta) - \psi(x)\} = \sup_{x \leq \xi}\{g(x, \theta) - \psi(x)\}$$

and the maximiser $x^*(\theta)$ satisfies $x^*(\theta) \leq \xi$.

For $0 \leq x \leq \xi$ define a measure m via

$$m(dx) = \frac{1}{2\rho}\frac{\phi''(x)}{\phi}dx = \frac{\psi''(x) + (\psi'(x))^2}{2\rho}dx. \tag{5.1}$$

Let $m(dx) = \infty$ for $x < 0$, and, if ξ is finite $m(dx) = \infty$ for $x > \xi$. We interpret (5.1) in a distributional sense whenever ϕ has a discontinuous derivative. In the language of strings ξ is the length of the string with mass distribution m. We assume that $\xi > 0$. The case $\xi = 0$ is a degenerate case which can be covered by a direct argument.

Let B be a Brownian motion started at 0 with local time process L_u^z and define $(\Gamma_u)_{u\geq 0}$ via

$$\Gamma_u = \int_{\mathbb{R}} m(dz) L_u^z = \int_0^t \frac{1}{2\rho} \frac{\psi''(B_s) + (\psi'(B_s))^2}{\psi(B_s)} ds.$$

Let A be the right-continuous inverse to Γ. Now set $X_t = B_{A_t}$. Then X is a local martingale (whilst away from zero) such that $d\langle X\rangle_t/dt = dA_t/dt = (dm/dx|_{x=X_t})^{-1}$. When $m(dx) = \sigma(x)^{-2}dx$, we have $d\langle X\rangle_t = \sigma(X_t)^2 dt$.

We want to conclude that $\mathbb{E}[e^{-\rho H_x}] = \exp(-\psi(x))$. Now, $\varphi_X(x) = (\mathbb{E}[e^{-\rho H_x}])^{-1}$ is the unique increasing solution to

$$\frac{1}{2}\frac{d^2 f}{dmdx} = \rho f$$

with the boundary conditions $f'(0-) = 0$ and $f(0) = 1$. Equivalently, for all $x, y \in (0, \xi)$ with $x < y$, φ_X solves

$$f'(y-) - f'(x-) = \int_{[x,y)} 2\rho f(z) m(dz).$$

By the definition of m above it is easily verified that $\exp(\psi(x))$ is a solution to this equation. Hence $\phi = \varphi_X$ and our candidate process solves the inverse problem. □

Remark 5.2. Since v is g-convex a natural candidate for ϕ is $e^{v^g(x)}$, at least if $v^g(0) = 0$ and e^{v^g} is convex. Then ϕ is the eigenfunction φ_X of a diffusion $X \in \mathcal{X}_0$.

Our next example is one where $\phi(x) = e^{v^g(x)}$ is convex but not twice differentiable, and in consequence the consistent diffusion has a sticky point. This illustrates the need to work with generalised diffusions. For related examples in a different context see Ekström and Hobson [6].

Example 5.3. *Let $\Theta = \mathbb{R}_+$ and let the objective function be $g(x, \theta) = \exp(\theta x)$. Suppose*

$$V(\theta) = \begin{cases} \exp(\frac{1}{4}\theta^2) & 0 \leq \theta \leq 2, \\ \exp(\theta - 1) & 2 < \theta \leq 3, \\ \exp(\frac{2}{3\sqrt{3}}\theta^{3/2}) & 3 < \theta. \end{cases}$$

Writing $\varphi = e^{v^g}$ we calculate

$$\varphi(x) = \begin{cases} \exp(x^2) & 0 \leq x \leq 1, \\ \exp(x^3) & 1 < x. \end{cases}$$

Note that φ is increasing and convex, and $\varphi(0) = 1$. Then φ' jumps at 1 and since

$$\varphi(1) = \varphi'(1+) - \varphi'(1-) = 2\rho\varphi(1)m(\{1\})$$

we conclude that $m(\{1\}) = \frac{1}{2\rho}$. Then Γ_u includes a multiple of the local time at 1 and the diffusion X is sticky there.

Theorem 5.1 converts a question about existence of a consistent diffusion into a question about existence of a log-eigenfunction with particular properties including $(\log\phi)^g = v$. We would

like to have conditions which apply more directly to the value function $V(\cdot)$. The conditions we derive depend on the value of x_-.

As stated in Remark 5.2, a natural candidate for ϕ is $e^{v^g(x)}$. As we prove below, if $x_- = 0$ this candidate leads to a consistent diffusion provided $v^g(0) = 0$ and $e^{v^g(x)}$ is convex and strictly increasing. If $x_- > 0$ then the sufficient conditions are slightly different, and e^{v^g} need not be globally convex.

Theorem 5.4. *Assume v is g-convex. Each of the following is a sufficient condition for there to exist a consistent diffusion:*

1. *$x_- = 0$, $v^g(0) = 0$ and $e^{v^g(x)}$ is convex and increasing on $[0, x_+)$.*
2. *$0 < x_- < \infty$, $v^g(x_-) > 0$, $e^{v^g(x)}$ is convex and increasing on $[x_-, x_+)$, and on $[0, x_-)$, $v^g(x) \leq f(x) = \log(F(x))$ where*

$$F(x) = 1 + x\frac{\exp(v^g(x_-)) - 1}{x_-}$$

   is the straight line connecting the points $(0, 1)$ and $(x_-, e^{v^g(x_-)})$.

3. *$x_- = \infty$ and there exists a convex, increasing function F with $\log(F(0)) = 0$ such that $f(x) \geq v^g(x)$ for all $x \geq 0$ and*

$$\lim_{x\to\infty} \{f(x) - v^g(x) = 0\},$$

   where $f = \log F$.

Proof. We treat each of the conditions in turn. If $x_- = 0$ then Theorem 5.1 applies directly on taking $\phi(x) = e^{v^g(x)}$, with $\phi(x) = \infty$ for $x > x_+$ (we use the fact that v is g-convex and so $v^{gg} = v$).

Suppose $0 < x_- < \infty$. The condition $e^{v^g(x)} \leq F(x)$ on $[0, x_-)$ implies $F'(x) = (e^{v^g(x_-)} - 1)/\,x_- \leq (e^{v^g(x_- -)})'$. Although the left-derivative $v^g(x-)'$ need not equal the right-derivative $v^g(x+)'$ the arguments in the proof of Proposition 3.11 show that $v^g(x-)' \leq v^g(x+)'$. This implies that the function

$$\phi_F(x) = \begin{cases} F(x) & x < x_- \\ \exp(v^g(x)) & x_- \leq x < x_+ \end{cases}$$

is convex at x_- and hence convex and increasing on $[0, x_+)$.

Setting $\phi_F(x_+) = \lim_{x\uparrow x_+} \phi_F(x)$ and $\phi_F = \infty$ for $x > x_+$ we have a candidate for the function in Theorem 5.1.

It remains to show that $(\log \phi_F)^g = v$ on Θ. We now check that ϕ_F is consistent with V on Θ, which follows if the g-convex dual of $\psi = \log(\phi_F)$ is equal to v on Θ.

Since $\psi \geq v^g$ we have $\psi^g \leq v$. We aim to prove the reverse inequality. By definition, we have for $\theta \in \Theta$

$$\psi^g(\theta) = \left(\sup_{x<x_-} \{g(x, \theta) - f(x)\}\right) \vee \left(\sup_{x_-\leq x\leq x_+} \{g(x, \theta) - v^g(x)\}\right). \tag{5.2}$$

Now fix $x \in [0, x_-)$. For $\theta < \theta_R$ we have by the definition of the g-subdifferential

$$g(x^*(\theta), \theta) - v^g(x^*(\theta)) \geq g(x, \theta) - v^g(x).$$

Hence $v(\theta) = \sup_{x \geq 0}\{g(x, \theta) - v^g(x)\} = \sup_{x \geq x_-}\{g(x, \theta) - v^g(x)\} \leq \psi^g(\theta)$.

Similarly, if $\theta \geq \theta_R$ we have for all $x' \in [0, x_-)$,

$$\limsup_{x \to \infty} g(x, \theta) - v^g(x) \geq g(x', \theta) - v^g(x').$$

and $v(\theta) = \limsup_x\{g(x, \theta) - v^g(x)\} = \sup_{x \geq x_-}\{g(x, \theta) - v^g(x)\} \leq \psi^g(\theta)$.

Finally, suppose $x_- = \infty$. By the definition of f^g and the condition $f \geq v^g$ we get

$$\begin{aligned} f^g(\theta) &= \sup_{x \geq 0}\{g(x, \theta) - f(x)\} \\ &\leq \sup_{x \geq 0}\{g(x, \theta) - v^g(x)\} \\ &= v(\theta). \end{aligned}$$

On the other hand

$$\begin{aligned} v(\theta) &= \limsup_{x \to \infty}\{g(x, \theta) - f(x) + f(x) - v^g(x)\} \\ &\leq \limsup_{x \to \infty}\{g(x, \theta) - f(x)\} + \lim_{x \to \infty}\{f(x) - v^g(x)\} \ \leq \ f^g(\theta). \end{aligned}$$

Hence $v(\theta) = f^g(\theta)$ on Θ. □

Remark 5.5. Case 1 of the Theorem gives the sufficient condition mentioned in the paragraph headed Inverse Problem in Section 2. If $\theta_- \in \Theta$ then $x_- = 0$ if and only if for all $x > 0$, $g(x, \theta_-) - v^g(x) < g(0, \theta_-)$, where we use the fact that, by supposition, $v^g(0) = 0$.

5.2 Non-Uniqueness

Given existence of a diffusion X which is consistent with the values $V(\theta)$, the aim of the next few sections is to determine whether such a diffusion is unique.

Fundamentally, there are two natural ways in which uniqueness may fail. Firstly, the domain Θ may be too small (in extreme cases Θ might contain a single element). Roughly speaking the g-convex duality is only sufficient to determine v^g (and hence the candidate ϕ) over (x_-, x_+) and there can be many different convex extensions of ϕ to the real line, for each of which $\psi^g = v$. Secondly, even when $x_- = 0$ and $x_+ = \infty$, if $x^*(\theta)$ is discontinuous then there can be circumstances in which there are a multitude of convex functions ϕ with $(\log \phi)^g = v$. In that case, if there are no θ for which it is optimal to stop in an interval I, then it is only possible to infer a limited amount about the speed measure of the diffusion over that interval.

In the following lemma we do not assume that ψ is g-convex.

Lemma 5.6. *Suppose v is g-convex and $\psi^g = v$ on Θ. Let $A(\theta) = \{x : g(x, \theta) - \psi(x) = \psi^g(\theta)\}$. Then, for each θ, $A(\theta) \subseteq \partial^g \psi^g(\theta) \equiv \partial^g v(\theta)$, and for $x \in A(\theta)$, $\psi(x) = \psi^{gg}(x) = v^g(x)$. Further, for $\theta \in (\theta_-, \theta_R)$ we have $A(\theta) \neq \emptyset$.*

Proof. Note that if ψ is any function, with $\psi^g = v$ then $\psi \geq \psi^{gg} = v^g$.

If $\hat{x} \in A(\theta)$ then

$$\psi^g(\theta) = g(\hat{x}, \theta) - \psi(\hat{x}) \leq g(\hat{x}, \theta) - v^g(\hat{x}) \leq v(\theta).$$

Hence there is equality throughout, so $\hat{x} \in \partial^g v(\theta)$ and $\psi(\hat{x}) = v^g(\hat{x}) = \psi^{gg}(\hat{x})$.

For the final part, suppose $\theta < \theta_R$ and fix $\tilde{\theta} \in (\theta, \theta_R)$. From the Spence-Mirrlees condition, if $x > \tilde{x} := x^*(\tilde{\theta})$,

$$g(x, \theta) - g(\tilde{x}, \theta) < g(x, \tilde{\theta}) - g(\tilde{x}, \tilde{\theta}),$$

and hence

$$\{g(x, \theta) - \psi(x)\} - \{g(\tilde{x}, \theta) - \psi(\tilde{x})\} < \{g(x, \tilde{\theta}) - \psi(x)\} - \{g(\tilde{x}, \tilde{\theta}) - \psi(\tilde{x})\} \leq 0.$$

In particular, for $x > \tilde{x}$, $g(x, \theta) - \psi(x) < g(\tilde{x}, \theta) - \psi(\tilde{x})$ and

$$\sup_{x \geq 0} g(x, \theta) - \psi(x) = \sup_{0 \leq x \leq \tilde{x}} g(x, \theta) - \psi(x).$$

This last supremum is attained so that $A(\theta)$ is non-empty.

□

5.3 Left extensions

In the case where $x_- > 0$ and there exists a diffusion consistent with V then it is generally possible to construct many diffusions consistent with V. Typically V contains insufficient information to characterise the behaviour of the diffusion near zero.

Suppose that $0 < x_- < \infty$. Recall the definition of the straight line F from Theorem 5.4.

Lemma 5.7. *Suppose that $0 < x_- < \infty$ and that there exists $X \in \mathcal{X}_0$ consistent with V.*

Suppose that $\theta_R > \theta_-$ and that v^g and is continuous and differentiable to the right at x_-. Suppose further that $x^*(\theta) > x_-$ for each $\theta > \theta_-$.

Then, unless either $v^g(x) = f(x)$ for some $x \in [0, x_-)$ or $(v^g)'(x_- +) = f'(x_-)$, there are many diffusions consistent with V.

Proof. Let ϕ be the log-eigenfunction of a diffusion $X \in \mathcal{X}_0$ which is consistent with V

If $\theta_- \in \Theta$ then $v^g(x_-) = \psi(x_-)$ by Lemma 5.6. Otherwise the same conclusion holds on taking limits, since the convexity of ϕ necessitates continuity of ψ.

Moreover, taking a sequence $\theta_n \downarrow \theta_-$, and using $\hat{x}(\theta_n) > x^*(\theta_n -) > x_-$ we have

$$\psi'(x_- +) = \lim_{\theta_n \downarrow \theta_-} \frac{\psi(\hat{x}(\theta_n)) - \psi(x_-)}{\hat{x}(\theta_n) - x_-} = \lim_{\theta_n \downarrow \theta_-} \frac{v^g(\hat{x}(\theta_n)) - v^g(x_-)}{\hat{x}(\theta_n) - x_-} = (v^g)'(x_- +)$$

In particular, the conditions on v^g translate directly into conditions about ϕ.

Since the straight line F is the largest convex function with $F(0) = 1$ and $F(x_-) = e^{v^g(x_-)}$ we must have $\phi \leq F$.

Then if $\phi(x) = F(x)$ for some $x \in (0, x_-)$ or $\phi'(x_- +) = F'(x_-)$, then convexity of ϕ guarantees $\phi = F$ on $[0, x_-]$.

Otherwise there is a family of convex, increasing $\tilde{\phi}$ with $\tilde{\phi}(0) = 1$ and such that $v^g(x) \leq \log \tilde{\phi}(x) \leq F(x)$ for $x < x_-$ and $\tilde{\phi}(x) = \phi(x)$ for $x \geq x_-$.

For such a $\tilde{\phi}$, then by the arguments of Case 2 of Theorem 5.1 we have $(\log \phi_F)^g = v$ and then $v^g \leq \log \tilde{\phi} \leq \phi_F$ implies $v \geq (\log \tilde{\phi})^g \geq (\log \phi_F)^g = v$.

Hence each of $\tilde{\phi}$ is the eigenfunction of a diffusion which is consistent with V. □

Example 5.8. *Recall Example 2.5, in which we have $x_- = 1$, $\varphi'(1) = 2$ and $\varphi(1) = 2$. We can extend φ to $x \in [0, 1)$ by (for example) drawing the straight line between $(0, 1)$ and $(1, 2)$ (so that for $x \leq 1$, $\varphi(x) = 1 + x$). With this choice the resulting extended function will be convex, thus defining a consistent diffusion on $\mathbb{R}^+$. Note that any convex extension of φ (i.e. any function $\hat{\varphi}$ such that $\hat{\varphi}(0) = 1$ and $\hat{\varphi}'(0-) = 0$, $\hat{\varphi}(x) = \varphi(x)$ for $x > 1$) solves the inverse problem, (since necessarily $\hat{\varphi}(x) \geq 2x = e^{v^g(x)}$ on $(0, 1)$). The most natural choice is, perhaps, $\varphi(x) = 1 + x^2$ for $x \in (0, 1)$.*

Our next lemma covers the degenerate case where there is no optimal stopping rule, and for all θ it is never optimal to stop. Nevertheless, as Example 5.10 below shows, the theory of u-convexity as developed in the article still applies.

Lemma 5.9. *Suppose $x_- = \infty$, and that there exists a convex increasing function F with $F(0) = 1$ and such that $\log F(x) \geq v^g(x)$ and $\lim_{x\to\infty}\{\log F(x) - v^g(x)\} = 0$.*

Suppose that $\lim_{x\to\infty} e^{v^g(x)}/x$ exists in $(0, \infty]$ and write $\kappa = \lim_{x\to\infty} e^{v^g(x)}/x$. If $\kappa < \infty$ then X is the unique diffusion consistent with V if and only if $e^{v^g(x')} = 1 + \kappa x'$ for some $x' > 0$ or $\limsup_{x\uparrow\infty}(1 + \kappa x) - e^{v^g(x)} = 0$. If $\kappa = \infty$ then there exist uncountably many diffusions consistent with V.

Proof. The first case follows similar reasoning as Lemma 5.7 above. Note that $x \mapsto 1 + \kappa x$ is the largest convex function F on $[0, \infty)$ such that $F(0) = 1$ and $\lim_{x\to\infty} \frac{F(x)}{x} = \kappa$.

If $e^{v^g(x')} = 1 + \kappa x'$ for any $x' > 0$, or if $\limsup_{x\uparrow\infty}(1 + \kappa x) - e^{v^g(x)} = 0$ then there does not exist any convex function lying between $1 + \kappa x$ and $e^{v^g(x)}$ on $[0, \infty)$. In particular $\phi(x) = 1 + \kappa x$ is the unique eigenfunction consistent with V.

Conversely, if e^{v^g} lies strictly below the straight line $1+\kappa x$, and if $\limsup_{x\uparrow\infty}(1+\kappa x)-e^{v^g(x)} > 0$ then it is easy to verify that we can find other increasing convex functions with initial value 1, satisfying the same limit condition and lying between e^{v^g} and the line.

In the second case define $F_\alpha(x) = F(x) + \alpha x$ for $\alpha > 0$. Then since $\lim_{x\to\infty} e^{v^g(x)}/x = \infty$ we have

$$\lim_{x\to\infty} \frac{F_\alpha(x)}{e^{v^g(x)}} = \frac{F(x)}{e^{v^g(x)}} = 1$$

Hence F_α is the eigenfunction of another diffusion which is consistent with V. We conclude that there exist uncountably many consistent diffusions. □

Example 5.10. *Suppose $g(x, \theta) = x^2 + \theta \tanh x$ and $v(\theta) = \theta$ on $\Theta = \mathbb{R}^+$. For this example we have that v is nowhere g-subdifferentiable and $x_- = \infty$. Then $v^g(x) = x^2$ and each of $\varphi(x) = e^{x^2}$,*

$$\bar{\varphi}(x) = \begin{cases} 1 + (e - 1)x & 0 \leq x < 1 \\ e^{x^2} & 1 \leq x, \end{cases}$$

and $\varphi_\alpha(x) = \varphi(x) + \alpha x$ for any $\alpha \in \mathbb{R}_+$ is an eigenfunction consistent with V.

5.4 Right extensions

The case of $x_+ < \infty$ is very similar to the case $x_- > 0$, and typically if there exists one diffusion $X \in \mathcal{X}_0$ which is consistent with V, then there exist many such diffusions. Given X consistent with V, the idea is to produce modifications of the eigenfunction φ_X which agree with φ_X on $[0, x_+]$, but which are different on (x_+, ∞).

Lemma 5.11. *Suppose $x_+ < \infty$. Suppose there exists a diffusion $X \in \mathcal{X}_0$ such that V_X agrees with V on Θ. If $v^g(x_+) + (v^g)'(x_+ +) < \infty$ then there are infinitely many diffusions in $\mathcal{X}_0$ which are consistent with V.*

Proof. It is sufficient to prove that given convex increasing $\hat{\phi}$ defined on $[0, x_+)$ with $\hat{\phi}(0) = 1$ and $(\log \hat{\phi})^g = v$ on Θ, then there are many increasing, convex ϕ with defined on $[0, \infty)$ with $\phi(0) = 1$ for which $(\log \phi)^g = v$.

The proof is similar to that of Lemma 5.7. □

Example 5.12. *Let $G(x, \theta) = \theta x/(\theta + x)$, and $\Theta = (0, \infty)$.*

Consider the forward problem when X is a reflecting Brownian motion, so that the eigenfunction is given by $\varphi(x) = \cosh(x\sqrt{2\rho})$. Suppose $\rho = 1/2$.

Then $\{g(x, \theta) - \log(\cosh x)\}$ attains its maximum at the solution $x = x^*(\theta)$ to

$$\theta = \frac{x^2 \tanh x}{1 - x \tanh x}. \tag{5.3}$$

It follows that $x_- = 0$ but $x_+ = \lim_{\theta \uparrow \infty} x^*(\theta) = \hat{\lambda}$ where $\hat{\lambda}$ is the positive root of $\mathcal{L}(\lambda) = 0$ and $\mathcal{L}(\lambda) = 1 - \lambda \tanh \lambda$.

Now consider an inverse problem. Let G and Θ be as above, and suppose $\rho = 1/2$. Let $x^*(\theta)$ be the solution to (5.3) and let $v(\theta) = g(x^*(\theta), \theta) - \log(\cosh x^*(\theta))$. Then the diffusion with speed measure $m(dx) = dx$ (reflecting Brownian motion), is an element of $\mathcal{X}_0$ which is consistent with $\{V(\theta) : \theta \in (0, \infty)\}$. However, this solution of the inverse problem is not unique, and any convex function φ with $\varphi(x) = \cosh x$ for $x \le \hat{\lambda}$ is the eigenfunction of a consistent diffusion. To see this note that for $x > x_+$, $v^g(x) = \lim_{\theta \uparrow \infty} \{g(x, \theta) - v(\theta)\} = \log(x \cosh(x_+)/x_+)$ so that any convex φ with $\varphi(x) = \cosh x$ for $x \le x_+$ satisfies $\varphi \ge e^{v^g}$.

Remark 5.13. If $x_+ + v^g(x_+) + (v^g)'(x_+ +) < \infty$ then one admissible choice is to take $\phi = \infty$. This was the implicit choice in the proof of Theorem 5.1.

Example 5.14. *The following example is 'dual' to Example 2.3.*

Suppose $\rho = 1/2$, $g(x, \theta) = \theta x$, $\Theta = (0, \infty)$ and $v(\theta) = \log(\cosh \theta)$. Then $v^g(x) = x \tanh^{-1}(x) + \frac{1}{2} \log(1 - x^2)$, for $x \le 1$. For $x > 1$ we have that v^g is infinite. Since v is convex, and g-duality is convex duality, we conclude that v is g-convex. Moreover, v^g is convex. Setting $\psi = v^g$ we have that $\psi(0) = 0$, $\varphi = e^\psi$ is convex and $\psi^g = v^{gg} = v$. Hence ψ is associated with a diffusion consistent with V, and this diffusion has an absorbing boundary at $\xi \equiv 1$.

For this example we have $x_+ = 1$ and $v^g(x_+) = \log 2$, but the left-derivative of v^g is infinite at x_+ and v^g is infinite to the right of x_+. Thus there is a unique diffusion in $\mathcal{X}_0$ which is consistent with V.

5.5 Non-Uniqueness on $[x_-, x_+)$

Even if $[x_-, x_+)$ is the positive real line, then if $x^*(\theta)$ fails to be continuous it is possible that there are multiple diffusions consistent with V.

Lemma 5.15. *Suppose there exists a diffusion $X \in \mathcal{X}_0$ which is consistent with $\{V(\theta) : \theta \in \Theta\}$.*

Suppose the g-subdifferential of v is multivalued, or more generally that $x^*(\theta)$ is not continuous on Θ. Then there exists an interval $I \subset (x_-, x_+)$ where the g-subdifferential of $\psi = v^g$ is constant, so that $\theta^*(x) = \bar{\theta}$, $\forall x \in I$. If $G(x, \bar{\theta}) = e^{g(x,\bar{\theta})}$ is strictly convex in x on some subinterval of I_0 of I then the diffusion X is not the unique element of $\mathcal{X}_0$ which is consistent with V.

Proof. First note that if $x^*(\theta)$, is continuous then $\theta^* = x^{*-1}$ is nowhere constant and hence strictly monotone and thus ψ=v^g is strictly g-convex (recall 3.10).

Suppose $\bar{G}(x) := G(x, \bar{\theta})$ is strictly convex on $I_0 \subseteq I$. Then we can choose $\hat{G}$ such that
• $\hat{G} = \bar{G}$ on I_0^c,
• $\hat{G}$ is linear on I_0,
• $\hat{G}$ is continuous.
Then $\hat{G}(x) \geq G(x, \bar{\theta})$.

By definition we have

$$\psi(x) = g(x, \bar{\theta}) - \psi^g(\bar{\theta}) \qquad x \in I.$$

Then $\varphi_X(x) = G(x, \bar{\theta})/V(\bar{\theta})$ on I, see Figure 2.

Let $\hat{\varphi}$ be given by

$$\hat{\varphi}(x) = \begin{cases} \varphi_X(x) & \text{on } I_0^c \ , \\ \frac{\hat{G}(x)}{V(\theta)} & \text{on } I_0 \end{cases}$$

Then $\hat{\varphi}$ is convex and $\hat{\varphi} \geq \varphi$, so that they are associated with different elements of $\mathcal{X}_0$. Let $\hat{\psi} = \ln \hat{\varphi}$.

It remains to show that $\hat{v} := \hat{\psi}^g = \psi^g = v$. It is immediate from $\hat{\psi} \geq \psi$ that $\hat{\psi}^g \leq \psi^g$. For the converse, observe that

$$\begin{aligned} v(\theta) &= \left(\sup_{x \in I_0} \{g(x, \theta) - \psi(x)\} \right) \vee \left(\sup_{x \in I_0^c} \{g(x, \theta) - \psi(x)\} \right) \\ &= \sup_{x \in I_0^c} \{g(x, \theta) - \psi(x)\} \\ &= \sup_{x \in I_0^c} \{g(x, \theta) - \hat{\psi}(x)\} \\ &\leq \hat{v}(\theta). \end{aligned}$$

□

Example 5.16. *Suppose $G(x, \theta) = e^{\theta x}$, and $\Theta = (0, \infty)$. Suppose that X is such that ψ is given by*

$$\psi(x) = \begin{cases} \frac{x^2}{4} & x < 2 \\ (x - 1) & 2 \leq x < 3 \\ \frac{x^2}{6} + \frac{1}{2} & 3 \leq x \end{cases}$$

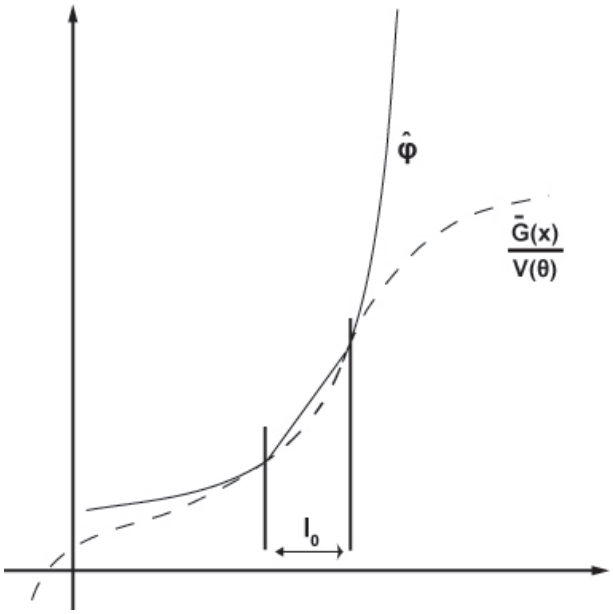

Figure 2: The dashed line represents a function $e^{g(x,\theta)+c}$ for some $\theta \in \Theta$ and some c. φ is given by the solid line on I_0^c and the dashed line on I_0. Then $\psi = \log \phi$ has a g-section over I_0 and G is convex there. $\hat{\varphi}$, given by the solid line is another eigenfunction consistent with V.

It follows that

$$v(\theta) = \begin{cases} \theta^2 & \theta \leq 1 \\ \frac{3\theta^2}{2} - \frac{1}{2} & 1 < \theta \end{cases}$$

Then $\partial^g v(1)$ is multivalued, and there are a family of diffusions $\tilde{X} \in \mathcal{X}_0$ which give the same value functions as X.

In particular we can take

$$\hat{\psi}(x) = \begin{cases} \frac{x^2}{4} & x < 2 \\ \log((e^2 - e)x + 3e - 2e^2) & 2 \leq x < 3 \\ \frac{x^2}{6} + \frac{1}{2} & 3 \leq x \end{cases}$$

Then $\hat{\psi}^g = v(\theta)$ and $\hat{\psi}$ is a log-eigenfunction.

5.6 Uniqueness of diffusions consistent with V

Proposition 5.17. *Suppose V is such that $x^*(\theta)$ is continuous on Θ, with range the positive real line.*

Then there exists at most one diffusion $X \in \mathcal{X}_0$ consistent with V.

Proof. The idea is to show that $v^g(x)$ is the only function with g-convex dual v. Suppose ψ is such that $\psi^g = v$ on Θ. For each x there is a θ with $x^*(\theta) = x$, and moreover $\partial^g v(\theta) = \{x\}$. Then by Lemma 5.6, $A(\theta) = \{x\}$ and $\psi(x) = v^g(x)$. □

Recall that we define $\theta_R = \sup_\theta \partial^g v(\theta) \neq \emptyset$ and if $\theta_R > \theta_-$ set $x_R = \lim_{\theta \uparrow \theta_R} x^*(\theta)$.

Theorem 5.18. *Suppose V is such that v is continuously differentiable on (θ_-, θ_R) and that $x_- = 0$ and $x_R = \infty$.*

Then there exists at most one diffusion $X \in \mathcal{X}_0$ consistent with V.

Proof. The condition on the range of $x^*(\theta)$ translates into the conditions on x_- and x_R, so it is sufficient to show that $x^*(\theta)$ is continuous at $\theta \in (\theta_-, \theta_R)$ if and only if v is differentiable there. This follows from Proposition 3.13. □

Corollary 5.19. *If the conditions of the Theorem hold but either $e^{v^g(x)}$ is not convex or $v^g(0) \neq 0$, then there is no diffusion $X \in \mathcal{X}_0$ which is consistent with $\{V(\theta), \theta \in \Theta\}$.*

Example 5.20. *Recall Example 2.6. For this example we have $x^*(\theta) = \theta^2 - 1$, which on $\Theta = (1, \infty)$ is continuous and strictly increasing. Then $e^{v^g(x)} = \sqrt{1+x}$ and by the above corollary there is no diffusion consistent with v.*

Remark 5.21. More general but less succinct sufficient conditions for uniqueness can be deduced from Lemma 5.7 or Lemma 5.11. For example, if $0 < x_- < x_+ = \infty$, but $(v^g)'(x_-) = (1 - e^{-v^g(x_-)})/x_-$ then there is at most one $X \in \mathcal{X}_0$ which is consistent with V.

6 Further examples and remarks

6.1 Birth-Death processes

We now return to $\mathcal{X}_0$ and consider the case when E is made up of isolated points only; whence X is a birth-death process on points $x_n \in E$ indexed by $n \in \mathbb{N}_0$, with associated exponential holding times λ_n. We assume $x_0 = 0$, x_n is increasing, and write $x_\infty = \lim_n x_n$.

For a birth-death process the transition probabilities are given by

$$\mathbb{P}_{n,n+1}(t) = p_n \lambda_n t + o(t),$$
$$\mathbb{P}_{n,n-1}(t) = q_n \lambda_n t + o(t),$$

where of course $q_n = 1 - p_n$, with $p_0 = 1$. By our assumption that, away from zero, $(X_t)_{t \geq 0}$ is a martingale, we must have $p_n = \frac{x_{n+1} - x_n}{x_{n+1} - x_{n-1}}$. Then we can write $x_n = x_{n-1} + \frac{\prod_{i=1}^{n-1} q_i}{\prod_{i=0}^{n-1} p_i}$. Let

$$m(x_n) = \frac{1}{\lambda_n} \frac{p_0 p_1 ... p_{n-1}}{q_1 q_2 ... q_n}.$$

Then it is easy to verify, but see [7], that (2.5) can be expressed in terms of a second-order difference operator

$$\frac{1}{m(x_n)} \left[\frac{f(x_{n+1}) - f(x_n)}{x_{n+1} - x_n} - \frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}} \right] - \rho f(x_n) = 0, \tag{6.1}$$

with boundary conditions $f(0) = 1$ and $f'(0-) = 0$.

Let $M(x) = \sum_{x_n < x} m(x)$. In the language of strings, the pair $(M, [0, x_\infty))$ is known as the Stieltjes String. If $x_\infty + M(x_\infty) < \infty$ the string is called regular and x_∞ is a regular boundary point, while otherwise the string is called singular, in which case we assume that x_∞ is natural (see Kac [9]).

In this section we consider the call option payoff, $G(x,\theta) = (x-\theta)^+$ defined for $\theta \in \Theta = [\theta_0, \infty)$. This objective function is straight-forward to analyse since the g-duality corresponds to straight lines in the original coordinates. It follows that for the forward problem V is decreasing and convex in θ. V is easily seen to be piecewise linear.

Our focus is on the inverse problem. Note that the solution of this problem involves finding the space E and the jump rates λ_n. Suppose that V is decreasing, convex and piecewise linear. Let $(\theta_n)_{n\in\mathbb{N}_0}$ be a sequence of increasing real valued parameters with $\theta_0 < 0$ and θ_n increasing to infinity, and suppose that V has negative slope s_i on each interval (θ_i, θ_{i+1}). Then s_i is increasing in i and

$$V(\theta) = V(\theta_n) + (\theta - \theta_n)s_n \qquad \text{for } \theta_n \le \theta < \theta_{n+1}. \tag{6.2}$$

We assume that $s_0 = \frac{\theta_0}{V(\theta_0)} < 0$.

Since V is convex, v is $\log((x-\theta)^+)$ convex. Let $\varphi(x) = \exp(v^g(x))$. By Proposition 3.11, for $\theta \in [\theta_n, \theta_{n+1})$

$$\frac{-1}{x^*(\theta) - \theta} = g_\theta(x^*(\theta), \theta) = \frac{s_n}{V(\theta_n) + (\theta - \theta_n)s_n}$$

so that $x_n := x^*(\theta_n) = \theta_n - V(\theta_n)/s_n$. Note that $x^*(\theta)$ is constant on $[\theta_n, \theta_{n+1})$. We find that for $\theta \in [\theta_n, \theta_{n+1})$

$$\psi(x^*(\theta)) = \log(\theta_n - \theta - V(\theta_n)/s_n) - v(\theta),$$

and hence $\varphi(x^*(\theta)) = \frac{-1}{s_n}$. Then, for $x \in [x^*(\theta_n), x^*(\theta_{n+1}))$,

$$\psi(x) = \log(x - \theta_n) - v(\theta_n). \tag{6.3}$$

We proceed by determining the Q-matrix for the process on $[x^*(0) = 0, \xi)$. For each n, let p_n denote the probability of jumping to state x_{n+1} and q_n the probability of jumping to x_{n-1}. Then p_n and q_n are determined by the martingale property (and $p_0 = 1$). Further λ_n is determined either through (6.1) or from a standard recurrence relation for first hitting times of birth-death processes:

$$\lambda_n = \frac{\rho\varphi(x_n)}{p_n\varphi(x_{n+1}) + (1-p_n)\varphi(x_{n-1}) - \varphi(x_n)}, \qquad n \ge 1.$$

Example 6.1. *Suppose that $\theta_n = n + 2^{-n} - 2$ so that $\theta_0 = -1$, and $V(\theta_n) = 2^{-n}$. It follows that $s_n = -(2^{n+1}-1)^{-1}$. We find $x_n = n$ (this example has been crafted to ensure that the birth-death process has the integers as state space, and this is not a general result). Also $\varphi(n) = 2^{n+1} - 1$ (φ is piecewise linear with kinks at the integers) and the holding time at x_n is exponential with rate $\lambda_n = 4\rho(1 - 2^{-(n+1)})$.*

6.2 Subsets of $\mathcal{X}$ and uniqueness

So far in this article we have concentrated on the class $\mathcal{X}_0$. However, the methods and ideas translate to other classes of diffusions.

Let $\mathcal{X}_{m,s}$ denote the set of all diffusions reflected at 0. Here m denotes the speed measure, and s the scale function. With the boundary conditions as in (2.5), $\varphi(x) \equiv \varphi_X(x)$ is the increasing, but not necessarily convex solution to

$$\frac{1}{2}\frac{d^2 f}{dmds} = \rho f. \tag{6.4}$$

In the smooth case, when m has a density $m(dx) = \nu(x)dx$ and s'' is continuous, (6.4) is equivalent to

$$\frac{1}{2}\sigma^2(x)f''(x) + \mu(x)f'(x) = \rho f(x), \tag{6.5}$$

where

$$\nu(x) = \sigma^{-2}(x)e^{M(x)}, \quad s'(x) = e^{-M(x)}, \quad M(x) = \int_{0-}^{x} 2\sigma^{-2}(z)\mu(z)dz,$$

see [3].

Now suppose $V \equiv \{V(\theta) : \theta \in \Theta\}$ is given such that $v^u(0) = 0$, $(v^u)'(0) = 0$ and v^u is increasing, then we will be able to find several pairs (σ, μ) such that $\exp(v^u)$ solves (6.5) so that there is a family of diffusions rather than a unique diffusion in $\mathcal{X}_{m,s}$ consistent with v.

It is only by considering subsets of $\mathcal{X}_{m,s}$, such as taking $s(x) = 0$ as in the majority of this article, or perhaps by setting the diffusion co-efficient equal to unity, that we can hope to find a *unique* solution to the inverse problem.

Example 6.2. *Consider Example* 2.6 *where we found* $\psi(x) = \sqrt{1+x}$. *Let* $\mathcal{X}_{1,s}$ *be the set of diffusions with unit variance and scale function* s *(which are reflected at 0). Then there exists a unique diffusion in* $\mathcal{X}_{1,s}$ *consistent with* V. *The drift is given by*

$$\mu(x) = \frac{1/4 + 2\rho(1+x)^2}{1+x}.$$

7 Applications to Finance

7.1 Applications to Finance

Let $\mathcal{X}_{stock}$ be the set of diffusions with the representation

$$dX_t = (\rho - \delta)X_t dt + \eta(X_t)X_t dW_t.$$

In finance this SDE is often used to model a stock price process, with the interpretation that ρ is the interest rate, δ is the proportional dividend, and η is the level dependent volatility. Let x_0 denote the starting level of the diffusion and suppose that 0 is an absorbing barrier.

Our aspiration is to recover the underlying model, assumed to be an element of $\mathcal{X}_{stock}$, given a set of perpetual American option prices, parameterised by θ. The canonical example is when θ is the strike, and $G(x, \theta) = (\theta - x)^+$, and then, as discussed in Section 6.2, the fundamental ideas pass over from $\mathcal{X}_0$ to $\mathcal{X}_{stock}$. We suppose ρ and δ are given and aim to recover the volatility η.

Let φ be the convex and decreasing solution to the differential equation

$$\frac{1}{2}\eta(x)^2 x^2 f_{xx} + (\rho - \delta)x f_x - \rho f = 0. \tag{7.1}$$

(The fact that we now work with decreasing φ does not invalidate the method, though it is now appropriate to use payoffs G which are monotonic decreasing in x.) Then η is determined by the Black-Scholes equation

$$\eta(x)^2 = 2\frac{\rho\varphi(x) - (\rho - \delta)x\varphi'(x)}{x^2\varphi''(x)}. \tag{7.2}$$

Let $G \equiv G(x,\theta)$ be a family of payoff functions satisfying assumption 3.6. Under the additional assumption that G is decreasing in x (for example, the put payoff) Lemma 2.1 shows that the optimal stopping problem reduces to searching over first hitting times of levels $x < x_0$. Suppose that $\{V(\theta); \theta \in \Theta\}$ is used to determine a smooth, convex $\varphi = \exp(\psi)$ on $[0,\xi)$ via the g-convex duality

$$\psi(x) = v^g(x) = \sup_{\theta\in\Theta}[g(x,\theta) - v(\theta)].$$

Then the inverse problem is solved by the diffusion with volatility given by the solution of (7.2) above. Similarly, given a diffusion $X \in \mathcal{X}_{stock}$ such that $\psi = \log(\varphi)$ is g-convex on $[0,\xi)$, then the value function for the optimal stopping problem is given exactly as in Proposition 4.4. See Ekström and Hobson [6] for more details.

Remark 7.1. The irregular case of a generalised diffusion requires the introduction of a scale function, see Ekström and Hobson [6]. That article restricts itself to the case of the put/call payoffs for diffusions $\mathcal{X}_{stock}$, using arguments from regular convex analysis to develop the duality relation. However, the construction of the scale function is independent of the form of the payoff function and is wholly transferable to the setting of g-convexity used in this paper.

7.2 The Put-Call Duality; g-dual Processes

In [2], Alfonsi and Jourdain successfully obtain what they term an American Put-Call duality (see below) and in doing so, they solve forward and inverse problems for diffusions $X \in \mathcal{X}_{stock}$. They consider objective functions corresponding to calls and puts: $G(x,\theta) = (x-\theta)^+$ and $G(x,\theta) = (\theta - x)^+$. In [1] the procedure is generalised slightly to payoff functions sharing 'global properties' with the put/call payoff. In our notation the assumptions they make are the following: [1, (1.4)]
Let $G : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous function such that on the space $\Phi = \{(x,\theta) : G(x,\theta) > 0\} \neq \emptyset$, G is C^2 and further for all $x, \theta \in \Phi$

$$G_x(x,\theta) < 0, \quad G_\theta(x,\theta) > 0, \quad G_{xx}(x,\theta) \leq 0 \quad G_{\theta\theta}(x,\theta) \leq 0.$$

Subsequently, they assume [1, (3.4)]

$$GG_{x\theta} > G_x G_\theta \quad \text{on } \Phi. \tag{7.3}$$

Condition (7.3) is precisely the Spence-Mirrlees condition applied to $g = \log(G)$. Note that unlike Alfonsi and Jourdain [1] we make no concavity assumptions on G. We also treat the case of the reverse Spence-Mirrlees condition, and we consider classes of diffusions other than $\mathcal{X}_{stock}$. Further, as in Ekström and Hobson [6] we allow for generalised diffusions. This is important if we are to construct solutions to the inverse problem when the value functions are not smooth. Moreover, even when v is smooth, if it contains a g-section, the diffusion which is consistent with v exists only in the generalised sense. When v contains a g-section we are able to address the question of uniqueness. Uniqueness is automatic under the additional monotonicity assumptions of [1].

In addition to the solution of the inverse problem, a further aim of Alfonsi and Jourdain [2] is to construct dual pairs of models such that call prices (thought of as a function of strike) under one model become put prices (thought of as a function of the value of the underlying stock) in the other. This extends a standard approach in optimal control/mathematical finance where pricing problems are solved via a Hamilton-Jacobi-Bellman equation which derives the value of

an option for all values of the underlying stock simultaneously, even though only one of those values is relevant. Conversely, on a given date, options with several strikes may be traded.

The calculations given in [2] are most impressive, especially given the complicated and implicit nature of the results (see for example, [1, Theorem 3.2]; typically the diffusion coefficients are only specified up to the solution of one or more differential equations). In contrast, our results on the inverse problem are often fully explicit. We find progress easier since the g-convex method does not require the solution of an intermediate differential equation. Furthermore, the main duality ([1, Theorem 3.1]) is nothing but the observation that the g-subdifferentials (respectively x^* and θ^*) are inverses of each other, which is immediate in the generalised convexity framework developed in this article. Coupled with the identity, $g_x(x^*(\theta), \theta) = \psi'(x^*(\theta))$ the main duality result of [2] follows after algebraic manipulations, at least in the smooth case where x^* and θ^* are strictly increasing and differentiable, which is the setting of [2] and [1].

A Proofs

A.1 The Forward and Inverse Problems

Proof of Lemma 2.1 Clearly $V \geq \hat{V}$, since the supremum over first hitting times must be less than or equal to the supremum over all stopping times.

Conversely, by (2.2), $\varphi(x) \geq \frac{G(x,\theta)}{\hat{V}(\theta)}$. Moreover, (2.5) implies that $e^{-\rho t}\varphi(X_t)$ is a non-negative local martingale and hence a supermartingale. Thus for stopping times τ we have

$$1 \geq \mathbb{E}_0[e^{-\rho\tau}\varphi(X_\tau)] \geq \mathbb{E}_0[e^{-\rho\tau}G(X_\tau, \theta)/\hat{V}(\theta)]$$

and hence $\hat{V}(\theta) \geq \sup_\tau E_0[e^{-\rho\tau}G(x_\tau, \theta)]$.

A.2 u-convexity

The methods and many of the results we use in this section are to be found in [11] and [4].

Lemma A.1. *For every function $f : D_z \to \mathbb{R}$, $(f^u)^u$ is the largest u-convex minorant of f.*

Proof. Note that $f^* \equiv -\infty$ is always a u-convex minorant and by Definition 3.1 a function defined as the supremum of a collection of u-convex functions is u-convex.

Hence we can define f^m, the largest u-convex minorant of f by

$$f^m(z) = \sup_\zeta\{\zeta(x); \zeta \leq f, \zeta \in U(D_z)\},$$

where $U(D_z)$ is the set of u-convex functions on D_z. Then by the Fenchel Inequality, $f \geq (f^u)^u$ and since $(f^u)^u$ is u-convex, it follows that $f^m \geq (f^u)^u$.

Since f^m is u-convex there exists $S' \subset D_z \times \mathbb{R}$ such that

$$f^m(y) = \sup_{(z,a)\in S'} [u(y, z) + a]$$

For fixed $(z, a) \in S'$, $l(y) = u(y, z) + a$ is u-convex and it is easy to check that $(l^u)^u(y) = l(y)$.

We note that if $f \leq \hat{f}$, then $f^u \geq \hat{f}^u$. Since $u(y, z) + a \leq f^m(y) \leq f(y)$, we therefore have $u(y, z) + a \leq (f^u)^u(y)$. Taking the supremum over pairs $(z, a) \in S'$, we get $f^m \leq (f^u)^u$. □

Proof of Lemma 3.3 If f is u-convex then f is its greatest u-convex minorant, so $f = (f^u)^u$. On the other hand if $(f^u)^u = f$ then since $(f^u)^u$ is u-convex, so is f.

Proof of Lemma 3.9 Let $y < \hat{y}$. For any $z \in \partial^u f(y)$ and any $\hat{z} \in \partial^u f(\hat{y})$, the following pair of inequalities follow from u-convexity (see especially the second part of Definition 3.4):

$$\begin{aligned} f(y) - f(\hat{y}) &\geq u(y, \hat{z}) - u(\hat{y}, \hat{z}), \\ f(\hat{y}) - f(y) &\geq u(\hat{y}, z) - u(y, z). \end{aligned}$$

Adding and re-arranging we get,

$$u(y, \hat{z}) - u(\hat{y}, \hat{z}) + u(\hat{y}, z) - u(y, z) \leq 0,$$

and hence,

$$\int_y^{\hat{y}} [u_y(v, z) - u_y(v, \hat{z})] dv \leq 0. \tag{A.1}$$

Since (by Assumption 3.6(b)) $u_y(v, w)$ is strictly increasing in the second argument it follows that $\hat{z} \geq z$.

Proof of Proposition 3.11 Suppose f is u-convex and u-subdifferentiable. Let y be a point of differentiability of f and $z \in \partial^u f(y)$. Then for $h > 0$, we have both $f(y+h) - f(y) = f'(y)h + o(h)$ and

$$f(\bar{y}) - f(y) \geq u(\bar{y}, z) - u(y, z) \qquad \forall \bar{y} \in D_y.$$

From the definition of u-convexity $f(y) = u(y, z) - f^u(z)$ and $f(y + h) \geq u(y + h, z) - f^u(z)$, so taking $\bar{y} = y + h$ and putting all this together,

$$f(y + h) - f(y) \geq u(y + h, z) - u(y, z) = u_y(y, z) + o(h).$$

Dividing by $h > 0$ on both sides and letting $h \to 0$ we get

$$f'(y) \geq u_y(y, z).$$

If instead we take $h < 0$ above then we get the reverse inequality and hence

$$f'(y) = u_y(y, z). \tag{A.2}$$

By the Spence-Mirrlees condition $u_y(y, z)$ is injective with respect to z. Hence, the last equation determines z uniquely; thus whenever f is differentiable the sub-differential $\partial^u f(y)$ is a singleton set. We set $\partial^u f(y) = \{z^*(y)\}$. Then, since $z^*(y) \in \partial^u f(y)$ we have $f(y) + f^u(z^*(y)) = u(y, z^*(y))$ as required.

To prove the converse statement, suppose that for any point of differentiability y of f,

$$f'(y) = u_y(y, z^*(y))$$

where z^* non-decreasing.

Define

$$\zeta(y) = \sup_v [f(v) + u(y, z^*(v)) - u(v, z^*(v))].$$

Then ζ is u-convex: to see this define

$$S = \{(z, a); \exists w \in D_y \mid z = z^*(w),\ a = f(w) - u(w, z^*(w))\}$$

and then

$$\zeta(y) = \sup_{(\theta,a)\in S} [u(y,\theta) + a].$$

Clearly $\zeta \geq f$. We want to show that $\zeta = f$. By the assumptions on f we have

$$f(y) = \int_v^y u_y(w, z^*(w))dw + f(v).$$

Hence

$$f(y) - (f(v) + u(y, z^*(v)) - u(v, z^*(v))) = \int_v^y [u_y(w, z^*(w)) - u_y(w, z^*(w))]dw \geq 0.$$

Thus $\zeta = f$ and so f is u-convex.

The following Corollary is immediate from 3.11.

Corollary A.2. *Suppose f is differentiable at y. Then f is twice-differentiable at y if and only if z^* is differentiable at y.*

Proof of Proposition 3.13

Suppose f is u-subdifferentiable in a neighbourhood of y. Then for small enough ϵ,

$$f(y+\epsilon) - f(y) \geq f(y+\epsilon, z^*(y)) - f(y, z^*(y))$$

and $\lim_{\epsilon\downarrow 0}\{f(y+\epsilon) - f(y)\}/\epsilon \geq f_y(y, z^*(y))$.

For the reverse inequality, if z^* is continuous at y then for ϵ small enough that $z^*(y+\epsilon) < z^*(y)+\delta$ we have

$$f(y+\epsilon) - f(y) \leq f(y+\epsilon, z^*(y+\epsilon)) - f(y, z^*(y+\epsilon)) \leq f(y+\epsilon, z^*(y)+\delta) - f(y, z^*(y)+\delta)$$

and $\lim_{\epsilon\downarrow 0}\{f(y+\epsilon) - f(y)\}/\epsilon \leq \lim_{\delta\downarrow 0} f_y(y, z^*(y)+\delta) = f_y(y, z^*(y))$.

Inequalities for the left-derivative follow similarly, and then $f'(y) = u_y(y, z^*(y))$ which is continuous.

Conversely, if $\partial^u f$ is multi-valued at y so that z^* is discontinuous at y, then

$$\lim_{\epsilon\downarrow 0}\{f(y+\epsilon) - f(y)\}/\epsilon \geq f_y(y, z^*(y)+) > f_y(y, z^*(y)-) \geq \lim_{\epsilon\downarrow 0}\{f(y) - f(y-\epsilon)\}/\epsilon$$

where the strict middle inequality follows immediately from Assumption 3.6.

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