Generalized Jacobian Analysis of Lower Mobility Manipulators

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Abstract
Mainly drawing on screw theory and linear algebra, this paper presents a general approach for Jacobian analysis of lower mobility manipulators. Given the definitions of twist/wrench spaces and their subspaces of the end-effector, the underlying relationships amongst these subspaces are identified using the virtual work principle. Using the orthogonal and dual properties of these subspaces and variational representation to account for the permitted and restricted instantaneous motions of the end-effector, a general and systematic procedure for the formulation of a generalized Jacobian is proposed. The merit of the generalized Jacobian allows the first order kinematic and static modeling (velocity, accuracy, force and stiffness) to be integrated into a unified mathematical framework. The generalized Jacobians for three well-known parallel manipulators are derived as examples to illustrate generality and effectiveness of this approach.
Keywords*: Jacobian analysis, Lower mobility Manipulators*

1. Introduction

Lower mobility manipulators having fewer than six degrees of freedom (DOF) have drawn continuous interest in both industry and academia for many years. Velocity, accuracy and stiffness are three important performance factors that should essentially be considered in design of the lower mobility manipulators, particularly in the many circumstances where high speed, high precision and high rigidity are basic requirements.

Mathematically, the common manipulation required for velocity, accuracy and stiffness modeling is to formulate a specific linear map between two vector spaces at a given configuration. For example, velocity modeling involves the linear map between actuator rates and velocity twist of the end-effector. The matrix form of this map is known as the *velocity Jacobian* or Jacobian, for simplicity [1-3]. For accuracy modeling the required linear map is between source errors (inclusive of geometric errors of the components and movement errors of the actuators, for instance) and pose accuracy of the end-effector. In stiffness modeling it becomes the linear map between the deformation twist and the externally-applied wrench imposed on the end-effector. This common feature makes it possible to formulate all these models under a unified mathematical framework.

In the last few decades, intensive efforts have been made toward Jacobian analysis, and several useful approaches are now at hand. The most straightforward method is to differentiate a set of vector-based constraint equations. Although Jacobian analysis in an implicit form can easily be carried out for general discussion purposes [4], it is by no means an easy task to achieve general and explicit expressions, as is shown by the case-by-case studies [5-21] though many others may not be included. The screw-based method [1, 22-29] is more powerful and easily gives deep insight into the constraints imposed upon of a robotic system thanks to the compact and meaningful representation. The initial work along this track can be traced back to the original contribution made by Hunt [22]: expressing velocity twist of the end-effector as a linear combination of joint screws leads easily to the Jacobian of 6-DOF parallel manipulators [1, 23-24, 28]. By exploiting the reciprocal properties of screw systems within a single open loop kinematic chain, the idea was then extended by Joshi and Tsai to develop a general and systemic approach for Jacobian analysis of non-overconstrained lower mobility parallel manipulators, resulting in a brand new 6×6 Jacobian known as the *overall Jacobian*, which accounts for both the actuation and constraint wrenches imposed upon the platform [27]. Use of the *overall Jacobian* can well explain the constraint singularity problem of a lower mobility parallel manipulator. More recently, the screw-based method has been employed to deal with the interconnected kinematic chains with serial and parallel architectures [30], and kinestatic and singularity analysis of lower mobility manipulators [31-32]. In addition, having an intrinsic nature similar to the screw-based method, the kinematic influence coefficient method [33-34] is also be worthy of consideration.

Despite the undoubted merit of the overall Jacobian and kinematic influence coefficient method, they are limited to velocity analysis because the instantaneous motions accessible to the end-effector are merely considered. Therefore, driven by considerable practical benefits to manipulator designers, there is a need to revisit the existing approaches and to develop a theoretical package for Jacobian analysis that enables the first order kinematic and static (i.e. velocity, accuracy, force and stiffness) modeling to be integrated in a consistent and systematic manner.

In order to achieve the goals mentioned above, we propose a general approach for Jacobian analysis of lower mobility

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manipulators by taking into account the instantaneous motions that are inaccessible in theory but accessible in practice to the end-effector due to the inevitable errors resulting from manufacture and assembly processes. Having outlined in Section 1 the significance of Jacobian analysis and its existing problems, the paper is organized as follows. In Section 2, with a brief review of relevant theory, the twist/wrench spaces and their subspaces of an *f*-DOF rigid body are defined. The virtual work principle is used to identify a set of underlying relationships amongst these subspaces. This is followed in Section 3 with a scheme to develop the generalized Jacobian of serial manipulators. In Section 4, this model is then extended to cover parallel manipulators by considering the loop closure constraints. In Section 5, the potential to use the generalized Jacobian to formulate velocity, accuracy and stiffness models of serial and parallel manipulators is discussed. Finally, three practical illustrations are presented in Section 6 and conclusions are drawn in Section 7.

2. Twist Space, Wrench Space, and Their Subspaces of a Constrained Rigid Body

Mainly drawing on linear algebra and screw theory, this section defines the twist/wrench spaces and their subspaces for a constrained rigid body at a specified configuration in 3D space and identifies them using the virtual work principle.

In order to enable velocity, accuracy and stiffness modeling to be formulated in a consistent manner, it is necessary to consider the realistic instantaneous motions (twists) of an *f*-DOF ($f<6$) body at a given configuration. On the one hand, it is well established that the entire set of the instantaneous motions theoretically accessible to the body forms an *f*-dimensional vector space, in which the motions belonging to the specific motion type and number of the body only occur. Whilst the entire set of instantaneous motions theoretically inaccessible to the body forms a 6-*f* dimensional vector space, in which the unexpected deviations not belonging to the specific motion type and number of the body may also occur due to the inevitable geometric errors in component manufacture and assembly, component compliances, *etc*. As a result, the entire set of realistic instantaneous motions of the body must form a six dimensional vector space known as the *twist space* $\mathcal{T} = \{\$_t \mid \$_t \in \mathbb{R}^6\}$ with $\$_t$ being its element, which is composed of a pair of complementary subspaces defined as follows.

Definition 1---The twist subspace of permissions $\mathcal{T}_a \subseteq \mathcal{T}$: An *f dimensional subspace of* $\mathcal{T}$ spanned by the twists permitted by the system constraints with $\$_{ta}$ being its element.

Definition 2---The twist subspace of restrictions $\mathcal{T}_c \subseteq \mathcal{T}$: A $6-f$ *dimensional subspace of* $\mathcal{T}$ spanned by the twists restricted by the system constraints with $\$_{tc}$ being its element, satisfying $\mathcal{T}_a \cap \mathcal{T}_c = \varnothing$.

Note that every element $\$_t \in \mathcal{T}$ can be written as a sum of the form $\$_t = \$_{ta} + \$_{tc}$ for any $\$_{ta} \in \mathcal{T}_a$ and $\$_{tc} \in \mathcal{T}_c$, and $\mathcal{T}_a \cap \mathcal{T}_c = \varnothing$. $\mathcal{T}$ is the direct sum of $\mathcal{T}_a$ and $\mathcal{T}_c$, i.e. $\mathcal{T} = \mathcal{T}_a \oplus \mathcal{T}_c$.

On the other hand, the external wrench applied onto the body must be equilibrated by the sum of the wrenches generated by *f* independent actuators and the wrenches produced by the $6-f$ independent joint constraints. Thus, the entire set of external wrench must form a six dimensional vector space known as the *wrench space* $\mathcal{W} = \{\$_w \mid \$_w \in \mathbb{R}^6\}$ with $\$_w$ being its element, which is also composed of a pair of complementary subspaces defined as follows.

Definition 3---The wrench subspace of actuations $\mathcal{W}_a \subseteq \mathcal{W}$: An *f dimensional subspace of* $\mathcal{W}$ spanned by the wrenches of actuations generated by *f* independent actuation forces and/or couples.

Definition 4---The wrench subspace of constraints $\mathcal{W}_c \subseteq \mathcal{W}$: A $6-f$ *dimensional subspace of* $\mathcal{W}$ spanned by the wrenches of constraints generated by $6-f$ independent reaction forces and/or couples in joints, satisfying $\mathcal{W}_a \cap \mathcal{W}_c = \varnothing$.

Similarly, since every element $\$_w \in \mathcal{W}_a$ can be written as a sum of the form $\$_w = \$_{wa} + \$_{wc}$ with $\$_{wa} \in \mathcal{W}_a$, $\$_{wc} \in \mathcal{W}_c$, and $\mathcal{W}_a \cap \mathcal{W}_c = \varnothing$, $\mathcal{W}$ is the direct sum of $\mathcal{W}_a$ and $\mathcal{W}_c$, i.e. $\mathcal{W} = \mathcal{W}_a \oplus \mathcal{W}_c$.

Theorem: $\mathcal{W}_a$ ($\mathcal{W}_c$) is the orthogonal complement of $\mathcal{T}_c$ ($\mathcal{T}_a$), and $\mathcal{T}_a$ ($\mathcal{T}_c$) is the dual of $\mathcal{W}_a$ ($\mathcal{W}_c$).

Proof: For any $\$_w \in \mathcal{W}$ and any $\$_t \in \mathcal{T}$, since $\mathcal{W} = \mathcal{W}_a \oplus \mathcal{W}_c$ and $\mathcal{T} = \mathcal{T}_a \oplus \mathcal{T}_c$, $\$_w = \$_{wa} + \$_{wc}$ and $\$_t = \$_{ta} + \$_{tc}$ are unique for $\$_{wa} \in \mathcal{W}_a$, $\$_{wc} \in \mathcal{W}_c$, $\$_{ta} \in \mathcal{T}_a$ and $\$_{tc} \in \mathcal{T}_c$. Expressing $\$_w$ ($\$_t$) in the form of axis-coordinate (ray-coordinate), the virtual work done by $\$_w$ on $\$_t$ can be repressed by the inner product as follows

$$\delta W = \langle \$_w, \$_t \rangle = \langle \$_{wa} + \$_{wc},\ \$_{ta} + \$_{tc} \rangle = \langle \$_{wa}, \$_{ta} \rangle + \langle \$_{wc}, \$_{tc} \rangle + \langle \$_{wa}, \$_{tc} \rangle + \langle \$_{wc}, \$_{ta} \rangle \tag{1}$$

Since any $\$'_t \in \mathcal{T}$ produced by $\$_{wa}$ ($\$_{wc}$) must be the twist of permissions (restrictions), i.e. $\$'_t \in \mathcal{T}_a$ ($\$'_t \in \mathcal{T}_c$), then $\$'_t \notin \mathcal{T}_c$ ($\$'_t \notin \mathcal{T}_a$) because $\mathcal{T}_a \cap \mathcal{T}_c = \varnothing$. Physically, this means that $\$_{wa} \in \mathcal{W}_a$ ($\$_{wc} \in \mathcal{W}_c$) does not do work on $\$_{tc} \in \mathcal{T}_c$ ($\$_{ta} \in \mathcal{T}_a$) and thus leads to $\langle \$_{wa}, \$_{tc} \rangle = 0$ ($\langle \$_{wc}, \$_{ta} \rangle = 0$). Also, note that $\$_{wa}$ ($\$_{wc}$) and $\$_{tc}$ ($\$_{ta}$) are arbitrary, so $\mathcal{W}_a$ ($\mathcal{W}_c$) is the orthogonal complement of $\mathcal{T}_c$ ($\mathcal{T}_a$), i.e. $\mathcal{W}_a = \mathcal{T}_c^{\perp}$ ($\mathcal{W}_c = \mathcal{T}_a^{\perp}$). This also means that at a singularity free configuration $\$_{wa} \in \mathcal{W}_a$ ($\$_{wc} \in \mathcal{W}_c$) does work on $\$_{ta} \in \mathcal{T}_a$ ($\$_{tc} \in \mathcal{T}_c$) and thus leads to $\langle \$_{wa}, \$_{ta} \rangle = \delta_a \in \mathbb{R}$ and $\langle \$_{wc}, \$_{tc} \rangle = \delta_c \in \mathbb{R}$. Therefore, $\mathcal{T}_a$ ($\mathcal{T}_c$) and $\mathcal{W}_a$ ($\mathcal{W}_c$) are a pair of dual subspaces [35], i.e. $\mathcal{W}_a = \mathcal{T}_a^{*}$ ($\mathcal{W}_c = \mathcal{T}_c^{*}$).

Fig.1 depicts the underlying relationships amongst the twist/wrench subspaces, which are invariant with respect to a change of coordinate system. It should be pointed out that from a projective geometry viewpoint and via a simple example Lipkin and Duffy attempted to explore these relationships when they investigated into the elliptic polarity of screws [36] and the hybrid control of twists and wrenched of a constrained rigid body [37]. Unfortunately, except for $\mathcal{W}_a = \mathcal{T}_c^{\perp}$ and $\mathcal{W}_c = \mathcal{T}_a^{\perp}$, the rest relationships seem incorrect or at least ambiguous. In fact, the result that $\mathcal{T}_c(\mathcal{W}_a)$ was an orthogonal complement of $\mathcal{T}_a(\mathcal{W}_c)$ does not hold, and that $\mathcal{T}_c(\mathcal{W}_a)$ and $\mathcal{W}_c(\mathcal{T}_a)$ were elliptic polars is meaningless in mechanics. In addition, the relationships $\mathcal{W}_a = \mathcal{T}_c^{\perp}$ and $\mathcal{W}_a = \mathcal{T}_a^{*}$ (drawn by solid lines in Fig.1) were proposed by other means in the existing literature [27] as only $\mathcal{T}_a$ was taken into account for ideal instantaneous motions; while, the relationships $\mathcal{W}_c = \mathcal{T}_a^{\perp}$ and $\mathcal{W}_c = \mathcal{T}_c^{*}$ (drawn by dashed lines in Fig. 1) are proposed for the first time, reflecting the other side of a coin. Therefore, we can conclude that exploring the correct and complete relationships amongst the twist/wrench subspaces shown in Fig.1 embody the major contribution of this article.

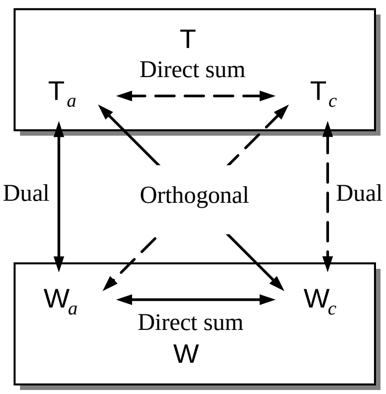

Fig.1 Relationships between the twist and wrench spaces, and their subspaces

In what follows the relationships amongst these subspaces will be used to develop a general and systematic approach for Jacobian analysis of the manipulators with serial or parallel architectures.

3. Generalized Jacobian of Serial Manipulators

3.1 Determination of the Basis Elements of Four Subspaces

Consider an $f < 6$ DOF serial manipulator. Let

$$\hat{\$}_{ta,1}, \cdots, \hat{\$}_{ta,f} \quad \text{--- the basis of } \mathcal{T}_a$$

$$\hat{\$}_{wa,1}, \cdots, \hat{\$}_{wa,f} \quad \text{--- the basis of } \mathcal{W}_a$$

$$\hat{\$}_{tc,1}, \cdots, \hat{\$}_{tc,6-f} \quad \text{--- the basis of } \mathcal{T}_c$$

$$\hat{\$}_{wc,1}, \cdots, \hat{\$}_{wc,6-f} \quad \text{--- the basis of } \mathcal{W}_c$$

be the bases of four vector subspaces of the end-effector or the "body" mentioned above. Here, $\hat{\$}$ represents a unit twist (wrench) in the form of axis-coordinate (ray-coordinate)[22].

A natural choice for a serial manipulator is to use the unit screws (i.e. the unit screws of permissions) of the actuated joints as the basis elements of $\mathcal{T}_a$ because the motion generators operate on these joints. As shown below, the process for obtaining the basis elements of $\mathcal{W}_c$ and $\mathcal{W}_a$ can be converted to that of determining 6-*f* orthogonal screws and *f* dual screws of an *f*-screw system; while the process for obtaining the basis elements of $\mathcal{T}_c$ can be converted to that of determining 6-*f* screws of an *f*-orthogonal screw system and a 6-*f* dual screw system. Several numerical approaches have been made available for determining the basis elements of $\mathcal{W}_c$, the Gram Schmidt algorithm [23] and augmentation matrix approach [38] for instance. Nevertheless, it is preferable to use the observation method [1] so that the explicit expressions of these basis elements can be obtained.

The process for obtaining the basis elements of $\mathcal{W}_c$, $\mathcal{W}_a$ and $\mathcal{T}_c$ can sequentially be carried out as follows.

Firstly, the property $\mathcal{W}_c = \mathcal{T}_a^{\perp}$ leads to

$$\left\langle \hat{\$}_{wc,k_c}, \hat{\$}_{ta,j_a} \right\rangle = \hat{\$}^{\mathrm{T}}_{wc,k_c} \hat{\$}_{ta,j_a} = 0 , \quad j_a = 1, 2, \cdots, f$$

$$k_c = 1, 2, \cdots, 6-f \tag{2}$$

Thus, the basis element $\hat{\$}_{wc,k_c}$ that is orthogonal to all $\hat{\$}_{ta,j_a}$ can be found by the observation method [1]. The physical meaning of $\hat{\$}_{wc,k_c}$ can be understood as the k_cth unit wrench of constraints imposed by the joints on the end-effector.

Secondly, the property $\mathcal{W}_a = \mathcal{T}_a^{*}$ gives

$$\left\langle \hat{\$}_{wa,k_a}, \hat{\$}_{ta,j_a} \right\rangle = \hat{\$}^{\mathrm{T}}_{wa,k_a} \hat{\$}_{ta,j_a} = \begin{cases} \delta_{k_a} & j_a = k_a \\ 0 & j_a \neq k_a \end{cases} ,$$

$$k_a, j_a = 1, 2, \cdots, f \tag{3}$$

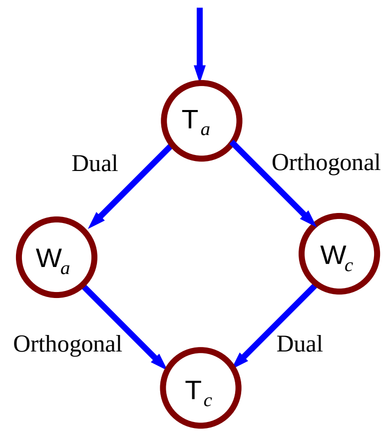

Fig.2 The routine to determine the basis elements of $\mathcal{T}_a$, $\mathcal{W}_c$, $\mathcal{W}_a$ and $\mathcal{T}_c$ by the observation method

where δ_{k_a} is a non-zero real number. In order to determine the basis element $\hat{\$}_{wa,k_a} \in \mathcal{W}_a$, let the k_ath actuated joint be locked for the time being. A unit wrench $\hat{\$}_{w,k_a} \in \mathcal{W}$ that is orthogonal to all $\hat{\$}_{ta,j_a}$ except for $\hat{\$}_{ta,k_a}$ can then be found by the observation method [1]. Although this process can be visualized as the addition of one dimension to $\mathcal{W}_c$, $\hat{\$}_{w,k_a} \notin \mathcal{W}_c$ because it does work on $\hat{\$}_{ta,k_a}$. As a result, $\hat{\$}_{w,k_a} = \hat{\$}_{wa,k_a} \in \mathcal{W}_a$ because $\mathcal{W} = \mathcal{W}_a \oplus \mathcal{W}_c$. Thus, the physical meaning of $\hat{\$}_{wa,k_a}$ can be explained as the unit wrench of actuations imposed by the k_ath actuated joint on the end-effector. The above processes have been well addressed in [27].

Thirdly, the properties $\mathcal{W}_a = \mathcal{T}_c^{\perp}$ and $\mathcal{W}_c = \mathcal{T}_c^{*}$ yields

$$\left\langle \hat{\$}_{wa,k_a}, \hat{\$}_{tc,j_c} \right\rangle = \hat{\$}^{\mathrm{T}}_{wa,k_a} \hat{\$}_{tc,j_c} = 0 , \; k_a = 1, 2, \cdots, f , \quad j_c = 1, 2, \cdots, 6-f \tag{4}$$

and

$$\left\langle \hat{\$}_{wc,k_c}, \hat{\$}_{tc,j_c} \right\rangle = \hat{\$}^{\mathrm{T}}_{wc,k_c} \hat{\$}_{tc,j_c} = \begin{cases} \delta_{k_c} & j_c = k_c \\ 0 & j_c \neq k_c \end{cases} , k_c, j_c = 1, 2, \cdots, 6-f \tag{5}$$

where δ_{k_c} is a non-zero real number. In order to determine the basis element $\hat{\$}_{tc,j_c} \in \mathcal{T}_c$, let the k_cth unit wrench of constraints, $\hat{\$}_{wc,k_c}$, be released for the time being. A unit screw $\hat{\$}_{t,k_c} \in \mathcal{T}$ that is orthogonal to $\hat{\$}_{wa,k_a}$ and to all unit wrenches of constraints except for $\hat{\$}_{wc,k_c}$ can also be found by the observation method. Although this process can be visualized as the addition of one dimension to $\mathcal{T}_a$, $\hat{\$}_{t,k_c} \notin \mathcal{T}_a$ because $\hat{\$}_{wc,k_c}$ does work on it. Thus, $\hat{\$}_{t,k_c} = \hat{\$}_{tc,k_c} \in \mathcal{T}_c$ because $\mathcal{T} = \mathcal{T}_a \oplus \mathcal{T}_c$ .The physical meaning of $\hat{\$}_{tc,k_c}$ can be explained as the k_cth unit screw of restrictions, produced by releasing the constraints imposed by the k_cth unit wrench of constraints. The process to obtain the basis elements of $\mathcal{T}_c$ is proposed in this article for the first time. Consequently, the routine to determine the basis elements of $\mathcal{W}_c$, $\mathcal{W}_a$ and $\mathcal{T}_c$ can be depicted in Fig.2, assuming that the basis elements of $\mathcal{T}_a$ are readily available.

3.2 The Generalized Jacobian

We now use the properties of four subspaces to develop an effective method for formulating the linear map that relates the twist of the end-effector to the joint intensities of serial manipulators.

As $\mathcal{T} = \mathcal{T}_a \oplus \mathcal{T}_c$, the basis of $\mathcal{T}$ can simply be spanned by that of $\mathcal{T}_a$ and $\mathcal{T}_c$, i.e. $\hat{\$}_{ta,1}, \cdots, \hat{\$}_{ta,f}, \hat{\$}_{tc,1}, \cdots, \hat{\$}_{tc,6-f}$.

Hence, the twist $\$_t$ of the end-effector can be expressed by the linear combinations of the basis elements of T

$$\$_t = \$_{ta} + \$_{tc} = \sum_{j_a=1}^{f} \delta\rho_{a,j_a} \hat{\$}_{ta,j_a} + \sum_{j_c=1}^{6-f} \delta\rho_{c,j_c} \hat{\$}_{tc,j_c} \tag{6}$$

where $\$_{ta} \in \mathsf{T}_a$ ($\$_{tc} \in \mathsf{T}_c$) is the twist of permissions (restrictions) and $\delta\rho_{a,j_a}$ ($\delta\rho_{c,j_c}$) is the coefficient or intensity of $\hat{\$}_{ta,j_a}$ ($\hat{\$}_{tc,j_c}$). Note that we here use variational symbol "δ" to encompass a broader sense in terms of joint rate, error and deflection.

Taking the inner product on both sides of Eq.(6) with $\hat{\$}_{wa,k_a}$ ($k_a = 1, 2, \cdots, f$) and $\hat{\$}_{wc,k_c}$ ($k_c = 1, 2, \cdots, 6-f$), and utilizing the properties given in Eqs.(2–5), leads to

$$\hat{\$}^{\mathrm{T}}_{wa,k_a} \$_t = \delta\rho_{a,k_a} \hat{\$}^{\mathrm{T}}_{wa,k_a} \hat{\$}_{ta,k_a}, \quad k_a = 1, 2, \cdots, f \tag{7}$$

$$\hat{\$}^{\mathrm{T}}_{wc,k_c} \$_t = \delta\rho_{c,k_c} \hat{\$}^{\mathrm{T}}_{wc,k_c} \hat{\$}_{tc,k_c}, \quad k_c = 1, 2, \cdots, 6-f \tag{8}$$

Rewriting Eqs.(7) and (8) in matrix form results in

$$\boldsymbol{J}_x \$_t = \boldsymbol{J}_\rho \delta\boldsymbol{\rho} \tag{9}$$

$$\boldsymbol{J}_x = \begin{bmatrix} \boldsymbol{J}_{xa} \\ \boldsymbol{J}_{xc} \end{bmatrix}, \quad \boldsymbol{J}_\rho = \begin{bmatrix} \boldsymbol{J}_{\rho a} & \\ & \boldsymbol{J}_{\rho c} \end{bmatrix}, \quad \delta\boldsymbol{\rho} = \begin{pmatrix} \delta\boldsymbol{\rho}_a \\ \delta\boldsymbol{\rho}_c \end{pmatrix}, \quad \delta\boldsymbol{\rho}_a = \begin{pmatrix} \delta\rho_{a,1} \\ \delta\rho_{a,2} \\ \cdots \\ \delta\rho_{a,f} \end{pmatrix}, \quad \delta\boldsymbol{\rho}_c = \begin{pmatrix} \delta\rho_{c,1} \\ \delta\rho_{c,2} \\ \vdots \\ \delta\rho_{c,6-f} \end{pmatrix}$$

$$\boldsymbol{J}_{xa} = \begin{bmatrix} \hat{\$}^{\mathrm{T}}_{wa,1} \\ \hat{\$}^{\mathrm{T}}_{wa,2} \\ \cdots \\ \hat{\$}^{\mathrm{T}}_{wa,f} \end{bmatrix}, \quad \boldsymbol{J}_{\rho a} = \begin{bmatrix} \hat{\$}^{\mathrm{T}}_{wa,1}\hat{\$}_{ta,1} & \cdots & \cdots & \cdots \\ \cdots & \hat{\$}^{\mathrm{T}}_{wa,2}\hat{\$}_{ta,2} & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots \\ \cdots & \cdots & \cdots & \hat{\$}^{\mathrm{T}}_{wa,f}\hat{\$}_{ta,f} \end{bmatrix}$$

$$\boldsymbol{J}_{xc} = \begin{bmatrix} \hat{\$}^{\mathrm{T}}_{wc,1} \\ \hat{\$}^{\mathrm{T}}_{wc,2} \\ \vdots \\ \hat{\$}^{\mathrm{T}}_{wc,6-f} \end{bmatrix}, \quad \boldsymbol{J}_{\rho c} = \begin{bmatrix} \hat{\$}^{\mathrm{T}}_{wc,1}\hat{\$}_{tc,1} & \cdots & \cdots & \cdots \\ \cdots & \hat{\$}^{\mathrm{T}}_{wc,2}\hat{\$}_{tc,2} & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots \\ \cdots & \cdots & \cdots & \hat{\$}^{\mathrm{T}}_{wc,6-f}\hat{\$}_{tc,6-f} \end{bmatrix}$$

where $\boldsymbol{J}_{xa}$ is an $f \times 6$ matrix and $\boldsymbol{J}_{\rho a}$ is an $f \times f$ diagonal matrix; $\boldsymbol{J}_{xc}$ is a $6-f \times 6$ matrix and $\boldsymbol{J}_{\rho c}$ is a $6-f \times 6-f$ diagonal matrix.

If $\boldsymbol{J}_x$ is nonsingular, we have

$$\$_t = \boldsymbol{J}_x^{-1} \boldsymbol{J}_\rho \delta\boldsymbol{\rho} = \boldsymbol{J} \delta\boldsymbol{\rho} \tag{10}$$

where $\boldsymbol{J}$ is known as the *generalized Jacobian* of serial manipulators with $f < 6$ DOF. The term "*generalized*" means that it has a broader sense than the *overall Jacobian*, which did not take the unit screws of restrictions into account.

Note that with this definition an explicit form for $\boldsymbol{J}$ can be obtained directly from Eq.(6)

$$\boldsymbol{J} = \begin{bmatrix} \boldsymbol{J}_a & \boldsymbol{J}_c \end{bmatrix}, \quad \boldsymbol{J}_a = \begin{bmatrix} \hat{\$}_{ta,1} & \cdots & \hat{\$}_{ta,f} \end{bmatrix}, \boldsymbol{J}_c = \begin{bmatrix} \hat{\$}_{tc,1} & \cdots & \hat{\$}_{tc,6-f} \end{bmatrix} \tag{11}$$

Also, an explicit form of $\boldsymbol{J}^{-1}$ can easily be obtained by comparison of Eq. (11) with Eq.(10):

$$J^{-1} = J_\rho^{-1} J_x = \begin{bmatrix} \hat{\$}_{wa,1}^{\mathrm{T}} \big/ \hat{\$}_{wa,1}^{\mathrm{T}} \hat{\$}_{ta,1} \\ \vdots \\ \hat{\$}_{wa,f}^{\mathrm{T}} \big/ \hat{\$}_{wa,f}^{\mathrm{T}} \hat{\$}_{ta,f} \\ \hat{\$}_{wc,1}^{\mathrm{T}} \big/ \hat{\$}_{wc,1}^{\mathrm{T}} \hat{\$}_{tc,1} \\ \vdots \\ \hat{\$}_{wc,6-f}^{\mathrm{T}} \big/ \hat{\$}_{wc,6-f}^{\mathrm{T}} \hat{\$}_{tc,6-f} \end{bmatrix} \tag{12}$$

It is easy to see that J^{-1} contains full information of the basis elements of all four subspaces.

4. Generalized Jacobian of Parallel Manipulators

Building upon the work in Section 3, the *generalized Jacobian* of parallel manipulators can be formulated with little effort. Let us consider an *f*-DOF ($2 \le f < 6$) parallel manipulator composed of l ($f \le l \le f+1$) limbs connecting the platform (end-effector) with the base. We model each limb by assuming it contains n_i ($i = 1, 2, \cdots, l$) 1-DOF joints with at most one of them actuated. Without losing generality, two families of parallel manipulators are considered. The first family covers fully parallel mechanisms with *f* constrained active limbs, i.e. $n_i < 6$ for all limbs; the Delta robot [39] and Sprint Z3 head [40] are examples. The second family contains those with *f* unconstrained active limbs (i.e. $n_i = 6$ for each of these *f* limbs) plus one properly constrained passive limb. Here, the term "properly constrained" means that the number and types of degrees of freedom of the limb are identical to those of the platform. The 3-DOF parallel module within the Tricept robot [41] is an example of this family. For convenience, the properly constrained passive limb is always designated by $l = f+1$. Any parallel manipulator not belonging to these two families can be dealt with in a manner similar to that used below.

For a parallel manipulator, the twist, $\$_t \in \mathrm{T}$, of the platform can be expressed as a linear combination of the basis elements of T_i ($i = 1, 2, \cdots, l$) because all limbs share the same platform, i.e.

$$\$_t = \$_{ta} + \$_{tc} = \$_{ta,i} + \$_{tc,i} = \sum_{j_a=1}^{n_i} \delta\rho_{a,j_a,i} \hat{\$}_{ta,j_a,i} + \sum_{j_c=1}^{6-n_i} \delta\rho_{c,j_c,i} \hat{\$}_{tc,j_c,i}, \; i = 1, 2, \cdots, l \tag{13}$$

where $\hat{\$}_{ta,j_a,i} \in \mathrm{T}_{a,i}$ and $\delta\rho_{a,j_a,i}$ ($\hat{\$}_{tc,j_c,i} \in \mathrm{T}_{c,i}$ and $\delta\rho_{c,j_c,i}$) are the j_ath (j_cth) unit screw of permissions (restrictions) and its intensity within the *i*th limb.

Let $\hat{\$}_{wa,g_k,k}$ be the unit wrench of actuations associated with the actuated joint, labeled g_k, in the *k*th ($k = 1, 2, \cdots, f$) limb. Note that $\hat{\$}_{wa,g_k,k}$ is dual to $\hat{\$}_{ta,g_k,k}$ but orthogonal to $\hat{\$}_{ta,j_a,k}$ ($j_a = 1, 2, \cdots, n_k$, $j_a \ne g_k$) and $\hat{\$}_{tc,j_c,k}$ ($j_c = 1, 2, \ldots, 6-n_k$). Similarly, let $\hat{\$}_{wc,k_c,i}$ be the k_cth unit wrench of constraints in the *i*th ($i = 1, 2, \cdots, l$) limb. Also, note that $\hat{\$}_{wc,k_c,i}$ is orthogonal to $\hat{\$}_{ta,j_a,i}$ ($j_a = 1, 2, \ldots, n_i$) and $\hat{\$}_{tc,j_c,i}$ ($j_c = 1, 2, \cdots, 6-n_i$, $j_c \ne k_c$) and dual to $\hat{\$}_{tc,k_c,i}$. Thus, taking the inner product on both sides of Eq. (13) with $\hat{\$}_{wa,g_k,k}$ and $\hat{\$}_{wc,k_c,i}$, respectively, leads to

$$\hat{\$}_{wa,g_k,k}^{\mathrm{T}} \$_t = \delta\rho_{a,g_k,k} \hat{\$}_{wa,g_k,k}^{\mathrm{T}} \hat{\$}_{ta,g_k,k}, \quad k = 1, 2, \cdots, f \tag{14}$$

$$\hat{\$}_{wc,k_c,i}^{\mathrm{T}} \$_t = \delta\rho_{c,k_c,i} \hat{\$}_{wc,k_c,i}^{\mathrm{T}} \hat{\$}_{tc,k_c,i}, \quad k_c = 1, 2, \cdots, 6-n_i, \quad i = 1, 2, \cdots, l \tag{15}$$

Rewriting Eqs.(14) and (15) in matrix form results in

$$J_x \$_t = J_\rho \delta\boldsymbol{\rho} \tag{16}$$

$$J_x = \begin{bmatrix} J_{xa} \\ J_{xc} \end{bmatrix}, \quad J_\rho = \begin{bmatrix} J_{\rho a} & \\ & J_{\rho c} \end{bmatrix}, \quad \delta\boldsymbol{\rho} = \begin{pmatrix} \delta\boldsymbol{\rho}_a \\ \delta\boldsymbol{\rho}_c \end{pmatrix}$$

$$\boldsymbol{J}_{xa}=\begin{bmatrix}\hat{\$}^{\mathrm{T}}_{wa,g_1,1}\\ \hat{\$}^{\mathrm{T}}_{wa,g_2,2}\\ \vdots\\ \hat{\$}^{\mathrm{T}}_{wa,g_f,f}\end{bmatrix},\quad \delta\boldsymbol{\rho}_a=\begin{pmatrix}\delta\rho_{a,g_1,1}\\ \delta\rho_{a,g_2,2}\\ \vdots\\ \delta\rho_{a,g_f,f}\end{pmatrix}$$

$$\boldsymbol{J}_{\rho a}=\begin{bmatrix}\hat{\$}^{\mathrm{T}}_{wa,g_1,1}\hat{\$}_{ta,g_1,1} & \cdots & \cdots & \cdots\\ \cdots & \hat{\$}^{\mathrm{T}}_{wa,g_2,2}\hat{\$}_{ta,g_2,2} & \cdots & \cdots\\ \vdots & \vdots & \vdots & \vdots\\ \cdots & \cdots & \cdots & \hat{\$}^{\mathrm{T}}_{wa,g_f,f}\hat{\$}_{ta,g_f,f}\end{bmatrix},\ \delta\boldsymbol{\rho}_c=\begin{pmatrix}\delta\boldsymbol{\rho}_{c,1}\\ \delta\boldsymbol{\rho}_{c,2}\\ \vdots\\ \delta\boldsymbol{\rho}_{c,l}\end{pmatrix},\quad \delta\boldsymbol{\rho}_{c,i}=\begin{pmatrix}\delta\rho_{c,1,i}\\ \delta\rho_{c.2,i}\\ \vdots\\ \delta\rho_{c,6-n_i,i}\end{pmatrix}$$

$$\boldsymbol{J}_{xc}=\begin{bmatrix}\boldsymbol{J}_{xc,1}\\ \boldsymbol{J}_{xc,2}\\ \vdots\\ \boldsymbol{J}_{xc,l}\end{bmatrix},\quad \boldsymbol{J}_{xc,i}=\begin{bmatrix}\hat{\$}^{\mathrm{T}}_{wc,1,i}\\ \hat{\$}^{\mathrm{T}}_{wc,2,i}\\ \vdots\\ \hat{\$}^{\mathrm{T}}_{wc,6-n_i,i}\end{bmatrix},\ \boldsymbol{J}_{\rho c}=\begin{bmatrix}\boldsymbol{J}_{\rho c,1} & \cdots & \cdots & \cdots\\ \cdots & \boldsymbol{J}_{\rho c,2} & \cdots & \cdots\\ \vdots & \vdots & \vdots & \vdots\\ \cdots & \cdots & \cdots & \boldsymbol{J}_{\rho c,l}\end{bmatrix},\quad \boldsymbol{J}_{\rho c,i}=\begin{bmatrix}\hat{\$}^{\mathrm{T}}_{wc,1,i}\hat{\$}_{tc,1,i} & \cdots & \cdots & \cdots\\ \cdots & \hat{\$}^{\mathrm{T}}_{wc,2,i}\hat{\$}_{tc,2,i} & \cdots & \cdots\\ \vdots & \vdots & \vdots & \vdots\\ \cdots & \cdots & \cdots & \hat{\$}^{\mathrm{T}}_{wc,6-n_i,i}\hat{\$}_{tc,6-n_i,i}\end{bmatrix}$$

where $\boldsymbol{J}_x$ is a $\left(f+\sum_{i=1}^{l}(6-n_i)\right)\times 6$ matrix and $\boldsymbol{J}_\rho$ is a $\left(\sum_{i=1}^{l}(6-n_i)+f\right)\times\left(\sum_{i=1}^{l}(6-n_i)+f\right)$ diagonal matrix.

Assuming that $\boldsymbol{J}_\rho$ is nonsingular, we have

$$\boldsymbol{J}\$_t=\delta\boldsymbol{\rho} \tag{17}$$

where $\boldsymbol{J}=\boldsymbol{J}_\rho^{-1}\boldsymbol{J}_x=\begin{bmatrix}\boldsymbol{J}_a^{\mathrm{T}} & \boldsymbol{J}_c^{\mathrm{T}}\end{bmatrix}^{\mathrm{T}}$ is known as the g*eneralized Jacobian* of parallel manipulators with $2\le f<6$ DOF. $\boldsymbol{J}_a$ is always an $f\times 6$ matrix. Moreover, if $\mathrm{rank}(\boldsymbol{J})=6$, we have

$$\$_t=\boldsymbol{G}\delta\boldsymbol{\rho} \tag{18}$$

where $\boldsymbol{G}=\boldsymbol{J}^{+}=\left(\boldsymbol{J}^{\mathrm{T}}\boldsymbol{J}\right)^{-1}\boldsymbol{J}^{\mathrm{T}}=\begin{bmatrix}\boldsymbol{G}_a & \boldsymbol{G}_c\end{bmatrix}$

For a non-overconstrained parallel manipulator, its number of degrees of freedom can be determined by the Grübler-Kutzbach formula i.e. $f=\sum_{i=1}^{l}(n_i-6)+6$. In this case, there are exactly $6-f$ linearly independent unit wrenches of constraints imposed on the platform by all the limbs, and $\boldsymbol{J}$ is a 6×6 matrix and $\boldsymbol{J}_c$ is a $(6-f)\times 6$ matrix. For an overconstrained parallel manipulator, however, the limbs impose on the platform a total of $\sum_{i=1}^{l}(6-n_i)$ unit wrenches of constraints, but only $6-f$ of them are linearly independent, i.e., $\mathrm{rank}(\boldsymbol{J}_c)=6-f$. Theoretically, this means that any $6-f$ unit wrenches of constraints can be grouped to restrict the motion of the platform if and only if they are linearly independent. The other $\sum_{i=1}^{l}(6-n_i)-6+f$ unit wrenches of constraints can be visualized as the 'virtual' constraints. For instance, consider a parallel manipulator having identical limbs, each with $6-f$ linearly independent unit wrenches of constraints. Then the number of limbs should equal the number of degrees of freedom of the manipulator, and the $6-f$ unit wrenches of an arbitrary limb can be considered to be responsible for restricting the motion of the platform. The remaining $(f-1)(6-f)$ unit wrenches become 'virtual' constraints. Practically, these 'virtual' constraints would become 'real' due to misalignment of joint axes, *etc*., and they thereby might affect the mobility of a system as whole.

5. Discussions

In order to illustrate the generality of the *generalized Jacobian* in terms of velocity, accuracy and stiffness modeling, rewrite Eqs.(10) and (18) in partitioned form:

$$\$_t=\$_{ta}+\$_{tc}=\boldsymbol{J}_a\delta\boldsymbol{\rho}_a+\boldsymbol{J}_c\delta\boldsymbol{\rho}_c \tag{19}$$

for the serial kinematic manipulators, and

$$\$_t = \$_{ta} + \$_{tc} = \boldsymbol{G}_a \delta\boldsymbol{\rho}_a + \boldsymbol{G}_c \delta\boldsymbol{\rho}_c \tag{20}$$

for the parallel manipulators.

The advantages of the generalized Jacobian are manifold. For instance, in velocity analysis, $\$_t$ turns into the velocity twist of the end-effector; and thus $\delta\boldsymbol{\rho}_a \to \dot{\boldsymbol{q}}_a$ and $\delta\boldsymbol{\rho}_c \to \boldsymbol{0}$, where $\dot{\boldsymbol{q}}_a$ denotes the actuated joint rates vector. Consequently, a velocity model having the same form as that proposed in [27] can be achieved. In accuracy analysis, $\$_t$ turns into pose error twist of the end-effector, and thus $\delta\boldsymbol{\rho}_a \to \Delta\boldsymbol{\rho}_a$ and $\delta\boldsymbol{\rho}_c \to \Delta\boldsymbol{\rho}_c$, where $\Delta\boldsymbol{\rho}_a$ and $\Delta\boldsymbol{\rho}_c$ represent respectively the linearized translational/rotational error vectors along/about the axes of unit screws of permissions/restrictions. This allows the source errors to be divided into two groups. One group affects $\Delta\boldsymbol{\rho}_a$, which is compensatable via kinematic calibration; the other affects $\Delta\boldsymbol{\rho}_c$, which is uncompensatable and thereby should be eliminated or at least minimized in the manufacturing and assembly processes. Therefore, one further step can be carried out to achieve the error model by formulating the relationship between the source errors and joint error intensities, i.e. $\Delta\boldsymbol{\rho}_a$ and $\Delta\boldsymbol{\rho}_c$. In stiffness analysis, $\$_t$ becomes deflection twist of the end effector, and thus $\delta\boldsymbol{\rho}_a \to \Delta\boldsymbol{\rho}_a$ and $\delta\boldsymbol{\rho}_c \to \Delta\boldsymbol{\rho}_c$, where $\Delta\boldsymbol{\rho}_a$ and $\Delta\boldsymbol{\rho}_c$ represent respectively the translational/rotational deformation vectors along/about the axes of unit screws of permissions/restrictions. This means that the component compliances affecting the deflection twist of the end-effector can also be divided into two groups, one affecting $\Delta\boldsymbol{\rho}_a$ and the other affecting $\Delta\boldsymbol{\rho}_c$. Therefore, the stiffness model can be achieved by generating the component stiffness matrices associated respectively with $\Delta\boldsymbol{\rho}_a$ and $\Delta\boldsymbol{\rho}_c$, and formulating the static equilibrium equations in which the externally applied wrench can be expressed as a linear combination of the unit wrenches of actuations and constraints.

In summary, the generalized Jacobian proposed in this paper enables velocity, accuracy and stiffness modeling of *f*-DOF lower mobility manipulators with either serial or parallel architectures to be integrated into a unified framework. The use of the generalized Jacobian for accuracy and stiffness modeling of serial and parallel manipulators will be detailed in separate articles.

6. Examples

In order to demonstrate the methodology developed in the preceding sections, we perform here a generalized Jacobian analysis for three typical lower mobility parallel manipulators. Each of them is essentially composed of a number of serial kinematic chains as shown in Fig.3.

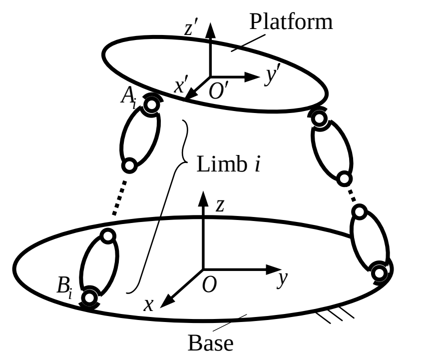

Fig.3 Schematic diagram of a parallel mechanism

Without losing generality, let the *i*th limb be connected to the platform at point A_i and to the base at point B_i as shown in Figure 4. A reference frame $O-xyz$ is placed at point O on the base and an instantaneous reference frame $O'-x'y'z'$ is placed at point O' with the x', y', and z' axes being parallel to the *x*, *y*, and *z* axes, respectively.

Each example will use the formal derivation process for the generalized Jacobian, because it emphasizes the exact relationships involved. Alternatively, having generated the relevant twists and wrenches, they could be substituted directly into Eq. (18) to obtain the same results.

6.1 The Sprint Z3 Head

Fig.4 shows the schematic diagram of a 3-PRS parallel manipulator which has been used as a 3-axis module named the Sprint Z3 [40] to build a 5-axis machine tool for high-speed machining. The manipulator consists of a base, a platform, and three identical limbs. Each limb connects the base to the platform in sequence by an actuated prismatic joint, a revolute joint, and a spherical joint. Therefore, we have $n_i = 5$ ($i = 1,2,3$). The unit screws of permissions, $\hat{\$}_{ta,j_a,i}$ ($j_a = 1,2,\ldots,5$), in the *i*th limb can be generated as

$$\hat{\$}_{ta,1,i} = \begin{pmatrix} \boldsymbol{s}_{1,i} \\ \boldsymbol{0} \end{pmatrix},\ \hat{\$}_{ta,2,i} = \begin{pmatrix} \boldsymbol{a}_i - l_3\boldsymbol{s}_{3,i} \times \boldsymbol{s}_{2,i} \\ \boldsymbol{s}_{2,i} \end{pmatrix},\ \hat{\$}_{ta,3,i} = \begin{pmatrix} \boldsymbol{a}_i \times \boldsymbol{s}_{3,i} \\ \boldsymbol{s}_{3,i} \end{pmatrix},\ \hat{\$}_{ta,4,i} = \begin{pmatrix} \boldsymbol{a}_i \times \boldsymbol{s}_{4,i} \\ \boldsymbol{s}_{4,i} \end{pmatrix},\ \hat{\$}_{ta,5,i} = \begin{pmatrix} \boldsymbol{a}_i \times \boldsymbol{s}_{5,i} \\ \boldsymbol{s}_{5,i} \end{pmatrix} \tag{21}$$

where $\boldsymbol{s}_{j_a,i}$ is a unit vector along the j_ath 1-DOF joint of the *i*th limb; $\boldsymbol{a}_i = \overline{O'A_i}$ and $l_3\boldsymbol{s}_{3,i} = \overline{P_iA_i}$. The joint axes are arranged such that $\boldsymbol{s}_{1,i} \perp \boldsymbol{s}_{2,i}$ and $\boldsymbol{s}_{2,i} \perp \boldsymbol{s}_{3,i}$; $\boldsymbol{s}_{3,i}$, $\boldsymbol{s}_{4,i}$, and $\boldsymbol{s}_{5,i}$ are coincident with three rotational axes of the spherical joint, with $\boldsymbol{s}_{3,i}$ aligned along the rod.

The unit wrench of constraints $\hat{\$}_{wc,1,i}$ is orthogonal to $\hat{\$}_{ta,j_a,i}$ ($j_a = 1,2,\ldots,5$), and so can be identified as a zero pitch screw passing through the center of the spherical joint and being parallel to $\boldsymbol{s}_{2,i}$:

$$\hat{\$}_{wc,1,i} = \begin{pmatrix} \boldsymbol{s}_{2,i} \\ \boldsymbol{a}_i \times \boldsymbol{s}_{2,i} \end{pmatrix} \tag{22}$$

Let the k_ath ($k_a = 1,2,\ldots,5$) joint be sequentially locked: the unit wrench, $\hat{\$}_{wa,k_a,i}$, which is orthogonal to $\hat{\$}_{ta,j_a,i}$ ($j_a = 1,2,\ldots,5$, $j_a \neq k_a$) and dual to $\hat{\$}_{ta,k_a,i}$, can be identified as

$$\hat{\$}_{wa,1,i} = \begin{pmatrix} \boldsymbol{s}_{3,i} \\ \boldsymbol{a}_i \times \boldsymbol{s}_{3,i} \end{pmatrix}, \quad \hat{\$}_{wa,2,i} = \begin{pmatrix} \boldsymbol{n}_{1,i} \\ \boldsymbol{a}_i \times \boldsymbol{n}_{1,i} \end{pmatrix},$$

$$\hat{\$}_{wa,3,i} = \begin{pmatrix} \boldsymbol{n}_{2,i} \\ \boldsymbol{a}_i - l_3 \boldsymbol{s}_{3,i} \times \boldsymbol{n}_{2,i} + h_i \boldsymbol{s}_{1,i} \end{pmatrix},$$

$$\hat{\$}_{wa,4,i} = \begin{pmatrix} \boldsymbol{n}_{3,i} \\ \boldsymbol{a}_i - l_3 \boldsymbol{s}_{3,i} \times \boldsymbol{n}_{3,i} \end{pmatrix} \hat{\$}_{wa,5,i} = \begin{pmatrix} \boldsymbol{n}_{4,i} \\ \boldsymbol{a}_i - l_3 \boldsymbol{s}_{3,i} \times \boldsymbol{n}_{4,i} \end{pmatrix} \tag{23}$$

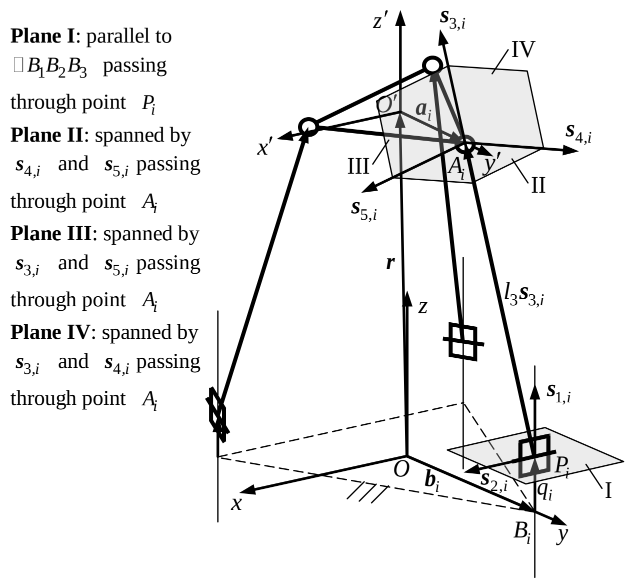

Fig.4 Schematic diagram of Sprint Z3 head

where $\hat{\$}_{wa,1,i}$ ($\hat{\$}_{wa,2,i}$) is a zero pitch screw passing through the center of the spherical joint and being parallel to $\boldsymbol{s}_{3,i}$ ($\boldsymbol{n}_{1,i} = \boldsymbol{s}_{1,i} \times \boldsymbol{s}_{2,i}$). $\hat{\$}_{wa,3,i}$ is a zero pitch screw coincident with the intersection line of plane I and plane II (see Figure 5) with $\boldsymbol{n}_{2,i}$ being its unit vector. h_i is the distance from point P_i to this line of intersection. In particular, if plane I is parallel to plane II, we have $\hat{\$}_{wa,3,i} = \mathbf{0}$ $\boldsymbol{s}_{1,i}^{\mathrm{T}}$ $^{\mathrm{T}}$ because $h_i \to \infty$. $\hat{\$}_{wa,4,i}$ ($\hat{\$}_{wa,5,i}$) is a zero pitch screw coincident with the intersection line of plane I and plane III (IV) with $\boldsymbol{n}_{3,i}$ ($\boldsymbol{n}_{4,i}$) being its unit vector. Noting that the prismatic joint, numbered 1, is the actuated joint, $\hat{\$}_{wa,1,i}$ ($i = 1,2,3$) can be used as the *i*th unit wrench of actuations of the parallel manipulator.

Finally, with the constraint provided by $\hat{\$}_{wc,1,i}$ being released, the unit screw of restrictions, $\hat{\$}_{tc,1,i}$, which is orthogonal to $\hat{\$}_{wa,k_a,i}$ ($k_a = 1,2,\ldots,5$) and dual to $\hat{\$}_{wc,1,i}$, can be identified as a zero pitch screw passing through point P_i and being parallel to $\boldsymbol{n}_{1,i}$.

$$\hat{\$}_{tc,1,i} = \begin{pmatrix} \boldsymbol{a}_i - l_3 \boldsymbol{s}_{3,i} \times \boldsymbol{n}_{1,i} \\ \boldsymbol{n}_{1,i} \end{pmatrix} \tag{24}$$

Hence, the twist of the platform can be written as

$$\$_t = \sum_{j_a=1}^{5} \delta\rho_{a,j_a,i} \hat{\$}_{ta,j_a,i} + \delta\rho_{c,1,i} \hat{\$}_{tc,1,i}, \quad i = 1,2,3 \tag{25}$$

Taking the inner product on both sides of Eq. (35) with $\hat{\$}_{wa,1,i}$ and $\hat{\$}_{wc,1,i}$, respectively, leads to

$$\boldsymbol{J}_x \$_t = \boldsymbol{J}_\rho \delta\boldsymbol{\rho} \tag{26}$$

$$\boldsymbol{J}_x = \begin{bmatrix} \boldsymbol{J}_{xa} \\ \boldsymbol{J}_{xc} \end{bmatrix}, \quad \boldsymbol{J}_\rho = \begin{bmatrix} \boldsymbol{J}_{\rho a} & \\ & \boldsymbol{J}_{\rho c} \end{bmatrix}, \quad \boldsymbol{J}_{xa} = \begin{bmatrix} \boldsymbol{s}_{3,1}^{\mathrm{T}} & (\boldsymbol{a}_1 \times \boldsymbol{s}_{3,1})^{\mathrm{T}} \\ \boldsymbol{s}_{3,2}^{\mathrm{T}} & (\boldsymbol{a}_2 \times \boldsymbol{s}_{3,2})^{\mathrm{T}} \\ \boldsymbol{s}_{3,3}^{\mathrm{T}} & (\boldsymbol{a}_3 \times \boldsymbol{s}_{3,3})^{\mathrm{T}} \end{bmatrix}, \quad \boldsymbol{J}_{xc} = \begin{bmatrix} \boldsymbol{s}_{2,1}^{\mathrm{T}} & (\boldsymbol{a}_1 \times \boldsymbol{s}_{2,1})^{\mathrm{T}} \\ \boldsymbol{s}_{2,2}^{\mathrm{T}} & (\boldsymbol{a}_2 \times \boldsymbol{s}_{2,2})^{\mathrm{T}} \\ \boldsymbol{s}_{2,3}^{\mathrm{T}} & (\boldsymbol{a}_3 \times \boldsymbol{s}_{2,3})^{\mathrm{T}} \end{bmatrix}$$

$$\boldsymbol{J}_{\rho a} = \begin{bmatrix} \boldsymbol{s}_{1,1}^{\mathrm{T}}\boldsymbol{s}_{3,1} & & \\ & \boldsymbol{s}_{1,2}^{\mathrm{T}}\boldsymbol{s}_{3,2} & \\ & & \boldsymbol{s}_{1,3}^{\mathrm{T}}\boldsymbol{s}_{3,3} \end{bmatrix}, \quad \boldsymbol{J}_{\rho c} = l_3 \begin{bmatrix} \boldsymbol{s}_{1,1}^{\mathrm{T}}\boldsymbol{s}_{3,1} & & \\ & \boldsymbol{s}_{1,2}^{\mathrm{T}}\boldsymbol{s}_{3,2} & \\ & & \boldsymbol{s}_{1,3}^{\mathrm{T}}\boldsymbol{s}_{3,3} \end{bmatrix}. \quad \delta\boldsymbol{\rho} = \begin{pmatrix} \delta\boldsymbol{\rho}_a \\ \delta\boldsymbol{\rho}_c \end{pmatrix}, \quad \delta\boldsymbol{\rho}_a = \begin{pmatrix} \delta\rho_{a,1,1} \\ \delta\rho_{a,1,2} \\ \delta\rho_{a,1,3} \end{pmatrix}, \quad \delta\boldsymbol{\rho}_c = \begin{pmatrix} \delta\rho_{c,1,1} \\ \delta\rho_{c,1,2} \\ \delta\rho_{c,1,3} \end{pmatrix}$$

6.2 The Tricept Robot

Fig.5 shows the schematic diagram of the 3-DOF module within the Tricept robot [41]. It is composed of three identical actuated U<u>P</u>S limbs and one properly constrained passive UP limb. Each U<u>P</u>S limb connects the base to the platform in sequence by a universal joint, an actuated prismatic joint, and a spherical joint. The UP limb connects the base to the platform by a universal joint followed by a prismatic joint. Therefore, we have $n_i = 6$ ($i = 1,2,3$) for the U<u>P</u>S limbs, and $n_4 = 3$ for the UP limb.

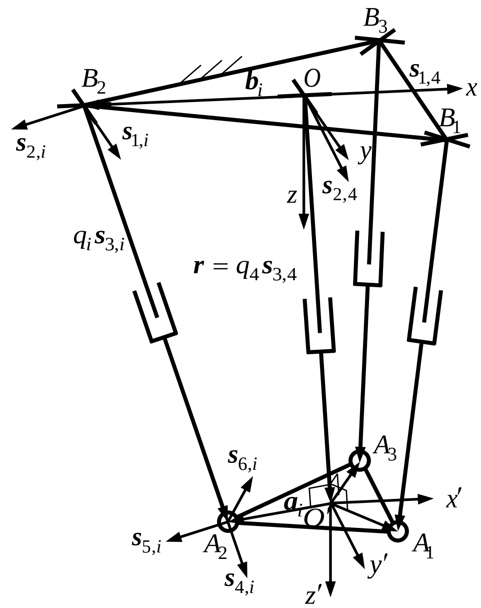

Fig. 5. Schematic diagram of Tricept robot

On one hand, the unit screws of permissions of the *i*th U<u>P</u>S limb, $\hat{\$}_{ta,j_a,i}$ ($j_a = 1,2,\ldots,6$), can be generated as

$$\hat{\$}_{ta,1,i} = \begin{pmatrix} (\boldsymbol{a}_i - q_i\boldsymbol{s}_{3,i}) \times \boldsymbol{s}_{1,i} \\ \boldsymbol{s}_{1,i} \end{pmatrix}, \quad \hat{\$}_{ta,2,i} = \begin{pmatrix} (\boldsymbol{a}_i - q_i\boldsymbol{s}_{3,i}) \times \boldsymbol{s}_{2,i} \\ \boldsymbol{s}_{2,i} \end{pmatrix}, \quad \hat{\$}_{ta,3,i} = \begin{pmatrix} \boldsymbol{s}_{3,i} \\ \boldsymbol{0} \end{pmatrix}$$

$$\hat{\$}_{ta,4,i} = \begin{pmatrix} \boldsymbol{a}_i \times \boldsymbol{s}_{4,i} \\ \boldsymbol{s}_{4,i} \end{pmatrix}, \quad \hat{\$}_{ta,5,i} = \begin{pmatrix} \boldsymbol{a}_i \times \boldsymbol{s}_{5,i} \\ \boldsymbol{s}_{5,i} \end{pmatrix}, \quad \hat{\$}_{ta,6,i} = \begin{pmatrix} \boldsymbol{a}_i \times \boldsymbol{s}_{6,i} \\ \boldsymbol{s}_{6,i} \end{pmatrix} \tag{27}$$

where $\boldsymbol{s}_{j_a,i}$ is a unit vector along the j_ath 1-DOF joint of the *i*th limb; $\boldsymbol{a}_i = \overline{O'A_i}$ and $q_i\boldsymbol{s}_{3,i} = \overline{B_iA_i}$. The joint axes are arranged such that $\boldsymbol{s}_{1,i} \perp \boldsymbol{s}_{2,i}$ and $\boldsymbol{s}_{2,i} \perp \boldsymbol{s}_{3,i}$; $\boldsymbol{s}_{4,i}$, $\boldsymbol{s}_{5,i}$, and $\boldsymbol{s}_{6,i}$ are coincident with three rotational axes of the spherical joint, with $\boldsymbol{s}_{3,i} = \boldsymbol{s}_{4,i}$.

Noting that $n_i = 6$ ($i = 1,2,3$), all that remains to be done in this case is to generate the unit wrench of actuations of the parallel mechanism. By locking the actuated prismatic joint, numbered 3, the unit wrench of actuations, $\hat{\$}_{wa,3,i}$, which is orthogonal to $\hat{\$}_{ta,j_a,i}$ ($j_a = 1,2,\ldots,6$, $j_a \neq 3$) and dual to $\hat{\$}_{ta,3,i}$, can be identified as a zero pitch screw passing through the center of the spherical joint and being parallel to $\boldsymbol{s}_{3,i}$

$$\hat{\$}_{wa,3,i} = \begin{pmatrix} \boldsymbol{s}_{3,i} \\ \boldsymbol{a}_i \times \boldsymbol{s}_{3,i} \end{pmatrix} \tag{28}$$

On the other hand, the unit screws of permissions of the UP limb, $\hat{\$}_{ta,j_a,4}$ ($j_a = 1,2,3$), can be generated as

$$\hat{\$}_{ta,1,4} = \begin{pmatrix} q_4\boldsymbol{s}_{3,4} \times \boldsymbol{s}_{1,4} \\ \boldsymbol{s}_{1,4} \end{pmatrix}, \quad \hat{\$}_{ta,2,4} = \begin{pmatrix} q_4\boldsymbol{s}_{3,4} \times \boldsymbol{s}_{2,4} \\ \boldsymbol{s}_{2,4} \end{pmatrix}, \quad \hat{\$}_{ta,3,4} = \begin{pmatrix} \boldsymbol{s}_{3,4} \\ \boldsymbol{0} \end{pmatrix} \tag{29}$$

where $q_4\boldsymbol{s}_{3,4} = \overline{OO'}$; $\boldsymbol{s}_{1,4}$ and $\boldsymbol{s}_{2,4}$ are coincident with the two axes of the universal joint; $\boldsymbol{s}_{3,4}$ is along the axial axis of the prismatic joint with $\boldsymbol{s}_{1,4} \perp \boldsymbol{s}_{2,4}$ and $\boldsymbol{s}_{2,4} \perp \boldsymbol{s}_{3,4}$.

The unit wrench of constraints of the UP limb, $\hat{\$}_{wc,k_c,4}$ ($k_c = 1,2,3$), which is orthogonal to $\hat{\$}_{ta,j_a,4}$ ($j_a = 1,2,3$), can then be identified as

$$\hat{\$}_{wc,1,4}=\begin{pmatrix}\mathbf{0}\\ \boldsymbol{n}_{1,4}\end{pmatrix},\ \ \hat{\$}_{wc,2,4}=\begin{pmatrix}\boldsymbol{s}_{2,4}\\ q_4\boldsymbol{s}_{3,4}\times\boldsymbol{s}_{2,4}\end{pmatrix},\ \ \hat{\$}_{wc,3,4}=\begin{pmatrix}\boldsymbol{n}_{2,4}\\ q_4\boldsymbol{s}_{3,4}\times\boldsymbol{n}_{2,4}\end{pmatrix} \tag{30}$$

where $\hat{\$}_{wc,1,4}$ is an infinite pitch screw parallel to $\boldsymbol{n}_{1,4}$=$\boldsymbol{s}_{1,4}$×$\boldsymbol{s}_{2,4}$; $\hat{\$}_{wc,2,4}$ ($\hat{\$}_{wc,3,4}$) is a zero pitch screw passing through the center of the universal joint and being parallel to $\boldsymbol{s}_{2,4}$ ($\boldsymbol{n}_{2,4}$=$\boldsymbol{s}_{2,4}$×$\boldsymbol{s}_{3,4}$).

By locking the k_ath ($k_a=1,2,3$) joint, the unit wrench, $\hat{\$}_{wa,k_a,4}$, which is orthogonal to $\hat{\$}_{ta,j_a,4}$ ($j_a=1,2,3$, $j_a\neq k_a$) and dual to $\hat{\$}_{ta,k_a,4}$, can then be identified as

$$\hat{\$}_{wa,1,4}=\begin{pmatrix}\mathbf{0}\\ \boldsymbol{s}_{1,4}\end{pmatrix},\ \ \hat{\$}_{wa,2,4}=\begin{pmatrix}\mathbf{0}\\ \boldsymbol{s}_{2,4}\end{pmatrix},\ \ \hat{\$}_{wa,3,4}=\begin{pmatrix}\boldsymbol{s}_{3,4}\\ \mathbf{0}\end{pmatrix} \tag{31}$$

where $\hat{\$}_{wa,1,4}$ ($\hat{\$}_{wa,2,4}$) is an infinite pitch screw parallel to $\boldsymbol{s}_{1,4}$ ($\boldsymbol{s}_{2,4}$); $\hat{\$}_{wa,3,4}$ is a zero pitch screw passing through point O' and being parallel to $\boldsymbol{s}_{3,4}$.

Finally, with the k_cth ($k_c=1,2,3$) constraints provided by $\hat{\$}_{wc,k_c,4}$ being sequentially released, the unit screw of restrictions, $\hat{\$}_{tc,j_c,4}$ ($j_c=1,2,3$), which is orthogonal to $\hat{\$}_{wa,k_a,4}$ ($k_a=1,2,3$) and $\hat{\$}_{wc,k_c,4}$ ($k_c=1,2,3$, $k_c\neq j_c$), i.e. except for $\hat{\$}_{wc,j_c,4}$, can be identified as

$$\hat{\$}_{tc,1,4}=\begin{pmatrix}q_4\boldsymbol{s}_{3,4}\times\boldsymbol{n}_{1,4}\\ \boldsymbol{n}_{1,4}\end{pmatrix},\ \ \hat{\$}_{tc,2,4}=\begin{pmatrix}\boldsymbol{s}_{2,4}\\ \mathbf{0}\end{pmatrix},\ \ \hat{\$}_{tc,3,4}=\begin{pmatrix}\boldsymbol{n}_{2,4}\\ \mathbf{0}\end{pmatrix} \tag{32}$$

where $\hat{\$}_{tc,1,4}$ is a zero pitch screw passing through the center of the universal joint and being parallel to $\boldsymbol{n}_{1,i}$. $\hat{\$}_{tc,2,4}$ ($\hat{\$}_{tc,3,4}$) is an infinite pitch screw parallel to $\boldsymbol{s}_{2,4}$ ($\boldsymbol{n}_{2,4}$).

Hence, the twist of the platform can be written as

$$\$_t=\sum_{j_a=1}^{6}\delta\rho_{a,j_a,i}\hat{\$}_{ta,j_a,i}\ ,\ \ i=1,2,3 \tag{33}$$

$$\$_t=\sum_{j_a=1}^{3}\delta\rho_{a,j_a,4}\hat{\$}_{ta,j_a,4}+\sum_{j_c=1}^{3}\delta\rho_{c,j_c,4}\hat{\$}_{tc,j_c,4} \tag{34}$$

Taking inner products on both sides of Eqs.(33) and (34) with $\hat{\$}_{wa,3,i}$ and $\hat{\$}_{wc,k_c,4}$, respectively, leads to

$$\boldsymbol{J}_x\$_t=\boldsymbol{J}_\rho\delta\boldsymbol{\rho} \tag{35}$$

$$\boldsymbol{J}_x=\begin{bmatrix}\boldsymbol{J}_{xa}\\ \boldsymbol{J}_{xc}\end{bmatrix},\ \ \boldsymbol{J}_\rho=\begin{bmatrix}\boldsymbol{J}_{\rho a} & \\ & \boldsymbol{J}_{\rho c}\end{bmatrix},\ \ \delta\boldsymbol{\rho}=\begin{pmatrix}\delta\boldsymbol{\rho}_a\\ \delta\boldsymbol{\rho}_c\end{pmatrix},\ \ \delta\boldsymbol{\rho}_a=\begin{pmatrix}\delta\rho_{a,3,1}\\ \delta\rho_{a,3,2}\\ \delta\rho_{a,3,3}\end{pmatrix},\ \ \delta\boldsymbol{\rho}_c=\begin{pmatrix}\delta\rho_{c,1,4}\\ \delta\rho_{c,2,4}\\ \delta\rho_{c,3,4}\end{pmatrix}$$

$$\boldsymbol{J}_{xa}=\begin{bmatrix}\boldsymbol{s}_{3,1}^{\mathrm{T}} & \boldsymbol{a}_1\times\boldsymbol{s}_{3,1}{}^{\mathrm{T}}\\ \boldsymbol{s}_{3,2}^{\mathrm{T}} & \boldsymbol{a}_2\times\boldsymbol{s}_{3,2}{}^{\mathrm{T}}\\ \boldsymbol{s}_{3,3}^{\mathrm{T}} & \boldsymbol{a}_3\times\boldsymbol{s}_{3,3}{}^{\mathrm{T}}\end{bmatrix},\ \ \boldsymbol{J}_{xc}=\begin{bmatrix}\mathbf{0} & \boldsymbol{n}_{1,4}^{\mathrm{T}}\\ \boldsymbol{s}_{2,4}^{\mathrm{T}} & q_4\boldsymbol{s}_{3,4}\times\boldsymbol{s}_{2,4}{}^{\mathrm{T}}\\ \boldsymbol{n}_{2,4}^{\mathrm{T}} & q_4\boldsymbol{s}_{3,4}\times\boldsymbol{n}_{2,4}{}^{\mathrm{T}}\end{bmatrix},\ \ \boldsymbol{J}_{\rho a}=\boldsymbol{J}_{\rho c}=\boldsymbol{I}_3$$

where $\boldsymbol{I}_3$ denotes a unit matrix of order 3.

6.3 The Delta Robot

Fig.6 shows the schematic diagram of the Delta robot with 3-DOF translational motion capabilities [39]. It consists of a base, a platform and three identical <u>R</u>RPaR limbs. In each <u>R</u>RPaR limb, the axes of three revolute joints are parallel to each other, and the axes of the revolute joints in the Pa joint are normal to the parallelogram $\square C_{1,i}C_{2,i}A_{2,i}A_{1,i}$. Since the Pa joint can be visualized as a 1-DOF compound joint, we have $n_i = 4$ ($i = 1,2,3$). Hence, the unit screws of permissions, $\hat{\$}_{ta,j_a,i}$ ($j_a = 1,2,\ldots,4$), in the ith limb can be generated as follows:

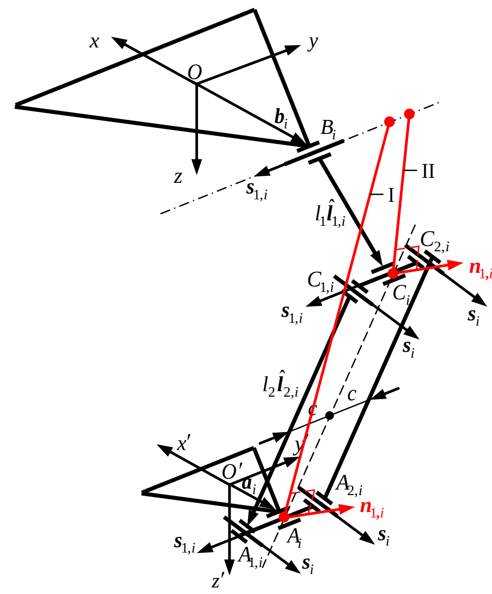

Fig.6 Schematic diagram of Delta robot

$$\hat{\$}_{ta,1,i} = \begin{pmatrix} \boldsymbol{a}_i - l_2\hat{\boldsymbol{l}}_{2,i} - l_1\hat{\boldsymbol{l}}_{1,i} \ \times \boldsymbol{s}_{1,i} \\ \boldsymbol{s}_{1,i} \end{pmatrix},\ \hat{\$}_{ta,2,i} = \begin{pmatrix} \boldsymbol{a}_i - l_2\hat{\boldsymbol{l}}_{2,i} \ \times \boldsymbol{s}_{1,i} \\ \boldsymbol{s}_{1,i} \end{pmatrix}$$

$$\hat{\$}_{ta,3,i} = \begin{pmatrix} \boldsymbol{n}_{1,i} \\ \boldsymbol{0} \end{pmatrix},\ \hat{\$}_{ta,4,i} = \begin{pmatrix} \boldsymbol{a}_i \times \boldsymbol{s}_{1,i} \\ \boldsymbol{s}_{1,i} \end{pmatrix} \tag{36}$$

where $\boldsymbol{s}_{1,i}$ is a unit vector along the axis of the first revolute joint of limb i; $\boldsymbol{a}_i = \overline{O'A_i}$, $l_1\hat{\boldsymbol{l}}_{1,i} = \overline{B_iC_i}$, and $l_2\hat{\boldsymbol{l}}_{2,i} = \overline{C_{1,i}A_{1,i}}$. The generation of $\hat{\$}_{ta,3,i}$ is given in the Appendix, with $\boldsymbol{n}_{1,i} = \boldsymbol{s}_i \times \hat{\boldsymbol{l}}_{2,i}$ and $\boldsymbol{s}_i$ being the normal vector to $\square C_{1,i}C_{2,i}A_{2,i}A_{1,i}$. The joint axes are arranged such that $\boldsymbol{s}_{1,i} \perp \hat{\boldsymbol{l}}_{1,i}$, $\boldsymbol{s}_{1,i} \perp \boldsymbol{s}_i$, and $\boldsymbol{s}_i \perp \hat{\boldsymbol{l}}_{2,i}$.

The unit wrench of constraints, $\hat{\$}_{wc,k_c,i}$ ($k_c = 1,2$), which is orthogonal to $\hat{\$}_{ta,j_a,i}$ ($j_a = 1,2,\ldots,4$), can then be identified as

$$\hat{\$}_{wc,1,i} = \begin{pmatrix} \boldsymbol{0} \\ \boldsymbol{s}_i \end{pmatrix},\ \hat{\$}_{wc,2,i} = \begin{pmatrix} \boldsymbol{0} \\ \boldsymbol{n}_{2,i} \end{pmatrix} \tag{37}$$

where $\hat{\$}_{wc,1,i}$ ($\hat{\$}_{wc,2,i}$) is an infinite pitch screw parallel to $\boldsymbol{s}_i$ ($\boldsymbol{n}_{2,i} = \boldsymbol{s}_i \times \boldsymbol{s}_{1,i}$).

Locking sequentially the k_ath ($k_a = 1,2,\ldots,4$) joint, the unit wrench $\hat{\$}_{wa,k_a,i}$, which is orthogonal to $\hat{\$}_{ta,j_a,i}$ ($j_a = 1,2,\ldots,4$, $j_a \neq k_a$) and dual to $\hat{\$}_{ta,k_a,i}$, can be identified as

$$\hat{\$}_{wa,1,i} = \begin{pmatrix} \hat{\boldsymbol{l}}_{2,i} \\ \boldsymbol{a}_i \times \hat{\boldsymbol{l}}_{2,i} \end{pmatrix},\ \hat{\$}_{wa,2,i} = \begin{pmatrix} \boldsymbol{n}_{3,i} \\ \boldsymbol{a}_i \times \boldsymbol{n}_{3,i} \end{pmatrix},\ \hat{\$}_{wa,3,i} = \begin{pmatrix} \boldsymbol{s}_{1,i} \\ \boldsymbol{a}_i \times \boldsymbol{s}_{1,i} \end{pmatrix},\ \hat{\$}_{wa,4,i} = \begin{pmatrix} \boldsymbol{n}_{4,i} \\ \boldsymbol{a}_i - l_2\hat{\boldsymbol{l}}_{2,i} \ \times \boldsymbol{n}_{4,i} \end{pmatrix} \tag{38}$$

where $\hat{\$}_{wa,1,i}$ ($\hat{\$}_{wa,3,i}$) is a zero pitch screw passing through point A_i and being parallel to $\hat{\boldsymbol{l}}_{2,i}$ ($\boldsymbol{s}_{1,i}$); $\hat{\$}_{wa,2,i}$ ($\hat{\$}_{wa,4,i}$) is a zero pitch screw coincident with the line I (II) that is perpendicular to $\boldsymbol{n}_{1,i}$ with $\boldsymbol{n}_{3,i}$ ($\boldsymbol{n}_{4,i}$) being its unit vector (see Fig. (6). Noting that the revolute joint, numbered 1, is the actuated joint, $\hat{\$}_{wa,1,i}$ ($i = 1,2,3$) can be used as the ith unit wrench of actuations of the parallel mechanism.

Finally, with the k_cth ($k_c = 1,2$) constraints provided by $\hat{\$}_{wc,k_c,i}$ being sequentially released, the unit screw of restrictions, $\hat{\$}_{tc,j_c,i}$ ($j_c = 1,2$), which is orthogonal to $\hat{\$}_{wa,k_a,i}$ ($k_a = 1,2,\ldots,4$) and $\hat{\$}_{wc,k_c,i}$ ($k_c = 1,2$, $k_c \neq j_c$) except for $\hat{\$}_{wc,j_c,i}$, can be identified as

$$\hat{\$}_{tc,1,i} = \begin{pmatrix} \boldsymbol{a}_i \times \boldsymbol{s}_i \\ \boldsymbol{s}_i \end{pmatrix},\quad \hat{\$}_{tc,2,i} = \begin{pmatrix} \boldsymbol{a}_i \times \hat{\boldsymbol{l}}_{2,i} \\ \hat{\boldsymbol{l}}_{2,i} \end{pmatrix} \tag{39}$$

where $\hat{\$}_{tc,1,i}$ ($\hat{\$}_{tc,2,i}$) is a zero pitch screw passing through point A_i and being parallel to $\boldsymbol{s}_i$ ($\hat{\boldsymbol{l}}_{2,i}$).

Hence, the twist of the platform can be written as

$$\$_t=\sum_{j_a=1}^{4}\delta\rho_{a,j_a,i}\hat{\$}_{ta,j_a,i}+\sum_{j_c=1}^{2}\delta\rho_{c,j_c,i}\hat{\$}_{tc,j_c,i}\ ,\quad i=1,2,3 \tag{40}$$

and taking inner products on both sides of Eq. (40) with $\hat{\$}_{wa,1,i}$ and $\hat{\$}_{wc,k_c,i}$, respectively, leads to

$$\boldsymbol{J}_x\$_t=\boldsymbol{J}_\rho\delta\boldsymbol{\rho} \tag{41}$$

$$\boldsymbol{J}_x=\begin{bmatrix}\boldsymbol{J}_{xa}\\ \boldsymbol{J}_{xc}\end{bmatrix},\quad \boldsymbol{J}_\rho=\begin{bmatrix}\boldsymbol{J}_{\rho a} & \\ & \boldsymbol{J}_{\rho c}\end{bmatrix}$$

$$\boldsymbol{J}_{xa}=\begin{bmatrix}\begin{pmatrix}\hat{\boldsymbol{l}}_{2,1}^{\mathrm{T}} & \boldsymbol{a}_1\times\hat{\boldsymbol{l}}_{2,1}\end{pmatrix}^{\mathrm{T}}\\ \begin{pmatrix}\hat{\boldsymbol{l}}_{2,2}^{\mathrm{T}} & \boldsymbol{a}_2\times\hat{\boldsymbol{l}}_{2,2}\end{pmatrix}^{\mathrm{T}}\\ \begin{pmatrix}\hat{\boldsymbol{l}}_{2,3}^{\mathrm{T}} & \boldsymbol{a}_3\times\hat{\boldsymbol{l}}_{2,3}\end{pmatrix}^{\mathrm{T}}\end{bmatrix},\quad \boldsymbol{J}_{xc}=\begin{bmatrix}\boldsymbol{J}_{xc,1}\\ \boldsymbol{J}_{xc,2}\\ \boldsymbol{J}_{xc,3}\end{bmatrix}\quad \boldsymbol{J}_{xc,i}=\begin{bmatrix}\boldsymbol{0} & \boldsymbol{s}_i^{\mathrm{T}}\\ \boldsymbol{0} & \boldsymbol{n}_{2,i}^{\mathrm{T}}\end{bmatrix},\quad \boldsymbol{J}_{\rho a}=l_1\begin{bmatrix}\hat{\boldsymbol{l}}_{2,1}^{\mathrm{T}}\ \boldsymbol{s}_{1,1}\times\hat{\boldsymbol{l}}_{1,1} & & \\ & \hat{\boldsymbol{l}}_{2,2}^{\mathrm{T}}\ \boldsymbol{s}_{1,2}\times\hat{\boldsymbol{l}}_{1,2} & \\ & & \hat{\boldsymbol{l}}_{2,3}^{\mathrm{T}}\ \boldsymbol{s}_{1,3}\times\hat{\boldsymbol{l}}_{1,3}\end{bmatrix}$$

$$\boldsymbol{J}_{\rho c}=\begin{bmatrix}\boldsymbol{J}_{\rho c,1} & & \\ & \boldsymbol{J}_{\rho c,2} & \\ & & \boldsymbol{J}_{\rho c,l}\end{bmatrix},\quad \boldsymbol{J}_{\rho c,i}=\begin{bmatrix}1 & \\ & \boldsymbol{n}_{2,i}^{\mathrm{T}}\hat{\boldsymbol{l}}_{2,i}\end{bmatrix},\quad \delta\boldsymbol{\rho}=\begin{pmatrix}\delta\boldsymbol{\rho}_a\\ \delta\boldsymbol{\rho}_c\end{pmatrix},\quad \delta\boldsymbol{\rho}_a=\begin{pmatrix}\delta\rho_{a,1,1}\\ \delta\rho_{a,1,2}\\ \delta\rho_{a,1,3}\end{pmatrix},\quad \delta\boldsymbol{\rho}_c=\begin{pmatrix}\delta\boldsymbol{\rho}_{c,1}\\ \delta\boldsymbol{\rho}_{c,2}\\ \delta\boldsymbol{\rho}_{c,3}\end{pmatrix},\quad \delta\boldsymbol{\rho}_{c,i}=\begin{pmatrix}\delta\rho_{c,1,i}\\ \delta\rho_{c,2,i}\end{pmatrix}$$

It is easy to see that the *generalized Jacobian* of the first two examples is a 6×6 matrix as the unit wrenches of constraints provided by all limbs are linearly independent for a non-overconstrained system. However, the *generalized Jacobian* of the last example is a 9×6 matrix. Out of six unit wrenches of constraints, only three of them are linearly dependent for a system with an overconstraint of 3.

7. Conclusions

Under the umbrella of linear algebra and screw theory, this paper has proposed an approach for the *generalized Jacobian* analysis of serial and parallel lower mobility manipulators. The following conclusions are drawn.

1) The underlying relationships amongst wrench/twist subspaces associated with the actuation/constraint forces and the permitted/restricted instantaneous motions of a constrained rigid body are found. These relationships reflect, in general, two sides of a coin, i.e. the mobility and immobility of a constrained mechanical system.

2) Exploiting the properties of these subspaces, a methodology for formulating the *generalized Jacobian* has been proposed. This approach is so general and systematic that it has a potential to enable velocity, accuracy and stiffness modeling of lower mobility serial and parallel manipulators to be integrated into a unified mathematical framework. The use of the *generalized Jacobian* for accuracy, kinestatic and stiffness modeling, even its extension to rigid body dynamics, will be addressed in separate articles, attempting to form a complete theoretical package for the unified parameter modeling of the lower mobility manipulators.

Appendix

The unit screw of permissions associated with the revolute joints of the *i*th limb at $A_{1,i}$, $A_{2,i}$, $C_{1,i}$, and $C_{2,i}$ (see Fig. 5) can be generated as

$$\hat{\$}_{ta,C1,i}=\begin{pmatrix}(\boldsymbol{a}_i+c\boldsymbol{s}_{1,i}-l_2\hat{\boldsymbol{l}}_{2,i})\times\boldsymbol{s}_i\\ \boldsymbol{s}_i\end{pmatrix},\quad \hat{\$}_{ta,C2,i}=\begin{pmatrix}(\boldsymbol{a}_i-c\boldsymbol{s}_{1,i}-l_2\hat{\boldsymbol{l}}_{2,i})\times\boldsymbol{s}_i\\ \boldsymbol{s}_i\end{pmatrix},\quad \hat{\$}_{ta,A1,i}=\begin{pmatrix}(\boldsymbol{a}_i+c\boldsymbol{s}_{1,i})\times\boldsymbol{s}_i\\ \boldsymbol{s}_i\end{pmatrix},\quad \hat{\$}_{ta,A2,i}=\begin{pmatrix}(\boldsymbol{a}_i-c\boldsymbol{s}_{1,i})\times\boldsymbol{s}_i\\ \boldsymbol{s}_i\end{pmatrix} \tag{42}$$

where $\boldsymbol{s}_i$ is coincident with the axis of the revolute joint, and is also the normal vector of $\square C_{1,i}C_{2,i}A_{2,i}A_{1,i}$; $\boldsymbol{a}_i=\overline{O'A_i}$, $2c\boldsymbol{s}_{1,i}=\overline{A_{2,i}A_{1,i}}$, and $l_2\hat{\boldsymbol{l}}_{2,i}=\overline{C_{1,i}A_{1,i}}$. Within the scope of velocity analysis, the instantaneous twist of the platform due solely to the twist of the parallelogram joint can be given as

$$\$_{ta,3,i}=\dot{\theta}_{3,i}\hat{\$}_{ta,C1(2),i}-\dot{\theta}_{3,i}\hat{\$}_{ta,A1(2),i} \tag{43}$$

where $\dot{\theta}_{3,i}$ represents the angular velocity of the revolute joint. Substituting Eq.(42) into Eq.(43) gives

$$\$_{ta,3,i} = \dot{\theta}_{3,i} l_2 \hat{\$}_{ta,3,i} = \dot{\theta}_{3,i} l_2 \begin{pmatrix} \boldsymbol{n}_{1,i} \\ \boldsymbol{0} \end{pmatrix} \tag{44}$$

where $\boldsymbol{n}_{1,i} = \boldsymbol{s}_i \times \hat{\boldsymbol{l}}_{2,i}$. Therefore, $\hat{\$}_{ta,3,i}$ can be considered to be the unit screw of permissions associated with the 1-DOF parallelogram joint of limb i.

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