THE 'MAXIMUM DELAY CONVENTION' IS A THEOREM

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TO MY FATHER AND MOTHER,

Dr. Joaquim de Figueiredo

and

D. Fernanda Maria V.F.M.R. de Figueiredo,

without whose help, throughout my life,

this Thesis would not have been possible.

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ABSTRACT

In this thesis we formalise the Maximum Delay Convention of Catastrophe theory.

We prove theorems concerning the genericity of the existence and uniqueness of lifts from the control space to the catastrophe manifold (see Chapter 1), according to the convention above mentioned.

Our methods of proof involve the application of transversality theory in a new context: that of higher order tangent bundles.

CONTENTS

LIST OF THE MAIN SYMBOLS

Symbol	Page
X^n	1.1.(1)
$\mathcal{F}$	
C	
M_f	
Π_x	
Π_c	
χ_f	
$\mathcal{F}^*$	
$\mathcal{V}^s(N)$	
$\mathcal{V}(N)$	
$M_f^k, k=1,\dots,n$	
M_f^d	1.1(2)
∂M_f^n	
$w[\Phi](x)$	
$\nabla f(g)(x)$	
$S(v)$	
inset/outset	1.1(3)
$V = \{v_y\}_{y \in C}$	
$w[\Phi](W)$	
sep Φ	
C_f	

Symbol	Page
$J_{f,x,y}$	1.1(4)
$M_{f,y}$	
$A_{f,y}$	
$\overline{\mathcal{V}(C)}$	
$O_y(v)$	
O_y	
$O_y^+(v)$	
O_y^+	
$O_y(\varepsilon,v)$	
$O_y(\varepsilon)$	
ϕ	1.2(1)
ψ	
H_1	
H_2	
$\mathcal{V}_f$	
$\mathbb{R}_0^+$	
$\mathcal{V}^*$	1.2(2)
$\#K$	2.1(8)
$\#s$	
$\#t$	
$B_\delta(x)$	
$D_\delta(x)$	
$S_\delta(x)$	

Symbol	Page
$ac_B(A)$	2.1(8)
$ac(A)$	
I_k	
$\chi_k, k=1,\ldots,n$	2.1(4)
T^e	3.1(1)
$\sim_e$	
$\hat{\alpha}$	
$T^e M$	
$T^e f$	
$\sim_*$	3.1(3)
$\tilde{\phi}$	3.1(4)
$\tilde{U}$	
$v[e]$	3.1(9)
S	3.2(1)
A	3.2(2)
A	3.2(3)
$v \Vdash_x S$	
$V \Vdash S$	
$\eta_1, \eta_2, \eta_3, \eta_4$	4.2(1)
g_1, g_2, g_3, g_4	
$S_1(\chi_f)$	4.2(3)
$S_{1,\ldots,1}(\chi_f)$	4.2(4)
M_1^d, M_2^d, M_3^d, etc	4.2(7)
$\widehat{Q, p}$	4.2(8)
$\widehat{\mathcal{M}, m}$	

Symbol	Page
$T^e(\widehat{Q,p})$	4.3(1)
$\mathcal{M}[e]$	
$\mathcal{N}[e]$	
$\mathcal{M}_i[e]$	
$\mathcal{N}_i[e]$	
$\mathcal{Z}_i, U_i^j$	4.3(6)
N_i^j, M_i^j	
$\mathcal{Z}$	4.3(7)
$\mathcal{M}_i^j[e]$	4.3(8)
$\mathcal{N}_i^j[e]$	
$\mathcal{P}[e]$	
$\mathcal{P}_i[e]$	
$\mathcal{P}_i^j[e]$	
A_i^j	4.3(11)
$A_i^{j,c}$	
W_i^j	
$W_i^{j,c}$	
$C_1^j[1]$	4.4(3)
$C_1[1]$	
$C[1]$	
$\tilde{I}$	4.4(6)
$C_1^j[2]$	
$C_2^j[2]$	
$C_1[2]$	
$C_2[2]$	
$C[2]$	
$Q_2[2]$	

Remark: Sometimes the letter f is dropped from some of the symbols above, when it is clear enough which function f we are referring to.

SOME ABBREVIATIONS AND NOTATIONS

A: = B	Means A is defined to be B
nghd	Neighbourhood
def	Definition
cod	Codimension
diff	Diffeomorphism
s.t.	such that
w.l.o.g.	Without loss of generality
r.h.s.	Right hand side
l.h.s.	Left hand side

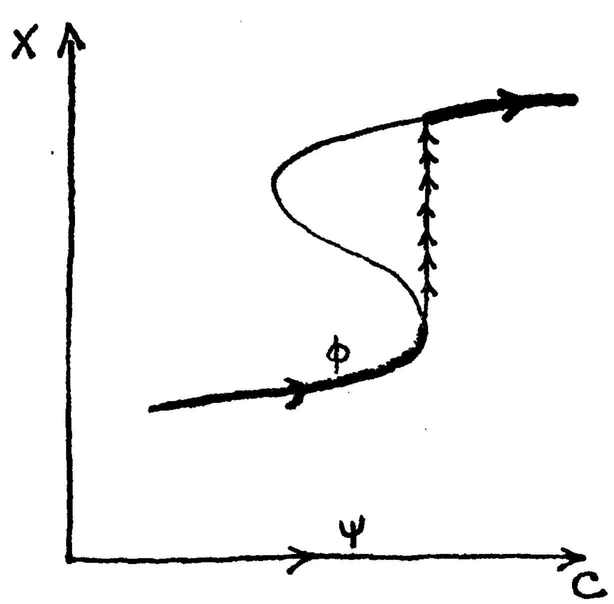
CHAPTER I

1.0. INTRODUCTION:

1.0.1. On 'what is' and 'why' the problem

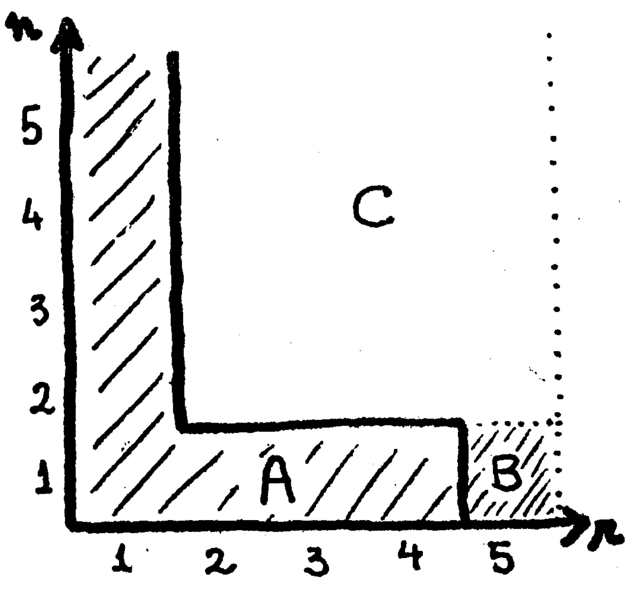
The problem we want to tackle here is that of giving mathematical substance to the so called 'maximum delay convention', as it is now known in Catastrophe Theory Literature.

We will briefly describe the general setting which will be considered here. Suppose we are given a state space X, a control space C, and a catastrophe manifold M, $\subset X \times C$, that is the critical set of some generic function on X parametrized by C; the minima of the function form a submanifold M^n of M. Suppose we now imagine some 'fast' dynamic on X parametrized by C, for which the minima are attractors, causing M^n to become the attracting manifold of the fast dynamic. If we also impose some 'slow' flow ψ on the control space, then this will induce a lifted flow ϕ 'near' the attracting manifold, M^n, which will be continuous most of the time, but will exhibit 'catastrophic jumps' when it comes to the boundary of M^n.



Our objective is to formalise this last statement, replacing the vague words 'fast', 'slow', 'near', and 'catastrophic jump' by the requirement that the lifted flow ϕ be on M^n. We shall give definitions of lift and precise generic conditions under which we shall prove the existence and uniqueness of lifts.

The technical difficulties may be summarized as follows: it is generally accepted that when the lift comes to the boundary of M^n then the state should 'jump to a neighbouring sink, the one into whose basin the original sink disappears' (see [15], page 156). An immediate objection to this sentence is that it does not always make sense, since the 'original sink' may find itself in a separatrix, and not in the basin of a 'neighbouring sink'. A natural question arises, as to whether it 'generically' makes sense. That is, can we give it a precise meaning, by restricting ourselves to open-dense sets $\mathcal{O}^* \subset \mathcal{O}$, the space of all objects determining the dynamics on X, parametrized by C (see Chapter 1, §2, for precise statements and Chapter 6 for a further discussion) and $\mathcal{V}^* \subset \mathcal{V}$, the space of all dynamics on C? Furthermore, can we prove the existence and uniqueness of a lift ϕ, under these circumstances? These are the problems we address ourselves here.



The picture below gives an idea of the present state of our research and also of our personal feelings on the subject, at the moment. We have completely solved the above questions in the region marked A. We believe that to extend this to region B is just a matter of some more technical work. As to region C, in certain cases (see Chapter 6), we have well defined conjectures, whereas in other all that we can (vaguely) say is that 'generic thinking' suggests that those questions should be answerable though we can not foresee at the moment, precise methods for its solution. This is basically due to the lack of mathematical development in areas closely related to these problems.

We would also like to comment on the relations of the questions above with catastrophe theory in a somewhat broader context. The central philosophical claim in qualitative dynamics is that observed processes in nature must be structurally stable, in the sense that they should 'remain', in some way, qualitatively the same, under small perturbations of whatever generates/parametrizes them, otherwise they would not be observable. Suppose now that the lift ϕ, according to some convention, is 'generically' existent and unique. It seems reasonable to 'identify' ϕ with the corresponding natural process under study, as far as the catastrophe theory method is concerned. Therefore, the solution of the questions proposed above would also allow one to consider in a precise mathematical context, through some 'natural' definition of 'similarity' among lifts the question of genericity - with respect to $\mathcal{O}/V$ - of the corresponding GLOBAL concept of structural stability. This seems, to my mind, a more satisfactory setting than a LOCAL concept (germ level) of structural stability.

1.0.2: On how we deal with the problem.

We assume the dynamics on the 'fibres', $X \times \{y\}$, $y \in C$, to be given by some $\sigma \in \mathcal{O}^*$, $\mathcal{O}^*$ open and dense in $\mathcal{O}$ (see 1-2 for precise statements) and then define $V^* \subset V$, which is subsequently proven to be open and dense, such that, for any fixed $v \in V^*$ the 'lift' ϕ exists and is unique. This appears to be the easiest approach to the problem formulated above.

The main results are stated in Chapter 1, where we also fix notations.

The solution corresponding to the case $n = 1$, $1 \leq r \leq 4$, is contained in Chapter 2-4. The proof that V^* is open and dense in V is based on transversality methods centered around the Thom Transversality Theorem on k-jet spaces; these are developed in Chapter 3 and applied in Chapter 4. Chapter 2 contains the proof that if $v \in V^*$ then ϕ exists and is unique.

Chapter 5 treats the case $r = 1$, $n \in \mathbb{N}$.

Chapter 6 contains some conjectures and concluding remarks. Each chapter is preceded by an introduction, where details of this general outline can be found.

1.1. DEFINITIONS:

Throughout this work $X = X^n$ will be a compact n-dimensional manifold, also referred to as the 'state space', $C = \mathbb{R}^r$, with $r \leq 5$, the 'control space'.

F denotes the set of all C^∞ functions $f: X \times C \to \mathbb{R}$, given the C^∞ Whitney topology.

DEFINITION 1:

$M_f = \{(x,y) \in X \times C | x$ is a critical point of $f_y, f_y(x) = f(x,y)\}$

DEFINITION 2:

Π_X and Π_C are the projections $X \times C \to X$ and C, respectively.

$\chi_f = \Pi_C / M_f$

DEFINITION 3:

We first remark that $\exists$ an open and dense set $F^* \subset F$ s.t. if $f \in F^*$ then M_f is an r-manifold and $\chi_f: M_f \to \mathbb{R}^r$ has only elementary catastrophes as singularities; this result is basically the same as in [16], Chapter 8 and is proved in Prop. 0, Chapter 2.

We call f 'generic' if $f \in F^*$

DEFINITION 4:

Let N be a differential manifold $V^s(N)$ is the space of C^s vector fields on N, $s \in \mathbb{N}$; $V(N)$ is the space of C^∞ vector-fields on N.

Note:

In the case $N = \mathbb{R}^r$, we identify $V^s(\mathbb{R}^r) \simeq C^s(\mathbb{R}^r, \mathbb{R}^r)$. We will use, in general, the letter 'v' to designate vector fields.

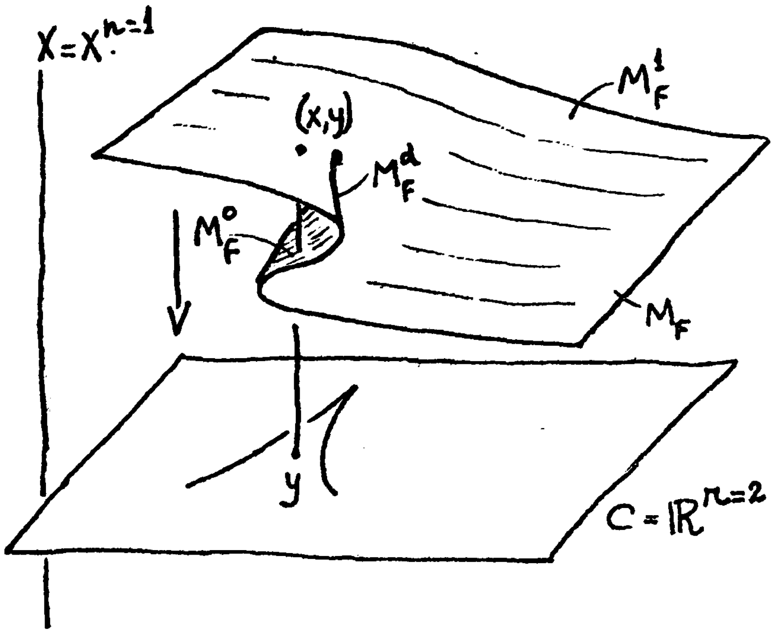
DEFINITION 5:

$M_f^k = \{(x,y) \in M_f | \exists$ chart (ϕ, U) around $x \in X$ s.t

$(f_y \, \phi^{-1})'' \, (\phi(x)): \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is diagonalizable to

$\sum_{i=1}^{n} \alpha_i x_i^2$, $\alpha_i = +1$ if $1 \leq i \leq k$, $\alpha_i = -1$ otherwise$\}$.

$M_f^d = \{(x,y) \in X \times C \mid \exists$ chart $(\phi,\mathcal{U})$ s.t. $(f_y\phi^{-1})''(\phi(x))$ is degenerate$\}$.



$\partial M_f^n = \overline{M_f^n} - M_f^n$

Note: Definitions above are the same if we substitute '$\exists$' by '$\forall$', since the relevant defining properties do not change under diffeomorphisms (see [3], pg. 105).

DEFINITION 6: Let be a dyn.system on N.

$\omega[\Phi](x)=\{y\in N \mid \exists\{\underset{\mathbb{R}^+}{t_n}\}, \underset{(\text{as } n\to\infty)}{t_n \to +\infty}$ s.t. $\lim\limits_{n\to\infty} \Phi(t_n,x)=y\}$

DEFINITION 7:

Let N as above, g a C^∞ Riemannian metric on TN(i.e. an element $g \in C^\infty(L_s^2(TN))$, compatible and pos. definite everywhere; to put it more explicitly, $\forall x \in N$, $\forall\, v \in T_xN$, fixed, $g_x \in L_s^2(T_x N,\mathbb{R})$, $g_x\,(v,v) \geq 0$, $g_x(v,v) = 0 \Leftrightarrow v=0$ and $(g_x(v,v)^{\frac{1}{2}}$ is a norm compatible with the original one in T_xN). Let $f:N \to \mathbb{R}$. Set $\nabla f(g)\,(x)$, or simply $\nabla f(x)$, when there is no possible confusion, as the unique vector in T_xN s.t.:

$$g_x(\nabla f(x);\omega) = df_x(w), \quad \forall\, \omega\in T_xN.$$

This defines a vector field, $\underset{(\nabla f(g))}{\underset{\parallel}{\nabla f}}$, on N, the gradient of f with respect to g.

DEFINITION 8:

Let $v \in \mathcal{V}(N)$.

$S(v) = \{x \in N \mid v(x) = 0 \in T_xN\}$.

DEFINITION 9:

Let N be compact, Φ_v the flow on N associated with $v \in \mathcal{V}(N)$.

Let P be a fixed point of Φ_v.

Define:

$$\begin{Bmatrix}\text{in}\\ \text{out}\end{Bmatrix}\text{set } [\Phi_v](P) = \{x \in N \mid \Phi_v(t,x) \to P \text{ as } t \to \begin{Bmatrix}+\infty\\ -\infty\end{Bmatrix}\}$$

We shall write $\begin{Bmatrix}\text{in}\\ \text{out}\end{Bmatrix}$ set (P) when it is clear enough what v is.

<u>DEFINITION 10:</u>

Let $f:N \to \mathbb{R}$, N as in Definition 9, g fixed. We say that $v \in \mathcal{V}(N)$ is subordinated to f if:

(A1) $S(v) = S(-\nabla f)$.

(A2) $\forall p \in S(v)$, fixed,

$$\begin{Bmatrix}\text{in}\\ \text{out}\end{Bmatrix}\text{ set } [\Phi_v](P) = \begin{Bmatrix}\text{in}\\ \text{out}\end{Bmatrix}\text{ set } [\Phi_{-\nabla f}](P).$$

<u>DEFINITION 11:</u>

Given $f:X \times C \to \mathbb{R}$, generic, a family $V = \{v_y\}_{y \in C}, v_y \in \mathcal{V}(X)$ is said to be compatible with f iff, $\forall y \in C$, fixed, v_y is subordinated to $f_y:X \to \mathbb{R}$, $f_y:x \to f(x,y)$ [Note: the reason for Definitions 10 and 11 is that we want to abstract those properties of gradients which we will use; that is, the nature of their sing. and in-sets].

<u>DEFINITION 12:</u>

If Φ is a dynamical system on N, define:

$$\omega[\Phi](W) = \bigcup_{x \in W} \omega[\Phi](x).$$

Write simply $\omega(W)$, if Φ is clearly fixed.

<u>DEFINITION 13:</u>

Let Φ be as above; then

separatrices of Φ = $\{x \in N \mid \nexists \text{ nghd } W \ni x \text{ s.t. } \omega(W) = \omega(x)\}$.

Write also sep Φ.

<u>DEFINITION 14:</u>

$y \in C$ is a bifurcation point for $f \Leftrightarrow \exists\, x \in X$ s.t. $(x,y) \in M_f^d$. C_f is the set of all such points.

DEFINITION 15:

Let f be given, V be a fixed compatible family.

Suppose $(x,y) \in \partial M_f^n$.

The local Maxwell set of f at (x,y) is the germ at y of:

$$J_{f,x,y} = \{\tilde{y} \in C \mid x \in \text{sep } \Phi_{\tilde{y}}\},$$

whereby $\Phi_{\tilde{y}}$ we mean the flow generated on X (compact) by $v_{\tilde{y}} \in V$. Please see page 4.1.1 for an illustrative example.

DEFINITION 16:

The f Maxwell set at y is the germ at y of

$$M_{f,y} = \bigcup_{(x_i,y)\in M_f^d} J_{f,x_i,y}.$$

We remark that the singularities (x_i,y) of the gradient field on X, compact, are isolated. Therefore, the union, as above, is finite.

DEFINITION 17:

Set $A_{f,y} = M_{f,y} \cup C_f$. Let $y \in C$, $v \in \overline{V(C)} = \{v \in V(C) \mid v \text{ is bounded}\}$. Let ψ_v be the associated flow (defined on $\mathbb{R} \times C$, since v is bounded). Then, set:

non-zero orbit of y under $v = O_y(v) = O_y = \bigcup_{\substack{t\in\mathbb{R}\\ t\neq 0}} \psi_v(t,y)$

positive orbit of y under $v = O_y^+(v) = O_y^+ = \bigcup_{t\in\mathbb{R}^+} \psi_v(t,y)$,

$$\mathbb{R}^+ = \{x\in\mathbb{R} \mid t>0\}.$$

ε-orbit of y under $v = O_y(\varepsilon,v) = O_y(\varepsilon) = \bigcup_{\substack{|t|<\varepsilon\\ t\neq 0}} \psi_v(t,y)$.

Note: We use the letter Φ for flows on X, and ψ for flows on C.

1.2. THE MAIN THEOREMS

Let f be generic, $f:X \times C \to \mathbb{R}$; let V, compatible with f, be fixed. Let $A_y = A_{f,y}$. (where a R-metric g has been fixed)

Set $\quad \mathcal{V}_f = \{v \in \overline{\mathcal{V}(C)} \mid v \text{ satisfies } H_1 \text{ and } H_2 \text{ below}\}$

H_1: $\forall y \in C_f$, Fixed, $\exists$ an $\varepsilon > 0$, s.t. $A_y \cap O_y(\varepsilon) = \emptyset$.

H_2: $\quad S(v) \cap C_f = \emptyset$.

THEOREM 1:

Let $v \in \mathcal{V}_f$, $\psi = \psi_v$ be the flow induced by v on C. $\exists$ a unique lift ϕ, with the following properties:

(1)

$$\begin{tikzcd} \mathbb{R}_0^+ \times \overline{M^n} \arrow[r, "\phi"] \arrow[d, "I\times\chi"'] & \overline{M^n} \arrow[d, "\chi"] \\ \mathbb{R}_0^+ \times C \arrow[r, "\psi"] & C \end{tikzcd}$$

is commutative, $\mathbb{R}_0^+ = \{t \in \mathbb{R} \mid t \geq 0\}$

(2) $\phi/_{\{0\}\times\overline{M^n}} \simeq I_{\overline{M^n}}$

(3) Let $(t,m) \in \mathbb{R}_0^+ \times \overline{M^n}$ be fixed. Then, $\exists\ \varepsilon = \varepsilon(t,m)$, $\varepsilon > 0$, such that

$\Pi_X\ \phi(t,m) \in \text{inset}\ (\Pi_X\phi(\tilde{t},m))$, $\forall \tilde{t} \in [t,t+\varepsilon)$.

The implicit vector field is $v_{\tilde{y},y} = \Pi_C\phi(\tilde{t},m)$.

(4) Define ϕ_m by: $\phi_m(t) = \phi(t,m)$, $m \in \overline{M^n}$ fixed. Then:

ϕ_m is left continuous at t, $\forall$ (t,m) fixed;

ϕ_m is continuous at t, provided $\psi(t,y) \notin C_f$,

$y = \Pi_C m$. Also, $\{t \mid \psi(t,y) \in C_f\}$ is a set of isolated points.

THEOREM 2:

Let $r \leq 4$, $n = 1$, f generic. $\exists\ \mathcal{V}^*$, open-dense in $\overline{\mathcal{V}(C)}$, $\mathcal{V}^* \subset \mathcal{V}_f$.

THEOREM 3:

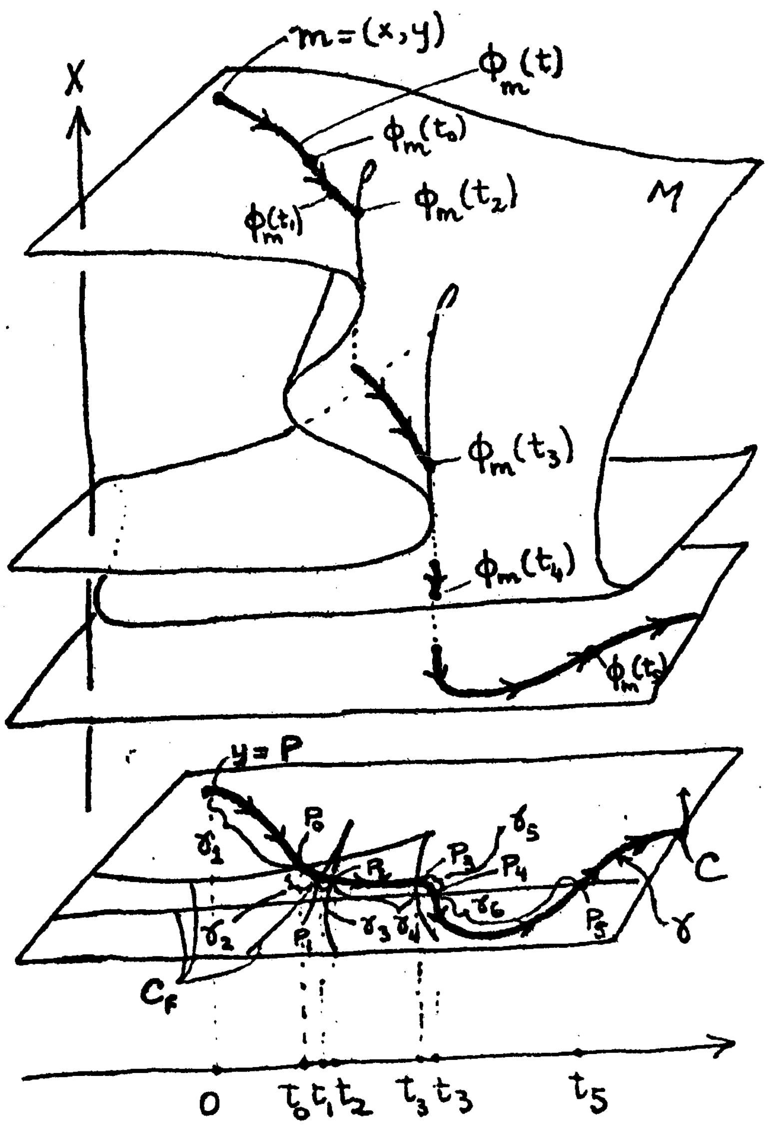
Let $r = 1$, $n \in \mathbb{N}$, f generic, $V = \{v_y\}$, the (one-parameter) compatible family be generic in the sense of [13] (see Theorem A in §4). Then, $\exists\ \mathcal{V}^*$, open-dense in $\overline{\mathcal{V}(C)}$,s.t, $\forall v \in \mathcal{V}^*$, fixed, $\exists$ a unique lift $\phi:\mathbb{R}_0^+ \times M^n \to \overline{M^n}$ with properties as in Theorem 1.

CHAPTER 2

2.0. INTRODUCTION

The aim, in this chapter, is to prove Theorem 1.

In §1 we collect some simple results, some of which also for later reference; the main reason for setting these propositions apart is, however, that they are just technicalities, needed in the proof of Theorem 1 (§2), and we felt that they might otherwise obscure that proof.



In §2, we construct the lifting, ϕ. Lemmas 1/3 show how to construct ϕ in 'easy' regions, i.e., where γ does not intersect C_f; in picture, see $\psi_y([0,t_0))$, which we denote by γ_1.

Lemma 4 is a technical assertion about the set $\{t_n\}$ of 'bad' points.

Lemma 5 tells how to extend the lift to $P_0 = \psi_y(t_0)$.

Lemma 6, which contains the central difficulty, shows how to uniquely do the jumping.

Finally, Lemmas 7 and 8 show how to inductively construct the rest of the lifting, extending first to γ_2, then P_1, then jumping again; to γ_3, then P_2, and so on.

$(P_i = \psi_y(t_i))$

<u>Note:</u> The 'jumps' at some of the P_i might be 'trivial' ('amplitude zero'), but this is irrelevant.

2.1. PRELIMINARY RESULTS

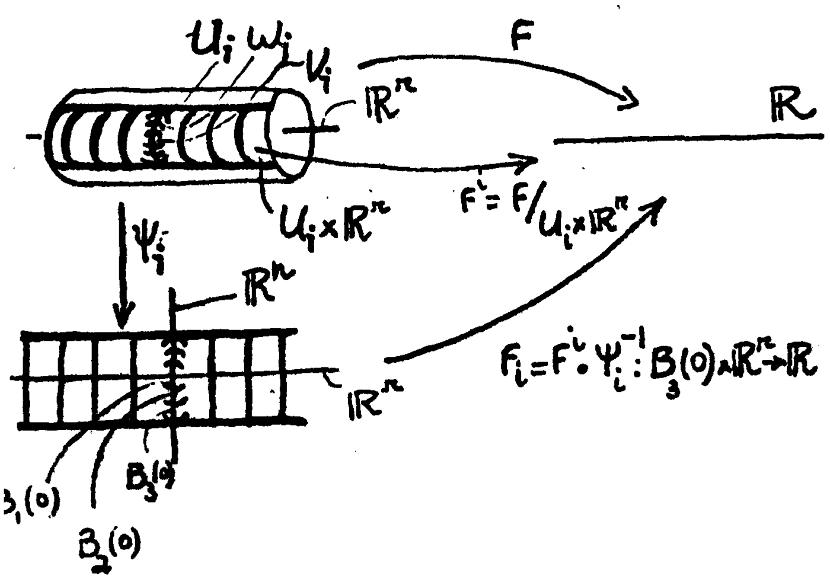
We initially prove Proposition 0, announced in Definition 3, Chapter 1; this generalizes Theorem 8.1 of [16] to the case where the state-space is an arbitrary n-dimensional compact manifold.

PROPOSITION 0:

Let X^n be a compact, C^∞, n-dimensional manifold, F be the set of all C^∞ Functions $X^n \times \mathbf{R}^r$ to $\mathbf{R}$, with the C^∞ Whitney topology, $r \in \{1,2,3,4,5\}$. $\exists$ an open-dense set $F^* \subset F$ such that M_f(see Definitio n 1) is a r-dimensional manifold and $\chi_f: M_f \subset X^n \times \mathbf{R}^r \to \mathbf{R}^r$ has only elementary catastrophes as singularities, where a point $(x,y) \in X^n \times \mathbf{R}^r$ is an elementary catastrophe for χ_f if $\exists$ a chart $\underset{(\phi \times I)}{\psi}$ for $X^n \times \mathbf{R}^r$ at (x,y) s.t. $\chi_{f\psi^{-1}}: M_{f\psi^{-1}} \subset \mathbf{R}^n \times \mathbf{R}^r \to \mathbf{R}^r$ has an elementary catastrophe (as definition in [1], Chapter 7) at $\psi(x,y)$.

Proof

We will initially prove two lemmas, from which the proposition easily follows.



First, we fix notation. Cover X^n with a finite number of charts, $\{U_i, \phi_i\}$ so that $\phi_i: U_i \to B_3(0) \subset \mathbf{R}^n$.

Let $\omega_i = \phi_i^{-1}(B_2(0))$.

$V_i = \phi_i^{-1}(B_1(0))$

$\psi_i = \phi_i \times I$

$f^i = f/U_i \times \mathbf{R}^r; f_i - f^i \circ \psi_i^{-1}$

Set then:

$F_i: B_3 \times \mathbf{R}^r \to J_n^7$ by:

$$F_i: (x,y) \to 7\text{-jet at } 0 \text{ of} \begin{cases} \mathbf{R}^n, 0 \to \mathbf{R}, 0 \\ x' \to f_i(x+x',y) - f_i(x,y). \end{cases}$$

Let $F_i = \{f \in F \mid F_i \pitchfork Q \text{ on } B_1 \times \mathbf{R}^r\}$, where Q is the stratification of J_n^7 as given in [16], Chapter 8. (Note: wlog, $\bigcup_i V_i$ covers X^n).

LEMMA 1

F_i is dense in F, for every fixed i.

Proof

Let $h \in F$, and A be an open set in $C^\infty(X^n \times \mathbb{R}^r, \mathbb{R})$ containing h.W.log., we can suppose that $A = B^k_\delta(h) = \{g \in F | d(j^k y(p), j^k h(p)) < \delta(p), \forall p \in X^n \times \mathbb{R}^r\}$, where k is a positive integer, d is a metric on $J^k(X^n \times \mathbb{R}^r, \mathbb{R})$, compatible with its topology, $\delta : X^n \times \mathbb{R}^r \to \mathbb{R}^+$ a continuous f^n, and $J^k(.,.)$, $j^k(.)$ are, respectively, the k-jet bundle, k-jet map [see [4] page 37].

Define $\delta[i] = \delta/U_i : U_i \times \mathbb{R}^r \to \mathbb{R}^+$, and

$$B_i = B_{\delta[i]}(h^i) = \{f^i \in C^\infty(U_i \times \mathbb{R}^r, \mathbb{R}) | d(j^k f^i(p), j^k h^i(p)) < \delta[i](p).\ \forall p \in U_i \times \mathbb{R}^r\},$$

where $h^i = h/U_i \times \mathbb{R}^r$, by definition.

B_i is an open nghd (in $C^\infty(U_i \times \mathbb{R}^r, \mathbb{R})$) of h^i.

Now ψ_i^{-1}: $B_3 \times \mathbb{R}^r \to U_i \times \mathbb{R}^r$ induces (see: note (1), page 49, [4]) a $(\psi_i^{-1})^* : C^\infty(U_i \times \mathbb{R}^r, \mathbb{R}) \to C^\infty(B_3 \times \mathbb{R}^r, \mathbb{R})$, given by $f^i \to f^i \circ \psi_i^{-1} \underset{\text{def.}}{=} f_i$.

Since ψ_i^{-1} is a diffeomorphism, $(\psi_i^{-1})^*$ is a homeomorphism (see note (2) page 49, [4]

Therefore, $C_i = (\psi_i^{-1})^* (B_i)$ is an open nghd of $h_i := h^i\ \psi_i^{-1}$, in $C^\infty(B_3 \times \mathbb{R}^r, \mathbb{R})$.

Let now $\xi : B_3 \times \mathbb{R}^r \to \mathbb{R}$ be a C^∞ bump function , s.t. $\xi \equiv 1$ on $B_1 \times \mathbb{R}^r$, $0 \le \xi \le 1$ everywhere and $\xi \equiv 0$ outside $B_2 \times \mathbb{R}^r$.

Let Γ: $(C^\infty(B_3 \times \mathbb{R}^r, \mathbb{R}))^4 \to C^\infty(B_3 \times \mathbb{R}^r, \mathbb{R})$ be given by:

$$(a,b,c,d) \longrightarrow a + b\ (c-d),$$

a continuous map.

The set $\{f_i \in C^\infty(B_3 \times \mathbb{R}^r, \mathbb{R}) | F_i \bar{\pitchfork} Q \text{ on } B_1 \times \mathbb{R}^r\}$ can be proven to be open and dense in $\{f_i | f_i \in C^\infty(B_3 \times \mathbb{R}^r, \mathbb{R})\}$; the proof is just the same as in [16] Chapter 8, except that $B_1 \times \mathbb{R}^r$ and not $\mathbb{R}^n \times \mathbb{R}^r$ has to be expressed as a union of compact sets.

Therefore, we can choose $\tilde{g}_i$ in this set, sufficiently close to h_i and so that:

$\Gamma(h_i,\xi,\tilde{g}_i,h_i) = h_i + \xi(\tilde{g}_i - h_i) =: g_i \in C_i$. This is because $\Gamma(h_i,\xi,h_i,h_i) = h_i$.

One then has: $\begin{cases} g_i \equiv h_i & \text{outside } B_2 \times \mathbb{R}^r \\ g_i \equiv \tilde{g}_i & \text{inside } B_1 \times \mathbb{R}^r \end{cases}$ Therefore $g_i \pitchfork Q$ on $B_1 \times \mathbb{R}^r$.

Therefore $g^i = (\psi_i)^* g_i = g_i \psi_i \in B_i$, and so

$$\underset{(C^\infty)}{g} = \begin{cases} h & \text{outside } \mathcal{U}_i \times \mathbb{R}^r \\ g_i & \text{on } \mathcal{U}_i \times \mathbb{R}^r \end{cases} \quad \text{is in } A \cap F_i \text{, as required.}$$

□

<u>LEMMA 2.</u>

Let $X \subset \mathcal{V}_i \times \mathbb{R}^r$ be a compact. $F_i^X = \{f \in F | F_i \pitchfork Q \text{ on } \psi_i(X)\}$ is C^{k+1} (hence C^∞) open.

Proof

Let $f \in F_i^X$. We will produce an open neighbourhood of f contained in F_i^X. Let d be a metric on $J^{k+1}(\mathcal{U}_i \times \mathbb{R}^r,\mathbb{R})$, compatible with its topology.

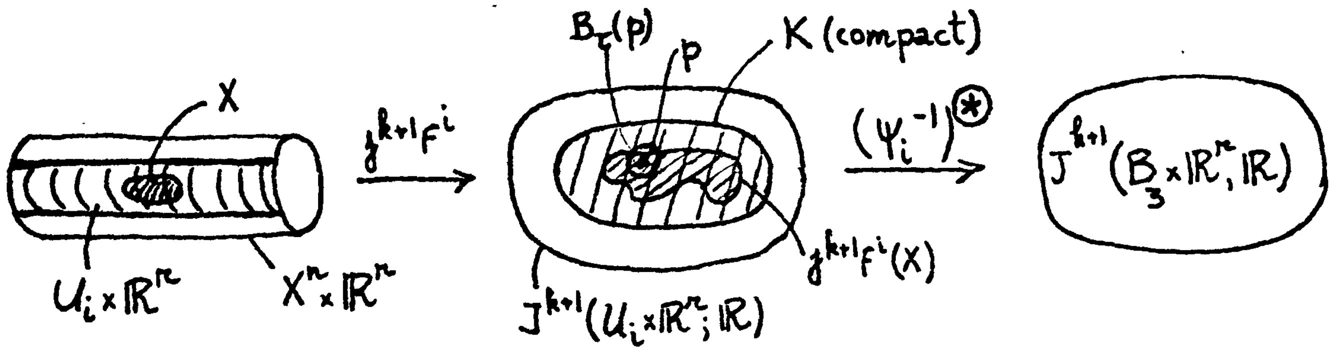
Claim:

Given $\varepsilon > 0$, $\exists\ \delta > 0$ s.t.

$$\boxed{d(j^{k+1}f^i(p),j^{k+1}g^i(p)) < \delta \Rightarrow \underline{d}(j^{k+1}f_i(q),j^{k+1}g_i(q)) < \varepsilon\ , \quad \forall p \in X}\ \circledast$$

where $q = \psi_i(p)$, $f^i,g^i \in C^\infty(\mathcal{U}_i \times \mathbb{R}^r,\mathbb{R})$, $f_i,g_i \in C^\infty(B_3 \times \mathbb{R}^r,\mathbb{R})$ as defined before. The distance $\underline{d}$, on the r.h.s., comes from the standard distance in $\mathbb{R}^s \cong J^{k+1}(\mathbb{R}^n \times \mathbb{R}^r,\mathbb{R})$ in a canonical way (see [4], pg.39).

Proof of Claim:



We first remark that $\exists$ K, compact, $\tau > 0$, $K \supset j^{k+1}f^i(X)$, also compact ($j^{k+1}f^i$ is continuous), s.t. $B_\tau(p) \subset K$, $\forall p \in j^{k+1}f^i(X)$.

Indeed: given $p \in j^{k+1}f^i(X) \subset J^{k+1}(U_i \times \mathbb{R}^r, \mathbb{R})$, $\exists$ nghd N_p and chart $\psi_p : N_p \to \psi_p(N_p) \subset$ some $\mathbb{R}^s$, $p \to 0$, $\psi_p(N_p)$ limited, w.l.o.g.; consider $B_{\xi(p)}(p) \subset N_p$ and cover $j^{k+1}f^i(X)$ with a finite number of such balls. Set

$U = \bigcup_{j(\text{finite})} B_{\xi(p_j)}(p_j)$ and construct λ,

$\lambda : j^{k+1}f^i(X) \to \mathbb{R}^+$ by $p \to d(p, C(U)) > 0$. Let $\tau = \min_{p \in j^{k+1}f^i(X)} \lambda(p) > 0$, where $C(U)$ means 'complement of U'.

Now, $\oplus$ $\boxed{\overline{\psi_{p_j}(B_{\xi(p_j)}(p_j))}}$ is compact; $K = \bigcup_j \psi_{p_j}^{-1}(\oplus) \supset U$ is compact, and $B_\tau(p) \subset K$, $\forall p \in j^{k+1}f^i(X)$. This concludes the remark.

ψ_i^{-1} induces naturally a $(\psi_i^{-1})^{\circledast} : J^{k+1}(U_i \times \mathbb{R}^r, \mathbb{R}) \to J^{k+1}(B_3 \times \mathbb{R}^r, \mathbb{R})$ (see (3), pg.39, [4]). Since ψ_i^{-1} is a diffeomorphism, so is $(\psi_i^{-1})^{\circledast}$ (see (3), pg.40, [4]). In particular, $(\psi_i^{-1})^{\circledast}$ is uniformly contiuous on K, Therefore $\exists\ \zeta$ s.t.

$d(p_1, p_2) < \zeta \Longrightarrow \underline{d}((\psi_i^{-1})^{\circledast}(p_1), (\psi_i^{-1})^{\circledast}(p_2)) < \varepsilon$, $\quad \forall (p_1, p_2) \in K \times K$.

By taking $\delta = \min\{\zeta, \tau\}\ \xi$, we get implication $\circledast$. $\square$ of claim

The proof of this lemma (and also of the rest of Proposition 0) now follows the same lines as those of the open lemmas in Chapter 8, [16] Fix $p \in X$. F_i is $\pitchfork$ to Q at $q = \psi_i(p)$. By continuity, $F_i \pitchfork Q$ in a nghd of q, $\tilde{N}$, say, which we assume to be compact, w.l.o.g. This remains true for suff. small changes of F_i and TF_i on $\tilde{N}$; so, for suff. small changes in $j^{k+1} f_i$ on $\tilde{N}$. Since $\tilde{N}$ is compact, $\exists\, \varepsilon > 0$ s.t. $d(j^{k+1}g_i(q);j^{k+1}f_i(q)) < \varepsilon \Rightarrow G_i \pitchfork Q$ on $\tilde{N}$. Therefore, from the claim above, $V^{k+1}_{\delta,N}(f) \subset F_i^N = \{h \in F \mid H_i \pitchfork Q \text{ on } \tilde{N}\}$, $N = \psi_i^{-1}(\tilde{N})$.

Cover the compact $\tilde{X} = \psi_i(X)$ by a finite number of $\tilde{N}_j$, $N_j = \psi_i^{-1}(\tilde{N}_j)$, at each stage choosing convenient ε_j, δ_j, so that $V^{k+1}_{\delta_j,N_j}(f) \subset F_i^{N_j}$. Let $\delta = \min.\ \delta_j$. One has: $V^{k+1}_{\delta,X}(f) = \bigcap_j V^{k+1}_{\delta,N_j} \subset \bigcap_j V^{k+1}_{\delta_j,N_j} \subset \bigcap_j F_i^{N_j} = F_i^X$, as required. □

LEMMA 3:

Let $X = \bigcup_{j=1}^{\infty} X_j$, a countable union of disjoint compacts X_j, with disjoint nghds Y_j, $X \subset V_i \times \mathbb{R}^r$.

$F_i^X = \{f \in F \mid F_i \pitchfork Q \text{ on } \psi_i(X)\}$ is C^{k+1} (therefore C^∞) open.

Proof

For each X_j, construct δ_j s.t. $V^{k+1}_{\delta_j,X_j}(f) \subset F_i^{X_j}$. Construct bump functions $\beta_j : X^n \times \mathbb{R}^r \to [0,1]$ s.t. $\beta_j \equiv 1$ on X_j, $\beta_j \equiv 0$ outside Y_j. Set $\mu : X^n \times \mathbb{R}^r \to \mathbb{R}^+$ by $\mu = 1 - \sum_{j=1}^{\infty} (1-\delta_j)\beta_j$.

$$V^{k+1}_{\mu}(f) \subset \bigcap_{j=1}^{\infty} V^{k+1}_{\delta_j,X_j}(f) \subset \bigcap_{j=1}^{\infty} F_i^{X_j} = F_i^X.$$

□

LEMMA 4:

$F_i = F_i^{V_i \times \mathbb{R}^r}$ is C^{k+1} open. (Therefore C^∞ open)

Proof

Follows easily, by expressing (see also Lemma 6, Chapter 8, [16]) $V_i \times \mathbb{R}^r$ as a (finite) union of sets with the properties of X as in Lemma 3. □

Proof of Proposition 0:

Set $F^* = \bigcap_{\text{finite}} F_i$. From the above lemmas, F^* is open and dense (in the Whitney C^∞ top.). Let $(x,y) \in X^n \times \mathbb{R}^r$ be in M_f, $f \in F^*$, fixed $(x,y) \in V_i \times \mathbb{R}^r$, for some i. Set $M^i :\!= M_{f/V_i \times \mathbb{R}^r} = M_f \cap (V_i \times \mathbb{R}^r)$.

Now, $M_i :\!= M_{f_i/B_1 \times \mathbb{R}^r} = \psi_i(M^i)$, and M_i is a r-submanifold, since $F_i \pitchfork Q$ on $B_1 \times \mathbb{R}^r$, from Theorem 8.1, ([16]). From this, M_f is an r-submanifold. Now, if (x,y) is singular for χ_f, $\psi_i(x,y)$ is singular for χ_{f_i}, hence an elementary catastrophe (Theorem 8.1 of [16]), as required. □

Throughout the rest of this chapter, $f\colon X^n \times \mathbb{R}^{r\leq 5} \to \mathbb{R}$ will be a fixed function in F^* (see Proposition 0 above), where X^n, compact, is given a Riemannian Metric g and V is a (fixed) family, compatible with f (see Definition 11).

We now show that, from a local point of view, and as far as gradients are concerned, one can assume that $f\colon \mathcal{O} \times \mathbb{R}^r \to \mathbb{R}$, $\mathcal{O}$ an open nghd of $0 \in \mathbb{R}^n$; we can actually prove the following:

<u>Remark 1:</u>

Suppose $(x,y) \in X^n \times \mathbb{R}^r, f_y\colon X^n \to \mathbb{R}$ given by $f_y(x) = f(x,y)$ and $\nabla f_y(g)$ the gradient field of f_y with respect to g (see Definition 7). Then $\exists$ chart $(\psi = \phi \times I;\ U \times \mathbb{R}^r)$ for $X^n \times \mathbb{R}^r$ around (x,y), $\phi(x) = 0 \in \mathbb{R}^n$, s.t. the vector field (on $\phi(U)$): $Z \overset{\nu}{\longmapsto} (T_{\phi^{-1}(Z)}\, \phi \circ \nabla f_y(g) \circ \phi^{-1})$. (i.e., just $\nabla f_y(g)$ on U 'transported' to $\phi(U) \subset \mathbb{R}^n$ by ϕ) equals $\nabla(f_y \phi^{-1})(g_\phi)$, where g_ϕ is a Riemannian metric on $\phi(U)$, with $g_\phi(0)$ being just the standard inner product of $\mathbb{R}^n$

To see this, we first note that, if $(\psi = \phi \times I;\ U \times \mathbb{R}^r)$ is any chart, then ν is equal to $\nabla(f_y\ \phi^{-1})(g_\phi)$, where g_ϕ is the Riemannian metric on $\phi(U)$ given by:

$$g_\phi(z) = g(\phi^{-1}(z)) \circ (T_z\phi^{-1} \times T_z\phi^{-1}).$$

Indeed: $g_\phi(Z)\ (\nu(Z);\omega) = g_\phi(z)((T_{\phi^{-1}(Z)}\ \phi \circ \nabla f_y(g) \circ \phi^{-1})(Z);\omega) =$

$= g(\phi^{-1}(Z))(\nabla f_y(g)(\phi^{-1}(Z));(T_Z\phi^{-1})\omega) = df_y(\phi^{-1}(Z)).\ [T_Z\phi^{-1}(\omega)] = d(f_y\phi^{-1})(Z).\omega$

If at $Z = 0$ one has that the matrix (with respect to the standard basis of $\mathbb{R}^n$) of $g_\phi(0)$, $G_\phi(0)$, is not the identity, then, by a further (linear) diffeomorphism, $\phi^*(0 \to 0)$, one gets a new ν, gradient of $f_y(\phi^{-1}\phi^{*-1})$ with respect to $g_{\phi\phi^*}$, with $G_{\phi\phi^*}(0) \overset{\text{Ⓐ}}{=} [\phi^{*-1}]^T\ G_\phi(0)[\phi^{*-1}] = I$, for convenient choice of ϕ^*. (This is so because $G_\phi(0)$ is symmetric, positive definite and therefore has only positive eigenvalues, being reducible to the identity - see pg. 310 [8]; the equality Ⓐ comes from linear algebra).

Summarizing: in the propositions that will follow, concerning local analysis of gradients, there is no loss of generality in supposing $f:\mathbb{R}^n \times \mathbb{R}^r \to \mathbb{R}$, $\mathbb{R}^n$ endowed with a Riemannian metric g, $g(0)$ = standard inner product of $\mathbb{R}^n$.

Let $(x,y) \in M\ (= M_f)$. We know that (via some chart ψ-see Proposition 0) f_y (germ of) is right equivalent (see [3] and [16]) to:

either (a) $h(x_1,\ldots,x_n) = \sum_{i=1}^{n} \varepsilon_i x_i^2$, where $\varepsilon_i = +1$ or -1

or (b) one of the polynomials which generates one of the elementary catastrophes.

It follows that ∇f_y has isolated singularities. Since $\chi^{-1}(\{y\})$ is the set of singularities of ∇f_y (see Definition 1), and X^n is compact, $\chi^{-1}(\{y\})$ is finite, $\forall y \in \mathbb{R}^r$.

We define $\#k:\mathbb{R}^r \to \mathbb{N}$ by setting $\#k(y)$ to be the number of elements in $\chi^{-1}(\{y\})$ which correspond to case (a), with $\varepsilon_i = +1$, if $1 \le i \le k$, $\varepsilon_i = -1$, if $k < i \le n$. Analogously, $\#s(y)$ is the number of elements in $\chi^{-1}(\{y\})$ corresponding to case (b). $\#t = \sum_{k=0}^{n} \#k + \#s$. We also use the notations

$$\begin{aligned} B_\delta(x) &= \{x' \in \text{some Banach space} \mid \|x'-x\| < \delta\}, \\ D_\delta(x) &= \{ \quad " \quad \mid \quad " \quad \le \delta\} \\ S_\delta(x) &= \{ \quad " \quad \mid \quad " \quad = \delta\} \end{aligned}$$

and $ac_B(A)$ = set of accumulation points of A in B, simply ac(A), when no confusion is possible.

PROPOSITION 1:

Let $h(x) = \sum_{i=1}^{n} \varepsilon_i x_i^2$, $x = (x_1,\ldots,x_n)$; $\varepsilon_i = \begin{cases} +1, & \text{if } 1 < i \le k \\ -1, & \text{if } k < i \le n. \end{cases}$

Suppose Φ is a diffeomorphism of $\mathbb{R}^n$, $\Phi(0) = 0$, $\mu = h\Phi$, g a Riemannian metric on $\mathbb{R}^n$, $g(0)$ = standard inner product.

Then $D[(-\nabla\mu)(g)(0)] = -2A^T I_k A$, where

$$I_k = \left[\begin{array}{cc} \left.\begin{matrix} 1 & & 0 \\ & \ddots & \\ & & 1 \end{matrix}\right\}k & \\ \hline & \begin{matrix} 0 & -1 & \\ & & \ddots \\ & & -1 \end{matrix} \end{array}\right], \quad A = \left\{ \frac{\partial \Phi_i}{\partial x_j}(0) \right\}.$$

Proof

Expanding Φ in Taylor Series around 0, we get:

$\Phi(x) = A.x$ + higher terms.

Hence, $\mu(x) = (h\Phi)(x) = \sum_{i=1}^{n} \varepsilon_i \left(\sum_{j=1}^{n} a_{ij} x_j\right)^2$ + higher terms.

Therefore, $\frac{\partial u}{\partial x_k}(x) = 2 \sum_{j=1}^{n} \sum_{i=1}^{n} (\varepsilon_i a_{ik}\ a_{ij})x_j$, so that:

$$(-\nabla u)(g)(X) = (-2 \sum_{k=1}^{n} \sum_{j=1}^{n} \sum_{i=1}^{n} (\varepsilon_i h_{k1}(x) a_{ik} a_{ij})x_j; \ldots; 2 \sum_{k=1}^{n} \sum_{j=1}^{n} \sum_{i=1}^{n} (\varepsilon_i h_{kn}(x) a_{ik} a_{ij})x_j) +$$

(see [3],pg.248) + higher terms,

where $(h_{ij}(x))$ is the matrix inverse to $(g_{ij}(x))$.

matrix of g(0)

Since, at 0, $g_{ij} (= g_{ij}''(0)) = I$, $(h_{ij}(0)) = I$ Therefore we get

$$(-\nabla u)(g)(0) = (-2 \sum_{j=1}^{n} \sum_{i=1}^{n} (\varepsilon_i a_{ij} a_{i1})x_j; \ldots; -2 \sum_{j=1}^{n} \sum_{i=1}^{n} (\varepsilon_i a_{ij} a_{in})x_j) + \text{higher}$$

Therefore $D(-\nabla u)(g)(0).x = -2A^T\ I_k A.x$, as wanted. □

PROPOSITION 2:

Let $(0,0) \in M^k$, w.l.o.g; let η be the germ at 0 of $f_o(f:\mathbb{R}^n \times \mathbb{R}^r \to \mathbb{R})$ and $g = f_o\ \Pi_n$(i.e., the germ at 0 of $f_o\ \Pi_n:\mathbb{R}^n \times \mathbb{R}^r \to \mathbb{R}$), Π_n be projection (on $\mathbb{R}^n$).

Then (r,g) is an universal unfolding of η (see [16] for the definition of universal unfolding).

Proof

As η has codimension 0 (see [16]), $(0,\eta)$ is an universal unfolding of η. Let (s,h) be an unfolding of η. By definition of universal unfolding, $\exists$ $(\emptyset;\overline{\Phi};\varepsilon)$, an unfolding morphism: $(s,h) \to (0,\eta)$. There also $\exists$ a morphism, $(\tilde{\emptyset};\tilde{\overline{\Phi}};\varepsilon)$: $(0,\eta) \to (r,g)$: just define $\begin{array}{c}\tilde{\emptyset}: x \to (x,0), \\ \mathbb{R}^n \quad \mathbb{R}^n \times \mathbb{R}^r\end{array}$

$\tilde{\overline{\Phi}} : \begin{array}{c}\mathbb{R}^0 \to \mathbb{R}^r \\ \{0\} \to 0\end{array}$ and $\tilde{\varepsilon}$: $\{0\} \to 0 \in \mathbb{R}$.

Thus $(\emptyset\tilde{\emptyset};\overline{\Phi}\tilde{\overline{\Phi}};\varepsilon)$ is a morphism $(s,h) \to (r,g)$. Therefore, (r,g) is universal. $\square$

COROLLARY:

(r,g) and (r,f) are isomorphic (where, by abuse, we write also f for the germ of f at 0).

Proof

Since f is generic (we are again thinking of f as from $\mathbb{R}^n \times \mathbb{R}^r \to \mathbb{R}$; as pointed out before, there is no loss of generality in this, since we are working with germs - see also Propostion 0), it is a 7-transversal unfolding of η. So (r,f) is an universal unfolding of η (see [16]). Corollary follows from Theorem 6.9 of [16] and Proposition 2. $\square$

REMARK 2:

(We follow the notation of [16].) A consequence of this corollary is that $\exists$ isomorphism, $(\emptyset,\overline{\Phi},\varepsilon)$: $(r,g) \to (r,f)$, with $\underbrace{g = f\emptyset + \varepsilon\Pi_r}_{\text{germ equation}}$. We recall (from [16]) that $\emptyset$ and $\overline{\Phi}$ are diffeomorphism germs. If $\emptyset(x,y)=(\emptyset^1(x,y);\emptyset^2(x,y))$ then our morphism "preserves fibres", i.e. $\Pi_r\,\emptyset = \overline{\Phi}\Pi_r$, or, equivalently, $\emptyset^2(x,y) = \overline{\Phi}(y)$. To simplify things we use the notation $\phi = \emptyset^1$, when referring to the above $\emptyset$.

Let $0 \in M^k \subset M_f$, as in Proposition 2. Since χ is not singular at 0, there is no loss of generality if we suppose that, in some (sufficiently small) neighbourhood of 0, $M_f \subset \mathbb{R}^r$. We shall assume this in Proposition 3 below; this implies $\phi_y(0) = 0$, if y is small enough so that $(0,y) \in$ that neighbourhood (see also Remark 1, 2.1(6)/(7)).

PROPOSITION 3:

Let $0 \in M^k$; $f_0 = h\Phi$, h, Φ as in Proposition 1; $g = f\emptyset + \epsilon\Pi_r$ and ϕ as

in Remark 2 (above).

Then, for y near 0,

$$D(-\nabla f_y)(0) = -2M^T I_k M, \text{ where } M = M(y) = D\Phi(0) \left\{\frac{\partial\phi^i}{\partial x_j}(\phi^{-1}(y),0)\right\}^{-1}$$

Proof

From the definition of g and properties of unfoldings and unfolding isomorphisms, the following (germ) equations hold:

$$\begin{cases} g_y = f_0 \\ g_y = \lambda_y f_{\bar{\Phi}(y)} \phi_y \end{cases}$$

Now, $h\Phi = f_0 = g_y = \lambda_y f_{\bar{\Phi}(y)}\phi_y$.

We just abandon λ_y, since $\lambda_y: \mathbb{R} \to \mathbb{R}$, $t \to t + \epsilon(y)$, a translation, does not affect the gradient field of $f_{\bar{\Phi}(y)}$.

So, $f_{\bar{\Phi}(y)} = h(\Phi\phi_y^{-1})$, and, by Proposition 1:

$$D(-\nabla f_{\bar{\Phi}(y)})(0) = -2(D\Phi(\underbrace{\phi_y^{-1}(0)}_{\overset{\shortparallel}{0}})D\phi_y^{-1}(0))^T I_k(D\Phi(\underbrace{\phi_y^{-1}(0)}_{\overset{\shortparallel}{0}})D\phi_y^{-1}(0))$$

and, since $D\phi_y^{-1}(0) = \left\{\frac{\partial\phi^i}{\partial x_j}(y,0)\right\}^{-1}$,

$$D(-\nabla f_y)(0) = -2\left(D\Phi(0)\left\{\frac{\partial\phi^i}{\partial x_j}(\bar{\Phi}^{-1}(y),0)\right\}^{-1}\right)^T I_k\left(D\Phi(0)\left\{\frac{\partial\phi^i}{\partial x_j}(\phi^{-1}(y),0)\right\}^{-1}\right),$$

as claimed. □

COROLLARY:

M^k is open in M, $\forall\, k$(i.e., $k = 0,\dots,n$).

Proof

Everything as above, Proposition 3 implies that, for y near 0, $D(-\nabla f_y)(0)$ has signature $n-2k$, hence $(0,y) \in M^k \subset M_f \subset \mathbb{R}^r$ for some (open in M_f) neighbourhood of 0. $\square$

REMARK 3:

The openness of M^k can be also obtained as a consequence of the local stability of hyperbolic fixed points. (see Theorem 3, page 82, of [10]; the point is that, when one has a r-parameter family of gradients of a generic f, an elementary proof, as above, is possible.

PROPOSITION 4:

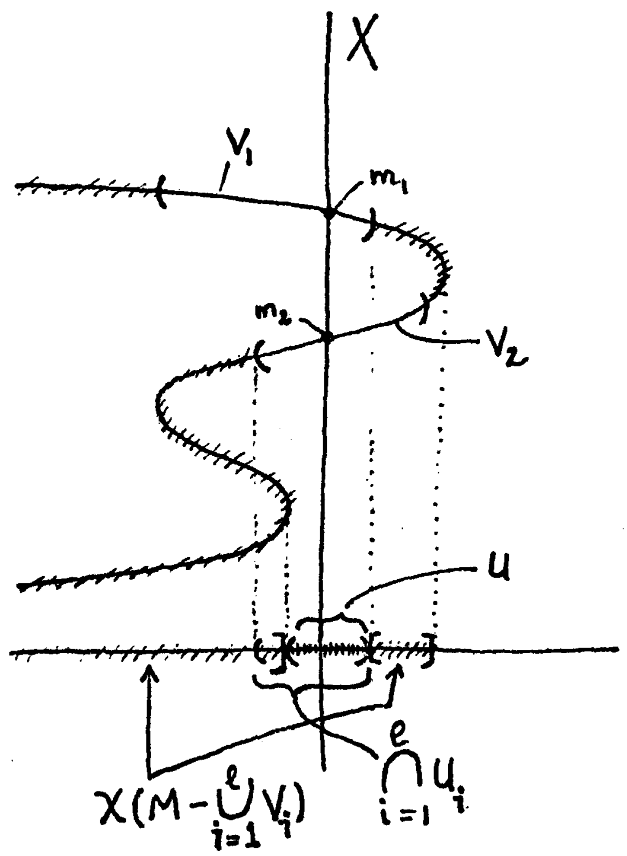
χ is closed.

Proof

M is closed in $X^n \times \mathbb{R}^r$ because it is locally algebraic (with respect to suitable local co-ordinates).

Given any closed k disk $D \subset \mathbb{R}^r$, then $X \times D$ is compact, hence $\chi^{-1}(D) = M \cap (X \times D)$ is compact, and hence $\chi/\chi^{-1}(D)$ closed and hence χ is closed. $\square$

PROPOSITION 5:



Suppose $y \notin C_f$. Then #t (see 2.1(8)), is locally constant at y.

Proof

Let $\#t(y) = \ell$, so that $\chi^{-1}(y) = \{m_1,\dots,m_\ell\}$. As y is regular value for χ, $D\chi(m_\ell)$ is an isomorphism, $i=1,\dots,\ell$. Hence, we can choose neighbourhoods V_i of m_i, open in M, disjoint from each other, s.t. X/V_i is a diffeomorphism on $\mathcal{U}_i$, open neighbourhood of y. Now $M - \bigcup_{i=1}^{\ell} V_i$ is closed in M therefore (from Proposition 4) $X(M - \bigcup_{i=1}^{\ell} V_i)$ is closed.

Set:

$$\mathcal{U} = \bigcap_{i=1}^{\ell} \mathcal{U}_i - X(M - \bigcup_{i=1}^{\ell} V_i).$$

This is open and $\neq \emptyset$, since $y \in \mathcal{U}$. Now, if $\tilde{m} \in X^{-1}(\tilde{y}), \tilde{y} \in \mathcal{U}$, then $\tilde{m} \in V_i$, some i; otherwise we would get $\tilde{m} \in X(M - \bigcup_{i=1}^{\ell} V_i)$, which implies $\tilde{m} \notin \mathcal{U}$.

Hence, the elements in $X^{-1}(\tilde{y})$ are precisely $\{(X/V_i)^{-1}(\tilde{y})\}_{i=1,\dots,\ell,}$ where $(X/V_i)^{-1}$ stands for the inverse of the diffeomorphism X/V_i; and so $\#t(\tilde{m}) = \ell, \forall \tilde{y} \in \mathcal{U}$.

We remark that the above argument also shows that $X^{-1}(\mathcal{U}) = \bigcup_{i}^{\ell} (X/V_i)^{-1}(\mathcal{U})$.

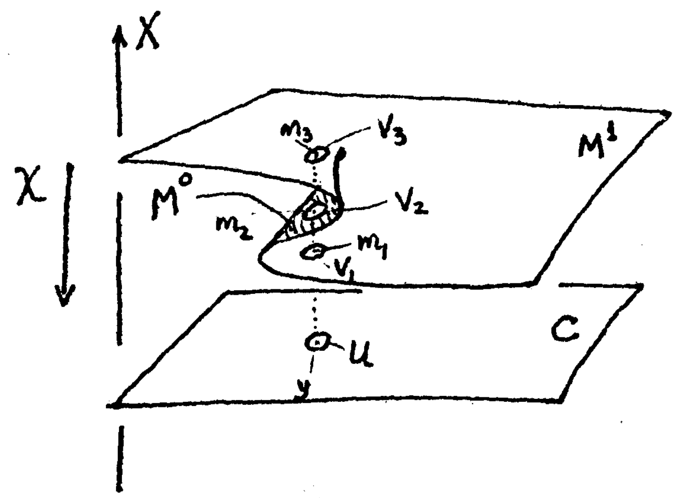
□

COROLLARY 1:

Suppose $y \notin C_f$, k fixed. Then #k is locally constant at y. ($k \in \{0,\dots,n\}$).

Proof

We first note that, if $I_k = \{i \in \{1,\dots,\ell\} | (\chi/V_i)^{-1}(y) \in M^k\}$, then we can suppose, $\forall\, i \in I_k, (\chi/V_i)^{-1}(u) \subset M^k$, w.l.o.g. This is so because M^k is open in M and $(\chi/V_i)^{-1}$ is a diffeomorphism.



The corollary follows immediately; in particular one also has:

$\chi_k^{-1}(u) = \chi^{-1}(u) \cap M^k = \bigcup_{i \in I_k} (\chi/V_i)^{-1}(u) = \bigcup_{i \in I_k} (\chi_k/V_i)^{-1}(u)$, where $\chi_k = \chi/_{M^k}$. □

COROLLARY 2:

Suppose $W \cap C_f = \emptyset$, $W \subset C$, path connected. Then #k, k = 0,...,n (and hence #t) is constant on W. Moreover $\chi^{-1}(W) \xrightarrow{\chi} W$ and $\chi_k^{-1}(W) \xrightarrow{\chi_k} W$ are covering spaces for W [see [5]].

Proof

If $y_1, y_2 \in W$, take a path joining them, and cover it by (a finite number of) open sets such that #k is constant in each one of them (Corollary 1). Corollary - first part of it - follows by taking points in the intersections. Last part is a re-statement of the equalities $\chi^{-1}(u) = \bigcup_1^{\ell} (\chi/V_i)^{-1}(u)$ and $\chi_k^{-1}(u) = \bigcup_{i \in I_k} (\chi_k/V_i)^{-1}(u)$. □

REMARK 4:

If $f: M \to N$, differentiable, M without boundary and compact, M and N of the same dimension, y regular value of f, then $\#f^{-1}(y)$ [in our case we denote $\#\chi^{-1}(y)$ by #t, omitting χ from the notation] is finite and locally constant.

$(\#\chi_y^{-1}(y))$ (#k) (χ_k)

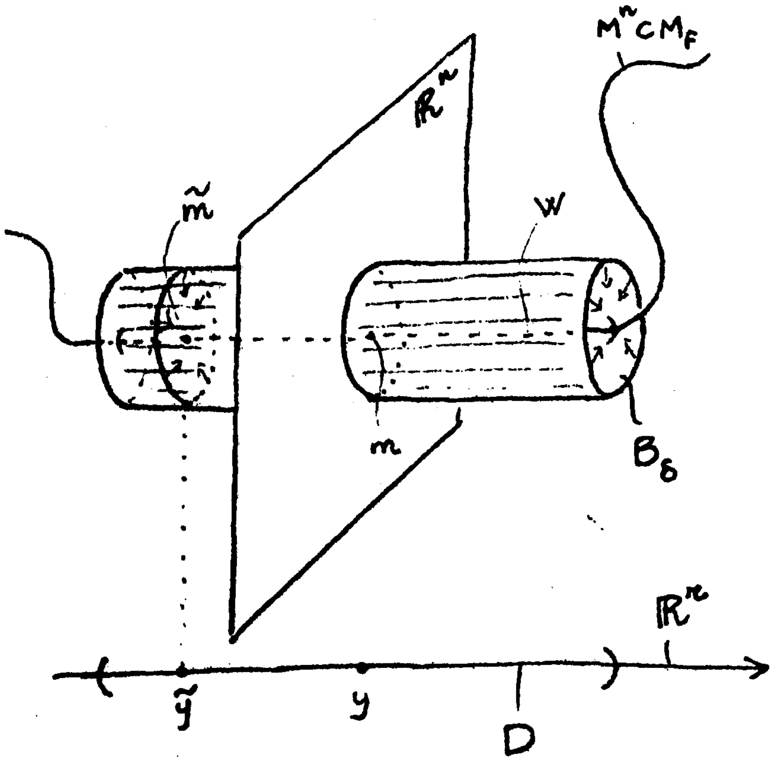
This is a standard result in differential topology. In the above, i.e, Proposition 5, Corollary 1, we have just proved that this extends to our case, although $M(M^k)$ is not necessarily compact without boundary.

We now prove a 'local' proposition, which will be used in the proof of Theorem 1.

PROPOSITION 6:

Let $m = (x,y) \in M^n$. Then, $\exists$ neighbourhood W of m, in M^n, and a $\delta > 0$, s.t., $\forall\ \tilde{m} = (\tilde{x},\tilde{y}) \in W$ fixed, $B_\delta(\tilde{x}) \subset$ inset $[\Phi_v](\tilde{x})$, $v = -\nabla f_y$.

Proof



Fix $(x,y) \in M^n$. $\exists$ a small closed disk neighbourhood B of x s.t.

(i) $-\nabla f_y$ has one generic fixed point in B.

(ii) $-\nabla f_y$ is transverse inwards to B.

These are open properties, and hence remain true, $\forall\ -\nabla f_{\tilde{y}}$, for $\tilde{y} \in$ some small neighbourhood D of y in C. Choose $\delta > 0$ s.t. $B_\delta \subset B$, and set $W = M \cap (D \times B_\delta)$. Clearly $B_\delta(x) \subset$ inset $[\Phi_v](\tilde{x})$, $\tilde{m} \in W$, proving our proposition.

2.2. PROOF OF THEOREM 1

Let V be a family compatible with f, and $v \in V_f$, fixed. The symbol ψ will be used for the flow induced by v.

LEMMA 1:

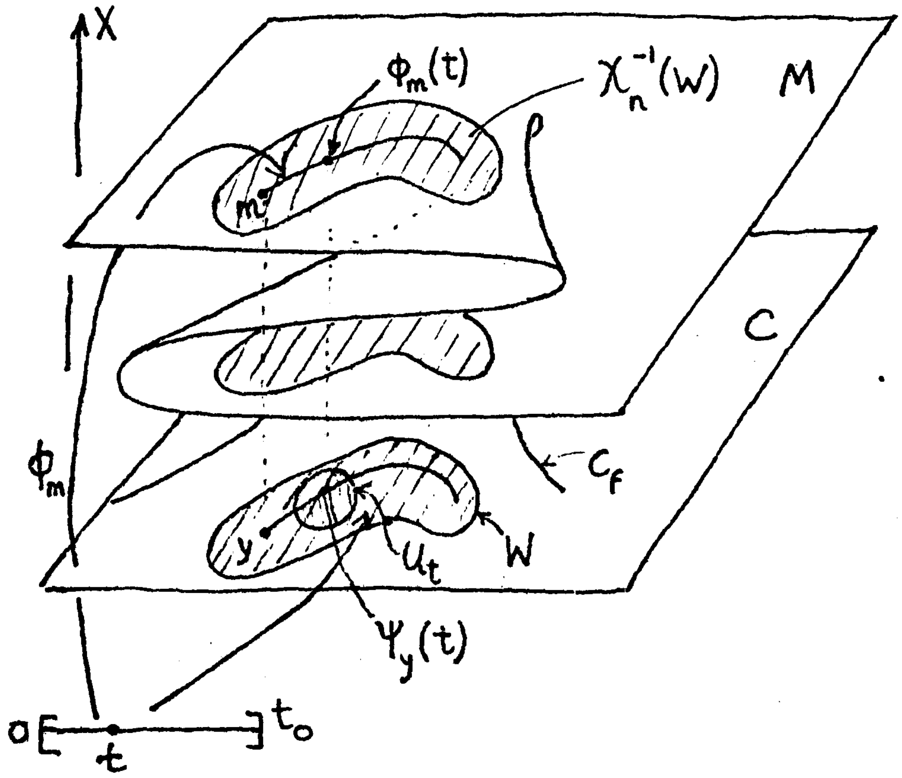
Let $m = (x,y) \in M^n$, $t_0 \in \mathbb{R}^+$, be fixed; suppose $\psi(t,y) \notin C_f$, $\forall t \in [0,t_0]$.

Then, $\exists$ a unique(continuous) $\phi_m = \phi_{m,t_o} : [0,t_o] \to M^n$, satisfying:

(1)' $\quad \chi\phi_m = \psi_y$

(2)' $\quad \phi_m(0) = m.$

Proof



Let $t \in [0,t_o]$ be fixed. We can construct $U = U_t$, neighbourhood of $\psi(t,y)$: ══════ $\psi_y(t)$, with U as in Proposition 5. In particular, $U_t \cap C_f = \emptyset$. Then, $W = \bigcup_{t\in[0,t_o]} U_t$ satisfies Corollary 2 of Proposition 5, since each U_t can be assumed path-connected, so that $\chi_n^{-1}(W) \xrightarrow{\chi_n} W$ is a covering space for W.

Now, ψ_y is a path in W, with initial point $y = \chi(m)$ and therefore, from the path lifting theorem in algebraic topology (see for instance [5], page 18) we conclude $\exists$ unique path, say ϕ_m, in M^n, with: (1)' $\chi\phi_m = \chi_n\phi_m = \psi_y$ ($\forall t \in [0,t_o]$)

(2)' $\phi_m(0) = m$

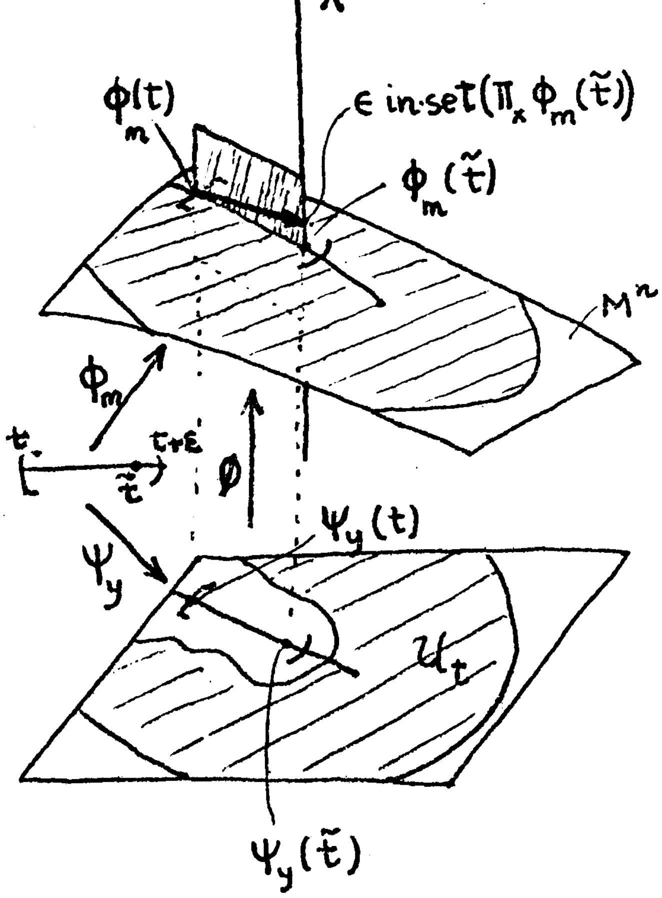
$\square$

<u>LEMMA 2:</u>

ϕ_m, given in Lemma 1, also satisfies:

(3)' For every fixed $t \in [0,t_o)$, $\exists\ \varepsilon = \varepsilon(m,t)$, such that: $\Pi_X\phi_m(t) \in$ inset $(\Pi_X\phi_m(\tilde{t}))$, $\forall\ \tilde{t} \in [t,t+\varepsilon]$, where the implicit vector field is $v_{\tilde{y}}$, $\tilde{y} = \Pi_C\phi_m(\tilde{t})$.

Proof



Let t be fixed and $\mathcal{U}_t$ as in Lemma 1. From Corollary 1 to Proposition 5 and definition of $\mathcal{U}_t$, we see (refer to [5], page 17) that $\mathcal{U}_t$ is evenly covered, where $\{(\chi_n/V_i)^{-1}(\mathcal{U}_t)\}_{i\in I_n}$, in the notation of Corollary 1, are the sheets over $\mathcal{U}_t$. So, for some fixed $i \in I_n$, $m \in (\chi_n/V_i)^{-1}(\mathcal{U}_t)$, with $(\chi_n/V_i)^{-1} := \emptyset$ a diffeomorphism from $\mathcal{U}_t$ to a neighbourhood of $m \in M^n$. The proof of the path lifting theorem referred above tells us that $\phi_m(\tilde{t}) = \emptyset_y(\tilde{t})$, $\tilde{t}$ suff. small so that $\psi_y(\tilde{t}) \in \mathcal{U}_t$.

By taking a smaller $\mathcal{U}_t$, if necessary, we can assume, by Proposition 6, that $\exists\,\delta$ such that, $\forall\,\tilde{m} = (\tilde{x};\tilde{y}) \in \emptyset\,(\mathcal{U}_t)$, $B_\delta(\tilde{x}) \subset$ in-set $(\tilde{x})$, where $-\nabla f_{\tilde{y}}$ is the implicit vector field.

Since ϕ_m and Π_X are continuous, $\exists\,\varepsilon > 0$ s.t.:

$|\tilde{t}-t| < \varepsilon \Longrightarrow \|\Pi_X\phi_m(\tilde{t}) - \Pi_X\phi_m(t)\| < \delta$, where $\tilde{m} = (\tilde{x},\tilde{y}) =$

$=\phi_m(\tilde{t}) = \emptyset\,(\psi_y(\tilde{t})) \in \emptyset\,(\mathcal{U}_t)$, $\tilde{x} = \Pi_X\phi_m(\tilde{t})$, and, so:

$$\Pi_X\phi_m(t) \in B_\delta(\tilde{x}) \Longrightarrow \Pi_X\phi_m(t) \in \text{inset}\,\underset{\Pi_X\phi_m(\tilde{t})}{\underset{\parallel}{(\tilde{x})}}\ ,$$

with $-\nabla f_{\tilde{y}}$ as implicit vector field; but this is the same as if the vector field where $\Phi_{\tilde{y}}$, since V is compatible, and we are done.

□

LEMMA 3:

Let $m = (x,y) \in M^n$ and suppose $O_y^+ \cap C_f = \emptyset$. There exists a unique (continuous) lift $\phi_m = \phi_{m,\infty}:\mathbb{R}^+ \to M^n$, satisfying (1)', (2)' as in Lemma 1, and (3)', $\forall\, t \in \mathbb{R}^+$, as in Lemma 2.

Proof

Let $t \in \mathbb{R}^+$ be fixed. Choose $t_o > t$, and define $\phi_m(t) = \phi_{m,t_o}(t)$, $\phi_{m,t_o}:[0,t_o] \to M^n$ as in Lemma 1. Claim: $\phi_m(t)$ is independent of the choice of t_o. To see this, let $t_1 > t$, $t_1 \neq t_o$, say $t_1 > t_o$. $\phi_{m,t_1}:[0,t_1] \to M^n$ satisfies (1)' and (2)' on $[0,t_1]$ and therefore so does $\phi_{m,t_1}/[0,t_o]$ on $[0,t_o]$. By unicity, in Lemma 1, $\phi_{m,t_1}/[0,t_o] \equiv \phi_{m,t_o}$, and so $\phi_{m,t_1}(t) = \phi_{m,t_o}(t)$. We remark that the above argument also shows that $\phi_m \equiv \phi_{m,t_o}$ on $[0,t_o]$, $\forall\, t_o \in R^+$ fixed. Therefore, ϕ_m is continuous and satisfies (1)', (2)' and (3)', $\forall\, t \in \mathbb{R}^+$; to prove this, we note that, given t, we can choose $t_o > t$ and use $\phi_m \equiv \phi_{m,t_o}$ on $[0,t_o)$. If we now define $W = \bigcup_{t\in\mathbb{R}^+} \mathcal{U}_t$, $\mathcal{U}_t$ as in Lemma 1, we see that, using Corollary 2 of Proposition 5, $\chi_n^{-1}(W) \to W$ is a covering space for W, and therefore the unique lifting theorem from algebraic topology (see for instance, Theorem 5.1, in [5]) shows that $\phi_m:\mathbb{R}^+,0 \to M^n$, m is unique. □

REMARK 1:

Suppose $v \in \mathcal{V}_f$, $y \in C_f$.

Then, $\exists\, \varepsilon > 0$ s.t. $|t| \leq \varepsilon$, $t \neq 0$ implies $\psi(t,y) \notin (C_f \cup M_{f,y})$. This is an immediate consequence of property H_1 (see page 1.2(1)) and of Definition 17 (see page 1.1(4)]

LEMMA 4:

Let $y \in C$, $O_y^+ \cap C_f \neq \emptyset$. Then $S_1 = \{t \in \mathbb{R}^+ | \psi_y(t) \in C_f\} = \{t_n\}_{n\in I}$, where either: (i) $I = \mathbb{Z}^+ = \{0,1...\}$ and $t_n \to \infty$ as $n \to \infty$, or: (ii) $I = \{0,1,...,N\}, N \in \mathbb{Z}$

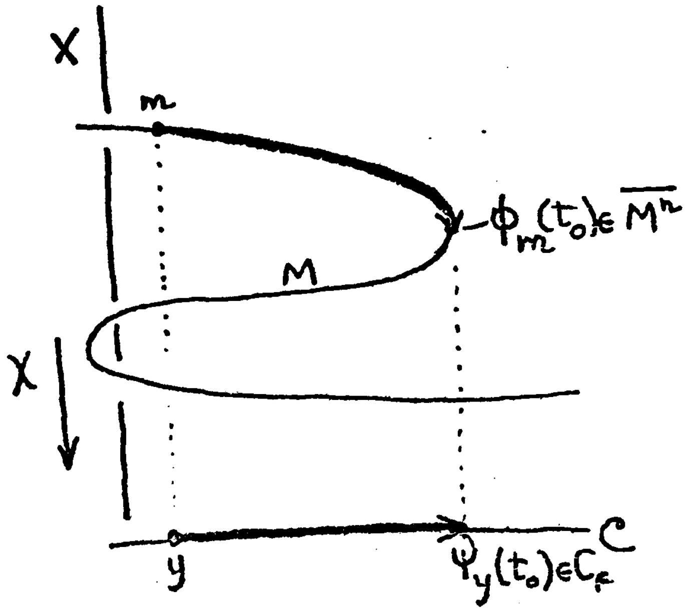
[Note: this accounts for the last line of (4), Theorem 1, page 1.2(1)]

Proof

This is clear because $\{t_n\}_{n\in I}$ can not accumulate by our hypothesis H_1 (page 1.2(1)).

LEMMA 5:

Let $m \in M^n = (x,y)$, $t_0 \in \mathbb{R}^+$, $\psi_y(t_0) \in C_f$, $\psi_y(t) \notin C_f$, $\forall\, t \in [0,t_0)$. Then, there exists a unique (continuous) function $\phi_m = \phi_{m,t_0} : [0,t_0] \to \overline{M^n}$, satisfying:



(1)', (2)' and (4) in $[0,t_o]$;

(3)' in $[0,t_o)$.

Proof

We first define ϕ_m in $[0,t_o)$. Let $t \in [0,t_o)$. Select $\tilde{t} \in (t,t_o)$, and define $\phi_m(t) = \phi_{m,\tilde{t}}(t)$, where $\phi_{m,\tilde{t}}:[0,\tilde{t}) \to M^n$ is constructed as in Lemma 1.

One can show that the definition of ϕ_m at t, as above, does not depend on the choice of $\tilde{t}$ (i.e., ϕ_m is well defined), and that ϕ_m is the unique continuous function (i.e., (4) is valid) satisfying (1)', (2)' and (3)' in $[0,t_o)$. The proof of this is a repetition of arguments as in Lemma 3.

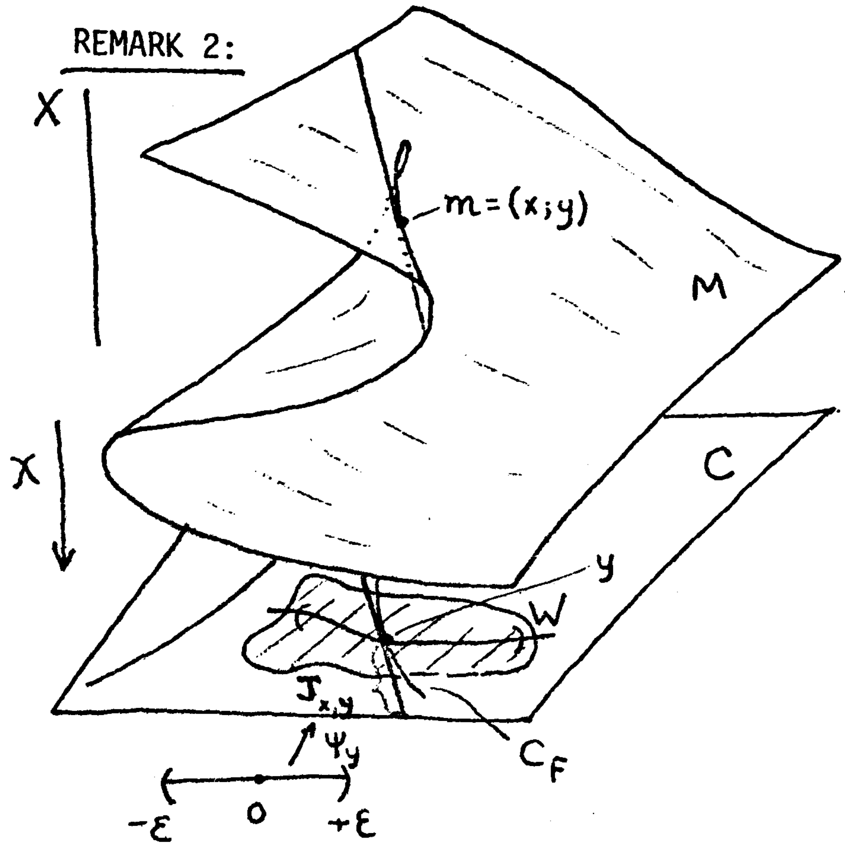
Any accumulation point of a sequence $\phi_m(t_n)$, $t_n \to t_o^-$ must be one of the finite number of points in $\chi^{-1}(y)$, by continuity of χ. If we take disjoint balls B_i about these points, (x_i,y), $i = 1,\ldots,r$, then, for some $t_1 < t_o$, $\phi_m(t)$ is in just one of these balls, B_j, for $t_1 < t < t_o$ and so every such accumulation point is $(x_j;y)$, i.e. $\phi_m(t) \to (x_j;y)$ as $t \to t_o^-$.

So $\phi_m(t_o) = \lim_{t\to t_o^-} \phi_m(t)$ is the unique way to make ϕ_m left continuous at t_o.

Note:

$\overline{M^n} \subset M^n \cup M^d$, or, equivalently, $\partial M^n \subset M^d$; this is so because $M = [\bigcup_{k=0}^{n} M^k] \cup M^d$, and M^k is open in M (closed), $M^i \cap M^j = \emptyset$ if $i \neq j$. □

REMARK 2:



Let $m = (x,y) \in M^d$, $y \in C_f$. Everything as in Remark 1, we note that $x \notin \text{sep } \Phi_{\tilde{y}}$, $\tilde{y} = \psi(t,y)$, $|t| \leq \varepsilon$. This is so because $\psi(t,y) \notin M_y = \bigcup_{(x_i,y)\in M^d} J_{x_i,y}$, $|t| \leq \varepsilon$, where $x = x_i$, for some i, hence $\psi(t,y) \notin J_{x,y} = \{y' \in C | x \in \text{sep } \Phi_{y'}\}$. If, on the other hand, $m = (x,y) \in M^n$, the constructions as in Proposition 6, 2.1(15), show that $\exists$ a neighbourhood of m, W, with $B(\tilde{x}) \subset \text{in-set } [\Phi_v](\tilde{x})$ ($\equiv$ in-set $[\Phi_{\tilde{y}}](\tilde{x})$, by compatibility), where $v = -\nabla f_{\tilde{y}}$, $\forall \tilde{m} = (\tilde{x},\tilde{y}) \in W$. Therefore, by restricting W so that $\|x-\tilde{x}\| < \delta$, $\forall (\tilde{x},\tilde{y}) \in W$, we get $x \in B_\delta(\tilde{x}) \subset \text{inset } [\Phi_{\tilde{y}}](\tilde{x})$. Hence, if $(\tilde{x},\tilde{y}) \in W$ is fixed, we can construct a neighbourhood Z of x, $x \in Z \subset B_\delta(\tilde{x}) \subset \text{in-set } [\Phi_{\tilde{y}}](\tilde{x})$, which implies $w(Z) = \tilde{x} = w(x)$, so that $x \notin \text{sep } \Phi_{\tilde{y}}$, $\forall \tilde{y} \in \chi(W)$, neighbourhood of $y \in C$. So, $\exists\ \varepsilon > 0$ s.t., $\forall t$ with $|t| \leq \varepsilon$. $x \notin \text{sep } \Phi_{\tilde{y}}$, $\tilde{y} = \psi(t,y)$; this ε can of course be taken so that $\psi(t,y) \notin C_f$, $\forall t$ with $|t| \leq \varepsilon$, since C_f is closed ($C_f = \chi(M - \bigcup_{k=0}^{n} M^k)$), and suits every $m \in \chi_n^{-1}(y)$.

From Remark 1 and above, we then conclude: if $m = (x,y) \in \overline{M^n}$, $y \in C_f$, $\exists\ \varepsilon > 0$ such that $\tilde{y} = \psi(t,y) \notin C_f$ and $x \notin \text{sep } \Phi_{\tilde{y}}$, $\forall t$ with $|t| \leq \varepsilon$, except perhaps $t = 0$. Also $x_i \notin \text{sep } \Phi_{\tilde{y}}$, $\forall x_i$ s.t. $m_i = (x_i,y) \in M^d$ or M^n, by construction; i.e., $m_i \in \chi^{-1}(y)$.

REMARK 3:

Let $(x,y) \in X \times C$ be fixed. Then $w[\Phi_{-\nabla f_y}](x) = w[\Phi_y](x)$, where Φ_y is the flow generated by v_y $\{v_y\}_{y \in C}$ on X. A trivial consequence of this, from the definition of separatrices, is: sep $\Phi_{-\nabla f_y} \doteq$ sep Φ_y. To show that equality, we first note that, as we are dealing with a gradient field (see [3],249), $w[\Phi_{-\nabla f_y}](x) = \{\tilde{x}\}$, where $\tilde{x}$ is a critical point of f_y. Now, $x \in$ in-set $[\Phi_{-\nabla f_y}](\tilde{x})$; if not, it would be possible to create a sequence $\{x_n\} \to x^*$, (X compact) $x_n \notin B_\varepsilon(\tilde{x})$, for some $\varepsilon > 0$ fixed, a contradiction, since in that case $x^* \in w(x)$, $x^* \neq \tilde{x}$. By compatibility, $x \in$ in-set $[\Phi_y](\tilde{x})$, and therefore $w[\Phi_y]\ (x) = \{\tilde{x}\}$.

REMARK 4:

Let $y \notin C_f$ be fixed, $\{x_i\}_{i \in I}$ be the set of singularities of $-\nabla f_y$, $\{x_i\}_{i \in J}$, $J \subset I$, the set of minimums of f_y. Then sep $\Phi_{-\nabla f_y} = X - \bigcup_{i \in J}$ in-set (x_i).

Proof

Let $x \in X$, $\tilde{x} = w[\Phi_{-\nabla f_y}](x)$; $X = \bigcup_{i \in I}$ in-set(x_i), all x_i hyperbolic, and the local form of a flow around a singularity easily imply $x \in$ sep $\Phi_{-\nabla f_y}$ iff $\tilde{x}$ is not a minimum for f_y, from which the equality above follows immediately.

LEMMA 6:

Let $m = (x,y) \in \overline{M^n}$, $y \in C_f$; let $\varepsilon > 0$, fixed, so that $x \notin$ Sep Φ_y and $\tilde{y} = \psi(t,y) \notin C_f$, $\forall t \neq 0$, $|t| \leq \varepsilon$ - we know that such an ε does exist (see last paragraph of Remark 2). There is a unique function $\phi_m = \phi_{m,\varepsilon}: [0,\varepsilon] \to \overline{M^n}$ satisfying (1)' in $[0,\varepsilon]$, (2)', (3)' in $[0,\varepsilon)$ and (4) in $(0,\varepsilon]$.

Proof

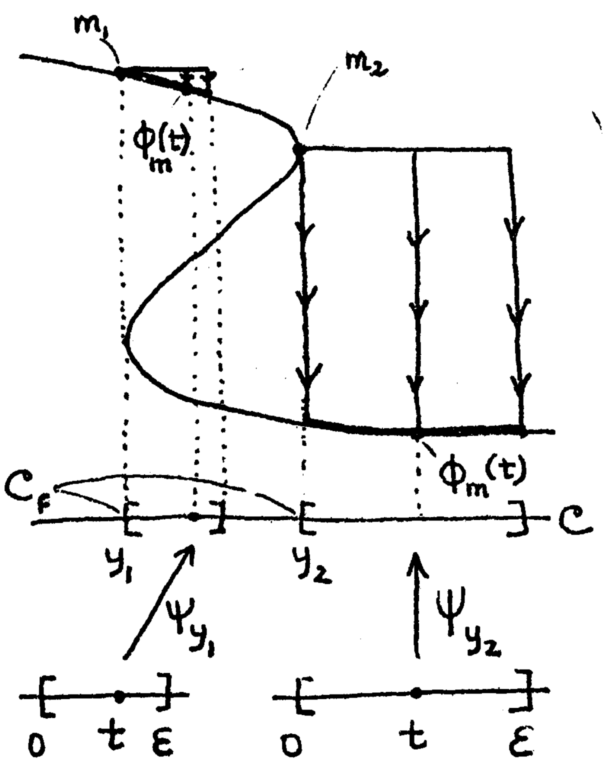
Existence

Let $t \in [0,\varepsilon]$. We define ϕ_m at t by: $\phi_m(t) = (w[\Phi_{\tilde{y}}](x);\tilde{y})$, with $\tilde{y} = \psi(t,y)$. ϕ_m is well defined, from Remark 3. From Remark 4, $x \notin \text{sep}\ \Phi_{\tilde{y}}$ $\tilde{x} = w[\Phi_{\tilde{y}}](x)$ is a minimum for $f_{\tilde{y}}$, hence $(\tilde{x},\tilde{y}) \in M^n$, $\tilde{y} = \psi(t,y)$, $t \neq 0$. Also $\phi_m(0) = (w[\Phi_y](x),y) = m$ so that $\phi_m:[0,\varepsilon] \to \overline{M^n}$ and (2)' is satisfied; $\chi\phi_m = \psi_y$ in $[0,\varepsilon]$ by construction. Now, $x \in \text{in-set}\ [\Phi_{\tilde{y}}](\tilde{x})$, from Remark 3; as $\tilde{x} = \Pi_x\phi_m(t)$, $x = \Pi_x\phi_m(0)$, we get that (3)' is verified at $t = 0$, just by taking $\varepsilon(m,0)$ to be ε in the statement of this lemma.

It remains therefore to prove that:

(i) (3)' holds at t, $\forall t \in (0,\varepsilon)$;

(ii) (4) holds at t, $\forall t \in (0,\varepsilon]$.

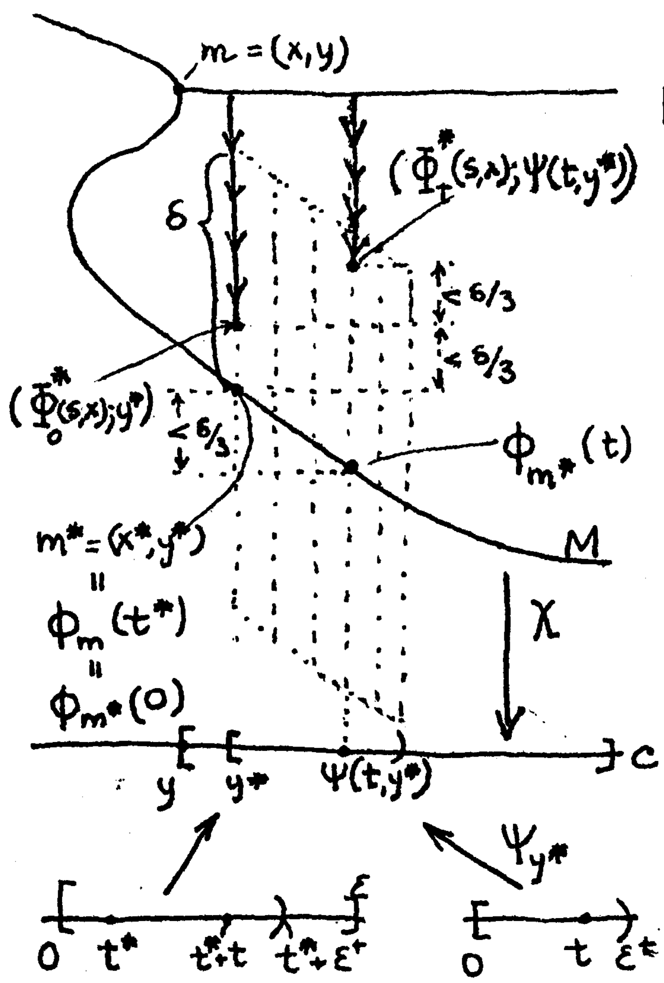


Picture of the two possible cases (y_1 and y_2) and of the corresponding lifts, which are being dealt with together in the proof.

Let $t^* \in (0,\varepsilon]$ be fixed, $\phi_m(t^*) = m^* = (x^*,y^*)$ $[\in M^n]$, so that $x^* = \Pi_x\phi_m(t^*) = w[\Phi_{y^*}]\ (x)$, $y^* = \psi(t^*,y)$.

We adopt here, for the rest of this proof, the following simplifying notation: Φ_t^* is the flow generated by the gradient of f restricted to $\psi(t,y^*) = \psi(t + t^*,y)$.

Define $\phi_{m*} = \phi_{m*,\varepsilon^+}$, from Lemmas 1 and 2, as the (unique) continuous function satisfying $\chi\phi_{m*}(t) = \psi_{y*}(t)$ and $\phi_{m*}(0) = m^*$, with $\varepsilon^+ = \varepsilon\,(m^*,0)$ taken as constructed in Lemma 2, so that (3)' is valid for $t = 0$ and $x^* = \Pi_X\phi_{m*}(0) \in B_\delta(\Pi_X\phi_{m*}(t)) \subset$ in-set $[\Phi_t^*]\ \Pi_X\phi_{m*}(t)$, $\forall\, t \in [0,\varepsilon^+)$, where compatibility was used in last step.



We claim that $\phi_{m*}(t) = \phi_m(t + t^*)$, $\forall t \in [0,\varepsilon^+)$.

Let s, fixed, be chosen so that:

$$\|\Phi_0^*(s,x) - x^*\| < \delta/3 \qquad \text{(I)}$$

By continuity of ϕ_{m*}, and reducing ε^+, if necessary, we can guarantee that:

$$\|\underset{\substack{\| \\ x^*}}{\Pi_X\phi_{m*}(0)} - \Pi_X\phi_{m*}(t)\| < \delta/3, \qquad \text{(II)}$$

$$\forall t \in [0,\varepsilon^+)$$

Finally, if ε^+ is small enough, we can also assure:

$$\|\Phi_0^*(s,x) - \Phi_t^*(s,x)\| < \delta/3, \qquad \text{(III)}.$$

s fixed as above, $|t| < \varepsilon^+$.

This is so because the continuity of ψ implies that the family of gradient vector fields, $\{-\nabla f_{\psi(t,y^*)}\}$, is continous on t. Therefore, for fixed (s,x), the well known result (from o.d.e) of continuity on parameters implies (III).

Let $t \in [0\,;\ \varepsilon^+)$. (I) + (II) + (III) $\Rightarrow \|\Phi_t^*(s,x) - \Pi_X\phi_{m*}(t)\| < \delta \Rightarrow$

$\Rightarrow \Phi_t^*(s,x) \in$ in.set $[\Phi_t^*]\ \Pi_X\phi_{m*}(t) \Rightarrow \Pi_X\phi_{m*}(t) = w[\Phi_t^*]\ (\Phi_t^*(s,x)) =$

$= w[\Phi_t^*](x) \Rightarrow \phi_{m*}(t) = (w[\Phi_t^*](x);\ \psi(t,y^*)) = (w[\Phi_{\psi(t,y^*)}](x);\psi(t,y^*))$,

last equality coming from Remark 3.

On the other hand, by definition $\phi_m(t + t^*) = (w[\Phi_{\tilde{y}}](x);\tilde{y})$, with $\tilde{y} = \psi(t + t^*,y) = \psi(t,y^*)$, proving the claim.

$\phi_{m*}(t) = \phi_m(t + t^*)$, $\forall\, t \in [0,\varepsilon^+)$, $t^* \in (0,\varepsilon]$ fixed, gives (i), and also shows ϕ_m to be right continuous where required by (ii). To see it is also left continuous, thus concluding existence, one just defines $\psi^-_{y*}(t) = \psi_{y*}(-t)$ (i.e., reverse the direction of ψ_{y*}), $\phi^-_{m*} = \phi^-_{m*,\varepsilon^-}$ to be the corresponding lift (from Lemmas 1 and 2) and repeat exactly the same constructions as above to show that, if $t \in [0,\varepsilon^-)$, then $\phi_m(t^*-t) = \phi^-_{m*}(t)$.

Uniqueness

Suppose $\tilde{\phi}_m$: $[0,\varepsilon] \to \overline{M^n}$ also satisfies the conditions in the statement of this lemma. By (3)' at 0, $\exists\, \tilde{\varepsilon} > 0$ such that $x = \Pi_x\tilde{\phi}_m(0) \in$ in-set $[\Phi_{\tilde{y}}]$ $(\Pi_x\tilde{\phi}_m(t))$, $t \in [0,\tilde{\varepsilon})$, $\tilde{y} = \psi(t,y)$, so that $\Pi_x\tilde{\phi}_m(t) = w[\Phi_{\tilde{y}}](x) = \Pi_x\phi_m(t)$. Therefore $\tilde{\phi}_m \equiv \phi_m$ on $[0,\tilde{\varepsilon})$. Pick $t^* \in [0,\tilde{\varepsilon})$. Define: $\tilde{\phi}_{m*}(t) = \tilde{\phi}_m(t^* + t)$. $\tilde{\phi}_{m*}$ is continuous on $[0,\varepsilon-t^*]$, and satisfies $\chi\tilde{\phi}_{m*} = \psi_{y*}$, by hypothesis, with $\tilde{\phi}_{m*}(0) = \tilde{\phi}_m(t^*) = \phi_m(t^*) = m^*$. Set $\phi^*_m(t) = \phi_m(t^* + t)$; ϕ^*_m satisfies the same properties as ϕ_{m*} (ϕ_m as above is defined from existence in this lemma). Therefore, from unicity in Lemma 1, $\phi^*_m \equiv \tilde{\phi}_{m*}$ on $[0,\varepsilon-t^*]$; so $\tilde{\phi}_m(t^* + t) = \tilde{\phi}_{m*}(t) = \phi^*_m(t) = \phi_m(t^*-t)$ on $[0,\varepsilon-t^*]$, hence $\tilde{\phi}_m \equiv \phi_m$ on $[t^*,\varepsilon]$ and we are done. □

LEMMA 7:

Let $m = (x,y) \in \overline{M^n}$; let $S_1 = \{t_n\}_{n\in I}$, as in Lemma 4, and, for each n, construct $\varepsilon = \varepsilon\,(n)$ as in Remark 2. Set $\bar{t}_n = t_n + \varepsilon(n)$. Then, if $n \in I$ is fixed:

(*) { There exists a unique $\phi^n_m:[0,\bar{t}_n] \to \overline{M^n}$ such that $\phi^n_m(0) = m$, and
{ ϕ^n_m satisfies (1)' and (4) on $[0,\bar{t}_n]$, (3)' on $[0,\bar{t}_n)$.

Proof

By induction.

Step 1: (*) is true for n = 0.

We have to show there is a unique function $\phi_m^o:[0,\bar{t}_o]\to\overline{M^n}$, such that $\phi_m^o(0) = m$, and ϕ_m^o satisfies (1)' and (4) on $[0,\bar{t}_o]$, (3)' on $[0,t_o)$.

Existence:

$$\text{Define } \phi_m^o(t) = \begin{cases} \phi_{m,t_o}(t), & t \in [0,t_o] \quad \text{(I)} \\ \phi_{m(0),\varepsilon(0)}(t-t_o), & t \in [t_o,\bar{t}_o] \quad \text{(II)}, \end{cases}$$

where $\phi_{m,t_o}:[0,t_o]\to\overline{M^n}$ is obtained from Lemma 5 and $\phi_{m(0),\varepsilon(0)} : [0,\varepsilon(0)]\to\overline{M^n}$ from Lemma 6, with $m(0) = \phi_{m,t_o}(t_o) = \phi_m^o(t_o)$. (I) and (II) show that ϕ_m^o is well defined and, just from the statements of the lemmas referred to, it follows trivially that ϕ_m^o satisfies (*).

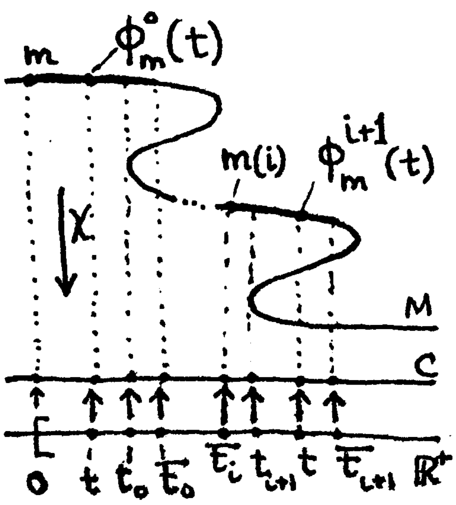
Uniqueness

Let $\tilde{\phi}_m^o:[0,\bar{t}_o]\to\overline{M^n}$ be another function satisfying (*).

Define $\tilde{\phi}_{m(0),\varepsilon(0)}:[0,\varepsilon(0)]\to\overline{M^n}$ by $\tilde{\phi}_{m(0),\varepsilon(0)}(t-t_o) = \tilde{\phi}_m^o(t)$, $\forall t \in [t_o,\bar{t}_o]$.

$\tilde{\phi}^o_{m/[0,t_o]} \equiv \phi_{m,t_o}$, by unicity in Lemma 5, hence $\tilde{\phi}_m^o(t_o) = \phi_{m,t_o}(t_o) = \phi_m^o(t_o)$.

So $\tilde{\phi}_{m(0),\varepsilon(0)}(0) = \tilde{\phi}_m^o(t_o = m(0)$. Therefore $\tilde{\phi}_{m(0),\varepsilon(0)} \equiv \phi_{m(0)\varepsilon(0)}$, by unicity in Lemma 6, so that $\tilde{\phi}_m^o \equiv \phi_m^o$, $\forall t \in [0,\bar{t}_o]$



Step 2:

(*) is true for $i \in I \Rightarrow$ (*) is true for i + 1.

By hypothesis, there is a unique function $\phi_m^i:[0,\bar{t}_i] \to \overline{M^n}$ satisfying (*). Set $m(i) = (x(i);y(i)) = \phi_m^i(\bar{t}_i)$. We are back to Step 1, with Lemmas 5 and 6 now applied m(i), $y(i) = \psi(\bar{t}_i,y)$, with $(t_{i+1}-\bar{t}_i),(\overline{t_{i+1}}-\bar{t}_i)$ now treated as the new $t_o,\bar{t}_o$, so that there is a unique $\phi_{m(i)}^o:[0,\bar{t}_{i+1} - \bar{t}_i] \to \overline{M^n}$ satisfying the required properties. We define

$$\phi_m^{i+1}(t) = \begin{cases} \phi_m^i(t), & t \in [0,\bar{t}_i] \\ \phi_{m(i)}^o(t - \bar{t}_i), & t \in [\bar{t}_i,\bar{t}_{i+1}] \end{cases}$$

It is trivial to verify that ϕ_m^{i+1} satisfies (*), since ϕ_m^i and $\phi_{m(i)}^o$ do; the unicity of these two functions imply the unicity of ϕ_m^{i+1}, as in Step 1. □

LEMMA 8:

Let $m = (x,y) \in \overline{M^n}$. There is a unique function $\phi_m:\mathbb{R}^+ \to \overline{M^n}$ satisfying (1)', (2)', (3)' and (4), $\forall t \in \mathbb{R}^+$.

Proof

Case 1 $\quad I = \{0,1,\ldots,N\}$.

By definition of I, $\psi_y(t) \notin C_f$, $\forall t \in [\bar{t}_N,\infty)$. Let $m(N) = (x(N);y(N)) = \phi_m^N(\bar{t}_N)$. We can then apply Lemma 3 to construct $\phi_{m(N)} = \phi_{m(N),\infty}:\mathbb{R}^+ \to \overline{M^n}$, satisfying the conditions required there.

Define $\phi_m:\mathbb{R}^+ \to \overline{M^n}$ by: $\phi_m(t) = \begin{cases} \phi_m^N(t), & \text{if } t \in [0,\bar{t}_N] \\ \phi_{m(N)}(t - \bar{t}_N), & \text{if } t \in [\bar{t}_N,\infty) \end{cases}$

ϕ_m satisfies (1)', (2)', (3)' and (4) and is unique by construction, since ϕ_m^N and $\phi_{m(N)}$ have these properties (proceed as in Step 1, Lemma 7).

Case 2: $I = \mathbb{Z}^+$

Let $t \in \mathbb{R}^+$. Then $t \in [\bar{t}_{n-1}, \bar{t}_n]$, some n. This is so because $t_n > t_{n-1}$, $\forall n \in \mathbb{Z}^+$, and $t_n \to \infty$ as $n \to \infty$. Define $\phi_m(t) = \phi_m^n(t)$. By definition, $\phi_m(t) = \phi_m^i(t)$, $\forall t \in [\bar{t}_{i-1}, \bar{t}_i]$, $i \le n$; but $\phi_m^n(t)_{/[0,\bar{t}_i]} \equiv \phi_m^i(t)$, by unicity of ϕ_m^i, hence $\phi_m \equiv \phi_m^n$ on $[0,\bar{t}_n]$. Therefore ϕ_m satisfies all required properties at t, $\forall t \in \mathbb{R}^+$ fixed; to see this, just choose n such that $t \in [0,\bar{t}_n)$ and use $\phi_m \equiv \phi_m^n$ and Lemma 7. Let $\tilde{\phi}_m$ be another function with the same properties, and t be fixed. Then $t \in [0,\bar{t}_n]$, some n, and by unicity in Lemma 7 $\tilde{\phi}_{m/[0,\bar{t}_n]} \equiv \phi_m^n$, hence $\tilde{\phi}_m(t) = \phi_m^n(t) = \phi_m(t)$. □

PROOF OF THEOREM 1

Let $(t,m) \in \mathbb{R}^+ \times \overline{M^n}$ be fixed, ϕ_m constructed as in Lemma 8; define $\phi(t,m) = \phi_m(t)$. The properties of ϕ_m as in Lemma 8 imply Theorem 1.

□

CHAPTER 3

3.0. INTRODUCTION

The reason why we could prove Theorem 1 was that we assumed hypothesis H (see Chapter 1). The question arises as to whether H is a generic property of vector fields. Our objective in Chapters 3 and 4 will be to prove Theorem 2, which affirmatively answers this question (see Chapter 1 for a precise statement).

In Chapter 1 we introduce the functor T^e, which generalises the tangent functor. Thus, T^eM is the higher dimensional analogue of TM. In paragraph 2 (§2) we define the notion of $e^{\underline{th}}$ expansion of a vector field $v \in V^k(\mathbb{R}^r), 1 \leq e \leq k$, $v[e]$. We then construct a submersion (off a certain set) S, such that:

$$\begin{tikzcd} & \mathbb{R}^r \arrow[dl, "j^{e-1}v"'] \arrow[dr, "v[e]"] & \\ J^{e-1}(\mathbb{R}^r,\mathbb{R}^r) \arrow[rr, "S"] & & T^e\mathbb{R}^r \end{tikzcd}$$

commutes.

This allows us to 'transfer' transversality theorems(§3) to a new context: we require instead that $v[e]$ be transversal to some submanifold of $T^e\mathbb{R}^r$. The reason for thinking about this at all is that it turns out that the notion of * transversality of $v[e]$ is closely related to that of 'isolated intersection' of a vector field with a set, which we also define in §3. And this last notion is, on the other hand, the basic idea behind the sufficient conditions for the lifting as presented in Chapter 2.

The following chapters, in which we prove, in certain cases, the genericity of the lifting property, will be dealing with the construction of the appropriate submanifold of some $T^e\mathbb{R}^r$.

3.1. THE FUNCTOR T^e

We now define T^e, from the category of C^k manifolds, $k \geq e$ fixed, with C^s, $e \leq s \leq k$, maps as morphisms, to the category of C^{k-e} fibre bundles (vector bundles if $e = 1$), with C^{s-e} fibre bundle (vector bundle if $e = 1$) maps as morphisms:

$$\begin{array}{llcl} \text{Objects} & M & \overset{T^e}{\rightsquigarrow} & T^eM \\ \text{Morphisms} & M \xrightarrow{f} N & & T^eM \xrightarrow{T^ef} T^eN \end{array}$$

Note:

T^1 coincides with T, the usual tangent functor; $k = \infty$ permitted.

We will first give the definitions of T^eM and T^ef, and then proceed to show that they are well defined and satisfy the required properties.

DEFINITION 1:

Let $\alpha,\beta\colon \mathbb{R} \to M$ be C^k. We say that $\alpha^* \sim_e \beta^*$ iff $\exists(\phi,U)$, chart for M, a sufficiently diff. manifold,s.t. $\frac{d^j(\phi\alpha)_i}{dt^j}(0) = \frac{d^j(\phi\beta)_i}{dt^j}(0)$, $\forall i = 1,\ldots,m$, $\forall j = 0,1,\ldots,e$, α^*, β^* the germs of α,β at 0.

DEFINITION 2:

Let $\hat{\alpha}$ denote the equivalence class generated by $\sim_e$ above. (we will shortly show that $\sim_e$ is an equivalence relation, independent of the choice of chart).

Call T^eM the set of all this equivalence classes.

DEFINITION 3:

Let M,N be C^k manifolds, $f\colon M \to N$ be C^s, $e \leq s \leq k$.

Define $T^ef\colon T^eM \to T^eN$

$$\hat{\alpha} \to \widehat{f\alpha}$$

where $\widehat{f\alpha}$ is the equivalence class of the germ at 0 of f, and where α is a representative of a germ in $\hat{\alpha}$.

PROPOSITION 1:

$\sim_e$ is well defined, an equivalence relation, and does not depend on the choice of chart as in Definition 1.

Proof

The definition of $\sim_e$ does not depend on representatives: if α_1, β_1 are other representatives for α^*, β^*, then $\frac{d^j(\phi\alpha_1)_i}{dt^j}(0) = \frac{d^j(\phi\alpha)_i}{dt^j}(0) = \frac{d^j(\phi\beta)_i}{dt^j}(0) = \frac{d^j(\phi\beta)_i}{dt^j}(0)$.

It is also clear that $\sim_e$ is an equivalence relation.

The rest of the proposition will result from:

Claim:

If $\gamma:\mathbb{R} \to \mathbb{R}^m$ is a C^k curve, $j \leq k$ is fixed, and $\emptyset$ is a C^k diffeomorphism from a neighbourhood of $\gamma(0)$ into its image, then

$$(*)\ \frac{d^j(\emptyset\gamma)_i}{dt^j}(t_o) = \Big(\sum_{1\leq q\leq j}\ \sum_{\substack{(h_1,\ldots,h_q)\\ \text{pos. integ.}\\ \text{s.t. } \sum_{s=1}^{q} h_s=j}}\ \sum_{i_1,\ldots,i_q=1}^{m} \sigma_j(h_1,\ldots,h_q)\frac{\partial^q\emptyset_i}{\partial x_{i_1}\ldots\partial x_{i_q}}(\gamma(t_o))$$

$$dx_{i_1}\ldots d_{x_{i_q}}\Big(\frac{d^{h_1}\gamma(t_o)}{dt^{h_1}};\ \frac{d^{h_q}\gamma(t_o)}{dt^{h_q}}\Big)\Big),$$

where $\sigma_j(h_1,\ldots,h_q)$ is an integer, which does not depend upon $\emptyset$.

This is a straightforward application of the composite mapping formula (see [1], 1.4), which states:

$$\underbrace{D^j(\emptyset\gamma)_i(t_o)}_{\substack{\in L(\underbrace{\mathbb{R}\times\ldots\times\mathbb{R}}_{j \text{ times}};\mathbb{R})}} = \sum_{1\leq q\leq j}\ \sum_{\substack{(h_1,\ldots,h_q)\\ \text{as above}}} \sigma_j(h_1,\ldots,h_q)\underbrace{D^q\emptyset_i(\gamma(t_o))}_{\in L(\underbrace{\mathbb{R}^m\times\ldots\times\mathbb{R}^m}_{q \text{ times}};\mathbb{R})}\ \underbrace{(D^{h_1}\gamma(t_o);\ldots;D^{h_q}(t_o))}_{\in L(\underbrace{\mathbb{R}\times\ldots\times\mathbb{R}}_{h_1 \text{ times}};\mathbb{R}^m),\text{etc}\ldots}$$

Since

$$D^q\emptyset_i(\gamma(t_o)) = \sum_{i_1,\dots,i_q=1}^{m} \frac{\partial^q \emptyset_i}{\partial x_{i_1}\cdots\partial x_{i_q}}(\gamma(t_o))dx_{i_1}\dots dx_{i_q}\text{ , and using}$$

the identifications $\frac{d^j(\emptyset\gamma)_i}{dt^j}(t_o) = D^j(\emptyset\gamma)_i(t_o)\cdot\underbrace{(1,\cdots,1)}_{j\text{ times}}$, $\frac{d^{h_s}\gamma}{dt^{h_s}}(t_o) = D^{h_s}\gamma(t_o)).\underbrace{(1,\dots}_{h^s\text{ ti}}$

$s = 1,\dots,q$, one gets (*). (The integer $\sigma_1(h_1,\dots,h_q)$ is actually defined in [1](1.4), but we only need what is stated above). This proves the claim. Let now α^*, β^* be as in Definition 1, and (ψ,V) be another chart for M, $\alpha(0) \in V$.

We have

$$\frac{d^j(\psi\alpha)_i}{dt^j}(0) = \frac{d^j(\overbrace{(\psi\phi^{-1})}^{\emptyset}\overbrace{(\phi\alpha)}^{\gamma})_i}{dt^j}(0) = \frac{d^j((\psi\phi^{-1})(\phi\alpha))_i}{dt^j}(0) = \frac{d^j(\psi\beta)_i}{dt^j}(0).$$

$\square$

Note:

We are using the following definition of germ: let $\alpha:I_1 \to M$, $\beta:I_2 \to M$, I_1,I_2 open. We say $\alpha \sim_* \beta \Leftrightarrow \exists$ open $I \subset I_1 \cap I_2$, $0 \in I$, s.t. $\alpha(t) \equiv \beta(t)$ on I. $\sim_*$ is an equivalence relation, and we denote the equivalence class of α by α^*.

PROPOSITION 2:

T^eM, as in Definition 2, can be made into a C^{k-e}, $m(e+1)$ dimensional manifold, which has the structure of a fibre-bundle.

Proof

We now produce 'local bijections', as defined below, for T^eM, from the charts on M.

So, let (ϕ,U) be a chart for M.

Define $\tilde{U} = \{\hat{\alpha} \in T^e M | \alpha(0) = x \in U\}$,

$\tilde{\phi} : \tilde{U} \to \mathbb{R}^{m(e+1)}$, by:

$\hat{\alpha} \to (\phi(\alpha(0); \frac{d(\phi\alpha)(0)}{dt}; \ldots; \frac{d^e(\phi\alpha)}{dt^e}(0))$

where α is some representative for $\alpha^* \in \hat{\alpha}$.

$\tilde{\phi}$ is well defined: if $\alpha^* \sim_e \beta^*$ and β represents β^*, then, by Proposition 1,

$$\frac{d^j(\phi\alpha)}{dt^j}i(0) = \frac{d^j(\phi\beta)}{dt^j}i(0),\ i = 1,\ldots,m; j = 0,1,\ldots,e$$

Claim 1:

$\tilde{\phi} : \tilde{U} \to \phi(U) \times \mathbb{R}^{me}$ is a bijection.

Proof of claim:

Define $\gamma:(y;v^1,\ldots,v^e) \to \hat{\alpha} \in \tilde{U}$, where

$(y;v^1,\ldots,v^e) \in \phi(U) \times \mathbb{R}^m \times \ldots \times \mathbb{R}^m$, by setting:

$\alpha: I \subset \mathbb{R} \to M$ to be $t \to \phi^{-1}(y + \sum_{j=1}^{e} \frac{v^j}{j!} t^j)$ (I conveniently small)

$\tilde{\phi}\gamma:(y;v^1,\ldots,v^e) \to (\underbrace{\phi\,\phi^{-1}(y)}_{=y}; \frac{d}{dt}\ \xi(0),\ldots, \frac{d^e}{dt^e}\ \xi(0))$,

where $\xi: t \to y + \sum_{j=1}^{e} \frac{v^j}{j!} t^j$, so that $\frac{d\xi}{dt}(t)_{/t=0} = v^1,\ldots, \frac{d^e\xi}{dt^e}(t)_{/t=0} = v^e$,

and therefore $\tilde{\phi}\gamma = Id_{/\phi(U)\times\mathbb{R}^{me}}$. It is also easy to check that $\gamma\tilde{\phi} = Id_{/\tilde{U}}$

Hence $\tilde{\phi}$ is a bijection, and $\tilde{\phi}^{-1} = \gamma$.

Let now (ϕ,U), (ψ,V) be two charts for M, as before. $\widetilde{\psi\phi}^{-1}$ is clearly a homeomorphism. We topologize $T^e M$ so that $\{(\tilde{\phi},\tilde{U})\}$, $(\phi,U) \in$ atlas for M, are homeomorphisms.

Claim 2:

$$\tilde\psi\tilde\phi^{-1}: \tilde\phi(\tilde U \cap \tilde V) \to \tilde\psi(\tilde U \cap \tilde V) \text{ is a } C^{k-e} \text{ diffeomorphism.}$$

Proof of claim:

Let $(y;v^1,\dots,v^e) \in \tilde\phi(\tilde U \cap \tilde V) = \phi(U \cap V) \times \mathbb{R}^{me}$.

$\tilde\phi^{-1}(y;v^1,\dots,v^e) = \hat\alpha$, where $\alpha : t \to \phi^{-1}\left(y + \sum_{j=1}^{e} \frac{v^j t^j}{j!}\right)$, so that $\alpha = \phi^{-1}\xi$, ξ as above.

Therefore:

$$\tilde\psi\tilde\phi^{-1} : (y;v^1,\dots,v^e) \to \left(\psi\phi^{-1}(y); \frac{d}{dt}(\psi\phi^{-1}\xi)(0);\dots;\frac{d^e}{dt^e}(\psi\phi^{-1}\xi)(0)\right).$$

Now, by (*) (Proposition 1), one has:

$$\frac{d^j}{dt^j}(\psi\phi^{-1}\xi)_i(0) = \sum_{1\le q\le j}\ \underbrace{\sum_{(h_1,\dots,h_q)}}_{\text{as before}}\ \sum_{i_1,\dots,i_q=1}^{m} \sigma_j(h_1,\dots,h_q)\frac{\partial^q(\psi\phi)^{-1}_i}{\partial x_{i_1}\cdots\partial x_{i_q}}(y)(dx_{i_1}\cdots dx_{i_q})\cdot(v^{h_1},\dots,v^{h_q})$$

Since the q in the formula above satisfies $1 \le q \le j \le e$, and $\psi\phi^{-1}$ is C^k, it follows that $\tilde\psi\tilde\phi^{-1}$ is C^{k-e}. So is its inverse, by an analogous argument, proving the claim.

Therefore, $\{(\tilde\phi,\tilde U)\}$ generates a maximal C^{k-e} atlas on T^eM, modelled on $\mathbb{R}^{m(e+1)}$.

Finally, to see that T^eM has indeed the structure of a fibre-bundle, we look at the diagram:

$$\begin{tikzcd}
\tilde U \arrow[r, "\tilde\phi"] \arrow[drr, "\Pi_e"'] & \phi(U)\times\mathbb{R}^{me} \arrow[r, "\phi^{-1}\times \mathrm{Id.}"] & U\times\mathbb{R}^{me} \arrow[d, "\Pi_U"] \\
 & & U
\end{tikzcd}$$

where $\Pi_e: \hat{\alpha} \to \alpha(0)$, and Π_U is the natural projection; $(\phi^{-1} \times \mathrm{Id})\tilde{\phi}$ is a diffeomorphism, $\tilde{U} = \Pi_e^{-1}(U)$, Π_e a submersion, hence T^eM is a fibre bundle. Its fibre is $\mathbf{R}^{me}$. □

PROPOSITION 3:

T^ef is well defined (we refer to Definition 3), C^{s-e} differentiable, and, furthermore: $T^e(\mathrm{id}_M) = \mathrm{id}_{T^eM}$. If $g:P \to M$, and g, P are C^k, then $T^e(fg) = T^ef.T^eg$. ($f:M \to N$; $\dim M = m$, $\dim N = r$)

Proof

To show that T^ef is well defined one just has to check, through (*) (3.1(2)) that if β represents $\beta^* \in \hat{\alpha}$, then $\widehat{f\alpha} = \widehat{f\beta}$.

As for the differentiability, consider charts $\tilde{\phi},\tilde{\psi}$, as below:

$$\begin{array}{ccc} T^eM & \xrightarrow{T^ef} & T^eN \\ \tilde{\phi}\downarrow & & \downarrow\tilde{\psi} \\ \mathbf{R}^{m(e+1)} & \xrightarrow[\gamma]{} & \mathbf{R}^{r(e+1)} \end{array}$$

By using (*) once more, we get:

$$\gamma_i^j:(a^o,a^1,..a^e) \to \Big(\sum_{1\le q\le j}\ \sum_{h's}\ \sum_{i_1,\dots,i_q=1}^{m} \sigma_j \frac{\partial^q \emptyset_i}{\partial x_{i_1}..\partial x_{i_q}}(a^o)dx_{i_1}..dx_{i_q}(a^{h_1};..;a^{h_q}),$$

if $j \ge 1$, $i=1,\dots,r$

and $\gamma^o(a^o,\dots,a^e) = \emptyset(a^o)$, where $\emptyset = \psi f\phi^{-1}$, C^s, and $\gamma = (\gamma^o,\dots,\gamma^e)$, $\gamma^j:\mathbf{R}^{m(e+1)} \to \mathbf{R}^r$, $\gamma^j = (\gamma_1^j,\dots,\gamma_i^j,\dots,\gamma_r^j)$

From this, T^ef is immediately C^{s-e}.

Now: $T^e\mathrm{id}_M = \mathrm{id}_{T^eM}$, since $T^e_{\mathrm{id}_M}:\hat{\alpha} \to \widehat{\mathrm{id}_M\alpha} = \hat{\alpha}$. Also $T^ef.T^eg(\hat{\alpha}) = T^ef(\widehat{g\alpha}) = \widehat{fg\alpha} = T^e(fg)(\hat{\alpha})$, as we wanted to prove. □

REMARK 1:

A quick look through definitions 1-3 shows that T^1M is - just TM, and T^1f is just Tf. In Proposition 3 above, if we set e = 1, γ turns out to be given by:

$$\gamma:(a^o,a^1) \to (\emptyset(a^o); \sum_{i=1}^{m} \frac{\partial\emptyset_1}{\partial x_i}(a^o)dx_i(a^1);\ldots; \sum_{1}^{m} \frac{\partial\emptyset_r}{\partial x_i}(a^o)dx_i(a^1)) =$$

$$= (\emptyset(a^o);\ d\emptyset(a^o)a^1), \text{ as it should be.}$$

Just to exemplify the case $e \neq 1$, fix e = 2.

Then, γ is given by: $\gamma(a^o,a^1,a^2) \to (\emptyset(a^o);d\emptyset(a^o)a^1;d\emptyset(a^o)a^2+d^2\emptyset(a^o)(a^1,a^1))$ (Note: we use both the notations, $d^j\emptyset$ and $D^j\emptyset$, with the same meaning).

REMARK 2:

We would like now to relate T^eM with the jet-spaces: let X, Y be manifolds, $J^e(X,Y)$ the manifold of e-jets from X to Y, defined in the usual way (see [4], page 37); then, setting $X = \mathbf{R}$, $Y = M$ and $J_0^e(\mathbf{R},M)$ = the subset of $J^e(\mathbf{R},M)$ constituted by the e-jets with source $0 \in \mathbf{R}$, one can easily check that T^eM is diffeomorphic to $J_0^e(\mathbf{R},M)$. (This last set has the structure of a manifold: it is a submanifold of $J^e(\mathbf{R},M)$).

REMARK 3:

For the rest of this section, we consider only the case $k = \infty$, for simplicity of exposition.

REMARK 4:

In our applications, the manifold M will sometimes appear, for a start, as a submanifold of $\mathbf{R}^r$. Then one can view 'naturally' 'T^eM' as a submanifold of $T^e\mathbf{R}^r$. We make these ideas precise.

DEFINITION 2':

Suppose $M \subset \mathbf{R}^r$ is a smooth (C^∞)m-dimensional submanifold of $\mathbf{R}^r$.

Define: $\overline{T^eM} = \{\hat{\alpha} \in T^e\mathbb{R}^r \,|\, \alpha(0) = x \in M,\ \exists \alpha \in \alpha^* \in \hat{\alpha}$ such that $\alpha(I) \subset M\}$.

This set can be given the structure of a manifold, as a submanifold of $T^e\mathbb{R}^r$, as follows: let $x \in M$ be fixed; $(\emptyset, U)$ be a chart for $\mathbb{R}^r$, $\emptyset: U \subset \mathbb{R}^r \to \mathbb{R}^r$, $x \in U$, s.t. $\emptyset(U \cap M) = \emptyset(U) \cap (\mathbb{R}^m \times \{0\})$, $0 \in \mathbb{R}^{m-r}$; it is easy to check that $\tilde{\emptyset}(\tilde{U} \cap \overline{T^eM}) = \tilde{\emptyset}(\tilde{U}) \cap V$, V a subspace of $\mathbb{R}^{r(e+1)}$, of dimension $m(e+1)$. This shows that $\overline{T^eM}$ is a smooth submanifold of $T^e\mathbb{R}^r$, whose smooth differential structure is given by the max. atlas generated by $\{(\widetilde{id}, \tilde{U})\}_{\substack{U \in \mathbb{R}^r \\ \text{open}}}$

We now show that $\overline{T^eM}$ and T^eM are 'the same' (take f = inclusion, in the next proposition); i.e. Definition 2 ≈ Definition 2'.

PROPOSITION 4:

Let M be a smooth manifold, $f: M \to \mathbb{R}$ a smooth embedding. Then f induces a diffeomorphism from T^eM to $\overline{T^ef(M)}$.

Proof

Define $h: T^eM \to \overline{T^ef(M)}$ by: $\hat{\alpha} \to \widehat{f\alpha}$. Let $x \in M$, (ϕ, U) be a chart for M, $x \in U$; $\emptyset = f\phi^{-1}: \phi(U) \to \mathbb{R}^r$ is an immersion, hence ([4], page7) $\exists$ open sets $U' \subset \phi(U)$, $\phi(x) \in U'$ and $V \subset \mathbb{R}^r$, with $\emptyset(U') \subset V$, and a diffeomorphism $\tau: V \to \tau(V) \subset \mathbb{R}^r$, s.t. $\tau\emptyset/U' : \mathbb{R}^m \to \mathbb{R}^m \times \mathbb{R}^{r-m}$ is the standard injection. Set $W = \phi^{-1}(U') \subset U$, $\psi = \phi/W$, and let $(\tilde{\psi}, \tilde{W})$ be a chart for T^eM induced from (ψ, W), chart for M. $\tau\emptyset(U') = \tau f(\phi^{-1}(U')) = \tau f(W) \subset \mathbb{R}^m \times \{0\}$; by restricting V (neighbourhood of f(x)) further, one can guarantee that $V \cap f(M) \subset f(W)$, since f(W) is open (in f(M))-f is a homeomorphism into its image. So, $\tau(V \cap f(M)) \subset \mathbb{R}^m \times \{0\}$. Setting $\eta = \tau/f(M) \cap V$, $Z = V \cap f(M)$, we therefore have that $\{\eta, Z\}$ is a chart for f(M), which generates $\{\tilde{\eta}, \tilde{Z}\}$, chart for $\overline{T^ef(M)}$, seen, as well as f(M), as a manifold on its own. We can assume, w.l.o.g, $W = f^{-1}(V \cap f(M))$. Since $\tau\emptyset = \tau f\phi^{-1}$ is the standard injection, we have

$nf\psi^{-1}$ = identity/U'. Now, $\tilde{\eta}h\tilde{\psi}^{-1}:\tilde{\psi}(\tilde{W}) \subset \mathbb{R}^{r(e+1)} \to \mathbb{R}^{r(e+1)}$ is given by (note: $\alpha:I \to M$, $\alpha(0) = x \in U'$):

$$(\psi\alpha(0);\ldots;\frac{d^e}{dt^e}(\psi\alpha)(0)) \xrightarrow{\tilde{\psi}^{-1}} \hat{\alpha} \xrightarrow{h} \widehat{f\alpha} \xrightarrow{\tilde{\eta}} (nf\alpha(0);\ldots;\frac{d^e}{dt^e}(nf\alpha)(0)) =$$

$$= (nf\psi^{-1}\psi\alpha(0);\ldots;\frac{d^e}{dt^e}(nf\psi^{-1})(\psi\alpha)(0)) = (\psi\alpha(0);\ldots;\frac{d^e}{dt^e}(\psi\alpha)(0))$$ Therefore

$\tilde{\eta}h\tilde{\psi}^{-1} = id/\tilde{\psi}(\tilde{W})$. Therefore $h/\tilde{W} = \tilde{\eta}^{-1}(id/\tilde{\psi}(\tilde{W}))\tilde{\psi}$ is a smooth diffeomorphism. Therefore $h:T^eM \to \overline{T^ef(M)}$ is a diffeomorphism.

3.2. THE e^{th} EXPANSION OF A VECTOR FIELD.

In this paragraph we will, given a vector field v in $\mathbb{R}^r$, define $v[e] : \mathbb{R}^r \to T^e\mathbb{R}^r$. We then construct a function S which makes the diagram below commutative.

$$\begin{tikzcd} & \mathbb{R}^r \arrow[dl, "j^{e-1}v"'] \arrow[dr, "v[e]"] & \\ J^{e-1}(\mathbb{R}^r,\mathbb{R}^r) \arrow[rr, "S"] & & T^e\mathbb{R}^r \end{tikzcd}$$

The important point here is that S turns out to be a submersion off a certain set, and this allows us (see next paragraph) to prove transversality theorems for submanifolds of $T^e\mathbb{R}^r$.

DEFINITION 4:

Let $v \in V^k(\mathbb{R}^r)$, $x \in \mathbb{R}^r$, $1 \leq k \leq \infty$.

Let $\alpha:I \to \mathbb{R}^r$ be a solution of v through x, $1 \leq e \leq k$, and $\hat{\alpha}$ be the equivalence class, under $\sim_e$, of α^*, the germ of α at 0.

Define the e^{th} expansion of v, v[e], by:

$$v[e]: \mathbb{R} \longrightarrow T^e\mathbb{R}^r$$
$$x \longmapsto \hat{\alpha}$$

In what follows we will be using the 'natural' identifications $T^e\mathbb{R}^r \cong \mathbb{R}^{r(e+1)}$, $\hat{\alpha} \longmapsto (\alpha(0), \ldots, \frac{d\alpha^e}{dt^e}(0))$, and $J^{e-1}(\mathbb{R}^r,\mathbb{R}^r) \cong \mathbb{R}^r \times \mathbb{R}^r \times B^{e-1}_{r,r}$, where $B^{e-1}_{r,r} = A^{e-1}_r(1) \oplus \ldots \oplus A^{e-1}_r(r)$, and each $A^{e-1}_r(i)$, $i = 1,\ldots,r$, is the space of polynomials in r variables and with degree $\leq e-1$. Choose as coordinates for $A^{e-1}_r(i)$ the coefficients of the polynomials.

We use the notation:

$$v[e]\ (x) = (x;v^0[e](x);\ldots;\ v^{e-1}[e](x)),$$

and $v^j[e] = (v^j_1[e],\ldots,v^j_r[e])$; when no ambiguity can result, we write v^j_i for $v^j_i[e]$.

PROPOSITION 5:

Each v^j_i is a polynomial P^j_i in partial derivatives of v, of order $\leq j-1$.

Proof

By induction. For $j = 0$, we just have $v^0_i = v_i$. Assume that our assertion is true for $j - 1$, $j \geq 1$. Then $P^j_i = \frac{d}{dt}(P^{j-1}_i(\alpha t)))_{/t=0} =$

$$= \sum_k \frac{\partial}{\partial x_k}(P^{j-1}_i(\alpha(t)))\frac{d\alpha_k}{dt}(t)_{/t=0} = \sum_k v_k \frac{\partial}{\partial x_k}(P^{j-1}_i),$$ proving the proposition.

COROLLARY:

$\{P^i_j\}$ determines a map S, such that

$$\begin{tikzcd} & \mathbb{R}^r \arrow[dl, "j^{e-1}v"'] \arrow[dr, "v[e]"] & \\ J^{e-1}(\mathbb{R}^r,\mathbb{R}^r) \arrow[rr, "S"] & & T^e\mathbb{R}^r \end{tikzcd}$$

commutes

Proof

This is just the map: $(x;v;\ldots) \in \mathbb{R}^r \times \mathbb{R}^r \times B^{e-1}_{r,r} \to (x;v;*) \in \mathbb{R}^{r(e+1)}$ where $*$ is determined by the polynomials in coordinates of $J^{e-1}(\mathbb{R}^r,\mathbb{R}^r)$ corresponding to the ones given in Proposition 5 above.

PROPOSITION 6:

If $v \neq 0$, then S is a submersion at v.

Proof

If $v \neq 0$, then $v_\lambda \neq 0$, for some λ. We will now order a sub-base of $J^{e-1}(\mathbb{R}^r,\mathbb{R}^r)$, by setting $q_i^j = \dfrac{\partial^i v_i}{\partial x_\lambda^i}$, $q^j = (q_1^j,\ldots,q_r^j)$, $q = (x,q^o,\ldots,q^{e-1})$.

Notice that by abuse of notation we are confusing an element of the base of $J^{e-1}(\mathbb{R}^r,\mathbb{R}^r)$ with the corresponding partial derivative of v, so that we can write $S_i^j = P_i^j$.

By induction, P_j^i contains a term $(v_\lambda)^j \dfrac{\partial^j v_i}{\partial x_\lambda^j} = (v_\lambda)^j q_i^j$. Indeed: if $j = 0$, $P_i^o = v_i = (v_\lambda)^o \dfrac{\partial^o v_i}{\partial x^o}$; suppose our claim is true for $j-1$. Then P_i^j contains the term :

$$\frac{d}{dt}\,(v_\lambda(\alpha(t))^{j-1}\;\frac{\partial^{j-1}v_i}{\partial x^{j-1}}\,(\alpha(t)))_{/t=0} \quad =$$

$$= \quad (v_\lambda(\alpha(t))^{j-1}\;\sum_k\,[\frac{\partial}{\partial x_k}\;(\frac{\partial^{j-1}v_i}{\partial x_\lambda^{j-1}}(\alpha(t)))\;\frac{d\alpha_k}{dt}(t)] + \ldots)_{/t=0} \quad =$$

$$= \quad (v_\lambda)^{j-1}\sum_k\frac{\partial}{\partial x_k}\,(\frac{\partial^{j-1}v_i}{\partial x_\lambda^{j-1}})v_k + \ldots = (v_\lambda)^j\,\frac{\partial^j v_i}{\partial x_\lambda^j} + \ldots\text{, as wanted.}$$

Furthermore,

$$\left.\begin{array}{l} P_k^j,\ k \neq i \\ P_k^s,\ \forall k,\ s < j \end{array}\right\} \quad \text{do not contain } q_i^j.$$

Also $S(x;-) = (x;-)$. Hence the Jacobian matrix $\frac{\partial S}{\partial q}$ is lower triangular, with either 1's or powers of v_λ down the diagonal. Hence $|\frac{\partial S}{\partial q}(v)| \neq 0$. Hence S has maximal rank at v. Hence S is a submersion at v.

COROLLARY:

Let $A = \{\sigma \in J^{e-1}(\mathbb{R}^r,\mathbb{R}^r) \mid \text{target of } \sigma = 0\}$.

Then S/A^c is a submersion.

3.3. SOME TRASVERSALITY THEOREMS:

In order to prove that, for fixed generic f, 'most' flows in $C = \mathbb{R}^r$ can be uniquely lifted (as in Chapter 2), we will need transversality theorems of the sort indicated in 3.§0. Proposition 7 below is a typical example of these; in Proposition 9 we show how it translates into the technical conditions related with the lifting theorems.

Let $A = \{\hat{\alpha} \in T^e\mathbb{R}^r \mid d\alpha/dt(0) = \ldots = d^e\alpha/dt^e(0) = 0\}$.

$\mathcal{A}$ as above, $N \subset T^e\mathbb{R}^r$ a submanifold, $\eta = S^{-1}(N)$,

$B = \{v \mid v[e] \pitchfork N\}$, $\mathcal{B} = \{v \mid j^{e-1}v \pitchfork \eta\}$.

PROPOSITION 7:

Let N be a (closed) smooth submanifold of $T^e\mathbb{R}^r$. Suppose $N \cap A = \emptyset$. Then, $\exists$ (open dense) a residual set (in the C^∞ Whitney topology) $\mathcal{B} \subset C^\infty(\mathbb{R}^r,\mathbb{R}^r) \simeq \mathcal{V}(\mathbb{R}^r)$, s.t. $v[e] \pitchfork N$, $\forall v \in \mathcal{B}$.

Proof

From the definitions of S, A and $\mathcal{A}$, it follows immediately that $S(\mathcal{A}) \subset A$. Hence $N \cap A = \emptyset \Longrightarrow S^{-1}(N) \cap \mathcal{A} = \emptyset$. Therefore, from Corollary on page (3.2(2)), η is a (closed) submanifold of $J^{e-1}(\mathbb{R}^r,\mathbb{R}^r)$. Hence $\mathcal{B} = \{v \mid j^{e-1}v \pitchfork \eta\}$ is (open dense) residual in $C^\infty(\mathbb{R}^r,\mathbb{R}^r)$, by Thom's theorem ([4], (page 54) / (page 56)). The proof will be finished by showing that $\mathcal{B} \subset B$.

Let $v \in \mathcal{B}$; choose (if possible) x s.t. $v[e](x) \in N$. So, $j^{e-1}v(x) \in \eta$. Now, since $j_v^{e-1} \pitchfork \eta$, one has, at x:

$$T_x j^{e-1}v\,(T_x\mathbb{R}^r) + T_y\eta = T_y J^{e-1}(\mathbb{R}^r,\mathbb{R}^r), \quad (*)$$

with $y = j^{e-1}v(x)$.

Now, S is a submersion, so that

$$T_yS(T_y\, J^{e-1}(\mathbb{R}^r,\mathbb{R}^r)) = T_{v[e](x)}(T^e\mathbb{R}^r)\,,\quad \text{Therefore}$$

$$T_x(\underbrace{S\, j^{e-1}v}_{v[e]})\, T_x\mathbb{R}^r + T_yS(T_y n) = T_{v[e](x)}\, (T^e\mathbb{R}^r), \text{ from } (*).$$

Also $S(n) = S(S^{-1}(N)) \subset N$. Therefore $T_yS(T_y n) \subset T_{S(y)}N = T_{v[e](x)}N$.

Therefore, $T_x(v[e])T_x\mathbb{R}^r + T_{v[e](x)}N = T_{v[e](x)}\, T^e\mathbb{R}^r$. This shows that $v[e] \pitchfork N$, Therefore $v \in B$, as wanted. □

PROPOSITION 8:

Let Q be a (closed) submanifold of $\mathbb{R}^r$, $c = \text{cod}.Q \geq 1$. Then, $\exists$ e and $\exists$ (open dense) residual $\mathcal{B} \subset C^\infty(\mathbb{R}^r,\mathbb{R}^r)$ s.t. $v[e](\mathbb{R}^r) \cap T^e Q = \emptyset$, $\forall\, v \in \mathcal{B}$.

We first prove some lemmas:

LEMMA 1:

Let X,Y be smooth manifolds, W a closed subset of Y.

Then $\{f \in C^\infty(X,Y) \mid f(X) \cap W = \emptyset\}$ is open in the Whitney C^0 topology (hence in the Whitney C^∞ topology as well).

Proof

Let $\mathcal{U} = \{\sigma \in J^0(X,Y) \mid y = \text{target } \sigma \notin W\}$, $\mathcal{V} = J^0(X,Y) - \mathcal{U}$.

Let $\{\sigma_i\}$ be a convergent sequence of 0-jets, $\sigma_i \in \mathcal{V}$, $\forall_i$, $\sigma = \lim_{i\to\infty} \sigma_i$.

Since target $\sigma_i \in W$, $\forall_i$, and W is closed, target $\sigma \in W$, therefore $\sigma \in \mathcal{V}$. Hence, $\mathcal{U}$ is open.

Now, $M(\mathcal{U}) = \{f \in C^\infty(X,Y) \mid j^0f(X) \subset \mathcal{U}\} = \{f \mid (x,f(x)) \notin \mathcal{U}, \forall x\} =$

$= \{f \mid f(x) \cap W = \emptyset\}$ is open in the C^0 Whitney Topology. □

LEMMA 2:

Let X,Y be smooth, W_α submanifold of $J^k(X,Y)$, $\forall\alpha \in I$, some index set, $\mathrm{cod}(W_\alpha) > \dim X$, $\forall\alpha$, and $W = \bigcup_{\alpha\in I} W_\alpha$ closed.

$$T_W = \{f \in C^\infty(X,Y) \mid j^kf \pitchfork W_\alpha, \forall\alpha\} \quad \text{is } C^0 \text{ open (and so, } C^\infty \text{ open).}$$

Furthermore, T_W is open-dense if I is denumerable.

Proof

$\{g \in C^\infty(X,J^k(X,Y)) \mid g(X) \cap W = \emptyset\}$ is open by Lemma 1. Now, $j^k: f \to j^kf$, $j^k: C^\infty(X,Y) \to C^\infty(X,J^k(X,Y))$ is continuous ([4] pg 46), and therefore $\{f \in C^\infty(X,Y) \mid j^kf \pitchfork W_\alpha, \forall\alpha\} = \{f \mid j^kf(X) \cap W = \emptyset\} =$
$= (j^k)^{-1}(\{g \in C^\infty(X,J^k(X,Y)) \mid g(X) \cap W = \emptyset\})$ is open.

Now $T_W = \bigcap_{\alpha\in I} T_{W_\alpha}$, $T_{W_\alpha} = \{f \mid f \pitchfork W_\alpha\}$, and each T_{W_α} is residual from Thom's Theorem. Hence T_W is dense, since $C^\infty(X,Y)$ is Baire. □

LEMMA 3:

Q closed $\Longrightarrow T^eQ$ closed, $\forall e$.

Proof

Assume $(T^eQ)^c$ is not open, by absurd. Let $\hat\alpha \in (T^eQ)^c$ be such that $\mathcal{O} \cap T^eQ \neq \emptyset$, $\forall\, \mathcal{O}$ containing $\hat\alpha$. If $x = \alpha(0) \notin Q$, and since Q^c is open, there would $\exists$ neighbourhood N of x with $N \cap Q = \emptyset$ and so, by setting $\mathcal{O} = \tilde N$, we would have $\mathcal{O} \cap T^eQ = \emptyset$, with $\hat\alpha \in \mathcal{O}$, a contradiction. So, we must have $x \in Q$. Let now U be a neighbourhood (in $\mathbb{R}^r$) of x, s.t. $(\tilde\emptyset,\tilde U)$ (see 3.1.(8)) satisfies $\tilde\emptyset(\tilde U \cap T^eQ) = \tilde\emptyset(\tilde U) \cap V$, V as before (3.1(8)). We have $\tilde\emptyset(\hat\alpha) \in V$, otherwise $\exists$ open W around $\tilde\emptyset(\hat\alpha) \in V$ with $W \cap V = \emptyset$, therefore $\tilde\emptyset^{-1}(W) \cap T^eQ = \emptyset$, contradictory. Finally, $\tilde\emptyset(\hat\alpha) \in V \Rightarrow \hat\alpha \in T^eQ$, contrary to assumption. Hence $(T^eQ)^c$ is open (see also Remark 4 and Definition 2'; Proposition 4 was implicitly used). □

PROOF OF PROPOSITION 8:

Choose e so that $e > \frac{r-c}{c}$ (*)

Set $N = T^eQ \cap A^c$; $W_1 = S^{-1}(N)$; $A_Q = T^eQ \cap A = \{\hat{\alpha} \in A \mid x = \alpha(0) \in Q\}$; $W_2 = S^{-1}(A_Q)$.
Since $N \cap A = \emptyset$, W_1 is a submanifold of $J^{e-1}(\mathbb{R}^r,\mathbb{R}^r)$ (see Proposition 7), with $\text{cod.}(W_1) = \text{cod.}(N) = c(e+1) > r$ (by (*)).

With the usual identification $J^{e-1}(\mathbb{R}^r,\mathbb{R}^r) \simeq \mathbb{R}^r \times \mathbb{R}^r \times B^{e-1}_{r,r}$ we have that $W_2 = Q \times \{0\} \times B^{e-1}_{r,r}$, hence a submanifold (closed, if Q is closed) of $J^{e-1}(\mathbb{R}^r,\mathbb{R}^r)$, with codimension $(W_2) = r + c > r$.

If Q is closed, so is T^eQ (Lemma 3) and also $W = W_1 \cup W_2 = S^{-1}(N \cup A_Q) = S^{-1}(T^eQ)$.
Hence, setting $\mathcal{B} = T_W = \{v \mid j^{e-1}v \pitchfork W_\alpha,\ \alpha = 1,2\} = (=\{v \mid j^{e-1}v(\mathbb{R}^r) \cap W = \emptyset\})$,

we get $\mathcal{B}$ open dense by Lemma 2. If Q is not closed, just apply usual Thom $\pitchfork$ Theorem ([4], page 54) to W_1, W_2 as above, to get the T_W residual.

DEFINITION 9:

Let $S \subset \mathbb{R}^r$ be a set, $v \in \mathcal{V}^k(\mathbb{R}^r)$, $k \geq 1$. Then, v has isolated intersection with S at $x \in \mathbb{R}^r$ iff, given $\alpha: I \to \mathbb{R}^r$, solution of v through x, $\exists\, \varepsilon > 0$ s.t. $\{t \mid \alpha(t) \in S,\ |t| < \varepsilon,\ t \neq 0\} = \emptyset$.

Notation: $v \,\bar{\pitchfork}_x\, S$. If $v \,\bar{\pitchfork}_x\, S$, $\forall\, x$, we say that v has the property of isolated intersection with respect of S: $v \,\bar{\pitchfork}\, S$. We write $v \,\mathring{\bar{\pitchfork}}\, S$ if $v \,\bar{\pitchfork}_x\, S$ for every x which is not singular for v(i.e. $v(x) \neq 0$).

PROPOSITION 9:

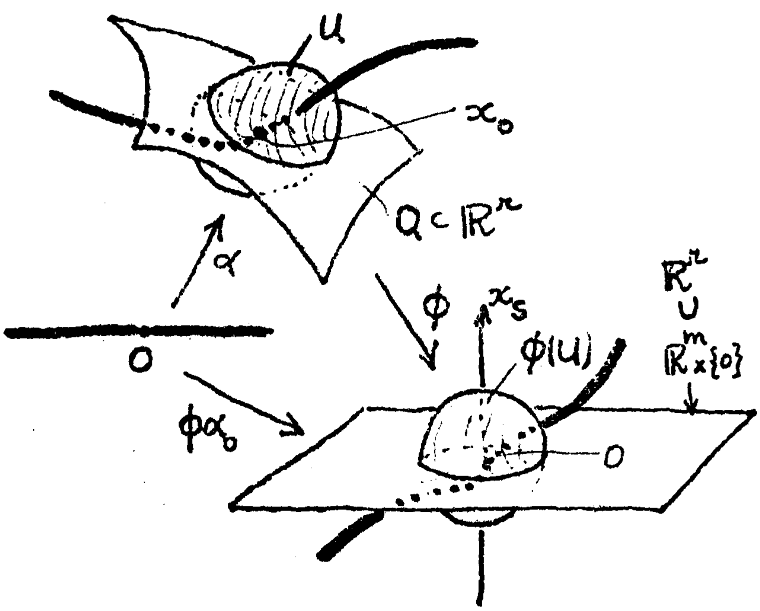
Let Q as in Proposition 8. $\exists\ \mathcal{B} \subset \mathcal{V}(\mathbb{R}^r)$, open and dense in the C^∞ Whitney topology, s.t. $v \,\bar{\pitchfork}\, Q$, $\forall\ v \in \mathcal{B}$.

Proof

Let e, $\mathcal{B}$ be chosen as in Proposition 8 above. Fix $v \in \mathcal{B}$ and $x_0 \in Q$.
Let $\mathcal{U}$ be a neighbourhood of x_0 (in $\mathbb{R}^r$), $\phi: \mathcal{U} \to \phi(\mathcal{U})$ a diffeomorphism s.t. $\phi(\mathcal{U} \cap Q) = \phi(\mathcal{U}) \cap (\mathbb{R}^m \times \{0\})$, where m=dim Q. Assume $\phi(x_0) = 0$, wlog. Denote $\phi/\mathcal{U} \cap Q$ also by ϕ, by abuse of notation, $T^e\phi: \hat{\alpha} \in T^e(\mathcal{U} \cap Q) \to \widehat{\phi\alpha} \in T^e(\phi(\mathcal{U} \cap Q))$

is a diffeomorphism, with inverse $T^e(\phi^{-1})$ (see 3.1(6), Proposition 3). Q is closed, $v \pitchfork_x Q$ is trivial if $x \notin Q$. Therefore we will prov theorem if we can show that $v \pitchfork_{x_0} Q$, x_0 as above. Let $\alpha_0 : I \to \mathbb{R}^r$ b solution of v through x_0. We seek to find an $\varepsilon > 0$ such that

$$\{t \mid \alpha_0(t) \in Q, |t| < \varepsilon, t \neq 0\} = \emptyset.$$



Now, $\widehat{\phi\alpha_0} \notin T^e(\phi(U) \cap (\mathbb{R}^m \times$

we would have $T^e(\phi^{-1})(\widehat{\phi\alpha_0})$

$\hat{\alpha}_0 = v[e](x_0) \in T^e(U \cap Q) \subset$

contrary to the hypothesis

with $\psi = \widetilde{\widehat{id}}$, one has: $\psi(T^e(\phi($

$= \phi(U) \cap (\mathbb{R}^m \times \{0\}) \times \underbrace{(\mathbb{R}^m \times}_{\hat{\mathbb{R}}^r}$.

Hence as $\psi(\widehat{\phi\alpha_0}) = (\phi\alpha_0(0)$

$\dfrac{d(\phi\alpha_0)}{dt}(0); \ldots ; \dfrac{d^e(\phi\alpha_0)}{dt^e}$ (0

$\exists\, s \in \{m+1, \ldots, r\}$ s.t. $\left(\dfrac{d(\phi\alpha_0)_s}{dt}(0); \ldots ; \dfrac{d^e(\phi\alpha_0)_s}{dt^e}(0)\right) \neq (0, \ldots, 0)$.

such that $a = 1/j! \dfrac{d^j(\phi\alpha_0)_s}{dt^j}(0) \neq 0$, $\dfrac{d(\phi\alpha_0)_s}{dt}(0) = \ldots = \dfrac{d^{j-1}(\phi\alpha_0)}{dt^{j-1}}$

Expanding $(\phi\alpha_0)_s : I \to \mathbb{R}$ in Taylor Series around 0, one has:

$$(\phi\alpha_0)_s : t \to (\phi\alpha_0)_s(0) + \frac{d(\phi\alpha_0)_s}{dt}(0)t + \ldots + 1/j! \frac{d^j(\phi\alpha_0)_s}{dt^j}(0)\, t^j + h$$

Let β_s be a local diffeomorphism (i.e. $\beta_s: \overset{\circ}{J}^{\text{open}} \to \beta_s(J)$) of $\mathbb{R}$, where $0 \in J$, $\beta_s(0) = 0$, and $\beta_s(\phi\alpha_0)_s: t \to t^j$. This is possible, because $(\phi\alpha_0)_s$ is j determined. Define $\beta:(x_1,x_2,\ldots,x_s,\ldots,x_r) \to (x_1,x_2,\ldots,\beta_s(x_s),\ldots,x_r)$, $\beta: U' = B_{\varepsilon_1}(0) \times\ldots\times B_{\varepsilon_{s-1}}(0) \times J' \times\ldots\times B_{\varepsilon_r}(0) \to \mathbb{R}^r$, $J' \subset J$ open, $\varepsilon_i \in \mathbb{R}^+$ $i = 1,\ldots,s-1,s+1,\ldots,r$, chosen so that $U' \subset \phi(U)$. Note that $\beta(U' \cap (\mathbb{R}^m \times \{0\})) \subset \mathbb{R}^m \times \{0\}$.

Finally, choose ε small enough so that $(\phi\alpha_0)((-\varepsilon,\varepsilon)) \subset U'$. We have $(\beta\phi\alpha_0)_s(t) = \beta_s(\phi\alpha_0)_s(t) = t^j$, $t \in (-\varepsilon,\varepsilon)$. Therefore $(\beta\phi\alpha_0)(t) = 0 \Leftrightarrow t = 0$ $(|t| < \varepsilon)$; since $|t| < \varepsilon$, $t \neq 0$ and $\alpha_0(t) \in Q$ would imply $(\beta\phi\alpha_0)_s(t) = 0$, we may conclude that $\alpha_0(t) \notin Q$, if $\begin{cases} t \neq 0 \\ |t| < \varepsilon \end{cases}$, as wished. □

<u>REMARK 6:</u>

The proof above also shows that: $x \in Q$, $v[e](x) \notin T^eQ \Rightarrow v \,\overline{\pitchfork}_x Q$.

We give now a last example, in which we examine a situation where Q is not necessarily smooth. Our intention is to illustrate once more how to interrelate the concepts developed here with standard transversality theorems.

From Levine's article, as in [14], we quote the following.

(*) 'The set of maps in $L(V,M,s)$ whose r-extensions are $\overline{\pitchfork}$ to W on V is dense, provided that $\underbrace{(n-q) < s-r, (s > r)}_{*}$, where $q = \mathrm{cod}(W)$, W is a C^{s-r} differential submanifold of $J^r(V,M)$, V and M at least s differentiable, $n = \dim V$'.

<u>Note</u>: The above is Theorem 1, in §7 of [14].

The topology on $L(V,M,s)$, with $V = \mathbb{R}^r$, $M = \mathbb{R}$, as in Proposition 10 below, is the topology of uniform convergence of all partial derivatives of orders $\le s$, including the 0^{th} (see §5, 5.3, of [14]).

PROPOSITION 10:

Let Q be a C^{k+1} submanifold of $\mathbb{R}^r$, $k \geq 0$. Let $c = \operatorname{cod} Q > 0$ and $k > \frac{r-c}{c}$. Then, $\exists\ \mathcal{B} \subset L(\mathbb{R}^r, \mathbb{R}, k)$, dense, s.t., $\forall\ v \in \mathcal{B}$ fixed, $v \tilde{\not\pitchfork} Q$.

Proof

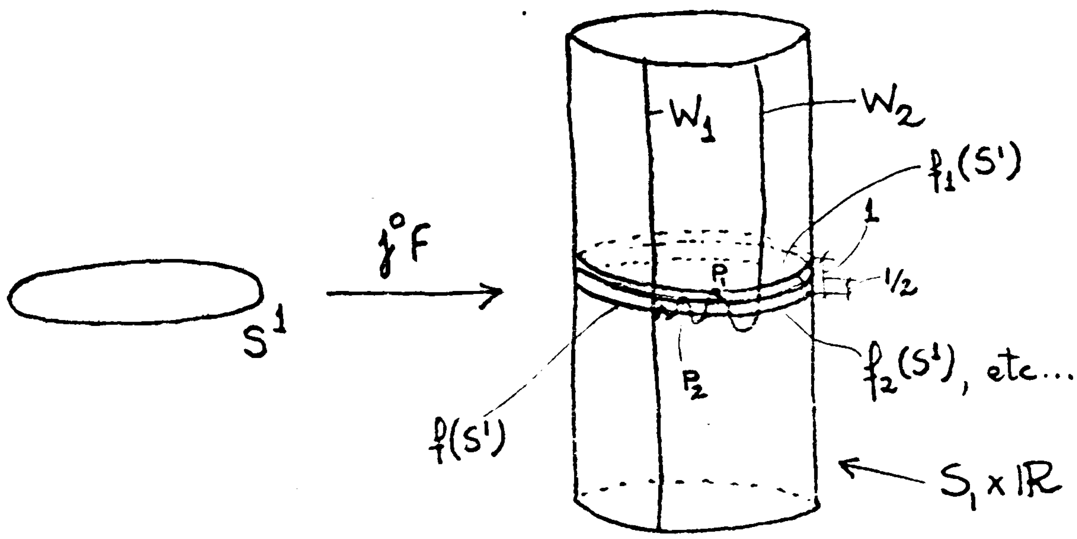
Construct $T^k Q$, C^1, and set $N = T^k Q \cap \psi^{-1}(\mathbb{R}^{r(k+1)} - A)$, and $\eta \subset J^{k-1}(\mathbb{R}^r, \mathbb{R}^r)$ as in Proposition 8. We have $q = \operatorname{cod}(\eta) = (k+1)c$, η of class C^1. Now, the condition $k > \frac{r-c}{c}$ implies $r-c(k+1) < 0$. i.e. $r-q < 0$. Applying (*) with $V = M = \mathbb{R}^r$, $s = k$, $r = k-1$ (hence $s-r=1$), $W = \eta$, we see that $\overset{*}{\smile}$ means, in our case, $r-q < 1$, which is just slightly less than we are requiring. So, $\mathcal{B} = \{v \mid j^{k-1}v \pitchfork \eta\} = \{v \mid j^{k-1}v(\mathbb{R}^r) \cap \eta = \emptyset\}$ is dense; the last equality comes from $\operatorname{cod} \eta = (k+1)c > r$, by hypothesis. As before, $v \in \mathcal{B} \Rightarrow v[k](\mathbb{R}^r) \cap N = \emptyset$. From Remark 6, $v \tilde{\not\pitchfork}_x Q$, $\forall$ x satisfying $v(x) \neq 0$, as wanted. □

(Note: If Q is closed, as in Proposition 8, and if one wants to prove the analogue of Proposition 9 in the non-smooth case, one just has to extend (*) to the situation as in the remark in the proof of Proposition 8. We will be concerned, however, with the smooth case; we will proceed, in Chapter 4, to extend the Propositions and Definitions above in yet another direction).

REMARK 7:

Lemma 2 is not valid if one removes the hypothesis $\operatorname{cod} W_\alpha >$ $\dim X, \forall \alpha$; though this result is mistakenly announced in [4], page 59. It does not hold even if the W_α's are disjoint and I is finite, as the following counter-example shows.

Let $X = S^1$, $Y = \mathbb{R}$, $k = 0$, W_1, W_2 as in picture below, and $f \equiv 0$.



Let $f_n \equiv 1/n$, $\forall\, x \in S^1$. Now, $\{f_n\} \to f$ in the C^∞ Whitney topology; $f_n \notin T_W$, $\forall\, n$ (because, by construction, our W_2 is such that the points $P_1, P_2, \ldots, P_n, \cdots$ (see picture) have coordinates $(x_n; 1/n)$, where the first coordinate refers to S^1, the second to $\mathbb{R}$). Therefore T_W is not open.

4.0 INTRODUCTION:

The purpose of this chapter is to show that the properties H_1 and H_2, necessary for the 'lift' as in Chapter 2, are generically met in $\overline{V(C)}$.

It is trivial to show that H_2 is generic (see 4.5), so that we will concentrate our comments on the genericity of H_1.

In Section 1 we show that the genericity approach is necessary, since the required properties are not always met.

In Section 2 we introduce some preliminary material, for later reference.

Sections 3 and 4 are devoted to the proof of genericity of H_1.

Section 3 deals with the problem of 'avoiding' separatrices 'immediately' after a 'catastrophe point'. This is in general a global problem. It can not be coped with if $n > 1$ and we use only that f is generic (in the sense of [16]). This is because [16] gives us only a local description 'around' singularities. However, if $n = 1$, only the local problem arises, because the 'separatrices' reduce to singularities. Therefore, in 4.3, the restriction $n = 1$ is fundamental (though - see conjectures, Chapter 6 - generalizations of our methods may be possible). A denumerable closed union of (sufficiently high codimensional) submanifolds of T^eC is built, and genericity is achieved through the transversality theory of Chapter 3.

Section 4 deals with 'avoiding' C_f, just after meeting it, from the point of view of our vector fields $v \in V(C)$. This is a problem independent of n, since it depends only on properties of C_f which do not depend on n. See 4.4.0 to a brief description of the methods used there.

Section 5 shows H_2 to be generic (one page), and Section 6 contains some final and brief technical remarks.

4.1. AN EXAMPLE

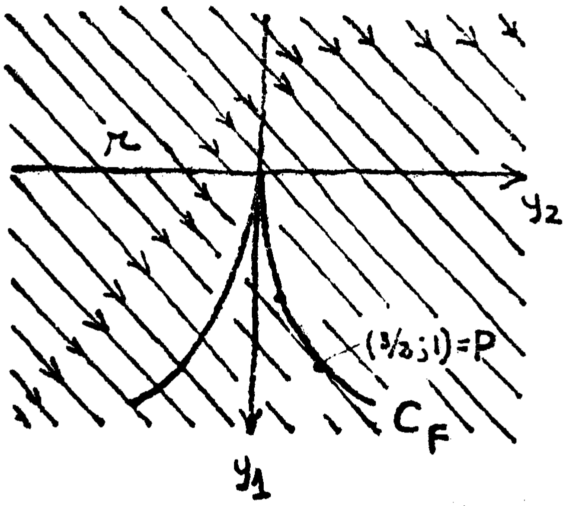
Before we give the proof of genericity of conditions H_1, H_2 (1.2(5)), we illustrate, through a particular example, what can go wrong. The vector field below violates H_1; this is equivalent to the fact that its second expansion, v[2], is not transversal to a certain submanifold of $\mathbf{R}^6$.

Since we just want to exemplify a local problem, let $X = \mathbf{R}$.

Let $C = \mathbf{R}^2$, $f:(x;y_1,y_2) \to x^4 - y_1 x^2 + y_2 x$, and $v \in V(\mathbf{R}^2)$,

$$X \times C \longmapsto \mathbf{R}$$

given by: $v(y_1,y_2) = (1;1)$.



We will show:

(1) $v \notin V_f$ (see 1.2(5))

(2) $\nexists$ lift ϕ with properties as in Theorem 1.

(the flow Φ_y on the state space is given by $-\nabla f_y$).

(1) Fix $(\bar{x},\bar{y}) \in \partial M_f^n$. $J_{f,\bar{x},\bar{y}} = \{y \in \mathbf{R}^2 | x \in \text{sep } \Phi_y\}$. We use the notation $J_{\bar{x},\bar{y}} = J_{f,\bar{x},\bar{y}}$. Now, sep Φ_y is, in this case, the set of points where f_y has a maximum (see 1.1.(3)). Therefore:

$$y \in J_{\bar{x},\bar{y}} \Leftrightarrow \bar{x} \text{ maximises } f_y \Leftrightarrow \left\{ \begin{array}{l} 4\bar{x}^3 - 2\bar{x}y_1 + y_2 = 0 \\ -12\bar{x}^2 + 2y_1 > 0 \end{array} \right\}$$

$$\left\{ \begin{array}{ll} y_2 = (-4\bar{x}^3) + (2\bar{x})y_1 & \text{(I)} \\ y_1 > 6\bar{x}^2 & \text{(II)} \end{array} \right\}$$

This is just the green straight line (open at $\bar{y}$).

For $(\bar{x};\bar{y}) = (1/2;\ 3/2,\ 1)$, one gets $\left\{ \begin{array}{l} y_2 = y_1 - 1/2 \\ y_1 > 3/2 \end{array} \right.$, which (see picture) is contained in $0_{\bar{y}=(3/2,1)}$, violating H_1.

(2) Suppose $\exists$ such a lift. Let $t = 0$, $\overline{m} = (1/2; \underbrace{3/2, 1}_{\overline{y}})$

From Theorem 1, (3), $\exists\, \varepsilon > 0$ s.t.

$1/2 = \Pi_X(\phi(0,\overline{m})) \in$ inset $\Pi_X(\phi(t,\overline{m}))$, $\forall\, t \in [0,\varepsilon)$, where the implicit vector field is $-\nabla f_y$, $y = \Pi_C(\phi(t,\overline{m})) = \psi(t,\overline{y}) = (t,t) + \overline{y} = (3/2 + t; 1 + t)$. Therefore, with $f_y = X^4 - (3/2 + t)X^2 + (1+t)x$, it is easy to check that f_y has a maximum at 1/2, $\forall\, t > 0$, so that $1/2 \in$ in-set $(*) \Rightarrow * = 1/2$. Hence, for $t[0,\varepsilon), \Pi_X(\phi(t,\overline{m})) = 1/2$, therefore $\phi(t,\overline{m}) = (1/2, t+3/2, t+1) \notin \overline{M^n}$, a contradiction.

The trouble with this example is that the orbit of v marked 'r' (see picture), after getting to $P = (3/2;1)$, runs into $J_{(\frac{1}{2};P)}$. The way to see that this can not happen generically is to associatc with each point, P in C_f, all the 'second-order equivalence classes' of curves through P and running into $J_{(\frac{1}{2};P)}$. This has dimension 2, and as we let P vary in C_f, we get in a natural way a stratified union of manifolds in $\mathbb{R}^6 \simeq T^2\mathbb{R}^2$. The higher strata has codim. 3, and therefore $v[2]$ generically misses our stratification. It is then possible to show that when this happens no orbit (through some P) can run into $J_{(.;P)}$.

These arguments will now be made precise, as we actually construct the required manifolds for a (generic) fixed f. We will also have to tackle the problem of avoiding C_f, which does not present itself in the context of the above example.

4.2. PRELIMINARY DEFINITIONS AND PROPOSITIONS

Let $f \in C^\infty(X \times C, \mathbb{R})$ be generic, in the sense of Proposition 0 (1.2); let $n = 1$, $r \leq 4$.

Let ξ_n be the set of germs at 0 of C^∞ functions from $\mathbb{R}^n$ to $\mathbb{R}$, which is a local ring (see [16]), and $m = m_n$ it's maximal ideal. Let $\eta \in m^2$, and (c,h) an unfolding of η, $h: \mathbb{R}^c \times \mathbb{R}^n, 0 \to \mathbb{R}, 0$.

DEFINITION 1:

Given (c,h) as above, we say that (c+d,g), as defined below, is (c,h) with d disconnected controls. ($d \geq 0$, an integer)

$$\begin{array}{ccccc} \mathbb{R}^n \times \mathbb{R}^c \times \mathbb{R}^d & \longmapsto & \mathbb{R}^n \times \mathbb{R}^c & \longmapsto & \mathbb{R} \\ \cup & & \cup & & \cup \\ g: (x, y, w) & \longrightarrow & (x,y) & \longrightarrow & h(x,y). \end{array}$$

REMARK:

It is easy to see ([16], pg. 39) that (c,h) is an universal unfolding of η iff (c+d,g) is.

DEFINITION 2:

The standard r-universal unfolding, (r,g), of η is the standard universal unfolding, (c,h), (where c = codimension η) of η with d = r-c disconnected controls. (For the definition of (c,h), see [16], pg.41; also $r \geq c$ - see [16], 51).

We will have a particular interest in the germs (justification below):

$$\eta_1(x) = \frac{x^3}{3}\ ;\ \eta_2(x) = \frac{x^4}{4}\ ;\ \eta_3(x) = \frac{x^5}{5} \text{ and } \eta_4(x) = \frac{x^6}{6}.$$

After convenient choice of base for m/J (see [17], pg 19), their standard universal unfoldings become:

$$g_1(x;u) = \frac{x^3}{3} + ux;\ g_2(x,u,v) = \frac{x^4}{4} + u\frac{x^2}{2} + vx;\ g_3(x,u,v,w) = \frac{x^5}{5} + u\frac{x^3}{3} + v\frac{x^2}{2} + wx;$$

$$g_4(x,u,v,w,z) = \frac{x^6}{6} + u\frac{x^4}{4} + v\frac{x^3}{3} + w\frac{x^2}{2} + z.$$

PROPOSITION 1:

Let $(x,y) \in \partial M^n_f \subset X^{n=1} \times \mathbb{R}^r$, $n = 1$, $r \leq 4$. There are diffeomorphism (fibre preserving) germs γ, Γ such that the diagram commutes and (r,g) is equal to (c,g_c) with $(r-c)$ disconnected controls, for some $c \in \{1,2,3,4\}$.

$$\begin{tikzcd} & \mathbb{R} \\ \mathbb{R}^n \times \mathbb{R}^r;(0,0) \arrow[ur, "g"] \arrow[r, "\gamma"'] \arrow[d, "\Pi_r"'] & X^n \times \mathbb{R}^r;(x,y) \arrow[u, "f"'] \arrow[d, "\Pi_r"] \\ \mathbb{R}^r \arrow[r, "\Gamma"] & \mathbb{R}^r \end{tikzcd}$$

Proof

From Proposition 0 (Chapter 2) we know $\exists$ some chart $(\phi,\mathcal{U})$ (V_i, for some i, in Proposition 0) around $x \in X^{n=1}$, $\psi = (\phi;\mathrm{id})\colon \underset{X \times \mathbb{R}^r}{\mathcal{U} \times \mathbb{R}^r} \to \underset{\mathbb{R}^n \times \mathbb{R}^r}{\phi(\mathcal{U}) \times \mathbb{R}^r}$, with $\psi(x,y) = (0,0)$, wlog, s.t. the extension map $F\colon\phi(\mathcal{U}) \times \mathbb{R}^r \to J^7_{n=1}$ induced by $f\psi^{-1}$ is $\pitchfork Q$ on $\mathcal{U}$, where Q is the stratification of J^7 as in [16], Chapter 8.

So (see [16], pg 51), $\operatorname{cod}\eta$ $(= f\psi^{-1}/_{\mathbb{R}^n\times\{0\}}) \leq r \leq 4$. Also, since $n = 1$, η is right equivalent to one of η_c ($c = 1,2,3,4$) above. (Note: we should consider $+\frac{x^4}{4}$, $-\frac{x^4}{4}$, $\frac{x^6}{6}$ and $-\frac{x^6}{6}$, but the distinction between the forms with + & − signs need not be made in this context - refer to Lemma 4.12 in [16]). One also has ([16], pg 51) that the germ of $f\psi^{-1}$ is a universal unfolding of $\eta = \eta_c\cdot\xi$, ξ given by the right equivalence above. Therefore ([16], pg 43), $h = f\psi^{-1}.\ (\xi^{-1} \times \mathrm{Id})$ is a universal unfolding of $\eta = \eta_c\,\xi$. Now, $(r,g) = (c,g_c)$ with $(r-c)$ disconnected controls is also an universal unfolding of η_c. (see remark in Definition 1), and from Theorem 6.9 ([16]), (r,g) is isomorphic - via some $(\emptyset,\bar{\Phi})$ - to (r,h). This allows us to write down the following diagram:

$$\begin{tikzcd}
\mathbb{R}^{n=1}\times\mathbb{R}^r,0 \arrow[rrr, "g"] \arrow[dr, "\emptyset"] \arrow[dd, "\Pi_r"'] & & & \mathbb{R} \\
& \mathbb{R}^{n=1}\times\mathbb{R}^r,0 \arrow[r, "\xi^{-1}\times\mathrm{Id}"'] \arrow[dd, "\Pi_r"] & \mathbb{R}\times\mathbb{R}^r,0 \arrow[r, "\psi^{-1}"'] \arrow[dd, "\Pi_r"] & X\times\mathbb{R}^r \arrow[u, "f"'] \arrow[ddl, "\Pi_r"] \\
\mathbb{R}^r,0 \arrow[dr, "\Phi"] & & & \\
& \mathbb{R}^r,0 \arrow[r, "\mathrm{Id}"] & \mathbb{R}^r,0 &
\end{tikzcd}$$

(y = 0, wlog).

The proposition then follows by taking $g = f\overbrace{\psi^{-1}(\xi^{-1}\times I)\emptyset}^{\gamma}$ and $\Gamma = \Phi$. γ is clearly fibre preserving, from the commutative of the above diagram. □

REMARK 1:

M_f^d $(\supset \partial M_f^n)$ = sing χ_f (see [17], pg.15). Also, M_f^d is closed in $X \times C$. Indeed: M^k is open in M, $k = 0,\dots,n$ (2.1(12)) therefore $M_f^d(\supset \partial M_f^n) = M_f - \bigcup_0^n M^k$ is closed in M and M is closed in $X \times C$ (see 2.1(12)).

PROPOSITION 2:

Let $S_1(\chi_f)$ = singularity set of χ_f (notation as in [4]) [$\overset{\text{remark1}}{=\!=}M_f^d$ = $\supset\partial M_f^n$]. Then, $S_1(\chi_f)$ is either $\emptyset$ or a cod. 1 submanifold of M_f.

Furthermore, suppose one has defined $S_{\underbrace{1,\dots,1}_{e}}(\chi_f)$ and it is a codimension e submanifold of M_f; then $S_{\underbrace{1,\dots,1}_{e+1}}(\chi_f) \overset{\text{def.}}{=\!=}$ sing $\chi_f/S_{\underbrace{1,\dots,1}_{e}}(\chi_f)$ is either $\emptyset$ or a cod $(e+1)$ submanifold of M_f.

In other words, one has a sequence $S_1(\chi_f) \supset \dots \supset S_{\underbrace{1,\dots,1}_{e}}(\chi_f) \supset \dots \supset S_{\underbrace{1,\dots,1}_{k}}($ each of which is a cod: $1,\dots,e,\dots,k-1$ (respectively) submanifold of M_f, the last set $(S_{\underbrace{1,\dots,1}_{k}}(\chi_f))$ being either $\emptyset$ or a codimension k submanifold of M_f.

Proof

Suppose $S_1(\chi_f) \neq \emptyset$. Let $m = (x,y) \in S_1(\chi_f)$. From Proposition 1, $\exists$ diffeomorphism germs (at 0) γ, Γ, with $g = f\gamma = g_c + (r-c)$ controls. Since γ is a diffeomorphism, one has $M_{\underbrace{f\gamma}_{g}} = \gamma^{-1}(M_f)$ (germ equation), and $S_1(\chi_g) = \gamma^{-1}(S_1(\chi_f))$.

Now (see [16], Lemma 7.6) $M_g = M_g^c \times \mathbb{R}^{r-c}$, where $M_g^c : \!=\!=\!= M_{g_c}$.

Construct the map $\theta : \underset{m/J}{\mathbb{R} \times \underset{\wr\wr}{\mathbb{R}^c}} \to \underset{m/mJ}{\mathbb{R} \times \underset{\wr\wr}{\mathbb{R}^c}}$ as in [17], pg 16; it is a diffeomorphism germ. One has the following diagram commuting ($h = f\gamma(\theta^{-1} \times I)$).

$$\begin{tikzcd}
\mathbb{R} \times \mathbb{R}^c \times \mathbb{R}^{r-c} \arrow[r, "\theta^{-1} \times I"] & \mathbb{R} \times \mathbb{R}^c \times \mathbb{R}^{r-c} \arrow[r, "\gamma"] & X \times \mathbb{R}^c \arrow[r, "f"] & \mathbb{R} \\
\cup & \cup & \cup & \\
M_h = \underset{\mathbb{R}^c}{\overset{\parallel}{M_h^c}} \times \mathbb{R}^{r-c} \arrow[r, "\theta^{-1} \times I/M_h"] \arrow[d, "\chi_h = \chi_h^c \times I"'] & M_g = M_g^c \times \mathbb{R}^{r-c} \arrow[r, "\gamma/M_g"] \arrow[d, "\chi_g = \chi_g^c \times I"'] & M_f \arrow[d, "\chi_f"'] & \\
\mathbb{R}^c \times \mathbb{R}^{r-c} \arrow[r, "I"] & \mathbb{R}^c \times \mathbb{R}^{r-c} \arrow[r, "\Gamma"] & \mathbb{R}^r &
\end{tikzcd}$$

By computation (see [17], pg 20 for the case c = 2), one gets $\chi_h^c = \chi_g^c \cdot \theta^{-1}/M_h^c : \mathbb{R}^c \circlearrowleft$ as:

$$* \begin{cases}
\text{(fold)} & c = 1: \quad a \longrightarrow -a^2 \\
\text{(cusp)} & c = 2: \quad (a,b) \longrightarrow (2a-3b^2;\ -2ab + 2b^3) \\
\text{(swallow tail)} & c = 3: \quad (a,b,c) \longrightarrow (3b-6c^2;\ 2a-6bc + 8c^3; 3bc^2-2ac-3c^4) \\
\text{(butter-fly)} & c = 4: \quad (a,b,c,d) \to (\underbrace{4c-10d^2}_{u}; \underbrace{3b-12cd+20d^3}_{v}; \underbrace{2a+12cd^2-6bd-15d^4}_{w}; \\
& \qquad \underbrace{2ad+3bd^2-4cd^3+4d^5}_{z})
\end{cases}$$

Since $\theta^{-1} \times I$ is a diffeomorphism, $S_1(\chi_h) = (\theta \times I)(S_1(\chi_g)) = (\theta \times I)\gamma^{-1}(S_1(\chi_f))$; i.e: $(\theta \times I)\gamma^{-1}/S_1(\chi_f) : S_1(\chi_f) \to S_1(\chi_h)$ diffeomorphically. Now $\chi_h = \chi_h^c \times \mathrm{Id}$, so that $S_1(\chi_h) = S_1(\chi_h^c) \times \mathbb{R}^{r-c}$. From $*$, by computation, one sees that a point in $\mathbb{R}^c$ is singular for χ_h^c $(c = 1,2,3,4) \iff a = 0$. That is, any case $S_1(\chi_h^c)$ is a cod. 1 vector subspace of $\mathbb{R}^c$ $\therefore$ $S_1(\chi_h)$ is a cod.1 vector subspace of $\mathbb{R}^r$. Therefore the chart $(\theta \times I)\ \gamma^{-1}/M_f$ takes M_f to $\mathbb{R}^r$ and $S_1(\chi_f)$ to a cod. 1 subspace of $\mathbb{R}^r$. Since $m \in S_1(\chi_f)$, this shows that $S_1(\chi_f)$ is a cod. 1 submanifold of M_f.

We have proved that $S_1(\chi_f)$ is either $\emptyset$ or a cod.1 submanifold of M_f. Suppose $S_{1,1}(\chi_f) \neq \emptyset$, let $m \in S_{1,1}(\chi_f)$. One can once more consider the diagram:

$S_1(\chi_h^c) = \mathbb{R}^{c-1} = M_h^{c-1}$, $\quad S_1(\chi_g^c) = M_g^{c-1}$

$$\begin{tikzcd}[column sep=large]
S_1(\chi_h)=M_h^{c-1}\times\mathbb{R}^{r-c} \arrow[r, "\boxed{(i)}\ \theta^{-1}\times I/S_1(\chi_h)"] \arrow[d, "\chi_h/S_1(\chi_h) = \chi_h^{c-1}\times I"'] & S_1(\chi_g)=M_g^{c-1}\times\mathbb{R}^{r-c} \arrow[r, "\boxed{(ii)}\ \gamma/S_1(\chi_f)"] \arrow[d, "\chi_g^{c-1}\times I"] & S_1(\chi_f) \arrow[d, "\chi_f/S_1(\chi_f)"] \\
\mathbb{R}^c\times\mathbb{R}^{r-c} \arrow[r, "I"] & \mathbb{R}^c\times\mathbb{R}^{r-c} \arrow[r, "\Gamma"] & \mathbb{R}^r
\end{tikzcd}$$

Since $\boxed{(i)}$ and $\boxed{(ii)}$ are again diffeomorphism germs, one has, by the same methods as above $\underbrace{S_{1,1}(\chi_h)}_{=\,S_1(\chi_h^{c-1})} \times \mathbb{R}^{r-c} = (\theta \times I)\ \gamma^{-1}(S_{1,1}(\chi_f))$

Now $S_1(\chi_h^{c-1})$ are computed by discarding <u>a</u> (i.e. setting a=0) from $*$ (this eliminates folds as candidates) and investigating where the Jacobian drops rank by one. This occurs iff $b = 0$ $(c = 2,3,4)$. Therefore $(\theta \times I)\ \gamma^{-1}/M_f$ sends $S_{1,1}(\chi_f)$ to $S_{1,1}(\chi_h) \times \mathbb{R}^{r-c} = \underbrace{S_1(\chi_h^{c-1})}_{=\,\mathbb{R}^{c-1}} \times \mathbb{R}^{r-c}$ = cod.1 subspace of

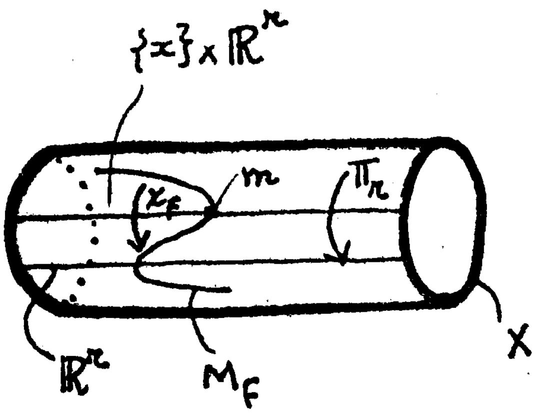
$\mathbb{R}^{c-1} \times \mathbb{R}^{r-c}$ = cod.2 subspace of $\mathbb{R}^r$, therefore $S_{1,1}(\chi_f)$ is a codimension 2 submanifold of M_f. The rest of the proof follows from the fact that setting: $a = b = 0 \Rightarrow$ Jacobian drops $\Leftrightarrow c = 0$; $a = b = c = 0 \Rightarrow$ Jacobian drops $\Leftrightarrow d = 0$ and a straightforward repetition of methods as above. □

PROPOSITION 3:

Given $m = (x,y) \in M^d$, $\exists$ Z, neighbourhood of m in $X^{n\geq 1} \times \mathbb{R}^r$, s.t. $Z \cap (\{x\} \times \mathbb{R}^r) \cap M_f$ is a submanifold of $X^{n\geq 1} \times \mathbb{R}^r$.

Proof

We have $\chi_f = \Pi_r/M_f$ singular (Remark 1), therefore $\exists\ v \neq 0 \in T_mM_f \subset T_m(X \times \mathbb{R}^r)$ such that $T_m\chi_f(v) = 0$, therefore $T_m\Pi_r(v) = 0$. Let $(v_1, v_2, \ldots, v_r)$ be a base for $T_m(\{x\} \times \mathbb{R}^r) \subset T_m(X \times \mathbb{R}^r)$. $\Pi_x = \Pi_r/\{x\} \times \mathbb{R}^r$ is a diffeomorphism. Therefore $T_m\Pi_x$ is an isomorphism.



If $v(\neq 0) = \sum_1^r \alpha_i v_i$, then, $T_m\Pi_x$ being isomorphic, one has: $T_m\Pi_r(v) = T_m\Pi_x(v) \neq 0$, a contradiction. Therefore $(v, v_1, \ldots, v_r)$ are l.i. in $T_m(X \times \mathbb{R}^r)$; so that:

$T_mM_f + T_m(\{x\} \times \mathbb{R}^r) = T_m(X \times \mathbb{R}^r)$, i.e.

$\{x\} \times \mathbb{R}^r \pitchfork M_f$ at m, hence in a neighbourhood Z of m; Therefore (from Theorem 4.1 of [4]), $Z \cap (\{x\} \times \mathbb{R}^r) \cap M_f$ is a submanifold of $X \times \mathbb{R}^r$.

□

PROPOSITION 4:

Let X be a Lindelöf manifold (i.e. every open cover of X admits a denumerable subcover), Y a manifold, $h: X \to Y$ an immersion. Then h(X) is a denumerable union of submanifolds of Y.

Proof

Let $x \in X$ be fixed. From Proposition 2.10 of [4], $\exists$ neighbourhood U_x of x s.t. $h(U_x)$ is a submanifold of Y. $\{U_x\}$ admits denumerable subcover $\{U_i\}$, $h(x) = \bigcup_1^\infty h(U_i)$ and each $h(U_i)$ is a submanifold of Y.

Note: $h/U_x : U_x \to h(U_x)$ is a diffeomorphism. (see [4]). $\square$

$\boxed{\text{NOTATION:}}$ We now fix notation through the following:

REMARK 2:

Fix f. Let $S_1 :=\!=\!= S_1(\chi_f), \ldots, S_{\underbrace{1,\ldots,1}_{k}} :=\!=\!= S_{\underbrace{1,\ldots,1}_{k}}(\chi_f)$ be as in Proposition 2, and define $M_e^d = S_{\underbrace{1,\ldots,1}_{e}} - S_{\underbrace{1,\ldots,1}_{e+1}}$ $(e \le k+1)$. Then

$\{M_e^d\}_{e=1,\ldots,k-1}$ is a stratification of $M^d (= S_1(\chi))$, in the sense that $M^d = \bigcup_{e=1}^{k-1} M_e^d$ (disjoint), and each M_e^d is a cod. e submanifold of M_f, with $\bigcup_{i=e+1}^{k-1} M_i^d = (\overline{M_e^d} - M_e^d)$, $e=1,\ldots,k-1$. To check this, let $m \in M_1^d = S_1 - S_{1,1}$. Then the c in g_c (Proposition 2) has to be 1, otherwise $m \in S_{1,1}$, and therefore the chart $(\theta \times I)\gamma^{-1}/M_f$ for M_f shows, as in Proposition 2, that M_1^d is a codimension 1 submanifold of M_f. The proof for M_e^d is similar. Now $\overline{S_1 - S_{1,1}} = S_1$, since our local charts in Proposition 1 show that $m \in S_{1,1} \Rightarrow m \in \overline{S}_1$, therefore $(\overline{M_1^d} - M_1^d) = S_1 - (S_1 - S_{1,1}) = S_{1,1} = \bigcup_2^{k-1} M_e^d$; again a similar proof shows that the result holds for $e = 2,\ldots,k-1$. So that

$$\{M_e^d\}_{e=1,\ldots,k-1 \le r},\ M_e^d = S_{\underbrace{1,\ldots,1}_{\text{e-times}}} - S_{\underbrace{1,\ldots,1}_{\text{(e+1)times}}} \quad \text{is a stratification}$$

of M^d.

REMARK 3:

Proposition 2 above is a straightforward consequence of the global fact that f is generic (Proposition 0 of Chapter 2), plus the local fact that at any given $m \in M_f$, the stratification germ induced by χ_f on the manifold germ of M_f at m is just the canonical stratification of m^2/m^k $(k = 3,\ldots,6)$, ([17], pgs. 14/21), since we are dealing with n=1.

It was to 'expect' that Proposition 2 should hold anyway, since it is generic for maps $\chi_f : M_f \to \mathbb{R}^r$ to have the $S_{1,\ldots,1}$ singularity occurring as k submanifold of M_f (see [4], Chapter VI, §5-Thom Boardman Strat.).

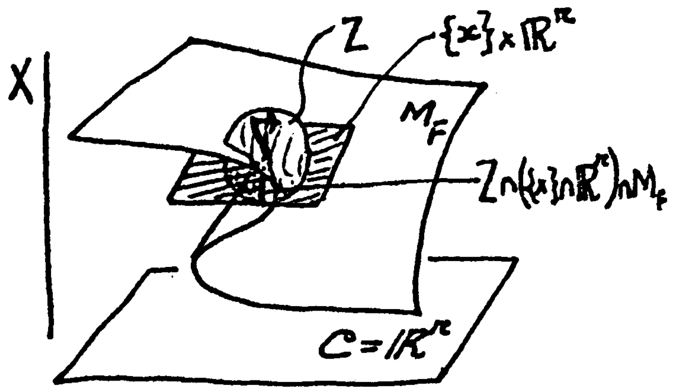
4.3. CONSTRUCTING THE SUBMANIFOLDS CORRESPONDING TO M_{f,y^-} (see 1.1.(4))

We will be interested in patching together fibres consisting of e-tangent bundles of submanifold germs, over a submanifold of X × C. We first need some definitions, to give the words above a precise mathematical meaning.

DEFINITION 3:

Let $\mathcal{Z}$ be a manifold. Two submanifolds $\mathcal{Z}_1$ and $\mathcal{Z}_2$ are equivalent at $p \in \mathcal{Z}$ iff $\exists$ N, neighbourhood of p in Z, s.t. $\mathcal{Z}_1 \cap N = \mathcal{Z}_2 \cap N$. This is easily seen to be an equivalence relation. A submanifold germ of $\mathcal{Z}$ near p is one of these equivalence classes. Notation: $\widehat{Q,p}$, where Q is some representative.

$\widehat{\mathfrak{M},m}$ will denote the submanifold germ of (X × C) at $m \in M^d$, m = (x,y), generated by $\mathfrak{M} = Z \cap (\{x\} \times \mathbb{R}^r) \cap M_f$, as given by Proposition 3 of 4.2.



REMARK 4:

We will call $\widehat{\mathfrak{M},m}$ the 'C-cross section of M_f at m'.

DEFINITION 4: Define

$T^e(\widehat{Q,p}) = \{ \hat{\alpha} \in T^e\mathcal{Z} \mid \exists$ representative, $\alpha : I \to \mathcal{Z}$, of $\hat{\alpha}$, Q of $\widehat{Q,p}$ s.t. $\alpha(I) \subset Q,\ p=\alpha(0)\}$

REMARK 5:

One can also use the definition $\widetilde{T^e}(\widehat{Q,p}) = \Pi_e^{-1}(p)$, where Π_e, $\Pi_e : T^eQ^* \to Q^*$, induced by the representative Q^* of $\widehat{Q,p}$, can be defined in a natural way (see 3.1(5)). It is easy to check that this definition is independent of representatives and that $\widetilde{T^e}(\widehat{Q,p}) = T^e(\widehat{Q,p})$.

REMARK 6:

If Q is a submanifold of $\mathcal{Z}$, then $T^eQ = \bigcup_{p \in Q} T^e(\widehat{Q,p})$, where Q itself is chosen as representative, everywhere. This is immediate from Remark 5.

DEFINITION 5:

$$\mathfrak{M}[e] = \{ \hat{\alpha} \in T^e(\widehat{\mathfrak{M},\overset{\alpha(0)}{m}}) \mid m \in M_f^d\} \underset{\text{since } \mathfrak{M}\subset M_f}{\subset} T^eM_f \subset T^e(X \times \underset{\mathbb{R}^r}{C}).$$

$$\mathfrak{N}[e] = T^e\chi_f(\mathfrak{M}[e]) \subset T^e(C) = T^e(\mathbb{R}^r).$$

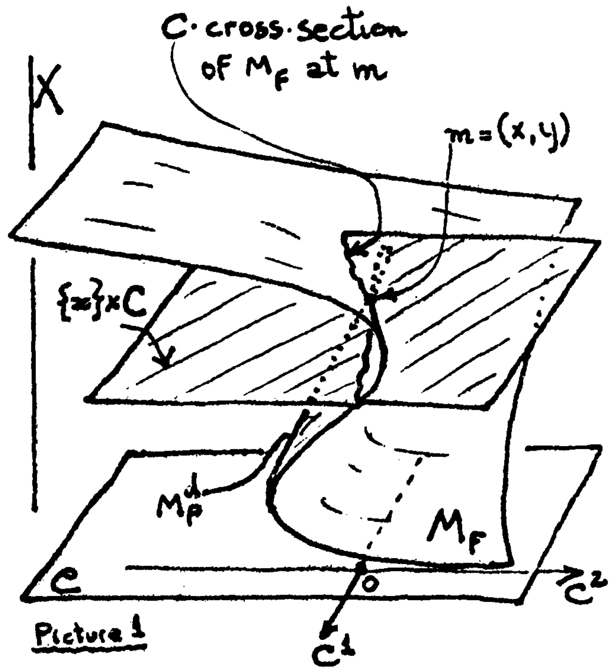
$$\left.\begin{aligned} \mathfrak{M}_i[e] &= \{\hat{\alpha} \in \mathfrak{M}[e] \mid \overset{\alpha(0)}{m} \in M_i^d\}\\ \mathfrak{N}_i[e] &= T^e\chi_f(\mathfrak{M}_i[e]). \end{aligned}\right\} 1 \le i \le r$$

PROPOSITION 5:

$\mathfrak{M}[e]$ and $\mathfrak{N}_i[e]$ $(1 \le i \le r)$ are submanifolds of $T^e(X \times C)$, of codimensions equal to $2(e+1)$ and $i+1+2e$, respectively.

Proof

Let $\alpha(0)=m,\ \hat{\alpha}\in \mathfrak{M}[e]$ $(\mathfrak{M}_i[e])$, $m\,(\in M_f^d) = (x,y)$, and, wlog, $y = 0 \in C = \mathbb{R}^r$. Our first aim will be to construct a local diffeomorphism,



Picture 1

$$\begin{array}{ccc} H : V & \longrightarrow & H(V) \\ \cap & & \cap \\ X\times C, m & & X\times C, m \end{array}$$, V a neighbourhood of m in $X \times C$, with the property of straightening up M_f, i.e.:

$H(V \cap M_f) = H(V) \cap (X \times (\text{linear subspace of } C))$

Let $C' = T_m(\chi_f)(T_m(M_f)) \subset T_0C \simeq C$ (from now on we will not distinguish between T_0C and C). Wlog, $C' = \{y \mid y_r = 0\}$, since C' is, in any case, a cod. 1 subspace of C. This is so because, $m \in M^d$ being arbitrarily fixed, $T_m(\chi_f)$ drops rank by precisely one. This is easy to check from the local forms as in $\star$,(4.2(4)). One gets the Jacobians:

$$\begin{bmatrix}-2a\end{bmatrix} \; ; \; \begin{bmatrix}\boxed{2} & \cdot \\ \cdot & \cdot\end{bmatrix} \; ; \; \begin{bmatrix}\boxed{\begin{matrix}0 & 3\\ 2 & \cdot\end{matrix}} & \cdot \\ \begin{matrix}\cdot & \cdot\end{matrix} & \cdot\end{bmatrix} \text{ and } \begin{bmatrix}\boxed{\begin{matrix}0 & 0 & 4\\ 0 & 3 & \cdot\\ 2 & \cdot & \cdot\end{matrix}} & \cdot \\ \begin{matrix}\cdot & \cdot & \cdot\end{matrix} & \cdot\end{bmatrix},$$

with the minors underlined having det. $\neq 0$, as wished ($\forall$ a,b,c,d)

Let $C^2 = \{y \mid y_1 = \dots = y_{r-1} = 0\}$, so that $C \simeq C^1 \times C^2$ (notation: $y = (\underset{C^1}{y_1}, \underset{C^2}{y_2})$)

Define $\xi: M_f \subset X \times C \to X \times C^1$ by

$$(x;y_1;y_2) \to (x;y_1).$$

We claim that (with $m \in M_f^d$) $T_m\,\xi$ is an isomorphism. First, we note that if $(v;v_1;\dots;v_{r-1};v_r) \in T_m(M_f) \subset T_xX \oplus T_yC^1 \oplus T_yC^2 = T_m(X \times C)$, then $v_r = 0$.

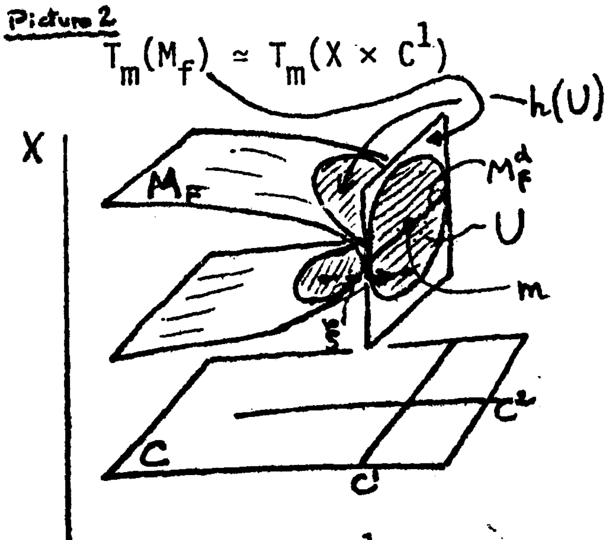
(Otherwise $T_m X_f(v_1;\ldots;v_r) = \quad (v_1;\ldots v_{r-1};\overbrace{v_r}^{\neq 0})$, contradicting the definition of C^1). Since dim. $(T_m(M_f)) = r$, it follows that $T_m(M_f) = T_x X \oplus T_y C^1 \oplus \{0\} \simeq T_m(X \times C^1)$.

Let $u \in T_m(X \times C^1)$. Therefore, $w = (u;0) \in T_m(M_f)$. Hence, if α represents $w, \alpha: I \to M_f, \alpha(t) = (\alpha_x(t);\alpha_1(t);\alpha_2(t))$, one has

$$(\alpha_x'(0);\alpha_1'(0);\alpha_2'(0)) = (u;0).$$

Therefore $T_m\ \xi(\omega) = ((\xi\alpha)_x'(0);(\xi\alpha)_1'(0)) = (\alpha_x'(0);\alpha_1'(0)) = u$ therefore $T_m\xi$ is surjective, hence an isomrophism, since dim $T_m(M_f) = \dim T_m(X \times C^1)$.

Picture 2



From the Inverse Function Theorem, $\exists$ neighbourhood U of m in $X \times C^1$ (which has been confused with $T_m(X \times C^1)$ in picture, because we are drawing X linear), and

$$h : U \to h(U) \subset M_f,$$

smooth and such that $h\xi = id/_{h(U)}$, $\xi h = id/U$.

Set $\Pi_1 : X \times C \to X \times C^1$ (so that $\Pi_{1/M_f} = \xi$) and $\Pi_2 : X \times C \to C^2$
$(x;y_1;y_2) \to (x_1;y_1)$ $\qquad (x,y_1,y_2) \to y_2$

Note: In the following M_i^d can be substituted everywhere by M^d; where 'codimension' appears set $i = 1$.

M_i^d is a cod.$(i+1)$ submanifold of $X \times C$, so that $\exists\ W$, neighbourhood of $m \in M_i^d$ in $X \times C$ and $\eta: W \to \mathbb{R}^{r+1}$ s.t. $\eta(W \cap M_i^d) = \eta(W) \cap A$, where A is a cod.$(i+1)$ linear subspace of $\mathbb{R}^{r+1}$.

Choose V, neighbourhood of m in $X \times C$, small enough so that $V \subset W$ and $V \subset U \times G^2 \subset X \times C$. (see Picture (3) next page).

Define $H : V \longrightarrow H(V)$ by:

$$(x;y_1;y_2) \longmapsto (x;y_1;y_2 - \underbrace{(\Pi_2 h \Pi_1)(x;y_1;y_2)}_{\bullet})$$

This is clearly smooth, since Π_1, Π_2 and h are. It is well defined, since $V \subset U \times C^2$. Let now: $\square : H(V) \longrightarrow \square(H(V))$ be defined by:

$$(x;y_1;y_2) \longmapsto (x;y_1;y_2 + (\Pi_2 h \Pi_1)(x;y_1;y_2)),$$

also well defined, since $H(V) \subset U \times C^2$, and smooth, for the same reasons.

Now

$$\square H(x;y_1;y_2) = (x;y_1;y_2 - (\bullet) + \Pi_2 h \Pi_1(x;y_1;y_2 - (\bullet))) = (x;y_1;y_2 - (\cdot) + \underbrace{\Pi_2 h(x;y_1)}_{=(\cdot)}) =$$

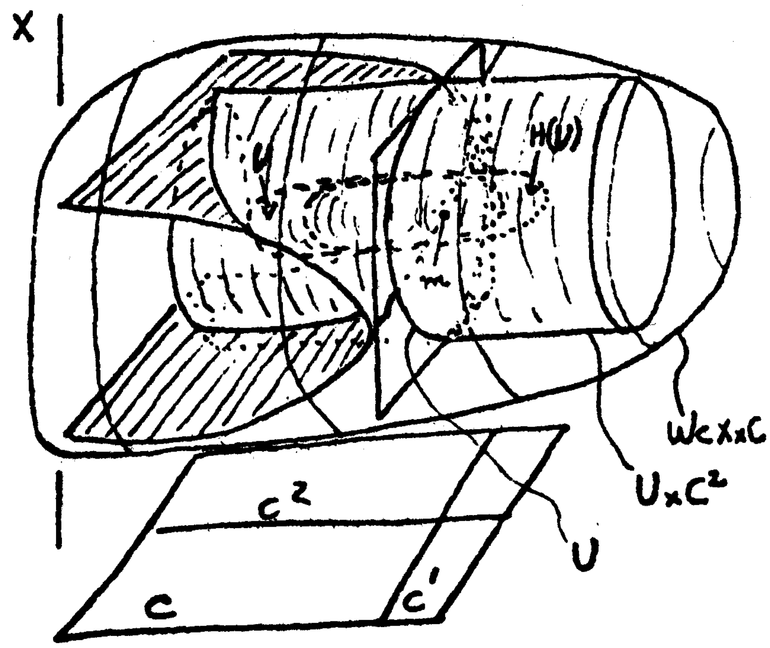
$$= (x;y_1;y_2) \text{ therefore } \square H = I_{/H(V)},\ \square H(V) = V.$$

Also $H\square = I_{/V}$ therefore H is a diffeomorphism $V \to H(V)$

Furthermore, if $(x;y_1;y_2) \in M_f$, then

$$\Pi_2 h \Pi_1 : \underbrace{(x;y_1;y_2) \xrightarrow[\xi]{\Pi_1/M_f} (x;y_1) \xrightarrow{h} (x;y_1;y_2)}_{\text{since } h\xi = I} \xrightarrow{\Pi} y_2 \text{ , so that}$$

$$H : (x;y_1;y_2) \longrightarrow (x;y_1;0), \text{ i.e. } \boxed{H(V \cap M_f) = H(V) \cap (X \times C^1 \times \{0\})} \ *$$



The rest of the proof is quite simple. By means of T^eH, plus a diffeomorphism to straighten $M_i^d(M^d)$ as well, we will be able to produce a local chart for $T^e(X \times C)$ sending $\mathfrak{M}_i[e]$ ($\mathfrak{M}[e]$) to a linear subspace of $\mathbb{R}^{r(e+1)}$, the model for $T^e(X \times C)$.

We first remark that since $H: V \to H(V)$ is a smooth diffeomorphism, then $T^eH:T^e(V) \to T^e(H(V))$ is also a smooth diffeomorphism (see Proposition 3, 3.1(6)).

Note: $T^e(V)$ is an open submanifold of $T^e(X \times C)$, containing $\hat{\alpha}$

Now, since X is a manifold, $\exists\ \mathcal{N}_b$, neighbourhood of x in X, and $\phi = \psi_{\times}\mathrm{id}$ with $\phi = \psi \times \mathrm{id}: \mathcal{N}_b \times C \longrightarrow \phi(\mathcal{N}_b \times C) \subset \mathbb{R} \times \mathbb{R}^r$. W.l.o.g., one can suppose $H(V) \subset \mathcal{N}_b \times C$ (otherwise reduce V conveniently). By abuse, denote $\phi/H(V)$ again by ϕ, so that, from now on, $\phi:H(V) \longrightarrow \phi H(V) \subset \mathbb{R} \times \mathbb{R}^r$.

Let $\tilde{\phi}$ (a diffeomorphism) be defined in the usual way (see 3.1(4)), i.e:

$$\tilde{\phi} : T^e(H(V)) \longrightarrow \mathbb{R}^{r+1} \times \underbrace{\mathbb{R}^{r+1} \times \ldots \times \mathbb{R}^{r+1}}_{e \text{ times}}$$

$$\hat{\alpha} \longmapsto \left(\phi(x;y_1;y_2);\ \frac{d(\phi\alpha)}{dt}(0);\ldots;\ \frac{d^e(\phi\alpha)}{dt^e}(0)\right)$$

We claim that:

$$\begin{array}{ccccc} \tilde{\phi}T^eH, & T^eV \longrightarrow & \underset{\smile}{\overset{\mathbb{R}^{r+1}}{}}\phi H(V) & \times & \overbrace{\mathbb{R}^{r+1} \times \ldots \times \mathbb{R}^{r+1}}^{e \text{ times}}, \text{ sends} \\ & \cup & \cup & & \cup \\ & T^eV \cap \mathcal{M}_i[e] \text{ to} & \phi H(M_i^d \cap V) & \times & (\{0\} \times \mathbb{R}^{r-1} \times \{0\})^e, \end{array}$$

i.e. $\underbrace{\tilde{\phi}.T^eH\ (T^eV \cap \mathcal{M}_i[e]}_{\text{LHS}} = \underbrace{\phi H(M_i^d \cap V) \times (\{0\} \times \mathbb{R}^{r-1} \times \{0\})^e}_{\text{RHS}}$

Indeed:

LHS $\subset$ RHS: Let $\hat{\alpha} \in T^eV \cap \mathcal{M}_i[e]$, $m = \alpha(C)$. Hence $T^eH(\hat{\alpha}) = \widehat{H\alpha}$, with

$\underbrace{H\alpha(I) \subset \{x\} \times C^1 \times \{0\}}_{\odot}$, Now $\tilde{\phi}\ (\widehat{H\alpha}) = \left(\phi H(m);\ \frac{d(\phi H\alpha)}{dt}(0);\ldots;\ \frac{d^e(\phi H\alpha)}{dt^e}(0)\right)$

If $\phi H\alpha: I \to \mathbb{R} \times \mathbb{R}^{r-1} \times \mathbb{R}$ is denoted by $((\phi H\alpha)_x;(\phi H\alpha)_1;(\phi H\alpha)_2)$

then $\odot$ implies
$$\begin{cases} \dfrac{d(\phi H\alpha)_x}{dt}(0) = \ldots = \dfrac{d^e(\phi H\alpha)_x}{dt^e}(0) = 0, \\ \dfrac{d(\phi H\alpha)_2}{dt}(0) = \ldots = \dfrac{d^e(\phi H\alpha)_2}{dt^e}(0) = 0 \end{cases}$$

so that $\tilde{\phi}.T^eH(\hat{\alpha}) = (\phi H(m);0,\ldots,-0;\ldots;0,-,\ldots-,0)$, as wanted

RHS $\subset$ LHS: Let $\tau = (\tilde{m},v_1,\ldots,v_e) \in \phi H(M_i^d \cap V) \times (\{0\} \times \mathbb{R}^{r-1} \times \{0\})^e$,

with $v_s = (0;-;\ldots;-;0)$, $s=1,\ldots,e$.

$\tilde{\phi}^{-1}(\tau) = \hat{\beta}$, where $\hat{\beta}$ is the equivalence class of

$\beta : I \to X \times C$ (I suff. small)

$$t \to \phi^{-1}(\tilde{m} + \sum_{j=1}^{e} \frac{v_j t^j}{j!}) \quad \text{(see 3.1(4))}.$$

Let $(x,y) = \phi^{-1}(\tilde{m})$

It is easy to check that $\beta(I) \subset \{x\} \times C^1 \times \{0\}$

Therefore $\tilde{\phi}^{-1}(\tau) = \hat{\beta} \in T^eH(T^eV \cap \mathfrak{M}_i[e])$ with $\beta(0) = \phi^{-1}(\tilde{m})$, therefore

$\tilde{\phi}(\tilde{\phi}^{-1}(\tau)) = \tau \in \tilde{\phi}.T^eH\ (T^eV \cap \mathfrak{M}_i[e])$.

Finally, denote η/V. $(V \subset W)$ also by η .

Then, we have:

$$\eta.H^{-1}\phi^{-1} \times \underbrace{I \times \ldots \times I}_{\text{e times}}: \phi H(M_i^d \cap V) \times (\{0\} \times \mathbb{R}^{r-1} \times \{0\})^e \to \eta(V) \cap A \times (\{0\} \times \mathbb{R}^{r-1}\ \{0\})^e$$

where A is a linear subspace of $\mathbb{R}^{r+1}$, of codimension i+1. Therefore, the local diffeomorphism

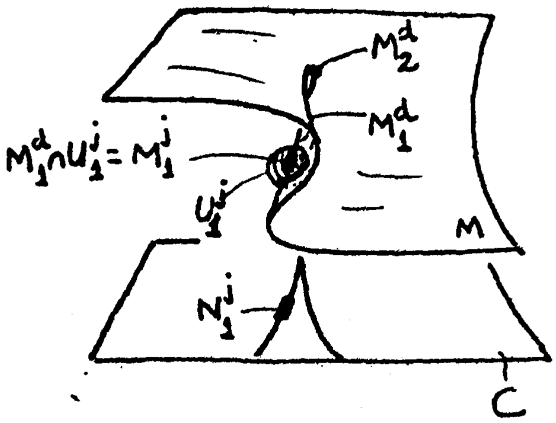
$\Phi = (\eta H^{-1}\phi^{-1} \times I^e).\tilde{\phi}.T^eH$ sends $T^e\ V \cap \mathfrak{M}_i[e]$ to a codimension $(i+1+2e)$ linear subspace of $\mathbb{R}^{(r+1)e}$, as we wanted to show. □

PROPOSITION 6:

There is a denumerable open cover, $\mathcal{F}_i = \{U_i^j\}_{j=1,\ldots,n,\ldots}$ of M_i^d (i fixed) such that, for every j:

$N_i^j := X_{f/M_i^d}(M_i^j)$ is a cod. i submanifold of C, where $M_i^j := M_i^d \cap U_i^j$.

Proof



This is an immediate consequence of Proposition 4 plus the following facts:

(1) M_i^d is Lindelöf. This is so because X (compact, metric, therefore Lindelöf) and $C = \mathbb{R}^r$ are Lindelöf, and therefore so is M_i^d, a (topological) subspace of $X \times C$.

(2) χ_f/M_i^d is an immersion: $M_i^d = \underbrace{S_{1,\ldots,1}}_{i \text{ times}} - \underbrace{S_{1,\ldots,1}}_{(i+1) \text{ times}} = \underbrace{S_{1,\ldots,1}}_{i \text{ times}} - \text{sing.}\chi_{f/S_{\underbrace{1,\ldots,1}_{i \text{ times}}}}$

so that χ_f has maximal rank on M_i^d. □

COROLLARY

$\exists$ a denumerable open cover $\mathcal{U} = \bigcup_i \mathcal{U}_i = \{U_i^j\}$ of M^d with the property that N_i^j (as above) is a cod. i submanifold of C, $\forall$ i,j. Also $\chi_{f/M_i^j}: M_i^j \to N_i^j$ is a diffeomorphism (see note to Proposition 4).

REMARK

W.l.o.g. U_i^j can be supposed to be so small as to satisfy Proposition 1 for some local diffeomorphism. ∎

Our next aim will now be to show that we can decompose $\mathcal{N}[e]$ in a denumerable number of (sufficiently high codim.) submanifolds of T^eC. For this we need some further definitions.

We recall that:

$$N_i^j = \chi_{f/M_i^d}(M_i^j),$$

$$M_i^j = M_i^d \cap \mathcal{U}_i^j$$

DEFINITION 6:

$$\mathfrak{M}_i^j[e] = \{\hat{\alpha} \in \mathfrak{M}_i[e] \mid \alpha(0) = m \in M_i^j\}; \mathfrak{N}_i^j[e] = T^e\chi_f(\mathfrak{M}_i^j[e])$$

Note 1: It follows immediately that $\mathfrak{M}[e] = \bigcup_{i,j} \mathfrak{M}_i^j[e], \mathfrak{N}[e] = \bigcup_{i,j} \mathfrak{N}_i^j[e]$.

$$\mathfrak{M}_i[e] = \bigcup_j \mathfrak{M}_i^j[e], \mathfrak{N}_i[e] = \bigcup_j \mathfrak{N}_i^j[e]. \blacksquare$$

$\mathfrak{P}[e] = \{\hat{\alpha} \in T^e(X \times C) \mid \alpha(0) = m \in (x,y) \in M^d$. $\hat{\alpha}$ admits representative $\alpha: I \to X \times C$ such that $\alpha(I) \subset \{x\} \times C\}$. $\mathfrak{P}_i[e]$ and $\mathfrak{P}_i^j[e]$ are defined analogously.

Note 2: $\mathfrak{M}[e] \subset \mathfrak{P}[e] \subset T^e(X \times C)$. $\blacksquare$

Note 3: It is easy to show that $\mathfrak{P}[e]$ $(\mathfrak{P}_i[e], \mathfrak{P}_i^j[e])$ is a submanifold of $T^e(X \times C)$. $\blacksquare$

PROPOSITION 7:

$\mathfrak{N}_i^j[e]$ is a submanifold of T^eC, $\forall$ i,j fixed. Cod. $(\mathfrak{N}_i^j[e]) = e+i$

Proof

The idea of the proof is to express $\mathfrak{N}[e]$ as $P(\mathfrak{M}[e])$, where P is defined below in a way which makes it easy to check that $P(\mathfrak{M}_i^j[e]) = \mathfrak{N}_i^j[e]$ is a submanifold.

Let P be given by the diagram:

$$\begin{tikzcd}
\mathcal{P}[e] \arrow[r, "G"] \arrow[ddr, "P"'] & M^d \times \overbrace{\mathbb{R}^r \times \ldots \times \mathbb{R}^r}^{e \text{ times}} \arrow[d, "\chi_f \times I/_{\mathbb{R}^{re}}"] \\
& \mathbb{R}^r \times \mathbb{R}^{re} \arrow[d, "\tilde{I}^{-1}/_{\mathbb{R}^r}"] \\
& T^eC
\end{tikzcd}$$

where G is defined as:

$$G: \mathcal{P}[e] \to M^d \times \overbrace{\mathbb{R}^r \times \ldots \times \mathbb{R}^r}^{e \text{ times}}$$

$$\hat{\alpha} \to (\beta(0); \frac{d\beta}{dt}(0); \ldots; \frac{d^e\beta}{dt^e}(0)),$$

where α represents $\hat{\alpha}$ and $\beta: I \to C$ is given by
$$\begin{tikzcd} I \arrow[r, "\alpha"] \arrow[rr, bend right, "\beta"'] & \{x\} \times C \arrow[r, "\Pi_{C/\{x\} \times C}"] & C. \end{tikzcd}$$

(easy to check that definition independs of representatives). It is easy to show that G is a diffeomorphism. And so is $G_i^j = G/\mathcal{P}_i^j[e] : \mathcal{P}_i^j[e] \to M_i^j \times (\mathbb{R}^r)^e$.
Also, from the corollary above, $\chi_f/M_i^j : M_i^j \to N_i^j$ is a diffeomorphism and so is $I/\mathbb{R}^{re}$ and $\tilde{I}^{-1}/\mathbb{R}^r : N_i^j \times \mathbb{R}^{re} \subset \mathbb{R}^r \times \mathbb{R}^{re}$ to its image. Therefore, $P_i^j = \tilde{I}^{-1}/\mathbb{R}^r.(\chi_f \times I)G_i^j : \mathcal{P}_i^j[e] \to P_i^j(\mathcal{P}_i^j[e])$ is a diffeomorphism.

Now, $\mathfrak{M}_i^j[e]$ is a submanifold of $\mathcal{P}_i^j[e]$, so that $P_i^j(\mathfrak{M}_i^j[e])$ is a submanifold of T^eC.

It remains to prove that $P_i^j(\mathfrak{M}_i^j[e]) = \mathfrak{N}_i^j[e]$. (The same argument as below also shows $P(\mathfrak{M}[e]) = \mathfrak{N}[e]$).

To see this, fix $\hat{\alpha} \in \mathfrak{M}_i^j[e]$, $\alpha(0) = \overset{(x,y)}{\overset{\shortparallel}{m}} \in M_i^j$, $G(\hat{\alpha}) = (m; \frac{d\beta}{dt}(0);\ldots;\frac{d^e\beta}{dt^e}(0))$, with

$\beta : I \xrightarrow{\alpha} \{x\} \times C \xrightarrow[\Pi_C]{} C$, i.e. $\beta = \chi_f\alpha$.

Therefore, $((\chi_f/M_i^j) \times I)G(\hat{\alpha}) = (\underset{N_i^j}{\underset{\wedge}{y}}\,; \frac{d(\chi_f\alpha)}{dt}(0);\ldots;\frac{d^e(\chi_f\alpha)}{dt^e}(0)) \xrightarrow{\tilde{I}^{-1}/\mathbb{R}^r} \widehat{\chi_f\alpha} =$

$= T^e\chi_f(\hat{\alpha})$, therefore $P_i^j(\mathfrak{M}_i^j[e]) = T^e\chi_f(\mathfrak{M}_i^j[e]) = \mathfrak{N}_i^j[e]$.

Finally, since P_i^j is a diffeomorphism $\dim(\mathfrak{N}_i^j[e]) = \dim(\mathfrak{M}_i^j[e]) =$ by Prop.5 $\dim(T^e(X \times C)) - (i+1+2e) = (r+1)(e+1)-(i+1+2e) = r(e+1)-(e+i)$. Hence $\operatorname{cod}(\mathfrak{N}_i^j[e])$ in T^eC is $r(e+1)-r(e+1)+(e+i) = e+i$. □

COROLLARY:

$\mathfrak{N}[e]$ is a denumerable union of submanifolds of T^eC, each one of which has codimension $\geq e+1$.

PROPOSITION 8:

$\mathfrak{M}[e]$, $\mathfrak{N}[e]$ are closed in $T^e(X \times C)$, $T^e(C)$, respectively.

Proof

First, we show that $\mathfrak{M}[e]$ is closed in $T^e(X \times C)$. Let $\{\hat{\alpha}_k\}$, $m_k = \alpha_k(0)$, be a sequence in $\mathfrak{M}[e]$, converging to $\hat{\alpha} \in T^e(X \times C)$. Now, let (ϕ, U) be some chart for $X \times C$ around m. $\tilde{U}$ (def. as usual) is a neighbourhood of $\hat{\alpha}$ in $T^e(X \times C)$, therefore $\hat{\alpha}_k \in \tilde{U}$ for k suff. big, therefore $m_k \in U$, therefore $m_k \to m$, since U can be taken arbitrarily small. Now, with η, Φ, V as in Proposition 5,

$$\Phi : T^e(V) \cap \mathfrak{M}[e] \to \underbrace{(\eta(V) \cap A) \times (\{0\} \times \mathbb{R}^{r-1} \times \{0\})^e}_{\circledast}, \text{ k suff. big}$$

$$\hat{\alpha}_k \quad \to \quad (\eta(m_k);(0;v_1^k;0);\ldots;(0;v_e^k;0))$$

So $\Phi(\hat{\alpha}_k) \to (\eta(m);(0;v_1;0);\ldots;(0;v_e;0))$, therefore $\hat{\alpha} \in \Phi^{-1}(\circledast)$, therefore $\hat{\alpha} \in \mathfrak{M}[e]$.

$\mathfrak{P}[e]$ is shown to be closed in $T^e(X \times C)$ in the same way, therefore $\mathfrak{M}[e] \underset{\text{closed}}{\subset} \mathfrak{P}[e]$. Now $P = \tilde{I}^{-1}/\mathbb{R}^r(\chi_f/M^d \times I)G$ is a closed map, since $\chi_f: M_f \to C$ is closed (chapter 2) and M^d closed in M_f, therefore $P(\mathfrak{M}[e]) \xrightarrow{\text{Prop.7}} \mathfrak{N}[e]$ is closed in T^eC. □

COROLLARY

Let $e \geq r$ be fixed. Then $\mathfrak{N}[e]$ is a denumerable closed union of submanifolds of T^eC, each one of which has codimension $\geq r+1$.

Proof

Use Proposition 7 and Proposition 8 above.

PROPOSITION 9:

Let $e \geq r$ be a fixed integer. There is an open and dense set B_e of vector fields with the property that $v[e](\mathbb{R}^r) \cap \mathfrak{N}[e] = \emptyset$, $\forall$ $v \in B_e$. $\hat{V}(\mathbb{R}^r)$.

Proof

Define $A_i^j = \mathfrak{N}_i^j[e] \cap A$, $A_i^{j,c} = \mathfrak{N}_i^j[e] \cap A^c$, $W_i^j = S^{-1}(A_i^j)$, $W_i^{j,c} = S^{-1}(A_i^{j,c})$, where A, S are defined as in Chapter 3.

Since $A_i^{j,c} \cap A = \emptyset$, $W_i^{j,c}$ is a cod$(e+i) > r$ submanifold of $J^{e-1}(\mathbb{R}^r,\mathbb{R}^r)$

As in the proof of Proposition 8, Chapter 3, we have $W_i^j = N_i^j \times \{0\} \times B_{r,r}^{e-1}$, where N_i^j has codimension i in $\mathbb{R}^r$, and $\{0\}$ codimension r in $\mathbb{R}^r$; therefore W_i^j is a cod.$(r+i) > r$ submanifold of $J^{e-1}(\mathbb{R}^r,\mathbb{R}^r)$.

Let $W = \bigcup_{i,j} (W_i^j \cup W_i^{j,c})$ (denumerable), each $W_i^j, W_i^{j,c}$ a submanifold of $J^{e-1}(\mathbb{R}^r,\mathbb{R}^r)$, with cod $> r$.

Now $W = \bigcup_{i,j} (S^{-1}(A_i^j) \cup S^{-1}(A_i^{j,c})) = \bigcup_{i,j} S^{-1}(A_i^j \cup A_i^{j,c}) = \bigcup_{i,j} S^{-1}(\mathfrak{N}_i^j[e]) = S^{-1}(\bigcup_{i,j} \mathfrak{N}_i^j[e] = S^{-1}(\mathfrak{N}[e])$, closed, from Proposition 8 above.

Set $B_e = T_W = \{v \mid j^{e-1}v \pitchfork (W_i^j \text{ and } W_i^{j,c}, \forall\, i,j)\}$. This is open and dense by Lemma 2 in 3.3(2). Transversality with these relative dimensions means $j^{e-1}v(\mathbb{R}^r) \cap \left\{ \frac{W_i^j}{W_i^{j,c}} \right\} = \emptyset$, therefore $j^{e-1}v\ (\mathbb{R}^r) \cap W = \emptyset$, where $W = S^{-1}(\mathfrak{N}[e])$.

Since
$$\begin{tikzcd} & \mathbb{R}^r \arrow[dl, "j^{e-1}v"'] \arrow[dr, "v[e]"] & \\ J^{e-1}(\mathbb{R}^r,\mathbb{R}^r) \arrow[rr, "S"] & & T^e\mathbb{R}^r \end{tikzcd}$$
commutes (3.2(1)), we therefore have $v[e](\mathbb{R}^r) \cap \mathfrak{N}[e] = \emptyset$, $\forall\, v \in B_e$. $\square$

PROPOSITION 10:

Let $B = B_r$, as above, $v \in B$, $y \in C_f$, arbitrarily fixed. Then $\exists\, \varepsilon > 0$ s.t. $M_{f,y} \cap O_y(\varepsilon) = \emptyset$.

(Note: this accounts for part of H_1; the 'rest' of H_1, i.e., the 'C_f part', will be dealt with in 4.4, so that we will conclude that H_1 is generic).

Proof

Since $n=1$ it is easy to see that $x \in \text{sep}\,(-\nabla f_{\tilde{y}}) \Rightarrow x$ is singular for $(-\nabla f_{\tilde{y}})$.

Therefore, if $\varepsilon > 0$ s.t.: $\bigstar [\Pi_c((\{x_t\} \times C) \cap M_f) \cap O_y(\varepsilon) = \emptyset]_{t=1,\dots,s}$, over all t such that $(x_t,y) \in M^d$, then one also has $M_{f,y} \cap O_y(\varepsilon) = \emptyset$.

It suffices to prove $\bigstar$ for a fixed $m = (x,y) \in M^d$, since $\{(x_t,y)\}_{t=1,\dots,s}$ is finite.

Let $m = (x,y) \in M^d$. $(\hat{\beta}) \in \mathfrak{N}[r]$, with $\beta(0) = y \Leftrightarrow \hat{\beta} = \widehat{\chi_f\alpha}$ where $\hat{\alpha}$ admits representative $\alpha: I \to X \times C$ s.t. $\alpha(I) \subset Z \cap (\{x\} \times C) \cap M_f$, Z some (open) neighbourhood of m in $X \times C$ (see Proposition 3). Since $Y = \Pi_c(Z \cap (\{x\} \times C) \cap M_f)$ is a submanifold of C (directly from Proposition 3), $y \in Y$, $\exists$ (open) neighbourhood $V (\subset \Pi_c(Z)$, wlog) of y in C and

$$\phi : V \subset \mathbb{R}^r \longrightarrow \phi(V) \subset \mathbb{R}$$
$$V \cap Y \longrightarrow \phi(V) \cap \underbrace{\{(y_1,\dots,y_r) \in C \mid y_r = 0\}}_{\cap \mathbb{R}^r}$$

Let us now consider $\hat{\gamma}$ (= $v[r](y)$), where $\gamma : I \to C$ be a solution of v through $\gamma(0) = y$, with $\gamma(I) \subset V$. Let $\phi\gamma :\!=\!=\!= ((\phi\gamma)_1;,,,;(\phi\gamma)_r)$.

Claim:

$$\frac{d^j(\phi\gamma)_r(0)}{dt^j} \neq 0, \text{ for some } 1 \leq j \leq r \text{ (may be more than one } j\text{)},$$

Proof

Suppose this is not so. Consider

$\eta(t) = ((\phi\gamma)_1(t);\dots;(\phi\gamma)_{r-1}(t);0)$; by supposition, $\eta \sim_r \phi\gamma$ therefore $\widehat{\eta} = \widehat{\phi\gamma}$. Hence $\hat{\gamma}$ (= $\widehat{\phi^{-1}\eta}$) admits representative $\phi^{-1}\eta$, satisfying $\phi^{-1}\eta\,(I) \subset V \cap Y$ (since $\eta(I) \subset \phi(V) \cap \{(y_1,\dots,y_r) \mid y_r = 0\}$). Setting $\alpha(t) = (\underset{\{x\}}{\underset{|||}{\alpha_x}}(t);\phi^{-1}\eta(t))$, we get $\alpha(I) \subset Z \cap (\{x\} \times C) \cap M_f$, with $\hat{\gamma} = \widehat{\chi_f\alpha}$

so that $\hat{\gamma} \in \mathfrak{N}[r]$, a contradiction to the hypothesis of $v \in \mathcal{B}$ (see Proposition 9).

It follows from the claim that $(\phi\gamma)_r$ is j-determined (if j is the smallest integer for which the claim is true). In the same way as in Proposition 9 (3.3(3)) it is easy to show that, wlog, we can suppose $(\phi\gamma)_r(t) = t^j$, for small enough t. Therefore, for conveniently small ε and $|t| < \varepsilon$,

$$\underset{\substack{t \neq 0 \\ |t| < \varepsilon}}{(\phi\gamma)(t)} \cap \{(y_1,\dots,y_r) \mid y_r = 0\} = \emptyset, \text{ hence}$$

$$\gamma(t)_{\substack{t\neq 0\\|t|<\varepsilon}} \cap [(V \cap Y)] = \emptyset \Rightarrow \gamma(t)_{\substack{t\neq 0\\|t|<\varepsilon}} \cap Y = \emptyset \Rightarrow O_y(\varepsilon) \cap \Pi_c(\{x\} \times C \cap M_f) = \emptyset,$$

as we wished to show. □

COROLLARY:

$\exists$ open and dense set, $B \subset V(\mathbb{R}^r)$, with the property that, $\left\{\begin{array}{l}\forall v \in B\\ \forall y \in C_f\end{array}\right\}$, fixed, $\exists\ \varepsilon > 0$ s.t. $M_{f,y} \cap O_y(\varepsilon) = \emptyset$.

4.4. CONSTRUCTING THE SUBMANIFOLDS CORRESPONDING TO C_f:

4.4.0 INTRODUCTION

Let $f:X \times \mathbb{R}^r \to \mathbb{R}$, (we won't be using that dim(X) = 1 in 4.4, see 4.0) where X is compact, $r \leq 4$ is fixed. We will now tackle the problem of proving that having orbits with the property of 'isolated intersection' with respect to $C_f (v \bar{\pitchfork} C_f)$ is a generic (open and dense) property of vector fields in $\mathbb{R}^r$.

To this purpose, we 'generate', from each of the different strata of C_f, a denumerable union of submanifolds of $T^r(\mathbb{R}^r) \simeq \mathbb{R}^{r(r+1)}$. In order to be able to apply our earlier results (see Chapter 3) we need to do this in such a way that the following conditions are met.

(1) Each submanifold has to have codimension bigger than r.

(2) The union of all submanifolds must be closed; this union, in the notation we use in the proofs below, will be the set C[r] (r = 1,2,3,4).

(3) If $v[r](\mathbb{R}^r) \cap C[r] = \emptyset$ (we will prove this to be generic) then $v \bar{\pitchfork} C_f$.

Before we give the formal proof, we would like to explain in a few words and in a very loose way how we have been led to the solution presented here; we feel that it is important not only to show that things work but also why they should.

We first tried to define our union of submanifolds of $T^r(\mathbb{R}^r)$ by crushing, via $T^r(\chi_f)$, what we knew to be a closed subset of $T^r(X \times \mathbb{R}^r)$, i.e. $T^r(M_f^d)$. This was good enough as far as condition (1) was concerned. But closeness failed.

Our next attempt was directed towards 'correcting' that definition. The idea would have been to work out the closure of each union of submanifolds, corresponding to each distinct strata, and perhaps try to 'close' those sets artificially. This, on one hand, proved to be an impossible task, since those closures were far too complicated; and, on the other hand, it seemed that the crushing process was too rough to preserve the property of isolated intersection. (i.e., one needs lifts to $X \times C$ to be able to prove (3)).

We therefore abandoned the whole method althogether, and tried the following strategy:

(I) Work out, on a case by case basis and 'up to the codimension required' [(r+1)] - hence satisfying condition (1) -, which conditions would be fulfilled if a curve α, through a point $y = \alpha(0)$ belonging to a certain strata of C_f, is to run into a smaller codimensional strata (or into this strata). See appendix for details.

(II) Try to show that if one has a sequence of curves $\{\alpha_n\} \to \alpha$ (this is made precise later), through points $y_n = \alpha_n(0)$ belonging to the smaller cod. strata referred above, with $y_n \to y$, then the conditions set up in (I) are met by α. From an intuitive point of view, it seems likely that one would get away with this proof; besides, this would take care of closeness - condition (2).

(III) From the set C[r] cooked up by avoiding local conditions as in (I), prove condition (3). This is a reasonable conjecture since in a sense a certain 'converse' is true: if a curve runs into the smaller cod. strata (which is the basic non-trivial problem that can happen) then it satisfies conditions as in (I).

This idea works. It actually allows us to fulfill (3) and, at the same time, force at each stage the union of submanifolds corresponding to each strata to 'close' the union of submanifolds relative to the strata of immediately smaller codimension, without ever having to work out its closure. Since we go 'up to the cod. required:- (r+1)' we are really exploiting to the limit the existing room in $\mathbb{R}^{r(r+1)}$ ($r = 1,\ldots,4$).

As to the way we present our results here, the solutions corresponding to $r = 1,\ldots,4$ are given in succession. It turns out that the proofs are in a certain way 'cumulative', each new r presenting the problems of the preceding r with a further degree of complexity, plus a new problem, inherent to the new dimension.

Item (I) is explained in an appendix, since we do not want to mix up the intuition which led to the method with the proof that it works. The definitions 'generated' by (I) (those of the $C_i^j[e]$ - see below - $1 \le e \le 4$, $i = 1,\ldots,e$, $j \in \mathbb{N}$) are given in the items 'A' of 4.4.1,...,4.4.4 below.

Items 'B' are essentially about (II); one needs, however, a certain amount of technical work to reduce the global problem to a number of local cases and then each one to canonical form. (III) is proved in items C.

The case r=5 is not done here, mainly because the amount of technical details would probably render it unbearably boring to read and to write, besides not throwing any specially new light into the problem. We remark that it is easy to work out (just use same methods as in appendix)what the 'intuitive' conditions coming from (I) should be in this case, though, of course, we make no claims of having proved this case.

4.4.1: The case r=1

A. Definition of C[1]

Let $\mathcal{P}$ be as in corollary to Proposition 6 (4.3(7)). Since r = 1, one has $\mathcal{P} = \mathcal{P}_1$, $N_1^j = \{y_j\}$, $\forall\, j \in \mathbb{N}$

Set: $C_1^j[1] = T^1(N_1^j) \subset T^1(\mathbb{R})$

Note: here we view N_1^j as a 0-dim. manifold; T^1 has the usual meaning

Define: $C_1[1] = \bigcup_{j\in\mathbb{N}} C_1^j[1]$

and $C[1] = C_1[1]$.

B. Closedness of C[1]

PROPOSITION 11:

C[1] is closed.

Proof

If ϕ is chart for a manifold M, we re-all that $\tilde{\phi}(=\tilde{\phi}^e)$ is a chart for T^eM (see 3.1(4)). Take ϕ = I, the identity on $\mathbb{R}$. Now, $\tilde{I}(C[1]) = C_f \times \{0\} \subset \mathbb{R}^r$, which is closed because C_f is closed; hence, the proposition is true. □

C. Genericity of $v \pitchfork C_f$

PROPOSITION 12:

$\exists$ open and dense set, $\mathcal{B} \subset \mathcal{V}(\mathbb{R})$, s.t. : $v \in \mathcal{B} \Rightarrow v[1](\mathbb{R}) \cap C[1] = \emptyset$

Proof

Define $V_1^j = S^{-1}(C_1^j[1])$ (see Chapter 3, for definition of S). Exactly as in Proposition 9 (4.3(11)), one sees that V_1^j has cod. 2. Hence $\mathcal{B} = \{v \mid j^0 v \pitchfork V_1^j, \forall_j\}$ is open and dense and $v \in \mathcal{B} \Rightarrow v[1](\mathbb{R}) \cap C[1] = \emptyset$, in a way similar to the above mentioned proposition. □

Note: The case r=1 is by far the most trivial case; the proof of theorems as above will be similar in the cases r = 2,3,4. We will give fuller details there.

PROPOSITION 13:

If $v \in \mathcal{B}$, as above, then $v \pitchfork C_f$.

Proof

Let $v \in \mathcal{B}$ be fixed.

$C_f = \{y_j\}_{j \in \mathbb{N}}$. Let $y \in C_f$. Hence $y = y_j$, some $j \in \mathbb{N}$. Now, $v[1](\mathbb{R}) \cap C[1] = \emptyset \Rightarrow$

$$\Rightarrow v[1](y_j) = \hat{\alpha} \underset{C_1^j[1]}{\pitchfork} \quad (\alpha \text{ solution of } v \text{ through } y_j)$$

Therefore $d\alpha/dt(0) \neq 0$, and so $\exists\ \varepsilon > 0$ s.t. $\{\alpha(t) \mid {}^{|t|<\varepsilon}_{t\neq 0}\} \cap \{y_j\} = \emptyset$, as wanted.

□

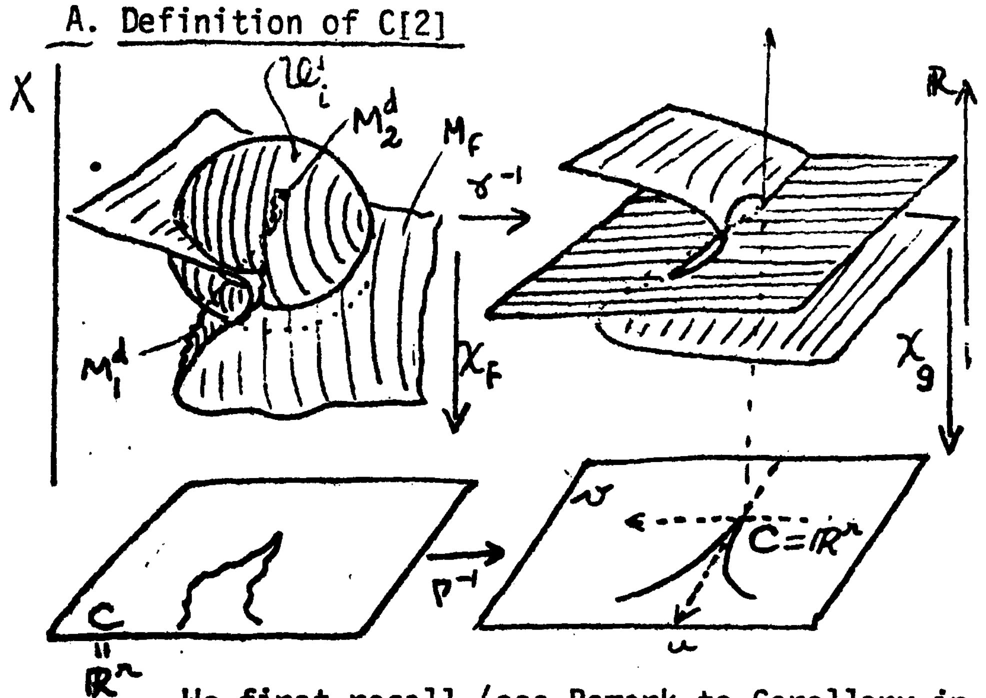
COROLLARY:

If $f: X \times \mathbb{R} \to \mathbb{R}$ is generic, $\exists$ open and dense set $B \subset V(\mathbb{R})$ s.t. $v \in B \Rightarrow$
$\Rightarrow v \pitchfork C_f$.

4.4.2: The case r = 2:

Let $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2$ (see 4.3(7)), $\mathcal{P}_1 = \{U_1^j\}_{j \in \mathbb{N}}$, $\mathcal{P}_2 = \{U_2^j\}_{j \in \mathbb{N}}$, and recall that $N_i^j = \chi_f/M_i^d\,(M_i^j)$ is a submanifold of $C = \mathbb{R}^2$, $\forall$ i,j fixed. In this case N_2^j, j fixed, is just a point, say $N_2^j = \{y_j\}$, while N_1^j is a submanifold of $\mathbb{R}^2$ of cod. 1, i.e., a 1-dimensional submanifold.

A. Definition of C[2]



Let i and j be fixed. [Picture illustrates the case r=2 and i=2, showing how $U_i^j \cap M_f$ it is mapped into the cusp, in its standard form - see also the definition of g_2, in 4.2(1)]

We first recall (see Remark to Corollary in 4.3(7)) that $\exists$ γ, Γ, diffeomorphisms (corresponding to (j,i)) making the above diagram commutative (for a precise statement, see Proposition 1, 4.2(1)). These are not, however, unique. This means that every definition which depends upon choosing γ, Γ s.t. the diagram commutes must be shown to be independent of that choice. For the rest of 4.4, the letters γ, Γ will be used for diffeomorphisms as indicated above.

We will give below a set of definitions which involve a choice of γ,Γ; we prove then that they do not depend on the choice.

DEFINITION 7:

We first recall the definition of $\tilde{I}$.

$$\tilde{I} : \hat{\alpha} \in T^e\mathbb{R}^r \to (\alpha(0); \frac{d\alpha}{dt}(0);\ldots; \frac{d\alpha^e}{dt^e}(0)) \in \mathbb{R}^{r(e+1)}$$

In this particular case,

$$\tilde{I} : \hat{\alpha} \in T^2\mathbb{R}^2 \to (\alpha(0); \frac{d\alpha}{dt}(0); \frac{d^2\alpha}{dt^2}(0)) \in \mathbb{R}^6$$

We now define, for fixed j:

$$C_1^j[2] = T^2(N_1^j) \subset T^2(\mathbb{R}^2)$$

$$C_2^j[2] = T^2\Gamma \quad \tilde{I}^{-1}(Q_2[2]),$$

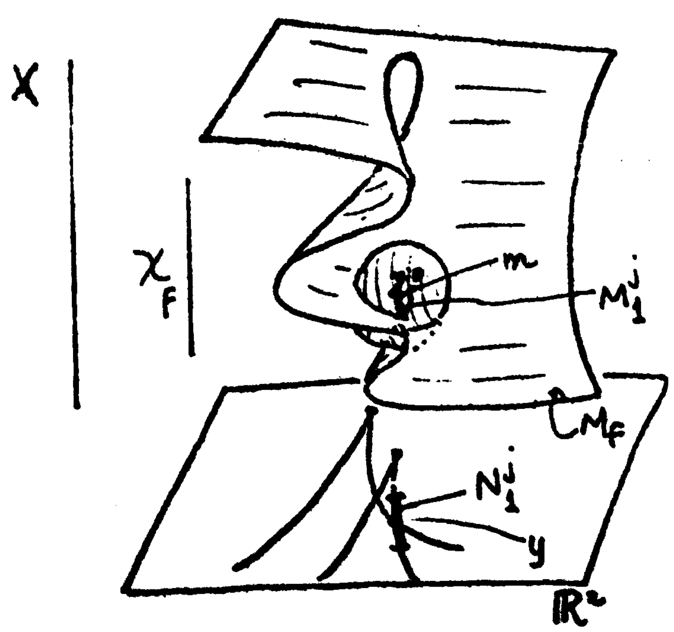
where Γ corresponds to $(j,2)$,

$$Q_2[2] = \{(x_1,\ldots,x_6)\in\mathbb{R}^6 \mid x_1=x_2=x_4=0\}$$

$$C_1[2] = \bigcup_{j\in\mathbb{N}} C_1^j[2]$$

$$C_2[2] = \bigcup_{j\in\mathbb{N}} C_2^j[2], \text{ and}$$

$$C[2] = C_1[2] \cup C_2[2]$$



The rest of 4.4.2,A, will be devoted to proving independence of choice in Definition 7.

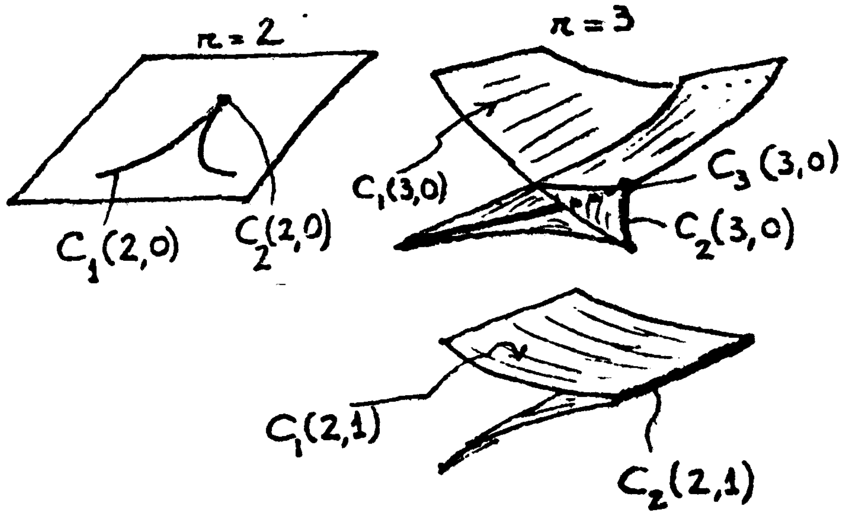
We will fix some notation, before we prove independence.

Let g_c be as in Definition 2 (4.2(1)), and let $g:\mathbb{R} \times \mathbb{R}^r \to \mathbb{R}$ be equal to g_c + (r-c) disconnected controls. Let M_i^d be defined as in 4.2(7).

DEFINITION 8:

$$C_i(c;r-c) = \chi_g(M_i^d)$$
$$\parallel$$
$$M^d_{i,g},$$

i.e. M_i^d corresponding to g.



REMARK 7:

Let now $f:\mathbb{R} \times \mathbb{R}^r \to \mathbb{R}$ be generic, $m \in M_1^d$ and $U_i^j \ni m$; and (γ_1,Γ_2), (γ_2,Γ_2) two pairs of local diffeomorphism making the diagram (see picture) in 4.4(5) commute. From Proposition 1, we know that $f\gamma_1 = \overbrace{g_{c_1} + (r-c_1)}^{\text{notation: } g_1}$ disconnected controls. Now, in 4.2(4)/(5) we have seen that $\gamma_1^{-1}(S_1(\chi_f)) = S_1(\chi_{g_1})$, $\gamma_1^{-1}(S_{1,1}(\chi_f)) = S_{1,1}(\chi_{g_1})$, etc.. . Hence, from the definition, as in 4.2(7), we get immediately $M^d_{i,g_1} = \gamma_1^{-1}(M^d_{i,f})$ (all these are germ equations, but we are not interested in making this explicit). By the commutativeness of the diagram (4.4.(5)), one therefore gets:

$$\Gamma_1(\chi_g(M^d_{i,g_1})) = \chi_f(M^d_{i,f}) = \Gamma_2(\chi_g(M^d_{i,g_2}))$$
$$\parallel \qquad\qquad\qquad\qquad \parallel$$
$$\Gamma_1(C_i(c_1;r-c_1)) \qquad \text{by analogous arguments} \qquad \Gamma_2(C_i(c_2;r-c_2)).$$

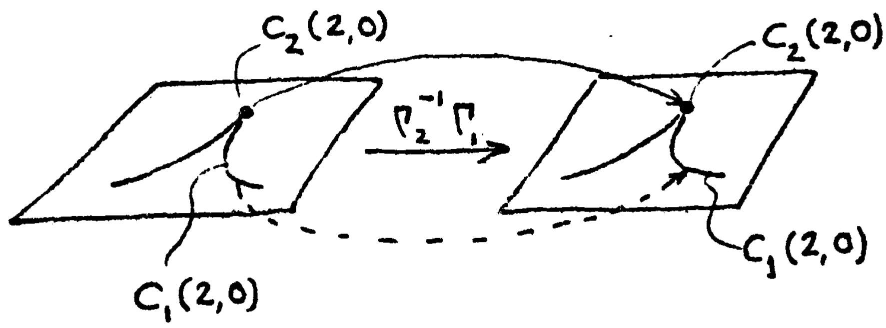
We have shown that

$$\Gamma_2^{-1}\Gamma_1 : C_i(c_1;r-c_1) \longrightarrow C_i(c_2;r-c_2) \qquad \forall\, i$$

(of course this is not defined on the whole of $C_i(c_1;r-c_1)$, since we are dealing with germs).

Note that, if in particular $c_1 = c_2$, we have proved that if Γ_1, Γ_2 are two choices of diffeomorphism, as above, then:

$\Gamma_2^{-1}\,\Gamma_1$ leaves, $\forall$ i, fixed, the i-strata $C_i(c,r-c)$ INVARIANT



REMARK 8:

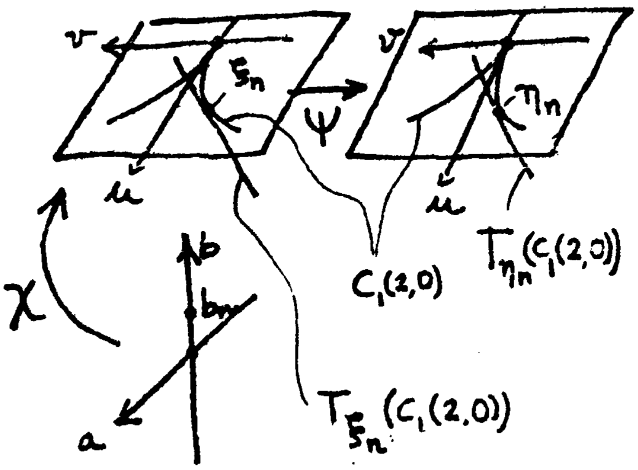
In Proposition 6,(4.3(6)), there is no loss of generality in taking U_i^j sufficiently small so that $U_i^j \cap \underset{\substack{\| \\ M^d_{e,f}}}{M_e^d} = \emptyset$, e > i (this is because M_e^d is closed in M^d_{e-1}, $\forall$ e).

This means that U_i^j contains points in $M^d_{i,f}$ (since $\mathcal{C}_i$ is a cover of $M^d_{i,f}$) but not in M_e^d, e > i. Therefore, if Γ,γ are diffeomorphic as above, this means that (from $\circledast$,4.4(7)) $\gamma^{-1}(U_i^j)$ contains points in $M^d_{i,g}$, but not in $M^d_{e,g}$, e > i, so that one must have c=i, with $g = g_c + (r-c)$ disc. cont. as in Proposition 1 of 4.2(1).

So: if Γ,γ are as above (corresponding to U_i^j), then $g = \gamma f = g_i + (r-i)$ d.c.

We now prove a proposition from which independence of choice in Definition 7 will follow easily. We will make common practice to identify: $T_y(\mathbb{R}^r) \simeq \mathbb{R}^r$.

PROPOSITION 14:



Let ψ: $\mathbb{R}^2 \circlearrowleft$ be a (germ of a) diffeomorphism, leaving the sets $C_i(2,0)$, $i = 1,2$ invariant. Then $T^2\psi$ leaves $(\tilde{I})^{-1}(Q_2[2])$ invariant.

Proof

It suffices to show that $T_o\psi(1,0) = (\tau_u, \tau_v)$ with $\tau_v = 0$. This is because $\hat{\alpha} \in \tilde{I}^{-1}(Q_2[2])$, $\alpha(0) = \xi$, means $\xi_u = \xi_v = \frac{d\alpha_v}{dt}(0) = 0$ and, since ψ preserves $C_2(2,0)$, $\psi(0) = 0$, therefore $(\psi(\xi))_u = (\psi(\xi))_v = 0$, and therefore all that is left to prove is that $\left(\frac{d(\psi\alpha)}{dt}u(0); \frac{d(\psi\alpha)}{dt}v(0)\right) = T_o\psi\left(\frac{d\alpha}{dt}u(0); \underset{\overset{\shortparallel}{0}}{\frac{d\alpha}{dt}v(0)}\right)$ satisfies $\frac{d(\psi\alpha)}{dt}v(0) = 0$. (Recall that $T^2\psi(\hat{\alpha}) = \widehat{\psi\alpha}$, therefore

$\tilde{I}(T^2\psi(\hat{\alpha})) = ((\psi(\xi))_u; (\psi(\xi))_v; \frac{d}{dt}(\psi\alpha)_u(0); \frac{d}{dt}(\psi\alpha)_v(0); \frac{d^2}{dt^2}(\psi\alpha)_u(0); \frac{d^2}{dt^2}(\psi\alpha)_v(0))$;

hence, to show that $T^2\psi(\hat{\alpha}) \in (\tilde{I})^{-1}(Q_2[2])$ or, equivalently, $\tilde{I}(T^2\psi(\hat{\alpha})) \in Q_2[2]$, one has to prove that $(\psi(\xi))_u = (\psi(\xi))_v = \frac{d}{dt}(\psi\alpha)_v(0) = 0$ — see the definition of $Q_2[2]$).

Suppose $T_o\psi(1,0) = (\tau_u; \tau_v)$, with $\tau_v \neq 0$. By continuity of $\xi \to T_\xi\psi$, $\exists\, \delta > 0$, $\varepsilon > 0$ s.t.: $T_\xi\psi(1,\zeta_v^*) = (\tau_u^*, \tau_v^*)$ satisfies $|\tau_v^*| > |\tau_{v/2}| > 0$, and $|\tau_u^*| < |(2\tau_u)|$ (or else $< \eta$, $\eta > 0$, if $\tau_u = 0$), so that $\left|\frac{\tau_u^*}{\tau_v^*}\right| <$ N, for some N $\in \mathbb{R}$, fixed, ξ, ζ_v^* s.t. $|\xi| < \delta$, $|\zeta_v^*| < \varepsilon$.

Let χ be constructed as in [17], $\chi:\mathbb{R} \to \mathbb{R}^2$, and

$$b \to (-3b^2;2b^3)$$

let $\{\xi_n\}$ be a sequence in $\mathbb{R}^2$, $\xi_n \in C_1(2,0), \forall n$, $\xi_n \to (0,0)$. Choose b_n ($\neq 0$, since $\xi_n \in C_1(2,0)$) s.t. $\chi(b_n) = \xi_n$. By computation, one gets $T_{\xi_n}(C_1(2,0)) = \{(\alpha;-\alpha b_n) \mid \alpha \in \mathbb{R}\}$. Notice that $b_n \to 0$ as $n \to \infty$ (from the definition of b_n, χ and the fact that $\xi_n \to (0,0)$). In particular, notice that, if $\underset{(\neq 0)}{(\zeta_u^n,\zeta_v^n)} \in T_{\xi_n}(C_1(2,0))$, for each fixed n, then $\boxed{\left|\frac{\zeta_u^n}{\zeta_v^n}\right| = \frac{1}{|b_n|} \to \infty}$ ⊛ as $n \to \infty$.

Let $\eta_n = \psi(\xi_n)$. $\{\eta_n\} \to (0,0)$ as $n \to \infty$, because ψ leaves $C_2(2,0)$ invariant. Hence, by the same arguments which led to ⊛, if $\underbrace{(\tau_u^n,\tau_v^n)}_{\neq 0} \in T_{\eta_n}(C_1(2,0))$, for each n fixed, then $\boxed{\left|\frac{\tau_u^n}{\tau_v^n}\right| \to \infty \quad \text{as } n \to \infty}$ ⊛⊛.

Finally, choose n sufficiently big so that:

$\left|\frac{\tau_u^n}{\tau_v^n}\right| > N$, $|\xi_n| < \delta$ and $|b_n| < \varepsilon$. ($(\tau_u^n,\tau_v^n) \in T_{\eta_n}(C_1(2,0))$.

Taking $\alpha = 1$, $(1;-b_n) \in T_{\xi_n}(C_1(2,0))$, $|\xi_n| < \delta$, hence, since $|-b_n| < \varepsilon$,

$T_{\xi_n}\psi\underbrace{(1,-b_n)}_{\neq 0} = \underbrace{(\tau_u^{*n};\ \tau_v^{*n})}_{\neq 0}$ satisfies $\left|\frac{\tau_u^{*n}}{\tau_v^{*n}}\right| < N$. But one also has

$T_{\xi_n}\psi(T_{\xi_n}(C_1(2,0)) = T_{\eta_n=\psi(\xi_n)}(C_1(2,0))$, because ψ leaves $C_1(2,0)$ invariant, and therefore $(\tau_u^{*n}, \tau_v^{*n}) \in T_{\eta_n}(C_1(2,0))$, therefore by our choice of n $|\tau_u^{*n}/\tau_v^{*n}| > N$, a contradiction. Therefore $\tau_v = 0$. □

Note: Proof above is just saying that the reason why $T_0\psi$ has to send the 'u-axis' into itself is that $T_{\xi_n}\psi$ sends $T_{\xi_n}(C_1(2,0))$ to $T_{\eta_n}(C_1(2,0))$, since ψ leaves $C_1(2,0)$ invariant, and, as it happens, $\{T_{\xi_n}(C_1(2,0))\}$ and $\{T_{\eta_n}(C_1(2,0))\}$ 'converge' to the 'u-axis' as $n \to \infty$.

PROPOSITION 15:

The definition of $C_2^j[2]$ above does not depend on the choice of Γ,γ.

Proof

By Remark 8, and if Γ_1,γ_1, Γ_2,γ_2 are two choices, $g(1) = \gamma_1.f$, $g(2) = \gamma_2.f$, then $g(1) = g(2) = g_2$ with 2-2) = 0 disc. controls. By Remark 7, $\psi = \Gamma_2^{-1}\Gamma_1$ leaves $C_i(2,0)$ invariant, $i = 1,2$.

Let $(C_2^j[2])_1 = T^2\Gamma_1.\tilde{I}^{-1}(Q_2[2])$, $(C_2^j[2])_2 = T^2\Gamma_2.\tilde{I}^{-1}(Q_2[2])$. Now, $T^2\Gamma_1.\tilde{I}^{-1}(Q_2[2]) = T^2\Gamma_2(T^2(\Gamma_2^{-1}\Gamma_1)(\tilde{I}^{-1}(Q_2[2]))) \underset{\text{Prop.14}}{=\!=\!=} T^2\Gamma_2(\tilde{I}^{-1}(Q_2[2]))$, as wished.

B. Closedness of C[2]

The aim of the definitions which now follow is to provide the framework for reducing the proof that C[2] (global) is closed to a number of local cases.

These are later reduced again to canonical forms.

DEFINITION 9:

We define below the total second bundle associated with (i,j), $TC_i^j[2]$

$$TC_1^j[2] = C_1^j[2]$$

$$TC_2^j[2] = C_2^j[2] \cup \Big(\bigcup_{m \in \mathcal{U}_2^j \cap M_1^d} C_{2,1}^j(m)[2] \Big), \text{ where:}$$

$$C_{2,1}^j(m)[2] = \{\hat{\beta} \in C_1^{j_o}[2] \mid \beta(0) = y = \chi_f(m)\}, \; j_o \text{ chosen so that } m \in \mathcal{U}_1^{j_o}$$

PROPOSITION 16:

Definition of $C_{2,1}^j(m)[2]$ (and hence that of $TC_2^j[2]$) is independent of choice of j_o.

Proof:

Let j_o, j_1 s.t. $m \in \mathcal{U}_1^{j_o}$, $m \in \mathcal{U}_1^{j_1}$. Recall that $\chi_f/\underbrace{\mathcal{U}_1^{j_o} \cap M_1^d}_{\substack{\parallel \\ M_1^{j_o}}} : M_1^{j_o} \to N_1^{j_o}$ diffeomorphically. Let B be a ball contained in $\mathcal{U}_1^{j_o} \cap \mathcal{U}_1^{j_1}$.

$P = \chi_f/M_1^{j_o} \underbrace{(B \cap M_1^d)}_{M_1^{j_o}}$ is open in $N_1^{j_o}$.

We claim that $\{\hat{\beta} \text{ with } \beta(0) = y \mid \hat{\beta} \in C_1^{j_o}[2]\} = \overbrace{\{\hat{\beta} \text{ with } \beta(0) = y \mid \hat{\beta} \in T^2P\}}^{*}$.

This is true since $\begin{bmatrix} \exists \text{ represent. } \beta \text{ of } \hat{\beta} \\ \text{s.t. } \beta(I) \subset N_1^{j_o},\ \beta(0) = y \end{bmatrix} \Longleftrightarrow \begin{bmatrix} \exists \beta, \text{represent. of } \hat{\beta}, \text{ s.t.} \\ \beta(I) \subset P,\ \beta(0) = y \end{bmatrix}$

P is open in $N_1^{j_o}$.

Similarly, $\{\hat{\beta} \in C_1^{j_1}[2] \mid \beta(0) = y\} = *$ proving the proposition. □

PROPOSITION 17: $\boxed{\text{(Reducing GLOBAL TO LOCAL)}}$

Suppose $\hat{\beta}_n \in C[2]$, $y_n = \beta_n(0)$, $\forall\, n \in \mathbb{N}$ and $\{\hat{\beta}_n\} \to \hat{\beta} \in T^2(\mathbb{R}^2)$, $y = \beta(0)$.

Then, $\exists\; i \in \{1,2\}$, $j \in \mathbb{N}$ and a subsequence $\{\hat{\beta}_{n(k)}\}$, with $y_{n(k)} = \beta_{n(k)}(0)$, which we will denote by $\{\hat{\beta}_k\}, (y_k = \beta_k(0))$, for simplicity's sake, s.t.:

$$\hat{\beta}_k \in TC_i^j[2],\ \forall\, k \in \mathbb{N} \text{ and } y \in \chi(\underbrace{\mathcal{U}_i^j \cap M_i^d}_{M_i^j}).$$

Proof

Since $(\hat{\beta}_n) \in C[2]$, choose (i_n, j_n) s.t. $\hat{\beta}_n \in C^{j_n}_{i_n}$ [2].

Recall that $\chi_f / M^{j_n}_{i_n} = \mathcal{U}^{j_n}_{i_n} \cap M^d_{i_n} : M^{j_n}_{i_n} \xrightarrow{\text{diffeom.}} N^{j_n}_{i_n}$; it is easy to see, from the definition of $C^{j_n}_{i_n}[2]$ that $\hat{\beta}_n \in C^{j_n}_{i_n}[2] \Rightarrow y_n \in N^{j_n}_{i_n}$.

Set $m_n = (\chi_f / M^{j_n}_{i_n})^{-1}(y_n)$. (in particular, $m_n \overset{\circledast}{\in} \mathcal{U}^{j_n}_{i_n} \cap M^d_{i_n}$).

Now, $(\underset{\underset{C_f}{\Uparrow}}{y_n} \to y) \Rightarrow y \in C_f$. Let $\chi_f^{-1}(y) = \{m_1, \ldots, m_p\}$. $\mathcal{C}$ covers M^d.

Choose (i_s, j_s), $s = 1, \ldots, P$, s.t. $m_s \in \mathcal{U}^{j_s}_{i_s}$, where $i_s = 1$ or 2 according to whether $m_s \in M^d_1$ or M^d_2.

The following lemma will immediately imply Proposition 17:

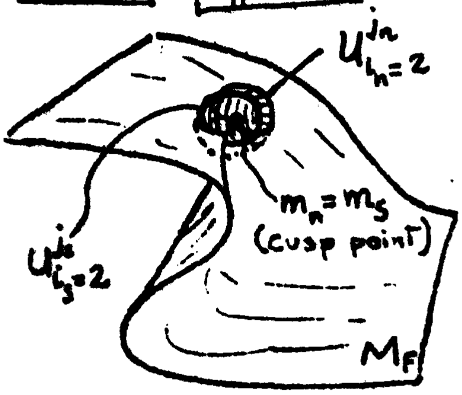
LEMMA:

Everything as above (hence $\hat{\beta}_n \in C^{j_n}_{i_n}$ [2]), one has:

$$m_n \in \mathcal{U}^{j_s}_{i_s} \Rightarrow \hat{\beta}_n \in TC^{j_s}_{i_s}[2]$$

PROOF OF LEMMA:

Case 1: $\boxed{i_n = 2.}$

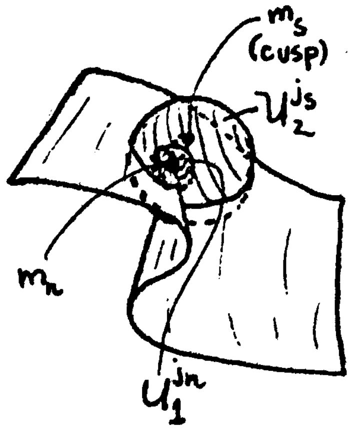


From $\circledast$ above and Remark 8, one gets $i_s = 2$. Since $m_s \in \mathcal{U}^{j_s}_{i_s=2} \Rightarrow m_s \in M^d_{i_s=2}$, $m_n \overset{\circledast}{\in} M^d_{i_n=2}$, $m_s = m_n$.

Therefore one can show, in precisely the same way as we did in Proposition 15, that $C^{j_n}_2[2] = C^{j_s}_2[2]$

Hence, $\hat{\beta}_n \in C^{j_s}_2[2] \subset TC^{j_s}_{i_s=2}[2]$.

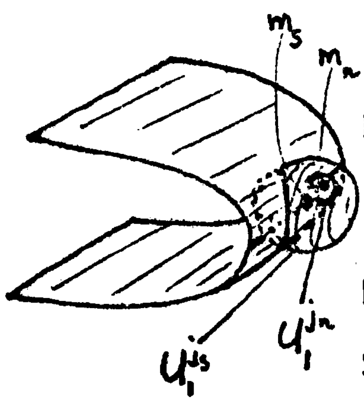
Case 2: $\boxed{i_n = 1}$: $\hat{\beta}_n \in C_1^{j_n}[2] = T^2(N_1^{j_n})$



Case 2.1: $i_s = 2$

$\hat{\beta}_n \in \{\hat{\beta} \in C_1^{j_n}[2],\ \beta(0) = y_n = \beta_n(0)\} =$

$= C^{j_s}_{2,1\,\hat{\mathcal{U}}_2^{j_s}}(m_n)[2] \subset TC_2^{j_s}[2]$, where this last equality comes from Definition 9 (we are also using Proposition 16), where j_0 has been taken as j_n.



Case 2.2 $i_s = 1$

$M_1^{j_n} \cap \mathcal{U}_1^{j_s}$ is open in $M_1^{j_n}$ (with the induced topology) therefore $\chi_f/M_1^{j_n}(M_1^{j_n} \cap \mathcal{U}_1^{j_s})$ is open in $N_1^{j_n}$ (induced topology)

Let B be then an open set of $\mathbb{R}^2$ s.t. $B \cap N_1^{j_n} = \chi_f/M_1^{j_n}(M_1^{j_n} \cap \mathcal{U}_1^{j_s})$. Since $\hat{\beta}_n \in T^2(N_1^{j_n})$, $\exists\, \beta_n \in \hat{\beta}_n$ s.t. $\beta_n(I) \subset N_1^{j_n}$, $\beta_n(0) = y_n \in B \cap N_1^{j_n}$; one gets $\beta_n(I) \subset B \cap N_1^{j_n}$, perhaps by reducing the original I, if necessary. Now, $\chi_f/M_1^{j_n}(M_1^{j_n} \cap \mathcal{U}_1^{j_s}) \subset \chi_f(M_1^d \cap \mathcal{U}_1^{j_s}) = \chi_f(M_1^{j_s}) = N_1^{j_s}$. Therefore $\beta_n(I) \subset N_1^{j_s}$, therefore $\hat{\beta}_n \in T^2 N_1^{j_s} = C_1^{j_s}[2] \subset TC_1^{j_s}[2]$. □

LEMMA $\Longrightarrow$ PROPOSITION 17:

Initially, we claim: $\exists\, N \in \mathbb{N}$ s.t. $m_n \in \bigcup_{s=1}^{p} \mathcal{U}_{i_s}^{j_s}$, $\forall\, n \geq N$. Otherwise, we would get a subsequence $\{m_r\}$ of $\{m_n\}$, contained in a compact, say $K \times \widehat{\mathbb{R}^2}$ (where K is some compact ball around y), hence converging to $m \notin \bigcup_{s=1}^{p} \mathcal{U}_{i_s}^{j_s}$.

So $m \notin \{m_1,\ldots,m_p\}$ and $y = \chi_f(m)$, a contradiction.

Therefore, $\exists s \in \{1,\ldots,p\}$ and subsequence $\{m_k\}_{k\in\mathbb{N}}$ s.t. $m_k \in \mathcal{U}_{i_s}^{j_s}$, $\forall k$. From lemma, $\hat{\beta}_k \in TC_{i_s}^{j_s}[2]$, $\forall\, k$. This settles the first part of Proposition 17: just take $j = j_s$, $i = i_s$.

As to the second part $\{m_k\} \to m\epsilon\{m_1,\ldots,m_p\}$ (same reasons as above) and, since there is no loss of generality in supposing $U_{i_s}^{j_s}$ two by two disjoint, $m \notin U_{i_1}^{j_1} \cup \ldots \cup U_{i_{s-1}}^{j_{s-1}} \cup U_{i_{s+1}}^{j_{s+1}} \cup \ldots \cup U_{i_p}^{j_p}$, therefore $m \notin \{m_1,\ldots,m_{s-1},m_{s+1},\ldots,m_p\}$,

therefore $m = m_s \in U_{i_s}^{j_s}$, therefore, by choice of (i_s,j_s), $m_s \in M_{i_s}^d$. □
(since $\chi_p(m)=y$)

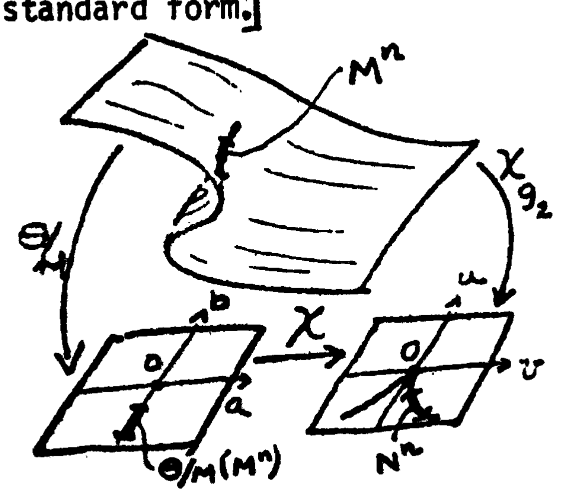
PROPOSITION 18: $\boxed{\text{('CUSP'S BUNDLE' CLOSES 'FOLD'S BUNDLE': STANDARD FORM)}}$

Let g_2 (see 4.2(1): $g_2\colon \underset{(u,v;x)}{\mathbb{R}^2 \times \mathbb{R}} \to \mathbb{R}$) denote the standard cusp (no disconnected controls) and let $\{\hat{\alpha}_n\}_{n\in N}$ be a sequence in $T^2(\mathbb{R}^2)$ converging to a point $\hat{\alpha}$, $\xi = \alpha(0) = 0$. Suppose that, for each n, fixed, $\exists\, M^n \underset{\text{subman.}}{\subset} M_1^d$ s.t.: (i) $\chi_{g_2/M^n}\colon M^n \to \chi_{g_2/M^n}(M^n)=N^m$ is a diffeomorphism; (ii) $\xi_n \in N^n \underset{C_1(2,0)}{\cap}$ and (iii) represent. α_n s.t. $\alpha_n(I) \subset N^n$. Then $\frac{d\alpha}{dt}v(0) = 0$.

[Closeness at Cusp's Point]

Proof

[Note: This proposition solves the non-trivial part of the proof that C[2] is closed; in Proposition 19 we show how to reduce the local cases to standard form.]



Construct $\theta/M : M = M_{g_2} \to \mathbb{R}^2$, $\chi = \chi_{g_2}\cdot(\theta/M)^{-1}$ as in [17] (pages 19/20); one has:

$\chi = (\chi^u;\chi^v)$, with: $\begin{cases}\chi^u(a,b) = 2a-3b^2\\ \chi^v(a,b) = -2ab+2b^3\end{cases}$ (see also 4.2(4))

Since θ/M is a diffeomorphism (see [17], so is θ/M^n (M^n is a submanifold of M). Now, χ_{g_2}/M^n is a diffeomorphism, by hypothesis. Therefore, one has that

$\chi_n \underset{\text{of } \chi_n}{\overset{\text{definition}}{=\!=}} \chi/\theta/M^n(M^n) \underset{\{(a,b)\mid a=0\}}{\cap} : \theta/M^n(M^n) \to N^n$ is a diffeomorphism.

Define:

$(a_n(t);\ b_n(t)) = \chi_n^{-1}(\alpha_n(t))$. Recall that $\alpha_n(I) \subset N^n$, therefore

$\theta/M(M^n) = \chi_n^{-1}(\alpha_n(t))$

therefore $a_n(t) \equiv 0$, therefore $\alpha_n(t) = \chi_n(\underset{\overset{\shortparallel}{0}}{a_n}(t);b_n(t)) = (\underset{(\alpha_n(t))_u}{\underset{\shortparallel}{-3b_n^2(t)}};\underset{(\alpha_n(t))_v}{\underset{\shortparallel}{2b_n^3(t)}})$

Therefore,

$$\tilde{I}(\hat{\alpha}_n)=(-3b_n^2(0);2b_n^3(0);-6(b_n(0)b_n'(0));\underbrace{-6(b_n^2(0)b_n'(0))}_{(\frac{d\alpha_n}{dt})v(0)};-6(b_n(0)b_n''(0)+(b_n'(0))^2;6(2b_n(0).(b_n'(0))^2+b_n^2(0)b_n''(0))$$

We want then to show (dropping the 0's):

$$\text{(I)}\quad \boxed{\begin{array}{c} -3b_n^2 \to 0 \\ 2b_n^3 \to 0 \end{array}} \Longrightarrow \boxed{-6(b_n^2 b_n') \to 0} \quad \text{(II)}$$

This is easy, since (I) $\Rightarrow b_n \to 0 \Rightarrow \overset{0}{\cancel{b_n}} \,.\, (-6\overset{\frac{d\alpha}{dt}u(0)}{\cancel{b_n b_n'}}) \to 0$, as wanted. □

PROPOSITION 19:

C[2] is closed in $T^2(\mathbb{R}^2)$.

Proof

Let $\{\hat{\beta}_n\}_{n\in\mathbb{N}}$, $y_n = \beta_n(0)$, be a sequence, with $\hat{\beta}_n \in C[2]$, $\forall$ n, converging to some $\hat{\beta} \in T^2(\mathbb{R}^2)$, $y = \beta(0)$. We will show that $\hat{\beta} \in C[2]$.

From Proposition 17 and its lemma, $\exists$ subsequence $\{\hat{\beta}_k\}$ $k \in \mathbb{N}$, $y_k = \beta_k(0)$ with $\hat{\beta}_k \in TC_{i_s}^{j_s}[2]$, $\forall$ $k \in \mathbb{N}$.

Case 1:

$\boxed{i_s = 1}$

In this case, $TC_{i_s}^{j_s}[2] = TC_1^{j_s}[2] = C_1^{j_s}[2] = T^2(N_1^{j_s})$. Let Γ, γ as usual. $\hat{\beta}_k \in T^2(N_1^{j_s}) \Longrightarrow \exists$ represent. β_k of $\hat{\beta}_k$ with $\beta_k(I) \subset N_1^{j_s}$, hence $\Gamma^{-1}\,\beta_k(I) \subset C_1(1,1)$ (see Definition 8 and Remark 7). Therefore $\tilde{I}(\widehat{\Gamma^{-1}\beta_k}) \in \{(x_1,\ldots,x_6)|x_1=x_3=x_5=0\}$

therefore (since $\tilde{I}$ and $T^2\,\Gamma^{-1}$ are continuous) $\tilde{I}(\widehat{\Gamma^{-1}\beta}) = \lim_{K\to\infty} \tilde{I}(\widehat{\Gamma^{-1}\beta_k}) \in$ $\{(x_1,\ldots,x_6)|x_1 = x_3 = x_5 = 0\}$, therefore $\exists$ represent. $(\Gamma^{-1}\beta)$ s.t. $\Gamma^{-1}(\beta(I)) \subset C(1,1)$

Hence (recall - see Proposition 17 - that

$y \in \chi_f(U_{i_{s=1}}^{j_s} \cap M_{i_{s=1}}^{d}) = \chi_f(M_1^{j_s}) = N_1^{j_s}$) we have $\beta(I) \subset N_1^{j_s}$, so that $\hat{\beta} \in T^2 N_1^{j_s} = C_1^{j_s}[2] \subset C[2]$.

Case 2:

$\boxed{i_s = 2}$

Case 2.1: $\exists$ subsequence, $\{\hat{\beta}_r\}_{r\in\mathbb{N}}$, with $y_r = \beta_r(0)$, of $\{\hat{\beta}_k\}_{k\in\mathbb{N}}$ s.t. $\hat{\beta}_r \in C_2^{j_s}[2]$, $\forall r \in \mathbb{N}$. If Γ,γ are as usual, $\hat{\alpha}_r = \widehat{\Gamma^{-1}\beta_r}$, then, by definition of $C_2^{j_s}[2]$, $\tilde{I}(\hat{\alpha}_r) \in Q_2[2] = \{(x_1,\ldots,x_6)|x_1 = x_2 = x_4 = 0\}$ therefore $\tilde{I}(\widehat{\Gamma^{-1}\beta}) = \lim_{r\to\infty} \tilde{I}(\hat{\alpha}_r) \in Q_2[2]$, therefore $\hat{\beta} \in C_2^{j_s}[2] \subset C[2]$.

Case 2.2: $\exists K \in \mathbb{N}$ s.t. $\hat{\beta}_k \in \bigcup_{m\in U_2^{j_s} M_1^d} C_{2,1}^{j_s}(m)[2], \forall k \geq K$, $y_k = \beta_k(0)$

Let $k \geq K$ fixed. Then $\hat{\beta}_k \in C_{2,1}^{j_s}(m_k)[2]$, for some $m_k \in U_2^{j_s} \cap M_1^d$, where $C_{2,1}^{j_s}(m_k)[2] = \{\hat{\beta} \in C_1^{j_o}[2]|\beta(0) = y_k = \chi_f(m_k)\}$ with j_o s.t. $m_k \in U_1^{j_o}$. Therefore, $\exists$ represent. β_k of $\hat{\beta}_k$ s.t. $\beta_k(I) \subset N_1^{j_o}$. We recall that $\chi_f/M_1^{j_o}: M_1^{j_o} \to N_1^{j_o}$ is a diffeomorphism; hence $\chi_{g=\gamma f/\gamma^{-1}(M_1^{j_o})} : \gamma^{-1}(M_1^{j_o}) \to \Gamma^{-1}(N_1^{j_o}) (\subset C_1(2,0))$, diffeomorphically ((i)'). Also $\Gamma^{-1}(\beta_k(0)) = \Gamma^{-1}(y_k) \in \Gamma^{-1}(N_1^{j_o})$ ((ii)') and, from above, $\Gamma^{-1}\beta_k(I) \subset \Gamma^{-1}(N_1^{j_o})$ (iii)').

By considering the sequence $\{\widehat{\Gamma^{-1}\beta_k}\}_{k\in\mathbb{N}, k\geq K} =\!=\!= \{\hat{\alpha}_k\}_{k\geq K}$ which converges to $\Gamma^{-1}\beta$, by continuity of $T^2\Gamma^{-1}$, and setting $M^k = \gamma^{-1}(M_1^{j_o})$, $N^k = \Gamma^{-1}(N_1^{j_o})$, one sees that:

(i)', (ii)' and (ii)' above $\Rightarrow$(i), (ii) and (iii) as in Proposition 18. Also, from Proposition 17, $y \in \chi_f(M_2^{j_s}) = N_2^{j_s}$, therefore $\Gamma^{-1}(y) = 0$, i.e., all conditions required in the hypothesis of Proposition 18 are met. Hence, $\frac{d(\Gamma^{-1}\beta)_v(0)}{dt} = 0$, therefore $\tilde{I}\,(\widehat{\Gamma^{-1}\beta}) \in \{(x_1,\ldots,x_6) | x_1 = x_2 = x_4 = 0\}$, so that $(\hat{\beta}) \in C_2^{j_s}[2] \subset C[2]$.

Since Cases: 2.1 and 2.2 cover all possibilities, case 2 is proved, so that Proposition 19 is proved. □

C. Genericity of $v \tilde{\pitchfork} C_f$

PROPOSITION 20:

$\exists$ open and dense set $\mathcal{B} \subset \mathcal{V}(\mathbb{R}^r)$ s.t. $v \in \mathcal{B} \Rightarrow v[2](\mathbb{R}^2) \cap C[2] = \emptyset$

Proof

The proof is again very similar to that of Proposition 9 in 4.3(11). One defines $B_i^j = C_i^j[2] \cap A$, $B_i^{j,c} = C_i^j[2] \cap A^c$, $V_i^j = S^{-1}(B_i^j)$, $V_i^{j,c} = S^{-1}(B_i^{j,c})$, $j \in \mathbb{N}$, $i = 1,2$ and A, S like def. in 3.2(3), 3.2(1).

We remark that, directly from their definitions, the $C_i^j[2]$'s are submanifolds of $T^2(\mathbb{R}^2)$, $T^2\Gamma \cdot \tilde{I}^{-1}$ being a chart which flattens then into a linear subspace of $\mathbb{R}^6$. They have all codimension > 2.

Now since $B_i^{j,c} \cap A = \emptyset$, $V_i^{j,c}$ is a (cod. > 2) submanifold of $J'(\mathbb{R}^2,\mathbb{R}^2)$ (8-dimensional, in this case).

On the other hand analogously to Proposition 8, Chapter 3, we have $V_i^j = N_i^j \times \{0\} \times \mathbb{R}^4$. Hence, since the codimension of N_i^j in $\mathbb{R}^2$ is > 0, we have codimension $(V_i^j) > 2$.

Setting $V = \bigcup_{i,j} (V_i^j \cup V_i^{j,c})$ (denumerable), and $\mathcal{B} = \{v | j'v \,\bar{\pitchfork}\, (V_i^j$ and $V_i^{j,c}) \forall i,j$

we get, in complete analogy with the referred above Proposition 9 in 4.3(11), the required open and dense set. The proof that $v[2](\mathbb{R}^2) \cap C[2] = \emptyset$ follows in precisely the same way as the proof that $v[e](\mathbb{R}^r) \cap \mathfrak{N}[e] = \emptyset$ follows, in that proposition, from the definition of $\mathcal{B}$. □

PROPOSITION 21:

(reducing GLOBAL to LOCAL)

Let $y \in C_f$, $m_s, (i_s, j_s)$, $U_{i_s}^{j_s}$, $s = 1, \ldots, p$ as in 4.4(13). $\exists\, V$, open neighbourhood of y in $\mathbb{R}^2$ s.t.:

$$V \cap C_f = V \cap \underbrace{\Big[\bigcup_{s=1}^{p} (\chi_f(U_{i_s}^{j_s} \cap M^d))\Big]}_{\bullet}$$

Proof

lhs $\supset$ rhs: let $\xi \in$ rhs; $\xi \in V$ and also $\xi \in \chi_f(U_{i_s}^{j_s} \cap M_d)$ some $s \in \{1, \ldots, p\}$.

Therefore $\xi = \chi_f(m)$, $m \in M^d$, therefore $\xi \in C_f$.

Suppose now that:

lhs $\not\subset$ rhs, $\forall V$, open neighbourhood of y. Let $V_n = B_{1/n}(y)$, $C_n = \overline{B_{1/n}(y)}$. By absurd hypothesis, $\exists\, y_n \in (V_n \cap C_f)$ s.t. $y_n \notin V_n \cap [\bullet]$ Hence $y_n \notin [\bullet]$. Since $y_n \in C_f$, $y_n = \chi_f(m_n)$, $m_n \in M^d$. Now $m_n \notin \bigcup_{s=1}^{p} (U_{i_s}^{j_s} \cap M^d)$ (otherwise $y_n \in [\cdot]$), hence $m_n \notin \bigcup_{s=1}^{p} U_{i_s}^{j_s}$. $\odot$ Now

$\{m_n\} \subset C_1 \times X$, compact. Let $\{m_r\} \to m$ be a subsequence converging to m. Immediately $\chi_f(m) = y$, and also $m \notin \bigcup_{s=1}^{p} U_{i_s}^{j_s}$ (otherwise, since the $U_{i_s}^{j_s}$'s are open, $\odot$ above is contradicted). Hence $m \notin \{m_1, \ldots, m_p\}$, a contradiction, therefore lhs $\subset$ rhs. □

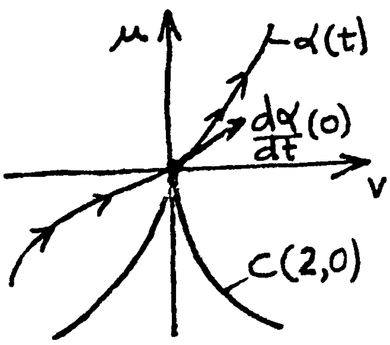
COROLLARY:

$$V \cap C_f \subset \bigcup_{s=1}^{p} \chi_f(U_{i_s}^{j_s} \cap M^d).$$

(V as above)

PROPOSITION 22:

(Genericity of $v \pitchfork$ cusp in STANDARD FORM: the 2-dimensional problem)



Let $\alpha(t) = (\alpha_u(t); \alpha_v(t))$ be a C^∞ curve through $0 \in \mathbb{R}^2$.

Suppose $\frac{d\alpha_v}{dt}(0) \neq 0$

Then, $\exists\ \varepsilon > 0$ s.t.:

$\left\{\alpha(t) \,\middle|\, \begin{smallmatrix} |t|<\varepsilon \\ t\neq 0 \end{smallmatrix}\right\} \cap C(2,0) = \emptyset$

Proof

We first remark that $C(2,0) = \{(-3b^2;2b^3) \mid b \in \mathbb{R}\}$. Suppose that this proposition is false: $\exists \{t_n\}_{n\in\mathbb{N}}$, $\underset{(t_n\neq 0)}{t_n} \to 0$ as $n \to \infty$ s.t. $\alpha(t_n) \in C(2,0)$. Choosing b_n conveniently, one has:

$\alpha(t_n) = (-3b_n^2;2b_n^3)$, and W.l.o.g. $b_n \neq 0$, $\forall n$ (since if there is a subsequence $\{t_r\}$ with $b_r = 0$, then $\alpha(t_r) = 0$, $\forall r$, $t_r \to 0$, therefore $d\alpha/dt(0) = 0$, false; and therefore we can just discard the (finite number of) n's for which $b_n = 0$).

Now $b_n \to 0$ as $n \to \infty$, since $\alpha(t_n) \to \alpha(0) = 0 \in \mathbb{R}^2$ as $n \to \infty$. Therefore

$$0 = \lim_{n\to\infty} b_n = \lim_{n\to\infty} -\frac{2b_n^3}{3b_n^2} = \lim_{\substack{n\to\infty \\ (t_n\neq 0,\ n)}} \frac{\dfrac{2b_n^3-0}{t_n-0}}{\dfrac{-3b_n^2-0}{t_n-0}} = \left.\frac{\dfrac{d\alpha_v(0)}{dt}}{\dfrac{d\alpha_u(0)}{dt}}\right\}\text{constant}$$

, therefore $\frac{d\alpha_v}{dt}(0) = 0$, a contradiction, therefore we are done. $\square$

PROPOSITION 23:

$v \in \mathcal{B}$ (as in Proposition 20) $\Longrightarrow v \pitchfork C_f$.

Proof

Let $y \in C_f$ and $v \in \mathcal{B}$ be fixed, and $V \ni y$ be as in Proposition 21.

$\exists\ \varepsilon^*$ s.t. $O_y(\varepsilon^*) \subset V$. Therefore, $O_y(\varepsilon^*) \cap C_f = O_y(\varepsilon^*) \cap (V \cap C_f) = O_y(\varepsilon^*) \cap (\bigcup_{s=1}^{p} \chi_f(U_{i_s}^{j_s} \cap M^d))$. If we prove that, for each choice of (i_s,j_s), $\exists\ \varepsilon_s$ s.t. $O_y(\varepsilon_s) \cap \chi_f(U_{i_s}^{j_s} \cap M^d) = \emptyset$, then, by choosing $\varepsilon = \min\{\varepsilon^*,\varepsilon_1,\dots,\varepsilon_p\}$, we will get $O_y(\varepsilon) \cap C_f = \emptyset$.

Case 1:

$\boxed{i_s = 1}$

In this case $\chi_f(\mathcal{U}_1^{j_s} \cap M^d) = \chi_f(M_1^{j_s}) = N_1^{j_s}$. Now, since $v[2](\mathbb{R}^2) \cap C_1^{j_s}[2] = \emptyset$, one has $v[2](\mathbb{R}^2) \cap (T^2(N_1^{j_s})) = \emptyset$, therefore by Remark 6 in 3.3(5), $v \pitchfork_y N_1^{j_s}$, as wanted.

Case 2:

$\boxed{i_s = 2}$

Let Γ, γ as usual. Since $\Gamma^{-1}(\chi_f(\mathcal{U}_2^{j_s} \cap M^d)) = \chi_{g=\gamma f}(\gamma(\mathcal{U}_2^{j_s} \cap M^d)) \subset C(2,0)$, one has that:

if $\varepsilon_s > 0$ is s.t. $\Gamma^{-1}(0_y(\varepsilon_s)) \cap C(2,0) = \emptyset \Longrightarrow 0_y(\varepsilon_s) \cap \chi_f(\mathcal{U}_2^{j_s} \cap M^d) = \emptyset$

Now, if $\beta: I \to \mathbb{R}^2$ is a solution curve of v through y, then $0_y(\varepsilon_s) := \{\beta(t) \mid |t| < \varepsilon_s,\ |t| \neq 0\}$. It suffices therefore to show that:

$\left[\exists \varepsilon_s > 0 \text{ s.t. } \{(\Gamma^{-1}\beta)(t) \mid |t| < \varepsilon_s,\ |t| \neq 0\} \cap C(2,0) = \emptyset\right]^{\circledast}$,

where $\alpha = \Gamma^{-1}\beta$, by definition. But, since $v[2](\mathbb{R}^2) \cap \underbrace{C_2^{j_s}[2]}_{T^2\Gamma.\tilde{I}^{-1}(Q_2[2])} = \emptyset \Longrightarrow$

$\Longrightarrow \tilde{I}(\hat{\alpha}) \notin Q_2[2] \Longrightarrow \frac{d\alpha}{dt}v(0) \neq 0$, we are done, because Proposition 22 $\Rightarrow \circledast$.

COROLLARY:

If $f: X \times \mathbb{R}^2 \to \mathbb{R}$ is generic, $\exists$ open and dense $\mathcal{B}$ s.t. $v \in \mathcal{B} \Rightarrow v \pitchfork C_f$.

4.4.3: The case r=3.

Let $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3$ (see 4.3(7)), N_i^j, M_i^j, $\mathcal{U}_i^j$ as before.

A. Definition of C[3]

DEFINITION 10: Define, for fixed j:

$C_1^j[3] = T^3(N_1^j) \subset T^3(\mathbb{R}^3)$.

$C_2^j[3] \overset{\oplus}{=} T^3\Gamma.\tilde{I}^{-1}(Q_2[3])$ (Γ,γ are diffeom. associated to (j,2)),

$$Q_2[3] = \{(x_1,\dots,x_{12}) \in \mathbb{R}^{12} | x_1 = x_2 = x_4 = x_5 = 0\}.$$

$C_3^j[3] = T^3\Gamma.\tilde{I}^{-1}(Q_3[3])$ (Γ,γ corresp. to (j,3)),

$$Q_3[3] = \{(x_1,\dots,x_{12}) \in \mathbb{R}^{12} | x_1 = x_2 = x_3 = x_6 = 0\}.$$

$C_i[3] = \bigcup_{j \in N} C_i^j[3]$ (i = 1,2,3) ; $C[3] = \bigcup_{i=1}^{3} C_i[3]$.

Note: Γ is a local diffeomorphism and therefore $T^3\Gamma$ is not defined on the whole of $\tilde{I}^{-1}(Q_2[3])$. Therefore the r.h.s. of $\oplus$ is meant to mean $\{T^3\Gamma(\cdot) \mid \cdot \in \tilde{I}^{-1}(Q_2[3])$ and $T^3\Gamma(\cdot)$ is defined$\}$. A similar remark also applies for the case r = 4.

PROPOSITION 24:

Let $\psi: \mathbb{R}^3 \circlearrowleft$ be a germ of a diffeomorphism, leaving $C_i(2,1)$ (i = 1,2) invariant. Then $T^3\psi$ leaves $\tilde{I}^{-1}(Q_2[3])$ invariant.

Proof

Let $\hat{\alpha} \in \tilde{I}^{-1}(Q_2[3])$, $\alpha(0) = \xi$, $\xi = (\overset{0}{\xi_u''};\overset{0}{\xi_v''};\xi_w)$, $\frac{d\alpha}{dt}u(0) = \frac{d\alpha}{dt}v(0) = 0$.

Now $\tilde{I}[T^3\psi(\hat{\alpha})] = (\psi_u(\xi); \psi_v(\xi); \psi_w(\xi); \frac{d(\psi\alpha)}{dt}u(0); \frac{d(\psi\alpha)}{dt}v(0); \frac{d(\psi\alpha)}{dt}w(0);\dots;\dots)$,

and $\psi_u(\xi) = \psi_v(\xi) = 0$, since ψ leaves $C_2(2,1)$ invariant. On the other hand,

$$(d(\psi\alpha)_u/dt(0);\ d(\psi\alpha)_v/dt(0);\ d(\psi\alpha)_w/dt(0)) = T_{(0,0,\xi_w)}\psi\ (\underset{\overset{\shortparallel}{0}}{\tfrac{d\alpha}{dt}u(0)};\ \underset{\overset{\shortparallel}{0}}{\tfrac{d\alpha}{dt}v(0)};\ \tfrac{d\alpha}{dt}w(0)).$$

The vector $\overset{\bullet}{\overbrace{(0,0,\alpha_w'(0))}}$ can be identified (as in the usual tangent bundle construction) with the equivalence class (under first tangency) of the curve $\gamma(t)$, $\gamma_u(t) \equiv 0$, $\gamma_v(t) \equiv 0$, $\gamma_w(t) = \xi_w + \alpha_w'(0)t$. Since ψ leaves $C_2(2,1)$ invariant, $(\psi\gamma)_u(t) \equiv (\psi\gamma)_v(t) \equiv 0$, therefore $T_{(0,0,\xi_w)}\psi(\bullet) = (0;0;*)$, therefore $T^3\psi(\hat{\alpha}) \in \tilde{I}^{-1}(Q_2[3])$, as wanted.

PROPOSITION 25:

Let $\psi:\mathbb{R}^3 \circlearrowleft$ a(germ of a)diffeomorphism, leaving $C_i(3,0)$, $i = 1,2,3$, invariant. Then $T^3\psi$ leaves $\tilde{I}^{-1}(Q_3[3])$ invariant.

Proof

We will show that $T_0\psi(1,0,0) = (\tau_u;\tau_v;\tau_w) \overset{\oplus}{\Longrightarrow} \tau_w = 0$, and that $T_0\psi(0,1,0) = (\tau_u^+;\tau_v^+;\tau_w^+) \overset{\oplus\!\oplus}{\Longrightarrow} \tau_w^+ = 0$. The rest of the proposition is trivial, since ψ preserves $C_3(3,0)$ (see also Proposition 14, 4.4(9)).

We initially prove $\oplus$. Suppose $\tau_w \neq 0$. In the same (analogous) way as in 4.4(10), one shows: $\delta > 0$, $\overset{①}{\varepsilon_v}, \varepsilon_w > 0$ s.t. $T_\xi\psi(1,\zeta_v^*,\zeta_w^*) \overset{②}{=} (\tau_u^*;\tau_v^*;\tau_w^*)$ satisfies $|\tau^*|/|\tau_w^*| < N$, N a fixed real, $\forall\, \xi$, ζ_v^* and ζ_w^* s.t. $|\xi|<\delta$, $|\zeta_v^*| \overset{③}{<} \varepsilon_v$ and $|\zeta_w^*| < \varepsilon_w$.

By computation, and using the χ (as in [17]) corresponding to g_3, $T_{\xi_n}(C_1(3,0)) = \{(\alpha;-2\alpha c_n + \beta;\alpha c_n^2 - \beta c_n) \mid \alpha,\beta \in \mathbb{R}\}$, where $\{\xi_n\}$ is a sequence in $\mathbb{R}^3$, $\xi_n \in C_1(3,0)$, $\forall\, n$, $\xi_n \to (0,0,0)$ as $n \to \infty$, b_n, c_n chosen so that $\chi(b_n,c_n) = \xi_n$ (hence $b_n,c_n \neq 0, \forall n$, since $\xi_n \in C_1(3,0)$). It is easy to prove that $b_n,c_n \to 0$ as $n \to \infty$. One also has $\overbrace{(\alpha c_n^2 - \beta c_n)}(\to 0 \text{ as } n \to \infty) \neq 0$ provided $(\alpha,\beta) \neq (0,0)$. Hence, if for each n, we choose $(\zeta_u^n,\zeta_v^n,\zeta_w^n) = \zeta^n \in T_{\xi n}$. $(C_1(3,0))$, then $\boxed{\dfrac{|\zeta^n|}{|\zeta_w^n|} \to \infty \text{ as } n \to \infty}$ ⊛ $(\neq (0,0,0))$

Setting $\eta_n = \psi(\xi_n)$, $\{\eta_n\} \to (0,0,0)$, since ψ leaves $C_3(3,0)$ invariant. Hence, by the same arguments which led to ⊛, one has: if $(\tau_u^m,\tau_v^n,\tau_w^n) \in T_{\eta_n}(C_1(3,0))$, for each n fixed, then $\dfrac{|\tau^n|}{|\tau_w^n|} \to \infty$ as $n \to \infty$.

Finally, choose n sufficiently big so that:

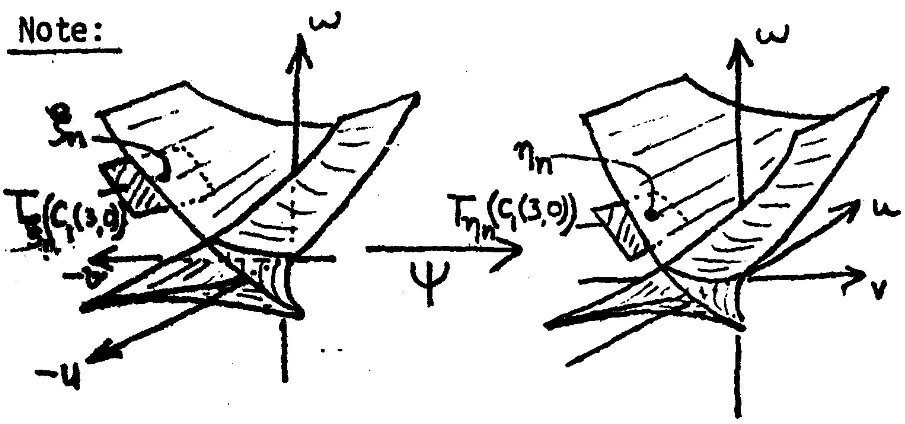
$\frac{|\tau^n|}{|\tau^n_w|} > N$ (where $(\tau^n_u,\tau^n_v,\tau^n_w) = \tau^n \in T_{\eta_n}(C_1(3,0))$; $|\xi_n| < \delta, |2c_n| < \varepsilon_v, |c_n^2| \overset{(4)}{<} \varepsilon_w$

Take $\overbrace{\alpha= 1,\ \beta= 0}^{(5)}$, $(1; -2c_n; c_n^2) \overset{(6)}{\in} T_{\xi_n}(C_1(3,0))$, with $|\xi_n|<\delta$, $|-2c_n| \overset{(7)}{<} \varepsilon_v$ and $|c_n^2| \overset{(8)}{<} \varepsilon_w$ therefore $\overset{*}{\tau}{}^n = \underbrace{(\overset{*}{\tau}{}^n_u;\ \overset{*}{\tau}{}^n_v;\ \overset{*}{\tau}{}^n_w)}_{\neq 0} \overset{(9)}{=} T_{\xi_n}\psi(1;-2c_n;c_n^2)$ satisfies $|\overset{*}{\tau}{}^n| / |\overset{*}{\tau}{}^n_w| < N$. But also $\overset{*}{\tau}{}^n \in T_{\eta_n}(C_1(3,0))$, therefore $|\overset{*}{\tau}{}^n| / |\overset{*}{\tau}{}^n_w| > N$, by choice of n, a contradiction.

As for $\oplus\!\!\oplus$, substitute (1),...,(9) by, respectively: $\varepsilon_u; T_\xi\psi(\overset{*}{\zeta}_u;1;\overset{*}{\zeta}_w) = (\overset{*}{\tau}_u;\overset{*}{\tau}_v;\overset{*}{\tau}_w)$; $|\overset{*}{\zeta}_u| < \varepsilon_u$; $|c_n| < \varepsilon_w$; $\alpha = 0$, $\beta = 1$; $(0,1,-c_n) \in T_{\xi_n}(C_1(3,0))$;

drop (7); $|c_n| < \varepsilon_w$ and $\overset{*}{\tau}{}^n = T_{\xi_n}(0;1;-c_n)$.

Note:



The proof above is saying that the reason why $T_o\psi$ sends the '(u × v) plane' into itself is that $T_{\xi_n}\psi$ sends $T_{\xi_n}(C_1(3,0)$ to $T_{\eta_n=\psi(\xi_n)}(C_1(3,0))$, since ψ leaves the cod. 1 strata $C_1(3,0)$ invariant and that $\{T_{\xi_n}(C_1(3,0))\}$, $\{T_{\eta_n}(C_1(3,0)\}$ converge to the '(u × v)-plane' as $n \to \infty$. □

PROPOSITION 26:

The definition of $C_2^j[3]$ above does not depend on the choice of Γ,γ.

Proof

Consider choices Γ_1,γ_1, Γ_2,γ_2. Set $\psi = \Gamma_2^{-1}\Gamma_1$ and apply Remark 7 and Proposition 24 above; arguments are as in Proposition 15. □

PROPOSITION 27:

Definition of $C_3^j[3]$ is independent of the choice of Γ,γ.

Proof

Remark 7 and Proposition 25 and arguments as in Proposition 15. □

B. Closedness of C[3]

DEFINITION 11:

We define below the total third bundle associated with (i,j), $TC_i^j[3]$.

$TC_1^j[3] = C_1^j[3]$

$TC_2^j[3] = C_2^j[3] \cup \left(\bigcup_{m\in \mathcal{U}_2^j\cap M_1^d} C_{2,1}^j(m)[3]\right)$, where

$C_{2,1}^j(m)[3] = \{\hat{\beta} \in C_1^{j_o}[3] \mid \beta(0) = y = \chi_f(m)\}$,

where j_o is chosen so that $m \in \mathcal{U}_1^{j_o}$.

$TC_3^j[3] = C_3^j[3] \cup \left(\bigcup_{m\in \mathcal{U}_3^j\cap M_2^d} C_{3,2}^j(m)[3] \cup \left(\bigcup_{m\in \mathcal{U}_3^j\cap M_1^d} C_{3,1}^j(m)[3]\right)\right)$,

$C_{3,1}^j(m)[3] = \{\hat{\beta} \in C_1^{j_o}[3] \mid \beta(0) = y = \chi_f(m)\}$, j_o chosen so that $m \in \mathcal{U}_1^{j_o}$.

$C_{3,2}^j(m)[3] = \{\hat{\beta} \in C_2^{j_o}[3] \mid \beta(0) = y = \chi_f(m)\}$, j_o chosen so that $m \in \mathcal{U}_2^{j_o}$.

PROPOSITION 28:

The definition of $C_{2,1}^j(m)[3]$, as above, independent of the choice of j_o.

Proof:

Let j_o, j, s.t. $m \in \mathcal{U}_1^{j_o}$, $m \in \mathcal{U}_1^{j_1}$. As in Proposition 16, choose $B \subset \mathcal{U}_1^{j_o} \cap \mathcal{U}_1^{j_1}$, and set $P = \chi_f/M_1^{j_o}\left(\underbrace{B \cap M_1^d}_{\cap M_1^{j_o}}\right)$, open in $N_1^{j_o}$. As in Proposition 16 it is easy to show that:

$\{\hat{\beta} \in C_1^{j_o}[3] \mid \beta(0) = y = \chi_f(m)\} = \{\hat{\beta} \in T^3P \mid \beta(0) = y\} =$

$= \{\hat{\beta} \in C_1^{j_1}[3] \mid \beta(0) = y\}$, proving the proposition. $\square$

PROPOSITION 29:

Definition of $C^j_{3,1}(m)[3]$ independs of choice of j_o.

Proof

As above.

PROPOSITION 30:

Definition of $C^j_{3,2}(m)[3]$ independs of choice of j_o.

Proof

Let j_o, j_1 be st. $m \in \mathcal{U}_2^{j_o}$, $m \in \mathcal{U}_2^{j_1}$. Let:

$\odot$ $C^j_{3,2}(m)[3](j_o) = \{\hat{\beta} \in C_2^{j_o}[3] \mid y = \beta(0) = \chi_f(m)\}$,

$\odot\!\odot$ $C^j_{3,2}(m)[3](j_1) = \{\hat{\beta} \in C_2^{j_1}[3] \mid y = \beta(0) = \chi_f(m)\}$.

Note: Γ_o, γ_o; Γ_1, γ_1 are diffeomorphisms corresponding to $(j_o,2)$; $(j_1,2)$, respectively.

We want to show that $\odot = \odot\!\odot$. Let $\hat{\beta} \in \odot$. Therefore, $\hat{\beta} \in T^3\Gamma_o\, \tilde{I}^{-1}(Q_2[3]) = T^3\Gamma_1\,(T^3(\Gamma_1^{-1}\Gamma_o)(\tilde{I}^{-1}(Q_2[3]))) = T^3\Gamma_1(\tilde{I}^{-1}(Q_2[3]))$, by Proposition 24, therefore $\hat{\beta} \in C_2^{j_1}[3]$, therefore $\hat{\beta} \in \odot\!\odot$, therefore $\odot \subset \odot\!\odot$. Analogously, $\odot\!\odot \subset \odot$. $\square$

PROPOSITION 31:

$\boxed{\text{(Reducing GLOBAL to LOCAL)}}$

Suppose that $\hat{\beta}_n \in C[3]$, $y_n = \beta_n(0)$, $\forall\, n \in \mathbb{N}$, and $\{\hat{\beta}_n\} \to \hat{\beta} \in T^3(\mathbb{R}^3)_{,y=\beta(0)}$. Then, $\exists\, i \in \{1,2,3\}$, $j \in \mathbb{N}$, and subsequence $\{\hat{\beta}_k\}$ (see 4.4(12)) such that $\hat{\beta}_k \in TC_i^j[3]$, $\forall\, k \in \mathbb{N}$. Furthermore, $y \in \chi(\mathcal{U}_i^j \cap M_i^d)$.

Proof

Very similar to that of Proposition 17; the only difference is that the local cases below correspond to $r = 3$.

Again, choose (i_n, j_n) s.t. $\hat{\beta}_n \in C_{i_n}^{j_n}[3]$, for each fixed $n \in \mathbb{N}$; recall: $\chi_f/M_{i_n}^{j_n} : M_{i_n}^{j_n} \xrightarrow{\text{diff.}} N_{i_n}^{j_n} \ni y_n$, and set $m_n = (\chi_f/M_{i_n}^{j_n})^{-1}(y_n)$. In particular, $m_n \overset{\oplus}{\in} M_{i_n}^{j_n}$.

Now, $y \in C_f$ (see 4.4(12)); let $\chi_f^{-1}(y) = \{m_1, \ldots, m_p\}$ and choose (i_s, j_s), $s = 1, \ldots, p$ s.t. $m_s \in \mathcal{U}_{i_s}^{j_s}$, $s = 1, 2$ or 3 according to whether $m_s \in M^d_{1,2 \text{ or } 3}$.

LEMMA:

Everything as above, $m_n \in \mathcal{U}_{i_s}^{j_s} \Longrightarrow \hat{\beta}_n \in TC_{i_s}^{j_s}[3]$

PROOF OF LEMMA:

Case 1:

$\boxed{i_n = 3:}$ $\hat{\beta}_n \in C_3^{j_n}[3]$. As in Case 1, Proposition 17, one sees that $i_s = 3$, $m_n = m_s$. One therefore can show, with precisely the same arguments as in Proposition 27, that $C_3^{j_n}[3] = C_3^{j_s}[3]$. Therefore $\hat{\beta}_n \in C_3^{j_s}[3] \subset TC_{i_s=3}^{j_s}[3]$.

Case 2:

$\boxed{i_n = 2.}$ $\hat{\beta}_n \in C_2^{j_n}[3]$, $y_n = \beta_n(0)$.

We may discard $i_s = 1$, from Remark 8 above (see also $\oplus$ above).

<u>Case 2.1:</u> $i_s = 2$

$\hat{\beta}_n \in T^3\Gamma_n(\tilde{I}^{-1}(Q_2[3]))$, where Γ_n corresponds to $(j_n, 2)$; hence $\widehat{\Gamma_n^{-1}\beta_n} \in \tilde{I}^{-1}(Q_1[3])$.

We notice that $(\Gamma_s^{-1}\Gamma_n)$ is well defined on $\Gamma_n^{-1}(y_n)$, since Γ_s^{-1} is defined on y_n

$$\begin{tikzcd} m_n \in U_{i_s}^{j_s} \arrow[r, "\gamma_s"] \arrow[d] & * \arrow[d] \\ y_n \in \chi_f(U_{i_s}^{j_s}) \arrow[r, "\Gamma_s"] & * \end{tikzcd}$$

); hence it makes sense to write:

$$\hat{\beta}_n = T^3\Gamma_s(T^3(\Gamma_s^{-1}\Gamma_n)(\Gamma_n^{-1}\beta_n)),\ \text{where } \widehat{\Gamma_n^{-1}\beta_n} \in \tilde{I}^{-1}(Q_2[3]),\ \text{hence}$$

$$\hat{\beta}_n \in T^3\Gamma_s(\tilde{I}^{-1}(Q_2[3])) = C_2^{j_s}[3] \subset TC_{i_s=2}^{j_s}[3].$$

<u>Note:</u> One can <u>not</u> write in general $\hat{\beta} \overset{\theta}{=} T^3\Gamma_s(T^3(\Gamma_s^{-1}\Gamma_n))(\widehat{\Gamma_n^{-1}\beta})$, $\hat{\beta} \in T^3\Gamma_n(\tilde{I}^{-1}(Q_2[3]))$, since Γ_s may not be defined on y. Otherwise, one would prove, via θ, that $C_2^{j_r}[3] = C_2^{j_s}[3]$, which is false. We just remark that, for the sake of notation, the fact that $T^3\Gamma_s$ is <u>not</u> defined on the whole of $\tilde{I}^{-1}(Q_2[3])$ has been pushed to the background by <u>Note</u> in 4.4(22), and that one must therefore be aware all the time that for expressions like $T^3\Gamma_s(\cdot)$ to make sense, $T^3\Gamma_s$ must be defined on $(\cdot)$.

<u>Case 2.2</u> $i_s = 3$

$\hat{\beta}_n \in \{\hat{\beta} \in C_2^{j_n}[3] \mid \beta(0) = y_n = \chi_f(m_n)\} =$

$C_{3,2}^{j_s}(m_n)[3] \subset TC_3^{j_s}[3]$, as wanted.

<u>Note</u>: $m_n \in U_{i_s=3}^{j_s} \cap M_2^d$, by the hypothesis of lemma, 4.4(27) θ and hypothesis of Case 2, so that the equality above then results by taking j_n as the j_o in Definition 11 on page 4.4(25).

Case 3: $\boxed{i_n = 1}$

$\hat{\beta}_n \in C_1^{j_n}[3]$

Case 3.1 $i_s = 1$

As in Case 2.2 (of 4.4(14)), $M_1^{j_n} \cap \mathcal{U}_1^{j_s}$ is open in $M_1^{j_n}$, therefore $\chi_f/M_1^{j_n}(M_1^{j_n} \cap \mathcal{U}_1^{j_s})$ open in $N_1^{j_n}$. Set

$B \underset{\text{open}}{\subseteq} \mathbb{R}^3$ s.t. $B \cap N_1^{j_n} = \chi_f/M_1^{j_n}(M_1^{j_n} \cap \mathcal{U}_1^{j_s})$

In the same way as in Case 2.2(4.4(14)) (just substitute 2 by 3 whenever it appears), one shows that $\hat{\beta}_n \in T^3 N_1^{j_s} \subset TC_1^{j_s}[3]$

Case 3.2 $i_s = 2$

$\hat{\beta}_n \in C_1^{j_n}[3]$, $\beta_n(0) = y_n$, and hence

$\hat{\beta}_n \in \{\hat{\beta} \in C_1^{j_n}[3] | \beta(0) = y_n = \chi_f(m_n)\} =$

$C_{2,1}^{j_s}(m_n)[3] \subset TC_2^{j_s}[3]$, as required.

This last equality results by taking j_n as the j_o in Definition 11 (notice that $m_n \in \mathcal{U}_{i_s=2}^{j_s} \cap M_{i_n=1}^{d}$).

Case 3.3 $i_s = 3$

$\hat{\beta}_n \in \{\hat{\beta} \in C_1^{j_n}[3] | \beta(0) = y_n = \chi_f(m_n)\} = C_{2,1}^{j_s}(m_n)[3] \subset TC_3^{j}[3]$,

as wanted. The equality results by taking j_n as the j_o in Definition 11; $m_n \in \mathcal{U}_{i_s=3}^{j_s} \cap M_{i_n=1}^{d}$.

□

LEMMA $\Longrightarrow$ PROPOSITION 31:

Precisely equal to the proof that Lemma to Proposition 17 $\Longrightarrow$ Proposition 17, eventually substituting 2 by 3 where necessary. □

We now solve, in the next three theorems, the problems which arise in the proof that C[3] is closed, in their standard form. We will later (Proposition 35) show that these local problems can be reduced to the canonical formulation as below.

PROPOSITION 32:

(CUSP'S BUNDLE CLOSES FOLD'S BUNDLE: THE STANDARD FORM)

[Closedness at Cusp's line]

Let g denote the standard cusp g_2 (see 4.2(1), $g_2: \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$) with one disconnected control. I.e. $g(x_1,u,v,w) = \frac{x^4}{4} + u\frac{x^2}{2} + vx$. Let $\{\hat{\alpha}_n\}_{n \in \mathbb{N}}$, $\xi_n = \alpha_n(0)$, be a sequence in $T^3(\mathbb{R}^3)$, converging to a point $\hat{\alpha}$, $\xi = \alpha(0)$, with $\xi_u = \xi_v = 0$. Suppose that, for each n fixed, $\exists\, M^n$, submanifold of M_1^d, s.t.:

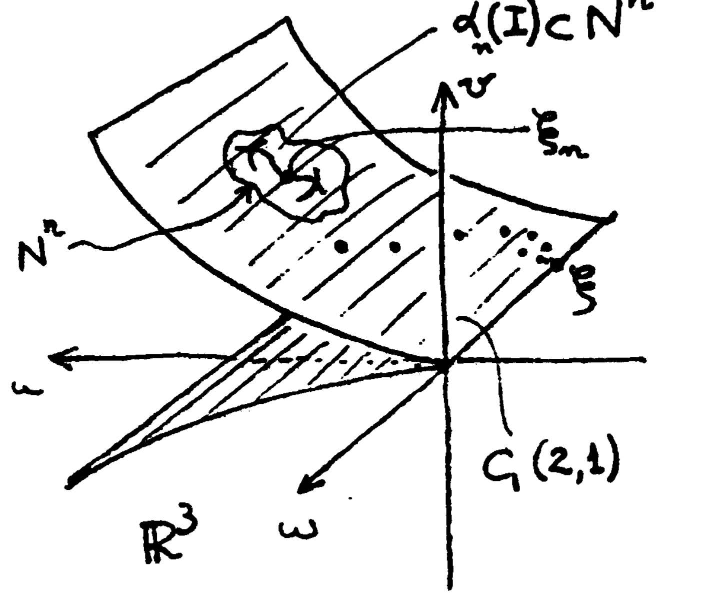
(i) $\chi_g/M^n : M^n \to N^n = \chi_g/M^n(M^n)$ is a diffeomorphism.

(ii) $\xi_n \in N^n \subset C_1(2,1)$.

(iii) $\exists$ representative $\alpha_n \in \hat{\alpha}_n$ s.t. $\alpha_n(I) \subset N^n$

Then:

$$\frac{d\alpha}{dt}u(0) = \frac{d\alpha}{dt}v(0) = 0.$$



Proof

Let $\theta/M: M_{g_2} \to \mathbb{R}^2$, $\chi :. \chi_{g_2} . (\theta/M^{-1})$, as mentioned in the proof of Proposition 18. Now, $M_g = M_{g_2} \times \mathbb{R}$ (see Lemma 7.6 of [16], so that we can define a map χ_1 (χ with 1 disconnected control) by the diagram:

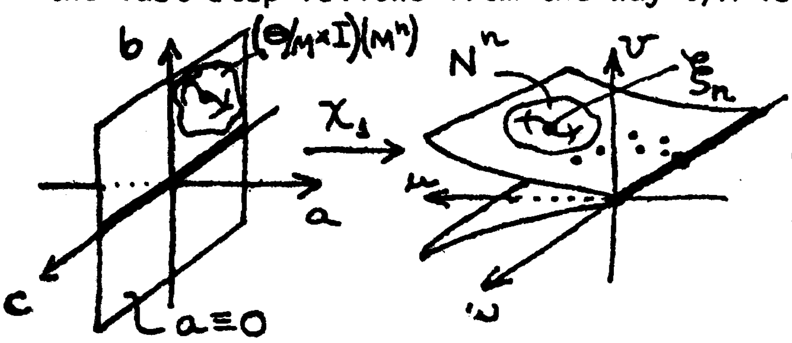
$$\begin{tikzcd} & M_g = M_{g_2} \times \mathbb{R} \arrow[dl, "\theta/M\times I"'] \arrow[dr, "\chi_g"] & \\ \mathbb{R}^3 \arrow[rr, "\chi_1"] & & \mathbb{R}^3 \end{tikzcd}$$

Like in Proposition 18, since $\chi_g/M^n: M^n \to N^n$, M^n a submanifold of M_1^d and $(\theta/M) \times I/M^n$ are diffeomorphisms, one has $\chi_{n,1} := \chi_1/((\theta/M) \times I)(M^n)$: $(\theta/M \times I)(M^n) \to \mathbb{R}^3$ is a diffeomorphism on its image, N^n. Now:

$$\chi_{n,1} = \chi_g((\theta/M)^{-1} \times I): \quad (a,b,c) \xrightarrow{(\theta/M)^{-1} \times I} (b;2a-3b^2;-2ab+2b^3;c) \longrightarrow$$

$$\xrightarrow{\chi_g} (2a-3b^2;\ -2ab + 2b^3;c).$$

Note that $(\theta/M \times I)(M^n) \subset (\theta/M \times I)(M^d_{1,g_2} \times \mathbb{R}) \subset \{(a,b,c)|a = 0\}$, where the last step follows from the way θ/M is constructed; also



$(\chi_{n,1})^{-1}(\alpha_n(t)) \subset (\theta/M \times I)(M^n) \subset$ $\subset \{(a,b,c)|a = 0\}$ Define: $a_n(t), b_n(t), c_n(t)$, $t \in I$, by $(a_n(t),b_n(t),c_n(t)) = \chi_{n,1}^{-1}(\alpha_n(t))$.

From observation above, $a_n(t) \equiv 0$. This allows us to rewrite $\alpha_n(t)$ as:

$$\alpha_n(t) = \chi_{n,1}(\underset{0}{a_n(t)};b_n(t);c_n(t)) = (-3b_n^2(t);2b_n^3(t);c_n(t))$$

Therefore, omitting the 0's (see 4.4(16)):

$$\tilde{I}(\hat{\alpha}_n) = \Big((\alpha_n)_u(0);\ (\alpha_n)_v(0);\ (\alpha_n)_w(0);\ \frac{d(\alpha_n)}{dt}u(0);$$

$$\frac{d(\alpha_n)}{dt}v(0);\ \frac{d(\alpha_n)}{dt}w(0);\ \frac{d^2(\alpha_n)}{dt^2}u(0);\ \text{etc...}\Big) =$$

$$= (-3b_n^2;\ 2b_n^3;\ c_n;\ -6b_n\, b_n';\ 6b_n^2 b_n';\ c_n';$$

$$-6(b_n b_n'' + (b_n')^2);\ 6(2b_n + (b_n')^2 + b_n^2\, b_n'');\ c_n'';$$

$$-6(b_n b_n''' + 3b_n' b_n'');\ 6(6b_n b_n' b_n'' + 2b_n^2 b_n''' + 2(b_n')^3;\ c_n''') \in \mathbb{R}^{12}.$$

We want then to show:

$$(\mathrm{I})\ \boxed{\begin{array}{l} -3b_n^2 \to 0 \\ 2b_n^3 \to 0 \end{array}} \Longrightarrow \boxed{\begin{array}{l} -6b_n b_n' \to 0 \\ 6b_n^2 b_n' \to 0 \end{array}}\ (\mathrm{II})$$

By computation, one sees that

$$\underbrace{\frac{d^3(\alpha_n)}{dt^3}v(0)}_{A} + \underbrace{b_n \frac{(d^3(\alpha_n)}{dt^3}u(0))}_{B} = \underbrace{-b_n' \cdot \left(6(b_n')^2 + 3\frac{d^2(\alpha_n)}{dt^2}u(0)\right)}_{C}$$

We claim that $\exists\, K, N \in \mathbb{N}$ s.t. $|b_n'| < K,\ \forall\, n \geq N$. This is so because

$$|c| = |b_n'|.\ \left| 6(b_n')^2 + \frac{3d^2(\alpha_n)_u}{dt^2}(0) \right| \leq |A| + |B|.$$

tends to a constant

as $n \to \infty$

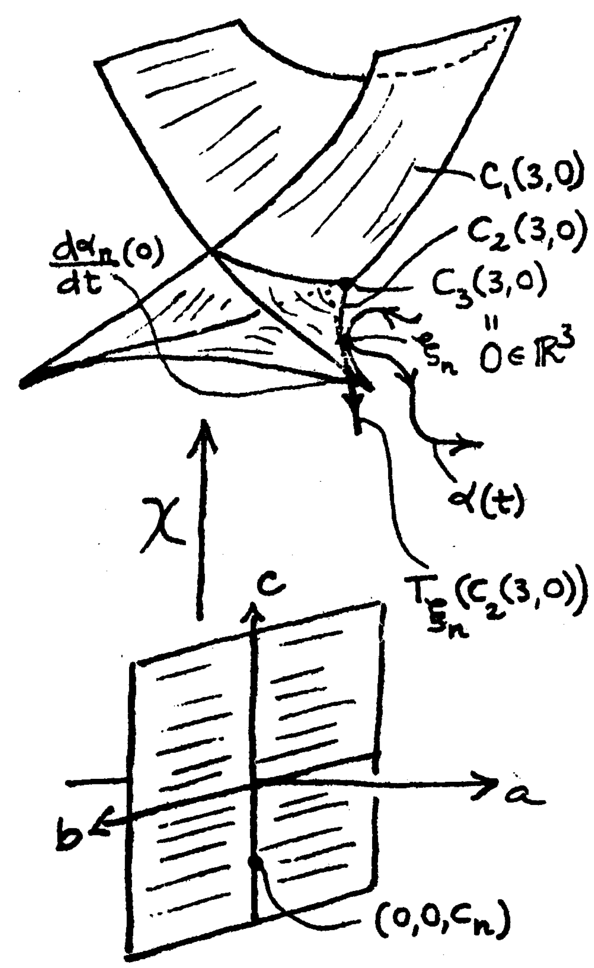
Therefore $\underbrace{-6b_n}_{\substack{0 \\ \text{see } 4.4(16)}} \underbrace{b_n'}_{\text{has limited module}} \to 0$. And so does $\underbrace{(-b_n)}_{0}\ \underbrace{(-6b_n b_n')}_{0}$, as wanted.

$\square$

PROPOSITION 33:

(SWALLOW TAIL'S BUNDLE CLOSES CUSP'S BUNDLE : STANDARD FORM)

[Closedness at Swallowtail's Point : case 1]



Let g_3 denote the swallowtail (no disconnected controls), $\{\hat{\alpha}_n\}_{n\in\mathbb{N}}$, $\alpha_n(0) = \xi_n$, sequence in $T^3(\mathbb{R}^3)$, converging to $\hat{\alpha}$, $\xi = \alpha(0) = 0$. Suppose that, for each n fixed:

(i) $\xi_n \in C_2(3,0)$; (ii) $d\alpha_n/dt(0) \in T_{\xi_n}(C_2(3,0))$.

Then: $\frac{d\alpha}{dt}v(0) = \frac{d\alpha}{dt}w(0) = 0$.

Proof

Let χ, corresponding to the swallowtail, be as computed from [17] (see 4.2(4): c = 3, bottom). Choose c_n s.t. $\chi(0,0,c_n) = \xi_n$ (therefore $c_n \neq 0$, because $\xi_n \in C_2(3,0)$). From the expression of χ, $c_n \to 0$ as $n \to \infty$, since $\xi_n \to 0$. Since χ preserves the codimension 2 strata, i.e. $\chi(\{(a,b,c)\,|\,a = b = 0,\ c \neq 0\}) = C_2(3,0)$, we compute

$$T_{\xi_n}(C_2(3,0)) = T_{(0,0,c_n)}\chi(\{(a,b,c)\,|\,a=b=0\}) =$$

$$= \underbrace{\{(r;-2rc_n;rc_n^2)\,|\,r \in \mathbb{R}\}}_{\oplus},\quad \text{n fixed.}$$

Since we know that $\underbrace{\left(\frac{d(\alpha_n)}{dt}u(0);\ \frac{d(\alpha_n)}{dt}v(0);\ \frac{d(\alpha_n)}{dt}w(0)\right)}_{\odot} \in \oplus$, $\forall$ n arbitrarily fixed, just choose, fixed n, r_n s.t. $\odot = (r_n;-2r_nc_n,r_nc_n^2)$.

Since $\{\hat{\alpha}_n\}$ converges, $\tilde{I}$ continuous, $\frac{d(\alpha_n)}{dt}u(0) = r_n \to r \in \mathbb{R}$ as $n \to \infty$.

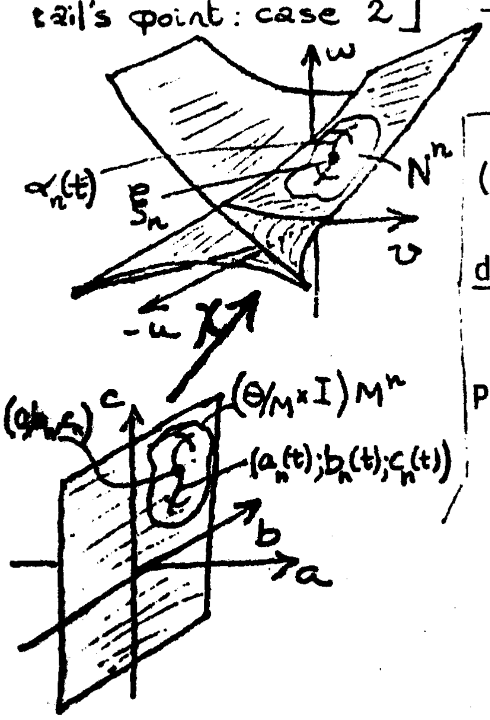
Therefore $\dfrac{d(\alpha_n)}{dt}w(0) = \underbrace{\cancel{2\gamma_n}}_{\text{constant}}\ \underbrace{c_n}_{0} \to 0$ as $n\to\infty$, and so does $\dfrac{d(\alpha_n)}{dt}w(0) = \underbrace{\gamma_n}_{\text{constant}}\ \underbrace{c_n^2}_{0}$,

therefore, by continuity of $\tilde{I}$, one has : $d\alpha_v(0)/dt = d\alpha_w(0)/dt = 0$. □

PROPOSITION 34:

(SWALLOW-TAIL'S BUNDLE CLOSES FOLD'S BUNDLE: STANDARD FORM)

[Closedness at Swallow tail's point: case 2]



Let g_3 denote the standard swallowtail, as in Proposition 33. Let $\{\hat{\alpha}_n\}_{n\in\mathbb{N}}$, $\xi_n = \alpha_n(0)$, be a sequence in $T^3(\mathbb{R}^3)$ converging to a point $\hat{\alpha}$, $\alpha(0) = \xi = 0$. Suppose that, for each m, fixed, $\exists M^n$, a submanifold of M_1^d, s.t.: (ii) $\xi_n \in N^n \cap C_1(3,0)$

(i) $\chi_{g_3/M^n}: M^n \to N^n = \chi_{g_3/M^n}(M^n)$ is a diff.;

(iii) $\exists$ representative α_n of $\hat{\alpha}_n$ s.t. $\alpha_n(I) \subset N^n$. Then $\dfrac{d\alpha}{dt}w(0) = 0$.

Proof

Let χ be as in Proposition 33, $\chi_n = \chi/_{\theta/M(M^n)}$, a local diffeomorphism. Set: $(a_n(t);b_n(t);c_n(t)) = \chi_n^{-1}(\alpha_n(t))$. It follows immediately that $a_n(t) \equiv 0$. This allows us to express, as before,

$\alpha_n(t)$ as a function of $b_n(t)$ and $c_n(t)$: $\alpha_n(t) = \chi_n(0;b_n(t);c_n(t)) =$

$= (3b_n(t) - 6c_n^2(t);\ -6b_n(t)c_n(t) + 8c_n^3(t);\ 3b_n(t)c_n^2(t) - 3c_n^4(t))$.

So that, omitting the 0's, as before, one has $\tilde{I}(\hat{\alpha}_n) =$

1st order expressions

$= (3b_n - 6c_n^2; -6b_nc_n + 8c_n^3; 3b_nc_n^2 - 3c_n^4;\ \underbrace{3b_n' - 12c_nc_n'}_{\frac{d(\alpha_n)}{dt}u(0)};\ \underbrace{-6(b_nc_n' + b_n'c_n) + 24c_n^2c_n'}_{\frac{d(\alpha_n)}{dt}v};\ \underbrace{3(b_n'c_n^2 + 2b_nc_nc_n') - 12c_n^3c_n'}_{\frac{d(\alpha_n)}{dt}w(0)}$;

2nd order expressions; 3rd order expressions)

We want therefore to show that:

$$\boxed{\begin{array}{l} 3b_n - 6c_n^2 \to 0 \\ -6b_nc_n+8c_n^3 \to 0 \\ 3b_nc_n^2+3c_n^4 \to 0 \end{array}} \quad (\mathrm{I}) \qquad \Longrightarrow \qquad \boxed{[3(b_n'c_n^2 + 2b_nc_nc_n') - 12c_n^3c_n'] \to 0} \quad (\mathrm{II})$$

We first prove that $c_n \to 0$ as $n \to \infty$. If this is false, $\exists$ subsequences $\{c_k\}$ of $\{c_n\}$ and $\varepsilon > 0$ s.t. $|c_k| > \varepsilon$, $\forall k$. ($c_k = c_{n(k)}$, $k \in \mathbb{N}$, to be more precise)

Now $\displaystyle\lim_{k\to\infty} (-6b_kc_k + 8c_k^3) = \lim_{k\to\infty} (-2c_k[\underbrace{(3b_k-6c_k^2)}_{0 \text{ as } k\to\infty} + 2c_k^2])$

Therefore, for k suff. big, $|\overbrace{[(3b_k - 6c_k^2)+2c_k^2]}^{\bullet}| \geq |c_k^2|$, therefore $|-2c_k||\bullet| \geq |c_k||\ c_k^2\ | > \varepsilon^3$, $\forall$ k suff. big, therefore $\displaystyle\lim_{k\to\infty} (-6b_kc_k + 8c_k^3) \neq 0$, a contradiction; therefore $c_n \to 0$ as $n \to \infty$

Now, by computation:

$$(\mathrm{II}) = \frac{d(\alpha_n)}{dt}w(0) = -\overset{0}{c_n}\,\overbrace{(\overset{0}{c_n} \cdot \overset{\text{fixed limit}}{\frac{d(\alpha_n)}{dt}}u(0)}^{0} + \overset{\text{fixed limit}}{\frac{d(\alpha_n)}{dt}}v(0)),$$ therefore $\displaystyle\lim_{n\to\infty} \frac{d(\alpha_n)}{dt}w(0) = 0$, as wanted. $\square$

PROPOSITION 35:

$C[3]$ is closed in $T^3(\mathbb{R}^3)$.

Proof

Let $\{\hat{\beta}_n\}_{n\in\mathbb{N}}$, $\beta_n(0) = y_n$, be a sequence converging to some $\hat{\beta} \in T^3(\mathbb{R}^3)$, $\beta(0) = y$, $\hat{\beta}_n \in C[3]$, $\forall\, n \in \mathbb{N}$ fixed. From Proposition 31 and its lemma, $\exists$ subsequence $\{\hat{\beta}_k\}_{k\in\mathbb{N}}$, $y_k = \beta_k(0)$, such that $\hat{\beta}_k \in TC_{i_s}^{j_s}[3]$, $\forall\, k \in \mathbb{N}$.

Case 1:

$\boxed{i_s = 1}$

$TC_{i_s=1}^{j_s}[3] = C_1^{j_s}[3] = T^3(N_1^{j_s})$. Let Γ,γ as usual. As in case 1,(4.4(16)), one shows that $\Gamma^{-1}\beta_k(I) \subset C(1,2)$, therefore $\tilde{I}(\widehat{\Gamma^{-1}\beta_k}) \in \{(x_1,\ldots,x_{12})|x_1=x_4=x_7=x_{10}=0\}$, therefore, by continuity of $\tilde{I}$ and $T^3\Gamma^{-1}$, $\tilde{I}(\widehat{\Gamma^{-1}\beta}) \in \{(\cdot)|x_1=x_4=x_7=x_{10}=0\}$, hence $\exists$ representative $\Gamma^{-1}\beta$ of $\widehat{\Gamma^{-1}\beta}$ s.t. $\Gamma^{-1}\beta(I) \subset C(1,2)$, therefore $\beta(I) \subset N_1^{j_s}$, therefore $\hat{\beta} \in C_1^{j_s}[3] \subset C[3]$.

Case 2:

$\boxed{i_s = 2}$

Case 2.1 $\exists$ subsequence, $\{\hat{\beta}_r\}_{r\in\mathbb{N}}$, $\beta_r(0) = y_r$, of $\{\hat{\beta}_k\}_{k\in\mathbb{N}}$, such that $\hat{\beta}_r \in C_2^{j_s}[3]$, $\forall\, r \in \mathbb{N}$. Let Γ,γ be as usual. By definition of $C_2^{j_s}[3]$, $\tilde{I}(\widehat{\Gamma^{-1}\beta_r}) \in Q_2[3] = \{(x_1,\ldots,x_{12})|x_1=x_2=x_4=x_5=0\}$. Therefore $\tilde{I}(\widehat{\Gamma^{-1}\beta}) = \lim_{r\to\infty} \tilde{I}(\widehat{\Gamma^{-1}\beta_r}) \in Q_2[3]$, therefore $\hat{\beta} \in C_2^{j_s}[3] \subset C[3]$.

Case 2.2 $\exists\, K \in \mathbb{N}$ s.t. $\hat{\beta}_k \in C_{2,1}^{j_s}(m_k)[3]$, some $m_k \in U_2^{j_s} \cap M_1^d$, $\forall\, k \geq K$, fixed. From the hypothesis, fixed $k \geq K$, one has $\hat{\beta}_k \in \{\hat{\beta} \in C_1^{j_o}[3]|\beta(0) = y_k = \chi_f(m_k)\}$ with j_o s.t. $m_k \in U_1^{j_o}$.

Therefore, $\exists\, \beta_k$, representative of $\hat{\beta}_k$, s.t. $\beta_k(I) \overset{\oplus}{\subset} N_1^{j_o}$.

Recall that $\chi_f/M_1^{j_o}: M_1^{j_o} \to N_1^{j_o}$ is a diffeomorphism. Therefore

$\chi_{g=\gamma f/\gamma^{-1}(M_1^{j_o})}: \gamma^{-1}(M_1^{j_o}) \to \Gamma^{-1}(N_1^{j_o})$ ($\subset C_1(1,2)$) diffeomorphically

((i)'). Also $\Gamma^{-1}(\beta_k(0)) = \Gamma^{-1}(y_k) \in \Gamma^{-1}(N_1^{j_o})$ ((ii)') and from $\oplus$

$\Gamma^{-1}\beta_k(I) \subset \Gamma^{-1}(N_1^{j_o})$ ((iii)').

By considering the sequence $\{\widehat{\Gamma^{-1}\beta_k}\}$ as in Case 2.2, 4.4(17), one gets (same arguments as there) $\hat{\beta} \in C_2^{j_s}[3] \subset C[3]$, this time via Proposition 32 above.

Case 3:

$\boxed{i_s = 3}$

Case 3.1.

$\exists$ subsequence $\{\hat{\beta}_r\}_{r\in\mathbb{N}}$, $y_r = \beta_r(0)$, such that $\hat{\beta}_r \in C_3^{j_s}[3]$

$\forall\, r \in \mathbb{N}$; with Γ,γ corresponding to $(j_s,3)$, as usual,

one gets $\tilde{I}(\widehat{\Gamma^{-1}\beta_r}) \in Q_3[3] = \{ (\cdot) \mid x_1=x_2=x_3=x_6 = 0\}$, therefore

$\tilde{I}(\widehat{\Gamma^{-1}\beta}) \in Q_3[3]$, therefore $\hat{\beta} \in C_3^{j_s}[3] \subset C[3]$

Case 3.2

$\exists$ subsequence $\{\hat{\beta}_r\}_{r\in\mathbb{N}}$, $y_r = \beta_r(0)$, such that, for each

fixed r, $\hat{\beta}_r \in C_{3,2}^{j_s}(m_r)[3]$, where $m_r \in U_3^{j_s} \cap M_2^d$. This menas

that $\hat{\beta}_r \in \{\hat{\beta} \in C_2^{j_o}[3] \mid \beta(0) = y_r = \chi_f(m_r)\}$ =

$= \{\hat{\beta} \in T^3\Gamma_o.\tilde{I}^{-1}(\{(\cdot)\mid x_1=x_2=x_4=x_5 = 0\})\}$, where j_o is

such that $m_r \in U_2^{j_o}$, and Γ_o corresponds to j_o,2.

That is: $\boxed{(\Gamma_o^{-1}(y_r))_u = (\Gamma_o^{-1}(y_r))_v = \frac{d(\Gamma_o^{-1}\beta_r)}{dt}u(0) = \frac{d(\Gamma_o^{-1}\beta_r)}{dt}v(0) = 0.}$ ✱

Now, if Γ corresponds to $(j_s,3)$ we know, from Remark 7 in 4.4(7), that:

$$\Gamma^{-1}\Gamma_o(C_2(2,1)) \overset{\boxtimes}{\subset} C_2(3,0)$$

(in that Remark $\Gamma \to \Gamma_2$, $\Gamma_o \to \Gamma$, $i = 2$, $r = 3$ and $c_1 = 2$, $c_2 = 3$)

Therefore, if $\xi_r = (\Gamma^{-1})(y_r)$, $\alpha_r = (\Gamma^{-1})(\beta_r)$, and by $\left\{\begin{matrix} ✱ \\ \boxtimes \end{matrix}\right.$ above:

$$\left(\frac{d(\alpha_r)}{dt}u(0); \frac{d(\alpha_r)}{dt}v(0); \frac{d(\alpha_r)}{dt}w(0)\right) =$$

$$= T_{\Gamma_o^{-1}(y_r)}(\Gamma^{-1}\Gamma_o)\ \overbrace{\left(\frac{d(\Gamma_o^{-1}\beta_r)}{dt}u(0),\ldots,\frac{d(\Gamma_o^{-1}\beta_r)}{dt}w(0)\right)}^{\bullet}$$

and hence, since $\bullet \in T_{\Gamma_o^{-1}(y_r)}(C_2(2,1))$, by ✱ , we have

$$\left(\frac{d(\alpha_r)}{dt}u(0);\ldots;\frac{d(\alpha_r)}{dt}w(0)\right) \in T_{\Gamma_o^{-1}(y_r)}(\Gamma^{-1}\Gamma_o)(T_{\Gamma_o^{-1}(y_r)}(C_2(2,1))) \underset{\boxtimes}{\subset}$$

$$\subset T_{\xi_r}(C_2(3,0))$$

Also, by $\boxtimes$, since $\Gamma_o^{-1}(y_r) \in C_2(2,1)$ [$y_r \in \chi_f(M_2^d)$, since $y_r = \chi_f(m_r)$; see also Remark 7], $\xi_r \in C_2(3,0)$.

Therefore, the conditions as in the hypothesis of Proposition 33 are met by $\{\hat{\alpha}_r\}$, hence $\frac{d\alpha}{dt}v(0) = \frac{d\alpha}{dt}w(0) = 0$, i.e.,

$\frac{d(\Gamma^{-1}\beta)}{dt}v(0) = \boxed{\frac{d(\Gamma^{-1}\beta)}{dt}w(0) = 0}$. We recall, from Proposition 31 and its lemma, that $y \in \chi(M^d_{i=i_s=3})$, therefore $\Gamma^{-1}(y) = (0;0;0)$.

This, together with θ (we don't need the whole of Proposition 33), shows that $\hat{\beta} \in T^3\Gamma.\tilde{I}^{-1}(Q_3[3] = C_3^{j_s}[3] \subset C[3]$

<u>Case 3.3</u>: $\exists K \in \mathbb{N}$ s.t. $\hat{\beta}_k \in C_{3,1}^{j_s}(m_k)[3]$, $\beta_k(0) = y_k$, some $m_k \in \mathcal{U}_3^{js} \cap M_1^d$,

$\forall k \geq K$, arbitrarily fixed.

The proof of Case 3.3 is entirely analogous to that of case 2.2 (4.4(17)). For $k \geq K$ fixed, $\hat{\beta}_k \in \{\hat{\beta} \in C_1^{j_o}[3] | \beta(0) = y_k = \chi_f(m_k)\}$, j_o s.t. $m_k \in \mathcal{U}_1^{j_o}$. Hence, $\exists$ representative β_k s.t. $\beta_k(I) \subset N_1^{j_o}$. One gets (as in 4.4(17)): $\chi_{g=\gamma_f/\gamma^{-1}(M_1^{j_o})} : \gamma^{-1}(M_1^{j_o}) \to \Gamma^{-1}(N_1^{j_o})$ diffeomorphically ($\Gamma^{-1}(N_1^{j_o}) \subset C_1(3,0)$) ((i)'); $\Gamma^{-1}(\beta_k(0)) \in \Gamma^{-1}(N_1^{j_o})$ ((ii)') and $\Gamma^{-1}\beta_k(I) \subset \Gamma^{-1}(N_1^{j_o})$ ((iii)')

By then considering the sequence $\{\hat{\alpha}_k\}_{k\in\mathbb{N},k\geq K}$, $\xi_k = \alpha_k(0)$, $\alpha_K = \Gamma^{-1}\beta_K$, and setting $M^k = \gamma^{-1}(M_1^{j_o})$, $N^k = \Gamma^{-1}(N_1^{j_o})$, one gets (as in 4.4(17))

$\frac{d(\overbrace{\Gamma^{-1}\beta}^{\alpha})}{dt}w(0) = 0$, from Proposition 34, and $\Gamma^{-1}(y) = 0$, since $y \in N_3^{j_s}$, hence $\hat{\beta} \in C_3^{j_s}[3] \subset C[3]$ □

<u>C. Genericity of $v \pitchfork C_f$:</u>

<u>PROPOSITION 36:</u>

$\exists$ open and dense set of vector fields, $\mathcal{B} \subset \mathcal{V}(\mathbb{R}^3)$, s.t. $v \in \mathcal{B} \Longrightarrow$ $\Rightarrow v[3](\mathbb{R}^3) \cap C[3] = \emptyset$.

Proof

The proof is analogous to that of Proposition 20; one sets $B_i^j = C_i^j[3] \cap A$, $B_i^{j,c} = C_i^j[3] \cap A^c$, $V_i^j = S^{-1}(B_i^j)$, $V_i^{j,c} = S^{-1}(B_i^{j,c})$, $j \in \mathbb{N}$, $i = 1,2,3$, A and S as before. $\mathcal{B} = \{v \mid j^2v \pitchfork (V_i^j \text{ and } V_i^{j,c}), \forall\, i,j\}$ is then proved to have the required properties. □

PROPOSITION 37: $\boxed{\text{(GLOBAL to LOCAL)}}$

Let $y \in C_f$, $m_s, (i_s, j_s)$, $U_{i_s}^{j_s}$, $s = 1,\ldots,p$ as in 4.4(27). $\exists\, V$, open neighbourhood of y in $\mathbb{R}^3$, s.t. $V \cap C_f = V \cap [\bigcup_{s=1}^{p} \chi_f(U_{i_s}^{j_s} \cap M^d)]$.

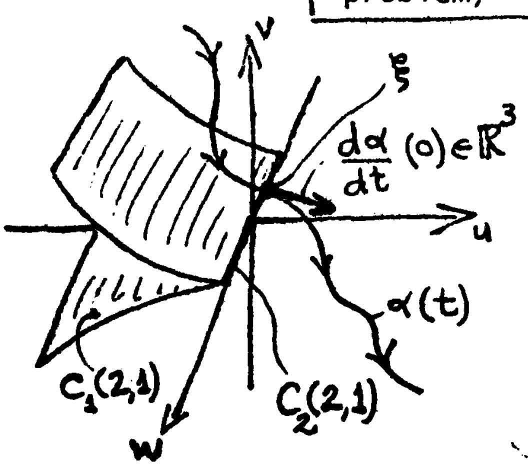
COROLLARY:

$$V \cap C_f \subset \bigcup_{s=1}^{p} \chi_f(U_{i_s}^{j_s} \cap M^d).$$

Proof

Same as that of Proposition 21. □

PROPOSITION 38: (Genericity of v cusp in STANDARD FORM: the 3 dimensional problem)



Let $\alpha(t) = (\alpha_u(t);\alpha_v(t);\alpha_w(t))$ be a C^∞ curve through $\xi = \alpha(0)$, $\xi = (\xi_u;\xi_v;\xi_w)$ satisfying $\xi_u = \xi_v = 0$. Suppose that:

$(\frac{d\alpha}{dt}u(0);\ \frac{d\alpha}{dt}v(0)) \neq (0,0)$. Then $\varepsilon > 0$ s.t.

$\{\alpha(t) \mid |t| < \varepsilon,\ t \neq 0\} \cap C(2,1) = \emptyset$.

Proof

Let $\alpha(t) = (\alpha_u(t);\alpha_v(t))$. If $\exists\ \varepsilon > 0$ s.t. $\{\alpha(t) \mid |t| < \varepsilon,\ t \neq 0\} \cap C(2,0) = \emptyset$ then our thesis would immediately follow from the fact that $C(2,1) = C(2,0) \times \mathbb{R}$.

Therefore our problem will be solved if we show if:

$\alpha = (\alpha_u,\alpha_v)$ is a curve in $\mathbb{R}^2$, $\alpha(0) = 0$, s.t. $(\alpha_u'(0);\alpha_v'(0)) \neq (0,0)$ (I)

then $\boxed{\exists\varepsilon > 0 \text{ s.t. } \{\alpha(t) \mid |t|<\varepsilon,\ t\neq 0\}\cap C(2,0)=\emptyset}$

Case 1: Suppose $\frac{d\alpha}{dt}v(0) \neq 0$. (II) Follows, from Proposition 22.

Case 2: Suppose $\frac{d\alpha}{dt}v(0) = 0$, $\frac{d\alpha}{dt}u(0) \neq 0$. In this case $\alpha_u(t) =$

$$= \frac{d\alpha}{dt}u(0)\, t + 0(t^2) \text{ Therefore } |\alpha_u(t)| \geq 1/2\, |\overbrace{\frac{d\alpha}{dt}u(0)}^{\neq 0}|\, |t|,$$ for t

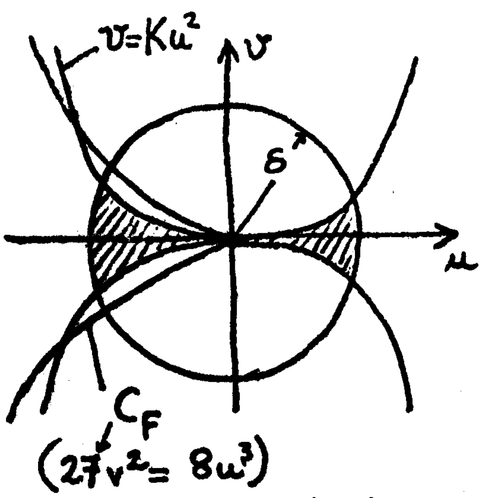
sufficiently small. Also $\alpha_v(t) = \frac{d^2\alpha}{dt^2}v(0)\, t^2 + 0(t^3)$; Therefore

$|\alpha_v(t)| \leq 2\varepsilon|t^2|$, where $\varepsilon = |\frac{d^2\alpha}{dt^2}v(0)|$, if $\frac{d^2\alpha}{dt^2}v(0) \neq 0$, $\varepsilon = 1$, if $\frac{d^2\alpha}{dt^2}v(0) = 0$.

Hence $|\alpha_v(t)| \leq A.\ |\alpha_u(t)|\,|t|$, for some $A \in \mathbb{R}^+$. Since $\frac{d\alpha}{dt}u(0) \neq 0$, we also

have $|\alpha_u(t)| \geq |t|\,|\frac{d\alpha}{dt}u(0)|\cdot\frac{1}{2}$, therefore $\boxed{|\alpha_v(t)| \leq K.\ |\alpha_u(t)|^2}$, $\oplus$

for some $K > 0$.



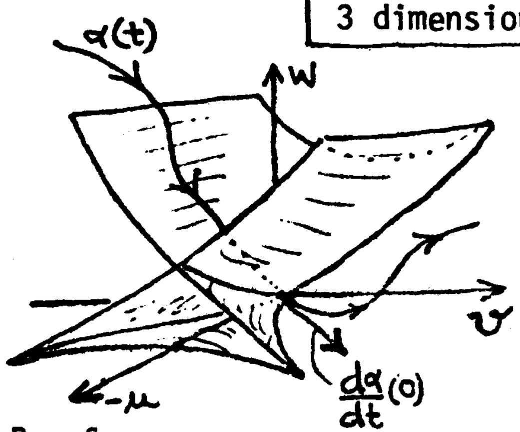
Let $\delta = 8/27K^2$. Choose ε_1 small enough so that $(\alpha_u(t),\alpha_v(t)) \in B_\delta(0)$, $\forall\ |t| < \varepsilon_1$. Suppose that $(\alpha_u(t),\alpha_v(t)) \in C_f$, $|t| < \varepsilon_1$. Then $27\alpha_v^2(t) = 8u_v^3(t)$, therefore, since $|\alpha_v^2(t)| \leq K^2|\alpha_u^4(t)|$, $27|\alpha_v^2(t)| = 8|u_v^3(t)| \leq$ $\leq 27K^2|\alpha_u^4(t)|$. If $\alpha_u(t) = 0$, this will lead to $8/27K^2 \leq |\alpha_u(t)|$, a contradiction; hence $\alpha_u(t) = 0$

therefore by $\oplus$ (above) $\alpha_v(t) = 0$. Now, since $\frac{d\alpha}{dt}u(0) \neq 0$, $\exists\ \varepsilon_2 > 0$ s.t. $\alpha_u(t) \neq 0$,

$\forall\ t$ s.t. $\underset{t\neq 0}{|t|} < \varepsilon_2$. Choose $\varepsilon = \min\ \{\varepsilon_1;\varepsilon_2\}$.

It follows from the above that if $\underset{t \neq 0}{|t|} < \varepsilon$, then $\alpha(t) \notin C_f = C(2,0)$, as required. $\square$

PROPOSITION 39: (Genericity of v $\pitchfork$ Swallowtail in STANDARD FORM: the 3 dimensional problem)



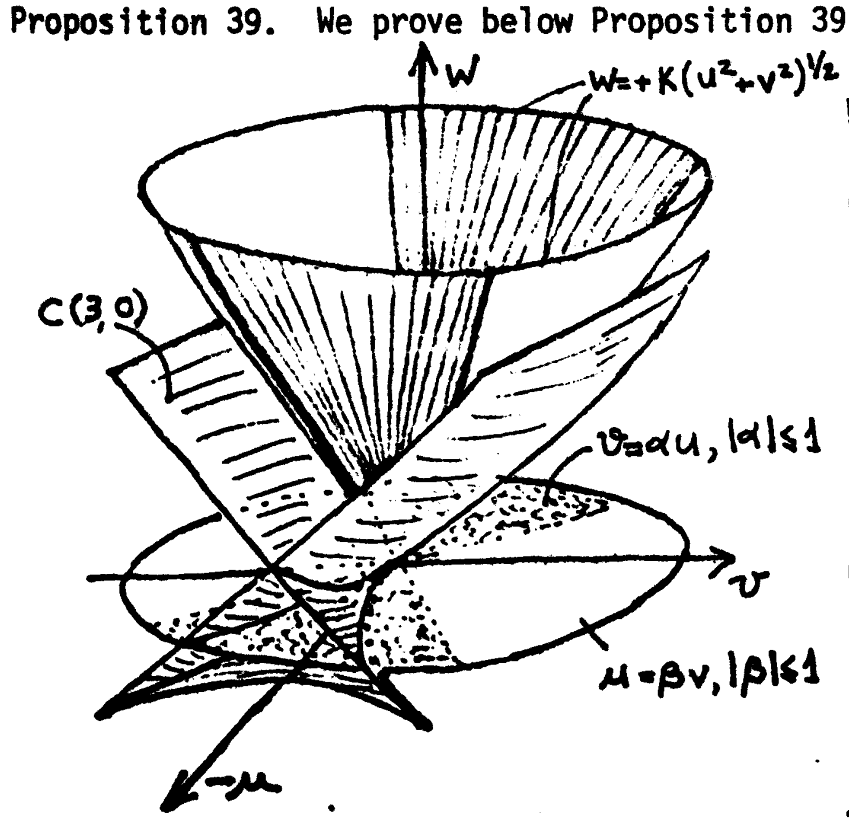
Let $\alpha = (\alpha_u;\alpha_v;\alpha_w)$ be a curve through $0 \in \mathbb{R}^3$. Suppose that $\frac{d\alpha}{dt}w(0) \neq 0$. Then, $\exists\ \varepsilon > 0$ s.t.

$\{\alpha(t) \mid |t| < \varepsilon,\ |t| \neq 0\} \cap C(3,0) = \emptyset$.

Proof

We first define
$$C^*(3,0) = \left\{ (u,v,w) \in \mathbb{R}^3 \;\middle|\; \begin{array}{l} 256w^3 - 27v^4 + 4u(32v^2w + 4u^3w - \\ -3uw^2 - u^2v^2) = 0 \end{array} \right\}$$

This is obtained by multiplying
$$\left[\begin{array}{ll} (1) & \dfrac{\partial g_3}{\partial x}(\cdot) = x^4 + ux^2 + vx + w = 0 \\ (2) & \dfrac{\partial^2 g_3}{\partial x^2}(\cdot) = 4x^3 + 2ux + v = 0 \end{array}\right]$$
(g_3 as in 4.2(1)) the equations (1) and (2) (as in brackets) by $x^2, x, 1$ and $x^3, x^2, x, 1$, respectively, and solving the 7x7 determinant for u,v,w. It follows immediately that $C^*(3,0) \supset C(3,0)$. (it is actually true that $C^*(3,0 \supsetneq C(3,0)$, but this will not concern us here). So, if we substitute, in the statement of Proposition 39, C(3,0) by $C^*(3,0)$, to get a Proposition 39', say, then Proposition 39' $\Longrightarrow$ Proposition 39. We prove below Proposition 39'.



We first give some definitions:

$C_u^k = \{(u,v,w) \in \mathbb{R}^3 \mid v \overset{1}{=} \alpha u,\ |\alpha| \leq 1,$ and $w \overset{2}{=} \pm k|u|(1+\alpha^2)^{1/2}, k \in \mathbb{R}^+\}$

(red part of the cone as in picture; we didn't draw the lower part of the cone)

$C_v^k = \{(u,v,w) \in \mathbb{R}^3 \mid u \overset{3}{=} \beta v, |\beta| \leq 1,$ $w \overset{4}{=} \pm k|v|(1+\beta^2)^{1/2}, k \in \mathbb{R}^+\}$

(white part of the cone)

$C^k = C_u^k \cup C_v^k$ $(= \{(u,v,w) \in \mathbb{R}^3 | w = \pm k(u^2 + v^2)^{\frac{1}{2}}\})$

$SC_u^k = \bigcup_{k' \geq k} C_u^{k'}$; $SC_v^k = \bigcup_{k' \geq k} C_v^{k'}$ 'S' stands for 'solid'; 'C' for 'cone'. Finally, $SC^k = SC_u^k \cup SC_v^k$. The proposition will follow from some lemmas.

LEMMA 1:

Let k be fixed. $\exists\ \delta = \delta(k)$ s.t: $B_\delta(0) \cap SC_u^k \cap C^*(3,0) = \{0\}$.

[Note: This says that the intersection of the red 'solid' cone with the swallowtail is locally $\emptyset$.]

Proof

Substituting (1) and (2) in the expression for $C^*(3,0)$, one gets:

$\pm 256k^3(1 + \alpha^2)^{3/2}|u|^3 - 27\alpha^4u^4 + 4u.(\pm 32\alpha^2u^2k|u|(1+\alpha^2)^{\frac{1}{2}} \pm 4u^3k|u|(1+\alpha^2)^{\frac{1}{2}} -$
$-3uk^2|u|^2(1 + \alpha^2) - \alpha^2u^4) = 0.$

From this, we have $k^3u^3 (A + |u|B) \overset{(5)}{=} 0$, where $|A| \geq 256$ and $B = B(k)$ is a positive constant ($B(k') < B(k)$ if $k' > k$). Therefore, by choosing u s.t. $|u| < \frac{256}{B(k)}$ (therefore $|u| < \frac{256}{B(k')}$, $\forall\ k' > k$), one guarantees that (5) is satisfied iff $u = 0$ ($\Rightarrow v = w = 0$). If we take $\delta = 256/B(k)$, then $B_\delta(0) \cap SC_u^k \cap C^*(3,0) = \{0\}$, as wanted. □

LEMMA 2:

Let k be fixed $\zeta = \zeta(k)$ s.t. $B_\delta(0) \cap SC_v^k \cap C^*(3,0) = \{0\}$

[Note: This says that the intersection of the white 'solid' cone with the swallowtail is locally $\emptyset$.]

Proof

Analogously, one gets $k^3v^3\ (A + |v|B)$ with $|A| \geq 256$ and $B = B(k)$ $(B(k') < B(k)$ if $k' > k)$. Choosing $\zeta = 256/B(k)$, one again gets $B_\zeta(0) \cap C_v^k \cap C^*(3,0) = \{0\}$, $\forall\, k' > k$, therefore $B_\zeta(0) \cap SC_v^k \cap C^*(3,0) = \{0\}$. □

LEMMA 3:

With the same hypothesis as those in Proposition 39, $\exists\ \varepsilon > 0$, $k \in \mathbb{R}^+$ such that $\{\alpha(t)|\ |t| < \varepsilon,\ t \neq 0\} \subset [B_\eta(0) \cap SC^k - \{0\}]$,

where $\eta = \min\limits_{\text{above}} \{\delta,\zeta\}$ [Therefore $B_\eta(0) \cap SC^k \cap C^*(3,0) = \{0\}$]

Proof

Let $\alpha_w' = C(\neq 0!)$; $\alpha_u'(0) = A$, $\alpha_v'(0) = B$. For small t, $\alpha_w'(t) \geq C/2$, $\alpha_u'(t) \leq 2A$ (or $\delta_A > 0$, if $A = 0$), $\alpha_v'(t) \leq 2B$ (or $\delta_B > 0$, if $B = 0$). For $\bar{t}$ fixed, $|\alpha_u(\bar{t})| = |\int_0^{\bar{t}} \alpha_u'(t)dt| \leq |2A\,\bar{t}|$; analogously, $|\alpha_v(\bar{t})| < |2B\bar{t}|$ and $|\alpha_w(\bar{t})| \geq |c/2|\,.\,|\bar{t}|$, i.e. $|\alpha_w(\bar{t})| \geq [\frac{1}{4}\frac{|c|}{\sqrt{A^2+B^2}}]\ (\alpha_u^2(\bar{t}) + \alpha_v^2(\bar{t}))^{\frac{1}{2}}$. Taking $k = \frac{1}{4}\ \frac{|c|}{A^2+B^2}$, we have that, for small t, say $|t| < \varepsilon_1$, one has $|\alpha_w(t)| \geq k\ (\alpha_u^2(t) + \alpha_v^2(t))^{\frac{1}{2}}$, therefore $(\alpha_u(t);\alpha_v(t);\alpha_w(t)) \in SC^k$; also $\alpha(t) \in B_\eta(0)$, t small ($|t| < \varepsilon_2$, say), hence $\{\alpha(t)|\ |t| < \varepsilon,\ \varepsilon = \min\{\varepsilon_1,\varepsilon_2\}, t \neq 0\} \subset$ $\subset (B_\eta(0) \cap SC^k)$. Also $\alpha(t) \neq 0$, $|t| < \varepsilon$, $t \neq 0$, as a consequence of $d\alpha_w/dt(0) \neq 0$: ε may be taken so small as to satisfy $\alpha(t) \neq 0$, $\forall t$ such that $|t| < \varepsilon$, $t \neq 0$. □

LEMMAS$(1 + 2 + 3) \Rightarrow$ PROPOSITION 39:

Choose ε as in Lemma 3. $\{\alpha(t)|\ \underset{t\neq 0}{|t|} < \varepsilon\} \subset B_\eta(0) \cap SC^k - \{0\}$. If $\alpha(t)(|t| < \varepsilon,\ t \neq 0) \in C^*(3,0)$ then $\alpha(t) \in [[C^*(3,0) \cap B_\eta(0) \cap SC^k] - \{0\}]$ (with $\alpha(t) \neq 0$ - see above) contradicting Lemma 1 or Lemma 2. Hence $\alpha(t) \notin C^*(3,0)$, therefore $\alpha(t) \notin C(3,0)$, $\forall\ t \left\{\begin{array}{l}|t| < \varepsilon.\\ t\neq 0\end{array}\right.$

PROPOSITION 40:

$v \in \mathcal{B}$ (as in Proposition 36) $\Rightarrow v \pitchfork C_f$.

Proof

As in Proposition 23, we have to show that, for fixed (arbitrarily) $y \in C_f$, $v \pitchfork_y C_f$, and this reduces (see 4.4(20)) to proving that $v \pitchfork_y \ \chi_f(U_{i_s}^{j_s} \cap M^d)$ in a number of separate cases, i.e.: $i_s = 1,2$ or 3.

Case 1: $\boxed{i_s = 1}$ This is like case 1 of Proposition 23: $\chi_f\ (U_1^{j_s} \cap M^d) = N_1^{j_s}$ and $v[3](\mathbb{R}^3) \cap C_1^{j_s}[3] = \emptyset \Rightarrow v \pitchfork_y N_1^{j_s}$.

Case 2: $\boxed{i_s = 2}$ Let Γ,γ as usual. Since $\Gamma^{-1}(\chi_f(U_2^{j_s} \cap M^d)) = \chi_{g=\gamma f}(\gamma(U_2^{j_s} \cap M^d)) \subset$

$\subset C(2,1)$, one has that:

$$\boxed{\varepsilon_s > 0 \text{ is s.t. } \Gamma^{-1}(0_y(\varepsilon_s)) \cap C(2,1) = \emptyset}^{\oplus} \Rightarrow \boxed{0_y(\varepsilon_s) \cap \chi_f(U_2^{j_s} \cap M^d) = \emptyset}^{\oplus}$$

i.e., $v \pitchfork_y (\chi_f(U_2^{j_s} \cap M^d))$. Hence, it suffices to prove $\oplus$. Set $\alpha = \Gamma^{-1}\beta$, where $\beta: I \to \mathbb{R}^3$ is a solution curve of v through y; $v[3](\mathbb{R}^3) \cap C_2^{j_s}[3] = \emptyset$ where $C_2^{j_s}[3] = T^3\Gamma\tilde{I}(Q_2[3])$, means $\tilde{I}(\hat{\alpha}) \overset{\boxtimes}{\notin} Q_2[3]$, since $v[3](y) \notin C_2^{j_s}[3]$. Therefore, since $\xi = \alpha(0) = \Gamma^{-1}(\beta(0)) = \Gamma^{-1}(y)$ satisfies $\xi_u = \xi_v = 0$ and, by $\boxtimes$, $(\xi_u;\xi_v;\frac{d\alpha}{dt}u(0);\frac{d\alpha}{dt}v(0)) \neq (0;0;0;0)$, one has $(d\alpha_u/dt(0); d\alpha_v/dt(0)) \neq 0$; hence, by Proposition 38, $\exists\, \varepsilon > 0$ s.t. $\{\alpha(t) \mid |t| < \varepsilon,\ t \neq 0\} \cap C(2,1) = \emptyset$, which is $\oplus$.

Case 3 : $\boxed{i_s = 3}$ Γ,γ as usual. As above, one has to prove only that

$\boxed{\exists \varepsilon_s > 0 \text{ s.t. } \Gamma^{-1}(0_y(\varepsilon_s)) \cap C(3,0) = \emptyset}$ $\oplus$. Now $v[3](\mathbb{R}^3) \cap C_3^{j_s}[3] = \emptyset \Rightarrow$

$(\xi_u;\xi_v;\xi_w;\frac{d\alpha}{dt}w(0)) \neq (0,0,0,0)$, where $\beta: I \to \mathbb{R}^3$ is a curve through y,

$\alpha = \Gamma^{-1}\beta$, $\xi = \alpha(0)$. Since $\xi = \Gamma^{-1}(y) = (0,0,0)$, $\frac{d\alpha}{dt}w(0) \neq 0$ therefore by Proposition 39 one gets $\oplus$. $\square$

COROLLARY:

If $f: X \times \mathbb{R}^3 \to \mathbb{R}$ is generic, $\exists$ open and dense $\mathcal{B}$ s.t. $v \in \mathcal{B} \Longrightarrow v \pitchfork C_f$.

4.4.4: The case $r = 4$:

Let $\mathcal{E} = \mathcal{E}_1 \cup \ldots \cup \mathcal{E}_4$ (see 4.3(7)), N_i^j, M_i^j, U_i^j as before

A. Definition of $C[4]$

DEFINITION 12:

Define, for fixed j:

$$C_1^j[4] = T^4(N_1^j) \subset T^4(\mathbb{R}^4)$$

$$C_2^j[4] = T^4\Gamma.\tilde{I}^{-1}.(Q_2[4]) \ (\Gamma,\gamma \text{ corresponding to } (j,2)),$$

$$Q_2[4] = \{(x_1,\ldots,x_{20}) \in \mathbb{R}^{20} | x_1 = x_2 = x_5 = x_6 = x_{10} = 0\}.$$

$$C_3^j[4] = T^4\Gamma.\tilde{I}^{-1}.(Q_3[4]) \ (\Gamma,\gamma \text{ corresponding to } (j,3)),$$

$$Q_3[4] = \{(x_1,\ldots,x_{20}) \in \mathbb{R}^{20} | x_1 = x_2 = x_3 = x_6 = x_7 = 0\}.$$

$$C_4^j[4] = T^4\Gamma.\tilde{I}^{-1}.(Q_4[4]) \ (\Gamma,\gamma \to (j,4)),$$

$$Q_4[4] = \{(x_1,\ldots,x_{20}) \in \mathbb{R}^{20} | x_1 = x_2 = x_3 = x_4 = x_8 = 0\}.$$

$$C_i[4] = \bigcup_{j \in N} C_i^j[4] \ (i = 1,\ldots,4).$$

$$C[4] = \bigcup_{i=1}^{4} C_i[4]$$

We prove below that these definitions are independentof the choice of Γ,γ.

PROPOSITION 41:

Let $\psi: \mathbb{R}^4 \circlearrowleft$ be a diffeomorphism (a germ of), leaving $C_i(2,2)$ $(i=1,2)$ invariant. Then $T^4\psi$ leaves $\tilde{I}^{-1}(Q_2[4])$ invariant.

Proof

Let $\hat{\alpha} \in \tilde{I}^{-1}(Q_2[4], \alpha(0) = \xi = (\xi_u, \xi_v, \xi_w, \xi_z)$, $\xi_u = \xi_v = 0$, $\frac{d\alpha}{dt}u(0) =$

$= \frac{d\alpha}{dt}v(0) = \frac{d^2\alpha}{dt^2}v(0) = 0$. Now $\tilde{I}(T^4\psi(\hat{\alpha})) = (\underbrace{\psi_u(\xi)}_{\bullet}; \underbrace{\psi_v(\xi)}_{\bullet}; \psi_w(\xi); \psi_z(\xi);$

$\underbrace{\frac{d(\psi\alpha)}{dt}u(0)}_{\bullet}; \underbrace{\frac{d(\psi\alpha)}{dt}v(0)}_{\bullet};.;.; \underbrace{\frac{d^2(\psi\alpha)}{dt^2}v(0)}_{\bullet};$ etc..). We would like to show that

the expressions marked with a dot are 0. By invariance of $C_2(2,2)$ one immediately gets $\psi_u(\xi) = \psi_v(\xi) = 0$.

The rest of the proposition follows from:

Claim:

$$\text{Let } P = \tilde{I}(\hat{\alpha}) = (\xi_u, \dots, \xi_z; \frac{d\alpha}{dt}u(0);..; \frac{d\alpha}{dt}z(0); \overset{k_1}{\boxed{\frac{d^2\alpha}{dt^2}u(0)}};\dots;\frac{d^2\alpha}{dt^2}z(0); \overset{k_2}{\boxed{\frac{d^3\alpha}{dt^3}u(0)}},$$

$$k_3 \boxed{\frac{d^3\alpha}{dt^3}v(0)};\dots;\frac{d^3\alpha}{dt^3}z(0)) \in \mathbb{R}^{16}, \hat{\alpha} \text{ as above.}$$

Then, $\exists$ a sequence $\widehat{\alpha^n}^3$, $\xi^n = \alpha^n(0)$ [where the symbol '3' in $\widehat{\alpha^n}^3$ denotes that the equivalence relation is $\sim_3$] such that:

(i) $P_n = \tilde{I}(\widehat{\alpha^n}^3) \to P$ as $n \to \infty$.

(ii) $\forall$ n fixed, $\exists$ representative α^n of $\widehat{\alpha^n}^3$ s.t. $\alpha^n(I) \subset C_1(2,2)$.

(hence $\xi_n \in C_1(2,2)$; a condition like (i) in Proposition 32 is easily met - see construction below - but we will leave this implicit for simplicity's sake).

To prove this claim, set $\alpha^n(t) = (\alpha^n_u(t); \alpha^n_v(t); \alpha^n_w(t); \alpha^n_z(t))$,

$\xi^n = \alpha^n(0)$ and define:

$$\text{(I)} \quad \left[\begin{aligned} \alpha^n_w(t) &= \xi_w + \frac{d\alpha}{dt}w(0)t + \frac{1}{2!}\frac{d^2\alpha}{dt^2}w(0)t^2 + \frac{1}{3!}\frac{d^3\alpha}{dt^3}w(0)t^3 \\ \alpha^n_z(t) &= \xi_z + \frac{d\alpha}{dt}z(0)t + \frac{1}{2!}\frac{d^2\alpha}{dt^2}z(0)t^2 + \frac{1}{3!}\frac{d^3\alpha}{dt^3}w(0)t^3. \end{aligned}\right\} \forall n \in \mathbb{N}$$

and

$$\text{(II)}\quad \left[\begin{array}{l} \alpha_u^n(t) = -\ 3b_n^2(t) \\ \alpha_v^n(t) = \ \ 2b_n^3(t) \end{array}\right\} \ \forall\, n \in \mathbb{N},$$ where $b_n(t) = b_n(0) + b_n'(0)\ t + \frac{1}{2!}\, b_n''(0)\ t^2 + \frac{1}{3!}\, b_n'''(0) t^3$, and $b_n(0), b_n'(0), b_n''(0), b_n'''(0)$ are defined below.

Set $b_n(0) \overset{\text{by def.}}{=\!=\!=} 1/n$, $\forall$ n. One then chooses, for every n arbitrarily fixed, $b_n'(0)$, $b_n''(0)$ and $b_n'''(0)$ s.t. (dropping the 0's):

$$\frac{d^2\alpha_u^n}{dt^2}(0) = -6\ (b_n b_n'' + (b_n')^2) = k_1 \qquad (1)$$

$$\frac{d^3\alpha_u^n}{dt^3}(0) = -6\ (b_n b_n''' + 3b_n' b_n'') = k_2 \qquad (2)$$

and

$$\frac{d^3\alpha_v^n}{dt^3}(0) = \ 6\ (2(b_n')^3 + b_n^2 b_n''' + 6b_n b_n' b_n'') = k_3 \quad (3)$$

This is done in the following way: choose b_n' to be a real root of the equation: $6(b_n')^3 + 3b_n' k_1 + (k_2/n + k_3) = 0$, and set $b_n'' = -n/6\ (k_1 + 6(b_n')^2)$, $b_n''' = -n/6\ (k_2 - 3nk_1 b_n' - 18(b_n')^3 n)$.

It is easy to check that with this choice (1), (2) and (3) are verified (by substitituion).

By definition, $(\alpha_u^n(t), \alpha_v^n(t))$ satisfy $(\alpha_u^n; \alpha_v^n)(I) \overset{\forall n}{\subset} C_1(2,0)$, therefore $\alpha^n(I) = (\alpha_u^n, \alpha_v^n, \alpha_w^n, \alpha_z^n)(I) \underset{\forall n}{\subset} C_1(2,2)$, since $C_1(2,2) = C_1(2,0) \times \mathbb{R}^2$.

Also (4) $\left[\begin{array}{l} \xi_u^n, \xi_v^n \to 0 \\ \dfrac{d(\alpha_u^n)}{dt}(0) = -6b_n b_n' \to 0 \\ \text{and } \dfrac{d(\alpha_v^n)}{dt}(0) = -6b_n^2 b_n' \to 0 \end{array}\right.$ $\Bigg\}$ as $n \to \infty$ (see 4.4(32)), since $b_n = 1/n \to 0$ as $n \to \infty$.

Finally, one can check, by computation, that:

$\dfrac{d^2(\alpha_v^n)}{dt^2}(0) = b_n \left(- \dfrac{d^2(\alpha_u^n)}{dt^2}(0) + 6(b_n')^2\right)$. Since $|b_n'|$ is limited. (see 4.4(32)), $\dfrac{d^2(\alpha_u^n(0))}{dt^2} = k_1$, $\forall$ n, and $b_n \to 0$ as $n \to \infty$, one has:

$$(5) \left[\frac{d^2(\alpha_v^n)}{dt^2}(0) \to 0 \text{ as } n \to \infty. \right.$$

(1),...,(5) and Definitions (I) and (II) imply immediately that $P_n \to P$ as $n \to \infty$. This proves the claim.

Now consider the sequence $\{\widehat{\psi\alpha^{n^3}}\}_{n\in\mathbb{N}}$. Since ψ is a (C^∞) diffeomorphism and $\tilde{I}$ is continuous, $\tilde{I}(\widehat{\psi\alpha^{n^3}}) \to \tilde{I}(\widehat{\psi\alpha^3})$ as $n \to \infty$, since $(\widehat{\alpha^{n^3}}) = \tilde{I}^{-1}(P_n) \to \tilde{I}^{-1}(P) = (\hat{\alpha}^3)$ as $n \to \infty$. Recall that

$\tilde{I}(\widehat{\psi\alpha^3}) = (\psi_u(\xi),\ldots,\psi_z(\xi); \dfrac{d(\psi\alpha)}{dt}u(0);\ldots; \dfrac{d^2(\psi\alpha)}{dt}u(0);\ldots; \dfrac{d^3(\psi\alpha)}{dt^3}u(0),\ldots)$

In the same way as in Proposition 32 and (5) above, it follows that

$\dfrac{d(\psi\alpha)}{dt}u(0) = \dfrac{d(\psi\alpha)}{dt}v(0) = \dfrac{d^2(\psi\alpha)}{dt}v(0) = 0$, as wanted, since $\psi\alpha^n(I) \subset C_1(2,2)$, because ψ leaves $C_1(2,2)$ invariant.

$\square$

PROPOSITION 42:

Let $\psi:\mathbb{R}^4 \circlearrowleft$ a (germ of) a diffeomorphism, leaving $C_i(3,1)$, $i = 1,2,3$, invariant. Then $T^4\psi$ leaves $\tilde{I}^{-1}(Q_3[4])$ invariant.

Proof

This is very similar to the situation we had in Proposition 25. The difference here is that the main argument exploits now the invariance (under ψ) of the cod. 2 strata, $C_2(3,1)$ - there the invariance of $C_1(3,0)$ was behind the main line of the proof.

Similarly to what was said in Note: (4.4(24)), the proof follows from the fact that ψ leaves the cod.2 strata, $C_2(3,1)$, invariant and that, if $\{\xi_n\} \to \xi \in C_3(3,1)$ is a sequence with $\xi_n \in C_2(3,1)$, then

$T_{\xi_n}(C_2(3,1))" \underset{\theta}{\to} "\{(\alpha,0,0,\beta)|\alpha,\ \beta \in \mathbb{R}\}$. (i.e., the $(u \times z)$ plane), as

$\xi_n \to \xi$. The rest of the proposition is trivial, following immediately from the invariance of the strata of higher (cod.3) codimension, $C_3(3,1)$. (see third line of proof of Proposition 25).

The technical details of the 'reduction to absurd proof' are very similar to those as in Proposition 25, so that we just verify θ. (Note: θ is correspondent, in Proposition 25, just to the fact that:
$T_{\xi_n}(C_1(3,0)) = \{(\alpha;-2\alpha c_n + \beta;\alpha^2 c_n^2 - \beta c_n)|\alpha,\ \beta \in \mathbb{R}\}" \to "\{(\alpha,\beta,0)|\alpha,\ \beta \in \mathbb{R}\}$,

as $n \to \infty$; the contradictions obtained there, in the reduction to absurd proof, are a direct result of this).

To work out what $T_{\xi_n}(C_2(3,1))$ is, we again refer to χ, corresponding to the swallowtail. $C_2(3,1) = (\chi \times I)\ \{(a,b,c,d)|a = b = 0\}$ where $\chi(0,0,c) = (-6c^2;8c^3;-3c^4)$, therefore $\chi \times I(0,0,c,d) = (-6c^2,8c^3;-3c^4;d)$.
By computation, one therefore gets: $T_{\xi_n}(C_2(3,1)) = \{(\alpha;-2\alpha c_n;\alpha c_n^2;\beta)|\alpha\ ,\ \beta \in \mathbb{R}$,
where, for each ξ_n, one chooses (c_n,d_n)s.t.$(\chi \times I)\ (0,0,c_n,d_n) = \xi_n$.

(Note: $c_n \neq 0, \forall\, n \in \mathbb{N}$.) Since $\xi_n \to \xi = (0,0,0,*)$ (since $\xi \in C_3(3,1)$) as $n \to \infty$, one has $(-6c_n^2) \to 0$, therefore $c_n \to 0$ as $n \to \infty$. Hence, $T_{\xi_n}(C_2(3,1))$ "$\to$" $\{(\alpha,0,0,\beta) | \alpha, \beta \in \mathbb{R}\}$, as wanted.

The conclusion is, therefore (similarly to Proposition 25), that $T_\xi\psi$ leaves $\{(\alpha,0,0,\beta) | \alpha, \beta \in \mathbb{R}\}$ invariant; this, together with the fact that ψ leaves $C_3(3,1)$ invariant, proves our proposition. □

PROPOSITION 43:

Let $\psi: \mathbb{R}^4 \circlearrowleft$, as in Proposition 42, leaving $C_i(4,0)$ invariant, $i = 1,2,3,4$. Then $T^4\psi$ leaves $\tilde{I}^{-1}(Q_4[4])$ invariant.

Proof

Idea is, as it was in Proposition 42, similar to that in Proposition 25. Part of the proof follows trivially from the fact that ψ preserves $C_4(4,0)$ (see Proposition 14, 4.4.(9)). The other part consists of a reduction to absurd argument, as in Proposition 25 and the details of which we will not write down explicitly, which depends (and follows immediately from) on the fact that $T_{\xi_n}(C_1(4,0)$ "$\to$" $\{(\beta,\alpha,\gamma,0) | \alpha,\beta, \gamma \in \mathbb{R}\}$, where $\{\xi_n\}$ is a sequence in $\mathbb{R}^4$, $\xi_n \in C_1(4,0), \forall n$, and $\xi_n \to (0,0,0,0)$ as $n \to \infty$.

To work out $T_{\xi_n}(C_1(4,0))$, one refers to χ, corresponding to the butterfly (see 4.2(4)). $C_1(4,0) = \chi(\{(a,b,c,d) | a = 0\})$, where χ:

$$(0,b,c,d) \to (\underbrace{4c-10d^2}_{u}; \underbrace{3b-12cd+20d^3}_{v}; \underbrace{12cd^2-6bd-15d^4}_{w}; \underbrace{3bd^2-4cd^3+4d^5}_{z}).$$

By computation, one gets $T_{\xi_n}(C_1(4,0)) = \{(\beta;\alpha-3\beta d_n;-2\alpha d_n+3\beta d_n^2 + \gamma;\alpha d_n^2 - \beta d_n^3 - \gamma d_n) | \alpha,\beta, \gamma \in \mathbb{R}\}$, where one chooses (b_n, c_n, d_n) (n fixed) s.t $\chi(0, b_n, c_n, d_n) = \xi_n$. One can show (see note below) that $\xi_n \to (0,0,0,0) \Rightarrow d_n \to 0$, therefore $T_{\xi_n}(C_1(4,0)) \to \{(\beta,\alpha,\gamma,0) | \alpha,\beta,\gamma \in \mathbb{R}\}$.

The conclusion is that $T_\xi\psi$ leaves $\{(\beta,\alpha,\gamma,0)|\alpha,\beta,\gamma \in \mathbb{R}\}$ invariant. This, together with the invariance of $C_4(4,0)$ (= $\{(0,0,0,0)\}$) under ψ, proves the proposition.

Note: Suppose $\xi_n \to (0,0,0,0)$, (b_n, c_n, d_n) as above. By computation, one has $w_n = -d_n(d_n(3u_n + 5d_n^2) + 2v_n)$, where v_n and $u_n \to 0$ as $n \to \infty$. An easy reduction to absurd argument shows that $d_n \not\to 0$ is impossible.

$\Box$

The following three propositions follow (in the same way as Propositions 26 and 27 followed from Propositions 24 and 25 + Remark 7 + arguments as in Proposition 15) from Propositions 41, 42 and 43, respectively:

PROPOSITION 44:

The definition of $C_2^j[4]$ is independent of choice of, Γ,γ.

PROPOSITION 45:

The definition of $C_3^j[4]$ is independent of choice of Γ, γ.

PROPOSITION 46:

The definition of $C_4^j[4]$ is independent of choice of Γ, γ.

B. Closedness of C[4]

DEFINITION 13:

We define the total fourth bundle associated with (i,j), $TC_i^j[4]$

$$TC_1^j[4] = C_1^j[4]$$

$$TC_2^j[4] = C_2^j[4] \cup \Big(\bigcup_{m \in U_2^j \cap M_1^d} C_{2,1}^j(m)[4]\Big), \text{ where}$$

$$C_{2,1}^j(m)[4] = \{\hat{\beta} \in C_1^{j_o}[4] \mid \beta(0) = y = \chi_f(m)\},$$

where j_o is chosen so that $m \in U_1^{j_o}$.

$$TC_3^j[4] = C_3^j[4] \cup \Big(\bigcup_{m \in U_3^j \cap M_2^d} C_{3,2}^j(m)[4]\Big) \cup \Big(\bigcup_{m \in U_3^j \cap M_1^d} C_{3,1}^j(m)[4]\Big), \text{ where}$$

$$C_{3,1}^j(m)[4] = \{\hat{\beta} \in C_1^{j_o}[4] \mid \beta(0) = y = \chi_f(m)\},$$

where j_o is chosen so that $m \in U_1^{j_o}$.

$$C_{3,2}^j(m)[4] = \{\hat{\beta} \in C_2^{j_o}[4] \mid \beta(0) = y = \chi_f(m)\},$$

where j_o is chosen so that $m \in U_2^{j_o}$.

$$TC_4^j[4] = C_4^j[4] \cup \Big(\bigcup_{m \in U_4^j \cap M_3^d} C_{4,3}^j(m)[4]\Big) \cup \Big(\bigcup_{m \in U_4^j \cap M_2^d} C_{4,2}^j(m)[4]\Big) \cup$$

$$\cup \Big(\bigcup_{m \in U_4^j \cap M_1^d} C_{4,1}^j(m)[4]\Big), \text{ where}$$

$$C_{4,1}^j(m)[4] = \{\hat{\beta} \in C_1^{j_o}[4] \mid \beta(0) = y = \chi_f(m)\},$$

where j_o is chosen so that $m \in U_1^{j_o}$.

$$C_{4,2}^j(m)[4] = \{\hat{\beta} \in C_2^{j_o}[4] \mid \beta(0) = y = \chi_f(m)\},$$

where j_o is chosen so that $m \in U_2^{j_o}$.

$$C_{4,3}^j(m)[4] = \{\hat{\beta} \in C_3^{j_o}[4] \mid \beta(0) = y = \chi_f(m)\},$$

where j_o is chosen so that $m \in U_3^{j_o}$.

PROPOSITION 47:

The definition of $C^j_{2,1}(m)[4]$ independs of the choice of j_o.

Proof

Identical to that of Proposition 28 (4.4(25)); just substitute 3 by 4 whenever necessary. □

PROPOSITION 48:

Definition of $C^j_{3,1}(m)[4]$ independs of the choice of j_o.

Proof

As above.

PROPOSITION 49:

Definition of $C^j_{4,1}(m)[4]$ independs of the choice of j_o.

Proof

As above.

PROPOSITION 50:

Definition of $C^j_{3,2}(m)[4]$ independs of the choice of j_o.

Proof

Let j_o, j_1 be s.t. $m \in \mathcal{U}_2^{j_o}$, $m \in \mathcal{U}_2^{j_1}$. Let

θ $\quad C^j_{3,2}(m)[4](j_o) = \{\hat\beta \in C_2^{j_o}[4] \mid \beta(0) = y = \chi_f(m)\}$,

j_o chosen so that $m \in \mathcal{U}_2^{j_o}$.

$\odot$ $\quad C^j_{3,2}(m)[4](j_1) = \{\hat\beta \in C_2^{j_1}[4] \mid \beta(0) = y = \chi_f(m)\}$,

j_1 chosen so that $m \in \mathcal{U}_2^{j_1}$.

Let Γ_o, γ_o; Γ_1, γ_1 be as usual, corresponding to $(j_o,2)$; $(j_1,2)$, respectively.

Let $\hat\beta \in \theta$. Therefore, $\hat\beta \in T^4\Gamma_o.\tilde{I}^{-1}(Q_2[4]) = T^4\Gamma_1(T^4(\Gamma_1^{-1}\Gamma_o)\tilde{I}^{-1}(Q_2[4])) =$ $\underset{\text{Prop.41}}{\overset{\text{by}}{=}}$ $T^4\Gamma_1(\tilde{I}^{-1}(Q_2[4]))$, therefore $\tilde\beta \in C_2^{j_1}[4]$, therefore $\hat\beta \in \odot$. $\odot \subset \theta$:analogous.

□

PROPOSITION 51:

Definition of $C^{j}_{4,2}(m)[4]$ independs of the choice of j_o.

Proof

As above (Proposition 50).

PROPOSITION 52:

Definition of $C^{j}_{4,3}(m)[4]$ independs of the choice of j_o.

Proof

Analogous to that of Proposition 50 above. Just substitute 2 by 3 everywhere, and use Proposition 42 instead of Proposition 41. □

PROPOSITION 53: $\boxed{\text{(Reducing GLOBAL to LOCAL)}}$

Suppose that $\hat{\beta}_n \in C[4]$, $y_n = \beta_n(0)$, $\forall\, n \in \mathbb{N}$, and $\{\hat{\beta}_n\}_{n\in\mathbb{N}} \to \hat{\beta} \in T^4(\mathbb{R}^4)$, $y = \ldots$

Then, $\exists\; i \in \{1,2,3,4\}$, $j \in \mathbb{N}$ and subsequence $\{\hat{\beta}_k\}_{k\in\mathbb{N}}$, $y_k = \beta_k(0)$, such that $\hat{\beta}_k \in TC^{j}_{i}[4]$, $\forall\; k \in \mathbb{N}$. Furthermore, $y \in \chi\,(\underbrace{U^{j}_{i} \cap M^{d}_{i}}_{= M^{j}_{i}})$.

Proof

Choose (i_n, j_n) s.t. $\hat{\beta}_n \in C^{j_n}_{i_n}[4]$, for each $n \in \mathbb{N}$; recall that $\chi_f/M^{j_n}_{i_n}: M^{j_n}_{i_n} \xrightarrow{\text{diff.}} N^{j_n}_{i_n} \ni y_n$. Set $m_n = (\chi_f/M^{j_n}_{i_n})^{-1}(y_n)$ In particular, $m_n \in M^{j_n}_{i_n}$.

Let $\chi_f^{-1}(y) = \{m_1,\ldots,m_p\}$, and choose (i_s, j_s), $s=1,\ldots,p$ s.t. $m_s \in U^{j_s}_{i_s}$, $s = 1,2,3$ or 4 according to whether $m_s \in M^{d}_{1,2,3 \text{ or } 4}$.

LEMMA:

Everything as above, $m_n \in U^{j_s}_{i_s} \Rightarrow \hat{\beta}_n \in TC^{j_s}_{i_s}[4]$

PROOF OF LEMMA:

<u>Case 1:</u> $\boxed{i_n = 4}$

$\hat{\beta}_n \in C_4^{j_n}[4]$. As in Proposition 17, one easily shows that $m_n = m_s$, $i_s = 4$. With the same arguments which lead to the proof of Proposition 46, one shows that $C_4^{j_n}[4] = C_4^{j_s}[4]$, therefore $\hat{\beta}_n \in C_4^{j_s}[4] \subset TC_4^{j_s}[4]$.

<u>Case 2:</u> $\boxed{i_n = 3}$

Cases $i_s = 1$ or 2 may be discarded (Remark 8 and ✱ above). 4.4.(55)

<u>Case 2.1</u>: $i_s = 3$

$\hat{\beta}_n \in C_3^{j_n}[4] = T^4\Gamma_n.\tilde{I}^{-1}.(Q_3[4])$ therefore $\widehat{\Gamma_n^{-1}\beta_n} \in \tilde{I}^{-1}(Q_3[4])$, therefore

$\hat{\beta}_n = T^4\Gamma_s(T^4(\Gamma_s^{-1}\Gamma_n)\ (\widehat{\Gamma_n^{-1}\beta_n}))$, [see <u>note</u> in 4.4(28)], where

$\widehat{\Gamma_n^{-1}\beta_n} \in \tilde{I}^{-1}(Q_3[4])$; hence by Proposition 42,

$\hat{\beta}_n \in T^4\Gamma_s(\tilde{I}^{-1}(Q_3[4])) = C_3^{j_s}[4] \subset T_3^{j_s}[4]$.

<u>Case 2.2</u> $i_s = 4$

$\hat{\beta}_n \in \{\hat{\beta} \in C_3^{j_n}[4] \mid \beta(0) = y_n = \chi_f(m_n)\} = C_{4,3}^{j_s}(m_n)[4] \subset TC_4^{j_s}[4]$.

[Note: $m_n \in U_{i_s=4}^{j_s} \cap M_3^d$, by the hypothesis of lemma, ✱ in 4.4(55), and hypothesis of case 2]. The equality above results by taking j_n as the j_o in Definition 13.

<u>Case 3:</u> $\boxed{i_n = 2}$

Case $i_s = 1$ may be discarded (Remark 8 + ✱, in 4.4(55)).

Case 3.1 $i_s = 2$

$\hat{\beta}_n \in C_2^{j_n}[4] = T^4\Gamma_n \tilde{I}^{-1}(Q_2[4])$. As in case 2.1 above, $\widehat{\Gamma_n^{-1}\beta_n} \in \tilde{I}^{-1}(Q_2[4])$, therefore, by Proposition 41, $\hat{\beta}_n \in T^4\Gamma_s\ (\tilde{I}^{-1}(Q_2[4])) = C_2^{j_s}[4] \subset TC_2^{j_s}[4]$.

Case 3.2: $i_s = 3$

$\hat{\beta}_n \in \ \{\hat{\beta} \in C_2^{j_n}[4] | \beta(0) = y_n = \chi_f(m_n)\} = C_{3,2}^{j_s}(m_n)[4] \subset TC_3^{j_s}[4]$. This equality results by setting j_n as the j_o in Definition 13. Note: $m_n \in \mathcal{U}_{i_s=3}^{j_s} \cap M_{i_n=2}^d$.

Case 3.3: $i_s = 4$

$\hat{\beta}_n \in \ \{\hat{\beta} \in C_2^{j_n}[4] | \beta(0) = y_n = \chi_f(m_n)\} = C_{4,2}^{j_s}(m_n)[4] \subset TC_4^{j_s}[4]$. Again set j_n as j_o, in Definition 13. Note: $m_n \in \mathcal{U}_{i_s=4}^{j_s} \cap M_{i_n=2}^d$.

Case 4: $\boxed{i_n = 1}$

Case 4.1: $i_s = 1$

As case 3.1 in 4.4(29); just change 3 by 4 everywhere.

Case 4.2: $i_s = 2$

Analogously as before, we get $\hat{\beta}_n \in C_{2,1}^{j_s}(m_n)[4] \subset TC_2^{j_s}[4]$.

Case 4.3: $i_s = 3$

Analogously, one gets $\hat{\beta}_n \in C_{3,1}^{j_s}(m_n)[4] \subset TC_3^{j_s}[4]$.

Case 4.4: $i_s = 4$

As above, it follows that $\hat{\beta}_n \in C_{4,1}^{j_s}(m_n)[4] \subset TC_4^{j_s}[4]$.

$\square$

LEMMA $\Longrightarrow$ PROPOSITION 53:

Equal to the proof that lemma to Proposition 17 $\Longrightarrow$ Proposition 17, substitute 2 by 4, whenever it appears. $\square$

PROPOSITION 54: $\boxed{\text{(CUSP'S BUNDLE CLOSES FOLD'S BUNDLE: STANDARD FORM)}}$

[Closedness at cusp's surface]

Let g denote the standard cusp $g_2:\mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$ with two disconnected controls [i.e.: $(u,v,x) \longmapsto x^2/2 + x^4/2 + ux^2/2 + vx$]

Let $\{\hat{\alpha}_n\}_{n\in\mathbb{N}}$, $\xi_n = \alpha_n(0)$, be a sequence in $T^4(\mathbb{R}^4)$, converging to a point , $\hat{\alpha} = \xi(0)$, with $\xi = (\xi_u;\xi_v;\xi_w;\xi_z)$, $\xi_u = \xi_v = 0$.

Suppose that, for each n fixed, $\exists\, M^n$, submanifold of M_1^d, such that

(i) $\chi_g/M^n : M^n \to N^n = \chi_g/M^n(M^n)$ is a diffeomorphism

(ii) $\xi_n \in N^n \subset C_1(2,2)$

(iii) $\exists$ representative $\alpha_n \in \hat{\alpha}_n$, s.t. $\alpha_n(I) \subset N^n$. Then

$$\frac{d\alpha}{dt}u(0) = \frac{d\alpha}{dt}v(0) = \frac{d^2\alpha}{dt^2}v(0) = 0.$$

In precisely the same way as done in Proposition 32 - with the only difference that we now have two disconnected controls - we can write:

$\alpha_n(t) = \chi_{n,2}(0;b_n(t);c_n(t);d_n(t)) = (-3b_n^2(t);2b_n^3(t);c_n(t);d_n(t))$, where

$\chi_{n,2} = \chi_2/(\theta/M \times I_{\mathbb{R}^2})(M^n)$, where χ_2 is defined by the diagram.

$$\begin{tikzcd} & M_g = M_{g_2} \times \mathbb{R}^2 \arrow[dl, "\theta/M\times I_{\mathbb{R}^2}"'] \arrow[dr, "\chi_g"] & \\ \mathbb{R}^4 \arrow[rr, "\chi_2"'] & & \mathbb{R}^4 \end{tikzcd}$$

Therefore, omitting the 0's, as before (see Proposition 32), we have:

$\tilde{I}(\widehat{\alpha_n^4}) = (-3b_n^2;2b_n^3;c_n;d_n;\ -6b_nb_n';6b_n^2b_n';c_n';d_n';\ -6(b_nb_n''+b_n^2);$

$6(2b_n+(b_n')^2+b_n^2b_n'');c_n'';d_n''$; "$3^{rd}$ and 4^{th} order" coordinates) $\in \mathbb{R}^{20}$,

where, like in Proposition 32, $b_n(t)$, $c_n(t)$, $d_n(t)$ are defined by $\chi_{n,2}^{-1}(\alpha_n(t)) =$ $= (a_n(t), b_n(t), c_n(t), d_n(t))$. Hence, since $a_n(t) = 0$

$(-3b_n^2;2b_n^3;\ c_n;\ d_n) = \alpha_n(0) = (\xi_u;\xi_v;\xi_w;\xi_z)$, we get $-3b_n^2 \to 0$ and $2b_n^3 \to 0$ as $n \to \infty$.

We want therefore to prove:

$$\text{(I)}\ \boxed{\begin{array}{l} -3b_n^2 \to 0 \\ 2b_n^3 \to 0 \end{array}} \Longrightarrow \boxed{\begin{array}{llll} \text{(a)} & \frac{d\alpha}{dt}u(0) & = & \lim_{n\to\infty}(-6b_n'b_n) = 0 \\ \text{(b)} & \frac{d\alpha}{dt}v(0) & = & \lim_{n\to\infty}(6b_n'b_n^2) = 0 \\ \text{(c)} & \frac{d^2\alpha}{dt^2}v(0) & = & \lim_{n\to\infty}(2b_n+(b_n')^2+b_n^2b_n'') = 0 \end{array}}\ \text{(II)}$$

(II) (a) and (b) have already been proved in Proposition 32. It remains to prove (c).

By computation:

$$\frac{d^2(\alpha_n)}{dt^2}v(0) = -b_n(0)\,\frac{d^2(\alpha_n)}{dt^2}u(0) + 6b_n(0)\,(b_n'(0))^2.$$

Now $b_n(0) \to 0$ as $n \to \infty$, $\dfrac{d^2(\alpha_n)}{dt^2}u(0)$ tends to a constant as $n\to\infty$ and $(b_n'(0))^2$ is limited (proved in Proposition 32). Therefore,

$$\lim_{n\to\infty} \frac{d^2(\alpha_n)}{dt^2}v(0) = \frac{d^2\alpha}{dt^2}v(0) = 0, \text{ as wanted.}$$

PROPOSITION 55:

SWALLOWTAIL'S BUNDLE CLOSES CUSP'S BUNDLE: CANONICAL FORM

[CLOSEDNESS AT SWALLOW-TAIL'S LINE: CASE 1]

Leg g_3 denote the swallowtail, $g = g_3$ + one disconnected control, $\{\hat{\alpha}_n\}_{n\in\mathbb{N}}$, $\xi_n = \alpha_n(0)$, be a sequence in $T^4(\mathbb{R}^4)$, converging to $\hat{\alpha}$, $\xi = \alpha(0)$, $\xi_u = \xi_v = \xi_w = 0$. Suppose that for each n arbitrarily fixed, one has:

(i) $\xi_n \in C_2(3,1)$, (ii) $d\alpha_n/dt(0) \in T_{\xi_n}(C_2(3,1))$.

Then: $d\alpha_v/dt(0) = d\alpha_w/dt(0) = 0$.

Proof

One first computes $T_{\xi_n}(C_2(3,1))$, as it was done in Proposition 33. In order to do this, one considers the map $\chi \times I_{\mathbb{R}}$, where χ corresponds to the swallowtail (see 4.2(4)), $\chi \times I_{\mathbb{R}}(0,0,c,d) = (-6c^2; 8c^3; -3c^4; d)$. Choose c_n, d_n s.t. $\chi \times I_{\mathbb{R}}(0,0,c_n,d_n) = \xi_n$ (possible since $\xi_n \in C_2(3,1)$), for each n arbitrarily fixed. Now χ preserves 2-dimensional strata, i.e.

$$\chi \times I_{\mathbb{R}}\ (\{(a,b,c,d)\,|\,\underset{c\neq 0}{a = b = 0}\}) = C_2(3,1),$$ so that, by computation

$$T_{\xi_n}(C_2(3,1)) = T_{(0,0,c_n,d_n)}\chi\times I_{\mathbb{R}}(\{(a,b,c,d)\,|\,a=b=0\}) \underset{c_n\neq 0,\forall n}{=\!=}$$

$= \{(r;-2rc_n;rc_n^2;s)\,|\,r,s \in \mathbb{R}\}$, n fixed. Since $d\alpha_n/dt(0) \in T_{\xi_n}(C_2(3,1))$, $\forall$ n arbitrarily fixed, choose $r = r_n$, $s = s_n$ so that $d\alpha_n/dt(0) = (r_n;-2r_nc_n;r_nc_n^2;s_n)$. Since $\xi_n = (-6c_n^2;8c_n^3;-3c_n^4;d_n) \to (0,0,0,*)$ as $n \to \infty$, $c_n \to 0$ as $n \to \infty$.

Also $\dfrac{d(\alpha_n)}{dt}u(0) = r_n \to$ some constant as $n \to \infty$, therefore one gets

$$\begin{bmatrix} d(\alpha_n)_v/dt(0) = -2r_nc_n \to 0 \\ d(\alpha_n)_w/dt(0) = -r_nc_n^2 \to 0 \end{bmatrix}$$ as $n \to \infty$, as wanted, precisely as in Proposition 33.

□

PROPOSITION 56:

SWALLOW-TAIL'S BUNDLE CLOSES FOLD'S BUNDLE: STANDARD FORM

[~~Closedness~~ Closeness at swallow-tail's line : case 2]

Let g denote the standard swallowtail g_3 with one disconnected control (see 4.2(1)). i.e.: $g(x,u,v,w,z) = x^5/5 + ux^3/3 + vx^2/2 + wx$. Let $\{\hat{\alpha}_n\}_{n\in\mathbb{N}}$, $\xi_n = \alpha_n(0)$ be a sequence in $T^4(\mathbb{R}^4)$, converging to a point $\hat{\alpha}$, $\xi = \alpha(0)$, with $\xi_u = \xi_v = \xi_w =$ Suppose that, $\forall n$ arbitrarily fixed, $\exists$ M^n, a manifold of M_1^d, such that:

(i) $\chi_g/M^n : M^n \to N^n = \chi_g/M^n(M^n)$ is a diffeomorphism.

(ii) $\xi_n \in N^n \subset C_1(3,1)$.

(iii) $\exists$ representative α of $\hat{\alpha}_n$ s.t. $\alpha_n(I) \subset N^n$.

Then: $\frac{d\alpha}{dt}v(0) = \frac{d\alpha}{dt}w(0) = 0$.

Proof

Construct χ_1 by the diagram:

$$\begin{array}{ccc} M_{g_3} \times \mathbb{R} & = & M_g(\subset \mathbb{R}^4 \times \mathbb{R}) \\ \downarrow{\scriptstyle \theta/M\times I} & & \downarrow{\scriptstyle \chi_g} \\ \mathbb{R}^4 & \xrightarrow{\chi_1} & \mathbb{R}^4 \end{array},$$

where θ/M is again as outlined in [17], and corresponds to the swallowtail. As previously (see for instance 4.4(31)), set $\chi_{n,1} = \chi_1/_{(\theta/M\times I)(M^n)}$, and define $a_n(t)$, $b_n(t)$, $c_n(t)$ and $d_n(t)$ by $\chi_{n,1}^{-1}(\alpha_n(t)) = (a_n(t); b_n(t); c_n(t); d_n(t))$.

Again $a_n(t) \equiv 0$, by (iii), and one can write:

$$\alpha_n(t) = \chi_{n,1}(0;b_n(t);c_n(t);d_n(t)) = (3b_n(t)-6c_n^2(t);-6b_n(t)c_n(t)+8c_n^3(t);$$
$$3b_n(t)c_n^2(t) - 3c_n^4(t);d_n(t)).$$

Omitting the 0's from notation below, our problem is reduced to show that:

$$\text{(I)}\ \boxed{\begin{array}{l} 3b_n - 6c_n^2 \to 0 \\ -6b_nc_n + 8c_n^3 \to 0 \\ 3b_nc_n^2 - 3c_n^4 \to 0 \end{array}} \Longrightarrow \boxed{\begin{array}{l} \text{(a)}\ \dfrac{d(\alpha_n)}{dt}v(0) = (-6(b_nc_n' + b_n'c_n) + 24c_n^2c_n') \to 0 \\ \text{(b)}\ \dfrac{d(\alpha_n)}{dt}w(0) = (3(b_n'c_n^2 + 2b_nc_nc_n') - 12c_n^3c_n') \to 0 \end{array}}\ \text{(II)}$$

(I) $\Longrightarrow$ (II)(b) has been proved in Proposition 34. It remains to show that (I) $\Longrightarrow$ (II)(a). We have already shown (4.4(35)) that $c_n \to 0$ as $n \to \infty$ Since $3b_n - 6\underset{\downarrow\atop 0}{c_n^2} \to 0$, one also gets $b_n \to 0$ as $n \to \infty$, from (I).

By computation, one has:

$$\text{(c)}\ \frac{d(\alpha_n)}{dt}v(0) = -2c_n(0).\ \frac{d(\alpha_n)}{dt}u(0) - 6b_n(0)c_n'(0).$$

and

$$\text{(d)}\ \frac{d^2(\alpha_n)}{dt^2}w(0) = -c_n^2(0).\ \frac{d^2(\alpha_n)}{dt^2}u(0) - c_n(0)\ \frac{d^2(\alpha_n)}{dt^2}v(0) - 6b_n(0)(c_n'(0))^2.$$

Suppose we do not have $\lim\limits_{n\to\infty} \dfrac{d(\alpha_n)}{dt}v(0) = 0.$

Hence, $\exists$ $\varepsilon > 0$, and a subsequence such that $\left|\dfrac{d(\alpha_k)}{dt}v(0)\right| > \varepsilon$, $\forall$ $k \in \mathbb{N}$

From (d):

$$K = \lim_{n\to\infty} \Big(\underbrace{\frac{d^2(\alpha_k)}{dt^2}w(0)}_{\text{constant}} + \underbrace{c_k^2(0)}_{0}\ \underbrace{\frac{d^2(\alpha_k)}{dt^2}u(0)}_{\text{constant}} +$$

$$+ \underbrace{c_k(0)}_{0}\ \underbrace{\frac{d^2(\alpha_k)}{dt^2}v(0)}_{\text{constant}}\Big) = \lim_{n\to\infty}\ (-6b_k(0)c_k'(0).c_k'(0)) =$$

<u>(From (c))</u> $$\lim_{n\to\infty}\ \Big(\Big(\underbrace{\frac{d(\alpha_k)}{dt}v(0)}_{\substack{\text{modulus}\\ \text{greater than}}} + 2\underbrace{c_k(0)}_{0}\ \underbrace{\frac{d(\alpha_k)}{dt}u(0)}_{\text{constant}}\Big)\ c_k'(0)\Big)$$

Hence, for all k sufficiently big,

$$|c_k'(0)| < \frac{4K}{\varepsilon}$$

Again, by (c):

$$\frac{d(\alpha_k)}{dt}v(0) = -2\underset{\substack{\downarrow\\ 0}}{c_k(0)}\ \underset{\substack{\downarrow\\ \text{constant}}}{\frac{d(\alpha_k)}{dt}u(0)} - 6\underset{\substack{\downarrow\\ 0}}{b_k(0)}\ \underset{\substack{\downarrow\\ \text{limited}}}{c_k'(0)},$$

and therefore $\left|\frac{d(\alpha_k)}{dt}v(0)\right| < \varepsilon$, for all k sufficiently big, a contradition.

Hence (I)$\Longrightarrow$ (II)(a). □

PROPOSITION 57:

BUTTERFLY'S BUNDLE Closes SWALLOW-TAIL'S BUNDLE: CANONICAL FORM

[Closedness at Butterfly's point: case 1]

Let g_4 denote the standard butterfly (no disconnected controls), $\{\hat{\alpha}_n\}_{n\in\mathbb{N}}$, $\xi_n = \alpha_n(0)$, be a sequence in $T^4(\mathbb{R}^4)$, converging to $\hat{\alpha}$, $\xi = \alpha(0)$, $\xi = 0 \in \mathbb{R}^4$. For each fixed n, let χ be as defined in [17], corresponding to the butterfly, and let $\xi_n \in C_3(4,0)$, so that we can choose (uniquely) $(0,0,0,d_n \in \mathbb{R})$ s.t. $\chi(0,0,0,d_n) = \xi_n$. Suppose that:

$$\frac{d\alpha_n}{dt}(0) \in [\xi_n(1);\xi_n(2)],$$ where by this we mean the space generated by the vectors $\xi_n(1)$ and $\xi_n(2)$, with

$$\xi_n(1) = \begin{bmatrix} 1 \\ -3d_n \\ 3d_n^2 \\ -d_n^3 \end{bmatrix}, \quad \xi_n(2) = \begin{bmatrix} 0 \\ 1 \\ -2d_n \\ d_n^2 \end{bmatrix}$$

Then $\frac{d\alpha}{dt}z(0) = 0$.

Proof

Since $\odot$ is true, we can choose, for every fixed n, r_n, s_n st.

$\frac{d\alpha}{dt}n(0) = (r_n; s_n - 3r_nd_n; 3r_nd_n^2 - 2s_nd_n; s_nd_n^2 - r_nd_n^3)$. Since $\{(\hat{\alpha}_n)\}_{n\in\mathbb{N}}$ converges,

$$\lim_{n\to\infty} r_n = \frac{d\alpha}{dt}u(0); \quad \lim_{n\to\infty} (s_n - 3\underset{\substack{\downarrow \\ \text{fixed} \\ \text{limit}}}{r_n}\underset{\substack{\downarrow \\ 0}}{d_n}) = \frac{d\alpha}{dt}v(0), \quad \text{hence} \quad \lim_{n\to\infty} s_n = \frac{d\alpha}{dt}v(0)$$

$$\text{hence } \lim_{n\to\infty} \frac{d\alpha}{dt}z(0) = \lim_{n\to\infty} (\overset{\substack{\text{fixed} \\ \text{limit} \\ \uparrow}}{s_n}\underset{\substack{\downarrow \\ 0}}{d_n^2} - \overset{\substack{\text{fixed} \\ \text{limit} \\ \nearrow}}{r_n}\underset{\substack{\downarrow \\ 0}}{d_n^3}) = 0, \text{ as wanted.}$$

□

PROPOSITION 58:

$\boxed{\text{BUTTERFLY'S BUNDLE CLOSES CUSP'S BUNDLE : THE STANDARD FORM}}$

[Closedness at Butterfly's Point: Case 2]

Let g_4 denote the butterfly, and let $\{\hat{\alpha}_n\}_{n\in\mathbb{N}}$, $\xi_n = \alpha_n(0)$ be a sequence in $T^4(\mathbb{R}^4)$, converging to $\hat{\alpha}$, $\xi = \alpha(0)$, $\xi = 0$. Suppose that, for each n fixed,

(i) $\xi_n \in C_2(4,0)$ (ii) $d\alpha/dt(0) \in T_{\xi_n}(C_2(4,0))$.

Then: $\frac{d\alpha}{dt}z(0) = 0$

Proof

Let χ be the one corresponding to the butterfly (as in Proposition 57 above). Choose c_n, d_n s.t. $\chi(0,0,c_n,d_n) = \xi_n$, possible since $\xi_n \in C_2(4,0)$. Now, since

$$\chi(0,0,c_n,d_n) = (\underbrace{4c_n - 10d_n^2}_{(\xi_u)_n}; \underbrace{-12c_nd_n + 20d_n^3}_{(\xi_v)_n}; 12c_nd_n^2 - 15d_n^4; 4d_n^5 - 4c_nd_n^3)$$

tends to (0,0,0,0) as $n \to \infty$ and:

$(\xi_v)_n = -3d_n((\xi_u)_n + 10/3\, d_n^2)$, if $d_n \not\to 0$, one would get a subsequence $\{d_r\}$ s.t. $|d_r| > \varepsilon$, $\forall\, r \in \mathbb{N}$, therefore

$|(\xi_v)_r| = 3|d_r| \cdot |\underbrace{(\xi_u)_r}_{0} + 10/3\, d_r^2| > 3\varepsilon^3, \forall\, r$ sufficiently big, an absurd.

Hence, $d_n \to 0$ as $n \to \infty$. As $(4c_n - 10d_n^2) \to 0$ as $n \to \infty$, one also has $c_n \to 0$ as $n \to \infty$.

We work out $T_{\xi_n} = \chi(0,0,c_n,d_n)(C_2(4,0)) = T_{(0,0,c_n,d_n)}\, \chi(\{(a,b,c,d)\ a=b=0$

$\underset{c_n \neq 0, \forall\, n}{=\!=\!=} [\xi_n(1);\xi_n(2)]$, where $\xi_n(1) = \begin{bmatrix} 1 \\ -3d_n \\ 3d_n^2 \\ -d_n^3 \end{bmatrix}$ and $\xi_n(2) = \begin{bmatrix} 0 \\ 1 \\ -2d_n \\ d_n^2 \end{bmatrix}$.

From (ii) above one therefore has, $\forall$ n, fixed, $d\alpha_n/dt(0) =$

$= (r_n; -3r_nd_n + s_n;\ 3r_nd_n^2 - 2s_nd_n;\ -r_nd_n^3 + s_nd_n^2)$, where $s_n, r_n \in \mathbb{R}$. By the convergence in the hypotheses, $\lim_{n\to\infty} r_n = \frac{d\alpha}{dt}u(0)$, and

$\lim_{n\to\infty} (-3\underbrace{r_n}_{\text{fixed limit}} \overset{0}{d_n} + s_n) = \lim_{n\to\infty} s_n = \frac{d\alpha}{dt}v(0)$, therefore $\lim_{n\to\infty}(-r_nd_n^3 + s_nd_n^2) = \frac{d\alpha}{dt}z(0) = 0$ □

PROPOSITION 59: **BUTTERFLY'S BUNDLE CLOSES FOLD'S BUNDLE: STANDARD FORM** [Closedness at Butterfly Point: Case 3]

Let g_4 denote the standard butterfly, $\{\hat{\alpha}_n\}_{n\in\mathbb{N}}$, $\xi_n = \alpha_n(0)$, be a sequence in $T^4(\mathbb{R}^4)$ converging to $\hat{\alpha}$, $\xi = \alpha(0)$, $\xi = 0$. Suppose that, $\forall$ n fixed, $\exists\, M^n$, submanifold of M^d, such that:

(i) $\chi_{g_4}/M^n : M^n \to N^n = \chi_{g_4}/M^n(M^n)$ is a diffeomorphism.

(ii) $\xi_n \in N^r \subset C_1(4,0)$

(iii) $\exists$ representative α_n s.t. $\alpha_n(I) \subset N^n$. Then, $d\alpha_z/dt(0) = 0$.

Proof (of Proposition 59):

It is very similar to that of Proposition 34. One sets $\chi_n = \chi/_{\theta/M(M^n)}$, where χ corresponds to the butterfly, $(a_n(t);b_n(t);c_n(t);d_n(t)) = \chi_n^{-1}\ (\alpha_n(t))$; $a_n(t) \equiv 0$, expressing $\alpha_n(t)$ as:

$$\alpha_n(t) = \chi_n(0;b_n(t);c_n(t);d_n(t)) = (4c_n(t) - 10d_n^2(t);\ 3b_n(t) - 12c_n(t)d_n(t) + 20d_n^3(t);12c_n(t)d_n^2(t) - 6b_n(t)d_n(t)-15d_n^4(t);4d_n^5(t)-4c_n(t)d_n^3(t) + 3b_n(t)d_n^2(t)).$$

Therefore the proof of proposition reduces to the proof of:

$$\boxed{\begin{array}{c} 4c_n - 10d_n^2 \to 0 \\ 3b_n-12c_nd_n+20d_n^3 \to 0 \\ 12c_nd_n^2-6b_nd_n-15d_n^4 \to 0 \\ 4d_n^5-4c_nd_n+3b_nd_n^2 \to 0 \end{array}}\ (\mathrm{I}) \Rightarrow \boxed{\underbrace{20d_n^4d_n'+3(2b_nd_nd_n' + b_n'd_n^2) -4(c_n'd_n+c_nd_n')}_{\frac{d(\alpha_z)_n(0)}{dt}} \to 0} \qquad (\mathrm{II})$$

First, one sees that $(\xi_w)_n = -d_n(2(\xi_v)_n + d_n(3(\xi_u)_n + 5d_n^2))$, and hence, since $\xi_n \to (0,0,0,0)$, $d_n \to 0$, as $n \to \infty$.

Then, by computation, we have:

$$\frac{d(\alpha_z)_n(0)}{dt} = -\underset{\swarrow \atop 0}{d_n^3}\ \Big(\underset{\downarrow \atop \frac{d\alpha_u}{dt}(0)}{\frac{d(\alpha_u)_n(0)}{dt}}\Big) - \underset{\downarrow \atop 0}{d_n^2}\ \Big(\underset{\downarrow \atop \frac{d\alpha_v}{dt}(0)}{\frac{d(\alpha_v)_n(0)}{dt}}\Big) - \underset{\downarrow \atop 0}{d_n}\ \Big(\underset{\downarrow \atop \frac{d\alpha_w}{dt}(0)}{\frac{d(\alpha_w)_n(0)}{dt}}\Big).$$

so that $\frac{d(\alpha_z)_n(0)}{dt} \to 0$ as $n \to \infty$, as wanted. □

<u>**PROPOSITION 60:**</u>

C[4] is closed in $T^4(\mathbb{R}^4)$.

Let $\{\hat{\beta}_n\}_{n\in\mathbb{N}}$, $y_n = \beta_n(0)$, be a sequence converging to some $\hat{\beta} \in T^4(\mathbb{R}^4)$, $y = \beta(0)$, and $\hat{\beta}_n \in C[4]$, $\forall\, n \in \mathbb{N}$ fixed. From Proposition 53 and its lemma, $\exists$ subsequence $\{\hat{\beta}_k\}_{k\in\mathbb{N}}$ such that $\hat{\beta}_k \in TC_{i_s}^{j_s}[4]$, $\forall\, k \in \mathbb{N}$.

<u>Case 1:</u> $\boxed{i_s = 1}$

$TC_{i_s=1}^{j_s}[4] = C_1^{j_s}[4] = T^4 N_1^{j_s}$. With Γ,γ as usual and as in case 1 of (4.4(16) and 4.4(36)), one shows that $\Gamma^{-1}\beta_k(I) \subset C(1,3)$ therefore $\tilde{I}(\widehat{\Gamma^{-1}\beta_k}) \subset \{(x_1,\dots x_{20}) \mid x_1 = x_5 = x_9 = x_{13} = x_{17} = 0\}$

As before (4.4(16)/(36)), one gets $\beta(I) \subset N_1^{j_s}$, hence $\hat{\beta} \in C[4]$.

<u>Case 2:</u> $\boxed{i_s = 2}$

<u>Case 2.1:</u> $\exists$ subsequence, $\{\hat{\beta}_r\}_{r\in\mathbb{N}}$, $y_r = \beta_r(0)$ of $\{\hat{\beta}_k\}_{k\in\mathbb{N}}$, such that $\hat{\beta}_r \in C_2^{j_s}[4]$, $\forall\, r \in \mathbb{N}$.

Proof

As that of case 2.1 (as in 4.4(36)); just substitute 3 by 4 whenever it appears.

<u>Case 2.2:</u> $\exists\, K \in \mathbb{N}$ s.t. $\hat{\beta}_k$, $y_k = \beta_k(0) \in C_{2,1}^{j}(m_k)[4]$, some $m_k \in M_1^d \cap \mathcal{U}_2^{j_s}$, $\forall\, k \geq K$, fixed.

Proof

Precisely as that of case 2.2 in 4.4(36); substitute 3 by 4 whenever it appears and apply Proposition 54 instead of Proposition 32.

<u>Case 3:</u> $\boxed{i_s = 3}$

Case 3.1: $\exists$ subsequence $\{\hat{\beta}_r\}_{r\in\mathbb{N}}$, $y_r = \beta_r(0)$, such that

$\hat{\beta}_r \in C_3^{j_s}[4]$, $\forall$ $r \in \mathbb{N}$; one gets, similarly as in case 3.1, (4.4(37)),

$\hat{\beta} \in C_3^{j_s}[4] \subset C[4]$.

Case 3.2: $\exists$ subsequence $\{\hat{\beta}_r\}_{r\in\mathbb{N}}$, $y_r = \beta_r(0)$, such that, $\forall$ r, fixed,

$\hat{\beta}_r \in C_{3,2}^{j_s}(m_r)[4]$, $m_r \in \mathcal{U}_3^{j_s} \cap M_2^d$. This means $\hat{\beta}_r \in \{\hat{\beta} \in C_2^{j_o}[4] \,|\, \beta(0) = y_r = \chi_f(m_r)\} =$

$\{\hat{\beta} \in T^4\Gamma_o\tilde{I}^{-1}(\{(\cdot) \,|\, x_1 = x_2 = x_5 = x_6 = x_{10} = 0\})\}$ (note: j_o is such that

$m_r \in \mathcal{U}_2^{j_o}$); i.e. in particular :

$$\boxed{(\Gamma_o^{-1}(y_r))_u = (\Gamma_o^{-1}(y_r))_v = \frac{d(\Gamma_o^{-1}\beta_r)_u(0)}{dt} = \frac{d(\Gamma_o^{-1}\beta_r)_v(0)}{dt} = 0} \quad \odot$$

Now, if Γ corresponds to $(j_s,3)$, we know, from Remark 7, that:

$$\Gamma^{-1}\Gamma_o(C_2(2,2)) \overset{\boxtimes}{\subset} C_2(3,1).$$

(in that remark, $\Gamma \to \Gamma_2$, $\Gamma_o \to \Gamma$, $i = 2$, $r = 4$, $c_1 = 2$, $c_2 = 3$).

Therefore, with $\xi_r = \Gamma^{-1}(y_r)$, $\alpha_r = \Gamma^{-1}\beta_r$, by $\odot$ and $\boxtimes$:

$$\left(\frac{d(\alpha_r)}{dt}u(0); \frac{d(\alpha_r)}{dt}v(0); \frac{d(\alpha_r)}{dt}w(0)\ \frac{d(\alpha_r)}{dt}z(0)\right) \in T_{\xi_r}(C_3(3,1)) \text{ (as in 4.4(38))}.$$

Also by $\boxtimes$, $\xi_r \in C_2(3,1)$. By Proposition 55, it follows that $\frac{d\alpha}{dt}v(0) = \frac{d\alpha}{dt}w(0) = 0$.

By Proposition 53, $y \in \chi(M_{i=3}^{j=j_s})$, therefore $\Gamma^{-1}(y) = (0;0;0;*)$. Hence

$\hat{\beta} \in T^4\Gamma\tilde{I}^{-1}(Q_3[4] \subset C[4]$

Case 3.3: $\exists$ $K \in \mathbb{N}$ s.t. $\hat{\beta}_k \in C_{3,1}^{j_s}(m_k)[4]$, $\beta_k(0) = y_k$, some

$m_k \in \mathcal{U}_3^{j_s} \cap M_1^d$, $\forall$ $k \geq K$.

The proof of case 3.3 is as proof of case 3.3 in 4.4(39). For $k \geq K$ fixed, $\hat{\beta}_k \in \{\hat{\beta} \in C_1^{j_0}[4] \mid \beta(0) = y_k = \chi_f(m_k)\}$, j_0 s.t. $m_k \in U_1^{j_0}$. Hence $\exists$ representative β_k with $\beta_k(I) \subset N_1^{j_0}$. One gets $\chi_{g=\gamma f/\gamma^{-1}(M_1^{j_0})} : \gamma^{-1}(M_1^{j_0}) \to \Gamma^{-1}(N_1^{j_0}) \subset C_1(3,1)$, diffeomorphically ((i)'); $\Gamma^{-1}(\beta_k(0)) \in \Gamma^{-1}(N_1^{j_0})$((ii)') and $\Gamma^{-1}\beta_k(I) \subset \Gamma^{-1}(N_1^{j_0})$((iii)')

By then considering the sequence $\{\hat{\alpha}_k\}_{k\in\mathbb{N}}$, $\xi_k = \alpha_k(0)$, $k \geq K$, with $\alpha_k = \Gamma^{-1}\beta_k$, and setting $M^k = \gamma^{-1}(M_1^{j_0})$, $N^k = (\Gamma^{-1}(N_1^{j_0}))$, one gets, from Proposition 56, $\frac{d\alpha}{dt}v(0) = \frac{d\alpha}{dt}w(0) = 0$, therefore $\hat{\beta} \in C_3^{j_s}[4] \subset C[4]$.

<u>Case 4:</u> $\boxed{i_s = 4}$

<u>Case 4.1:</u> $\exists$ subsequence $\{\hat{\beta}_r\}_{r\in\mathbb{N}}$, $\beta_r(0) = y_r$ s.t. $\hat{\beta}_r \in C_4^{j_s}[4]$, $\forall\, r \in \mathbb{N}$.

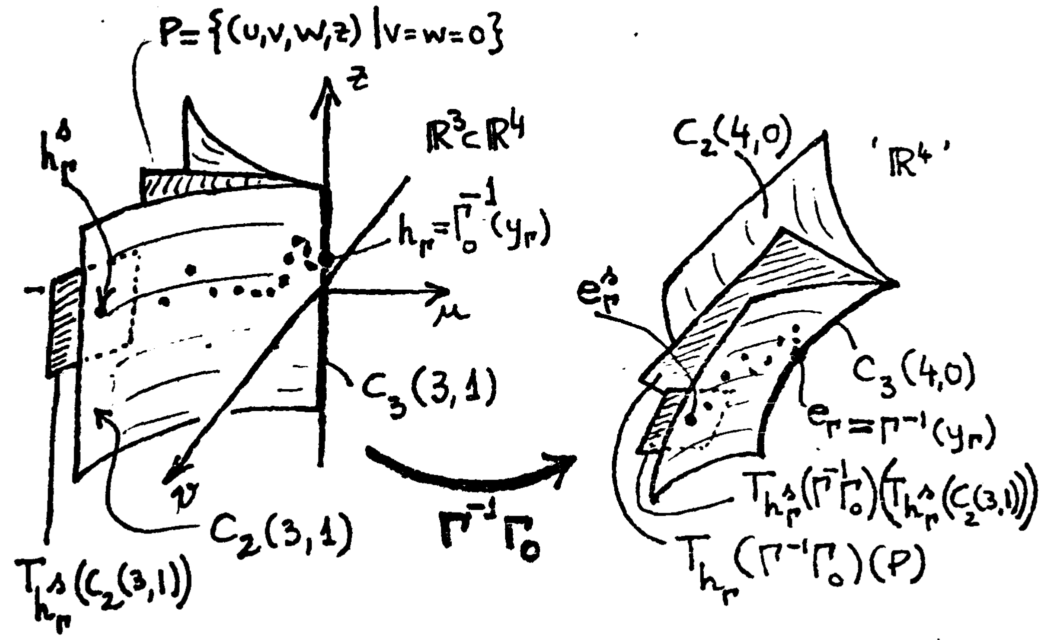
With Γ,γ as usual, one gets $\tilde{I}(\widehat{\Gamma^{-1}\beta_r}) \in Q_4[4] = \{(\cdot) \mid x_1=x_2=x_3=x_4=x_8= 0\}$, therefore $\tilde{I}(\widehat{\Gamma^{-1}\beta}) \in Q_4[4]$, therefore $\hat{\beta} \in C_4^{j_s}[4] \subset C[4]$.

<u>Case 4.2:</u> $\exists$ subsequence $\{\hat{\beta}_r\}_{r\in\mathbb{N}}$, $\beta_r(0) = y_r$, such that for each fixed r, $\hat{\beta}_r \in C_{4,3}^{j_s}(m_r)[4]$, $m_r \in U_4^{j_s} \cap M_3^d$.

This means $\hat{\beta}_r \in \{\hat{\beta} \in C_3^{j_0}[4] \mid \beta(0) = y_r = \chi_f(m_r)\} =$

$= \{\hat{\beta} \in T^4\Gamma_0\tilde{I}^{-1}(Q_3[4]) \mid \beta(0) = y_r = \chi_f(m_r)\}$, where

$Q_3[4] = \{(\cdot) \mid x_1 = x_2 = x_3 = x_6 = x_7 = 0\}$, and where j_0 is s.t. $m_r \in U_3^{j_0}$.



(w-axis missing; we actually draw the projection of C(3,1) on the (z×u×v) space)

Therefore, we know that

$$\frac{d(\Gamma_0^{-1}\beta_r)}{dt}v(0) = \frac{d(\Gamma_0^{-1}\beta_r)}{dt}w(0) = 0,$$

i.e. $\boxed{\frac{d(\Gamma_0^{-1}\beta_r)}{dt}(0) \in P}$, (I)

P as described in the pict

The idea of the proof will

to find out what

$\boxed{T_{h_r}(\Gamma^{-1}\Gamma_0)(P)}$ (II) is, where $h_r = \Gamma_0^{-1}(y_r) \in C_3(3,1)$ (since $y_r \in \mathcal{X}_f(m_r)$, $m_r \in M_3^d$).

Once we work out (II), we then use:

$$\boxed{\left(\frac{d(\Gamma^{-1}\beta_r)}{dt}u(0);\ldots;\frac{d(\Gamma^{-1}\beta_r)}{dt}z(0)\right) = T_{h_r}(\Gamma^{-1}\Gamma_0)\left(\frac{d(\Gamma_0^{-1}\beta_r)}{dt}u(0);\ldots;\frac{d(\Gamma_0^{-1}\beta_r)}{dt}z(0)\right)} \quad \text{(III)}$$

where $\left(\frac{d(\Gamma_0^{-1}\beta_r)}{dt}u(0);\ldots;\frac{d(\Gamma_0^{-1}\beta_r)}{dt}z(0)\right) \in P$, by (I)

Let $e_r = \Gamma^{-1}\,\Gamma_0(h_r) = \Gamma^{-1}(y_r)$. Let χ be correspondent to the butterfly, and let d_r be the unique number s.t. $\chi(0,0,0,d_r) = (-10d_r^2;20d_r^3;-15d_r^4;4d_r^5) = e_r$ (it is easy to prove unicity).

LEMMA (to case 4.2):

$$T_{h_r}(\Gamma^{-1}\Gamma_0)(P) = \underbrace{[\xi_r(1),\xi_r(2)]}_{\text{space generated by}},\quad \xi_r(1) = \begin{bmatrix} 1 \\ -3d_r \\ 3d_r^2 \\ -d_r^3 \end{bmatrix},\quad \xi_r(2) = \begin{bmatrix} 0 \\ 1 \\ -2d_r \\ d_r^2 \end{bmatrix}$$

PROOF OF LEMMA:

The idea here is to exploit the invariance of the two dimensional strata, i.e., the fact that (see Remark 7) $\Gamma^{-1}\Gamma_0(C_2(3,1)) \overset{\boxtimes}{\subset} C_2(4,0)$. As we have pointed out in Proposition 42, the proof will consist in considering sequences $\{h_r^s\}_{s\in\mathbb{N}} \in C_2(3,1) \to h_r (\in C_3(3,1))$ and $\{e_r^s\}_{s\in\mathbb{N}}$, $e_r^s = \Gamma^{-1}\Gamma_0(h_r^s) \in$ $\in C_2(4,0) \to e_r \in C_3(4,0)$ and showing that $\boxed{T_{h_r^s}(C_2(3,1) \to P}$ (a), $\boxed{T_{e_r^s}(C_2(4,0)) \to [\xi_r(1),\xi_r(2)]}$ (b) (as above) as $s \to \infty$; a reduction to absurd proof, using $\boxtimes$ (and continuity of $\bullet \in \mathbb{R}^4 \to T_\bullet(\Gamma^{-1}\Gamma_0)$ - see also 4.4(9)) easily proves (see Prop. 14 Propositions 14 and 25) that $T_{h_r}(\Gamma^{-1}\Gamma_0)(P) = [\xi_r(1),\xi_r(2)]$.

We first prove (a). For this, we compute $T_{h_r^s}(C_2(3,1))$. Let $n_r^s = \Pi_{(u,v,w)}(h_r^s)$, where $\Pi_{(u,v,w)}: \begin{array}{c}(u,v,w,z) \to (u,v,w).\\ \mathbb{R}^4 \to \mathbb{R}^3\end{array}$ Since $C_2(3,1) = C_2(3,0) \times$ [z-axis]

$T_{h_r^s}(C_2(3,1))$ will be generated by the (one dimensional) generator, $\mathcal{J}_r^s(1)$, of $T_{h_r^s}(C_2(3,0))$ and $\begin{bmatrix}0\\0\\0\\1\end{bmatrix} = \mathcal{J}_r^s(2)$. To find $\mathcal{J}_r^s(1)$, consider χ (corresponding to the swallowtail), and let c_r^s be s.t. $\chi(c_r^s) = n_r^s$ ($c_r^s \in \mathbb{R}^3$). Since $h_r^s \to h_r = (0,0,0,*)$, then $n_r^s \to n_r = (0,0,0)$ (as $s\to\infty$), therefore $c_r^s \to 0$ as $s \to \infty$, and $c_r^s \neq 0$, $\forall$ s, since $h_r^s \in C_2(3,1)$. $\mathcal{J}_r^s(1)$ is easily computed to be $\begin{bmatrix}1\\-2c_r^s\\(c_r^s)^2\end{bmatrix} \simeq \begin{bmatrix}1\\-2c_r^s\\(c_r^s)^2\\0\end{bmatrix}$ (identifying $\mathbb{R}^3 \simeq \underbrace{\mathbb{R}^3 \times \{0\}}_{\mathbb{R}^4?}$). Therefore

$$T_{h_r^s}(C_2(3,1)) = \left[\begin{bmatrix}1\\-2c_r^s\\(c_r^s)^2\\0\end{bmatrix} ; \begin{bmatrix}0\\0\\0\\1\end{bmatrix}\right] \to P,$$

as $s \to \infty$, since $c_r^s \to 0$ (as we have commented above, the whole argument is made precise by a reduction to absurd proof, as in Proposition 14 and 25, for example; we allow ourselves the somewhat loose use of ' $\to$ ', as above, in view of that)

As to (b), we start by working out $T_{e_r^s}(C_2(4,0))$. Let χ be the one corresponding to the butterfly. Choose $m_r^s = (a_r^s, b_r^s, c_r^s, d_r^s)$, s.t. $\chi(m_r^s) = e_r^s$. Since $e_r^s \overset{\odot}{\in} C_2(4,0)$, $a_r^s = b_r^s = 0$. Now: $\chi(0;0;c_r^s;d_r^s) =$

$= (4c_r^s - 10(d_r^s)^2; - 12c_r^s d_r^s + 20(d_r^s)^3; 12c_r^s(d_r^s)^2 - 15(d_r^s)^4; 4(d_r^s)^5 - 4c_r^s(d_r^s)^3)$, and one gets $T_{e_r^s}(C_2(4,0))$ as generated by

$$\xi_r^s(1) = \begin{bmatrix} 1 \\ -3d_r^s \\ 3(d_r^s)^2 \\ -(d_r^s)^3 \end{bmatrix} \quad \& \quad \xi_r^s(2) = \begin{bmatrix} 0 \\ 1 \\ -2d_r^s \\ (d_r^s)^2 \end{bmatrix}$$

since $\odot \Rightarrow c_r^s \neq 0, \ \forall s$.

One can show that $c_r^s \to 0$ as $s \to \infty$, since $e_r^s \to e_r \in C_3(4,0)$, from which it easily follows that $d_r^s \to d_r$ as $s \to \infty$, hence (b). (end of proof of lemma) $\square$

From lemma, $\dfrac{d(\Gamma^{-1}\beta_r)}{dt}(0) = (r_r; s_r - 3r_r d_r; 3r_r d_r^2 - 2s_r d_r; s_r d_r^2 - r_r d_r^3)$;

therefore $\lim\limits_{r\to\infty} r_r = \dfrac{d(\Gamma^{-1}\beta)}{dt} u(0)$, $\lim\limits_{r\to\infty} s_r = \dfrac{d(\Gamma^{-1}\beta)}{dt} v(0)$, since $d_r \to 0$ as $r \to \infty$ ($d_r \to 0$ $r \to \infty$ because, as $e_r = \Gamma^{-1}(y_r) \to \Gamma^{-1}(y) = (0,0,0,0)$, as $r \to \infty$, and $\chi(0,0,0,d_r) = e_r$)

Therefore $\lim\limits_{r\to\infty} \dfrac{d(\Gamma^{-1}\beta_r)}{dt} z(0) = \lim\limits_{r\to\infty} (\underset{\substack{\downarrow \\ \text{fixed} \\ \text{limit}}}{s_r} \underset{\substack{\downarrow \\ 0}}{d_r^2} - \underset{\substack{\downarrow \\ \text{fixed} \\ \text{limit}}}{r_r} \underset{\substack{\searrow \\ 0}}{d_r^3}) = 0$ hence

$\dfrac{d(\Gamma^{-1}\beta)}{dt} z(0) = 0$, therefore $\tilde{I}(T^4\Gamma^{-1}(\hat{\beta})) \in Q_4[4]$,

Since $\Gamma^{-1}(y) = 0 \in \mathbb{R}^4$. So $\hat{\beta} \in C_4^{j_s}[4] \subset C[4]$.

<u>Case 4.3</u> $\exists$ subsequence $\{\hat{\beta}_r\}_{r\in N}$, $\beta_r(0) = y_r$, s.t., for each fixed r, $\hat{\beta}_r \in C^{j_s}_{4,2}(m_r)[4]$ $(m_r \in \mathcal{U}^j_4 \cap M^d_2)$.

We have $\hat{\beta}_r \in \{\hat{\beta} \in T^4\Gamma_0\tilde{I}^{-1}\{(\cdot)|x_1 = x_2 = x_5 = x_6 = x_{10} = 0\} \mid \beta(0) = y_r\}$

From Remark 7, $\Gamma^{-1}\Gamma_0(\underset{\in C_2(2,2)}{C_2(2,2)}) \subset C_2(4,0)$, therefore $\dfrac{d(\Gamma^{-1}\beta_r)}{dt}(0) \overset{\odot}{=}$

$T_{\Gamma_0^{-1}(y_r)}(\Gamma^{-1}\Gamma_0)\cdot\left(\dfrac{d(\Gamma_0^{-1}\beta_r)}{dt}(0)\right) \in T_{\Gamma^{-1}(y_r)}(C_2(4,0))$, where we use the notation

$h_r = \Gamma_0^{-1}(y_r)$, $e_r = \Gamma^{-1}(y_r)$

We work out $T_{e_r}(C_2(4,0))$ by using χ corresponding to the butterfly, as usual: $T_{e_r}(C_2(4,0)) = T_{(0,0,c_r,d_r)}\chi(\{(a,b,c,d)|a = b = 0\}) = [\xi_r(1);\xi_r(2)]$

where $m_r = (0,0,c_r,d_r)$ is chosen so that $\chi(m_r) = e_r$ and $\xi_r(1) = \begin{bmatrix} 1 \\ -3d_r \\ 3d_r^2 \\ d_r^3 \end{bmatrix}$ $\xi_r(2) = \begin{bmatrix} 0 \\ 1 \\ -2d_r \\ d_r^2 \end{bmatrix}$

(Note: $c_r \neq 0, \forall r$, since $y_r \in \chi_f(M^d_2)$)

As $y_r \to y$, $\Gamma^{-1}(y_r) \to 0$; it is easy to show that $c_r, d_r \to 0$. Now, from $\odot$ above, one has

$$\left(\frac{d(\Gamma^{-1}\beta_r)}{dt}u(0);\ldots;\frac{d(\Gamma^{-1}\beta_r)}{dt}z(0)\right) = (r_r;s_r-3r_rd_r;3r_rd_r^2-2s_rd_r;s_rd_r^2-r_rd_r^3),$$

$s_r, d_r \in \mathbb{R}$; therefore $\lim_{r\to\infty} r_r = \dfrac{d(\Gamma^{-1}\beta)}{dt}u(0)$, $\lim_{r\to\infty} s_r = \dfrac{d(\Gamma^{-1}\beta)}{dt}v(0)$; it follows that

$$\lim_{r\to\infty}\frac{d(\Gamma^{-1}\beta_r)}{dt}z(0) = \lim_{r\to\infty}(\underset{\text{constant}}{s_r}\ \underset{0}{d_r^2} - \underset{\text{constant}}{r_r}\ \underset{0}{d_r^3}) = 0; \text{therefore } \hat{\beta} \in C_4^{j_s}[4] \subset C[4].$$

Case 4.4: $\exists\, K \in \mathbb{N}$ s.t. $\hat{\beta}_k \in C_{4,1}^{j_s}(m_k)[4]$, where $\beta_k(0) = y_k = \chi_f(m_k)$, some $m_k \in U_4^{j_s} \cap M_1^d$, $\forall\, k \geq K$.

For $k \geq K$ fixed, $\hat{\beta}_k \in \{\hat{\beta} \in C_1^{j_o}[4] \mid \hat{\beta}(0) = y_k = \chi_f(m_k)\}$, j_o s.t. $m_k \in U_1^{j_o}$. Hence, $\exists$ represent. β_k s.t. $\beta_k(I) \subset N_1^{j_o}$, so that, as in case 3.3 (4.4(68)), one gets $\chi_{g=\gamma g/\gamma^{-1}(M_1^{j_o})} : \gamma^{-1}(M_1^{j_o}) \to \Gamma^{-1}(N_1^{j_o})$ diffeomorphically, $\Gamma^{-1}(\beta_k(0)) \in \Gamma^{-1}(N_1^{j_o})$ and $\Gamma^{-1}\beta_k(I) \subset \Gamma^{-1}(N_1^{j_o})$; so that, considering the sequence $\{\hat{\alpha}_k\}_{k \geq k}$, $\xi_k = \alpha_k(0)$, with $\alpha_k = \Gamma^{-1}\beta_k$ and setting $M^k = \gamma^{-1}(M_1^{j_o}), N^k = \Gamma^{-1}(N_1^{j_o})$, as before, we have, from Proposition 59, $d\alpha_z/dt(0) = 0$; therefore $\hat{\beta} \in C_4^{j_s}[4] \subset C[4]$.

□

C. Genericity of $v \pitchfork C_f$:

PROPOSITION 61:

$\exists$ open and dense set of vector fields, $\mathcal{B} \subset \mathcal{V}(\mathbb{R}^4)$, s.t. $v \in \mathcal{B} \Rightarrow v[4](\mathbb{R}^4) \cap C[4] = \emptyset$

Proof

Like the proof of Proposition 36; just substitute 3 by 4 everywhere, and j^2v by j^3v in the definition of $\mathcal{B}$. □

PROPOSITION 62: (GLOBAL to LOCAL)

Let $y \in C_f$, m_s, (i_s, j_s), $U_{i_s}^{j_s}$, $s=1,\ldots,p$ as in 4.4(27). $\exists\, V$, open, neighbourhood of y in $\mathbb{R}^4$, s.t. $V \cap C_f = V \cap [\bigcup_{s=1}^{p} \chi_f(U_{i_s}^{j_s} \cap M^d)]$.

COROLLARY:

$$V \cap C_f \subset \bigcup_{s=1}^{p} \chi_f\, (U_{i_s}^{j_s} \cap M^d).$$

Proof:

Same as that of Proposition 21. □

PROPOSITION 63: $\boxed{\text{(Genericity of v-}\Xi\text{-cusp in STANDARD FORM: the 4 dimensional problem)}}$

Let $\alpha(t) = (\alpha_u(t);\alpha_v(t);\alpha_w(t);\alpha_z(t))$ be a (C^∞) curve through $\xi = \alpha_o$, $\xi = (\xi_u;\xi_v;\dot{\xi}_w;\xi_z)$, satisfying $\xi_u = \xi_v = 0$. Suppose that $(\frac{d\alpha}{dt}u(0);\frac{d\alpha}{dt}v(0);\frac{d^2\alpha}{dt}v(0)) \neq$ $\neq (0,0,0)$. Then, $\exists\ \varepsilon > 0$ s.t. $\{\alpha(t)|\ \ |t| < \varepsilon\ ,\ t \neq 0\} \cap C(2,2) = \emptyset$.

Proof

Since $C(2,2) = C(2,0) \times \mathbb{R}^2$, we see, like in Proposition 38, that we will be done if we can prove:

if $\boxed{\alpha = (\alpha_u,\alpha_v) \text{ is a curve in } \mathbb{R}^2,\ \alpha(0)=0,\ (\alpha_u'(0);\alpha_v'(0);\alpha_v''(0)) \neq (0,0,0)}$ (I) then $\boxed{\exists\ \varepsilon > 0 \text{ s.t. } \{\alpha(t)|\ |t| < \varepsilon,\ t \neq 0\} \cap C(2,0) = \emptyset}$ (II)

Case 1:

Suppose $\frac{d\alpha}{dt}v(0) \neq 0$. (II) follows from Proposition 22.

Case 2:

Suppose $\frac{d\alpha}{dt}v(0) = 0$, $\frac{d\alpha}{dt}u(0) \neq 0$. (II) follows from Proposition 38

Case 3:

Suppose $\frac{d\alpha}{dt}v(0) = \frac{d\alpha}{dt}u(0) = 0$, $\frac{d^2\alpha}{dt^2}v(0) \neq 0$.

In this case
$$\begin{cases} \alpha_u(t) = \alpha_u''(0)t^2 + r_u(t), & |r_u(t)|/t^2 \to 0 \quad \text{as } t \to 0. \\ \alpha_v(t) = \underbrace{\alpha_v''(0)}_{\neq 0}t^2 + r_v(t), & |r_v(t)|/t^2 \to 0 \quad \text{as } t \to 0. \end{cases}$$

From this, $\exists\ \varepsilon_1 > 0$ s.t. $|t| < \varepsilon \Rightarrow \begin{cases} |\alpha_v(t)| \geq |\alpha_v''(0)/2|t^2 & \text{and} \\ |\alpha_u(t)| \leq Kt^2,\ K = \min\{\alpha_v''(0);\ 1\}. \end{cases}$

setting $c = 2K/\alpha_v''(0)$, we then have $|\alpha_u(t)| \leq c|\alpha_v(t)|$. Let $\varepsilon_2 > 0$ be s.t. $|\alpha(t)| < 27/8c^2$ (possible, since α is continuous and $\alpha(0) = 0$), if $|t| < \varepsilon_2$.

Let $\varepsilon^* = \min\{\varepsilon_1;\varepsilon_2\}$.

Let t be s.t. $|t| < \varepsilon^*$. Suppose t is s.t. $\alpha(t) \in C(2,0)$. Then, $8\alpha_u^3(t) = 27\,\dot{\alpha}_v^2(t) \geq 27/c^2.\alpha_u^2(t)$; therefore $\alpha_u(t) \geq 27/8c^2$ or $\alpha_u(t) = 0$. The first inequality is impossible, since $|t| < \varepsilon^* \leq \varepsilon_2$. So, $\alpha_u(t) = 0$; therefore $\alpha_v(t) = 0$. But then, one can choose ε_3 s.t. $\left\{\begin{array}{l}|t| < \varepsilon_3\\ t \neq 0\end{array}\right\} \Rightarrow \alpha_v(t) \neq 0$, because $|\alpha_v(t)| \geq \underbrace{|\alpha_v'(0)/2}_{\neq 0}|t^2$, t sufficiently small; hence, if $\varepsilon = \min\{\varepsilon^*,\varepsilon_3\}$ and $\left\{\begin{array}{l}|t| < \varepsilon\\ t \neq 0\end{array}\right.$, one concludes then that $\alpha(t) \notin C(2,0)$. This ε settles case 3. □

PROPOSITION 64 (Genericity of v/⊥\ swallowtail in STANDARD FORM: the 4 dimensional problem)

Let $\alpha = (\alpha_u;\alpha_v;\alpha_w;\alpha_z)$ be a (C^∞) curve through $\xi = \underset{\mathbb{R}^4}{\alpha(0)}$, $\xi_u = \xi_v = \xi_w = 0$

Suppose that $(\frac{d\alpha}{dt}v(0); \frac{d\alpha}{dt}w(0)) \neq (0,0)$. Then,

$$\exists\ \varepsilon > 0 \text{ s.t } \{\alpha(t) \mid |t| < \varepsilon,\ t \neq 0\} \cap C(3,1) = \emptyset.$$

Proof

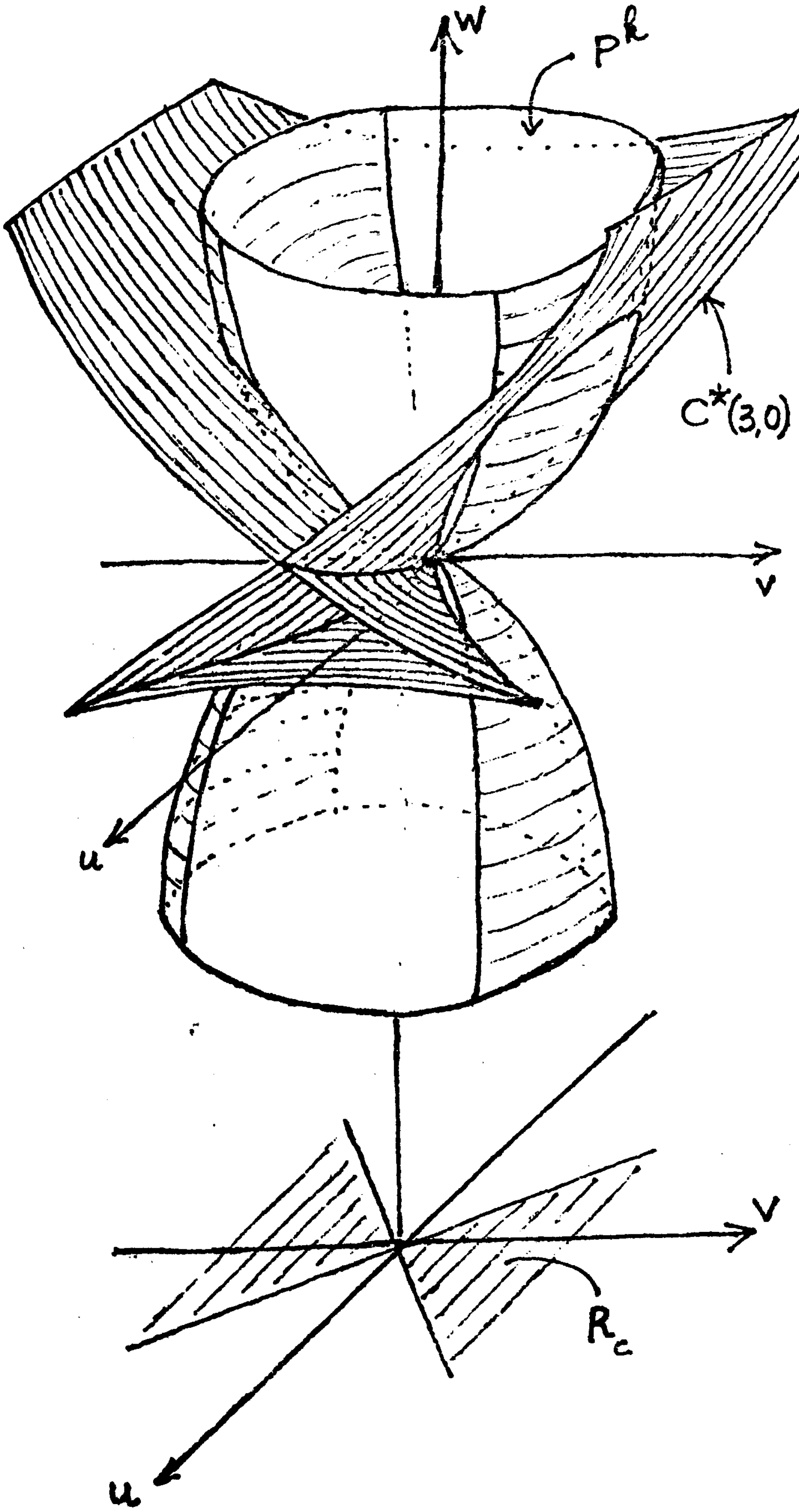
Since $C(3,1) = C(3,0) \times \mathbb{R}$, we will be done if we can show:

if $\boxed{\alpha = (\alpha_u,\alpha_v,\alpha_w) \text{ is a curve in } \mathbb{R}^3, \text{ through } 0, \text{ with } (\alpha_v'(0);\alpha_w'(0)) \neq (0,0)}$ (I) then $\boxed{\exists \varepsilon > 0 \text{ s.t. } \{\alpha(t) \mid |t| < \varepsilon, t \neq 0\} \cap C(3,0) =}$ (II

Case 1:

Suppose $\alpha_w'(0) \neq 0$. (II) follows from Proposition 39.

Case 2: Suppose $\alpha_w'(0) = 0$, $\alpha_v'(0) \neq 0$.



Instead of proving (II), we will actually show that:

$$\boxed{\varepsilon > 0 \text{ s.t. } \{\alpha(t) \mid {}^{|t|<\varepsilon}_{t\neq 0}\} \cap C^*(3,0) = \emptyset} \quad (\text{I}$$

where $C^*(3,0)$ is as defined in 4.4(42). (II)' $\Rightarrow$ (II), since, as pointed out in 4.4(42), $C^*(3,0) \supset C(3,0)$.

We define, given $c, k \in \mathbb{R}^+$, the sets:

$R_c \quad = \{(u,v) \in \mathbb{R}^2 \mid u = \alpha v, |\alpha| \leq c\}$,

$R_c^3 \quad = \{(u,v,w) \in \mathbb{R}^3 \mid (u,v) \in R_c\}$,

$P^k \quad = \{(u,v,w) \in \mathbb{R}^3 \mid w = \pm k(u^2+v^2)\}$,

$SP^k \quad = \bigcup_{k' \lessgtr k} P^{k'}$,

$SP_c^k \quad = SP^k \cap R_c^3$.

To prove (II)', we adopt a method similar to that used in the proof of Proposition 39. We first show that, for η suff. small, $B_\eta(0) \cap SP_c^k \cap C^*(3,0) = \{($ (see 4.4(43)), and then prove that, if $|t| < \varepsilon$, suff. small, $\varepsilon \neq 0$, the orbit of α has to be inside $B_\eta(0) \cap SP_c^k - \{0\}$ (for convenient k, $c \in \mathbb{R}$).

Lemma 1:

Let c, k as above be fixed.

$\exists \eta > 0 (= \eta(c,k))$, s.t. $B_\eta(0) \cap SP_c^k \cap C^*(3,0) = \{0\}$.

Proof

Set $\eta = \min\{1;1/K\}$, $K = 256k^3(C^2+1)^3 + 128k(C^2+1)C + 16kC^4(C^2+1) + $
$+ 12k^2C^2(C^2+1)^2 + 4C^3$.

Suppose $\exists(u,v,w) \in \mathbb{R}^3$ s.t. $(u,v,w) \in B_\eta(0) \cap SP_C^k \cap C^*(3,0)$. Substituting $u = \alpha v$, $w = \pm k'(u^2+v^2)$, with $k' \leq k$, in the expression for $C^*(3,0)$ (see 4.4(42)), one gets:

$$v^4(-27+v.\overbrace{(\pm 256(k')^3v(\alpha^2+1)^3 \pm 128\alpha k'(\alpha^2+1) \pm 4k'\alpha^3(\alpha^2+1)4\alpha v - 3\alpha(k')^2(\alpha^2+1)^2 4\alpha v - 4\alpha^3)}^{\odot}) = 0$$

Now $|v.\odot| = |v||\odot| \leq |v|(|256(k')^3v(\alpha^2+1)^3| + |128\alpha k'(\alpha^2+1)| + |4k'\alpha^3(\alpha^2+1)4\alpha v| +$
$+|3\alpha(k')^2(\alpha^2+1)^2 4\alpha v| + |4\alpha^3|) \leq |v|.K$, since $k' \leq k$, $|\alpha| \leq C$ and $|v| \leq 1$ (since $(u,v,w) \in B_\eta(0)$). Hence, $|v.\odot| < \frac{1}{K}.K = 1$; therefore $(-27 + v.\odot) \neq 0$; therefore $v = 0$; therefore $u = v = 0$; therefore, from the expression of $C^*(3,0)$, $w = 0$. This proves the lemma.

<u>LEMMA 2:</u>

$$\exists\, k, C \in \mathbb{R}^+, \varepsilon > 0, \text{ s.t. } \overbrace{\{\alpha(t) \mid |t| < \varepsilon, t \neq 0\}}^{\boxtimes} \subset [B_\eta(0) \cap SP_C^k - \{0\}].$$

Proof

We first choose ε_1 s.t. $\boxtimes \subset R_C^3$, where $C = \max.\{1; 4\left|\frac{\alpha_u'(0)}{\underbrace{\alpha_v'(0)}_{\neq 0}}\right|\}$:

If $\alpha_u'(0) \neq 0$, choose ε_1', s.t: $\begin{cases} |\alpha_u(t)| = |\alpha_u'(0)t + r_u(t)| \leq |2\alpha_u'(0)|t \\ |\alpha_v(t)| = |\overbrace{\alpha_v'(0)}^{\neq 0}t + r_v(t)| \geq \dfrac{|\alpha_v'(0)|t}{2} \end{cases}$

Hence $|\alpha_u(t)| \leq \left|\dfrac{4\alpha_u'(0)}{\alpha_v'(0)}\right| . |\alpha_v(t)|$; therefore $|\alpha_u(t)| \leq C|\alpha_v(t)|$; therefore $\alpha(t) \in R_C^3$, $\forall\, t$ s.t. $|t| < \varepsilon_1'$.

If $\alpha_u'(0) = 0$, choose ε_1'' s.t.: $\begin{cases} |\alpha_u(t)| \le |2/\alpha_v'(0)|t \\ |\alpha_v(t)| \ge |\alpha_v'(0)/2|t \end{cases}$; therefore

$$|\alpha_u(t)| \le |\alpha_v(t)| \text{ ; therefore } \alpha(t) \in R_c^s, \ |t| < \varepsilon_1''.$$

Set $\varepsilon_1 = \min.\{\varepsilon_1', \varepsilon_1''\}$.

We now choose ε_2 s.t. $\boxtimes \subset SP^k$, where $k = \max\{ \dfrac{8\alpha_w''(0)}{(\alpha_u'(0))^2+\alpha_v'(0))^2}, \dfrac{4}{\alpha_u'(0)^2+\alpha_v'(0)^2} \}$:

If $\alpha_w''(0) \neq 0$, choose ε_2', s.t.: $\begin{cases} |\alpha_w(t)| \le 2|\alpha_w''(0)|t^2 \\ |\alpha_u^2(t)| \ge \dfrac{(\alpha_u'(0))^2}{4} t^2, |\alpha_v^2(t)| \ge \dfrac{(\alpha_v'(0))^2 t^2}{4} \end{cases}$

Hence, $\dfrac{|\alpha_w(t)|}{\alpha_u^2(t)+\alpha_v^2(t)} \le k$; therefore $|\alpha_w(t)| \le k(\alpha_u^2(t)+\alpha_v^2(t))$; therefore

$$\alpha(t) \in R_c^3, \ |t| < \varepsilon_2'.$$

If $\alpha_w''(0) = 0$, choose $\varepsilon_2''(0)$ s.t.: $\left.\begin{cases} |\alpha_w(t)| \le t^2 \\ |\alpha_u^2(t)| \ge \alpha_u'(0)^2/4.t^2, |\alpha_v^2(t)| \ge \alpha_v'(0)^2/4.t^2 \end{cases}\right] \Rightarrow$

$$\Rightarrow \alpha(t) \in R_c^3, \ |t| < \varepsilon_2''.$$

Set $\varepsilon_2 = \min \{\varepsilon_2', \varepsilon_2''\}$.

Choose ε_3 s.t. $\boxtimes \subset B_\eta(0)$. This is possible because $\alpha(0) = 0$, and α is continuous.

Choose ε_4 s.t. $|t| < \varepsilon_4 \Rightarrow |\alpha(t)| \neq 0$, possible because $\alpha_v'(0) \neq 0$.

Set $\varepsilon = \min.\{\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4\}$; this will do.

Lemmas (1 + 2) $\Rightarrow$ case 2 immediately, therefore Proposition 64 is proved.

PROPOSITION 65: (Genericity of v butterfly in STANDARD FORM: the 4 dimensional problem)

Let $\alpha = (\alpha_u;\alpha_v;\alpha_w;\alpha_z)$ be a curve through $0 \in \mathbb{R}^4$. Suppose that $d\alpha_z/dt(0) \neq 0$. Then $\exists\, \varepsilon > 0$ s.t.: $\{\alpha(t) \mid |t| < \varepsilon,\ t \neq 0\} \cap C(4,0) = \emptyset$.

Proof

The idea is very similar to that of Proposition 39 (see 4.4(42). One first defines $C^*(4,0) = \{(u,v,w,z) \in \mathbb{R}^4 \mid P(u,v,w,z) \equiv 0\}$, where $P(u,v,w,z)$ is a polynomial in u,v,w and z, by multiplying

$$\begin{bmatrix} (1) & \partial g_4/\partial x\,(\cdot) = x^5 + ux^3 + vx^2 + wx + z \\ (2) & \partial^2 g_4/\partial x^2(\cdot) = 5x^4 + 3ux^2 + 2vx + w \end{bmatrix}$$

(1) and (2) by x^3, x^2, x,1 and x^4, x^3, x^2, x,1, respectively, and solving the 9×9 determinant for u,v,w,z. It follows that $C^*(4,0) \supset C(4,0)$.

$P(u,v,w,z)$ is a polynomial containing the following monomials, with coefficients in $\mathbb{R}$ (these coefficients are irrelevant - from a qualitative point of view - in the proof): $z^4, w^5, v^2z^2w, vzw^3, v^5z, v^4w^2; uvz^3, uw^2z^2, uv^3wz, uv^2w^3; u^2v^2z^2, u^2vzw^2, u^2v^4w; u^3z^2w, u^3v^3z, u^3v^2w^2; u^4vzw, u^4w^3; u^5z^2, u^5v^2w; u^6vz, u^6w^2$.

As in Proposition 39, it suffices to prove a Proposition 65', obtained by substituting $C(4,0)$ by $C^*(4,0)$ in Proposition 65.

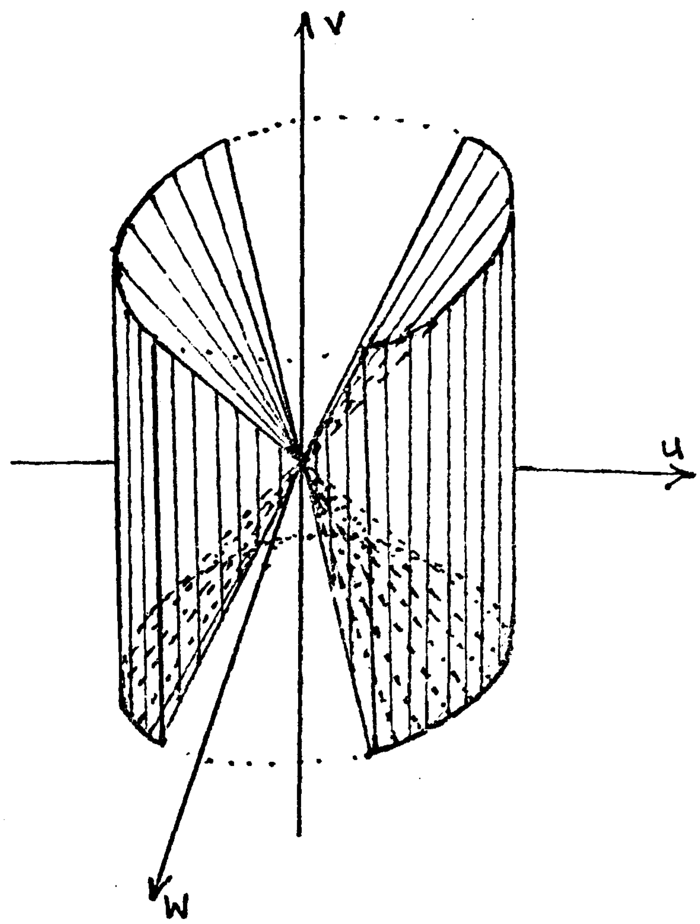
We then give the following definitions:

$$C_u^k = \left\{(u,v,w,z) \in \mathbb{R}^4 \;\middle|\; \begin{array}{l} u \overset{1}{=} \alpha w,\ v \overset{2}{=} \beta(u^2+w^2)^{1/2},\ z \overset{3}{=} \pm k(u^2+v^2+w^2)^{1/2}, \\ \text{with } |\alpha| \leq ||\beta| \leq 1 \end{array}\right\}$$

$$C_{w(1)}^k = \{(u,v,w,z) \in \mathbb{R}^4 \mid w \overset{4}{=} \alpha u,\ v \overset{2}{=} \beta(u^2+w^2)^{1/2}, z \overset{3}{=} \pm k(u^2+v^2+w^2)^{1/2}, |\alpha| \text{ and } |\beta| \leq 1\}$$

$$C_v^k = \{(u,v,w,z) \in \mathbb{R}^4 \mid v \overset{5}{=} \alpha w,\ u \overset{6}{=} \beta(v^2+w^2)^{1/2}, z \overset{3}{=} \pm k(u^2+v^2+w^2)^{1/2}, |\alpha|\ ,\ |\beta| \leq 1\}.$$

$$C_{w(2)}^k = \{(u,v,w,z) \in \mathbb{R}^4 \mid w \overset{7}{=} \alpha v,\ u \overset{6}{=} \beta(v^2+w^2)^{1/2}, z \overset{3}{=} \pm k(u^2+v^2+w^2)^{1/2}, |\alpha|\ ,\ |\beta| \leq 1\}$$



This picture represents the region defined by 4 and 2 in $\mathbb{R}^3$

A $\Pi/2$ rotation around the v-axis gives region reg.def.by 1 and 2.

A $\Pi/2$ rotation of the two above cases, around the w-axis, gives reg.def. by 7 and 6 and 5 and 6, respectively.

Therefore the complement of [(reg.def. by 4 and 2) $\cup$ (reg.def. by 1 and 2)] = reg.def. by 2 is just the interior of the cone shown in above picture.

Also, one has that the complement of [(reg.def. by 7 and 6) $\cup$ (reg.def. 5 & 6)] = reg.def. by 6 is the interior of above cone rotated by $\Pi/2$ around w-axis.

We claim that if $(u,v,w) \in \mathbb{R}^3$ then it belongs to one of the regions below, defined by the equations:

1 and 2; 4 and 2; 5 and 6; 7 and 6 (see picture and note below it for immediate geometrical intuitive proof).

To see this, suppose that $(u,v,w) \notin$ (4 and 2) $\cup$ (1 and 2). Immediately $|v| > |(u^2 + u^2)^{1/2}|$. Suppose also $(u,v,w) \notin$ (5 and 6) $\cup$ (7 and 6). Then $|u| > |(v^2 + w^2)^{1/2}|$. Therefore $v^2 + w^2 < u^2 < v^2 - w^2$, therefore $2w^2 < 0$, absurd.

From the above:

$$C^k = C^k_u \cup C^k_{w(1)} \cup C^k_v \cup C^k_{w(2)}$$

$$= \{(u,v,w,z) \in \mathbb{R}^4 \mid z = \pm k(u^2+v^2+w^2)^{1/2}\}$$

One further defines:

$$SC^k_u = \bigcup_{k' \geq k} C^{k'}_u \text{ and } SC^k_{w(1)}, SC^k_v \text{ and } SC^k_{w(2)}$$

analogously.

Finally,

$$SC^k = SC^k_u \cup SC^k_{w(1)} \cup SC^k_v \cup SC^k_{w(2)} \underset{\text{by comment above.}}{\overset{8}{=}} \{(u,v,w,z) \in \mathbb{R}^4 \mid z = \pm k'(u^2+v^2+w^2)^{1/2}, k' \geq k\}$$

LEMMA 1:

Let k be fixed. $\exists\, \delta_u = \delta_u(k)$ s.t. : $B_{\delta_u}(0) \cap SC_u^k \cap C^*(4,0) = \{0\}$.

Proof

By substituting (1), (2) and (3) in the polymial P, one gets, as in 4.4(43), $k^3 w^3\ (A + |u|\ B) \underset{(9)}{=} 0$, where $|A| \geq$ coef. of z^4 in P, B(k) is a positive constant (B(k') < B(k) if k' > k). Therefore, by choosing u, s.t. $|u| < \dfrac{\text{coef. of } z^4}{B(k)}$ (therefore $|u| < \dfrac{\text{coef.of } z^4}{B(k')}, \forall k' > k$), one guarantees that (9) is satisfied iff $u = 0 (\Rightarrow v = w = z = 0)$. Take δ_u = coef. of $z^4/B(k)$.

Let k be fixed. Then:

LEMMA 2: $\exists\ \delta_{w(1)}$ s.t. $B_{\delta_{w(1)}}(0) \cap SC_{w_1}^k \cap C^*(4,0) = \{0\}$

LEMMA 3:

$\exists\ \delta_v$ s.t. $B_{\delta_v}(0) \cap SC_v^k \cap C^*(4,0) = \{0\}$

LEMMA 4:

$\exists\ \delta_{w(2)}$ s.t. $B_{\delta_{w(2)}}(0) \cap SC_{w(2)}^k \cap C^*(4,0) = \{0\}$

LEMMAS's 2,3 and 4 are proved as Lemma 1.

LEMMA 5:

Let k be fixed. $\exists\ \delta = \delta(k)$ s.t. $B_\delta(0) \cap SC^k \cap C^*(4,0) = \{0\}$.

Proof

Immediate from Lemmas 1/4 above.

LEMMA 6:

$\exists\ k \in \mathbb{R}^+, \varepsilon > 0$, s.t. $\{\alpha(t) |\ |t| < \varepsilon,\ t \neq 0\} \subset ([B\ (0) \cap SC^k] - \{0\})$.

Proof

Let $\alpha_z'(0) = D\ (\neq 0)$; $\alpha_u'(0) = A$, $\alpha_v'(0) = B$, $\alpha_w'(0) = C$. For small t, $\alpha_z'(t) \geq D/2$, $\alpha_u'(t) \leq 2A$ (or $\delta_A > 0$, if $A = 0$), $\alpha_v'(t) \leq 2B$ (or $\delta_B > 0$, if $B = 0$), $\alpha_w'(t) \leq 2C$(or $\delta_c > 0$, if $C = 0$). Like in Lemma 3 (4.4(44)), one gets

$$|\alpha_z(t)| \geq k\,(\alpha_u^2(t) + \alpha_v^2(t) + \alpha_w^2(t))^{\frac{1}{2}}, \text{ for } \begin{cases} |t| < \varepsilon_1, \text{say, and } \varepsilon_1 \text{ suff. small;} \\ k = 1/4 \quad \dfrac{|D|}{A^2+B^2+C^2} \end{cases}$$

therefore, by 8, $\alpha(t) \in SC^k$, $|t| < \varepsilon_1$.

Choose ε_2 s.t. $\{\alpha(t) \mid |t| < \varepsilon_2\} \subset B_\delta(0)$, ε_3 s.t. $\alpha(t) \neq 0$ if $|t| < \varepsilon_3$, $t \neq 0$ (possible since $\alpha_z'(0) \neq 0$), and $\varepsilon = \min\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$. This will do.

LEMMAS 5 & 6 $\Longrightarrow$ PROPOSITION 65' $\Longrightarrow$ Proposition 65 immediately.

□

PROPOSITION 66:

$v \in \mathcal{B}$ (as in Proposition 61) $\Longrightarrow$ $v \pitchfork C_f$.

Proof

Just like Propositions 23 and 40. We have to show that, for fixed (arbitrarily) $y \in C_f$, $v \pitchfork_y C_f$, and this reduces to proving that $v \pitchfork_y \chi_f(\mathcal{U}_{i_s}^{j_s} \cap M^d)$ in a number of separate cases, i.e. $i_s = 1,2,3$ or 4.

Case 1: $\boxed{i_s = 1}$

This is like cases 1 in Propositions 23 and 40: $\chi_f(\mathcal{U}_1^{j_s} \cap M^d) = N_1^{j_s}$ and $v[4](\mathbb{R}^4) \cap C_1^{j_s}[4] = \emptyset \Rightarrow v \pitchfork_y N_1^{j_s}$.

Case 2: $\boxed{i_s = 2}$

Let Γ, γ as usual. Since $\Gamma^{-1}(\chi_f(\mathcal{U}_2^{j_s} \cap M^d)) = \chi_{g=\gamma f}(\gamma(\mathcal{U}_2^{j_s} \cap M^d)) \subset C(2,2)$, one has:

$$\boxed{\varepsilon_s > 0 \text{ is s.t. } \Gamma^{-1}(0_y(\varepsilon_s)) \cap C(2,2) = \emptyset}_{\ominus} \quad \boxed{0_y(\varepsilon_s) \cap \chi_f(\mathcal{U}_2^{j_s} \cap M^d) = \emptyset}\,\star$$

i.e. $v \pitchfork_y (\chi_f(\mathcal{U}_2^{j_s} \cap M^d))$. Hence, it suffices to prove $\ominus$.

Let $\beta: I \to \mathbf{R}^4$ be a solution curve of v through y, $\alpha \stackrel{\text{by def.}}{=} \Gamma^{-1}\beta$. Now, $v[4](\mathbf{R}^4) \cap C_2^{j_s}[4] = \phi$ means $\tilde{I}(\hat{\alpha}) \not\stackrel{\odot}{\pitchfork} Q_2[4]$, since $v[4](y) \notin C_2^{j_s}[4]$

Therefore, since $\xi = \Gamma^{-1}(y)$ satisfies $\xi_u = \xi_v = 0$ and, by $\odot$, we have $(\xi_u, \xi_v, d\alpha_u/dt(0), d\alpha_v/dt(0), d^2\alpha_v/dt^2(0)) \neq (0,0,0,0,0)$, it follows that $(d\alpha_u/dt(0), d\alpha_v/dt(0), d^2\alpha_v/dt^2(0)) \neq (0,0,0)$ and hence by Proposition 63, $\oplus$ follows.

Case 3: $\boxed{i_s = 3}$

Let Γ, γ as usual. It follows, as above, that

$$\boxed{\varepsilon_s > 0 \text{ is s.t. } \Gamma^{-1}(0_y(\varepsilon_s)) \cap C(3,1) = \emptyset} \underset{\oplus}{\Rightarrow} \boxed{0_y(\varepsilon_s) \cap \chi_f(U_3^{j_s} \cap M^d) = \emptyset}$$

The proof of $\oplus$ is immediate from Proposition 64 and our hypothesis.

Case 4: $\boxed{i_s = 4}$

Like cases above (see also case 3, 4.4(45)), follows directly from Proposition 65. □

COROLLARY:

If $f: X \times \mathbf{R}^4 \to \mathbf{R}$ is generic, $\exists$ open and dense $\mathcal{B}$ s.t. $v \in \mathcal{B} \Rightarrow v \pitchfork C_f$.

4.4.5. Appendix to 4.4

This is not an integral part of any proof in this thesis, as it was pointed out in 4.4.0 (see 4.4.2, (I)) we just show below what is the motivation behind the definitions of $Q_i[r]$ $(r = 2,3,4,\ i = 2,\dots,r)$.

Cusp's case: $(Q_2[r],\quad r = 2,3,4)$

Cusp's equation: $\boxed{27v^2 = 8u^3}$ $\oplus$

As our curve (see 4.4.0,(I) α is constricted to $\alpha_u(0) = \alpha_v(0) = 0$, we will find (r+1)-2=(r-1) conditions on α'_u, α'_v, etc., imposed by the supposition that α runs into the cod. 1 strata, since the total number of conditions one needs, from 不 considerations, is r+1.

$\boxed{r = 2}$

(0 disc. controls)

$$\text{(i)}\begin{cases} \alpha_u(t) = \underset{\overset{\parallel}{\alpha'_u(0)}}{\alpha'_u} t + \alpha''_u t^2 + 0_3 \\ \alpha_v(t) = \alpha'_v t + \alpha''_v t^2 + 0_3 \end{cases}$$

(i) in $\theta \Rightarrow 27(\alpha'_v)^2 t^2 + 0_3 = 0$, therefore $\boxed{\alpha'_v = 0}$

Unique condition: $\boxed{\alpha'_v = 0}$; this generates the definition of $Q_2[2]$.

$\boxed{r = 3}$ substituting back $\alpha'_v = 0$ in (i),

(1 disc. controls)

$$\text{(ii)}\begin{cases} \alpha_u(t) = \alpha'_u t + \alpha''_u t^2 + 0_3 \\ \alpha_v(t) = \alpha''_v t^2 + 0_3 \end{cases}$$

and (ii) in $\theta \Rightarrow 8(\alpha'_u)^3 t^3 + 0_4 = 0$

therefore $\boxed{\alpha'_u = 0}$

Conditions: $\boxed{\alpha'_v = \alpha'_u = 0} \longrightarrow Q_2[3]$

$\boxed{r = 4}$ substituting $\alpha'_u = 0$ in (ii):

(2 disc. controls)

$$\text{(iii)}\begin{cases} \alpha_u(t) = \alpha''_u t^2 + 0_3 \\ \alpha_v(t) = \alpha''_v t^2 + 0_3 \end{cases}$$

and (iii) in $\theta \Rightarrow 27(\alpha''_v)^4 t^4 + 0_5 = 0$

therefore $\boxed{\alpha''_v = 0}$

Conditions: $\boxed{\alpha'_v = \alpha'_u = \alpha''_v = 0} \rightarrow Q_2[4]$

Swallowtail's case: $(Q_3[r];\ r = 3,4)$

Swallowtail's equation: $\boxed{256w^3 - 27v^4 + 4u(32v^2w + 4u^3w - 3uw^2 - u^2v^2) = 0}$

(actually contains it, but this isn't relevant here)

Curve α satisfies 3 conditions, $\alpha_u(0) = \alpha_v(0) = \alpha_w(0)$, therefore we need $(r+1)-3 = (r-2)$ conditions.

$\boxed{r = 3}$

(0 disc. controls)

$$(i)\begin{cases} \alpha_u(t) = \alpha_u' t + \alpha_u'' t^2 + 0_3 \\ \alpha_v(t) = \alpha_v' t + \alpha_v'' t^2 + 0_3 \\ \alpha_w(t) = \alpha_w' t + \alpha_w'' t^2 + 0_3 \end{cases}$$

(i) in $\theta \Rightarrow 256(\alpha_w')^3 t^3 + 0_4 = 0$, therefore $\boxed{\alpha_w' = 0}$

Unique condition: $\boxed{\alpha_w' = 0} \to Q_3[3]$

$\boxed{r = 4}$

(1 disc. controls)

Substituting back in (i):

$$(ii)\begin{cases} \alpha_u(t) = \alpha_u'(t) + \alpha_u'' t^2 + 0_3 \\ \alpha_v(t) = \alpha_v' t + \alpha_v'' t^2 + 0_3 \\ \alpha_w(t) = \alpha_w'' t^2 + 0_3 \end{cases}$$

(ii) in $\theta \Rightarrow 27(\alpha_v')^4 t^4 + 0_5 = 0$ therefore $\boxed{\alpha_v' = 0}$

Conditions: $\boxed{\alpha_v' = \alpha_w' = 0} \to Q_3[4]$

Butterfly's case: $(Q_4[4])$ Equation: $KZ^4 + \text{higher terms} = 0$ θ

$\boxed{r = 4}$

(0 disc. controls)

$$(i)\begin{cases} \alpha_u(t) = \alpha_u' t + 0_2, & \alpha_v(t) = \alpha_v' t + 0_2 \\ \alpha_w(t) = \alpha_w' t + 0_2, & \alpha_z(t) = \alpha_z' t + 0_2 \end{cases}$$

(i) in $\theta \Rightarrow K(\alpha_z')^4 t^4 + 0_5 = 0$ therefore $\boxed{\alpha_z' = 0}$

Condition: $\boxed{\alpha_z' = 0} \to Q_4[4]$

Note: Case $r = 5$: just go one step further in each of the above cases; for instance, in the one parameter family of butterflies we would get $\boxed{\alpha'_w = \alpha'_z = 0}$, giving $Q_4[5]$ as $\{x_1,\ldots,x_{30}) \mid x_1 = x_2 = x_3 = x_4 = x_8 = x_9 = 0\}$.

4.5. H_2(see 1.2(1)) is generic

PROPOSITION 67:

Let f be generic as before. $\exists$ an open and dense, $\mathcal{N} \subset V(\mathbb{R}^r)$ s.t. $v \in \mathcal{N} \Rightarrow S(v) \cap C_f = \emptyset$.

Proof

We have $C_f = \bigcup_{i,j} N_i^j$, a closed denumerable union of cod $i \geq 1$ submanifolds (see Proposition 6 in 4.3(6)). Set $\underset{\mathbb{R}^r \times \mathbb{R}^r}{\overset{\cap}{C_f^*}} = \underset{\mathbb{R}^r}{C_f \times \overset{\cap}{\{0\}}}$, which is therefore a denumerable (closed) union of manifolds with cod. $(i+r) > r$.

Set $\mathcal{N} = \{v \mid j^0 v \pitchfork (N_i^j \times \{0\}), \forall i,j\}$, open and dense from lemma 2 in (3.3(2)). Finally $v \in \mathcal{N} \Rightarrow j^0 v(x) = (x, v(x)) \notin C_f^*$, i.e. $x \in C_f \Rightarrow v(x) \neq 0$, therefore $S(v) \cap C_f = \emptyset$.

4.6. CONCLUSIONS:

PROPOSITION 68

$\exists$ and open and dense (in $V(\overset{\mathbb{R}^r}{\overset{\|}{C}})$) set $\mathcal{A}$, s.t. $v \in \mathcal{A} \Rightarrow$ v satisfies H_1 and H_2 in 1.2(1), $\forall\ r \in \{1,2,3,4\}$ fixed.

Proof

Follows immediately from Corollary in 4.3(14), Corollaries at the end of Sections 4.4.1 - 4.4.4 and Proposition 67 above.

PROPOSITION 69:

$\overline{\mathcal{V}(\mathbb{R}^r)}$ is open in $\mathcal{V}(\mathbb{R}^r)$

Proof

Let $v \in \overline{\mathcal{V}(\mathbb{R}^r)}$. $\exists$ K s.t. $|v(x)| < K$, $\forall x \in \mathbb{R}^r$. Consider the open set $B_1(v) = \{v' \mid d(\underbrace{j^0v'(x)}_{=(x,v'(x))};\underbrace{j^0v(x)}_{=(x,v(x))}) < 1, \forall x\}$. If $v' \in B_1(v)$, then

$|v'(x) - v(x)| < 1$, $\forall\, x \in \mathbb{R}^r$, therefore $v' \in \mathcal{V}(\mathbb{R}^r)$

THEOREM 2:

Let $r \leq 4$ be fixed, $n = 1$.

$\exists$ $\mathcal{V}^*$, open and dense in $\overline{\mathcal{V}(\mathbb{R}^r)}$, $\mathcal{V}^* \subset \mathcal{V}_f$

Proof

Set $\mathcal{V}^* = \mathcal{A} \cap \overline{\mathcal{V}(\mathbb{R}^r)}$. By definition and Proposition 67, $\mathcal{V}^* \subset \mathcal{V}_f$. It is also immediate that $\mathcal{V}^*$ is open and dense in $\overline{\mathcal{V}(\mathbb{R}^r)}$, from Propositions 68 and 69.

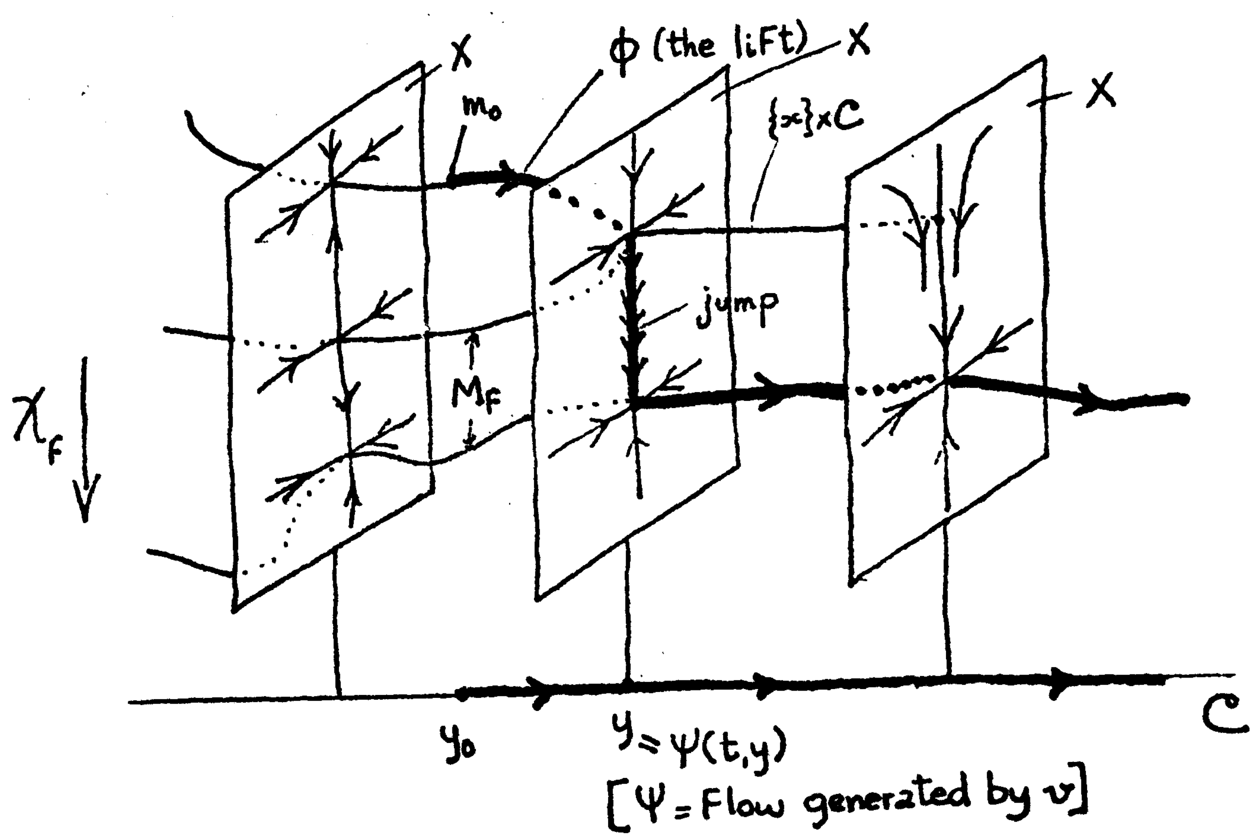
CHAPTER 5

5.1. PROOF OF THEOREM 3

The purpose of this chapter is to prove Theorem 3 (see Chapter 1)

LEMMA A: Let r = 1, n = 2, f generic (see Chapter 1); suppose that $V = \{v_y\}$, the (one-parameter) compatible family, is generic in the sense of [12] (in particular, $v_y \in [\overset{\text{Kupka}}{K}.\overset{\text{Smale}}{S}] \cup \Sigma_1$, $\Sigma_1 = Q_1^1 \cup Q_1^2 \cup Q_2 \cup Q_3$ - see [12], pages 35, 19, 25, 9 and 26). Then, $\exists\ V^*$, open and dense in $V(c)$, s.t., $\forall\ v \in V^*$, fixed, $\exists$ unique lift $\phi: \mathbb{R}_0^+ \times M^n \to \overline{M^n}$, with properties as in Theorem 1.

We would like to comment that the proof of the existence and uniqueness of the lift is exactly as before (Chapter 2) with the only difference that, to perform the 'jumps' (see picture), we use a global description of the change of the phase space of v_y, 'around' a singularity of χ_f, obtained as a direct consequence of [12]: see picture below. See 5.2 for a counter-example showing that f generic only is not sufficient.



We make these ideas precise:

PROOF OF LEMMA A:

Let X be a compact 2-dimensional manifold, $v \in \mathcal{V}(X)$, $x \in X$ a saddle node of v (see [12], page 16); we can suppose, w.l.o.g., that the flow of v, around x (in a ball $B_\delta(x) \subset X$, which can locally be supposed to be $\mathbb{R}^2$), looks like (see [12]), Figure 1 below.

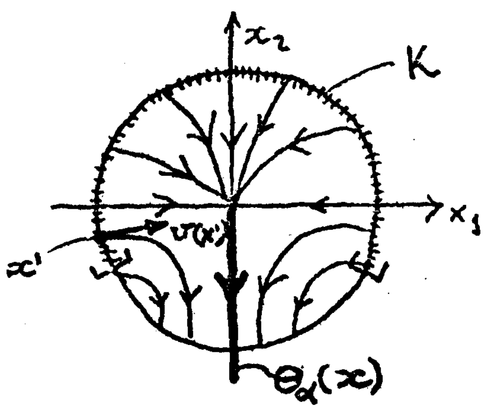


Fig.1

In particular, there is a unique non-trivial (i.e., $\neq$ from x itself) orbit θ - which we will call $\theta_\alpha(x)$ - s.t. x is the α-limit of θ. Also, a set $K \subset S_\delta(x)$, as in Figure 1, s.t., at every point x' of K, v(x') "enters" $B_\delta(x)$. (with "enters" defined in the obvious way).

We first establish some lemmas, before proving Lemma A.

LEMMA 1:

Let X, v, x as above, $v = v_y$, $V = \{v_y\}$ as in Lemma A. Then the w-limit of $\theta_\alpha(x)$ is a sink.

Proof

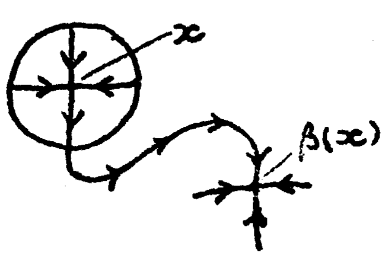


Fig.2

First, from Remark 3, in 2.2.(7), we know that the w-limit of $\theta_\alpha(x)$ is just a point, a singularity of v_y. Now, since V is generic, in the sense of [12], $v_y \in [K.S] \cup Q_1^1 \cup Q_1^2 \cup Q_2 \cup Q_3$ (see [12]); from the definitions of these sets, one sees immediately that $v_y \in Q_1^1$.

Hence, in particular, all other singular points of v_y are hyperbolic (see:1),pg 19, of [12]) and there are no saddle connections, proving our lemma.

□

LEMMA 2:

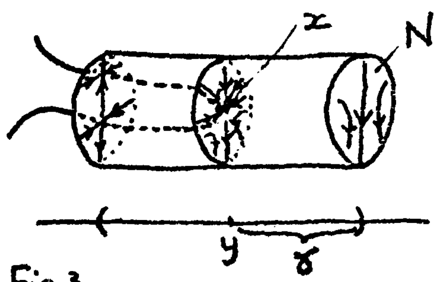


Fig.3

Let X, v, x, V as in Lemma 1. Then $\exists$ neighbourhood N of x (w.l.o.g., $N \supset B_\delta(x)$ and a $\gamma \in \mathbb{R}^+$ s.t. v_{y+t} satisfies either [(i)+(ii)] or [(i)+(ii)'] below, $\forall t$ s.t. $|t| < \gamma$.

(i) v_{y+t} has a unique singularity in N $\Leftrightarrow t = 0$.

(ii) v_{y+t} has two sing. points in N, one saddle and one node - i.e., sink or source - if $t < 0$; no sing. points in N, if $t > 0$.

(ii)' As (ii) but with $t < 0$ and $t > 0$ interchanged.

Proof

From Lemma 3.2 of [12], $\exists$ N, neighbourhood of x, B, neighbourhood of v_y (in $V(X) - \mathcal{X}$ in the notation of [12]-,with the topology as defined in [12] and a function $f: B \to \mathbb{R}$ s.t., $\forall$ $v \in B$:

(i) $f(v) = 0 \Leftrightarrow t = 0$

(ii) $f(v) > 0$ if v has two sing. in N, one saddle and one node;

$f(v) < 0$ if v has no sing. in N.

Take γ small enough so that $v_{y+t} \in B$, $\forall t$ s.t. $|t| < \gamma$, and also so that v_{y+t} has a saddle node on N iff $t = 0$ (C_f consists of isolated points). Therefore $f^*(t) = f(v_{y+t})$ has no zeroes on $(-\gamma,0)$ and $(0,+\gamma)$. It has to change sign at 0, otherwise it is very easy to perturb the family so to avoid this intersection with Σ_1 (this means it is non-transversal to Σ_1, violating 2), page 37, [12]). Therefore, either $f^* > 0$ if $t < 0$ and $f^* < 0$ if $t > 0$ (which is (ii)) or $f^* < 0$ if $t < 0$ and $f^* > 0$ if $t > 0$ (which is (ii)' proving the lemma.

□

Let $X, v, x, V,\ s(x) \in M^2_f$ (see 1.1(1)) as in Lemma 1 above, f generic. From Proposition 6 (2.1(15)) $\exists$ neighbourhood $\mathcal{W}$ of s(x) in M^2 and a $\delta > 0$ s.t. $B_\delta(\tilde{x}) \subset$ in-set $[\Phi_{-\nabla f_{\tilde{y}}}](\tilde{x})$ (= in-set $[\Phi_{v_{\tilde{y}}}](\tilde{x})$, by the definition of compatibility in 1.1(3)), $\forall \tilde{m} = (\tilde{x},\tilde{y}) \in \mathcal{W}$. Let $\mathcal{U} = \mathcal{U}(s(x),\mathcal{W}) =$

$= \bigcup_{\tilde{m}\in\mathcal{W}}$ in-set $[\Phi_{v_{\tilde{y}}}](\tilde{x})$.

LEMMA 3:

$\mathcal{U}$, as defined above, is open in $X \times C$.

Proof

Let $\emptyset$ be the C^∞ flow induced on $X \times C$ by f, by $\emptyset(t(x,y)) = \emptyset[y](t,x)$, and Ψ its time 1 diffeomorphism.

Set $B_\delta(\mathcal{W}) = \bigcup_{\tilde{m}=(\tilde{x},\tilde{y})\in\mathcal{W}} B_\delta(\tilde{x})$, open. It is easy to check that

$\mathcal{U} = \bigcup_{k=1}^{\infty} \psi^{-k}\ (\underbrace{B_\delta(\mathcal{W})}_{\text{open}})$, hence the lemma.

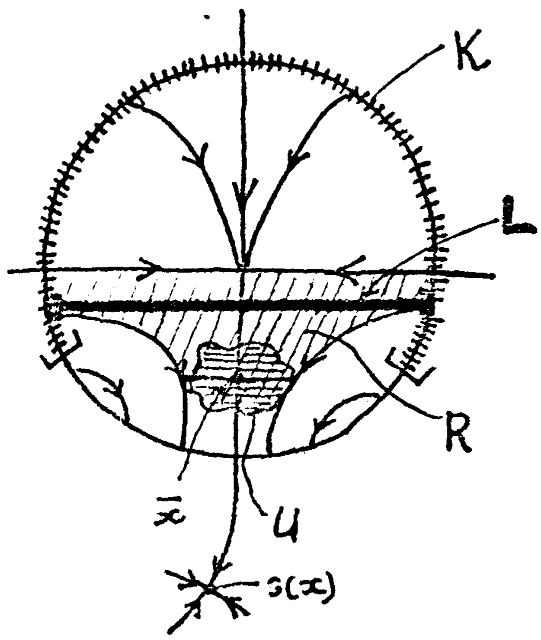
LEMMA 4:

$\exists\ \eta \in \mathbb{R}^+$ s.t. v_{y+t} enters $B_\delta(x)$ on K (as in note previous to Lemma 1) $\forall$ t with $|t| < \eta$.

Proof

Immediate, since K is compact and $v_{y'}(x')$ is continuous on x' and y' $\square$

LEMMA 5:



Given K as above (everthing as before), $\exists$ a compact L, as in picture, s.t. $x^* \in L \Rightarrow x^* \in$ in-set $[\Phi_{v_y}](s(x))$.

Proof

Since in-set $[\Phi_{v_y}](s(x))$ is an open submanifold of X, given $\bar{x} \in \theta_\alpha(x) \cap B_\delta(x)$, $\exists$ $U \subset B_\delta(x)$, $\bar{x} \in U$, $U \subset$ in-set $[\Phi_{v_y}](s(x))$. Therefore the region R, and in particular L, as claimed, is contained in this set (see picture). $\square$

LEMMA 6:

$$\exists\, \zeta \in \mathbb{R}^+ \text{ s.t. } L \times (y-\zeta;y+\zeta) \subset \mathcal{U}.$$

Proof

$L \times \{y\} \subset \mathcal{U}$, by Lemma 5, and $\mathcal{U}$ is open, (in $X \times C$), by Lemma 3. Hence, for each (x^*,y), $x^* \in L$, $\exists$ a neighbourhood of (x^*,y) contained in $\mathcal{U}$. Their union covers the compact $L \times \{y\}$. Extract a finite sub-cover; it is easy to see that their union contains a set of the form $L \times (y-\zeta;y+\zeta)$, as required.

$\square$

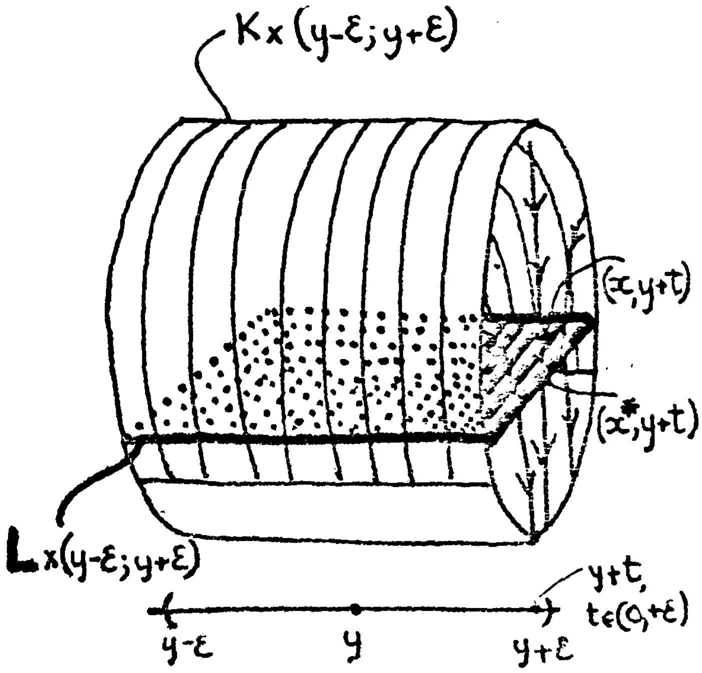
LEMMA 7:

Let $\varepsilon = \min\{\gamma,\zeta\}$ (γ,ζ defined in Lemmas 2 and 6 above). Then

either [I] $(x,y+t) \in \mathcal{U}$, $t \in (0,+\varepsilon)$

or [II] $(x,y+t) \in \mathcal{U}$, $t \in (-\varepsilon,0)$.

Proof

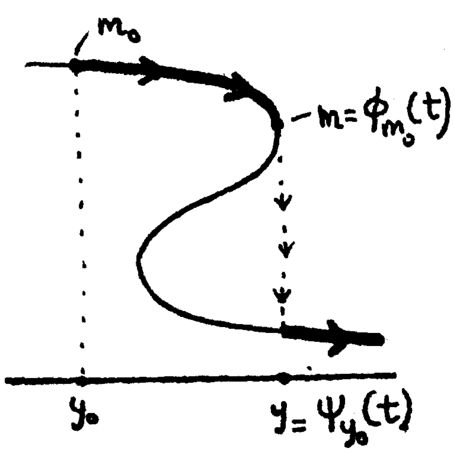


Suppose that one has [(i)+(ii)] satisfied in Lemma 2. Let $t \in (0,+\varepsilon)$ be fixed. By Lemma 2, v_{y+t}, $t \in (0,+\varepsilon)$ $(\varepsilon \leq \gamma)$, has no singularities in $B_\delta(x)(\subset N)$. Therefore the orbit of x (under v_{y+t}) must leave $B_\delta(x)$. It has to do so outside K, and has therefore to cross L at a point $(x^*,y+t)$. By Lemma 6, $(x,y+t) \in U$; this is [I]. Case [II] comes from supposing [(i)+(ii)'] satisfied in Lemma 2. □

PROOF OF LEMMA A:

Construct, as before, an open and dense set V^* s.t., $\forall v \in V^*$ fixed, one has $S(v) \cap C_f = \emptyset$ and, if $y \in C_f$ is fixed, $\exists$ $\varepsilon > 0$ s.t. $O_y(\varepsilon) \cap C_f = \emptyset$. We want to show that, if $v \in V^*$ is fixed, then there is a unique lift $\phi:\mathbb{R}_0^+ \times M^n \to \overline{M^n}$ satisfying the properties as in Theorem 1.

A quick look at the proof of Theorem 1 shows that the only point where one needs more than the above hypothesis is in Lemma 6. This is 'jumping' lemma, in the sense that one has already constructed ϕ_{m_0} up to m, in Lemmas 1→5 (see Chapter 2), and wants then to perform the 'jump' (see picture) in a unique well-defined way.



We will therefore outline how the proof of Lemma 6 would go in the present situation.

Let $m = (x,y) \in \overline{M^n}$, $y \in C_f$. If $(x,y) \in M^n$, $\exists\ \varepsilon > 0$, s.t. $x \notin \operatorname{sep} \Phi_y$ (see Lemma 6), as before, via Proposition 6 in 2.1(15), so that the proof of Lemma 6 is exactly the same. Assume therefore that $m = (x,y) \in M^d$ (see 2.2(6)), $m = \phi(t, \underset{\in M^n}{m_0})$, say. We also assume, w.l.o.g., that $v(y) > 0$, so that

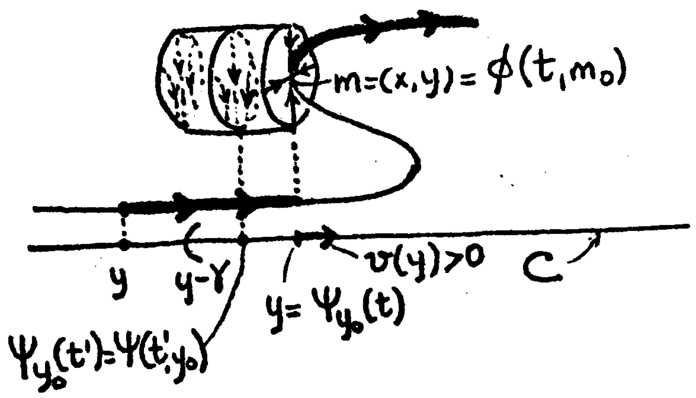
ψ_{y_0} (ψ stands for the flow generated by v) $\underbrace{\text{is strictly crescent at } t}_{(a)}$. (see 1.2(1) for the notation).

From Lemma 2, above, $\exists$ neighbourhood N ($\supset B_\delta(x)$) and a $\gamma > 0$ s.t. either [(i)+(ii)] or [(i)+(ii)'] hold, if $|t| < \gamma$.

Suppose [(i)+(ii)'] holds. In particular, $\nexists$ sing. of v_{y*} in N,

$\forall\ y^*$ s.t. $\underbrace{0 < y - y^* < \gamma}_{(b)}$. Therefore, since the x-component of $\phi(t';m_0)$,

$\Pi_x(\phi(t',m_0))$, must be a singular point of $v_{\psi(t',y_0)}$ and since, for $0 < t-t' < \xi$ (some small ξ)



we have, from (a),

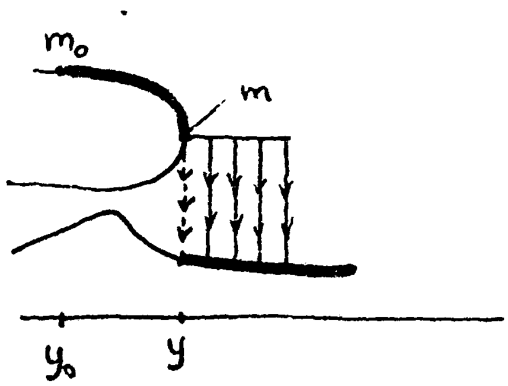
$$0 < \underbrace{\psi_{y_0}(t)}_{\overset{\shortparallel}{y}} - \psi_{y_0}(t') < \gamma$$

it follows, from (b), that:

$$|\Pi_x(\phi(t',m_0)) - \underset{\Pi_x(\phi(t,m_0))}{\underset{\shortparallel}{x}}| > \delta, \ \forall\ t' \text{ s.t. } 0 < t-t' < \xi,$$

therefore ϕ_{m_0} is not left continuous at t; this contradicts (4), 1.2(1), so that our supposition is false.

Therefore, [(i)+(ii)] holds. Since ψ_y is strictly crescent at $0(v(y) > 0)$, $\exists \varepsilon^* > 0$ s.t. $0 < t^* < \varepsilon^* \Rightarrow$ $\Rightarrow 0 < \psi_y(t^*) - \underbrace{\psi_y(0)}_{=y} < \varepsilon$, i.e. $\psi_y(t^*) \in (y, y+\varepsilon)$,



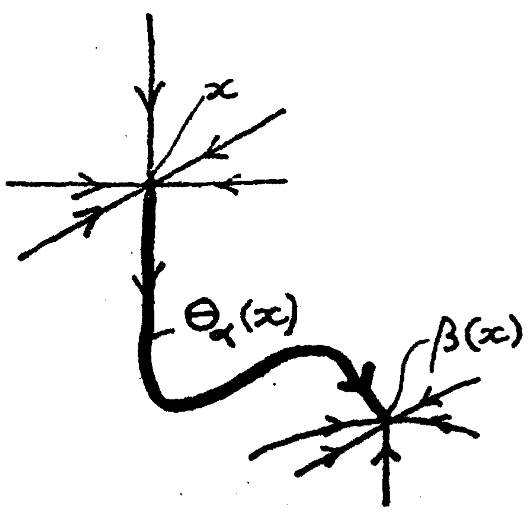
ε, as in Lemma 7, given. By Lemma 7, $x \notin \text{sep}\, \Phi_{\psi_y(t^*)}$, $t^* \in (0,\varepsilon^*)$. This is precisely what is used in the proof of Lemma 6 (see 2.2(7)) and therefore we are done. □

LEMMA B:

Let n be arbitrary, fixed, $n \in \mathbb{N}$, $r = 1$, V be C^1 generic in the sense of Theorem A of [13] (see §4, page 579), everything else as in Lemma A. Suppose that $v = v_y$ has, at x, a saddle node of type 2 with dim. (stable man.) = $(n-1)$, dim. (centre man.) = 1 (see [13] 2.1.a, pg. 564 for these definitions). As before, $\exists$ a unique non-trivial orbit $\theta_\alpha(x)$ (which is, in this case, the 'expanding' part of the one-dimensional W^c - see page 564 of [13] and picture below). Then the w-limit of $\theta_\alpha(x)$ is a sink.

Proof.

This is the equivalent to Lemma 1 above in the n-dimensional case (n not necessarily equal to 2). This is an immediate consequence of (3) in the above mentioned theorem. That is, since the (V)-unfolding-unstable (denoted by ξ, in notation of [13]) manifold of the saddle node (see 570, of [13], for this definition), has to meet the (V)-unf. stable of $\beta(x)$ (i.e. associated to $\beta(x)$) transversally, as stratified sets (see 571 of [13] for the stratification



of the saddle node) in particular the strata $\theta_\alpha(x)$ (corresponding to

$W_o^u - W_o^{uu}$ in Soto's notation) has to meet the stable manifold of $\beta(x)$
$\overset{\shortparallel}{\{x\}}$

transversally; i.e., there can be no saddle node connections; therefore $\beta(x)$ is a sink.

□

PROOF OF THEOREM 3

Lemma 2 as above carries on as in the case n = 2. (see (A) and (B) in 3.1, pages 569/570, of [13]). Lemma 3 was not dependent on n. Lemmas 4→7 admit the obvious generalizations, so that the proof of Theorem 3 is then carried in precisely the same way as outlined in the proof of Lemma A.

Appendix: We prove an alternative version of Theorem 3.

THEOREM 3'

As Theorem 3, but with the assumption that the family $V = \{v_y\}$ is generated by generic f substituted by the assumption that V is a C^1 family of gradient vector fields.

Proof

One first writes down the 'natural' equivalences as follows: $M \underset{\text{def.}}{=} M_v$ (to replace old $M = M_f$) is the set of singularities of $\{v_y\}$, $y \in C$; M^k, the set of hyperbolic singularities, s.t. dim (stable man.) = k; $\chi \overset{\text{def}}{=} \chi_v$ (to replace old χ_f) is the restriction of Π_C to M, as before. From [12]/[13], one has that M is a cod. n (i.e. 1 dimensional) sub-manifold of X × C, and the set C_v (critical values of χ, as before) is a cod.1 submanifold of C, i.e., a set of isolated points, in our case (the 'fold' points).

We remark that Proposition 1→6 (in 2.1) carry out without any problems. Therefore, the proof of the lifting theorem in 2.2 can be repeated up to Lemma 6, as explained in the proof of Theorem 3 above; the rest can be carried out by repeating the proofs of Lemmas A, B above. These are absolutely the same; the only crucial detail is that the gradient character of the dynamics has to be re-used in Lemma 1, above, otherwise Remark 3 in 2.2(7) can not be applied (as a matter of fact Lemma 1 is false if we drop the gradient hypothesis). □

Note:

Theorem 3' is perhaps a more 'natural' one, in the sense that it deals only with one type of genericity. The imposition of 'gradient' may not be too restrictive. See comment 4, on page 98 of [6].

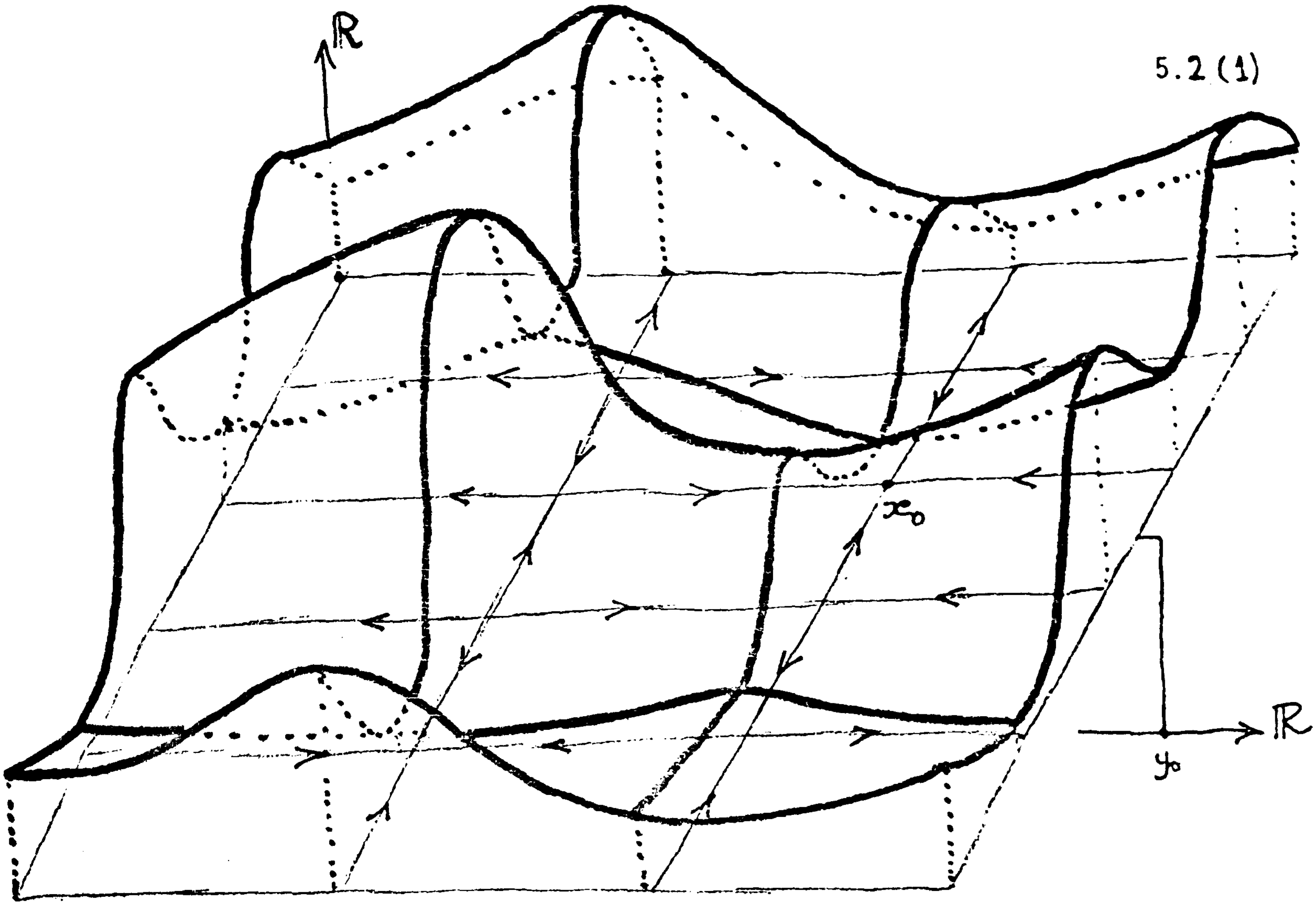
5.2. An example

The purpose of the example below is to show that, if $n > 1$, there is no hope that 'f generic', in the sense of [16] - i.e. in Thom's sense, would be enough, as far as proving theorems 1 and 2 (see Chapter 1) is concerned.

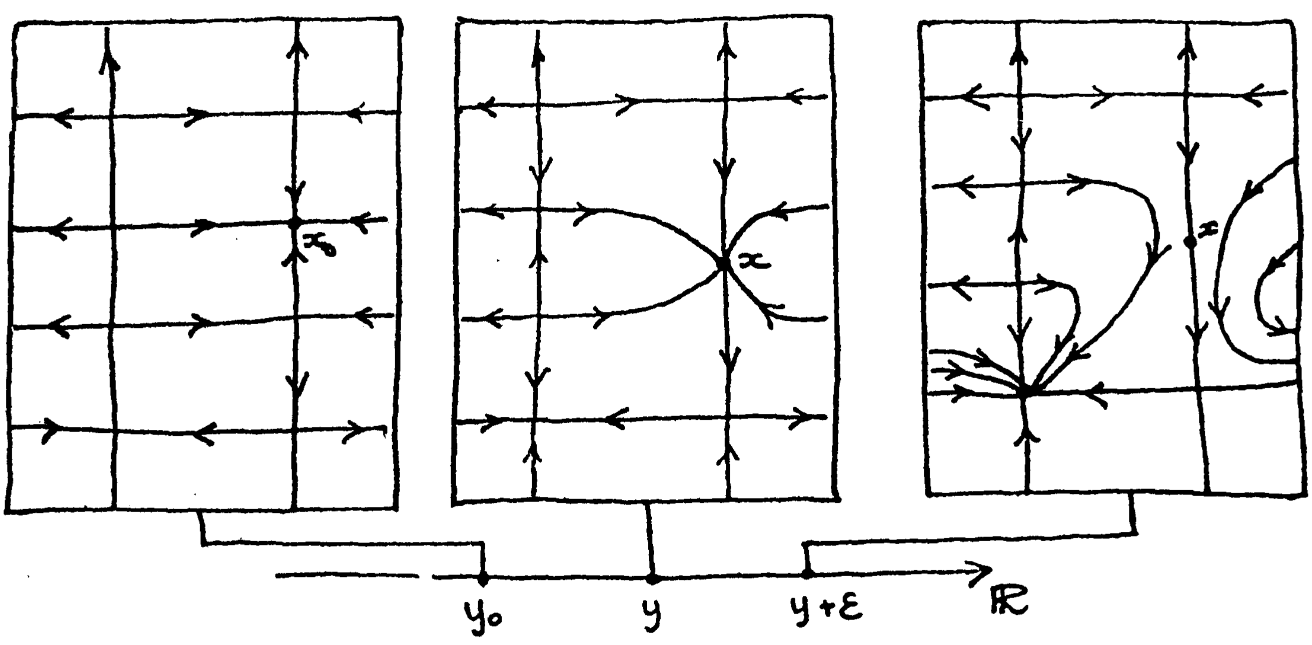
The reason for this is that 'f-generic' is a concept related with the singularities of $-\nabla f_y$, $y \in C$, at a germ level, whereas in the 'separatrix' problem one has to deal with (in general) a global problem, if $n > 1$.

Our example is a function $f: T^2 \times \mathbb{R} \to \mathbb{R}$, generic, but such that the conditions necessary for the existence and uniqueness of ϕ in the sense of Theorem 1 are not met.

We draw f/y_0, below, $y_0 \in \mathbb{R}$ fixed.



We now show what happens when one increases the y; i.e., we will draw some pictures to illustrate how f is defined to the 'right' of y_0:



To the left of y_0 and to the right of $y+\varepsilon$, f_y is defined so that the phase space is not altered. f is clearly generic. However ϕ_{m_0} ($m_0 = (x_0,y_0)$) can not be continued beyond m = (x,y), since x 'finds itself' in a separatrix.

CHAPTER 6

In this chapter we will make comments of a speculative nature.

We first would like to consider the problem of choice of $\mathcal{O}$, the space of objects determining the dynamics in the state space. This is a most important problem, because it deals with the question of deciding the context in which genericity (of those objects) is going to be considered.

We recall that the possibilities we have been considering here are:

(I) to look at $\mathcal{O}$ as a space of potential functions.

(II) to look at $\mathcal{O}$ as a space of r-parameter families of gradient dynamical systems.

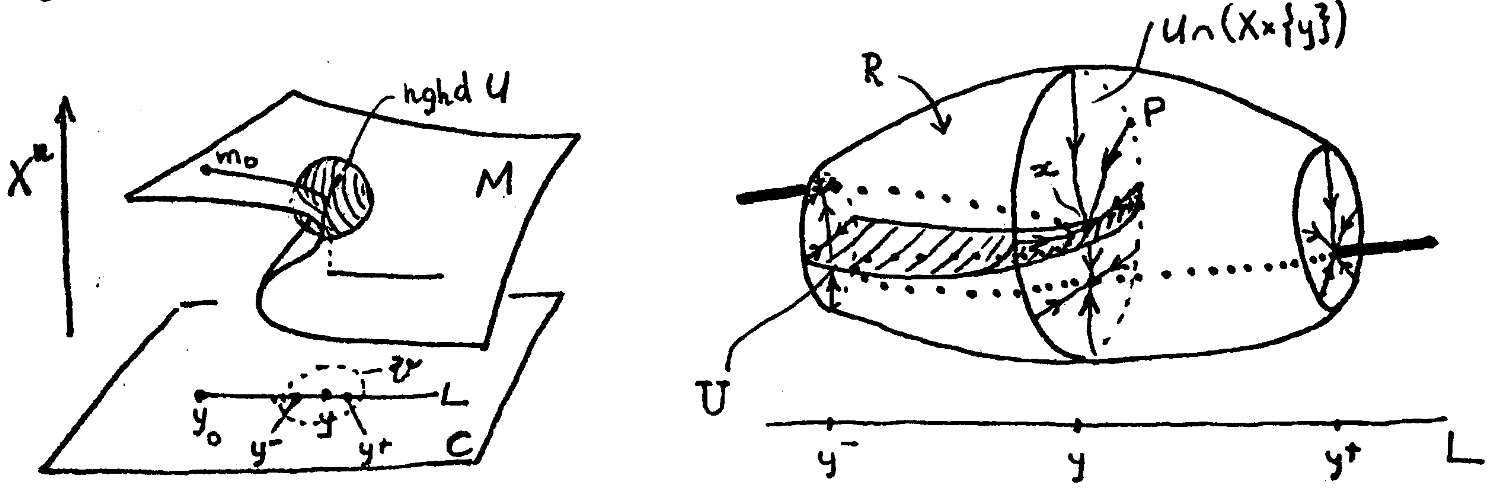
As J. Guckenheimer has pointed out in [6], (I) and (II) are not equivalent, even at the local level; he shows this through an example, with $n = 2$, $r = 3$. He further comments 'Thom assumes that one can pass from the bifurcation of gradient dynamical systems to the unfolding of their potential functions in studying catastrophes. The point which we raise here is that the maths of the situation is not sufficient to justify this assumption' (see [6], page 96).

We show in Chapters 2-4 that, if $n = 1$, the potential function approach is completely justifiable, as far as the problem we considered is concerned. If $n > 1$, however, genericity related to universal unfolding of potential functions at map-germ level is not sufficient, because the 'separatrix problem' is global, in the first place, and, even at a local level, the definitions of universal unf. of map germs relate to diffeomorphisms, and separatrices of gradients of potential functions are not 'preserved' under diffeomorphisms.

This suggests that in this case, as we already did in Chapter 5, the context as in (II) should be considered.

The problems here seem to be two-fold. First, one does not have at hand (as far as we know) a theory of bifurcation of r-parameter gradient dynamical systems for $r > 1$, n arbitrary. Second, even if Soto's results ([12],[13]) have a 'natural' generalization for $r > 1$, it is not clear that vector fields 'generic' in this sense would be well behaved with respect to the delicate transversality (of union of in-sets of saddles with $\{x\}\times C$ 'type' sets) condition needed to generalize Theorem 2.

The second comment we would like to make is that, in spite of the general observations as above, there is a case where we can solve the 'separatrix problem' within the context of Chapter 2-4 (i.e. that of (I)), even if $n > 1$, $r > 1$. This is when, at points where 'jumps' have to be performed, one knows that the only scpatrices one has to worry about are 'generated' in a neighbourhood of the jump point itself; i.e., there is no 'global' sepatrix problem.



[Cross section across L; notice that at any P the vector field enters R. We suppose that this happens for all L with non $\emptyset$ intersection with V-see picture - so that no 'global' separatrix problem arises]

From the picture above one sees that the set S one has to 'avoid' is the ('locally generated') 3 dimensional union of sections (as the one in picture) $U^{\cdot}$(2 dimensional). In general, we will have to 'avoid' a $[(m+r)-1]$ dimensional manifold. In this case, it seems likely that invariant manifold theory will show that S is transversal to $\{x\} \times C$. This would allow one to define the germ manifolds of 4.3 and hopefully proceed in the same way as there, solving the problem of 'avoiding separatrices' (which is the only one which depends upon n).

Thirdly, one can remark that generically in θ', in some sense, it is reasonable to expect intersections of S, as above, with $\{x\} \times C$ to be transversal; so that germ manifolds of codimension at least 1 could be defined, and the problem solved. The difficulty is how to express that condition mathematically and prove its genericity.

Finally, we remark that the question of choice of θ' has been considered within the framework of the 'max.delay convention'; to other conventions would correspond other 'natural' choices.

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