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CIRCUIT SIZE IS NONLINEAR IN DEPTH

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Abstract.

Two fundamental complexity measures for a Boolean function f are its circuit depth d(f) and its circuit size c(f). It is shown that $c \geq \frac{1}{4} d.\log_2 d$ for all f.

1. Introduction.

We consider acyclic Boolean circuits for $B_n = \{f \mid f : \{0,1\}^n \to \{0,1\}\}$ over the basis B_2. Two complexity measures can be defined for $f \in B_n$ as follows :

$c(f)$ = minimum number of gates of a circuit to compute f,

$d(f)$ = minimum depth of a circuit to compute f,

where the depth of a circuit is the maximum number of gates in a path of the circuit.

These measures satisfy the obvious relations

$$2^d > c \geq d,$$

and the example of n-argument conjunction demonstrates the optimality of the first inequality. In this paper we improve the second to $c \geq \frac{1}{4} d.\log_2 d$ as $d \to \infty$.

McColl and Paterson [1] have shown that $d(f) \leq n + 1$ for all $f \in B_n$. If f depends on all its n arguments then clearly $c(f) \geq n-1$ and therefore our result can be useful for only a small range of complexities. However, we hope that the novel construction introduced in the proof will be of interest.

2. Preliminaries.

With any circuit Z, a directed acyclic graph can be associated in the usual way. Nodes of the graph correspond either to inputs or to logical gates. Let $e(Z)$ be the number of arcs joining pairs of gate nodes in the graph of Z. We define

$$D(z) = \max \{d(f) \mid f \text{ is computed by a circuit } Z \text{ with } e(Z) \leq z\} ,$$

and

$$A(d) = \max \{z \mid D(z) \leq d\} .$$

Lemma. For all $z > 0$, $D(z) \leq 1 + D(z-1)$.

Proof. For any circuit Z with $e(Z) = z > 0$, consider one of its gates that has only input variables as inputs, and replace this gate by a new input variable. The resulting new function is computed by a circuit Z_1 with $e(Z_1) \leq z-1$ and so has depth at most $D(z-1)$. It follows that $D(z) \leq 1 + D(z-1)$ since the original function requires a depth of at most one more □

3. Main result.

Theorem. For all Boolean functions,

$$c \geq \tfrac{1}{4}\, d.\log_2 d - O(d)$$

Proof. Suppose Z is a circuit of minimum size computing a function f at gate g_o, and suppose $e(Z) = z > 0$. We consider partitions of the gates of Z into sets X and Y such that no gate of Y precedes a gate of X. If Y is non-empty then $g_o \in Y$. Let $M \subseteq X$ be the set of gates of X adjacent to gates of Y and let $m = |M|$. We denote by X the circuit with X as the set of gates, and arcs and inputs as in Z. We denote by Y the circuit with Y as the set of gates, with the inputs of Z together with new inputs corresponding to each node of M as its set of inputs and arcs as in Z. If $e(X) = x$ and $e(Y) = y$ then we have

$$x + y + m \leq z \qquad (1)$$

since each node of M accounts for at least one arc from X to Y.

We wish to select a partition so that x and y are nearly equal. The transference of one gate from X to Y reduces x by at most two and m by at most one, therefore we may choose a partition such that

$$|2x + m - z| \leq 2 \tag{2}$$

If we define $v = \max\{x, y\}$, then from (1) and (2) we deduce that

$$2v + m \leq z + 2 \tag{3}$$

Since X is a circuit for each of the functions computed at nodes of M, each of these may be computed in depth D(x), by the definition of D. By composing these circuits with a minimal depth circuit equivalent to Y we can construct a circuit for f which establishes

$$d(f) \leq D(x) + D(y) \leq 2D(v) \tag{4}$$

An alternative circuit for f is designed as follows. For each vector $\underset{\sim}{c} \in \{0,1\}^m$, replace each node of M (under some fixed ordering) in Y by the corresponding constant in $\underset{\sim}{c}$ and simplify Y by absorbing these constants into the gates. Let $Y_{\underset{\sim}{c}}$ be the resulting circuit and $f_{\underset{\sim}{c}}$ the function it computes at g_o. Thus $d(f_{\underset{\sim}{c}}) < D(y)$. For each $\underset{\sim}{c}$, let $\delta_{\underset{\sim}{c}}$ be the function which is 1 if and only if the nodes of M in X have the values corresponding to $\underset{\sim}{c}$. These are just conjunctions of the (possibly negated) functions computed by X at the m nodes of M and so require depth at most $D(x) + \lceil \log m \rceil$. Using the identity

$$f = \bigvee_{\underset{\sim}{c}} \delta_{\underset{\sim}{c}} \wedge f_{\underset{\sim}{c}}$$

we may produce a circuit for f establishing

$$d(f) \leq \max\{D(x) + \lceil \log m \rceil , D(y)\} + 1 + m$$

since the disjunction requires just m parallel steps.

Hence

$$d(f) \leq 2v + \lceil \log m \rceil + 1 + m$$

$$\leq 2v + z + 3 + \lceil \log m \rceil \quad \text{from (3)} \qquad (5)$$

In (4) and (5) we have two inequalities for d(f) each defined in terms of the same Z-block partition of Z.

Now we can suppose that f and Z were chosen for some r so that $z = A(r)$ and $d(f) > r$.

Then (4) implies

$$v \geq \lfloor r/2 \rfloor \quad \text{or, equivalently,} \quad v > A(\lfloor r/2 \rfloor) \qquad (6)$$

By the lemma, the right hand side of (5) is a decreasing function of v, and it is positive with m. Therefore it may be bounded above by taking $v = A(\lfloor r/2 \rfloor) + 1$ and $m = z + 2 - 2v$. Whence

$$r < d(f) \leq 2(A(\lfloor r/2 \rfloor) + 1) + (z + 2 - 2(A(\lfloor r/2 \rfloor) + 1)) + z + 3 + \lceil \log z \rceil$$

or

$$z + \lceil \log z \rceil > A(\lfloor r/2 \rfloor) + \lceil r/2 \rceil - 2 \qquad (7)$$

Since $z = A(r) + 1$, we have a recurrence inequality for the function A(r). For a suitably [illegible]

$$A(r) = \tfrac{1}{2}r.\log_2 r + 2\log_2 r - kr$$

then since

$$A(\lfloor r/2 \rfloor) + \lceil r/2 \rceil - 2 > H(r) + 1 + \lceil \log(H(r) + 1) \rceil$$

for all $k > 0$ and sufficiently large r, we can prove by induction on r that [illegible]

$$d(f) > H(r) = \tfrac{1}{2}r.\log_2 r - O(r)$$

For any f, $d(f) > \tfrac{1}{2}n\log_2 d - O(d)$ and so we have proved

$$\ldots \geq \tfrac{1}{2}n\log_2 d - O(d) \qquad \square$$

Corollary 1. For all functions f

$$d \leq O(c/\log c)$$

Corollary 2. For any sequence of functions f_2, f_3,... where $f_n \in B_n$ with linear circuit complexity, that is $c(f_n) = O(n)$, there is a constant A such that for all n, f_n can be represented by a formula of size $A^{n/\log n}$.

Reference

[1] W.F. McColl and M.S. Paterson, "Depth of all Boolean functions", Theory of Computation Report 7, University of Warwick 1975.