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POWERDOMAINS

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1. Introduction. If the meaning of a deterministic program may be considered to be a function from D to D, where D is some domain of "states", then it would seem that the meaning of a non-deterministic program is a function from D to 2^D, or perhaps from 2^D to 2^D. To apply the methods of fixpoint semantics, then, we should find some way to construe the power-set of a domain as itself a domain, with a suitable ordering.

Actually the position is more complex than this. Consider the operation par, where π_1 par π_2 performs an arbitrary interleaving of the elementary operations of the programs π_1 and π_2. If we are to accomodate par, we cannot take the meaning of a program to be a function from D to 2^D. For, although the programs $\pi_1 = (x:= 0;\ x:= x + 1)$, $\pi_2 = (x:= 1)$ define the same function, π_1 par π_2 and π_2 par π_2 clearly do not. (The example is due to Milner). As Plotkin indicates, we can model the situation better by taking meanings to be resumptions, where the domain R of resumptions satisfies

$$R \doteq S \longrightarrow \mathcal{P}[S + (S \times R)]$$

where S is a domain of states and $\mathcal{P}[\ ]$ is the powerdomain-forming operation. The detailed properties of R do not concern us here; what is important is the fact that we need to be able to solve recursive domain equations involving $\mathcal{P}[\ ]$.

This paper derives its inspiration from Plotkin (1975). In fact, our main purpose is to derive Plotkin's results in a simple and concise way. The simplification can be attributed mainly to the new approach to defining the orderings in the powerdomain (Sec.4 below). As to content, the main innovations in the present work are: the definition of a "weak" powerdomain, which appears to be adequate for most purposes, and which has a particularly simple theory; and the material on categories in Sec. 8 (algebraic categories; fixpoints of ω-colimit preserving functors).

2. Domains, predomains. The following definition is standard:

Definition 1. A poset $(P, \sqsubseteq)$ is a cpo provided that (i) P has a least element, and (ii) every directed subset X of P has a lub $\sqcup X$ in P. An element a of a cpo P is finite (= isolated = compact) provided that, for every directed $X \subseteq P$, if $a \sqsubseteq \sqcup X$, then $a \sqsubseteq x$ for some $x \in X$. P is said to be countably algebraic if (i) the set of finite elements of P is countable, and (ii) every element of P is the lub of a directed set of finite elements of P.

We shall refer to countably algebraic cpo's simply as domains (they are the only domains with which we are concerned). If D is a domain, the set of the finite elements of D will be denoted D^0.

The criterion for "domainhood" which we shall use in practice is given in the following theorem. The proof of the theorem is routine, and is omitted:

Theorem 1. Let P be a poset, and B a countable subset of P. P is a countably algebraic cpo, with B as the set of finite elements of P, iff the following conditions are satisfied:

(0) P has a least element;

(1) Every increasing sequence in B has a lub in P;

(2) Every element of P is a lub of some increasing sequence in B;

(3) For any sequence S in B with lub $x \in P$, and any $a \in B$, if $a \sqsubseteq x$ then $a \sqsubseteq s_i$ for some term s_i of S.

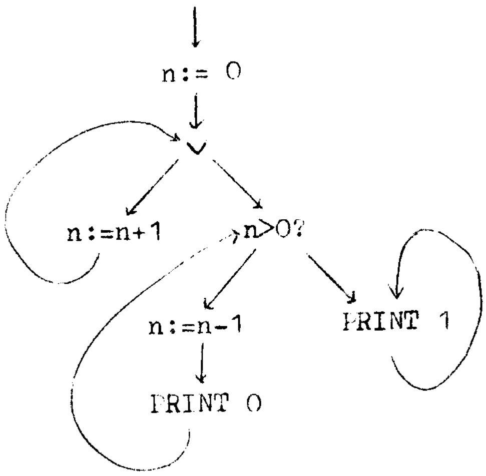
It will be convenient to have some special notation for pre-ordered sets (most of the structures with which we are concerned arise in the form of preorders rather than partial orders). If $(P, \leq)$ is a preorder, we denote by $[P]$ the (quotient) poset $(P/\equiv_P, \leq/\equiv_P)$. Furthermore: if $x \in P$, then $[x]$ is the equivalence-class of x; if $S \subseteq P$, then $[S] \subseteq [P]$ is

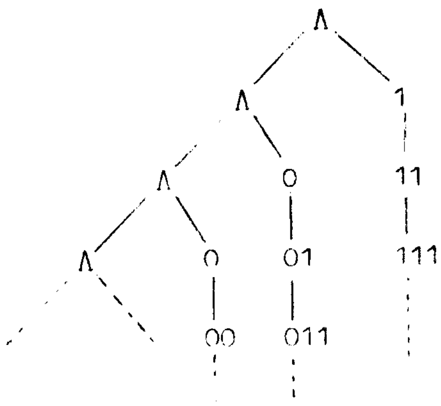
$\{[x] \mid x \in S\}$; and if $f:P \to P'$ (P,P' preorders) is monotone, then $[f]:[P] \to [P']$ is given by $[f][x] = [f(x)]$. We say that P is a <u>pre-domain</u> if $[P]$ is a domain.

Any notion defined for domains yields automatically a corresponding notions for predomains. Thus: if P,P' are predomains, we say that $a \in P$ is <u>finite</u> if $[f]$ is finite in $[P]$; a monotone function $f:P \to P'$ is <u>continuous</u> provided $[f]$ is continuous; and so on. Usually, there are simple direct criteria as well. The conditions (0) - (3) of Theorem 1, for example, have been formulated so that they can be applied directly to preorders: if P is pre-ordered, with countable subset B, then (P,B) satisfies (0)-(3) iff $([P],[B])$ does. (Proof trivial.)

<u>3. Finitely generable sets.</u> Not every subset of an output domain D can occur as the set of possible outcomes of a non-deterministic computation. Following Plotkin, we restrict attention to processes having only finite non-deterministic branching. Thus the set of possible execution sequences (for a given input) can be arranged in a finitary tree. If the nodes of the tree are labelled with the intermediate results attained in the appropriate execution sequence, then the labels along any branch form an increasing sequence of finite elements of D.

<u>Example</u>. There follows an example of a flowchart program with a simple non-deterministic choice node ($\vee$), together with the appropriate tree of intermediate results. The possible "outputs" of the program are strings in $\{0,1\}$. (The output domain is the domain Ω of finite and infinite strings in $\{0,1\}$, with the subsequence ordering). Λ is the null string.





The set of possible outcomes is the set of "limits" along paths of the tree, viz. $\{\Lambda\} \cup \{0^n 1^\omega \mid n \geq 0\}$. This suggests:

<u>Definition 2</u>. Let D be a domain, and T a (node-)labelled finitary tree satisfying (i) for each node t the label $l(t) \in D$; (ii) T has no terminating branches; and (iii) if t' is a descendant of t in T, then then $l(t) \sqsubseteq l(t')$. Let L be the function which assigns to each (infinite) path π through T the lub of the labels occurring along π. We say that T is a <u>generating tree</u> over D, which <u>generates</u> the set $S = \{ L(\pi) \mid \pi \text{ is a path through } T \}$. A set $S \subseteq D$ is <u>finitely generable</u> (f.g.) if it is generated by some tree T. The class of f.g. subsets of D is denoted $\mathcal{F}(D)$.

If the labels of a generating tree are thought of as (possible) partial results of a non-deterministic computation, these labels should be <u>finite</u> elements of the output domain D. Let us call the tree a <u>strict</u> generating tree if all its labels are finite. The next result shows that requiring trees to be strict would not alter the class of sets generated:

Theorem 2. For any generating tree T over D there is a strict generating tree T', which generates the same set as T.

Proof-outline. Let $\perp = e_0, e_1, \ldots$ be an enumeration of the finite elements of D. Let T' be the tree with the same arcs and nodes as T, but with labelling l', defined by induction on the depth n of node t, as follows:

For $n = 0$: $l'(t) = \perp$

For $n > 0$: $l'(t) = e_k$, where k is the least integer such that (i) $l'(\mathrm{father}(t)) \sqsubset e_k \sqsubseteq l(t)$, and (ii) $\forall i \leq n.\ e_i \sqsubseteq l(t) \rightarrow e_i \sqsubseteq e_k$.

Then T' is strict, and generates the same set as T.

Theorem 3. (1) If $f:D \rightarrow D'$ is continuous, and X is a f.g. subset of D, then $f(X)$ is a f.g. subset of D'. (2) If X,Y are f.g. subsets of D, then so is $X \cup Y$.

Proof. (1) Let T' be the tree obtained by applying f to all the labels of T, where T is any generating tree for X. Then T' generates $f(X)$

(2) Let T,T' be generating trees for X,Y resp., and let T'' be the tree obtained by "grafting" the trees T,T' onto a common root (labelled $\perp$). Then T'' generates $X \cup Y$.

Notation. If T is a generating tree, we denote by T_n the cross-section of T at depth n (that is, the set of labels of nodes at depth n)

4. Orderings. Our approach is to ask: What is a "finite piece of "information " about the result of a non-deterministic computation? Having decided an ordering on the set M of such pieces (indicating, for $a,b \in M$, whether b provides "at least as much" information as a), we could then

simply define the powerdomain as the <u>completion</u> of M - that is, the (essentially unique) domain having M as basis. As a slight variant of this, we can say that we already know what the elements of the powerdomain are - the f.g. sets; and the ordering between them should be given by:

$S \sqsubseteq S' =_{df}$ every (finite) piece of information that is true of the result of a computation, given that S is the set of possible outcomes, is also true when S' is the set of possible outcomes ... (1)

It will turn out that the two variants are equivalent. We shall take the second variant as the basic one, since it gives more insight into Plotkin's results (although it would have been technically more convenient to formulate everything in terms of the first variant).

As a "finite piece of information" , it seems appropriate to take a non-empty finite set of finite elements of the output domain D (that is, a possible cross-section of a generating tree); let M(D) be the collection of such sets. What, exactly, is the information that is conveyed by an element A of M(D) ? It appears that this may be construed in more than one way, and that (1) is ambiguous. Specifically, A may be considered

(i) as information about the outcome (that is, information which must be true of the actual outcome); or

(ii) as information about the f.g. set S of all possible outcomes.

According to (i), the information given by A is:

$$\forall x \in S \; \exists a \in A \quad a \sqsubseteq x \tag{2}$$

which we abbreviate as $A \sqsubseteq_0 S$. Version (ii) can be formalized as:

$$A \sqsubseteq_0 S \;\; \& \;\; \forall a \in A \; \exists x \in S \quad a \sqsubseteq x \tag{3}$$

which is abbreviated as $A \sqsubseteq_M S$ (the "Milner ordering").

By way of further explanation of (3), we note: if A is regarded as a cross-section of a generating tree at, say , depth n, then (3) gives <u>all</u> the information which can be gleaned about the set S of outcomes by analysing the computation to depth n.

In accordance with this analysis, we have two preorders $\sqsubseteq_O, \sqsubseteq_M$ for $\mathcal{F}(D)$, defined by:

$$S \sqsubseteq_O S' \;=_{df}\; \forall A\in M(D).\; A \sqsubseteq_O S \rightarrow A \sqsubseteq_O S'$$

$$S \sqsubseteq_M S' \;=_{df}\; \forall A\in M(D).\; A \sqsubseteq_M S \rightarrow A \sqsubseteq_M S' .$$

<u>Theorem 4</u>. Under each of the preorders $\sqsubseteq_O, \sqsubseteq_M$, $\mathcal{F}(D)$ is a predomain, with $M(D)$ as the set of finite elements.

<u>Proof</u>. The proof proceeds by way of two lemmas:

<u>Lemma 1</u>. Suppose that X is a f.g. set, generated by tree T, that $A \in M(D)$, and $A \sqsubseteq_O X$. Then $A \sqsubseteq_O T_m$ for some cross-section T_m. The same holds with $\sqsubseteq_O$ replaced by $\sqsubseteq_M$.

<u>Proof</u>. Choose an integer m so that for every node(-label) b of T at depth $\geq m$ there is an element a of A such that $a \sqsubseteq b$. (This is possible, since if there are nodes b at arbitrary depth such that $\forall a\in A.\; a \not\sqsubseteq b$, then by Konig's lemma there is an infinite branch all of whose nodes have this property - contradicting the fact that $A \sqsubseteq_O X$). Then $A \sqsubseteq_O T_m$. For the second part of the lemma, we assume that $A \sqsubseteq_M X$. We choose m as before, then continue by choosing $n \geq m$ such that $\forall a\in A\; \exists c\in T_n.\; a \sqsubseteq c$. Then $A \sqsubseteq_M T_n$.

<u>Lemma 2</u>. If X is generated by tree T, then X is a lub of the set of cross-sections of T (with respect to each of the preorders $\sqsubseteq_O, \sqsubseteq_M$).

<u>Proof</u>. Trivially, each $T_n \sqsubseteq X$ (Subscripts O,M are omitted, since the proof is the same in each case). Suppose that $\forall n\; T_n \sqsubseteq Y$, where $Y \in \mathcal{F}(D)$. We have to show that $\forall A\in M(D).\; A \sqsubseteq X \rightarrow A \sqsubseteq Y$. But this follows from Lemma 1:

$$A \sqsubseteq X \rightarrow \exists n.\, A \sqsubseteq T_n \rightarrow A \sqsubseteq Y .$$

Returning to the proof of Theorem 4, we show that $\mathcal{F}(D)$ satisfies clauses (0)-(3) of Theorem 1; the result then follows by the remark at the end of Sec.2.

(0) The least element of $\mathcal{F}(D)$ is $\{\perp_D\}$.

(1) Let $\langle A_i \rangle_{i=1,2,..}$ be an increasing sequence in $M(D)$ (under either of the orders $\sqsubseteq_0, \sqsubseteq_M$). Construct a tree T as follows. Label the root with $\perp_D$. If v is a node at depth n, labelled with $b \in D$, take as the successors (if any) of v, one node for each $c \in A_{n+1}$ such that $b \sqsubseteq c$. (Thus, the sets A_i are to be the successive cross-sections of T). Then, let T' be the tree which results from keeping only the nodes and arcs of T which lie on <u>infinite</u> branches of T. (T has at least one infinite branch. If the sequence $\langle A_i \rangle$ is $\sqsubseteq_M$-ordered, then T has no terminating branches, and $T = T'$). T' is a generating tree; let $X \subseteq D$ be the f.g. set generated by T'. We claim that X is a lub of $\langle A_i \rangle$. In case the ordering of $\langle A_i \rangle$ is $\sqsubseteq_M$, this is just Lemma 2. For $\sqsubseteq_0$ we argue as follows. X is an upper bound of the A_i (trivially). By Lemma 2, X is a lub of the T_i' (cross-sections of T'). But, by an pplication of König's lemma (as in the proof of Lemma 1), each $T_i' \sqsubseteq_0 A_n$,for some n. Hence X is a lub of $\langle A_i \rangle$. - The same argument establishes (3), since if $A \sqsubseteq X$ (where $A \in M(D)$), then by Lemma 1 $A \sqsubseteq T_m'$ for some m, and so $A \sqsubseteq A_n$ for some n.

(2) Lemma 2.

The domain $[(\mathcal{F}(D), \precsim_0)]$ (i.e. $(\mathcal{F}(D), \precsim/\equiv)$) will be denoted $\mathcal{F}_0[D]$; similarly for $\mathcal{F}_M[D]$.

Theorem 2 confirms that the two "variants" mentioned at the beginning of the section are equivalent; more precisely, $\mathcal{F}_0[D]$ is isomorphic tothe completion of $[(M(D), \precsim_0)]$ (noting that the restriction of $\precsim_0$ to $M(D)$ is $\sqsubseteq_0$); and similarly for $\mathcal{F}_M[D]$.

The next theorem shows that, for f.g. subsets of a domain, $\lesssim_M$ coincides with the preorder $\lesssim$ defined by Plotkin (p.11).

Theorem 5. Let Φ be the two element domain $\{\bot, \top\}$ (with $\bot \sqsubseteq \top$). For any domain D, define the preorder $\lesssim$ on $\mathcal{F}(D)$ by:

$$X \lesssim Y \quad \equiv_{df} \quad \forall \text{ continuous } f:D \to \Phi.\ f(X) \sqsubseteq_M f(Y).$$

Then $X \lesssim Y$ iff $X \lesssim_M Y$.

Proof. Note that, for subsets S, S' of Φ, the relation $S \sqsubseteq_M S'$ reduces to: $\top \in S \to \top \in S'$ & $S = \{\top\} \to S' = \{\top\}$. Now, suppose that $A \in M(D)$, $A \sqsubseteq_M X \lesssim Y$, and $a \in A$. Define $f:D \to \Phi$ by: $f(x) =$ if $a \sqsubseteq x$ then $\top$ else $\bot$. Then $\top \in f(X)$, so $\top \in f(Y)$; thus $\exists y \in Y.\ a \sqsubseteq y$. Next, define f' by: $f'(x) =$ if $(\exists a \in A)\ a \sqsubseteq x$ then $\top$ else $\bot$. Then $f'(X) = \{\top\}$, so $f'(Y) = \{\top\}$; hence $\forall y \in Y\ \exists a \in A.\ a \sqsubseteq y$. Thus $A \sqsubseteq_M Y$.

Conversely, suppose $\forall A \in M(D).\ A \sqsubseteq_M X \to A \sqsubseteq_M Y$, and $f:D \to \Phi$ is continuous. Suppose $\top \in f(X)$. Then, for some finite $a \in D$ we have: $a \sqsubseteq x$ for some $x \in X$, and $f(a) = \top$. Since $\{\bot, a\} \sqsubseteq_M X$ we have $\{\bot, a\} \sqsubseteq_M Y$, so that $\top \in f(Y)$. Next, suppose that $f(X) = \{\top\}$. Let T be a strict generating tree for X. For some n, $f(T_n) = \{\top\}$. Since $T_n \sqsubseteq_M Y$, we have $f(Y) = \{\top\}$.

The final theorem of the section lists some elementary properties of the orderings.

Notation. For $X \subseteq D$, let $RC(X) = \{y \mid \exists x \in X\ x \sqsubseteq y\}$ and $Con(X) = \{y \mid \exists x, z \in X\ x \sqsubseteq y \sqsubseteq z\}$.

Theorem 6. (i) $X \sqsubseteq_0 Y \to X \lesssim_0 Y$; $X \sqsubseteq_M Y \to X \lesssim_M Y$

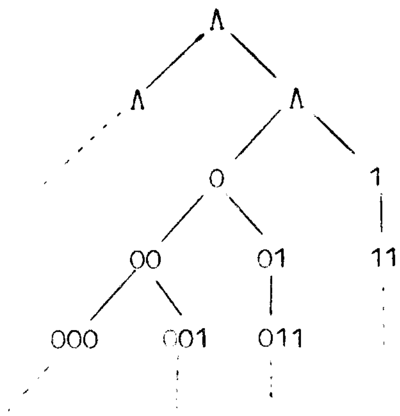
(ii) a) $X \eqsim_0 RC(X)$; $X \eqsim_0 Y$ iff $RC(X) = RC(Y)$

b) $X \eqsim_M Con(X)$; $X \eqsim_M Y$ iff $Con(X) = Con(Y)$

Proof. Obvious.

From this theorem we see that any f.g. set over D which contains $\perp_D$ is equivalent, in the "weak" ($\sqsubseteq_0$) ordering, to $\{\perp_D\}$. If this seems unsatisfactory, it should be recalled that an analysis in terms of $\sqsubseteq_0$ is intended to give us information about the outcome of which we can be certain (alternative (i), p.7 above); from this point of view a computation which may fail to yield any result is as good as worthless.

The preorder $\sqsubseteq_M$ also requires us to make some identifications which may seem unwelcome. For example, the f.g. set $X_0 = \{\perp\} \cup \{0^n 1^\omega \mid n \geq 0\}$, discussed in Sec.3, must be identified with $X_1 = X_0 \cup \{0^\omega\}$ - as we see by comparing the tree previously given for X_0 with the following generating tree for X_1:



If a non-deterministic process P has X_0 as its set of possible outcomes, then we know, as soon as 0 has been output, that a 1 will be subsequently output; with X_1 as the set of outcomes, we do not have this assurance. Is not this an important difference between X_0 and X_1 ? The situation is puzzling, since it is hard to see how a more refined ordering than $\sqsubseteq_M$ could be computationally meaningful: $\sqsubseteq_M$ has been designed to take account of all information about an output set which can be attained in a finite time.

The answer seems to be that a mere ordering of information is not sufficient; we need a more refined analysis of the ways in which information may be improved. This means in effect that we should take account of the arcs of generating trees (instead of only their cross-sections). The natural framework in which to develop this idea is category theory: the cross-sections of generating trees (over a given domain), for example, will be objects in a category, in which morphisms correspond to the different ways in which the connecting links between successive cross-sections can be drawn. A theory of this kind was suggested by Egli(unpublished), and is currently being developed at Warwick (principally by Daniel Lehmann).

5. Special functions. A number of special functions is needed for the interpretation of programs admitting parallea and non-deterministic operations. The following result (a slight generalization of Plotkin's Lemma 4) will be useful in establishing their contiuity.

Lemma 3. Let D,E be predomains. A mapping $f:D \to E$ is continuous iff

(i) The restriction of f to D^o is monotone; and

(ii) For each $x \in D$ there is an increasing sequence $\Sigma(x)$ of elements of D^o, having x as lub, such that $f(x)$ is a lub of $f(\Sigma(x))$.

Proof. Only if: trivial. If: Suppose that (i),(ii) are satisfied, and that X is a directed of D, having lub y. For each term b of $\Sigma(y)$ there exists (by finiteness of b) $x \in X$ such that $b \sqsubseteq x$, and (for the same reason) a term a of $\Sigma(x)$ such that $b \sqsubseteq a$. Similarly, for each term a of $\Sigma(x)$ (for each $x \in X$), there exists b in $\Sigma(y)$ such that $a \sqsubseteq b$. We can express this by saying that the families $\Sigma(y)$ and $\cup\{\Sigma(x) \mid x \in X\}$ are cofinal. Since f is monotonic on D^o, the images y_f, X_f of these families under f are cofinal. Hence y_f, X_f have the same lub(s). Hence $f(y)$ is a lub of X_f, and so of $f(X)$.

For the first three of the four special functions to be considered, continuity is established by showing that the sufficient conditions given in Lemma 3 are satisfied. (Actually, we will verify only (ii), since the monotonicity condition is trivial in each case.) Subscripts $0,M$ are omitted, since the proofs are the same for each.

1). Function extension. If $f:D \to E$ is continuous, then $\hat{f}: \mathcal{F}(D) \to \mathcal{F}(E)$, where $\hat{f}(X) = f(X)$, is continuous. Proof: For $X \in \mathcal{F}(D)$, take $\Sigma(X)$ as $\langle T_n \rangle$, where T is a strict generating tree for X. Since the $f(T_n)$, $n=0,1,..$ are cross-sections of a generating tree for $\hat{f}(X)$, $\hat{f}(X)$ is a lub of $\langle f(T_n)\rangle$.

2). Union: $(\mathcal{F}(D))^2 \to \mathcal{F}(D)$: $\langle X,Y\rangle \mapsto X \cup Y$. The argument is similar to 1) (utilizing the construction of Theorem 3(2)).

3) $\{| \ |\}: D \to \mathcal{F}(D)$: $x \mapsto \{x\}$. If $\langle a_i \rangle$ is a sequence in D^0 having x as lub, then $\langle \{a_i\}\rangle$ is the sequence of cross-sections of a generating tree for $\{x\}$.

4). Big union, $\dot{\bigcup}: \mathcal{F}^2[D] \to \mathcal{F}[D]$. We define this first on the basis of $\mathcal{F}^2[D]$, namely $M^2[D]$ (here we presuppose the notation $M[D]$ for $[M(D)]$). Any $P \in M^2[D]$ has a representation $[[A_1],\ldots,[A_m]]$, where $A_1,\ldots,A_m \in M(D)$. Define $\dot{\bigcup}(P)$ as $[A_1 \cup \ldots \cup A_m]$. We have to check that this value is independent of the representation chosen for P. Suppose that $P = [[B_1],\ldots,[B_n]]$, and that $a \in \bigcup A_i$ - say $a \in A_i$. Since $\{B_j \mid 1 \leq j \leq n\} \sqsubseteq_0 \{A_i \mid 1 \leq i \leq m\}$, we have $B_j \sqsubseteq_0 A_i$ for some B_j; thus $b \sqsubseteq a$ for some $b \in \bigcup B_j$. The remaing conditions (for equivalence of $\cup A_i$ and $\cup B_j$) are verified similarly. - Finally, take the (unique) continuous extension to $\mathcal{F}^2[D]$ of the function so defined.

There is a technical difficulty in giving a direct definition for $\mathcal{F}^2[D]$: we are unable to show that the union of a "f.g. set of f.g. sets" is itself f.g. Plotkin escapes this difficulty, since he works with a special class of <u>closed</u> f.g. sets (rather than arbitrary f.g. sets). Closed sets will be discussed in Sec. 7.

6. Closure properties. In order to handle recursive domain equations, we must ensure that the class of domains considered be closed under suitable sum, product, function-space and powerdomain constructions. Because of the requirement of closure under function space, the class of arbitrary (countable) algebraic cpo's is not suitable (since without some restriction on D, D', we cannot find a basis for the space $[D \to D']$ of continuous functions). It is known that a suitable restriction is that bounded joins exist in the domains in question (domain D has bounded joins provided that, for each $A \in M(D)$, if A is bounded, then A has a lub). As, e.g., Constable and Egli(1975) show, if D, D' are domains having bounded joins, then $[D \to D']$ has the same property. The bounded join property is preserved also by $\mathcal{F}_0[\ ]$. For suppose that $S = \{A_0, \ldots, A_n\} \subseteq M(D)$, and that S is bounded w.r.t. $\sqsubseteq_0$. Then it is readily verified that $[\{\bigsqcup_i a_i \mid a_i \in A_i \ (i=0,\ldots,n) \ \& \ \{a_0,\ldots,a_n\}$ is bounded $\}]$ is the lub of $[S]$. Thus, in case we take $\mathcal{F}_0[\ ]$ as the powerdomain constructor, the problem is solved by taking the class of domains having bounded joins; and we can proceed at once to the solution of recursive domain equations.

For $\sqsubseteq_M$ (which is in effect Plotkin's powerdomain constructor) the situation is more difficult. $\mathcal{F}_M[D]$ need not have bounded joins even when D has (see Plotkin, Sec.3, p.15 for an example). To handle this case we will introduce, following Plotkin, the SFP objects ("SFP" is an abbreviation for "sequence of finite partial orders").

Definition 3. An injection $f: D \to D'$, where D, D' are cpo's, is called an embedding if f has a continuous adjoint $f': D' \to D$.

- Equivalently, f is an embedding if there is a continuous $f': D' \to D$ such that $\langle f, f' \rangle$ is a projection pair, i.e.: $f' \circ f = I_D$ and $f \circ f' \sqsubseteq I_{D'}$.

An <u>embedding sequence</u> is a sequence $\langle D_n, p_n \rangle$, where each $p_n : D_n \to D_{n+1}$ is an embedding. CPO is the category of cpo's and continuous maps; CPO_E has the same objects as CPO, but with maps restricted to be embeddings. An ω-system in a category C is a functor from the (partially) ordered set $\omega = 0 \leq 1 \leq \ldots$ into C.

<u>Notation</u>. If p is an embedding, we denote the adjoint of p by p'. If $\langle A_m, p_m \rangle$ is an embedding sequence, define the maps $p_{mn} : A_m \to A_n$ by:

$$p_{mn} = \begin{cases} p_{n-1} \circ \ldots \circ p_m & \text{if } n > m \\ I_{A_m} & \text{if } n = m \\ p'_n \circ \ldots \circ p'_{m-1} & \text{if } n < m \end{cases}$$

- Thus the embedding sequence $\langle A_m, p_m \rangle$ determines the ω-system (in CPO_E) $\langle A_m, p_{mn} \rangle_{m \leq n}$.

The following theorem summarizes some well-known facts about embedding sequences:

<u>Theorem 7</u>. (i) Let $\Sigma = \langle D_m, p_m \rangle$ be an embedding sequence of cpo's. Let D_∞ be the inverse limit of Σ; that is, D_∞ is the set $\{\langle x_m \rangle \mid \forall m.\ x_m \in D_m \ \&\ p'_m(x_{m+1}) = x_m\}$ with the ordering defined componentwise by the orderings of the D_m. Then D_∞, together with the embeddings $i_m : D_m \to D_\infty$ defined by $i_m(x) = \langle p_{mn}(x) \rangle_{n \in \omega}$, is a colimit of Σ (strictly, of the ω-system associated with Σ) in CPO_E. If each D_m is a domain with basis (set of finite elements) D_m^0, then D_∞ is a domain with basis $D_\infty^0 = \bigcup_m i_m(D_m^0)$.

(ii) Let $\langle D_m, p_m \rangle$, $\langle E_m, q_m \rangle$ be embedding sequences of cpo's. For each m, define $F_m : [D_m \to E_m] \to [D_{m+1} \to E_{m+1}]$ by $F_m(f) = q_m \circ f \circ p'_m$. Then $\langle [D_m \to E_m], F_m \rangle$ is an embedding sequence, and its colimit (as constructed in (i)) is isomorphic with $[D_\infty \to E\,]$.

By Theorem 7(ii), the operator $\to$ commutes with the taking of colimits of ω-systems in CPO_E. The same is easily shown to be true for suitably-defined sum and product operators. That it holds also for the powerdomain operator $\mathcal{P}_M$ is the content of:

Theorem 8. If D is a colimit of $\langle D_m, p_m\rangle$, then $\mathcal{F}_M[D]$ is a colimit of $\langle\mathcal{F}_M[D_m],[p_m]\rangle$.

Proof. The basis of $\mathcal{F}_M[D]$ is $B = [M(D)]$, which is (Theorem 7) $[\bigcup_m i_m(M(D_m))]$, while the basis of colim$\langle\mathcal{F}_M[D_m],[p_m]\rangle$ is $B' = \bigcup_m [i_m][M(D_m)]$. But there is an obvious order-preserving bijection between B and B'; hence $\mathcal{F}_M[D] \cong$ colim$\langle\mathcal{F}_M[D_m],[p_m]\rangle$.

Definition 4. Colimits of ω-systems of finite cpo's in CPO_E are called SFP objects

A finite cpo is trivially a domain, and so by Theorem 7(i) every SFP object is a domain. The sum, product, function space and powerdomain of finite domains are obviously finite, and so (cf. the remarks preceding Theorem 8) the class of SFP objects is also closed under these operations.

7. Maximal representatives. Plotkin shows that, instead of working with equivalence classes of f.g. sets, we can use certain distinguished (actually, maximal) elements of these classes. We present a simplified version of this theory. (We mention - as of course does Plotkin - only the case $\mathcal{F}_M[D]$. But our account applies, with trivial modifications, also to $\mathcal{F}_O[D]$).

Lemma 4. Suppose that $i:D \to D'$ is an embedding, $A \subseteq D$, and $B \subseteq D'$. Then $i(A) \sqsubseteq_M B$ iff $A \sqsubseteq_M i'(B)$.

Proof. Obvious.

Definition 5. Let D be a SFP object, $\langle D_n, p_n\rangle$ a fixed embedding sequence of finite domains having D as colimit (with embeddings $i_n:D_n \to D$). If $X \subseteq D$, define

$$X^+ = \{x \mid i_n'(x) \in i_n'(X) \text{ for all } n\}.$$

Remark. It is readily checked that $^+$ is a closure operation on the power-set $\mathcal{P}(D)$. For a description of the associated topology, see the appendix.

Lemma 5. (1) $X \simeq_M X^+$ (2) $X \sqsubseteq_M Y$ iff $X^+ \sqsubseteq_M Y^+$.

Proof. (1) Let $A \in M(D)$, and let n be large enough so that $i_n \circ i_n'(A) = A$. We have:

$$A \sqsubseteq_M X \leftrightarrow i_n'(A) \sqsubseteq_M i_n'(X) \text{ (Lemma 4)} \leftrightarrow i_n'(A) \sqsubseteq_M i_n'(X^+) \leftrightarrow A \sqsubseteq_M X^+ .$$

(2) IF: $X \simeq_M X^+ \sqsubseteq_M Y^+ \simeq_M Y \rightarrow X \sqsubseteq_M Y$. ONLY IF: Notice that $X \sqsubseteq_M Y$ iff $\forall n\ i_n'(X) \sqsubseteq_M i_n'(Y)$. Suppose that $x \in X^+$. For each n, let $Y_n = \{b \in i_n'(Y) \mid i_n'(x) \sqsubseteq b\}$. Each Y_n is finite non-empty, and $i_n'|Y_{n+1}$ is a (decreasing) map of Y_{n+1} into Y_n. Thus (by a version of König's lemma) the inverse limit $Y_x = \lim \langle Y_n, i_n'|Y_{n+1} \rangle$ is non-empty; if $\langle y_n \rangle$ is any element of Y_x, then $x \sqsubseteq \bigsqcup_n i_n(y_n)$. - Similarly, for any $y \in Y^+$ we find $x \in X^+$ such that $x \sqsubseteq y$.

Note that X^+ is f.g., even if X is not (there is an obvious finitary tree, having the $i_n'(X)$ as cross-sections, which generates X^+). $\mathrm{Con}(X^+)$ - which we denote by X^* - is also f.g.; the appropriate generating tree has the sets $\mathrm{Con}(i_n'(X))$ as cross-sections.

<u>Theorem 9</u>, (1) $X \subseteq X^*$ (2) $X \simeq_M X^*$ (3) $X \simeq_M Y$ iff $X^* = Y^*$ (4) $X \sqsubseteq_M Y$ iff $X^* \sqsubseteq_M Y^*$.

<u>Proof</u>. (1): Obvious. (2)-(4): Theorem 6 and Lemma 5.

These results show that each X^* is the greatest (w.r.t. set-inclusion) element of its equivalence-class; and that we can (as an alternative to $\mathcal{Y}_M[D]$) define the powerdomain as $\{X^* \mid X \text{ non-empty}\}$ ordered by $\sqsubseteq_M$.

<u>8. Categories, domain equations</u>. In this section we show that several notions and results about cpo's/domains generalize to categories. The main application is an improved account of the category-theoretic solution of recursive domain equations, previously developed by Reynolds, Wand, and Plotkin (see Plotkin 1975 for references).

In fact, the notions: poset, least element, monotone function, increasing sequence, continuous function, finite element, (countably) algebraic cpo generalize respectively to: category, initial object, functor, ω-system, ω-continuous functor, finite object, (countably) algebraic category. The first four pairs in this comparison are familiar, the others are explained by:

Definition 6. Let C,C' be categories admitting ω-colimits. A functor $F:C \to C'$ is weakly ω-continuous if, whenever X is a colimit object for an ω-system Q in C, then FX is a colimit object for FQ. An object $A \in C$ is finite if, for any ω-system $\langle A_n \rangle$ in C with colimit $\langle X, i_n : A_n \to X \rangle$, the following holds: for any arrow $u:A \to X$ and for any sufficiently large n, there is a unique arrow $v:A \to A_n$ such that $u = i_n \circ v$. Let K be a category having an initial object and at most countably many finite objects. We say that K is (countably) algebraic provided (1) every object of K is a colimit of an ω-system of finite objects, and (2) every ω-system of finite objects has a colimit in K.

Remarks. (Strong) ω-continuity of F would require preservation of colimit diagrams (not just objects). Strictly, finiteness should be formulated in terms directed systems (not just ω-systems); what we have defined is ω-finiteness. The name "algebraic category" is provisional (it conflicts with established usage). We have adopted the analogue of the characterization of "algebraic cpo" given in Theorem 1, rather than that of Definition 1. This is purely for convenience: it is usually easier to verify that a given category fulfils the conditions laid down in Definition 6, than would be the case if we had used the analogue of Definition 1

Examples. The category SFP_E of SFP objects and embeddings is countably algebraic, with the finite domains as finite objects. The functor $Fun:(SFP_E)^2 \to SFP_E$, defined on objects by $Fun(D,E) = [D \to E]$ and on arrows $p:D \to D'$, $q:E \to E'$ by $Fun(p,q) = \lambda f:D \to E.\ q \circ f \circ p'$, is weakly ω-continuous, by Theorem 7. The functor $P:SFP_E \to SFP_E$, defined on objects by $P(D) = \mathcal{F}_M[D]$ and on arrows by $P(f) = [\hat{f}]$, is weakly ω-continuous, by Theorem 8. (With a little more effort we could show that these functors are (strongly) ω-continuous). Continuous Sum and Product functors are readily defined. Compositions of ω-continuous functors are again ω-continuous

Theorem 10. Every algebraic category admits ω-colimits.

Proof. Suppose that $\langle A_m, p_m\rangle$ is an ω-system in an algebraic category C. Each A_m is the colimit of an ω-system of finite objects of C, say $\langle A_m^n, p_m^n\rangle_{n\in\omega}$, via arrows $i_m^n : A_m^n \to A_m$. We will define, by induction r, a sequence $\langle A_r^{s(r)}\rangle$ with a (canonical) arrow from each A_m^n to $A_r^{s(r)}$ (for any sufficiently great r); the colimit of this sequence will be the desired colimit of $\langle A_m, p_m\rangle$. Put $s(0) = 0$. $s(r)$ having been defined, define $s(r+1)$ as follows: For each $A_m^{s(r)}$ $(m=0,\ldots,r)$ let $q_m : A_m^{s(r)} \to A_{r+1}$ be $p_{rm} \circ i_m^{s(r)}$. For sufficiently great t we have (Definition 6) unique arrows $q_m^t : A_m^{s(r)} \to A_{r+1}^t$ such that $i_{r+1}^t \circ q_m^t = q_m$. Let t_0 be the least t, such that q_m^t exists for all $m \in \{0,\ldots,r\}$. Finally, put $s(r+1) = \max\{t_0, s(r)+1\}$.

We now have an (infinite) commuting diagram G, with arrows as follows: from each A_m^n to $A_m^{n'}$ (all $n' \geq n$); from each A_m^n to A_m; from each A_m^n such that $m \leq r$ and $n \leq s(r)$ to $A_{r+1}^{s(r+1)}$; and from A_m to $A_{m'}$ (whenever $m' \geq m$). Any cone from the ω-system $\langle A_m\rangle$ to an object X yields a cone from $\langle A_r^{s(r)}\rangle$ to X (by composing with the arrows in G). Conversely, let V be a cone from $\langle A_r^{s(r)}\rangle$ to X. For each fixed m, V yields a cone from the ω-system $\langle A_m^n\rangle_{n\in\omega}$ to X (since G has arrows from each A_m^n to $A_r^{s(r)}$, for any sufficiently great r). By the colimit property of A_m, this yields a (unique) arrow from A_m to X such the (augmented) diagram commutes. By varying m, we get a cone from $\langle A_m\rangle$ to X. It is immediate from the commuting properties of the (augmented) diagram that these constructions determine an isomorphism between the category of cones from $\langle A_r^{s(r)}\rangle$ and the category of cones from $\langle A_m\rangle$, under which corresponding cones have the same vertex. It follows that the ω-systems $\langle A_r^{s(r)}\rangle$, $\langle A_m\rangle$ share the same colimits; the proof is complete.

Theorem 11. Let C be a category admitting ω-colimits, with initial object Ω. Let $F : C \to C$ be weakly ω-continuous. Then there is an object X such that $FX \cong X$ and such that for any Y with arrow $p : FY \to Y$ there is an arrow from X to Y.

<u>Proof.</u> Take X as colimit of the ω-system $\Sigma = \Omega \xrightarrow{f} F\Omega \xrightarrow{Ff} F^2\Omega \xrightarrow{F^2f} \dots$. It is clear that Σ has the same colimit(s) as $F\Sigma$; hence (by weak continuity of F) $X \cong FX$. If $g:\Omega \to Y$, the square

$$\begin{tikzcd}
\Omega \arrow[r, "f"] \arrow[d, "g"'] & F\Omega \arrow[d, "Fg"] \\
Y & FY \arrow[l, "p"]
\end{tikzcd}$$

commutes. By repeated translation of this square by F, we obtain a cone from Σ to Y; and hence (by the colimit property of X) an arrow from X to Y.

It follows from this theorem that any equation of the form $D \cong F(D)$, where F is weakly continuous (for example, the resumption-domain equation, Sec.1), has a solution in SFP_E which is minimal, in the sense that it may be embedded into any other solution.

From a categorical poimt of view, Theorem 11 leaves something to be desired. In fact, under the assumption of full ω-continuity of F, the conclusion can be strengthened, so as to characterize the <u>least</u> fixpoint of F by a universal property; this of course yields unique (up to isomorphism) minimal solutions for equations.

REFERENCES

Plotkin, G. (1975), A Powerdomain Construction, D.A.I. Research Report No.3, University of Edinburgh.

Constable, R., and Egli, H. (1975), Computability Concepts for Programming Language Semantics, 7th Annual ACM Symposium on Theory of Computing, 98-106.

APPENDIX

The Cantor topology. We noted in Sec. 7 that $^{+}$ is a closure operation. Plotkin makes considerable use of the topology associated with this closure. Topological considerations were not needed in our treatment of the powerdomains, and so this topic was omitted from the main text. The topology is interesting in its own right, however, and a brief account is appended here.

Theorem. With the notation of Definition 5, let $\mathcal{T}$ be the topology on D got by considering D as the inverse limit of the D_n (each endowed with the discrete topology). Then the operation $^{+}$ (on D) coincides with closure in $\mathcal{T}$.

Proof. $\mathcal{T}$ is (by definition) the coarsest topology which makes all the maps i'_n continuous. Equivalently, it is the topology obtained by taking $\bigcup_n \{i'^{-1}_n(a) \mid a \in D_n\}$ as a basis for the open sets. On the other hand, note that $X^{+} = \bigcap_n i'^{-1}_n \circ i_n(X)$. Then (denoting the complement by C) we have:

$$\begin{aligned}\mathrm{Closure}_{\mathcal{T}}(X) = C(\mathrm{Interior}_{\mathcal{T}}(CX)) &= C(\bigcup_n \{i'^{-1}_n(a) \mid a \in D_n \ \&\ a \notin i'_n(X)\}) \\ &= \bigcap_n \{i'^{-1}_n(a) \mid a \in i'_n(X)\} \\ &= X^{+}.\end{aligned}$$

As Plotkin points out, we can also define the topology in an intrinsic way (not involving the arbitrary choice of sequence D_n), via "positive and negative information":

Theorem. The topology on D obtained by taking as a subbasis all sets of the form $P_e = \{x \mid e \sqsubseteq x\}$ and $N_e = \{x \mid e \not\sqsubseteq x\}$ (for finite $e \in D$) coincides with $\mathcal{T}$.

Proof. (i) For $a \in D_n$, $i'^{-1}_n(a) = P_{i_n(a)} \cap \bigcap\{N_{i_n(e)} \mid e \in D_n \ \&\ a \sqsubsetneq e\}$.
(ii) Each finite element of D is $i_n(e)$ for some n and $e \in D_n$. Now $P_{i_n(e)} = \bigcup\{i'^{-1}_n(a) \mid a \in D_n \ \&\ e \sqsubseteq a\}$, while $N_{i_n(e)} = \{i'^{-1}_n \mid a \in D_n \ \&\ e \not\sqsubseteq a\}$.