# The University of Warwick

# THEORY OF COMPUTATION

# REPORT NO. 22

DATA TYPES AS

EFFECTIVE OBJECTS

BY

AKIRA KANDA

Department of Computer Science
University of Warwick
COVENTRY CV4 7AL
ENGLAND.

December 1977

# Data Types as Effective Objects

Akira Kanda
Department of Computer Science
University of Warwick
Coventry, England

## §0. Introduction

A key feature of the so called fix-point approach for theoretical computer science is the idea of abstract data types and continuous functions. In this approach we treat the domains of computation as abstract data types which are in fact cpo's. Then every computable function is modeled as a function from a cpo to a cpo, which is continuous in the sense of directed limit preservation. Finally observing that continuous functions of the same functionality form an abstract data type, we specify every computable function as a least fix-point of some continuous function. Substantial development of this approach can be found in Scott [2].

This approach has been successfully used to prove some interesting properties of computable objects with this continuity property, as can be seen in Park [1], Manna & Vuillemin [17], and Milner [7]. Scott extended this idea to the functions of domains rather than on objects. Then he developed the so called retract calculus on a universal domain to provide solutions of self-referential domain equations, as in Scott [4] and Plotkin [15]. Also Scott suggested a categorical treatment for this kind of problem [3], and following his suggestion Reynold [8], Plotkin [20], Wand [9], Lehmann [10], and Smyth [illegible] developed a categorical theory of self-referential domain equations. These theories provided a mathematical model for denotational semantics [16].

But from the birth of this approach, the relevance of continuity to computability has been a big question. Actually in the initial development of this approach, Scott [22] tried to answer this question by pointing out which elements of a domain are computable, and gave an essential idea of computability and effectively given algebraic domains.

But soon after that D.Scott discovered that interval lattices could not be treated as algebraic domains. Thence he started to work on more general domains, namely continuous domains, and tried to define computability in them as in Scott [2]. Motivated from Scott's pioneer work on effectively given continuous domains, Tang [23] obtained the first successful theory of effectively given continuous domains for restricted basis. Also introducing algebraic constraint to Scott's idea of effectively given domains, Egli & Constable [13] and Markowsky & Rosen [14] obtained a successful theory of effectively given algebraic domains. Smyth [12] generalized all of these ideas and obtained the most general class of effectively given continuous domains. Actually Smyth generalized Tang's idea of recursive basis to r.e. basis. Just recently Scott and Plotkin quite independently obtained a universal domain for effectively given domains as in [5]. For the purpose of simplicity, we will talk about only algebraic domains.

The importance of effectively given domain is that we can distinguish computable elements from non-computable elements. One may claim that, as long as we have a concise criterion to check computability, we are O.K. But we must admit that we still have domains which are too big, i.e. most elements of an effectively given domain are not effective at all.

There have been several suggestions to exclude noncomputable objects for special cases as can be seen in Scott [4] and Egli & Constable [13]. But so far nobody seems to have taken this problem seriously in a general setting. Presumably for the purpose of making discussion simpler, effectiveness of the construction was omitted, and we got the problem of non-computable elements. But now, with sharp distinction between computable elements and non-computable elements, it is time to return back to the beginning and rebuild the theory taking effectiveness into account throughout the construction of our theory.

The following summarises our amendment to the basic assumptions in the classical theory [2] for this purpose.

(1) In the classical theory, the notion of computational approximation was treated simply in terms of directed sets. By doing so, we threw away the effective feature of the computational approximation. Our claim is that we should take effective chains rather than directed sets for this purpose.

(2) Therefore data types should be effective chain complete rather than directed complete. Note that we are claiming that data types are not cpo's any more.

(3) Also the continuity which is really required is not the directed limit preserving continuity but the effective limit preserving continuity.

(4) Every data type must be effectively enumerable. Also the graph of every function must be effectively enumerable.

One could suspect that if we took effectiveness in throughout the construction of theory, it would be extremely difficult or even impossible to obtain a desirably strong theory. In this paper, we will see that we will not fail to obtain a statisfactory theory of domains even if we take this effective approach. In fact it will be observed that our effective theory is at least as strong as the classical theory of self-referential domain equations, and is just as complicated as the classical theory. Furthermore there are several points, due to effectiveness, which are stronger than the classical theory, like the effective isomrophism and the notion of effective functors.

One interesting observation of this paper is a successful investigation of the effectiveness of functors, i.e. domain constructors. So far all classical results essentially used $\omega$-continuous functors which do not adequately represent the effective nature of domain constructors. In fact, Smyth has been working on the computability of $\omega$-continuous functors for the category of effectively given domains and more generally effectively given categories.

He told me, privately, the difficulty of doing so. But in our approach, since every data type is effectively enumerable and the effectiveness of functions is treated in terms of effective chain preservation, we can simplify the situation and can naturally introduce the effectiveness of functors. This is one of the advantage of our theory of effective data types.

Also an effective completion theorem which is an effective version of the traditional completion theorem, and the construction of functional data types from data types which are not cpo's are of mathematical interest by themselves, and suggest a relevant operational theory of domains.

## §1. An overview of effectively given domains

Our theory of effctive data types is motivated mainly from the criticism of redundancy of effectively given domains. Therefore we will review the theory of effectively given domains briefly here. For details see Scott [2,22,5], Tang [23], Egli-Constable [13], Rosen-Markowsky [14], Smyth [12].

A poset $(D,\sqsubseteq)$ is a set D with a partial ordering $\sqsubseteq$. A directed subset S of a poset $(D,\sqsubseteq)$ is a subset $S \subseteq D$ s.t. every finite subset of S has a least upper bound (lub) in S. A poset $(D,\sqsubseteq)$ is said to be directed complete iff every directed subset of D has a lub in D. A directed complete poset with the least element $\perp$ is called a cpo. Historically the initial discussion of recursive domain definition was carried out on cpo's. But to investigate effectiveness (or computability), recent theory involves more sophisticated structure than cpo.

Our intuition tells us that all finite objects should be computable. So the question which arises next concerns a proper mathematical characterization of finiteness. Scott [22] and Egli-Constable [13] adopted this idea from algebra. We say that an element x of a poset D is finite iff for every directed subset $S \subseteq D$ s.t. the lub $\bigsqcup S$ exists in D,

$$x \sqsubseteq \bigsqcup S \quad \text{implies} \quad x \sqsubseteq s \text{ for some } s \in S.$$

We will write $E_D$ to denote the set of all finite elements in D. Then to make the theory more interesting, we have to obtain some more interesting effective objects. Scott [2] claimed that for every effective object, there should be arbitarily good finite approximations. Following this claim, we admit that every effective object must be a lub of some chain of finite elements. This observation leads us to the idea of what algebraists call countably algebraic posets. A direted complete poset $(D,\sqsubseteq)$ is said to be <u>countably algebraic</u> iff $E_D$ is countable and for every $x \in D$, there exists a directed subset $J_x \subseteq E_D$ s.t. $x = \bigsqcup J_x$. We call such $E_D$ a basis of D.

So far we have established that each data type should be a countably algebraic poset with bottom, i.e. a countably algebraic cpo. As mentioned in the introduction, one of the crutial point of our theory is the data types of functions. We have to be able to form a countably algebraic cpo for the set of all continuous functions from a countably algebraic cpo to a countably algebraic cpo. Markowsky-Rose pointed out that for this purpose, for technical reason, we had to introduce a bounded join condition to data types. We say that a poset $(D,\sqsubseteq)$ has a <u>bounded join</u> iff every finite bounded subset of D has a lub in D. Also if every bounded subset has a lub in D, we say that D is <u>bounded complete</u>. The following can be readily seen.

<u>FACT 1.1</u>

A countably algebraic poset $(D,\sqsubseteq)$ is bounded complete iff D has a bounded join iff $E_D$ has a bounded join.

To capture the effective nature of finite elements, we have to introduce some effective properties of finite objects. Hence we have the following definition of effectively give domains.

<u>DEFINITION 1.2 (effectively given domains)</u>

An <u>effectively given domain</u> $(D,\sqsubseteq)$ is a bounded complete countably algebraic cpo s.t. the following relations are recursive in indices:

(1) $\sqcup\{\epsilon_{i_1},\ldots,\epsilon_{i_n}\}$ exists in $E_D$,

(2) $\epsilon_j = \sqcup\{\epsilon_{i_1},\ldots,\epsilon_{i_n}\}$

(3) $\{\epsilon_{i_1},\ldots,\epsilon_{i_n}\}$ is bounded in $E_D$.

where $\{\epsilon_i \mid i\in N\}$ is an enumeration of $E_D$. We call $E_D$ an <u>effective basis</u> of D. Note that by the previois fact $(D,\sqsubseteq)$ effectively given implies that $E_D$ has bounded joins, therefore (1) is logically equivalent to (3). Also $\epsilon_i \sqsubseteq \epsilon_j$ is recursive in indices by the above definition.

Now what is good about taking effectively given domains as data types? The point is that we can point out which are computable. Roughly speaking, we claim that the computable elements of an effctively given domain are exactly those which can be reached by the limit of effective finite approximation sequences. The following is a mathematical formulation of this idea due to Scott [22].

<u>DEFINITION 1.3</u> (<u>computable elements</u>)

An element x of an effctively given domain D is <u>computable</u> iff there exists a recursive function $\rho: N \to N$ s.t. $\epsilon_{\rho(n)} \sqsubseteq \epsilon_{\rho(n+1)}$ for all $n\in N$ and $x = \sqcup\{\epsilon_{\rho(n)} \mid n\in N\}$.

It is quite reasonable to call such kind of chain like $\{\epsilon_{\rho(n)} \mid n\in N\}$ as an <u>effective chain</u> and the least upper bound of it as an <u>effective limit</u>. Actually we can define these ideas more precisely in more general setting.

<u>DEFINITION 1.4</u> (<u>effective chains</u>, <u>effective limits</u>)

(1) An $\omega$-chain $x_0 \sqsubseteq x_1 \sqsubseteq \ldots$ in a countable poset $X = \{\chi_i \mid i\in N\}$, is an <u>effective chain</u> iff there exists a recursive function $r: N \to N$ s.t.

$$x_i = \chi_{r(i)} \quad \text{for all } i\in N.$$

(2) Let $X = \{\chi_i \mid i\in N\}$ be a countable poset with the property s.t. every finite join, if exists is recursive in indices. Let $x_0 \sqsubseteq x_1 \sqsubseteq \ldots$ be an effective chain in X. Then $\bigsqcup_i x_i$, if exists in some superset X' of X, is called an <u>effective limit</u> (<u>in</u> X' <u>of an effective chain in</u> X).

Therefore an element $x \in D$ is computable iff it is an effective limit of an effective chain in $E_D$.

Egli-Constable [13] gave an alternative defintion of computability which looks odd but is usefull.

DEFINITION 1.5 (simgle-valued r.e.sets)

Given any effectively given domain D with effective basis $E_D = \{\epsilon_i \mid i \in N\}$, and any r.e. set $W_j = \text{range } \phi_j$, where j is the recursive index of the r.e. set: the single-valued version $W_{s(j)}$ of $W_j$ with respect to $E_D$ is defined by the following procedure:

--- enumerate $W_j$: $x_1, x_2, \ldots$

--- $W_{s(j)}$ is obtained by:

$$y_1 = x_1$$

$$y_{n+1} = \begin{cases} x_{n+1} & \text{if } \{\epsilon_{y_1}, \ldots, \epsilon_{y_n}, \epsilon_{y_n+1}\} \text{ is bounded in } E_D \\ y_n & \text{otherwise.} \end{cases}$$

Obviously $W_{s(j)}$ is recursively enumerable, since it is defined by the above procedure. Also evidently $W_{s(j)}$ is boudned in D. Furtheremore it can be readily seen that $W_{s(j)} = W_j$ whenever $W_j$ is already single-valued, i.e. $\{\epsilon_n \mid n \in W_j\}$ is bounded in D. Also since the construction of $W_{s(j)}$ in the above definition is uniform in j, we can think of the s in $W_{s(j)}$ as a recursive function. Thence the Egli-Constable's version of computability is as follows:

THEOREM 1.6 (Egli-Constable's version of computablity)

$x \in D$ is computable iff $x = \bigsqcup \{\epsilon_i \mid i \in W_{s(j)}\}$ for some $j \in N$, where D is an effectively given domain and $E_D = \{\epsilon_i \mid i \in N\}$.

proof (if part): Let $\rho_j$ be a recursive function enumerating $W_{s(j)}$. Define a recursive function $r_j : N \longrightarrow N$ by:

$$\epsilon_{r_j(0)} = \epsilon_{\rho_j(0)}$$

$$\epsilon_{r_j(n+1)} = \epsilon_{r_j(n)} \sqcup \epsilon_{\rho_j(n+1)}.$$

Obviously $\epsilon_{r_j(0)} \sqsubseteq \epsilon_{r_j(1)} \sqsubseteq \ldots$ is an effective chain in $E_D$. Also obviously x is the lub of this effective chain. Therefore x is computable.

(only if part): Let $x = \bigsqcup_i \epsilon_{r(i)}$ for some recursive function $r:N \to N$ s.t. $\epsilon_{r(0)} \sqsubseteq \epsilon_{r(1)} \sqsubseteq \ldots$ . Then range r is obviously r.e. and single-valued. But $x = \bigsqcup \{\epsilon_i \mid i \in \text{range } r\}$. **Q.E.D.**

The technique used in the if part of the above proof will be repeatedly used throughout this paper.

The advantage of this alternative definition is that this definition enables us to index the set $C_D$ of all computable elements of D by:

$$C_D = \{\gamma_i \mid \gamma_i = \bigsqcup \{\epsilon_j \mid j \in W_{s(i)}\},\ i \in N\}.$$

Furtheremore we will observe that $C_D$ can be enumerated effectively in some sense. But before examining the effective enumerability of $C_D$, we have to obtain an adequate notion of effective enumeration of such abstract objects. To do this we will start to examine an analogue in classical recursive function theory, that is the well-known recursive eneumeration of partial recursive functinions and recursively enumerable sets.

In classical recursive function theory, there is a hidden point in the recursive enumeration of infinite objects like partial recursive functions or r.e. sets. In fact we have to observe the following:

(1) Eventually we recursively enumerate finite representations (i.e. programs) of partial recursive functions rather than function themselves.

(2) From each program P, the corresponding partial recursive function is obtained by the following effective limiting process:

step 1 : execute the first step of P(0)

step n : execute the nth step of P(0)

execute the (n-1)th step of P(1)

-----------------------

execute the first step of P(n-1)

Once P(m) terminates output (m,P(m)).

Exactly the same discussion can be done for the recursive enumeration of r.e. sets. Actually the above (1) and (2) are typical schema involved in the effective enumeration of infinitary objects. Also note that this schema can be applied even for the effective enumeration of finite objects like natural numbers, because finite process can be considered as an effective limiting process.

The above observation suggests the following notion of effective enumeration in a general setting. A family of objects is said to be <u>effectively enumerable</u> iff there exists a corresponding set of finitary representations s.t. there is a recursive function from N to the set of representations and there is an effective limiting process which is an onto map from the set of representations to the family of objects.

By taking $\bigsqcup\{\epsilon_n | n \in W_{s(i)}\}$ as the finitary representation of $\gamma_i \in C_D$, and using the effective limiting process described in the if-part of the proof of 1.6, we can easily observe:

<u>THEOREM 1.7</u>

For every effectivly given domain $(D, \sqsubseteq)$, the set $C_D$ of all computable elements of D is effectively enumerable.

We shall call the effective enumeration:

$$C_D = \{\gamma_i | i \in N\}, \ \gamma_i = \bigsqcup\{\epsilon_n | n \in W_{s(i)}\}$$

as the <u>standard enumeration</u> of $C_D$. Egli-Constable used this standard enumeration for the purpose of establishing essential equivalence between computable functions and operational functions. In this paper we will use this indexing for some other interesting purpose later.

The effectively given domains enjoy much more interesting properties, some of which will be listed in the following:

<u>FACT 1.8</u> (product and disjoint union)

Let D, D' be effectively given domains with effective basis $E_D = \{\epsilon_i | i \in N\}$, $E_{D'} = \{\epsilon'_i \mid i \in N\}$ respectively. Then we have:

(1) $D \times D'$ is an effectively given domain with the effective basis

$$E_{D \times D'} = \{(\epsilon x \epsilon')_m | m \in N\}$$

where $(\epsilon x \epsilon')_m = \langle \epsilon_{\pi_1(m)}, \epsilon'_{\pi_2(m)} \rangle$.

(2) $D+D'$ is an effectively given domain with the effective basis

$$E_{D+D'} = \{(\epsilon+\epsilon')_m | m \in N\}$$

where $(\epsilon+\epsilon')_0 = \perp$

$$(\epsilon+\epsilon')_{2n+1} = \langle \epsilon_n, 0 \rangle$$

$$(\epsilon+\epsilon')_{2n} = \langle \epsilon'_n, 1 \rangle$$

FACT 1.9 (function space theorem)

Let D and D' be effectively given domains with effective basis $E_D$ and $E_{D'}$ respectively. For every $(e,e') \in E_D x E_{D'}$, define a step function $[e,e']: D \dashrightarrow D'$ by: $[e,e'](d) = \underline{\text{if}}\ d \sqsupseteq e\ \underline{\text{then}}\ e'\ \underline{\text{else}}\ \perp_{D'}$.

Also let $F = \{[e,e'] | e \in E_D, e' \in E_{D'}\}$. Then the set $[D \dashrightarrow D']$ of all continuous functions from D to D' with point-wise ordering also is an effectively given domain with effctive basis:

$$E_{[D \dashrightarrow D']} = \{\bigsqcup A |\ A \text{ is a finite subset of } F,\ \bigsqcup A \in [D \dashrightarrow D']\}$$
$$= \{(\epsilon \dashrightarrow \epsilon')_i | i \in N\}$$

where $(\epsilon \dashrightarrow \epsilon')_i = \underline{\text{if}}\ \sigma(i)$ has a lub in $[D \dashrightarrow D']$ $\underline{\text{then}}\ \bigsqcup \sigma(i)\ \underline{\text{else}}\ \perp_{[D \dashrightarrow D']}$ where $\sigma(k) = \{[\epsilon_i, \epsilon_j] |\ (i,j) \in p(k)\}$ and p is the standard enumeration of finite subsets of NxN.

FACT 1.10

Let D, D' be effectively given domains then $f \in [D \dashrightarrow D']$ is a computable object iff the graph of f

$$\Gamma(f) = \{(n,m) |\ \epsilon'_n \sqsubseteq f(\epsilon_m)\}$$

is r.e. Note that since f is continuous, $\Gamma(f)$ precisely determines f.

FACT 1.11

(1) Every computable function sends each computable element to a computable element, i.e. $f \in C_{[D \dashrightarrow D']}$ & $x \in C_D$ implies $f(x) \in C_{D'}$. Furtehermore if the

index of f is j and the index of x is k then there exists a recursive function $t:N\times N \to N$ s.t.: $f(x) = \gamma'_{t(j,k)}$.

Also there exists a recursive function $r:N \to N$ s.t.: $f(x) = \gamma_{r(k)}$, namely $r(k) = t(j,k)$.

(2) If $f \in C_{[D \to D']}$ and $g \in C_{[D' \to D'']}$ then $g \circ f \in C_{[D \to D'']}$. Furtheremore if the indices of f and g are i and j respectively then there exists a recursive function $v:N\times N \to N$ s.t. $v(i,j)$ is an index of $g \circ f$.

Park [21] discovered that for the construction of continuous lattices from basis, we could use a generalization of Dedekind Cut. Using quite similar idea, Markowsky-Rosen [14] and Smyth [12] gave a more elegant specification of effctively given domains. We will see this in the following:

<u>DEFINITION 1.12</u>

Let $(E,\sqsubseteq)$ be a countable poset. By <u>completion</u> of $(E,\sqsubseteq)$, we mean a poset $(\overline{E},\subseteq)$ s.t. $\subseteq$ is the set theoretical inclusion and $\overline{E}$ is the set of all subsets $X \subseteq E$ s.t.:

(1) every finite subset of X has a lub in X (directed)

(2) $x \sqsubseteq y$ & $y \in X$ implies $x \in X$ (downward closed)

<u>FACT 1.13</u> (<u>the completion theorem</u> I)

(1) Let $(\overline{E}, \subseteq)$ be the completion of a countable poset $(E,\sqsubseteq)$. Define $\tau:E \to \overline{E}$ by $\tau(x) = \{a \in E \mid a \sqsubseteq x\}$. Then $\tau$ is an isomorphic embedding of E to $\overline{E}$.

(2) For every $x \in \overline{E}$, $J_x = \{\tau(a) \mid a \in E, \tau(a) \subseteq x\}$ is directed and $x = \bigcup J_x$.

(3) $\tau(E)$ is exactly the finite elements of $\overline{E}$.

(4) $\tau$ satisfies the extension property s.t. for each monotone $m:E \to Q$ there exists a unique continuous extenstion $f_m:\overline{E} \to Q$ of m s.t.

$$\begin{tikzcd} E \arrow[rr, "\tau"] \arrow[dr, "m"'] & & \overline{E} \arrow[dl, "f_m"] \\ & Q & \end{tikzcd}$$ commutes

where Q is an arbitary cpo.

Furtheremore if $(E,\sqsubseteq)$ has the least element $\perp$, then so does $(\overline{E}, \subseteq)$.

FACT 1.14 (the completion theorem II)

(1) Let $(E,\sqsubseteq)$ be a countable poset with bounded joins and the least element s.t. the following relations are recursive in indices:

(i) $\bigsqcup\{\epsilon_{i_1},\ldots,\epsilon_{i_m}\}$ exists in E,

(ii) $\epsilon_j = \bigsqcup\{\epsilon_{i_1},\ldots,\epsilon_{i_m}\}$.

Then $\overline{E}$ is an effectively given domain with the effective basis $\mathcal{T}(E)$ and the extension property described in the above theorem.

(2) Let $(D,\sqsubseteq)$ be an effectively given domain with the effective basis $(E_D,\sqsubseteq)$. Then $(\overline{E}_D, \subseteq) \cong (D,\sqsubseteq)$ and $(E_D,\sqsupseteq) \cong (\mathcal{T}(E), \subseteq)$.

This idea has been characterized much more elegantly by Scott [5], using algebraic closure operators. Also Scott [5] discovered a universal domain for effectively given domains and developed a retract calculus for this doamin. But this theory is beyond the scope of this paper.

## §2. From effectively given domains to effective domains

Even though the theory of effectively given domains looks very satisfactory, there are quite odd things in it. First the way to extend effective basis to an effectively given domain does not have any effective flavor at all. As can be seen in definition 1.2, we extend $E_D$ to D using limits of directed sets, which is not effective. Secondly even though we are mainly interested in the computable elements, i.e. the effectively enumerable set $C_D$, we took the huge D as a data type. Thirdly we took the set of all continuous functions as a functional data type, even though continuity does not represent computability quite well.

On the other hand, the following observation of the nature of $C_D$ will convince us that $C_D$ could be a satisfactory data type.

THEOREM 2.1

For every effectively given domain $(D,\sqsubseteq)$, the set of all computable elements of D, $(C_D, \sqsubseteq)$ is effecive chain complete, i.e. is closed under the limit of effective chains.

<u>proof</u> Let $\{\gamma_{\rho(n)} | n \in N\}$ be an effective chain in $C_D$. For each n, we have $\gamma_{\rho(n)} = \bigsqcup\{\epsilon_m | m \in W_{s(\rho(n))}\}$. Therefore by the proof of 1.6, there exists a recursive function $t_{\rho(n)}: N \to N$ s.t. $\epsilon_{t_{\rho(n)}(i)}$, $i \in N$ is an effective chain and $\gamma_{\rho(n)} = \bigsqcup_i \epsilon_{t_{\rho(n)}(i)}$. Define $e_i$ by:

$$e_0 = \epsilon_{t_{\rho(n)}(0)}$$

$$e_i = \epsilon_{t_{\rho(0)}(i)} \sqcup \epsilon_{t_{\rho(1)}(i-1)} \sqcup \dots \sqcup \epsilon_{t_{\rho(i)}(0)}$$

Since all possible finite joins are recursive in indices and $\{\gamma_{\rho(n)} | n \in N\}$ is an effective chain, evidently $e_0 \sqsubseteq e_1 \sqsubseteq \dots$ is an effective chain in $E_D$. Therefore $\bigsqcup_n \gamma_{\rho(n)} = \bigsqcup_n e_n$ is a computable element of D. <u>Q.E.D.</u>

Note that if $\gamma_{\rho(n)}$ is not an effective chain then $e_i$, $i \in N$ fails to be an effective chain. Therefore $C_D$ is not a cpo.

<u>TEOREM 2.2</u>

Let D and D' be effectively given domains with effective basis $E_D = \{\epsilon_i | i \in N\}$ and $\{\epsilon'_i | i \in N\}$ respectively. Then:

(1) $C_D \times C_{D'} = C_{D \times D'}$

(2) $C_D + C_{D'} = C_{D+D'}$

<u>proof</u> (1) Let $\langle c, c' \rangle \in C_D \times C_{D'}$. Then for some recursive functions $r, r': N \to N$:

$$\langle c, c' \rangle = \langle \bigsqcup_n \epsilon_{r(n)}, \bigsqcup_n \epsilon'_{r'(n)} \rangle$$

$$= \bigsqcup_n \langle \epsilon_{r(n)}, \epsilon'_{r'(n)} \rangle = \bigsqcup_n (\epsilon \times \epsilon')_{t(n)}$$

where $t(n) = (r(n), r'(n))$. Since r, r' and the pairing function are recursive, t evidently is recursive. Therefore $\langle c, c' \rangle \in C_{D \times D'}$. Conversely let $\langle c, c' \rangle \in C_{D \times D'}$. Then for some recursive function $r: N \to N$,

$$\langle c, c' \rangle = \bigsqcup_n (\epsilon \times \epsilon')_{r(n)}$$

$$= \bigsqcup_n \langle \epsilon_{\pi_1 r(n)}, \epsilon'_{\pi_2 r(n)} \rangle = \langle \bigsqcup_n \epsilon_{\pi_1 r(n)}, \epsilon'_{\pi_2 r(n)} \rangle$$

By recursiveness of $\pi_1$ and $\pi_2$, obviously $\pi_1 r$ and $\pi_2 r$ are recursive. Therefore we have:

$$\langle c,c'\rangle = \langle \bigsqcup_n \epsilon_{\pi_1 r(n)},\ \bigsqcup_n \epsilon'_{\pi_2 r(n)}\rangle \in C_D \times C_{D'}.$$

(2) Let $\langle x,i\rangle \in C_D + C_{D'}$, where $i = 0$ or $1$. In case $i = 0$, $x \in C_D$. Therefore we have:

$$\langle x,i\rangle = \langle \bigsqcup_n \epsilon_{r(n)},\ 0\rangle = \bigsqcup_n \langle \epsilon_{r(n)},\ 0\rangle = \bigsqcup_n (\epsilon+\epsilon')_{2r(n)+1}$$

for some recursive function $r:N \to N$. Therefore $\langle x,i\rangle \in C_{D+D'}$. Similarly for $i = 1$. Conversely let $\langle x,i\rangle \in C_{D+D'}$. In case $i = 0$, $x \in C_D$. Therefore $\langle x,i\rangle \in C_D + C_{D'}$. Similarly for $i = 1$.

Q.E.D.

The problem for function space is more difficult. If we have a computable function $g:D \to D'$, we are actually interested in only the restriction $\bar{g}$ of $g$ to $C_D$. But does this sort of object have any interesting mathematical property at all? The answer is positive. First, fact 1.11 tells us that $\bar{g}$ has the functionality $C_D \to C_{D'}$. In the next section we will observe that for each such restricted function form $C_D$ to $C_{D'}$, there is a unique computable extension from D to D'. Therefore we can see that all computable elements of $[D \to D']$, i.e. $C_{[D \to D']}$ is essentially equivalent to the set $[C_D \to C_{D'}]$ of all restriction of elements of $C_{[D \to D']}$ to $C_D$. Roughly speaking we have:

$$C_{[D \to D']} \cong [C_D \to C_{D'}].$$

These results will almost convince us that $C_D$ should be taken as a data type rather than D. Because not only the basis but also all elements are effective, we shall call such data types as $C_D$ <u>effective data types</u> or <u>effective domains</u>.

## §3. Effective domains

One outstanding point in the privious section is that the notion of effective domains is totally dependant upon the notion of effectively given domains. Fortunately we can solve this problem quite satisfactory.

First we observe that both D and $C_D$ share the same basis $E_D$. The difference between them is the way to extends $E_D$. D is extended by taking directed limits of $E_D$, while $C_D$ is extended by taking the effective limits of $E_D$. (Also this is why $C_D$ is more favorable than D for computer scienctists as mentioned in the introduction.) Thus essentially $C_D$ does depend not on D but on $E_D$. These observation will lead us to the following effectively given domain independent notion of effective domains which is due to Smyth.

DEFINITION 3.1 (effective domains)

A poset $(X,\sqsubseteq)$ is an effective domain iff

(1) $E_X$ is countable with an indexing $E_X = \{\epsilon_i | i \in N\}$ and has bounded joins.

(2) The following relations are recursive in indices:

(i) $\bigsqcup\{\epsilon_{i_1},\ldots,\epsilon_{i_m}\}$ exists in $E_X$,

(ii) $\epsilon_k = \bigsqcup\{\epsilon_{i_1},\ldots,\epsilon_{i_m}\}$.

(3) Every effective chain of $E_X$ has a lub in X.

(4) Every element of X is a lub of an effective chain of $E_X$.

Evidently the following enumeration of X is an effective enumeration and we will call it the standard enumeration of X:

$$X = \{\chi_i | i \in N\}, \quad \chi_i = \bigsqcup\{\epsilon_n | n \in W_{s(i)}\}.$$

It is intuitively clear that an effective domain is the set of all computable elements of the effectively given domain which is obtained by extending the effective basis of the effective domain using directed limit. This will be formally observed in the following discussion.

DEFINITION 3.2 (effective completion)

(1) An effective poset $(E,\sqsubseteq)$ is a countable poset $E = \{\epsilon_n | n \in N\}$ with bounded joins s.t. the following relations are recursive in indices:

(i) $\{\epsilon_{n_1},\ldots,\epsilon_{n_k}\}$ is bounded in E,

(ii) $\epsilon_j = \bigsqcup\{\epsilon_{n_1},\ldots,\epsilon_{n_k}\}$.

(2) Given an effective poset E, the relation "$\leq$" among effective chains of E, defined by: $\{x_n | n \in N\} \leq \{y_m | m \in N\}$ iff $\forall n \exists m. x_n \sqsubseteq y_m$, is a pre-ordering. Let R be the equivalence relation defined by "$\{x_n\}$ R $\{y_n\}$ iff $\{x_n\} \leq \{y_n\}$ & $\{y_n\} \leq \{x_n\}$". Then by the effective completion of an effective poset $(E,\sqsubseteq)$, in symbol $(\tilde{E},\sqsubseteq)$, we mean the set of equivalent classes of effective chains of E w.r.t. the

equivalence relation R, with the following partial ordering:

$$[\{x_n\}] \sqsubseteq [\{y_m\}] \text{ iff } \forall\{x'_n\}\in[\{x_n\}].\exists\{y'_m\}\in[\{y_m\}].\ \{x'_n\} \leq \{y'_m\}.$$

By definition, obviously $\{x_n\} \leq \{y_m\}$ iff $[\{x_n\}] \sqsubseteq [\{y_m\}]$. Therefore we can introduce the following convention: (1) identify $[\{x_n\}]$ to $\{x_n\}$, i.e. to its representative, (2) identify $\leq$ to $\sqsubseteq$ . This convention allows us to consider the pre-ordered set of effective chains of E as the effective completion of E. Reminding this convention, we have the following theorem:

<u>THEOREM 3.3</u> (<u>the effective completion theorem</u>)

(1) Let $(\tilde{E},\sqsubseteq)$ be the effective completion of an effective poset $(E,\sqsubseteq)$. Define $\tau':E \dashrightarrow \tilde{E}$ by $\tau'(e) = \{e\}$ $(= [\{e\}] = \{\{e\}\})$. Then evidently $\tau'$ is an isomorphic embedding of E to $\tilde{E}$.

(2) For every $c\in\tilde{E}$, there exists an effective chain $J_c$ in $\tau'(E)$ s.t. $c = \bigsqcup J_c$.

(3) Let $(\bar{E}, \subseteq)$ be the completion of $(E,\sqsubseteq)$ and $\tau:E \dashrightarrow \bar{E}$ be the natural embedding $\tau(e) = \{x\in E\,|\,x \sqsubseteq e\}$. Then evidently $\tau(E)$ is the set of all finite elements of $(\bar{E}, \subseteq)$ and $\tau(E) \cong \tau'(E)$.

(4) $(\tilde{E}, \subseteq) \cong (C_{\bar{E}}, \subseteq)$.

(5) $(\tilde{E},\sqsubseteq)$ is an effective domain.

<u>proof</u> (1) trivial

(2) Let $c = \{c_0 \sqsubseteq c_1 \sqsubseteq \ldots\}\in\tilde{E}$. Since $x\in\tau'(x)$, $\tau'(x) \sqsubseteq c$ implies $x \sqsubseteq c_n$ for some $n\in N$. Conversely $x \sqsubseteq c_n$ obviously implies $\tau'(x) \sqsubseteq c$. Thus $\tau'(x) \sqsubseteq c$ iff $x \sqsubseteq c_n$ for some n. Depending on the indexing $E = \{\epsilon_i\,|\,i\in N\}$, we index $\tau'(E)$ by $\tau'(E) = \{t'_i = \tau'(\epsilon_i)\,|\,i\in N\}$. Since $\epsilon_k \sqsubseteq c_n$ is recursive in k for each n, $\{k\,|\,t'_k\in J'_c\}$ is r.e. where $J'_c = \{\tau'(x)\,|\,x\in E, \tau'(x) \sqsubseteq c\}$. Since finite join is recursive in indices in E, finite join is recursive in indices in $\tau'(E)$. Therefore we have the following effective chain:

$$t'_{r(0)} = t'_{p(0)}$$

$$t'_{r(n+1)} = t'_{r(n)} \sqcup t'_{p(n+1)}$$

where p is a recursive enumeration function of $\{k\,|\,t'_k\in J'_c\}$. But $\tau'(a),\tau'(b)\in J'_c$ implies $\tau'(a\sqcup b) = \tau'(a)\sqcup\tau'(b)\in J'_c$, since $a \sqsubseteq c_n$ and $b \sqsubseteq c_m$ implies

$a|\_|b \sqsubseteq c_n|\_|c_m \sqsubseteq c_k$ for some $c_k \in c$. Therefore $J_c = \{t'_{r(n)}|n\in N\}$ satisfies $c = |\_|J_c$.

(3) Evidently $E \cong \tau'(E)$. Thereofre $\tau'(E) \cong \tau(E)$ by 1.13.

(4) Think of the following correspondence:

$$f:\tilde{E} \dashrightarrow C_{\tilde{E}}:\{e_n|n\in N\} \longmapsto \cup\{\tau(e_n)|n\in N\}$$

$$g:C_{\tilde{E}} \dashrightarrow \tilde{E}:\cup\{\tau(e_n)|n\in N\} \longmapsto \{e_n|n\in N\}.$$

Note that since we index $\tau(E)$ by $\tau(E) = \{t_i = \tau(e_i)|i\in N\}$, we have: $\{e_n|n\in N\}$ is an effective chain in E iff $\{\tau(e_n)|n\in N\}$ is an effective chain in $\tau(E)$. Therefore f and g are well-defined. Obviously $f\circ g = id$ and $g\circ f = id$. Also $\{e_n|n\in N\} \sqsubseteq \{e'_m|m\in N\}$ implies $\forall n.\exists m.\tau(e_n) \subseteq \tau(e'_m)$. Therefore f is monotone. By finiteness of $\tau(e_n)$, $\tau(e_n) \subseteq \underset{m}{\cup}\tau(e_m) \subseteq \underset{m}{\cup}\tau(e'_m)$ implies $\tau(e_n) \subseteq \tau(e'_m)$ for some $m\in N$. Therefore $e_n \sqsubseteq e'_m$ for some $m\in N$. Therefore g also is monotone.

(5) By (1), $\tau'(E)$ obviously is an effective poset. Therefore all we need to show is that $\tau'(E)$ is the set of all finite elements of $(\tilde{E},\sqsubseteq)$. To do this , first note that since E is effective, given $\{e_n|n\in N\} \sqsubseteq \{e'_m|m\in N\}$, we can effectively find m for each n s.t. $e_n \sqsubseteq e_m$. Now let $X \subseteq \tilde{E}$ be directed and assume that $|\_|X \in \tilde{E}$. Since $\tilde{E}$ is countable, X also is countable. Therefore using the following effective procedure, we can effectively enumerate the effective chain $|\_|X$. First we denote by $r_{ij}$, the function which, given $\{e^i_n|n\in N\} \sqsubseteq \{e^j_m|m\in N\}$, effectively gives us the minimal m s.t. $e^i_n \sqsubseteq e^j_m$, for every $m\in N$. Then $|\_|X$ is $\{x_n|n\in N\}$ where

$$x_0 = e^0_0$$

$$x_n = e^n_{r_{(n-1)n}(r_{(n-2)(n-3)}(\ldots(r_{01}(n))\ldots))}$$

where $X = \{\{e^i_n|n\in N\}|i\in N\}$. Now assume $\tau'(e) \sqsubseteq |\_|X$. Then $e \sqsubseteq x_n$ for some $n\in N$. Therefore $\{e\} \sqsubseteq \{e^n_k|k\in N\} \in X$. Thus $\tau'(e)$ is finite. Conversely let $\{e_k|k\in N\} \in \tilde{E}$ be finite. By (2), $\{e_k|k\in N\} = |\_|J_{\{e_k|k\in N\}}$. By finiteness of $\{e_k|k\in N\}$, $\{e_0 \sqsubseteq e_1 \sqsubseteq \ldots\} \sqsubseteq \{e'_0 \sqsubseteq e'_1 \sqsubseteq \ldots\}$ for some $\{e'_0 \sqsubseteq e'_1 \sqsubseteq \ldots\}$ for some $\{e'_0 \sqsubseteq e'_1 \sqsubseteq \ldots\} \in J_{\{e_0 \sqsubseteq e_1 \sqsubseteq \ldots\}}$. But since

$\{e'_0 \sqsubseteq e'_1 \sqsubseteq \dots\} \in J_{\{e_0 \sqsubseteq e_1 \sqsubseteq \dots\}}$, we have:

$$\{e'_0 \sqsubseteq e'_1 \sqsubseteq \dots\} = \tau'(e) \sqsubseteq \{e_0 \sqsubseteq e_1 \sqsubseteq \dots\}$$

for some $e \in E$. Therefore $\{e_0 \sqsubseteq e_1 \sqsubseteq \dots\} = \tau'(e)$ for some $e \in E$. Thus $\tau'(E)$ is exactly the set of all finite elements of $\tilde{E}$. Q.E.D.

Intuitive interpretation of this elaborate theorem is as follows. Given an effective basis $\tau(E) \cong \tau'(E)$, the classical directed limit completion gives us the effectively given domain $(\bar{E}, \subseteq)$ and the effective chain completion gives us the effective domain $(\tilde{E}, \sqsubseteq)$. Furthermore $(\tilde{E}, \sqsubseteq) \cong (C_{\bar{E}}, \subseteq)$. Therefore this theorem justifies the intuitive argument following 3.1.

In the above discussion we have observed a relation between directed completion and effective chain completion , starting from the same (isomorphic) effective basis. Now we will examine the other way around. That is , given an effective doamin, what happens if we take the effective completion of its basis.

<u>THEOREM 3.4</u> (alternative definition of effective domains)

(1) Let $X$ be an effective domain with the effective basis $E_X$. Then $X \cong \tilde{E}_X$ and $E_X$ is an effective poset.

(2) A poset is an effective domain iff it is isomorphic to an effective completion of some effective poset.

(3) A poset is an effective domain iff it is isomorphic to a set of all computable elements of some effectively given domain.

<u>proof</u> (1) Define $f: X \to \tilde{E}_X: \bigsqcup_n e_n \longmapsto \{e_n | n \in N\}$ where $\{e_n | n \in N\}$ is an effective chain and $g: \tilde{E}_X \to X: \{e_n | n \in N\} \longmapsto \bigsqcup_n e_n$. Evidently $f \circ g = id$ and $g \circ f = id$. Let $\bigsqcup_n e_n \sqsubseteq \bigsqcup_m e'_m$. Then $e_n \sqsubseteq \bigsqcup_m e'_m$ for every $n \in N$. By compactness of $e_n$, $e_n \sqsubseteq e'_m$ for some $m \in N$. Therefore $\{e_n | n \in N\} \sqsubseteq \{e'_m | m \in N\}$. Thus $f$ is monotone. Now let $\{e_n | n \in N\} \sqsubseteq \{e'_m | m \in N\}$. Then for each $n$, there exists $m$ s.t. $e_n \sqsubseteq e_m$. therefore $\bigsqcup_n e_n \sqsubseteq \bigsqcup_m e'_m$. Thus $g$ also is monotone.

(2) (only if) : by (1)-3.4

(if-part) : Let $(i: X \to \tilde{E}, j: \tilde{E} \to X)$ be the isomorphism. Then $(X, \sqsubseteq)$ is an effective domain with $E_X = j(\tau'(E))$.

(3) (only if) : by (2)-theorem 3.4 and (4)-theorem 3.3.

(if-part) : Let $(X,\sqsubseteq) \cong (C_D,\sqsubseteq)$ where $(D,\sqsubseteq)$ is an effectively given domain. Then $(\tilde{E}_D,\sqsubseteq) \cong (C_D,\sqsubseteq) \cong (X,\sqsubseteq)$. Thus by (2)-Theorem 3.4. Q.E.D.

(3)-theorem 3.4 tells us that our notion of effective domains defined in 3.1 is exactly what we wanted. Also this fact ensures us that we can safely regard each effective domain as the set of all computable elements of the effectively given domain which is extended from the same effective basis by directed completion. More precisely, given an effective doamin $(X,\sqsubseteq)$, we can regard this as $(C_{\overline{E}_X}, \subseteq)$ since $(X,\sqsubseteq) \cong (\tilde{E}_X,\sqsubseteq) \cong (C_{\overline{E}_X}, \subseteq)$. Therefore discussing effective domains as the sets of all computable elements of effectively given domains gives us techincal simplification without harming generality and the independence of the notion of effective domains on that of effectively given doamins. Thus directly from theorem 2.1 and theorem 2.2, we have:

THEOREM 3.5

(1) Every effective data type is effective chain complete.

(2) The class of effective data types is closed onder x and +.

Now what the functional data types are. As mentioned in the introduction, effective functions must staisfy the following:

DEFINITION 3.6 (effective functions)

Let $C_{D'}$ and $C_{D'}$ be effective domains, then $f:C_D \dashrightarrow C_{D'}$ is an effective function iff

(1) $\overline{\Gamma}(f) = \{(n,m) \mid \epsilon'_n \sqsubseteq f(\epsilon_m)\}$ is r.e., and

(2) f is effectively continuous i.e., for every effective chain $\{e_n \mid n \in N\}$,

$$f(\bigsqcup_n e_n) = \bigsqcup_n f(e_n).$$

We will denote the set of all effective functions from $C_D$ to $C_{D'}$ with point-wise ordering by $[C_D \dashrightarrow C_{D'}]$.

Now a question arises. Does $[C_D \dashrightarrow C_{D'}]$ form an effective domain, and what is the relevance of $[C_D \dashrightarrow C_{D'}]$ to $C_{[D \dashrightarrow D']}$? We will see positive answer in the following. First of all we will observe that each effective function $C_D \dashrightarrow C_{D'}$ is a restriction to $C_D$ of some computable function from D to D'.

THEOREM 3.7

A function $f:C_D \dashrightarrow C_{D'}$ is effective iff it is a restriction to $C_D$ of some computable function from D to D'.

proof (if part):trivial.

(only if part): Let $f:C_D \dashrightarrow C_{D'}$ be effective and $\tilde{f}$ be the restriction of f to $E_D$. Then evidently $\tilde{f}:E_D \dashrightarrow C_{D'}$ is monotone. By fact 1.13, $\tilde{f}$ has a unique continuous extension $\Phi_{\tilde{f}}:D \dashrightarrow D'$ s.t.:

$$\Phi_{\tilde{f}}(\bigsqcup_n e_n) = \bigsqcup_n \tilde{f}(e_n).$$

By (1)-3.6, $\Phi_{\tilde{f}}$ is evidently computable, because:

$$\bar{\Gamma}(f) = \Gamma(\Phi_{\tilde{f}}).$$

Therefore $\Phi_{\tilde{f}}$ is the unique computable extension of f. Let $\bar{\Phi}_{\tilde{f}}$ be the restriction to $C_D$ of $\Phi_{\tilde{f}}$. Then for every $c = \bigsqcup_n \epsilon_{r(n)} \in C_D$ with r recursive function,

$$\begin{aligned}\bar{\Phi}_{\tilde{f}}(c) = \Phi_{\tilde{f}}(c) &= \bigsqcup_n f(\epsilon_{r(n)}) \\ &= f(\bigsqcup_n \epsilon_{r(n)}) \quad (\because (2)\text{-}3.6) \\ &= f(c).\end{aligned}$$

Therefore f is the restriction to $C_D$ of the computable function $\Phi_{\tilde{f}} : D \dashrightarrow D'$.

Q.E.D.

The above proof has further implication which is crucial in the theory of effective domains. First define

$$i:[C_D \dashrightarrow C_{D'}] \dashrightarrow C_{[D \dashrightarrow D']}:f \mapsto \Phi_{\tilde{f}}, \text{ and}$$

$$j:C_{[D \dashrightarrow D']} \dashrightarrow [C_D \dashrightarrow C_{D'}]:f \mapsto \text{the restriction to } C_D \text{ of } f.$$

Then evidently (i,j) is an isomorphism. Therefore we have:

THEOREM 3.8

(1) $C_{[D \dashrightarrow D']} \cong [C_D \dashrightarrow C_{D'}]$

(2) $[C_D \dashrightarrow C_{D'}]$ is an effective domain.

proof (1) as above.

(2) by (1)-3.8 and (3)-3.4 Q.E.D.

We can restate this without using the conventional identification of $(X,\sqsubseteq)$ to $(C_{\overline{E}_X}, \subseteq)$.

COROLLARY 3.9

Given effective domains X and X', we have:

(1) $C_{[\overline{E}_X \dashrightarrow \overline{E}_{X'}]} \cong [X \dashrightarrow X']$

(2) $[X \dashrightarrow X']$ is an effective domain.

Note that (1)-3.8 and 3.9 are claiming that as long as computable, or equivalently effective functions are concerned, both effectively given domains and effective domains are essentially the same. In other words, the theory of effectively given domains does contain properly the theory of effective domains. Thus we can regard the theory of effectively given domains as a redundant theory of effective domains. This observation justifies the claim of effectively given domain theorists that as long as we have a concise criterion to distingulish computable elements from noncomutable elements we are O.K. But without our observation of 3.8 and 3.9, their claim should be hardly justified.

## §4. Effective functions and operations

This section is essentially due to Egli-Constable [13]. This is nothing more than a simplification of their works. Our main goal in this section is to establish that a function is effective iff it is operational in some suitable sense, hence the equivalence of descriptive power of denotational semantics and operational semantics.

Given any r.e. set $W_j$, we can effectively single-value it w.r.t. an effective basis $E_X$ of an effective domain X and (n+1) tuples by:

--- enumerate $W_j$ : $x_1, x_2, \ldots$

--- $y_1 = x_1$

$$y_{k+1} = \begin{cases} y_k & \text{if } x_{k+1} = (p_1,\ldots,p_n,q) \text{ and } \exists i \leq k.\, y_i = (p_1,\ldots,p_n,q') \,\& \\ & \{\epsilon_q, \epsilon_{q'}\} \text{ not bounded} \\ x_{k+1} & \text{otherwise.} \end{cases}$$

Since $\{y_i \mid i\in N\}$ is effectively enumerable, there exists a recursive function $s_n^X$ s.t. $W_{s_n^X(j)} = \{y_i \mid i\in N\}$.

FACT 4.1

(1) For each $x_1,\ldots,x_n \in N$, $\bigsqcup\{\epsilon_k \mid (x\ ,\ldots,x\ ,k)\in W_{s_n^X(j)}\}$ exists in X. Therefore there is a function ${}_X\psi_j^n : N^n \dashrightarrow N$ s.t.:

$${}_X\psi_j^n(x_1,\ldots x_n) = \bigsqcup\{\epsilon_k \mid (x_1,\ldots,x_n,k)\in W_{s_n^X(j)}\}.$$

Also note that $\{{}_X\psi_j^n \mid j\in N\}$ is an effective enumeration of the family of such functions.

(2) Such families satisfy S-m-n property:

$${}_X\psi_j^{m+n}(x_1,\ldots,x_m,y_1,\ldots,y_n) = {}_X\psi^n_{s_n^m(j,x_1,\ldots,x_m)}(y_1,\ldots,y_n).$$

(3) In case n = 0, $\{{}_X\psi_j^0 \mid j\in N\}$ is an effective enumeration of X. We shall write $\chi_j = {}_X\psi_j^0$.

DEFINITION 4.2

Let X and X' be effective domains with effective basis $E_X$ and $E_{X'}$, respectively. Then a function ${}_{X'}\psi_j^0 : N \dashrightarrow E_{X'}$ satisfying:

$${}_X\psi_n^0 = \chi_n = {}_X\psi_m^0 = \chi_m \quad \text{implies} \quad {}_{X'}\psi_j^1(n) = {}_{X'}\psi_j^0(m)$$

determines a unique function $\Psi_j : X \dashrightarrow X'$ by ${}_{X'}\psi_j^1 = \Psi_j \circ (\lambda n.\chi_n)$. We shall call this $\Psi_j$ the operation of the oprational index j. Then evidently the set of operational index is a nonrecursive subset of N, because function equality is not a recursive relation.

Note that $\Psi_j$ is operational in the sense that it is induced by the function which sends the text n of $\chi_n \in X$ into an element of X'. The next fact due to Egli-Constable shows the equivalence (even recursive equivalence) of operations and effective functions.

FACT 4.3

Let X and X' be effective domains with effective basis $E_X$ and $E_{X'}$, respectively. Let $\Phi_i$, $i \in N$ be the effective enumeration of [X -->X']. Then :

(1) There exists a recursive function g:N -->N s.t.:

$$\Phi_i = \Psi_{g(i)}.$$

(2) There exists a recursive function h:N -->N s.t. for every operational index j,

$$\Phi_{h(j)} = \Psi_j.$$

Note that 4.3 is claiming that a function f:X -->X' is effective iff it is operational.

## §5. The category of effective domains

So far we have obtained three major methods of (effectively) constructing new effective domains from given effective domains, namely "x", "+", and "-->". What is more interesting among (effective) constructions of effective domains is not any of these but rather self-referential definition of reflexible domains. There are two known ways to handle self-referential definition in classical domain theory, namely the universal domain approach developed by Scott [4,5] and Plotkin [15]; and the categorical approach proposed by Scott [3] and developed by Reynolds [8], Wand [9], Plotkin [20], Lehmann [10], Smyth [11]. In this paper we will follow the latter approach. Therefore we will study the category of effective domains in this section. We assume that readers have basic knowledge of category theory throughout this paper. Actually only a small piece of category theory is needed and one of the best introduction to category theory might be "Arrows, Structures, and Functors" by Arbib & Manes [18].

DEFINITION 5.1

(1) Let D and D' be posets. A pair of monotone functions (f:D -->D',g:D'-->D') s.t. $f\circ g \sqsubseteq id_{D'}$, and $g\circ f = id_D$ is called an embedding (pair). Occasionally we call f an embedding and g an adjoint of f.

(2) In case D and D' are effectively given domains, a computable embedding (pair) is an embedding pair (f,g) s.t. both f and g are computable.

(3) In case D and D' are effective domains, an effective embedding (pair) is an embedding pair (f,g) s.t. both f and g are effective.

Evidently if (f:D -->D',g:D' -->D) and (f':D' -->D",g':D" -->D') are embeddings then so is ($f'\circ f$:D-->D",$g\circ g'$:D" -->D). We shall call this the composition of embeddings. Therefore the class of all $\omega$-chain complete posets with bottom and the class of all embeddings form a category which is usually called the category of $\omega$-cpo's, and denoted by $\omega\text{-CPO}^E$. Historically the first categorical work on the theory of domains took solutions for self-referential domain equations within this category [3,11]. But in this paper we will use a category which is more desirable for computer science.

As for computable functions the composition of two composable effective functions is again effective and this composition is recursive in indices. Therefore the composition of two computable embeddings is again a computable embedding, and the composition of two effective embeddings is again an effective embedding. Thus we have:

THEOREM 5.2

The class of effective domains and the class of effective embeddings form a category. We call such category the category of effective domains and denote it by $ED^E$.

In the classical categorical theory of domains, the so called inverse limit construction was proved to be the $\omega$-colimit construction in the category $\omega\text{-CPO}^E$, and played a crucial role in the solution of self-referential domain definition [11]. Therefore we will briefly outline inverse limit construction.

DEFINITION 5.3

Let
$$D_0 \underset{g_0}{\overset{f_0}{\rightleftarrows}} D_1 \underset{g_1}{\overset{f_1}{\rightleftarrows}} D_2 \rightleftarrows \cdots$$

be an $\omega$-sequence of embeddings. The inverse limit of this sequence is the following poset:

$$\varprojlim D_i = D^{\infty} = \{\langle x_0, x_1, \ldots \rangle \mid x_n = g_n(x_{n+1})\}$$

with component-wise ordering. Define $(f_{n\infty}: D_n \longrightarrow D^{\infty}, g_{\infty n}: D^{\infty} \longrightarrow D_n)$ by:

$$f_{n\infty}(x) = \langle g_0 g_1 \ldots g_{n-1}(x), \ldots, g_{n-1}(x), x, f_n(x), f_{n+1} f_n(x), \ldots \rangle$$

$$g_{\infty n}(\langle x_0, x_1, \ldots \rangle) = x_n.$$

Then evidently $(f_{n\infty}, g_{\infty n})$ is an embedding pair for every $n \in N$.

FACT 5.4

Let
$$D_0 \underset{g_0}{\overset{f_0}{\rightleftarrows}} D_1 \underset{g_1}{\overset{f_1}{\rightleftarrows}} D_2 \rightleftarrows \cdots$$

be an embedding sequence.

(1) In case $D_i$ are $\omega$-chain complete cpo's and $(f_i, g_i)$ are $\omega$-chain continuous embeddings, then $D^{\infty}$ is also an $\omega$-chain complete cpo.

(2) In case $D_i$ are cpo's and $(f_i, g_i)$ are continuous embeddings then $D^{\infty}$ is also a cpo.

(3) In case $D_i$ are effectively given domains and $(f_i, g_i)$ are computable embeddings then $D^{\infty}$ is also an effectively given domain.

For (3) the effective basis of $D^{\infty}$ is given by :

$$E_{D^{\infty}} = \{\epsilon_n^{\infty} \mid n \in N\} \qquad \text{and} \qquad E_{D_n} = \{\epsilon_m^n \mid m \in N\}$$

where
$$\epsilon_0^{\infty} = f_{0\infty}(\epsilon_0^0)$$
$$\epsilon_1^{\infty} = f_{0\infty}(\epsilon_1^0)$$
$$\epsilon_2^{\infty} = f_{1\infty}(\epsilon_0^1)$$
$$\epsilon_3^{\infty} = f_{0\infty}(\epsilon_2^0)$$
$$\epsilon_4^{\infty} = f_{1\infty}(\epsilon_1^1)$$
$$\epsilon_5^{\infty} = f_{2\infty}(\epsilon_0^2)$$
--------- .

This fact can be restated as follows: The class of $\omega$-chain cpo's is closed under the inverse limit of $\omega$-sequence of $\omega$-chain continuous embeddings, the class of cpo's is closed under the inverse limit of $\omega$-sequence of continuous embeddings, and the class of effectively given domains is closed under the inverse limit of $\omega$-sequence of computable embeddings. But unfortunately, the class of effective domains is not closed under the inverse limit of $\omega$-sequence of effective embeddings. To obtain a desirable closure property, we have to introduce the notion of effective inverse limit.

DEFINITION 5.5

An effective sequence of effective embeddings is an $\omega$-sequence of effective embeddings

$$X_0 \underset{g_0}{\overset{f_0}{\rightleftarrows}} X_1 \underset{g_1}{\overset{f_1}{\rightleftarrows}} X_2 \rightleftarrows \cdots\cdots$$

s.t. $f_i = \chi^{(i,i+1)}_{r(i)}$, $g_i = \chi^{(i+1,i)}_{t(i)}$ for some recursive functions r and t, where $\{\chi^{(i,i+1)}_n \mid n \in N\}$ is the standard enumeration of $[X_i \dashrightarrow X_{i+1}]$ and $\{\chi^{(i+1,i)}_n \mid n \in N\}$ is the standard enumeration of $[X_{i+1} \dashrightarrow X_i]$.

The effective inverse limit of the above effective sequence is the set $\{\langle x_0, x_1, \ldots \rangle \mid x_i = \chi^i_{u(i)},\ x_i = g_i(x_{i+1})$ for some recursive function u$\}$ with the component-wise ordering where $\{\chi^i_n \mid n \in N\}$ is the standard enumeration of $X_i$. We shall denote it by ef-$\varprojlim X_n$.

Remember that by virture of the isomorphism $X \cong C_{\overline{E}_X}$, we can regard each effective domain as $C_D$ for some suitable effectively given domain D. This convention drastically reduces non-essential technical complications without harming the generality. Therefore we will use this convention in the following.

THEOREM 5.6

The class of effective domains is closed under the effective inverse limit of effective sequences of effective embeddings.

proof Each effective sequence of effective embeddings can be considered as a restriction of an effective sequence of computable embeddings,

$$\begin{tikzcd} D_0 \arrow[r, bend left, "f_0"] & D_1 \arrow[l, bend left, "g_0"] \arrow[r, bend left, "f_1"] & D_2 \arrow[l, bend left, "g_1"] \arrow[r, bend left] & \cdots \arrow[l, bend left] \end{tikzcd}$$

where $f_n = \gamma_{r(n)}^{(n,n+1)}$ , $g_n = \gamma_{t(u)}^{(n+1,n)}$ for some recursive function r and t, where $\{\gamma_k^{(n,n+1)} | k\in N\}$ and $\{\gamma_k^{(n+1,n)} | k\in N\}$ are the standard enumerations of $C_{[D_n \to D_{n+1}]}$ and $C_{[D_{n+1} \to D_n]}$ respectively. More precisely each effective sequence of effective embeddings can be considered as the following effective sequence:

$$\begin{tikzcd} C_{D_0} \arrow[r, bend left, "\overline{f}_0"] & C_{D_1} \arrow[l, bend left, "\overline{g}_0"] \arrow[r, bend left, "\overline{f}_1"] & C_{D_2} \arrow[l, bend left, "\overline{g}_1"] \arrow[r, bend left] & \cdots\cdots \arrow[l, bend left] \end{tikzcd}$$

where $\overline{f}_n = \overline{\gamma}_{r(n)}^{(n,n+1)}$, $\overline{g}_n = \overline{\gamma}_{t(u)}^{(n+1,n)}$ for some recursive r and t, where $\{\overline{\gamma}_k^{(n,n+1)} | k\in N\}$ and $\{\overline{\gamma}_k^{(n+1,n)} | k\in N\}$ are the standard enumerations of $[C_{D_n} \to C_{D_{n+1}}]$ and $[C_{D_{n+1}} \to C_{D_n}]$ respectively. We will show **ef-$\underleftarrow{\lim}$** $C_{D_i} = C_{\underleftarrow{\lim} D_i}$.

First of all assume $d = \langle d_0, d_1, \ldots \rangle \in$ ef-$\underleftarrow{\lim}$ $C_{D_i}$. Then $d_n = \gamma_{t(n)}^n$ for some recursive function t. Also $d = \bigsqcup_n f_{n\infty}(d_n)$. But since the sequence is effective, $f_{n\infty}$ is computable and $f_{n\infty} = \gamma_{r(n)}^{(n,\infty)}$ for some recursive function r, where $\{\gamma_k^{(n,\infty)} | k\in N\}$ is the standard enumeration of $C_{[D_n \to D_\infty]}$. Therefore we have:

$$\begin{aligned} d &= \bigsqcup_n f_{n\infty}(d_n) \\ &= \bigsqcup_n \gamma_{r(n)}^{(n,\infty)}(\gamma_{t(n)}^n) = \bigsqcup_n \gamma_{u(n)}^\infty \quad \text{for some recursive u.} \end{aligned}$$

Therefore $d \in C_{\underleftarrow{\lim} D_i}$. Thus we have established ef-$\underleftarrow{\lim}$ $C_{D_i} \subseteq C_{\underleftarrow{\lim} D_i}$.

Conversely let $c = \langle c_0, c_1, \ldots \rangle \in C_{\underleftarrow{\lim} D_i}$. Then by computability of $g_{\infty n}$, $c_n = g_{\infty n}(c) \in C_{D_n}$ for every $n\in N$. Therefore $\overline{g}_n(c_{n+1}) = c_n$ for every n. By the effectiveness of the embedding sequence, $g_{\infty n} = \gamma_{t(n)}^{(\infty,n)}$ for some recursive function t. Let $c = \gamma_k^\infty$. Then $c_n = g_{\infty n}(c) = \gamma_{t(n)}^{(\infty,n)}(\gamma_k) = \gamma_{v(t(n),k)}^n = \gamma_{u(n)}^n$ for some recursive function u. Therefore $c \in$ ef-$\underleftarrow{\lim}$ $C_{D_i}$. <u>Q.E.D.</u>

Now the question is; "What is the categorical characterization of ef-$\varprojlim X_i$ within the category of effective domains and effective embeddings?". Classical results showed that the inverse limits were $\omega$-colimits within the category $\omega$-$CPO^E$ [11]. But the traditional $\omega$-colimt construction is not strong enough to characterize effective inverse limits. What we need is the idea of effective categories and effective colimits which we will discuss in the next section.

## §6. Effective categories and effective colimits

In the previous section, we encountered a funny category $ED^E$, subject to constraints of "effectiveness". Traditional category is not concerned with this sort of category. We will study this here in this section.

DEFINITION 6.1 (E-category)

An E-category is a category $\underline{K}$ s.t. for all $K_1,K_2 \in OB(\underline{K})$, there exists an associated countable set $E(K_1,K_2) = \{\epsilon_n^{(K_1,K_2)} | n \in N\}$ s.t. $Hom(K_1,K_2) \subseteq E(K_1,K_2)$ and the composition of morphisms is effective in indices.

Remember that $[X_1 \dashrightarrow X_2]$ is countable, the composition of effective functions is recursive in indices, and the set of all embeddings from $X_1$ to $X_2$ is a non-recursive subset of $[X_1 \dashrightarrow X_2]$. Therefore the idea of E-category captures quite well the effective nature of $ED^E$. But to characterize more interesting effective nature of $ED^E$, namely effective inverse limits, we need a stronger notion. We will study this in the following.

DEFINITION 6.2 (effective categories)

(1) $\underline{\omega}$ is the category of non-negative integers and $\underline{\leq}$. Pictorially:

$$0 \leq 1 \leq 2 \leq \dots$$

(2) An effective sequence in an E-category $\underline{K}$ is a functor $G: \underline{\omega} \dashrightarrow \underline{K}$ s.t.

$$G(n \leq n+1) = g_n : K_n \dashrightarrow K_{n+1}$$
$$= \epsilon_{r(n)}^{(K_n,K_{n+1})} \quad \text{for some recursive function } r.$$

(3) Given an effective sequence G, an effective cocone of G is a cocone

$$D_0 \underset{g_0}{\overset{f_0}{\rightleftarrows}} D_1 \underset{g_1}{\overset{f_1}{\rightleftarrows}} D_2 \rightleftarrows \cdots$$

where $f_n = \gamma^{(n,n+1)}_{r(n)}$, $g_n = \gamma^{(n+1,n)}_{t(u)}$ for some recursive function r and t, where $\{\gamma_k^{(n,n+1)} | k \in N\}$ and $\{\gamma_k^{(n+1,n)} | k \in N\}$ are the standard enumerations of $C_{[D_n \to D_{n+1}]}$ and $C_{[D_{n+1} \to D_n]}$ respectively. More precisely each effective sequence of effective embeddings can be considered as the following effective sequence:

$$C_{D_0} \underset{\bar{g}_0}{\overset{\bar{f}_0}{\rightleftarrows}} C_{D_1} \underset{\bar{g}_1}{\overset{\bar{f}_1}{\rightleftarrows}} C_{D_2} \rightleftarrows \cdots\cdots$$

where $\bar{f}_n = \bar{\gamma}^{(n,n+1)}_{r(n)}$, $\bar{g}_n = \bar{\gamma}^{(n+1,n)}_{t(u)}$ for some recursive r and t, where $\{\bar{\gamma}_k^{(n,n+1)} | k \in N\}$ and $\{\bar{\gamma}_k^{(n+1,n)} | k \in N\}$ are the standard enumerations of $[C_{D_n} \to C_{D_{n+1}}]$ and $[C_{D_{n+1}} \to C_{D_n}]$ respectively. We will show **ef-$\underleftarrow{\lim}$** $C_{D_i} = C_{\underleftarrow{\lim}\, D_i}$.

First of all assume $d = \langle d_0, d_1, \ldots \rangle \in$ ef-$\underleftarrow{\lim}\, C_{D_i}$. Then $d_n = \gamma^n_{t(n)}$ for some recursive function t. Also $d = \bigsqcup_n f_{n\infty}(d_n)$. But since the sequence is effective, $f_{n\infty}$ is computable and $f_{n\infty} = \gamma^{(n,\infty)}_{r(n)}$ for some recursive function r, where $\{\gamma_k^{(n,\infty)} | k \in N\}$ is the standard enumeration of $C_{[D_n \to D_\infty]}$. Therefore we have:

$$\begin{aligned} d &= \bigsqcup_n f_{n\infty}(d_n) \\ &= \bigsqcup_n \gamma^{(n,\infty)}_{r(n)}(\gamma^n_{t(n)}) = \bigsqcup_n \gamma^\infty_{u(n)} \quad \text{for some recursive } u. \end{aligned}$$

Therefore $d \in C_{\underleftarrow{\lim}\, D_i}$. Thus we have established ef-$\underleftarrow{\lim}\, C_{D_i} \subseteq C_{\underleftarrow{\lim}\, D_i}$.

Conversely let $c = \langle c_0, c_1, \ldots \rangle \in C_{\underleftarrow{\lim}\, D_i}$. Then by computability of $g_{\infty n}$, $c_n = g_{\infty n}(c) \in C_{D_n}$ for every $n \in N$. Therefore $\bar{g}_n(c_{n+1}) = c_n$ for every n. By the effectiveness of the embedding sequence, $g_{\infty n} = \gamma^{(\infty,n)}_{t(n)}$ for some recursive function t. Let $c = \gamma^\infty_k$. Then $c_n = g_{\infty n}(c) = \gamma^{(\infty,n)}_{t(n)}(\gamma_k) = \gamma^n_{v(t(n),k)} = \gamma^n_{u(n)}$ for some recursive function u. Therefore $c \in$ ef-$\underleftarrow{\lim}\, C_{D_i}$. <u>Q.E.D.</u>

$\lambda_n = \epsilon_{t(n)}^{(K_n, K_{n+1})}$ for some recursive function t. Pictorially,

$$\begin{tikzcd} G = K_0 \arrow[r, "g_0"] \arrow[drr, "\lambda_0"'] & K_1 \arrow[r, "g_1"] \arrow[dr, "\lambda_1"] & K_2 \arrow[r, "g_2"] \arrow[d, "\lambda_2"] & {} \\ & & K & \cdots \end{tikzcd} \quad \text{commutes}$$

(4) We say that an effective sequence G has an <u>effective</u> <u>colimit</u>, in symbols ef-colim G, iff there exists an effective cocone $\delta_n : K_n \dashrightarrow$ ef-colim G of G s.t. for every effective cocone $\lambda_n : K_n \dashrightarrow K$ of G, there exists a unique morphism $\sigma$:ef-colim G -->K s.t. the following diagram commutes

$$\begin{tikzcd} G = K_0 \arrow[r, "g_0"] \arrow[d, "\lambda_0"'] \arrow[drr, "\delta_0"] & K_1 \arrow[r, "g_1"] \arrow[dl, "\lambda_1"] \arrow[dr, "\delta_1"] & K_2 \arrow[r, "g_2"] \arrow[dll, "\lambda_2"] \arrow[d, "\delta_2"] & \cdots \\ K & & \text{ef-colim } G \arrow[ll, dashed, bend left, "\sigma"] \end{tikzcd}$$

We shall call the effective cocone $\delta_n$, the <u>effective</u> <u>colimiting</u> <u>cocone</u>.

(5) We say that an E-category is an <u>effective</u> <u>category</u> iff every effective sequence in it has an effective colimit.

We will see that this abstract category, effective category, will capture the effective nature of the concrete category $ED^E$ in the following discussion. First note that given an effective sequence of effective embeddings,

$$\begin{tikzcd} X_0 \arrow[r, "f_0"] \arrow[drr, "f_{0\infty}"'] & X_1 \arrow[r, "f_1"] \arrow[dr, "f_{1\infty}"] & X_2 \arrow[r, "f_2"] \arrow[d, "f_{2\infty}"] & \cdots \\ & & \text{ef-}\underleftarrow{\lim}\, X_i \end{tikzcd}$$

we have:

$$\left.\begin{aligned} f_{n\infty} &= f_{(n+1)\infty} \circ f_n \\ g_{\infty n} &= g_n \circ g_{\infty(n+1)} \end{aligned}\right\} \quad \text{for all } n.$$

Therefore $f_{n\infty}$, $n \in N$ is an effective cocone. Thus we have the following theorem:

<u>THEOREM 6.3</u>

Every effective inverse limit is an effective colimit in the category

$ED^E$. Therefore by 5.6, $ED^E$ is an effective category.

<u>proof</u> Evidently an effective sequence of effective embeddings $X_i$, $i \in N$,

$$\begin{tikzcd}
X_0 \arrow[r, "f_0"] \arrow[d, "\lambda_0"'] \arrow[drr, "f_{0\infty}"] & X_1 \arrow[r, "f_1"] \arrow[dl, "\lambda_1"] \arrow[dr, "f_{1\infty}"] & X_2 \arrow[r, "f_2"] \arrow[dll, "\lambda_2"] \arrow[d, "f_{2\infty}"] & \cdots \\
X & & \text{ef-}\varprojlim X_i \arrow[ll, dashed, "\psi"]
\end{tikzcd}$$

is an effective sequence in $ED^E$ and $f_{n\infty}: X_n \dashrightarrow \text{ef-}\varprojlim X_i$, $n \in N$ is an effective cocone of this effective sequence. Think of an effective cocone $\lambda_n : X_n \dashrightarrow X$, $n \in N$. Let $f_n = \chi^{(n,n+1)}_{r(n)}$, $g_n = \chi^{(n+1,n)}_{s(n)}$, $f_{n\infty} = \chi^{(n,\infty)}_{t(n)}$, $g_{\infty n} = \chi^{(\infty,n)}_{u(n)}$, $\lambda_n = \chi^{(n,X)}_{v(n)}$, and $\delta_n = \chi^{(X,n)}_{w(n)}$ for some recursive functions r,s,t,u,v,and w, where $g_n$, $g_{\infty n}$, $\delta_n$ are adjoints of $f_n$, $f_{n\infty}$, $\lambda_n$ respectively. Define $\psi$: $\text{ef-}\varprojlim X_i \dashrightarrow X$ by: $\psi = \bigsqcup_n \lambda_n \circ g_{\infty n}$, and $\phi$: $X \dashrightarrow \text{ef-}\varprojlim X_i$ by : $\phi = \bigsqcup_n f_{n\infty} \circ \delta_n$. Then evidently $(\psi,\phi)$ is an embedding pair. Also:

$$\psi = \bigsqcup_n \lambda_n \, g_{\infty n} = \bigsqcup_n \chi^{(n,X)}_{v(n)} \circ \chi^{(\infty,n)}_{u(n)}$$

$$= \bigsqcup_n \chi^{(\infty,X)}_{\theta(n)} \quad \text{for some recursive } \theta \text{, and}$$

$$\phi = \bigsqcup_n f_{n\infty} \circ \delta_n = \bigsqcup_n \chi^{(n,\infty)}_{r(n)} \circ \chi^{(X,n)}_{w(n)}$$

$$= \bigsqcup_n \chi^{(X,\infty)}_{\alpha(n)} \quad \text{for some recursive } \alpha.$$

Therefore $(\psi,\phi)$ is an effective embedding pair. Uniqueness of $(\psi,\phi)$ is evident , and obviously the above diagram commutes. <u>Q.E.D.</u>

## §7. <u>Effective functors and initial solutions</u>

Smyth showed that for each functor from an $\omega$-category, i.e. a category with $\omega$-colimit for each its $\omega$-sequence, to itself, we could obtain initial solutions of the isomorphic equation

$$X \cong F(X)$$

within this $\omega$-category as long as F preserves each $\omega$-colimitng diagram [11].

We are going to play quite similar game here in this section. As mentioned repeatedly, the category $\omega\text{-CPO}^E$ is not relevant to computer science, and the $\omega$-colimit preserving functor does not have an effective flavor. What we are going to do is an effective version of Smyth's work. The author thinks that the idea of effective functors which will be discussed in the following is original.

<u>DEFINITION 7.1</u>

Let $\underline{K}$, $\underline{K}'$ be effective categories. A functor $F:\underline{K} \to \underline{K}'$ is said to be <u>effective</u> iff the arrow function of F is effective in the sense:

$$F(\epsilon_n^{(K_1,K_2)}) = \epsilon'^{(FK_1,FK_2)}_{f(n)} \quad \text{for some recursive function } f:N \to N,$$

and F preserves effective colomiting diagrams.

Evidently the following properties about effective functors are true, and will convince us the adequacy of the notion of effective functors.

<u>LEMMA 7.2</u>

(1) For every effective sequence G of an effective category $\underline{K}$, effective functor $F:\underline{K} \to \underline{K}'$ gives effective sequence FG of $\underline{K}'$. In other words, effective functors map effective sequences to effective sequences.

(2) Every effective functor maps effective colimits and effective colimiting cocones to effective colimits and effective colimitng cocones respectively.

Pictorially,

$$\begin{tikzcd}
G = K_0 \arrow[r, dash, "g_0"] \arrow[drr, "c_0"'] & K_1 \arrow[r, dash, "g_1"] \arrow[dr, "c_1"] & K_2 \arrow[r, dash, "g_2"] \arrow[d, "c_2"] & \cdots \\
& & \text{ef-colim } G &
\end{tikzcd}$$

commutes

implies

$$\begin{tikzcd}
FG = FK_0 \arrow[r, "Fg_0"] \arrow[drr, "Fc_0"'] & FK_1 \arrow[r, "Fg_1"] \arrow[dr, "Fc_1"] & FK_2 \arrow[r, "Fg_2"] \arrow[d, "Fc_2"] & \cdots \\
& & F(\text{ef-colim } G) = \text{ef-colim } FG. &
\end{tikzcd}$$

commutes

Note that the structure of the diagrams for $\omega$-continuous functors and $\omega$-sequences is exactly the same as this and the only difference is that in effective case, we have to take care of effectiveness. More precisely, for example, (3)-7.2 for an $\omega$-continuous functor F is as follows:

$$\begin{tikzcd} G = K_0 \arrow[r, "g_0"] \arrow[drr, "c_0"'] & K_1 \arrow[r, "g_1"] \arrow[dr, "c_1"] & K_2 \arrow[r, "g_2"] \arrow[d, "c_2"] & \cdots \\ & & \operatorname{colim} G & \end{tikzcd}$$ commutes

implies

$$\begin{tikzcd} FG = FK_0 \arrow[r, "Fg_0"] \arrow[drr, "Fc_0"'] & FK_1 \arrow[r, "Fg_1"] \arrow[dr, "Fc_1"] & FK_2 \arrow[r, "Fg_2"] \arrow[d, "Fc_2"] & \cdots \\ & & F(\operatorname{colim} G) = \operatorname{colim} FG & \end{tikzcd}$$ commutes

where G, FG are not effective sequences but $\omega$-sequences, and $c_n$, $Fc_n$ are not effective colimiting cocones but just $\omega$-colimiting cocones.

Therefore we can proceed exactly as Smyth did for $\omega$-continuous functoers; the only extra requirement is the check of effectiveness which is tedious but easy. This fact can be recognized once we compare the proof of 6.3 with the Smyth's proof of the similar claim that the category $\omega$-CPO$^E$ is an $\omega$-category. Hence we can safely leave the proofs of the following dixcussions to the readers by refering them to Smyth [11].

DEFINITION 7.3

Let $\underline{K}$ be an effective category and $F:\underline{K} \dashrightarrow \underline{K}$ be an effective functor. Also assume $k \in \underline{K}$ and $\theta: k \dashrightarrow Fk$. Define an effective sequence $\Delta_{(F,k,\theta)} : \underline{\omega} \dashrightarrow \underline{K}$ by:

$$\Delta_{(F,k,\theta)} = k \xrightarrow{\theta} Fk \xrightarrow{F\theta} F^2k \xrightarrow{F^2\theta} \cdots$$

Since $\underline{\omega}$ is an effective sequence in $\underline{\omega}$ and F is effective, $\Delta_{(F,k,\theta)}$ is an effective sequence. Therefore $\Delta_{(F,k,\theta)}$ is well-defined. Also by the effectiveness of the category $\underline{K}$, ef-colim $\Delta_{(F,k,\theta)}$ evidently exists. The next theorem shows us the importance of $\Delta_{(F,k,\theta)}$.

THEOREM 7.4

Let $\underline{K}$ be an effective category and $F:\underline{K} \dashrightarrow \underline{K}$ be an effective functor. Then for every $k \in \underline{K}$ and $\theta: k \dashrightarrow Fk$ we have:

$$\text{ef-colim}\,\Delta_{(F,k,\theta)} \cong F(\text{ef-colim}\,\Delta_{(F,k,\theta)}).$$

Note that in the classical theory this sort of isomorphism is just continuous one because the isomorphism is taken in $\omega\text{-CPO}^E$. But in our theory the isomorphism is taken in the category $ED^E$, therefore it is an effective isomorphism, which is more desirable. This is one of the advantage of our theory.

Now we can see the initiality of ef-colim $\Delta_{(F,k,\theta)}$ within some suitable category induced by the effective functor F.

DEFINITION 7.5

Let $F:\underline{K}\dashrightarrow\underline{K}$ be a functor.

(1) $x\in\underline{K}$ is a post-fix-point of F via $\gamma$ iff $\gamma:Fx\dashrightarrow x$.

(2) A $\underline{K}$-morphism $\pi:x\dashrightarrow y$ is a post-fix-arrow iff both x and y are post-fix-point of F via $\gamma$ and $\delta$ respectively and,

$$\begin{tikzcd} x \arrow[d, "\pi"'] & Fx \arrow[l, "\gamma"'] \arrow[d, "F\pi"] \\ y & Fy \arrow[l, "\delta"] \end{tikzcd}$$

commutes.

DEFINITION 7.6

Let $F:\underline{K}\dashrightarrow\underline{K}$ be a functor and $\theta:k\dashrightarrow Fk$ be a $\underline{K}$-morphism. Define a category of post-fix-points of F, in symbols $PF(\underline{K},F,\theta)$ as follows:

(1) the objects are the triples $\langle\alpha,x,\gamma\rangle$ s.t.

$$\begin{tikzcd} k \arrow[r, "\theta"] \arrow[d, "\alpha"'] & Fk \arrow[d, "F\alpha"] \\ x & Fx \arrow[l, "\gamma"] \end{tikzcd}$$

commutes.

Essentially each object is a post-fix-point of F with associated $\underline{K}$-morphism $\alpha$ s.t. the above diagram commutes.

(2) a morphism between $\langle\alpha,x,\gamma\rangle$ and $\langle\alpha',x',\gamma'\rangle$ is a $\underline{K}$-morphism $\pi:x\dashrightarrow x'$ s.t.

$$\begin{tikzcd} & k \arrow[r, "\theta"] \arrow[d, "\alpha"] \arrow[dl, "\alpha'"'] & Fk \arrow[d, "F\alpha"] \arrow[dl, "F\alpha'"'] \\ x' & x \arrow[l, "\pi"'] & Fx \arrow[l, "\gamma"'] \arrow[dl, "F\pi"] \\ & Fx' \arrow[ul, "\gamma'"] & \end{tikzcd}$$

commutes.

Note that each PF-morphism is essentially a fix-point-arrow from $\gamma$ to $\gamma'$.

Therefore essentially PF($\underline{K}$,F,$\theta$) is the category of post-fix-points of F and post-fix-arrows. Now we can show that $\rho$:F(ef-colim$\Delta_{(F,k,\theta)}$) --> ef-colim$\Delta_{(F,k,\theta)}$ , which is an effective isomorphism established in 7.4, is the initial object of PF($\underline{K}$,F,$\theta$), in case $\underline{K}$ and F are effective.

THEOREM 7.7

Let $\underline{K}$ be an effective category and F:$\underline{K}$ -->$\underline{K}$ be an effective functor. Let k$\in\underline{K}$ and $\theta$:k -->Fk. Also let $c_n$,n$\in$N be the effective colimiting cocone of $\Delta_{(F,k,\theta)}$. Then <$c_0$,ef-colim$\Delta_{(F,k,\theta)}$,$\rho$> is the initial object in the category PF($\underline{K}$,F,$\theta$) where $\rho$ is as above.

With this theorem, we have established that for every effective functor F:$\underline{K}$ -->$\underline{K}$, and for every $\theta$:k -->Fk, ef-colim$\Delta_{(F,k,\theta)}$ is an initial solution to X $\cong$ F(X) within the effective category $\underline{K}$. Therefore for every effective functor F:$ED^E$ -->$ED^E$,

$$\text{ef-colim}\,\Delta_{(F,k,\theta)} = \text{ef-}\underleftarrow{\lim}\,\Delta_{(F,k,\theta)}(n)$$
$$= \text{ef-}\underleftarrow{\lim}\,F^n k$$

is an initial solution to X $\cong$ F(X) within the category $ED^E$.

Now we shall observe that the domain constructors "x", "+", and "-->" are effective functors.

DEFINITION 7.8 (product category)

Let $\underline{K}$ and $\underline{L}$ be categories, then the product category $\underline{K}$x$\underline{L}$ is the following category: Ob($\underline{K}$x$\underline{L}$) = Ob($\underline{K}$)xOb($\underline{L}$), $\underline{K}$x$\underline{L}$-morphisms are pairs (f, g):(K, L) -->(K', L') where f:K -->L and g:K' -->L', $id_{(K,L)}$ = ($id_K$, $id_L$):(K, L) -->(K , L ), and (f', g')$\circ$(f, g) = (f'$\circ$f, g'$\circ$g).

DEFINITION 7.9 (arrow, product, and sum functors)

(1) The arrow functor $\circ$-->:$ED^E$x$ED^E$-->$ED^E$ is defined on objects by: $\circ$-->(X,X') = [X -->X'], and on morphisms by: for (p,q):($X_1$,$X_1'$) -->($X_2$,$X_2'$), $\circ$-->(p,q) = $\lambda f\in[X_1 \text{-->} X_2].q\circ f\circ p'$ where p' is the adjoint of p.

(2) The product functor $\otimes$ :$ED^E$x$ED^E$-->$ED^E$ is defined on objects by: $\otimes$ (X,X') = XxX', and on morphisms by : for (p,q):($X_1$,$X_1'$) -->($X_2$,$X_2'$),

$\otimes(p,q) = \lambda(x_1,x_2).(px_1,qx_2)$.

(3) The sum functor $\oplus: ED^E \times ED^E \to ED^E$ is defined on objects by: $\oplus(X_1,X_2) = X_1+X_2$, and on morphisms by: for $(p,q):(X_1,X_1') \to (X_2,X_2')$, $\oplus(p,q) = \lambda x \in X_1+X_2$. if $x$ is in $X_1$ then $p(x)$ embeded in $X_1'+X_2'$ else $q(x)$ embeded in $X_1'+X_2'$.

THEOREM 7.10 (arrow, product, and sum theorem)

The arrow, product, and sum functors are effective functors.

THEOREM 7.11

A functor $F:\underline{K}\times\underline{L} \to \underline{M}$ is effective iff it is effective in both $\underline{K}$ and $\underline{L}$, where $\underline{K}$, $\underline{L}$, and $\underline{M}$ are effective categories.

THEOREM 7.12

Let $F:\underline{K} \to \underline{L}$ and $G:\underline{L} \to \underline{M}$ be effective functors, then $G \bullet F:\underline{K} \to \underline{M}$ also is an effective functor.

Summarising these results, finally we obtain the following very important result.

THEOREM 7.13

Every self-referential domain equation has initial solutions within the category of effective domains, as long as the equation involves "x", "+", and "-->" as domain constructors.

Note that we have presented effective (thus denumerable) solutions to self-referential domain equations which appear in denotational semantics. Thus we have given, in part, effective models to denotational semantics. The thing which is left to be observed is the effectiveness of functions used in the semantic specification. This could be done easily for each cases returning back to the definition of effectiveness. But the development of some language for denoting effective functions is much more desirable.

## §8. Conclusion

The idea of effective chains was adopted as a better model for computational finite approximation processes than $\omega$-chains or directed sets. Hence we obtained a notion of effective domains which are the extensions of effective basis by means of effective limits rather than downward closed directed limits as in the theory of effectively given domains. The important fact that every element of each effective domain was computable was observed. Furthermore the effective nature of domain constructors "x", "+", "-->" was studied as the effective inverse limit preserving functors. Thence we showed the existence of initial solutions to each self-referential domain equations involving "x", "+", "-->" as domain constructors, within the category of effective domains.

Park indicated to me that the notion of effective directed sets could be technically more suitable than the idea of effective chains. Actually an effective directed subset of an effective basis can be defined as an r.e. subset of the effective basis. Thus evidently the effective completion theorem and its proof can be technically simplified. Details of this fact can be safely left to the readers as an exercise. The only advantage of effective chains is intuitive clearness.

Even though not mentioned in this paper, the weak power domain construction preserves the effectiveness of domains, and can be proved to be an effective functor. Thus we can solve self-referential domain equations which involve power domain constructors.

References


[1] D. Park, Fix-point induction and proofs of program semantics, Machine Intelligence, Vol 5.

[2] D. Scott, Outline of the mathematical theory of computation, Proc. 4th Annual Princeton Conference on Information Science and Systems.

[3] D. Scott, Continuous lattices, Lecture Notes in Mathematics, No. 274, Springer-Verlag.

[4] D. Scott, Data types as lattices, SIAM Journal of Computing, Sept. 1976.

[5] D. Scott, Mathematical theory of computation, unpublished lecture note, Oxford, 1977.

[6] D. Scott, Data types as lattices, unpublished lecture note, Amsterdam 1972.

[7] R. Milner, Implementation and application of Scott logic for computable functions, Proc. ACM Conference on Proving Assertions about Programs 1972.

[8] J. Reynold, Note on a lattice theoretic approach to the theory of computation, Syracuse University, 1972

[9] M. Wand, Fix-point construction in order enriched category, TR-23, Indiana University, Computer Science Dept. 1975

[10] D. Lehmann, Categories of mathematical semantics, Proc. 17th Annual IEEE Symposium on Foundations of Computer Science, 1975.

[11] M. Smyth, Category theoretic solution of recursive domain equations, Theory of Computation Report, No. 14, Dept. of Computer Science, Univ. of Warwick, 1976

[12] M. Smyth, Effectively given domains, Theory of Computation Report, No. 9, University of Warwick, 1976.

[13] H. Egli, R. Constable, Computability concepts for programming language semantics, Theoretical Computer Science, Vol 2, 1976.

[14] Markowsky, Rosen, Basis for chain complete posets, IBM Journal of R&D, Vol.20, No. 2, 1976.

[15] G. Plotkin, $T^{\omega}$ as a universal domain, Dept. of Artificial Intelligence Research Report, No. 28, Edinburgh Univ., 1977

[16] Scott-Strachey, Towards a mathematical semantics of computer langages, Proc. of Symposium on Computer and Automata, Polytechnic Institute of Brooklyn, 1971

[17] Manna-Vuillemin, Fixpoint approach to the theory of computation, CACM Vol 15., No. 7, 1972.

[18] Arbib-Manes, Arrows structures and functors, Academic Press, 1975.

[19] M. Smyth, Power domains, Theory of Computation Report, No.12, Univ. of Warwick, 1976.

[20] G. Plotkin, A power domain construction, D.A.I. Research Report, No.3, Univ. of Edinburgh, 1975.

[21] D. Park, unpublished notes, Warwick Univ., 1970.

[22] D. Scott, A theory of computable functions of higher types, unpublished notes, Oxford Univ.,1969.

[23] A. Tang, Recursion theory and descriptive set theory in effectively given $T_0$-spaces, Ph.D. thesis, Princeton University, 1974

## Acknowledgement

I am very grateful to M. Smyth and D. Scott for their patience to introduce me to the world of effectively given domains. I also have to thank to M. Smyth for his continuous encouragement and very stimulating discussions. M. Smyth and D. Park motivated me to study one part of the core of this theory, namely the effective completion theorem. Several other people helped me very much during the course of this research. Especially I have benefited from discussions with W. Wadge, M. Paterson, and M. Beynon. Also thanks are due to D. Park for his comments on the historical background of this field.