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ON THE COMPUTATION OF THE STRUCTURE OF AN ABELIAN GROUP REPRESENTED BY A SET OF DEFINING RELATIONS

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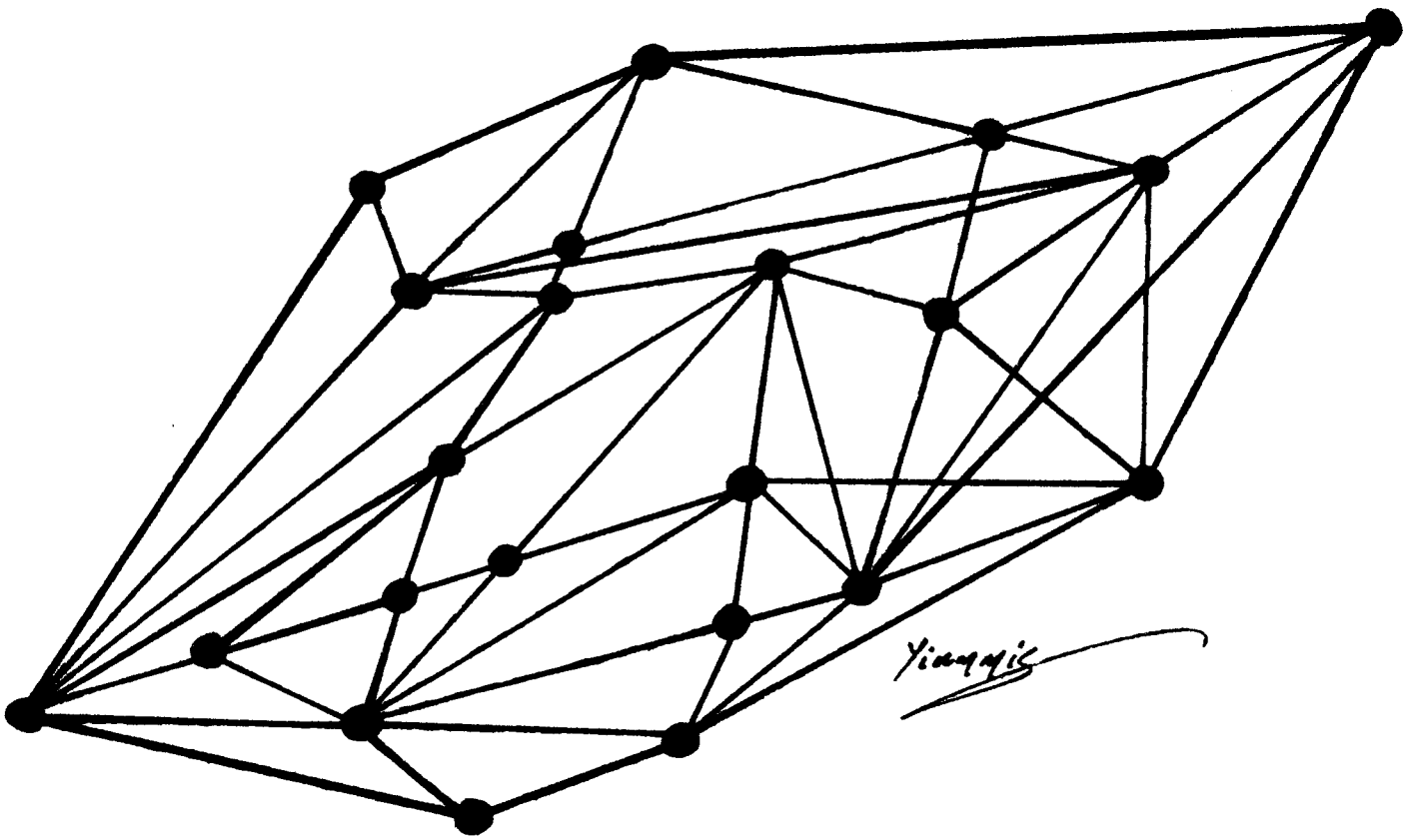
To My Parents



1. INTRODUCTION

Suppose that the set

$$R = \{x_1^{c_{1i}} x_2^{c_{2i}} \dots x_n^{c_{ni}} = 1,\ 1 \leq i \leq m,\ c_{ij} \in \mathbb{Z},\ x_j = x_j x_i,\ \forall i,j\}$$

is a set of defining relations for the abelian group G and let C be an $m \times n$ matrix with $C = (c_{ij})$. In this paper, the computational complexity of the problem of computing the order and the structure of a group G, given R, is examined.

The complexity of an algorithm is measured in elementary operations. An elementary operation is a Boolean operation on a single binary bit or pairs of bits. The size s of an $m \times n$ matrix $C = (c_{ij})$, is the number

$$m + n + \log\|C\|$$

where $\|C\| = \max_{i,j}\{|c_{ij}|\}$.

It is shown that in the case of a finite group G, represented by a matrix C of size s, the order and the structure of G can be computed in polynomial time. This computation requires $O(s^5\ M(s^2))$ elementary operations, where M(n) denotes the number of elementary operations required for the multiplication of two integers of length at most n bits.

Also it is given an upper bound for the time required to compute the structure of an infinite group G is also given. This computation requires $O(2^{r+\varepsilon})$ elementary operations, where r is the rank of the matrix representing the group G.

Sims in [4] formulated the classical algorithm for the above problem without analysing its computational complexity.

NOTATION

$Row_C(i)$ and $COL_C(i)$ will be used to denote the i-th row and the i-th column of a matrix C respectively.

An integer row-column operation (IRC operation) is said to be:

(i) The multiplication of all the elements of a row(column) by -1

or (ii) The interchange of two rows(columns)

or (iii) The addition of an integer multiple of row(column) to a different row(column).

2. THE COMPUTATION OF THE DETERMINANT

Suppose that an abelian group G is represented by an m×n matrix C with integer entries, where m is the number of defining relations and n is the number of generators. Then it is not difficult to show that the group G is finite if and only if

$$\text{rank}_{\mathbb{Q}}(C) = n$$

where $\mathbb{Q}$ denotes the field of rationals.

Hence one can decide whether or not G is finite by means of Gaussian elimination on C.

2.1 PROCEDURE DET

```
Procedure    DET(C)
Comment  This procedure computes the determinant (for square matrix only)
and the rank of an m×n matrix C(W.l.o.g. m ≥ n is assumed)
Begin
   i ← 0;
1. while < C is not in echelon form > do
      Begin
         i ← i+1;
           if c_li = 0 then < interchange ROW(i) and ROW(j) where c_ji≠0, j>i >
         μ_j ← c_ji / c_ii for i < j ≤ m;
3.       ROW(j) ← ROW(j) - μ_j ROW(i) for i < j ≤ m;
4.    end
   Return    DET(C) = Π_i c_ii (if C is square), RANK(C) = i;
end □
```

PROPOSITION 2.2 The procedure DET correctly computes the determinant and the rank of a matrix C in $O(s^3M(s^2))$ elementary operations, where s is the size of the matrix C.

Proof

Let $C^{(i)}$ denote C at the beginning of the i-th iteration of loop 1-4. Step 3 requires at most mn multiplications or $O(mn\ M(\log\|C^{(i)}\|))$ elementary operations. Hence the procedure requires at most $O(\sum_{i=1}^{m} mn\ M(\log\|C^{(i)}\|))$ elementary operations.

Now using the formula (12) in [2] (p.26), one can see that

$$\|C^{(i)}\| = O((i\|C\|)^i)$$

Hence the procedure terminates in $O(m^2n\ M(\log(m\|c\|^m))) = O(s^3\ M(s^2))$.

REMARK Step 3 of the procedure DET can be expressed as the matrix multiplication R.C, where

$$R = \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & \mu_{i+1} & \ddots & \\ 0 & & \vdots & & 1 \\ & & \mu_m & & \end{pmatrix}$$

i-th column

Hence it may be helpful to use the fast matrix multiplication algorithms (see [3] for the computation of step 3. It is worth mentioning here that Strassen's algorithm for matrix multiplication (see [1]) is not suitable for matrix multiplication over the rationals, since it uses a large number of additions and an addition over the rationals costs three integer multiplications.

COROLLARY 2.3 There exists an algorithm which decides whether or not an abelian group G represented by a matrix C, is finite or infinite in $O(s^3\ M(s^2))$ elementary operations. □

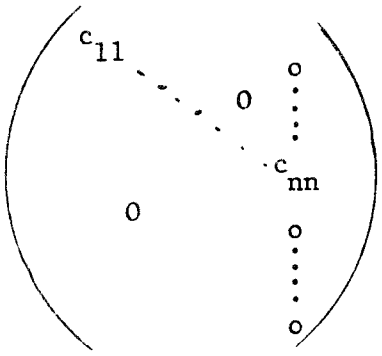
3. THE ALGORITHM FOR FINITE GROUPS

3.1 THE I.R.C. OPERATIONS

The I.R.C. operation transforms the matrix C of the set of defining relations of an abelian group G to a matrix C* which represents the same group. The interchange of rows yields the set of defining relations reordered, the multiplication of a row by -1 substitutes the relation $\prod_i x_i^{c_{ij}}=1$ for the relation $\prod_i x_i^{c_{ij}}=1$ and the addition of an integer multiple of a row to a different row substitutes the relations $\{\prod_i x_i^{c_{ji}}=1, \prod_i x_i^{c_{ki}}=1\}$. The interchange of columns corresponds to the renaming of the generators, the multiplication of a column by -1 is the substitution of the generator x^{-1} for the generator x and the addition of an integer multiple of a column to a different column substitutes $\{xy^p, y\}$ for $\{x, y\}$.

3.2 BACKGROUND FOR THE ALGORITHM

A. Given an m×n matrix C representing a finite abelian group G, the algorithm transforms C via IRC operations to the matrix of the form



where $c_{ii} \mid c_{i+1,\,i+1}$, $1 \leq i \leq n$. The form of the matrix implies that

$$G \approx \ell_{c_{11}} \times \ldots \times \ell_{c_{nn}}$$

where $\ell_{c_{ii}}$ is a cyclic group of order c_{ii}, for $1 \leq i \leq n$.

B. In the case m = n = rank(C), the order $|G|$ of the group G can be computed by means of Gaussian elimination, since

$$|G| = \det(C).$$

In the case $m > n = \text{rank}(C)$, there exist n linearly independent rows of C over the rationals, say, $\text{ROW}(j_i)$, $1 \leq i \leq n$. If $C' = (\text{ROW}(j_1), \ldots, \text{ROW}(j_n))^T$ represents an abelian group F, then the order of the group F is

$$|F| = \det(C')$$

Moreover, since G is a subgroup of F, it follows that

$$|F| \equiv 0 \pmod{|G|} \qquad (3.1)$$

Hence, if one knows the order $|G|$ of group G represented by C, then it is not difficult to see that the matrix $C'' = (c''_{ij})$, with $c''_{ij} = c_{ij} \pmod{|G|}$ representing the same group. And similarly, if one knows a multiple of $|G|$ (e.g. $|F|$ above), then the matrix $C''' = (c'''_{ij})$ with $C'''_j = c_{ij} \pmod{|F|}$ represents the same group.

3.3 THE ALGORITHM

INPUT : An m×n matrix C representing a finite abelian group G

OUTPUT : The canonical structure and the order of the group G

PROCEDURE ELIMINATEROW(C,p,D)

Comment The procedure transforms C via IRC operations to a matrix having $\text{ROW}(p) = (a_1, a_2, \ldots, a_p, 0, \ldots, 0)$. The parameter D is a multiple of the order of the group if one is known, else $D = \infty (x \bmod \infty := x, \forall\, x \in \mathbb{Z})$

Begin

while $c_{pi} \neq 0$ for some $p < i \leq n$ do

Begin

$\lambda \leftarrow \text{index}\{|c_{p\lambda}| = \min_{p \leq j \leq n} \{|c_{pj}| > 0\}\}$

< Interchange COL(λ) and COL(p) >;

< Compute e_k, f_k, $p < k \leq n$: $c_{pk} = e_k\, c_{pp} + f_k$ with $|f_k| < c_{pp}$ 2 >;

COL(k) ← COL(k) − e_k COL(p) (mod D) for $p < k \leq n$;

end

Return C;

end .

In a similar way the procedure ELIMINATECOL(C ,p,D) is defined.

PROCEDURE DIAG(C,D)

Comment This procedure transforms the matrix C via IRC operations to a matrix $C' = (c'_{ij})$ with $c'_{ij} = 0$ for $i \neq j$.

Begin

p ← 0;

(i) while $c_{ij} \neq 0$ for some $i \neq j$ do

Begin

p ← p+1;

(ii) while $< c_{pi} \neq 0$ for some $p < i \leq n >$ do

Begin

(iii) ELIMINATEROW(C,p,D);

(iv) ELIMINATECOL(C,p,D);

(v) end (step (ii))

(vi) end (step (i))

Return C;

end .

The main algorithm is the following:

Begin

1. D ← det(C) (If C non-square, D ← 0);
   If D = 0 then
2. D ← < a minor determinant of C of rank r = rank(C) >;
3. G ← DIAG(C,D);
4. < Transform C via IRC operations to a matrix C such that the sequence $\{c_{11},\ldots,c_{rr}\}$ is sorted by increasing order > ;
5. $D \leftarrow \prod_{i=1}^{r} c_{ii}$;
6. while $< c_{ii} \nmid c_{i+1,i+1}$ for some $1 \leq i \leq r >$ do

Begin

< let α: $c_{\alpha\alpha} \nmid c_{\alpha+1,\alpha+1}$ for some $1 \leq \alpha \leq r$ and $c_{ii} | c_{i+1,i+1}$ $i<\alpha >$

7. < compute v,t : $c_{\alpha+1,\alpha+1} = t\, c_{\alpha\alpha} + v$ with $|v| < |c\alpha\alpha|/2 >$;
8. $\begin{pmatrix} c_{\alpha\alpha} & 0 \\ 0 & c_{\alpha+1,\alpha+1} \end{pmatrix} \leftarrow DIAG\left(\begin{pmatrix} c_{\alpha\alpha} & c_{\alpha\alpha} \\ -t\, c_{\alpha\alpha} & v \end{pmatrix}\right)$
9. end (step 6)

end. □

PROPOSITION 3.4 Algorithm 3.3 correctly computes the order and the structure of the finite abelian group G in $O(s^5\, M(s^2))$, where s is the size of the matrix C.

Proof

The correctness of the algorithm follows from observations in §3.1 and §3.2.

Steps 1-2 require $O(s^3 M(s^2))$ elementary operations by Proposition 2.2 and the remarks of §3.2B. Moreover

$$\log|D| \leq \log(m!\, \|c\|^m) \leq s^2 \qquad (3.2)$$

Since $D \neq 0$, it is not difficult to show that the procedure ELIMINATEROW (respectively ELIMINATECOL) requires $O(\mu s^2 M(\log|D|)) = O(\mu s^2 M(s^2))$ elementary operations, where μ is the number of iterations required by the loop of the procedure. Moreover

$$|c_{pp}^{(i)}| \leq |c_{pp}^{(i-1)}|/2 \qquad (3.3)$$

where $c_{pp}^{(i)}$ denotes c_{pp} at the i-th iteration of the loop of the procedure ELIMINATEROW (respectively ELIMINATECOL).

Suppose now that in the procedure DIAG the loop (ii)-(v) requires g iterations, and that μ_j, ν_j, $1 \leq j \leq g$, denotes the number of iterations of the loop of the procedure ELIMINATEROW and ELIMINATECOL respectively at the j-th iteration of the loop (ii)-(v). Then using (3.3) we have

$$\sum_{j=1}^{g} (\mu_j + \nu_j) \leq \log|D| \leq s^2$$

Hence loop (ii)-(v) requires $O(s^4\, M(s^2))$ elementary operations and moreover procedure DIAG terminates in $O(s^5\, M(s^2))$ elementary operations.

Now step 4 requires $O(s^3 \log s)$ elementary operations by using an a sorting algorithm which requires $O(s \log s)$ comparisons (see [1]).

In a similar way, as above, one can show that steps 7-8 require $((s^2\, M(s^2))$ elementary operations. Moreover, loop 6-9 requires at most

$\sum_{j=1}^{n} \log D = O(s^3)$ iterations and thus loop 6-9 requires $O(s^5\, M(s^2))$ elementary operations.

From the above analysis the proposition follows. □

4. THE ALGORITHM FOR INFINITE GROUPS

Suppose that an infinite abelian group G is represented by an m×n matrix C, with r = rank(C) < n. In this case, the matrix C is transformed via IRC operations to an m×n matrix of the form

$$\left(C^* \;\middle|\; 0 \right) \qquad (4.1)$$

where C* is an m×r matrix with rank(C*) = r. Then one can apply algorithm 3.3 to C* to obtain the structure of G.

The transformation can be done in a similar way as in algorithm 3.3 using the following algorithm:

ALGORITHM 4.1

INPUT : an m×n matrix C representing the infinite abelian group G

OUTPUT : The order and the canonical structure of G.

```
Begin
    r←rank(C);
1.  For p = 1 to r do
      Begin
        if < c_pj = 0, for all p ≤ j ≤ n > then
        Begin
          λ ← index{ROW(λ) : c_λj ≠ 0 for some p ≤ j ≤ n};
          < Interchange ROW(λ) and ROW(p) >;
        end
        ELIMINATEROW(C,p,∞);
2.    end
      C* ← as it is in (4.1)
      < Apply algorithm 3.3 on C* >
end. □
```

PROPOSITION 4.2 Algorithm 4.1 correctly computes the structure of the group G in $O(2^{\tau+\epsilon})$ elementary operations.

Proof

It is not difficult to conclude its correctness.

Let $C^{(i)}$ denote the matrix C at the i-th iteration of the loop of the procedure ELIMINATEROW. Then one can show

$$\|C^{(i)}\| \leq i \|C\|^2$$

Now since

$$|c_{pp}^{(i)}| < |c_{pp}^{(i-1)}| /2$$

the number of iterations of the loop of the procedure ELIMINATEROW is at most $\log\|C\|$.

Hence one can show that after r iterations of loop 1-2 the size of the matrix C is bounded by

$$m + n + \log(\|C\|^{2^r})$$

and thus the proposition follows. □

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