JAMES J KAPUT (1942-2005) IMAGINEER AND FUTUROLOGIST OF MATHEMATICS EDUCATION

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Jim Kaput lived a full life in mathematics education and we have many reasons to be grateful to him, not only for his vision of the use of technology in mathematics, but also for his fundamental humanity. This paper considers the origins of his 'big ideas' as he lived through the most amazing innovations in technology that have changed our lives more in a generation than in many centuries before. His vision continues as is exemplified by the collected papers in this tribute to his life and work.

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INTRODUCTION

Jim Kaput was a mathematician born into an age of change where he seized on the new technology as a means of making mathematics available to the whole population, as a 'process of knowing' rather than as an abstract platonic body of knowledge. His personal mathematical education occurred in an era of pencil and paper arithmetic, writing algebraic manipulations on paper and drawing geometrical figures with ruler and compass; arithmetic was performed to two or three significant figures using a slide rule. He lived his whole life in Massachusetts, with his adulthood in a house in North Dartmouth (dating from early pioneer days) not far from the University of Massachusetts at Darmouth where he spent his entire career as a mathematics professor. Born into a Polish family, he did not speak English until he went to school, but was a voracious reader, taking his ideas from a wide range of literature and building his own unique way of seeing into the future development of society. By looking back to recent times as well as forwards to the possibilities for the future, he was able to see how the new technologies were fundamentally changing our lives. From his origins in Massachusetts, he steadily enlarged his arena of operation to the wider reaches of the USA and subsequently spread his ideas to mathematics education communities around the whole world. He not only reflected on the future development and use of technology, he also designed new curricular materials based on software which he then made widely available as a free download from the internet. He is perhaps less publicly known for his quiet support for many fellow academics around the world who sought his assistance and benefited from his gentle diplomacy. Jim was a scholar and a gentleman.

THE HISTORY OF KAPUT'S 'BIG IDEAS'

Jim Kaput was fortunate to flourish in an era when computers developed from huge machines with tiny memories in the forties to personal computers with great power in the seventies and eighties, and on to hand-held devices and interpersonal communication in the nineties and early twenty-first century.

Jim began his mathematical life in a pre-computer era of paper, pencil and mental mathematics. As a young mathematics professor at Southern Massachusetts University, he took a special interest in teaching the calculus. In the mid-seventies, the Apple II computer made it possible for individuals to use the power of the computer in their own homes for the first time in history. However, at this time, his energies were still directed to traditional teaching methods in the calculus. His growing involvement led him to co-author a typical massive calculus text (Fleming & Kaput, 1979). It was replete with all that was necessary for a complete year-long course on differentiation, integration, differential equations and on to techniques in two and three dimensions. He told me it sold seven thousand copies, which may seem impressive, but it was a flea-bite in the huge market of American students taking college calculus.

It is a salutary experience to read that book today. It is compendious (as all American calculus texts had to be to guarantee widespread take-up) but there is already a distinction between visual geometric ideas, marked with a special symbol to set them apart, numerical calculations to calculate good approximations and symbolic manipulations to give precise expression to the limit concept. His view at this time was the use of visual ideas to give human insights as a gentle introduction to the formal mathematics and the epsilon-delta definition of limit. It also marked the beginnings of ideas of multiple-linked representations which featured in his conversations.

He first contacted me by letter in the late seventies when he learnt of my own interest in the calculus, and we shared experiences by airmail. I still have a photocopy draft of his first 'official' paper on 'mathematics and learning' (Kaput, 1979) which began with what he called a 'Polemical Abstract':

> This paper concerns itself with certain vital aspects of the platonism-constructivism issue and how they reflect themselves in our everyday work of teaching and learning. It is suggested that we are confronted with a pair of universes: one being the stark, atemporal, formal universe of ideal knowledge; the other being the organic, fluid, processual universe of human knowing. The former, Plato's, has monopolized status and power, and is responsible for the fundamental dominance of product over process, the values by which legitimacy of knowledge is conferred, and the rules, particularly linguistic rules, under which any inquiry occurs. We illustrate how the manifold consequences of this status dominance permeate our academic lives at all levels, concentrating mainly on its exclusionary function in mathematics and mathematics knowing/learning.

Here, in the first paragraph of his first paper is his distinction between the state of formal knowledge and the process of learning, which set the foundation for a life-time of campaigning for the 'democratization' of mathematical knowledge, making it freely available to all, as a human process of learning and knowing rather than an austere body of knowledge that tended to freeze out many potential participants.

In 1985, Jim invited me to visit him at his home in Massachusetts to share ideas and demonstrate my *Graphic Calculus* software. He astounded me with his insights. He was grappling with his early ideas of symbol systems, but what made me sit up was the way in which he viewed the symbols typed into the computer as cybernetic entities that could carry out the computations internally, leaving the individual to think about broader issues. Listening to him talk about his vision was intoxicating.

We had a lot to talk about concerning the calculus, as my research into students' thinking about limits (Tall, 1975; Tall & Vinner, 1981) had led me to program *Graphic Calculus* software to enable the learner to zoom in on a curve to 'see' it becoming less curved, until it looked 'locally straight'. This approach was validated both in terms of research into students' known difficulties and also through formal theorems in non-standard analysis which proved that, under high magnification, the part of a graph that could be seen looked precisely like an infinitely straight line.

I related graph-drawing to the continuous action of a pencil on paper, and while the theoretical graph had no thickness, the pencil drawing *covered* the underlying points on the graph (Stewart & Tall, 1977). It was necessary to learn to look *through* such pictures, as John Mason was later to observe. If one magnified a physical pencil graph, the thickness of the graph would also be magnified. However, if one magnified a graph on a computer screen, the magnified graph could be drawn to the same level of detail, whatever the scale; this enabled the user to 'zoom in' on a graph and 'see' the graph of a differentiable function becoming less curved until lt looked straight. I had great success in using this idea to encourage students to look along the graph to 'see' its changing slope and to be able to sketch the graph of the slope function. Thus the learner could imagine the slope graph in the mind's eye and seek numeric approximations or symbolic precision. The embodiment therefore gave a meaning to the concept of changing slope while the use of the symbolism gave a reason for constructing the notion of limit. I saw this as a cognitive approach to the calculus which built on human perception and reflection, which could develop to be related later to formal developments in both standard and non-standard analysis.

However, Jim's vision was not just about the whole of calculus and analysis, not even just the whole of mathematics, but the broader issues of human life in a technological age. All graph-plotting software at the time, including my own, allowed the user to type in a symbolic formula and then to explore the properties of the resulting graph. Jim wanted to go further than that. A one-way link from symbol to graph was not enough. He wanted the student to be able to personally draw the graph and he imagined this as part of a framework of multiple-linked representations that dynamically connected real world phenomena to graphs and symbols that represented the rate of change, and the rate of growth.

To enable the learner to be in full control of the activity required software both to receive input from sensors that measured real world data and also the facility for the learner to draw a graph with appropriate accuracy. For Jim, this entailed pointing and clicking the mouse at successive points with the computer joining the points as a piecewise linear graph. This approximation would please a mathematician who already used piece-wise linear functions as practical approximations to continuous curves, enabling straightforward computations. But it introduced a new element into the initial conceptions of the calculus, in which the slope function consisted of a discontinuous sequence of piece-wise functions, one for each linear part of the graph. This gave a concept image of slope with a very different meaning from that which would occur later. Unlike my approach, which envisaged a locally straight curve which looked straight under magnification, his curves had corners and his derivatives were disconnected flat pieces, not locally linear curves that could possibly be differentiated again and again.

Jim's concern was to make the calculus meaningful to everyone by using technology to link the learner's actions to a range of interconnected represetntatons. For me the long-term development of enormous power lay not just in the connection between specific real world events such as 'distance', 'velocity', 'acceleration', and 'jerk', in which each successive derivative had a new meaning. I even saw these disparate meanings as potential obstacles to the heart of the calculus where the derivative of a function is simply another function that may itself be differentiable again and again, leading to power series expansions and the magical relationship between the complex exponential function and the trigonometric functions. I saw this uniquely embodied in the notion of 'local straightness' that acted as a 'generic organiser' for the whole of the calculus, from the initial meanings in elementary calculus through to the embodied local flatness of differentiable manifolds.

For Jim this was far off from his interests in democratizing the fundamental ideas of calculus for younger children and we each followed our own path while continuing to take a major interest in the work of the other; I built a long-term learning theory relating embodiment, symbolism and formal mathematics and he focused on using technology to make fundamental mathematical ideas available to everyone. As time passed we continued to meet regularly at conferences, never missing one without spending an evening together, honing our ideas by sharing them with a friend with a different, but sympathetic, perspective.

Jim's future became increasingly enriched by his vision of using the new technological changes to revolutionize the very ways in which we thought about and taught mathematics:

> Anyone who presumes to describe the roles of technology in mathematics education faces challenges akin to describing a newly active volcano – the mathematical mountain is changing before our eyes… (Kaput, 1992, p. 515.)

He focused on the way in which younger children have intuitive sense of concepts such as distance, velocity, acceleration, which could be utilised in conjunction with computer simulations to study aspects of calculus at a far earlier age. He imagined computer representations far beyond the capacity of the computers available at the time involving simulations such as driving a car along a highway—linked to numeric and graphic displays of distance and velocity against time—allow a study of change which is not limited to functions given by standard formulae (figure 1).

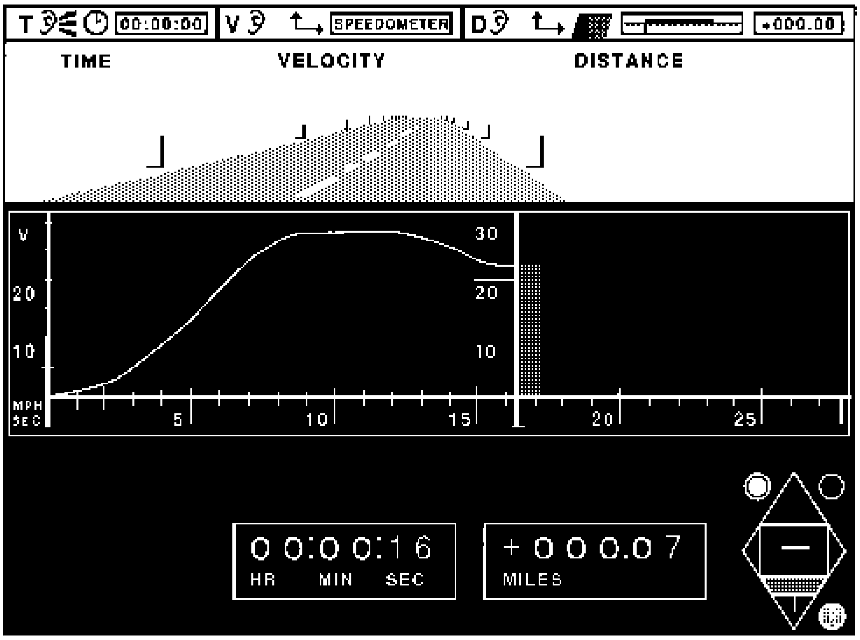

Figure 1: An early vision of *MathCars*: simulating relationships between time, distance and velocity

Jim wisely never took the step of programming his own ideas, instead he went around at conferences showing mock-up simulations of them for several years before the technology reached a stage where it was powerful enough to develop *SimCalc* as a fully grown software package, programmed professionally.

In the meantime, he tirelessly fought the case for new ways of conceptualizing algebra and calculus, dismissing the current ways of teaching mathematics, as being fundamentally flawed:

> School algebra in the US is institutionalized as two or more highly redundant courses, isolated from other subject matter, introduced abruptly to post-pubescent students, and often repeated at great cost as remedial mathematics at the post secondary level. Their content has evolved historically into the manipulation of strings of alphanumeric characters guided by various syntactical principles and conventions, occasionally interrupted by "applications" in the form of short problems presented in brief chunks of highly stylized text. All these are carefully organized into small categories of very similar activities that are rehearsed by category before introduction of the next category, when the process is repeated. The net effect is a tragic alienation from mathematics for those who survive this filter and an even more tragic loss of life-opportunity for those who don't. (Kaput, 1995, p.71)

This led to him broadening the scope of algebra into five strands:

i. Algebra as generalizing and formalizing patterns and regularities, in particular, algebra as generalised arithmetic;
ii. Algebra as syntactically guided manipulations of symbols;
iii. Algebra as the study of structure and systems abstracted from computations and relations;

iv. Algebra as the study of functions, relations and joint variations;
v. Algebra as modelling. (Kaput 1998, p.26)

He saw traditional school algebra mainly in terms of the second of these (symbol manipulation), currently taught in a way that made little sense to many students, and proposed the need to 'algebrafy' the mathematics curriculum to include earlier aspects of generalising patterns as generalised arithmetic, and later broadening algebra to include modelling, functions, and abstract structures. He followed an algebraic thread throughout his life, from studying the ways in which students (mis)conceptualised algebra, such as the student-professor problem (Kaput & Sims-Knight 1983) to work on intensive quantities with Judah Schwartz (Kaput, Schwartz & Poholsky, 1985) through to recent developments in 'emergent algebra' that occur in the transition from arithmetic to algebra (Kaput, Carraher & Blanton, 2007).

The major exemplification of his vision lay in his development of the software *SimCalc*, an approach using computer-based graphs and animations as well as symbols and tables to make mathematical rates of change and accumulation conceptually available to elementary and middle school students. It has progressed over time as a beautiful interactive computer program (figure 2) implemented as an interconnected system with children using hand-held calculators to interact wirelessly

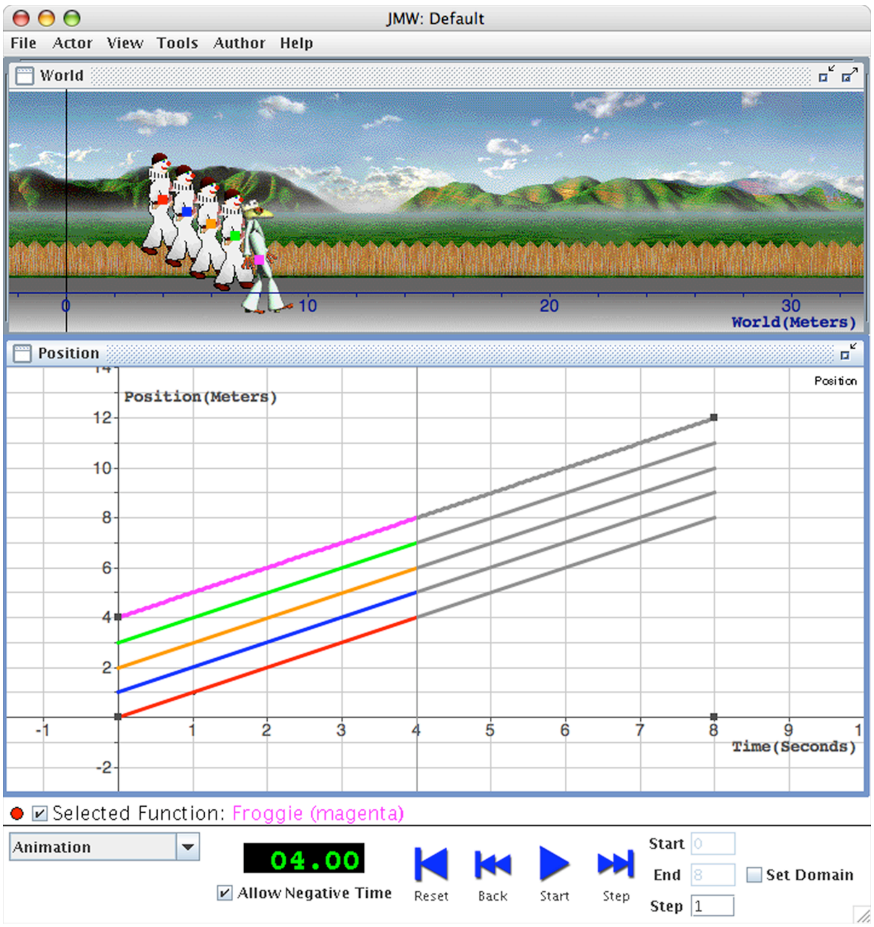

Figure 2: *SimCalc MathWorlds*® linking imagined motion to time-distance graphs.

with a central display sharing individual operations on a communal screen.

Kaput described this as ‘a harbinger of a whole new style of instruction that mixes “first person action” and “third person” observation and adjustment of mathematical objects.’ In figure 2, he saw the “froggie” character in the foreground being used to enact the student’s physical motion (imported from a sensor as the individual moved) as a template to cause the clowns in the parade to follow (or even move in some interesting counterpoint to the froggie’s motion). He saw this as one stage in a development in which students could ‘exploit their understandings in other settings, with quantities other than velocity, position, etc.’

Over the years we met regularly for a shared meal at several conferences a year (usually accompanied by suitable quantities of good quality local beer and the inevitable malt whisky) and continued to discuss our consonant but differing viewpoints. I saw the power of his implementation of *SimCalc*, though we continued to differ over whether to begin with piece-wise linear graphs, with their corners and piece-wise flat derivatives, or locally-straight graphs that embodied longer-term notions met in more advanced calculus. We had differences in our approach to symbolism too, with my work on procept theory with Eddie Gray (Gray & Tall, 1991) seeking a flexible approach to number operations as process and concept while American commentators in general preferred a broader kind of ‘flexibility’ with the use of multi-linked representation.

In more recent times Jim began to see the fruits of his earlier labours as his work on algebra and calculus came together in developing projects to share his ideas in the classroom. It was a great blow to us all when this intellectual giant of a man was taken from us with so much still to do. But the foundations of his vision are there to build upon. Although he had no personal religious persuasion, he continues to live in the hearts and minds of those who remain and those who continue his vision.

THE JOURNEY CONTINUES

The papers in this journal show that Jim’s vision continues to grow. The first is one of Jim’s recent collaborations with fellow co-workers, formulating developments of his insight into how technology allows symbols to carry the burden of calculation, allowing the learner to focus on the bigger, foundational ideas (Hegedus, Moreno & Kaput, 2007). This remains a continuing vision of the expressive use of symbolism in mathematics afforded by technology which occupied his attention in the last quarter of a century and is destined to remain a focus in years to come.

Jim’s belief that technology will not simply be used to support the mathematics that we have now, but to develop entirely new ways of thinking mathematically, is taken up as a thought experiment in the second paper (Lesh, Caylor, Gupta & Middleton, 2007). Here the focus is on statistics—largely used unthinkingly by the community as tools to support analysis of data. The paper considers the much-used least squares statistic and considers the alternative of using least distances in a new technological approach that is both powerful and meaningful. Kaput's ideas of rethinking mathematics using technology are at the very centre of this work.

Schorr and Goldin (2007) offer a first-hand account of the use of Kaput's software *SimCalc* in an inner city classroom. A small-scale study reports the joy and excitement of disadvantaged, even disaffected, children as they take an active part in their own learning using the software in meaningful ways. This confirms Jim's vision of democratizing mathematics through technology actually works in a disadvantaged classroom.

Noss and Hoyles (2007) view the wide programme of development suggested by Kaput (1992) in which he elucidated four major principles for the future use of technology: *attend to representational infrastructure* (i.e. look at how technology can represent ideas to help us think better), *work for infrastructural change*, *outsource processing to the computer but attend to the implications*, and *exploit connectivity to encourage sharing and discussion*. They illustrate their interpretation of all four principles, focusing in particular on the second and third, using technological infrastructure that helps organise our lives and outsourcing mathematical processing to the computer enabling the learner to focus on the wider picture. The four principles also permeate the other papers in the collection, vindicating the strength and pourpose of Kaput's considered theoretical synthesis.

Kaput's desire, not only to think through the changes made possible by technology, but also to implement them on a wider scale, is taken up by Roschelle, Tatar, Shechtman & Knudsen (2007), with a study of *SimCalc* scaled up for use in the state of Texas. While mathematics education often involves carrying out small scale research and disseminating curriculum materials through publication, Jim saw the necessity of matching good ideas with curriculum support and interchange of experiences on a wide scale. Here we see Jim's ideas continuing in an increasingly broadening framework to move from theoretical ideas and work in individual classrooms to in-depth analysis of larger-scale implementation.

The final paper returns us to the continuing research in the James J Kaput Centre for Research and Innovation in Mathematics Education, working on the opportunities of a wireless-connected classroom where children can work together and in groups, submitting their work on hand-held devices for a central sharing of ideas for discussion amd development. Now we are seeing Jim's vision of technology being used collaboratively for children to work actively with multi-linked representations to construct insightful shared ideas. His vision remains and continues into the future.

REFERENCES

Fleming, J., & Kaput, J. J. (1979). *Calculus and analytic geometry*. New York: Harper & Row.

Gray, E. M., & Tall, D. O. (1991). Duality, ambiguity and flexibility in successful mathematical thinking. In F. Furinghetti (Ed.), *Proceedings of the 15th International*

Conference for the Psychology of Mathematics Education (Vol. 2, pp. 72-79). Assisi, Italy.

Kaput, J. J. (1979). Mathematics and learning: Roots of epistemological status. In J. Clement & J. Lochhead (Eds.), *Cognitive process instruction* (pp. 289–303). Philadelphia, PA: Franklin Institute Press.

Kaput, J. J. (1992). Technology and mathematics education. In D. Grouws (Ed.), *Handbook on research in mathematics teaching and learning* (pp. 515-556). New York: Macmillan.

Kaput, J. J. (1995). A research base for algebra reform: Does one exist? In D. Owens, M. Reed, & G. M. Millsaps (Eds.), *Proceedings of the 17th Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 71–94). Columbus, OH: The Eric Clearinghouse for Science, Mathematics and Environmental Education.

Kaput, J. J. (1998). Transforming algebra from an engine of inequity to an engine of mathematical power by "algebrafying" the K-12 curriculum. In the National Council of Teachers of Mathematics & the Mathematical Sciences Education Board (Eds.), *The nature and role of algebra in the K-14 curriculum: Proceedings of a national symposium* (pp. 25-26). Washington, DC: National Research Council, National Academy Press.

Kaput, J. J., Carraher, D. W. & Blanton, M. L. (Eds.) (2008). *Algebra in the early grades*. Mahwah, NJ: Lawrence Erlbaum Associates.

Kaput, J. J., Schwartz, J., & Poholsky, J. (1985). Extensive and intensive quantities in multiplication and division word problems: Preliminary report. In S. K. Damarin & M. Shelton (Eds.), *Proceedings of the 7th Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Columbus, OH: The Eric Clearinghouse for Science, Mathematics and Environmental Education.

Sims-Knight, J. E., & Kaput, J. J. (1983). Misconceptions of algebraic symbols: Representations and component processes. In H. Helm & J. D. Novak (Eds.), *Proceedings of the international seminar: Misconceptions in science and mathematics* (pp. 477-488). Ithaca, NY: Cornell University.

Stewart, I. N., & Tall, D. O. (1977). *The Foundations of Mathematics*. Oxford, UK: Oxford University Press.

Tall, D. O., & Vinner, S. (1981). Concept image and concept definition in mathematics, with special reference to limits and continuity. *Educational Studies in Mathematics*, *12*, 151–169.

Tall, D. O. (1975). A long-term learning schema for calculus/analysis. *Mathematical Education for Teaching*, *2*(5), 3–16.