

University of Warwick institutional repository: <http://go.warwick.ac.uk/wrap>

A Thesis Submitted for the Degree of PhD at the University of Warwick

<http://go.warwick.ac.uk/wrap/50011>

This thesis is made available online and is protected by original copyright.

Please scroll down to view the document itself.

Please refer to the repository record for this item for information to help you to cite it. Our policy information is available from the repository home page.

AUTHOR: **Nicholas Korpelainen** DEGREE: **Ph.D.**

TITLE: **Boundary Properties of Graphs**

DATE OF DEPOSIT:

I agree that this thesis shall be available in accordance with the regulations governing the University of Warwick theses.

I agree that the summary of this thesis may be submitted for publication.

I **agree** that the thesis may be photocopied (single copies for study purposes only).

Theses with no restriction on photocopying will also be made available to the British Library for microfilming. The British Library may supply copies to individuals or libraries. subject to a statement from them that the copy is supplied for non-publishing purposes. All copies supplied by the British Library will carry the following statement:

“Attention is drawn to the fact that the copyright of this thesis rests with its author. This copy of the thesis has been supplied on the condition that anyone who consults it is understood to recognise that its copyright rests with its author and that no quotation from the thesis and no information derived from it may be published without the author’s written consent.”

AUTHOR’S SIGNATURE:

USER’S DECLARATION

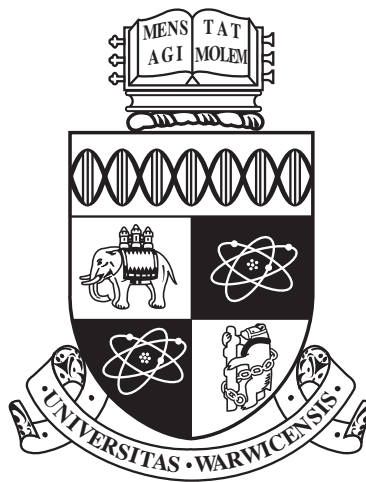
1. I undertake not to quote or make use of any information from this thesis without making acknowledgement to the author.
2. I further undertake to allow no-one else to use this thesis while it is in my care.

DATE

SIGNATURE

ADDRESS

.....
.....
.....
.....
.....



Boundary Properties of Graphs

by

Nicholas Korpelainen

Thesis

Submitted to the University of Warwick

for the degree of

Doctor of Philosophy

Mathematics Institute

June 2012

THE UNIVERSITY OF
WARWICK

Contents

List of Figures	iv
Acknowledgments	v
Declarations	vi
Abstract	viii
Chapter 1 Introduction	1
1.1 Graphs: Basic Definitions and Conventions	1
1.2 A Quick Introduction to Partial Orders	4
1.3 Partial Orders on Sets of Graphs: Hereditary Graph Properties . . .	6
1.4 Boundary Ideals	9
1.4.1 Motivations	10
1.4.2 Definitions and tools	11
Chapter 2 Algorithmic Graph Problems	15
2.1 The Time Complexity of Algorithmic Graph Problems	15
2.2 Boundary Classes for Algorithmic Graph Problems	16
2.3 The Hamiltonian Cycle Problem	18
2.3.1 Approaching a limit class	18
2.3.2 Limit class	20
2.3.3 Minimality of the limit class	21
2.3.4 One more boundary class	24
2.3.5 Concluding remarks and related open problems	24
2.4 The k -Path Partition Problem	25
2.4.1 A boundary class	26
2.4.2 Concluding remarks and related open problems	31
2.5 The Dominating Induced Matching Problem	32

2.5.1	A boundary class	33
2.5.2	Polynomial-time algorithms	36
2.5.3	Concluding remarks and related open problems	51
Chapter 3	Clique-Width	52
3.1	Clique-Width: A Short Introduction	52
3.2	Building Bipartite Graphs of Large Clique-Width	55
3.2.1	Building blocks and building operations	55
3.2.2	A minimal class of unbounded clique-width	57
3.2.3	The class \mathcal{M} and a boundary subclass	61
3.3	Bipartite Double Bichain Graphs and Split Permutation Graphs . .	64
3.3.1	An overview of some properties of split graphs	64
3.3.2	Split permutation graphs of large clique-width	70
3.4	Open Problems	73
Chapter 4	Induced Subgraphs and Well-Quasi-Orderability	74
4.1	Definitions and Examples	74
4.2	Well-Quasi-Orderability of Classes of Bipartite Graphs	75
4.2.1	The class of $(2P_3, \text{Sun}_4)$ -free bipartite graphs is not WQO . .	78
4.2.2	The class of (P_8, \widetilde{P}_8) -free biconvex graphs is not WQO	82
4.2.3	The class of double bichain graphs is not WQO	86
4.2.4	The class of $(P_7, S_{1,2,3})$ -free bipartite graphs is WQO	88
4.2.5	The class of (P_7, Sun_1) -free bipartite graphs is WQO	89
4.2.6	The class of P_k -free bipartite permutation graphs is WQO . .	91
4.2.7	Characterisation of all monogenic classes of bipartite graphs .	93
4.3	Bigenic Classes of Graphs	94
4.4	Bigenic Classes of Graphs Which Are Well-quasi-ordered	96
4.4.1	Well-quasi-order and k -uniform graphs	96
4.4.2	Well-quasi-order, k -letter graphs and modular decomposition	102
4.5	Bigenic Classes of Graphs Which Are Not Well-quasi-ordered	105
4.6	A Summary for Bigenic Classes	107
4.7	Boundary Classes for Well-Quasi-Orderability	112
4.7.1	On the number of boundary classes	113
4.7.2	The well-quasi-orderability of finitely defined classes	117
4.7.3	Remarks and open problems	119
Chapter 5	Conclusion	120

Index	123
References	125

List of Figures

1.1	A 3-edge graph on 3 vertices.	2
1.2	A graph and its complement.	3
1.3	From left to right: $4K_1$, K_4 , P_4 and C_4	3
2.1	A tribranch $Y_{i,j,k}$	19
2.2	Transformation F_p	20
2.3	A caterpillar with hairs of arbitrary length	21
2.4	Transformation R	24
2.5	The computational complexity of the k -path partition problem . . .	27
2.6	Graphs $S_{i,j,k}$ (left) and H_i (right)	28
2.7	Graphs X (left), Y (middle), and Z (right)	39
2.8	An example of a τ -caterpillar	42
2.9	The graph $S_{i,j,k}$	45
2.10	A diamond (left) and a butterfly (right).	46
3.1	The tree representing the expression defining a C_5	53
3.2	A bipartite chain graph.	65
4.1	Graphs H_i (left) and Sun_4 (right)	75
4.2	Inclusion relationships between subclasses of bipartite graphs	77
4.3	Diagram representing the permutation $(2, 3, 5, 1, 7, 4, 9, 6, 12, 8, 11, 10)$. .	77
4.4	The permutation π_{10}^* (left) and the permutation graph $G_{\pi_{10}^*}$ (right) .	79
4.5	The graph $B'(S_{\pi_n^*}) = G_{\pi_n^*}$	86
4.6	The graph $H_{5,5}$	92
4.7	The graphs Φ (left) and T (right)	115

Acknowledgments

Most importantly, I would like to thank my supervisor Vadim V. Lozin for his exceptional patience, inspiring enthusiasm and the consistent clarity of his advice. Of course, I also wish to express my gratitude to him for our fruitful collaboration in research. These thanks extend to all other co-authors of joint research papers.

I want to thank all of the staff members and fellow students at DIMAP (Centre for Discrete Mathematics and its Applications) at the University of Warwick, for creating a positive and welcoming atmosphere throughout my studies. Similarly, I wish to thank all other staff members working at the Mathematics Institute and the University of Warwick for their dedicated care.

EPSRC and DIMAP also deserve my appreciation for their generous financial contributions towards funding.

Malgosia Senderek, my mathematics and physics teacher at high school, is the person who first inspired me to study mathematics at university, and so I send to her a very special thank you. Of course, the professors and staff at the University of Cambridge also have my gratitude for sustaining and nurturing my interest in this wonderful subject throughout my undergraduate years.

I'd like to thank any and all readers of this thesis for their interest.

Finally, I want to thank my mother for her never-ending love, warmth, wisdom and immeasurable support. Thank you.

Declarations

All uncited content (proofs, conjectures etc.) in this thesis is either my own original work or a collaborative effort that I shared as a main author together with my supervisor Vadim V. Lozin and other co-authors of my research papers. All of these papers have either been published or submitted for publication after the start of my PhD studies at University of Warwick in October 2008.

This thesis contains material from the following journal/conference papers:

- N. Korpelainen. A Polynomial-time Algorithm for the Dominating Induced Matching Problem in the Class of Convex Graphs, *Electronic Notes in Discrete Mathematics*, 32:133, 2009
- N. Korpelainen and V. V. Lozin. Bipartite Graphs of Large Clique-Width, *Lecture Notes in Computer Science*, 5874:385, 2009
- N. Korpelainen, V. V. Lozin and A. Tiskin. Hamiltonian Cycles in Subcubic Graphs: What Makes the Problem Difficult, *Lecture Notes in Computer Science*, 6108:320, 2010
- D. M. Cardoso, N. Korpelainen and V. V. Lozin. On the complexity of the dominating induced matching problem in hereditary classes of graphs, *Discrete Applied Mathematics*, 159:521, 2011
- N. Korpelainen, V. V. Lozin, D. S. Malyshev and A. Tiskin. Boundary properties of graphs for algorithmic graph problems, *Theoretical Computer Science*, 412:3545, 2011

- N. Korpelainen and V. V. Lozin. Bipartite induced subgraphs and well-quasi-ordering, *Journal of Graph Theory*, 67:235, 2011
- N. Korpelainen and V. V. Lozin. Two forbidden induced subgraphs and well-quasi-ordering, *Discrete Mathematics*, 311:1813, 2011
- N. Korpelainen, V. V. Lozin and C. Mayhill, Split Permutation Graphs, submitted to *Graphs and Combinatorics*.
- N. Korpelainen, V. V. Lozin and I. Razgon. Boundary properties of well-quasi-ordered sets of graphs, submitted to *Order*.

The work in Section 2.4 is my independent unpublished work. The same is true of all uncited proofs in Sections 3.3, 4.2.3 and 4.2.7. Any unpublished proofs in Section 2.5.2 were obtained in collaboration with my supervisor Vadim V. Lozin.

Abstract

A set of graphs may acquire various desirable properties, if we apply suitable restrictions on the set. We investigate the following two questions: How far, exactly, must one restrict the structure of a graph to obtain a certain interesting property? What kind of tools are helpful to classify sets of graphs into those which satisfy a property and those that do not?

Equipped with a containment relation, a graph class is a special example of a partially ordered set. We introduce the notion of a *boundary ideal* as a generalisation of a notion introduced by Alekseev in 2003, to provide a tool to indicate whether a partially ordered set satisfies a desirable property or not. This tool can give a complete characterisation of lower ideals defined by a finite forbidden set, into those that satisfy the given property and to those that do not. In the case of graphs, a lower ideal with respect to the induced subgraph relation is known as a *hereditary graph class*.

We study three interrelated types of properties for hereditary graph classes: the existence of an efficient solution to an algorithmic graph problem, the boundedness of the graph parameter known as *clique-width*, and well-quasi-orderability by the induced subgraph relation.

It was shown by Courcelle, Makowsky and Rotics in 2000 that, for a graph class, boundedness of clique-width immediately implies an efficient solution to a wide range of algorithmic problems. This serves as one of the motivations to study clique-width. As for well-quasi-orderability, we conjecture that every hereditary graph class that is well-quasi-ordered by the induced subgraph relation also has bounded clique-width.

We discover the first boundary classes for several algorithmic graph problems, including the Hamiltonian cycle problem. We also give polynomial-time algorithms for the dominating induced matching problem, for some restricted graph classes.

After discussing the special importance of bipartite graphs in the study of clique-width, we describe a general framework for constructing bipartite graphs of large clique-width. As a consequence, we find a new minimal class of unbounded clique-width.

We prove numerous positive and negative results regarding the well-quasi-orderability of classes of bipartite graphs. This completes a characterisation of the well-quasi-orderability of all classes of bipartite graphs defined by one forbidden induced bipartite subgraph. We also make considerable progress in characterising general graph classes defined by two forbidden induced subgraphs, reducing the task to a small finite number of open cases. Finally, we show that, in general, for hereditary graph classes defined by a forbidden set of bounded finite size, a similar reduction is not usually possible, but the number of boundary classes to determine well-quasi-orderability is nevertheless finite.

Our results, together with the notion of boundary ideals, are also relevant for the study of other partially ordered sets in mathematics, such as permutations ordered by the pattern containment relation.

Chapter 1

Introduction

The theory of graphs is rich, active and has expanded rapidly in the recent years. It has applications and surprising connections to a large array of disciplines, including topology, computer science and to seemingly unrelated subjects such as psychology.

The structure of general graphs is usually rather complex, both algorithmically and combinatorially. However, under some restrictions, it may acquire some desirable properties. A natural question arises: How far, exactly, must one restrict the structure of a graph to obtain a certain interesting property? What kind of tools are helpful to classify sets of graphs into those that satisfy a property and those that do not?

In this thesis, we explore the above questions with respect to various desirable properties, such as polynomial-time solvability of some algorithmic graph problems, boundedness of clique-width and well-quasi-orderability. The reader is assumed to have a basic working knowledge of functions, sets, and binary relations.

1.1 Graphs: Basic Definitions and Conventions

Definition 1.1.1. A graph G is defined by the ordered pair of sets $(V(G), E(G))$, where each member of $E(G)$ is a subset of $V(G)$ of cardinality 2.

- Members of $V(G)$ are called vertices and members of $E(G)$ are called edges .
- An edge $\{x, y\}$ is often denoted xy for short, and it is said to be adjacent to (or incident at) x and y .
- For a vertex $x \in V(G)$, we denote by $N(x)$ the set of vertices in $V(G)$ that are adjacent to x . The cardinality of the set $N(x)$ is called the degree of x . A graph whose every vertex has degree k is called k -regular .

- For a vertex subset $W \subset V(G)$, the notation $G[V(G) \setminus W]$ (or sometimes just $G \setminus W$) will refer to the graph with vertex set $V(G) \setminus W$ and edge set $\{xy \in E(G) : x, y \notin W\}$. Such a graph is called an induced subgraph of G .
- For an edge subset $F \subset E(G)$, the notation $G \setminus F$ will refer to the graph with vertex set $V(G)$ and edge set $E(G) \setminus F$.
- For an edge $e = xy \in E(G)$, the edge contraction G/e will have vertex set $(V(G) \setminus \{x, y\}) \cup \{z\}$ and edge set

$$\{x'y' \in E(G) : x', y' \notin \{x, y\}\} \cup \{z'z : z'x \in E(G) \text{ or } z'y \in E(G)\}$$

- By $G + H$ we denote the disjoint union of two graphs G and H . In particular, $mG = G + \dots + G$ is the disjoint union of m copies of G .

Unless we state otherwise, $V(G)$ will be assumed to be finite, in which case the graph G is said to be finite.

A graph is most often visualised by plotting its vertices as points on a plane and drawing each edge as a line segment that identifies a pair of vertices by its end-points.

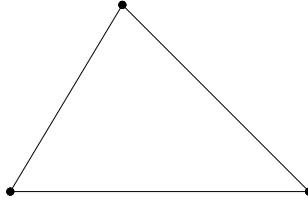


Figure 1.1: A 3-edge graph on 3 vertices.

There is a natural way to define what it means for two graphs to be 'identical':

Definition 1.1.2. *Graphs G and H are said to be isomorphic if there exists a bijection $f : V(G) \rightarrow V(H)$ such that $xy \in E(G)$ if and only if $f(x)f(y) \in E(H)$. A set of graphs is called a graph class or a graph property, when any two isomorphic graphs are taken to be equal.*

Definition 1.1.3. *The complement graph \overline{G} of G is defined as the graph with vertex set $V(G)$ and edge set $\overline{E(G)} := \{e \subseteq V(G) : |e| = 2 \text{ and } e \notin E(G)\}$.*

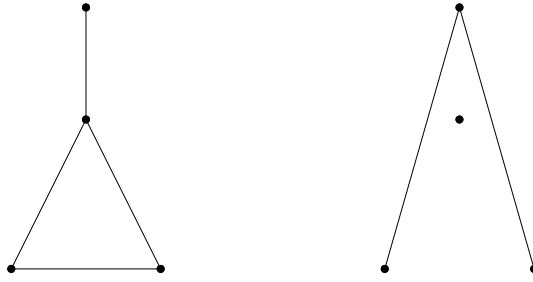


Figure 1.2: A graph and its complement.

There are a few basic graphs that will be referred to repeatedly:

- An *empty (edgeless) graph on n vertices*, denoted by nK_1 , is a graph on n vertices with no edges.
- A *complete graph on n vertices*, denoted by K_n is the graph $\overline{nK_1}$.
- The *chordless path on n vertices* will be denoted by P_n .
- The *chordless cycle on n vertices* will be denoted by C_n .

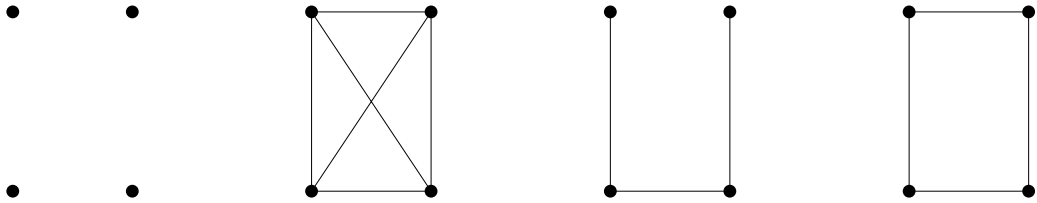


Figure 1.3: From left to right: $4K_1$, K_4 , P_4 and C_4

We need a few more definitions.

Definition 1.1.4. Let G be a graph.

- A subset of $V(G)$ that induces an empty graph is called an independent set.
- A subset of $V(G)$ that induces a complete graph is called a clique.

Definition 1.1.5. A graph G is said to be connected if any two vertices of G are the end-vertices of a chordless path in G . If a graph G is not connected, its maximal connected induced subgraphs are called the connected components of G .

1.2 A Quick Introduction to Partial Orders

In this thesis, we will consider various partial orders on sets of graphs. With this in mind, we need to introduce some basic concepts about partially ordered sets.

Definition 1.2.1. A binary relation on a set X is a subset of X^2 .

Definition 1.2.2. A partial order \leq on a set X is a binary relation on X with the following properties:

1. reflexivity. For each $x \in X$, we have $x \leq x$.
2. anti-symmetry. For any $x, y \in X$, if $x \leq y$ and $y \leq x$, then $x = y$.
3. transitivity. For any $x, y, z \in X$, if $x \leq y$ and $y \leq z$, then $x \leq z$.

A set $X := (X, \leq)$, considered under a specific partial order \leq , is called a partially ordered set or a poset.

Definition 1.2.3. A binary relation that satisfies reflexivity and transitivity, but not necessarily anti-symmetry, is called a quasi-order.

Example 1.2.1. The set \mathbb{Z} of integers is partially ordered by each of the following two binary relations: \leq ('less than or equal to') and divisibility.

Example 1.2.2. The set of permutations on n items can be given a partial order, called pattern containment (see e.g. [Atkinson et al., 2002]).

To define this order, suppose the permuted set is $[n] := \{1, 2, \dots, n\}$. With each permutation π on $[n]$, we can naturally associate a 'word', given by the sequence $\pi(1)\pi(2)\dots\pi(n)$. By a subword of π , we mean any subsequence of entries of π (not necessarily consecutive). We say that 'permutation π contains permutation σ as a pattern' if there is an order-preserving bijection from σ to a subword of π . (A map f is called order-preserving if $f(x) \leq f(y)$ whenever $x \leq y$.)

We will be interested in some special subsets of posets:

Definition 1.2.4. A subset X of a poset (Y, \leq) is a *chain* if for any $x, y \in X$, the elements x and y are comparable, i.e. we have $x \leq y$ or $y \leq x$.

Example 1.2.3. (\mathbb{Z}, \leq) is a chain in itself.

Definition 1.2.5. A subset X of a poset (Y, \leq) is an *antichain* if for any $x, y \in X$, the elements x and y are incomparable, i.e. we have $x \not\leq y$ and $y \not\leq x$.

Example 1.2.4. The set of prime numbers is an antichain in the set \mathbb{N} of positive integers partially ordered by divisibility.

The following rather intuitive theorem illuminates the relationship between chains and antichains:

Theorem 1.2.1 ([Dilworth, 1950]). Let Y be a finite poset. Then the maximum cardinality of an antichain in Y is equal to the minimum number of chains in any partition of Y into chains.

Most of the posets considered in this thesis will be *lower ideals*:

Definition 1.2.6. A subset X of a poset (Y, \leq) is a *lower ideal* of Y if for any $x \in X$ and $y \in Y$ such that $y \leq x$, we have $y \in X$.

Definition 1.2.7. For an antichain M of a poset (Y, \leq) , we define the set $Free_{\leq}(M) := \{y \in Y : x \not\leq y \text{ for all } x \in M\}$.

The following simple proposition suggests an alternative way to define a lower ideal, by considering the minimal elements excluded from it:

Proposition 1.2.2. Let (Y, \leq) be a poset that does not contain any infinite descending chains. If X is a lower ideal of Y , there is a unique antichain N such that $X = Free_{\leq}(N)$, given by setting N to be the set M of minimal elements in $Y \setminus X$. Moreover, X is a lower ideal if and only if $X = Free_{\leq}(N)$ for an antichain N .

Proof. Suppose that X is a lower ideal of Y and let M be the set of minimal elements in $Y \setminus X$. Then clearly M is an antichain, by minimality of its elements. To see that $X \subseteq \text{Free}_{\leq}(M)$, it suffices to show that for any $y \in X$, we have $x \not\leq y$ for all $x \in M$. Suppose, for contradiction, that $x \leq y$ for some $x \in M$. Then $x \in X$ by definition of a lower ideal, giving a contradiction. To see that $\text{Free}_{\leq}(M) \subseteq X$, it suffices to show that if $x \in \text{Free}_{\leq}(M)$, then $x \in X$. Suppose, for contradiction, that $x \in Y \setminus X$. Then there must be a minimal element $z \in Y \setminus X$ such that $z \leq x$, contradicting the fact that $x \in \text{Free}_{\leq}(M)$. Thus $X = \text{Free}_{\leq}(M)$ for the antichain M of minimal elements in $Y \setminus X$.

Now let us show the uniqueness of M as an antichain N such that $X = \text{Free}_{\leq}(N)$. Suppose that N is an antichain such that $X = \text{Free}_{\leq}(N)$. Clearly $N \subseteq Y \setminus X$, by definition of $\text{Free}_{\leq}(N)$. First we claim that $N \subseteq M$. Suppose that for some $x \in N$, there exists a $z \in Y \setminus X$ such that $z \neq x$ and $z \leq x$. But then, since N is an antichain, we must have $x' \not\leq z$ for all $x' \in N$ (otherwise $x' \leq z \leq x$). But then $z \in X$, contradicting our assumption. To see that $M \subseteq N$, we suppose, for contradiction, that $x \in M \setminus N$. But then $x \in X$, by minimality of x in $Y \setminus X$ together with the definition of $\text{Free}_{\leq}(N)$. This is a contradiction to the assumption that $x \in M$. Thus $N = M$, proving uniqueness of M .

Now suppose that $X = \text{Free}_{\leq}(N)$ for an antichain N . By the previous paragraph, $M \subseteq N$ (this part of the proof never used the assumption that X is a lower ideal). Let us show that X is a lower ideal. Suppose $x \in X$ and pick some $y \in Y$ such that $y \leq x$. Then $y \in X$, since otherwise there would exist some $z \in M$ for which $z \leq y \leq x$, contradicting the fact that $x \in \text{Free}_{\leq}(N)$. Thus X is a lower ideal. \square

Proposition 1.2.2 gives rise to the following definition:

Definition 1.2.8. *For any lower ideal X of a poset (Y, \leq) , the set M of forbidden elements of X , denoted by $M := \text{Forb}(X)$, is defined as the unique set of minimal elements in $Y \setminus X$. The set M is also the unique antichain such that $X = \text{Free}_{\leq}(M)$.*

1.3 Partial Orders on Sets of Graphs: Hereditary Graph Properties

There are three basic partial orders that are commonly applied to graphs:

1. G is an *induced subgraph* of H if G it can be obtained from H by a sequence of vertex deletions.

2. G is a *subgraph* of H if G can be obtained from H by a sequence of edge and vertex deletions.
3. G is a *minor* of H if G can be obtained from H by a sequence of edge and vertex deletions and edge contractions.

We will focus on the induced subgraph relation, which will be denoted by \leq without further notice. It is easy to check that it is a partial order on graphs. In Definition 1.2.6, we defined lower ideals on posets. For a graph class ordered by induced subgraphs, we use the following terminology:

Definition 1.3.1. *A graph property (or graph class) X is called hereditary if it is a lower ideal (of the class of all graphs) with respect to the induced subgraph relation. That is, whenever $G \in X$ and $H \leq G$, we have $H \in X$.*

Definition 1.3.2. *The corresponding lower ideals with respect to the subgraph and minor relations will be called monotone properties and minor-closed properties, respectively. With these definitions, it is immediately clear that any minor-closed property is monotone, and any monotone property is hereditary.*

The following proposition is a direct corollary of Proposition 1.2.2:

Proposition 1.3.1.

- *For any hereditary graph class X there is a unique set M of minimal (forbidden) graphs not in X .*
- *Equivalently, for any antichain M , we may define X as the maximal hereditary graph class not containing any graph in M .*
- *We use the notation $X := \text{Free}(M)$ and talk about the class of M -free graphs. We use the term set of forbidden elements to describe M . If M is finite, we say that X is finitely defined.*

In general, the problem of finding the forbidden induced subgraph characterization of a hereditary class is far from being trivial, as the example of perfect graphs shows [Chudnovsky et al., 2006].

The importance of the forbidden induced subgraph characterization of a hereditary class of graphs can be illustrated by the following example. In 1969, “Journal of Combinatorial Theory” published a paper entitled “An interval graph is a comparability graph” [Jean, 1969]. One year later, the same journal published another

paper entitled “An interval graph is not a comparability graph” [Fishburn, 1970], revealing a mistake in the earlier paper. With the induced subgraph characterization this mistake could not occur, because it is not difficult to check the following:

Proposition 1.3.2. *Consider two hereditary graph classes $X := \text{Free}(M)$ and $Y := \text{Free}(N)$. Then X is a subclass of Y if and only if for every $H \in N$, there exists some $G \in M$ such that $G \leq H$.*

Proof. First suppose that X is a subclass of Y . Suppose, for contradiction, that for some $H \in N$, there exists no $G \in M$ such that $G \leq H$. Then, by definition of X , we have $H \in X$. Since $H \notin Y$, we deduce that X is not a subclass of Y . This is a contradiction.

Conversely, suppose that for every $H \in N$, there exists some $G \in M$ such that $G \leq H$. Suppose, for contradiction, that some graph belongs to X , but not to Y . Pick a minimal such graph H . Then $H \in N$, by definition. Thus there exists some $G \in M$ such that $G \leq H$. This implies that $H \notin X$, which is a contradiction. \square

Therefore, given two hereditary classes of graphs and the induced subgraph characterization for both of them, it is a simple task to decide the inclusion relationship between them. Apparently, in 1969 the induced subgraph characterization was not available for interval or comparability graphs. Nowadays, it is available for both classes.

Definition 1.3.3. *For a graph class X , we define the complement class \overline{X} (or co- X) as follows: $\overline{X} := \{\overline{G} : G \in X\}$.*

The proof of the following proposition is trivial:

Proposition 1.3.3. *For any hereditary graph class $X := \text{Free}(M)$, we have $\overline{X} = \text{Free}(\overline{M})$.*

Let us give examples of some important hereditary graph classes:

Example 1.3.1.

- If $X = \text{Free}(C_3, C_4, C_5, \dots)$, then X is known as the class of forests. A connected forest is called a tree.
- If $X = \text{Free}(C_3, C_5, C_7, \dots)$, then X is known as the class of bipartite graphs. The vertex set V of any bipartite graph can be partitioned into two independent sets (A, B) .

- If $X = \text{Free}(C_4, C_5, C_6, \dots)$, then X is known as the class of chordal graphs. Another term for chordal graphs is triangulated graphs.
- If $X = \text{Free}(C_3, C_5, C_6, C_7, \dots)$, then X is known as the class of chordal bipartite graphs. Note that the only induced cycle permissible in a chordal bipartite graph is a C_4 .
- If $X = \text{Free}(P_4)$, then X is known as the class of cographs. This important and well-studied class is the closure of $\{K_1\}$ under the two operations of graph complementation and disjoint union.

1.4 Boundary Ideals

In Section 1.3, we pointed out that a hereditary graph class can be characterised in terms of minimal graphs that do not belong to the class. Let us ask the following question: is it possible to characterise a family of hereditary graph classes in terms of minimal classes that do not belong to the family? More formally, assume we are given a family of hereditary graph classes \mathcal{U} (the universe) and consider a subfamily $\mathcal{A} \subseteq \mathcal{U}$ with the property that if a class X belongs to \mathcal{A} then any subclass of X from the same universe also belongs to \mathcal{A} .

(Q) Is it possible to characterise the family \mathcal{A} in terms of minimal classes from \mathcal{U} that do not belong to \mathcal{A} ?

We will attempt to attack the question (Q) for various families \mathcal{A} of lower ideals of a poset.

Notation. We will refer to the following notations throughout this section:

- Let \leq^* be a partial order on a countable set S .
- Let \mathcal{U} (the *universe*) be the family of all subsets of S that are lower ideals with respect to \leq^* .
- Recall that by Proposition 1.2.2, any lower ideal X in \mathcal{U} is defined by a unique set M of forbidden elements. We denote $X := \text{Free}_{\leq^*}(M)$.
- We will consider a subfamily $\mathcal{A} \subseteq \mathcal{U}$ closed under taking subsets in \mathcal{U} .

1.4.1 Motivations

In order to give motivation for the main notion of this section, the notion of 'boundary ideals', let us first discuss some previously studied cases where the answer to question (Q) is known to be positive or negative.

Our first example is of combinatorial nature and deals with the notion of the speed of a hereditary property. The *speed of a hereditary property* is the number X_n of n -vertex graphs in a hereditary class X studied as a function of n . It is known [Balogh et al., 2000] that the family \mathcal{U} of all hereditary graph classes is partitioned with respect to the speed of classes into discrete layers. The lowest layer of this hierarchy contains finite classes of graphs, i.e. classes with finitely many graphs.

Let us quote Ramsey's theorem:

Theorem 1.4.1. *For any pair (m, n) of positive integers, there exists a positive integer $R(m, n)$ (called a Ramsey number) such that any graph on $R(m, n)$ vertices either contains a clique of size m or an independent set of size n .*

From Ramsey's theorem, it follows that there are two minimal classes of graphs that do not belong to the layer of finite classes: complete graphs and their complements (edgeless graphs). Both of these classes are infinite, and any class excluding at least one complete graph and one edgeless graph (i.e. any class of the form $\text{Free}(K_n, \overline{K}_m)$) is finite. All classes in all other layers are infinite, and there are infinitely many such layers. The first four lower layers containing infinite classes of graphs are [Scheinerman and Zito, 1994]:

- *constant* layer contains classes X with $\log_2 |X_n| = O(1)$,
- *polynomial* layer contains classes X with $\log_2 |X_n| = \Theta(\log_2 n)$,
- *exponential* layer contains classes X with $\log_2 |X_n| = \Theta(n)$,
- *factorial* layer contains classes X with $\log_2 |X_n| = \Theta(n \log_2 n)$.

Each of these layers contains a finite collection of minimal classes. For instance, in the factorial layer there are exactly nine minimal classes [Balogh et al., 2000]. Therefore, the family of subfactorial classes can be characterised by nine minimal classes that do not belong to this family, which gives an example of a positive answer to question (Q).

If we now move to the factorial layer, the question becomes much harder, because this layer is substantially richer. It contains plenty of graph classes of theoretical and practical importance, such as forests, interval, permutation, chordal bipartite,

threshold graphs, cographs, and even more generally, all minor-closed graph classes (other than the class of all graphs) [Norine et al., 2006]. Therefore, it would be interesting to characterize the factorial layer in terms of minimal superfactorial classes. However, none of such classes have been identified so far, and possibly, no such class exists. To better explain this phenomenon, let us consider the following example.

It is known that the class of bipartite graphs is superfactorial. Moreover, subclasses of bipartite graphs defined by forbidding

- (1) either large cycles, such as (C_{10}, C_{12}, \dots) -free bipartite graphs or (C_8, C_{10}, \dots) -free bipartite graphs,
- (2) or small cycles, such as C_4 -free bipartite graphs or (C_4, C_6) -free bipartite graphs,

are superfactorial. The first sequence can be extended by adding to it the class of chordal bipartite graphs, i.e. $(C_6, C_8, C_{10}, \dots)$ -free bipartite graphs, which is still superfactorial [Spinrad, 1995]. However, by adding to the set of forbidden graphs one more cycle, i.e. C_4 , we obtain the class of forests, which is factorial. On the contrary, the second sequence of graph classes can be extended to an infinite chain by forbidding more and more cycles. In other words, for any $k \geq 2$, the class of $(C_4, C_6, \dots, C_{2k})$ -free bipartite graphs is superfactorial [Lazebnik et al., 1995], and only the limit of this sequence, which is again the class of forests, is factorial. Therefore, in this sequence there is no minimal superfactorial class, which gives an example of a negative answer to question (Q).

Another important negative instance to question (Q) concerns the computational complexity of the so-called MAXIMUM INDEPENDENT SET problem in hereditary graph classes. The attempt to identify the 'minimal graph classes' for which the problem is not computationally simple (i.e. 'not polynomial-time solvable') is what originally motivated Alekseev to introduce the definition of 'boundary classes' in [Alekseev, 2003]. We will postpone the more detailed discussion of this family of graphs until Section 2.1, where we define algorithmic notions more formally. For now, let us extend the definition of boundary classes from graphs to ideals of arbitrary nature.

1.4.2 Definitions and tools

We now introduce a definition and a lemma that are prerequisites for defining the notion of a boundary ideal.

Definition 1.4.1. A non-empty subset X of S is called a *limit ideal* for the family \mathcal{A} (\mathcal{A} -limit for short) if and only if $X = \bigcap_{i=1}^{\infty} X_i$, where $X_1 \supseteq X_2 \supseteq \dots$ is a sequence of ideals that belong to $\mathcal{U} \setminus \mathcal{A}$.

If X is the limit ideal of a sequence $X_1 \supseteq X_2 \supseteq X_3 \supseteq \dots$, we say that the sequence *converges* to X . Observe that we do not require the ideals in the sequence $X_1 \supseteq X_2 \supseteq \dots$ to be distinct, which means that every ideal from $\mathcal{U} \setminus \mathcal{A}$ is \mathcal{A} -limit. On the other hand, this definition also allows some ideals that belong to \mathcal{A} to be limit for this family.

Alekseev showed (for sets of graphs) that every ideal $Y \in \mathcal{U} \setminus \mathcal{A}$ contains a minimal \mathcal{A} -limit ideal. We extend Alekseev's notion of 'boundary classes' (and the relevant proofs) to 'boundary ideals' within the more general framework of partially ordered sets [Alekseev, 2003]. To properly define this notion, we first need some lemmata.

Lemma 1.4.2. A finitely defined non-empty ideal is a limit ideal if and only if it belongs to $\mathcal{U} \setminus \mathcal{A}$.

Proof. Every ideal in $\mathcal{U} \setminus \mathcal{A}$ is a limit ideal by definition. Now let $X = \text{Free}(G_1, \dots, G_k)$ be a limit ideal and let $X_1 \supseteq X_2 \supseteq X_3 \supseteq \dots$ be a sequence of ideals from $\mathcal{U} \setminus \mathcal{A}$ converging to X . Obviously, there must exist a positive integer n such that X_n is (G_1, \dots, G_k) -free. But then for each $i \geq n$, we have $X_i = X$ and therefore X belongs to $\mathcal{U} \setminus \mathcal{A}$. \square

Lemma 1.4.3. If an ideal Y contains a limit ideal X , then Y is also a limit ideal.

Proof. Let $X_1 \supseteq X_2 \supseteq X_3 \dots$ be a sequence of ideals from $\mathcal{U} \setminus \mathcal{A}$ converging to X . Then the sequence $(X_1 \cup Y) \supseteq (X_2 \cup Y) \supseteq (X_3 \cup Y) \dots$ consists of ideals from $\mathcal{U} \setminus \mathcal{A}$ and it converges to Y . \square

Lemma 1.4.4. If a sequence $X_1 \supseteq X_2 \supseteq X_3 \dots$ of limit ideals converges to a non-empty ideal X , then X is also a limit ideal.

Proof. Let $\mathcal{G} := \{G_1, G_2, \dots\}$ be the set of minimal elements of S that do not belong to the ideal X . For each natural k , define $X^{(k)}$ to be the ideal $\text{Free}(G_1, \dots, G_k)$. Since no element in \mathcal{G} belongs to X , for every k there is an n such that X_n does not contain G_1, \dots, G_k , which means $X_n \subseteq X^{(k)}$. Therefore, by Lemma 1.4.3, $X^{(k)}$ is a limit ideal, and by Lemma 1.4.2, $X^{(k)}$ does not belong to \mathcal{A} . This is true for all natural k , and therefore, $X^{(1)} \supseteq X^{(2)} \supseteq X^{(3)} \dots$ is a sequence of ideals from $\mathcal{U} \setminus \mathcal{A}$ converging to X , i.e. X is a limit ideal. \square

Lemma 1.4.5. *Every ideal X from $\mathcal{U} \setminus \mathcal{A}$ contains a minimal limit ideal Y . Moreover, there exists a sequence of ideals from $\mathcal{U} \setminus \mathcal{A}$ converging to Y , such that each ideal in the sequence is a subset of X .*

Proof. Let X be an ideal from $\mathcal{U} \setminus \mathcal{A}$. To reveal a minimal limit ideal contained in X , let us fix an arbitrary linear order \mathcal{L} of elements of S and let us define a sequence $X_1 \supseteq X_2 \supseteq \dots$ of ideals as follows. We define X_1 to be equal X . For $i > 1$, let G be the first element of S in the order \mathcal{L} such that G belongs to X_{i-1} and $X_{i-1} \cap \text{Free}(G)$ is a limit ideal. If there is no such element G , we define $X_i := X_{i-1}$. Otherwise, $X_i := X_{i-1} \cap \text{Free}(G)$.

Denote by Y the intersection of ideals $X_1 \supseteq X_2 \supseteq X_3 \dots$. Clearly, $Y \subseteq X$. By Lemma 1.4.4, Y is a limit ideal. Let us show that Y is a minimal limit ideal contained in X . By contradiction, assume there exists a limit ideal Z which is properly contained in Y . Let H be an element of Y which does not belong to Z . Then $Z \subseteq Y \cap \text{Free}(H) \subseteq X_k \cap \text{Free}(H)$ for each k . Therefore, by Lemma 1.4.3, $X_k \cap \text{Free}(H)$ is a limit ideal for each k . For some k , the element H becomes the first element in the order \mathcal{L} such that $X_k \cap \text{Free}(H)$ is a limit ideal. But then $X_{k+1} := X_k \cap \text{Free}(H)$, and H belongs to no ideal X_i with $i > k$, which contradicts the fact that H belongs to Y . \square

Lemma 1.4.5 motivates the following key definition.

Definition 1.4.2. *A minimal limit ideal for \mathcal{A} is called a boundary ideal, boundary class for \mathcal{A} .*

The importance of this notion is due to the following theorem.

Theorem 1.4.6. *A finitely defined ideal belongs to \mathcal{A} if and only if it contains no boundary ideal for \mathcal{A} .*

Proof. From Lemma 1.4.5 we know that every ideal from $\mathcal{U} \setminus \mathcal{A}$ contains a boundary ideal. To prove the converse, consider a finitely defined ideal X containing a boundary ideal. Then, by Lemma 1.4.3, X is a limit ideal, and therefore, by Lemma 1.4.2, X does not belong to \mathcal{A} . \square

As a tool to detect minimality of a limit ideal, we will use the following helpful minimality criterion.

Lemma 1.4.7. *An \mathcal{A} -limit ideal $X = \text{Free}(M)$ is minimal (i.e. boundary) if and only if for every element $x \in X$ there is a finite set $T \subseteq M$ such that $\text{Free}(\{x\} \cup T) \in \mathcal{A}$.*

Proof. Suppose X is a boundary ideal for \mathcal{A} , and assume for contradiction that there is an element $x \in X$ such that for every finite set $T \subseteq M$ we have $\text{Free}(\{x\} \cup T) \in \mathcal{U} \setminus \mathcal{A}$. Let $M := \{m_1, m_2, \dots\}$ and $Z_i := \text{Free}(x, m_1, m_2, \dots, m_i)$. Then $Z_i \in \mathcal{U} \setminus \mathcal{A}$ for each i and therefore $Z := \cap_i Z_i$ is an \mathcal{A} -limit ideal. It contains no element from M and it does not contain x . Therefore, it is a proper subset of X , contradicting the minimality of X .

Conversely, assume that for every element $x \in X$ there is a finite set $T \subseteq M$ such that $\text{Free}(\{x\} \cup T) \in \mathcal{A}$, and suppose for contradiction that there exists an \mathcal{A} -limit ideal Z which is properly contained in X . Since Z is a limit ideal, there exists a sequence $Z_1 \supseteq Z_2 \supseteq \dots$ of ideals from $\mathcal{U} \setminus \mathcal{A}$ converging to Z . Pick any element $x \in X \setminus Z$ and a finite set $T \subseteq M$ such that $\text{Free}(\{x\} \cup T) \in \mathcal{A}$. Then there must exist a Z_n which is $(\{x\} \cup T)$ -free, in which case $Z_n \in \mathcal{A}$, since \mathcal{A} is closed under taking subsets in \mathcal{U} . This contradiction finishes the proof. \square

Convention

In this thesis, we will usually choose S to be the set of all graphs and \leq^ to be the induced subgraph relation \leq . In this case, \mathcal{U} is the family of all hereditary graph classes and \mathcal{A} is a subfamily closed under taking hereditary subclasses.*

Note. In further chapters, we will search for boundary ideals for three special types of families \mathcal{A} :

1. \mathcal{A} is the family of hereditary graph classes for which an algorithmic graph problem P is solvable in polynomial time.
2. \mathcal{A} is the family of hereditary graph classes for which the graph parameter called 'clique-width' is bounded.
3. \mathcal{A} is the family of hereditary graph classes that are well-quasi-ordered by the induced subgraph relation \leq .

Definitions and further details will be provided in the respective chapters.

It is non-trivial to decide the number of boundary ideals for a family \mathcal{A} and, indeed, to determine the structure of the boundary ideals. As we shall see in further chapters, Lemma 1.4.5 and Theorem 1.4.6 can often prove the existence of a boundary ideal, even when its structure remains unknown. Some families may even have an uncountable number of boundary ideals [Malyshev, 2009].

Chapter 2

Algorithmic Graph Problems

2.1 The Time Complexity of Algorithmic Graph Problems

We start with a few basic definitions:

Definition 2.1.1. *An algorithm (for a problem) is called polynomial-time if for an input of size n , the algorithm solves the problem in $p(n)$ elementary steps for some polynomial $p(n)$. The class of problems for which there exists a polynomial-time algorithm is usually denoted by P .*

Intuitively speaking, polynomial-time problems tend to have 'fast' algorithms, and the problems are thus considered to be 'easy'. Of course, the definitions of 'fast' and 'easy' are somewhat subjective and the truth of the matter depends on the degree of the polynomial in the algorithm for solving the problem. The typical size of the input could also be a concern. In any case, polynomial-time algorithms are often practical for real-world applications.

NP-complete problems have, for decades, defied all attempts to produce polynomial-time algorithms. Thus, it is commonly assumed and hypothesised that $NP \neq P$. Proving this hypothesis remains one of the biggest and most important open challenges in computer science. From the assumption that $NP \neq P$, we deduce that NP-complete problems are not polynomial-time solvable. In this thesis, we will generally make this assumption. There exist other useful time complexities in addition to P and NP , (indeed there exist entire hierarchies of them), however we will not consider them in this thesis.

2.2 Boundary Classes for Algorithmic Graph Problems

One of the typical ways to relax a difficult algorithmic graph problem is to restrict the class of input graphs. Literature contains thousands of results analyzing particular problems on various classes of graphs.

The main idea of the notion of boundary classes is to turn from the study of individual classes of graphs to that of families of graph classes. In such families, certain classes are critical in the sense that they separate “difficult” instances of a problem from “simple” ones. To give an example, consider the family of minor-closed classes. An important representative of this family is the class of planar graphs, (the class of graphs embeddable into the Euclidean plane), which is of interest both from theoretical and practical point of view. The theoretical importance of this class is partially due to the fact that many algorithmic problems, such as MAXIMUM INDEPENDENT SET or MINIMUM DOMINATING SET, are NP-hard in planar graphs. On the other hand, if a minor-closed class X excludes at least one planar graph, then many algorithmic problems, including the two mentioned before, are polynomial-time solvable for graphs in X . (This is because both the tree-width and clique-width are then bounded by a constant in such classes X . We will return to these notions in the next chapter.) Thus, the family of minor-closed graph classes that are “simple” in the above sense can be characterised by the unique minimal class which does not belong to this subfamily, namely the class of planar graphs.

Unfortunately, the restriction to minor-closed classes is not always justified in the study of algorithmic problems, since many classes that are important from an algorithmic point of view, such as bipartite graphs or graphs of bounded vertex degree, are hereditary but not minor-closed.

In this thesis, we attempt to classify hereditary graph classes according to whether certain algorithmic graph problems are polynomial-time solvable or not, when restricted to the classes. Due to Lemma 1.4.5 and Theorem 1.4.6, the notion of boundary classes can aid us in this pursuit. In this chapter, we will discuss boundary classes for families \mathcal{A} of hereditary classes for which an algorithmic graph problem R is solvable in polynomial time. We will talk about ‘boundary classes for R ’.

To increase the reader’s familiarity with the notion of boundary classes, let us consider an example that deals with the MAXIMUM INDEPENDENT SET problem. In this example, the universe \mathcal{U} is the family of all hereditary classes of graphs, and \mathcal{A} is the family of hereditary graph classes where the problem is polynomial-time solvable. In this example, and any other example of algorithmic nature, we assume

that $P \neq NP$, since otherwise the notion of boundary classes is not applicable.

It is known (see e.g. [Murphy, 1992]) that the problem is NP-complete in the class of (C_3, C_4, \dots, C_k) -free graphs for any particular value of k . By pushing the parameter k to infinity, we obtain a limit class for this problem, which is the class of graphs without cycles, i.e. the class of forests. However, the class of forests is not a minimal limit class for this problem, because from the same paper [Murphy, 1992] we know that the problem is NP-complete for graphs of vertex degree at most 3 in the class $\text{Free}(C_3, C_4, \dots, C_k)$. Therefore, the class of forests of degree at most three is a limit class for the problem in question. But this class again is not a minimal limit class, because Alekseev found in [Alekseev, 2003] a smaller limit class: the class of forests every connected component of which has at most 3 leaves. This class is of special interest in the study of boundary properties. Let us introduce a special notation for it:

Y : the class of forests every connected component of which has at most 3 leaves.

Alekseev also proved in [Alekseev, 2003] that Y is a minimal limit class, i.e. a boundary class, for the independent set problem. So far, this is the only boundary class known for this problem. But the importance of this class is not only due to this fact. This class also appears in many other problems.

For instance, Y is a boundary class for the MINIMUM DOMINATING SET problem. In terms of boundary classes, currently this is the most explored problem. The paper [Alekseev et al., 2004] describes three boundary classes for this problem. One of them is Y , the other is the class of line graphs of graphs in Y , and the third class is also related to the class Y . Remember that Y is a class of bipartite graphs, i.e. graphs partitionable into two independent sets. By replacing one of these sets by a clique we obtain a split graph, i.e. a graph partitionable into an independent set and a clique. The class of split graphs obtained in this way from graphs in Y is the third boundary class for the dominating set problem.

Y also is a boundary class for some other graph problems, not necessarily of algorithmic nature (see e.g. [Alekseev et al., 2004, 2007; Lozin, 2008]). However, this class is not boundary for every graph problem. For instance, the HAMILTONIAN CYCLE problem is not of this type. In Section 2.3, we discover the first two boundary classes for this problem.

In addition to the HAMILTONIAN CYCLE problem, we study two other algorithmic graph problems in this chapter: the k -PATH PARTITION problem and the DOMINATING INDUCED MATCHING problem. We identify some boundary classes for these problems and find polynomial-time solutions in some restricted classes of graphs.

2.3 The Hamiltonian Cycle Problem

In a graph, a Hamiltonian cycle is a cycle containing each vertex of the graph exactly once. Determining whether a graph has a Hamiltonian cycle is an NP-complete problem. Moreover, it remains NP-complete even if restricted to subcubic graphs, i.e. graphs of vertex degree at most 3. However, under some further restrictions, the problem may become polynomial-time solvable. A trivial example of this type is the class of graphs of vertex degree at most 2. Our goal is to distinguish boundary graph properties that make the problem difficult in subcubic graphs. In our study, we restrict ourselves to the properties that are hereditary in the sense that whenever a graph possesses a certain property the property is inherited by all induced subgraphs of the graph.

In [Alekseev et al., 2007], it was observed that there must exist at least five boundary classes of graphs for the Hamiltonian cycle problem, but none of them has been identified so far. We will discover the first two boundary classes for the problem in question.

If the degree of each vertex of G is exactly 3, we call G a *cubic* graph, and if the degree of G is at most 3, we call G *subcubic*. A vertex of degree 3 will be called a cubic vertex.

As in the study of other algorithmic problems, we will assume that $P \neq NP$, since otherwise the notion of boundary classes is not applicable. Our goal is to identify a boundary class of graphs for the family of hereditary classes where the HAMILTONIAN CYCLE problem is polynomial-time solvable. The hereditary classes of graphs that do not belong to this family will be called *HC-tough*.

2.3.1 Approaching a limit class

As we mentioned earlier, the HAMILTONIAN CYCLE problem is NP-complete for subcubic graphs [Itai et al., 1982]. Recently, it was shown in [Alekseev et al., 2007] and [Arkin et al., 2007] that the problem is NP-complete for graphs of large girth, i.e. graphs without small cycles. In this section, we strengthen both these results. First, we show that the problem is NP-complete in the class of subcubic graphs, in which every cubic vertex has a non-cubic neighbor. Throughout the section, we denote this class by Γ .

Lemma 2.3.1. *The HAMILTONIAN CYCLE problem is NP-complete in the class Γ .*

Proof. It was proved in [Plesník, 1979] that the HAMILTONIAN CYCLE problem is NP-complete in the class of directed graphs, where every vertex has either indegree

1 and outdegree 2, or indegree 2 and outdegree 1. The lemma is proved by a reduction from the HAMILTONIAN CYCLE problem on such graphs, which we call *Plesník graphs*. Given a Plesník graph H , we associate with it an undirected graph from Γ as follows. First, we consider all the *prescribed edges* of H , i.e. directed edges $u \rightarrow v$, such that either u has outdegree 1, or v has indegree 1 (or both). We replace every such edge by a *prescribed path* $u \rightarrow w \rightarrow v$, where w is a new node of indegree and outdegree 1. Then, we erase orientation from all edges, and denote the resulting undirected graph by G .

Clearly, $G \in \Gamma$. Assume H has a directed Hamiltonian cycle. Then the corresponding edges of G form a Hamiltonian cycle in G . Conversely, if G has a Hamiltonian cycle, then it must contain all the prescribed paths, and therefore the corresponding Hamiltonian cycle in H respects the orientation of the edges. \square

Now we strengthen Lemma 2.3.1 as follows. Denote by $Y_{i,j,k}$ the graph represented in Figure 2.1 and call any graph of this form a tribranch. Also, denote $\mathcal{Y}_p = \{Y_{i,j,k} : i, j, k \leq p\}$ and $\mathcal{C}_p = \{C_k : k \leq p\}$. Finally, let \mathcal{X}_p be the class of $\mathcal{C}_p \cup \mathcal{Y}_p$ -free graphs in Γ .

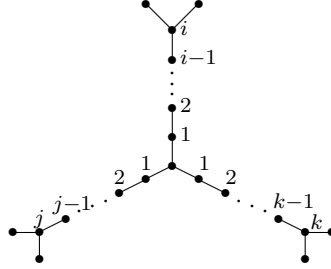


Figure 2.1: A tribranch $Y_{i,j,k}$

Lemma 2.3.2. *For any $p \geq 1$, the HAMILTONIAN CYCLE problem is NP-complete in the class \mathcal{X}_p .*

Proof. We reduce the problem from the class Γ to $\mathcal{C}_p \cup \mathcal{Y}_p$ -free graphs in Γ . Let G be a graph in Γ . Obviously, every edge of G incident to a vertex of degree 2 must belong to any Hamiltonian cycle in G (should G have any). Therefore, by subdividing each of such edges with p new vertices we obtain a graph $G' \in \Gamma$ which has a Hamiltonian cycle if and only if G has. It is not difficult to see that G' is \mathcal{Y}_p -free. Moreover, G' has no small cycles (i.e. cycles from \mathcal{C}_p) containing at least one vertex of degree 2. If G' has a cycle $C \in \mathcal{C}_p$ each vertex of which has degree 3, we apply to any vertex a_0

of C the transformation F_p represented in Figure 2.2, where a_3 denotes a non-cubic neighbor of a_0 . We claim that F_p transforms G' into a new graph in Γ , which has a Hamiltonian cycle if and only if G has. To see this, note that any Hamiltonian cycle in G' contains exactly one of the three paths (a_1, a_0, a_3) , (a_1, a_0, a_2) and (a_2, a_0, a_3) . Then, clearly, G' has a Hamiltonian cycle if and only if the transformed graph has a Hamiltonian cycle containing exactly one of the following three paths:

- $(a_1, x_{p+1}, x_p, \dots, x_2, x_1, x_0, y_{p+1}, y_p, \dots, y_2, y_1, y_0, a_0, a_3)$
- $(a_1, x_{p+1}, x_p, \dots, x_2, x_1, x_0, a_0, y_0, y_1, y_2, \dots, y_p, y_{p+1}, a_2)$
- $(a_2, y_{p+1}, y_p, \dots, y_2, y_1, y_0, x_{p+1}, x_p, \dots, x_2, x_1, x_0, a_0, a_3)$

Moreover, the transformation F_p increases the length of C without producing any new cycle from \mathcal{C}_p or any tribranch from \mathcal{Y}_p . Repeated applications of this transformation allow us to get rid of all small cycles. Thus, any graph G in Γ can be transformed in polynomial time into a $\mathcal{C}_p \cup \mathcal{Y}_p$ -free graph in Γ , which has a Hamiltonian cycle if and only if G has. Together with the NP-completeness of the problem in the class Γ , this proves the lemma. \square

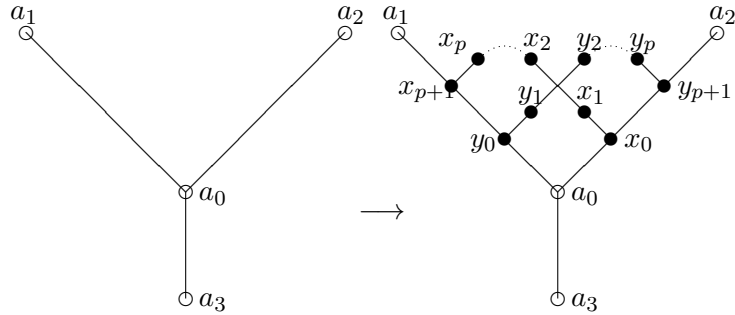


Figure 2.2: Transformation F_p

2.3.2 Limit class

The results of the previous section show that $\bigcap_{p \geq 1} \mathcal{X}_p$ is a limit class for the HAMILTONIAN CYCLE PROBLEM. Throughout the section we will denote this class by \mathcal{X} . In the present section, we describe the structure of graphs in the class \mathcal{X} . Let us define a *caterpillar with hairs of arbitrary length* to be a subcubic tree in which all cubic vertices belong to a single path. An example of a caterpillar with hairs of arbitrary length is given in Figure 2.3.

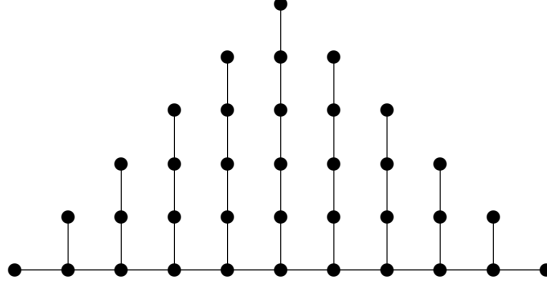


Figure 2.3: A caterpillar with hairs of arbitrary length

Lemma 2.3.3. *A graph G belongs to the class \mathcal{X} if and only if every connected component of G is a caterpillar with hairs of arbitrary length.*

Proof. If every connected component of G is a caterpillar with hairs of arbitrary length, then G is a subcubic graph without induced cycles or tribranches. Therefore, G belongs to \mathcal{X} .

Conversely, let G be a connected component of a graph in \mathcal{X} . Then, by definition, G is a subcubic tree without tribranches. If G has at most one cubic vertex, then obviously G is a caterpillar with hairs of arbitrary length. If G has at least two cubic vertices, then let P be an induced path of maximum length connecting two cubic vertices, say v and w . Suppose there is a cubic vertex u that does not belong to P . The path connecting u to P meets P at a vertex different from v and w (since otherwise P would not be maximum). But then a tribranch arises. This contradiction shows that every cubic vertex of G belongs to P , i.e., G is a caterpillar with hairs of arbitrary length. \square

In the next section, we will prove that \mathcal{X} is a minimal limit class for the Hamiltonian cycle problem. Without loss of generality, we will restrict ourselves to those graphs in \mathcal{X} every connected component of which has the following “canonical” form: T_d ($d \geq 2$) is a caterpillar with a path of length $2d$ (containing all cubic vertices) and $2d - 1$ consecutive hairs of lengths $1, 2, \dots, d - 1, d, d - 1, \dots, 2, 1$. Figure 2.3 represents the graph T_5 . The following lemma is obvious.

Lemma 2.3.4. *Every graph in \mathcal{X} is an induced subgraph of T_d for some $d \geq 2$.*

2.3.3 Minimality of the limit class

The proof of minimality of the class \mathcal{X} will follow from the following application of Lemma 1.4.7:

Lemma 2.3.5. *If for every graph G in \mathcal{X} , there is a constant $p = p(G)$, such that the HAMILTONIAN CYCLE problem can be solved in polynomial time for G -free graphs in \mathcal{X}_p , then \mathcal{X} is boundary for the problem.*

We apply Lemma 2.3.5 to prove the key result of this section.

Lemma 2.3.6. *For each graph $T \in \mathcal{X}$, there is a constant p such that the HAMILTONIAN CYCLE problem can be solved in polynomial time for T -free graphs in \mathcal{X}_p .*

Proof. By Lemma 2.3.4, T is an induced subgraph of T_d for some d . We define $p = 3 \times 2^d$, and will prove the lemma for T_d -free graphs in \mathcal{X}_p . Obviously, this class contains all T -free graphs in \mathcal{X}_p .

Let G be a T_d -free graph in \mathcal{X}_p . First we check if G has a vertex of degree 1. If such a vertex exists, there is no Hamiltonian cycle in G . Now suppose that G has no vertices of degree 1, so we assume that every vertex of G has degree 2 or 3.

For each vertex v of degree 2 in G , note that both edges incident to v must belong to all Hamiltonian cycles of G , should any exist. We label an edge of G *good* if we can argue that it must belong to all Hamiltonian cycles of G . Conversely, we label an edge *bad* if we can argue that it cannot belong to any Hamiltonian cycle of G . So we start by labelling edges good whenever they are incident to a vertex of degree 2 in G .

For each cubic vertex v in G , we claim that there is a polynomial-time algorithm to label at least two edges incident to v to be good (or the algorithm returns as output that the graph has no Hamiltonian cycles). Let us first show that this suffices to prove the lemma. Note that if for some vertex v of G , we determine three good edges incident to v , then clearly G has no Hamiltonian cycle. Otherwise, if for every vertex $v \in G$, we determine exactly two good edges incident to v , the good edges clearly partition the graph into a collection of disjoint cycles. If this collection contains exactly one cycle, then this is a Hamiltonian cycle contained in G . If the collection contains more than one cycle, then by our definition of good, there are no Hamiltonian cycles in G .

For an arbitrary cubic vertex v of G , we now attempt to label at least two edges incident to v to be good. It suffices to repeat these steps for each cubic vertex v of G . If, at any step, there is a labelling conflict, i.e. if we relabel a good edge to be bad, or vice versa, it is clear that the graph G does not contain any Hamiltonian cycles and we can stop the procedure.

In order to apply the procedure at a cubic vertex v of G , we start by showing that the graph has a simple structure locally, around v . Denote by H the subgraph

of G induced by the set of vertices of distance at most d from v . Since the degree of each vertex of H is at most 3, the number of vertices in H is less than p . Since H belongs to \mathcal{X}_p , it cannot contain small cycles and small tribranches (i.e. graphs from the set $\mathcal{C}_p \cup \mathcal{Y}_p$). Moreover, H cannot contain large cycles and large tribranches, because the size of H is too small (less than p). Therefore, H belongs to \mathcal{X} , and obviously H is connected. Thus, H is a caterpillar with hairs of arbitrary length. Observe that each leaf u in H is at distance exactly d from v , since otherwise u has degree 1 in G . We now start the procedure:

1. Let P be a path in H connecting two leaves and containing all vertices of degree 3 in H .
2. If every vertex of P (except the endpoints) has degree 3, then $H = T_d$, which is impossible because G is T_d -free. Therefore, P must contain a vertex of degree 2. Let v_i be such a vertex closest to v , and let $(v = v_0, v_1, \dots, v_i)$ be the path connecting v_i to $v = v_0$ (along P).
3. The edge $v_i v_{i-1}$ has already been labelled good, as it is incident to a vertex of degree 2. By the choice of v_i , the vertex v_{i-1} has degree 3, and hence it has a neighbor u of degree 2 that does not belong to P . Therefore, the edge $u v_{i-1}$ has already been labelled good.
4. If the edge $v_{i-1} v_{i-2}$ belonged to a Hamiltonian cycle in G , there would be three edges incident to v_{i-1} , all belonging to the same Hamiltonian cycle. This would be impossible, so we label $v_{i-1} v_{i-2}$ to be bad.
5. Now we label other edges incident to v_{i-2} to be good, since if G contains a Hamiltonian cycle, the cycle must contain the vertex v_{i-2} , without containing the edge $v_{i-1} v_{i-2}$. So in particular, $v_{i-2} v_{i-3}$ is then labelled good.
6. We then label $v_{i-3} v_{i-4}$ to be bad, similarly to step 4. Inductively, we label the edges of the path $(v = v_0, v_1, \dots, v_i)$ good and bad, alternately.
7. If the edge $v_0 v_1$ is labelled bad, then any other edges incident to $v = v_0$ can be labelled good, and so we have labelled two edges incident to v to be good. Otherwise, both $v_0 v_1$ and the edge connecting v to the vertex of degree 2 outside P are both labelled good, so again, we have labelled two edges incident to v to be good.

Repeating this consideration for each cubic vertex v of G , we either reach a labelling conflict (relabelling a good edge to be bad, or vice versa) or we have labelled at least

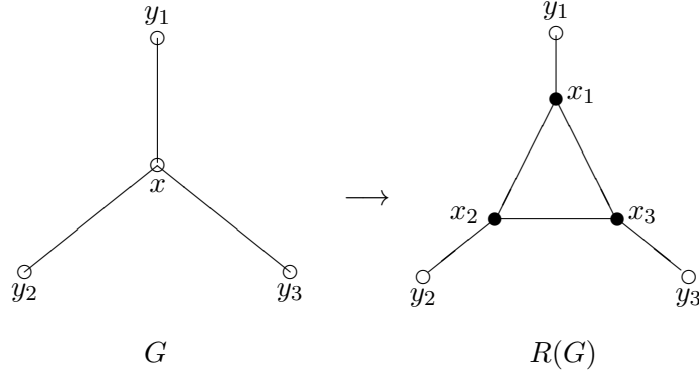


Figure 2.4: Transformation R

two edges incident to each vertex v to be good, in polynomial time. This completes the proof of the lemma. \square

From Lemmas 2.3.5 and 2.3.6 we conclude that

Theorem 2.3.7. *\mathcal{X} is a boundary class for the HAMILTONIAN CYCLE problem.*

2.3.4 One more boundary class

To obtain one more boundary class, we use the transformation R represented in Figure 2.4. It is not difficult to see that a graph G has a Hamiltonian cycle if and only if $R(G)$ has. Let us denote by $R(\mathcal{X})$ the class of graphs obtained from graphs in \mathcal{X} by application of transformation R to each cubic vertex.

Theorem 2.3.8. *$R(\mathcal{X})$ is a boundary class for the HAMILTONIAN CYCLE problem.*

2.3.5 Concluding remarks and related open problems

We revealed the first two boundary classes of graphs for the HAMILTONIAN CYCLE problem. The existence of one more boundary class for this problem arises from the fact that HAMILTONIAN CYCLE is NP-complete in the class of chordal bipartite graphs (i.e. in the class $\text{Free}(C_3, C_5, C_6, C_7 \dots)$) [Müller, 1996]. This fact implies that there must exist a boundary subclass of chordal bipartite graphs, i.e. a class Z together with a sequence $Z_1 \subseteq Z_2 \subseteq Z_3 \dots$ of subclasses of chordal bipartite graphs such that $Z = \cap Z_i$ and the problem is NP-complete in each class in the sequence $Z_1 \subseteq Z_2 \subseteq Z_3 \dots$. In fact, \mathcal{X} is a subclass of chordal bipartite graphs. But we claim that \mathcal{X} is not equal to Z . Indeed, each class Z_i in the sequence must contain a

C_4 , since otherwise Z_i is a subclass of forests where the problem is polynomial-time solvable. But if each class contains a C_4 , then Z also must contain a C_4 , which is not the case for the class \mathcal{X} . Some hints regarding the structure of graphs in a boundary class of chordal bipartite graphs are given in the following two observations.

Observation. Let $Z_1 \subseteq Z_2 \subseteq Z_3 \dots$ be a sequence of subclasses of chordal bipartite graphs such that the HAMILTONIAN CYCLE problem is NP-complete in each class in the sequence. Then the class $Z = \cap Z_i$ must contain a fork F_p (the graph obtained from a star $K_{1,p}$ by subdividing one edge exactly once) for all values of p and a domino (the graph obtained from a chordless cycle C_6 by adding an edge connecting two vertices of distance 3).

Proof. Every connected domino-free chordal bipartite graph is distance-hereditary [Bandelt and Mulder, 1986], and the clique-width of distance-hereditary graphs is at most 3 [Golumbic and Rotics, 2000]. Also, the clique-width is bounded by a constant in the class of F_p -free chordal bipartite graphs for any value of p [Lozin and Rautenbach, 2004a]. It is known [Borie et al., 2009] that the HAMILTONIAN CYCLE problem can be solved for graphs of bounded clique-width in polynomial time. Therefore, each class in the sequence $Z_1 \subseteq Z_2 \subseteq Z_3 \dots$ must contain a domino and all forks F_p . Consequently, the class $Z = \cap Z_i$ must contain a domino and all forks F_p . \square

Finally, we observe that for each boundary class of bipartite graphs, there must exist a respective class of split graphs. Indeed, a bipartite graph $G = (V_1, V_2, E)$ has a Hamiltonian cycle only if $|V_1| = |V_2|$. If in such a graph we replace V_1 (or V_2) by a clique, then the split graph obtained in this way has a Hamiltonian cycle if and only if G has. Therefore, any result on the HAMILTONIAN CYCLE problem in bipartite graphs can be transformed into a respective result in split graphs.

2.4 The k -Path Partition Problem

In this section we study an algorithmic graph problem known as the k -path partition problem:

Definition 2.4.1. *The k -path partition problem (k -PP) is, given a graph G , the problem of finding a minimum number of vertex-disjoint (not necessarily induced) paths of length at most k that partition $V(G)$.*

The k -path partition problem has several real-life applications, for instance in the field of broadcasting in computer and communication networks [Yan et al., 1997]. The problem is known to be NP-complete in the class of all graphs [Garey and Johnson, 1979]. To get an intuition for possible applications, one might consider the problem of minimising the number of postal delivery vans needed to service a city, where each van can only service a limited amount of customers (or can only drive a limited distance) on its daily route, visiting each customer at most once. Clearly this problem can be made to correspond to minimising the number of vans needed to service all customers.

Let us also introduce a useful variant of this problem, called the P_k -partition problem:

Definition 2.4.2. *The P_k -partition problem is, given a graph G , the decision problem of deciding whether $V(G)$ can be partitioned into vertex-disjoint subgraphs isomorphic to P_k .*

Each of the above two algorithmic problems has an 'induced variant', (i.e. the induced k -path partition problem and the induced P_k -partition problem), each defined by the additional requirement that the paths in the partitions must be induced subgraphs of the underlying graph.

In order to highlight the usefulness of the P_k -partition problem, we note that whenever this problem is NP-hard on a graph class X , then the k -path partition problem must also be NP-hard on X . A similar statement obviously holds for the induced variants of the two problems, respectively.

An overview of the complexity status of the k -path partition problem for various graph classes is given in Figure 2.5. It is of particular note that although the problem is known to be NP-complete on the class of convex graphs [Asdre and Nikopoulos, 2007] (a superclass of biconvex graphs), and polynomial-time solvable for bipartite permutation graphs [Steiner, 2003] (a subclass of biconvex graphs), the complexity status remains an open problem for the class of biconvex graphs.

2.4.1 A boundary class

In a paper by Steiner, the author used a reduction from EXACT COVER BY 3-SETS to show that the 3-path partition problem is NP-complete on comparability graphs [Steiner, 2003]. Later, similar ideas were used in [Monnot and Toulouse, 2007], with a reduction from k -DM (the k -dimensional matching problem), to prove that the P_k -partition problem (and the induced P_k -partition problem) remains NP-complete

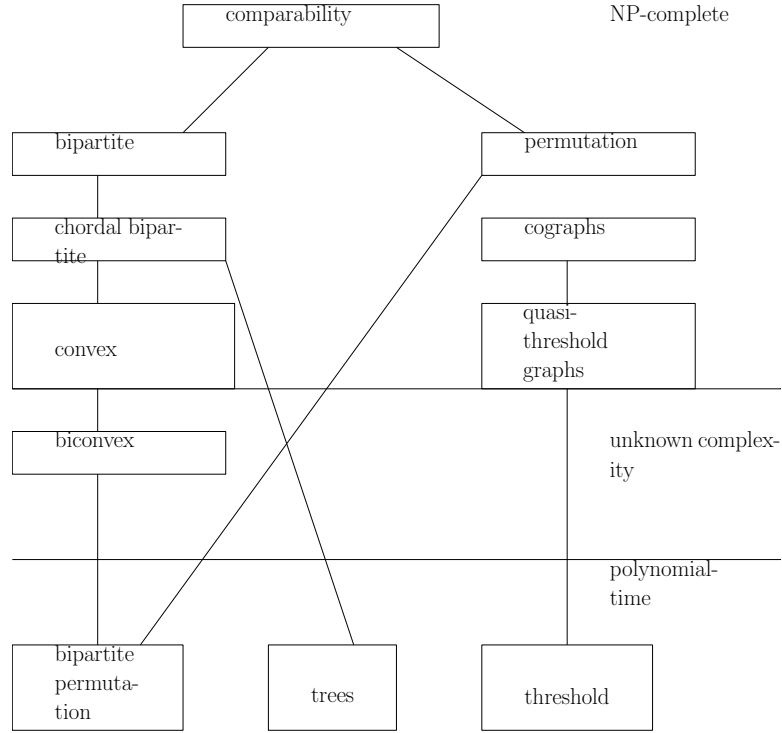


Figure 2.5: The computational complexity of the k -path partition problem

on bipartite graphs of maximum degree 3, for any fixed $k \geq 3$. As discussed in the previous section, this is enough to show NP-completeness of the k -path partition problem for the same graph class. In this section we will extend the latter proof with the aim of discovering the first boundary class for the k -path partition problem (k -PP).

A graph class X will be called k -PP-*easy* if the k -path partition problem is polynomial-time solvable for graphs in X , and k -PP-*tough* otherwise. If $P \neq NP$, the family of k -PP-tough classes is disjoint from that of k -PP-easy classes, in which case the problem of characterisation of these two families arises. By analogy with the induced subgraph characterisation of hereditary classes, we want to characterise the family of k -PP-easy classes in terms of minimal classes that do not belong to this family. Unfortunately, a k -PP-tough class may contain infinitely many k -PP-tough subclasses, which makes the task of finding minimal k -PP-tough classes impossible. To overcome this difficulty, we employ the notion of a boundary class, which can be defined (in this context) as follows.

A class of graphs \mathcal{S} will be called a *limit class* for the k -path partition problem if $\mathcal{S} = \bigcap_{i=1}^{\infty} \mathcal{S}_i$, where $\mathcal{S}_1 \supseteq \mathcal{S}_2 \supseteq \dots$ is a sequence of k -PP-tough classes. A minimal

limit class will be a *boundary* class for the problem in question.

We define H_i and $S_{i,j,k}$ as the graphs represented in Figure 2.6.

Definition 2.4.3. We define S_i to be the class of $(C_3, C_4, \dots, C_i, H_1, H_2, \dots, H_i)$ -free bipartite graphs of maximum degree 3.

Lemma 2.4.1. Let G be a graph and e an edge in G . If G' is the graph obtained from G by subdividing the edge e exactly by mk times, for some positive integers k and m , then G has a P_k -partition if and only if G' has a P_k -partition.

Proof. Denote the endpoints of e by a and b . In G' , we denote the subdivided e by $S := (a, s_1, s_2, \dots, s_{mk}, b)$.

First suppose that G has a P_k -partition \mathcal{P} . If e does not belong to any subgraph P_k in the partition, then G' has a P_k -partition \mathcal{P}' , which we define as the union of \mathcal{P} with the m disjoint copies of P_k that cover S in G' . So we may assume that e belongs to some P_k in \mathcal{P} , say P .

We claim that one can construct a P_k -partition \mathcal{P}' of G' by replacing P with $m + 1$ disjoint copies of P_k . Suppose $P = (p_1, p_2, \dots, p_i, a, b, q_1, q_2, \dots, q_j)$, where $i + j + 2 = k$. Then we let

$$(p_1, p_2, \dots, p_i, a, s_1, s_2, \dots, s_{j+1}) \text{ and } (s_{mk-i}, s_{mk-i+1}, \dots, s_{mk}, b, q_1, q_2, \dots, q_j)$$

be two of the $m + 1$ paths to replace P . It remains to find a P_k -partition of the path $(s_{j+2}, s_{j+3}, \dots, s_{mk-i-1})$, i.e. a path on $mk - (i + 1) - (j + 1) = (m - 1)k$ vertices. There is a unique way to partition $P_{(m-1)k}$ into $m - 1$ copies of P_k .

Conversely, suppose that G' has a P_k -partition \mathcal{P}' . If \mathcal{P}' contains a P_k -partition of S , we can just delete its members from \mathcal{P}' to construct a P_k -partition \mathcal{P} of G . Otherwise, \mathcal{P}' must contain two disjoint k -paths of the form

$$(p_1, p_2, \dots, p_i, a, s_1, s_2, \dots, s_{j+1}) \text{ and } (s_{mk-i}, s_{mk-i+1}, \dots, s_{mk}, b, q_1, q_2, \dots, q_j)$$

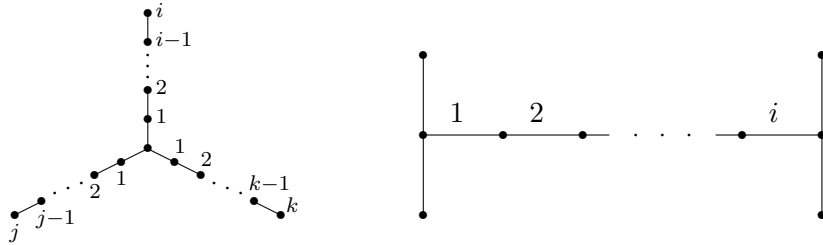


Figure 2.6: Graphs $S_{i,j,k}$ (left) and H_i (right)

where $i + j + 2 = k$ (if the equation did not hold, it would be impossible to partition the rest of the vertices in S into copies of P_k , contradicting the existence of \mathcal{P}'). In this case, we just delete these two paths, as well as the $m - 1$ paths contained in S from \mathcal{P}' . Finally we add to \mathcal{P}' a single path $P := (p_1, p_2, \dots, p_i, a, b, q_1, q_2, \dots, q_j)$. This gives us a P_k -partition \mathcal{P} of G . \square

Lemma 2.4.2. *S_i is k -PP-tough for each $i \geq 3$.*

Proof. Assuming that $\text{NP} \neq \text{P}$, it suffices to show that the P_k -partition problem is NP-complete on S_i . To this end, choose any positive integer m such that $mk \geq i$. Since we know that the P_k -partition problem is NP-complete on bipartite graphs of maximum degree 3, it suffices to reduce each instance of that problem to an instance of the P_k -partition problem on S_i . Given any bipartite graph G of maximum degree 3, we perform mk subdivisions on each edge of G , resulting in a new graph. Denote this new graph by G'' . By repeated applications of Lemma 2.4.1, we know that G'' has a P_k -partition if and only if G does. Furthermore, G'' clearly belongs to S_i . This completes the reduction. \square

Lemma 2.4.2 implies that S_i is a k -PP-tough class for any i . Therefore, $\mathcal{S} := \bigcap_{i \geq 3} S_i$ is a limit class for the k -path partition problem. It is easy to see that the graphs in \mathcal{S} are precisely the graphs of maximum degree at most 3, each connected component of which is a tree with at most one cubic vertex, i.e. a graph of the form $S_{i,j,k}$ displayed in Figure 2.6.

Our aim is to show that \mathcal{S} is a minimal limit class. To this end, we use Lemma 1.4.7, which is reproduced here for the convenience of the reader.

Lemma 2.4.3. *An \mathcal{A} -limit ideal $X = \text{Free}(M)$ is minimal (i.e. boundary) if and only if for every element $x \in X$ there is a finite set $T \subseteq M$ such that $\text{Free}(\{x\} \cup T) \in \mathcal{A}$.*

We will apply Lemma 2.4.3 in the case where $X = \mathcal{S}$ and \mathcal{A} is the family of k -PP-easy graph classes.

Lemma 2.4.4. *Let $G \in \mathcal{S}$ and suppose G has s connected components. Choose a positive integer constant t such that each connected component of G is an induced subgraph of $S_{t,t,t}$, i.e. $G \leq sS_{t,t,t}$. Then the class*

$$\mathcal{F} := \text{Free}(G, K_{1,4}, C_3, \dots, C_{2t+1}, H_1, \dots, H_{2t+1})$$

is k -PP-easy.

Proof. For the purposes of our proof, we may assume that $G = sS_{t,t,t}$. We claim the following:

Claim 2.4.5. *Let \mathcal{T} be the class of graphs whose each connected component contains at most one cycle. If the k -path partition problem is polynomial-time solvable for \mathcal{T} , then the k -path partition problem is polynomial-time solvable for \mathcal{F} .*

Let us first show that the claim suffices to imply the Lemma. To do this, we prove that for any $T \in \mathcal{T}$, it is possible to find a minimum k -path partition of T in polynomial time. For this purpose, we can clearly assume that T is connected (we could otherwise consider each connected component of T in turn). If T is a tree, we can apply a result from [Yan et al., 1997] stating that the k -path partition problem is polynomial-time solvable for trees. If T contains exactly one cycle, this cycle must be an induced subgraph of T . Choose any vertex v on the cycle. For any possible k -path partition of T , its members must avoid at least one edge on the cycle which is at distance of at most $k/2$ from v . By altering which one of these $k+1$ edges is deleted, we can create $k+1$ different trees. We may assume that $k+1 \leq n := |V(T)|$ (since T certainly cannot have a path of length greater than n). Thus there are at most n different trees to check, each of which can be checked in polynomial time. Thus the claim implies the lemma.

We proceed to prove the claim, with the aim of inductively reducing each graph $F \in \mathcal{F}$ to at most $c(s) := 3^s$ graphs whose each connected component has at most one cycle. Suppose that $F \in \mathcal{F}$ has a connected component with at least two cycles. Then, by assumption, the connected component must contain two distinct induced cycles $C := C_r$ and $C' := C_l$ such that $r, l \geq 2t+2$.

Choose a vertex w of C' that does not lie in C . Suppose v is a vertex of C that minimises $d(v, w)$, and let P' be the minimal induced path joining v and w . We claim that there exists a copy of $S_{t,t,t}$ in F , centered at v .

Clearly P' is disjoint from $C \setminus \{v\}$, by definition of v . If $d(v, w) \geq t$, then it is easy to see that F contains an induced copy of $S_{t,t,t}$, centered at v . Now assume that $d(v, w) < t$. Clearly there are two disjoint induced copies of P_t in C' , each starting at w . Let us denote these two paths by P_1 and P_2 . At least one of the two paths, say P_1 , is disjoint from P' (otherwise F would contain an induced cycle on less than $2t+2$ vertices, contradicting our assumption). So there exists a subpath P'' of $P' \cup P_1$, of length $t+1$ and starting at v . Then P'' is disjoint from $C \setminus \{v\}$ (otherwise F would contain an induced cycle on less than $2t+2$ vertices, contradicting our assumption). Also in this case, F clearly contains an induced copy of $S_{t,t,t}$, centered at v . Thus, in any case, there exists a copy of $S_{t,t,t}$ centered at v .

For any possible k -path partition of F , its members must avoid at least one neighbor of v . By altering which of these k edges is deleted, we can create 3 graphs F_2 .

Now for each of the three choices of F_2 , supposing that F_2 has a connected component containing at least two cycles, we can similarly find a cycle $C_2 \in F_2$ and a vertex $v_2 \in C_2$ such that there is a copy of $S_{t,t,t}$ centered at v_2 . Furthermore we may assume $v_2 \neq v$, since v is of degree less than 3 in F_2 , by construction. We can then proceed to create 3 graphs F_3 .

Inductively, for each possible sequence (F, F_2, \dots, F_i) , supposing that F_i has a connected component containing at least two cycles, we can find a cycle $C_i \in F_i$ and a vertex $v_i \in C_i$ such that there is a copy of $S_{t,t,t}$ centered at v_i . Furthermore we may assume $v_i \notin \{v_1, \dots, v_{i-1}\}$, since the vertices of $\{v_1, \dots, v_{i-1}\}$ are all of degree less than 3 in F_i , by construction. We can then proceed to create 3 graphs F_{i+1} .

We note that any two copies of $S_{t,t,t}$ with different central vertices in F are disjoint and without any edges between them. This follows directly from the fact that F is (H_1, \dots, H_{2t+1}) -free. Furthermore, since F is (C_3, \dots, C_{2t+1}) -free, edge deletions cannot create any new induced copies of $S_{t,t,t}$ in F ; i.e. whenever F contains $S_{t,t,t}$ as a subgraph, it must contain it as an induced subgraph. In any sequence (F, F_2, \dots, F_{s+1}) , we have found s disjoint induced copies of $S_{t,t,t}$, such that there are no edges between any two of them, contradicting the assumption that F is G -free. Thus there are at most 3^s sequences (F, F_2, \dots, F_j) , where $j \leq s$, and F_j is a graph whose each connected component contains at most one cycle.

This concludes the proof of the claim, which in turn implies the Lemma. \square

Lemmata 2.4.3 and 2.4.4 together imply the following theorem.

Theorem 2.4.6. *\mathcal{S} is a boundary class for the k -path partition problem.*

2.4.2 Concluding remarks and related open problems

We revealed the first boundary class of graphs for the k -path partition problem. The existence of one more boundary class for this problem arises from the fact that the problem is NP-complete in the class of convex graphs (which is a subclass of chordal bipartite graphs, i.e. the class $Free(C_3, C_5, C_6, C_7 \dots)$) [Asdre and Nikopoulos, 2007]. This fact implies that there must exist a boundary subclass of convex graphs, i.e. a minimal class X defined by a sequence $X_1 \supseteq X_2 \supseteq X_3 \dots$ of subclasses of convex graphs such that $X = \cap X_i$ and the problem fails to be polynomial-time solvable in each class of the sequence $X_1 \supseteq X_2 \supseteq X_3 \dots$.

There also remain some interesting graph classes for which the complexity status of k -PP is open. The path partition problem is different from the k -path partition problem in that there is no upper bound on the lengths of the paths in the desired partition. In [Yaeh and Chang, 1998], it was shown that the path partition problem is polynomial-time solvable in the class of bipartite distance-hereditary graphs. (The proof uses a similar technique to that of the proof that k -PP is polynomial-time solvable for trees [Yan et al., 1997].) The k -path partition problem, however remains of unknown complexity on this class. Also, as mentioned in the previous section, the complexity of k -PP is also unknown for the class of biconvex graphs.

2.5 The Dominating Induced Matching Problem

Given an edge e in a graph G , we say that e *dominates* itself and every edge sharing a vertex with e . An *induced matching* in G is a subset of edges such that each edge of G is dominated by at most one edge of the subset. In this section, we study the problem of determining whether a graph has a dominating induced matching, i.e., an induced matching that dominates *every* edge of the graph. This problem is also known in the literature as EFFICIENT EDGE DOMINATION. Alternatively, the problem can be viewed as a restricted version of VERTEX 3-COLORABILITY, i.e., the problem of determining whether the vertices of a graph can be partitioned into three independent sets. In the DOMINATING INDUCED MATCHING problem we are looking for a partition of a graph into three independent sets such that two of them induce a 1-regular graph.

One more related problem is that of finding in a graph an induced matching of maximum cardinality. Recently, it was shown in [Cardoso et al., 2008] that an induced matching in a graph is dominating only if it is maximum in terms of its size. Finding a maximum induced matching is a well-studied problem, which is NP-hard in general graphs and in many particular classes such as bipartite graphs of degree at most three [Lozin, 2002] or line graphs [Kobler and Rotics, 2003]. On the other hand, the problem is known to be polynomial-time solvable for chordal graphs and interval graphs [Cameron, 1989], circular-arc graphs [Golumbic and Laskar, 1993], weakly chordal graphs [Cameron et al., 2003], convex graphs [Brandstädt et al., 2007] and many other special classes (see e.g. [Cameron, 2004; Chang, 2004; Golumbic and Lewenstein, 2000; Kobler and Rotics, 2003]).

The complexity of the DOMINATING INDUCED MATCHING problem in special graph classes is less explored. It is known that the problem is NP-complete in general [Grinstead et al., 1993] and in some particular classes such as planar bipartite graphs

[Lu et al., 2002] and d -regular graphs [Cardoso et al., 2008] (also see [Kratochvíl, 1994] for the case $d = 3$). Polynomial-time solutions are available only for bipartite permutation graphs [Lu and Tang, 1998], chordal graphs [Lu et al., 2002] and claw-free graphs [Cardoso and Lozin, 2009].

Our contribution to the topic is as follows. In Section 2.5.1, we identify the first boundary class for this problem, and in Section 2.5.2 we extend two polynomial-time results to larger classes. In particular, we show how to solve the problem in polynomial-time for convex (bipartite) graphs (extending the result for bipartite permutation graphs) and for E -free graphs (extending the result for claw-free graphs).

A graph class X will be called *DIM-easy* if the DOMINATING INDUCED MATCHING problem is polynomial-time solvable for graphs in X , and *DIM-tough* otherwise. If $P \neq NP$, the family of *DIM-tough* classes is disjoint from that of *DIM-easy* classes, in which case the problem of characterization of these two families arises. By analogy with the forbidden induced subgraph characterization of hereditary classes, we want to characterize the family of *DIM-easy* classes in terms of minimal classes that do not belong to this family. Unfortunately, a *DIM-tough* class may contain infinitely many *DIM-tough* subclasses, which makes the task of finding minimal *DIM-tough* classes impossible. To overcome this difficulty, we employ the notion of a boundary class, which can be defined (in this context) as follows.

A class of graphs X will be called a *limit class* for the DOMINATING INDUCED MATCHING problem if $X = \bigcap_{i=1}^{\infty} X_i$, where $X_1 \supseteq X_2 \supseteq \dots$ is a sequence of *DIM-tough* classes. A minimal limit class will be a *boundary* class for the problem in question.

2.5.1 A boundary class

Throughout the rest of the section we denote by \mathcal{S}_k the class of $(C_3, \dots, C_k, H_1, \dots, H_k)$ -free bipartite graphs of vertex degree at most 3 and by \mathcal{S} the intersection $\bigcap_{k \geq 0} \mathcal{S}_k$. (See Figure 2.6 for the definition of H_i .)

The main result of this section is that the class \mathcal{S} is a boundary class for the DOMINATING INDUCED MATCHING problem. First, we show that \mathcal{S} is a limit class for the problem and then we prove its minimality.

From [Grinstead et al., 1993] we know that determining if G has a dominating induced matching is an NP-complete problem. Moreover, it is NP-complete even for bipartite graphs [Lu et al., 2002] and graphs of vertex degree at most three [Kratochvíl, 1994]. In this section, we strengthen these results by showing that the

problem is NP-complete in the class \mathcal{S}_k for any value of k . To this end, we use the following technical lemma of Cardoso & Lozin:

Lemma 2.5.1. *Let G be a graph and e an edge in G . If G' is the graph obtained from G by subdividing the edge e exactly three times, then G has a dominating induced matching if and only if G' has.*

Proof. Denote the endpoints of e by a and b , and the three vertices subdividing the edge e by x, y, z . Assume first that G has a dominating induced matching M . If $e = ab \in M$, then the set $M' = (M \cup \{ax, zb\}) - \{ab\}$ is a dominating induced matching in G' . If $e = ab \notin M$ and e is dominated by a certain edge of M incident to a , then $M' = M \cup \{yz\}$ is a dominating induced matching in G' .

Conversely, suppose G' has a dominating induced matching M' . If neither xy nor yz belong to M' , then $ax, zb \in M'$ and hence $M = (M' - \{ax, zb\}) \cup \{ab\}$ is a dominating induced matching in G . Assume now without loss of generality that $yz \in M'$. Then the set $M = M' - \{yz\}$ is a dominating induced matching in G . \square

A direct consequence of this lemma is the following result, again by Cardoso & Lozin.

Lemma 2.5.2. *For any k , the DOMINATING INDUCED MATCHING problem is NP-complete in the class \mathcal{S}_k .*

Proof. We prove the lemma by reducing the problem from graphs of vertex degree at most three, where the problem is known to be NP-complete.

Let G be a graph of vertex degree at most 3 and G' a graph obtained from G by a triple subdivision of an edge of G . Then G' is also of degree at most three and it has a dominating induced matching if and only if G has. If we subdivide each edge $e := ab$ of G three times, transforming e into $e' := axyzb$, then we obtain a bipartite graph. One can easily verify this by putting $\{a, y, b\}$ in one color class and $\{x, z\}$ in the other, for each edge e of G . Applying this operation repeatedly, we can get rid of small induced cycles and small induced graphs of the form H_i . The resulting graph is bipartite, of maximum degree three and it has a dominating induced matching if and only if G has. This proves the lemma. \square

Lemma 2.5.2 implies that \mathcal{S}_k is a DIM-tough class for any k . Therefore, $\mathcal{S} = \bigcap_{k \geq 0} \mathcal{S}_k$ is a limit class for the DOMINATING INDUCED MATCHING problem.

Next, we show that \mathcal{S} is a minimal limit class for this problem.

In general, the proof of minimality is not a trivial task. However, for the class \mathcal{S} some helpful tools have been developed in [Alekseev et al., 2007]. In particular, it was shown that in the proof of minimality of the class \mathcal{S} for an algorithmic graph problem Π the following lemma plays a key role, where a monotone class is a hereditary class closed under deletion of edges from graphs in the class. (This lemma will only hold for specific algorithmic graph problems Π .)

Lemma 2.5.3. *If X is a monotone graph class such that $\mathcal{S} \not\subseteq X$, then Π is polynomial-time solvable for graphs in X .*

The crucial role of the above lemma in the proof of minimality of \mathcal{S} is based on the following conclusion derived in [Alekseev et al., 2007].

Lemma 2.5.4. *Let Π be a problem for which Lemma 2.5.3 holds. Then \mathcal{S} is a boundary class for Π whenever it is a limit class for the problem.*

In order to show that Lemma 2.5.3 holds for the DOMINATING INDUCED MATCHING problem, we will use the following result from [Boliac and Lozin, 2002]. Knowing the precise definition of the graph parameter called *clique-width* is not vital here, so we postpone the relevant discussion until the following chapter.

Lemma 2.5.5. *If X is a monotone graph class such that $\mathcal{S} \not\subseteq X$, then the clique-width of graphs in X is bounded by a constant.*

Now all we have to do to prove the minimality of the class \mathcal{S} for the DOMINATING INDUCED MATCHING problem is to show that the problem is polynomial-time solvable for graphs of bounded clique-width.

Lemma 2.5.6. *The DOMINATING INDUCED MATCHING problem can be solved in polynomial time in any class of graphs where clique-width is bounded by a constant.*

Proof. In [Courcelle et al., 2000], it was shown that any decision problem expressible in $\text{MSOL}(\tau_1)$ (Monadic Second-Order Logic with quantification over subsets of vertices, but not of edges) can be solved in linear time in any class of graphs of bounded clique-width. The DOMINATING INDUCED MATCHING problem can be expressed in $\text{MSOL}(\tau_1)$ in the following way:

$$\exists B, W (Partition(B, W) \wedge InducedMatching(B) \wedge IndependentSet(W)),$$

where $Partition(B, W)$, $InducedMatching(B)$ and $IndependentSet(W)$ are defined by

$$\begin{aligned} Partition(B, W) &= \forall v(B(v) \vee W(v)) \wedge \neg \exists u(B(u) \wedge W(u)), \\ IndependentSet(W) &= \forall u, v((W(u) \wedge W(v)) \rightarrow \neg \exists E(u, v)), \\ InducedMatching(B) &= \forall u(B(u) \rightarrow \exists! v(B(v) \wedge E(u, v))). \end{aligned}$$

Summarizing the above discussion we conclude that

Theorem 2.5.7. *The class \mathcal{S} is a boundary class for the DOMINATING INDUCED MATCHING problem.*

2.5.2 Polynomial-time algorithms

In this section, we attack the problem from the polynomial side. Some partial results of this type follow from Lemma 2.5.6. It is known that the clique-width is bounded for P_4 -free graphs and some of their generalizations [Makowsky and Rotics, 1999], distance-hereditary graphs [Golumbic and Rotics, 2000], and some other classes (see e.g. [Lozin and Rautenbach, 2004b]). Together with Lemma 2.5.6, this implies polynomial-time solvability of the problem in all those classes. On the other hand, let us observe that boundedness of the clique-width is sufficient but not necessary for polynomial-time solvability of the problem. Indeed, the clique-width is bounded neither in chordal graphs [Makowsky and Rotics, 1999] nor in bipartite permutation graphs [Brandstädt and Lozin, 2003], the only two previously known classes with polynomial-time solvable DOMINATING INDUCED MATCHING problem. The NP-completeness result proved in the previous section suggests directions for further steps in the search for DIM-easy classes.

Unless $P = NP$, according to Lemma 2.5.2, the problem is solvable in polynomial time in a class of graphs $X = Free(M)$ only if X excludes graphs from all classes \mathcal{S}_k , i.e., only if

- (1) $M \cap \mathcal{S}_k \neq \emptyset$ for each k .

On the other hand, if the problem is solvable in polynomial time in any class $X = Free(M)$ satisfying (1) then obviously \mathcal{S} is the only boundary class for the problem. Proving or disproving uniqueness of the class \mathcal{S} is a challenging research problem. In this section, we restrict ourselves to distinguishing three major ways to satisfy (1).

One way to satisfy (1) is to include in M a graph G belonging to \mathcal{S} , which means G has no induced cycles, no induced graphs of the form H_i and no vertices of degree more than three. In other words, every connected component of G is of the form

$S_{i,j,k}$ represented in Figure 2.6. Cardoso & Lozin studied the class of $S_{1,1,1}$ -free graphs, also known as the claw-free graphs, and proved that the problem is solvable in polynomial time in this class [Cardoso and Lozin, 2009].

If we do not include in M a graph $G \in \mathcal{S}$, then to satisfy (1) M must contain infinitely many graphs. Two basic ways to satisfy (1) with infinitely many graphs are $M \supseteq \{C_p, C_{p+1}, \dots\}$ and $M \supseteq \{H_p, H_{p+1}, \dots\}$ for a constant p . Both polynomially solvable cases mentioned in the introduction (bipartite permutation [Lu and Tang, 1998] and chordal graphs [Lu et al., 2002]) deal with graphs that do not contain large induced cycles. We present a result of this type by extending polynomial-time solvability of the problem from the class of bipartite permutation graphs to the class of convex graphs. Finally, we consider classes $\text{Free}(M)$ with $M \supseteq \{H_p, H_{p+1}, \dots\}$ and prove solvability of the problem in such classes whenever the degree of vertices is bounded by a constant.

In our solution, we will use an alternative definition of the DOMINATING INDUCED MATCHING problem which asks to determine if the vertex set of a graph G admits a partition into two subsets W and B such that W is an independent set and B induces a 1-regular graph. Throughout the section we will call the vertices of W *white* and the vertices of B *black*, and the partition $V(G) = B \cup W$ *black-white partition* of G . In other words, a graph G has a dominating induced matching if and only if G admits a black-white partition. We will use these two notions interchangeably.

An assignment of one of the two possible colors to each vertex of G will be called a *coloring* of G . A coloring is *partial* if only part of the vertices of G are assigned colors, otherwise it is *total*. A partial coloring is *valid* if no two white vertices are adjacent and no black vertex has more than one black neighbor. A full coloring is *valid* if no two white vertices are adjacent and every black vertex has exactly one black neighbor.

Before we proceed to give specific solutions, let us make a few observations valid for arbitrary graphs. First, without loss of generality we will assume that

- (A1) all of our graphs are connected, because for a disconnected graph G the problem is solvable if and only if it is solvable for every connected component of G .

We can also assume that

- (A2) G has no induced path with three consecutive vertices of degree 2, because any three consecutive vertices of degree 2 can be replaced by an edge and the modified graph has a dominating induced matching if and only if the original one has (Lemma 2.5.1).

The assumption *A2* implies in particular that any vertex of degree 1 is connected to the nearest vertex of degree more than 2 by a chordless path of length at most 3. Moreover, it is not difficult to see that if the length of the path is 3, we can delete this path and the new graph has a dominating induced matching if and only if the original one has. Therefore, in what follows we assume that

(A3) any vertex of degree 1 is connected to the nearest vertex of degree more than 2 by a chordless path of length at most 2.

Definition 2.5.1. *A vertex of degree 1 will be called a leaf and the only neighbor of a leaf will be called a preleaf.*

It is not difficult to see that

Lemma 2.5.8. *In any black-white partition of G , each preleaf is black.*

This simple observation shows that analysis of local properties of a graph G may lead to a partial coloring of G . With a more involved analysis, some stronger conclusions can be made.

Application of Lemma 2.5.8 may lead either to the conclusion that the input graph has no dominating induced matching or to a partial coloring of the graph. We will assume that any partial coloring is maximal (i.e., cannot be extended to a larger coloring) under some simple rules. The three obvious rules are

R1 : each neighbor of a white vertex must be colored black;

R2 : all neighbors of two black adjacent vertices must be colored white;

R3 : each vertex that has two black neighbors (not necessarily adjacent) must be colored white.

Three other rules that will be used in our solutions are not so obvious, but are also simple:

R4 : if a vertex v belongs to a triangle T and has a neighbor w outside T , then v and w must be colored differently;

R5 : in any induced C_4 , any two adjacent vertices must be colored differently;

R6 : if a preleaf v is adjacent to more than one leaf, then all but one leaf adjacent to v can be colored white.

The main strategy in our polynomial-time solutions is the following. The algorithm starts by finding an initial partial coloring of the input graph G by analyzing local properties of G . Then the algorithm incrementally extends the partial coloring by application of the above rules and some more specific considerations. At each step of the algorithm, we delete from G those colored vertices that have no neighbors among uncolored ones (as they have no importance for the completion of the procedure) and denote the resulting graph G_0 . By Rule $R1$, any colored vertex of G_0 is black, and by Rule $R2$, the set of colored vertices of G_0 is independent. Application of the above strategy either leads to a conflict (two adjacent vertices colored white or a black vertex with more than one black neighbor) or reduces the problem to a graph G_0 for which the solution is simple.

A polynomial-time algorithm for convex graphs

Definition 2.5.2. A convex graph is a bipartite graph $G = (V_1, V_2, E)$ in which at least one of the parts, V_1 or V_2 , has the adjacency property, i.e., the vertices in that part can be ordered so that for any vertex v in the opposite part, $N(v)$ forms an interval (the vertices of $N(v)$ appear consecutively in the order).

The class of convex graphs generalizes several important subclasses such as bi-convex graphs and bipartite permutation graphs (see e.g. [Brandstädt et al., 1999]). In the latter class, the DOMINATING INDUCED MATCHING problem has a polynomial-time solution [Lu et al., 2002]. In the present section, we extend this result to convex graphs.

It is known (and can be easily seen) that no cycle of length more than 4 is convex. Three other non-convex graphs that play an important role in our solution are X , Y and Z , represented in Figure 2.7.

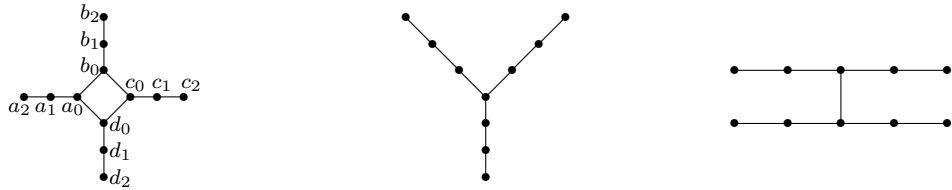


Figure 2.7: Graphs X (left), Y (middle), and Z (right)

Lemma 2.5.9. The graphs X , Y and Z are not convex.

Proof. To prove the lemma for the graph X , assume by symmetry that the part of X containing a_0 has the adjacency property. Then both triples a_0, b_1, c_0 and a_0, d_1, c_0

must create intervals, which means the vertices b_1, a_0, c_0, d_1 create an interval with a_0, c_0 being in the middle. But then a_0, a_2 cannot create an interval. Therefore, X is not convex.

Let v be the vertex of degree 3 in Y . The the part of Y containing v cannot have the adjacency property, since otherwise v would be consecutive with three different vertices in its part. Suppose the other part of Y has the adjacency property. Then the three vertices adjacent to v must create an interval, and the middle vertex of this interval must be also consecutive with one more vertex, which is impossible. Therefore, Y is not convex.

Let v be a vertex of degree 3 in Z . By symmetry, we may assume that the part of Z containing v has the adjacency property. But then v must be consecutive with three different vertices in its part, which is impossible. Hence Z is not convex. \square

To solve the problem for a convex graph G , we start by coloring the vertices of each C_4 in G . According to Rule *R4*, the colors must alternate along the cycle in any valid coloring of a C_4 . So, in general, an induced C_4 admits two possible colorings. However, as we prove below, in a convex graph only one coloring is possible, and this coloring can be determined in a polynomial time.

Lemma 2.5.10. *In a convex graph any C_4 is uniquely colorable, and the only possible coloring of a C_4 can be determined in polynomial time.*

Proof. Let G be a convex graph and let vertices a_0, b_0, c_0, d_0 induce a C_4 . We will illustrate the proof with the help of the picture of the graph X in Figure 2.7. The algorithm that determines a coloring of the $C_4 = G[a_0, b_0, c_0, d_0]$ can be described as follows.

Algorithm C_4

1. If $G[a_0, b_0, c_0, d_0]$ cannot be extended to an induced subgraph of G isomorphic to $X[a_0, b_0, c_0, d_0, a_1, c_1]$, then color a_0 white.
2. If $G[a_0, b_0, c_0, d_0, a_1, c_1]$ cannot be extended to an induced subgraph of G isomorphic to $X[a_0, b_0, c_0, d_0, a_1, b_1, c_1, d_1]$, then color b_0 white.
3. If $G[a_0, b_0, c_0, d_0, a_1, b_1, c_1, d_1]$ cannot be extended to an induced subgraph of G isomorphic to $X[a_0, b_0, c_0, d_0, a_1, b_1, c_1, d_1, a_2, c_2]$, then color a_0 black, otherwise color a_0 white.

Clearly, the algorithm has a polynomial running time. Now let us prove the correctness of the algorithm.

Suppose $G[a_0, b_0, c_0, d_0]$ cannot be extended to $X[a_0, b_0, c_0, d_0, a_1, c_1]$ and assume by contradiction that there is a valid coloring of G in which a_0, c_0 are black and b_0, d_0 are white. Denoting by a_1 the unique black neighbor of a_0 and by c_1 the unique black neighbor of c_0 , we conclude that $G[a_0, b_0, c_0, d_0, a_1, c_1]$ is isomorphic to $X[a_0, b_0, c_0, d_0, a_1, c_1]$, which contradicts the assumption. This contradiction proves the correctness of Step 1 of the algorithm.

Suppose $G[a_0, b_0, c_0, d_0, a_1, c_1]$ cannot be extended to $X[a_0, b_0, c_0, d_0, a_1, b_1, c_1, d_1]$ and assume by contradiction that there is a valid coloring of G in which b_0, d_0 are black and a_0, c_0 are white. Then a_1, c_1 are black (Rule $R1$). Denoting by b_1 the unique black neighbor of b_0 and by d_1 the unique black neighbor of d_0 and remembering that a black vertex cannot have more than one black neighbor, we conclude that $G[a_0, b_0, c_0, d_0, a_1, b_1, c_1, d_1]$ is isomorphic to $X[a_0, b_0, c_0, d_0, a_1, b_1, c_1, d_1]$, which contradicts the assumption. This contradiction proves the correctness of Step 2 of the algorithm.

To show the correctness of Step 3, suppose $G[a_0, b_0, c_0, d_0, a_1, b_1, c_1, d_1]$ cannot be extended to $X[a_0, b_0, c_0, d_0, a_1, b_1, c_1, d_1, a_2, c_2]$ and assume by contradiction that there is a valid coloring of G in which a_0, c_0 are white and b_0, d_0 are black. Then a_1, c_1 are black (Rule $R1$). Denoting by a_2 the unique black neighbor of a_1 and by c_2 the unique black neighbor of c_1 and remembering that a black vertex cannot have more than one black neighbor and that G has no induced cycles except C_4 , we conclude that $G[a_0, b_0, c_0, d_0, a_1, b_1, c_1, d_1, a_2, c_2]$ is isomorphic to $X[a_0, b_0, c_0, d_0, a_1, b_1, c_1, d_1, a_2, c_2]$, which contradicts the assumption. This contradiction proves the correctness of the first part of Step 3 of the algorithm.

To prove the second part of Step 3, suppose that $G[a_0, b_0, c_0, d_0, a_1, b_1, c_1, d_1]$ admits an extension to $X[a_0, b_0, c_0, d_0, a_1, b_1, c_1, d_1, a_2, c_2]$ and assume by contradiction that there is a valid coloring of G in which a_0, c_0 are black and b_0, d_0 are white. Then b_1, d_1 are black (Rule $R1$). Denoting by b_2 the unique black neighbor of b_1 and by d_2 the unique black neighbor of d_1 and remembering that a black vertex cannot have more than one black neighbor and that G has no induced cycles except C_4 , we conclude that $G[a_0, b_0, c_0, d_0, a_1, b_1, c_1, d_1, a_2, b_2, c_2, d_2] = X[a_0, b_0, c_0, d_0, a_1, b_1, c_1, d_1, a_2, b_2, c_2, d_2]$, which is not possible because the latter graph is not convex. This contradiction completes the proof of the lemma. \square

Lemma 2.5.10 and assumption $A1$ reduce the problem from convex graphs to connected graphs without cycles, i.e., trees. Moreover, we will show that with the help of Lemma 2.5.8 and rules $R1 - R6$ the problem further reduces to trees of a special form which we call τ -caterpillars.

Definition 2.5.3. A τ -caterpillar is a tree of vertex degree at most 3 in which

- all vertices of degree 3 lie on a single path,
- no two vertices of degree 3 are adjacent,
- the distance between any vertex of degree 3 and a nearest leaf is at most 2.

As before, we denote by G_0 the subgraph of G obtained by deletion of those colored vertices that have no neighbors among uncolored ones.

Claim 2.5.11. Let v be a vertex of degree at least 3 in G_0 . Then

- v has degree 3,
- each neighbor of v has degree at most 2,
- either v is a preleaf or v is adjacent to a preleaf.

Proof. Assume first that v is not adjacent to a leaf. To avoid an induced Y (Figure 2.7), at least one of the neighbors of v , say w , is a preleaf. By Rule $R6$, w has degree 2 and by Lemma 2.5.8, w is colored black. Therefore, by Rule $R3$, no other neighbor of v is a preleaf (otherwise v is white and hence does not belong to G_0). This implies that no neighbor of v has degree more than 2 (since otherwise an induced Z arises) and the degree of v is exactly 3 (since otherwise an induced Y arises).

Suppose now that v is adjacent to a leaf u . Then, by Lemma 2.5.8, v is black and by Rule $R6$, u is the only leaf adjacent to v . No neighbor x of v is a preleaf, since otherwise neither x nor v belong to G_0 (Rule $R2$). This implies that the degree of v is exactly 3 (since otherwise an induced Y arises) and no neighbor x of v has degree more than 2 (since otherwise we are in the conditions of the previous paragraph with respect to x , in which case x cannot be adjacent to a vertex of degree at least 3). \square

Lemma 2.5.12. G_0 is a τ -caterpillar.

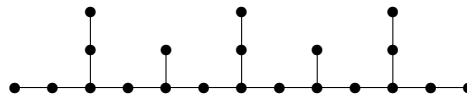


Figure 2.8: An example of a τ -caterpillar

Proof. The lemma is obviously true if G_0 has at most 2 vertices of degree 3. Assume now that G_0 has at least three vertices of degree 3 and suppose by contradiction that there is no path containing all of them. Then G_0 must contain three vertices u, v, w of degree 3 with no path containing them. Denote by P be the unique path connecting u to w in G_0 and by P' a shortest path connecting v to a vertex x of P . By assumption $x \neq u, v$ (otherwise $P \cup P'$ is a path containing all three vertices). Then x is also a vertex of degree 3. By Claim 2.5.11, x is adjacent to none of the vertices u, v, w , but then G_0 contains Y as an induced subgraph. This contradiction proves the lemma. \square

We denote by $P = (v_0, v_1, \dots, v_p)$ a maximal path containing all vertices of degree three of G_0 . The maximality implies that both v_0 and v_p have degree 1 in G_0 . According to the definition of a τ -caterpillar, there are two types of vertices of degree 3 in G_0 : preleaves (type 1) and vertices adjacent to a preleaf (type 2). No vertex v of degree three can be simultaneously of type 1 and type 2, since otherwise v must be colored black and one of its neighbors must be colored black, in which case neither v nor its black neighbor belong to G_0 .

If v_i is of type 1, we denote by $v_{i,1}$ the leaf adjacent to v_i , and if v_i is of type 2, we denote by $v_{i,1}$ and $v_{i,2}$, respectively, the preleaf adjacent to v_i and the leaf adjacent to $v_{i,1}$.

To complete the procedure of coloring of G_0 , we will use, in addition to rules $R1 - R6$, one more rule:

R7 : if v_i is of type 2, then color v_{i-2} and v_{i+2} black. To prove correctness of this rule, assume that v_{i+2} is colored white. Then v_{i+1} must be black. Remembering that v_{i+1} has degree 2, we conclude that v_i must be black as well, since otherwise v_{i+1} has no black neighbor. But now the black vertex v_i has two black neighbors $v_{i,1}$ and v_{i+1} . This contradiction shows that black is the only possible color for v_{i+2} , and similarly for v_{i-2} .

Lemma 2.5.13. *G_0 admits a total valid coloring.*

Proof. According to Rules $R2$ and $R3$, between any two nearest black vertices v_i and v_j ($i < j$) of P , there are at least 2 uncolored vertices. According to assumption $A2$, the number of uncolored vertices between v_i and v_j is exactly 2, unless one of the uncolored vertices is of type 2, in which case $j = i + 4$ (Rule $R7$).

We prove the lemma by induction on the number of vertices of type 2 in G_0 . If there are no vertices of type 2, then $p = 3k + 2$ for some k and vertices v_{3i+1}

($i = 0, \dots, k$) are black. There are two possible ways to extend this partial coloring to a total valid coloring:

- W1: vertices v_{3i} ($i = 0, \dots, k$) are colored black and all the other vertices of G_0 are colored white,
- W2: vertices v_{3i+2} ($i = 0, \dots, k$) are colored black and all the other vertices of G_0 are colored white.

Assume now that G_0 has at least one vertex of type 2, and let v_t be such a vertex with minimum index t . Let G'_0 be the subgraph of G_0 induced by vertices v_0, \dots, v_{t-1} , and G''_0 the subgraph of G_0 induced by the remaining vertices. By the inductive hypothesis, G''_0 admits a total valid coloring ϕ , and G'_0 has no vertices of type 2. If v_t is colored black in ϕ , apply coloring W1 to G'_0 , otherwise apply coloring W2 to G'_0 . It is not difficult to see that in both cases we obtain a total valid coloring of G_0 . \square

We now summarize the above discussion in Algorithm \mathcal{B} below. This algorithm is robust in the sense that it does not require the input graph G to be convex. The algorithm either finds a black-white partition of G or reports that G has no such partition or G is not convex.

Algorithm \mathcal{B}

Input: a graph G

Output: a black-white partition of G or report “ G has no black-white partition or G is not convex”

1. As long as G has an induced C_4 , apply *Algorithm C_4* to color the vertices of the C_4 . If the partial coloring obtained in this way is not valid or the subgraph G_0 of G is not a τ -caterpillar, then STOP and output “ G has no black-white partition or G is not convex”.
2. Apply Rule $R7$ to G_0 . If the partial coloring obtained by this application is not valid, then STOP and output “ G has no black-white partition or G is not convex”.
3. Extend the partial coloring of G_0 to a full coloring according to Lemma 2.5.13 and output the black-white partition of G .

Theorem 2.5.14. *Algorithm \mathcal{B} correctly solves the DOMINATING INDUCED MATCHING problem for convex graphs in polynomial time.*

Correctness of the algorithm and its polynomial running time follow directly from the results preceding the algorithm.

It is known [Cameron et al., 2003] that finding a maximum induced matching is polynomial-time solvable in the class of weakly chordal graphs. This class generalizes simultaneously two polynomially solvable cases for the DOMINATING INDUCED MATCHING problem, namely, chordal graphs and convex graphs. It would be interesting to investigate whether these two cases can be extended to the larger class of weakly chordal graphs.

A polynomial-time algorithm for E -free graphs

Cardoso & Lozin gave a polynomial-time algorithm for the DOMINATING INDUCED MATCHING problem in the class of claw-free graphs [Cardoso and Lozin, 2009]. We extend this solution to E -free graphs, where E is the graph $S_{1,2,2}$ (see Figure 2.9, where this type of graph is again reproduced for the benefit of the reader). The graph $E = S_{1,2,2}$ contains a claw (i.e. $S_{1,1,1}$) as an induced subgraph, and therefore the class of E -free graphs extends the class of claw-free graphs. So this result extends the result of Cardoso & Lozin on claw-free graphs.

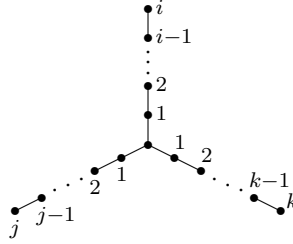


Figure 2.9: The graph $S_{i,j,k}$

As with the case of convex graphs, our strategy in solving the problem is to incrementally extend a partial valid coloring according to certain rules. This strategy suggests a more general framework for the problem, in which the graph is given together with a partial valid coloring. The question is to determine if the partial coloring can be extended to a total valid coloring. We will refer to this more general version of the problem as EXTENSION TO DOMINATING INDUCED MATCHING (EDIM for short).

Let us recite the rules that we will use:

R1: each neighbor of a white vertex must be colored black;

R2: each neighbor of a matched black vertex must be colored white;

R3: each vertex that has two black neighbors must be colored white;

R4: if a vertex v belongs to a triangle T and has a neighbor w outside T , then v and w must be colored differently;

R5: in any induced C_4 , any two adjacent vertices must be colored differently.

Given a graph G and a partial coloring of its vertices, we can obviously ignore those colored vertices that have no neighbors among uncolored ones. We shall call such vertices *irrelevant*. Removing irrelevant vertices from the graph can reduce the problem to a more specific instance. In particular, the following reduction is valid for arbitrary graphs.

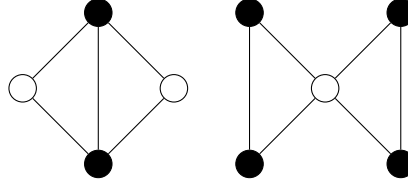


Figure 2.10: A diamond (left) and a butterfly (right).

Lemma 2.5.15. *The EDIM problem can be reduced in polynomial time from an arbitrary graph G to an induced subgraph G' of G such that G' is $(\text{diamond}, \text{butterfly}, K_4)$ -free and any every vertex of G' has at most one neighbor of degree 1.*

Proof. Since K_4 is not 3-colorable, no graph G containing a K_4 has a black-white partition. This immediately reduces the problem from general graphs to K_4 -free graphs. Also, by direct inspection the reader can easily check that the diamond and butterfly have unique valid coloring represented in Figure 2.10. Therefore, if a graph G contains a copy of an induced diamond or butterfly, the vertices of this copy can be colored and removed from the graph, since they become irrelevant after coloring all their neighbors.

Finally, assume G contains a vertex that has more than one neighbor of degree 1. If G admits a black-white partition, then all these neighbors, except possibly one, are white. Moreover, if one of these neighbors must be black, then *any one* of them can be assigned this color. Therefore, all but one neighbor of degree 1 can be colored white and removed from the graph. \square

The following lemma provides a useful characterization of $(\text{diamond}, \text{butterfly}, K_4)$ -free graphs.

Lemma 2.5.16. *Let G be a $(\text{diamond}, \text{butterfly}, K_4)$ -free graph and v a vertex of G . Then the neighborhood of v contains at most one edge.*

Proof. Assume $N(v)$ contains two edges e_1 and e_2 . If these edges share a vertex, then their endpoints together with v induce either a *diamond* or a K_4 . If neither e_1, e_2 nor any other two edges share a vertex, then the endpoints of e_1 and e_2 together with v induce a *butterfly*. \square

We conclude this section with a result that will be critical for solving the problem in the class of E -free graphs.

Theorem 2.5.17. *The EDIM problem can be solved in polynomial time in any class of graphs where clique-width is bounded by a constant.*

Proof. In [Courcelle et al., 2000], it was shown that any decision problem expressible in $\text{MSOL}(\tau_1, p)$ can be solved in linear time in any class of graphs of bounded clique-width. $\text{MSOL}(\tau_1)$ is a Monadic Second-Order Logic with quantification over subsets of vertices, but not of edges. $\text{MSOL}(\tau_1, p)$ is the extension of $\text{MSOL}(\tau_1)$ by unary predicates representing labels attached to vertices. Therefore, to prove that EXTENSION TO DOMINATING INDUCED MATCHING is expressible in $\text{MSOL}(\tau_1, p)$ all we have to do is to show that DOMINATING INDUCED MATCHING is expressible in $\text{MSOL}(\tau_1)$. This was done in the proof of Lemma 2.5.6. \square

The solution for E -free graphs is based on a reduction of the problem to graphs of bounded clique-width. The reduction consists of two steps. In the first step, we reduce the problem from the entire class of E -free graphs to graphs of bounded vertex degree in this class. In the second step, we further reduce the problem to graphs of bounded chordality, i.e., graphs without long induced cycles. Together, bounded vertex degree and bounded chordality imply bounded clique-width.

The first step of the reduction is valid even for the larger class of $S_{2,2,2}$ -free graphs.

Lemma 2.5.18. *The EDIM problem in the class of $S_{2,2,2}$ -free graphs can be reduced in polynomial time to graphs of vertex degree of at most 11 in this class.*

Proof. Let G be an $S_{2,2,2}$ -free graph. According to Lemma 2.5.15, we may assume without loss of generality that G is $(\text{diamond}, \text{butterfly}, K_4)$ -free and every vertex of G has at most one neighbor of degree 1. Suppose G has a vertex v of degree at least 12.

Assume G admits a black-white partition in which v is colored white. Then every neighbor of v is colored black. By Lemma 2.5.16, the neighborhood of v contains at most one edge. Therefore, at least 3 neighbors of v are isolated in the subgraph of G induced by $N(v)$. Moreover, each of these three vertices must have its own black neighbor. But then G contains an induced $S_{2,2,2}$. This contradiction shows that every vertex of degree more than 11 in an $S_{2,2,2}$ -free graph must be colored black in any black-white partition of G (if there exists any).

From now on, we assume that v (and every other vertex of degree at least 12) is colored black. If two nonadjacent neighbors of v , say x and y , have another common neighbor, say z , then v, x, y , and z form an induced C_4 , in which case both x and y must be colored white (Rule $R5$) and can be removed from G . Implementing this rule with respect to each vertex of degree at least 12 reduces the problem to the case when no two nonadjacent neighbors of a vertex of degree at least 12 have another common neighbor. This also reduces the degree of v . If the degree is still at least 12, then the graph has no black-white partition. Indeed, if the degree of v is at least 12, then $N(v)$ contains at least nine vertices each of which is isolated in the subgraph of G induced by $N(v)$ and each of which has a private neighbor different from v . Since the graph is $S_{2,2,2}$ -free, the set of 9 private neighbors does not contain an independent set of size 3. Therefore, by Ramsey Theorem, it contains a clique of size 4, in which case the graph has no black-white partition.

The above discussion shows that, given an $S_{2,2,2}$ -free graph G , we either reduce the problem to an induced subgraph of G of vertex degree at most 11 or conclude that G has no dominating induced matching. The polynomiality of the reduction is obvious. \square

The next lemma implements the second step of the reduction in our solution.

Lemma 2.5.19. *The EDIM problem in the class of E -free graphs of vertex degree at most 11 can be reduced in polynomial time to $(C_9, C_{10}, C_{11}, \dots)$ -free graphs in this class.*

Proof. Let G be a connected E -free graph of vertex degree at most 11. By Lemma 2.5.15, we also assume that G is $(diamond, butterfly, K_4)$ -free. Suppose G contains a chordless cycle $C = (1, 2, 3, \dots, k-1, k)$ of length $k \geq 9$. If G coincides with C , then the problem is trivial. Otherwise, G contains a vertex v which has at least one neighbor on C . Keeping in mind that the graph is $diamond$ - and $butterfly$ -free, we conclude that v has at most 3 neighbors on the cycle, since otherwise v is the center of an induced E . Also if v has exactly one neighbor on C , or two neighbors of minimum

distance at least 3 along the cycle, or three neighbors, then there is an induced E centered at a neighbor of v on C . From the above discussion, it follows that v has exactly two neighbors on C , either $i, i+2$ or $i, i+1$. Let us show that

R6': if v is adjacent to i and $i+2$, then v and $i+1$ must be colored white.

Indeed, since $v, i, i+1, i+2$ create a C_4 , vertices v and $i+1$ must have the same color (Rule *R5*). Assume v and $i+1$ are colored black, which implies $i, i+2$ are white and therefore $i-1, i+3$ are black. If the graph admits a black-white partition, v has a black neighbor, say w . If w is adjacent neither to i nor to $i+2$ then $G[i-1, i, i+2, i+3, v, w] = E$, and if w is adjacent both to i and to $i+2$ then $G[i, i+2, v, w] = \text{diamond}$. Therefore, w has exactly one neighbor in $\{i, i+2\}$, say i . Observe that replacing $i+1$ by v creates another cycle C' of length k , and from the above discussion we know that w cannot have more than 2 neighbors on C' . Therefore, w is not adjacent to $i-2$. But then $G[i-2, i-1, i, i+1, i+2, w] = E$. This contradiction proves validity of Rule *R6'*.

Applying Rule *R6'* as long as possible and removing irrelevant vertices from the graph leaves us with the case when every vertex outside C that has a neighbor on C is adjacent to exactly two consecutive vertices of C . Also, since the graph is $(K_4, \text{diamond}, \text{butterfly})$ -free, we conclude that every vertex of C that has a neighbor outside C is adjacent to exactly one vertex outside C . Moreover, the problem can be further reduced to the case when *every* vertex of C has a neighbor outside C . This can be done according to the following rules. Assume $i, i+1, \dots, i+p, i+p+1$ is a list of consecutive vertices on C such that i and $i+p+1$ have neighbors outside C , while $i+1, \dots, i+p$ have no neighbors outside C .

R7': If $p \geq 3$, then replacing the path $i, i+1, i+2, i+3, i+4$ by an edge $(i, i+4)$ transforms G into a graph G' which has a black-white partition if and only if G has.

To see this, assume first that G has a black-white partition. We know that i is adjacent to a vertex outside C , while $i+1$ is not, i.e., there is a triangle containing i but not $i+1$. Therefore, by Rule *R4*, i and $i+1$ must be colored differently. Suppose i is black, then $i+1$ is white, implying that $i+2$ and $i+3$ are black and $i+4$ is white. Therefore, by deleting from G the vertices $i+1, i+2, i+3$ and connecting i to $i+4$ we obtain a graph G' which also has a black-white partition. If i is white, then $i+1$ and $i+2$ are black, $i+3$ is white and $i+4$ is black, and again G' has a black-white partition. The converse statement (that a black-white partition of G' implies a black-white partition of G) can be shown by analogy.

R8: If $p = 2$, then $i, i + 3$ must be colored white and $i + 1, i + 2$ must be colored black.

Indeed, if $i + 1$ is white, then $i + 2$ is black (Rule *R1*) and therefore $i + 3$ is white (Rule *R4*). But then black vertex $i + 2$ has no black neighbors in G . This contradiction shows that $i + 1$ must be colored black. Symmetrically, $i + 2$ must be colored black. This implies that $i, i + 3$ must be colored white.

R9: If $p = 1$, then $i + 1$ must be colored white.

Indeed, if $i + 1$ is black, then, by Rule *R4*, i and $i + 2$ are white. But then black vertex $i + 1$ has no black neighbors in G . Therefore, $i + 1$ must be colored white.

Applying rules *R7'*, *R8*, *R9* as long as possible and removing irrelevant vertices from the graph reduces the problem to the case when every vertex outside C with a neighbor on C is adjacent to exactly two consecutive vertices of C , and every vertex of C has exactly one neighbor outside C , i.e., C is of even length. Moreover, without loss of generality, every even edge belongs to a triangle and every odd edge does not belong to any triangle. By Rule *R4*, the endpoints of odd edges must be colored differently, which in turn implies that the endpoints of even edges must be colored differently. In other words, the colors of the vertices alternate along the cycle, while all its neighbors outside the cycle are black. This means that we can choose arbitrarily one of the two possible ways to color the vertices of the cycle. By coloring, for instance, the odd vertices of C white and removing them from the graph, and repeating this procedure for each cycle of length at least 9, we reduce the problem to graphs without long induced cycles.

Finding an induced cycle of length at least 9 can be done in $O(n^9)$ time. All other operations of the reduction can also be implemented in polynomial time. \square

We now summarize the above discussion in the following conclusion.

Theorem 2.5.20. *The (EXTENSION TO) DOMINATING INDUCED MATCHING PROBLEM can be solved in the class of E -free graphs in polynomial time.*

Proof. By Lemmas 2.5.18 and 2.5.19, the EDIM problem can be reduced from E -free graphs to graph of degree at most 11 and of chordality (the length of a longest induced cycle) at most 8. It has been shown in [Bodlaender and Thilikos, 1997] that if a graph has chordality at most c and maximum degree at most k , then its tree-width is at most $k(k - 1)^{c-3}$. Also, in [Corneil and Rotics, 2005] it was shown that for any graph G , the clique-width of G does not exceed $3 \cdot 2^{\text{tw}(G)-1}$, where

$\text{tw}(G)$ denotes the tree-width of G . Therefore, Lemmas 2.5.18 and 2.5.19 reduce the problem from E -free graphs to graphs of bounded clique-width. Together with Theorem 2.5.17 this implies a polynomial-time solution to the problem in the class of E -free graphs. \square

2.5.3 Concluding remarks and related open problems

Further narrowing the gap between NP-complete and polynomially solvable cases of the DOMINATING INDUCED MATCHING problem is an interesting direction for future research. In this respect, classes of graphs without long induced paths are of particular interest. Indeed, by forbidding a path P_k we simultaneously exclude a graph from the class \mathcal{S} , long cycles, and long graphs of the form H_k , which are the three major ways to satisfy condition (1) stated in the beginning of Section 2.5.2. It is known that the clique-width of P_4 -free graphs is at most 2, which implies polynomial-time solvability of many algorithmic graph problems in this class, including DOMINATING INDUCED MATCHING and MAXIMUM INDUCED MATCHING. Furthermore, one of these two problems, DOMINATING INDUCED MATCHING, has recently been shown to be linear-time-solvable in the class of P_7 -free graphs [Brandstädt and Mosca, 2011]. Apart from this resolved case, for $k \geq 5$, the complexity of the two problems in the class of P_k -free graphs is unknown.

DOMINATING INDUCED MATCHING and MAXIMUM INDUCED MATCHING are solvable in polynomial time for $(P_k, K_{1,s})$ -free graphs, for any fixed k and s . For the MAXIMUM INDUCED MATCHING problem, this was proved in [Lozin and Rautenbach, 2004a], while for the DOMINATING INDUCED MATCHING problem, this trivially follows from Lemma 2.5.21 below (by [Cardoso and Lozin, 2009]), because the vertex degree in the input graph must be bounded by a constant (depending on s), in which case the number of vertices of the graph is bounded by a constant (assuming the graph is connected).

Lemma 2.5.21. *If a graph G has a dominating induced matching, then the neighborhood of each vertex of G induces a subgraph each connected component of which is a star $K_{1,s}$ for some s .*

Chapter 3

Clique-Width

3.1 Clique-Width: A Short Introduction

The notion of *clique-width* of a graph was introduced in [Courcelle et al., 1993] and is defined as the minimum number of labels needed to construct the graph by means of the four graph operations:

1. creation of a new vertex v with label i (denoted $i(v)$);
2. disjoint union of two labeled graphs G and H (denoted $G \oplus H$);
3. connecting vertices with specified labels i and j (denoted $\eta_{i,j}$);
4. renaming label i to label j (denoted $\rho_{i \rightarrow j}$)

The clique-width of a graph G will be denoted $\text{cwd}(G)$.

Every graph can be defined by an algebraic expression using the four operations above. This expression will be called a k -expression if it uses k different labels. For instance, the cycle C_5 on vertices a, b, c, d, e (listed along the cycle) can be defined by the following 4-expression:

$$\eta_{4,1}(\eta_{4,3}(4(e) \oplus \rho_{4 \rightarrow 3}(\rho_{3 \rightarrow 2}(\eta_{4,3}(4(d) \oplus \eta_{3,2}(3(c) \oplus \eta_{2,1}(2(b) \oplus 1(a))))))))).$$

Alternatively, any algebraic expression defining G can be represented as a rooted tree, called a *parse tree*, whose leaves correspond to the operations of vertex creation, the internal nodes correspond to the \oplus -operations, and the root is associated with G . The operations η and ρ are assigned to the respective edges of the tree. Figure 3.1 shows the tree representing the above expression defining a C_5 .

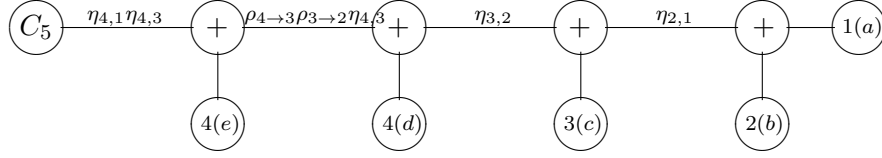


Figure 3.1: The tree representing the expression defining a C_5

Clique-width is a relatively new notion compared to another important graph parameter, *tree-width*. The notion of clique-width generalizes that of tree-width in the sense that graphs of bounded tree-width have bounded clique-width.

The importance of these graph invariants is due to the fact that numerous problems that are NP-hard in general admit polynomial-time solutions when restricted to graphs of bounded tree- or clique-width (see e.g. [Arnborg and Proskurowski, 1989; Courcelle et al., 2000]). The celebrated result due to Robertson and Seymour states that for any planar graph H there is an integer N such that the tree-width of graphs containing no H as a minor is at most N [Robertson and Seymour, 1986]. In other words, the planar graphs constitute a unique minimal minor-closed class of graphs of unbounded tree-width. A special role in this class is assigned to rectangle grids, because every planar graph is a minor of some large enough grid and grids can have arbitrarily large tree-width. Therefore, grids form the only “unavoidable minors” in graphs of large tree-width. In the study of the notion of tree-width, the restriction to the graph minor relation is justified by the fact that the tree-width of a graph cannot be less than the tree-width of its minor. This is not the case with respect to the notion of clique-width. Therefore, in the study of this notion the restriction to the graph minor relation is not valid anymore. Instead, we restrict ourselves to the induced subgraph relation, because the clique-width of a graph cannot be less than the clique-width of any of its induced subgraphs [Courcelle and Olariu, 2000]. The family of hereditary graph classes is much richer than that of minor-closed graph classes, and we believe that the set of “unavoidable induced subgraphs” of large clique-width is more diverse than the set of “unavoidable minors” of large tree-width.

We’d like to find some boundary classes for a family \mathcal{A} of hereditary classes for which the clique-width is bounded. This would help us classify graph classes according to whether they have bounded clique-width or not, due to Lemma 1.4.5 and Theorem 1.4.6. In this thesis, we restrict to bipartite graphs and related classes. This restriction can be motivated as follows.

In [Boliac and Lozin, 2002], it was shown that if X is a hereditary class of graphs of bounded clique-width, then the number X_n of n -vertex labelled graphs in X (also known as the speed of X) is bounded by n^{cn} for a constant c . Also, in [Allen et al., 2009] it was shown that if X_n is strictly less than n^{cn} for every constant $c > 0$, then the clique-width of graphs in X is necessarily bounded. Therefore, each “non-trivial” hereditary class X of bounded clique-width satisfies $n^{c_1 n} \leq X_n \leq n^{c_2 n}$ for some constants c_1 and c_2 , i.e. X is a factorial class in the terminology of [Balogh et al., 2000].

In [Lozin et al., 2011], it was conjectured that a hereditary graph property X is factorial if and only if the fastest of the following three properties is factorial: bipartite graphs in X , co-bipartite graphs in X and split graphs in X . It is known that the “only if” part of this conjecture is valid, because all minimal factorial classes are subclasses of bipartite, co-bipartite or split graphs.

The above discussion shows the importance of bipartite, co-bipartite and split graphs in the study of factorial classes, and hence in the study of the notion of clique-width, since all non-trivial hereditary classes of bounded clique-width are factorial. Moreover, speaking of the notion of clique-width of graphs in these three classes, we may restrict ourselves, without loss of generality, to bipartite graphs only for the following reason. Every bipartite graph G can be transformed into a split or co-bipartite graph by applying “local” complementation (i.e. complementation of an induced subgraph of G) at most twice. In [Kaminski et al., 2009], it was shown that local complementation does not change the clique-width of a graph “too much”. In other words, the clique-width of graphs in a subclass X of bipartite graphs is bounded if and only if it is bounded in the respective subclasses of split and co-bipartite graphs (i.e. those obtained from X by local complementations). This observation shows the exceptional role of bipartite graphs in the study of the notion of clique-width.

In this chapter, motivated by the importance of bipartite graphs to the study of clique-width, we propose a general framework for constructing bipartite graphs of large clique-width. Suggested by this framework, we identify a new boundary class for the family of graph classes of bounded clique-width and a new minimal hereditary graph class of unbounded clique-width. In addition, we discover one more candidate for being a minimal class of bipartite graphs of unbounded clique-width. This class and a related class of split graphs are discussed in Section 3.3.

3.2 Building Bipartite Graphs of Large Clique-Width

Recently, several constructions of bipartite graphs of large clique-width have been discovered in the literature (see e.g. [Brandstädt and Lozin, 2003; Lozin and Rautenbach, 2006; Lozin and Volz, 2008]). We propose a general framework for developing such constructions and use it to obtain new results on this topic.

As we mentioned in the previous section, in the study of clique-width we may restrict to hereditary graph classes. If a class of graphs X is not hereditary, we can extend it to a hereditary class by adding to it all induced subgraphs of graphs in X . The hereditary closure of X will be denoted $[X]$.

Recently, the clique-width has been shown to be unbounded in several hereditary classes of bipartite graphs, such as chordal bipartite graphs [Boliac and Lozin, 2002], bipartite permutation graphs [Brandstädt and Lozin, 2003], P_7 -free bipartite graphs [Lozin and Volz, 2008] and bipartite graphs of bounded vertex degree and large girth [Lozin and Rautenbach, 2006]. Our goal is to identify minimal hereditary classes of graphs of unbounded clique-width. From this perspective, the class of chordal bipartite graphs is of no interest, because it properly contains another class of unbounded clique-width, namely, bipartite permutation graphs. On the contrary, in any proper hereditary subclass of bipartite permutation graphs the clique-width is bounded by a constant (see [Lozin, 2008] for a related result), i.e., the role of bipartite permutation graphs in the family of hereditary classes is analogous to the role of planar graphs in the family of minor-closed graph classes. This makes the class of bipartite permutation graphs critically important in the study of the notion of clique-width. In the attempt to identify more critical classes with respect to this notion, we propose in the next section a general framework for constructing bipartite graphs of large clique-width.

3.2.1 Building blocks and building operations

There are exactly three minimal factorial classes of bipartite graphs: the class of graphs of vertex degree at most 1, the class of “bipartite complements” of graphs of vertex degree at most 1, and the class of $2K_2$ -free bipartite graphs (also known as chain graphs). Graphs in these three classes will be used as building blocks in our construction of bipartite graphs of large clique-width, as follows:

Building blocks:

B_n : the graph B_n has $2n$ vertices x_1, \dots, x_n and y_1, \dots, y_n and edges connecting, for each $i = 1, \dots, n$, vertex x_i to vertices y_j with $j \geq i$.

M_n : the graph M_n has $2n$ vertices x_1, \dots, x_n and y_1, \dots, y_n and edges connecting vertex x_i to y_i for each $i = 1, 2, \dots, n$, i.e., M_n is a collection of n disjoint edges.

F_n : the graph F_n is the bipartite complement of M_n .

Let X_n denote any of the building blocks described above. Notice that the clique-width of X_n is at most 3 regardless of the choice of the block. Now we define two building operations by means of which we will create graphs of large clique-width out of X_n . In the description of the operations, we use the following terminological convention: given a set $C = \{c_{i,j} \mid 1 \leq i, j \leq n\}$ of n^2 elements, we call the elements $c_{i,1}, \dots, c_{i,n}$ the i -th row of C , and we call the elements $c_{1,j}, \dots, c_{n,j}$ the j -th column of C .

Building operations:

- * *n -concatenation* $n * X_n$ is the graph with n^2 vertices $C = \{c_{i,j} \mid 1 \leq i, j \leq n\}$ such that any two consecutive rows of C induce a copy of X_n , and there are no other edges in the graph.
- * *orthogonal concatenation* $X_n^{(2)}$ is the graph with $2n + n^2$ vertices $A = \{a_1, \dots, a_n\}$, $B = \{b_1, \dots, b_n\}$ and $C = \{c_{i,j} \mid 1 \leq i, j \leq n\}$ such that
 - $A \cup B$ and C are independent sets;
 - in the subgraph of $X_n^{(2)}$ induced by A and C , the vertices of the same row of C have the same neighborhood, and by contracting each row of C to a single vertex we obtain an X_n ;
 - in the subgraph of $X_n^{(2)}$ induced by B and C , the vertices of the same column of C have the same neighborhood, and by contracting each column of C to a single vertex we obtain an X_n .

Examples.

1. The graph $n * B_n$ was studied in [Brandstädt and Lozin, 2003; Lozin and Rudolf, 2007]. In particular, in [Brandstädt and Lozin, 2003] it was shown that the clique-width of $n * B_n$ is at least $n/6$ and that $n * B_n$ is a bipartite permutation graph. Moreover, in [Lozin and Rudolf, 2007] it was proved that $n * B_n$ is an n -universal bipartite permutation graph, i.e., it contains every bipartite permutation graph with n vertices as an induced subgraph. In other words, the role of the graph $n * B_n$ in the class of bipartite permutation graphs is analogous to the role of the grids in the class of planar graphs. We also repeat that the role of bipartite permutation graphs

in the family of hereditary classes is analogous to the role of planar graphs in the family of minor closed graph classes, as $[\{n * B_n : n \geq 1\}]$ is a minimal hereditary class of unbounded clique-width.

2. The graph $B_n^{(2)}$ was introduced in [Brandstädt et al., 2006] and was shown there to have clique-width at least n . Therefore, the clique-width of graphs in the class $[\{B_n^{(2)} : n \geq 1\}]$ is unbounded. However, whether this is a minimal hereditary class of unbounded clique-width is an open question.

In the next two sections we will study more constructions obtained by means of the above operations. Not each of them leads to a graph of large clique-width. For instance, $n * M_n$ is the disjoint union of n chordless paths and therefore the clique-width of $n * M_n$ is at most 3. However, n -concatenation of the bipartite complement of M_n , i.e., the graph $n * F_n$, has large clique-width, as we show in Section 3.2.2. Moreover, in the same section we show that $\mathcal{F} := [\{n * F_n : n \geq 1\}]$ is a minimal hereditary class of unbounded clique-width.

In Section 3.2.3, we study the class $\mathcal{M} := [\{M_n^{(2)} : n \geq 1\}]$ and show that it is also of unbounded clique-width. However, this class is not a minimal hereditary class of unbounded clique-width. Moreover, we discover an infinite decreasing sequence $\mathcal{M}_1 \supset \mathcal{M}_2 \supset \dots$ of subclasses of \mathcal{M} of unbounded clique-width. We also show that the limit class of this sequence, i.e., the class $\bigcap_{i \geq 1} \mathcal{M}_i$, is unique in the sense that by excluding any graph from this class we obtained a subclass of \mathcal{M} of bounded clique-width.

3.2.2 A minimal class of unbounded clique-width

Recall that \mathcal{F} is the hereditary closure of the set $\{n * F_n : n \geq 1\}$. Throughout the section, we denote the set of vertices of the graph $n * F_n$ by $V = \{v_{i,j} : 1 \leq i, j \leq n\}$. Also, the subgraph of $n * F_n$ induced by any k consecutive rows of V will be denoted $k * F_n$, i.e. $2 * F_n = F_n$.

Theorem 3.2.1. *The clique-width of the graph $n * F_n$ is at least $\lfloor n/2 \rfloor$.*

Proof. Let $\text{cwd}(n * F_n) = t$. Denote by τ a t -expression defining $n * F_n$ and by $\text{tree}(\tau)$ the rooted tree representing τ . The subtree of $\text{tree}(\tau)$ rooted at a node x will be denoted $\text{tree}(x, \tau)$. This subtree corresponds to a subgraph of $n * F_n$, which will be denoted $F(x)$. The label of a vertex v of the graph $n * F_n$ at the node x is defined as the label that v has immediately prior to applying the operation x .

Let a be a lowest \oplus -node in $\text{tree}(\tau)$ such that $F(a)$ contains a full row of V . Denote the children of a in $\text{tree}(\tau)$ by b and c . Let us color all vertices in $F(b)$ blue

and all vertices in $F(c)$ red, and the remaining vertices of $n * F_n$ yellow. Note that, by the choice of a , the graph $n * F_n$ contains a non-yellow row (i.e. a row each vertex of which is non-yellow), but none of its rows is entirely red or blue. We denote the number of a non-yellow row of $n * F_n$ by r . Without loss of generality, we assume that $r \leq \lfloor n/2 \rfloor$ and that the row r contains at least $n/2$ red vertices, since otherwise we could consider the rows in reverse order and swap colors red and blue.

Observe that edges of $n * F_n$ between differently colored vertices are not present in $F(a)$. Therefore, if a non-red vertex distinguishes two red vertices u and v , then u and v must have different labels at the node a . We will use this fact to show that $F(a)$ contains a set U of at least $\lfloor n/2 \rfloor$ vertices with pairwise different labels at the node a . Such a set can be constructed by the following procedure.

1. Set $i = r$, $U = \emptyset$ and $J = \{j : v_{r,j} \text{ is red}\}$.
2. Set $K = \{j \in J : v_{i+1,j} \text{ is non-red}\}$.
3. If $K \neq \emptyset$, add the vertices $\{v_{i,k} : k \in K\}$ to U . Remove members of K from J .
4. If $J = \emptyset$, terminate the procedure.
5. Increase i by 1. If $i = n$, choose an arbitrary $j \in J$, put $U = \{v_{m,j} : r \leq m \leq n-1\}$ and terminate the procedure.
6. Go back to Step 2.

It is not difficult to see that this procedure must terminate. To complete the proof, it suffices to show that whenever the procedure terminates, the size of U is at least $\lfloor n/2 \rfloor$ and the vertices in U have pairwise different labels at the node a .

First, suppose that the procedure terminates in Step 5. Then U is a subset of red vertices from at least $\lfloor n/2 \rfloor$ consecutive rows of column j . Consider two vertices $v_{l,j}, v_{m,j} \in U$ with $l < m$. According to the above procedure, $v_{m+1,j}$ is red. Since $n * F_n$ does not contain an entirely red row, the vertex $v_{m,j}$ must have a non-red neighbor w in row $m+1$. But w is not a neighbor of $v_{l,j}$, trivially. We conclude that $v_{l,j}$ and $v_{m,j}$ have different labels. Since $v_{l,j}$ and $v_{m,j}$ have been chosen arbitrarily, the vertices of U have pairwise different labels.

Now suppose that the procedure terminates in Step 4. By analyzing Steps 2 and 3, it is easy to deduce that U is a subset of red vertices of size at least $\lfloor n/2 \rfloor$. Suppose that $v_{l,j}$ and $v_{m,k}$ are two vertices in U with $l \leq m$. The procedure certainly guarantees that $j \neq k$ and that both $v_{l+1,j}$ and $v_{m+1,k}$ are non-red. If $m \in \{l, l+2\}$,

then it is clear that $v_{l+1,j}$ distinguishes vertices $v_{l,j}$ and $v_{m,k}$, and therefore these vertices have different labels. If $m \notin \{l, l+2\}$, we may consider vertex $v_{m-1,k}$ which must be red. Since $n * F_n$ does not contain an entirely red row, the vertex $v_{m,k}$ must have a non-red neighbor w in row $m-1$. But w is not a neighbor of $v_{l,j}$, trivially. We conclude that $v_{l,j}$ and $v_{m,k}$ have different labels, and therefore, the vertices of U have pairwise different labels. The proof is complete. \square

By Theorem 3.2.1, the clique-width of graphs in \mathcal{F} is unbounded. Now let us show that \mathcal{F} is a minimal hereditary class of unbounded clique-width. To this end, we will employ a technical lemma proved in [Lozin, 2008]. First we need a definition.

Definition 3.2.1. *For a graph G and a vertex subset $W \subset V(G)$, two vertices of W are said to be W -similar if they have the same set of neighbors in $V \setminus W$. The number of equivalence classes in W with respect to W -similarity is denoted by $\mu(W)$.*

Lemma 3.2.2. *If the vertices of a graph G can be partitioned into subsets V_1, V_2, \dots in such a way that for each i ,*

- (1) $\text{cwd}(G[V_i]) \leq l$ with $l \geq 2$,
- (2) $\mu(V_i) \leq m$ and $\mu(V_1 \cup \dots \cup V_i) \leq m$,

then $\text{cwd}(G) \leq lm$.

In particular, Lemma 3.2.2 implies the following corollary.

Corollary 3.2.3. *The clique-width of $k * F_n$ is at most $2k$.*

Proof. Denote by V_i the i -th column of $k * F_n$. Since each column induces an independent set, it is clear that $\text{cwd}(G[V_i]) \leq 2$ for every i . Trivially, $\mu(V_i) \leq k$, since $|V_i| = k$. Also, denoting $W_i := V_1 \cup \dots \cup V_i$, it is not difficult to see that $\mu(W_i) \leq k$ for every i , since the vertices of the same row in W_i are W_i -similar. Now the conclusion follows from Lemma 3.2.2.

Now we use Lemma 3.2.2 and Corollary 3.2.3 to prove the following result.

Lemma 3.2.4. *For any fixed $k \geq 1$, the clique-width of $k * F_k$ -free graphs in the class \mathcal{F} is bounded by a function of k .*

Proof. Let k be a fixed number and G be a $k * F_k$ -free graph in \mathcal{F} . By definition of \mathcal{F} , the graph G is an induced subgraph of $n * F_n$ for some n . For convenience, assume that n is a multiple of k , say $n = tk$. The vertices of $n * F_n$ that induce G

will be called *black* and the remaining vertices of $n * F_n$ will be called *white*. Also, we will refer to the set of vertices of G in the same row of $n * F_n$ as a layer of G .

For $1 \leq i \leq t$, let us denote by W_i the subgraph of $n * F_n$ induced by the k consecutive rows $(i-1)k+1, (i-1)k+2, \dots, ik$. For simplicity, we will use the term 'row r of W_i ' when referring to the row $(i-1)k+r$ of $n * F_n$. We partition the vertices of G into subsets V_1, V_2, \dots, V_t according to the following procedure:

1. Set $V_j = \emptyset$ for $1 \leq j \leq t$. Add every black vertex of W_1 to V_1 . Set $i = 2$.
2. For $j = 1, \dots, n$,
 - if column j of W_i is entirely black, then add the first vertex of this column to V_{i-1} and the remaining vertices of the column to V_i .
 - otherwise, add the (black) vertices of column j preceding the first white vertex to V_{i-1} and add the remaining black vertices of the column to V_i .
3. Increase i by 1. If $i = t+1$, terminate the procedure.
4. Go back to Step 2.

Let us show that the partition V_1, V_2, \dots, V_t given by the procedure satisfies the assumptions of Lemma 3.2.2 with l and m depending only on k .

The procedure clearly assures that each $G[V_i]$ is an induced subgraph of $W_i \cup W_{i+1}$. By Corollary 3.2.3, we have $\text{cwd}(W_i \cup W_{i+1}) = \text{cwd}(2k * F_n) \leq 4k$. Since the clique-width of an induced subgraph cannot exceed the clique-width of the parent graph, we conclude that $\text{cwd}(G[V_j]) \leq 4k$, which shows condition (1) of Lemma 3.2.2.

To show condition (2) of Lemma 3.2.2, let us call a vertex $v_{m,j}$ of V_i boundary if either $v_{m-1,j}$ belongs to V_{i-1} or $v_{m+1,j}$ belongs to V_{i+1} (or both). It is not difficult to see that a vertex of V_i is boundary if it belongs either to the second row of an entirely black column of W_i or to the first row of an entirely black column of W_{i+1} . Since the graph G is $k * F_k$ -free, the number of columns of W_i which are entirely black is at most $k-1$. Therefore, the boundary vertices of V_i introduce at most $2(k-1)$ equivalence classes in V_i .

Now consider two non-boundary vertices $v_{m,j}$ and $v_{m,p}$ in V_i from the same row. It is not difficult to see that $v_{m,j}$ and $v_{m,p}$ have the same neighborhood outside V_i . Therefore, the non-boundary vertices of the same row of V_i are V_i -similar, and hence the non-boundary vertices give rise to at most $2k$ equivalence classes in V_i . Thus, $\mu(V_i) \leq 4k-2$ for all i .

An identical argument shows that $\mu(V_1 \cup \dots \cup V_i) \leq 3k-1 \leq 4k-2$ for all i . Therefore, by Lemma 3.2.2, we conclude that $\text{cwd}(G) \leq c(k) := 16k^2 - 8k$, which completes the proof. \square

Theorem 3.2.5. *\mathcal{F} is a minimal hereditary class of graphs of unbounded clique-width.*

Proof. Let X be a proper hereditary subclass of \mathcal{F} and $H \in \mathcal{F} - X$. Since H is an induced subgraph of $k * F_k$ for some k , each graph in X is $k * F_k$ -free. Therefore, by Lemma 3.2.4, the clique-width of graphs in X is bounded by a constant.

3.2.3 The class \mathcal{M} and a boundary subclass

Recall that \mathcal{M} is the hereditary closure of the set $\{M_n^{(2)} : n \geq 1\}$. By analogy with Theorem 3.2.1, one can prove the following result.

Theorem 3.2.6. *The clique-width of $M_n^{(2)}$ is at least $n/4$.*

Theorem 3.2.6 shows that the clique-width of graphs in the class \mathcal{M} is unbounded. However, unlike the class \mathcal{F} studied in the previous section, \mathcal{M} is not a minimal hereditary class of unbounded clique-width. To show this, let us provide an alternative definition of the graph $M_n^{(2)}$. Let $K_{n,n}$ be the complete bipartite graph with vertices a_1, \dots, a_n in one part and vertices b_1, \dots, b_n in the other part. Denote by $M_{n,n}$ the graph obtained from the $K_{n,n}$ by subdividing each edge $a_i b_j$ by a new vertex c_{ij} (i.e., by introducing vertex c_{ij} on the edge $a_i b_j$). It is not difficult to see that $M_{n,n}$ coincides with $M_n^{(2)}$. Therefore, every graph in \mathcal{M} is obtained from a bipartite graph by subdividing each of its edges exactly once (or is an induced subgraph of such a graph). We will call the vertices of type a_i or b_i in $M_{n,n}$ *black* and the vertices of type $c_{i,j}$ *white*.

We intend to show that \mathcal{M} is not a minimal hereditary class of unbounded clique-width. To this end denote by \mathcal{S}_k be the class of $(C_3, \dots, C_k, H_1, \dots, H_k)$ -free bipartite graphs of vertex degree at most 3 and by \mathcal{M}_k the intersection $\mathcal{S}_k \cap \mathcal{M}$. (H_i is defined as in Figure 2.6.)

Lemma 3.2.7. *For any natural k , the clique-width of graphs in \mathcal{M}_k is unbounded.*

Proof. It is known that both the clique-width and tree-width are unbounded in the class \mathcal{S}_k for any value of k [Lozin and Rautenbach, 2006]. Since subdivision of an edge does not change the tree-width of a graph (see e.g. [Lozin and Rautenbach, 2006]), by subdividing each edge of graphs in \mathcal{S}_k exactly once we obtain a class of graphs X of unbounded tree-width. Moreover, it is known that for graphs of bounded vertex degree, the tree-width is bounded if and only if the clique-width is bounded [Courcelle and Olariu, 2000]. Therefore, the clique-width of graphs in X also is unbounded, and obviously $X \subseteq \mathcal{S}_k \cap \mathcal{M}$, which proves the lemma. \square

Lemma 3.2.7 shows that \mathcal{M} is not a minimal hereditary class of unbounded clique-width. Indeed, for any k , the class \mathcal{M}_k is a subclass of \mathcal{M} simply because in \mathcal{M}_k the vertex degree is bounded by 3, while in \mathcal{M} it is not. Moreover, it is not difficult to see that $M_{2,2} = M_2^{(2)}$ is a C_8 , i.e., \mathcal{M}_8 is a subclass of $M_{2,2}$ -free graphs in \mathcal{M} .

Let us denote the limit class of the sequence $\mathcal{S}_1 \supset \mathcal{S}_2 \supset \mathcal{S}_3 \dots$ by \mathcal{S} , i.e., $\mathcal{S} = \bigcap_{k \geq 1} \mathcal{S}_k$. It is not difficult to see that \mathcal{S} is the class of graphs every connected component of which is of the form $S_{i,j,k}$ represented in Figure 2.6. Obviously, \mathcal{S} is a subclass of \mathcal{M}_k for each k . Therefore, \mathcal{S} is a limit class of the sequence $\mathcal{M}_1 \supset \mathcal{M}_2 \supset \mathcal{M}_3 \dots$ as well. In the rest of the section, we show that \mathcal{S} is a minimal limit subclass of \mathcal{M} , i.e., for any graph $H \in \mathcal{S}$, the clique width of graphs in $\mathcal{M} \cap \text{Free}(H)$ is bounded by a constant. This will be done through a sequence of auxiliary lemmas. The first lemma in this sequence was proved in [Lozin and Rautenbach, 2004b].

Lemma 3.2.8 (Lozin and Rautenbach [2004b]). *For a class of graphs X and an integer ρ , let $[X]_\rho$ be the class of graphs G such that $G - U$ belongs to X for some subset $U \subseteq V(G)$ of cardinality at most ρ , and let $[X]_B$ be the class of graphs every block of which belongs to X . If the clique-width of graphs in X is bounded by p , then the clique-width of graphs in $[X]_\rho$ is bounded by $2^\rho(p + 1)$, and the clique-width in $[X]_B$ is bounded by $p + 2$.*

In the proofs of the next lemmas we will frequently use the following obvious observation.

Observation. Any cycle in any graph $G \in \mathcal{M}$ is chordless.

Lemma 3.2.9. *For $k \geq 3$, the clique-width of graphs in $L_k := \mathcal{M} \cap \text{Free}(C_k, C_{k+1}, \dots)$ is bounded by a function of k .*

Proof. For $k = 3$, the proposition follows from the fact that every graph in L_3 is a forest. For $k > 3$, we use induction on k .

Let G be a graph in L_{k+1} . By Lemma 3.2.8, we can assume without loss of generality that G is 2-connected. If G contains no cycles of length k , then $G \in L_k$ in which case the lemma follows by induction. Now let C be a cycle of length k in G . We will show that any other cycle C' of length k in G (if any) has a common vertex with C . Assume the contrary: C and C' are vertex disjoint. Consider two edges $e \in C$ and $e' \in C'$. Since G is 2-connected, there is a cycle containing both e and e' . In this cycle, one can distinguish two disjoint paths P and Q , each of

which contains the endpoints in C and C' , and the remaining vertices outside the cycles. The endpoints of the paths P and Q partition each of the cycles C and C' into two parts. The larger parts in both cycles together with paths P and Q form a cycle of length at least $k + 2$, contradicting the assumption that $G \in L_{k+1}$. This contradiction shows that any two cycles of length k in G have a vertex in common. Therefore, removing the vertices of any cycle of length k from H results in a graph in L_k , as required. \square

Lemma 3.2.10. *For each $k \geq 1$, the clique-width of graphs in $\mathcal{M} \cap \text{Free}(S_{k,k,k})$ is bounded by a function of k .*

Proof. Let $G \in \mathcal{M} \cap \text{Free}(S_{k,k,k})$. Consider a chordless path P of length $2k - 2$ and a chordless cycle C of length at least $2k + 2$ in G . If G does not contain such P or C , the the clique-width of G is bounded according to Lemma 3.2.9. Assume P and C are vertex disjoint. Since G is connected, there must exist a chordless path P' connecting C to P . Since only black vertices of C can have neighbors outside C , the vertex of P' that has a neighbor on C is white, and therefore this vertex has exactly one neighbor on C . Similarly, it is not difficult to see that the vertex of P' that has a neighbor on P is adjacent to exactly one vertex of P . But now the reader can easily find an induced $S_{k,k,k}$. This contradiction shows that P and C contain a vertex in common. Therefore, the graph obtained from G by deletion of the vertices of P belongs to L_{2k+2} , and the proposition follows from Lemmas 3.2.8 and 3.2.9. \square

Theorem 3.2.11. *For any graph $H \in \mathcal{S}$, the clique-width of graphs in $\mathcal{M} \cap \text{Free}(H)$ is bounded by a constant.*

Proof. Without loss of generality we will assume that every connected component of H is of the form $S_{k,k,k}$ for some *even* $k \geq 2$ (obviously every graph in \mathcal{S} is an induced subgraph of a graph of this form). Let p be the number of connected components of H , i.e., $H = pS_{k,k,k}$. We will show that the clique-width of any graph G in $\mathcal{M} \cap \text{Free}(H)$ is bounded by a function of k and p . The proof will be given by induction on the minimum number $m \leq p$ such that G is $mS_{k,k,k}$ -free. If $m = 1$, then the clique-width of G is bounded according to Lemma 3.2.10. If G contains an induced copy of $S_{k,k,k}$, then by deleting this copy we obtain a graph G' which is $(m - 1)S_{k,k,k}$ -free. Indeed, if G' contains an induced copy of $(m - 1)S_{k,k,k}$, then there are no edges between this copy and the deleted copy of $S_{k,k,k}$ in G , because k is even, which means white vertices in both copies have no neighbors outside the copies. By the induction hypothesis, the clique-width of G' is bounded by a function of k and p . Therefore, by Lemma 3.2.8, the clique-width of G is bounded as well, since the number of deleted vertices is $3k + 1$. \square

3.3 Bipartite Double Bichain Graphs and Split Permutation Graphs

In this section, we identify one more class of bipartite graphs of unbounded clique-width. Moreover, we conjecture that this is a minimal class of bipartite graphs of unbounded clique-width. The related class of split graphs (i.e. graphs obtained by replacing one of the two parts of a bipartite graph with a clique) is known in the literature as split permutation graphs. We reveal the relationship between the two classes in Section 3.3.1. Then in Section 3.3.2, we prove that both of them have unbounded clique-width.

3.3.1 An overview of some properties of split graphs

In this subsection we survey and extend results on some special graph classes that will be of interest to us. We start with a few definitions:

Definition 3.3.1. *A graph is a comparability graph if its edges admit a transitive orientation. In other words, one can direct the edges in such a way that for any directed path xyz , there is a directed edge (arc) xz . A graph is a co-comparability graph if it is the complement of a comparability graph.*

We will define permutation graphs in more detail later. For now, we will use the following characterisation:

Theorem 3.3.1 ([Dushnik and Miller, 1941]). *A graph is a permutation graph if it is comparability and co-comparability.*

Definition 3.3.2. *For a graph G and vertices $v, w \in G$, we say that v dominates w if $N(w) \setminus \{v\} \subseteq N(v)$. It is easy to check that domination defines a quasi-order on $V(G)$. This quasi-order will be called the vicinal order.*

Definition 3.3.3. *The Dilworth number $dilw(G)$ of a graph G is defined as the minimum number of chains in any partition of $V(G)$ into chains with respect to the vicinal order. The same definition can be adapted to subsets of $V(G)$.*

Remark. By Theorem 1.2.1 [Dilworth, 1950], $dilw(G)$ is the maximum size of an antichain in $V(G)$ with respect to the vicinal order.

Example 3.3.1. A bipartite graph such that each part has Dilworth number 1 is called a bipartite chain graph. It is trivial to show that the class of bipartite chain graphs is precisely the class of $2K_2$ -free bipartite graphs.

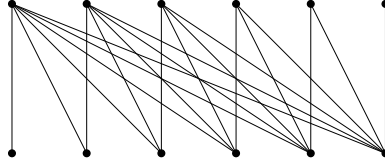


Figure 3.2: A bipartite chain graph.

Definition 3.3.4. A split graph is a graph G whose vertex set $V(G)$ is partitionable into a clique and an independent set.

The class of split graphs is clearly hereditary and has a well-known characterisation in terms of forbidden induced subgraph, due to Földes and Hammer:

Theorem 3.3.2 ([Földes and Hammer, 1977a]). The class of split graphs is given by $\text{Free}(2K_2, C_4, C_5)$. In other words a graph is a split graph iff it is chordal and co-chordal.

There is an obvious analogy between split graphs and bipartite graphs. In order to relate properties of these two classes, we will define the following notation to be used in this section:

Definition 3.3.5. For a split graph G , denote by $\beta(G)$ the bipartite graph obtained from G by replacing its clique-part with an independent set.

Remark. If we think of β as a function from the class of split graphs to the class of bipartite graphs, then β is clearly surjective. Also note that for an arbitrary bipartite graph H , the set $\beta^{-1}(H)$ will be a well-defined subset of split graphs with cardinality 1 or 2. For a set of split graphs \mathcal{G} we will denote $\beta(\mathcal{G}) := \{\beta(G) : G \in \mathcal{G}\}$. For a set of bipartite graphs \mathcal{H} , we will denote $\beta^{-1}(\mathcal{H}) := \cup_{H \in \mathcal{H}} \beta^{-1}(H)$.

Let us define an important class of graphs:

Definition 3.3.6. *A graph G is a threshold graph if there exists a real number S and a real weight $w(v)$ for each $v \in V(G)$ such that*

$$E(G) = \{((u, v) \in V(G)^2 : u \neq v \text{ and } w(u) + w(v) \geq S)\}.$$

Threshold graphs are motivated by applications in several disciplines such as psychology, computer science and scheduling [McKee and McMorris, 1999]. The following characterisation is due to Chvátal and Hammer:

Theorem 3.3.3 ([Chvátal and Hammer, 1973]). *A graph is a threshold graph iff it is a split cograph, i.e. a graph belonging to $\text{Free}(2K_2, P_4, C_4, C_5)$.*

Claim 3.3.4. *The class \mathcal{G} of threshold graphs corresponds to bipartite chain graphs in the sense that $\mathcal{H} := \beta(\mathcal{G})$ is the class of $2K_2$ -free bipartite graphs and $\beta^{-1}(\mathcal{H}) = \mathcal{G}$.*

Proof. This follows trivially from the observation that a split graph G is P_4 -free iff $\beta(G)$ is $2K_2$ -free. \square

Given the claim, the following theorem will not come as a surprise to the reader. Threshold graphs are precisely those graphs that form a chain with respect to the vicinal order:

Theorem 3.3.5 ([Földes and Hammer, 1978]). *A graph G is a threshold graph iff $\text{dil}w(G) = 1$.*

A natural question is to ask for a characterisation of graphs G with Dilworth number at most two.

Definition 3.3.7. *A graph is a threshold signed graph (TS-graph) if there exist real numbers S, T and a real weight $w(v)$ for each $v \in V(G)$ such that*

$$E(G) = \{((u, v) \in V(G)^2 : u \neq v \text{ and } (w(u) + w(v) \geq S \text{ or } |w(u) - w(v)| \leq T))\}$$

Benzaken et al. proved that this slight generalisation of threshold graphs gives the class of graphs with Dilworth number at most two:

Theorem 3.3.6 ([Benzaken et al., 1985a]). *G is a TS-graph iff $\text{dil}w(G) \leq 2$.*

Recall that class of split graphs is the intersection of chordal graphs and co-chordal graphs. Also, the class of permutation graphs is the intersection of comparability graphs and co-comparability graphs [Dushnik and Miller, 1941]. We quote one more result:

Proposition 3.3.7 ([Gilmore and Hoffman, 1964]). *Interval graphs are exactly the chordal co-comparability graphs.*

There are many equivalent ways to characterise split TS-graphs:

Theorem 3.3.8. *The following are equivalent:*

1. *G is a split graph with $\text{dilw}(G) \leq 2$.*
2. *G is a split TS-graph.*
3. *G is a split graph that is both an interval graph and a comparability graph.*
4. *G is a split permutation graph.*
5. *G is both an interval graph and a co-interval graph.*
6. *G is a (3-sun, co-3-sun, rising sun, co-rising sun)-free split graph.*

The equivalence of 1. and 2. is just a restatement of Theorem 3.3.6. The equivalence of 4. and 5. follows directly from Proposition 3.3.7. Since the same proposition implies that a split graph is an interval graph iff it is a co-comparability graph, the equivalence of 3. and 4. is also immediate. The proof of the equivalence of 1. and 3. is given in [Földes and Hammer, 1977b]. Finally, the proof of the equivalence of 1. and 6. is from [Akiyama et al., 1983].

In a much-quoted paper [Benzaken et al., 1985b], the authors claim to give a shorter proof to the result in [Akiyama et al., 1983], having found a forbidden induced subgraph characterisation for the class of TS-graphs. For the reader's benefit, we note that in the same paper (Theorem 5), the authors misquote a result from the paper by Földes and Hammer, making an error in justifying the equivalence of 2. and 5.

From this point forward, we will generally refer to split TS-graphs as split permutation graphs.

Definition 3.3.8. • *A bichain graph is a bipartite graph such that at least one part can be partitioned into at most two chains with respect to the vicinal order.*

- *A double bichain graph is a bipartite graph such that each part can be partitioned into at most two chains with respect to the vicinal order.*

We claim that there is a natural correspondence between split permutation graphs and double bichain graphs:

Proposition 3.3.9. *Let \mathcal{G} denote the class of split permutation graphs. Then $\beta(\mathcal{G})$ is the class of double bichain graphs and $\beta^{-1}(\beta(\mathcal{G}))$ is the class of split permutation graphs. Furthermore, double bichain graphs are precisely the $(3K_2, C_6, P_7)$ -free bipartite graphs.*

Proof. Let G be a split permutation graph. By Theorems 3.3.5 and 3.3.8, G can be partitioned into two threshold graphs G_1 and G_2 , each of which is a chain in G with respect to the vicinal order. But then $\beta(G_1)$ and $\beta(G_2)$ will offer a suitable partition of $\beta(G)$, proving that it is a double bichain graph. Conversely, for any double bichain graph H , one can easily partition any graph in $\beta^{-1}(H)$ into two threshold graphs, each forming a chain with respect to the vicinal order.

To prove the forbidden induced subgraph characterisation for double bichain graphs, it suffices to refer to Theorem 3.3.8, noting that $\beta(3\text{-sun, co-3-sun, rising sun, co-rising sun}) = (3K_2, C_6, P_7)$ and $\beta^{-1}(3K_2, C_6, P_7) = (3\text{-sun, co-3-sun, rising sun, co-rising sun})$. Clearly a split graph G is $(3\text{-sun, co-3-sun, rising sun, co-rising sun})$ -free iff $\beta(G)$ is $(3K_2, C_6, P_7)$ -free.

For completeness, let us give a direct proof for the forbidden induced subgraph characterisation of double bichain graphs. This will also allow us to make some observations about the respective characterisation for bichain graphs.

Proposition 3.3.10. *The class of double bichain graphs is the class of $(3K_2, C_6, P_7)$ -free bipartite graphs.*

Proof. To prove the theorem, we will show that a bipartite graph $G = (U, V, E)$ with a bipartition $U \cup V$ is $(P_7, C_6, 3K_2)$ -free if and only if the vertices of each part of the graph can be partitioned into at most 2 vicinal chains.

One direction of the proof is simple, because at least one part in each of the graphs P_7, C_6 and $3K_2$ contains three vertices which are incomparable with respect to the vicinal order.

Now assume that G is $(P_7, C_6, 3K_2)$ -free. Suppose, for contradiction, that in one part of G , say U , there is an antichain of three vertices a, b, c with respect to the vicinal order. Then, in the part V , there exists a vertex d which is adjacent to a but not b , and a vertex e which is adjacent to b but not a . We will split the proof into three cases:

Case 1. Suppose c is adjacent to both d and e . Then there must exist a vertex f which is adjacent to a but not c . Vertex b must be non-adjacent to f , otherwise $afbecd$ would form an induced C_6 . So there must exist a vertex g which is adjacent

to b but non-adjacent to c . Again, to avoid an induced C_6 , the vertex g must also be non-adjacent to a . But then $fadcebg$ forms an induced P_7 , which is a contradiction.

Case 2. Now suppose that c is adjacent to exactly one of d and e , say e . Then there must exist a vertex g which is adjacent to b but not c , and a vertex h which is adjacent to c but not b . If a were adjacent to neither of g and h , then $adbgch$ would form an induced $3K_2$. If a were adjacent to exactly one of g and h , say g , then $dagbech$ would form an induced P_7 . Finally, if a is adjacent to both g and h , then $agbech$ would form an induced C_6 . Each of these possibilities is a contradiction.

Case 3. Finally, suppose c is non-adjacent to d and e . Then there must exist a vertex h which is adjacent to c but not b , and a vertex i which is adjacent to c but not a . Note that h and i must not be the same vertex, since otherwise $adbech$ would form an induced $3K_2$. The vertex a must be adjacent to h , otherwise $adbech$ would form an induced $3K_2$. Similarly, b must be adjacent to i . Now $dahcibe$ forms an induced P_7 , which is a contradiction.

We have exhausted all possible cases, each leading to a contradiction. Thus our proof is complete. \square

Note that all of the contradicting copies of P_7 found in the proof of Proposition 3.3.10 have the same bi-coloring with respect to the considered $(P_7, C_6, 3K_2)$ -free bipartite graph G . Thus we obtain the following immediate corollary:

Corollary 3.3.11. *The class of (bichain) graphs $G := (U, V, E)$ such that U has Dilworth number at most 2 is precisely the class of $(P_7^U, C_6, 3K_2)$ -free bipartite graphs, where P_7^U is a copy of P_7 containing exactly three vertices in U .*

Trivially, the class of double bichain graphs $G := (U, V, E)$ is merely the intersection of the two classes of $(P_7^U, C_6, 3K_2)$ -free bipartite graphs and $(P_7^V, C_6, 3K_2)$ -free bipartite graphs. We can make an analogous statement for the class of bichain graphs:

Corollary 3.3.12. *The class of bichain graphs $G := (U, V, E)$ is the union of the two classes of $(P_7^U, C_6, 3K_2)$ -free bipartite graphs and $(P_7^V, C_6, 3K_2)$ -free bipartite graphs, where P_7^U and P_7^V are defined as in the previous corollary. Thus the class of bichain graphs is the class of $\{3K_2, C_6\} \cup \mathcal{S}$ -free bipartite graphs, where \mathcal{S} is the set of minimal graphs containing copies of both P_7^U and P_7^V .*

Determining the set \mathcal{S} in Corollary 3.3.12 would result in a forbidden induced subgraph characterisation of bichain graphs.

In terms of split graphs $G := (U, V, E)$, with clique-part U , we obtain:

- $\beta(\text{co-rising sun}) = P_7^U$
- $\beta(\text{rising sun}) = P_7^V$
- $\beta(3\text{-sun}) = C_6$
- $\beta(\text{co-3-sun}) = 3K_2$

In [Földes and Hammer, 1977b], it is shown that split interval graphs (i.e. split co-comparability graphs) are the (3-sun, co-3-sun, rising sun)-free split graphs. Analogously, the split co-interval graphs (i.e. split comparability graphs) are the (3-sun, co-3-sun, co-rising sun)-free split graphs.

In other words, split interval graphs are the split graphs whose 'independent set'-part has Dilworth number at most 2. Analogously, the split co-interval graphs are the split graphs whose clique-part has Dilworth number at most 2.

We may thus deduce that determining the set \mathcal{S} in Corollary 3.3.12 would also result in a forbidden induced subgraph characterisation of the class of all split graphs that are either comparability or co-comparability.

3.3.2 Split permutation graphs of large clique-width

In this section, we prove that the clique-width of split permutation graphs can be arbitrarily large. We quote the following theorem by Courcelle [Courcelle, 2004] which deals with infinite countable graphs (i.e. graph whose vertex set is countable):

Theorem 3.3.13. *If a countable graph G has clique-width greater than 2^{2k+1} , then some finite induced subgraph of G has clique-width greater than k .*

We consider a countable grid of vertices $\{v(i, j) : i, j \in \mathbb{N} \cup \{0\}\}$. We say that vertex $v(i, j)$ belongs to row i and column j . We define the following sets of vertices for all $i, j \in \mathbb{N} \cup \{0\}$ and all $n \in \mathbb{N}$.

- $X_{i,n} := \{v(i, in + 1), v(i, in + 2), \dots, v(i, (i + 1)n)\}$ (*A horizontal block*)
- $Y_{j,n} := \{v(jn + 1, j), v(jn + 2, j), \dots, v((j + 1)n, j)\}$ (*A vertical block*)
- $X_0 := \{x_0, x_1, x_2, \dots\}$
- $Y_0 := \{y_0, y_1, y_2, \dots\}$

It is not difficult to see that the blocks are pairwise disjoint for fixed n . Now let us define the graph G_n by

$$V(G_n) = X_0 \cup Y_0 \cup (\cup_{i=0}^{\infty} X_{i,n}) \cup (\cup_{j=0}^{\infty} Y_{j,n})$$

and $E(G_n) = E_0 \cup E_x \cup E_y$, where

$$E_0 = \{(x_i, x_j) : i, j \in \mathbb{N} \cup \{0\}\} \cup \{(x_i, y_j) : i, j \in \mathbb{N} \cup \{0\}\} \cup \{(y_i, y_j) : i, j \in \mathbb{N} \cup \{0\}\}$$

$$E_x = \{(x_i, v(r, s)) : i, r, s \in \mathbb{N} \cup \{0\}, r \geq i \text{ and } v(r, s) \in V(G_n)\}$$

$$E_y = \{(y_j, v(t, u)) : j, t, u \in \mathbb{N} \cup \{0\}, u \geq j \text{ and } v(t, u) \in V(G_n)\}$$

Lemma 3.3.14. G_n is a split permutation graph.

Proof. Since $X_0 \cup Y_0$ is a clique (the edge set E_0) and the remaining vertices form an independent set, the graph G_n is a split graph. Also, if $i < j$, then $N(x_j) \subset N(x_i) \cup \{x_i\}$ and $N(y_j) \subset N(y_i) \cup \{y_i\}$. Therefore, the set $X_0 \cup Y_0$ can be split into two vicinal chains. Finally, it is not difficult to see that the set of vertices in horizontal blocks and the set of vertices in vertical blocks each forms a vicinal chain. Therefore, the Dilworth number of G_n is at most 2 and hence by Theorem 3.3.8, G_n is a split permutation graph. \square

Lemma 3.3.15. $cwd(G_n) \geq n/4$.

Proof. Let T be a parse tree defining G_n . For a node a in T , we denote by $T(a)$ the subtree of T rooted at a . The label of a vertex v of the graph G_n at the node a is defined as the label that v has immediately prior to applying the operation a .

Let a be a lowest \oplus -node in T such that $T(a)$ contains a full block (i.e. a full set of form $X_{i,n}$ or $Y_{j,n}$) of G_n , and denote by b and c the two sons of a in T . We colour the vertices of $T(b)$ and $T(c)$ by red and blue, respectively, and all the other vertices by white. If u and v are red vertices and there exists a non-red vertex w which is adjacent to u but not to v , we say that w *distinguishes* between u and v . The following fact is easy to deduce:

- If w distinguishes between u and v , then u and v have different labels at node a .

In order to justify this fact, we simply note that the operation of type η for introducing the edge (w, u) is located outside $T(a)$, so u and v must have different labels to avoid creating an edge (w, v) under the same operation. Of course, the respective fact holds for blue vertices u and v , where w is non-blue.

Due to the choice of a , the graph G_n does not contain an entirely red block or an entirely blue block, but it contains a block with no white vertex. We may assume without loss of generality that there exists such a block which is horizontal and

contains at least $n/2$ red vertices. (Otherwise we could swap the roles of columns and rows and/or the colors blue and red.) Let $X_{k,n}$ be such a horizontal block.

To prove the lemma, we will show that G_n contains a subset of red vertices of size at least $n/4$ whose members have pairwise different labels at node a . Any subset of this type will be called *good*.

Consider the set $Y'_0 := \{y_j \in Y_0 : \text{there exists a red vertex of } X_{k,n} \text{ in column } j\}$. We split our proof into two cases.

Case 1. At least half of the vertices in Y'_0 are non-red. Let us denote the set of non-red vertices of Y'_0 by Y_0^* and let $X^* := \{v(k, j) \in X_{k,n} : y_j \in Y_0^*\}$. Note that $|X^*| = |Y_0^*| \geq n/4$. It suffices to show that X^* is a good set of red vertices. Choose any pair of vertices $v(k, j_1), v(k, j_2) \in X^*$ such that $j_1 < j_2$. Then, by definition, the non-red vertex y_{j_2} distinguishes between $v(k, j_1)$ and $v(k, j_2)$. By our earlier observation, this implies that $v(k, j_1)$ and $v(k, j_2)$ have different labels at node a of T . Thus X^* is a good set of red vertices.

Case 2. At least half of the vertices in Y'_0 are red. Let us denote the set of red vertices of Y'_0 by Y_0^{**} . For each $y_j \in Y_0^{**}$, choose a non-red vertex $v(i(j), j) \in Y_{j,n}$. This is possible, since by minimality of node a , there exist no entirely red blocks in G_n .

We denote $X^{**} := \{v(i(j), j) : y_j \in Y_0^{**}\}$. Note that $|Y_0^{**}| = |X^{**}| \geq n/4$. It suffices to show that Y_0^{**} is a good set of red vertices. Choose any pair of vertices $y_{j_1}, y_{j_2} \in Y_0^{**}$ such that $j_1 < j_2$. Then, by definition, the non-red vertex $v(i(j_1), j_1)$ distinguishes between y_{j_1} and y_{j_2} . By our earlier observation, this implies that y_{j_1} and y_{j_2} have different labels at node a of T . Thus Y_0^{**} is a good set of red vertices.

Since we attained a good set of red vertices in both cases, we are done. \square

Combining Theorem 3.3.13 and Lemmas 3.3.14 and 3.3.15, we obtain the following conclusion:

Theorem 3.3.16. *The class of (finite) split permutation graphs has unbounded clique-width.*

With each split graph $G = (K, I, E)$ we can associate a bipartite graph $\beta(G)$, as was done in Definition 3.3.5.

From Propositions 3.3.9 and 3.3.10, we can derive the following conclusion:

Corollary 3.3.17. *The class of $(P_7, C_6, 3K_2)$ -free bipartite graphs (i.e. the class of double bichain graphs) has unbounded clique-width.*

3.4 Open Problems

An open problem arises from the fact that bipartite permutation graphs constitute a minimal hereditary class (and thus a boundary class) of unbounded clique-width, meaning that in all proper hereditary subclasses of bipartite permutation graphs the clique-width is bounded by a constant. We now ask:

- (1) Is the class of split permutation graphs a minimal hereditary class of graphs of unbounded clique-width?

We believe that the answer to this question is affirmative and leave this as a conjecture for future research.

Recalling that planar graphs form the unique boundary class with respect to boundedness of tree-width, and rectangle grids are canonical planar graphs with respect to minor inclusion, it is not very surprising that most (if not all) of the graphs of unbounded clique-width that have been found and studied also have a grid-like structure. It is an interesting challenge to construct graphs of large clique-width that do not adhere to this kind of structure, if possible.

Chapter 4

Induced Subgraphs and Well-Quasi-Orderability

4.1 Definitions and Examples

For definitions of a *quasi-order* and an *antichain*, we refer the reader to Section 1.2.

Definition 4.1.1. *A quasi-order (X, \leq) is a well-quasi-order if X contains no infinite strictly decreasing sequences and no infinite antichains.*

According to the celebrated Graph Minor Theorem [Robertson and Seymour, 2004], the set of all graphs is well-quasi-ordered by the minor relation. As a consequence, any minor-closed graph class must be finitely defined (by a minimal set of forbidden minors). This, however, is not the case for the more restrictive relations such as subgraphs or induced subgraphs. Clearly, the cycles C_3, C_4, C_5, \dots form an infinite antichain with respect to both relations. Except for this example, only a few other infinite antichains are known with respect to the subgraph or induced subgraph relations. One of them is the sequence of graphs H_1, H_2, H_3, \dots reproduced once more in Figure 4.1(left). Moreover, in [Ding, 1992], the author proved that, in a sense, the cycles C_3, C_4, C_5, \dots and the graphs H_1, H_2, H_3, \dots are the only two infinite antichains with respect to the subgraph relation. More formally, Ding proved that a class of graphs closed under taking subgraphs is well-quasi-ordered by the subgraph relation if and only if it contains finitely many graphs C_n and H_n .

The situation with induced subgraphs is less explored. One of the first non-trivial results in this area was proved by Damaschke, who showed that the class of cographs is well-quasi-ordered by induced subgraphs [Damaschke, 1990]. A cograph is a graph whose every induced subgraph with at least two vertices is either disconnected or

the complement of a disconnected graph. The class of cographs is precisely the class of P_4 -free graphs, i.e., graphs containing no P_4 as an induced subgraph. Damaschke also showed that the class of P_4 -free graphs is the only maximal monogenic class (i.e. a class defined by a single forbidden induced subgraph) which is well-quasi-ordered by induced subgraphs.

An attempt to prove a similar result for bipartite graphs was made by Ding, who studied monogenic classes of bipartite graphs (i.e. classes of bipartite graphs defined by a single forbidden induced subgraph) [Ding, 1992]. For some of these classes, he proved well-quasi-orderability; and for some others, he found infinite antichains.

In this chapter, we provide a complete characterisation of monogenic classes of bipartite graphs with respect to their well-quasi-orderability by induced subgraphs. We also make considerable progress towards completing a wqo-characterisation of classes of graphs that are defined by forbidding exactly two induced subgraphs. We show that for finitely defined graph classes with larger forbidden sets, one cannot expect a similar characterisation without relying on the notion of a boundary class. Nevertheless, we prove that by bounding the size of the forbidden set, the (generally infinite) number of boundary classes becomes finite.

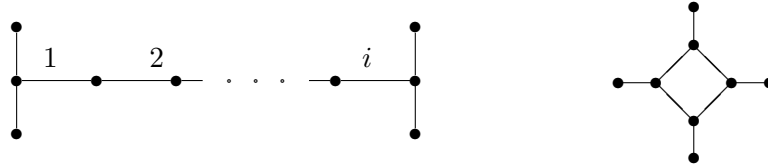


Figure 4.1: Graphs H_i (left) and Sun_4 (right)

4.2 Well-Quasi-Orderability of Classes of Bipartite Graphs

The exclusion of an induced linear forest (disjoint union of paths) is obviously a necessary condition for a class of graphs defined by finitely many forbidden induced subgraphs to be well-quasi-ordered by induced subgraphs, since otherwise the class contains infinitely many cycles. It is also necessary for such classes to exclude the complement of an induced linear forest, since the complements of cycles also form an antichain with respect to the induced subgraph relation.

As we mentioned in the previous section, Damaschke proved that the class of cographs (P_4 -free graphs) is well-quasi-ordered by induced subgraphs [Damaschke, 1990]. It is not difficult to show that Damaschke's result gives a complete charac-

terisation of the well-quasi-orderability of monogenic graph classes (classes defined by one forbidden induced subgraph). In this section, we achieve a similar characterisation of monogenic classes of bipartite graphs, i.e. classes of bipartite graphs defined by forbidding exactly one induced *bipartite* subgraph.

As a gateway from general graphs to bipartite graphs, Ding studied bi-cographs, i.e., the bipartite analog of cographs: these are bipartite graphs whose every induced subgraph with at least two vertices is either disconnected or the *bipartite complement* of a disconnected graph [Ding, 1992]. Ding proved that the class of bi-cographs is also well-quasi-ordered by induced subgraphs. In terms of forbidden induced subgraphs this is precisely the class of $(P_7, \text{Sun}_4, S_{1,2,3})$ -free bipartite graphs [Giakoumakis and Vanherpe, 1997], where Sun_4 is the graph represented in Figure 4.1(right) and $S_{1,2,3}$ is a tree with 3 leaves being of distance 1,2,3 from the only vertex of degree 3.

In order to attain well-quasi-orderability in a class of bipartite graphs, one must exclude not only an induced path, but also the *bipartite complement* \tilde{P}_k of an induced path P_k . Excluding an induced path and the bipartite complement of an induced path is not, however, sufficient for a class of bipartite graphs to be well-quasi-ordered. In [Ding, 1992], the author found an infinite antichain of (P_8, \tilde{P}_8) -free bipartite graphs. On the other hand, he proved that (P_6, \tilde{P}_6) -free bipartite graphs are well-quasi-ordered by induced subgraphs. Observe that the bipartite complement of a P_7 is a P_7 again. The question whether the class of P_7 -free bipartite graphs is well-quasi-ordered remained open for about 20 years. In this section, we answer this question negatively by exhibiting an antichain of $2P_3$ -free bipartite graphs. Moreover, we show that this antichain is also Sun_4 -free. On the other hand, we show that (P_7, Sun_1) -bipartite graphs are well-quasi-ordered by the induced subgraph relation, where Sun_1 is the graph obtained from Sun_4 by deleting 3 vertices of degree 1. We also obtain two other positive results. First, we show that $(P_7, S_{1,2,3})$ -free bipartite graphs are well-quasi-ordered by induced subgraphs, generalizing both the bi-cographs and P_6 -free graphs. Second, we prove that P_k -free bipartite permutation graphs are well-quasi-ordered by induced subgraphs for any value of k . The latter fact is in contrast with one more negative result of the section: by strengthening the Ding's idea, we show that (P_8, \tilde{P}_8) -free bipartite graphs are not well-quasi-ordered even when restricted to *biconvex graphs*, a class generalizing bipartite permutation graphs. The relationship between the classes of graphs under consideration is represented in Figure 4.2.

Let us repeat that Ding proved that the class of (P_8, \tilde{P}_8) -free bipartite graphs is not well-quasi-ordered by the induced subgraph relation [Ding, 1992]. In this section, we strengthen this result in two ways. First, as we mentioned earlier, we

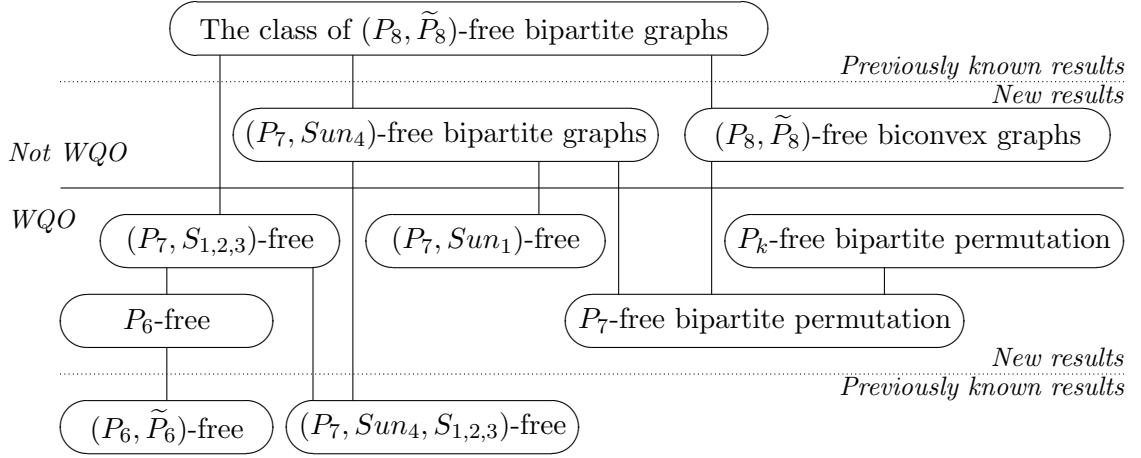


Figure 4.2: Inclusion relationships between subclasses of bipartite graphs

show that $2P_3$ -free bipartite graphs are not wqo. Then we prove that (P_8, \tilde{P}_8) -free *biconvex* graphs are not wqo. To prove the results, in both cases we use the notion of a permutation, i.e., a bijection of the set $[n] := \{1, 2, \dots, n\}$ to itself. To represent a permutation $\pi : [n] \rightarrow [n]$, we use one of the following two ways:

- one-line notation, which is the ordered sequence $(\pi(1), \pi(2), \dots, \pi(n))$.
- a diagram (see Figure 4.3 for an example).

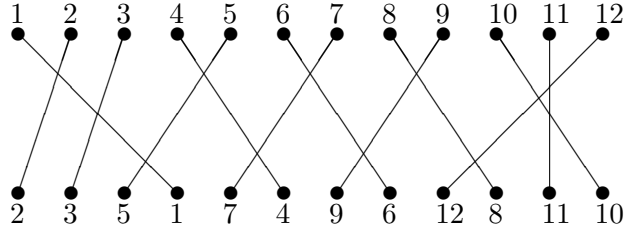


Figure 4.3: Diagram representing the permutation $(2, 3, 5, 1, 7, 4, 9, 6, 12, 8, 11, 10)$.

The permutation graph G_π of a permutation π is the intersection graph of the diagram representing π . Figure 4.4 gives an example of a permutation and its permutation graph.

The composition $\mu \circ \rho$ of two permutations μ and ρ is a permutation π such that $\pi(i) = \mu(\rho(i))$. The inverse of a permutation π is a permutation π^{-1} such that $\pi^{-1}(\pi(i)) = i$.

Let π and ρ be two permutations given in one-line notation. We say that π is contained in ρ if ρ has a subsequence which is order-isomorphic to π . It is not difficult to see that if G_π is not an induced subgraph of G_ρ , then π is not contained in ρ .

4.2.1 The class of $(2P_3, Sun_4)$ -free bipartite graphs is not WQO

We start by introducing a special class of bipartite graphs defined as follows:

Definition 4.2.1. *For each permutation $\pi := \pi_n$ on $[n]$, the graph $T := T_\pi$ is a bipartite graph with parts $A \cup C$ and $B \cup D$, where:*

1. *The vertex set of T is the disjoint union of four independent vertex sets*
 - $A := \{a_1, a_2, \dots, a_n\}$,
 - $B := \{b_1, b_2, \dots, b_n\}$,
 - $C := \{c_1, c_2, \dots, c_n\}$,
 - $D := \{d_1, d_2, \dots, d_n\}$.
2. *$X(T) := T[A \cup B]$ is a 1-regular graph with $e_i := a_i b_{\pi(i)}$ being an edge for each $i \in [n]$.*
3. *$Y(T) := T[C \cup D]$ is a biclique (i.e., a complete bipartite graph).*
4. *Each of $Z'(T) := T[A \cup D]$ and $Z''(T) := T[B \cup C]$ is a $2K_2$ -free bipartite graph defined as follows: for $i = 1, 2, \dots, n$,*
 - $N_{Z'}(a_i) = \{d_1, \dots, d_i\}$,
 - $N_{Z''}(b_i) = \{c_1, \dots, c_i\}$.

Any graph of the form T_π will be called a T -graph.

In order to derive the main result of this section, we will show that every T -graph is $(2P_3, Sun_4)$ -free and that the set of T -graphs is not well-quasi-ordered by induced subgraphs.

Lemma 4.2.1. *Any T -graph is $(2P_3, Sun_4)$ -free.*

Proof. Suppose, for contradiction, that T contains an induced $2P_3$. Then it is easy to check that each of the two P_3 must contain at least one vertex in each of $X(T)$ and $Y(T)$. Note that the vertices in $2P_3 \cap Y(T)$ must all belong to the same part of the biclique $Y(T)$. We may assume without loss of generality that this part is D .

It is clear that each P_3 has an edge from A to D . But then $Z'(T)$ is not $2K_2$ -free, a contradiction showing that T is $2P_3$ -free.

Now suppose, for contradiction, that T contains an induced Sun_4 . Note that any two vertices in the same part of $Y(T)$ have nested neighborhoods. Therefore, no two vertices of degree 3 in the Sun_4 can belong to the same part of $Y(T)$. This implies that no two vertices of degree 3 in the Sun_4 can belong to the same part of $X(T)$. Therefore, each of A, B, C and D must contain exactly one vertex of degree 3 in the Sun_4 . Suppose that these vertices are a, b, c and d , respectively. The leaf attached to a in the Sun_4 cannot belong to B (since otherwise a has degree more than 1 in $X(T)$) and cannot belong to D (since otherwise $Y(T)$ is not a biclique). This contradiction shows that T is Sun_4 -free. \square

Now we turn to showing that the set of T -graphs is not well-quasi-ordered by the induced subgraph relation. To this end, for each even $n \geq 6$ we define a specific permutation π_n^* , as follows:

$$\pi_n^* := (4, 2, \dots, 2j, 2j-5, \dots, n-1, n-3) \quad j = 3, \dots, n/2.$$

For instance, $\pi_6^* = (4, 2, 6, 1, 5, 3)$ and $\pi_8^* = (4, 2, 6, 1, 8, 3, 7, 5)$. For $n = 10$, we use the diagram to represent π_{10}^* (see Figure 4.4 (left)). This diagram can also be interpreted as the subgraph $X(T)$ of $T_{\pi_{10}^*}$, which can be seen by labeling the vertices in the upper part of the diagram by a_1, \dots, a_{10} consecutively from left to right and the vertices in the lower part of the diagram by b_1, \dots, b_{10} consecutively from left to right. The permutation graph $G_{\pi_{10}^*}$ of the permutation π_{10}^* is represented in Figure 4.4 (right).

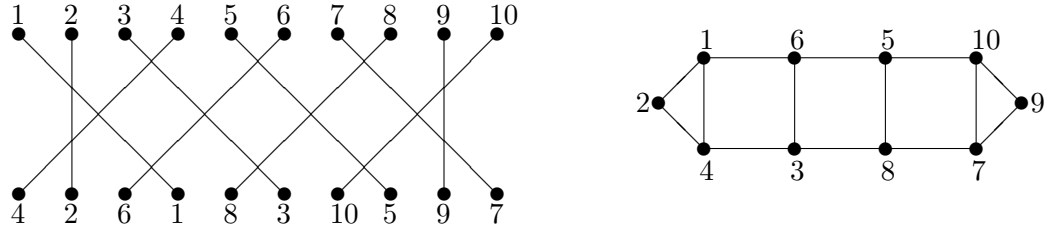


Figure 4.4: The permutation π_{10}^* (left) and the permutation graph $G_{\pi_{10}^*}$ (right)

The important fact about the permutations π_n^* is that

Claim 4.2.2. *The sequence $\pi_6^*, \pi_8^*, \pi_{10}^* \dots$ is an antichain of permutations with respect to the containment relation.*

This claim follows directly from the observation that no graph $G_{\pi_n^*}$ is an induced subgraph of $G_{\pi_m^*}$ with $n \neq m$. Indeed, in one of the two graphs, the length of the shortest induced path between the two disjoint triangles is strictly larger than in the other graph, and this value cannot be decreased by vertex deletions. We now use Claim 4.2.2 in order to prove the following result.

Lemma 4.2.3. *The sequence $T_{\pi_6^*}, T_{\pi_8^*}, T_{\pi_{10}^*}, \dots$ is an antichain with respect to the induced subgraph relation.*

Proof. Suppose by contradiction that there is a graph $H := T_{\pi_m^*}$ which is an induced subgraph of a graph $G := T_{\pi_n^*}$ for some even $6 \leq m < n$. We fix an arbitrary embedding of H into G , i.e., we assume that $V(H) \subset V(G)$. We will denote the vertex subsets A, B, C, D of the graph H by $A(H), B(H), C(H), D(H)$ and of the graph G by $A(G), B(G), C(G), D(G)$. Since both graphs are connected and in both graphs the role of the parts $A \cup C$ and $B \cup D$ is symmetric, we may assume that

Claim 4.2.4. $A(H) \cup C(H) \subseteq A(G) \cup C(G)$ and $B(H) \cup D(H) \subseteq B(G) \cup D(G)$.

Keeping Claim 4.2.4 in mind, we derive a series of conclusions. First, we show that

Claim 4.2.5. $|A(H) \cap C(G)| \leq 1$, $|C(H) \cap A(G)| \leq 1$, $|B(H) \cap D(G)| \leq 1$, and $|D(H) \cap B(G)| \leq 1$.

Proof. Suppose $|A(H) \cap C(G)| \geq 2$, and pick two distinct vertices $a_i, a_j \in A(H)$ that belong to $C(G)$. Let $\pi := \pi_m^*$. Since $Y(G)$ is a biclique, both $b_{\pi(i)}$ and $b_{\pi(j)}$ must lie in $B(G)$, which contradicts the $2K_2$ -freeness of $Z''(G)$. Thus $|A(H) \cap C(G)| \leq 1$.

Similarly, suppose $|C(H) \cap A(G)| \geq 2$, and pick two distinct vertices $a_i, a_j \in A(G)$ that belong to $C(H)$. Let $\pi := \pi_n^*$. Both of $b_{\pi(i)}$ and $b_{\pi(j)}$ cannot lie in $B(H)$, since this would contradict the $2K_2$ -freeness of $Z''(H)$.

Now suppose that $b_{\pi(i)}$ belongs to $B(H)$, but $b_{\pi(j)}$ does not. The vertex a_j must have some neighbour $b'_j \in B(H) \cap D(G)$, whereas $b_{\pi(i)}$ has a neighbour $a'_i \in A(H) \cap C(G)$. But now there exists an edge $a'_i b'_j$, contradicting the 1-regularity of $X(H)$.

Finally, suppose that neither of $b_{\pi(i)}$ and $b_{\pi(j)}$ belong to $B(H)$. Then a_i and a_j must have neighbours $b'_i, b'_j \in B(H) \cap D(G)$, $i \neq j$. But the inequality $|B(H) \cap D(G)| \leq 1$ follows from $|A(H) \cap C(G)| \leq 1$, by symmetry.

The rest of the proof follows by symmetry. \square

Now we prove that

Claim 4.2.6. $|X(H) \cap Y(G)| \leq 1$ and $|Y(H) \cap X(G)| \leq 1$.

Proof. By Claim 4.2.5 and the definition of $Y(G)$, if the intersection $X(H) \cap Y(G)$ contains two vertices, then these vertices must be adjacent. Let $\pi := \pi_m^*$ and suppose an edge $a_i b_{\pi(i)}$ of $X(H)$ belongs to $Y(G)$. By Claim 4.2.5, $|D(H) \cap B(G)| \leq 1$, which means that a_i is adjacent to all but at most one vertex of $D(H)$. According to the definition of H , we conclude that $i \in \{m-1, m\}$. Similarly, $b_{\pi(i)}$ is adjacent to all but at most one vertex of $C(H)$, implying that $\pi(i) \in \{m-1, m\}$. Together $i \in \{m-1, m\}$ and $\pi(i) \in \{m-1, m\}$ imply $i = \pi(i) = m-1$. From this and Claim 4.2.5 we conclude that both $a_m \in A(H)$ and $b_m \in B(H)$ belong to $X(G)$. Also, since

- $a_{m-1} \in A(H)$ belongs to $Y(G)$,
- a_{m-1} is not adjacent to $d_m \in D(H)$ in H and
- $Y(G)$ is a biclique,

we conclude that $d_m \in D(H)$ belongs to $X(G)$. Similarly, $c_m \in C(H)$ belongs to $X(G)$. By the definition of $Y(H)$, c_m is adjacent to d_m , and by the definition of $Z''(H)$, c_m is adjacent to b_m . But now c_m , being a vertex of $X(G)$, is adjacent to two other vertices of $X(G)$, i.e., d_m and b_m , contradicting the 1-regularity of this graph. This contradiction shows that $|X(H) \cap Y(G)| \leq 1$.

The second inequality can be reduced to the first. Indeed, suppose we have an edge $c_i d_i$ of $Y(H)$ belonging to $X(G)$. Then c_i must have a neighbor in $B(H) \cap D(G)$. Similarly, d_i must have a neighbour in $A(H) \cap C(G)$. This contradicts $|X(H) \cap Y(G)| \leq 1$. \square

Next, we show that

Claim 4.2.7. $X(H) \cap Y(G) = Y(H) \cap X(G) = \emptyset$.

Proof. Assume first that $X(H) \cap Y(G)$ is not empty, and suppose without loss of generality that a vertex a_i of $A(H)$ belongs to $Y(G)$. Then, by Claim 4.2.6, all vertices of $B(H)$ belong to $X(G)$. By Claim 4.2.5, $|D(H) \cap B(G)| \leq 1$, which means that a_i is adjacent to all but at most one vertex of $D(H)$. According to the definition of H , we conclude that $i = m-1$ or $i = m$. In either case, vertex b_m is not adjacent to a_i , and the neighborhood of b_m in the graph $Z''(H)$ is strictly greater than the neighborhood of $b_{\pi(i)}$.

Suppose $i = m$. By Claim 4.2.5, at least one of $c_{m-1}, c_m \in C(H)$ belongs to $C(G)$, say $c_m \in C(G)$. But then a_m, b_{m-3}, b_m, c_m induce a $2K_2$, contradicting the $2K_2$ -freeness of $Z''(G)$.

Suppose now that $i = m - 1$. By definition, the vertex a_{m-1} of $A(H)$ has a non-neighbor in $D(H)$. Therefore, the set $D(H)$ must have a vertex in $X(G)$. This implies by Claim 4.2.6 that $C(H) \subset C(G)$, and hence the vertices $a_{m-1}, b_{m-1}, c_m, b_m$ induce a $2K_2$, contradicting the $2K_2$ -freeness of $Z''(G)$. This completes the proof of the fact that $X(H) \cap Y(G) = \emptyset$.

Now assume that $Y(H) \cap X(G) \neq \emptyset$ and suppose without loss of generality that a vertex d_i of $D(H)$ belongs to $X(G)$. By the definition of T -graphs, d_i must have at least one neighbor in $A(H)$. Since $A(H) \subset A(G)$ (by the previous fact) and no vertex of $X(G)$ can have more than 1 neighbor in $X(G)$, we conclude that d_i has exactly one neighbor $a \in A(H)$. On the other hand, by the definition of $X(H)$, vertex a must have exactly one neighbor in $B(H)$, which is a subset of $B(G)$ (by the previous fact). But now a , being a vertex of $X(G)$, has two neighbors in $X(G)$, contradicting the definition of this graph. Therefore, $Y(H) \cap X(G) = \emptyset$. \square

Claims 4.2.7 and Claim 4.2.4 together imply the following conclusion.

Claim 4.2.8. $A(H) \subseteq A(G)$, $B(H) \subseteq B(G)$, $C(H) \subseteq C(G)$ and $D(H) \subseteq D(G)$.

Assuming that H is an induced subgraph of G , we must conclude that the ordering of vertices of $A(H)$ respects the ordering of vertices of $A(G)$, and similarly, the ordering of vertices of $B(H)$ respects the ordering of vertices of $B(G)$. But then we must conclude that π_m^* is contained in π_n^* which is a contradiction to Claim 4.2.2. This contradiction completes the proof of the lemma. \square

Lemmas 4.2.1 and 4.2.3 imply the main result of this section.

Theorem 4.2.9. *The class of $(2P_3, \text{Sun}_4)$ -free bipartite graphs is not well-quasi-ordered by the induced subgraph relation.*

4.2.2 The class of (P_8, \tilde{P}_8) -free biconvex graphs is not WQO

A bipartite graph is *biconvex* if the vertices of the graph can be linearly ordered so that the neighborhood of each vertex forms an interval, i.e., the neighborhood consists of consecutive vertices in the order. Strengthening the result from [Ding, 1992], we show in this section that the class of (P_8, \tilde{P}_8) -free *biconvex* graphs is not wqo by the induced subgraph relation. We start by introducing two special types of permutations.

Definition 4.2.2. A permutation π_n is convex if for any $1 \leq i \leq n$ the set $\{i, i+1, \dots, n-1, n\}$ forms an interval, i.e., the elements of the set occupy consecutive positions in the permutation.

For instance, the permutation $\rho = (1, 2, 3, 5, 7, 9, 10, 8, 6, 4)$ is convex. Indeed, the elements of the set $\{5, 6, 7, 8, 9, 10\}$ occupy positions 4, 5, 6, 7, 8, 9, the elements of the set $\{6, 7, 8, 9, 10\}$ occupy positions 5, 6, 7, 8, 9, and the same is true for any other set of the form $\{i, i+1, \dots, n-1, n\}$. The permutation $\mu = (2, 3, 5, 7, 10, 9, 8, 6, 4, 1)$ is another example of a convex permutation.

Definition 4.2.3. A permutation π_n is biconvex if there are two convex permutations μ and ρ such that $\pi = \mu \circ \rho^{-1}$.

To give an example, consider the following permutation:

$$\pi = (2, 3, 5, 1, 7, 4, 10, 6, 9, 8)$$

It is not difficult to verify that $\pi = \mu \circ \rho^{-1}$, where μ and ρ are the two convex permutations given above. For instance, $\pi(1) = \mu(\rho^{-1}(1)) = 2$, $\pi(2) = \mu(\rho^{-1}(2)) = 3$, $\pi(3) = \mu(\rho^{-1}(3)) = 5$, etc. Therefore, π is a biconvex permutation.

Now we introduce a special class of bipartite graphs defined as follows:

Definition 4.2.4. For a biconvex permutation $\pi := \pi_n$ such that $\pi = \mu \circ \rho^{-1}$, where μ and ρ are two convex permutations, the graph $S := S_\pi$ is a bipartite graph with parts $A \cup C$ and B , where:

1. $V(S)$ is the disjoint union of three independent vertex sets
 - $A := \{a_1, a_2, \dots, a_n\}$,
 - $B := \{b_1, b_2, \dots, b_n\}$,
 - $C := \{c_1, c_2, \dots, c_n\}$,
2. Each of $X(S) := S[A \cup B]$ and $Y(S) := S[B \cup C]$ is a $2K_2$ -free bipartite graph defined as follows: for $i = 1, 2, \dots, n$,
 - $N_X(b_i) = \{a_1, \dots, a_{\rho(i)}\}$,
 - $N_Y(b_i) = \{c_1, \dots, c_{\mu(i)}\}$.

Any graph of the form S_π will be called an S -graph.

Claim 4.2.10. Any S -graph is a (P_8, \widetilde{P}_8) -free biconvex graph.

Proof. Let $S := S_\pi$ be an S -graph associated with a biconvex permutation $\pi := \pi_n$ such that $\pi = \mu \circ \rho^{-1}$, where μ and ρ are two convex permutations. The (P_8, \tilde{P}_8) -freeness of S follows from the $2K_2$ -freeness of $X(T)$ and $Y(T)$. Now let us prove that S is biconvex. To this end, we need to show that the vertices in each part of the graph can be linearly ordered so that the neighborhood of any vertex in the opposite part forms an interval. To achieve this goal we keep the natural order of the vertices in the B -part, i.e., $B = (b_1, \dots, b_n)$. The vertices of the $A \cup C$ -part are ordered under inclusion of their neighborhoods, increasingly for the A -vertices and decreasingly for the C -vertices, i.e., the vertices with the largest neighborhood in A and C are in the middle of the order. Now let us show that the defined order is biconvex.

Let b be any vertex from B . If b is adjacent to any vertex a from A , then b is adjacent to any vertex from A with larger neighborhood than $N(a)$, i.e., b is adjacent to any vertex of A following a . Similarly, if b is adjacent to any vertex c from C , then b is adjacent to any vertex from C with larger neighborhood than $N(c)$, i.e., b is adjacent to any vertex of C preceding c . Therefore, $N(b)$ is an interval.

Now let a_i be a vertex from A . Let I be the interval (i.e., the set of positions) of length $n - i + 1$ containing the elements $\{i, \dots, n\}$ of the permutation ρ . Then $N(a_i) = \{b_j : j \in I\}$, i.e., $N(a_i)$ is an interval. Similarly, if c_i is a vertex from C and I is the interval of length $n - i + 1$ containing the elements $\{i, \dots, n\}$ of the permutation μ , then $N(c_i) = \{b_j : j \in I\}$, i.e., $N(c_i)$ is an interval. \square

Now we define a specific permutation π_n^* in the following way: for each even $n \geq 8$,

$$\pi_n^* := (2, 3, 5, 1, \dots, 2j+3, 2j, \dots, n, n-4, n-1, n-2) \quad j = 2, \dots, n/2 - 4.$$

For instance, $\pi_8^* = (2, 3, 5, 1, 8, 4, 7, 6)$ and $\pi_{10}^* = (2, 3, 5, 1, 7, 4, 10, 6, 9, 8)$. The permutation π_{12}^* is represented in Figure 4.3.

Let us show that π_n^* is a biconvex permutation. To this end, we define two convex permutations ρ_n^* and μ_n^* in the following way:

$$\rho_n^* := (1, 2, 3, 5, \dots, \text{odds}, \dots, n-3, n-1, n, n-2, \dots, \text{evens}, \dots, 6, 4).$$

$$\mu_n^* := (2, 3, 5, \dots, \text{odds}, \dots, n-3, n, n-1, n-2, n-4, \dots, \text{evens}, \dots, 6, 4, 1).$$

It is not difficult to verify that for $n = 10$ the permutations π_n^* , ρ_n^* and μ_n^* coincide with the permutations π , ρ and μ defined in the beginning of the section.

Claim 4.2.11. $\pi^* = \mu^* \circ \rho^{*-1}$.

Proof. For small and large values of i , one can verify by direct inspection that $\pi^*(i) = \mu^*(\rho^{*-1}(i))$. Now let $4 < i < n - 3$. If i is odd, then $\pi^*(i) = \mu^*(\rho^{*-1}(i)) = i + 2$, and if i is even, then $\pi^*(i) = \mu^*(\rho^{*-1}(i)) = i - 2$. \square

Lemma 4.2.12. *The sequence $S_{\pi_8^*}, S_{\pi_{10}^*}, S_{\pi_{12}^*}, \dots$ is an antichain with respect to the induced subgraph relation.*

Proof. Suppose by contradiction that there is a graph $H := S_{\pi_m^*}$ which is an induced subgraph of a graph $G := S_{\pi_n^*}$ for some even $8 \leq m < n$. We fix an arbitrary embedding of H into G , i.e., we assume that $V(H) \subset V(G)$. Since both graphs are connected, we may assume that exactly one of the following two possibilities holds:

1. $A(H) \cup C(H) \subseteq A(G) \cup C(G)$ and $B(H) \subseteq B(G)$.
2. $A(H) \cup C(H) \subseteq B(G)$ and $B(H) \subseteq A(G) \cup C(G)$

We claim that the first possibility holds.

Claim 4.2.13. $A(H) \cup C(H) \subseteq A(G) \cup C(G)$ and $B(H) \subseteq B(G)$.

Proof. Note that, by definition, $A(G) \cup C(G)$ can be partitioned into two chains with respect to the neighborhood inclusion. On the other hand, the set $B(H)$ does not have this property, since $b_{n/2}, b_{n/2+1}, b_{n/2+2}$ is an antichain of length 3 with respect to the same relation. Indeed, $\rho^*(n/2) = n - 3$, $\rho^*(n/2 + 1) = n - 1$, $\rho^*(n/2 + 2) = n$ and $\mu^*(n/2) = n$, $\mu^*(n/2 + 1) = n - 1$ and $\mu^*(n/2 + 2) = n - 2$. This proves the claim. \square

We make the following helpful remark:

- If two vertices of $B(H)$ are incomparable with respect to the neighborhood inclusion in $B(H)$, then these two vertices must also be incomparable with respect to the neighborhood inclusion in $B(G)$.

Let $B'(H)$ be the incomparability graph for the relation of neighborhood inclusion on the vertex set $B(H)$. In other words, two vertices of $B(H)$ are adjacent

in $B'(H)$ precisely when they are incomparable with respect to the neighborhood inclusion. We define $B'(G)$ similarly.

Clearly, by the above remark, $B'(H)$ must be a subgraph of $B'(G)$. But for any even $n \geq 8$, the graph $B'(S_{\pi_n}^*)$ is simply the permutation graph $G_{\pi_n^*}$ of π_n^* and this graph is represented in Figure 4.5.

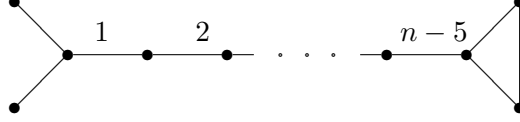


Figure 4.5: The graph $B'(S_{\pi_n}^*) = G_{\pi_n^*}$

It is not difficult to see that the sequence of graphs $G_{\pi_n^*}$, $n \geq 8$, forms an antichain with respect to the (induced) subgraph relation. Therefore, $B'(H)$ is not a subgraph of $B'(G)$. As a result, H is not an induced subgraph of G . This contradiction completes the proof of Lemma 4.2.12. \square

Lemma 4.2.12 and Claim 4.2.10 together imply the main result of this section:

Theorem 4.2.14. *The class of (P_8, \tilde{P}_8) -free biconvex graphs is not well-quasi-ordered by the induced subgraph relation.*

4.2.3 The class of double bichain graphs is not WQO

Recall Definition 3.3.8:

Definition 4.2.5. *A double bichain graph is a bipartite graph such that each part is partitionable into two chains with respect to neighbourhood inclusion.*

By Proposition 3.3.9, there is a natural correspondence between double bichain graphs and split permutation graphs.

We will use a characterization of bipartite permutation graphs from [Lozin and Rudolf, 2007].

Definition 4.2.6. *Let us define a canonical bipartite permutation graph as follows. For $n > m$, let $H_{m,n} := (X, Y, E)$, where*

- $X = \{x_1, \dots, x_n\}$ and $Y = \{y_1, \dots, y_n\}$
- $N(x_i) = \{y_i, y_{i+1}, \dots, y_{i+m}\}$ for $1 \leq i \leq n - m$

- $N(x_i) = \{y_i, y_{i+1}, \dots, y_n\}$ for $i > n - m$

Proposition 4.2.15. *A graph is a bipartite permutation graph if and only if it is an induced subgraph of $H_{m,n}$ for some m, n .*

Canonical bipartite permutation graphs are revisited in Section 4.2.6.

Definition 4.2.7. *Define $\phi := \phi_{m,n}$ to be the following linear ordering on the vertices of $H_{m,n}$:*

- $\phi(x_i) = 2i - 1$ for each i .
- $\phi(y_i) = 2i$ for each i .

Definition 4.2.8. *For each graph $H_{m,n} := (X, Y, E)$, define a double bichain graph $\Phi(H_{m,n}) := (X', Y', E')$ as follows:*

- $X' = \{x_1, y_1, x_2, y_2, \dots, x_n, y_n\}$ and $Y' = \{x'_1, y'_1, x'_2, y'_2, \dots, x'_n, y'_n\}$
- $E' = \{(x_i, x'_i), (y_i, y'_i) : 1 \leq i \leq n\} \cup \{(a, b') : a, b \in V(H_{m,n}), (a, b) \notin E \text{ and } \phi(a) < \phi(b)\}$

Remark. Note that $\Phi(H_{m,n})$ is a double bichain graph. This can be seen from the definition of $H_{m,n}$ by partitioning vertices v according to whether $\phi(v)$ is even or odd.

Definition 4.2.9. *Let G be any bipartite permutation graph. We define a double bichain graph $\Phi(G) := \Phi_{m,n}(G)$ by embedding G into some $H_{m,n}$ and by letting the ordering ϕ induce an ordering on the vertices of G . We define $\Phi(G)$ as the corresponding induced subgraph of $\Phi(H_{m,n})$.*

Theorem 4.2.16. *Any bipartite permutation graph G is the incomparability graph (with respect to neighbourhood inclusion) of each part of the double bichain graph $\Phi(G) := (A, B, E')$.*

Proof. Suppose that G is embedded into $H_{m,n} := (X, Y, E)$ and so we have $A \subseteq X'$ and $B \subseteq Y'$. We will denote the vertices of $\Phi(G)$ by the same convention as the corresponding vertices of $\Phi(H_{m,n})$.

We begin by showing that G is the incomparability graph of A . Let $a, b \in A$ and suppose that $(a, b) \in E$. Then $(a, b'), (b, a') \notin E'$, but $(a, a'), (b, b') \in E'$. Thus a and b are incomparable by neighbourhood inclusion.

Now let $a, b \in A$ such that $(a, b) \notin E$. We will show that a and b are comparable by neighbourhood inclusion. We may assume that $\phi(a) < \phi(b)$. Suppose that $(b, c') \in E'$. It suffices to show that $(a, c') \in E'$.

If $c' = b'$, then $(a, c') \in E'$ by definition of E' . Otherwise, by the same definition, $(b, c) \notin E$ and $\phi(b) < \phi(c)$. Now, by the definition of $H_{m,n}$ and the fact that $\phi(a) < \phi(b) < \phi(c)$, we must have that $(a, c) \notin E$ and $\phi(a) < \phi(c)$. But this implies that $(a, c') \in E'$.

The proof that G is the incomparability graph of B is symmetric. This can be seen by applying the reverse ordering on B . \square

Corollary 4.2.17. *The class of double bichain graphs is not WQO by the induced subgraph relation.*

Proof. We will show that the sequence $\Phi(H_1), \Phi(H_2), \dots$ of double bichain graphs forms an infinite antichain with respect to the induced subgraph relation, where H_i is defined as in Figure 4.1. Suppose, for contradiction, that $\Phi(H_i)$ is the induced subgraph of $\Phi(H_j)$ for some $i < j$.

By the previous theorem, H_j is the incomparability graph of each part of $\Phi(H_j)$. From the definition of $\Phi(H_j)$, it is easy to check that deleting a vertex v of $\Phi(H_j)$ will correspond to deleting a vertex from the incomparability graph of the part in which v resides and (possibly) deleting edges from the incomparability graph of the opposite part. Thus the incomparability graph of each part of $\Phi(H_i)$ must be a subgraph of H_j . In other words, H_i must be a subgraph of H_j , which is impossible. This contradiction proves the corollary. \square

Note that the same class was found to be of unbounded clique-width in Corollary 3.3.17.

4.2.4 The class of $(P_7, S_{1,2,3})$ -free bipartite graphs is WQO

Ding showed that $(P_7, S_{1,2,3}, \text{Sun}_4)$ -free bipartite graphs and (P_6, \tilde{P}_6) -free bipartite graphs are well-quasi-ordered by the induced subgraph relation [Ding, 1992]. Now we extend both results to the larger class of $(P_7, S_{1,2,3})$ -free bipartite graphs. To this end, let us introduce the following notation.

Given a set of bipartite graphs \mathcal{F} , we denote by $[\mathcal{F}]$ the set of graphs constructed from graphs in \mathcal{F} by means of the following three binary operations defined for any two disjoint bipartite graphs $G_1 = (X_1, Y_1, E_1)$ and $G_2 = (X_2, Y_2, E_2)$:

- the disjoint union is the operation that creates out of G_1 and G_2 the bipartite graph $G = (X_1 \cup X_2, Y_1 \cup Y_2, E_1 \cup E_2)$,
- the join is the operation that creates out of G_1 and G_2 the bipartite graph which is the bipartite complement of the disjoint union of \tilde{G}_1 and \tilde{G}_2 ,
- the skew join is the operation that creates out of G_1 and G_2 the bipartite graph $G = (X_1 \cup X_2, Y_1 \cup Y_2, E_1 \cup E_2 \cup \{xy : x \in X_1, y \in Y_2\})$.

The importance of these operations is due to the following theorem.

Theorem 4.2.18. *If \mathcal{F} is a set of bipartite graphs well-quasi-ordered by the induced subgraph relation, then so is $[\mathcal{F}]$.*

For the proof of this theorem, we refer the reader to Theorems 4.1 and 4.4 from [Ding, 1992], where the author used this result (without formulating it implicitly) in his proof that $(P_7, S_{1,2,3}, Sun_4)$ -free bipartite graphs and (P_6, \tilde{P}_6) -free bipartite graphs are well-quasi-ordered by the induced subgraph relation. Now we combine Theorem 4.2.18 with a result from [Fouquet et al., 1999] that can be formulated as follows.

Theorem 4.2.19. *The class of $(P_7, S_{1,2,3})$ -free bipartite graphs is precisely $[\{K_1\}]$.*

Together, Theorem 4.2.18 and Theorem 4.2.19 imply the following conclusion.

Theorem 4.2.20. *The class of $(P_7, S_{1,2,3})$ -free bipartite graphs is well-quasi-ordered by the induced subgraph relation.*

4.2.5 The class of (P_7, Sun_1) -free bipartite graphs is WQO

The graph Sun_1 is obtained from Sun_4 (Figure 4.1) by deleting three vertices of degree 1. Therefore, the class of (P_7, Sun_1) -free bipartite graphs is a proper subclass of (P_7, Sun_4) -free bipartite graphs. In contrast to the result of Section 4.2.1, below we prove that (P_7, Sun_1) -free bipartite graphs are well-quasi-ordered by the induced subgraph relation. According to Theorem 4.2.18, it suffices to show that the set of *connected* (P_7, Sun_1) -free bipartite graphs is well-quasi-ordered by this relation. The following lemma shows that the structure of connected graphs in this class containing a C_4 is rather simple.

Lemma 4.2.21. *Every connected (P_7, Sun_1) -free bipartite graph containing a C_4 is complete bipartite.*

Proof. Let H be a (P_7, Sun_1) -free bipartite graph containing a C_4 . Denote by H' any maximal complete bipartite subgraph of H containing the C_4 . If $H' \neq H$, there must exist a vertex v outside H' that has a neighbor in H' . If v is adjacent to every vertex of H' in the opposite part, then H' is not maximal, and if v has a non-neighbor in the opposite part of H' , the reader can easily find an induced Sun_1 . The contradiction in both cases shows that $H' = H$, i.e., H is a complete bipartite graph. \square

It is not difficult to see that there is no infinite antichain of complete bipartite graphs, which follows, for instance, from the fact that every complete bipartite graph is P_4 -free and the class of P_4 -free (not necessarily bipartite) graphs is well-quasi-ordered. This observation, together with Lemma 4.2.21, reduces the problem from (P_7, Sun_1) -free bipartite graphs to (P_7, C_4) -free bipartite graphs. The proof that the class of (P_7, C_4) -free bipartite graphs is well-quasi-ordered is based on the following lemma.

Lemma 4.2.22. *No (P_7, C_4) -free bipartite graph contains P_9 as a subgraph (not necessarily induced).*

Proof. Let G be a (P_7, C_4) -free bipartite graph. To prove the lemma, we first derive the following helpful observation.

Claim 4.2.23. *If $P := (a_1, a_2, \dots, a_7)$ is a copy of P_7 contained in G as a subgraph, then P has exactly one chord in G , either a_1a_6 or a_2a_7 .*

Proof. Since G is P_7 -free, P must contain a chord, and since G is bipartite, any chord of P connects an even-indexed vertex to an odd-indexed one. Among 6 possible chords of P only a_1a_6 and a_2a_7 do not produce a C_4 , and these two chords cannot be present in the graph simultaneously, since otherwise the vertices a_1, a_2, a_7, a_6 induce a C_4 . Therefore, P must contain exactly one of a_1a_6 or a_2a_7 as a chord. \square

Suppose now that $Q := (b_1, b_2, \dots, b_9)$ is a copy of P_9 contained as a subgraph in G , and for $1 \leq i \leq 3$, let $Q_i := (b_i, b_{i+1}, \dots, b_{i+6})$. If b_1b_6 is a chord of Q , then Claim 4.2.23 applied to each of Q_1, Q_2 and Q_3 implies that Q contains exactly two chords, namely b_1b_6 and b_3b_8 . But then the vertices $b_1, b_6, b_5, b_4, b_3, b_8, b_9$ induce a P_7 , a contradiction.

The case when b_1b_6 is not a chord of Q is symmetric and also leads (with the help of Claim 4.2.23) to an induced P_7 in G . The contradiction in both cases shows that G does not contain P_9 as a subgraph. \square

Now we combine Lemma 4.2.22 with the following result from [Ding, 1992].

Theorem 4.2.24. *For any fixed $k \geq 1$, the class of graphs containing no P_k as a (not necessarily induced) subgraph is well-quasi-ordered by the induced subgraph relation.*

Together Lemma 4.2.22 and Theorem 4.2.24 imply the main conclusion of this section.

Theorem 4.2.25. *The class of (P_7, C_4) -free bipartite graphs is well-quasi-ordered by the induced subgraph relation.*

4.2.6 The class of P_k -free bipartite permutation graphs is WQO

The class of bipartite permutation graphs is the intersection of bipartite graphs and permutation graphs. This class is a subclass of biconvex graphs (see e.g. [Brandstädt et al., 1999]). In contrast to the result of Section 4.2.2, we show that P_k -free bipartite permutation graphs are well-quasi-ordered by the induced subgraph relation for any fixed value of k . In general, bipartite permutation graphs are not well-quasi-ordered by this relation, since they contain the antichain of graphs of the form H_i (Figure 4.1). Our proof is based on a number of known results.

Denote by $H_{n,m}$ the graph with nm vertices which can be partitioned into n independent sets $V_1 = \{v_{1,1}, \dots, v_{1,m}\}, \dots, V_n = \{v_{n,1}, \dots, v_{n,m}\}$ so that for each $i = 1, \dots, n-1$ and for each $j = 1, \dots, m$, vertex $v_{i,j}$ is adjacent to vertices $v_{i+1,1}, v_{i+1,2}, \dots, v_{i+1,j}$ and there are no other edges in the graph. In other words, every two consecutive independent sets induce in $H_{n,m}$ a universal chain graph. An example of the graph $H_{n,n}$ with $n = 5$ is given in Figure 4.6.

It is not difficult to see that the graph $H_{n,n}$ is a bipartite permutation graph. Moreover, it was proved in [Lozin and Rudolf, 2007] that $H_{n,n}$ is an n -universal bipartite permutation graph in the sense that every bipartite permutation graph with n vertices is an induced subgraph of $H_{n,n}$. This characterisation can be seen to correspond naturally to the one given in Definition 4.2.6. If a connected bipartite permutation graph is P_k -free, it occupies at most k consecutive levels of $H_{n,n}$. In other words, every connected P_k -free bipartite permutation graph is an induced subgraph of $H_{k,n}$.

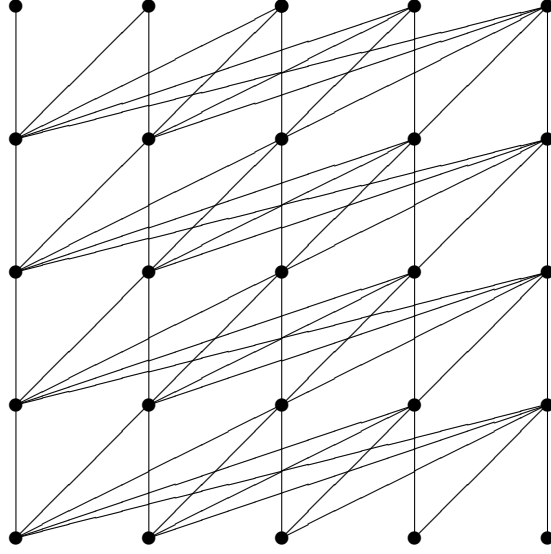


Figure 4.6: The graph $H_{5,5}$

In order to prove that P_k -free bipartite permutation graphs are well-quasi-ordered, we will show that any connected graph in this class is a k -letter graph. This notion was introduced in [Petkovšek, 2002] and its importance for our study is due to the following result, also proved in [Petkovšek, 2002].

Theorem 4.2.26. *For any fixed k , the class of k -letter graphs is well-quasi-ordered by the induced subgraph relation.*

The k -letter graphs have been characterized in [Petkovšek, 2002] as follows. (For a more complete discussion with definitions, see Section 4.4.2.)

Theorem 4.2.27. *A graph $G = (V, E)$ is a k -letter graph if and only if*

1. *there is a partition V_1, \dots, V_p of $V(G)$ with $p \leq k$ such that each V_i is either a clique or an independent set in G ,*
2. *there is a linear ordering L of $V(G)$ such that for each pair of indices $1 \leq i, j \leq p$, $i \neq j$, the intersection of E with $V_i \times V_j$ is one of*
 - (a) $L \cap (V_i \times V_j)$,
 - (b) $L^{-1} \cap (V_i \times V_j)$,
 - (c) $V_i \times V_j$,
 - (d) \emptyset .

Corollary 4.2.28. *Connected P_k -free bipartite permutation graphs are k -letter graphs.*

Proof. From Theorem 4.2.27, it follows that an induced subgraph of a k -letter graph is again a k -letter graph. In addition, we have seen already that any connected P_k -free bipartite permutation graph is an induced subgraph of $H_{k,n}$. Therefore, all we have to do is to prove that $H_{k,n}$ is a k -letter graph. To this end, we define a partition V_1, \dots, V_k of the vertices of $H_{k,n}$ by defining V_i to be the i -th row of $H_{k,n}$. Thus the first condition of Theorem 4.2.27 is satisfied. Then we define a linear ordering L of the vertices of $H_{k,n}$ by listing first the vertices of the first column consecutively from bottom to top, then the vertices of the second column, and so on. Now let's take any two subsets V_i and V_j with $i \neq j$. If they are not consecutive rows of the graph, then the intersection of E with $V_i \times V_j$ is empty. If they are consecutive, then the intersection of E with $V_i \times V_j$ is either $L \cap (V_i \times V_j)$ (if $i > j$) or $L^{-1} \cap (V_i \times V_j)$ (if $i < j$). Thus the second condition of Theorem 4.2.27 is satisfied, which proves the corollary. \square

Combining Corollary 4.2.28 with Theorems 4.2.18 and 4.2.26 we conclude that

Corollary 4.2.29. *For any fixed k , the class of P_k -free bipartite permutation graphs is well-quasi-ordered by the induced subgraph relation.*

4.2.7 Characterisation of all monogenic classes of bipartite graphs

By Theorem 4.2.9, the class of $2P_3$ -free bipartite graphs is not well-quasi-ordered by the induced subgraph relation. On the other hand, by Theorem 4.2.20, the class of $(P_7, S_{1,2,3})$ -free bipartite graphs is well-quasi-ordered by the same relation. For a complete characterisation of the well-quasi-orderability of classes of bipartite graphs defined by forbidding exactly one induced subgraph, it suffices to decide well-quasi-orderability of the classes of H -free bipartite graphs for which H is a linear forest such that $(H \not\leq P_7$ or $H \not\leq S_{1,2,3})$ and $2P_3 \not\leq H$. Let us consider two examples of such graphs H :

- If $H = K_2 + 3K_1$, then the class of H -free bipartite graphs is not WQO since it contains the non-WQO class of $\tilde{S}_{1,1,1}$ -free bipartite graphs.
- If $H = 3K_2$, then the class of H -free bipartite graphs is not WQO since it contains the non-WQO class of \tilde{C}_6 -free bipartite graphs.

It is a routine exercise to check that any linear forest H not containing one of these two examples must satisfy either $H = nK_1$ or $(H \leq P_7$ and $H \leq S_{1,2,3})$ or

$2P_3 \leq H$. Note that for $H = nK_1$, the class of H -free bipartite graphs is finite (by Ramsey's theorem) and thus WQO. Thus we have a complete characterisation of all monogenic classes of bipartite graphs.

4.3 Bigenic Classes of Graphs

As we discussed in the previous section, there is a complete characterisation of well-quasi-orderability by the induced subgraph relation in the case of monogenic graph classes (due to Damaschke). We also completed a corresponding characterisation for monogenic classes of bipartite graphs. Very little is known about well-quasi-ordered classes of graphs defined by more than one forbidden induced subgraph.

In this section, we study the induced subgraph relation on *bigenic* graph classes, i.e. graph classes defined by two forbidden induced subgraphs. We characterize most of them (except finitely many specified cases) as being or not being wqo with respect to this relation. One outcome of this analysis is that in this family there are finitely many minimal classes which are not well-quasi-ordered by the induced subgraph relation.

In [Damaschke, 1990], the following results were proved.

Theorem 4.3.1.

- (A) *A monogenic class $\text{Free}(G)$ is WQO if and only if G is a (not necessarily proper) induced subgraph of P_4 .*
- (B) *The classes $\text{Free}(K_3, P_5)$ and $\text{Free}(K_3, K_2 + 2K_1)$ are WQO.*

Part (A) of this theorem provides complete characterization of monogenic classes of graphs in terms of their well-quasi-orderability. In this section, we study bigenic classes and extend part (B) of Theorem 4.3.1 in various ways. To this end, let us first recall a few helpful results.

For an arbitrary set M , denote by M^* the set of all finite sequences of elements of M . If \leq is a partial order on M , the elements of M^* can be partially ordered by the following relation: $(a_1, \dots, a_m) \leq (b_1, \dots, b_n)$ if and only if there is an order-preserving injection $f : \{a_1, \dots, a_m\} \rightarrow \{b_1, \dots, b_n\}$ with $a_i \leq f(a_i)$ for each $i = 1, \dots, m$. The celebrated Higman's lemma states [Higman, 1952]

Lemma 4.3.2. *If (M, \leq) is a WQO, then (M^*, \leq) is a WQO.*

Kruskal extended this result to the set of finite trees partially ordered under homeomorphic embedding [Kruskal, 1960]. In other words, Kruskal's tree theorem restricted to paths becomes Higman's lemma. Moreover, Kruskal proved his theorem

under the additional assumption that the vertices of trees are equipped with labels from a well-quasi-ordered set.

From Higman's lemma it is not difficult to derive the following conclusion (see [Damaschke, 1990] for a more general result).

Claim 4.3.3. *A set of graphs X is well-quasi-ordered (by the induced subgraph relation) if and only if connected graphs in X are well-quasi-ordered.*

Since two graphs G and H are isomorphic if and only if their complements are isomorphic, we conclude that

Claim 4.3.4. *A set of graphs is a WQO if and only if the set of their complements is a WQO.*

From the Ramsey theory we know that for all values of n and m the class $\text{Free}(K_n, mK_1)$ is finite. As an immediate corollary from this observation we obtain the following conclusion.

Claim 4.3.5. *The class $\text{Free}(K_n, mK_1)$ is WQO for all n and m .*

The following result will also be useful in our study of bigenic classes of graphs.

Claim 4.3.6. *The class $\text{Free}(\text{paw}, H)$ is WQO if and only if the class $\text{Free}(K_3, H)$ is WQO. (Paw is the name for a complement of $P_3 + K_1$.)*

Proof. The claim follows by combining Theorem 4.3.1 (A), Claim 4.3.3 and the following fact proved in [Olariu, 1988]: every connected paw-free graph is either K_3 -free or \overline{P}_3 -free. \square

In our analysis of bigenic classes two antichains will play a key role. These are:

- $\mathcal{F} = \{K_{1,3}, K_3, C_4, C_5, C_6, \dots\}$
- $\overline{\mathcal{F}} = \{\overline{K_{1,3}}, \overline{K_3}, \overline{C_4}, \overline{C_5}, \overline{C_6}, \dots\}$

Note that $\text{Free}(\mathcal{F})$ is the class of *linear forests*, i.e. graphs every connected component of which is a path. Similarly, $\text{Free}(\overline{\mathcal{F}})$ is the class of complements of linear forests. The importance of the classes $\text{Free}(\mathcal{F})$ and $\text{Free}(\overline{\mathcal{F}})$ is due to the following result.

Claim 4.3.7. *Let $X = \text{Free}(G, H)$ be a bigenic class of graphs.*

- *If neither of G and H belong to $\text{Free}(\mathcal{F})$, then X is not WQO.*

- If neither of G and H belong to $\text{Free}(\overline{\mathcal{F}})$, then X is not WQO.
- If one of G and H belongs to both $\text{Free}(\mathcal{F})$ and $\text{Free}(\overline{\mathcal{F}})$, then X is WQO.

Proof. If neither of G and H belong to $\text{Free}(\mathcal{F})$, then it is easy to see that X contains infinitely many cycles, i.e. an infinite antichain. The second statement follows by symmetry.

To prove the third statement, suppose G belongs to both $\text{Free}(\mathcal{F})$ and $\text{Free}(\overline{\mathcal{F}})$. It is not difficult to verify that G is an induced subgraph of P_4 . But then X is a subclass of $\text{Free}(P_4)$, which is WQO by Theorem 4.3.1 (A). \square

According to Claim 4.3.7, in what follows we consider bigenic classes of graphs $\text{Free}(G, H)$ with $G \in \text{Free}(\overline{\mathcal{F}})$ and $H \in \text{Free}(\mathcal{F})$.

4.4 Bigenic Classes of Graphs Which Are Well-quasi-ordered

In this section, we reveal a number of bigenic classes which are well-quasi-ordered by induced subgraphs. In fact, we prove stronger results that deal with a binary relation which we call *labelled-induced subgraphs*. Assume (W, \leq) is an arbitrary WQO. We call G a *labelled graph* if each vertex $v \in V(G)$ is equipped with an element $l(v) \in W$ (the label of v), and we say that a graph G is a labelled-induced subgraph of H if G is isomorphic to an induced subgraph of H and the isomorphism maps each vertex $v \in G$ to a vertex $w \in H$ with $l(v) \leq l(w)$. We split the results of this section into two parts depending on the technique we use to prove well-quasi-orderability.

4.4.1 Well-quasi-order and k -uniform graphs

Let k be a natural number, K a symmetric 0-1 square matrix of order k , and F_k a simple graph on the vertex set $\{1, 2, \dots, k\}$. Let H be the disjoint union of infinitely many copies of F_k , and for $i = 1, \dots, k$, let V_i be the subset of $V(H)$ containing vertex i from each copy of F_k . Now we construct from H an infinite graph $H(K)$ on the same vertex set by connecting two vertices $u \in V_i$ and $v \in V_j$ if and only if $uv \in E(H)$ and $K(i, j) = 0$ or $uv \notin E(H)$ and $K(i, j) = 1$. Finally, let $\mathcal{P}(K, F_k)$ be the hereditary class consisting of all the finite induced subgraphs of $H(K)$.

Definition 4.4.1. A graph G will be called k -uniform if there is a number k such that $G \in \mathcal{P}(K, F_k)$ for some K and F_k .

Theorem 4.4.1. *For any fixed k , the set of k -uniform graphs is well-quasi-ordered by the labelled-induced subgraph relation.*

Proof. For a fixed k , there are only finitely many matrices K of order k and finitely many graphs on the set $\{1, \dots, k\}$. Therefore, it suffices to prove the theorem for an arbitrary matrix K and an arbitrary graph F_k , i.e. for a fixed property $\mathcal{P}(K, F_k)$. Moreover, without loss of generality, we will identify each graph $G \in \mathcal{P}(K, F_k)$ with an arbitrary embedding of G into $H(K)$.

Since G is a finite graph, there is a finite number m of copies of the graph F_k (i.e. of the graph which is used in the construction of $H(K)$) that contain at least one vertex of G . We represent G by a binary $k \times m$ matrix $M = M_G$ whose (i, j) entry contains 1 if the i -th vertex of the j -th copy of F_k belongs to G , and 0 otherwise.

Now assume the vertices of G are labelled by the elements of a WQO set (W, \leq) . We replace each non-zero entry of M by the label of the respective vertex of G , which transforms M into a matrix $M^* = M_G^*$ in the alphabet $W_0 = W \cup \{0\}$. We extend (W, \leq) to a WQO (W_0, \leq) by defining $0 \leq x$ for each element $x \in W$.

Let us denote the set $\{M_G^* \mid G \in \mathcal{P}(K, F_k)\}$ by \mathcal{M}_k and define a binary relation \leq^* on this set in two steps, as follows:

- for two words $x = (x_1 \dots x_k) \in W_0^k$ and $y = (y_1 \dots y_k) \in W_0^k$, we define $x \leq_k y$ if and only if $x_i \leq y_i$ for each $i = 1, \dots, k$.
- for two matrices $M_1^* \in \mathcal{M}_k$ and $M_2^* \in \mathcal{M}_k$, we define $M_1^* \leq^* M_2^*$ if and only if there is an injection mapping each column x of M_1^* to a column y of M_2^* with $x \leq_k y$.

From the definition of k -uniform graphs and the matrices of the form M_G^* it follows that in order to show that $\mathcal{P}(K, F_k)$ is well-quasi-ordered by the labelled-induced subgraph relation it is enough to show that the set (\mathcal{M}_k, \leq^*) is a WQO. This easily follows by a double application of Higman's lemma [Higman, 1952]. The first application implies that (W_0^k, \leq_k) is a WQO (since (W_0, \leq) is a WQO), and the second application implies that (\mathcal{M}_k, \leq^*) is WQO (since (W_0^k, \leq_k) is a WQO). \square

Lemma 4.4.2. *Let G be a graph and v a vertex of G . If $G - v$ is a k -uniform graph, then G is $2k + 1$ -uniform.*

Proof. Let $G - v$ be a k -uniform graph given together with an embedding into $H(K)$. We call the sets V_1, \dots, V_k of $H(K)$ *color classes* of the graph. First, we split each of the k colour classes of $G - v$ into two subsets (of vertices adjacent and non-adjacent to v), which makes $G - v$ a $2k$ -uniform graph. Then we add an extra colour class,

containing vertex v only, and connect it to the rest of the graph accordingly. More formally, assume $G - v \in \mathcal{P}(K, F_k)$. Viewing K as a graph with loops on the vertex set $\{w_1, \dots, w_k\}$, split every looped vertex w_i into two adjacent looped vertices w'_i and w''_i , split every loopless vertex w_i into two non-adjacent loopless vertices w'_i and w''_i , and add an extra vertex (no matter with or without a loop) which is adjacent to exactly one vertex in each pair w'_i, w''_i . Also, split every vertex of F_k into two non-adjacent vertices, and then add to F_k an isolated vertex. Denoting the resulting graphs by K', F'_{2k+1} , we conclude that $G \in \mathcal{P}(K', F'_{2k+1})$. \square

Corollary 4.4.3. *Let X be a class of graphs and c, k constants. If every graph G in X has a subset W of at most c vertices such that $G - W$ is k -uniform, then every graph of G is $(2^c(k+1) - 1)$ -uniform.*

Now we apply Lemma 4.4.1 and Corollary 4.4.3 to derive well-quasi-orderability for some particular bigenic classes. In the proof of the next three theorems, $S_{1,2,3}$ denotes a tree with three leaves being of distance 1, 2 and 3 from the only vertex of degree 3.

Theorem 4.4.4. *The class $\text{Free}(K_3, P_3 + 2K_1)$ is WQO.*

Proof. Note that $P_3 + 2K_1$ is an induced subgraph of the following graphs: $P_7, S_{1,2,3}$ and C_i for $i \geq 8$. Since $(P_7, S_{1,2,3})$ -free bipartite graphs are WQO (see Section 4.2.4), we may restrict ourselves to graphs in $\text{Free}(K_3, P_3 + 2K_1)$ containing a C_5 or a C_7 . Let G be such a graph. By Claim 4.3.3 we may assume that G is connected.

Assume first that G contains a copy of C_7 , say $C = (v_1, v_2, \dots, v_7)$. Suppose G has a vertex u that does not belong to C . Due to the K_3 -freeness, u cannot have more than 3 neighbours in C . If u has exactly three neighbours, then the only (up to symmetry) possibility to avoid a K_3 is when u is adjacent to v_1, v_3, v_6 , in which case vertices v_2, v_4, v_6, v_7, u induce $P_3 + 2K_1$. If u has fewer than 3 neighbors in C , finding one of the two forbidden graphs is a trivial task. Therefore, if G contains a copy of C_7 , then $G = C_7$.

Now we assume that G contains an induced copy of C_5 , say $C = (v_1, v_2, v_3, v_4, v_5)$. Let u be a vertex of G outside the cycle. Since G is K_3 -free, u can be adjacent to at most two vertices of C , and if u has two neighbours in C , they are non-consecutive vertices of the cycle. We denote the set of vertices in $V(G) \setminus V(C)$ that have exactly i neighbours on C by N_i , $i \in \{0, 1, 2\}$. Also, for $i = 1, \dots, 5$, we denote by V_i the set of vertices in N_2 adjacent to $v_{i-1}, v_{i+1} \in V(C)$ (throughout the proof subscripts i are taken modulo 5). We call two different sets V_i and V_j *consecutive* if v_i and v_j are

consecutive vertices of C , and *opposite* otherwise. The proof will be given through a series of claims.

- (1) *Each V_i is an independent set, and vertices in opposite sets V_i and V_j are non-adjacent*, which follows directly from the K_3 -freeness of G .
- (2) *Each vertex in V_i is adjacent to all but at most one vertex in V_{i+1}* , since otherwise a vertex $x \in V_i$ together with any of its two non-neighbours $y_1, y_2 \in V_{i+1}$ and vertices v_{i-1}, v_{i+1} would induce a $P_3 + 2K_1$.
- (3) $|N_1| \leq 5$. Indeed, if $|N_1| > 5$, then it contains two vertices x, y adjacent to the same vertex v_i of C . Then either $G[v_i, x, y] = K_3$ (if x is adjacent to y) or $G[v_{i+1}, v_{i+2}, v_{i+3}, x, y] = P_3 + 2K_1$ (if x is not adjacent to y).
- (4) $|N_0| \leq 1$. Indeed, assume N_0 contains two vertices x, y . If x is not adjacent to y , then $G[v_1, v_2, v_3, x, y] = P_3 + 2K_1$. Suppose now that x is adjacent to y . Since the graph is connected, there must exist a path connecting x, y to the cycle. Without loss of generality we may assume that x is adjacent to a vertex z that has a neighbour on C . Then z is not adjacent to y (since G is K_3 -free) and z has at least two non-adjacent non-neighbours on C , say v_1 and v_3 . But now $G[z, x, y, v_1, v_3] = P_3 + 2K_2$.
- (5) *If V_i and V_j are opposite, then at least one of them is empty*. Indeed, assume without loss of generality that V_1 contains a vertex x and V_3 contains a vertex y , then $G[v_3, v_4, y, x, v_1] = P_3 + 2K_1$.

By Claim (5), G contains at most two non-empty sets V_i and V_j and these sets are consecutive. By Claims (1) and (2) these two sets induce a 2-uniform graph. Therefore, by Claims (3) and (4) and Corollary 4.4.3 G is a k -uniform graph for a constant k . \square

Theorem 4.4.5. *The class $\text{Free}(K_3, \text{co-gem})$ is WQO.*

Proof. Note that a co-gem $P_4 + K_1$ is an induced subgraph of P_6 and therefore of any cycle C_i with $i \geq 7$. Since P_6 -free bipartite graphs are WQO (see Section 4.2.4), we may restrict our attention to graphs in $\text{Free}(K_3, P_4 + K_1)$ that contain a C_5 .

Let G be a graph in $\text{Free}(K_3, P_4 + K_1)$ containing an induced copy of C_5 , say $C := (v_1, v_2, \dots, v_5)$. Every vertex outside C must have at least two neighbours on the cycle (since otherwise an induced *co-gem* arises) and at most two neighbours on the cycle (since otherwise a K_3 arises). Therefore, every vertex outside C has

exactly two neighbours on C and due to K_3 -freeness of G these neighbours are non-consecutive vertices of the cycle. We denote the vertices outside C that are adjacent to v_{i-1} and v_{i+1} by V_i . Then each V_i is an independent set and vertices in opposite sets V_i and V_j are non-adjacent, since G is K_3 -free. In addition, every vertex in V_i is adjacent to every vertex in V_{i+1} , since otherwise two non-adjacent vertices $x \in V_i$ and $y \in V_{i+1}$ together with $v_{i-2}, v_{i-1}, v_{i+1}$ would induce a copy of $P_4 + K_1$. Therefore, G is a 5-uniform graph, and hence, by Theorem 4.4.1, $\text{Free}(K_3, P_4 + K_1)$ is a well-quasi-ordered class. \square

Theorem 4.4.6. *The class $\text{Free}(K_3, P_3 + P_2)$ is WQO.*

Proof. Note that a $P_3 + P_2$ is an induced subgraph of P_6 and therefore of any cycle C_i with $i \geq 7$. Since P_6 -free bipartite graphs are WQO (see Section 4.2.4), we may restrict ourselves to those graphs in the class $\text{Free}(K_3, P_3 + P_2)$ that contain a C_5 .

Let G be a connected $(K_3, P_3 + P_2)$ -free graph and let $C = (v_1, v_2, v_3, v_4, v_5)$ be an induced cycle of length five in G . Let v be a vertex of G outside the cycle. Since G is K_3 -free, v can be adjacent to at most two vertices of C , and if v has two neighbours on C , they are non-consecutive vertices of the cycle. We denote the set of vertices in $V(G) \setminus V(C)$ that have exactly i neighbours on C by N_i , $i \in \{0, 1, 2\}$. Also, for $i = 1, \dots, 5$, we denote by V_i the set of vertices in N_2 adjacent to $v_{i-1}, v_{i+1} \in V(C)$ (throughout the proof subscripts i are taken modulo 5). We call two different sets V_i and V_j *consecutive* if v_i and v_j are consecutive vertices of C , and *opposite* otherwise. Finally, we call V_i *large* if $|V_i| \geq 2$, and *small* otherwise. The proof of the theorem will be given through a series of claims.

- (1) N_0 is an independent set, since otherwise any edge connecting two vertices $x, y \in N_0$ together with v_1, v_2, v_3 would induce a $P_3 + P_2$.
- (2) No vertex $x \in N_1$ has a neighbour in N_0 . Indeed, if $x \in N_1$ is adjacent to v_i and $z \in N_0$, then $G[x, z, v_{i+1}, v_{i+2}, v_{i+3}]$ is isomorphic to $P_3 + P_2$.
- (3) Any vertex $x \in N_2$ has at most one neighbour in N_0 . Indeed, if $x \in V_i$ is adjacent to $z, z' \in N_0$, then $G[x, z, z', v_{i+2}, v_{i+3}]$ is isomorphic to $P_3 + P_2$.
- (4) $|N_1| \leq 5$. Indeed, if there are two vertices $x, x' \in N_1$ which are adjacent to the same vertex $v_i \in V(C)$, then $G[x, x', v_i, v_{i+2}, v_{i+3}]$ is isomorphic to $P_3 + P_2$.
- (5) If V_i and V_j are opposite sets, then no vertex of V_i is adjacent to a vertex of V_j , since G is K_3 -free.

- (6) If V_i and V_j are consecutive, then every vertex x of V_i has at most one non-neighbour in V_j . Indeed, if $x \in V_i$ has two non-neighbours $y, y' \in V_{i+1}$, then $G[x, y, y', v_{i-1}, v_{i-3}]$ is isomorphic to $P_3 + P_2$.
- (7) Each V_i is an independent set, since G is K_3 -free.
- (8) If V_i and V_j are two opposite large sets, then no vertex in N_0 has a neighbour in $V_i \cup V_j$. Assume without loss of generality that $i = 1$ and $j = 4$, and suppose for contradiction that a vertex $x \in N_0$ has a neighbour $y \in V_1$. Obviously x has either at least one non-neighbor or at least two neighbors in V_4 . If x is non-adjacent to a vertex $z \in V_4$, then $G[x, y, z, v_3, v_4]$ is isomorphic to $P_3 + P_2$, and if x is adjacent to vertices $z, z' \in V_4$, then $G[x, z, z', v_1, v_2]$ is isomorphic to $P_3 + P_2$.

Since G is connected and N_0 is an independent set, every vertex of N_0 has a neighbour in N_2 (see Claim (2)). Let us denote by V_0 those vertices of N_0 at least one neighbour of which belongs to a large set V_i and by G_0 the subgraph of G induced by V_0 and the large sets. From Claims (3) and (4), it follows that at most 15 vertices of G do not belong to G_0 . We will show that G_0 is a k -uniform graph for some constant k , which will imply by Corollary 4.4.3 that G is c -uniform for a constant c . We may assume that G has at least one large set, since otherwise G_0 is empty. We will show that G_0 is k -uniform by examining all possible combinations of large sets.

Case 1: Assume that for every large set V_i there is an opposite large set V_j . Then it follows from Claim (8) that $V_0 = \emptyset$. Suppose there are two consecutive large sets V_i and V_{i+1} such that V_i contains a vertex x nonadjacent to a vertex $y \in V_{i+1}$. Then V_{i-1} is small. Indeed, if V_{i-1} is large, then, by Claim (6), it must contain a vertex z adjacent to x . But then vertices $x, y, z, v_{i-1}, v_{i+2}$ induce in G a $P_3 + P_2$. Therefore, G_0 does not contain vertices of V_{i-1} . Symmetrically, G_0 does not contain vertices of V_{i+2} . Therefore, if G contains a couple of non-adjacent vertices in two consecutive large sets, then G_0 consists of at most three sets: V_i, V_{i+1} and V_{i+3} . By Claim (6), V_i and V_{i+1} induce a 2-uniform graph, and therefore, G_0 is 3-uniform. If every two vertices of G_0 in consecutive large sets are adjacent, then G_0 is 5-uniform.

Case 1 allows us to assume that G contains a large set such that the opposite sets are small. Without loss of generality we let V_1 be large, and V_3 and V_4 be small. The rest of the proof is based on the analysis of the size of the sets V_2 and V_5 .

Case 2: V_2 and V_5 are large. Then, by Claim (8), there are no edges between V_0 and $V_2 \cup V_5$. As a result, if V_0 has at least two vertices, then each vertex of V_0

has exactly one neighbour in V_1 . Indeed, assume vertex $a \in V_0$ has at least two neighbours $b, c \in V_1$. Let d be any other vertex of V_0 and e its neighbour in V_1 . By Claim (3), e must be different from b and c . But then a, b, c, d, e induce a $P_3 + P_2$. Therefore, if V_0 has at least two vertices, G_0 is a 4-uniform graph. If V_0 has at most 1 vertex, we can neglect it by Corollary 4.4.3, which makes G_0 a 3-uniform graph.

Case 3: V_2 and V_5 are small. Then G_0 is a bipartite graph with bipartition (V_1, V_0) , and as in Case 2 if V_0 has at least two vertices, then each vertex of V_0 has exactly one neighbour in V_1 , i.e. G_0 is a 2-uniform graph.

Case 4: V_2 is large and V_5 is small, i.e. G_0 is induced by $V_0 \cup V_1 \cup V_2$. Denote by V_{01} the vertices of V_0 that have no neighbours in V_2 , by V_{02} the vertices of V_0 that have no neighbours in V_1 , and by V_{012} the vertices of V_0 that have neighbours both in V_1 and V_2 . Without loss of generality, we assume that each of V_{01} and V_{02} has at least 2 vertices, since otherwise these sets can be neglected by Corollary 4.4.3. Therefore, as in Case 2, each vertex of V_{01} has exactly one neighbor in V_1 , and each vertex of V_{02} has exactly one neighbor in V_2 . This means that if V_{012} is empty, then G_0 is 4-uniform.

Suppose now that V_{012} contains a vertex x and let y be a neighbour of x in V_1 and z be a neighbour of x in V_2 . Then y and z are non-adjacent (since G is K_3 -free) and therefore, by Claim (6), y is adjacent to every vertex of $V_2 \setminus \{z\}$ and z is adjacent to every vertex of $V_1 \setminus \{y\}$. From the K_3 -freeness of G it follows that x has no neighbours in $(V_1 \cup V_2) \setminus \{y, z\}$. Thus, each vertex V_{012} has exactly one neighbour in V_1 and exactly one neighbour in V_2 . We denote the vertices of V_1 that have neighbours in V_{012} by V_1' , and the vertices of V_2 that have neighbours in V_{012} by V_2' . Also, for $i = 1, 2$ let $V_i'' = V_i - V_i'$.

It follows from the above discussion and Claims (3) and (6) that

- vertices of V_{012} have no neighbours in $V_1'' \cup V_2''$,
- there are all possible edges between V_1' and V_2'' , and between V_2' and V_1'' .
- there are no edges between $V_{01} \cup V_{02}$ and $V_1' \cup V_2'$.

Therefore, G_0 is a 7-uniform graph. □

4.4.2 Well-quasi-order, k -letter graphs and modular decomposition

To reveal more classes of graphs well-quasi-ordered by the induced subgraph relation, we need to introduce more notions. We already mentioned a particular characterisation of k -letter graphs in Section 4.2.6, but we define the notion here for completeness. This class of graphs was introduced in [Petkovšek, 2002]:

Definition 4.4.2. A k -letter graph G is a graph defined by a finite word $x_1x_2 \dots x_n$ on alphabet X of size k together with a subset $S \subseteq X^2$ such that:

- $V(G) = \{x_1, x_2, \dots, x_n\}$
- $E(G) = \{x_i x_j : i \leq j \text{ and } (x_i, x_j) \in S\}$

For any fixed sets X and $S \subseteq X^2$, the subsequence relation on words corresponds precisely to the induced subgraph relation on k -letter graphs. Since there are only finitely many different choices for S , the following is an immediate corollary of Higman's lemma:

Corollary 4.4.7 (Petkovšek, 2002). For any fixed k , the class of k -letter graphs is WQO by induced subgraphs.

Using Higman's lemma in all its generality (which is just a special case of Kruskal's tree theorem), the above corollary can be extended in the following way.

Corollary 4.4.8. For any fixed k , the class of k -letter graphs is WQO by the labelled-induced subgraph relation.

Together, the two notions, k -uniform graphs and k -letter graphs, give a wide range of hereditary classes well-quasi-ordered by the labelled-induced subgraph relation. To further extend this family let us introduce more definitions.

Given a graph $G = (V, E)$, a subset of vertices $U \subseteq V$ and a vertex $x \in V$ outside U , we say that x *distinguishes* U if x has both a neighbour and a non-neighbour in U . A subset $U \subseteq V$ is called a *module* of G if no vertex in $V \setminus U$ distinguishes U . A module U is *nontrivial* if $1 < |U| < |V|$, otherwise it is *trivial*. A graph is called *prime* if it has only trivial modules.

An important property of maximal modules is that if G and the complement of G are both connected, then the maximal modules of G are pairwise disjoint. Moreover, from the above definition it follows that if U and W are maximal modules, then either there are all possible edges between them or no edges at all. Therefore, by contracting each maximal module of G into a single vertex we obtain an induced subgraph G^0 of G which is prime. Sometimes this graph is called the characteristic graph of G (alternatively, one can think of G as being obtained from G^0 by substituting its vertices by maximal modules of G). This property allows to recursively decompose the graph into connected components, co-components or maximal modules. This decomposition can be described by a rooted tree and is known in the literature under various names such as modular decomposition [McConnell and Spinrad, 1999] or *substitution decomposition*.

The importance of the notion of modular decomposition for our study is due to the following theorem.

Theorem 4.4.9. *If the set of prime graphs in a hereditary class X is well-quasi-ordered by the labelled-induced subgraph relation, then the class X is well-quasi-ordered by the induced subgraph relation.*

Proof. Assume to the contrary that X is not a WQO and let $\mathcal{G} = \{G_1, G_2, \dots\}$ be an infinite antichain. By Higman's lemma, we can assume that every graph in \mathcal{G} is connected and co-connected. We also assume that this antichain is minimal in the sense that there is no infinite antichain G'_1, G'_2, \dots with $|V(G_1)| = |V(G'_1)|, \dots, |V(G_{i-1})| = |V(G'_{i-1})|$ and $|V(G_i)| > |V(G'_i)|$ for some $i \geq 1$. Obviously, if X has an infinite antichain, then it has a minimal infinite antichain.

Since for each $i \geq 1$, the graph G_i is both connected and co-connected, the maximal modules of G_i are pairwise disjoint. We contract each maximal module of G_i into a single vertex, obtaining in this way the characteristic graph G_i^0 , and assign to each vertex of G_i^0 the subgraph of G_i induced by the respective module. In this way, the antichain \mathcal{G} transforms into an antichain \mathcal{G}^0 of prime graphs whose vertices are labelled by some graphs from X . Due to minimality of \mathcal{G} we may assume that the set of labels is WQO by induced subgraphs. But then \mathcal{G}^0 must be WQO by labelled-induced subgraphs, according to our assumption about prime graphs in X . This contradiction shows that X is WQO by induced subgraphs. \square

We now use Theorem 4.4.9 to prove the following result.

Theorem 4.4.10. *The classes $\text{Free}(\text{diamond}, P_5)$ and $\text{Free}(\text{diamond}, \text{co-diamond})$ are WQO. (Diamond is the name for the complement of a $K_2 + 2K_1$.)*

Proof. To prove the theorem, we define several special types of graphs:

- A *thin spider* is a graph partitionable into a clique C and an independent set S , with $|C| = |S|$ or $|C| = |S| + 1$, such that the edges between C and S are a matching and at most one vertex of C is unmatched.
- A *matched co-bipartite graph* is a graph partitionable into two cliques C_1 and C_2 , with $|C_1| = |C_2|$ or $|C_1| = |C_2| + 1$, such that the edges between C_1 and C_2 are a matching and at most one vertex of C_1 is unmatched.

- An *enhanced co-bipartite chain graph* is a graph partitionable into two cliques C_1 and C_2 , inducing the complement of a bipartite chain graph together with at most three additional vertices a, b, c for which $N(a) = C_1 \cup C_2$, $N(b) = C_1$ and $N(c) = C_2$.
- An *enhanced (bipartite) chain graph* is the complement of an enhanced co-bipartite chain graph.

It is not difficult to see that any thin spider or matched co-bipartite graph is 2-uniform graph. and a chain bipartite graph is 2-letter graph.

It was proved in [Brandstädt, 2004] that every connected and co-connected prime graph in the class $Free(diamond, P_5)$ is either a thin spider or a matched co-bipartite graph or an enhanced chain graph or a graph with at most 9 vertices. In [Brandstädt and Mahfud, 2002], it is shown that for a connected and co-connected prime graph G in the class $Free(diamond, co-diamond)$, either G or \overline{G} is a matched co-bipartite graph or G has at most 9 vertices. Together with Theorems 4.4.1, 4.4.9 and Corollary 4.4.8, this proves the theorem. \square

4.5 Bigenic Classes of Graphs Which Are Not Well-quasi-ordered

Let us start by recalling a few known or easy results about infinite antichains and classes which are not WQO. First we repeat that the set of cycles

$\mathcal{C} = \{C_3, C_4, \dots\}$ is an infinite antichain.

This example leads to several more infinite antichains. Denote by \tilde{C}_{2k} the bipartite complement of an even cycle C_{2k} . Then obviously

$\tilde{\mathcal{C}} = \{\tilde{C}_{2k} : k = 3, 4, \dots\}$ is an infinite antichain.

Also, denote by C_{2k}^* the graph obtained from an even cycle C_{2k} by creating a clique on the set of even-indexed vertices. It is easy to see that

$\mathcal{C}^* = \{C_{2k}^* : k = 2, 3, \dots\}$ is an infinite antichain.

Finally, denote by C_{3k}^Δ the graph obtained from a cycle C_{3k} by connecting every two vertices at distance 0 mod 3 from each other. In this way, we form three big cliques of size k each. For $k > 1$, any triangle in C_{3k}^Δ must belong to one of the three created cliques, and therefore it is not difficult to see that

$\mathcal{C}^\Delta = \{C_{3k}^\Delta : k = 2, 3, \dots\}$ is an infinite antichain.

To reveal more infinite antichains, let us note that the class of $3K_2$ -free bipartite graphs is not WQO [Ding, 1992] (see also Section 4.2 for a stronger result). This class contains an infinite antichain \mathcal{B} consisting of graphs partitionable into three independent sets A, B, C so that each of $A \cup B$ and $B \cup C$ induces a $2K_2$ -free bipartite graph, with no other edges present. By creating a biclique between the sets A and B (i.e. by creating all possible edges between these sets), we transform \mathcal{B} into a new sequence of graphs which will be denoted \mathcal{B}^* . Also, by replacing A and C with cliques (i.e. creating all possible edges inside the sets) we transform \mathcal{B} into a new sequence which will be denoted \mathcal{B}^{**} . With the same proof that shows that \mathcal{B} is an antichain, one can show \mathcal{B}^* and \mathcal{B}^{**} are infinite antichains.

We now use the infinite antichains described above to prove the following results.

Theorem 4.5.1. *The classes $\text{Free}(C_4, 2K_2)$, $\text{Free}(K_3, 2P_3)$, $\text{Free}(K_3, K_2 + 3K_1)$, $\text{Free}(\text{diamond}, 4K_1)$ and $\text{Free}(K_4, 2K_2)$ are minimal bigenic classes which are not well-quasi-ordered by the induced subgraph relation.*

Proof. The class $\text{Free}(C_4, 2K_2)$ contains $\text{Free}(C_5, C_4, 2K_2, \cdot)$, i.e. the class of split graphs, which in turn contains the antichain \mathcal{C}^* . If we delete any vertex from C_4 or $2K_2$, then we obtain an induced subgraph of P_4 . Since P_4 -free graphs are WQO, $\text{Free}(C_4, 2K_2)$ is a minimal not WQO class.

The class $\text{Free}(K_3, 2P_3)$ contains $2P_3$ -free bipartite graphs, which are not well-quasi-ordered, as we found in Section 4.2. To show the minimality, let us call a bigenic class trivial if one of its forbidden graphs has fewer than 3 vertices. Obviously, any trivial class is WQO. The class $\text{Free}(K_3, 2P_3)$ contains two maximal non-trivial bigenic subclasses: $\text{Free}(K_3, P_3 + 2K_1)$ and $\text{Free}(K_3, P_3 + P_2)$. Both of them are WQO by Theorem 4.4.4 and Theorem 4.4.6, respectively. Thus, $\text{Free}(K_3, 2P_3)$ is a minimal bigenic class which is not WQO.

It is not difficult to see that the bipartite complement of $K_2 + 3K_1$ contains either $K_{1,3}$ or C_4 for any bipartition of this graph. Therefore, if B is a bipartite complement of $K_2 + 3K_1$, then the class of B -free bipartite graphs contains the antichain C_6, C_8, \dots . As a result, the class of $K_2 + 3K_1$ -free bipartite graphs contains the antichain $\tilde{\mathcal{C}}$, which implies that $\text{Free}(K_3, K_2 + 3K_1)$ is not WQO. To see the minimality, observe that this class contains two maximal non-trivial bigenic subclasses: $\text{Free}(K_3, 4K_1)$ and $\text{Free}(K_3, K_2 + 2K_1)$. The first of them contains finitely many graphs by Ramsey's Theorem, the second is WQO by Theorem 4.4.6.

To see that $\text{Free}(\text{diamond}, 4K_1)$ is not WQO, observe first that every graph in \mathcal{C}^Δ is partitionable into three cliques and therefore is $4K_1$ -free. Also, any triangle in a C_{3k}^Δ must belong to one of the three cliques created in the construction of this

graph, and therefore, every graph in \mathcal{C}^Δ is *diamond*-free. Thus, $\text{Free}(\text{diamond}, 4K_1)$ contains the antichain \mathcal{C}^Δ and therefore is not WQO. This class contains three maximal bigenic classes: $\text{Free}(K_3, 4K_1)$, $\text{Free}(P_3, 4K_1)$ and $\text{Free}(\text{diamond}, 3K_1)$. The first of them contains finitely many graphs (Ramsey's Theorem), the second is a subclass of P_4 -free graphs and the last one is a subclass of $\text{Free}(\text{gem}, 3K_1)$ which is WQO by Theorem 4.4.5.

Finally, it is not difficult to see that every graph in \mathcal{B}^* is $(K_4, 2K_2)$ -free and therefore $\text{Free}(K_4, 2K_2)$ is not WQO. The set of maximal bigenic subclasses of $\text{Free}(K_4, 2K_2)$ consists of $\text{Free}(K_3, 2K_2)$ and $\text{Free}(K_4, K_2 + K_1)$. The first of them is a subclass of $(P_5, \text{diamond})$ -free graphs, which are WQO by Theorem 4.4.10, while the second is a subclass of P_4 -free graphs and therefore is WQO as well. \square

Theorem 4.5.2. *The classes $\text{Free}(K_3, 3K_2)$, $\text{Free}(\text{gem}, P_4 + K_2)$ and $\text{Free}(\text{gem}, P_6)$ are not WQO.*

Proof. The class $\text{Free}(K_3, 3K_2)$ contains the antichain \mathcal{B} , which is easy to see. Now let us show that $\text{Free}(\text{gem}, P_4 + K_2)$ and $\text{Free}(\text{gem}, P_6)$ contain the antichain \mathcal{B}^{**} . From the definition of graphs in the set \mathcal{B}^{**} it follows that both $A \cup B$ and $B \cup C$ induce P_4 -free graphs. Now it is not difficult to see that each graph in \mathcal{B}^{**} is P_6 -free and $P_4 + K_2$ -free. To see *gem*-freeness, note that any P_4 must contain at least one vertex in each of A, B and C , in which case there obviously cannot exist a vertex dominating such a P_4 . \square

4.6 A Summary for Bigenic Classes

In the two previous sections we discovered a number of bigenic classes which are well-quasi-ordered by the induced subgraph relation and a number of those which are not. In the present section, we summarize the results obtained in Sections 4.4 and 4.5, and reveal all bigenic classes for which the question of well-quasi-orderability is open. The first two columns of Table 4.1 contain a summary of the obtained results. For convenience, we also include in the first column classes $\text{Free}(P_4)$ and $\text{Free}(K_n, mK_1)$.

Proposition 4.6.1. *Let $X = \text{Free}(G, H)$ be a bigenic class containing neither $\text{Free}(\mathcal{F})$ nor $\text{Free}(\overline{\mathcal{F}})$. If*

- (1) *X contains none of the non-WQO classes listed in Table 4.1 and none of their complements, and*

WQO Classes	Thm
$Free(K_3, P_3 + 2K_1)$	4.4.4
$Free(K_3, P_4 + K_1)$	4.4.5
$Free(K_3, P_3 + P_2)$	4.4.6
$Free(P_5, diamond)$	4.4.10
$Free(diamond, co-diamond)$	4.4.10
$Free(P_4)$	4.3.1 (A)
$Free(K_n, mK_1)$	Claim 4.3.5

Not WQO	Thm	Not WQO	Clm
$Free(C_4, 2K_2)$	4.5.1	$Free(2K_2, C_5)$	4.3.7
$Free(K_3, 2P_3)$	4.5.1	$Free(C_4, C_5)$	4.3.7
$Free(K_3, K_2 + 3K_1)$	4.5.1	$Free(C_3, C_4)$	4.3.7
$Free(diamond, 4K_1)$	4.5.1	$Free(C_3, C_5)$	4.3.7
$Free(K_4, 2K_2)$	4.5.1	$Free(C_3, C_6)$	4.3.7
$Free(K_3, 3K_2)$	4.5.2	$Free(C_3, C_7)$	4.3.7
$Free(gem, P_4 + K_2)$	4.5.2	$Free(C_3, K_{1,3})$	4.3.7
$Free(gem, P_6)$	4.5.2	$Free(C_4, K_{1,3})$	4.3.7

Table 4.1: Some bigenic classes of graphs

(2) X is not contained in any of the WQO classes listed in Table 4.1 or their complements,

then $Free(G, H)$ is one the following 14 classes or one of their complements:

$$\begin{array}{lll}
Free(K_3, 2K_2 + K_1), & Free(K_3, P_4 + K_2), & Free(K_3, P_5 + K_1), \\
Free(K_3, P_6), & Free(diamond, co - gem), & Free(diamond, 2K_2 + K_1), \\
Free(diamond, P_3 + K_2), & Free(diamond, P_4 + K_2), & Free(diamond, P_6), \\
Free(gem, 2K_2), & Free(gem, co - gem), & Free(gem, 2K_2 + K_1), \\
Free(gem, P_3 + K_2), & Free(gem, P_5) &
\end{array}$$

Proof. If $X = Free(G, H)$ contains neither $Free(\mathcal{F})$ nor $Free(\overline{\mathcal{F}})$, then $Free(\mathcal{F})$ contains G or H and $Free(\overline{\mathcal{F}})$ contains G or H . If one of G and H belongs to both $Free(\mathcal{F})$ and $Free(\overline{\mathcal{F}})$, then by Claim 4.3.7, X is a subclass of $Free(P_4)$. Therefore, we may assume without loss of generality that

- $G \in Free(\overline{\mathcal{F}})$ and $H \in Free(\mathcal{F})$.

Since $\overline{C}_3, C_5, 2K_2 \notin Free(\overline{\mathcal{F}})$, we know that G is $(\overline{C}_3, C_5, 2K_2)$ -free. If additionally G is (K_3, C_4) -free, then G is an induced subgraph of P_4 . Therefore, we may assume that

- G contains either K_3 or C_4 .

By symmetry, we assume that

- H contains either $3K_1$ or $2K_2$.

If G contains C_4 and H contains $2K_2$, then X contains $\text{Free}(C_4, 2K_2)$, in which case assumption (2) fails. Since C_4 is the complement of $2K_2$, we may assume without loss of generality that

- G is a C_4 -free graph containing K_3 .

Now if H contains a graph from the set $\{K_2 + 3K_1, 3K_2, 2P_3\}$, then X contains one of the non-WQO classes from Table 4.1. Therefore, we assume that

- H is $(K_2 + 3K_1, 3K_2, 2P_3)$ -free.

Let H be a linear forest in $\text{Free}(K_2 + 3K_1, 3K_2, 2P_3)$ that is not an induced subgraph of P_4 .

- Since $2P_3$ is an induced subgraph of P_7 , we know that every connected component of H is a path on at most 6 vertices.
- If H contains a P_6 , then $H = P_6$, since otherwise $K_2 + 3K_1$ is an induced subgraph of H .
- If H is a P_6 -free graph containing a P_5 , then H is either P_5 or $P_5 + K_1$, since otherwise $3K_2$ or $K_2 + 3K_1$ is an induced subgraph of H .
- If H is a P_5 -free graph containing a P_4 , then H is either $P_4 + K_1$ or $P_4 + K_2$, since otherwise $K_2 + 3K_1$ is an induced subgraph of H .
- If H is a P_4 -free graph containing a P_3 , then H is one of $P_3 + K_1$, $P_3 + 2K_1$ or $P_3 + K_2$, since otherwise at least one of $K_2 + 3K_1, 3K_2$ or $2P_3$ is an induced subgraph of H .
- If P_2 is the longest path belonging to H , then H is one of $K_2 + 2K_1$, $2K_2$ or $2K_2 + K_1$, since otherwise $K_2 + 3K_1$ or $3K_2$ is an induced subgraph of H .
- Otherwise $H = nK_1$ for some $n \geq 3$.

Since $\text{Free}(K_n, mK_1)$ is in Table 4.1, we may assume that either G is different from a complete graph or H is different from an edgeless graph. Without loss of generality, we will assume that $H \neq nK_1$. Moreover, by Claim 4.3.6, a class $\text{Free}(G, P_3 + K_1)$ is WQO if and only if the class $\text{Free}(G, 3K_1)$ is WQO. Therefore, we may assume that $H \neq P_3 + K_1$. This reduces the analysis to the case when

- $H \in R = \{co\text{-diamond}, co\text{-gem}, 2K_2, 2K_2 + K_1, P_3 + 2K_1, P_3 + K_2, P_4 + K_2, P_5, P_5 + K_1, P_6\}$

Every graph in the set R contains either *co-diamond* or $2K_2$. Therefore, H contains *co-diamond* or $2K_2$. If additionally G contains a K_4 , then X contains one of the non-WQO classes from Table 4.1 or one of their complements. Therefore, we may assume that

- G is K_4 -free.

Let G be a (C_4, K_4) -free graph containing a triangle in the class $Free(\overline{\mathcal{F}})$, or alternatively, \overline{G} is a linear forest in $Free(2K_2, 4K_1)$ containing \overline{K}_3 .

- Since $2K_2$ is an induced subgraph of P_5 , we know that every connected component of \overline{G} is a path on at most 4 vertices.
- If \overline{G} contains P_4 and \overline{K}_3 , then $\overline{G} = P_4 + P_1$, since otherwise $4K_1$ or $2K_2$ is an induced subgraph of \overline{G} .
- If \overline{G} is a P_4 -free graph containing P_3 and \overline{K}_3 , then $\overline{G} = P_3 + K_1$, since otherwise $4K_1$ or $2K_2$ is an induced subgraph of \overline{G} .
- If \overline{G} is a P_3 -free graph containing P_2 and \overline{K}_3 , then $\overline{G} = K_2 + 2K_1$, since otherwise $4K_1$ or $2K_2$ is an induced subgraph of \overline{G} .
- \overline{G} is a P_2 -free graph containing \overline{K}_3 , then $\overline{G} = \overline{K}_3$, since otherwise $4K_1$ is an induced subgraph of \overline{G} .

Again, we may assume that $\overline{G} \neq P_3 + K_1$, i.e. $G \neq paw$, since this case reduces to the case $G = K_3$ by Claim 4.3.6. Therefore,

- $G \in Q = \{K_3, diamond, gem\}$

It is not difficult to verify that if $Free(G, H)$ is a bigenic class with $G \in Q$ and $H \in R$ satisfying (1) and (2), then $Free(G, H)$ is one of the following 14 classes:

$$\begin{array}{lll}
Free(K_3, 2K_2 + K_1), & Free(K_3, P_4 + K_2), & Free(K_3, P_5 + K_1), \\
Free(K_3, P_6), & Free(diamond, co - gem), & Free(diamond, 2K_2 + K_1), \\
Free(diamond, P_3 + K_2), & Free(diamond, P_4 + K_2), & Free(diamond, P_6), \\
Free(gem, 2K_2), & Free(gem, co - gem), & Free(gem, 2K_2 + K_1), \\
Free(gem, P_3 + K_2), & Free(gem, P_5) &
\end{array}$$

□

From Proposition 4.6.1 it follows, in particular, that there are finitely many minimal bigenic non-WQO classes containing neither $Free(\mathcal{F})$ nor $Free(\overline{\mathcal{F}})$. We prove that

Theorem 4.6.2. *There are finitely many minimal non-WQO classes of graphs defined by at most two forbidden induced subgraphs.*

To verify this theorem, we have to show that there are finitely many minimal bigenic non-WQO classes containing either $Free(\mathcal{F})$ or $Free(\overline{\mathcal{F}})$.

Proposition 4.6.3. *If a class $X = Free(G, H)$ contains either $Free(\mathcal{F})$ or $Free(\overline{\mathcal{F}})$, then X contains one of the non-WQO classes listed in Table 4.1 or one of their complements.*

Proof. Assume, without loss of generality, that X contains $Free(\mathcal{F})$. This means that neither of G and H belong to $Free(\mathcal{F})$, or alternatively, both G and H contain a graph from \mathcal{F} as an induced subgraph.

(1) If G contains a cycle C_i of length $i \geq 6$ and

H contains C_3 , then X contains one of $Free(K_3, 2P_3)$, $Free(C_3, C_6)$, $Free(C_3, C_7)$.

H contains C_4 , then X contains $Free(C_4, 2K_2)$.

H contains C_5 , then X contains $Free(2K_2, C_5)$.

H contains \overline{C}_3 , then X contains the complement of $Free(C_3, C_4)$.

(2) If G contains a C_5 and

H contains C_3 , \overline{C}_3 or C_5 , then X contains $Free(C_3, C_5)$ or its complement.

H contains C_4 , then X contains $Free(C_4, C_5)$.

(3) If G contains a C_4 and

H contains C_3 or C_4 , then X contains $Free(C_3, C_4)$.

H contains $K_{1,3}$, then X contains $Free(C_4, K_{1,3})$.

(4) If G contains a C_3 and H contains $K_{1,3}$ or C_3 , then X contains $Free(C_3, K_{1,3})$.

(5) If both G and H contain $K_{1,3}$, the X contains $Free(C_3, K_{1,3})$.

Since every graph in \mathcal{F} contains one of C_3 , C_4 , C_5 or \overline{C}_3 , items (1) and (2) in the above analysis prove the theorem in the case when one of the forbidden graphs contains a cycle of length at least 5. Items (3), (4) and (5) prove the theorem in the case when one of the forbidden graphs contains C_4 , C_3 or $K_{1,3}$. \square

4.7 Boundary Classes for Well-Quasi-Orderability

Let \mathcal{Y}_k be the family of hereditary classes of graphs defined by k forbidden induced subgraphs. In Theorem 4.6.2, we showed that there are only finitely many minimal non-WQO classes defined by two forbidden induced subgraphs. However, in the case of more than two forbidden induced subgraphs, the situation changes dramatically.

Theorem 4.7.1. *For every $k \geq 3$, the family \mathcal{Y}_k contains infinitely many minimal non-WQO classes.*

Proof. Consider the class $\text{Free}(K_{1,3}, C_3, C_t)$ for any $t \geq 4$. This class is non-WQO, since it contains infinitely many cycles. Assume that it is not a minimal non-WQO class in \mathcal{Y}_3 , and let $X \in \mathcal{Y}_3$ be a proper subclass of $\text{Free}(K_{1,3}, C_3, C_t)$ which is non-WQO. Then the set of forbidden induced subgraphs for X contains a graph G which is a proper induced subgraph of one of $K_{1,3}, C_3, C_t$. If G is a proper induced subgraph of $K_{1,3}$ or C_3 , then either G is an induced subgraph of P_4 , in which case X must be WQO, or G consists of three isolated vertices, in which case X is WQO too, because it is finite (every graph in X is (\overline{K}_3, K_3) -free and hence has at most 5 vertices, by Ramsey's theorem).

Assume now that G is a proper induced subgraph of C_t with $t \geq 4$. Then $X \subseteq \text{Free}(K_{1,3}, C_3, P_t)$. We claim that in this case X is WQO again.

For any natural t , the class $\text{Free}(K_{1,3}, C_3, P_t)$ is well-quasi-ordered. Since $K_{1,3}$ is forbidden and C_3 is forbidden, the degree of each vertex of any graph in this class is at most 2, and since P_t is forbidden, every connected graph in this class has at most t vertices. We know from Claim 4.3.3 that a class of graphs is well-quasi-ordered if and only if the set of connected graphs in the class is well-quasi-ordered. Since $\text{Free}(K_{1,3}, C_3, P_t)$ contains finitely many connected graphs, it is well-quasi-ordered.

Thus, the class $\text{Free}(K_{1,3}, C_3, C_t)$ contains no proper subclass from \mathcal{Y}_3 which is non-WQO, i.e. $\text{Free}(K_{1,3}, C_3, C_t)$ is a minimal non-WQO class for all $t \geq 4$.

For $k > 3$, the proof is similar, i.e. we consider the class $\text{Free}(K_{1,3}, C_3, \dots, C_k, C_t)$ and show that it is a minimal non-WQO class for any $t > k$. \square

The finiteness of the number of minimal non-WQO classes in the family $\mathcal{Y}_1 \cup \mathcal{Y}_2$ implies, in particular, that the problem of deciding whether a class in this family is WQO or non-WQO is polynomial-time solvable. For larger values of k , this approach does not work, as is shown by Theorem 4.7.1. In the attempt to overcome this difficulty, we will use the notion of boundary classes as a helpful tool to investigate finitely defined classes of graphs.

We start by discovering a specific boundary class for the family of graph classes that are WQO by the induced subgraph relation. This special case will be useful in finding more boundary classes later.

Theorem 4.7.2. *The class of linear forests is a boundary class for well-quasi-orderability.*

Proof. Let F be a linear forest. Without loss of generality, we may assume that $F = P_t$, since every linear forest is an induced subgraph of P_t for some value of t . Clearly, the class $\text{Free}(P_t, K_{1,3}, C_3, C_4, \dots, C_t)$ is a subclass of linear forests, and obviously, the class linear forests is well-quasi-ordered. Therefore, by Lemma 1.4.7, the class of linear forests is a minimal limit class. \square

In the proof of Theorem 4.7.2, we observed that the class of linear forests is well-quasi-ordered by the induced subgraph relation. In fact, any boundary class must be WQO.

Lemma 4.7.3. *Every boundary class is well-quasi-ordered.*

Proof. If a boundary class X is non-WQO, it must contain an infinite antichain G_1, G_2, \dots . Then for any G_i , the class $\text{Free}(G_i) \cap X$ is a proper limit subclass of X , contradicting the minimality of X . \square

4.7.1 On the number of boundary classes

In the previous section, we revealed one boundary class, the class of linear forests. We denote this class by \mathcal{F} . Are there other boundary classes? Yes, because for any boundary class X , the class of complements of graphs in X is also boundary. Therefore, the complements of linear forests form a boundary class; we denote this class by $\overline{\mathcal{F}}$. As we shall see later, there are many other boundary classes. Moreover, in this section we show that the family of boundary classes is infinite. To this end, for any natural number $k \geq 1$, we define the following graph operation. Given graph G , we subdivide each edge of G by exactly k ‘new’ vertices and then create a clique on the set of ‘old’ vertices. Let us denote the graph obtained in this way by

$G^{(k)}$. Also, for an arbitrary hereditary class X , we define $X^{(k)}$ to be the class of all induced subgraphs of the graphs $G^{(k)}$ formed from graphs $G \in X$.

It is not difficult to see that classes $\mathcal{F}^{(k)}$, for various values of k , are pairwise incomparable, i.e. none of them is a subclass of another. We will show that for any $k \geq 3$, the class $\mathcal{F}^{(k)}$ is a boundary class. To this end, let us prove a few auxiliary results.

Throughout the section, we denote by \mathcal{D} the class of graphs of vertex degree at most 2. Clearly, this is a hereditary class. The set of minimal forbidden induced subgraphs for this class consists of 4 graphs (each of them has a vertex of degree 3 and the three neighbours of that vertex induce all possible graphs on 3 vertices). We will show that for the class $\mathcal{D}^{(k)}$, the situation is similar, in the sense that the set of minimal forbidden induced subgraphs for it is finite, regardless of the value of k .

Lemma 4.7.4. *For each $k \geq 3$, the set of minimal forbidden induced subgraphs for the class $\mathcal{D}^{(k)}$ is finite.*

Proof. First of all, let us observe that the class $\mathcal{D}^{(k)}$ is a subclass of the class $\mathcal{M}^{(k)}$ of graphs whose vertices can be partitioned into a clique A and a set B of vertices inducing a P_{k+1} -free linear forest (i.e. a graph every connected component of which is a path on at most k vertices). $\mathcal{M}^{(k)}$ is a wider class than $\mathcal{D}^{(k)}$, since by definition we do not specify what is happening between the two parts A and B for graphs in $\mathcal{M}^{(k)}$, while for graphs in $\mathcal{D}^{(k)}$ there are severe restrictions on the edges between A and B (these restrictions are described below). Therefore, the set of minimal forbidden induced subgraphs for $\mathcal{D}^{(k)}$ consists of the set M of minimal forbidden induced subgraphs for $\mathcal{M}^{(k)}$ and the set D of graphs from $\mathcal{M}^{(k)}$ that restrict the behavior of edges between A and B . We will show that both sets M and D are finite.

For the finiteness of M we refer the reader to [Zverovich, 2002], where the following result was proved: Let P and Q be two hereditary classes of graphs such that both P and Q are defined by finitely many forbidden induced subgraphs, and there is a constant bounding the size of a maximum clique for all graphs in P and the size of a maximum independent set for all graphs in Q . Then the class of all graphs whose vertices can be partitioned into a set inducing a graph from P and a set inducing a graph from Q has a finite characterization in terms of forbidden induced subgraphs. For the class $\mathcal{M}^{(k)}$, we have $Q = \text{Free}(\overline{K}_2)$ is the class of complete graphs, in which case the size of a maximum independent set is 1, and $P = \text{Free}(K_{1,3}, C_3, \dots, C_{k+1}, P_{k+1})$, in which case the size of a maximum clique is

at most 2. Therefore, M is a finite set.

In order to show that the size of D is bounded, let us describe the restrictions on the behavior of edges connecting vertices of A to the vertices of B in graphs in the class $\mathcal{D}^{(k)}$. To simplify the task, we are working under the assumption that $k \geq 3$.

- (1) Every vertex of B has at most one neighbour in A ;
- (2) Only an end-vertex of a path in B can have a neighbour in A ;
- (3) If both end-vertices of a path in B have neighbours in A , then these neighbours are different and the path has exactly k vertices;
- (4) Let P and P' be two paths in B such that each contains exactly k vertices and both end-vertices in both paths have neighbours in A . Then the pair of neighbours of P in A and the pair of neighbours of P' in A are different, i.e. they share at most one vertex.
- (5) Every vertex of A has at most two neighbours in B .

It is not difficult to see that a graph $G \in \mathcal{M}^{(k)}$ belongs to $\mathcal{D}^{(k)}$ if and only if G satisfies restrictions (1) – (5). The first four restrictions are common for any graph $G^{(k)}$ (or an induced subgraph of $G^{(k)}$) and they completely specify the behavior of the edges connecting ‘new’ vertices to ‘old’ ones. Restriction (5) is specific for graphs in $\mathcal{D}^{(k)}$.

Now we translate restrictions (1) – (5) to the language of forbidden induced subgraphs. We denote by Φ and T the two graphs represented in Figure 4.7. Also, C''_{k+2} stands for the graph consisting of two cycles C_{k+2} sharing an edge, and *diamond* for K_4 without an edge. It is a routine task to verify that the graphs *diamond*, $K_{1,4}$, $C_4, \dots, C_{k+1}, C''_{k+2}, \Phi, T$ belong to $\mathcal{M}^{(k)}$ but do not belong to $\mathcal{D}^{(k)}$ (for $k \geq 3$). Moreover, they are minimal graphs that do not belong to $\mathcal{D}^{(k)}$.

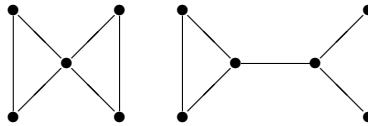


Figure 4.7: The graphs Φ (left) and T (right)

Now let G be a graph in $\mathcal{M}^{(k)}$ which is free of *diamond*, $K_{1,4}$, $C_4, \dots, C_{k+1}, C''_{k+2}, \Phi, T$. We assume that

- every vertex of B has at least one non-neighbour in A , since otherwise this vertex can be moved to A ,
- A contains at least 3 vertices, because there are finitely many connected $K_{1,4}$ -free graphs in $\mathcal{M}^{(k)}$ with $|A| \leq 2$, and a minimal graph in $\mathcal{M}^{(k)}$ which does not belong to $\mathcal{D}^{(k)}$ must be connected.

Under these assumptions, the *diamond*-freeness of G guarantees that (1) is satisfied.

Suppose that the part B of G contains a path in which a non-end-vertex v has a neighbour x in A . Since $|A| \geq 3$, there must exist two other vertices $y, z \in A$, and these vertices must be non-adjacent to v , by (1). Since v is a non-end-vertex of a path in B , it must have two distinct neighbours in the path, say u and w , with u being non-adjacent to w . By (1), each of u and w has at most one neighbour among x, y, z . If one of them is adjacent to x , then the forbidden graph Φ arises. If u or w is adjacent to y or z , then a C_4 arises, and if the vertices u, w have no neighbours among x, y, z , then the graph T arises. This discussion shows that restriction (2) is satisfied.

Assume both end-vertices of a path P in B have neighbours in A . Together with (2), this gives rise to a chordless cycle C consisting of P and its neighbours in A . If P has less than k vertices, then C is of size at most $k + 1$, which is forbidden. If P has exactly k vertices and just one neighbour in A , then the size of C is $k + 1$, which is impossible. Therefore, P has k vertices and 2 neighbours in A . Therefore, restriction (3) is satisfied.

Let P and P' be two paths in B such that each contains exactly k vertices and both end-vertices in both paths have neighbours in A . If the neighbours of P in A coincide with the neighbours of P' in A , then G contains the forbidden graph C''_{k+2} . Therefore, restriction (4) is satisfied.

Finally, if a vertex x of A has at least three neighbours in B , say u, v, w , then from the previous discussion, we know that u, v, w belong to different connected components of B , and therefore, x, u, v, w together with any vertex $y \in A$ different from x induce a $K_{1,4}$. This shows that restriction (5) is satisfied.

From the above discussion, we conclude that D must be finite, which completes the proof of the lemma. \square

Lemma 4.7.5. *Let G be a graph with at least 4 vertices, and let $G^{(k)}$ be an induced subgraph of $H^{(k)}$. Then G is a subgraph of H .*

Proof. Observe that in the graphs $G^{(k)}$ and $H^{(k)}$, every ‘new’ vertex has degree 2, while every ‘old’ vertex has degree at least 3. Therefore, if $G^{(k)}$ is an induced subgraph of $H^{(k)}$, then ‘new’ vertices of $G^{(k)}$ are mapped to ‘new’ vertices of $H^{(k)}$ and ‘old’ vertices of $G^{(k)}$ are mapped to ‘old’ vertices of $H^{(k)}$. Let U be the set of vertices whose deletion from $H^{(k)}$ results in $G^{(k)}$. If U contains a ‘new’ vertex v subdividing an edge e of H , then U must contain all new vertices subdividing e , since otherwise a pendant vertex appears, which is not possible for $G^{(k)}$. Obviously, deletion of all new vertices subdividing e from $H^{(k)}$ is equivalent to deletion of the edge e from H . Also, if U contains an ‘old’ vertex v of H , then U must contain all new vertices subdividing all edges incident to v (in H), since otherwise again a pendant vertex appears. Clearly, deleting from $H^{(k)}$ an ‘old’ vertex v together with all new vertices subdividing all edges incident to v (in H) is equivalent to deleting from H vertex v together with all edges incident to v . Therefore, if $G^{(k)}$ is an induced subgraph of $H^{(k)}$, then G is a subgraph of H . \square

Theorem 4.7.6. *For any natural number $k \geq 3$, the class $\mathcal{F}^{(k)}$ is a boundary class.*

Proof. Let \mathcal{D} be the class of graphs of vertex degree at most 2 and $k \geq 3$ a natural number. First, we show that $\mathcal{F}^{(k)}$ is a limit class. To this end, define the sequence $\mathcal{F}_3^{(k)}, \mathcal{F}_4^{(k)}, \dots$ of graph classes by $\mathcal{F}_i^{(k)} := \text{Free}(C_3^{(k)}, C_4^{(k)}, \dots, C_i^{(k)}) \cap \mathcal{D}^{(k)}$. It is not difficult to see that the sequence $\mathcal{F}_3^{(k)}, \mathcal{F}_4^{(k)}, \dots$ converges to $\mathcal{F}^{(k)}$. Also, for each i , the class $\mathcal{F}_i^{(k)}$ contains an infinite antichain, namely $C_{i+1}^{(k)}, C_{i+2}^{(k)}, \dots$, which follows from Lemma 4.7.5 and the obvious observation that cycles form an antichain with respect to the subgraph relation.

The proof of minimality of $\mathcal{F}^{(k)}$ is similar to Theorem 4.7.2. We consider a graph G in $\mathcal{F}^{(k)}$ and without loss of generality assume that $G = P_t^{(k)}$, since every graph in $\mathcal{F}^{(k)}$ is an induced subgraph of $P_t^{(k)}$ for some t . Then the class $\text{Free}(P_t^{(k)}, C_3^{(k)}, C_4^{(k)}, \dots, C_t^{(k)}) \cap \mathcal{D}^{(k)}$ is a subclass of $\mathcal{F}^{(k)}$. By Lemma 4.7.4, this class is finitely defined, and since $\mathcal{F}^{(k)}$ is well-quasi-ordered, this class is well-quasi-ordered too. Therefore, by Lemma 1.4.7, $\mathcal{F}^{(k)}$ is a minimal limit class. \square

4.7.2 The well-quasi-orderability of finitely defined classes

In this section, we show that for any $k \geq 1$, the set of boundary classes essential for determining well-quasi-orderability of classes in \mathcal{Y}_k (the family of graph classes defined by k forbidden induced subgraphs) is finite.

We start with the initial case $k = 1$, in order to apply induction for the general case.

Theorem 4.7.7. *A monogenic class of graphs is wqo if and only if it contains neither \mathcal{F} nor $\overline{\mathcal{F}}$.*

Proof. Let $X = \text{Free}(G)$ be a monogenic class of graphs. If X contains \mathcal{F} or $\overline{\mathcal{F}}$ then X is not wqo, by Theorem 1.4.6. Assume now that X contains neither \mathcal{F} nor $\overline{\mathcal{F}}$, i.e. G belongs both to \mathcal{F} and to $\overline{\mathcal{F}}$. The intersection $\mathcal{F} \cap \overline{\mathcal{F}}$ is the class of graphs free of

$$K_{1,3}, \overline{K}_{1,3}, C_3, \overline{C}_3, C_4, \overline{C}_4, C_5, \overline{C}_5, C_6, \overline{C}_6, \dots$$

$K_{1,3}$ and every cycle C_i with $i > 5$ contain a \overline{C}_3 . Therefore,

$$\mathcal{F} \cap \overline{\mathcal{F}} = \text{Free}(C_3, \overline{C}_3, C_4, \overline{C}_4, C_5).$$

It is not difficult to verify that $\mathcal{F} \cap \overline{\mathcal{F}}$ consists of P_4 and its induced subgraphs. In [Damaschke, 1990], it was shown that P_4 -free graphs are wqo by induced subgraphs. \square

Theorem 4.7.8. *For any natural k , there is a finite set \mathcal{B}_k of boundary classes such that a class $X = \text{Free}(G_1, \dots, G_k)$ is wqo if and only if it contains none of the boundary subclasses from the set \mathcal{B}_k .*

Proof. We prove the theorem by induction on k . For $k = 1$, the result follows from Theorem 4.7.7.

To make the inductive step, we assume that the theorem is true for $k - 1$. Let \mathcal{C} be the set of graph classes $\text{Free}(G_1, \dots, G_k)$ such that

- each of the graphs G_1, \dots, G_k belongs to one of the boundary classes in \mathcal{B}_{k-1} ,
- $\text{Free}(G_1, \dots, G_k)$ is not wqo.

Since the set \mathcal{B}_{k-1} is finite and each class in this set is well-quasi-ordered (Lemma 4.7.3), we conclude (by Higman's Lemma) that \mathcal{C} is well-quasi-ordered by subclass inclusion, and thus the set of minimal classes in \mathcal{C} is finite; we denote this set by \mathcal{C}_k .

For each class in \mathcal{C}_k , we arbitrarily choose a boundary subclass contained in it (such a boundary subclass must exist, by Theorem 1.4.6), and denote the set of boundary classes chosen in this way by \mathcal{B} . Since \mathcal{C}_k is finite, \mathcal{B} is finite too. Now we claim that the theorem holds with $\mathcal{B}_k = \mathcal{B}_{k-1} \cup \mathcal{B}$. To see this, consider a class of graphs $X = \text{Free}(G_1, \dots, G_k)$. If it is wqo, then it does not contain any boundary subclass from \mathcal{B}_k , since it contains no boundary subclasses, by Theorem 1.4.6.

Suppose now that $X = \text{Free}(G_1, \dots, G_k)$ is not wqo. If each of the graphs G_1, \dots, G_k belongs to one of the boundary classes in \mathcal{B}_{k-1} , then it must contain

a class from \mathcal{C}_k by definition of \mathcal{C}_k and therefore it must contain a boundary class from $\mathcal{B} \subseteq \mathcal{B}_k$. If one of the forbidden graphs, say G_i , does not belong to any class in \mathcal{B}_{k-1} , then we consider the class $\text{Free}(G_1, \dots, G_{i-1}, G_{i+1}, \dots, G_k)$. By induction, it contains a boundary class from \mathcal{B}_{k-1} . But then X contains the same boundary class. \square

4.7.3 Remarks and open problems

We proved that for each k , there is finite collection of boundary properties that allow us to determine whether a class of graphs defined by k forbidden induced subgraphs is wqo or not. This conclusion is in a sharp contrast with the fact that the number of boundary properties is generally infinite, for which we also gave a proof. The proof of this fact is obtained with the help of a simple graph operation applied to linear forests. More graph operations (complementation, “bipartite” complementation, etc.) can produce more boundary classes related to the class of linear forests. However, it is not clear whether there exist boundary properties that are not derived from linear forests. Identifying such properties is a natural open question.

To formulate one more open problem related to this topic, we extend the induced subgraph relation to the more general notion known as the labelled-induced subgraph relation, as in Section 4.4. For the reader’s convenience, we re-introduce this notion here briefly. Assume (W, \leq) is an arbitrary well-quasi-order. We call G a *labelled graph* if each vertex $v \in V(G)$ is equipped with an element $l(v) \in W$ (the label of v), and we say that a graph G is a *labelled-induced subgraph* of H if G is isomorphic to an induced subgraph of H and the isomorphism maps each vertex $v \in G$ to a vertex $w \in H$ with $l(v) \leq l(w)$.

It is interesting to observe that the class of linear forests, although well-quasi-ordered by the induced subgraph relation, is not well-quasi-ordered by the labelled-induced subgraph relation. On the other hand, all finitely defined classes which are known to be well-quasi-ordered also are well-quasi-ordered by the labelled-induced subgraph relation. This observation motivates the following conjecture.

Conjecture 4.7.1. *Let X be a hereditary class which is well-quasi-ordered by the induced subgraph relation. Then X is well-quasi-ordered by the labelled-induced subgraph relation if and only if the set of minimal forbidden induced subgraphs for X is finite.*

Chapter 5

Conclusion

We began with two initial questions: How far, exactly, must one restrict the structure of a graph to obtain a certain interesting property? What kind of tools are helpful to classify sets of graphs into those which satisfy a property and those that do not?

With these two questions in mind, we have studied three main types of useful properties that can be attained by classes of graphs: the efficient solvability of algorithmic graph problems, the relative structural simplicity that comes with boundedness of clique-width, and the elegance of being well-quasi-ordered (lacking infinite antichains of graphs with respect to a binary relation).

It is worth noting that these three types of properties have some interrelationships. For example, the boundedness of clique-width has implications for the efficient solvability of a large number of algorithmic graph problems, as we discussed in Section 3.1. So although Chapter 2 is mainly dedicated to the discovery of boundary properties and polynomial-time algorithms in relation to algorithmic graph problems, the later chapters also have algorithmic incentives, among other motivations.

We found that bipartite graphs play a crucial role in the study of notions such as clique-width (Section 3.2), and we proceeded to set up a general framework to construct bipartite graphs of large clique-width. This led to several new discoveries, for instance the discovery of a new minimal hereditary graph class of unbounded clique-width.

In our study of well-quasi-orderability, we managed to complete a characterisation of hereditary classes of bipartite graphs defined by one forbidden induced bipartite subgraph, into those classes which are WQO and those which are not. This was achieved by proving the non-well-quasi-orderability of the class of P_7 -free bipartite graphs. A similar characterisation already existed for general graph classes defined by one forbidden induced subgraph [Damascchke, 1990]. We made major progress

towards establishing such a characterisation for general graph classes defined by two forbidden induced subgraphs, by determining the WQO-status of several interesting graph classes, and by narrowing the open cases down to a small finite number. After studying many cases of well-quasi-orderability with respect to the induced subgraph relation in Chapter 4, evidence suggests that it is natural to make the following conjecture relating well-quasi-orderability of graph classes to boundedness of clique-width:

Conjecture 5.0.2. *If a graph class X is well-quasi-ordered with respect to the induced subgraph relation, then X has bounded clique-width.*

To the best of our knowledge, there are no known counter-examples to this conjecture.

The notion of boundary properties is a useful way of identifying whether a graph class has a desirable property, in the case where it is not possible to simply give a list of minimal classes that do not possess the given property. In particular, we discovered new boundary properties in relation to various algorithmic graph problems: for example, we found the first boundary properties for the family of hereditary graph classes for which the Hamiltonian cycle problem is polynomial-time solvable. Although it is a non-trivial task to find and list boundary properties, or even to determine the number of boundary properties for a specific family of graph classes, the task is worthwhile in order to study our two initial questions in a more systematic way.

In the case of more restricted graph classes, such as those defined by finitely many forbidden induced subgraphs, the notion of boundary properties is especially important, since it can provide us with an exact tool for determining whether such graph classes belong to a certain family (due to Theorem 1.4.6). If one can show that for graph classes defined by a forbidden set of bounded size, the number of boundary properties for a family \mathcal{A} also becomes bounded, then this has obvious implications for the complexity of the decidability problem: Does \mathcal{A} include a specific finitely defined class? We presented such a result for well-quasi-orderability in Section 4.7.

In addition to the challenge of identifying more boundary properties of graphs, the notion of ‘boundary ideals’ can be applied more generally to the study of other partial orders. For instance, recently there has been considerable interest in the pattern containment relation on permutations (see e.g. [Atkinson et al., 2002; Brignall, 2012; Murphy and Vatter, 2003; Vatter and Waton, 2011]). The problem of deciding whether a permutation class given by a finite set of “forbidden” permutations is wqo

or not was proposed in [Brignall et al., 2008]. The notion of boundary ideals could be helpful in finding an answer to this question.

Boundary ideals seem to be a promising concept with which to relate various important partially ordered structures to each other, some of which have been studied in relative isolation before. In this sense, this pleasingly generalised notion could be seen to have a 'unifying' effect as a mindset or framework for mathematical research.

Index

- 3-colorability, 32
- adjacent, 1
- antichain, 5
- bi-cograph, 76
- bichain graph, 67
- biconvex, 82
- bigenic, 94
- binary relation, 4
- bipartite, 8
- bipartite permutation graph, 86, 91
- boundary, 13
- caterpillar, 20
- chain, 5
- chordal, 9
- chordal bipartite, 9
- class, 2
- claw, 45
- clique, 3
- clique-width, 52
- cograph, 9
- comparability graph, 64
- comparable, 5
- complement, 2
- complete graph, 3
- composition, 77
- concatenation, 56
- connected, 3
- connected component, 3
- convex, 39
- countable grid, 70
- cubic, 18
- cycle, 3
- degree, 1
- diamond, 46, 104
- Dilworth number, 64
- disjoint union, 2
- dominating induced matching, 32
- domination, 32
- double bichain graph, 67, 86
- edge, 1
- edge contraction, 2
- empty graph, 3
- factorial, 10, 54
- finitely defined, 7
- forbidden set, 6
- forest, 8
- gem, 99
- graph, 1
- graph minor theorem, 74
- Hamiltonian cycle, 18
- hereditary, 7
- Higman's lemma, 94
- incident, 1
- independent set, 3
- induced matching, 32
- induced subgraph, 2, 6

- interval graph, 67
- isomorphism, 2
- k-letter graph, 92
- k-path partition, 25
- k-uniform graph, 96
- Kruskal's tree theorem, 94
- labelled graph, 96
- labelled-induced subgraph, 96, 104, 119
- leaf, 38
- limit ideal, limit class, 12
- lower ideal, 5
- matched co-bipartite graph, 104
- minimality criterion, 13
- minor, 7
- minor-closed, 7
- modular decomposition, 103
- module, 103
- monadic second order logic, 35
- monogenic, 75
- monotone, 7
- neighborhood, 1
- NP-complete, 15
- order-preserving, 4
- orthogonal concatenation, 56
- parse tree, 52
- partial order, 4
- partially ordered set, poset, 4
- path, 3
- pattern containment, 4
- paw, 95
- permutation graph, 64, 77
- polynomial-time, 15
- preleaf, 38
- prime graph, 103
- property, 2
- quasi-order, 4
- Ramsey number, 10
- regular, 1
- speed, 10
- split graph, 65
- split permutation graph, 67
- subcubic, 18
- subgraph, 7
- superfactorial, 11
- thin spider, 104
- threshold graph, 66
- tree, 8
- tree-width, 53
- TS-graph, 66
- vertex, 1
- vicinal order, 64
- well-quasi-order (WQO), 74

References

- J. Akiyama, K. Ando, and F. Harary. A graph and its complement with specified properties viii: interval graphs. *Mathematica Japonica*, 1983.
- V. E. Alekseev. On easy and hard hereditary classes of graphs with respect to the independent set problem. *Discrete Applied Mathematics*, 132:17, 2003.
- V. E. Alekseev, D. V. Korobitsyn, and V. V. Lozin. Boundary classes of graphs for the dominating set problem. *Discrete Mathematics*, 285:1, 2004.
- V. E. Alekseev, R. Boliac, D. V. Korobitsyn, and V. V. Lozin. NP-hard graph problems and boundary classes of graphs. *Theoretical Computer Science*, 389: 219, 2007.
- P. Allen, V. V. Lozin, and M. Rao. Clique-width and the speed of hereditary properties. *Electronic Journal of Combinatorics*, 16, 2009.
- E. M. Arkin, J. S. B. Mitchell, and V. Polishchuk. Two new classes of hamiltonian graphs (extended abstract). *Electronic Notes in Discrete Mathematics*, 29:565, 2007.
- S. Arnborg and A. Proskurowski. Linear time algorithms for NP-hard problems restricted to partial k -trees. *Discrete Applied Mathematics*, 23:11, 1989.
- K. Asdre and S.D. Nikopoulos. NP-completeness results for some problems on subclasses of bipartite and chordal graphs). *Theoretical Computer Science*, 381: 248, 2007.
- M. D. Atkinson, M. M. Murphy, and N. Ruskuc. Partially well-ordered closed sets of permutations. *Order*, 19:101, 2002.
- J. Balogh, B. Bollobas, and D. Weinreich. The speed of hereditary properties of graphs. *Journal of Combinatorial Theory, B*, 79:131, 2000.

- H. J. Bandelt and H. M. Mulder. Distance-hereditary graphs. *Journal of Combinatorial Theory, B*, 41:182, 1986.
- C. Benzaken, P. L. Hammer, and D. De Werra. Threshold characterizations of graphs with dilworth number two. *Journal of Graph Theory*, 9:245, 1985a.
- C. Benzaken, P. L. Hammer, and D. De Werra. Threshold characterizations of graphs with dilworth number two. *Discrete Mathematics*, 55:123, 1985b.
- H. L. Bodlaender and D. M. Thilikos. Treewidth for graphs with small chordality. *Discrete Applied Mathematics*, 79:45, 1997.
- R. Boliac and V. V. Lozin. On the clique-width of graphs in hereditary classes. *Lecture Notes in Computer Science*, 2518:44, 2002.
- R. B. Borie, R. G. Parker, and C. A. Tovey. Solving problems on recursively constructed graphs. *ACM Comput. Surv.*, 41(1):4:1–4:51, January 2009.
- A. Brandstädt. $(p_5, \text{diamond})$ -free graphs revisited: structure and linear time optimization. *Discrete Applied Mathematics*, 138:13, 2004.
- A. Brandstädt and V. V. Lozin. On the linear structure and clique-width of bipartite permutation graphs. *Ars Combinatoria*, 67:273, 2003.
- A. Brandstädt and S. Mahfud. Maximum weight stable set on graphs without claw and co-claw (and similar graph classes) can be solved in linear time. *Information Processing Letters*, 84:251, 2002.
- A. Brandstädt and R. Mosca. Dominating induced matchings for P_7 -free graphs in linear time. *Lecture Notes in Computer Science*, 7074:100, 2011.
- A. Brandstädt, V. B. Le, and J. P. Spinrad. *Graph Classes: a Survey*. SIAM Monographs on Discrete Math. Appl. Vol. 3, 1999.
- A. Brandstädt, J. Engelfriet, H.-O. Le, and V.V. Lozin. Clique-width for four-vertex forbidden subgraphs. *Theory of Computing Systems*, 34:561, 2006.
- A. Brandstädt, E.M. Eschen, and R. Sritharan. The induced matching and chain subgraph cover problems for convex bipartite graphs. *Theoretical Computer Science*, 381:260, 2007.
- R. Brignall. Grid classes and partial well order. *J. Combin. Theory Ser. A*, 119:99, 2012.

- R. Brignall, N. Ruskuc, and V. Vatter. Simple permutations: decidability and unavoidable substructures. *Theoretical Computer Science*, 391:150, 2008.
- K. Cameron. Induced matchings. *Discrete Applied Mathematics*, 24:97, 1989.
- K. Cameron. Induced matchings in intersection graphs. *Discrete Mathematics*, 278:1, 2004.
- K. Cameron, R. Sritharan, and Y. Tang. Finding a maximum induced matching in weakly chordal graphs. *Discrete Mathematics*, 132:67, 2003.
- D. M. Cardoso and V. V. Lozin. Dominating induced matchings. *Lecture Notes in Computer Science*, 5420:77, 2009.
- D. M. Cardoso, J. O. Cerdeira, C. Delorme, and P. C. Silva. Efficient edge domination in regular graphs. *Discrete Applied Mathematics*, 2008.
- J. M. Chang. Induced matchings in asteroidal triple-free graphs. *Discrete Applied Mathematics*, 132:67, 2004.
- M. Chudnovsky, N. Robertson, P. D. Seymour, and R. Thomas. The strong perfect graph theorem. *Annals of Mathematics*, 164:51, 2006.
- V. Chvátal and P. L. Hammer. Set-packing and threshold graphs. *Research Report, University of Waterloo*, 1973.
- D. G. Corneil and U. Rotics. On the relationship between clique-width and treewidth. *SIAM J. Comput.*, 34:825, 2005.
- B. Courcelle. Clique-width of countable graphs: a compactness property. *Discrete Mathematics*, 276:127, 2004.
- B. Courcelle and S. Olariu. Upper bounds to the clique-width of a graph. *Discrete Applied Mathematics*, 101:77, 2000.
- B. Courcelle, J. Engelfriet, and G. Rozenberg. Handle-rewriting hypergraph grammars. *Journal of Computer and System Sciences*, 46:218, 1993.
- B. Courcelle, J. A. Makowsky, and U. Rotics. Linear time solvable optimization problems on graphs of bounded clique-width. *Theory Comput. Syst.*, 33:125, 2000.
- P. Damaschke. Induced subgraphs and well-quasi-ordering. *J. Graph Theory*, 14:427, 1990.

- R. P. Dilworth. A decomposition theorem for partially ordered sets. *Annals of Mathematics*, 51(1):161, 1950.
- G. Ding. Subgraphs and well-quasi-ordering. *J. Graph Theory*, 16:489, 1992.
- B. Dushnik and E. W. Miller. Partially ordered sets. *American Journal of Mathematics*, 63(3):600, 1941.
- P. C. Fishburn. An interval graph is not a comparability graph. *Journal of Combinatorial Theory*, 8:442, 1970.
- S. Földes and P. L. Hammer. Split graphs. *Congressus Numerantium*, 19:311, 1977a.
- S. Földes and P. L. Hammer. Split graphs having dilworth number two. *Canadian Journal of Mathematics*, 29:666, 1977b.
- S. Földes and P. L. Hammer. The dilworth number of a graph. *Annals of Discrete Mathematics*, 2:211, 1978.
- J.-L. Fouquet, V. Giakoumakis, and J.-M. Vanherpe. Bipartite graphs totally decomposable by canonical decomposition. *Inter. J. Foundations of Computer Science*, 10:513, 1999.
- M. R. Garey and D. S. Johnson. *Computers and Intractability: A Guide to the Theory of NP-completeness*. Freeman, New York, 1979.
- V. Giakoumakis and J.-M. Vanherpe. Bi-complement reducible graphs. *Advances in Applied Mathematics*, 18:389, 1997.
- P. C. Gilmore and A. J. Hoffman. The dilworth number of a graph. *Canadian Journal of Mathematics*, 16:539, 1964.
- M. C. Golumbic and R. C. Laskar. Irredundancy in circular arc graphs. *Discrete Applied Mathematics*, 44:79, 1993.
- M. C. Golumbic and M. Lewenstein. New results on induced matchings. *Discrete Applied Mathematics*, 101:157, 2000.
- M. C. Golumbic and U. Rotics. On the clique-width of some perfect graph classes. *International J. Foundations of Computer Science*, 11:423, 2000.
- D.L. Grinstead, P. J. Slater, N. A. Sherwani, and N. D. Holmes. Efficient edge domination problems in graphs. *Information Processing Letters*, 48:221, 1993.

- G. Higman. Ordering by divisibility of abstract algebras. *Proc. London Math. Soc.*, 2:326, 1952.
- A. Itai, C.H. Papadimitriou, and J.L. Szwarcfiter. Hamilton paths in grid graphs. *SIAM J. Computing*, 11:676, 1982.
- M. Jean. An interval graph is a comparability graph. *Journal of Combinatorial Theory*, 7:189, 1969.
- M. Kaminski, V. V. Lozin, and M. Milanic. Recent developments on graphs of bounded clique-width. *Discrete Applied Mathematics*, 157:2747, 2009.
- D. Kobler and U. Rotics. Finding maximum induced matchings in subclasses of claw-free and P_5 -free graphs, and in graphs with matching and induced matching of equal maximum size. *Algorithmica*, 37:327, 2003.
- J. Kratochvíl. Regular codes in regular graphs are difficult. *Discrete Mathematics*, 133:191, 1994.
- J. B. Kruskal. Well-quasi-ordering, the tree theorem, and vazsonyi's conjecture. *Transactions of the American Mathematical Society*, 95:210, 1960.
- F. Lazebnik, V.A. Ustimenko, and A.J. Woldar. A new series of dense graphs of high girth. *Bulletin of the AMS*, 32:73, 1995.
- V. V. Lozin. On maximum induced matchings in bipartite graphs. *Information Processing Letters*, 81:7, 2002.
- V. V. Lozin. Boundary classes of planar graphs. *Combinatorics, Probability and Computing*, 17:287, 2008.
- V. V. Lozin and D. Rautenbach. Chordal bipartite graphs of bounded tree- and clique-width. *Discrete Mathematics*, 283:151, 2004a.
- V. V. Lozin and D. Rautenbach. On the band-, tree- and clique-width of graphs with bounded vertex degree. *SIAM J. Discrete Mathematics*, 18:195, 2004b.
- V. V. Lozin and D. Rautenbach. The tree- and clique-width of bipartite graphs in special classes. *Australasian J. Combinatorics*, 34:57, 2006.
- V. V. Lozin and G. Rudolf. Minimal universal bipartite graphs. *Ars Combinatoria*, 84:345, 2007.

- V. V. Lozin and J. Volz. The clique-width of bipartite graphs in monogenic classes. *International Journal of Foundations of Computer Science*, 19:477, 2008.
- V.V. Lozin, C. Mayhill, and V. Zamaraev. A note on the speed of hereditary graph properties. *Electronic Journal of Combinatorics*, 18, 2011.
- C. L. Lu and C. Y. Tang. Solving the weighted efficient edge domination problem on bipartite permutation graphs. *Discrete Applied Mathematics*, 87:203, 1998.
- C. L. Lu, M.-T. Ko, and C. Y. Tang. Perfect edge domination and efficient edge domination in graphs. *Discrete Applied Mathematics*, 119:227, 2002.
- J. A. Makowsky and U. Rotics. On the clique-width of graphs with few P_4 's. *International J. Foundations of Computer Science*, 10:329, 1999.
- D. S. Malyshev. On infinity of the set of boundary classes for the 3-edge-colorability problem. *Diskretn. Anal. Issled. Oper.*, 16:37, 2009.
- R. McConnell and J. P. Spinrad. Modular decomposition and transitive orientation. *Discrete Mathematics*, 201:189, 1999.
- T. A. McKee and F. R. McMorris. *Topics in Intersection Graph Theory*. SIAM Monographs on Discrete Math. Appl. Vol. 2, 1999.
- J. Monnot and S. Toulouse. The path partition problem and related problems in bipartite graphs. *Operations Research Letters*, 35:677, 2007.
- H. Müller. Hamiltonian circuits in chordal bipartite graphs. *Discrete Mathematics*, 156:291, 1996.
- M. N. Murphy and V. Vatter. Profile classes and partial well-order for permutations. *Electronic Journal of Combinatorics*, 9:17, 2003.
- O. J. Murphy. Computing independent sets in graphs with large girth. *Discrete Applied Mathematics*, 35:167, 1992.
- S. Norine, P. Seymour, R. Thomas, and P. Wollan. Proper minor-closed families are small. *Journal of Combinatorial Theory, B*, 96:754, 2006.
- S. Olariu. Paw-free graphs. *Information Processing Letters*, 28:53, 1988.
- M. Petkovšek. Letter graphs and well-quasi-order by induced subgraphs. *Discrete Mathematics*, 244:375, 2002.

- J. Plesník. The NP-completeness of the hamiltonian cycle problem in planar digraphs with degree bound two. *Information Processing Letters*, 8:199, 1979.
- N. Robertson and P. D. Seymour. Graph minors, v. excluding a planar graph. *Journal of Combinatorial Theory, B*, 41:92, 1986.
- N. Robertson and P. D. Seymour. Graph minors, xx. wagner's conjecture. *Journal of Combinatorial Theory, B*, 92:325, 2004.
- E. R. Scheinerman and J. Zito. On the size of hereditary classes of graphs. *Journal of Combinatorial Theory, B*, 61:16, 1994.
- J. P. Spinrad. Nonredundant 1's in γ -free matrices. *SIAM J. Discrete Math*, 2:251, 1995.
- G. Steiner. On the k -path partition of graphs. *Theoretical Computer Science*, 290:2147, 2003.
- V. Vatter and S. Waton. On partial well-order for monotone grid classes of permutations. *Order*, 28:193, 2011.
- H-G. Yach and G.J. Chang. The path-partition problem in bipartite distance-hereditary graphs. *Taiwanese Journal of Mathematics*, 2:353, 1998.
- J. Yan, G.J. Chang, S. M. Hedetniemi, and S. T. Hedetniemi. k -path partitions in trees. *Discrete Applied Mathematics*, 78:227, 1997.
- I. E. Zverovich. r -bounded k -complete bipartite bihypergraphs and generalized split graphs. *Discrete Mathematics*, 247:261, 2002.