One dimensional dynamics: cross-ratios, negative Schwarzian and structural stability

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Declaration

I declare that to the best of my knowledge the contents of this thesis are original and my own work, except where indicated otherwise.

Summary

This thesis concerns the behaviour of maps with a unique critical point which is either a maximum or a minimum: so-called unimodal maps. Our first main result proves that for $C^{2+\eta}$ unimodal maps with non-flat critical point we have good control on the behaviour of cross-ratios on small scales. This result, an improvement on a result of Kozlovski in [**K2**], proves that in many cases the negative Schwarzian condition (which is not even defined if a map is not C^3) is unnecessary. This result follows recent work of Shen, van Strien and Vargas. The main tools are standard cross-ratio estimates, the usual principal nest, the Koebe Lemma, the real bounds from [**SV**] and the 'Yoccoz Lemma'.

Our second main result concerns questions of structural stability. Prompted by the final section of Kozlovski's thesis [**K1**], we prove that in some cases we can characterise those points at which a small local perturbation changes the type of the map. We prove for these cases that this set of 'structurally sensitive points' is precisely the postcritical set. The main tools are the Koebe Lemma, the real bounds of [**LS1**], and the quasiconformal deformation argument of [**K3**].

The thesis is arranged in the form of two chapters dealing with each of the main results separately, followed by an appendix to prove an auxiliary result. The chapters may be read independently of each other.

Chapter 1

Cross-ratio bounds

In this chapter we will show that the negative Schwarzian condition, which is only defined for C^3 maps, is often unnecessary for $C^{2+\eta}$ (for any $\eta > 0$) unimodal maps with non-flat critical points. Our main result is that for such maps, we can choose our intervals in such a way that we have good bounds on how cross ratios are affected. That is, for such maps, for all $0 < K < 1$ there exists some nice interval I_0 around the critical point such that for any intervals $J \subset T$, if $f^n|_T$ is a diffeomorphism and $f^n(T) \subset I_0$ then

$$B(f^n, T, J) > K,$$

$$A(f^n, T, J) > K.$$

This is principally an extension of the results of [**K2**] and [**SV**].

1.1 Introduction to cross-ratio estimates

For maps with critical points we can have no bound on non-linearity. However, for C^3 maps with negative Schwarzian derivative we have some nice properties on how intervals behave under iteration, for example there are some useful extensions of the Koebe Lemma which do give us some bounds on non-linearity in certain cases. As demonstrated in [**MS**], such properties tend to arise from the fact that maps with negative Schwarzian derivative increase cross-ratios.

In fact, negative Schwarzian seems a somewhat synthetic condition. A smooth change of coordinate can destroy this property. Or, indeed for C^3 maps, an analytic coordinate change can create a map which has first return maps with negative Schwarzian, as proved by Graczyk, Sands and Świątek in [**GSS2**]. So we might look for other conditions to give us properties like increasing cross-ratios.

We note that many theorems do not even need cross-ratios to increase, but for them to be bounded away from zero. Probably the most important such result in one-dimensional dynamics is the Koebe Lemma. Kozlovski proved in [**K2**] that for C^3 unimodal maps with non-flat critical point, lower bounds for the change in cross ratios under iteration do exist, provided the intervals are small. So he was able to transfer the properties of maps with negative Schwarzian to 'small scales' for C^3 unimodal maps with non-flat critical points.

So we might say that negative Schwarzian condition is only really meaningful on large scales, for example limiting the number of attracting cycles. (However, it is also interesting to note that Palamore [**P**] has some results which apply on small scales too. There it is shown that the sign of the Schwarzian is significant in finding the speed of convergence for Newton's method of finding the zeroes of a map.)

It is a natural question to ask whether the result of [**K2**] can be extended to the C^2 case, as most of the results in one-dimensional dynamics are proved for either C^1 maps, C^2 maps, or analytic maps. It is often the absence of non-wandering domains for C^2 maps which lies behind the C^2 results (in fact this can be weakened slightly to C^{1+Z} maps too, see [**MS**]). In order to deal with central cascades, here we prove the result of [**K2**] for $C^{2+\eta}$ maps. It is likely that many proofs which currently rely on negative Schwarzian for cross-ratio estimates and so are currently proved only for C^3 maps (for example the decay of geometry in [**GSS1**]) can be extended to the $C^{2+\eta}$ case by our results here.

In [**K2**], real bounds were found for all unimodal maps, even when there were cascades of central returns: saddle node cascades or Ulam-Neumann cascades. Some real bounds were proved in the C^2 multimodal case without inflection points by [**V**] and also in [**Sh1**]. While these bounds suffice for us, the successor to that paper [**SV**] is more general and is closer in spirit to the following approach, although we restrict ourselves to the unimodal case. (For example, they provide a generalisation of the following result of [**K2**] to the multimodal case: there exists some neighbourhood U of the critical values of the map f such that if $f^n(x) \in U$ for $n > 0$ then the Schwarzian derivative of f^n at x is negative.)

However, these bounds were proved away from central cascades so we must find a way of extending some of their useful properties into the cascades. We return to methods applied in [**K1**] to bound a sum of intervals which was to be used in the proof of the main result in [**K3**] (this method not required in the final version). Finding bounds there relied on 'decay of geometry'. We don't, however, require decay of geometry. Instead, we show that we can decompose the sums of intervals into blocks determined by the central returns. These blocks are then shown to decay in size in a uniform way.

1.2 The cross-ratio theorem

We will generally assume that we are dealing with maps that are merely C^2 and then point out when we are restricting our results to the $C^{2+\eta}$ case.

We explain the terminology in the following definitions.

Definition 1.2.1. For some interval T, suppose that $J \subset T$ is a subinterval. If we denote the left-hand and right-hand components of $T \setminus J$ by L and R respectively, then a *cross-ratio* for J and T is the value of

$$\frac{|T||J|}{|L||R|}.$$

We denote this by $B(T, J)$. We also have the cross-ratio

$$A(T, J) := \frac{|T||J|}{|L \cup J||J \cup R|}.$$

Suppose that $g : T \to \mathbb{R}$ is a diffeomorphism. We let $B(g, T, J) := \frac{B(g(T),g(J))}{B(T,J)}$ and $A(g, T, J) := \frac{A(g(T),g(J))}{A(T,J)}$ be our estimates of how the map acts on cross-ratios. Observe that for diffeomorphisms $g : T \to g(T)$ and $h : g(T) \to hg(T)$ we have

$$B(hg, T, J) = B(h, g(T), g(J))B(g, T, J).$$

Similarly for $A(g, T, J)$.

For a C^3 map g of an interval, the *Schwarzian derivative* Sg is defined for non-critical points x as

$$Sg(x) = \frac{D^3g(x)}{Dg(x)} - \frac{3}{2}\left(\frac{D^2g(x)}{Dg(x)}\right)^2.$$

Section IV.1a of [**MS**] shows that cross-ratios are increased by maps with negative Schwarzian.

Definition 1.2.2. We say that T is a *δ-scaled neighbourhood of J* if $\frac{|L|}{|J|}, \frac{|R|}{|J|} > \delta$.

When we have some 'universal' lower bound on δ for some such pairs of intervals J, T we say that J is *well inside* T. (Our bound, which will be denoted by χ, will depend only on f.)

We suppose throughout that our functions map from I to I where I is the unit interval $[0, 1]$. Furthermore, we will assume that $f(\partial I) \subset \partial I$ (these are not meaningful restrictions).

Definition 1.2.3. We say that a map $g : [a, b] \to \mathbb{R}$ is in the class $C^{k+\eta}$ for some $0 < \eta < 1$ if $D^k g$ is continuous and, furthermore, there is some constant C such that $|D^k g(x) - D^k g(y)| \leq C|x - y|^\eta$ for all $x, y \in [a, b]$.

Definition 1.2.4. We say that a unimodal C^k map has *non-flat* critical point c if in some neighbourhood U of c, there exists some C^k diffeomorphism $\phi : U \to I$ with $\phi(c) = 0$ and $g(x) = \pm|\phi(x)|^\alpha + g(c)$ for some $\alpha > 1$. α is known as the *critical order* for f. We denote the set of such maps by NF^k. We also denote this neighbourhood by U_ϕ.

Such maps have many useful properties. For example, such maps have no wandering intervals (that is, there is no non-trivial interval U such that $f^n(U) \cap U = \emptyset$ for $n \geq 1$), see for example Chapter IV of [**MS**]. More importantly for us here is how such maps act on cross-ratios. In particular, how iterates of such maps act on cross-ratios. Our main result is as follows.

Theorem 1.2.5. *For any $\eta > 0$, let $f \in NF^{2+\eta}$ be a unimodal map with a critical point whose iterates do not converge to a periodic attractor. Then for any $0 < K < 1$, there is a nice interval V around the critical point such that if, for an interval T and some $n > 0$,*

- *$f^n|_T$ is monotone; and*
- *each interval from the orbit $T, f(T), \ldots, f^n(T)$ belongs to a domain of the first entry map $F_V : \bigcup_j V^j \to V$ and $f^n(T) \subset V$,*

then

$$B(f^n, T, J) > K,$$

$$A(f^n, T, J) > K$$

where J is any subinterval of T.

This theorem is proved for C^3 maps in [**K2**].

We have the following corollary which we prove in Section 1.3.

Corollary 1.2.6. *For any $\eta > 0$, let $f \in NF^{2+\eta}$ be a unimodal map with a critical point whose iterates do not converge to a periodic attractor. Then for all $0 < K' < 1$ there exists some nice interval I_0 around the critical point such that for any intervals $J \subset T$ and $n > 0$, if $f^n|_T$ is a diffeomorphism and $f^n(T) \subset I_0$ then*

$$B(f^n, T, J) > K',$$

$$A(f^n, T, J) > K'.$$

Our setup will involve first return maps, as outlined below.

Definition 1.2.7. For a map f, we say that an open interval V is *nice for f* if $f^n(\partial V) \cap V = \emptyset$ for $n \geq 1$. (When f is clear we just refer to such interval as nice.)

Suppose that c is recurrent. Let $I_0 \ni c$ be a nice interval. Given some $x \in I$ which eventually iterates by f into I_0, there exists some minimal $n(x) > 0$ such that $f^{n(x)}(x) \in I_i$. For every $x \in I_i$ there is some domain $I_i^j \subset I_i$ around x which maps diffeomorphically to I_i. We thus obtain the *first return map* $F_0 : \bigcup_j I_0^j \to I_0$ to I_0. We label the interval which contains c by I_0^0. This branch is referred to as the *central* branch. Observe that F_0 is diffeomorphic on all branches I_0^j when $j \neq 0$ and is unimodal on I_0^0. For the next step of this inducing process, we now let I_0^0 also be denoted by I_1. Then we derive the first return map $F_1 : \bigcup_j I_1^j \to I_1$, where we denote the central branch by I_1^0. Thus we obtain maps $F_i : \bigcup_j I_i^j \to I_i$. We refer to this process as *inducing on* I_0. The sequence of intervals $I_0 \supset I_1 \supset \cdots$ is known as a *principal nest*.

For every $x \notin I_i$ which eventually iterates by f into I_0 there is some domain U_i^j around x which maps diffeomorphically to I_i. So we may extend F_i by letting $F_i|_{U_i^j} : U_i^j \to I_i$. Then letting $\bigcup_j U_i^j$ consist of all such intervals added to $\bigcup_j I_i^j$, we let $F_i : U_i^j \to I_i$ be known as the *first entry map to* I_i. We will often switch between these two very similar types of map.

Given $F_i : \bigcup_j I_i^j \to I_i$, we will generally assume that $F_i(c)$ is a maximum for $F_i|_{I_{i+1}}$ (this assumption is merely for purposes of exposition, it plays no role in the proofs). We say that F_i is *low* if $F_i(c)$ lies to the left of c and F_i is *high* if $F_i(c)$ lies to the right of c. F_i is *central* if $F_i(c)$ is inside I_{i+1} (if this is not the case, then F_i is *non-central*). A representative of some F_i where F_i is high and central is given in Figure 1.1.

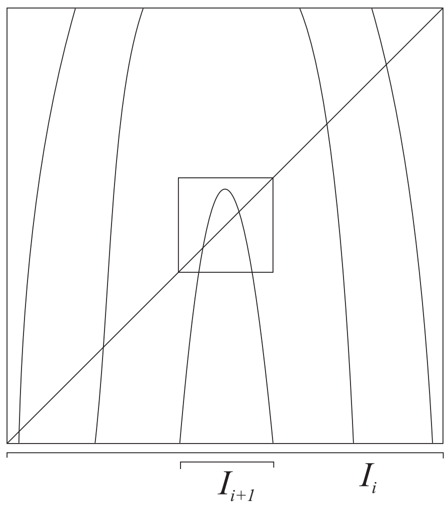

Figure 1.1: F_i is high and central.

Note that we will use capitals to refer to first entry maps or first return maps throughout this paper. For example G_J will denote the first entry map of g to J.

Strategy of proof

It can be shown that we can get a lower bound on $B(f^n, T, J)$ if we can find some bound on $\sum_{k=0}^{n-1} |f^k(T)|$. We will split up this sum in a manner determined by the principal nest explained above.

We will suppose that I_0 is some small interval and T is an interval and n is some integer such that $f^n|_T$ is a diffeomorphism such that $f^n(T) \subset I_0$. Let $n_0 = n$. For a given $i > 0$, suppose that some iterate $f^j(T)$ enters I_i. Now we let n_i be the last time that $f^j(T)$ is in I_i, i.e. $f^{n_i}(T) \subset I_i$ and $f^{n_i+j}(T) \cap I_i = \emptyset$, $j = 0, 1, \ldots, n - n_i$. If $f^j(T)$ never enters I_i, let $k \geq 1$ be minimal such that $f^j(T)$ does enter I_{i-k}. Then we let $n_i = n_{i-k}$. We will be interested in estimating $\sum_{k=1}^{n_i - n_{i+1}} |f^{k+n_{i+1}}(T)|$ for different i. We refer to this as the sum for F_i. Note that if we don't have an infinite cascade, as $i \to infty$ the intervals I_i shrink down to c. Thus we are able to bound $\sum_{k=0}^{n-1} |f^k(T)|$ by bounding the sums for all F_i. We will find another method in the infinite cascade case.

In order to prove the main theorem, we will consider the following cases.

- F_{i-2} is non-central. We consider the sum for F_i as follows.

  Proposition 1.2.8. *There exists some $C_{wb} > 0$ such that*

  $$\sum_{k=1}^{n_i - n_{i+1}} |f^{k+n_{i+1}}(T)| < C_{wb}\sigma_i \frac{|f^{n_i}(T)|}{|I_i|}$$

  where $\sigma_i := \sup_{V \in \mathrm{dom} F_i} \sum_{j=1}^{n(V)} |f^j(V)|$ (and $n(V)$ is defined as k where $F_i|_V = f^k$).

  We call this the well bounded case. It is dealt with in Section 1.4;

- F_{i-2} is non-central and $F_i, \ldots, F_{i+m-1}$ are central. We consider the sums for $F_i, F_{i+1}, \ldots, F_m$ as follows.

  Proposition 1.2.9. *For all $\xi > 0$ there exists some $C_{casc} > 0$ such that*

  $$\sum_{k=1}^{n_i - n_{i+m+1}} |f^{k+n_{i+m+1}}(T)|^{1+\xi} < C_{casc}\sigma_{i,m}$$

  where $\sigma_{i,m}$ is defined as follows. Let $\sigma_i := \sup_{V \in \mathrm{dom} F_i} \sum_{j=1}^{n(V)} |f^j(V)|$ (and $n(V)$ is defined as k where $F_i|_V = f^k$). Let $\hat{V} \subset I_i \setminus I_{i+1}$ be an interval such that $f^{\hat{n}}(\hat{V})$ is one of the connected components of $I_i \setminus I_{i+1}$ for some $\hat{n} > 0$ and $f^j(\hat{V})$ is disjoint from both $I_i \setminus I_{i+1}$ and I_m for $0 < j < \hat{n}(\hat{V})$. Then $\sigma_{i,m}$ is the supremum of all such sums $\sum_{j=1}^{\hat{n}(\hat{V})} |f^j(\hat{V})|$ and σ_i.

  We call this the cascade case. It is dealt with in Section 1.5;

- F_{i-2} is central and F_{i-1} is non-central. We consider the sum for F_i as follows.

  Proposition 1.2.10. *There exists some $C_{ex} > 0$ and $n_{i+1} < n_{i,3} < n_{i,2} < n_i$ such that*

  $$\sum_{k=1}^{n_i - n_{i+1}} |f^{k+n_{i+1}}(T)| < C_{ex}\sigma_i \left(\frac{|f^{n_i}(T)|}{|I_i|} + \frac{|f^{n_{i,2}}(T)|}{|I_i|} + \frac{|f^{n_{i,3}}(T)|}{|I_i|} \right)$$

  where $\sigma_i := \sup_{V \in \mathrm{dom} F_i} \sum_{j=1}^{n(V)} |f^j(V)|$ (and $n(V)$ is defined as k where $F_i|_V = f^k$).

(In many cases, the latter two summands are not required.)

We call this the exceptional branches case. It is dealt with in Section 1.6;

- we have some interval I_0 such that F_i are all central for $i = 0, 1, \ldots$. We call this the infinite cascade case. We prove a similar proposition to those above in Section 1.7.1.

For the case where c is non-recurrent see [**St**]. Finally, the proof of Theorem 1.2.5 is given in Section 1.7.

With these propositions we can decompose the sum $\sum_{k=0}^{n-1} |f^k(T)|$ into blocks of sums $\sum_{k=1}^{n_i - n_{i+1}} |f^{k+n_{i+1}}(T)|$. These blocks will then be shown to decay in size in a uniform way.

The first two cases use real bounds of Theorem 1.3.5. These bounds imply that the cross-ratio for any branch of F_i and I_i is bounded above. This will also be the case for all, but possibly two, branches of F_i in the third case. Then cross-ratios are used to bound the sums. The main tool here is Lemma 1.4.3, which gives us decay of cross-ratios when we have these real bounds.

The final case, which arises in the infinitely renormalisable case, is different from the other three. It uses the Koebe Lemma and a lemma of [**K2**] to find some uniform expanding property which helps bound the sums.

In all cases except the infinite cascade case we must ensure that we have some initial interval which has a first return map which is well bounded. To do this we can simply pick some nice interval to begin with and then induce until we find a map which is well-bounded. This is always possible when there is not an infinite cascade.

Note that we need extra smoothness to deal with the cascade case. This ensures that we can deal with the case when we have many consecutive low central returns, a 'saddle node cascade'. Furthermore, note that the sum for F_i in the third case only arises when F_{i-1} is high and central. The sum for F_i when F_{i-1} is low and central is the same as the well bounded case.

In Kozlovski's proof for C^3 maps he was able to use the fact that there exists some $C > 0$ depending only on f such that for interval $J \subset T$ we have $B(f, T, J) > \exp\{C|T|^2\}$ and, denoting the left and right components of $T \setminus J$ by L and R respectively, $A(f, T, J) > \exp\{C|L||R|\}$. See Chapter IV.2 of [**MS**]. In particular this means that there exist such real bounds as in our Theorem 1.3.5 for all i, not just those for which F_{i-1} is a non-central return. So the long central cascades we encounter in Section 1.5 present much less of a problem in the C^3 case. Indeed, the work done in Section 1.6 is also unnecessary in the C^3 case.

We will deal with the well bounded case first. It is the simplest and gives us a good idea about how we may proceed in general.

We will use J to refer to a general interval from here until Section 1.7. This allows us to use less new notation.

When we use the constant $C > 0$, we mean some constant depending only on f.

1.3 Introductory results

Note that some of the more basic definitions which are used more widely in Chapter 2 are defined there (for example, recurrence, periodic points and so on).

We start by restating some results of Chapter IV of [**MS**]. We introduce the following definition.

Definition 1.3.1. For an interval $[a, b]$, let $g : [a, b] \to \mathbb{R}$ be a continuous function. Then define for $x, y \in [a, b]$,

$$w_g(\delta) := \sup_{|x-y|<\delta} |g(x) - g(y)|.$$

The function w_g is known as the *modulus of continuity of g*.

For a reference on this function, see [**D**]. There it is proved in Theorem 1.4.3 that for any continuous g, $w_g(0) = 0$ and w_g is continuous on $[0, b - a]$. We will be concerned here with the modulus of continuity of $D^2f : I \to \mathbb{R}$. For the duration of this chapter we let $w := w_{D^2f}$.

We suppose throughout this chapter that our maps have a maximum at the critical point. This is not a meaningful restriction. We also suppose that f is symmetric about c. That is, $f(c - \epsilon) = f(c + \epsilon)$ for all ϵ. This assumption is useful for simplifying proofs (particularly in Section 1.6, which is already quite technical), but is not crucial since on small scales our maps will be essentially symmetric (in particular, for small ϵ, $|Df(c - \epsilon)|$ is like $|Df(c + \epsilon)|$).

The following lemma and theorem are proved for a more general case in Chapter IV of [**MS**]. We include a proof here both for completeness and to give some insight into how, firstly, the non-flatness of the critical point and, secondly, the behaviour of the second derivatives of our maps affect our results.

Lemma 1.3.2. *For a unimodal map $f : I \to I$, $f \in NF^2$, there exists $C_1 > 0$, $\beta > 0$ and $\epsilon_f > 0$ such that if T is an interval such that $f|_T$ is a diffeomorphism, $|T| < \epsilon_f$ and $J \subset T$, then*

$$B(f, T, J) \geq \exp\{-C_1|T|[w(|T|) + \beta|T|]\}.$$

Proof: We suppose that U_ϕ is symmetric about c and let U'_ϕ denote the interval centred at c with size $\frac{|U_\phi|}{2}$. We let $\epsilon_f := \frac{|U_\phi|}{5}$. If $|T| < \epsilon_f$ then T is either contained in U_ϕ or is outside U'_ϕ. Suppose first that T is contained in U_ϕ. Then for $x \in T$, $f(x) = \pm|\phi(x)|^\alpha + f(c)$. Since the map $g_\alpha(x) := x^\alpha$ has negative Schwarzian for any $\alpha > 1$ and thus expands cross-ratios, $B(f, T, J) = B(\phi, T, J)B(g_\alpha, T, J) \geq B(\phi, f, T, J)$. So we need only deal with ϕ.

Let $T = [a_1, a_4]$, $J = [a_2, a_3]$. Then,

$$\begin{aligned}
|B(\phi, T, J) - 1| &= \left| \frac{\frac{\phi(a_3)-\phi(a_2)}{a_3-a_2} \cdot \frac{\phi(a_4)-\phi(a_1)}{a_4-a_1} - \frac{\phi(a_2)-\phi(a_1)}{a_2-a_1} \cdot \frac{\phi(a_4)-\phi(a_3)}{a_4-a_3}}{\frac{\phi(a_2)-\phi(a_1)}{a_2-a_1} \cdot \frac{\phi(a_4)-\phi(a_3)}{a_4-a_3}} \right| \\
&\leq \frac{1}{|D\phi|^2_{-\infty}} \left| \frac{\phi(a_3) - \phi(a_2)}{a_3 - a_2} \cdot \frac{\phi(a_4) - \phi(a_1)}{a_4 - a_1} \right. \\
&\qquad \left. - \frac{\phi(a_2) - \phi(a_1)}{a_2 - a_1} \cdot \frac{\phi(a_4) - \phi(a_3)}{a_4 - a_3} \right|
\end{aligned}$$

where $|\cdot|_{-\infty}$ gives the minimum value of a function in the domain in which it is defined.

We estimate this by applying the Mean Value Theorem repeatedly. $\theta_i \in T$ will denote points obtained from this theorem.

$$\begin{aligned}
&\left| \frac{\phi(a_3) - \phi(a_2)}{a_3 - a_2} \cdot \frac{\phi(a_4) - \phi(a_1)}{a_4 - a_1} - \frac{\phi(a_2) - \phi(a_1)}{a_2 - a_1} \cdot \frac{\phi(a_4) - \phi(a_3)}{a_4 - a_3} \right| \\
&\qquad = |D\phi(\theta_1)D\phi(\theta_2) - D\phi(\theta_3)D\phi(\theta_4)| \\
&\qquad = |D\phi(\theta_1)[D\phi(\theta_4) + D^2\phi(\theta_5)(\theta_2 - \theta_4)] \\
&\qquad\quad - D\phi(\theta_4)[D\phi(\theta_1) + D^2\phi(\theta_6)(\theta_3 - \theta_1)]| \\
&\qquad = |D\phi(\theta_1)D^2\phi(\theta_5)(\theta_2 - \theta_4) - D\phi(\theta_4)D^2\phi(\theta_6)(\theta_3 - \theta_1)| \\
&\qquad = |[D\phi(\theta_4) + D^2\phi(\theta_7)(\theta_1 - \theta_4)]D^2\phi(\theta_5)(\theta_2 - \theta_4)
\end{aligned}$$

$$
\begin{aligned}
&-D\phi(\theta_4)D^2\phi(\theta_6)(\theta_3-\theta_1)| \\
&= |D\phi(\theta_4)[D^2\phi(\theta_5)(\theta_2-\theta_4) - D^2\phi(\theta_6)(\theta_3-\theta_1)] \\
&\quad -D^2\phi(\theta_7)D^2\phi(\theta_5)(\theta_1-\theta_4)(\theta_2-\theta_4)| \\
&\leq |D\phi|_\infty |D^2\phi(\theta_5)(\theta_2-\theta_4) - D^2\phi(\theta_5)(\theta_3-\theta_1)| \\
&\quad +|D\phi|_\infty |D^2\phi(\theta_5)(\theta_3-\theta_1) - D^2\phi(\theta_6)(\theta_3-\theta_1)| + |D^2\phi|_\infty^2 |T|^2 \\
&\leq |D\phi|_\infty |D^2\phi|_\infty |T| + |D\phi|_\infty w(|T|)|T| + |D^2\phi|_\infty^2 |T|^2
\end{aligned}
$$

where $|\cdot|_\infty$ is the usual sup norm.

Therefore there exist $C_1, \beta > 0$ depending on $D\phi$ and $D^2\phi$ such that

$$|B(\phi, T, J) - 1| \leq C_1 |T| [w(|T|) + \beta |T|]$$

i.e. $B(\phi, T, J) \geq \exp\{-C_1|T|[w(|T|) + \beta|T|]\}$.

The case where $T \subset U'_\phi$ follows similarly.

□

We may simply extend the above lemma to the following theorem.

Theorem 1.3.3. *For a unimodal map $f : I \to I$, $f \in NF^2$, if T is an interval such that $f^n|_T$ is a diffeomorphism and $J \subset T$ is a subinterval, then there exists a continuous function $\sigma : [0, |I|] \to [0, \infty)$ depending on f such that*

$$B(f^n, T, J) \geq \exp\left\{-\sigma(S(n,T)) \sum_{i=0}^{n-1} |f^i(T)|\right\}$$

where $S(n,T) := \max_{0 \leq i \leq n-1} |f^i(T)|$.

Proof: In the above lemma we let $\sigma(\epsilon) = C_1[w(\epsilon) + \beta\epsilon]$.

□

The following lemma, a consequence of the absence of wandering intervals, is Lemma 5.2 in [**K2**].

Lemma 1.3.4. *Suppose that $f \in NF^2$, $f : I \to I$. Then there exists a function $\tau : [0, |I|] \to [0, \infty)$ such that $\lim_{\epsilon \to 0} \tau(\epsilon) = 0$ and for any interval V for which*

$f^n|_V$ is a diffeomorphism and $f^n(V)$ is disjoint from the immediate basins of periodic attractors, we have

$$\max_{0\leq i\leq n} |f^i(V)| < \tau(|f^n(V)|).$$

We may use this lemma to extend the estimate of Theorem 1.3.3 as

$$B(f^n,T,J) \geq \exp\left\{-\sigma'(|f^{n-1}(T)|)\sum_{i=0}^{n-1}|f^i(T)|\right\}$$

where $\sigma'(|f^m(T)|) = \sigma\tau(|f^m(T)|)$.

We can now prove our main corollary.

Proof of Corollary 1.2.6: Let $0 < K' < 1$ be as in the corollary.

We will fix some small I_0 from Theorem 1.2.5 and assume that $f^n(T) \subset I_0$. Just how small I_0 must be is specified below.

Let $0 \leq m < n$ be maximal such that $f^m(T) \subset I_0$. Then $f^m(T)$ is contained in a domain of the first return map to I_0. Then Theorem 1.2.5 says that $B(f^m,T,J) > K$ for some $0 < K < 1$ which can be very close to 1 if I_0 is sufficiently small.

We also have

$$\begin{aligned} B(f^n,T,J) &= B(f^m,T,J)B(f^{n-m},f^m(T),f^m(J)) \\ &> KB(f^{n-m},f^m(T),f^m(J)). \end{aligned}$$

By the above theorem and lemma we have

$$B(f^{n-m},f^m(T),f^m(J)) > \exp\left\{-\sigma'(|f^{n-1}(T)|)\sum_{i=m}^{n-1}|f^i(T)|\right\}.$$

Since $f^n(T) \subset I_0$, by Lemma 1.3.4 we have $\sigma'(|f^{n-1}(T)|) \leq \sigma'(|I_0|)$. Furthermore, $\sum_{i=m}^{n-1}|f^i(T)|$ is a sum of disjoint intervals and so is less than 1. Therefore $\exp\{-\sigma'(|f^{n-1}(T)|)\sum_{i=m}^{n-1}|f^i(T)|\} > \exp\{-\sigma'(|I_0|)\}$ and

$$B(f^n,(T),(J)) > K\exp\{-\sigma'(|I_0|)\}.$$

So if I_0 is small enough then $B(f^n,T,J) > K'$. There is an analogous result to Theorem 1.3.3 for $A(f,T,J)$, see, for example [**Sh3**], which allows us to prove the second part of the corollary.

□

We will use the following result of [**SV**] throughout. (In fact it is stated there in greater generality, as Theorem A.)

Theorem 1.3.5. *If $g \in NF^2$ is a unimodal map with recurrent critical point, then the following hold.*

1. *For all $k \geq 0$ there exists $\xi(k) > 0$ such that if $G_{i-1} : \bigcup_j I_{i-1}^j \to I_{i-1}$ is non-central, then I_{i+k} is a $\xi(k)$-scaled neighbourhood of I_{i+k+1}.*
2. *For each $\xi > 0$ there is some $\xi' > 0$ such that if I_i is a ξ-scaled neighbourhood of I_{i+1} then I_{i+1} is a ξ'-scaled neighbourhood of any domain of G_{i+1}.*

This result gives us *real bounds* for some of our first return maps. We let $\chi := \xi(1) > 0$ from the above theorem for our map f.

We now state a version the Koebe Lemma. The following is Theorem IV.3.1 of [**MS**].

Theorem 1.3.6. *For each $S, \delta > 0$ and each map $g \in NF^2$ there exists a constant $C(S,\delta) > 0$ with the following property. If T is an interval such that $g^n|_T$ is a diffeomorphism and if $\sum_{i=0}^{n-1} |g^i(T)| \leq S$, then for each interval $J \subset T$ for which $g(T)$ contains a δ-scaled neighbourhood of $g(J)$ we have*

$$\frac{1}{C(S,\delta)} \leq \frac{Dg^n(x)}{Dg^n(y)} \leq C(S,\delta)$$

for any $x, y \in J$, where $C(S,\delta) = \frac{(1+\delta)^2}{\delta^2} e^{C_g S}$ where $C_g \geq 0$ depends only on g.

The proof of this theorem uses a more basic Koebe Lemma added to Theorem 1.3.3. We sometimes wish to deal with intervals $J \subset T$ such that $\sum_{i=0}^{n-1} |f^i(T)|$ has no good bound, but $\sum_{i=0}^{n-1} |f^i(J)|$ *is* well bounded. The following improvement of Theorem 1.3.6 deals with these cases. It is presented in more generality in [**SV**] as Proposition 2: 'a Koebe principle requiring less disjointness'.

Theorem 1.3.7. *Suppose that $g \in NF^2$. Then there exists a function $\nu : [0, |I|] \to [0, \infty)$ such that $\nu(\epsilon) \to 0$ as $\epsilon \to 0$ with the following properties. Suppose that for some intervals $J \subset T$ and a positive integer n we know that $g^n|_T$ is a diffeomorphism. Suppose further that $g^n(T)$ is a δ-scaled neighbourhood of $g^n(J)$ for some $\delta > 0$. Then,*

- *for every* $x, y \in J$,

$$\frac{|Dg^n(x)|}{|Dg^n(y)|} \leq \exp\left\{\nu(S(n,T))\sum_{i=0}^{n-1}|g^i(J)|\right\}\left[\frac{1+\delta}{\delta}\right]^2;$$

- T *is a* δ' *scaled neighbourhood of* J *where*

$$\delta' = \frac{1}{2}\exp\{-\theta\}\left[\frac{1+\delta}{\delta}\right]^2\left(\frac{-2\theta+\delta(1-2\theta)}{2+\delta}\right),$$

and we let $\theta := \nu(S(n,T))\sum_{i=0}^{n-1}|g^i(J)|$.

We will often use this theorem when weaker versions would also suffice.

Again we may use Lemma 1.3.4 to substitute $\nu(S(n,T))$ with $\nu'(|f^n(T)|)$ where we define $\nu'(|f^m(V)|) := \nu\tau(|f^m(V)|)$.

We will often use the first part of this theorem. We mostly deal with the case when δ is the χ we obtained following Theorem 1.3.5.

We will sometimes be in a situation where we wish to estimate the derivative of a function in between two repelling fixed points. The following two well known results allow us to do this. The following is known as the *Minimum Principle*; see, for example, Theorem IV.1.1 of [**MS**].

Theorem 1.3.8. *Let* $T = [a,b] \subset I$ *and* $g : T \to g(T) \subset I$ *be a* C^1 *diffeomorphism. Let* $x \in (a,b)$. *If for any* $J^* \subset T^* \subset T$,

$$B(g, T^*, J^*) \geq \hat{\mu} > 0$$

then

$$|Dg(x)| \geq \hat{\mu}^3 \min(|Dg(a)|, |Dg(b)|).$$

The following result is Theorem IV.B of [**MS**].

Theorem 1.3.9. *For* $g \in NF^2$ *there exist* $n_0 \in \mathbb{N}$ *and* $\rho_g > 1$ *such that*

$$|Dg^n(p)| > \rho_g$$

for every periodic point p *of period* $n \geq n_0$.

We are now ready to begin the proof of Theorem 1.2.5.

1.4 Well bounded case

We deal with the case where F_i is well bounded. Let $n'_i > n_{i+1}$ be minimal such that $f^{n'_i}(T) \subset I_i$. We will initially assume that we have some $\kappa > 0$ such that for the 'return sum',

$$\sum_{k=0}^{j_i} |F^k(f^{n'_i}(T))| < \kappa |f^{n_i}(T)| \tag{1.1}$$

where j_i is such that $F^{j_i}|_{f^{n'_i}(T)} = f^{n_i - n'_i}|_{f^{n'_i}(T)}$. (We prove this proposition before bounding this return sum in order to try to give an idea why we need bounds on return sums.) We recall Proposition 1.2.8.

Proposition 1.2.8 *There exists some $C_{wb} > 0$ such that*

$$\sum_{k=1}^{n_i - n_{i+1}} |f^{k+n_{i+1}}(T)| < C_{wb}\sigma_i \frac{|f^{n_i}(T)|}{|I_i|}$$

where $\sigma_i := \sup_{V \in \mathrm{dom} F_i} \sum_{j=1}^{n(V)} |f^j(V)|$ (and $n(V)$ is defined as k where $F_i|_V = f^k$).

This proposition is taken from Lemma 5.3.4 of [**K1**] which assumes that $f \in C^3$. There, the bound on the sum $\sum_{k=0}^{j_i - 1} |F^k(f^{n'_i}(T))|$ is obtained using methods which fail in the C^2 case.

Proof of Proposition 1.2.8 assuming (1.1): Let $n_{i+1} < m_1 < \cdots < m_{j_i} = n_i$ be all the integers between n_{i+1} and n_i such that $f^{m_j}(T) \subset I_i \setminus I_{i+1}$ for $j = 1, \ldots, j_i - 1$ and let $m_0 = n_{i+1}$. Let $F_i : \bigcup_j U_i^j \to I_i$ be the first entry map to I_i. We will decompose $\sum_{i=n_{i+1}+1}^{n_i} |f^i(T)|$ as $\sum_{j=0}^{j_i - 1} \sum_{k=1}^{m_{j+1} - m_j} |f^{m_j + k}(T)|$.

Suppose, for $1 \le j \le j_i - 1$, that $f^{m_j+1}(T) \subset U_i^j$ where U_i^j is some domain of first entry to I_i. Suppose further, that $F_i|_{U_i^j} = f^{i_j}$. Then there exists an extension to $V_i^j \supset U_i^j$ so that $f^{i_j} : V_i^j \to I_{i-1}$ is a diffeomorphism. Then from Theorem 1.3.7 we have the distortion bound: $\frac{|f^k(f^{m_j+1}(T))|}{|f^k(U_i^j)|} \le C(\chi) \frac{|f^{m_j+1}(T)|}{|I_i|}$. Whence

$$\sum_{k=1}^{m_{j+1} - m_j} |f^{m_j + k}(T)| \le C(\chi) \frac{|f^{m_j+1}(T)|}{|I_i|} \sum_{k=0}^{m_{j+1} - m_j - 1} |f^k(U_i^j)| \le C(\chi)\sigma_i \frac{|f^{m_{j+1}}(T)|}{|I_i|}.$$

Therefore

$$\sum_{j=n_{i+1}+1}^{n_i} |f^i(T)| \le C(\chi) \frac{\sigma_i}{|I_i|} \sum_{j=1}^{j_i} |f^{m_i}(T)|.$$

I.e. we are interested in the sum $\sum_{j=1}^{j_i} |f^{m_j}(T)|$, that is, $\sum_{k=0}^{j_i - 1} |F^k(\hat{T})|$ where $\hat{T} = f^{m_1}(T)$. But we have assumed in (1.1) that this sum is bounded by $\kappa |F^{n_i}(T)|$.

□

We will next prove that (1.1) does hold, i.e. that the return sum is indeed bounded.

1.4.1 Bounding return sums

We note that no element of $f^{n_{i+1}+1}(T),\dots,f^{n_i}(T)$ is contained inside I_{i+1}. Such elements will have been summed in the sum for F_{i+j} for some $j \geq 1$. If some $f^{n_{i+1}+k}(T)$ for $1 \leq k \leq n_i - n_{i+1}$ intersects ∂I_{i+1} then $f^{n_{i+1}+k}(T)$ could never iterate by f inside I_i as elements of ∂I_{i+1} never return to I_i. Therefore $f^{n_{i+1}+k}(T) = f^{n_i}(T)$.

We derive some useful estimates regarding cross-ratios.

Suppose that $B(f^n, U, J) > \theta$ and $B(f^n(U), f^n(J)) < \Delta$. Then

$$\frac{|L||R|}{|U||J|} = \frac{1}{B(U,J)} > \frac{\theta}{B(f^n(U), f^n(J))} > \frac{\theta}{\Delta}.$$

Whence $\frac{|L|}{|J|}, \frac{|R|}{|J|} > \frac{\theta}{\Delta}$.

Since $|J| = |U| - |L| - |R|$ we have $|J| \leq |U| - 2|J|\frac{\theta}{\Delta}$ so $|J| \leq \frac{|U|}{1+2\frac{\theta}{\Delta}}$.

Lemma 1.4.1. *For all $\delta > 0$ there exists $\Delta = \Delta(\delta) > 0$ such that $\Delta(\delta) \to 0$ as $\delta \to \infty$. Suppose that U is an interval, $J \subset U$ is a subinterval and that the left and right components of $U \setminus J$ are denoted by L and R respectively. Suppose further that $|L|, |R| > \delta|J|$. Then*

$$B(U, J) < \Delta.$$

Proof: We know that $\frac{|J|}{|L|}, \frac{|J|}{|R|} < \frac{1}{\delta}$. Suppose, without loss of generality that $|L| \leq |R|$. Then

$$\frac{|U||J|}{|L||R|} < \frac{|U|}{\delta|R|} = \frac{|L| + |J| + |R|}{\delta|R|} < \frac{2}{\delta} + \frac{1}{\delta^2}.$$

So, letting $\Delta := \frac{2}{\delta} + \frac{1}{\delta^2}$ the lemma is proved.

□

From the above we have an upper bound on $B(I_i, I_{i+1})$ depending on χ.

We suppress the i notation for now. Let I', I'' denote I_i and I_{i-1} and let I^j denote the first return domains to I'.

Let D_1 denote the set of non-central domains $F^{-1}(I')$, i.e. $D_1 = \bigcup_{j\neq 0} I^j$. Let D_2 denote the set of domains $F^{-1}(D_1)$ which are disjoint from the central domain. Inductively, we let D_k denote the set of domains $F^{-1}(D_{k-1})$ which are disjoint from the central domain. Therefore, for any element $J_k \in D_k$, $F^k : J_k \to I'$ is a diffeomorphism. We will try to bound $\sum_{j=0}^{k-1} |F^j(J_k)|$ for some $J_k \in D_k$. We will prove this by showing that there exists some $\lambda < 1$ such that $B(I', J) \leq \lambda B(I', F(J))$ where λ is independent of i. We let

$$\mu := \exp\{-\sigma'(|I_0|)\}.$$

Supposing that $n(j)$ is the return time of I^j to I' and $J \subset I^j$ then by Theorem 1.3.3 we have $B(f^{n(j)}, I^j, J) > \mu$ for all j. We will use this value of μ repeatedly. It is not an optimal bound, but for any intervals $J' \supset J$ such that $f^m|_{J'}$ is a diffeomorphism; $f^m(J')$ is disjoint from the basin of immediate attractors; $|f^m(J')| \leq |I_0|$; and the intervals $J', f(J'), \ldots, f^m(J')$ are disjoint, then we have $B(f^m, J', J) > \mu$.

The following lemma is Lemma 2.3 of [**GK**]. For completeness, we provide a proof here.

Lemma 1.4.2. *For every $\delta > 0$ there is some $\lambda' = \lambda'(\delta) < 1$ such that if $J \subset V \subset U$ are three intervals and U is a δ-scaled neighbourhood of V then*

$$B(U, J) < \lambda' B(V, J).$$

Proof: Denote the left and right components of $V \setminus J$ by L and R respectively. Further, denote the left and right components of $U \setminus V$ by L' and R' respectively. We have $|L'|, |R'| > \delta|V|$. Denote the closed interval which is left-adjacent to $|V|$ and has size $\delta|L|$ by $\hat{L}$. Denote the interval which is right-adjacent to $|V|$ and has size $\delta|R|$ by $\hat{R}$. Let $\hat{U} := \hat{L} \cup V \cup \hat{R}$. Clearly $\hat{U} \subset U$. Then,

$$B(\hat{U}, J) = B(V, J)\frac{|\hat{U}|}{|V|}\left(\frac{|L|}{|\hat{L}| + |L|}\right)\left(\frac{|R|}{|\hat{R}| + |R|}\right).$$

But,

$$\frac{|\hat{U}|}{|V|}\left(\frac{|L|}{|\hat{L}| + |L|}\right)\left(\frac{|R|}{|\hat{R}| + |R|}\right) = \left(\frac{|\hat{L}| + |J| + |\hat{R}|}{|J|}\right)\left(\frac{1}{1+\delta}\right)^2 = \frac{1+2\delta}{(1+\delta)^2}.$$

Letting $\lambda' := \frac{1+2\delta}{(1+\delta)^2}$ we have $B(\hat{U}, J) = \lambda' B(V, J)$. Note that $\lambda' < 1$ since $(1+\delta)^2 = (1+2\delta) + \delta^2$. Since $\hat{U}$ is a strict subset of U,

$$B(U, J) < \lambda' B(V, J)$$

as required.

$\square$

We add this lemma to the real bounds of Theorem 1.3.5 and the distortion of cross-ratios in Theorem 1.3.3.

Lemma 1.4.3. *If I_0 small enough, there exists some $\lambda < 1$, depending only on f such that for any $J \subset D_k$,*

$$B(I', J) < \lambda B(I', F(J)).$$

Observe from the proof that λ depends strongly on χ, but χ depends only on f.

Proof: From Theorem 1.3.5 we know that for each j we have $B(I_i, I_i^j) < \chi$. So from the previous lemma there exists some $\lambda' = \lambda'(\chi) < 1$ such that

$$B(I', J) < \lambda' B(I^j, J).$$

Now from Theorem 1.3.3 we obtain

$$B(I', J) < \lambda' \frac{B(I', F(J))}{\mu}$$

where μ is defined above in terms of $|I_0|$. If I_0 is chosen small enough then $\frac{\lambda'}{\mu} < 1$. We let $\lambda := \frac{\lambda'}{\mu}$. Thus $B(I', J) < \lambda B(I', F(J))$.

$\square$

In fact we shall adjust λ again in both Sections 1.5 and 1.6, but it will remain strictly less than 1.

So for $k \geq 2$, $B(I', J_k) < \lambda^{k-1} B(I', F^{k-1}(J_k))$. Suppose that $F^{k-1}(J_k) \subset I^j$. Then by Lemma 1.4.1, using Theorems 1.3.5 and 1.3.7, $B(I', I^j) < \Delta$ where $\Delta = \Delta(\chi')$.

Therefore, it is easy to see that $B(I', F^{k-1}(J_k)) < \Delta\frac{|F^{k-1}(J_k)|}{|I^j|}$. Since we have $|F^{k-1}(J_k)| < C(\chi)|F^k(J_k)|\frac{|I^j|}{|I'|}$, we know that $B(I', F^{k-1}(J_k)) < C(\chi)\Delta\frac{|F^k(J_k)|}{|I'|}$.

We apply these estimates to the sizes of J_k:

$$|J_k| < \frac{|I'|}{1 + \frac{2|I'|}{\lambda^{k-1}C(\chi)\Delta|F^k(J_k)|}}.$$

Then $|J_k| < C\lambda^{k-1}|F^k(J_k)|$. So $\sum_{j=0}^{k-1}|F^j(J_k)| < C\frac{|F^k(J_k)|}{1-\lambda}$. Whence

$$\sum_{j=0}^{k}|F^j(J_k)| < |F^k(J_k)|\left(1 + \frac{C}{1-\lambda}\right).$$

This holds for any sum of returns which never lands in the central domain. It is independent of i. Thus prove that (1.1) holds by letting $\left(1 + \frac{C}{1-\lambda}\right)$ be κ. Thus, we complete the proof of Proposition 1.2.8.

1.5 Cascade case

Note that Theorem 1.3.5 implies that if there is some I_0 with a bound on the number of consecutive central returns then we have uniform real bounds for all F_i. However, there may be arbitrarily long chains of consecutive central returns; the case we deal with here.

We suppose that there is some i such that $f^{n_i}(T) \subset I_i$ where F_{i-2} has a non-central return and F_i has a central return, possibly followed by more central returns. We assume that F_{i+j} all have central returns for $j = 0, 1, \ldots, m-1$ and that F_{i+m} has a non-central return. We aim to bound the sum

$$\sum_{k=1}^{n_i - n_{i+m+1}} |f^{k+n_{i+m+1}}(T)|.$$

As in the well bounded case, we focus on the 'final consecutive sequence' of j for which $f^j(T)$ lie in $I_i \setminus I_{i+m+1}$.

Let $m_0 = n_{i+m+1}$ and let $n_i \geq m_1 > n_{i+1}$ be the minimal integer such that $f^{m_1}(T) \subset I_i \setminus I_{i+1}$. Let $n_i \geq m_2 > m_1$ be the next integer for which $f^{m_2}(T) \subset I_i \setminus I_{i+1}$ if such m_2 exists. Proceeding in this manner, we obtain a sequence, $n_{i+1} < m_1 < m_2 < \cdots < m_N = n_i$.

So

$$\sum_{k=1}^{n_i-n_{i+m+1}} |f^{k+n_{i+m+1}}(T)| = \sum_{j=0}^{N-1} \sum_{k=1}^{m_{j+1}-m_j} |f^{k+m_j}(T)|.$$

We denote the left and right components of $I_j \setminus I_{j+1}$ by L_j and R_j respectively. We know from Theorem 1.3.5 that $\frac{|L_i|}{|I_{i+1}|}, \frac{|R_i|}{|I_{i+1}|} > \chi$.

Observe that if we only consider the central branches, the elements of $F^{-1}(L_{i+j})$ and $F^{-1}(R_{i+j})$ are L_{i+j+1}, R_{i+j+1} for $j = 0, \ldots, m-1$. So any pullback of L_i, R_i looks like a procession of intervals $J_0, J_1, J_2, \ldots$ where $J_j \in \{L_{i+j}, R_{i+j}\}$ for $j \in \{0, \ldots, m-1\}$, followed by a pullback by a non-central branch of F_i into $I_i \setminus I_{i+1}$. The picture is similar to that in Figure 1.5.

Let D_1 be the set of I_i^j for $j \neq 0$. Let D_2 be the set of domains $J \subset I_i \setminus I_{i+1}$ such that $F^k|_{I_{i+1}} F|_{I_i^j}(J) \in D_1$ for some $j \neq 0$ and $0 \leq k \leq m$. Similarly we define D_k. We then let $\hat{F}$ be the map such that for $J \subset D_k$, $\hat{F}(J) \in D_{k-1}$.

We can't use Lemma 1.4.3 since for $J \in D_1 \cap I_{i+1}$ there is not an extension to I_{i-1}. So we use Lemma 1.5.1 instead to say that for $J_k \in D_k$, for $k \geq 3$, if $\hat{F}(J_k) \subset J_2 \subset J_1$ for some $J_2 \in D_2$, $J_1 \in D_1$ then we have an extension to J_1. So we can apply the same ideas of cross-ratio decay.

Observe that for $J_1 \in D_1$ and $J_2 \in D_2$ where $J_1 \subset J_2$, we have $B(J_1, J_2) < \frac{B(I_i, F(J_2))}{\mu}$. Then by Lemma 1.4.2 there exists $\lambda'_C < 1$ such that for $J \subset J_2$ then $B(J_1, J) < \lambda'_C B(J_2, J)$. Then we can prove that for $J_k \in D_k$ for $j \geq 2$ where $J_k \subset J_2$ for $J_2 \in D_2$,

$$B(J_1, J_k) < \frac{\lambda'_C}{\mu} B(\hat{F}(J_2), \hat{F}(J_k)) < \frac{\lambda'_C}{\mu} B(J'_1, \hat{F}(J_k))$$

for some $J'_1 \in D_1$. We can assume that $\frac{\lambda'_C}{\mu} < 1$ and can adjust $\lambda < 1$ slightly, if necessary, so that $\frac{\lambda'_C}{\mu} < \lambda$. So we can prove the following lemma.

Lemma 1.5.1. *For $J_k \in D_k$ and $k \geq 2$ where $J_k \subset J_1$ and $\hat{F}^{k-2}(J_k) \subset J'_1$ we have*

$$B(J_1, J_k) < \lambda^{k-2} B(J'_1, \hat{F}^{k-2}(J_k)).$$

Clearly we also have, $B(J_1, J_k) < \frac{\lambda^{k-2}}{\mu} B(I_i, \hat{F}^{k-2}(J_k))$.

We recall Proposition 1.2.9.

Proposition 1.2.9 *There exists some $C_{casc} > 0$ such that*

$$\sum_{k=1}^{n_i - n_{i+m+1}} |f^{k+n_{i+m+1}}(T)|^{1+\xi} < C_{casc}\sigma_{i,m}$$

where $\sigma_{i,m}$ is defined as follows. Let $\sigma_i := \sup_{V\in \mathrm{dom} F_i} \sum_{j=1}^{n(V)} |f^j(V)|$ (and $n(V)$ is defined as k where $F_i|_V = f^k$). Let $\hat{V} \subset I_i \setminus I_{i+1}$ be an interval such that $f^{\hat{n}}(\hat{V})$ is one of the connected components of $I_i \setminus I_{i+1}$ for some $\hat{n} > 0$ and $f^j(\hat{V})$ is disjoint from both $I_i \setminus I_{i+1}$ and I_m for $0 < j < \hat{n}(\hat{V})$. Then $\sigma_{i,m}$ is the supremum of all such sums $\sum_{j=1}^{\hat{n}(\hat{V})} |f^j(\hat{V})|$ and σ_i.

Observe that since we have supposed that our map f is on the unit interval, $\sigma_{i,m} < 1$.

Proof: We will consider $\sum_{j=0}^{N-1}\sum_{k=1}^{m_{j+1}-m_j} |f^{k+m_j}(T)|$. We begin by considering the sum $\sum_{k=1}^{m_{j+1}-m_j} |f^{k+m_j}(T)|$ for $0 \leq j \leq N-2$. We will initially assume that $f^{n_i}(T) \in I_i$ will lie inside some $I_{i+j} \setminus I_{i+j+1}$ for $0 \leq j \leq m-1$.

Denote $f^{m_j}(T)$ by $\hat{T}$. Then there is some domain $J_1 \in D_1$ which contains $\hat{T}$. J_1 must be some I_i^j where $j \neq 0$. We have two cases for $\hat{F}|_{\hat{T}}$. Firstly, $\hat{T}$ could be mapped by a non-central branch straight back into $I_i \setminus I_{i+1}$, i.e. $\hat{F}|_{\hat{T}} = F|_{\hat{T}\cap I_i^j}$. Secondly, $\hat{T}$ could be mapped by a non-central branch inside $I_{i+1} \setminus I_m$ and then it will be mapped out to $I_i \setminus I_{i+1}$ by iterations of the central branch, i.e. $\hat{F}|_{\hat{T}} = (F^{\hat{k}}|_{I_{i+1}} F|_{I_i^j})|_{\hat{T}}$ for some $0 < \hat{k} \leq m$. Therefore, in general supposing that $F|_{J_1} = f^{j(J_1)}$ and that $F|_{I_{i+1}} = f^s$ our sum is

$$\sum_{k=1}^{m_{j+1}-m_j} |f^k(\hat{T})| = \sum_{k=0}^{j(J_1)-1} |f^k(\hat{T})| + \sum_{k=0}^{\hat{k}}\sum_{p=0}^{s-1} |f^{ks+p}(F(\hat{T}))|$$

for some $0 \leq \hat{k} \leq m$.

We consider these two terms separately.

Step 1: consider first $\sum_{k=0}^{j(J_1)-1} |f^k(\hat{T})|$. Since $j \leq N-2$ we know that $\hat{T}$ is contained in some $J_2 \in D_2$. We suppose that $\hat{F}(J_2) = J_1'$ where $J_1' \in D_1$. Now $B(f^k(J_2), f^k(\hat{T})) < \frac{B(F(J_2),F(\hat{T}))}{\mu}$ since all intermediate intervals are disjoint. If $F|_{\hat{T}} = \hat{F}|_{\hat{T}}$ (i.e. $\hat{k} = 0$) then we have $B(f^k(J_2), f^k(\hat{T})) < \frac{B(J_1',\hat{F}(\hat{T}))}{\mu}$. If not, then $B(f^k(J_2), f^k(\hat{T})) < \frac{B(J_1',\hat{F}(\hat{T}))}{\mu^2}$ by disjointness again. So in either case, $|f^k(\hat{T})| < \frac{|f^k(J_2)|}{1+\frac{2\mu^2}{B(J_1',\hat{F}(\hat{T}))}}$.

Therefore,

$$\sum_{k=0}^{j(J_1)-1} |f^k(\hat{T})| < \frac{\sum_{k=0}^{j(J_1)-1} |f^k(J_2)|}{1+\frac{2\mu^2}{B(J_1',\hat{F}(\hat{T}))}} < \frac{\sigma_i}{1+\frac{2\mu^2}{B(J_1',\hat{F}(\hat{T}))}} \le \frac{\sigma_{i,m}}{1+\frac{2\mu^2}{B(J_1',\hat{F}(\hat{T}))}}.$$

Step 2: supposing that $\hat{k} > 0$, we consider $\sum_{k=0}^{\hat{k}} \sum_{p=0}^{s-1} |f^{ks+p}(F(\hat{T}))|$. Clearly, $F(\hat{T}) \subset L_{i+\hat{k}}$. Since $\frac{|f^{ks+p}(F(\hat{T}))|}{|f^{ks+p}(J_2)|} < \frac{\sigma_{i,m}}{1+\frac{2\mu^2}{B(J_1',\hat{F}(\hat{T}))}}$ for any $0 \le k \le \hat{k}$, we have

$$\sum_{k=0}^{\hat{k}} \sum_{p=0}^{s-1} |f^{ks+p}(F(\hat{T}))| < \frac{\sum_{k=0}^{\hat{k}} \sum_{p=0}^{s-1} |f^{ks+p}(L_{i+\hat{k}})|}{1+\frac{2\mu^2}{B(J_1',\hat{F}(\hat{T}))}} \le \frac{\sigma_{i,m}}{1+\frac{2\mu^2}{B(J_1',\hat{F}(\hat{T}))}}.$$

The two steps above give us

$$\sum_{k=1}^{m_{j+1}-m_j} |f^{k+m_j}(T)| < \frac{2\sigma_{i,m}}{1+\frac{2\mu^2}{B(J_1',\hat{F}(\hat{T}))}}.$$

Note that $f^{m_k}(T) \subset J_k$ for some $J_k \subset D_k$. Then $\frac{2\sigma_{i,m}}{1+\frac{2\mu^2}{B(J_1',\hat{F}(\hat{T}))}} < \frac{2\sigma_{i,m}}{1+\frac{2\mu^2|J_k|}{B(J_1',J_k)|\hat{F}(\hat{T})|}}$. Since $k \ge 2$, supposing $\hat{F}^{k-2} \subset J_2$ for $J_2 \in D_2$ then by Lemma 1.5.1 we have

$$\frac{2\sigma_{i,m}}{1+\frac{2\mu^2}{B(J_1',\hat{F}(\hat{T}))}} < \frac{2\sigma_{i,m}}{1+\frac{2\mu^2|J_2|}{\lambda^{k-2}B(J_1',J_2)|\hat{F}^{k-2}(\hat{T})|}} < \frac{2\sigma_{i,m}}{1+\frac{2\mu^2|J_2|}{\lambda^{k-2}\Delta|\hat{F}^{k-2}(\hat{T})|}}$$

for $\Delta = \Delta(\chi)$ as usual.

Then if $\hat{F}^{N-3}(\hat{T}) \subset J_2$,

$$\begin{aligned}\sum_{j=0}^{N-4} \sum_{k=1}^{m_{j+1}-m_j} |f^{k+m_j}(T)| &< \sum_{j=0}^{N-3} \frac{2\sigma_{i,m}}{1+\frac{2\mu^2|J_2|}{\lambda^i\Delta|\hat{F}^{N-3}(\hat{T})|}} \\ &< \frac{\lambda\Delta\sigma_{i,m}}{\mu^2(1-\lambda)} \frac{|\hat{F}^{N-3}(\hat{T})|}{|J_2|}.\end{aligned}$$

We have $|\hat{F}^{N-3}(\hat{T})| < C(\chi)\frac{|J_2|}{|\hat{F}(J_2)|}|\hat{F}^{N-2}(\hat{T})|$. Similarly,

$$|\hat{F}^{N-2}(\hat{T})| < C(\chi)|\hat{F}^{N-1}(\hat{T})|\frac{|\hat{F}(J_2)|}{|\hat{F}^2(J_2)|} = C(\chi)|\hat{F}^{N-1}(\hat{T})|\frac{|\hat{F}(J_2)|}{|I_i|}.$$

Therefore,

$$\sum_{j=0}^{N-3}\sum_{k=1}^{m_{j+1}-m_j} |f^{k+m_j}(T)| < C\sigma_{i,m}\frac{|\hat{F}^{N-1}(\hat{T})|}{|I_i|} = C\sigma_{i,m}\frac{|f^{n_i}(T)|}{|I_i|}.$$

Also, using the method for the well bounded case we have

$$\sum_{k=1}^{m_N-m_{N-1}} |f^{k+m_{N-1}}(T)| < C(\chi)\sigma_{i,m}\frac{|f^{n_i}(T)|}{|I_i|}$$

and

$$\sum_{k=1}^{m_{N-1}-m_{N-2}} |f^{k+m_{N-2}}(T)| < C(\chi)\sigma_{i,m}\frac{|f^{m_{N-1}}(T)|}{|I_i|} < C\sigma_{i,m}\frac{|f^{n_i}(T)|}{|I_i|}.$$

So

$$\sum_{j=0}^{N-1}\sum_{k=1}^{m_{j+1}-m_j} |f^{k+m_j}(T)| < C\sigma_{i,m}\frac{|f^{n_i}(T)|}{|I_i|}.$$

Now we consider the case where $f^{n_i}(T) \cap \partial I_{i+1} \neq \emptyset$. Suppose that there is some maximal $n_{i+1} \leq j < n_i$ such that $f^j(T) \subset I_i \setminus I_{i+1}$. Then let $\hat{T} := f^{j+1}(T)$. If no such j exists, let $n_{i+1} \leq j \leq n_i$ be minimal such that $f^j(T) \subset I_i$ and let $\hat{T} := f^j(T)$. There exists some $M \geq 0$ such that $F_i^M(\hat{T}) = f^{n_i}(T)$. We will bound $\sum_{k=0}^{M} |F_i^k(\hat{T})|$.

If $\hat{T}$ intersected a uniformly bounded number of boundary points ∂I_{i+j} for $1 \leq j < m$ then we would be able to find some bound easily. But we may have m very large and also many of these intersections. We relabel $F|_{I_{i+1}}$ as F and I_i as I_0. We have two cases to consider. For some background on this dichotomy see [**Ly2**].

The high case

We first assume that F_i are high and central for $i = 0, \ldots, m$. This is known as an Ulam-Neumann cascade. Let $I_i = (a_i, a_i')$. We are assuming that $F_0(c)$ is a maximum for $F_0|_{I_1}$. Denote F_0 on the central branch by F. We know that I_0 is a χ-scaled neighbourhood of I_1.

We will use the Minimum Principle (Theorem 1.3.8) and Theorem 1.3.9 to estimate derivatives. The idea here is that either we have derivative greater than one in (a_1, a_m) and we can bound $\sum_{k=0}^{M} |F_i^k(\hat{T})|$ as a geometric sum; or we have a small derivative in some region, in which case we find have a bound on the number of a_i that are in this region.

We fix some integer $r \geq 1$ and some $\gamma > 1$ such that $\chi \sum_{i=0}^{r} \gamma^{-i} > 1$. Observe that there is a fixed point $p \in (a_1, c)$. We can choose I_0 to be so small that the return time to it is greater than n_0 given in Theorem 1.3.9. Therefore, by this theorem, $|DF(p)| > \rho_f$. If $|DF(a_1)| \geq \gamma$ then from the Minimum Principle, $|DF|_{(a_1,p)} > \hat{\gamma}$ where $\hat{\gamma} = \mu^3 \min(\gamma, \rho_f)$ where μ is defined in terms of $|I_0|$ in Subsection 1.4.1. We fix I_0 to be small so that $\hat{\gamma} > 1$. Therefore, we can easily bound $\sum_{k=0}^{M} |F_i^k(\hat{T})|$.

Suppose now that there is some $u \in (a_1, c)$ such that $|DF|_{(a_1,u)} < \gamma$. We will show that this must mean that the region (a_1, u) must be small in terms of the cascade. We have $|a_{i+1} - a_i| > \frac{|a_i - a_{i-1}|}{\gamma}$. Therefore, if $(a_1, a_N) \subset U$ then

$$|c - a_0| > \sum_{i=0}^{N-1} |a_{i+1} - a_i| > |a_1 - a_0| \sum_{i=0}^{N} \gamma^{-i}.$$

We know that $|a_1 - a_0| > \chi|c - a_0|$. If $N \geq r$ then $|c - a_0| > \chi|c - a_0| \sum_{i=0}^{r} \gamma^{-i}$ which is impossible by the definition of r and γ. Thus N is bounded.

So we will have $|DF|_{(a_N,p)} > \hat{\gamma}$

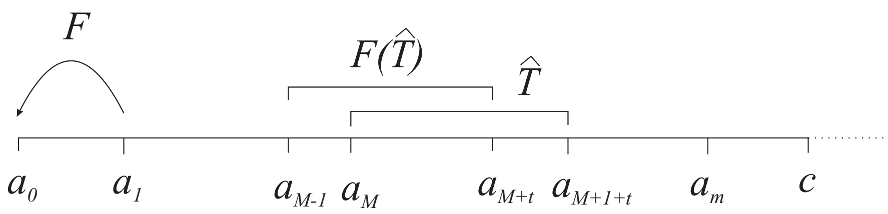

Figure 1.2: When $\hat{T}$ intersects the boundary points ∂I_j.

This helps us bound $\sum_{k=0}^{M} |F^k(\hat{T})|$ where $F^k(\hat{T}) \subset I_0 \setminus I_m$. We suppose that $F^M(\hat{T}) = (a_0, a_t)$ for $t \leq m$. (In fact the worst case would be if $F^M(\hat{T}) = (a_0, F^t(c))$ for some large $t > 0$ but we will obtain essentially the same bound.) See Figure 1.2. Then

$$\begin{aligned}\sum_{k=0}^{M} |F^k(\hat{T})| &= |a_1 - a_0| + \min(2, M-1)|a_2 - a_1| + \min(3, M-2)|a_3 - a_2| + \\ &\quad \cdots + \min(i, M-(i-1))|a_i - a_{t-i}| + \cdots + |a_{M+t} - a_{M+t-1}|.\end{aligned}$$

This is bounded above by

$$N|a_N - a_0| + |a_N - a_{N+1}| \sum_{i=0}^{\infty} \frac{\min(i, M-(i-1))}{\hat{\gamma}^i}.$$

This second summand is clearly bounded.

The low case

We assume that we are in the same setting as above, but with F_0 central and low. This is known as a saddle node cascade. Again we would like to bound $\sum_{k=0}^{M} |F^k(\hat{T})|$ defined as above. However, as we shall see, we are only able to bound $\sum_{k=0}^{M} |F^k(\hat{T})|^{1+\xi}$. We will apply the following result, a form of the Yoccoz Lemma, see for example [**FM**].

Lemma 1.5.2. *Suppose that $f \in NF^2$. Then for all $\delta, \delta' > 0$ there exists $C > 0$ such that if I_0 is a nice interval such that*

1. *I_0 is a δ-scaled neighbourhood of I_1;*
2. *F_i is low and central for $i = 0, \dots, m$;*
3. *there is some $0 < i < m$ with $\frac{|I_i|}{|I_{i+1}|} < 1 + \delta'$,*

then for $1 \leq k < m$,

$$\frac{1}{C}\frac{1}{\min(k, m-k)^2} < \frac{|I_{i+k-1} \setminus I_{i+k}|}{|I_i|} < \frac{C}{\min(k, m-k)^2}.$$

This lemma was suggested by W. Shen. For the proof, see the appendix. (For comparison with other statements of the Yoccoz Lemma, note that we will prove that one consequence of our conditions for the lemma is that we have a lower bound on $\frac{|I_m \setminus I_{m+1}|}{|I_1|}$.)

Suppose that I_0 satisfies all the conditions of the lemma. In particular we assume that for some fixed $\delta' > 0$, we have $\frac{|I_i|}{|I_{i+1}|} < 1 + \delta'$ for some $0 < i < m$. Then,

$$\begin{aligned}
\sum_{k=0}^{M} |F^k(\hat{T})| &= |a_1 - a_0| + \min(2, M-1)|a_2 - a_1| + \min(3, M-2)|a_3 - a_2| + \\
&\quad \cdots + \min(i, M-(i-1))|a_i - a_{p-i}| + \cdots + |a_{M+p} - a_{M+p-1}| \\
&< C|I_0| \sum_{k=0}^{M} \frac{1}{\min(k, M-(k-1))}.
\end{aligned}$$

This is bounded above by $C|I_0| \sum_{k=0}^{M} \frac{1}{k}$. If M is allowed to be large then this bound becomes large too. So instead of bounding $\sum_{k=0}^{M} |F^k(\hat{T})|$, we recall that the sum we want to bound is $\sum_{k=0}^{M} |F^k(\hat{T})|^{1+\xi}$. Since $\sum_{k=0}^{\infty} \frac{1}{k^{1+\xi}}$ is bounded for any $\xi > 0$ we are finished in this case.

Now suppose that $\frac{|I_i|}{|I_{i+1}|} > 1 + \delta'$ for $i = 0, \ldots, m$. Note that $|I_0| > (1+\delta')|I_1| > (1+\delta')^2|I_2| > \cdots > (1+\delta')^M|I_M|$. Therefore

$$\sum_{k=0}^{M} |F^k(\hat{T})| < \frac{1}{2}\sum_{k=0}^{M} k|I_k| < \frac{|I_0|}{2}\sum_{k=0}^{M} \frac{k}{(1+\delta')^k}.$$

Clearly this sum is bounded.

So in either case $\sum_{k=0}^{M} |F^k(\hat{T})| < C|I_0|$. We apply the usual method to say that this means that $\sum_{k=1}^{n_i-j} |f^{k+j}(\hat{T})| < C\sigma_{i,m}$. So there is some C_{casc} such that

$$\sum_{k=1}^{n_i-n_{i+m+1}} |f^{k+n_{i+m+1}}(T)|^{1+\xi} < C_{casc}\sigma_{i,m}$$

as required.

□

1.6 Exceptional case

In the last section we dealt completely with the saddle node cascade. It is easily shown, for example applying Lemma 1.6.1 to all branches, that following a saddle node cascade we essentially have a well bounded case. An Ulam-Neumann cascade, however, is not always followed by a well bounded case. We estimate the sum for F_i in this alternative case here. Most of the sum is dealt with using the methods for the well bounded case, but we need some new techniques to deal with two of the branches of F_i.

We suppose that we have F_i where F_{i-1} has a non-central return, but F_{i-2} has a central return. We assume that $F_{i-1}(c)$ is a maximum for $F_{i-1} : I_i \to I_{i-1}$. We will also assume that F_{i-1} has a high return (otherwise, we may proceed as in the well bounded case). The situation is only slightly different to the case where F_{i-2} has a non-central return. We can prove that all branches of F_i are well inside I_i, except possibly two. These branches I_i^L and I_i^R are the largest subintervals I_i^j of I_i which have $F_i|_{I_i^j} = F_{i-1}|_{I_i^j}$. We let the left-hand such interval be denoted by I_i^L and the right-hand one by I_i^R. These are the exceptional branches, see

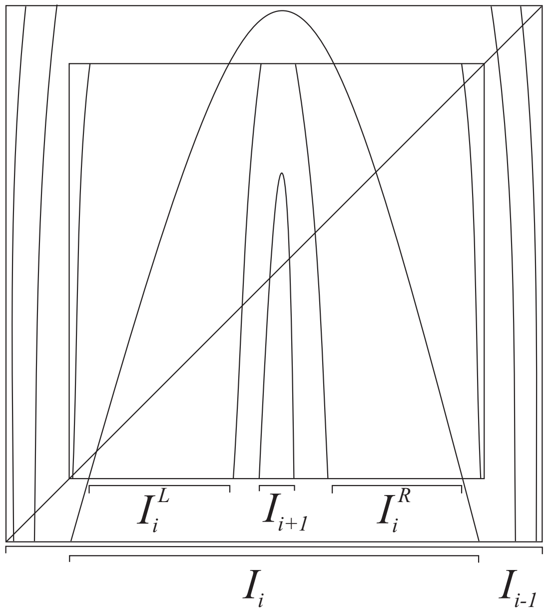

Figure 1.3: The exceptional case.

Figure 1.3. If I_i is well inside I_{i-1} then by Theorem 1.3.5 we know that I_i^L and I_i^R are well inside I_i and we may proceed as in the well bounded case. But this won't always be so if I_{i-1} is at the end of a long Ulam-Neumann cascade. So we will assume that I_i is not well inside I_{i-1}. We recall Proposition 1.2.10.

Proposition 1.2.10 *There exists some $C_{ex} > 0$ and $n_{i+1} < n_{i,3} < n_{i,2} < n_i$ such that*

$$\sum_{k=1}^{n_i - n_{i+1}} |f^{k+n_{i+1}}(T)| < C_{ex}\sigma_i \left(\frac{|f^{n_i}(T)|}{|I_i|} + \frac{|f^{n_{i,2}}(T)|}{|I_i|} + \frac{|f^{n_{i,3}}(T)|}{|I_i|} \right).$$

The strategy for the proof is as follows.

- Show there is some upper bound on $B(I_i, I_i^j)$ for $j \neq L, R$.
- State our main result in the proof, Proposition 1.6.4. We suppose that we have some interval $J \subset I_i^j$ for $j \neq L, R, 0$; $F_i(J), \ldots, F_i^m(J) \subset I_i^L \cup I_i^R$; and $F_i^{m+1}(J) \subset I_i^{j'}$ for $j' \neq L, R, 0$. Then there exists some $\lambda < 1$ such that $B(I_i, J) < \lambda B(I_i, F_i^{m+1}(J))$. Furthermore, $\sum_{k=1}^{m} |F_i^k(J)| < B(I_i, F_i^{m+1}(J))$. We can prove Proposition 1.2.10 with this result. In the rest of this section we prove Proposition 1.6.4.

- We look for an upper bound on $\sum_{k=1}^{m}|F_i^k(J)|$. In Lemma 1.6.5 we show that for some interval V there is some $\gamma > 1$ such that

$$|DF_i|_{(I_i^L \cup I_i^R)\setminus V} > \gamma.$$

- We next focus on V. We take first return maps to V and use decay of cross-ratios again to estimate sums of intervals in V, see Lemma 1.6.7.

We first show that we have uniform bounds on how deep the branches of F_i are in I_i for all branches except I_i^L, I_i^R.

Lemma 1.6.1. *In the exceptional case outlined above, if j not equal to L, R or 0, we know that I_i is a χ''-scaled neighbourhood of I_i^j.*

In fact, the above result holds for the central domain too by Theorem 1.3.5, but this isn't important for us here.

As we shall see, the proof of this lemma is reminiscent of the cascade case since we follow iterates of intervals along the central branch of some $F_{i'}$.

Proof: There exists some maximal $i' < i$ such that $F_{i'-2}$ is non-central. Then by Theorems 1.3.5 and 1.3.7, $I_{i'-1}$ is a χ-scaled neighbourhood of $I_{i'}$ and $I_{i'}$ is a χ'-scaled neighbourhood of all domains of $F_{i'}$.

For $j \neq L, R$ we will find $F_i|_{I_i^j}$ as a composition of some branches of $F_{i'}$ in order to find some extensions. $F_{i'}|_{I_{i'+1}}$ maps I_i^j out of I_i along the cascade, through the sets $I_{i-1} \setminus I_i$, $I_{i-2} \setminus I_{i-1}$ and so on, until it maps to some interval in $I_{i'+1} \setminus I_{i'+2}$. Then this interval is mapped into some $I_{i'}^{j'}$. This then maps back into $I_{i'+1}$. The process may be repeated many times before I_i^j is finally mapped back to I_i.

So know that $F_i|_{I_i^j}$ is a composition of maps as follows. Let $j_1 \neq 0$ be such that $(F_{i'}^{i-i'}|_{I_{i'+1}})(I_i^j) \subset I_{i'}^{j_1}$. Let $k_1 = i - i'$. If $F_i|_{I_i^j} = (F_{i'}|_{I_{i'}^{j_1}})(F_{i'}^{(i-i')}|_{I_{i'+1}})|_{I_i^j}$ then we stop here; we say $r = 1$. Otherwise, let $k_2 \geq 0$ be minimal such that $F_{i'}^{k_1+1+k_2}(I_i^j) \subset I_{i'} \setminus I_{i'+1}$. Let $j_2 \neq 0$ be such that $F_{i'}^{k_1+1+k_2}(I_i^j) \subset I_{i'}^{j_2}$. If $F_i|_{I_i^j} = F_{i'}^{k_1+1+k_2+1}|_{I_i^j}$ then we stop here; we say $r = 2$. Otherwise, we continue this process until we finally return to I_i and obtain k_r.

Suppose that $r = 1$. That is,

$$F_i|_{I_i^j} = F_{i'}^{(i-i')+1}|_{I_i^j}.$$

(This is the simplest case; in the other cases, we must deal with compositions of such maps.) Let U denote $F_{i'}^{(i-i')}(I_i^j)$ and U' denote $I_{i'}^{j_1}$. Then $F_{i'}(U) = I_i$ and

$F_{i'}(U') = I_{i'}$. From Theorem 1.3.5 we know that $I_{i'}$ is a χ'-scaled neighbourhood of I_i. So if we can show that, taking the appropriate branch, $(F_{i'}^{-(i-i')}|_{I_{i'+1}})(U') \subset I_i$, we know by Theorem 1.3.7 that I_i is a χ''-scaled neighbourhood of I_i^j (since all the intervals we are concerned with are disjoint). It is easy to see that $(F_{i'}^{-(i-i')}|_{I_{i'+1}})(U') \subset I_i$ by the structure of the saddle node-cascade: we have $(F_{i'}^{-1}|_{I_{i'+1}})(U') \subset I_{i'+1} \setminus I_{i'+2}$, $(F_{i'}^{-2}|_{I_{i'+1}})(U') \subset I_{i'+2} \setminus I_{i'+3}$ and so on. So the lemma is proved when $r = 1$.

In the more general case, where $r > 1$ and

$$F_i|_{I_i^j} = F_{i'}^{\sum_{l=1}^{r}(k_l+1)}|_{I_i^j}$$

we may apply the same idea, again using the disjointness of the first return map, to prove that I_i is a χ''-scaled neighbourhood of I_i^j.

□

Remark 1.6.2. If we have a good upper bound on $B(I_i, I_i^L), B(I_i, I_i^R)$ then we can proceed with the method from the well bounded case to prove Proposition 1.2.10. But this is not generally the case. So for our work here, we may assume that $B(I_i, I_i^L), B(I_i, I_i^R)$ are large.

To prove Proposition 1.2.10 we must deal with the case where some iterate of J enters $I_i^L \cup I_i^R$.

Remark 1.6.3. In the previous sections we had uniform upper bounds on the cross-ratio $B(I_i, I_i^j)$ for all j and so we obtained estimates on the decay of cross-ratios directly. This was used to estimate the sums of intervals. The problem we often encounter in this section is that sometimes we have good estimates on how cross-ratios decay and sometimes we have good estimates for the decay of the sizes of intervals. But these estimates are difficult to marry together directly, so we will have to split up such cases. This type of argument is used three times in this section. The process is first described in the proof of Proposition 1.2.10 and again in the proof of Lemma 1.6.7. (As we will see later, this scheme deals with the cases where we enter $I_i^L \cup I_i^R$ from I_i; V from $I_i^L \cup I_i^R$; and Λ from V.)

Denote the smallest interval containing both I_i^L and I_i^R by I'. We suppress the i notation when it is clear that we are in I_i. (In fact we will refer to I^L, I^R in place of I_i^L, I_i^R and F in place of F_i; but we keep the same notation for I_i so as not to confuse it with the original interval I.) To gain some intuition as to the properties of $F|_{I'}$, we may consider the model $g_2 : (-2, 2) \to (-2, 2)$ where $g_2(x) = 2 - x^2$. (In fact, since $g_2(0) = 2$ this is, in a sense, a 'worst case' model for our situation.)

The principal result in this section is the following proposition.

Proposition 1.6.4. *If $J, F(J), \ldots, F^m(J) \subset I^L \cup I^R$ then*

1. *there exists some $0 \leq \hat{m} < m$ such that $\sum_{k=0}^{m} |F^k(J)| < C(|F^m(J)| + |F^{\hat{m}}(J)|)$;*
2. *$|J| < C(|F^m(J)| + |F^{\hat{m}}(J)|)$;*
3. *if $F^{m+1}(J) \subset I^j$, $j \neq L, R$ then*
   - *(a) $\sum_{k=0}^{m} |F^k(J)| < CB(I_i, F^{m+1}(J))$;*
   - *(b) letting J' be some element of $F^{-1}(J)$ inside some interval $I^{j'}$ for $j' \neq L, R$ then we have $B(I_i, J') < \lambda B(I_i, F^{m+2}(J'))$ where $\lambda < 1$ depends only on f.*

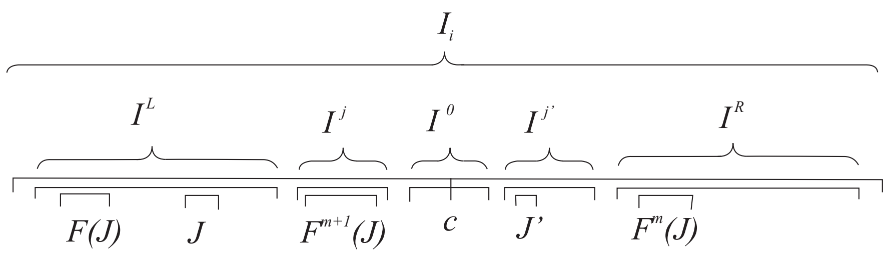

Figure 1.4: An illustration of Proposition 1.6.4.

To see a schematic representation of the situation of this proposition, see Figure 1.4. Once we have proved this then we can prove Proposition 1.2.10 as follows.

Proof of Proposition 1.2.10: As in the proof in the well bounded case, we first show that we are principally concerned with the intervals inside I_i. Again, the proof of this fact is a slightly modified version of the proof in the well bounded case.

Let $n_{i+1} < m_1 < \cdots < m_{j_i} = n_i$ be all the integers between n_{i+1} and n_i such that $f^{m_j}(T) \subset I_i \setminus I_{i+1}$ for $j = 1, \ldots, j_i - 1$ and let $m_0 = n_{i+1}$. Let $F_i : \bigcup_i U_i^j \to I_i$

be the first entry map to I_i. We will decompose the sum $\sum_{i=n_{i+1}+1}^{n_i} |f^i(T)|$ as $\sum_{j=0}^{j_i-1} \sum_{k=1}^{m_{j+1}-m_j} |f^{m_j+k}(T)|$.

Suppose that $f^{m_j+1}(T) \subset U_i^j$ for some U_i^j. Suppose further, that $F_i|_{U_i^j} = f^{i_j}$. Then there exists an extension to $V_i^j \supset U_i^j$ so that $f^{i_j} : V_i^j \to I_{i'-1}$ is a diffeomorphism, where i' is defined in the proof of Lemma 1.6.1. Then we have distortion bounds as usual: $\frac{|f^k(f^{m_j+1}(T))|}{|f^k(U_i^j)|} \le C(\chi)\frac{|f^{m_j+1}(T)|}{|I_i|}$.

$$\sum_{k=1}^{m_{j+1}-m_j} |f^{m_j+k}(T)| \le C(\chi)\frac{|f^{m_{j+1}}(T)|}{|I_i|} \sum_{k=0}^{m_{j+1}-m_j-1} |f^k(U_i^j)| \le C(\chi)\sigma_i \frac{|f^{m_{j+1}}(T)|}{|I_i|}.$$

Therefore, $\sum_{j=n_{i+1}+1}^{n_i} |f^i(T)| \le C(\chi)\frac{\sigma_i}{|I_i|}\sum_{j=1}^{j_i} |f^{m_i}(T)|$. I.e. we are principally interested in the sum $\sum_{j=1}^{j_i} |f^{m_i}(T)|$, that is $\sum_{k=0}^{j_i-1} |F^k(\hat{T})|$ where $\hat{T} = f^{m_1}(T)$. In fact, we focus on bounding $\sum_{k=0}^{j_i-2} |F^k(\hat{T})|$.

We split $\sum_{k=0}^{j_i-2} |F^k(\hat{T})|$ into two sums. The first sum is that concerning summands outside $I^L \cup I^R$. The problem clearly comes when we must deal with some J such that for some $k \ge 0$, we have $F^k(J) \subset I^j$ for some $j \ne L, R$; then $F^{k+1}(J), F^{k+2}(J), \ldots, F^{k'}(J) \subset I^L \cup I^R$ for some $k' > k$; and finally $F^{k'+1}(J) \subset I^{j'}$ for some $j' \ne L, R$. From the last part of Proposition 1.6.4 we have

$$B(I_i, F^k(J)) < \lambda B(I_i, F^{k'+1}(J)).$$

Therefore, we can bound the sums of intervals which lie in the intervals I_i^j for all $j \ne L, R$ in a similar manner to that for the well bounded case as follows, independently of those intervals inside $I^L \cup I^R$.

Given $k \ge 0$ such that $F^k(\hat{T}) \subset I^j$ for some $j \ne L, R$ we wish to estimate $|F^k(\hat{T})|$. Let $0 \le \hat{k} \le j_i - 2$ be maximal such that $F^{\hat{k}}(\hat{T}) \subset I^{j'}$ for some $j' \ne L, R$. Then we apply Proposition 1.6.4 repeatedly to obtain $B(I_i, F^k(\hat{T})) < \lambda^l B(I_i, F^{\hat{k}}(\hat{T}))$ for some $l \ge 0$. This l counts the number of times that $F^{k+r}(\hat{T})$ lies outside $I^L \cup I^R$ for $0 < r \le \hat{k}$. Then

$$|F^k(\hat{T})| < \frac{|I_i|}{1 + \frac{2}{\lambda^l B(I_i, F^{\hat{k}}(\hat{T}))}}.$$

We have two cases. In the first case we have $\hat{k} = j_i - 2$. Then

$$\begin{aligned} B(I_i, F^{j_i-2}(\hat{T})) &< B(I_i, I^{j'})\frac{|F^{j_i-2}(\hat{T})|}{|I^{j'}|} < \Delta(\chi)\frac{|F^{j_i-2}(\hat{T})|}{|I^{j'}|} \\ &< \Delta(\chi)C(\chi)|F^{j_i-1}(\hat{T})|. \end{aligned}$$

Therefore, $|F^k(\hat{T})| < C\lambda^l |F^{j_i-1}(\hat{T})|$. This suffices to prove a bound of the form $C|F^{j_i-1}(\hat{T})|$ for the iterates of T outside $I^L \cup I^R$ in this case. Then we can show in the usual way that including all the intermediate f-iterates as well we have a bound of the form $C\sigma_i \frac{|F^{j_i-1}(\hat{T})|}{|I_i|}$ in this case.

In the second case $\hat{k} < j_i - 2$. We have

$$B(I_i, F^{\hat{k}}(\hat{T})) < B(I_i, I^{j'}) \frac{|F^{\hat{k}}(\hat{T})|}{|I^{j'}|} < \frac{\Delta |F^{\hat{k}}(\hat{T})|}{|I^{j'}|}.$$

Since $|F^{\hat{k}}(\hat{T})| < C(\chi)|F^{\hat{k}+1}(\hat{T})| \frac{|I_i|}{|I^{j'}|}$ and by the second part of Proposition 1.6.4 we have $|F^{\hat{k}+1}(\hat{T}))| < C(|F^{j_i-2}(\hat{T})| + |F^{\hat{m}}(\hat{T})|)$ for some $\hat{k} < \hat{m} < j_i - 2$, we obtain

$$B(I_i, F^{\hat{k}}(\hat{T})) < CC(\chi)\Delta |I_i| (|F^{j_i-2}(\hat{T})| + |F^{\hat{m}}(\hat{T})|).$$

Therefore, in this case we have a bound of the form $C(|F^{j_i-2}(\hat{T}) + |F^{\hat{m}}(\hat{T})|)$ for the iterates of T outside $I^L \cup I^R$. Again we can show in the usual way that including all the intermediate f-iterates as well we have a bound of the form $\frac{C\sigma_i(|F^{j_i-2}(\hat{T})+|F^{\hat{m}}(\hat{T})|)}{|I_i|}$ in this case.

We need to use the above information about sizes of intervals outside $I^L \cup I^R$ to bound the sums of intervals inside $I^L \cup I^R$ too. In the first case above, we have a bound of the form $C|F^{j_i-1}(\hat{T})|$ for the iterates of T in $I^L \cup I^R$. In the second case above, we have bounds of the form $C(|F^{j_i-2}(\hat{T})| + |F^{\hat{m}}(\hat{T})|)$ for the iterates of T in $I^L \cup I^R$.

So in the worst case we have the bound

$$C_{ex}\sigma_i \left(\frac{|f^{n_i}(T)|}{|I_i|} + \frac{|f^{n_{i,2}}(T)|}{|I_i|} + \frac{|f^{n_{i,3}}(T)|}{|I_i|} \right)$$

for the sum $\sum_{k=n_{i+1}+1}^{n_i} |f^k(T)|$, as required.

□

1.6.1 Proof of Proposition 1.6.4

Recall that we are assuming that the critical point is a maximum for $F|_{I'}$. Note that whenever we refer to $F|_{I'}$, we mean $F_{i-1}|_{I'}$. This means that there is some fixed point p of F in I^R. Clearly, there also exists a point $p' \in I^L$ such that $F(p') = p$. Let $V := (p', p)$.

We outline the proof of Proposition 1.6.4 as follows. Let $0 \leq s_1 \leq s_2 \leq s_3$ be defined as follows. $F^k(J) \subset I' \setminus V$ for $1 \leq k \leq s_1$; $F^{s_1+1}(J) \subset V \cap (I^L \cup I^R)$; and $F^{s_2+k}(J) \subset I' \setminus V$ for $1 \leq k \leq s_3 - s_2$. Any sum of the form $\sum_{k=0}^{m} |F^k(J)|$ can be broken up into blocks consisting of such sums.

The scheme for proving Proposition 1.6.4 is to firstly to show that $|DF|_{I' \setminus V}$ is uniformly large. This is proved in Lemma 1.6.5 and helps to deal with the sums $\sum_{k=0}^{s_1} |F^k(J)|$ and $\sum_{k=1}^{s_3-s_2} |F^{s_2+k}(J)|$. Then we have to prove that we have bounds on the sums of intervals which return to V. This, proved in Lemma 1.6.7, helps to deal with $\sum_{k=1}^{s_2-s_1} |F^{s_1+k}(J)|$.

Lemma 1.6.5. *There exists some $\gamma > 1$ depending only on f such that*

$$|DF|_{I' \setminus V} > \gamma.$$

Proof: We start by using Theorem 1.3.9 along with Theorem 1.3.8, the Minimum Principle. Clearly, if I_0 is small enough then the first return map to I_0 must have large return times. So we may choose I_0 so small that the return time to I_0 is greater than n_0 given in Theorem 1.3.9. Since $I' \subset I_0$ we know by Theorem 1.3.9 that $|DF(p)| > \rho_f$. By symmetry, $|DF(p')| > \rho_f$ too. Observe that I^L also contains a fixed point q of F. We have $|DF(q)| > \rho_f$ too. Furthermore, there exists a point $q' \in I^R$ such that $F(q') = q$. From symmetry, $|DF(q')| > \rho_f$.

We can estimate $|DF|_{(p,q')}$ using the Minimum Principle as follows. We use our μ defined in Subsection 1.4.1 in place of $\hat{\mu}$. Then $|DF|_{(p,q')} > \mu^3 \rho_f$. By Theorem 1.3.3, when I_0 is small enough, μ is close to 1. Thus we may ensure that our intervals are so small that $|DF|_{(p,q')} > \rho$ for some $\rho > 1$. (To fix precisely how small our intervals must be, we can let $\rho = \sqrt{\rho_f}$.) By symmetry, $|DF|_{(q,p')} > \rho$.

We deal with the remaining part of the proof of Lemma 1.6.5 by showing that F has large derivative when $x \in I'$ has either $x < q$ or $x > q'$. We use the following consequence of Theorem 1.3.5 and the Minimum Principle.

Claim 1.6.6. *There exists some $\gamma' > 1$ such that, denoting $I^L = (l^-, l^+)$ and $I^R = (r^-, r^+)$, if I_0 is sufficiently small and $B(I_i, I^L), B(I_i, I^R)$ are sufficiently large then*

$$|DF|_{(l^-,q)}, |DF|_{(q',r^+)} > \gamma'.$$

Proof: We will use the Minimum Principle to prove the claim. Let $\mathcal{L}$ and $\mathcal{R}$ denote the left and right components of $I_{i'+1} \setminus I'$ respectively. Then $F(\mathcal{L})$ and $F(\mathcal{R})$ are the left and right components respectively of $I_{i'} \setminus I_i$. Using symmetry,

$$\frac{|F(\mathcal{L})|}{|\mathcal{L}|} = \frac{|I_{i'} \setminus I_i|}{|I_{i'+1} \setminus I'|}.$$

We suppose that $|I'| = \delta |I_i|$ for some $0 < \delta < 1$ which tends to 1 as $B(I_i, I^L)$ increases. We can calculate that

$$\frac{|I_{i'} \setminus I_i|}{|I_{i'+1} \setminus I'|} = \frac{|I_{i'}|}{|I_{i'+1}|}\left(\frac{1 - \frac{|I_i|}{|I_{i'}|}}{1 - \frac{|I'|}{|I_{i'+1}|}}\right) > (1 + 2\chi)\left(\frac{1 - \frac{|I_i|}{|I_{i'+1}|}}{1 - \frac{\delta|I_i|}{|I_{i'+1}|}}\right).$$

From the second part of Theorem 1.3.5 we have an upper bound on $\frac{|I_i|}{|I_{i'+1}|}$, so if $B(I_i, I^L)$ is sufficiently large then

$$(1 + 2\chi)\left(\frac{1 - \frac{|I_i|}{|I_{i'+1}|}}{1 - \frac{\delta|I_i|}{|I_{i'+1}|}}\right) > \gamma''$$

for some $\gamma'' > 1$.

We deduce that there must be some $x_0 \in I_{i'+1} \setminus I'$ such that $|DF(x_0)| \geq \gamma''$. Therefore, by Theorems 1.3.8 and 1.3.3 we have

$$|DF|_{(x_0,p)} > \exp\{-3\sigma'(|I_0|)|I_0|\} \min(\gamma'', \rho_f).$$

Choosing $|I_0|$ small we have some $\gamma' > 1$ such that $|DF|_{(x_0,q)} > \gamma'$. In particular $|DF|_{(l^-,q)} > \gamma'$. Similarly we can show $|DF|_{(q',r^+)} > \gamma'$.

□

Letting $\gamma := \min(\rho, \gamma')$, the lemma is proved.

□

By the above, we will be able to estimate the sizes of iterates of T inside $(I^L \cup I^R) \setminus V$ as a geometric sum. The next step is to consider what happens to intervals inside V. We denote the first return map to V by $\hat{F} : \bigcup_j V^j \to V$. We first wish to find some control on the sizes of the domains of $\hat{F}$. Let $m_{V,j}$ be such that $\hat{F}|_{V^j} = F^{m_{V,j}}|_{V^j}$.

Lemma 1.6.7. *If $F^{l_1}(J), \ldots, F^{l_m}(J) \subset V \cap (I^L \cup I^R)$ are all the iterates of J up to l_m which lie in $V \cap (I^L \cup I^R)$, and all intermediate iterates $F^k(J)$ for $k = 0, 1, \ldots, l_m$ lie in $I^L \cup I^R$ then*

$$\sum_{k=0}^{l_m} |F^k(J)| < C|F^{l_m}(J)|.$$

Furthermore, there exists $\lambda_V < 1$ such that $|J| < C\lambda_V^{l_m - m}|F^{l_m}(J)|$.

Before we can prove this Lemma, we need some real bounds for V. The following lemma, which contrasts with Lemma 1.6.5, will later be used to obtain these bounds.

Lemma 1.6.8. *There exists some $\hat{C} = \hat{C}(\chi, |I'|) > 0$, where $\hat{C}(\chi, |I'|)$ tends to some constant $\hat{C}(\chi)$ as $|I'| \to 0$, such that*

$$|DF|_{I'} < \hat{C}.$$

Proof: We work with $F_{i'} : I_{i'+1} \to I_{i'}$ where i' is defined in the proof of Lemma 1.6.1. There exists some $m \geq 1$ such that $F_{i'}|_{I_{i'+1}} = f^m|_{I_{i'+1}}$. We can decompose this map into two maps so that $F_{i'} = Lg$ where $g = f|_{U_\phi}$, i.e $g(x) = f(c) - |\phi(x)|^\alpha$, and $L = f^{m-1} : f(I_{i'+1}) \to I_{i'}$.

By Theorems 1.3.7 and 1.3.5 we have $\frac{DL(x)}{DL(y)} < C(\chi)$ for $x, y \in f(I_{i+1})$. So

$$|DL(x)| \leq C(\chi)\frac{|I_i|}{|f(I_{i+1})|} = C(\chi)\frac{|I_i|}{\left|\phi\left(\frac{|I_{i+1}|}{2}\right)^\alpha\right|}$$

for $x \in f(I_{i+1})$. Also

$$|Dg(x)| = \alpha|D\phi(x)||\phi(x)^{\alpha-1}| < \alpha \sup_{x \in I_{i'+1}} |D\phi(x)| \left|\phi\left(\frac{|I_{i+1}|}{2}\right)\right|^{\alpha-1}.$$

For $\hat{U} \subset U_\phi$ a small neighbourhood of c, let $\mathrm{Dist}(\phi, \hat{U})$ denote $\sup_{x,y\in\hat{U}} \frac{|D\phi(x)|}{|D\phi(y)|}$. Observe that as I' becomes smaller, $\mathrm{Dist}(\phi, I')$ tends to 1. For $x \in I_{i+1}$,

$$|DF(x)| < \alpha C(\chi)\frac{\sup_{x\in I_{i+1}} |D\phi(x)||I_i|}{\left|\phi\left(\frac{|I_{i+1}|}{2}\right)\right|} < 2\alpha C(\chi)\mathrm{Dist}(\phi, I')\frac{|I_i|}{|I_{i+1}|}.$$

Since we have assumed that $\frac{|I_i|}{|I_{i+1}|}$ is bounded, there is some constant $C > 0$ such that

$$|DF(x)| < C\mathrm{Dist}(\phi, I').$$

Letting $\hat{C}(\chi, |I'|) := C\mathrm{Dist}(\phi, I')$ we have proved the lemma.

□

We use this below.

Proof of Lemma 1.6.7: We split the sum as follows

$$\sum_{k=0}^{l_m} |F^k(J)| = \sum_{j=0}^{m-1} \sum_{k=1}^{l_{j+1}-l_j} |F^{l_j+k}(J)|$$

where we let $l_0 = -1$. We know from Lemma 1.6.5 that $|DF|_{I' \setminus V} > \gamma$ so

$$\sum_{k=1}^{l_{j+1}-l_j} |F^{l_j+k}(J)| < |F^{l_{j+1}}(J)| \sum_{k=0}^{l_{j+1}-l_j-1} \left(\frac{1}{\gamma}\right)^k < \frac{|F^{l_{j+1}}(J)|}{1-\left(\frac{1}{\gamma}\right)}.$$

Whence,

$$\sum_{k=0}^{l_m} |F^k(J)| < \frac{1}{1-\left(\frac{1}{\gamma}\right)} \sum_{j=0}^{m} |F^{l_j}(J)|.$$

So we only need bound the sum of returns to V.

Denote the rightmost element of $\bigcup^j V^j$ by V^1 and the leftmost element by V^2 (observe that $\hat{F}|_{V^1} = F^2|_{V^1}$ and $\hat{F}|_{V^2} = F^2|_{V^2}$). We get an estimate on how deep each V^j is inside V for $j > 2$ because V^1 and V^2 have some definite size compared to $|V|$; since by Lemma 1.6.8 we know that $|V^1|, |V^2| > \frac{|V|}{\hat{C}^2}$. Therefore, there exists some δ_0' depending only on f such that V is a δ_0'-scaled neighbourhood of V^j for all $j > 2$. So by Lemma 1.4.2, there exists some $\lambda_V' < 1$ depending on δ_0' such that for any interval $J' \subset V^j$, $B(V, J') < \lambda_V' B(V^j, J')$ for $j > 2$ (in fact this is also shown in Claim 1.6.9 below). As usual we can use Theorem 1.3.3 to conclude that there exists some $\lambda_V < 1$ such that $B(V, J') < \lambda_V B(V, \hat{F}(J'))$. If we remain away from V^1 and V^2, this fact and the usual argument would be sufficient to obtain the required bound on sums.

We must deal with the case where iterates enter V^1, V^2. The idea is to split the situation into the case where intervals land in a region where $|D\hat{F}|$ is large and the case when the intervals land in a region where we don't have good estimates on $|D\hat{F}|$.

We firstly focus on V^2. We know from Theorem 1.3.9 that $|DF(p')| > \rho_f$ and so $|D\hat{F}(p')| > \rho_f^2$. There must also exist some fixed point r of $\hat{F}$ in V^2 with $|D\hat{F}(r)| > \rho_f$. Letting $\Lambda_2 := (p', r)$ and applying the Minimum Principle as before, we obtain $|D\hat{F}|_{\Lambda_2} > \rho$. Let r' be the point in V^1 such that $\hat{F}(r') = r$. Then there exists some $\rho > 1$ such that $|D\hat{F}|_{(r',p)} > \rho$. We define Λ_1 to be the interval in V^1 which has $\hat{F}(\Lambda_1) = V \setminus V^2$. Clearly $\Lambda_1 \subset (r', p)$, so $|D\hat{F}|_{\Lambda_1} > \rho$. For convenience later, we let $\Lambda := \Lambda_1 \cup \Lambda_2$. The reason for fixing these intervals in this way is explained below when we consider the case when $\hat{F}^{m-1}(J) \subset V^1 \cup V^2$.

We are now ready to deal with bounding $\sum_{k=0}^{m-1} |\hat{F}^k(J)|$. Observe that $\hat{F}^{m-1}(J)$ must be contained in some V^j. Suppose first that $j > 2$; we deal with the case where $j = 1$ or 2 later. Suppose further that $J \subset V^{j'}$ and $j' > 2$; here the other case is similar. We will again split up the sum. Let $N_0' = 0$. Let N_1 be minimal such that $\hat{F}^{N_1}(J) \cap \Lambda = \emptyset$ and $\hat{F}^{N_1+1}(J) \subset \Lambda$. Let $N_1' > N_1$ be minimal such that $\hat{F}^{N_1'}(J) \subset \Lambda$ and $\hat{F}^{N_1'+1}(J) \cap \Lambda = \emptyset$. In a similar fashion we obtain $N_0' < N_1 < N_1' < \cdots < N_{M-1} < N_{M-1}'$ so that

$$\begin{aligned}\sum_{k=0}^{m-1} |\hat{F}^k(J)| &= \sum_{j=0}^{M-1} \left(\sum_{k=1}^{N_{j+1}-N_j'} |\hat{F}^{N_j'+k}(J)| + \sum_{k=1}^{N_{j+1}'-N_{j+1}} |\hat{F}^{N_{j+1}+k}(J)| \right) \\ &\quad + \sum_{k=1}^{N_M - N_{M-1}'} |\hat{F}^{N_{M-1}'+k}(J)|\end{aligned}$$

where $N_M = m-1$. Observe that the first sum in the brackets concerns intervals which land inside Λ and the second sum in the brackets concerns intervals in $V \setminus \Lambda$. Then

$$\sum_{k=1}^{N_{j+1}'-N_{j+1}} |\hat{F}^{N_{j+1}+k}(J)| < |\hat{F}^{N_{j+1}'}(J)| \sum_{k=0}^{N_{j+1}-N_{j+1}'-1} \left(\frac{1}{\rho}\right)^k < \frac{C}{1-\left(\frac{1}{\rho}\right)} |\hat{F}^{N_{j+1}'}(J)|$$

for some C.

Now we consider $\sum_{k=1}^{N_{j+1}-N_j'} |\hat{F}^{N_j'+k}(J)|$. In fact we learn most from estimating the sum $\sum_{k=1}^{N_M-N_{M-1}'} |\hat{F}^{N_{M-1}'+k}(J)|$. We adjust $\lambda_V < 1$ so that for $J \subset V^j \setminus \Lambda_j$ for $j = 1, 2$ we have $B(V, J) < \lambda_V B(V, F(J))$. Then for $1 \leq k < N_m - N_{M-1}'$,

$$B(V, \hat{F}^{N_{M-1}'+k}(J)) < \lambda_V^{N_M - N_{M-1}'-k} B(V, \hat{F}^{N_M}(J)).$$

Recalling that $M = m - 1$ we calculate $B(V, \hat{F}^{m-1}(J)) < B(V, V^j)\frac{|\hat{F}^{m-1}(J)|}{|V^j|}$. Letting $B_V := \max\left\{\sup_{j>2} B(V, V^j), B(V, V^1 \setminus \Lambda_1), B(V, V^2 \setminus \Lambda_2)\right\}$, we obtain

$$|\hat{F}^{N'_{M-1}+k}(J)| < \frac{|V|}{1 + \frac{2|V^j|}{\lambda_V^{N_M - N'_{M-1} - k} B_V |\hat{F}^{m-1}(J)|}}.$$

Letting $\hat{B}_V := \frac{B_V}{B_V + 2}$ we have

$$|\hat{F}^{N'_{M-1}+k}(J)| < \hat{B}_V \lambda_V^{N_M - N'_{M-1}+k} \frac{|V|}{|V^j|} |\hat{F}^{m-1}(J)|.$$

We wish to bound $|\hat{F}^{m-1}(J)|$ in terms of $|\hat{F}^m(J)|$. We do this by constructing an extension. Let $V = (a, a')$. Let the left-hand and right-hand members of $F^{-1}(a)$ be denoted by b and b' respectively. Denote (b, b') by V'. By Lemma 1.6.8, V' is a $\delta_{V'}$-scaled neighbourhood of V where $\delta_{V'}$ depends only on f.

Claim 1.6.9. *For all branches V^j, $j > 2$ there exists an extension to some interval $U^j \supset V^j$ such that $U^j \subset V$ and $F^{m_{V,j}} : U^j \to V'$ is a diffeomorphism.*

Proof: For $j > 2$ the return maps are a composition of $F|_V$ followed by $F|_{I^R}$ and then some number of iterates of $F|_{I^L}$. So $\hat{F}^{-1}$ must pull V' back into I^L. Observe that this element of $F^{-1}(V')$ is below p' (and clearly away from $F(c)$). Any further pullbacks in I^L remain below p' also. Therefore when some element $F^{-k}(V')$ is finally pulled back into I^R, it is mapped above p and remains away from $F(c)$. Therefore we have elements of $F^{-k-2}(V')$ mapping inside V which don't contain c.

□

By the above claim and Theorem 1.3.7 we have some $C > 0$ depending only on f such that if $j > 2$,

$$\frac{1}{C}\frac{|V|}{|V^j|} \leq |D\hat{F}|_{V^j} \leq C\frac{|V|}{|V^j|}.$$

(Recall that we are assuming that $F^{m-1}(V) \cap \Lambda = \emptyset$.)

Letting $\tilde{B} = C\hat{B}_V$ we have $|\hat{F}^{N'_{M-1}+k}(J)| < \lambda_V^{N_M - N'_{M-1}+k} \tilde{B} |\hat{F}^m(J)|$. Therefore,

$$\sum_{k=1}^{N_M - N'_{M-1}} |\hat{F}^{N'_{M-1}+k}(J)| < \frac{\tilde{B}|\hat{F}^m(J)|}{1 - \lambda_V}.$$

So we can find an estimate for the sums of lengths of the iterates of J which land in Λ in terms of $|\hat{F}^m(J)|$. We now proceed to estimate the other sums concerning intervals outside Λ as follows. Let $\hat{\mu} := \exp\left\{-\nu'(I_0)\frac{|I_0|}{1-\left(\frac{1}{\rho}\right)}\right\}$. Suppose that $F^{N_{M-2}}(J) \subset V^j$. then taking the appropriate branch, $\hat{F}^{N_{M-2}-N'_{M-1}-1}(V) \subset V^j$ and

$$\begin{aligned}
B(V, \hat{F}^{N_{M-2}}(J)) &< \lambda'_V B(\hat{F}^{N_{M-2}-N'_{M-1}-1}(V), \hat{F}^{N_{M-2}}(J)) \\
&< \frac{\lambda'_V}{\hat{\mu}} B(\hat{F}^{-1}(V), \hat{F}^{N'_{M-1}}(J)) \\
&< \frac{\lambda'_V}{\mu\hat{\mu}} B(V, \hat{F}^{N'_{M-1}+1}(J))
\end{aligned}$$

Modifying $\mu, \hat{\mu}$ if necessary, as usual, so that $\frac{\lambda'_V}{\mu\hat{\mu}} =: \lambda_V < 1$, we obtain

$$B(V, \hat{F}^{N_{M-2}}(J)) < \lambda_V B(V, \hat{F}^{N'_{M-1}+1}(J)).$$

Clearly then we can proceed in bounding the sum using the usual method of decaying cross-ratios. So we have bounded $\sum_{k=0}^{m-1} |\hat{F}^k(J)|$ for this case.

There remains a further complication to consider. Above we assumed that $\hat{F}^{m-1}(J) \subset V^j$ where $j > 2$. But if $j \in \{1, 2\}$ we have two cases. We first note that if $F^{l_m}(J) \cap \{r, r'\} = \emptyset$ then the intervals we are concerned with are either completely inside Λ_2, Λ_1 or completely inside $V \setminus (\Lambda_2 \cup \Lambda_1)$. Then we may proceed as above. But if $F^k(J)$ contains r or r' then we split $F^k(J)$ into two intervals, with this periodic point at their intersection. We may then apply the procedure above to estimate the size of each interval. We need only apply this splitting argument once since if we intersect a periodic point of $\hat{F}$ once, we must stay there for all time under iteration by $\hat{F}$. Thus we need only alter our constants by a factor of 2 to deal with this case. Note that we only have one sum where this problem could occur: $\sum_{k=1}^{N'_M-N_M} |\hat{F}^{N_M+k}(J)|$ where $N'_M = m$. This is because r is a fixed point for $\hat{F}$.

Clearly, we can use the cross-ratio argument as usual to obtain the estimate $|F^{l_1}(J)| < \lambda_V^{m-1} C |F^{l_m}(J)|$, so $|J| < \lambda_V^{m-1} C |F^{l_m}(J)|$.

□

We may adjust our usual λ so that $\lambda_V \leq \lambda < 1$.

Proof of Proposition 1.6.4: Suppose first that $F^{m+1}(J) \subset I^j$ for $j \neq L, R$. Then, in particular, we can be sure that $F^m(J)$ does not contain p or p'. Then we also know that none of $F^k(J)$ contain p or p' for $0 \leq k \leq m-1$. This means that we can be sure that all the intervals we consider are either contained in V or are disjoint from V.

Recall that $0 \leq s_1 < s_2 \leq s_3 = m$ are defined as follows. $F^k(J) \subset I' \setminus V$ for $1 \leq k \leq s_1$; $F^{s_1+1}(J) \subset V \cap (I^L \cup I^R)$; and $F^{s_2}(J) \subset V \cap (I^L \cup I^R)$, $F^{s_2+k}(J) \subset I' \setminus V$ for $1 \leq k \leq s_3 - s_2$.

Then if $s_3 > s_2$,

$$\sum_{k=1}^{s_3-s_2} |F^{s_2+k}(J)| < |F^{s_3}(J)| \sum_{k=0}^{s_3-s_2-1} \left(\frac{1}{\rho}\right)^k < \frac{|F^{s_3}(J)|}{1-\left(\frac{1}{\rho}\right)},$$

by Lemma 1.6.5.

From Lemma 1.6.7,

$$\sum_{k=1}^{s_2-s_1} |F^{s_1+k}(J)| < C|F^{s_2}(J)|$$

and $|F^{s_1+1}(J)| < C|F^{s_2}(J)|$.

Also

$$\begin{aligned}\sum_{k=0}^{s_1} |F^k(J)| &< |F^{s_1}(J)| \sum_{k=0}^{s_1-1} \left(\frac{1}{\gamma}\right)^k < \frac{1}{\gamma\left(1-\left(\frac{|F^{s_1+1}(J)|}{\gamma}\right)\right)} \\ &< \frac{C|F^{s_2}(J)|}{\gamma\left(1-\left(\frac{1}{\gamma}\right)\right)}.\end{aligned}$$

Therefore,

$$\sum_{k=0}^{s_2} |F^k(J)| < \frac{C|F^{s_2}(J)|}{\gamma\left(1-\left(\frac{1}{\gamma}\right)\right)} + C|F^{s_2}(J)| < C|F^{s_2}(J)|.$$

If $s_3 > s_2$ then

$$\sum_{k=0}^{s_3} |F^k(J)| < \frac{|F^{s_3}(J)|}{1-\left(\frac{1}{\gamma}\right)} + C|F^{s_2}(J)|.$$

Therefore, the first two parts of the proposition are proved.

Now if $F^{m+1}(J) \subset I^j$ for $j \neq L, R, 0$ then recalling that $s_3 = m$ we will obtain an estimate for $|F^{s_2}(J)|$ in terms of $B(I_i, F^{m+1}(J))$.

$$B(I_i, F^{s_2}(J)) < B(F^{-(s_3-s_2)}(I_i), F^{s_2}(J)) < \frac{B(I_i, F^m(J))}{\mu} < \frac{B(I_i, F^{m+1}(J))}{\mu^2}.$$

We are allowed to use μ here since all intermediate intervals must be disjoint (otherwise we would have to pass through V again). Therefore $|F^{s_2}(J)| < \frac{|I_i|}{1+\frac{2\mu^2}{B(I_i,F^{m+1}(J))}} < C|I_i|B(I_i,F^{m+1}(J))$. Similarly we can show that $|F^m(J)| < C|I_i|B(I_i,F^{m+1}(J))$. Therefore

$$\sum_{k=0}^{s_3}|F^k(J)| < C|I_i|B(I_i,F^{m+1}(J))$$

We now prove the final part of the proposition.

Let $\tilde{\mu} := \exp\{-\nu'(|I_0|)2C|I_0|\}$. Clearly for any run of intervals $F(J),\ldots,F^k(J) \subset I^L \cup I^R$, considering the branch of F^{-k} which follows the iterates of J, we know that $B(F^k, F^{-k}(I_i), J) > \tilde{\mu}$. We consider the branch of F^{-m-2} which follows the backward orbit of $F^{m+1}(J)$. Clearly, $F^{-m-2}(I_i)$ is strictly inside I^j. Thus,

$$\begin{aligned} B(I_i,J') &< \lambda' B(I^j,J') < \lambda' B(F^{-m-2}(I_i),J') \\ &< \frac{\lambda'}{\tilde{\mu}}B(F^{-1}(I_i),F^{m+1}(J')) < \frac{\lambda'}{\tilde{\mu}\mu}B(I_i,F^{m+2}(J')). \end{aligned}$$

We can alter the usual λ slightly so that $\frac{\lambda'}{\tilde{\mu}\mu} \leq \lambda$ and still ensure that $\lambda < 1$. Thus, $B(I_i,J') < \lambda B(I_i,F^{m+2}(J'))$ as required.

When we do not escape $I^L \cup I^R$ then we may have some intersection with p or p'. In this case, we spilt our interval in two and estimate the size of each piece as above. We need only apply this idea once, so we can change our constants to cater for this case too. Then we needn't find the cross-ratio $B(I_i,J')$ in terms of $B(I_i,F^{m+2}(J'))$.

□

1.7 Proof of the main result

We will first prove Theorem 1.2.5 when we are not in the infinite cascade case.

When $f \in C^{2+\eta}$ there is some $C_2 > 0$ such that $w_{Df^2}(\epsilon) = C_2\epsilon^\eta$. In fact, we change C_2 so that $C_1(\epsilon^\eta + \beta\epsilon) < C_2\epsilon^\eta$ where C_1, β come from Lemma 1.3.3. Recall that $B(f^n,T,J) > \exp\{-C_2\sum_{k=0}^{n-1}|f^k(T)|^{1+\eta}\}$. We will find a bound on the sum $\sum_{k=0}^{n-1} |f^k(T)|^{1+\eta}$ by using the main propositions above and also finding some decay property for the size of the intervals $f^{n_i}(T)$ for some values of i.

Let $F_i : \bigcup_j U_i^j \to I_i$ be the first entry map to I_i (we include the branches of the first return map in this case too). For $i < j$, we define the function $S(i, j, J)$ to be the maximum of $|f^{i+1}(J)|, |f^{i+2}(J)|, \dots, |f^j(J)|$. We will consider $S(n_{i+1}, n_i, T)$. Let $n(j)$ be such that $F_i|_{U_i^j} = f^{n(j)}|_{U_i^j}$. Now let $U_i^{s(i)}$ be the interval for which $S\left(0, n(j), U_i^j\right)$ is maximal. Clearly,

$$S(n_{i+1}, n_i, T) \leq S\left(0, n(s(i)), U_i^{s(i)}\right).$$

We would like to show that for certain i, this quantity decays with i in a controlled way.

First suppose that F_{i-1} is in the well bounded case. We have two cases. Firstly, suppose that $U_i^{s(i)} \subset I_i$. Then since F_{i-1} is in the well bounded case, we have $|U_i^{s(i)}| < \frac{|I_{i-1}|}{1+2\chi}$. Since I_i is a domain of the first return map to I_{i-1} we have

$$|U_i^{s(i)}| < \frac{S\left(0, n(s(i-1)), U_{i-1}^{s(i-1)}\right)}{1+2\chi}.$$

Now assume that $U_i^{s(i)} \cap I_i = \emptyset$. Then there exists some extension $V_i \supset U_i^{s(i)}$ such that $f^{n(s(i))} : V_i \to I_{i-1}$ is a diffeomorphism. We will show that $U_i^{s(i)}$ is uniformly smaller than V_i. By Theorem 1.3.3 we know that $B(V_i, U_i^{s(i)}) < \frac{B(I_{i-1}, I_i)}{\mu}$. Thus, by Lemma 1.4.1, $|U_i^{s(i)}| < \frac{|V_i|}{1+\frac{2\mu}{\Delta(\chi)}}$. Since V_i is a first return domain to I_{i-1} we have

$$|U_i^{s(i)}| < \frac{S\left(0, n(s(i-1)), U_{i-1}^{s(i-1)}\right)}{1+\frac{2\mu}{\Delta(\chi)}}.$$

Let $\gamma := \max\left(\frac{1}{1+2\chi}, \frac{1}{1+\frac{2\mu}{\Delta(\chi)}}\right)$. Clearly $\gamma < 1$. So

$$S\left(0, n(s(i)), U_i^{s(i)}\right) < \gamma S\left(0, n(s(i-1)), U_{i-1}^{s(i-1)}\right).$$

We let $C_{all} = \max(C_{wb}, C_{casc}, 3C_{ex})$. If $f \in C^{2+\eta}$ and F_{i-1} is well bounded, we have

$$B(f^{n_i-n_{i+1}}, f^{n_{i+1}+1}(T), f^{n_{i+1}+1}(J)) \\ \geq \exp\left\{-C_2\left(S(n_{i+1}, n_i, T)\right)^\eta \sum_{k=1}^{n_i-n_{i+1}} |f^{k+n_{i+1}}(T)|\right\}$$

$$
\begin{aligned}
&> \exp\left\{-C_2\left(S\left(0, n\left(s\left(i\right)\right), U_i^{s(i)}\right)\right)^{\eta} C_{all}\right\} \\
&> \exp\left\{-C_2\left(\gamma S\left(0, n\left(s\left(i-1\right)\right), U_{i-1}^{s(i-1)}\right)\right)^{\eta} C_{all}\right\}.
\end{aligned}
$$

(For the case of a long saddle node cascade we are able to bound the sum $\sum_{k=1}^{n_i-n_{i+1}} |f^{k+n_{i+1}}(T)|^{1+\xi}$ for any $\xi > 0$. So we can alter the above calculation by choosing some $\xi < \eta$ and considering

$$\left(\gamma S\left(0, n\left(s\left(i-1\right)\right), U_{i-1}^{s(i-1)}\right)\right)^{\eta-\xi} \sum_{k=1}^{n_i-n_{i+1}} |f^{k+n_{i+1}}(T)|^{1+\xi}$$

instead.)

If we are not in the infinite cascade case then the sums for $F_i, F_{i+1}, \ldots$ can be broken into blocks consisting of a cascade; possibly followed by an exceptional case; followed by one or more well bounded cases. So suppose that F_i is well bounded, $F_i, F_{i+1}, \ldots, F_{i+m-1}$ have central returns, F_{i+m} has a non-central return and F_{i+m+1} is an exceptional case. So note that, in particular, F_{i+m+2} must be well bounded. Then,

$$
\begin{aligned}
&S\left(0, n(s(i+m+3)), U_{i+m+3}^{s(i+m+3)}\right) < \gamma S\left(0, n(s(i+m+2)), U_{i+m+2}^{s(i+m+2)}\right), \ldots \\
&\qquad \ldots, \gamma S\left(0, n(s(i+1)), U_{i+1}^{s(i+1)}\right) < \gamma^2 S\left(0, n(s(i)), U_i^{s(i)}\right).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
B(f^n, T, J) &> \exp\left\{-C_2 \sum_{k=0}^{n-1} |f^k(T)|^{1+\eta}\right\} \\
&> \exp\left\{-C_2 C_{all}\left(S\left(0, n(s(0)), U_0^{s(0)}\right)\right)^{\eta} \sum_{k=0}^{\infty} \gamma^{k\eta}\right\} \\
&> \exp\left\{-C_2 C_{all}\frac{(\sigma'(|I_0|))^{\eta}}{1-\gamma^{\eta}}\right\}.
\end{aligned}
$$

Therefore, it is easy to see that for any $0 < K < 1$, if I_0 is the central domain of a first return map to some I_{-1}, I_0 is sufficiently small and F_{-1} is in the well bounded case, then we may bound $B(f^n, T, J)$ below by K.

It remains to show that we can always begin with a well-bounded case when we don't have an infinite cascade. If $\omega(c)$ is non-minimal, then arguing as in the proof of Theorem A' of [**LS1**], we can immediately find an interval I_{-1} which has the first return map F_{-1} in the well bounded case. Otherwise, when we don't have an infinite cascade, we simply induce on a nice interval finitely many times until we obtain a non-central return and thus obtain a suitable I_{-1}. We consider the infinite cascade case in the next section.

The second part of Theorem 1.2.5, which concerns $A(f^n, T, J)$ is proved similarly with an analogous result to Theorem 1.3.3 for $A(f, T, J)$ (see [**Sh3**] or [**St**] for example).

Remark 1.7.1. Note that the above method of proof relies on the modulus of continuity $w = w_{D^2 f}$ having some smoothness property. In fact, what we actually require is that there is some $\epsilon > 0$ such that $\sum_{i=1}^{\infty} w(\epsilon^i)$ is bounded. In general, this is not the case. For example, if w is of the form $w(x) = \frac{-\omega}{\log x}$, for some parameter $\omega > 0$, for x in a neighbourhood of 0.

Remark 1.7.2. In [**K1**] there is a way of proving the above theorem for the C^3 case by showing that σ_i decays to 0 in a good way as i increases. Indeed, suppose that we only have C^2 smoothness, but that all the other conditions above are met. If we can then show that $\sum_{i=0}^{\infty} \sigma_i$ is bounded then the result follows. However, this may not be the case in general.

1.7.1 Infinite cascade case

Here we deal with the case where we have some I_0 such that F_i are central for $i = 0, 1, \ldots$. In this case we will find that $\frac{|I_{i+1}|}{|I_i|}$ gets very close to 1. See Figure 1.5 to see what a typical such map will look. In particular, I_i will not shrink down to a point (the critical point c) as i increases so we can't use the method above to bound sums of intervals which get very close to c.

When f is only finitely renormalisable, we don't encounter this phenomenon. The principal tool is an extension given to us by a result of [**K2**].

We start by letting I_0 be any nice interval about c. We assume that we have some infinite cascade. This means that F_i is central for all i, where F_i is defined in the usual way. The main idea here is that we can still find good bounds on some interval $I_{0,0}$ and then apply the methods of Section 1.5 to it. Then we need to find another interval $I_{1,0}$ around c which is smaller than all $I_{0,i}$, also has good bounds and is uniformly smaller than $I_{0,0}$. In such a way, we obtain a sequence of intervals $I_{i,0}$ which can each be treated as in the cascade case above and which shrink down uniformly to the critical point. We will assume that $F_{i,j}$ is always

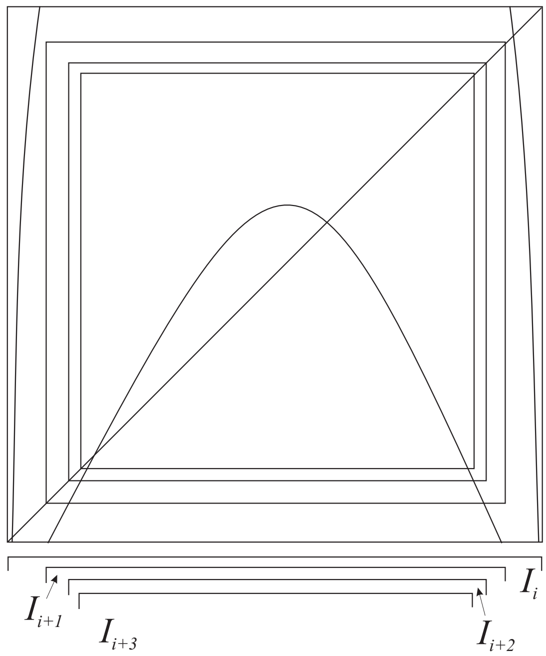

Figure 1.5: An infinite cascade.

central for all $i, j \geq 0$. Otherwise we can simply choose I_0 such that we never have an infinite cascade.

For all i the central branch of F_i has two fixed points, q_0 and p_0 to the left and right of c respectively (as usual, we assume that $F_i(c)$ is a maximum for $F_i|_{I_{i+1}}$). We let q_0' be the point in I_{i+1} not equal to q_0 which maps by F_i to q_0. We define p_0' similarly. We define $I_{0,0}$ to be (p_0', p_0). Let $F_{0,0} : \bigcup_j I_{0,0}^j \to I_{0,0}$ be the first return map to $I_{0,0}$ (where $I_{0,0}^0 = I_{0,1}$ is the central domain). We have the following lemma.

Lemma 1.7.3. *There exists some $\hat{\chi} > 0$ depending only on f such that $I_{0,0}$ is a $\hat{\chi}$-scaled neighbourhood of every domain $I_{0,0}^j$.*

Proof: Clearly, I_i tends to (q_0, q_0'). So we denote (q_0, q_0') by I_∞. We will first show that I_∞ is uniformly larger than $I_{0,0}$ and then show that all non-central domains of the first entry map to $I_{0,0}$ have an extension to I_∞ and show what this means for $I_{0,0}^j$.

In a similar manner to the exceptional case, we will find an upper bound for $|DF_i|_{I_{i+1}}$. This will allow us to get good bounds for the first return map to $I_{0,0}$

For large i, the ratio I_i has $\frac{|I_{i+1}|}{|I_i|}$ close to 1. The following lemma, an adaptation of Lemma 7.2 of [**K2**], allows us to bound $|DF_i|_{I_{i+1}}$.

Lemma 1.7.4. *If $f \in NF^2$ then there exist constants $0 < \tau_2 < 1$ and $\tau_3 > 0$ with the following property. If T is any sufficiently small nice interval around the critical point, R_T is the first entry map to T and its central domain J is sufficiently big, i.e. $\frac{|J|}{|T|} > \tau_2$, then there is an interval W which is a τ_3-scaled neighbourhood of the interval T such that if $c \in R_T(J)$ then the range of any branch of $R_T : V \to T$ can be extended to W provided that V is not J.*

This lemma is only given as a C^3 result in [**K2**], but it easily extends to our C^2 case.

In the infinite cascade case we always have this $c \in R_T(J)$ condition.

It is straightforward to see that the above lemma is sufficient to prove a version of Lemma 1.6.8 in our case. That is, for large i, there exists some $\hat{C}'$ such that $|DF_i|_{I_{i+1}} < \hat{C}'$. This implies that there exists some $0 < \theta < 1$ depending only on f such that $|I_{0,0}| < \theta |I_\infty|$ and, equivalently, some $\tilde{\chi} > 0$ such that I_∞ is a $\tilde{\chi}$-scaled neighbourhood of $I_{0,0}$.

Now, for the moment we let $F_{0,0}$ also denote the first entry map and $\bigcup_j I_{0,0}^j$ also denote the first entry domains. Suppose that there exists a domain $I_{0,0}^j$ disjoint from $I_{0,0}$ which does not have an extension to I_∞. That is, supposing $F_{0,0}|_{I_{0,0}^j} = f^{n(j)}|_{I_{0,0}^j}$ there is no interval V such that $f^{n(j)} : V \to I_\infty$ is a diffeomorphism. Let $0 \leq k < n(j) - 1$ be maximal such that $f^{n(j)-k} : f^k(I_{0,0}^j) \to I_{0,0}$ has no extension to I_∞ (clearly if $I_{0,0}$ is small $f : f^{n(j)-1}(I_{0,0}^j) \to I_{0,0}$ always has an extension so $k < n(j) - 1$). Then there exists some interval $V \supset f^{k+1}(I_{0,0}^j)$ such that $f^{n(j)-k-1} : V \to I_\infty$ is a diffeomorphism and the element of $f^{-1}(V)$ containing $f^k(I_{0,0}^j)$ contains c.

Since I_∞ is a nice interval, $V \subset I_\infty$. We also know that $f^k(I_{0,0}^j) \subset I_\infty \setminus I_{0,0}$. Therefore V contains either p_0 or p_0'. But then either $f^{n(j)-k-1}(p_0)$ or $f^{n(j)-k-1}(p_0')$ is contained in $I_\infty \setminus I_{0,0}$ which is not possible.

Now consider $f(I_{0,0}^j)$ for some $j \neq 0$ where $I_{0,0}^j \subset I_{0,0}$ is a domain of the first return map. There exists some $V \supset f(I_{0,0}^j)$, where $f^{n(j)} : V \to I_\infty$ is a diffeomorphism and V is a $\tilde{\chi}$-scaled neighbourhood of $f(I_{0,0}^j)$. Let $V(f(c))$ denote the maximal interval around c which pulls back by f^{-1} to $I_{0,0}$. If V is not contained in $V(f(c))$ then either p_0 or p_0' is contained in V' the respective pullback by f^{-1} of V (the one which contains $I_{0,0}^j$). Thus, $f^{n(j)}(p_0)$ or $f^{n(j)}(p_0')$ lies in $I_\infty \setminus I_{0,0}$, a contradiction.

So $V' \subset I_{0,0}$ and $I_{0,0}$ is a $\hat{\chi}$-scaled neighbourhood of $I_{0,0}^j$ where $\hat{\chi} = \min\left(\tilde{\chi}', \frac{1}{2}\right)$. The case of the central branch follows in the usual manner.

□

So we are in a type of well bounded case for $F_{0,0}$. Furthermore, we may assume that $F_{0,0}$ has an infinite cascade too. We sum for $F_{0,0}, F_{1,0}, \ldots$ as in the cascade case. We let q_1, q_1', p_1, p_1' be defined as above for the fixed points of $F_{0,0}|_{I_{0,1}}$. We let $I_{0,\infty}$ denote (q_1, q_1'). We may apply the same ideas as above to find some new interval $I_{1,0} := (p_1, p_1')$ which has $|I_{1,0}| < \theta |I_{0,\infty}|$. We may define $I_{i,j}$ for $i \geq 2$, and $0 \leq j \leq \infty$ in a similar way.

Let $f^{N_i}(T)$ be the last iterate of T which lies inside $I_{0,i}$. Let $N_i' > N_{i+1}$ be the last time that $f^{N_i'}(T)$ contains p_i (if no such integer exists, set $N_i' = N_{i+1}$). Then we can prove the following proposition.

Proposition 1.7.5. *For all $\xi > 0$ there exists some $C_{inf} > 0$ such that*

$$\sum_{k=1}^{N_i - N_i'} |f^{k+N_i'}(T)|^{1+\xi} < C_{inf}\hat{\sigma}_i$$

where $\hat{\sigma}_i$ is defined as follows. Let $\sigma_i := \sup_{V \in \mathrm{dom} F_{i,0}} \sum_{j=1}^{n(V)} |f^j(V)|$ (and $n(V)$ is defined as k where $F_{i,0}|_V = f^k$). Let $\hat{V} \subset I_{i,0} \setminus I_{i,1}$ be an interval such that $f^{\hat{n}}(\hat{V})$ is one of the connected components of $I_{i,0} \setminus I_{i,1}$ and $f^j(\hat{V})$ is disjoint from both $I_{i,0} \setminus I_{i,1}$ and $I_{i+1,0}$ for $0 < j < \hat{n}(\hat{V})$. Then $\hat{\sigma}_i$ is the supremum of all such sums $\sum_{j=1}^{\hat{n}(\hat{V})} |f^j(\hat{V})|$ and σ_i.

It is clear that we can prove the proposition when we don't enter $I_{i,\infty}$. When we enter $I_{i,\infty} \setminus I_{i+1,0}$ we simply use the Minimum Principle as usual to show that $|DF_i|_{I_{i,\infty} \setminus I_{i+1,0}}$ is uniformly greater then 1. Then this gives us decay of the size of these pullbacks.

In order to bound the whole sum, we must bound $\sum_{k=1}^{N_i' - N_{i+1}} |f^{k+N_{i+1}}(T)|$ too. To do this we split $f^{N_i'}(T)$ into the intervals $f^{N_i'}(T_i^-)$ and $f^{N_i'}(T_i^+)$ to the left and right of p_i respectively. If we denote $M_i \geq 0$ to be the last time that an iterate of T_i^- contains p_{i+1}. Then using Proposition 1.7.5, we have

$$\sum_{k=1}^{N_i' - M_i} |f^{k+M_i}(T)| < C_{inf}\hat{\sigma}_{i+1}.$$

Such logic then leads us to conclude that

$$\sum_{k=0}^{N_i} |f^{k+N_{i+1}}(T)| < C_{inf} \sum_{k=i}^{\infty} (k-i)\hat{\sigma}_k.$$

While this sum is not obviously bounded, this is not the sum we need to bound when $f \in NF^{2+\xi}$. As in the cascade case, we bound $\sum_{k=1}^{N_i - N_i'} |f^{k+N_i'}(T)|^{1+\xi}$. This is bounded by a sum of the form $C_{inf}|I_{0,0}| \sup_{0 \le i} \sigma_i \sum_{i=0}^{\infty} i\theta^{\xi i}$. Clearly this is bounded.

1.8 Possible Extensions

To continue these studies we could:

- extend our result to all maps in NF^2. Most of our bounds only require the maps to be C^2. It is only in Section 1.7 when we require extra smoothness. Furthermore, in many cases it is not hard to improve our bounds on return sums to bounds like $CB(I_i, f^{n_i}(T))$ and indeed, in some cases to obtain even better bounds which relate some cross ratios for $f^{n_i}(T)$ and $f^{n_i+1}(T)$;
- extend our results to the case of $C^{2+\eta}$ multimodal maps with non-flat critical points. The key results we use here come from [**SV**] and they relate to multimodal maps;
- use our result to find some extension to results already existing relating to the 'decay of geometry' for maps with non-flat critical points. See for example, [**GSS1**] (which requires negative Schwarzian to obtain decay of geometry in C^3 maps). This would possibly help us to show some decay of σ_i with i which could, by Remark 1.7.2 help to prove Theorem 1.2.5 in the C^2 case.
- use our bounding methods to simplify and possibly extend the estimates on the Hausdorff dimension of attractors for unimodal maps in [**GK**]. This process will be a little more complicated than our situation above as we raise the summands to some powers. If we consider T is as in Theorem 1.2.5, we will be trying to bound sums of the form $\sum_{k=0}^{n} |f^k(T)|^\beta$ where $\frac{1}{2} < \beta < 1$.

Chapter 2

Structurally sensitive points

Kozlovski proved in [**K3**] that Axiom A maps are dense in the space of C^k unimodal maps for $k = 1, 2, \ldots, \infty, \omega$. So we know what form 'most maps' in the unimodal class take. The proof involves some global perturbation of a unimodal map (to make it analytic) followed by a local perturbation at the critical value to reach an Axiom a map. The consequence of this is that almost all small *global* C^k perturbations of non-Axiom A unimodal maps produce Axiom A maps. Here we go some way to showing that near some points almost all small *local* C^k perturbations produce Axiom A maps too. Conversely, at other points no small local C^k perturbation changes the type of the map. Thus we are able to characterise the effect of a large class of perturbations on a large class of maps.

In fact, we prove that in some cases we know that a specific perturbation about any point in the postcritical set will give us an Axiom A map. Any small perturbation outside this set will not change the type of the map.

This type of result has been proved in the C^2 case around $f(c)$ in [**BM**].

In [**A**] a lamination of the space of C^k unimodal maps with the same topological type is constructed. So our results show that for some cases we can find families of maps which are transverse to this lamination.

There are, unfortunately, only some maps which we can apply our ideas to. We will give some idea as to the difficulties in other cases.

2.1 Finding structurally sensitive points

In this section we introduce our main result. See the following section for definitions.

We will try to find local perturbations which lead to a change of the combinatorial type of the unimodal map. We may only address this problem in the C^k setting, $k \neq \omega$ (see Section 2.5 for the definition of this class of maps), as there are no local perturbations of analytic maps.

The class of maps we deal with is defined as follows.

Definition 2.1.1. We say that a unimodal C^k map has *non-flat* critical point c if in some neighbourhood U of c, there exists some C^k diffeomorphism $\phi : U \to I$ with $\phi(c) = 0$ and $g(x) = \pm|\phi(x)|^\alpha + g(c)$ for some $\alpha > 1$. α is known as the *critical order* for f. We denote the set of such maps by NF^k. We also denote this neighbourhood by U_ϕ. When the critical order is 2 we say that we have a *quadratic critical point*. We denote this class by Q^k.

In this chapter we deal with maps in NF^k which have no parabolic cycles and are not Axiom A. We denote this class by NFH^k. Denote the set of maps in both NFH^k and Q^k by QH^k.

We introduce a definition.

Definition 2.1.2. For $f \in C^k$ where $f : I \to I$, a point x is called a *structurally sensitive point of the map* if for any neighbourhood U of x and any $\varepsilon > 0$ there is some map $g \in C^k$ s.t. $\|f - g\|_{C^k} < \varepsilon$ and $g|_{I \setminus U} = f|_{I \setminus U}$, but f and g are not combinatorially equivalent.

That is, in an arbitrarily small neighbourhood of such points, an arbitrarily small perturbation of the map changes the combinatorial type. In fact, in some cases this concept will be stronger as we will see in Lemma 2.1.5.

As we explain in detail in Section 2.2, we let $B(f)$ denote the basin of attraction for f. We are interested in finding the following type of maps.

Definition 2.1.3. We say that C^1 map $f : I \to I$ is *Axiom A* if:

- f has finitely many hyperbolic attractors;
- the set $\Sigma(f) = I \setminus B(f)$ is hyperbolic.

We let $PC(f)$ denote the postcritical set $\bigcup_{n\geq 0} f^n(c)$. Our main theorem is as follows.

Theorem 2.1.4. *We find perturbations to prove the following.*

1. *If $f \in NFH^k$ for $k \geq 2$ and c is preperiodic or non-recurrent then $\overline{PC(f)}$ is the set of C^k structurally sensitive points for $k = 1, 2, \ldots$.*
2. *If $f \in NFH^2$ then the set of C^1 structurally sensitive points is $\overline{PC(f)}$.*
3. *If $f \in QH^\omega$ then $\overline{PC(f)}$ is the set of C^k structurally sensitive points where $k = 1, 2 \ldots$.*
4. *If $f \in QH^k$ and $\omega(c)$ is minimal then $\overline{PC(f)}$ is the set of C^k structurally sensitive points where $k = 1, 2 \ldots$.*
5. *If $f \in QH^k$ and $\overline{\{x \in \omega(c) : \omega(x) \neq \omega(c)\}} \cap \{c\} = \emptyset$ then $\overline{PC(f)}$ is the set of C^k structurally sensitive points where $k = 1, 2 \ldots$.*

Usually this type of perturbation is done at $f(c)$, see for example, [**BM**] or [**K3**]. In fact, in [**BM**] it is proved that any $f \in NF^2$ is C^2 structurally sensitive at $f(c)$. The method of proof there does not easily extend to $f^n(c)$ for $n > 1$.

For any point x_0 outside $\overline{PC(f)}$, there is an open set $U_{x_0} \ni x_0$ which also lies outside $\overline{PC(f)}$. Any small C^1 perturbation supported on U_{x_0} will clearly not change the combinatorial type of a map f (we need merely to ensure that we don't create a new critical point). So to prove any of the parts of Theorem 2.1.4 we only need to prove that points in $\overline{PC(f)}$ are structurally sensitive. Below we will let U_{x_0} be some small neighbourhood of the point x_0 in which we will perturb. Observe that if $f \in Q^k$, for any point not equal to c, a small enough local perturbation will yield a map in Q^k.

Also, if c is preperiodic then it is easy to see that there are small perturbations which change the itinerary of c.

In Section 2.2.1 we prove the following.

Lemma 2.1.5. *Let $f \in NFH^3$. Then for any $x_0 \notin \overline{PC(f)}$ there exists some $\epsilon > 0$ such that any C^3 perturbation in an ϵ-small neighbourhood of x_0 which is ϵ-small in the C^3 topology yields a map which is conjugate to f.*

Therefore, whenever we can prove that the set of C^3 structurally sensitive points for f is precisely $\overline{PC(f)}$ then we can also be sure that these are the only points about which small local perturbations change the *topological type* (clearly if two maps are not combinatorially equivalent then they are not conjugate). So we can talk of these points as *topologically structurally sensitive points.*

The main idea behind this lemma, as we will see in Subsection 2.2.1, is a result of [**K3**] that states that small C^3 perturbations outside $\overline{PC(f)}$ cannot create attracting cycles. We are unable to prove this for all small C^2 perturbations.

Strategy of the proof of Theorem 2.1.4

- We prove Theorem 2.1.4 in the case where c is non-recurrent in Section 2.3.
- We make C^1 perturbations in Section 2.4.
- The analytic case is proved in Section 2.5.
- The minimal case is dealt with in Section 2.6.
- A simple non-minimal case is considered in Section 2.7.

Our theorem could extend to the following conjecture, a version of which appears in the final section of [**K1**].

Conjecture 2.1.6. *Let $f : I \to I$ have $f \in QH^k$. Then the set of C^k structurally stable points is the closure of the postcritical set. Here $k = 1, 2, \ldots$.*

To prove the conjecture, it would be necessary to deal with C^k-structurally sensitive points for $k \geq 2$ where f is non-Axiom A, recurrent and non-minimal.

The following theorem provides the main technique for our proofs for C^k structurally sensitive points for $k = 2, \ldots$. It is taken from Theorem C of [**K3**]. We don't use it directly, but we are able to make minor adjustments to the proof in order to obtain a similar result in our case. For definitions, see below.

Theorem 2.1.7. *Let $f_\lambda : I \to I$ be an analytic family of analytic unimodal maps with a non-degenerate critical point and no parabolic cycles, $\lambda \in \Omega \subset \mathbb{R}^N$ where Ω is a closed set. Suppose that the family f is non-degenerate in the sense that there exist some $\lambda, \lambda' \in \Omega$ such that f_λ and $f_{\lambda'}$ are not combinatorially equivalent. Then for each $\lambda \in \Omega$, either f_λ is Axiom A or there is a map which is not combinatorially equivalent arbitrarily close in the C^k topology to f_λ. Here $k = 1, \ldots, \infty, \omega$.*

In [**K3**] for a given unimodal map $f \in C^k$, firstly f is approximated by a map in QH^ω. This is a *global* perturbation. This is the map then denoted by f_0 in the theorem. In our case, we can often find our map f_0 by a *local* perturbation of f. In fact, we don't require f_0 to be analytic. We just need f_0 to induce some such polynomial-like map. Then in [**K3**] f is embedded in a non-degenerate family by perturbing at $f(c)$ rather than at any point of $PC(f)$. Note that in both their case and our case, this second perturbation can be large in the C^2 sense. However, rigidity theorems for holomorphic maps show that, in fact, such a perturbation can be very small in the C^k topology for $k = 1, \ldots$.

So the fundamental difference for us is that we make purely local perturbations: firstly to get the polynomial-like map and secondly to create a non-degenerate family by perturbing a point in the postcritical set.

We sketch the proof of Theorem 2.1.7, observing how it can be adapted to our case. For more details, see [**K3**]. A family of maps are created as follows. Given a non-Axiom A map f_0 with non-flat critical point, a polynomial-like map $F_0 : B_0 \to A$ is induced where $B_0 = \bigcup_i B_0^i$ and $A = \bigcup_i A^i$ (there are a finite number of such B_0^i and A^i).

We suppose that f_λ for λ in a neighbourhood of 0 is a family which depends holomorphically on λ. If we know that for all neighbourhoods D of 0 there is some $\lambda \in D$ such that f_0 and f_λ are not combinatorially equivalent then we are finished.

Next we obtain $F_\lambda : B_\lambda \to A$. By the construction of F_λ we can see that, in our case, this must be a polynomial-like mapping. This is because the structure of the set $\partial A \cap \mathbb{R}$ persists for f_λ. (In fact, by a change of coordinates, we may assume that A is fixed for all f_λ.) Now given $F : B_0^i \to A^{j(i)}$, a quasi-conformal mapping ϕ_λ is defined as mapping $\phi_\lambda : \partial B_0^i \cup \partial A \to \partial B_\lambda^i \cup A^{j(i)}$ where for $z \in \partial B_0^i$, $\phi_\lambda(z) = F_\lambda^{-1} F_0(z)$ and for $z \in \partial A$, $\phi_\lambda(z) = z$.

The map ϕ_λ can be extended to $A \setminus B_0$ in a quasiconformal way since for most of the finite number of domains B^i we have, $\mathrm{mod}(A \setminus B^i)$ bounded away from zero (there is a further argument when $\partial B_0 \cap \partial A$ intersect). The Beltrami coefficient of this map can be pulled back by F_0. The measurable Riemann mapping Theorem provides a family of maps with this Beltrami coefficient $h_\lambda : A \to A$ such that the map

$$G_\lambda : h_\lambda F_0 h_\lambda^{-1} : B_\lambda \to A$$

is holomorphic for each λ close to 0. Furthermore this family depends holomorphically on λ.

The next step is to show that f_0 and f_λ are combinatorially equivalent if and only if $F_\lambda = G_\lambda$. The proof of this requires the rigidity theorem and the straightening theorem.

Finally, if we assume that 0 is an accumulation point of parameters λ such that f_λ is combinatorially equivalent to f_0 then $F_\lambda = G_\lambda$ in an open neighbourhood of 0. But by kneading theory we know that such a set of parameters must be closed. So *all* parameters must give combinatorially equivalent maps. But it is assumed that for some λ we have f_0 and f_λ not combinatorially equivalent so this is a contradiction. Whence there are maps arbitrarily close to f_0 which are not combinatorially equivalent to f_0.

Therefore, if we are able to find a C^k small local perturbation of f which induces a polynomial-like map for some k, and we are then able to change the combinatorics by another local perturbation, then the method of Theorem 2.1.7 yields a map which is a C^k small local perturbation of f and is not topologically conjugate to f.

2.2 The setting for local perturbations

We fix the notation I to be the unit interval $[0, 1]$. We will also assume that the unimodal maps $f : I \to I$ which we deal with have $f(\partial I) \subset \partial I$. (Throughout, for a set $U \subset \mathbb{R}$, we let $\overline{U}$ denote the closure of U, let $\mathring{U}$ denote the interior of U and ∂U denote the set $\overline{U} \setminus \mathring{U}$.)

Definition 2.2.1. Two non-recurrent unimodal maps f and $\tilde{f}$ with critical points c and $\tilde{c}$ respectively are *combinatorially equivalent* if there exists an order preserving bijection,

$$h : \bigcup_{n\in\mathbb{Z}} f^n(c) \to \bigcup_{n\in\mathbb{Z}} \tilde{f}^n(\tilde{c}),$$

which conjugates f and $\tilde{f}$, where $c, \tilde{c}$ are the critical points of f and $\tilde{f}$ respectively and $h(c) = \tilde{c}$.

We can also refer to such maps being of the same combinatorial type.

Supposing that we have a unimodal map $f : I \to I$ with critical point c, then we may divide I into the intervals $L = [0, c)$ and $R = (c, 1]$. So $I = L \cup \{c\} \cup R$. We may associate to a point $x \in I$ a sequence $\underline{i}(x) = (i_0(x), i_1(x), \ldots)$ where

$$i_k(x) = \begin{cases} L & \text{if } f^k(x) \in L \\ c & \text{if } f^k(x) = c \\ R & \text{if } f^k(x) \in R \end{cases}$$

This is called the *itinerary* of x.

We say that a point x is periodic with period n if there is some minimal $n > 0$ such that $f^n(x) = x$. We say that a point x is preperiodic if there is some $m \geq 0$ such that $f^m(x)$ is periodic. Observe that this implies that here we allow periodic points to be classed as preperiodic too (which is not always the case elsewhere).

If $f \in C^1$ and for some point $x \in I$, there is some $p \geq 1$ such that $f^p(x) = x$. Then we say that x is *attracting* if $|Df^p(x)| < 1$, *repelling* if $|Df^p(x)| > 1$ and *parabolic* if $|Df^p(x)| = 1$. $\{x, f(x), \ldots, f^{p-1}(x)\}$ is referred to an attracting, repelling or parabolic cycle respectively for each of the above cases. $Df^p(x)$ is referred to as the *multiplier* for the periodic cycle $\{x, f(x), \ldots, f^{p-1}(x)\}$.

We say that a sequence $(i_0, i_1, \ldots)$ is *periodic* with period n if there exists some $n \geq 1$ with $i_{j+np} = i_j$ for any $0 \leq j < n$ and $p \geq 0$. Such a sequence is *eventually periodic* if there is some $m > 0$ such that $(i_m, i_{m+1}, \ldots)$ is periodic.

Remark 2.2.2. For the unimodal map f, any point x which is not preperiodic, but which does have an eventually periodic itinerary must be attracted to an attracting periodic orbit. For example suppose that f has periodic itinerary. Suppose that $\underline{i}(x) = (R, R, \ldots)$. Then since $f|_R$ is continuous, the only way that x can always land in R is if it is attracted to a periodic point (see, for example, Lemma 2.2.12 below). It is easy to see how this extends to all periodic and eventually periodic itineraries.

Definition 2.2.3. We call the sequence given by $\lim_{y \downarrow c} \underline{i}(y)$ the *kneading invariant* of the unimodal map f.

Note that if two unimodal maps are combinatorially equivalent then they have the same kneading invariant. We are interested in changing the kneading invariant of a non-Axiom A unimodal map f by some perturbation. We will create a continuous family f_λ where $f = f_0$ which has some λ' such that $f_{\lambda'}$ has a different combinatorial type to f_0. We can use the continuity of the kneading invariant to show that some Axiom A map $f_{\lambda''}$ has $\lambda'' \in (0, \lambda']$. We see this from Corollary II.10.1 of [**MS**] as follows.

Theorem 2.2.4. *Let $f_\lambda : I \to I$ be a unimodal family consisting of C^1 maps depending continuously on λ. Suppose that maps in our family satisfy the following condition. If $f_\lambda, f_{\lambda'}$ have preperiodic critical points and their kneading invariants are equal then $\lambda = \lambda'$. Then*

$$\lambda \to \nu(f_\lambda)$$

is monotone.

We can choose our families to be so small that there is no region where this map is discontinuous, but the critical point is preperiodic for some member of the family. Then, in our case we will obtain a map f_λ with

$$\nu(f_\lambda) = (i_{0,\lambda}(c), i_{1,\lambda}(c), \ldots, i_{m-1,\lambda}(c), i_{m,\lambda}(c), \ldots)$$

where $i_m(c) \in \{L, R\}$ and

$$\nu(f_{\lambda'}) = (i_{0,\lambda}(c), i_{1,\lambda}(c), \ldots, i_{m-1,\lambda}(c), i_{m,\lambda'}(c), \ldots)$$

where $i_{m,\lambda'}(c) \neq i_{m,\lambda'}(c)$. Then by Theorem 2.2.4 there must be some $\lambda'' \in [\lambda, \lambda']$ such that $i_{m,\lambda''}(c) = c$. Thus, c must be periodic for $f_{\lambda''}$ and $f_{\lambda''}$ is Axiom A.

We will often need the following definition.

Definition 2.2.5. Define the *omega limit set* $\omega(x)$ of a point $x \in I$ as the following:

$$\omega(x) = \{y : \text{there exists a sequence } n_i \to \infty \text{ with } f^{n_i}(x) \to y\}.$$

We say that x is recurrent if it is not periodic and $x \in \omega(x)$.

Definition 2.2.6. For a C^1 map $f : I \to I$, if x is an attracting periodic point of period n then we call the set of points $y \in I$ such that $x \in \omega(y)$ the *basin of* x. We let the *immediate basin of* x be set of n connected components of the basin of x, each containing an element $f^j(x)$ for $0 \leq j < n$. We denote the union of all the basins by $B(f)$ and the union of all the immediate basins by $B_0(f)$.

Definition 2.2.7. Given a map $f : X \to X$, a set $A \subset X$ is said to be *minimal with respect to* f (or simply *minimal*) if A is a closed, f-invariant set (i.e. $f(A) \subset A$) which has no closed f-invariant subsets

(Note that when we say A is f-invariant, we mean

For $f : I \to I$ a unimodal map with critical point c, we say that f *is minimal* if $\omega(c)$ is a minimal set.

Definition 2.2.8. Let $f : I \to I$. A closed proper subinterval J of I is called *restrictive* with period $n \geq 1$ for f if

1. the interiors of $J, \dots, f^{n-1}(J)$ are disjoint;
2. $f^n(J) \subset J$, $f^n(\partial J) \subset \partial J$;
3. at least one of $J, \dots, f^{n-1}(J)$ contains a turning point;
4. J is maximal with these properties.

A map with restrictive intervals of arbitrarily large periods is called *infinitely renormalisable.*

Definition 2.2.9. Given some map of an interval $f : I \to I$ an open interval T is called *nice* for f if the elements of ∂T never map into the interior of T under iteration by f. Further, given some interval T, we say that the interval T' is *nice with respect to* T if the elements of $\partial T'$ never map into the interior of T.

For a given $x \in I$ we may choose a nice interval containing x as follows. Given a periodic cycle, choose a^- to be the element of the cycle which is nearest to x and less than x; similarly choose a^+ greater than x. Typically, we will be looking for very small nice intervals. We can usually find such intervals around x if preperiodic points accumulate on x.

For a nice open interval T, we let $F : \bigcup_j U^j \to T$ be a first return map to T for the map f. If T is nice then we know that U^j are all disjoint. We see this by letting $U^j = (a_j, a'_j)$ and $n(j)$ be some integer such that $F|_{U^j} = f^{n(j)}|_{U^j}$. Then suppose that $U^j \cap U^{j'} \neq \emptyset$. We may assume that $a_j \in U^j \cap U^{j'}$ and $n(j) < n(j')$. Then since $f^{n(j)}(a_j) = b$ for some $b \in \partial T$, we must have $f^{n(j')-n(j)}(b) \in T$ which contradicts niceness. Similarly we can show that all U^j are strictly contained in T.

If $f^k(U^j)$ never meets c for $0 \leq k \leq n(j)-1$ then $F : U^j \to T$ is a diffeomorphism. Clearly, there will be a periodic point x_j in each branch U^j on which F is a diffeomorphism. If T is a nice interval around c then we let U^0 denote the domain which contains c. Note that $F : U^0 \to T$ is unimodal. On all other domains F is clearly a diffeomorphism. U^0 is known as the *central domain.*

Lemma 2.2.10. *For $f \in NFH^2$ be such that c is not preperiodic, there exist arbitrarily small nice intervals around c which have preperiodic boundary points.*

We prove this lemma using the following results.

Definition 2.2.11. We say that U is a *wandering interval* for a map $f : I \to I$ if $U, f(U), f^2(U), \ldots$ are pairwise disjoint.

The following is Lemma II.3.1 of [**MS**].

Lemma 2.2.12. *Let $f : I \to I$ be a continuous map. Suppose that $J \subset I$ is an interval such that $f^n|_J$ is monotone for all $n \geq 0$. Then there are two possibilities,*

1. *J is a wandering interval.*
2. *every point of J is contained in the basin of a periodic orbit.*

The following is a well known result, see, for example, Theorem IV.A of [**MS**].

Theorem 2.2.13. *If $f \in NF^2$ then f has no wandering intervals.*

Thus for $f \in NF^2$, we only need consider case 2 in Lemma 2.2.12. Therefore every point $x \in I$ is accumulated by either the preimages of c or of B_0 or both.

We will need the following result, Proposition IV.18 of [**BC**]

Proposition 2.2.14. *For $f : I \to I$ a continuous map, any point x such that there is some sequence $x_k \to x$ and a sequence of integers $n_k \to \infty$ such that $f^{n_k}(x_k) \to x$ then x is contained in the closure of the set of preperiodic points.*

We also need the following claim.

Claim 2.2.15. *For an attracting periodic point $x \in I$ with period m, there exists some interval B_x which is a connected component of B_0 containing x. Then the elements of ∂B_x are periodic.*

Proof: For $B_x = (a, a')$ suppose that a_n is some sequence of points in B_x such that $a_n \to a$. Then $f^m(a_n) \in B_x$ for all n. Furthermore, by continuity, $\lim_{n\to\infty} f^m(x_n) \in \overline{B_x}$. Thus $f^m(a) \in \overline{B_x}$. If $f^m(a) \in \overset{\circ}{B}_x$ then $a \in \overset{\circ}{B}_x$ which is a contradiction. Hence $f^m(a) \in \partial B_x$. Since $f^m|_{B_x}$ is a diffeomorphism then either $f^m(a) = a$ and $f^m(a') = a'$ or $f^m(a) = a'$ and $f^m(a') = a$. In either case $f^{2m}(a) = a$ and $f^{2m}(a') = a'$.

□

Proof of Lemma 2.2.10: First we want to find preperiodic points near c. If c is accumulated by preimages of B_0 then since, by the above claim, the elements of ∂B_0 are preperiodic we are finished. If c is accumulated by preimages of c then by Proposition 2.2.14 we are finished.

Now we construct our interval using these points. Let U be an arbitrarily small neighbourhood of c. For some preperiodic point b of f in U we let a be the element of $\bigcup_{n\geq 0} f^n(b)$ which is closest to c (since c is not preperiodic, we can be sure that $a \neq c$). Let a' be the point not equal to a for which $f(a') = f(a)$. Then let $U' = (a, a')$. So $\partial U'$ consists of preperiodic points. Furthermore, it is easy to show that since a is close to c, then a' must be close to c too. Therefore U' satisfies the lemma.

□

2.2.1 Topologically structurally sensitive points

Here we will prove Lemma 2.1.5. We first define what we mean by a conjugacy.

Definition 2.2.16. Given two continuous interval maps $f : I \to I$ and $g : I' \to I'$ then we say that a continuous map $h : I \to I'$ a *conjugacy between f and g* if the following diagram commutes.

$$\begin{array}{ccc} I & \xrightarrow{f} & I \\ {\scriptstyle h}\big\downarrow & & \big\downarrow{\scriptstyle h} \\ I' & \xrightarrow{g} & I' \end{array}$$

We say that f and g are *conjugate*, or of the same *topological type*.

We recall the lemma.

Lemma 2.1.5. *Let $f \in NFH^3$. Then for any $x_0 \notin \overline{PC(f)}$ there exists some $\epsilon > 0$ such that any C^3 perturbation in an ϵ-small neighbourhood of x_0 which is ϵ-small in the C^3 topology yields a map which is conjugate to f.*

Proof of Lemma 2.1.5: Suppose that $x_0 \notin \overline{PC(f)}$. Then we choose some open interval U_{x_0} containing x_0 which is also disjoint from $\overline{PC(f)}$. We apply the following result, Theorem II.3.1 of [**MS**].

Theorem 2.2.17. *Suppose that f, g are two unimodal maps with turning points c and $\tilde{c}$ for f and g respectively. Assume that the map*

$$h : \bigcup_{n\geq 0} f^n(c) \to \bigcup_{n\geq 0} g^n(\tilde{c})$$

defined by $h(f^n(c)) = g^n(\tilde{c})$ is an order preserving bijection. We suppose that the following properties are satisfied.

1. *If c is periodic then the 'conjugacy' h maps $\bigcup_{n\geq 0} f^n(c) \cap B_0(f)$ into the corresponding set for g.*
2. *Assume that i) f and g have no wandering intervals; ii) there are no intervals consisting of periodic points of constant period; and iii) the restriction of the map h to $B_0(f)$, i.e.*

   $$h : \bigcup_{n\geq 0} f^n(c) \cap B_0(f) \to \bigcup_{n\geq 0} g^n(\tilde{c}) \cap B_0(g),$$

   extends to a congugacy from $B_0(f)$ to $B_0(g)$. Then h extends to a congugacy on I.

Note that any perturbation inside U_{x_0} to a map g will give g the same combinatorial type as f. This is because no forward iterate of the critical point ever enters U_{x_0}. Clearly, if our perturbation is small enough it does not affect the immediate basins of the attractive periodic points $B_0(f)$. We first assume that we have not created a periodic attractor, so we may begin to apply the theorem above. We will assume that $x_0 \notin B_0(f)$ since this case is obvious.

Our two maps already have the same combinatorial type, so our h satisfies the first part of the theorem. Furthermore, 2i) is satisfied trivially and ii) is satisfied since there are no parabolic cycles. If our perturbation doesn't create a new attracting cycle then and iii) holds since we haven't changed the basins; so we let h be the identity on $B_0(f)$. Therefore the maps *are* topologically conjugate.

It remains to show that we don't create an attracting cycle. We apply the following result, Lemma 4.6 of [**K3**]

Lemma 2.2.18. *Let $f \in NFH^3$. Then there is a C^3 neighbourhood of f which is contained in NFH^3.*

This lemma relies on Theorem A of [**K2**] which says that for such maps there must exist some neighbourhood Z of the critical value such that if $f^n(x) \in Z$ for some $x \in I$ then $Sf^n(x) < 0$ where $Sf(x)$ denotes the Schwarzian derivative of f at x. This is then coupled with a well known theorem of [**Sin**] that parabolic cycles near the critical point attract the critical point when the Schwarzian is negative.

This means that when we are in the C^3 case, our perturbation cannot create a parabolic cycle and thence an attracting cycle. So Lemma 2.1.5 is proved.

□

2.3 Non-Recurrent Case

We want to prove that the post-critical set is structurally sensitive. Here we deal with the Misiurewicz case, that is, when c is not preperiodic and there is a neighbourhood W around c such that $f^n(c) \notin W$ for any $n > 0$. We must restrict ourselves to the C^2 case since we can't prove our result in the case that c is in a wandering interval. Theorem 2.2.13 means than we can be sure that in the C^2 case we don't have any wandering intervals.

Lemma 2.3.1. *Let $f \in NFH^2$ be a map for which c is non-recurrent and let $\epsilon > 0$. Then for any $n \geq 0$ there exists a C^k perturbation in an ϵ-small neighbourhood of $f^n(c)$ which is ϵ-small in the C^k topology which maps $f^n(c)$ onto a preperiodic point. Here $k = 0, 1, \ldots$.*

Proof: Given $x_0 \in \overline{PC(f)}$ we take some small neighbourhood U_{x_0} around x_0. Clearly there will be some $n \geq 0$ such that $f^n(c) \in U_{x_0}$. We assume that $n > 0$ as the case when $x_0 = c$ follows similarly and more simply. We will make our perturbation in small interval $\hat{U} \subset U_{x_0}$ around $f^n(c)$.

By Lemma 2.2.10 we can find preperiodic points arbitrarily close to c, so we can find preperiodic points arbitrarily close to $f^n(c)$. Let U be a small nice interval around c with preperiodic boundary points as in Lemma 2.2.10. Also, let $\hat{U}$ be a small nice interval around x_0, disjoint from U, with preperiodic boundary points such that both U and the intervals of B_0 are nice with respect to $\hat{U}$ and such that $f^j(c) \cap \hat{U} = \emptyset$ for $j < n$. Let $V \ni f^n(c)$ be a small interval deep inside $\hat{U}$. We have two cases.

Case 1: Suppose that there is some $m > 0$ such that $f^m(V) \cap (U \cup B_0) \neq \emptyset$ and $f^j(V) \cap \hat{U} = \emptyset$ for $0 < j < m$. We know that $f^m(f^n(c))$ cannot be contained in U or B_0. Thus $f^m(V)$ is not strictly contained in either U or B_0 and must intersect some boundary point of $U \cup B_0$. So by the construction of U and the above claim, $f^m(V)$ contains a preperiodic point. We let p be the corresponding preperiodic point in V. Observe that p never returns to $\hat{U}$ under iteration by f. Then we show below that there is a perturbation which perturbs $f^n(c)$ to p.

Case 2: If we are not in case 1 then we must return to $\hat{U}$ before entering $U \cup B_0$. We use the following well known result of Mañé , see for example Theorem III.2.1 of [**MS**].

Theorem 2.3.2. *Let $f : I \to I$ be a C^2 map and U be a neighbourhood of the set of critical points of f. Then*

1. *all periodic points of f contained in $I \setminus U$ and of sufficiently large period are hyperbolic and repelling;*
2. *if all periodic orbits of f which are contained in $I \setminus U$ are hyperbolic then there exists $C = C(U)$ and $\lambda = \lambda(U) > 1$ such that*
$$|Df^n(x)| > C\lambda^n$$
whenever $f^i(x) \in I \setminus (U \cup B_0(f))$ for all $0 \leq i \leq n-1$.

Therefore, $|Df^m|_V > C\lambda^m$ where $C = C(U)$ and $\lambda = \lambda(U)$. Whence V expands under iteration if it doesn't meet $U \cup B_0$ and so we must either be in case 1 or we must return to $\hat{U}$. So again we have the preperiodic point p (a point which maps to $\partial\hat{U}$) as required.

We let $\hat{U} = (a^-, a^+)$ and let
$$p_{\lambda,k}(x) := \lambda \frac{(x-a^-)^{k+1}(a^+ - x)^{k+1}}{(a^+ - a^-)^{2k+2}} \tag{2.1}$$

We choose λ such that $p_{\lambda,k}(f^n(c)) = p$. Observe that $D^j p_{\lambda,k}(a^\pm) = 0$ for $0 \leq j \leq k$. When we fix $\hat{U}$ this perturbation has $\|p_{\lambda,k}\|_{C^k}$ governed by the size of $|f^n(c) - p|$. But since we can make this distance arbitrarily small, the lemma is proved.

□

We will use this perturbation below also.

2.4 C^1 structurally sensitive points

In this section we prove part 2 of Theorem 2.1.4: the case where we are only looking for C^1 structurally sensitive points, which should be the simplest case. Note that such a perturbation at c is done in Exercise III.2.4 of [**MS**]. However, that result relies heavily on the derivative of f being small in the support of the perturbation, so it doesn't generalise to our case.

2.4.1 Direct C^1 perturbation for non-minimal C^2 maps

We will apply an explicit perturbation which is C^1 small. Not only will this process prove part 2 of Theorem 2.1.4, but it will also provide us with a family of maps which is non-trivial in the sense that the combinatorics change through the family. This is not so important to notice for this part of the proof of Theorem 2.1.4, but we will use this family in later parts of the proof. For an overall idea what's happening here, see Figure 2.1.

The idea here is that the 'decay of geometry' gives us a large space between some domain of a first return map and the boundary of a range of the first return map. The C^1 perturbation can then be supported on that range. This first return map won't be the usual type of map to one domain, but will be a return map to two domains. The following definition makes the concept of space in nested intervals rigorous.

Definition 2.4.1. Suppose that $J \subset T$ are two intervals. Denote the left-hand and right-hand elements of $T \setminus J$ by L and R respectively. We say that T is a *δ-scaled neighbourhood of J* if $\frac{|L|}{|J|}, \frac{|R|}{|J|} > \delta$.

We will assume that c is recurrent. Observe that since f is non-Axiom A and recurrent, the itinerary and the kneading sequence for f coincide. Then by the Remark 2.2.2 we know that the kneading sequence must not be preperiodic. We will perturb f to some map which does have a periodic kneading sequence.

Given a nice interval U around c, we will choose some nice interval $\hat{U}$ such that U and $\hat{U} \subset U_{x_0}$ are *nice with respect to each other*. That is to say, $f^n(\partial U) \cap \hat{U} = \emptyset$ for all $n \geq 0$ and $f^n(\partial \hat{U}) \cap U = \emptyset$ for all $n \geq 0$. In fact, we will also require that the boundary points of these intervals are preperiodic points, as in Lemma 2.2.10. Then we will consider the first return map $R_{U \cup \hat{U}} : \left(\bigcup_j U^j\right) \cup \left(\bigcup_j \hat{U}^j\right) \to U \cup \hat{U}$

where U^j are the domains in U and $\hat{U}^j$ are in $\hat{U}$. As usual, we denote the central domain by U^0. Then all maps $R_{U\cup\hat{U}} : U^j \to U \cup \hat{U}$ for $j \neq 0$ and $R_{U\cup\hat{U}} : \hat{U}^j \to U \cup \hat{U}$ are diffeomorphisms.

We can show that $U^j \subset U$ and $\hat{U}^j \subset \hat{U}$ as follows. Suppose that $U = (a, a')$ and $\hat{U} = (b, b')$. We have shown above that any U^j for which $R_{U\cup\hat{U}}(U^j) = U$ will be contained in U. Suppose instead that $R_{U\cup\hat{U}}(U^j) = \hat{U}$ and $U^j \cap \partial U \neq \emptyset$. Assume that a is in this intersection. Then, $R_{U\cup\hat{U}}(a) \in \hat{U}$. But this is a contradiction by the niceness of these intervals with respect to each other. The case when $R_{U\cup\hat{U}}(\hat{U}^j) = U$ for some j follows similarly.

The following can be proved as in Theorem A' in [**LS1**].

Theorem 2.4.2. *Suppose that $f \in NFH^2$ has a recurrent critical point c such that $\omega(c)$ is non-minimal. Then there exists a sequence $\delta_i > 0$ and a sequence of nested nice intervals $I_i \supset I_{i+1} \supset \cdots$ containing the critical point such that*

1. *I_{i+1} is the central domain I_i^0 of the first return map $F_i : \bigcup_j I_i^j \to I_i$;*
2. *$|I_i| \to 0$ as $i \to \infty$;*
3. *the elements of ∂I_i eventually map onto repelling periodic points;*
4. *I_{i-1} is a δ_i-scaled neighbourhood of I_i;*
5. *there exists a subsequence $i_m \to \infty$ as m increases such that $\delta_{i_m} \to \infty$.*

The proof relies principally on finding a partition of I by repelling periodic points and using distortion estimates. In fact, it can be shown that for all branches of F_{i_m} except two diffeomorphic branches, which we denote by I_i^L and I_i^R, and the central branch, the range of $F_{i_m}|_{I_{i_m}^j} : I_{i_m}^j \to I_{i_m}$ extends to I_{i_m-1}. See Section 1.6 for details of such maps. In the following we generally refer to F_i and only mention F_{i_m} when necessary.

For interest, if $f \in C^3$ we could use the following result: Theorem 1 of [**GSS1**] which applies to the both minimal and non-minimal cases.

Theorem 2.4.3. *Suppose that $f \in NF^3$ has a recurrent critical point and is not infinitely renormalisable. Then for every $\delta > 0$ there exists some arbitrarily small nice interval T containing c such that for the central domain U of the first return map to T we know that T is a δ-scaled neighbourhood of U.*

We will only focus on the non-minimal case for the moment. The proof for the minimal case follows in Section 2.6. This means that we only need f to be C^2.

Consider $\bigcup_{n\geq 0} f^n(\partial I_i)$ for some I_i from Theorem 2.4.2. Let $\hat{U}_i \subset U_{x_0}$ containing some $f^n(c)$ for $c > 0$ have boundary points consisting of two adjacent members of $\bigcup_{n\geq 0} f^n(\partial I_i)$. If I_i is sufficiently small then $f^n(I_i)$ is very small and so $\hat{U}_i \subset U_{x_0}$ must be very small too. Then, as before, $\hat{U}_i$ is nice and, furthermore, I_i and $\hat{U}_i$ are nice with respect to each other. Again we consider the first return map $R_{I_i \cup \hat{U}_i} : \left(\bigcup_j U^j\right) \cup \left(\bigcup_j \hat{U}_i^j\right) \to I_i \cup \hat{U}_i$. We wish to find a domain $\hat{U}_i^j$ containing an iterate of c which is deep inside $\hat{U}_i$ which allows us to change the combinatorial type of the map by supporting a perturbation on $\hat{U}_i$. We will use the following theorem, presented in more generality in [**SV**] as Proposition 2: 'a Koebe principle requiring less disjointness'.

Theorem 2.4.4. *Suppose that $g \in NF^2$. Then there exists a function $\nu : [0, |I|] \to [0, \infty)$ such that $\nu(\epsilon) \to 0$ as $\epsilon \to 0$ with the following properties. Suppose that for some intervals $J \subset T$ and a positive integer n we know that $g^n|_T$ is a diffeomorphism. Suppose further that $g^n(T)$ is a δ-scaled neighbourhood of $g^n(J)$ for some $\delta > 0$. Then,*

- *for every $x, y \in J$,*
$$\frac{|Dg^n(x)|}{|Dg^n(y)|} \leq \exp\left\{\nu(S(n,T)) \sum_{i=0}^{n-1} |g^i(J)|\right\} \left[\frac{1+\delta}{\delta}\right]^2;$$
- *T is a δ' scaled neighbourhood of J where*
$$\delta' = \frac{1}{2}\exp\{-\theta\}\left[\frac{1+\delta}{\delta}\right]^2 \left(\frac{-2\theta + \delta(1-2\theta)}{2+\delta}\right),$$
where we let $\theta := \nu(S(n,T)) \sum_{i=0}^{n-1} |g^i(J)|$.

Let $n > 0$ be minimal such that $f^n(c) \in \hat{U}_i$. Denote the domain of $R_{I_i \cup \hat{U}_i}$ containing $f^n(c)$ by $\hat{U}_i^0$. We have two cases.

Case 1: $R_{I_i \cup \hat{U}_i}(\hat{U}_i^0) \in I_i$. Suppose that $g : I \to \mathbb{R}$ is C^1 and that $g|_V : V \to T$ is a diffeomorphism onto the interval T. If there is some larger interval $V' \supsetneq V$ such that $g|_{V'} : V' \to g(V')$ is a diffeomorphism then we say that the range of the map $g|_V : V \to g(V)$ can be *extended* to $g(V')$. We use the following well known result, see for example Lemma 1.1 of [**K3**].

Lemma 2.4.5. *Let f be unimodal map, T a nice interval around the critical point and U be the central domain of the first return map to T. If V is a domain of the first entry map R_U to U and V is disjoint from U then the range of $R_U : V \to U$ can be extended to T.*

Suppose that $R_{I_i \cup \hat{U}_i}|_{U^0} = f^k|_{U^0}$. Then by this lemma, there is an extension $V^0 \supset \hat{U}_i^0$ such that $f^k : V^0 \to I_{i-1}$ is a diffeomorphism. We can show that $V^0 \subset \hat{U}_i$ as follows. If $V^0 \cap \partial\hat{U}_i \neq \emptyset$ then there exists a point of $\bigcup_{n\geq 0} f^n(\partial I_i)$ which maps inside $I_{i-1} \setminus \overline{I_i}$. But the first time that I_i maps back into I_{i-1}, its boundary points map to ∂I_{i-1}. So by the niceness of I_{i-1} we are finished.

Since I_{i-1} is a δ_i-scaled neighbourhood of I_i we have from Theorem 2.4.4 that $\hat{U}_i$ is a δ_i'-scaled neighbourhood of $\hat{U}_i^0$.

For simplicity, we will support our perturbation, $p_{\lambda,1}$ on $V = (a^-, a^+)$ where $V \subset \hat{U}_i$ is defined as follows. Its boundary meets one boundary point of $\hat{U}_i$ and, defining the left-hand and right-hand components of $V \setminus \hat{U}_i^0$ as L and R respectively, we have $|L| = |R|$. Note that since $\hat{U}_i$ is a δ_i'-scaled neighbourhood of $\hat{U}_i^0$ then V is a δ_i'-scaled neighbourhood of $\hat{U}_i^0$ too. This process extends so that we can let $V = \hat{U}_i$ if necessary. This extension will be useful in proving later parts of the proof of Theorem 2.1.4, but for the moment the V that we have defined above is sufficient.

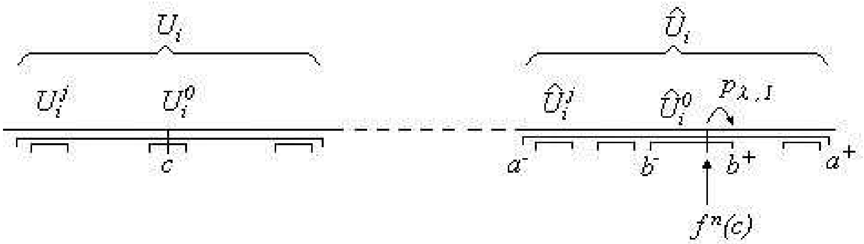

Figure 2.1: How $p_{\lambda,1}$ acts on $\hat{U}_i$ in case 1.

Letting $\hat{U}_i^0 = (b^-, b^+)$, we require that $p_{\lambda,1}(c_n) = b^+$. See Figure 2.1 Thus $\lambda = \frac{(b^+-c_n)(a^+-a^-)^4}{(c_n-a^-)^2(a^+-x)^2}$. To bound $\|p_{\lambda,1}\|_{C^1}$, we must first bound $|\lambda|$. We have

$$|\lambda| < \frac{(b^+ - b^-)(a^+ - a^-)^4}{(b^- - a^-)^2(a^+ - b^+)^2}.$$

Note that by Theorems 2.4.2 and 2.4.4 we have $\frac{(b^+-b^-)}{(b^--a^-)} < \frac{1}{\delta_i'}$. So

$$\frac{(a^+-a^-)}{(b^--a^-)} = \frac{(a^--b^+)+(b^+-b^-)+(b^--a^-)}{(b^--a^-)} < 2+\frac{1}{\delta_i'}$$

and similarly $\frac{(a^+-a^-)}{(b^--a^-)} < 2+\frac{1}{\delta_i'}$. Thus

$$|\lambda| < (b^+-b^-)\left(2+\frac{1}{\delta_i'}\right)^4.$$

We may suppose that $\left(2+\frac{1}{\delta_i'}\right)^4$ is bounded above by a constant $C_1 > 16$. When $i = i_m$ for large m, this is close 16. Then for $x \in (a^-, a^+)$,

$$|p_{\lambda,1}(x)| = \frac{|\lambda||x-a^-|^2|a^+-x|^2}{|a^+-a^-|^4} < \frac{(b^+-b^-)C_1}{16}$$

and

$$\begin{aligned} |Dp_{\lambda,1}(x)| &= \frac{2|\lambda||(x-a^-)(a^+-x)^2-(x-a^-)^2(a^+-x)|}{|a^+-a^-|^4} \\ &< \frac{2C_1|b^+-b^-|}{|a^+-a^-|} < \frac{2C_1}{1+2\delta_i'}. \end{aligned}$$

So by choosing a large m, we obtain a large δ_{i_m} and we may make $\|p_{\lambda,1}\|_{C^1}$ as small as we wish.

Since $f^n(c)$ is the first time that an iterate of c enters $\hat{U}_i$, once we perturb $f^n(c)$ to b^+, which in turn maps to ∂I_i, this image will never be perturbed again and c becomes preperiodic as required.

Case 2: $R_{I_i\cup\hat{U}_i}(\hat{U}_i^0) \in \hat{U}_i$. Then $\hat{U}_i^0$ and I_i are nice with respect to each other and so we may instead consider the first return map to $I_i \cup \hat{U}_i^0$. If $f^n(c)$ is in some domain which maps by $R_{I_i\cup\hat{U}_i^0}$ to I_i then we proceed as in case 1. Otherwise, we take a new first return map again. This procedure can be repeated many times, but since c is recurrent we must eventually find ourselves in case 1 after a finite number of applications of this process.

So we have proved part 2 of Theorem 2.1.4.

Observe that we can use these ideas to create a C^k family of maps $f_\lambda = f + p_{\lambda,k}$. We can support $p_{\lambda,k}$ on $\hat{U}_i$ in case 1 and on the corresponding domain in case 2.

2.4.2 General method shown through C^1

This section is concerned with exploring a possible route to proving Conjecture 2.1.6. The fundamental problem in proving Theorem 2.1.4 is that $\|p_{\lambda,k}\|_{C^2}$ defined above is generally very large. One way around this is to use the methods of [**K3**]. There, rigidity results relating to analytic maps are employed. So we would like to perturb our first return maps so that they have some extension to $\mathbb{C}$. As proved in Lemma 2.4.7, we can sometimes find an extension of the first return map into $\mathbb{C}$. In fact we will find the following type of map. See Figure 2.2.

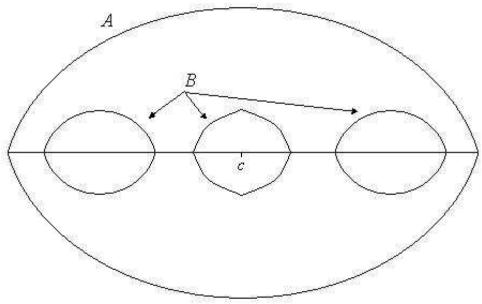

Figure 2.2: A holomorphic box mapping (with three domains)

Definition 2.4.6. Let $A \subset \mathbb{C}$ be a simply connected Jordan Domain, $B \subset A$ be a domain such that each connected component is a simply connected Jordan domain, and let $G : B \to A$ be a holomorphic map. Then we call G a *holomorphic box mapping* if the following conditions are satisfied:

- G maps the boundary of a connected component of B onto the boundary of A;
- there is one component of B (which we will call *a central domain*) which is mapped in a 2-to-1 fashion onto A;
- all other components of B are mapped univalently onto A by the map G;
- the iterates of the critical point of G never leave the domain B.

A holomorphic box mapping G is *induced* by the map g if, on each of the domains of B, the mapping is some iterate g^n.

If the domains B and A are symmetric with respect to the real line, the holomorphic box mapping is called *real*.

Note that a *Jordan domain* is a subset of $\mathbb{C}$ which is homeomorphic to the open unit disk. For an interval $J \subset \mathbb{R}$, we let $\rho_J(\cdot,\cdot)$ denote the Poincaré metric on the domain $\mathbb{C}_J := \mathbb{C} \setminus (\mathbb{R} \setminus J)$. It can be shown, for example in Section VI.5 of [**MS**], that the set $D_k(J) := \{z \in \mathbb{C} : \rho_J(z,J) < k\}$ is the intersection of two Euclidean disks, symmetric to each other with respect to the real axis. (Furthermore, it can be shown that the exterior of D_k intersects the right-hand boundary of J in the upper half plane with an angle of θ such that $k = \log\tan\left(\frac{\pi}{2} - \frac{\theta}{4}\right)$. Therefore, $D_{\log\tan\left(\frac{3\pi}{8}\right)}(J)$ is the disk symmetric with respect to the real line intersecting $\mathbb{R}$ at J.) We refer to such sets as *lenses*.

Our holomorphic box maps will have such lenses for their range. The following lemma is the main result in this subsection.

Lemma 2.4.7. *Let $f \in QH^2$ be non-minimal and let $\epsilon > 0$. Then there exists an ϵ-small C^1 perturbation of f in an ϵ-small neighbourhood of either c or $f(c)$ such that for our perturbed map there is an induced holomorphic box map to some complex neighbourhood of the critical point.*

Proof: We will start with c. The case for $f(c)$ follows similarly. This proof also helps to illustrate why making local perturbations in the C^k case is problematic.

The idea is that we will make a perturbation on each branch of a first return map to make the first return map into a piecewise linear map. Then it might be possible to apply the techniques of [**K3**].

We take some I_{i-1}, I_i from Theorem 2.4.2 such that i is some i_m. There exists some $K_i = K_i(\delta_i) > 1$ such that for x, y in some branch of the first return map $F_i : \bigcup_j I_i^j \to I_i$, we have $\frac{1}{K_i} \le \frac{DF(x)}{DF(y)} \le K_i$. By Theorem 2.4.4 as m increases, $K_{i_m} \to 1$. We have

$$|DF_i(x) - DF_i(y)| \le |DF_i(y)|(K_i - 1). \tag{2.2}$$

This helps us estimate f by some map $\tilde{f}$ which induces a piecewise linear first return map on most branches as follows.

We deal with the branches $F_i|_{I_i^L}, F_i|_{I_i^0}, F_i|_{I_i^R}$ later. For some $j \notin \{L, 0, R\}$, let $L_j(x) = \frac{|I_i|}{|I_i^j|}x$. We ignore translation and show that this helps us approximate f in a useful way. This is our first approximation for $F_i|_{I_i^j}$. Let $n(j)$ be the integer such that $F_i|_{I_i^j} = f^{n(j)}|_{I_i^j}$. We could approximate f on I_i^j by $f_j = f^{-(n(j)-1)}L_j$ which would give us a first return map to I_i which is linear on I_i^j. However, if we were to approximate in a similar way on all branches, we would obtain a map which is not globally C^1. Therefore, we must add on some perturbation q_j as explained below.

From here until Lemma 2.4.9, when we refer to F_i, we mean the branch $F_i : I_i^j \to I_i$ for fixed $j \neq 0$. Supposing that $I_i^j = (0, |I_i^j|)$ and $I_i = (0, |I_i|)$ (so we can 'ignore translation'), we let $d_0 = DF_i(0) - \frac{|I_i|}{|I_i^j|}$ and $d_1 = DF_i(|I_i^j|) - \frac{|I_i|}{|I_i^j|}$. Observe that by (2.2), $|d_0|, |d_1| < (K_i - 1)\frac{|I_i|}{|I_i^j|}$. We let

$$q_j(x) := d_0 x - \left(\frac{2d_0 + d_1}{\alpha}\right) x^2 + \left(\frac{d_0 + d_1}{\alpha^2}\right) x^3.$$

Then $q_j(0), q_j(|I_i^j|) = 0$ and $Dq_j(0) = d_0, Dq_j(|I_i^j|) = d_1$. We let $\phi_j := L_j + q_j$. Then $\phi_j(0) = 0,\ \phi_j(|I_i^j|) = |I_i|$ and $D\phi_j(0) = DF_i(0),\ D\phi_j(|I_i^j|) = DF_i(|I_i^j|)$.

We must show that $\|\tilde{f} - f\|_{C^1}$ is small and that the first return map has an extension into the complex plane which is a holomorphic box mapping.

Lemma 2.4.8. *For any $\epsilon > 0$, we can adjust the above situation so that*

$$\|(fF_i^{-1}\phi_j - f)|_{I_i^j}\|_{C^1} < \epsilon.$$

Proof: For $x \in I_i^j$,

$$|fF_i^{-1}\phi_j(x) - f(x)| < |Df|_\infty |F_i^{-1}\phi_j(x) - x| < \frac{|Df|_\infty}{|DF_i|_{-\infty}} |\phi_j(x) - F_i(x)|$$

where $|\cdot|_\infty$ and $|\cdot|_{-\infty}$ denote the maximum and minimum respectively of a function (in its domain of definition).

Now, $|\phi_j(x) - F_i(x)| < \left|\frac{|I_i|}{|I_i^j|}x - F_i(x)\right| + |q_j(x)|$. For the first term, since for every $x \in I_i^j$, there exists some $\theta_x \in I_i^j$ such that $\frac{|I_i|}{|I_i^j|}\theta_x = F_i(x)$, we have $\left|\frac{|I_i|}{|I_i^j|}x - F_i(x)\right| < \frac{|I_i|}{|I_i^j|}|x - \theta_x| < |I_i|$.

For the second term,

$$|q_j(x)| < |d_0 x| + \left|\left(\frac{2d_0 + d_1}{\alpha}\right) x^2\right| + \left|\left(\frac{d_0 + d_1}{\alpha^2}\right) x^3\right| < 6(K_i - 1)|I_i|.$$

Therefore,

$$|fF_i^{-1}\phi_j(x) - f(x)| < |Df|_\infty \frac{7K_i|I_i|}{|DF_i|_{-\infty}} < |Df|_\infty 7K_i|I_i^j|.$$

Now

$$\begin{aligned}|D(fF_i^{-1}\phi_j)(x) - Df(x)| &= |Df(F_i^{-1}\phi_j(x))DF_i^{-1}(\phi_j(x))D\phi_j(x) - Df(x)| \\ &= |Df(x)[DF_i^{-1}(\phi_j(x))D\phi_j(x)] - Df(x)| \\ &\quad + |(x - F_i^{-1}\phi_j(x))D^2f(\theta)[DF_i^{-1}(\phi_j(x))D\phi_j(x)]|\end{aligned}$$

for some $\theta \in I_i^j$. For the term in square brackets we have $D\phi_j(x) = \frac{|I_i|}{|I_i^j|} + Dq_j(x)$. But

$$|Dq_j(x)| < |d_0| + \left|\frac{2}{|I_i^j|}(2d_0 + d_1)x\right| + \left|3\left(\frac{d_0 + d_1}{|I_i^j|^2}\right)x^2\right| < 13(K_i - 1)\frac{|I_i|}{|I_i^j|}.$$

Therefore, $|DF_i^{-1}(\phi_j(x))D\phi_j(x)| < K_i + 13K_i(K_i - 1)$ and

$$|Df(x)[DF_i^{-1}(\phi_j(x))D\phi_j(x)] - Df(x)| < 14K_i(K_i - 1).$$

Furthermore,

$$\begin{aligned}&|(x - F_i^{-1}\phi_j(x))D^2f(\theta)[DF_i^{-1}(\phi_j(x))D\phi_j(x)]| \\ &\qquad < 6K_i(K_i - 1)|I_i^j||D^2f|_\infty(K_i + 13K_i(K_i - 1)).\end{aligned}$$

So we have $\|(fF_i^{-1}\phi_j - f)|_{I_i^j}\|_{C^1} < \epsilon$

□

Lemma 2.4.9. *There exists some $\phi > 0$ such that if we choose i_m correctly then for each $j \notin \{L, 0, R\}$ there exists some domain A^j in $\mathbb{C}$ such that $A^j \cap I_{i_m}^j = I_{i_m}^j$ and $A^j \subset D_\phi(I_{i_m})$ has $\phi_j(A^j) = D_\phi(I_{i_m})$.*

Proof: To ease notation, we assume that $i = i_m$. We consider $D\phi_j(x + iy)$ for $x \in I_i^j$.

$$|D\phi_j(x + iy) - D\phi_j(x)| = \left|\frac{-2}{|I_i^j|}(2d_0 + d_1)(iy) + 3\left(\frac{d_0 + d_1}{|I_i^j|}\right)(2xiy - y^2)\right|.$$

If we suppose that $|y| < |I_i^j|$ then

$$|D\phi_j(x + iy) - D\phi_j(x)| < 24(K_i - 1)\frac{|I_i|}{|I_i^j|}.$$

Therefore,
$$|D\phi_j(x+iy)| > \frac{|I_i|}{|I_i^j|}\left(\frac{1}{K_i} - 24(K_i - 1)\right),$$
and so for K_i close to 1, this is strictly positive and ϕ_j is univalent in $D(j) := \{x + iy : x \in I_i^j, |y| < |I_i^j|\}$. We can then show that this has a uniformly large range in $\mathbb{C}$ too (in terms of $|I_i|$).

Recalling that i is some i_m, note that when we increase m we will have $K_{i_m} \to 1$ and so we have everything as small as we wish.

□

We may approximate each branch I_i^j for $j \notin \{L, 0, R\}$ in this way. We approximate on I_i^L, I_i^0, I_i^R by polynomials. For details on how we obtain a holomorphic box mapping from this see Section 2.5. So we obtain a map $\tilde{f}$ such that there is a first return map $\tilde{F}_i$ which extends to the complex plane to give us a holomorphic box mapping as required. We may shift this entire construction to $f(c)$ too.

□

The problem with extending this construction to $f^n(c)$ for $n > 1$ is that some domains return to $f^n(I_i)$ and so might be perturbed arbitrarily many times by this method (this will occur in the non-renormalisable case). So we are able to prove Lemma 2.4.8 by this method for finitely many domains, but not for infinitely many domains (which we have to deal with when f is non-minimal). Hence the resulting map $\tilde{f}$ would not be arbitrarily close to f.

Observe also that we cannot use this process as a step in showing that c and $f(c)$ are C^2 structurally sensitive because the perturbation from $F_i|_{I_i^j}$ to ϕ_j is large in the C^2 sense.

To prove Conjecture 2.1.6, we would hope now to apply the methods of [**K2**] directly to the holomorphic box map which we have constructed. However, that method currently requires such a map to have a finite number of domains (for example the 'polynomial-like map' given in the following section). This would allow us to use quasi-conformal deformation techniques to construct holomorphic families of such maps. It is possible that extending such ideas to maps with an infinite number of ranges might be the next step in proving structural sensitivity.

2.5 Analytic case

Definition 2.5.1. We say that $f \in C^{\omega}(\Delta)$ if f is defined on the real interval I and can be holomorphically extended to a $\Delta-$neighbourhood of I in the complex plane.

We shall simply refer to this case as the analytic case.

We will deal with the minimal case for more general maps later, so we restrict ourselves to maps which are non-minimal. We will next obtain the following type of map, similar to a holomorphic box map. The difference is that the range of the map is that the range is not necessarily connected (see Figure 2.3 below).

Definition 2.5.2. A holomorphic map $G : B \to A$ is called *polynomial-like* if the following conditions are satisfied:

- B and A are subsets of the complex plane, they each have finitely many connected components and each connected component is a simply connected Jordan domain. B is a subset of A and the intersection of the boundaries of A and B is either empty or consists of a forward invariant set of finitely many points;
- G maps the boundary of a connected component of B onto the boundary of some connected component of A;
- there is one component $B(c)$ of B (which we will call *a central domain*) which is mapped in a 2-to-1 fashion onto A and $B(c)$ is relatively compact in A (i.e. $\overline{B(c)} \subset A$). Here $B(x)$ and $A(x)$ denote connected components of B and A which contain x;
- all other components of B are mapped univalently onto A by the map G;
- the iterates of the critical point of G never leave the domain B.

A polynomial-like map G is *induced* by the map g if, on each of the domains of B, the mapping is some iterate g^n.

If the domains B and A are symmetric with respect to the real line, the map is called *real*.

The following result is Theorem 3.1 of [**K3**].

Theorem 2.5.3. *Let $f \in QH^\omega$ be an analytic unimodal not infinitely renormalisable map. Then for any $\epsilon > 0$ there exists a polynomial-like map $F : B \to A$ induced by the map f, and satisfying the following properties:*

- *the forward orbit of the critical point under iterations of F is contained in B;*
- *A is a union of finitely many lenses of the form $D_k(J)$ where J is an interval on the real line, $|J| < \epsilon$ and $0 < k < \log\tan\left(\frac{3\pi}{8}\right)$;*
- *if $F(x) \in A(c)$, then $B(x)$ is compactly contained in $A(x)$ (i.e. $\overline{B(x)} \subset A(x)$);*
- *if $a \in \partial A \cap \partial B$, then the boundaries of A and B at a are smooth; however if we consider a smooth piece of the boundary of A containing a and the corresponding smooth of piece of the boundary of B, then these pieces have the second order of tangency;*
- *if $B(x) \cap B(x') = \emptyset$ and $b \in \partial(B(x) \cap B(x'))$, then the boundaries of $B(x)$ and $B(x')$are not smooth at the point b and not tangent to each other;*
- *for any $x \in B$ we have*
$$\frac{|B(x)|}{|A(x)|} < \epsilon,$$
where $|B(x)|$ denotes the length of the real trace of $B(x)$;
- *if $x \in B$ and $F|_{B(x)} = f^n$, then $f^i(x) \notin A(c)$ for $i = 1, \dots, n-1$;*
- *$f(c) \notin A$;*
- *let $a \in \partial A$ be a point closest to the critical value $f(c)$, then*
$$\frac{|f(B(c))|}{|a - f(c)|} < \epsilon.$$

So we obtain the kind of map shown in Figure 2.3.

Note that we want to create a perturbation which yields a map in Q^k. Clearly, if we apply a small local perturbation in $\overline{PC(f)} \setminus \{c\}$ then we remain in Q^k.

To show that maps are structurally sensitive at c we will perturb as follows. Since we are assuming here that c is recurrent, for any small neighbourhood U of c there

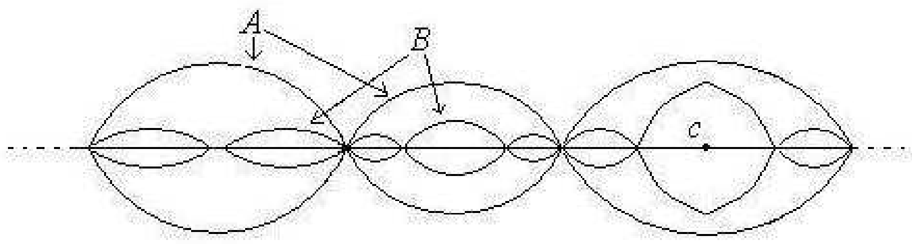

Figure 2.3: A fragment of our polynomial-like map.

is some $n > 0$ such that $f^n(c) \in U$. We may apply our perturbation at $f^n(c)$ rather than at c, and so we remain in Q^k.

The key difference between applying the methods of Theorem 2.1.7 to the above polynomial-like mapping in [**K3**] and our case here is that we create our family in a different way. In the construction of the polynomial-like map, partitions are defined as follows. Firstly, a partition $P_0 \subset I$ of periodic points, some of which are boundary points of a nice interval T_0 around c, is defined. Then this partition is pulled back m times to give a partition P_m which defines the elements of $\partial A \cap \mathbb{R}$ in the theorem. Importantly, intervals defined by this partition are small and the partition points all eventually map to repelling periodic points.

In [**K3**] the perturbation to create the family f_λ is at $f(c)$: the partition P_0 is stable under this perturbation. That is, there is a conjugacy between f and f_λ on P_0 and the perturbed partition. This ensures that when the map G_λ is created, it is a polynomial-like map. In our case we wish find some partition P_0' which is stable under local perturbation at $f^n(c)$ for some $n \geq 0$.

We adapt the original partition as follows. We include the elements of $\partial\hat{U} \cup \partial U$, which are defined in Section 2.4, in the partition P_0'. ∂U will behave as T_0 does in the original theorem. Since our added points eventually map to repelling periodic points, the adapted partition P_0' gives us the same type of map as in Theorem 2.5.3. Next we show that we can extend our perturbation $p_{\lambda,k}$ to act upon our polynomial-like mapping.

Letting $\hat{U} = (a^-, a^+)$, we extend $p_{\lambda,k}$ to the complex plane:

$$p_{\lambda,k}(z) = \begin{cases} \lambda \frac{(z-a^-)^{k+1}(a^+-z)^{k+1}}{(a^+-a^-)^{2k+2}} & \text{if } \Re(z) \in [a^-, a^+] \\ 0 & \text{if } \Re(z) \notin [a^-, a^+] \end{cases}$$

(Where $\Re(z)$ is the real part of z.)

Therefore, for a given $k \geq 1$, the map $f_\lambda = f + p_{\lambda,k}$ extends holomorphically to $\Delta \setminus \{z \in \Delta : \Re(z) \in \{a^-, a^+\}\}$. Observe that our partition P_0' is stable under this perturbation for small λ. Whence we find a non-degenerate family of polynomial-like maps F_λ which depend holomorphically on λ; so we can apply the methods of Theorem 2.1.7.

2.6 The minimal case

We will first show that we can use the fact that minimal maps give us first return maps with finitely many domains to construct a holomorphic extension to some domain in $\mathbb{C}$. Then we will use this to construct a holomorphic box mapping. This construction will initially take place around c, but we will show that we can push it forward to any $f^n(c)$ for $n \geq 0$. Finally we will make some perturbation of this box mapping to give us a suitable family of maps.

We will use the following well known claim which provides an equivalent definition of minimality. We provide a proof for completeness.

Claim 2.6.1. *The minimality of $\omega(x)$ is equivalent to all $y \in \omega(x)$ having $\omega(y) = \omega(x)$.*

Proof: Let $y \in \omega(x)$. We can prove that $\omega(y)$ is f-invariant: given $z \in \omega(y)$ there must be some sequence n_k such that $f^{n_k}(y) \to z$ as $n_k \to \infty$. Then $f(z) = f(\lim_{k\to\infty} f^{n_k}(y)) = \lim_{k\to\infty} f(f^{n_k}(y))$ by continuity. This is clearly in $\omega(y)$ so $\omega(y)$ is f-invariant. Since $\omega(x)$ is minimal, any f-invariant subset must be equal to $\omega(x)$; whence $\omega(y) = \omega(x)$.

□

We next state and prove another well known result.

Lemma 2.6.2. *Suppose that $\omega(c)$ is minimal and T is some nice interval around c. Then the first return map to T has finitely many domains intersecting $\omega(c)$.*

Proof: Given a nice interval T, let $R_T : \bigcup_j U^j \to T$ be the first return map to T. Note that $\bigcup_j U^j$ cover all elements of $\omega(c) \cap T$. This is because every $x \in \omega(c)$ has $\omega(x) = \omega(c)$ and so all $x \in \omega(c) \cap T$ must return to T. Therefore the first return domains form an open cover of $\omega(c) \cap T$. Since $\omega(c) \cap T$ is closed, it is compact, so there is a finite subcover. Because each domain is disjoint, this subcover is precisely the set $\bigcup_j U^j$ of domains of the first return map.

□

We obtain the first return map $R_T : \bigcup_{j=0}^N U^j \to T$ as above. Given $j \geq 0$, we approximate $R_T|_{U^j} : U^j \to T$ by a polynomial p_j. We must choose p_j to be close to F on U^j and have its derivatives match up with those for R_T on ∂U^j. We can do this using, for example, the Bernstein polynomials of Section 6 of [**P**]. Furthermore, we can assume that we are looking for C^k-structurally sensitive points (as we have already dealt with the C^1 case) and so we can ensure that c is non-flat for p_0.

For each $j \neq 0$ there exists some constant $\kappa_j > 0$ such that there is some domain $\hat{A}^j \subset \mathbb{C}$ with $|\hat{A}^j| = U^j$ such that $p_j : \hat{A}^j \to D_{\kappa_j}(T)$ is univalent. Furthermore, there exists some $\kappa_0 > 0$ such that $p_0 : A^0 \to D_{\kappa_0}(T)$ is a holomorphic double covering. Then we let

$$\kappa := \min_{0 \leq j \leq N} \kappa_j$$

and $A := D_\kappa(T)$. Then for each j there exists some $A^j \subset \mathbb{C}$ such that for $j \neq 0$, $p_j : A^j \to A$ is univalent and $p_0 : A^0 \to A$ is a holomorphic double covering. So letting $\tilde{F}$ be equal to p_j on A^j, $\tilde{F} : \bigcup_{j=0}^N A^j \to A$ is a holomorphic map. Indeed, if $\overline{A^j} \subset A$ for $0 \leq j \leq N$ then this is a holomorphic box mapping. However, this may not be the case. So we must do more work.

We will need the domains of the map that we are dealing with to be compactly contained in their ranges in order to guarantee that we have a holomorphic box mapping. To this end we induce on $\tilde{F}$ as follows. Denote $\tilde{F}$ by $\tilde{F}_0$ and $|A^0|$ by I_0. and the first return map by $\tilde{F}_0$ to I_1 by $\tilde{F}_1 : \bigcup_j I_1^j \to I_1$. Similarly we obtain $\tilde{F}_i : \bigcup_j I_i^j \to I_i$. Each of these maps extends to a map $\tilde{F}_i : \bigcup_j A_i^j \to A_i$ where A_i is some complex neighbourhood of I_i such that $|A_i| = I_i$. Similarly for A_i^j.

We use the following result, Lemma V.I.5.2 of [**MS**]. Recall that we let $\rho_J(\cdot, \cdot)$ denote the Poincaré metric on the domain $\mathbb{C}_J := \mathbb{C} \setminus (\mathbb{R} \setminus J)$.

Lemma 2.6.3. *Given $a > 0$ and $r_0 > 0$ there exist $s = s(r_0, a)$ and $l_0 > 0$ with the following property. If Φ satisfies the following conditions:*

1. *Φ is holomorphic and univalent on a Euclidean disk of radius a centred at a point of an interval $J \subset \mathbb{R}$ and $D_{r_0}(J)$ is contained in this Euclidean disk;*

2. *Φ maps the real axis into the real axis;*

3. *$|J| \leq l_0$.*

Then, provided $k \leq r_0$,

$$\rho_{\Phi(J)}(\Phi(x), \Phi(y)) \leq (1 + s|J|)\rho_J(x, y)$$

for all $x, y \in D_k(J)$; so in particular $\Phi(D_k(J)) \subset D_{(1+s|J|)k}(\Phi(J))$.

This extends in our case to the following result.

Lemma 2.6.4. *For i large enough there is some $k > 0$ and some domains $\hat{A}_i^j$ such that $\tilde{F}_i$ extends to some holomorphic box mapping $\tilde{F}_i : \bigcup_j \hat{A}_i^j \to D_k(I_i)$.*

This lemma is an adaptation of Lemma 3.2 of [**K3**]. See [**LS2**] for the infinitely renormalisable case. For the proof of this we introduce some notation. For an interval J and $a > 0$ let $D(a, J)$ denote the round disk in $\mathbb{C}$ centred at the midpoint of J with radius a. Let $D(J)$ be $D(\frac{|J|}{2}, J)$.

Proof: Suppose that $n(j) > 0$ is such that $\tilde{F}_i|_{I_i^j} = \tilde{F}_0^{n(j)}|_{I_i^j}$. Pulling back by the appropriate branch, we see that by Lemma 2.4.5 we have an extension of $\tilde{F}_0^{-1}$ on $F^{-k}(I_i)$ to some domain $I(k)_i^j$ where $k < n(j) - 1$. In fact, if we select a suitable i then we have an extension of $\tilde{F}_0^{-1}$ on the appropriate pullbacks of $F^{-(n(j)-1)}(I_i)$ too (we just ensure that $\tilde{F}_{i-1}$ is 'non-central'). By Theorem 2.4.4 the size of this extension is dictated by the size of $\frac{|I_i|}{|I_{i-1}|}$. So for a given $\epsilon > 0$ there exists some i such that $I(k)_i^j$ is an ϵ-scaled neighbourhood of $\tilde{F}_0^{-k}(I_i)$.

For large i we see that $\tilde{F}_0$ is holomorphic on the round disk $D(\hat{I}_i^j)$. Then we can choose some $a > 0$, $r_0 > 0$ as in Lemma 2.6.3 depending on ϵ such that $D(a, \tilde{F}_0^{-k}(I_i)) \subset D(I(k)_i^j)$ (and therefore each branch of $\tilde{F}_0^{-1}$ is univalent on $D(a, \tilde{F}_0^{-k}(I_i)))$ and $D_{r_0}(\tilde{F}_0^{-k}(I_i)) \subset D(a, \tilde{F}_0^{-k}(I_i))$. So we have satisfied all but the last condition of Lemma 2.6.3. By inducing some more we can satisfy that condition too. Observe that by [**SV**], we can choose an i such that our inducing does not alter the condition for ϵ, so the above assertions still hold.

It can be seen from rescaling that we can do this for all iterates of branches. Then if k is small enough,

$$\tilde{F}_0^{-n}[D_k(\tilde{F}_0^n(I_i^j))] \subset \tilde{F}_0^{-(n-1)}\left[D_{\left(1+s|\tilde{F}_0^n(I_i^j)|\right)k}(\tilde{F}_0^{n-1}(I_i^j))\right] \subset \cdots \subset D_{\left(1+s\sum_{i=0}^n|\tilde{F}_0^i(I_i^j)|\right)k}(I_i^j).$$

So choosing $0 < k_0 < \frac{r_0}{1+s\sum_{i=0}^{n(j)}|\tilde{F}_0^i(I_i^j)|}$ and noting that $\sum_{i=0}^{n(j)}|\tilde{F}_0^i(I_i^j)| < |A_0|$, we complete the lemma by fixing some $k \leq k_0$ and letting $\hat{A}_i^j$ be the pullback by $\tilde{F}_i$ of $D_k(I_i^j)$. Finally, we denote $D_k(I_i^j)$ by $\hat{A}_i$.

□

Letting $\tilde{f}|_{U^j} := f(F^{-1}p_j)$, we obtain $\tilde{F}$ as the first return map by $\tilde{f}$ to T. Furthermore, we have ensured that $\tilde{f} \in C^k$.

We can now apply the methods of [**K3**] to $\tilde{f}$.

We can transfer this construction to $f^n(c)$ as follows. We denote $f^n(c)$ by x_0. We choose some neighbourhood U_{x_0} of x_0 which is disjoint from T. We may choose T and U_{x_0} so that $f^k(T)$ are disjoint from $T \cup U_{x_0}$ for $k = 1, \ldots, n-1$. Let $1 \leq M < \infty$ be the maximum number of times any domain of F iterates by f into U_{x_0} before returning to T. Then similar calculations to those in the proof of Lemma 2.4.8 can be used to prove the following lemma.

Lemma 2.6.5. *For the setting above, for $\epsilon > 0$ there exists some $\epsilon' > 0$ such that for $j = 0, \ldots N$, if $\|p_j - F\|_{C^k} < \epsilon'$ then, choosing appropriate branches, on U^j we have $\|f(f^{-(n(j)-n)}p_j f^n) - f\|_{C^k} < \frac{\epsilon}{M}$.*

We let f_0 be equal to f outside $f^n(T) \setminus f^n(U^0)$ and equal to $f(f^{-(n(0)-n)}p_0 f^n)$ on U^0. For $i = 1, \ldots, N-1$ we let f_{i+1} be equal to f_i outside $f^n(T) \setminus f^n(U^{i+1})$ and equal to $f_i(f_i^{-(n(i+1)-n)}p_i f_i^n)$ on U^{i+1}. So if we choose our p_j as in the above lemma we have $\|f_N - f\|_{C^k} < \epsilon$.

Note that if any interval $f^n(U^j)$ iterates back into $f^n(T)$ then it must coincide with some other $f^n(U^{j'})$ for $j' \neq j$. Thus for $x \in \partial\left(\bigcup_{j=0}^N U^j\right)$ we have $D^i f_N(x) = D^i f_N(x)$ for $i = 0, \ldots k$. Therefore $f_N \in C^k$.

We can therefore obtain a holomorphic box mapping in this case as above.

We next define a perturbation to create a family of holomorphic box maps. Given our holomorphic box mapping $\tilde{F}_i : \bigcup_j \hat{A}_i^j \to \hat{A}_i$, we consider the first time that

$f^n(\hat{A}_i)$ lands in U_{x_0}. We then support our perturbation on the $f^n(\hat{A}_i^0)$, mapping $f^n(c)$ to a boundary point of $f^n(\hat{A}_i^0)$ by some $p_{\lambda,k}$. Our partition is unchanged by $p_{\lambda,k}$, so despite the fact that the perturbation is large even in the C^1 sense, we can apply the methods of [**K3**] to this family.

2.7 A straightforward non-minimal case

Here we prove part 5 of Theorem 2.1.4.

Let $E_f := \{x \in \omega(c) : \omega(x) \neq \omega(c)\}$. We consider the cases in which it is simple to perturb appropriately in this set.

Lemma 2.7.1. *If $\overline{E_f} \cap \{c\} = \emptyset$ then there exists a small nice interval U around c such that the first return map to U has a finite number of domains which intersect $\omega(c)$.*

Proof: Let U be a small interval around c as in Lemma 2.2.10 such that $U \cap E_f = \emptyset$. Then any point $y \in \omega(c) \cap U$ must have $\omega(y) = \omega(c)$. In particular, y must return to U under iteration by f. Thus, as in Lemma 2.6.2 the first return map to U has a finite number of domains intersecting $\omega(c)$ as required.

□

So we may proceed as in the proof of the minimal case.

Appendix A

Proof of the Yoccoz Lemma

We recall the lemma.

Lemma 1.5.2 *Suppose that $f \in NF^2$. Then for all $\delta, \delta' > 0$ there exists $C > 0$ such that if I_0 is a nice interval such that*

1. *I_0 is a δ-scaled neighbourhood of I_1;*
2. *F_i is low and central for $i = 0, \ldots, m$;*
3. *there is some $0 < i < m$ with $\frac{|I_i|}{|I_{i+1}|} < 1 + \delta'$,*

then for $1 \leq k < m$,

$$\frac{1}{C}\frac{1}{\min(k, m-k)^2} < \frac{|I_{i+k-1} \setminus I_{i+k}|}{|I_i|} < \frac{C}{\min(k, m-k)^2}.$$

For similar statements see [**FM**] and [**Sh2**].

Proof: We firstly use the following claim.

Claim A.0.2. *For f as in the lemma, there exists some $C(f, \delta, \delta') > 0$ such that*

$$\frac{|I_m|}{|I_0|} > C(f, \delta, \delta').$$

This is proved in Section 5 of [**Sh2**]. Immediate consequences of this are that $\frac{|I_m \setminus I_{m+1}|}{|I_1|}$ is bounded below (this is one of the assumptions in the statement of the Yoccoz Lemma in [**FM**]) and that F_0 has bounded distortion in (a_1, a_m).

Our proof now involves using the bound δ and the small size of I_0 to find a nearby map in the Epstein class. The structure of such maps, particularly at parabolic fixed points, along with some new coordinates, give us estimates for $\frac{|I_{i+k-1} \setminus I_{i+k}|}{|I_i|}$.

We suppose that $s > 0$ is such that $F|_{I_1} = f^s|_{I_1}$. We observe that f^{s-1} has uniformly bounded distortion depending on δ. We will denote $F|_{I_1}$ by F. Letting $\psi : [a_m, a_1] \to [0,1]$ be an affine diffeomorphism we will work with the map $\psi F \psi^{-1}$. For the rest of the appendix we will denote this map by F too.

Previously we assumed that $F|_{I_1}$ had a maximum at c. It will be convenient to suppose now for this section that c is a minimum for $F|_{I_1}$. Also we let $I_i = (a_i', a_i)$. So in particular, $F(a_{i+1}) = a_i$. We firstly define a point which allows us to partition $[a_m, a_1]$ in another way.

Let $x_0 \in [a_m, a_1]$ be defined so that $|F(x_0) - x_0| = \min_{a_m \le x \le a_0} |F(x) - x|$. We can show that $DF(x_0) = 1$ as follows. Note that $F(x) > x$ for all $x \in [a_m, a_1]$ and that F is increasing. Then let $h(x) := F(x) - (F(x_0) - x_0)$. By the definition of x_0, we have $h(x) - x > 0$. Then we express h as $h(x) = x_0 + (x - x_0)Dh(x_0) + O(|x - x_0|^2)$ and so $h(x) - x = (x - x_0)(Dh(x_0) - 1) + O(|x - x_0|^2)$. Therefore, if $Dh(x_0) > 1$ then there exists some $x < x_0$ near x_0 such that $h(x) - x < 0$. Similarly, if $Dh(x_0) < 1$ then there exists some $x > x_0$ near x_0 such that $h(x) - x < 0$. In either case we have a contradiction.

We are able to estimate the shape of F near x_0 using the following definition and lemma.

Definition A.0.3. Let $a > 0$. We say that the real analytic map $f : [0,1] \to [0,1]$ is in the *Epstein class* $\mathcal{E}_a$ if $f(x) = \varphi Q \psi$ where Q is the quadratic map $Q(z) = z^2$, ψ is an affine map and $\varphi : [0,1] \to [0,1]$ is a diffeomorphism whose inverse has a holomorphic extension which is univalent in the domain $\mathbb{C}_{(-a,1+a)} := \mathbb{C} \setminus (\mathbb{R} \setminus (-a, 1+a))$.

For more details on maps in this class see [**MS**].

Lemma A.0.4. *Let $f \in NF^2$. Suppose that I is a nice interval around c and J is a first entry domain which is disjoint from I and with entry time s. Suppose that $\delta > 0$ is some constant such that there exists some $\hat{J} \supset J$ such that $f^s : \hat{J} \to I'$ is a diffeomorphism where I' is a δ-scaled neighbourhood of I and $\sum |f^j(\hat{J})| \le 1$. Let $\tau_0 : J \to [0,1]$ and $\tau_s : I \to [0,1]$ be affine diffeomorphisms. Then for all $\epsilon > 0$ there exists $\delta > 0$ such that $|I| < \delta$ implies that there exists some function $G : I \to I$ in the Epstein class $\mathcal{E}_\delta$ such that $\|\tau_s f^s \tau_0^{-1} - G\|_{C^2} < \delta$ and G^{-1} extends* *univalently to* $\mathbb{C}_{(-\delta,1+\delta)}$.

We use this to prove the following claim.

Claim A.0.5. *There exists some* $0 < A < B$ *such that*

$$F(x_0) + (x - x_0) + A(x - x_0)^2 \leq F(x) \leq F(x_0) + (x - x_0) + B(x - x_0)^2.$$

Proof: We know that $f^s : I_2 \to I_1$ has the following property. The map $f^{s-1} : f(I_2) \to I_1$ has an extension to I_0. Furthermore, since I_0 is a δ-scaled neighbourhood of I_1 we use Lemma A.0.4 to obtain a C^2 close map G_∞ which is in the Epstein class.

In fact we can choose different starting intervals I_n with the same real bounds which are smaller and smaller and which are then rescaled to maps F_n which map from the unit interval. For each such map we obtain the nearby map G_n in the Epstein class where $\|F_n - G_n\|_{C^2} \to 0$ as $n \to \infty$. We let x_0^n denote a point which is equivalent to x_0 for F. We suppose $|F_n(x_0^n) - x_0^n|$ becomes very small; otherwise our proof is simpler. Then our limit map G_∞ has a parabolic fixed point x_0^∞. Also $D^2 G_\infty(x_0^\infty) > 0$. Thus, there exist $0 < A < B$ depending only on f such that for all $x \in [0, 1]$ we have

$$G_\infty(x_0^\infty) + (x - x_0^\infty) + A(x - x_0^\infty)^2 \leq G_\infty(x) \leq G_\infty(x_0^\infty) + (x - x_0^\infty) + B(x - x_0^\infty)^2.$$

Clearly, for large n, we have the same condition for G_n. Therefore, if we take I_0 small enough, we may assume that it holds for F too.

□

We denote $\epsilon := F(x_0) - x_0$. Then we have

$$\epsilon + A(x - x_0)^2 \leq F(x) - x \leq \epsilon + B(x - x_0)^2.$$

We suppose that N is such that $x_0 \in [a_N, a_{N+1})$. Then for $0 \leq i \leq N - 1$ we let $x_i := F^i(x_0)$. We will use this equation to find estimates for $a_j - a_{j+1}$. Throughout we will find estimates of the following type: 'there exist $C' > C > 0$ such that $C\beta < \alpha < C'\beta$'. We will write this as $\alpha \asymp \beta$. We will use C, C' where necessary too. These constants will not all be the same, but to ease notation they will denote some constants depending only on δ.

Claim A.0.6.
$$N \asymp \frac{1}{\sqrt{\epsilon}}.$$

Proof: Let $N' = \max\{1 \leq j \leq N-1 : x_j - x_0 \leq \sqrt{\epsilon}\}$. We will first show that N' satisfies the claim. For $j \leq N'$, we have

$$\epsilon \leq x_{j+1} - x_0 \leq \epsilon(B+1).$$

Therefore,

$$N'\epsilon \leq \sum_{j=0}^{N'-1} x_{j+1} - x_j \leq N'\epsilon(B+1).$$

Since $\sum_{j=0}^{N'-1} x_{j+1} - x_j = x_j - x_0 \leq \sqrt{\epsilon}$ we have $N' \leq \frac{1}{\sqrt{\epsilon}}$. Furthermore $x_{j+1} - x_0 > \sqrt{\epsilon}$ so $(N'+1)\epsilon(B+1) > \sqrt{\epsilon}$ and $N' > \frac{1}{(B+1)\sqrt{\epsilon}} - 1$. I.e. $N' \asymp \frac{1}{\sqrt{\epsilon}}$.

Next we find estimates for $N - N'$. For $N' < j \leq N$ we consider again the equation

$$\epsilon + A(x_j - x_0)^2 \leq x_{j+1} - x_j \leq \epsilon + B(x_j - x_0)^2.$$

But note that here $B(x_j - x_0)^2 > \epsilon$ so we can write instead

$$A(x_j - x_0)^2 \leq x_{j+1} - x_j \leq 2B(x_j - x_0)^2.$$

We make a change of coordinates. We let $y_j := \frac{1}{x_j - x_0}$. Then we have

$$y_j - y_{j+1} = \frac{x_{j+1} - x_j}{(x_j - x_0)(x_{j+1} - x_0)}.$$

By the above bounds we have

$$\frac{A(x_j - x_0)}{x_{j+1} - x_0} < y_j - y_{j+1} < \frac{2B(x_j - x_0)}{x_{j+1} - x_0} < 2B.$$

Furthermore,

$$y_j - y_{j+1} > \frac{A(x_j - x_0)}{(x_{j+1} - x_j) + (x_j - x_0)} > \frac{A(x_j - x_0)}{2B(x_j - x_0)^2 + (x_j - x_0)} = \frac{A}{2B+1}.$$

Observe that $x_N \in (a_1, a_0)$ and $|a_0 - a_1| > \delta$. So since $|x_N - x_{N-1}|$ is approximately $|a_0 - a_1|$ we know that $y_N \asymp 1$. Also note that $y_{N'} \asymp \frac{1}{\sqrt{\epsilon}}$ and so $y_{N'} - y_N \asymp \frac{1}{\sqrt{\epsilon}}$. Summing we obtain

$$\frac{C}{\sqrt{\epsilon}} < y_{N'} - y_N = \sum_{j=N-1}^{N'} y_j - y_{j+1} < 2B(N - N')$$

and

$$\frac{C'}{\sqrt{\epsilon}} > y_{N'} - y_N = \sum_{j=N-1}^{N'} y_j - y_{j+1} > \frac{A(N-N')}{2B+1}.$$

So $N - N' \asymp \frac{1}{\sqrt{\epsilon}}$ too. Adding this to the estimates for N' we prove the claim.

□

To prove the Lemma 1.5.2, we use the above claim added to the fact that, since Claim A.0.2 gives us bounded distortion, $a_j - a_{j+1}$ is like $x_{N-j} - x_{N-j-1}$.

Firstly we will use that above coordinate change again. For $j > N'$ we have

$$y_j > y_j - y_N = \sum_{j=N-1}^{j} y_i - y_{i+1} > \frac{A(N-j)}{2B+1}$$

and so $\frac{1}{x_j - x_0} > \frac{A(N-j)}{2B+1}$ and $x_{j+1} - x_j < 2B\left(\frac{2B+1}{A(N-j)}\right)^2$.

We have proved that if $0 \leq j \leq N'$ then

$$\epsilon < x_{j+1} - x_j < C'\epsilon$$

and if $N' < j \leq N$ then

$$\epsilon < x_{j+1} - x_j < \frac{C'}{(N-j)^2}.$$

Similarly we can define $x_j = F^j(x_0)$ for negative j where $0 \leq |j| < m - N$. We define some M' analogously to the definition for N' and so if $|j| \leq M'$ then

$$\epsilon < x_{j+1} - x_j < C'\epsilon.$$

And if $M' < |j| \leq m - N$ then

$$\frac{C}{(m-N+j)^2} < x_{j+1} - x_j < \frac{C'}{(m-N+j)^2}.$$

(In the step of the proof where estimates on y_{N-m} are required, we use the fact that $|a_{m-1} - a_m| > \delta$ and $|x_{-m-1} - x_m|$ is approximately $|a_{m-1} - a_m|$.)

Note also that we can show that $m - M' \asymp \frac{1}{\sqrt{\epsilon}}$.

Since $a_j - a_{j+1}$ is essentially the same as $x_{N-j} - x_{N-j-1}$. So if $N \geq j \geq N - N'$, we have

$$C\epsilon < a_j - a_{j+1} < C'\epsilon.$$

Note that $\frac{1}{N-N'} \geq \frac{1}{j} \geq \frac{1}{N}$. Since $\epsilon \asymp \frac{1}{N^2}$ and $\epsilon \asymp \frac{1}{(N-N')^2}$ this implies that we have

$$\frac{C}{j^2} < a_j - a_{j+1} < \frac{C'}{j^2}.$$

Now if $N - N' \geq j \geq 1$ then

$$\frac{C}{j^2} < a_j - a_{j+1} < \frac{C'}{j^2}$$

(where the lower inequality comes from δ). If $N \leq j \leq m - M'$ then we may again derive

$$C\epsilon < a_j - a_{j+1} < C'\epsilon.$$

Note that we also have $m - N \geq m - j \geq m - M'$. Since $m - N, m - M' \asymp \frac{1}{\sqrt{\epsilon}}$ we have

$$\frac{C}{(m-j)^2} < a_j - a_{j+1} < \frac{C'}{(m-j)^2}.$$

If $m - M' \leq j \leq m - 1$ we have

$$\frac{C}{(m-j)^2} < a_j - a_{j+1} < \frac{C'}{(m-j)^2}$$

where the lower inequality comes from δ and δ'.

To conclude, if $1 \leq j \leq N$ then we have some constant C such that $j \leq C(m-j)$ and $a_j - a_{j+1} \asymp \frac{1}{j^2}$. If $N \leq j \leq m - 1$ then we have some constant C' such that $m - j \leq C'j$ and $a_j - a_{j+1} \asymp \frac{1}{(m-j)^2}$. So in either case we have

$$a_j - a_{j+1} \asymp \frac{1}{(\min(j, m-j))^2}$$

as required.

□

Appendix B

C^2 convergence

It remains to prove Lemma A.0.4, which we recall below.

Lemma A.0.4. *Let $f \in NF^2$. Suppose that I is a nice interval around c and J is a first entry domain which is disjoint from I and with entry time s. Suppose that $\delta > 0$ is some constant such that there exists some $\hat{J} \supset J$ such that $f^s : \hat{J} \to I'$ is a diffeomorphism where I' is a δ-scaled neighbourhood of I and $\sum |f^j(\hat{J})| \leq 1$. Let $\tau_0 : J \to [0,1]$ and $\tau_s : I \to [0,1]$ be affine diffeomorphisms. Then for all $\epsilon > 0$ there exists $\delta > 0$ such that $|I| < \delta$ implies that there exists some function $G : I \to I$ in the Epstein class $\mathcal{E}_\delta$ such that $\|\tau_s f^s \tau_0^{-1} - G\|_{C^2} < \delta$ and G^{-1} extends univalently to $\mathbb{C}_{(-\delta,1+\delta)}$.*

This lemma was suggested by W. Shen. Such arguments go back to Sullivan, see [**Sul**], but usually the convergence is only $C^{1+\eta}$ for some $\eta > 0$. We are able to show here that we have convergence in the C^2 topology.

Proof: We will assume that we can fix some open neighbourhood U of c such that $f(x) = f(c) + |x - c|^\alpha$ for $x \in U$. We may ignore the usual function ϕ for the moment as the lemma extends to that general case too. We also fix some open set U' such that $\overline{U'} \subset U$. Let $J_0 = J$ and $J_i = f^i(J)$. For every $0 \leq i < j \leq s-1$ we have a diffeomorphism $f^{j-i} : f^i(\hat{J}) \to f^j(\hat{J})$. Thus each such map has distortion bounded by some $K = K(\delta)$.

We will rescale our maps as follows. Let $\tau_i : J_i \to [0,1]$ be the affine homoeomorphism such that $f_i = \tau_{i+1} f \tau_i^{-1}$ is monotone increasing. Then the following diagram commutes.

$$\begin{array}{ccccccccc}
J_0 & \xrightarrow{f} & J_1 & \xrightarrow{f} & \cdots & \xrightarrow{f} & J_{s-1} & \xrightarrow{f} & J_s \\
\tau_0 \big\downarrow & & \big\downarrow \tau_1 & & \big\downarrow & & \big\downarrow \tau_{s-1} & & \big\downarrow \tau_s \\
[0,1] & \xrightarrow{f_0} & [0,1] & \xrightarrow{f_1} & \cdots & \xrightarrow{f_{s-2}} & [0,1] & \xrightarrow{f_{s-1}} & [0,1]
\end{array}$$

When I is sufficiently small then by Lemma 1.3.4 each J_i is either inside U or is disjoint from U'. We then approximate f_i as follows. For $x \in J_i$ and $0 \leq i \leq s-1$ let

$$g_i(x) = \begin{cases} f_i(x) & \text{if } J_i \subset U, \\ \left(1 - \frac{\delta_i}{2}\right) x + \frac{\delta_i}{2} x^2 & \text{if } J_i \cap U' = \emptyset \end{cases}$$

where $\delta_i = \int_0^1 D^2 f_i(t) dt$. We observe that $Sg_i < 0$ for all $0 \leq i \leq s-1$. If $J_i \subset U$ for some $0 \geq j \leq s-1$ we know that g_i extends holomorphically to $\mathbb{C}_{(-\delta, 1+\delta)}$. Also, if $J_i \cap U' = \emptyset$ then $Dg_i(x) = 1 - \frac{\delta_i}{2} + \frac{\delta_i x}{2}$. As we will see later, even when the modulus of x is very large, this is non-zero. Therefore $(g_n \cdots g_0)^{-1}$ extends univalently to a holomorphic map defined on $\mathbb{C}_{(-\delta,1+\delta)}$ for $0 \leq n \leq s-1$. Furthermore, we can prove that for any $k \geq 0$ there exists some $M(\delta, k)$ such that $\|g_n \cdots g_0\|_{C^k} < M(\delta, k)$ for $0 \leq n \leq s-1$. (For details of the proof see the Koebe Bieberbach result in [**Mi**].) We will only be concerned with $M(\delta) := M(\delta, 3)$.

We will prove that $\|g_{s-1} \cdots g_0 - f_{s-1} \cdots f_0\|_{C^2}$ is small. We first prove the following lemma.

Lemma B.0.7. *There exist some $M_1, M_2 > 0$ such that*

$$\|g_i - f_i\|_{C^2} < M_1 w(|J_i|)|J_i|$$

where $w = w_{D^2 f}$ as defined in Section 1.3. Furthermore, $|Df_i(x) - 1|, |D^2 f_i(x)| < M_2 |J_i|$ for $x \in [0,1]$.

Proof: Assume J_i is not in U, otherwise there is nothing to prove. We will first estimate $|D^2 g_i(x) - D^2 f_i(x)|$ for $x \in [0,1]$. Observe that $D^2 g_i(x) = \delta_i = \int_0^1 D^2 f_i(t) dt$ and $D^2 f_i(x) = \frac{|J_i|^2}{|J_{i+1}|} D^2 f(\tau_i^{-1}(x))$. There exists $y \in [0,1]$ such that $\int_0^1 D^2 f_i(t) dt = D^2 f_i(y)$ so $D^2 g_i(x) = D^2 f_i(y)$ and

$$\begin{aligned}
|D^2 g_i(x) - D^2 f_i(x)| &= |D^2 f_i(y) - D^2 f_i(x)| \\
&= \frac{|J_i|^2}{|J_{i+1}|} |D^2 f(\tau_i^{-1}(y)) - D^2 f(\tau_i^{-1}(x))| \\
&< \frac{|J_i|^2}{|J_{i+1}|} w(|J_i|) < M_1 |J_i| w(|J_i|)
\end{aligned}$$

for some $M_1 > \frac{1}{\inf_{x \notin U} |Df(x)|}$.

We use this to make the remaining derivative estimates. By the Mean Value Theorem there exists some $x_1 \in [0,1]$ such that $f_i(x_1) = g_i(x_1)$. Then for $x \in [0,1]$,

$$|Dg_i(x) - Df_i(x)| \leq \int_{x_1}^{x} |D^2 g_i(t) - D^2 f_i(t)| dt < M_1 |J_i| w(|J_i|).$$

Similarly,

$$|g_i(x) - f_i(x)| \leq \int_0^x |Dg_i(t) - Df_i(t)| dt < M_1 |J_i| w(|J_i|).$$

Also for $x \in [0,1]$ we have

$$|D^2 f_i(x)| = \frac{|J_i|^2}{|J_{i+1}|} |D^2 f(\tau_i^{-1} x)| < M_2 |J_i|$$

where $M_2 > \frac{\sup_{x \notin U} |D^2 f(x)|}{\inf_{x \notin U} |Df(x)|}$.

By the Mean Value Theorem there exists some $x_0 \in [0,1]$ such that $Df_i(x_0) = 1$. Then for $x \in [0,1]$ we have

$$|Df_i(x) - 1| \leq \int_{x_0}^{x} |D^2 f_i(t)| dt < M_2 |J_i|.$$

$\square$

For our calculations, we introduce some new notation. We let A_{s-1} and B_0 be the identity map; for $0 \leq j < s-1$ we let $A_j = g_{s-1} \cdots g_j$; and for $0 < j \leq s-1$ we let $B_j = f_j \cdots f_0$. We have

$$\begin{aligned} (g_{s-1} \cdots g_0) - (f_{s-1} \cdots f_0) &= \sum_{j=0}^{s-1} (g_{s-1} \cdots g_{j+1} g_j f_{j-1} \cdots f_0) \\ &\qquad -(g_{s-1} \cdots g_{j+1} f_j f_{j-1} \cdots f_0) \\ &= \sum_{j=0}^{s-1} A_{j+1} g_j B_{j-1} - A_{j+1} f_j B_{j-1}. \end{aligned}$$

If we let $S_j := A_{j+1} g_j B_{j-1}$ and $T_j := S_j - S_{j+1}$ then $(g_{s-1} \cdots g_0) - (f_{s-1} \cdots f_0) = \sum_{j=0}^{s-1} T_j$.

If $J_j \subset U$ then $g_j = f_j$ and so $T_j = 0$. We assume that J_j is disjoint from U'. Then

$$\begin{aligned}
|DT_j(x)| &= |[DA_{j+1}(g_jB_{j-1}(x))Dg_j(B_{j-1}(x))DB_{j-1}(x)] \\
&\quad -[DA_{j+1}(f_jB_{j-1}(x))Df_j(B_{j-1}(x))DB_{j-1}(x)]| \\
&\leq |DB_{j-1}|_\infty |DA_{j+1}(g_j(y))Dg_j(y) - DA_{j+1}(f_j(y))Df_j(y)|
\end{aligned}$$

where $y = B_{j-1}(x)$ and $|\cdot|_\infty$ denotes the maximal value of a function on its domain. Note also that $|DB_j|_\infty, |DA_j|_\infty$ are bounded since f^s has bounded distortion. So

$$\begin{aligned}
|DT_j(x)| &\leq C\,(|DA_{j+1}g_j(y) - DA_{j+1}f_j(y)||Dg_j(y)| \\
&\qquad + |Dg_j(y) - Df_j(y)||DA_j(f_j(y))|) \\
&\leq C(|g_j - f_j|_\infty + |Dg_j - Df_j|_\infty) < CM_1 w(|J_j|)|J_j|.
\end{aligned}$$

Therefore,

$$|D(g_{s-1}\cdots g_0) - D(f_{s-1}\cdots f_0)|_\infty < CM_1 \sum_{j=0}^{s-1} w(|J_j|)|J_j| < CM_1 w(\tau(|I|)) \sum_{j=0}^{s-1} |J_j|$$

where the function τ is defined in Lemma 1.3.4. So our result is proved for the C^1 norm. The result in the C^0 norm follows easily. Bounding the C^2 norm is more complicated.

We rewrite the summands as follows.

$$\begin{aligned}
|D^2S_j(x) - D^2S_{j+1}(x)| &= \left| \frac{D^2S_j(x)}{DS_j(x)} DS_j(x) - \frac{D^2S_{j+1}(x)}{DS_{j+1}(x)} DS_{j+1}(x) \right| \\
&\leq \left| \frac{D^2S_j}{DS_j} - \frac{D^2S_{j+1}}{DS_{j+1}} \right|_\infty |DS_j|_\infty \\
&\quad + \left| \frac{D^2S_{j+1}}{DS_{j+1}} \right|_\infty |DS_j - DS_{j+1}|_\infty.
\end{aligned}$$

We will show that

$$\sum_{j=0}^{s-1} \left| \frac{D^2S_j}{DS_j} - \frac{D^2S_{j+1}}{DS_{j+1}} \right|_\infty \tag{B.1}$$

is small. Since $|D^2S_0|$ is bounded above and $|DS_0|$ is bounded below we have an upper bound on $\left|\frac{D^2S_0}{DS_0}\right|$. Therefore $\left|\frac{D^2S_{j+1}}{DS_{j+1}}\right|_\infty$ is bounded by some $M_2 > 0$ for $0 \leq j \leq s-1$. So once we have shown that (B.1) is small then we know that $|D^2(g_{s-1}\cdots g_0) - D^2(f_{s-1}\cdots f_0)|_\infty$ is small also.

To find the bounds for (B.1) we use the fact that for increasing C^2 functions g we have $D(\log(Dg))(x) = \frac{D^2g(x)}{Dg(x)}$. Then since

$$\log(DS_j(x)) = \log(DA_{j+1}(g_jB_{j-1}(x))) + \log(Dg_j(B_{j-1}(x))) + \log(DB_{j-1}(x))$$

and

$$\log(DS_{j+1}(x)) = \log(DA_{j+1}(f_jB_{j-1}(x))) + \log(Df_j(B_{j-1}(x))) + \log(DB_{j-1}(x)),$$

we have

$$\begin{aligned}
&\log(DS_j(x)) - \log(DS_{j+1}(x)) \\
&\qquad = \log(DA_{j+1}(g_jB_{j-1}(x))) - \log(DA_{j+1}(f_jB_{j-1}(x))) \\
&\qquad\quad + \log(Dg_j(B_{j-1}(x))) - \log(Df_j(B_{j-1}(x))).
\end{aligned}$$

Differentiating we obtain

$$\begin{aligned}
\frac{D^2S_j}{DS_j}(x) - \frac{D^2S_{j+1}}{DS_{j+1}}(x) \quad = \quad & \frac{D^2A_{j+1}(g_jB_{j-1}(x))Dg_j(B_{j-1}(x))DB_{j-1}(x)}{DA_{j+1}(g_jB_{j-1}(x))} \\
& -\frac{D^2A_{j+1}(f_jB_{j-1}(x))Df_j(B_{j-1}(x))DB_{j-1}(x)}{DA_{j+1}(f_jB_{j-1}(x))} \\
& +\frac{D^2g_j(B_{j-1}(x))DB_{j-1}(x)}{Dg_j(B_{j-1}(x))} \\
& -\frac{D^2f_j(B_{j-1}(x))DB_{j-1}(x)}{Df_j(B_{j-1}(x))}.
\end{aligned}$$

Denoting $\frac{D^2S_j}{DS_j}(x) - \frac{D^2S_{j+1}}{DS_{j+1}}(x)$ by P_j, we can define θ_j as being the function obtained by postcomposing P_j with B_{j-1}^{-1} and dividing by the function $x \mapsto DB_{j-1}(B_{j-1}^{-1}(x))$. Thus,

$$\begin{aligned}
\theta_j(x) \quad = \quad & \left[\frac{D^2A_{j+1}(g_j(x))Dg_j(x)}{DA_{j+1}(g_j(x))} - \frac{D^2A_{j+1}(f_j(x))Df_j(x)}{DA_{j+1}(f_j(x))}\right] \\
& + \left[\frac{D^2g_j(x)}{Dg_j(x)} - \frac{D^2f_j(x)}{Df_j(x)}\right].
\end{aligned}$$

We now denote the first summand in square brackets as $\theta_j^1(x)$ and the second summand in square brackets as $\theta_j^2(x)$.

We calculate

$$\begin{aligned}
|\theta_j^2(x)| \quad &= \quad \left|\frac{D^2g_j(x)Df_j(x) - D^2f_j(x)Df_j(x)}{Dg_j(x)Df_j(x)}\right| \\
&= \quad \left|\frac{(D^2g_j(x) - D^2f_j(x))Df_j(x) + D^2f_j(x)(Df_j(x) - Dg_j(x))}{Dg_j(x)Df_j(x)}\right| \\
&\leq \quad M_1^2[M_1w(J_j)|J_j|(1 + M_2|J_j|) + M_2(M_1w(J_j)|J_j|)] < M_3w(|J_j|)|J_j|
\end{aligned}$$

for some $M_3 > M_1^2[M_1(1 + M_2|J_j|) + M_1M_2]$.

Also $\theta_j^1(x) = \theta_j^{1,1}(x)(\theta_j^{1,2}(x) + \theta_j^{1,3}(x))$ where

$$\theta_j^{1,1}(x) = \frac{1}{[DA_{j+1}(g_j(x))][DA_{j+1}(f_j(x))]},$$

$$\theta_j^{1,2}(x) = [DA_{j+1}(f_j(x)) - DA_{j+1}(g_j(x))][D^2A_{j+1}(g_j(x))Dg_j(x)]$$

and

$$\theta_j^{1,3}(x) = [DA_{j+1}(g_j(x))][D^2A_{j+1}(g_j(x))Dg_j(x) - D^2A_{j+1}(f_j(x))Df_j(x)].$$

Clearly $|\theta_1^{1,1}(x)| < K^2$ and $|\theta_1^{1,2}(x)| < |D^2A_{j+1}|_\infty^2(1 + M_2|J_j|)M_1w(|J_j|)|J_j|$. Finally, $\theta_1^{1,3}(x) = [DA_{j+1}(g_j(x))](\gamma_1(x) + \gamma_2(x))$ where

$$\gamma_1(x) = (D^2A_{j+1}g_j(x))(Dg_j(x) - Df_j(x))$$

and

$$\gamma_2(x) = (D^2A_{j+1}(g_j(x)) - D^2A_{j+1}(f_j(x)))Df_j(x).$$

Clearly, $DA_{j+1}(g_j(x))$ is bounded. Also

$$|\gamma_2(x)| \leq |D^2A_{j+1}|_\infty|D^3A_{j+1}|_\infty|Df_j|_\infty|g_j(x) - f_j(x)|$$

and so there is some $M_4 > 0$ such that $|P_j(x)| < M_4w(|J_j|)|J_j|$.

We conclude the proof of the lemma by observing that this means that the value of (B.1) is small.

□

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