<u>Some New Surfaces with</u> $p_g = 0$

by

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Summary

We give families of examples of surfaces of general type X with $p_g = 0$, $K^2 = 1$ double covered by surfaces T with $p_g = 0$, $K^2 = 2$.

In Chapter 2 we classify all such constructions with $|\pi_1(T)| = 8$, giving 4-parameter families of surfaces X for which $\pi_1(X) = \mathbb{Z}_2$ and $\mathbb{Z}_4$. There is a complete description of surfaces with $p_g = 0$, $K^2 = 1$, $\pi_1 = \mathbb{Z}_4$ in [R1]. There was one example S with $H_1(S,\mathbb{Z}) = \mathbb{Z}_2$ in [O&P]. The most interesting construction is the one in Chapter 3, for which $\pi_1 X = \{1\}$. This answers negatively the following question "are all simply connected surfaces with $p_g = 0$ $K^2 > 0$ rational" coming from Severi's conjecture.

These constructions were motivated by Reid's conjecture that if a given fundamental group H occurs, there should be examples $X = T/\mathbb{Z}_2$ with $\pi_1(X) = H$.

In the Appendix we give an alternative proof of a formula for the arithmetic genus of a quotient surface, based on a remark of Hirzebruch.

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I would like to thank my family for staying friends with me!

Introduction

For a survey of surfaces with $p_g = 0$ giving classical references see [Dolg]. Several classical examples are also presented in [Be].

Rationality conditions

After Clebsh proved that all curves with $p_g = 0$ are rational, there was interest in finding characterizations of rational surfaces. In 1894 Enriques found an irrational surface X with $p_g = q = 0$, but $p_2 = h^0(X,\omega_X^{\otimes 2}) \neq 0$ (obtained as a quotient by a fixed-point-free involution of a certain K3 surface; see [G&H]). In 1896 Castelnuovo showed that the ammended conditions $p_2 = q = 0$ do imply rationality. Several more surfaces with $p_g = q = 0$ were given but they were all elliptic surfaces until 1931-2, when Godeaux and Campadelli each gave examples of general type. Godeaux's was the quotient $X = Q/\mathbb{Z}_5$ where $\mathbb{Z}_5$ acts freely on the Fermat quintic $Q \subset \mathbb{P}^3$, and has $K^2 = 1$. Campadelli's was a double cover of the plane, with $K^2 = 2$ and $H_1(S,\mathbb{Z}) \cong \mathbb{Z}_2^3$. So far all irrational surfaces with $p_g = q = 0$ had $H_1(S,\mathbb{Z}) \neq 0$. In 1949 Severi conjectured that the conditions $H_1(S,\mathbb{Z}) = 0$, $p_g = 0$ should imply rationality. This was refuted in 1966 by Dolgachev's simply connected irrational elliptic surfaces with $p_g = q = 0$ $(K^2 = 0)$ (see [Dolg]). He classified these surfaces and asked if they are the only counterexamples to Severi's conjecture. Equivalently, do Severi's conditions together with $K^2 > 0$ imply rationality? The example of Chapter 3 of a simply connected surface of general type with $p_g = 0$ $K^2 = 1$ shows this is not the case.

Classification of surfaces with $p_g = 0$ $K^2 = 1$

Bombieri showed that surfaces of general type with $p_g = 0$ have $q = 0$ and $|\mathrm{Tors}| \leq 5$ (where $\mathrm{Tors} = \mathrm{Tors}\ H_1(S,\mathbb{Z})$) . Miyaoka showed that for $\mathrm{Tors}\ S = \mathbb{Z}_5$ the moduli space of S is connected; in particular $\mathrm{Tors} = \pi_1$ by Godeaux's example. He gave an example with $\mathrm{Tors} = \mathbb{Z}_4$, and Reid gave a complete description of the cases $\mathrm{Tors} = \mathbb{Z}_3, \mathbb{Z}_4, \mathbb{Z}_5$ in [R1]. The construction shows the moduli spaces are connected, and that $\mathrm{Tors} = \pi_1$ again for $\mathbb{Z}_4$. For $\mathrm{Tors} = \mathbb{Z}_3$ it is conjectured that $\mathrm{Tors} = \pi_1$ but this is unknown. In [O&P] an example with $\mathrm{Tors} = \mathbb{Z}_2$ is constructed using Campadelli's double plane method. Reid and Catanese have almost proved the moduli space is connected in this case, so the family of examples with $\pi_1 = \mathbb{Z}_2$ given in Chapter 2 are deformations of this example and show $\pi_1 = \mathrm{Tors}$.

The case $\mathrm{Tors} = \mathbb{Z}_2^2$ cannot occur (see [R1]). The example of Chapter 3 shows S can be torsion free - it is unknown whether the moduli space is connected.

Method

The examples given in Chapters 2 and 3 are obtained as quotients $X = T/\mathbb{Z}_2$, where T is a minimal surface of general type with $p_g = 0$ $K^2 = 2$ and $\mathbb{Z}_2$ has 4 fixed points. By a conjecture of Reid (see below), families of such surfaces $X = T/\mathbb{Z}_2$ should exist.

To find them we look for subfamilies of the family of Galois étale covers $Y \to T$ corresponding to $\pi_1^{alg}(T)$ (shown to be finite by Beauville) for which there is a suitable extension of the action of $\pi_1^{alg}(T)$ to an action of G on Y with $G/\pi_1 = \mathbb{Z}_2$. To decide where to look we use integrality conditions coming from the Holomorphic Lefschetz fixed point formula.

The hard part is to check that a proposed construction is nondegenerate, i.e. the existence of the G-action with the required fixed loci does not force Y to be singular.

This is expressed formally in §11. It generalizes Reid's method of finding and classifying examples with unramified $\mathbb{Z}_3$, $\mathbb{Z}_4$ and $\mathbb{Z}_5$ covers [R1].

Motivation

Let S be a surface with $p_g = 0$, $K^2 = 1$. By Kuranishi's theorem S has a local deformation space Z of dimension $d \geq -\chi(T_S) = \frac{1}{6}(7K_S^2 - 5e(S))$. Since $K_S^2 = 1$ we have $e(S) = 11$ by Noether's formula, hence $d \geq 8$.

By considering the structure of $H^2(S,\mathbb{Z})^+$, Reid conjectured that there should exist a subspace $Z_0 \subset Z$ of $\dim \geq 4$ corresponding to deformations S_0 of S whose canonical model X is a quotient $X = T/Z_2$ as above, ("one condition on Z is imposed per node")

Although the conjecture does not have a precise justification, it is verified (incidentally) in the cases Tors $= Z_2$ (subject to irreducibility of moduli) and Z_4 by the 4-parameter families of Chapter 2. Also for Tors $= Z_5$ there is a 4-parameter family $X = T/Z_2$ associated to Catanese's surfaces T (see 18.8.2). This was additional incentive for trying the torsion free case. The construction of Chapter 3 depends on at least 2 parameters but the full family has not yet been described.

CHAPTER 1 : Preliminaries

0. Notation

A *variety* V is a quasiprojective variety over an algebraically closed field k. We will assume $k = \mathbb{C}$, although most of what follows holds for fields k of characteristic $p > 0$.

We will consider both the Zariski topology and the analytic topology on V.

The singular locus of V is written $\operatorname{Sing} V$. We write:

$k(V)$ = function field of V

$\mathcal{O}_V$ = sheaf of regular functions $V \to k$

$\Omega^1_{k(V)/k}$ = sheaf of regular 1-forms of V over k.

$\Omega^n_{k(V)/k} = \Lambda^n\, \Omega^1_{k(V)/k}$

$T_v V$ = Tangent space to $v \in V$.

$\pi_1(V)$ = Topological fundamental group of V.

$e(V)$ = Euler number of V

$$= \sum_i (-1)^i\, h^i(V,\mathbb{C}) \quad .$$

For a coherent sheaf L on V,

$$h^i(L) = h^i(V,L) = \dim H^i(V,L)$$

and

$$\chi(L) = \Sigma(-1)^i\, h^i(V,L) \quad .$$

If L is an invertible sheaf corresponding to a Cartier divisor D on V, we also write

$$h^i(D) = h^i(\mathcal{O}_V(D)) = h^i(L) \quad .$$

1. Formulas for Nonsingular surfaces

Let S be a nonsingular projective surface. For divisors D_1, D_2, $D_1.D_2$ denotes the intersection number.

The *canonical divisor* K_S and sheaf ω_S are defined by

$$\omega_S = \Lambda^2 \, \Omega_{k(S)/k} = \mathcal{O}_S(K_S) \quad .$$

The *geometric genus* p_g *of* S is defined

$$p_g(S) = h^0(S,\omega_S) \quad ,$$

and the *irregularity* q of S is defined

$$q(S) = h^1(S,\omega_S) = h^0(S,\Omega_{k(S)/k}) \quad .$$

Since $h^0(S,\mathcal{O}_S) = 1$ (the only regular functions $S \to \mathbb{C}$ are the constant functions) and $h^0(S,\mathcal{O}_S) = h^2(S,\mathcal{O}_S)$ (by Serre Duality) we have

$$\chi(\mathcal{O}_S) = p_g(S) - q(S) + 1 \quad .$$

1.1. Riemann-Roch Formula:

$$\chi(\mathcal{O}_S(D)) = \chi(\mathcal{O}_S) + \tfrac{1}{2}\,(D^2 - D.K_S)$$

for any divisor D on S .

1.2. Noether's Formula:

$$\chi(\mathcal{O}_S) = \frac{1}{12}\,(K_S^2 + e(S)) \quad .$$

1.3. Adjunction Formula:

For a nonsingular curve C of genus g on S :

$$C^2 + C.K_S = 2g - 2 \; .$$

2. Group Actions and Quotients

2.1. Definitions

A finite group G is said to *act* on the variety Y if it acts by algebraic automorphisms. For $y \in Y$, $g \in G$ we write $g(y)$ for the image of y under the action of g on Y.

The *fixed locus* of $g \in G$ is the set $Y^g = \{y \in Y : g(y) = y\}$.

The *stabilizer* of $y \in Y$ is the subgroup $G_y = \{g \in G : g(y) = y\}$.

The *elliptic* elements $g \in G$ are those with nonempty fixed loci. The *elliptic subgroup* of G (which will be labelled E thoughout most of this thesis) is the subgroup generated by elliptic elements.

2.2. Remarks

i) $\forall\, h,g \in G$, $h:Y \to Y$ induces an isomorphism:

$$Y^g \to Y^{hgh^{-1}} \quad .$$

In particular this implies that the elliptic subgroup E is normal in G .

ii) $\forall\, h,g \in G$, $hG_y h^{-1} = G_{hy}$.

iii) $Y^g \subseteq Y^{g^k} \quad \forall\, k \in \mathbb{Z}$.

iv) The action of G_y at y induces an action of G_y on T_yY . Throughout, we will write r_y for the representation of G_y on T_yY .

If $\forall\, g \in G_y$, $\mathrm{Det}(r_y(g)) = 1$, we will write

$$r_y : G_y \subset SL(n) \quad ,$$

where $n = \dim Y$.

2.3. <u>Quotients</u> ([M] p.66)

Suppose the finite group G acts on the quasiprojective variety Y . Then there exists a finite separable surjective morphism of varieties $\pi : Y \to X$ such that

i) As a topological space, X is the quotient for the G-action. We write $X = Y/G$.

ii) $\mathcal{O}_X = (\pi_* \mathcal{O}_Y)^G$. In particular, if Y is normal then X is normal.

iii) π is étale at $y \in Y$ if and only if $G_y \cong \{1\}$. So if Y is nonsingular, X is singular at most under points $y \in Y$ with $G_y \neq \{1\}$.

2.4. <u>Lemma on Normalization in a Galois extension</u> (well known)

Let X be a normal variety, and let $\phi : k(X) \hookrightarrow L$ be a Galois extension of its function field. Let $\pi : Y \to X$ be the normalization of X in L ([Shaf] p.266) and let $G = \mathrm{Gal}(L:k(X))$. Then G acts on Y and π is the quotient map.

Proof

For an affine subset $U = \operatorname{Spec} A$ of X, the normalization $\pi^{-1}(U)$ of U in L is equal to Spec B, where B is the integral closure of A in L. The action of G on L restricts to an action on B. Now $A \subseteq L^G \cap B = B^G$, but B^G is integral over A which is integrally closed in $L^G = k(X)$ (since X is normal). Hence $B^G \subseteq A$, giving $B^G = A$. Thus π is the quotient for a G action over U. To prove the lemma it suffices to check that if $U_1 \subset U_2$ are affine subsets of X then the G action on $\pi^{-1}(U_2)$ restricts to the G-action on $\pi^{-1}(U_1)$. Since both are given by restrictions of the G action on L, this is clear.

3. The Canonical Sheaf of a Normal Variety

3.1. Definitions

In ([R3], Appendix to §1), Reid discussed a definition of the *canonical sheaf* ω_V where V is a normal variety of dimension n:

$$\omega_V = \{s \in \Omega^n_{k(V)/k} : s \text{ regular in codim. } 1\} .$$

Note that if V is nonsingular this is the usual definition $\omega_V = \Omega^n_{k(V)/k}$.

By ([R3] §1 Thm. 7), ω_V is a divisional sheaf. By ([R3] §1 Thm. 3), this implies that there exists a unique Weil divisor K_V such that $\mathcal{O}_V(K_V) = \omega_V$, called the *canonical divisor*.

3.2. Remarks

i) In chapters 2 and 3 all surfaces X will have invertible ω_X (see (4.3)).

ii) In the Appendix we also consider quotient surfaces $X = Y/G$ where Y is nonsingular. We will see that $|G|.K_X$ is then Cartier (7.1(ii)).

iii) ω_V is the dualizing sheaf for Serre duality. So for a surface X ,

$$h^0(X,\omega_X) = h^2(X,\mathcal{O}_X) \ .$$

4. Some Surface Singularities

Let X be a normal surface, $p \in X$ an (isolated) singularity.

4.1. $p \in X$ is *rational* if for a resolution $f:S \to X$, $R^1 f_* \mathcal{O}_S = 0$. This implies that $h^i(S,\mathcal{O}_S) = h^i(X,\mathcal{O}_X)$ $(i = 0,1,2)$.

4.2. p is of *type* (q,n) , where q,n are positive coprime integers with $q < n$, if $p \in X$ is locally analytically isomorphic to $\mathbb{A}^2/\mathbb{Z}_n$, where $\mathbb{Z}_n$ acts by $(x,y) \to (\varepsilon x,\varepsilon^q y)$ for ε a primitive n^{th} root of 1.

This is a special kind of rational singularity ([Br]) and will be considered in the Appendix.

Another important kind of rational singularities are Du Val points (also known as rational double points, Klein singularities, etc.):

4.3. Du Val Points

Let X be a surface. The Du Val points $p \in X$ are isolated singularities which can be defined in several ways (see [D]). We use the following two:

i) $p \in X$ is Du Val if and only if $p \in X$ is locally analytically isomorphic to $\mathbb{A}^2/G$, where G is a finite subgroup of $SL(2)$.

ii) $p \in X$ is Du Val if and only if ω_X is invertible at p , and for a minimal resolution $f:S \to X$ we have $f^*\omega_X = \omega_S$.

From (ii) it follows that if C is an irreducible curve in the exceptional set $f^{-1}(p)$ then $C.K_S = 0$. Since C is exceptional, $C^2 < 0$. The adjunction formula implies that C is a nonsingular rational curve with $C^2 = -2$, called a -2 *curve*.

The exceptional set $f^{-1}(p)$ is easily seen to be a configuration A_n, D_n, E_6, E_7 or E_8 of (-2) curves. This can be taken as another characterization of Du Val points. For tables showing the relation between the type of configuration and the group G in (i), see ([Pl] p. 4 & 11).

4.4. Example

A *node* p on the surface X is a rational double point given locally by $x^2 + y^2 + z^2 = 0$ or $xy = z^2$.

It is also an A_1 singularity: the minimal resolution $f:S \to X$ is a single blow up at p, with $f^{-1}(p) = C$ a (-2) curve. The group G in (i) is $\mathbb{Z}_2$ acting by $(s,t) \to (-s,-t)$. Hence $p \in X$ is locally analytically isomorphic to Spec $k[s^2,t^2,st]$. Putting $x = s^2$, $y = t^2$, $z = st$ gives the second of the above equations.

5. Minimal Surfaces of General Type

5.1. Definitions

Let S be a nonsingular projective surface.

i) A divisor D on S is *nef* (numerically effective) if for all curves C on S,

$$D.C \geq 0 .$$

ii) S is a *minimal surface of general type* if K_S is nef and $K_S^2 > 0$.

5.2. Canonical Models ([B])

If S is a minimal surface of general type then the only curves C with $K_S.C = 0$ are (-2) curves, and there are a finite number of these. There is a map $f:S \to X$ contracting all the (-2) curves to Du Val points. X is called the *canonical model* of S, and S is the minimal resolution of X.

Since X has Du Val points, K_X is Cartier and $f^*K_X = K_S$; so $K_X^2 = K_S^2$. We have contracted all curves C with $K_S.C = 0$, so for any irreducible curve $\Gamma \subset X$, $K_X.\Gamma > 0$. By Nakai's criterion ([Ha.2] p.29), it follows that K_X is ample.

Let $R(S) = \bigoplus_{m\geq 0} H^0(S,mK_S) = \bigoplus_{m\geq 0} H^0(X,mK_X)$. This is the *canonical ring* of S ; and $X \cong \mathrm{Proj}\, R(S)$.

6. Fundamental Groups

6.1. Definitions

The *algebraic fundamental group* $\pi_1^{alg}(X)$ of a variety X is the inverse limit of the Galois groups of the finite étale covers of X ([SGA], [Mi] (Introduction)).

If X is a complex variety, every finite topological covering $\pi:Y \to X$ has a complex structure making π a finite étale cover, ([G&R], p.267). This means that $\pi_1^{alg}(X)$ can be identified with the profinite completion of $\pi_1(X)$.

Both π_1 and π_1^{alg} will turn up later.

6.2. Birational Invariance of π_1

It is well known that if X, X' are nonsingular complex varieties and $f:X \dashrightarrow X'$ is a birational map between them then $\pi_1(X) \cong \pi_1(X')$, ([G&H], p.494).

The same is not true for singular varieties in general. However, in certain cases Van Kampen's theorem can be used to show that π_1 is still invariant. The following is a well known example.

6.3. Birational Invariance of π_1 for surfaces with rational singularities

Let x be a rational singularity of the surface X. Let $f:S \to X$ be a minimal resolution. Then $\pi_1(S) = \pi_1(X)$.

Proof (pointed out by I. Nakamura)

Let V be a closed analytic neighbourhood of x, and let $U = f^{-1}(V)$. Then U has the homotopy type of the exceptional set $E = f^{-1}(x)$. By a criterion for rationality given in [P1], E is a tree of rational curves. So $\pi_1(E) = \{1\}$. Since a path $\gamma \in \pi_1(V,x)$ lifts to a path in $\pi_1(U)$, the map $\pi_1(U) \to \pi_1(V)$ is surjective. Hence $\pi_1(V) = \{1\}$.

Van Kampen's theorem applied to X,V and S,U gives:

$$\pi_1(S) = \pi_1(U)*\pi_1(S - (U-\partial U))/\sim$$

$$\pi_1(X) = \pi_1(V)*\pi_1(X - (V-\partial V))/\sim.$$

Hence $\pi_1(S) \cong \pi_1(X)$.

6.4. Fundamental Group of a Quotient (well known)

Let Y be a normal variety with $\pi_1^{alg}(Y) = \{1\}$, (respectively, $\pi_1(Y) = \{1\}$). Let G be a finite group acting on Y with elliptic subgroup E, (recall that $E \triangleleft G$). Let $X = Y/G$.

Then $\pi_1^{alg}(X) = G/E$ (respectively, $\pi_1(X) = G/E$).

Proof

Let $T \to X$ be any finite étale cover. Take the fibre product with Y :

$$\begin{tikzcd} T \times_X Y \arrow[r] \arrow[d] & Y \arrow[d] \\ T \arrow[r] & X \end{tikzcd}$$

Since $T \to X$ is étale, $T \times_X Y \to Y$ is étale. Since $\pi_1^{alg}(Y) = \{1\}$, this implies that $T \times_X Y \cong \coprod Y_i$ = (disjoint copies of Y) . Hence $Y \to X$ factorizes as $Y \to T \to X$. By Galois theory and (2.4) this gives the result.

The proof for π_1 is similar, using homotopy lifting.

6.5. A Lifting Lemma

Consider the tower of morphisms of varieties:

$$\begin{tikzcd} Y \arrow[d, "h"] \arrow[dd, bend right=60, "g"'] \\ S \arrow[d, "f"] \\ X \end{tikzcd}$$

where i) Y is nonsingular and $\pi_1^{alg}(Y) = \{1\}$.

ii) h is Galois with group H acting freely.

iii) f is Galois with group F , and is unramified in codimension 1.

Then $g:Y \to X$ is Galois with group G such that $H \lhd G$ and $G/H = F$.

Proof

Let S_0 = fixed point free locus of F. Let $X_0 = f(S_0)$ and $Y_0 = h^{-1}(S_0)$. Then $g:Y_0 \to X_0$ is an étale cover. Since codim $(Y-Y_0) \geq 2$, by (iii), we have $\pi_1(Y_0) = \{1\}$. Hence $g:Y_0 \to X_0$ is Galois. By (2.4) the group $G = \mathrm{Gal}(Y_0,X_0)$ acts on Y_0 with quotient X_0. Since $k(Y) = k(Y_0)$ and $k(X) = k(X_0)$, we can apply (2.4) to $g:Y \to X$. Hence G acts on Y with quotient X. By Galois theory $H \lhd G$ with $G/H = F$.

7. Weil Divisors on a Quotient Variety

7.1. Proposition

Let G be a finite group acting on the nonsingular variety Y. Let $\pi:Y \to X$ be the quotient map, and let D be a Weil divisor on X. Then

i) We can define a Cartier divisor π^*D on Y which agrees with π^*D_0 on Y_0 (where D_0 is the restriction of D to the nonsingular locus X_0 of X, and $Y_0 = \pi^{-1}(X_0)$). Hence π^* defines a homomorphism $\mathrm{Div}X \to \mathrm{Div}Y$, which agrees with that already defined on Cartier divisors.

ii) If $n = |G|$ then nD is a Cartier divisor on X.

Proof (pointed out by Reid)

i) Define π^*D to be the closure in the Zariski topology of π^*D_0 (considered as a Weil divisor on Y_0) . Since X is normal (by 2.3(ii)), $\operatorname{codim}(X-X_0,X) = \operatorname{codim}(Y-Y_0,Y) \geq 2$. So π^*D is the unique divisor on Y which agrees with π^*D_0 on Y_0 .

ii) Let $p \in X$, and let $s \in k(Y)$ be a defining equation for π^*D at $q \in \pi^{-1}(p)$. Then $t = \prod_{g \in G} g^*s$ is a defining equation for π^*nD at q , and $t \in k(Y)^G \cong k(X)$. So $t = \pi^*t_0$ for some $t_0 \in k(X)$, and t_0 is a defining equation for nD at $p \in X$. Hence nD is Cartier.

7.2. Applications to Quotient Surfaces

With notation as in (7.1), let Y be a surface. Then (7.1) implies the following.

7.2.1.

For Weil divisors D_1, D_2 on X , there is a well defined intersection pairing given by

$$D_1 . D_2 = \frac{1}{n^2} (nD_1).(nD_2) \in \mathbb{Q} .$$

This follows from (7.1.(i)).

7.2.2.

Assume all fixed loci of G are finite, so that $\pi : Y \to X$ is unramified in codimension 1.

Then

$$\pi^*K_X = K_Y .$$

This follows from the definition of π^*K_X (K_X is a Weil divisor by (3.1) since X is normal).

Since $(\pi^*(nK_X))^2 = n.(nK_X)^2$, (7.2.1) gives

$$K_Y^2 = nK_X^2 .$$

7.2.3.

Assume that for all $y \in Y$, $r_y:G_y \subset SL(2)$. (See (2.2(iv))).

Then:

i) Fixed loci are finite, because $r_y(g) = \begin{pmatrix} \varepsilon_y & 0 \\ 0 & \varepsilon_y^{-1} \end{pmatrix}$ in suitable coordinates, where ε_y is a primitive n^{th} root of 1.

(N.B. This is something which fails in higher dimensions.)

ii) The singularities of X are Du Val (by characterization (i) of (4.3)). Hence K_X is Cartier. Let $f:S \to X$ be the minimal resolution.

Then $\qquad f^*K_X = K_S \qquad$ (by (4.3(ii)))

and $\qquad \pi^*K_X = K_Y \qquad$ (by (7.2.2)).

So $K_S^2 = K_X^2 = \frac{1}{n} K_Y^2$, and if K_Y is nef then so is K_S .

<u>Corollary</u>

If Y is a minimal surface of general type (5.1.(ii)) then so is S .

8. A Lefschetz Formula and Applications

8.1. Theorem ([A&B] p.458)

Let G be a finite group acting on the nonsingular variety Y with finite fixed loci. Let r_Y be the representation of G_y induced on T_yY. For $F = \mathcal{O}_Y$ or $\omega_Y^{\otimes m}$, G acts naturally on the cohomology groups $H^i(Y,F)$. Let

$$L(F,g) = \Sigma(-1)^i \operatorname{Tr}(g|H^i(Y,F)) ,$$

for $g \in G$. Then

i) $L(\mathcal{O}_Y,g) = \sum\limits_y \det(1-r_y(g))^{-1}$,

where the sum runs over the fixed points of g.

We abbreviate $L(\mathcal{O}_Y,g) = L(g)$.

ii) If for all $y \in Y$ with $G_y \neq \{1\}$ we have $r_y : G_y \subset SL(n)$, where $n = \dim Y$, then (for $g \neq 1$)

$$L(\omega_y^{\otimes m},g) = L(g) \qquad \forall\, m \geq 0 .$$

8.1.1. Remarks

(4.1)(i) and (ii) follow easily from the Holomorphic Lefschetz formula for finite fixed loci given in ([A&B] p.458).

For more general formulas, evaluating $L(\mathcal{F},g)$ for an arbitrary G-sheaf $\mathcal{F}$ (defined in ([M] p.69)) with arbitrary fixed loci, see ([A&S] p.566). In particular the last reference evaluates $L(g)$ explicitly in the case of a group acting on a surface Y with fixed curves and points.

8.1.2. Examples

i) If g acts freely, then $L(g) = 0$.

ii) If $g^2 = 1$ and g fixes t points, then

$$L(g) = \sum_y \det(1 - \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix})^{-1}$$

$$= \sum_y \det\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}^{-1} = \frac{t}{4} .$$

iii) If $r_y(g) = \begin{pmatrix} \varepsilon_y & 0 \\ 0 & \varepsilon_y^{-1} \end{pmatrix}$ $\forall\ y \in Y^g$, then

$$L(g) = \Sigma_y\ (1-\varepsilon_y)^{-1}(1-\varepsilon_y^{-1})^{-1} .$$

8.2. A G-equivariant Riemann-Roch Formula([A&S] p.566)

Let $\pi: Y \to X$ be the quotient map for the action of G in (8.1). Then

$$\chi(\mathcal{O}_X) = \frac{1}{|G|}\ (\chi(\mathcal{O}_Y) + \sum_{\substack{g \neq 1 \\ g \in G}} L(g))$$

8.2.1. Derivation from 8.1.

By definition of the quotient X , we have

$$H^i(Y,\mathcal{O}_Y)^G \cong H^i(X,\mathcal{O}_X) \qquad \forall\, i \;.$$

The dimensions of the invariant subspaces are given by the following well known lemma.

Lemma ([H&Z] p.21)

Let G be a finite group acting linearly on the $\mathbb{C}$-vector space V , and let V^G be the subspace on which G acts trivially. Then

$$\dim V^G = \frac{1}{|G|} \sum_{g\in G} \mathrm{Tr}(g|V) \;.$$

It follows that

$$|G|.\chi(\mathcal{O}_X) = \sum_{g\in G} L(g) = \chi(\mathcal{O}_Y) + \sum_{\substack{g\neq 1\\ g\in G}} L(g)$$

since $L(1) = \chi(\mathcal{O}_Y)$, and this is (8.2).

8.3. Integrality Conditions

A necessary condition for the existence of an action of G on Y with prescribed fixed loci is that the functions $L(g)$ they determine as in (8.1), are virtual characters of G .

This implies (by Lemma 4.5) that

$$\sum_{g \in G} L(g) \in |G|.\mathbb{Z} \quad .$$

We use this condition as a first check in Chapters 2 and 3.

The following definition is convenient:

8.3.1. Weight of a group action

For a group G acting on Y with prescribed fixed loci, we define the *weight* $W(G)$ by

$$W(G) = \sum_{g \neq 1} L(g) \quad .$$

By (8.2) we have

$$W(G) = |G|.\chi(\mathcal{O}_X) - \chi(\mathcal{O}_Y) \quad .$$

Notice that if G acts freely then $W(G) = 0$.

8.3.2. Example

Suppose $G \cong \mathbb{Z}_n$ acts on the surface Y , fixing t points. If at each fixed point $\mathbb{Z}_n$ acts as a subgroup of $SL(2)$ on the tangent space, then by (8.1.2(iii)) we have

$$\begin{aligned} W(G) &= t \sum_{k=1}^{n-1} (1-\varepsilon^k)^{-1}(1-\varepsilon^{-k})^{-1} \\ &= t.\frac{(n^2-1)}{12} \quad . \end{aligned}$$

So $W(G) \geq 0$, with equality if and only if G acts freely.

The condition

$$\chi(\mathcal{O}_{Y/G}) = \frac{1}{|G|}(\chi(\mathcal{O}_Y) + W(G)) \in \mathbb{Z}$$

places restrictions on t .

8.4. Application of (8.2) to the resolution of a quotient surface

Let Y be a projective surface, with a finite group G acting as in 4.1. Let $S \to X$ be a resolution of $X = Y/G$. Then

$$\chi(\mathcal{O}_S) = \chi(\mathcal{O}_X)$$

$$= \frac{1}{|G|}(\chi(\mathcal{O}_Y) + \sum_{\substack{g \neq 1 \\ g \in G}} (\sum_{y \in Y^g} \det(1 - r_y(g))^{-1})) \quad .$$

Proof

By [Br], X has rational singularities. Hence by (4.1) $\chi(\mathcal{O}_S) = \chi(\mathcal{O}_X)$.

The above formula comes from putting the values of $L(g)$ given by (8.1) into formula (8.2).

8.4.1. Example (used in Chapters 2 and 3)

Let T be a minimal surface of general type with $p_g = q = 0$, $K^2 = 2$. Suppose there is an action of $\mathbb{Z}_2$ on T with exactly 4 fixed points. Then the quotient X has minimal resolution S a minimal surface of general type with $p_g = q = 0$, $K^2 = 1$.

Proof

By (8.1.2) and (8.2) we have

$$\chi(\mathcal{O}_S) = \chi(\mathcal{O}_X) = \frac{1}{2}\,(\chi(\mathcal{O}_T) + \frac{t}{4})$$

(where t is the number of fixed points of $\mathbb{Z}_2$, so $t = 4$)

$$= 1 \quad .$$

Since $p_g(S) = p_g(X) \leq p_g(Y) = 0$, this gives $p_g(S) = q(S) = 0$. By (7.2.2) we have $K_X^2 = K_S^2$ and $K_X^2 = \frac{1}{2}\,K_T^2 = 1$, and S is a minimal surface of general type by (7.2.3).

8.5. Lemma (used only in Appendix)

In the appendix we give an alternative proof of the formula for $\chi(\mathcal{O}_S)$ of (8.4), based on calculating K_S^2 , $e(S)$ and hence $\chi(\mathcal{O}_S)$. For cyclic quotient singularities we have Hirzebruch's resolution for which the arithmetic is nice. The following Lemma shows this is all we need.

Lemma

Suppose (8.4) holds whenever G is cyclic. Then it holds in general.

Proof

Let a,b be two functions $G \to C$. Assume that for cyclic subgroups $H < G$ we have

$$\sum_{h \in H} a(h) = \sum_{h \in H} b(h) .$$

We show by induction on r that for any $S = \bigcup_{i=1}^{r} H_i \subset G$ with H_i cyclic subgroups of G, we have

$$\sum_{s \in S} a(s) = \sum_{s \in S} b(s) . \qquad (*)$$

This holds for $r = 1$ by assumption. Assume $(*)$ holds for $r < R$. Let

$$S = \bigcup_{1}^{R} H_i = S_1 \cup S_2 ,$$

where $S_1 = H_R$ and $S_2 = \bigcup_{1}^{R-1} H_i$.

Then $$\sum_{s \in S} a(s) = \sum_{s \in S_1} a(s) + \sum_{s \in S_2} a(s) - \sum_{s \in S_1 \cap S_2} a(s) .$$

By the inductive hypothesis we have $\sum_{s \in S_i} a(s) = \sum_{s \in S_i} b(s)$ $(i = 1 \text{ or } 2)$, and also $\sum_{s \in S_1 \cap S_2} a(s) = \sum_{s \in S_1 \cap S_2} b(s)$, since $S_1 \cap S_2 = \bigcup_{i=1}^{R-1} (H_R \cap H_i)$.

Hence $\sum_{s \in S} a(s) = \sum_{s \in S} b(s)$.

Putting $a(g) = L(g)$

$$b(g) = \sum_{y:gy=y} \det(1 - r_y(g))^{-1}$$

with $S = G$ gives the Lemma.

CHAPTER 2 : Groups of order 16 acting on complete intersections of 4 quadrics

9. Involutions on Godeaux-Reid Surfaces

9.1. Godeaux Reid Surfaces ([G] and [R2])

Let T be a minimal surface of general type with $p_g = q = 0$, $K^2 = 2$, $|\pi_1^{alg}| = 8$. Let $Y \to T$ be the Galois étale cover corresponding to $\pi_1^{alg}(T) \cong H$, so that $\pi_1^{alg}(Y) = \{1\}$ and $T = Y/H$ (with H acting freely).

The first examples of such surfaces were Godeaux's quotients $T = Y/H$, where Y is a complete intersection of 4 quadrics in $\mathbb{P}^6$, and $H = \mathbb{Z}_8$ acts freely on Y, [G]. In [R2] Reid showed that conversely all surfaces T with the above invariants can be obtained in this way. He gave families of examples $T = Y/H$ for all groups H of order 8 except the dihedral group, which cannot occur (see (15.5)). In particular the surfaces T have $\pi_1(T) = \pi_1^{alg}(T)$.

For proof that T has the claimed invariants, see proof of (10.1).

9.2. Involutions

Suppose there is an involution on T with just 4 fixed points. By (8.4.1.) the minimal resolution S of the quotient $X = T/\mathbb{Z}_2$ is a minimal surface of general type with $p_g = q = 0$, $K^2 = 1$. According to (6.4) there is an action of a group G of order 16 on Y extending the action of H and inducing the original involution on T.

Thus a classification of surfaces $X = T/\mathbb{Z}_2$ (where $\mathbb{Z}_2$ acts with just 4 fixed points) is equivalent to a classification of surfaces

$X = Y/G$, where $Y = \cap Q_i$ is a complete intersection of 4 quadrics in $\mathbb{P}^6$ and G is a group of order 16 with action on Y satisfying:

1) Some subgroup $H \lhd G$ of order 8 acts freely on Y ;

and 2) G acts freely outside a set of 32 points, each of which is fixed by just one involution.

10. Groups of order 16 acting on complete intersections of quadrics

10.1. The condition (*) and its relation to (9.2) ((1) and (2))

Throughout this section, let G be a group of order 16 acting on a complete intersection $Y = \cap Q_i$ of 4 quadrics in $\mathbb{P}^6$, such that conditions (*) hold:

$$* \begin{cases} \text{(i) } \forall\, y \in Y \text{ with } G_y \neq \{1\} \text{, we have} \\ \qquad r_y : G_y \subset SL(2) \text{ .} \\ \text{(ii) } W(G) = 8 \text{ , or equivalently} \\ \qquad \text{(see (10.2)), } \chi(\mathcal{O}_{Y/G}) = 1 \text{ .} \end{cases}$$

The notation r_y , W can be recalled from (2.2), (8.3) respectively.

The motivation for considering (*) is that it is (at first sight) considerably weaker than (1) and (2), but preserves all the properties of X except that of having a double cover $T = Y/H$ with $p_g = 0$, (see (10.2)). For example, condition (2) alone implies (*).

However, in §12-§14 we give three examples $X = Y/G$ for which (1) and (2) hold, and in §15 we show that these are the only ones for which (*) holds. So in fact (*) is equivalent to (1) and (2).

To prove the Theorem of §15 we obtain contradictions to the existence of any other examples using an algorithm (11.2) for 'realizing prescriptions' of group actions. This is the same algorithm as is used to obtain the families of examples in §12-§14.

10.2. Proposition

Let $X = Y/G$ and let $S \to X$ be the minimal resolution. Then S is a minimal surface of general type with $p_g = q = 0$, $K^2 = 1$, and $\pi_1 = G/E$ (where E is the elliptic subgroup defined in (2.2)).

Proof

i) Invariants of Y : It is well known that complete intersections are simply connected (as a consequence of the Lefschetz Hyperplane theorem), so $\pi_1(Y) = \{1\}$ and hence $q(Y) = 0$.

The formula for the canonical divisor of a complete intersection ([Hal] p.188) shows that Y is canonically embedded. Hence

$$p_g(Y) = 7 , \qquad \chi(\mathcal{O}_Y) = 8 ,$$

and $$K_Y^2 = \deg Y = 16 .$$

By (5.1), Y is a minimal surface of general type.

ii) Invariants of S : By (7.2), S is a minimal surface of general type with

$$K_S^2 = K_X^2 = \frac{1}{|G|} \cdot K_Y^2 = 1 .$$

By (4.1) we have $h^i(S,\mathcal{O}_S) = h^i(X,\mathcal{O}_X)$.

Since $\chi(\mathcal{O}_X) = \frac{1}{16}(\chi(\mathcal{O}_Y) + W(G)) = 1$

and $h^1(X,\mathcal{O}_X) \leq h^1(Y,\mathcal{O}_Y) = 0$, this gives $p_g(S) = q(S) = 0$.

By (6.3), $\pi_1(S) = \pi_1(X)$.

By (6.4), $\pi_1(X) = G/E$.

10.3. The Action of G on the Canonical Ring $R(Y)$

Since Y is a canonically embedded complete intersection, the action of G induced on $R(Y) = \bigoplus_{m\geq 0} H^0(Y,mK_Y)$ determines the action of G on $\mathbb{P}^6$ which in turn restricts to the original action on Y .

i) Let $\omega_m(g) = T_r(g|H^0(Y,mK_Y))$, where $g \in G$ and $m \geq 1$. It is well known that

$$h^i(Y,\mathcal{O}_Y(m)) = 0 \qquad \forall\ i \geq 1\ ,\ m \geq 2$$

because Y is a complete intersection.

Since $\mathcal{O}_Y \cong \omega_Y$, this gives for $g \neq 1$:

$$\omega_m(g) = L(\omega_Y^{\otimes m}, g) \qquad \text{(see §8)}.$$

By (*i) we can apply (8.1ii) to give

$$\omega_m(g) = L(g) \quad \forall\ m \geq 2 .$$

and $$\omega_1(g) = L(g) - 1 .$$

We have $\omega_1(1) = p_g(Y) = 7$, and the values $\omega_m(1) = \chi(\omega_Y^{\otimes m})$ are given by the Riemann-Roch formula:

$$\chi(\omega_Y^{\otimes m}) = \chi(\mathcal{O}_Y) + \binom{m}{2} K_Y^2 = 8\ (1+m(m-1)) .$$

ii) There is an exact sequence of representations of G :

$$0 \to \Delta \to S^2H^0(Y,\omega_Y) \to H^0(Y,\omega_Y^{\otimes 2}) \to 0 ,$$

where Δ is the 4-dimensional subspace of $S^2H^0(Y,\omega_Y) \cong H^0(\mathbb{P}^6,\mathcal{O}_{\mathbb{P}^6}(2))$ consisting of quadrics vanishing on Y .

So the function $d:G \to \mathbb{C}$ defined

$$d(g) = S^2\omega_1(g) - \omega_2(g)$$

gives the character of the action of G induced on Δ . Here $S^2\omega_1$ denotes the character of the action induced on $S^2H^0(K_Y)$ by that on $H^0(K_Y)$. It is given by

$$S^2\omega_1(g) = \frac{1}{2}\,(\omega_1(g^2) + (\omega_1(g))^2) ,$$

(see [S] p.11.)

11. Method of Classifying surfaces $X = Y/G$ (as in §10)

11.1. Definitions

i) Let G be a finite group.

A *prescription* for an action of G on a surface is a set

$$\underline{G} = \{(i, H_i, r_i)\}_{i \in I}$$

where

I is a finite indexing set

H_i is a subgroup of G

r_i is an (isomorphism class of) representation of H_i on $\mathbb{C}^2$.

ii) A prescription is *realized* on the surface Y if G acts on Y such that

i) G acts freely outside a set $\{y_i\}_{i \in I}$ with $i_1 \neq i_2 \Rightarrow y_{i_1} \neq y_{i_2}$

ii) $G_{y_i} = H_i$

iii) $r_{y_i} = r_i$.

(Recall notation from (2.2).)

iii) A prescription $\underline{G}$ *satisfies* * if the action of G on a realization Y satisfies * of §10.

(This could also be stated purely in terms of the prescription:

$$r_i : H_i \subset SL(2) \qquad \forall\, i \in I$$

and
$$\sum_{\substack{g \neq 1 \\ \in G}} L(g) = 8 \,,$$

where
$$L(g) = \sum_{\{i : g \in H_i\}} \frac{1}{\det(1 - r_i(g))} \,.)$$

The following algorithm will be used for deciding which prescriptions can be realized and classifying realizations of a given prescription. It is used implicitly for fixed-point-free prescriptions in [R1] and [R2] and [C].

It will be stated for prescriptions satisfying (*), but can be generalized in several ways (see (11.3)).

11.2. Algorithm

Let $\underline{G}$ be a prescription for a group G of order 16, satisfying (*) .

1) Assume $\underline{G}$ has a realization on a complete intersection Y of 4 quadrics in $\mathbb{P}^6$.

2) The action of G induced on $R(Y)$ is given in (10.3). If the functions ω_m , d given by L are not characters of G , we have a contradiction to (1). Otherwise, they determine the action of G

on $\mathbb{P}^6$ and Δ (up to isomorphism; see (10.3)).

Choose a basis for $\mathbb{P}^6$ and an action of G on $\mathbb{P}^6$ with character ω_1 on $H^0(\mathbb{P}^6, \mathcal{O}_{\mathbb{P}^6}(1))$.

3) In practise it is now easy to write down the set S = {4-dimensional subspaces $\Delta \subset H^0(\mathbb{P}^6, \mathcal{O}_{\mathbb{P}^6}(2))$ such that G-action given by (2) restricts to an action on Δ with character d} .

4) Let $\mathcal{K}'$ = {subvarieties (possibly reducible) $Y \subset \mathbb{P}^6$ such that $I(Y)$ is generated by some $\Delta \in S$} .

Then by construction, $Y \in \mathcal{K}'$ if and only if $T_r(g|H^0(Y, \mathcal{O}_Y(m))) = \omega_m(g)$, where the values $\omega_m(g)$ for $g \in G$ are as in (2).

Let $\mathcal{K} = \{Y \in \mathcal{K}'$: Y is a nonsingular surface and a realization of $\underline{G}$ }. Clearly $\mathcal{K}$ is an open subset of $\mathcal{K}'$:

5) <u>Hard Part</u>: Decide whether or not $\mathcal{K}$ is empty. Since it is open in $\mathcal{K}'$, this can be done using generic arguments. Contradictions to (1) can occur at this stage too.

11.3. <u>Remarks on generalizations</u>

(1) The definition of prescription can be generalized to higher dimensions, and can be made to work for higher dimensional fixed loci.

(2) The algorithm can be adapted to look for realizations on surfaces S in any class of surfaces of general type for which an n-canonical model is known.

This is because if ϕ_{nK_S} is birational onto its image $\overline{S} \subset \mathbb{P}^N$, then $\overline{S}$ is projectively normal and $\mathcal{O}_{\overline{S}}(1)$ is a G-sheaf. So as in (10.3) the action of G on S determines an action on $\mathbb{P}^N$ restricting to the original action on $\overline{S}$. Furthermore the characters $\omega_{mn}(g)$ $(m \geq 1)$ are given by $L(\omega_Y^{\otimes mn}, g)$ since $h'(nK_Y) = 0 \ \forall\ n \geq 2$ (by [M3]).

If we do not impose condition (*i), then we need a stronger version of the Lefschetz formula than (8.1); for example ([A&B] p.458, or [A&S] p.566 if we allow fixed curves).

(3) Warning: If $H_i \neq \mathbb{Z}_2$, then $H_{i_1} = H_{i_2}$ does not imply $r_{i_1} = r_{i_2}$, even if both r_{i_1} and r_{i_2} embed H_i in SL(2). (cf. (8.1.2).) So the number of possibilities to consider explodes.

12. Construction I

A 4-dimensional family of minimal surfaces of general type with $p_g = q = 0$, $K^2 = 1$, $\pi_1 = \mathbb{Z}_2$, double covered by Godeaux surfaces with $p_g = q = 0$, $\pi_1 = \mathbb{Z}_8$.

12.0. Corollary of Construction

All minimal surfaces of general type with $p_g = q = 0$, $K^2 = 1$, $\mathrm{Tors} = \mathbb{Z}_2$ have $\pi_1 = \mathbb{Z}_2$.

This is because the Moduli space for the above surfaces with $\mathrm{Tors} = \mathbb{Z}_2$ has been shown to be connected by Catanese & Reid.

12.1. Description of the construction

In this paragraph we will write down a family $\mathcal{K}$ of complete intersections Y of 4 quadrics in $\mathbb{P}^6$ such that the group

$$G = \langle a,t : a^2 = t^8 = 1, ata = t^3 \rangle$$

acts on Y , satisfying

i) The subgroup $H = \langle t \rangle \cong \mathbb{Z}_8$ acts freely.

ii) The elements at^{2k} $(k = 0,..,3)$ fix exactly 8 points each. These 4 involutions generate the subgroup $E = \langle a,t^2 \rangle \cong D_8$.

Remarks

Since the squares $(at^{2k+1})^2 \in H$, condition (i) implies that at^{2k+1} acts freely on Y .

Since the product of at^{2k_1} and at^{2k_2} lies in H , condition (ii) implies that their fixed loci are disjoint, so that G acts freely outside a set of 32 points each fixed by just one involution; hence condition (1) and (2) of (9.2) hold.

Hence (by (10.2)), the minimal resolution S of the quotient $X = Y/G$ is a minimal surface of general type with $p_g = q = 0$, $K^2 = 1$, and $\pi_1 \cong G/E = \mathbb{Z}_2$. The surface X has an even set of 4 nodes and the double cover $T = Y/H$ ramified over the nodes is a Godeaux surface with $p_g = q = 0$, $K^2 = 2$, $\pi_1 = H \cong \mathbb{Z}_8$.

The family $\mathcal{X}$ will be obtained by the algorithm of (11.2), and is unique up to isomorphism of $\mathbb{P}^6$.

12.2. The Action of G on $R(Y)$

Suppose G acts on Y as in (12.1). Then by (8.1.2) and (10.3) we have the following character table for the action of G induced on $R(Y)$.

$g \in G$	number of fixed points of g	$L(g)$	$\omega_1(g)$	$S^2\omega_1(g)$	$\omega_2(g)$	$d(g)$
$t^k (k \neq 4)$	0	0	-1	0	0	0
t^4	0	0	-1	4	0	4
at^{2k}	8	2	1	4	2	2
at^{2k+1}	0	0	-1	4	0	4
1	-	8	7	28	24	4

Since the characters determine the representations (up to isomorphism), we can choose bases $\{X_1,\dots,X_7\}$ for $H^0(Y,K_Y) \cong H^0(\mathbb{P}^6,\mathcal{O}_{\mathbb{P}^6}(1))$ and $\{Q_0,Q_2,Q_4,Q_6\}$ for Δ such that

1) The action of G on $\mathbb{P}^6$ is given by:

i) $t(X_i) = t^i X_i$ (writing t for a primitive 8^{th} root of 1)

ii) $a(X_i) = X_{3i(\text{mod } 8)}$.

2) The action induced on Δ by the above action on $\mathbb{P}^6$ is given by:

i) $t(Q_i) = t^i Q_i$

ii) $a(Q_i) = Q_{3i(\text{mod } 8)}$.

12.3. The Families $\mathcal{X}$ and $\mathcal{X}'$

Let G act on $\mathbb{P}^6$ as in (1) above.

Let $L_i = \{Q \in H^0(\mathbb{P}^6, \mathcal{O}_{\mathbb{P}^6}(2)) : t(Q) = t^i Q\}$.

Then $L_0, \ldots, L_6$ are linear systems generated as follows:

$$L_0 = \langle X_1X_7, X_2X_6, X_3X_5, X_4^2\rangle$$

$$L_2 = \langle X_1^2, X_3X_7, X_4X_6, X_5^2\rangle$$

$$L_4 = \langle X_1X_3, X_2^2, X_5X_7, X_6^2\rangle$$

$$L_6 = \langle X_1X_5, X_2X_4, X_3^2, X_7^2\rangle \ .$$

Now condition (2(i)) above is satisfied if and only if $Q_i \in L_i$ $(i = 0,2,4,6)$.

Condition (2(ii)) is satisfied if and only if the following relations (†) among the coefficients of the Q_i's hold. Let r_{ij}

be the coefficient of X_iX_j in the quadric $Q_{i+j(\mathrm{mod}\ 8)}$.

$$(\dagger)\ \begin{cases} r_{17} = r_{33}\ ,\ r_{22} = r_{66} \\ \\ r_{11} = r_{33}\ ,\ r_{37} = r_{15}\ ,\ r_{55} = r_{77}\ ,\ r_{46} = r_{24} \end{cases} .$$

Let $\mathcal{X}' = \{Y = \bigcap_{i=0,2,8,6} Q_i \mid Q_i \in L_i$ and $(\dagger)$ holds$\}$.

Then by construction:

$\mathcal{X}' = \{Y = \cap\, Q_i \subset \mathbb{P}^6$ such that the G action of on $\mathbb{P}^6$ restricts to Y , and $\mathrm{Tr}(g|H^0(Y,\mathcal{O}_Y(m)) = \omega_m(g)$ $\forall\, g \in G$, where the values $\omega_m(g)$ are given by Table (12.2)$\}$.

Let $\mathcal{X}$ be the subfamily given by:

$\mathcal{X} = \{Y \in \mathcal{X}'$: Y is nonsingular and the action of G on Y satisfies

i) at^{2k} $(k=0,..,3)$ fixes just 8 points

ii) the remaining elements of G act freely$\}$.

Then clearly $\mathcal{X}$ is an open subfamily of $\mathcal{X}'$.

Note: condition (ii) implies that $Y \in X$ is a surface because t^4 has eigenspaces $\mathbb{P}^2 \cup \mathbb{P}^3$ (see (12.6)).

12.4. Godeaux's surfaces

Godeaux showed that for generic $Q_i \in L_i$ $(i = 0,2,4,6)$, the intersection $Y = \bigcap_{i=0,2,4,6} Q_i$ is a nonsingular surface with fixed point free action of $\mathbb{Z}_8$ given by ((1)(i)) of (12.2).

This is unfortunately no guarantee that $\mathcal{X}$ is nonempty.

12.5. Theorem

$\mathcal{X}$ is dense in $\mathcal{X}'$.

Proof

Since $\mathcal{X}$ is open in $\mathcal{X}'$ it would suffice to exhibit an example $Y \in \mathcal{X}$. However, the nonsingularity calculations etc., are apparently harder for a special case than for the general case, for which we have Bertini's theorem.

So instead we show that for generic r_{ij} , $Y \in \mathcal{X}$. The theorem follows from Propositions (12.6) and (12.7)

12.6. Proposition

For generic $Y \in \mathcal{X}'$, the fixed loci of the G action on Y are as in (12.1).

The proposition follows from the lemma below:

Lemma

For generic $Y \in \mathcal{X}'$,

i) t^4 acts freely on Y , and

ii) a has exactly 8 fixed points on Y .

From (i) it is clear that t^k acts freely for $k = 1,..,7$. (cf. (2.2.iii)). The elements at^{2k} $(k = 1,2,3)$ are all conjugates of a in G . So if (ii) holds then they each have 8 fixed points. Since $(at^{2k+1})^2 = t^4$ $(\forall\ k)$, (i) implies that the elements at^{2k+1} act freely.

Hence the Lemma implies Proposition (12.6).

Proof of Lemma

i) Let $E = E^+ \cup E^-$ be the fixed spaces for the action of t^4 on $\mathbb{P}^6$. Then

$$E^+ = (X_1 = X_3 = X_5 = X_7 = 0)$$

$$E^- = (X_2 = X_4 = X_6 = 0) \ .$$

In coordinates $(X_2:X_4:X_6)$ for E^+ , $Y \cap E^+$ is given by

$$*X_2X_6 + *X_4^2 = *X_4X_6 = *(X_2^2+X_6^2) = *X_2X_4 = 0 \ ,$$

where $*$ denotes a general coefficient. If all the $*$'s are nonzero, this gives

$$X_2 = X_4 = X_6 = 0 .$$

Hence $Y \cap E^+ = \phi$.

A similar argument shows that $E^- \cap Y = \phi$. Hence t^4 acts freely on Y .

ii) Let $D = D^+ \cup D^-$ be the fixed space for the action of a on $\mathbb{P}^6$. Then

$$D^+ = (X_1 = X_3 , X_2 = X_6 , X_5 = X_7)$$

$$D^- = (X+X_3 = X_2+X_6 = X_5+X_7 = X_4 = 0) .$$

As for E^+ we get $D^- \cap Y = \phi$.

In coordinates $(X_1:X_2:X_4:X_5)$ for D^+ , $Y \cap D^+$ is given by equations $Q_i' = Q_i \cap D$. It is clear that $Q_2' = Q_6'$, so $Y \cap D^+ \neq \phi$. In the special case

$$Q_0' = X_2^2 - X_4^2$$

$$Q_2' = X_1^2 - X_5^2$$

$$Q_4' = X_1^2 + X_2^2 + X_5^2 ,$$

$Y \cap D^+ = \cap Q_i'$ consists of 8 distinct points $(X_1 = \pm X_5, X_2 = \pm X_4 = \pm\sqrt{2}X_1)$.

The condition that $Y \cap D^+$ is finite and consists of 8 distinct points is open, so this example shows that for generic coefficients $Y \cap D^+$ consists of 8 distinct points.

12.7. Proposition

For generic $Y \in \mathcal{X}'$, Y is nonsingular.

Proof

We will use the following version of the 2nd Bertini Theorem [Ha.1]:

Let L be a linear system on a projective variety V. Let Σ be the base locus of L. Then the general divisor $D \in L$ is nonsingular outside S, where

$$S = \operatorname{Sing} V \cup \Sigma .$$

Let $Y = \bigcap_{i=0,2,4,6} Q_i$. We apply the above to the linear systems L_0 and L_4 on the variety $V = Q_2 \cap Q_6$. To find $\operatorname{Sing}V$ we consider the Jacobian at a point $p \in V$:

$$J_pV = \begin{pmatrix} 2r_{11}X_1 & 0 & r_{37}X_7 & r_{46}X_6 & 2r_{55}X_5 & r_{46}X_4 & r_{37}X_3 \\ r_{37}X_5 & r_{46}X_4 & 2r_{11}X_3 & r_{46}X_2 & r_{37}X_1 & 0 & 2r_{55}X_7 \end{pmatrix}$$

where $p = (X_1:X_2:X_3:X_4:X_5:X_6:X_7)$

It is easily checked that for general coefficients, $rkJ_pV < 2$ if and only if $X_1 = X_3 = X_4 = X_5 = X_7 = 0$.

Thus $\operatorname{Sing} V$ is a line $\mathbb{P}^1$ parametrized by X_2, X_6 .

Since $\mathbb{P}^1 \cap Y = \phi$, two applications of Bertini's theorem gives:

$$\operatorname{Sing} Y \subset F_0 \cap F_4 ,$$

where F_i is the fixed locus of L_i :

$$F_0 = \{X_1X_7 + X_3X_5 = X_2X_6 = X_4 = 0\}$$

$$F_4 = \{X_1X_3 = X_5X_7 = X_2^2 + X_6^2 = 0\} .$$

To complete the proof of Proposition (12.7), we now write down the Jacobian J_pY for a point $p = (X_1 : \ldots : X_7) \in Y$, and check that for generic coefficients $\operatorname{Rk} J_pY = 4$ for all $p \in F_0 \cup F_4$.

$$J_pY = \begin{pmatrix} r_{17}X_7 & r_{26}X_6 & r_{17}X_5 & 2r_{44}X_4 & r_{17}X_3 & r_{26}X_2 & r_{17}X_1 \\ 2r_{11}X_1 & 0 & r_{37}X_7 & r_{46}X_6 & 2r_{55}X_5 & r_{46}X_4 & r_{37}X_3 \\ r_{13}X_3 & 2r_{22}X_2 & r_{13}X_1 & 0 & r_{57}X_7 & 2r_{22}X_6 & r_{57}X_5 \\ r_{37}X_5 & r_{46}X_4 & 2r_{11}X_3 & r_{46}X_2 & r_{37}X_1 & 0 & 2r_{55}X_7 \end{pmatrix}$$

i) Suppose $p \in F_0$.

Then $X_2X_6 = 0$. Assume $X_2 = 0$.

Let A_i $(i = 1,3,5,7)$ be the 4×4 matrix whose columns are columns $i,2,4,6$ of J_pY . If $\mathrm{Rk}J_pY < 4$, then $\det A_i = 0$ (for all i) . For generic r_{ij} this gives

$$X_iX_6 = 0 \qquad (i = 1,3,5,7) .$$

Together with $X_2 = X_4 = 0$ (for $p \in F_0$) , this contradicts Prop. (12.6) (t^4 acts freely on Y).

Hence for generic coefficients $\mathrm{rk}A_i = \mathrm{rk}J_pY = 4$.

A similar argument shows that if $X_6 = 0$ then $\mathrm{rk}J_pY = 4$.

ii) Suppose $p \in F_4$. Assume $X_1 = X_5 = 0$.

Let A be the 4×4 matrix whose columns are the odd-numbered columns of J_pY . Then (rearranging):

$$\det A = \det \begin{pmatrix} r_{17}X_7 & r_{17}X_3 & 0 & 0 \\ r_{13}X_3 & r_{57}X_7 & 0 & 0 \\ 0 & 0 & r_{37}X_7 & r_{37}X_3 \\ 0 & 0 & 2r_{11}X_3 & 2r_{55}X_7 \end{pmatrix}$$

Suppose $\det A = 0$.

There are two factors for $\det A$, one of which must vanish. Suppose the first does. Then

$$r_{57}r_{17}X_7^2 - r_{17}r_{13}X_3^2 = 0 \quad . \tag{1}$$

Since $p \in Q_2 \cap Q_6$, we also have

$$r_{37}X_3X_7 \pm (r_{11}X_3^2 + r_{55}X_7^2) = 0 \ . \tag{2}$$

For generic r_{ij}, equations (1) and (2) have no nontrivial common solutions.

A similar argument for the other factor of $\det A$ completes the proof of the proposition, and hence of Theorem (12.5).

12.8. Moduli for $X = Y/G$

The surfaces $Y \in \mathcal{X}$ are paramterized by a dense open subset U of $\mathbb{P}^2 \times \mathbb{P}^3 \times \mathbb{P}^2$ of coordinates of Q_0, Q_2, Q_4.

For $r \neq r' \in U$, surfaces $X_r = Y_r/G$ and $X_{r'}$ are isomorphic if and only if there is an isomorphism $\phi: \mathbb{P}^6 \to \mathbb{P}^6$ commuting with the G-action, such that $\phi(Y_r) = Y_{r'}$.

The group A of such isomorphisms ϕ consists of diagonal matrices $(a_1:\ldots:a_7)$ subject to $a_1 = a_3, a_2 = a_6, a_5 = a_7$. A can be shown to act effectively on U.

Corollary

The construction for X depends on 4 parameters. More precisely the moduli space for the construction is connected, 4-dimensional and unirational.

13. Construction II

A 4-dimensional family of minimal surfaces of general type with $p_g = q = 0$, $K^2 = 1$, $\pi_1 = \mathbb{Z}_4$ double covered by Godeaux surfaces with $p_g = q = 0$, $K^2 = 2$, $\pi_1 = \mathbb{Z}_8$.

13.1. Description of the construction

We will write down a family $\mathcal{X}$ of complete intersections Y of 4 quadrics in $\mathbb{P}^6$ such that the group

$$G = \langle a,t : a^2 = t^8 = 1,\ ata = t^5\rangle$$

acts on Y satisfying

i) The subgroup $H = \langle t\rangle \cong \mathbb{Z}_8$ acts freely.

ii) The elements a and at^4 fix 16 points each.
Thus the elliptic subgroup $E = \langle a,t^4\rangle \cong \mathbb{Z}_2^2$.

As in (12.1) we see that (i) and (ii) imply (1) and (2) of (9.2), so the resolution S of the quotient $X = Y/G$ has the required properties, (by (10.2)).

13.2. Action of G on $R(Y)$

Suppose G acts on the complete intersection Y of 4 quadrics in $\mathbb{P}^6$ as in (13.1). Calculating the characters as in (12.2), we find we can choose bases $\{X_1..X_7\}$ for $H^0(\mathbb{P}^6, {}_{\mathbb{P}^6}(1))$ and $\{Q_0,Q_2,Q_4,Q_6\}$ for such that G acts by

1) (i) $t(X_i) = t^i X_i$ (writing t for primitive 8^{th} root of 1)

(ii) $a(X_i) = X_{5i(\mathrm{mod}\ 8)}$

2) (i) $t(Q_i) = t^i Q_i$

(ii) $a(Q_i) = Q_{5i(\mathrm{mod}\ 8)}$

13.3. <u>The Families $\mathcal{X}$ and $\mathcal{X}'$</u>

Let L_i be as in (12.3).

The action of G on $\mathbb{P}^6$ written in (1) induces the action (2) on Δ if and only if

$$Q_i \in L_i \qquad i = 0,..,6 ,$$

and conditions $(\dagger)$ on the r_{ij}'s hold:

$$r_{17} = r_{35},\ r_{11} = r_{55},\ r_{13} = r_{57},\ r_{33} = r_{77} \qquad (\dagger) .$$

Let $\mathcal{X}' = \{Y = \bigcap_{i=0,2,4,6} Q_i : Q_i \subset L_i \text{ subject to } (\dagger)\}$.

Let $\mathcal{X} = \{Y \in \mathcal{X}' : Y \text{ is nonsingular and } G \text{ acts as in (13.1)}\}$.

Then (as in 12.3), $\mathcal{X}$ is open in $\mathcal{X}'$.

The Godeaux surfaces $T = Y/H$ are as in (12.4).

13.4. Theorem

$\mathcal{X}$ is dense in $\mathcal{X}'$. This is proved in the same way as Theorem (12.5).

13.5. Moduli

Counting as in (12.8), we find that the moduli space for surfaces $X = Y/G$ is connected, 4-dimensional, and unirational.

14. Construction III

A 4 dimensional family of minimal surfaces of general type with $p_g = q = 0$, $K^2 = 1$, $\pi_1 = \mathbb{Z}_4$, double covered by Reid surfaces with $p_g = q = 0$, $K^2 = 2$, $\pi_1 = \mathbb{Z}_2 \times \mathbb{Z}_4$.

14.1. Description of the construction

We will write down a family $\mathcal{X}$ of complete intersections of 4 quadrics $Y = \cap Q_i$ in $\mathbb{P}^6$ such that the group

$$G = \langle a,b,t : a^2 = b^2 = t^4 = 1\ ,\ aba = b,\ ta = bt \rangle$$

acts on Y satisfying

i) The subgroup $H = \langle ab,t \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_4$ acts freely.

ii) The elements a and b fix 16 points each.

Thus $E = \langle a,b \rangle \cong \mathbb{Z}_2^2$.

Arguing as in (12.1) we see that (i) and (ii) imply (1) and (2) of (9.2), so the resolution S of the quotient $X = Y/G$ is as required. In this case the double cover of X is a surface $T = Y/H$ such as constructed by Reid in [R2]; see (14.4).

14.2. Action of G on $R(Y)$

Suppose G acts on Y as in (14.1). Then by (10.3) we can choose bases

$$\{X_{nm} : n = 0,1;\ m = 0,1,2,3;\ nm \neq 00\}$$

for $H^0(\mathcal{O}_{\mathbb{P}^6}(1))$, and

$$\{Q_n(k) : n = 0,2;\ k = 1,2\}$$

for Δ , such that G acts by:

1) (i) $\begin{cases} t(X_{nm}) = t^n X_{nm} \\ ab(X_{nm}) = (-1)^n X_{nm} \end{cases}$ (writing t for a primitive 4^{th} root of 1)

(ii) $\begin{cases} a(X_{10}) = X_{12} \\ a(X_{11}) = X_{13} \\ a(X_{0i}) = X_{0i} \quad \text{(for } i = 1,2,3) . \end{cases}$

2) i) $\begin{cases} t(Q_n(k)) = t^n\, Q_n(k) \\ ab(Q_n(k)) = Q_n(k) \end{cases}$

ii) $a(Q_n(k)) = Q_n(k)$.

14.3. The families $\mathcal{X}$ and $\mathcal{X}'$

Let $L_{nm} = \{Q \in H^0(\mathbb{P}^6, \mathcal{O}_{\mathbb{P}^6}(2)) : t^i(ab)^j(Q) = t^{ni}(-1)^{mj}Q\}$.

These are generated as follows:

$$L_{00} = \langle X_{02}^2, X_{10}^2, X_{12}^2, X_{01}X_{03}, X_{11}X_{13}\rangle$$

$$L_{02} = \langle X_{01}^2, X_{03}^2, X_{11}^2, X_{13}^2, X_{10}X_{12}\rangle .$$

Then condition (2i) holds if and only if $Q_n(k) \in L_{on}$ $(k=1,2;\ n=0,2)$ and condition (2ii) is equivalent to conditions (†) on coefficients:

Let $r_{ij,lm}$ be the coefficient of $X_{ij}X_{lm}$ in $Q_n(k)$.

$$\left.\begin{array}{lll} & r_{10,10} = r_{12,12} & (\text{for } k = 1,2) \\ \text{and} & r_{11,11} = r_{13,13} & \quad ''\qquad '' \end{array}\right\} \quad (\dagger)$$

(This gives 4 conditions in all.)

Let $\mathcal{X}' = \{Y = \bigcap_{\substack{n=0,2\\k=1,2}} Q_n(k) \ : Q_n(k) \in L_{on}$ subject to (†)}

Let $\mathcal{X} = \{Y \in \mathcal{X}'$ such that Y is nonsingular and G acts as in 14.1$\}$.

Then $\mathcal{X}$ is an open subset of $\mathcal{X}'$.

14.4. Reid's surfaces

In [R2] Reid stated that for generic coefficients the variety

$$Y = \bigcap_{\substack{n=0,2\\k=1,2}} Q_n(k)$$

where $Q_n(k) \in L_{0n}$ $(k = 1,2)$ is a nonsingular surface, and the action of $\mathbb{Z}_2 \times \mathbb{Z}_4 = \langle ab,t\rangle$ on $\mathbb{P}^6$ given by (14.2(1(i))) restricts to a fixed point free action on Y .

So the quotients $T = Y/\mathbb{Z}_2\times\mathbb{Z}_4$ are minimal surfaces of general type with $p_g = q = 0$, $K^2 = 2$, $\pi_1 = \mathbb{Z}_2\times\mathbb{Z}_4$.

Again this does not guarantee that $\mathcal{X}$ is nonempty.

14.5. Theorem

$\mathcal{X}$ is dense in $\mathcal{X}'$.

Proof

Like (12.5).

14.6. Moduli

Counted as in (12.8), but this time the parameter space U for $\mathcal{X}$ is a dense open subset of

$$Gr(2,4) \times Gr(2,4)$$

since we take pairs of quadrics from the same linear system.

15. Theorem

Let G be a group of order 16 acting on a complete intersection $Y = \cap Q_i$ of 4 quadrics in $\mathbb{P}^6$ such that condition (*) of (10.1) holds.

Then G and its action on Y are as in Construction I, II or III.

15.0. Proof

Let $\underline{G}$ be a prescription for a group G of order 16, satisfying (*). Assume $\underline{G}$ can be realized on a complete intersection $Y = \cap Q_i$ of 4 quadrics in $\mathbb{P}^6$.

Step 1, using properties of $X = Y/G$, places initial restrictions on $\underline{G}$.

The remaining steps show that we get contradictions at stage (2) of (11.2) for all $\underline{G}$ except those corresponding to constructions I-III.

If we omit Step 1, what happens is that some prescriptions get past stage (2) of (11.2) but give contradictions at stage (5).

It is convenient to be able to identify a group G from its configuration of subgroups. This can be done by elementary group theory, but its easier to consult [T&W] (see Steps 7 and 8).

15.1. Step 1:

$$G/E = \{1\},\ \mathbb{Z}_2\ ,\quad \text{or}\quad \mathbb{Z}_4\ .$$

Proof

For a minimal surface of general type S with $p_g = q = 0$ $K^2 = 1$ we have

$$|\pi_1^{alg}(S)| \leq 6 \qquad ([B]),$$

and $$\pi_1^{alg}(S) \neq \mathbb{Z}_2^2 \qquad ([R1]).$$

Since the resolution $S \to X = Y/G$ is such a surface and has $\pi_1 = G/E$ (by Proposition (10.2)), this implies

$$G/E = \{1\},\ \mathbb{Z}_2 \ \text{or}\ \mathbb{Z}_4\ .$$

15.2. Step 2:

G must act freely outside a set of 32 points, each fixed by just one involution.

Proof

Condition (*i) implies that G acts freely outside a finite set by (7.2.3), and if $G_y \neq \{1\}$ then

$$G_y \cong \mathbb{Z}_2,\ \mathbb{Z}_4,\ \mathbb{Z}_8,\ \mathbb{Z}_{16}\ ,\quad \text{or}\quad Q\ .$$

(Q = quarternions of order 8.)

Ignoring the fact that the geometry of X probably excludes most of these, we apply (8.3.2), (with $W(G) = 8$) , to get

$$\Sigma \frac{(n^2-1)}{12} S_n + \frac{13}{4} S_8' = 8 , \qquad (\dagger)$$

where $S_n = \# \{y \in Y : G_y \cong \mathbb{Z}_n\}$

and $S_8' = \# \{y \in Y : G_y \cong Q\}$.

Now for a subgroup $A \leq G$, the set $\{y \in Y : G_y \cong A\}$ is a union of G - orbits each of size $|G:A|$. Hence S_n is divisible by $(\frac{16}{n})$. It is easy to check that the equation $(\dagger)$ has only one positive integer solution subject to this condition, namely

$$S_2 = 32 , \quad S_n = 0 \quad \forall\, n > 2 .$$

15.3. Step 3:

Conditions on weights of subgroups:

Let $A \leq G$. From (8.3.2) we obtain

i) $8 + W(A) \in |A| . \mathbb{Z}$

ii) $W(A) \geq 0$, with equality if and only if A acts freely on Y .

iii) $W(A) \leq W(G) = 8$, with equality if and only if $E \leq A$.

15.4. Step 4:

Let $g \in G$ be an elliptic element, so that $g^2 = 1$ (by (15.2)).

Then g fixes either 8 or 16 points of Y .

Proof

If g fixes t points then $L(g) = \frac{t}{4}$, (by (8.1.2)).
By (15.3), $8 + L(g) \in 2.\mathbb{Z}$. Hence $t = 8k$ for some $k \in \mathbb{Z}$.

Calculating $d(g)$ as in (10.3) gives

$$d(g) = 2k^2 - 4k+4 .$$

Since $d(1) = 4$, $d(g) \leq 4$. This gives $k = 1$ or 2 .

15.5. Step 5: (due to Reid [R2])

The dihedral group $D_8 = \langle a,b : a^2 = b^4 = 1, aba = b^3 \rangle$ cannot act freely on Y .

Proof

Suppose D_8 acts freely. We calculate d as in (10.3):

$$d(ab^k) = 4 \quad (k = 0,..,3)$$

$$d(1) = 4 .$$

Since the involutions ab^k generate D_8 , this means its action on Δ should be trivial. But $d(b) = 0$, contradicting this.

15.6. Step 6:

We cannot have $G = E$, (so X is not simply connected).

Proof

Let $g \in G$ be an elliptic involution. By Sylow's theorem there exists a subgroup $A < G$ of order 8 containing G . By (15.3), $W(A) = 8$. Hence $E \leq A$, so $E \neq G$.

15.7. Step 7:

If $|E| = 8$ then $E \cong D_8$ and G and its action are as in Construction I.

Proof

Since E is generated by involutions, $E \cong \mathbb{Z}_2^3$ or D_8 .

If $E \cong \mathbb{Z}_2^3 = \langle X,Y,Z \rangle$, then applying (15.3) to subgroups of order 4 and (15.4) gives that (up to isomorphism) we must have

$$L(X) = L(Y) = L(Z) = L(XYZ) = 2 .$$

But the function ω_1 (see (10.3)) turns out then not to be a character of $\mathbb{Z}_2^3$, so this is impossible.

So $E \cong D_8 = \langle a,b : a^2 = b^4 = 1, aba = b^3 \rangle \triangleleft G$.
Since conjugate elements have the same value of L , and we wish the elliptic involutions to generate the whole of D_8 , (15.4) implies

$$L(ab^k) = 2 \qquad (k = 0,1,2,3) \quad .$$

i.e. each element ab^k fixes 8 points and the remaining elements of G act freely.

Let H be a subgroup of G of order 8 distinct from E (these exist because G is noncyclic). By (15.3) we must have $W(H) = 0$. So by (15.5), $H \not\cong D_8$. There is exactly one group G of order 16 having a unique dihedral subgroup, namely the group G of Construction I. This can be checked by elementary group theory or by looking at [T&W].

15.8. <u>Step 8</u>:

If $|E| = 4$, then $E \cong \mathbb{Z}_2^2$ and G and its action are as in II or III.

<u>Proof</u>

E is generated by elliptic involutions (15.2), so $E \cong \mathbb{Z}_2^2 = \langle a,b \rangle$.

The condition that ω_1 is a character together with (15.4) implies that up to isomorphism we must have

$$L(a) = L(b) = 4 \ ,$$

i.e. a and b fix 16 points each and the remaining elements of G act freely.

By (15.1), $G/E \cong \mathbb{Z}_4$. So

$$G = \langle a,b,t : t^4 \in E \rangle \quad .$$

There are two possibilities:

i) $t^4 = 1$. Then $L(tat^{-1}) = L(a) = 4$, so

$$tat^{-1} = a \text{ or } b .$$

If $tat^{-1} = a$, then $A = \langle a,t\rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_4$ and $W(A) = 4$, contradicting (15.3). Hence $tat^{-1} = b$. Similarly $tbt^{-1} = a$.

So G is some nonabelian group of order 16 containing $\langle a,b,t^2\rangle \cong \mathbb{Z}_2^3$ and $\langle ab,t\rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_4$.

The tables [T&W] reveal two such groups, called (16/9) and (16/6). The first of these is the group G of Construction III. The second contains four copies of D_8 , at least one of which, A , does not contain E . By (15.3) $W(A) = 0$, so A acts freely. This contradicts (15.5).

ii) $t^4 \neq 1$. This implies $t^4 = ab$, since otherwise $L(t^4) = 4$, and the group $A = \langle t\rangle$ has $W(A) = 4$ contradicting (15.3).

If $tat^{-1} = a$, then $G \cong \mathbb{Z}_2 \times \mathbb{Z}_8$ and the subgroup $A = \langle t^2,a\rangle \cong \mathbb{Z}_4 \times \mathbb{Z}_2$ has $W(A) = 4$ contradicting (15.3) again.

Hence $tat^{-1} = b$. So $ata = abt = t^5$, giving G as in Construction II.

CHAPTER 3

A minimal surface of general type with $p_g = q = 0$, $K^2 = 1$, $\pi_1 = \{1\}$.

16. Introduction

In [V&Z] a surface V with $p_g = 4$, $q = 0$, $K^2 = 10$, $\pi_1 = \{1\}$, with an action of $S_5 \times \mathbb{Z}_2$ is obtained from the Hilbert Modular group for $\mathbb{Q}(\sqrt{21})$. This result is summarized in §17.

In [C], Catanese showed that a subgroup $\mathbb{Z}_5 \subset S_5$ acts freely on V giving a surface $T = V/\mathbb{Z}_5$ with $p_g = q = 0$, $K^2 = 2$, $\pi_1 = \mathbb{Z}_5$.

The aim of this chapter is to point out that T has an involution with just 4 fixed points given by a subgroup $D_{10} \subset S_5 \times \mathbb{Z}_2$ acting on V, extending the fixed point free action of $\mathbb{Z}_5$ (Theorem (18.1)). It follows that the minimal resolution of the quotient $X = T/\mathbb{Z}_2 = V/D_{10}$ is a minimal surface of general type with $p_g = q = 0$, $K^2 = 1$, $\pi_1 = \{1\}$ (18.2).

16.1. Parameters for the example

The surface V is a double cover of a 20-nodal quintic Q. By considering the canonical rings of such surfaces V Catanese counted 4 parameters for his construction $T = V/\mathbb{Z}_5$, ([C] §4.). A similar argument gives 2 parameters for $X = V/D_{10}$.

Using Ciliberto's recent description [Ci] of the canonical ring of a general surface Y with $p_g = 4$, $q = 0$, $K^2 = 10$, (which has ϕ_{K_Y}

birational onto a hypersurface of degree 10 instead of 2-1 onto a quintic in $\mathbb{P}^3$), we hope to obtain larger families for T and X .

16.2. Remark

Arguing as in (15.6) we can show that if T is minimal surface of general type with $p_g = q = 0$, $K^2 = 2$ having an involution with just 4 fixed points such that $X = T/\mathbb{Z}_2$ is simply connected, then $|\pi_1^{alg}(T)| \neq 2,4$ or 6 . Any surface T with the above invariants has $|\pi_1^{alg}(T)| \leq 8$ (the bound 10 given by Beauville was improved to 8 by Reid in [R2]).

This leaves the possibilities $\pi_1^{alg}(T) = \{1\}, \mathbb{Z}_3, \mathbb{Z}_5, \mathbb{Z}_7$. The only confirmed examples of such surfaces T are Catanese's (with $\pi_1 = \mathbb{Z}_5$) .

Reid has proposed a construction for $\pi_1^{alg}(T) = \mathbb{Z}_7$, but the corresponding construction $X = T/\mathbb{Z}_2 = Y/D_{14}$ where $T = Y/\mathbb{Z}_7$ can be shown not to work (because character theory (c.f. (10.3)) would imply $\phi_{K_Y}(Y)$ lies in a quadric, which is not the case).

17. Summary of Results from [V&Z]

17.1. The Surface V

The surface V is the minimal model of the surface Y of ([V&Z] §4 p.106), which is the minimal resolution of the compactification of $H \times H/\Gamma$. Here $H = C : \mathrm{Im} z > 0$ and Γ is the 2-congruence subgroup of the Hilbert Modular group $SL(2,\mathcal{O}_K)$ where $\mathcal{O}_K$ is the ring of integers of the field $K = Q(\sqrt{21})$ (see (17.4)). Standard calculations (as in [V&Z] §1) show that V is a minimal surface of general type with $p_g = 4$, $q = 0$, $K^2 = 10$.

The fact that $\pi_1(Y) = \{1\}$ is due to a result of Schvartsman [Sch] generalising the well known result that Hilbert Modular Surfaces are simply connected.

17.2. The surface Q and the action of A_5 on Q

A calculation as in ([V&Z] §1 and 3) shows that the canonical map of V is 2-1 onto the 20-nodal quintic Q given in coordinates $s_1 \dots s_5$ for $\mathbf{P}^4$ by

$$\sum_{i=1}^{5} s_i = 0$$

$$\sum_{i=1}^{5} s_i^5 - \frac{5}{4} \sum_{i=1}^{5} s_i^2 \sum_{i=1}^{5} s_i^3 = 0 .$$

([V&Z] Theorem 3: Q is the canonical model of the surface called Y_e^0 .)

The coordinates s_i correspond to sections of $H^0(K_V)$ coming from the 5 cusp cycles on V .

The action of $SL(2,\mathcal{O}_K)/\Gamma \cong A_5$ induced on V acts by permutation on the set of cusp cycles, and hence on $\mathbf{P}^4$ by

$$a \in A_5 : a(S_i) = S_{a(i)} \qquad i = 1,..,5 .$$

This is how the equation of Q is determined ([V&Z] Theorem 2 : there is only one 20-nodal quintic invariant under such an A_5-action).

The 20 nodes of Q are the A_5-orbit of the point

$$(2,2,2,-3-\sqrt{-7},-3+\sqrt{-7}) .$$

17.3. <u>The Action of $A_5 \times \mathbb{Z}_2$ on V</u>

Using the fact that $\pi_1(V) = \{1\}$ and a generalization of (6.5) we can show that the action of A_5 on Q lifts to an action of $A_5 \times \mathbb{Z}_2$ on V , where the second factor is given by the canonical involution σ of V (with $Q = V/\langle\sigma\rangle$) .

This can also be obtained directly by considering the extended Hilbert Modular Group (see (17.4(iv)).

17.4. <u>Some Details on the Hilbert Modular Groups and their actions on H×H</u>

i) The group $GL(2,\mathcal{O}_K)$ acts on $H\times H$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : (z_1, z_2) \longmapsto \left(\frac{az_1 + b}{cz_1 + d} \, , \, \frac{a'z_2 + b'}{c'z_2 + d'} \right)$$

where $a \to a'$ denotes conjugation in K .

This gives effective actions of the groups $B = SL(2,\mathcal{O}_K)/\{\pm 1\}$ and $\hat{B} = \{A \in GL(2,\mathcal{O}_K) : \det A \in U^+\}/\{\begin{pmatrix} e & 0 \\ 0 & e \end{pmatrix} : e \in U\}$ on $H \times H$, where U is the group of units in $\mathcal{O}_K$ and $U^+ = \{e \in U : e > 0\}$.

Note: Some authors call B, $\hat{B}$ the (respectively, extended) Hilbert Modular groups for K .

ii) There is an exact sequence

$$1 \to B \underset{i}{\to} \hat{B} \underset{d}{\to} U^+/U^2 \to 1$$

where i is given by inclusion $SL(2) \subset GL(2)$ and d is given by the determinant.

iii) The 2-congruence subgroups Γ , C, $\hat{C}$ of $SL(2,\mathcal{O}_K)$, B and $\hat{B}$ (respectively) are the subgroups given by matrices

$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $a \equiv d \equiv 1$, $c \equiv b \equiv 0 \pmod{2\mathcal{O}_K}$.

Then $SL(2,\mathcal{O}_K)/\Gamma \cong B/C \cong \hat{B}/\hat{C}$

$$\cong SL(2,\mathcal{O}_K/2\mathcal{O}_K)$$

$$\cong SL(2,\mathbb{F}_4) \cong A_5$$

where $\mathbb{F}_4$ is the field with 4 elements.

<u>Claim</u>: $\hat{B}/C = B/C \times \hat{C}/C = A_5 \times \mathbb{Z}_2$

(not S_5 as stated in ([C] §5)) .

<u>Proof</u>.

Let $\bar{\varepsilon} \in \hat{C}$ be the element given by the matrix $\begin{pmatrix} \varepsilon & 0 \\ 0 & 1 \end{pmatrix}$ where $\varepsilon = 55 + 12\sqrt{21} \in U^+$. Then $\bar{\varepsilon} \notin C$, so $\hat{C}/C \cong \mathbb{Z}_2$, (by (ii)). Let $t \in B$, represented by the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then $\bar{\varepsilon}\, t\, \bar{\varepsilon}^{-1}$ is represented by

$$\begin{pmatrix} \varepsilon & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \varepsilon' & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & \varepsilon b \\ \varepsilon' c & d \end{pmatrix}$$

Since $\varepsilon \equiv \varepsilon' \equiv 1 \pmod{2\mathcal{O}_K}$, this gives

$$\bar{\varepsilon}\, t\, \bar{\varepsilon}^{-1} \equiv t \pmod{C} .$$

Hence $\hat{B}/C \cong B/C \times \hat{C}/C \cong A_5 \times \mathbb{Z}_2$.

iv) Actions of $A_5 \times \mathbb{Z}_2$ and $S_5 \times \mathbb{Z}_2$ on V

The action of $\hat{B}/C$ on $H\times H/\Gamma = H\times H/C$ induces an action of $A_5 \times \mathbb{Z}_2$ on V .

The second factor $\hat{C}/C \cong \mathbb{Z}_2$ is shown to have just 20 fixed points on V in ([V&Z] Theorem 3), and the quotient Q is birational to $H\times H/\hat{C}$, which has an action of A_5 given by $\hat{B}/\hat{C}$ (see (17.2)).

The action of S_5 on Q which naturally extends the A_5-action described in (17.2) comes from the extension of the $\hat{B}/C$ action on $H\times H/C$ given by the involution $\tau:(Z_1,Z_2) \to (Z_2,Z_1)$ on $H\times H$. Let $C^+ = \langle C,\tau\rangle$. Then $C < C^+$ and

$$\langle C^+/C \ , \ \hat{B}/C\rangle \cong S_5 \times \mathbb{Z}_2 \ .$$

18. The Surface $X = V/D_{10}$

18.1. Theorem

There is a subgroup $D_{10} = \langle \alpha, \beta : \alpha^2 = \beta^5 = 1, \alpha\beta\alpha = \beta^4 \rangle$ of $S_5 \times \mathbb{Z}_2$ acting on V such that

$*\begin{cases} \text{i) } \mathbb{Z}_5 = \langle\beta\rangle \text{ acts freely (as shown by [C])} \\ \text{ii) The involution } \alpha \text{ (and hence each of its conjugates } \alpha\beta^k \end{cases}$

$(k = 1,\ldots,4)$) fixes just 4 points of V. Equivalently, the involution induced by α on Catanese's surface $T = V/\mathbb{Z}_5$ has just 4 fixed points.

18.2. Corollary

The minimal resolution S of the quotient $X = V/D_{10}$ is a minimal surface of general type with $p_g = q = 0$, $K^2 = 1$, $\pi_1 = \{1\}$.

Proof of Corollary

Catanese showed that $T = V/\mathbb{Z}_5$ is a minimal surface of general type with $p_g = q = 0$, $K^2 = 2$, $\pi_1 = \mathbb{Z}_5$. The proof is similar to (10.2).

So by (8.4.1), *(ii) gives that S is minimal of general type with $p_g = q = 0$, $K^2 = 1$.

By (6.3), $\pi_1(S) = \pi_1(X)$.

By (6.4), $\pi_1(X) = D_{10}/E$, where E is the elliptic subgroup defined in (2.1).

Since in this case the elliptic involutions $\alpha\beta^k$ generate D_{10} , we have $E = D_{10}$.

Hence $\pi_1(X) = \{1\}$.

18.4. Proof of (18.1)

Let $a = (25)(34)$, $b = (12345)$.

Consider the subgroup $\langle a,b,\sigma\rangle \cong D_{10} \times \mathbb{Z}_2$ of $A_5 \times \mathbb{Z}_2$ acting on V .

By looking at the action of $\langle\bar{a},\bar{b}\rangle$ induced on Q , we will show:

18.5. Proposition

Exactly one of the two dihedral subgroups $\langle a\sigma,b\rangle$ and $\langle a,b\rangle$ of $D_{10} \times \mathbb{Z}_2$ acts on V as in (18.1*).

18.6. Lemma

The action of $D_{10} = \langle\bar{a},\bar{b}\rangle$ on Q satisfies

i) $\langle\bar{b}\rangle = \mathbb{Z}_5$ acts freely.

ii) The fixed locus of $\bar{a}, Q^{\bar{a}}$, consists of 5 distinct points $\{P_i\}_{i=1}^5$ and one line L . Furthermore $Q^{\bar{a}} \cap \operatorname{Sing} Q = \phi$.

18.7. Proof of Lemma

The action of D_{10} on $\mathbb{P}^4$ is given by $\bar{a}(s_i) = s_{a(i)}$

$$\bar{b}(s_i) = s_{b(i)} .$$

i) The 4 fixed points of the action of $\bar{b}$ on the hyperplane $(\Sigma s_i = 0)$ are easily seen not to lie on Q ; this gives (18.6(i)).

ii) The fixed locus of the action of $\bar{a}$ on $\mathbf{P}^4$ splits into Eigenspaces $E^+ \cup E^-$, where

$$E^+ = \{s_2 = s_5 , s_3 = s_4\}$$

$$E^- = \{s_1 = 0 , s_2 + s_5 = s_3 + s_4 = 0\} .$$

The line $L = E^- \cap (\Sigma s_i = 0)$ lies in Q .

The line $L^+ = E^+ \cap (\Sigma s_i = 0)$ can be given coordinates s_2, s_3 . The intersection $L^+ \cap Q$ is then given by a quintic Q^+ in s_2 and s_3 , which splits as follows:

$$Q^+ = (s_2+s_3)(s_2+2s_3)(2s_2+s_3)(s_2^2+3s_2s_3+s_3^2) .$$

There are 5 distinct roots to $(Q^+ = 0)$. So $L^+ \cap Q$ consists of 5 distinct points. It is easy to check that none of the nodes of Q lie in $E^+ \cup E^-$. This proves (18.6(ii)).

18.8. Proof of (18.5) from Lemma (18.6)

i) By (17.7.(i)) the map $Q \to Q/\langle\bar{b}\rangle = Y/\langle b,\sigma\rangle$ is unramified. Hence σ is the only elliptic element of the action of $\langle b,\sigma\rangle$ so $\langle b\rangle$ acts freely.

ii) Since the fixed locus of $\bar{a}$ avoids the nodes of Q, the stabilizer of any $p \in Y$ in $\langle a,\sigma\rangle \cong \mathbb{Z}_2^2$ has order at most 2.

Let $\pi : Y \to Q$ be the quotient by $\langle\sigma\rangle$; by above we have

$$\pi(Y^a \cup Y^{a\sigma}) = Q^{\bar{a}} = \{P_i\}_1^5 \cup L$$

and $$Y^a \cap Y^{a\sigma} = \phi .$$

Now $\pi^{-1}(L) = L_1 \cup L_2$, where L_1, L_2 are 2 lines interchanged by σ . In particular if $aL_1 = L_1$ then $aL_2 = a\sigma L_1 = \sigma L_1 = L_2$.

Hence $\pi^{-1}(L)$ lies entirely in Y^a or $Y^{a\sigma}$. So one of Y^a , $Y^{a\sigma}$ lies entirely in $\pi^{-1}(\{P_i\}_1^5)$, say $Y^{a\sigma}$ (see Note below). Hence $Y^{a\sigma}$ is a finite set of $\mu \le 10$ points.

Since $\chi(\mathcal{O}_{Y/\langle a\sigma\rangle}) = \frac{1}{2}(\chi(\mathcal{O}_Y) + \frac{\mu}{4}) \in \mathbb{Z}$ (by (8.2))

we must have $\mu = 4$.

This completes the proof of (18.5) and hence of (18.1).

18.8.1. Note:

The subgroups $\langle a\sigma,b\rangle$, $\langle a,b\rangle$ are indistinguishable in $D_{10} \times \mathbb{Z}_2$. However, they lie differently in $S_5 \times \mathbb{Z}_2$:

there is no copy of S_5 between $\langle a\sigma, b\rangle$ and $S_5 \times \mathbb{Z}_2$.

By considering the action of $S_5 \times \mathbb{Z}_2$ on the canonical ring $R(V)$, it is possible to see that the finite fixed locus is in fact $Y^{a\sigma}$ (see [R4]).

18.8.2. Corollary of Lemma (18.6) : A Godeaux Surface W

The surface $W = Q/\langle\bar{b}\rangle = Y/\langle\sigma, b\rangle = T/\langle\bar{\sigma}\rangle$ also has an even set of 4 nodes with double cover T . Its minimal resolution is a minimal surface of general type with $p_g = q = 0$, $K^2 = 1$, $\pi_1 = \mathbb{Z}_5$.

Counting parameters for the construction of $W = Y/\mathbb{Z}_2 \quad \mathbb{Z}_5 = T/\mathbb{Z}_2$ as in ([C] §4), we find it depends on the same number as the construction for T , namely 4 .

APPENDIX

A Topological proof of (8.4) (a G-equivariant Riemann-Roch Formula)

A.0. Introduction

In ([H 3] p.61) Hirzebruch stated that he and Zagier had calculated the 'signature defect' due to a cyclic quotient singularity by resolving it explicitly. This gives an elementary proof of the 'G-signature Theorem' (for a group G acting with finite fixed locus on a surface Y) given in [H2].

In this chapter we do the analagous calculation for 'genus defect'. However, as pointed out in (A4.2) we convert to a signature calculation half way through in order to make use of the properties of Dedekind sums. So this chapter should be viewed as an attempted reconstruction of a calculation of Hirzebruch.

A.1. Theorem (see (8.4))

Let G be a finite group acting on the nonsingular projective surface Y, with finite fixed loci. Let r_y be the representation of G_y induced on T_yY. Let $\pi : Y \to X$ be the quotient map, and let $f : S \to X$ be a resolution. Then

$$\chi(\mathcal{O}_S) = \frac{1}{|G|}\left(\chi(\mathcal{O}_Y) + \sum_{x \in \mathrm{Sing}X} \gamma_x\right) ,$$

where

$$\gamma_x = \sum_{y \in \pi^{-1}(x)} \; \sum_{\substack{g \in G_y \\ g \neq 1}} \det\,(1 - r_y(g))^{-1} .$$

A.1.1. Outline of Proof

By (8.5) we may assume G is cyclic, so that a singularity $x \in X$ is of type (q,n) (for some n dividing G ; see (4.2)).

Using Hirzebruch's resolution of such a singularity (A2), we obtain a formula for the contribution μ_x of x to $\chi(\mathcal{O}_S)$ (in (A3)).

In (A4) we use arithmetic of Dedekind sums and continued fractions to prove that $\mu_x = \gamma_x$, which gives (A1).

A.2. Toric Geometry and Hirzebruch's resolution of a singularity of type (q,n)

A.2.1. Proposition

Let $T = \mathbb{A}^2/\mathbb{Z}_n$, where $\mathbb{Z}_n$ acts on $\mathbb{A}^2$ by $(x,y) \to (\varepsilon x, \varepsilon^q y)$ for some positive integer $q < n$ prime to n. Let $\pi: \mathbb{A}^2 \to T$ be the quotient map. Then

1) (Hirzebruch [H1])

There exists a resolution $f:S \to T$ of $t = \pi(0) \in T$, such that

i) $f^{-1}(t) = \bigcup_{i=1}^{r} C_i$ with C_i nonsingular rational curves.

ii) $C_i^2 = -b_i$

$C_i . C_j = 1$ if $|i-j| = 1$

$= 0$ otherwise.

The numbers b_i are integers coming from the continued fraction expansion for $\frac{n}{q}$ explained in (A2.3) and $b_i \geq 2$.

Picture: $C_0 = \pi(y=0)$, $C_{r+1} = \pi(x=0)$

2) The n-canonical divisor of S is given by

$$nK_S = f^* nK_T + \sum_{i=1}^{r} (\lambda_i + \mu_i - n)C_i \quad .$$

The numbers λ_i, μ_i are integers related to $\frac{n}{q}$ and will be defined in (A2.3).

A.2.2. <u>Background in Toric Geometry for proof of Proposition (A2.1)</u>

(See [Da])

Since $\mathbb{Z}_n$ acts on $K[x,y]$ by $\varepsilon:(x,y) \to (\varepsilon x, \varepsilon^q y)$, we have $T = \text{Spec } R$, where $R = K[x,y]^{\mathbb{Z}_n}$ is generated by the monomials $\{x^a y^b : a \geq 0,\ b \geq 0,\ \frac{(a+qb)}{n} \in \mathbb{Z}\}$.

This is expressed in [D] by writing

$$T = \text{Spec}[\sigma \cap M] \quad ,$$

where σ is the positive cone in the lattice M :

$$M = \{(a,b) \in \mathbb{R}^2 \ , \ \frac{a+qb}{n} \in \mathbb{Z}\} \quad .$$

The points of the lattice M correspond to monomials in R ; $(n,0)$ and $(0,n)$ correspond to x^n , y^n respectively.

According to ([D] §5), a toric resolution $S \to T$ corresponds to a subdivision $\Sigma \to \sigma$ of σ into cones basic for the lattice N , where

$N = \mathbb{Z}$ - dual of M

$= \{(h,k) \in \mathbb{R}^2 : ah + kb \in \mathbb{Z} \ , \quad \forall\ (a,b) \in M.\}$.

$= \mathbb{Z}.(0,1) \oplus \mathbb{Z}.(\frac{1}{n} , \frac{q}{n})$.

A.2.3. The continued fraction expansion for $\frac{n}{q}$ ([H1])

The sequences $\{\mu_0,\dots,\mu_{r+1}\}$ and $\{b_1,\dots,b_r\}$ are found simultaneously as follows:

Set $\mu_0 = n$, $\mu_1 = q$.

Given μ_{i-1} and μ_i , we define μ_{i+1} and b_i by:

$$\mu_{i+1} = b_i\mu_i - \mu_{i-1}$$

$$b_i,\ \mu_i \in \mathbb{Z}$$

$$0 \le \mu_{i+1} < \mu_i \ ,$$

continuing until $\mu_r = 1$, $\mu_{r+1} = 0$.

Notice that $b_i \geq 2$ and

$$\frac{n}{q} = b_1 - \cfrac{1}{b_2 - \cfrac{1}{b_3 - \ddots - \cfrac{1}{b_r}}} = [b_1 \dots . b_r] \ .$$

The sequence $\{\lambda_0,\dots,\lambda_{r+1}\}$ is then defined by:

$$\lambda_0 = 0 \ , \ \lambda_1 = 1$$

$$\lambda_{i+1} = \lambda_i b_i - \lambda_{i-1} \ .$$

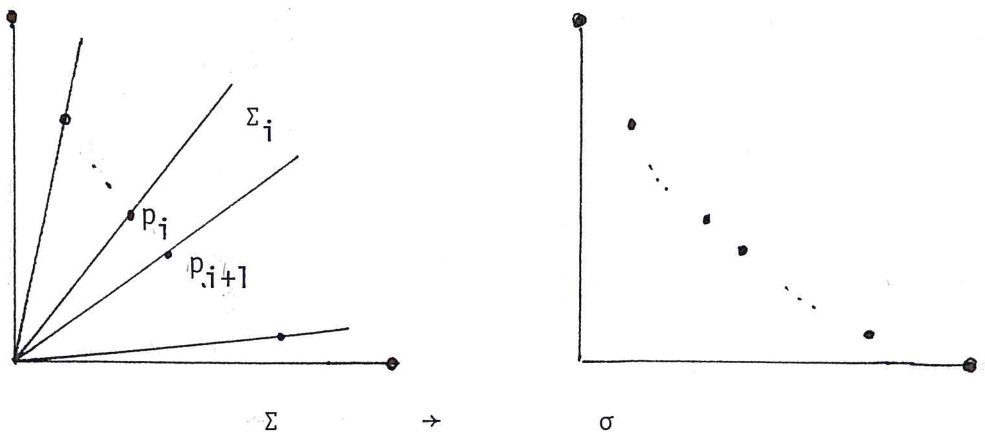
Since $b_i \geq 2$ and $\lambda_1 > \lambda_0$, we also have

$$\lambda_{i+1} > \lambda_i > 0 .$$

A.2.4. A Decomposition for σ

Claim: The cones Σ_i $(i = 0,..,r)$ spanned by p_i, p_{i+1} where $p_i = \frac{1}{n}(\lambda_i, \mu_i)$, (as in (A2.3)), are basic for the lattice N and give a decomposition of σ.

$p_0 = (0,1)$



Proof

$$\det\begin{pmatrix} \lambda_i & \mu_i \\ \lambda_{i+1} & \mu_{i+1} \end{pmatrix} = \det\begin{pmatrix} \lambda_{i-1} & \mu_{i-1} \\ \lambda_i & \mu_i \end{pmatrix}$$

by definition of λ_i, μ_i.

So $$\det\begin{pmatrix}\lambda_i & \mu_i \\ \lambda_{i+1} & \mu_{i+1}\end{pmatrix} = \det\begin{pmatrix}\lambda_0 & \mu_0 \\ \lambda_1 & \mu_1\end{pmatrix} = 1 \quad \forall\, i .$$

Hence Σ_i is basic. Since $\lambda_{i+1} > \lambda_i > 0$ and $\mu_i > \mu_{i+1} \geq 0$, the cones Σ_i give a decomposition of σ .

A.2.5. Proof of Proposition (A2.1)

Hirzebruch's resolution $f:S \to T$ corresponds to the subdivision $\Sigma = \cup\, \Sigma_i \to \sigma$.

The properties of S can be read off from the picture of Σ using the rules of toric geometry in [Da]. We get

1) i) The 1-dimensional cones spanned by the p_i's correspond to nonsingular rational curves $C_i \subset S$, and $f^{-1}(t) = \bigcup_{i=1}^{r} C_i$.

ii) Curves C_i, C_j intersect in a point if the cone spanned by p_i, p_j is in Σ , i.e. if $|i-j| = 1$, and are disjoint otherwise.

The principal divisors $\mathrm{Div}(X^n), \mathrm{Div}(Y^n)$ are given by

$$\left.\begin{aligned} \mathrm{Div}(X^n) &= \sum_{i=0}^{r+1} \lambda_i C_i \\ \mathrm{Div}(Y^n) &= \sum_{i=0}^{r+1} \mu_i C_i \end{aligned}\right\} *$$

Since $C_i.(\mathrm{Div}(X^n)) = 0$, we have

$$\lambda_i C_i^2 + \lambda_{i-1} C_i.C_{i-1} + \lambda_{i+1}\, C_i.C_{i+1} = 0 \quad .$$

Hence $\quad C_i^2 = - \dfrac{(\lambda_{i-1} + \lambda_{i+1})}{\lambda_i} = -b_i \quad .$

2) $$K_S = - \sum_{i=0}^{r+1} C_i \quad .$$

Using the relations (*) this gives

$$nK_S \sim \sum_{i=1}^{r} (\lambda_i + \mu_i - n) C_i$$

where $\sim$ denotes linear equivalence.

Since

$$nK_S = f^* nK_T + nD_X \quad , \text{ where}$$

D_X is a divisor in $\mathrm{Div}S \otimes Q$ supported on the exceptional set $f^{-1}(t) = \bigcup_{i=1}^{r} C_i$ and $nD_X \in \mathrm{Div}S$. Since the coefficients of D_X are determined by the r equations

$$C_i^2 + K\,.C_i = -2$$

from the adjunction formula, we actually have equality

$$nK_S = f^* nK_T + \sum_{i=1}^{r} (\lambda_i + \mu_i - n) C_i \quad .$$

Since $nK_T = 0$, this gives

$$nK_S = \sum_{i=1}^{r} (\lambda_i + \mu_i - n) C_i \quad .$$

This completes the proof of Proposition (A.2.1).

A.3. The contribution to $\chi(\mathcal{O}_S)$ due to a singularity $x \in X$ of type (q,n)

A.3.1. A Resolution $S \to X$

Let $X = Y/G$ as in (A1). Let $x \in X$ be a singularity of type (q,n). Since a resolution $S_x \to X$ of $x \in X$ is a sequence of blow ups at x, and these are constructed using a small analytic neighbourhood of x, we can use Hirzebruch's sequences (as in (A2)) to obtain a resolution

$$f : S \to X$$

such that

i) $f^{-1}(x) = E_x = \bigcup_{i=1}^{r} C_i$ with

intersection matrix as in (A2).

ii) $NK_S = f^*(NK_X) + N \sum_{x \in \operatorname{Sing} X} D_x$

where $N = |G|$, and D_x is as in (A2.4); $f^*(NK_X)$ is defined because NK_X is Cartier (by (7.1)).

A.3.2. The invariants K_S^2 and $e(S)$

i) K_X^2 is defined (7.2.2) and equals $\frac{1}{N} K_Y^2$.

So by above we have

$$NK_S^2 = K_Y^2 + N \sum_{x \in \mathrm{Sing}X} D_x^2$$

$$D_x = \sum_{i=1}^{r} \left(\frac{\lambda_i+\mu_i-n}{n}\right) C_i \qquad \text{(as in (A2));}$$

so
$$D_x^2 = \sum_{i=1}^{r} \left(\frac{\lambda_i+\mu_i-n}{n}\right) C_i . K_S$$

$$= \sum_{i=1}^{r} \left(\frac{\lambda_i+\mu_i-n}{n}\right) (b_i - 2) \quad .$$

$$nD_x^2 = \sum_{i=1}^{r} (\lambda_{i-1}+\lambda_{i+1} + \mu_{i-1}+\mu_{i+1}) - 2 \sum_{i=1}^{r} (\lambda_i+\mu_i)$$
$$- n \sum_{i=1}^{r} (b_i - 2)$$

$$= (\lambda_0+\mu_0+\lambda_{r+1}+\mu_{r+1}) - (\lambda_1+\mu_1+\lambda_r+\mu_r) - n \sum_{i=1}^{r} (b_i - 2) \quad .$$

To evaluate this we need to know λ_r and λ_{r+1} .

Lemma

$\lambda_r = q'$ and $\lambda_{r+1} = n$, where q' is the "*socius*" of q (defined to be the positive integer q' less than n with $qq' \equiv 1 \bmod n$).

Proof of Lemma

In (A2.4) we saw that the cones Σ_i are basic for N .

Hence
$$\det \begin{pmatrix} \lambda_r & \mu_r \\ \lambda_{r+1} & \mu_{r+1} \end{pmatrix} = n \ .$$

Since $\mu_{r+1} = 0$ and $\mu_r = 1$, this gives $\lambda_{r+1} = n$.

Since $\forall\, i$, $p_i = \frac{1}{n}(\lambda_i, \mu_i) \in N$

and $N = \mathbb{Z}.(0,1) \oplus \mathbb{Z}(\frac{1}{n}, \frac{q}{n})$, we have $\mu_i \equiv \lambda_i q \pmod n$.

In particular, $\lambda_r q \equiv \mu_r \equiv 1 \pmod n$.

Since $0 < \lambda_r < \lambda_{r+1} = n$, this implies $\lambda_r = q'$.

<u>Note</u>:

The λ's and b's in reverse order give the continued fraction expansion

$$\frac{n}{q'} = [b_r \dots b_1] .$$

The lemma gives us the following expression for D_X^2 :

$$nD_X^2 = 2n - (2+q+q'+n \sum_{i=1}^{r} (b_i - 2)) .$$

ii) The Euler number $e(S)$:

Let $X_0 = X - \operatorname{Sing} X = \pi(Y_0)$, where Y_0 = fixed point free locus for the G-action on Y.

Since e is additive, we have:

$$e(S) = e(X_0) + \sum_{x \in \operatorname{Sing} X} e(E_x) ,$$

$$Ne(X_0) = e(Y_0) ,$$

and $$e(Y) = e(Y_0) + \sum_{x \in \mathrm{Sing} X} t_x \quad ,$$

where t_x = number of points $y \in \pi^{-1}(x)$.

If x is a singularity of type (q,n) , E_x is a chain of r rational curves; hence $e(E_x) = r+1$. Also for $y \in \pi^{-1}(x)$ we have $G_y \cong \mathbb{Z}_n$; hence $t_x = \frac{N}{n}$.

Combining these gives

$$Ne(S) = e(Y) - \Sigma t_x + \Sigma Ne(E_x)$$

$$= e(Y) + \sum_{x \in \mathrm{Sing} X} e_x \quad ,$$

where if x is of type (q,n) then

$$e_x = N(r+1) - \frac{N}{n} \quad .$$

A.3.3. <u>The contribution μ_x to $\chi(\mathcal{O}_S)$</u>

Applying Noether's formula to S and Y , (A3.2) gives

$$N\chi(\mathcal{O}_S) = \frac{N}{12}(e(S) + K_S^2)$$

$$= \chi(\mathcal{O}_Y) + \sum_{x \in \mathrm{Sing} X} \mu_x \quad ,$$

where if x is of type (q,n) then $\mu_x = \frac{1}{12}(ND_x^2 + e_x)$

so $$\frac{12n}{N}\mu_X = 2n - (2+q+q'+n\sum_{i=1}^{r}(b_i-2)) + n(r+1) - 1$$

$$= 3(n-1) - (q+q' + n\sum_{i=1}^{r}(b_i-3))$$

$$= 3(n-1) - h(q,n) \quad ,$$

where $h(q,n) = q+q'+n\sum_{i=1}^{r}(b_i-3)$.

A.4. <u>Proof of Theorem</u> (A1)

By (A3.3) it suffices to prove

$$\mu_X = \gamma_X \ . \qquad (**)$$

First we expand γ_X :

Since $\forall\, y \in \pi^{-1}(x)$, $G_y \cong \mathbb{Z}_n$ acting on T_yY by $\begin{pmatrix} \varepsilon_y & 0 \\ 0 & \varepsilon_y^q \end{pmatrix}$

in suitable coordinates, we have

$$\gamma_X = \frac{N}{n}\sum_{k=1}^{(n-1)}(1-\varepsilon^k)^{-1}(1-\varepsilon^{kq})^{-1}$$

$$\frac{n}{N}\gamma_X = \sum_{k=1}^{(n-1)}\varepsilon^{-(q+1)k/2}(\varepsilon^{-k/2} - \varepsilon^{k/2})^{-1}(\varepsilon^{-kq/2} - \varepsilon^{kq/2})^{-1}$$

$$= \sum_{k=1}^{(n-1)}\frac{\cos(q+1)k\pi/n}{(-2i\sin k\pi/n)(-2i\sin kq\pi/n)}$$

(since the imaginary part vanishes)

$$= -\frac{1}{4}\sum_{k=1}^{(n-1)}\left(\cot\frac{k\pi}{n}\cot\frac{kq\pi}{n} - 1\right) \quad .$$

So $\frac{12n}{N}\gamma_X = 3(n-1) - 12ns(q,n)$,

where $s(q,n) = \frac{1}{4n}\sum_{k=1}^{(n-1)} \cot\frac{k\pi}{n}\cot\frac{kq\pi}{n}$.

By (A3.3) it now suffices to prove:

$$h(q,n) = 12ns(q,n) \quad . \qquad (*)$$

This is done in (A4.3) using the following:

A.4.1. Properties of the Dedekind Sum s

The number $s(q,n)$ defined above for any pair of coprime integers (q,n) with $n > 0$ is called the Dedekind sum. It can be defined in several ways, (see [H&Z] p.92 and [H2]), and has the following properties (see [H&Z] p.92-100 for proofs):

i) $s(q,n) = s(p,n)$ whenever $p \equiv q \pmod n$.

ii) $s(-q,n) = -s(q,n)$.

iii) Reciprocity: if q,n are both positive then

$$s(q,n) + s(n,q) = \frac{1}{12nq}(n^2 + q^2 + 1 - 3nq)$$

iv) (Alternative definition of s, used in [H&Z]):

$$s(q,n) = \sum_{k=1}^{(n-1)} \left(\left(\frac{k}{n}\right)\right)\left(\left(\frac{kq}{n}\right)\right) ,$$

where for $x \in \mathbb{R}$, $((x)) = x - [x] - \frac{1}{2}$ if $x \notin \mathbb{Z}$

$= 0$ if $x \in \mathbb{Z}$

($[x]$ denotes the integer part of x).

A.4.2. Remark

In [H2] Hirzebruch remarked that the formula for "signature defect" due to a cyclic quotient singularity (given by the G-signature theorem) could probably be proved using properties of Dedekind sums.

By converting equation (**) to (*) we are in fact switching to a calculation for signature defect.

A.4.3. Proof of (*) and hence of (A1)

For any pair q,n of coprime positive integers with $1 \le q < n$, we have

$$h(q,n) = 12n\, s(q,n) \qquad (*) .$$

Proof

We use induction on r = length of continued fraction $\frac{n}{q}$.

If $r = 1$ then $q = q' = 1$ and $b_1 = n$.

So $h(q,n) = 2 + n(n-3) = n^2 - 3n + 2$,

and $s(q,n) = \sum_{k=1}^{n-1} ((\frac{k}{n}))^2$ (by (A4.1)(iv)),

$$= \sum_{k=1}^{n-1} (\frac{k}{n} - \frac{1}{2})^2 = \sum_{1}^{n-1} \frac{k^2}{n^2} - \sum_{1}^{n-1} \frac{k}{n} + \frac{(n-1)}{4}$$

$$= \frac{(n-1)n(2n-1)}{6n^2} - \frac{n(n-1)}{2n} + \frac{(n-1)}{4} .$$

Hence $12ns(q,n) = (n-1)(n-2) = h(q,n)$.

Now assume (*) holds whenever $r < R$, and we have $\frac{n}{q} = [b_1 .. b_R]$.

Then $\frac{n}{q} = b_1 - \frac{m}{q}$, where $0 < m < q$, m and q are coprime, and $\frac{q}{m} = [b_2 ... b_R]$.

So by the inductive hypothesis,

$$h(m,q) = 12qs(m,q) .$$

Since $m \equiv -n \pmod q$, (ii) gives:

$$h(m,q) = - 12q(n,q) .$$

By the reciprocity law (A4.1 (iii)), we obtain:

$$12qns(q,n) = q^2 + n^2 + 1 - 3qn + nh(m,q) .$$

Putting $h(m,q) = m + m' + q \sum_{i=2}^{R} (b_i-3)$ and $n^2 = n(b_1q-m)$,

we have:

$$12qns(q,n) = q^2 + 1+nm' + nq \sum_{i=1}^{R} (b_i-3) \quad .$$

To complete the proof of (*) it remains to check that

$$1 + nm' = qq' \ .$$

Now $1 + nm' = 1 - mm' \equiv 0 \pmod q$, and $1 + nm' \equiv 1 \pmod n$. So $1 + nm' = q(q' + kn)$ some $k \in \mathbb{Z}$. But $0 < 1 + nm' < 1 + qn$, so $k = 0$. This proves (*) .

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